# The Segal model as a ring completion and a tensor product of permutative categories

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# Introduction

Classically, algebraic $K$-theory of rings is the study of modules over a ring $R$ and their automorphisms. This study started with the definition of functors

$$K_n\colon \mathbf{Rings} \to \mathbf{Groups}, \quad n = 0, 1, 2$$

around the sixties of the 20th century. There are various applications connecting these groups to invariants in many fields of mathematics. For instance, $K_0$ of a Dedekind domain $R$ is isomorphic to $\mathbb{Z} \oplus Cl(R)$ where $Cl(R)$ denotes the ideal class group of $R$ and $K_2$ of a field $F$ is related to the Brauer group of $F$ which classifies central simple $F$-algebras. Moreover, given a nice enough manifold $M$ with fundamental group $\pi$, the Whitehead group $Wh(\pi)$, a quotient of $K_1(\mathbb{Z}\pi)$, classifies $h$-cobordisms built on $M$.
There were several results, such as the existence of a product and an exact sequence in nice cases, suggesting that these groups should be part of a more general theory. Around 1970, Daniel Quillen succeeded in constructing a space $KR$ such that

$$\pi_n(KR) = K_n(R), \quad n = 0, 1, 2. \tag{0.1}$$

The higher $K$-groups of $R$ were consequently defined as the higher homotopy groups of $KR$. Other constructions of a space satisfying (0.1) followed (e.g. [Qui72a], [Wal85]) and were shown to be equivalent to Quillen's original one. The study of these spaces is now called *higher algebraic $K$-theory*.
Higher algebraic $K$-theory is important in many branches of mathematics. One of the most prominent conjectures involving higher $K$-theory is the Farrell-Jones Conjecture. Given a group $G$, it relates $K$-theory of the group ring $RG$ to equivariant homology of certain classifying spaces.

Comparing the different constructions of the $K$-theory space $KR$ in retrospect, the main idea bonding them is taking a symmetric monoidal category $\mathcal{C}$ and associating to it another symmetric monoidal category $\mathcal{C}'$ such that its classifying space $B\mathcal{C}'$ is the group completion of the space $B\mathcal{C}$. (In fact, this concept was used to show that the different constructions are equivalent, cf. e.g. [Gra76].) In this sense, we can talk about *algebraic $K$-theory of symmetric monoidal categories*. There are again several constructions of the category $\mathcal{C}'$ and a unified way to describe them is to associate to a given category $\mathcal{C}$ a connective $\Omega$-spectrum $Spt(\mathcal{C})$. Then, algebraic $K$-theory of $\mathcal{C}$ can be defined as

$$K_i(\mathcal{C}) = \pi_i^s(Spt(\mathcal{C})) = \pi_i(Spt(\mathcal{C})_0).$$

Robert Thomason gave an axiomatic description of the functor Spt in [Tho82]. In particular, the zeroth space of the spectrum $Spt(\mathcal{C})$, $Spt(\mathcal{C})_0$, is the group completion of $B\mathcal{C}$.

In the last decades, the theory was extended to other "ringlike" objects such as ring spectra (cf. [Wal78]) and strictly bimonoidal categories (cf. [BDR04]). Recently, Nils Baas, Bjørn Dundas, Birgit Richter and John Rognes showed that algebraic $K$-theory of strictly bimonoidal categories is equivalent to $K$-theory of the associated ring spectra (cf. [BDRRb], Theorem 1.1). This establishes a connection between cohomology theories (spectra) and geometric interpretation (categories). For instance, their motivating example is the category of finite dimensional complex vector spaces $\mathcal{V}$. The space $|BGL_n(\mathcal{V})|$ classifies 2-vector bundles of rank $n$ and $K(\mathcal{V})$ is the algebraic $K$-theory of the 2-category of 2-vector spaces (cf. [BDR04]). Their theorem establishes an equivalence between $K(\mathcal{V})$ and $K(ku)$ where $ku$ is the connective complex $K$-theory spectrum with $\pi_*(ku) = \mathbb{Z}[u]$, $|u| = 2$.
A keypoint in the proof of this theorem is the notion of a multiplicative group completion. To construct this, the above-named use a version of the Grayson-Quillen model which requires certain conditions on the category they are working with. We present a different model of a multiplicative group completion that does not require these conditions.

Algebraic $K$-theory is very hard to compute. One way to do it is to make use of a so called trace map from $K$-theory to (topological) Hochschild homology. Having established a good algebraic $K$-theory of strictly bimonoidal categories, one is tempted to ask for a model of Hochschild homology of strictly bimonoidal categories that would simplify trace map-calculations. To be more precise: We have a trace map in mind that models the classical one for rings in appropriate cases and is more accessible than the existing ones (see for example [BHM93] and [Dun00]).
However, when working on this we were missing a key ingredient: a tensor product of

permutative categories. People have been thinking about it for quite a while (cf. [Tho95], Introduction and [EM06], Introduction), but to our knowledge there is no elaborate treatment of this subject in the literature. John Gray's tensor product of 2-categories does apply to strict monoidal categories but his construction is not very explicit and many questions remain unanswered. Regrettably, the tensor product we construct does not help to define Hochschild homology since it does not support a reasonable multiplicative structure.

**Outline**

The first chapter is the theoretical foundation of this thesis. We explain monoidal and bimonoidal categories and examples we will refer to later on. The concept of a free permutative category (Definition 1.14) will be very important in our definition of the tensor product of permutative categories. Furthermore, we establish notation we use throughout this thesis.

In chapter two, we specify most of what we mentioned in the introduction. Algebraic $K$-theory of monoidal and strictly bimonoidal categories is explained, the Grayson-Quillen model in particular. Moreover, we define the term group completion (Definition 2.2). We end the chapter with citing the theorem of Baas, Dundas, Richter and Rognes, that connects $K$-theory of strictly bimonoidal categories with $K$-theory of ring spectra.

Chapter three is dedicated to the Segal model. In the first section, we explain the idea of a graded category which is vital to the notion of a multiplicative group completion. In section two and three, we define the model and prove its main properties. Finally, section four contains the proof that the Segal model defines a multiplicative group completion (Theorem 3.23). Moreover, we show that it is equivalent to the Grayson-Quillen model in appropriate cases (Prop. 3.24).

The tensor product of permutative categories is developed in the fourth chapter. We start with a discussion of quotient categories which are crucial in the construction of the tensor product. In the second section, we define the tensor product and prove its main properties. In particular, the tensor product fulfills a universal property with respect to certain bifunctors (Prop. 4.12). This universal property discloses the main flaws of the tensor product (cf. discussion after Prop. 4.12). We continue with a comparison of our tensor product to those defined by John Gray and Anthony Elmendorf and Michael Mandell respectively. The conclusion at the end of the chapter comprehends a proposal for an alternative ansatz.

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# 1 Bimonoidal categories

Monoidal and bimonoidal categories are the basis of all constructions in this thesis and thus this first chapter serves to recall their main features and to establish notation. Moreover, we explain examples which we will refer to in the sequel. The important definition of the free permutative category on a category $\mathcal{C}$ can be found at the end of this chapter.
The main source are sections VII and XI in [Mac98], others are stated explicitly.

Conventions: Throughout this thesis, all categories are assumed to be small - unless blatantly otherwise - and for an object $c$ in a category $\mathcal{C}$ we often use the notation $c \in \mathcal{C}$.

**Definition 1.1.** A monoidal category is a category $\mathcal{C}$ together with a bifunctor $\Box\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, an object $e \in \mathcal{C}$ and three natural isomorphisms

$$\alpha = \alpha_{a,b,c}\colon a\Box(b\Box c) \cong (a\Box b)\Box c,$$
$$\lambda = \lambda_a\colon e\Box a \cong a,\ \ \rho = \rho_a\colon a\Box e \cong a$$

for all $a, b, c \in \mathcal{C}$ such that the pentagonal diagram

$$\begin{tikzcd}
a\Box(b\Box(c\Box d)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_a\Box\alpha"'] & (a\Box b)\Box(c\Box d) \arrow[r, "\alpha"] & ((a\Box b)\Box c)\Box d \\
a\Box((b\Box c)\Box d) \arrow[rr, "\alpha"] & & (a\Box(b\Box c))\Box d \arrow[u, "\alpha\Box\mathrm{id}_d"']
\end{tikzcd}$$

and the triangular diagram

$$\begin{tikzcd}
a\Box(e\Box c) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_a\Box\lambda"'] & (a\Box e)\Box c \arrow[d, "\rho\Box\mathrm{id}_c"] \\
a\Box c \arrow[r, equal] & a\Box c
\end{tikzcd}$$

commute for all $a, b, c, d \in \mathcal{C}$ and such that $\lambda_e = \rho_e\colon e\Box e \cong e$.

**Definition 1.2.** A monoidal category $\mathcal{C}$ is called symmetric if it is equipped with natural isomorphisms

$$\gamma = \gamma_{a,b} \colon a \Box b \cong b \Box a$$

for all $a, b \in \mathcal{C}$ such that $\gamma_{a,b} \circ \gamma_{b,a} = \mathrm{id}_{b\Box a}$, $\rho_b = \lambda_b \circ \gamma_{b,e} \colon b\Box e \cong b$ and the diagram

$$\begin{tikzcd}
a\Box(b\Box c) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_a\Box\gamma_{b,c}"'] & (a\Box b)\Box c \arrow[r, "\gamma_{a\Box b,c}"] & c\Box(a\Box b) \arrow[d, "\alpha"] \\
a\Box(c\Box b) \arrow[r, "\alpha"] & (a\Box c)\Box b \arrow[r, "\gamma_{a,c}\Box\mathrm{id}_b"] & (c\Box a)\Box b
\end{tikzcd}$$

commutes for all $a, b, c \in \mathcal{C}$. These conditions imply that $\alpha$ and $\gamma$ are coherent (cf. [Mac63], section 4).

**Definition 1.3.** A lax monoidal functor is a functor $F \colon \mathcal{C} \to \mathcal{C}'$ between monoidal categories together with morphisms

$$\phi = \phi_{a,b} \colon F(a)\Box F(b) \longrightarrow F(a\Box b), \quad \psi \colon e_{\mathcal{C}'} \longrightarrow F(e_{\mathcal{C}})$$

in $\mathcal{C}'$ that are natural in $a, b \in \mathcal{C}$ and such that the diagrams

$$\begin{tikzcd}
e\Box F(a) \arrow[r, "\psi\Box\mathrm{id}"] \arrow[d, "\lambda"'] & F(e)\Box F(a) \arrow[d, "\phi"] \\
F(a) & F(e\Box a), \arrow[l, "F(\lambda)"']
\end{tikzcd}$$

$$\begin{tikzcd}
(F(a)\Box F(b))\Box F(c) \arrow[r, "\phi\Box\mathrm{id}"] \arrow[d, "\alpha"'] & F(a\Box b)\Box F(c) \arrow[r, "\phi"] & F((a\Box b)\Box c) \arrow[d, "F(\alpha)"] \\
F(a)\Box(F(b)\Box F(c)) \arrow[r, "\mathrm{id}\Box\phi"] & F(a)\Box F(b\Box c) \arrow[r, "\phi"] & F(a\Box(b\Box c))
\end{tikzcd}$$

commute for all $a, b, c \in \mathcal{C}$. We call $F$ a strong (strict) monoidal functor if $\phi$ and $\psi$ are isomorphisms (identities).

**Definition 1.4.** A lax (strong, strict) symmetric monoidal functor is a lax (strong, strict) monoidal functor $F \colon \mathcal{C} \to \mathcal{C}'$ such that in addition the diagram

$$\begin{tikzcd}
F(a)\Box F(b) \arrow[r, "\phi"] \arrow[d, "\gamma"'] & F(a\Box b) \arrow[d, "F(\gamma)"] \\
F(b)\Box F(a) \arrow[r, "\phi"] & F(b\Box a)
\end{tikzcd}$$

commutes for all $a, b \in \mathcal{C}$.

**Definition 1.5.** A symmetric monoidal natural transformation $\eta\colon F \to G$ of lax symmetric monoidal functors is a natural transformation such that the following diagrams commute for all $a, b \in \mathcal{C}$:

$$\begin{tikzcd} F(a)\Box F(b) \arrow[r, "\phi"] \arrow[d, "\eta\Box\eta"'] & F(a\Box b) \arrow[d, "\eta"] \\ G(a)\Box G(b) \arrow[r] & G(a\Box b), \end{tikzcd} \qquad \begin{tikzcd} & F(e) \arrow[d, "\eta"] \\ e \arrow[ur, "\psi"] \arrow[r, "\psi"'] & G(e). \end{tikzcd}$$

**Proposition 1.6** ([Seg74], sections 1+2.)**.** *The monoidal structure on a symmetric monoidal category $\mathcal{C}$ induces a continous map $B\mathcal{C} \times B\mathcal{C} \to B\mathcal{C}$ that turns its classifying space into a homotopy-associative and homotopy-commutative $H$-space.*

**Definition 1.7.** A permutative category is a symmetric monoidal category $\mathcal{C}$ with $\alpha = \mathrm{id}$, i.e. $a\Box(b\Box c) = (a\Box b)\Box c$ (strict associativity) and $\lambda = \rho = \mathrm{id}$, i.e. $e\Box a = a\Box e = a$ (strict unit) such that the diagrams

$$\begin{tikzcd} a\Box e \arrow[rr, "\gamma_{a,e}"] \arrow[dr, "="'] & & e\Box a \arrow[dl, "="] \\ & a & \end{tikzcd}, \qquad \begin{tikzcd} a\Box b\Box c \arrow[rr, "\gamma_{a\Box b,c}"] \arrow[dr, "\mathrm{id}_a\Box\gamma_{b,c}"'] & & c\Box a\Box b \\ & a\Box c\Box b \arrow[ur, "\gamma_{a,c}\Box\mathrm{id}_b"'] & \end{tikzcd}$$

commute for all $a, b, c \in \mathcal{C}$.

Any symmetric monoidal category is naturally equivalent to a permutative one (see [May77], VI, Prop. 3.2).

We denote the category of permutative categories and lax/strong/strict symmetric monoidal functors with **Perm/Strong/Strict**.

**Example 1.8.** We consider two categories of finite sets: $\mathbf{I}$ and $\Sigma$. In both cases, the objects are given by $[0] = \emptyset$ and finite sets $[n] = \{1, 2, \ldots, n\}$ without basepoint. Morphisms are

- injective maps in $\mathbf{I}$ and
- permutations in $\Sigma$, i.e. $\Sigma([m],[n]) = \begin{cases} \Sigma_n & m = n, \\ \emptyset & \text{otherwise.} \end{cases}$

The permutative structure is given by disjoint union, that is $[m] \oplus [n] = [m+n]$, with unit $[0]$ and twist

$$\gamma^{\oplus}_{m,n} = \chi^{m,n}(i) = \begin{cases} n+i & i \leq m, \\ i-m & i > m. \end{cases}$$

**Definition 1.9.** A strictly bimonoidal category is a permutative category $(\mathcal{R}, \oplus, 0, \gamma)$ together with a second strict monoidal structure $(\mathcal{R}, \otimes, 1)$, natural identities

$$\lambda^* = \lambda^*_a \colon 0 \otimes a = 0, \quad \rho^* = \rho^*_a \colon a \otimes 0 = 0$$

for all $a \in \mathcal{R}$ and natural distributivity maps

$$\delta_l = \delta_{l\,a,b,c} \colon (a \otimes b) \oplus (a \otimes c) \longrightarrow a \otimes (b \oplus c),$$
$$\delta_r = \delta_{r\,a,b,c} \colon (a \otimes c) \oplus (b \otimes c) \longrightarrow (a \oplus b) \otimes c$$

for all $a, b, c \in \mathcal{R}$ which we require to be an isomorphism in case of $\delta_l$ and an identity in case of $\delta_r$. Morevover, these maps are subject to appropriate coherence conditions which can be taken from Definition 3.2 in the case of a $*$-graded category with $*$ being the one-point category.

**Definition 1.10.** A symmetric bimonoidal category is a symmetric monoidal category $(\mathcal{R}, \oplus, 0, \gamma^{\oplus})$ together with a second symmetric monoidal structure $(\mathcal{R}, \otimes, 1, \gamma^{\otimes})$, natural isomorphisms $\lambda^*, \rho^*$ and natural distributivity isomorphisms $\delta_l, \delta_r$ which are subject to several coherence conditions spelled out in [Lap72], pp. $31-35$.

**Definition 1.11.** A bipermutative category is a permutative category $(\mathcal{R}, \oplus, 0, \gamma^{\oplus})$ together with a second permutative structure $(\mathcal{R}, \otimes, 1, \gamma^{\otimes})$ such that there are natural identities $0 \otimes a = 0 = a \otimes 0$ for all $a \in \mathcal{R}$, right distributivity holds strictly and left distributivity is defined by use of the following diagram:

$$\begin{tikzcd} (a \otimes c) \oplus (b \otimes c) \arrow[r, "\gamma^{\otimes} \oplus \gamma^{\otimes}"] \arrow[d, "\delta_r"'] & (c \otimes a) \oplus (c \otimes b) \arrow[d, "\delta_l"] \\ (a \oplus b) \otimes c \arrow[r, "\gamma^{\otimes}"] & c \otimes (a \oplus b). \end{tikzcd} \tag{1.1}$$

The necessary coherence conditions can be taken from Definiton 3.1 in the case of a $*$-graded category.

Unless the twist map is an identity, the definiton of $\delta_l$ in terms of $\delta_r = \mathrm{id}$ and $\gamma$ implies that $\delta_l$ is usually not an identitity map.

Any symmetric bimonoidal category is equivalent to a bipermutative one (see [May77], VI, Prop. 3.5).

**Example 1.12.** Let $\Sigma$ be the category of finite sets and permutations with permutative structure given by $(\oplus, [0], \gamma^{\oplus})$ as above. There is a second permutative structure given by $[m] \otimes [n] = [mn]$ with unit $[1]$ and twist

$$\gamma^{\otimes}_{m,n}((i-1)n + j) = (j-1)m + i, \; \forall\, 0 \leq i \leq m, \; 1 \leq j \leq n.$$

Permutations $\sigma \otimes \tau \colon [mn] \to [mn]$ are defined as

$$(\sigma \otimes \tau)((i-1)n + j) = (\sigma(i) - 1)n + \tau(j), \; \forall\, 0 \leq i \leq m, \; 1 \leq j \leq n,$$

which corresponds to $(i, j) \mapsto (\sigma(i), \tau(j))$ if we think of the object $[mn]$ as $\{(1,1), \dots, (1,n), \dots, (m,1), \dots, (m,n)\}$.

**Example 1.13.** The (topological) bipermutative category of finite dimensional complex vector spaces, $\mathcal{V}_{\mathbb{C}}$, is defined as follows: Objects are given by the set $\mathbb{N} = \{0, 1, 2, \dots\}$ with $d \in \mathbb{N}$ interpreted as the complex vector space $\mathbb{C}^d$. Morphisms from $d$ to $e$ are given by the space

$$\mathcal{V}_{\mathbb{C}}(d, e) = \begin{cases} U(d) & \text{if } d = e, \\ \emptyset & \text{otherwise.} \end{cases}$$

The sum functor $\oplus$ takes $(d, e)$ to $d + e$ and embeds $U(d) \times U(e)$ into $U(d+e)$ by the block sum of matrices. The tensor functor $\otimes$ takes $(d, e)$ to $de$ and maps $U(d) \times U(e)$ to $U(de)$ via the tensor product of matrices by identifying $\{1, \dots, d\} \times \{1, \dots, e\}$ with $\{1, \dots, de\}$ by means of the left lexicographic ordering. The zero and unit objects are 0 and 1 respectively.

There is another variant of $\mathcal{V}_{\mathbb{C}}$ with

$$\mathcal{V}_{\mathbb{C}}(d, e) = \begin{cases} GL_d(\mathbb{C}) & \text{if } d = e, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $BU(d) \simeq BGL_d(\mathbb{C})$, both versions are equivalent.

The following definition is taken from [Tho82].

**Definition 1.14.** Let $\mathcal{C}$ be a small category. The free permutative category on $\mathcal{C}$, $P\mathcal{C}$, has as objects all entities $n[c_1, \dots, c_n]$ where $n$ is a natural number and $c_1, \dots, c_n$ are objects of $\mathcal{C}$. A morphism $n[c_1, \dots, c_n] \to n'[c'_1, \dots, c'_{n'}]$ exists only if $n = n'$ and then is specified as $\sigma[f_1, \dots, f_n]$ where $\sigma \in \Sigma_n$ and each $f_i \colon c_i \to c'_{\sigma(i)}$ is a morphism in $\mathcal{C}$. Composition is induced by composition in $\mathcal{C}$ and $\Sigma_n$. Thus, there is the formula

$$\tau[f'_1, \dots, f'_n] \circ \sigma[f_1, \dots, f_n] = \tau\sigma[f'_{\sigma(1)} f_1, \dots, f'_{\sigma(n)} f_n].$$

$P\mathcal{C}$ is a permutative category with $\Box = +$ given by

$$n[c_1, \dots, c_n] + k[d_1, \dots, d_k] = (n+k)[c_1, \dots, c_n, d_1, \dots, d_k]$$

and unit element $0[\,]$. The twist map

$$\gamma \colon n[c_1, \dots, c_n] + k[d_1, \dots, d_k] \xrightarrow{\cong} k[d_1, \dots, d_k] + n[c_1, \dots, c_n]$$

is $\chi(n,k)[\mathrm{id}, \dots, \mathrm{id}]$ where $\chi(n,k) \colon [n+k] \to [n+k]$ is the permutation that shuffles the first $n$ elements to the last $n$ and the last $k$ elements to the first $k$.

One word on notation: We think of $n[\quad]$ as the $n$-ary operation which is built up by iterated $+$'s in a permutative category and of $n[c_1, \dots, c_n]$ as the result of applying this operation to the objects $c_1 = 1[c_1], \dots, c_n = 1[c_n]$. In different words, we think of $n[c_1, \dots, c_n]$ as a finite sum and this is why we use $+$.

Note that there is a canonical embedding $j \colon \mathcal{C} \to P\mathcal{C}$ sending an object $c$ to $1[c]$. This embedding induces a bijection

$$\mathbf{Strict}(P\mathcal{C}, \mathcal{D}) \cong \mathbf{Cat}(\mathcal{C}, U\mathcal{D}),\ F \mapsto F \circ j.$$

Hence, the functor $P \colon \mathbf{Cat} \to \mathbf{Strict}$, sending a category $\mathcal{C}$ to $P\mathcal{C}$, is left adjoint to the forgetful functor $U$. In particular, the composite $UP$ is a monad on $\mathbf{Cat}$ with $\mathbf{Strict}$ being equivalent to the category of $UP$-algebras (cf. [Tho82] and [Mac98], VI, for an introduction to monads in a category).

# 2 Algebraic K-theory

We give a short overview of the beginning of algebraic $K$-theory and then concentrate on a recent development, algebraic $K$-theory of strictly bimonoidal categories.

## 2.1 Classical constructions

Throughout this section, $R$ is an associative ring with unit.

**Definition 2.1.** Let $M$ be a commutative semi-group and consider $M \times M$ with coordinate-wise addition. Define the Grothendieck group $\mathrm{Gr}M = M \times M/\sim$ where $(m_1, m_2) \sim (n_1, n_2)$ if for some $k \in M$, $m_1 + n_2 + k = m_2 + n_1 + k$. Thus, the identity element is of the form $(m, m)$ and the inverse of $(m_1, m_2)$ is $(m_2, m_1)$.

The Grothendieck group fulfills a universal property: There exists a monoid homomorphism $i \colon M \to \mathrm{Gr}M$ such that for any monoid homomorphism $f \colon M \to A$ from the commutative monoid $M$ to an abelian group $A$, there is a unique group homomorphism $g \colon \mathrm{Gr}M \to A$ such that $f = gi$.

Consider the monoid $Proj(R)$ of isomorphism classes of finitely generated projective $R$-modules, together with direct sum and identity 0. Then $K_0(R)$ is the Grothendieck group of this monoid,

$$K_0(R) = \mathrm{Gr}(Proj(R), \oplus).$$

For $i = 1, 2$, the groups $K_i$ study the automorphism group of such modules. The first $K$-group

$$K_1(R) = GL(R)/E(R),$$

is the abelianisation of $GL(R)$ where $E(R)$ denotes the commutator subgroup of $GL(R)$ which is generated by the elementary matrices. The group $E(R)$ has a universal central extension, the Steinberg Group, $St(R)$, together with a surjective homomorphism

$\varphi\colon St(R) \to E(R)$. The second $K$-group is given by the kernel of this map,

$$K_2(R) = ker\varphi,$$

and consists of the non-trivial relations between elementary matrices.

As mentioned in the introduction, there are different constructions of the algebraic $K$-theory space associated to $R$. We present the one that serves best to motivate the upcoming definition of algebraic $K$-theory of strictly bimonoidal categories. It is based on the notion of the group completion of an $H$-space which is a generalization of the concept of the universal group associated to a monoid.

Michael G. Barratt and Stewart Priddy introduced the following concept of group completion in [BP72]. It was generalized and developed further by many authors (e.g. [May74]) and the following formulation is taken from [Wei].

**Definition 2.2.** Let $X$ be a homotopy-commutative, homotopy-associative $H$-space. A group completion of $X$ is an $H$-space $Y$ together with an $H$-space map $X \to Y$, such that $\pi_0(Y) = \mathrm{Gr}\pi_0(X)$ and the homology ring $H_*(Y;k)$ is isomorphic to the localization $\pi_0(X)^{-1}H_*(X;k)$ of $H_*(X;k)$ by $\pi_0(X) \to H_0(X;\mathbb{Z}) = \mathbb{Z}[\pi_0(X)] \to H_0(X;k)$ for all commutative rings $k$.

**Theorem 2.3.** *Let $M$ be a homotopy-commutative topological monoid. Then the map of $H$-spaces*

$$\iota\colon M \longrightarrow \Omega BM$$

*is a group completion. If $\pi_0(M)$ is a group, the map $\iota$ is a homotopy equivalence.*

(Cf. [Aus01] or - for a full discussion - [Ada78], §3.2.)
Consequently, $M$ is its own group completion if it is group-like (i.e., $\pi_0(M)$ is a group).

If the classifying space of a symmetric monoidal category $\mathcal{S}$ is group-like, we call $\mathcal{S}$ group complete.

The first definition of higher $K$-theory of rings is due to Quillen (cf. [Qui72b]). The following is a reformulation of his definition and §3.2 in [Ada78] contains a nice discussion of why both formulations are equivalent.
Consider the following category: The objects are given by based free $R$-modules

$\{0, R, R^2, \dots, R^n, \dots\}$. There are no morphisms $R^m \to R^n$ for $m \neq n$ and the self-maps of $R^n$ are given by $GL_n(R)$. The classifying space of this symmetric monoidal category is equivalent to $\coprod_{n\geq 0} BGL_n(R)$.

**Definition 2.4.** The algebraic $K$-theory space of $R$ is the group completion of the monoid $\coprod_{n\geq 0} BGL_n(R)$ and its $K$-groups are the homotopy groups of this space:

$$K_i R = \pi_i \Omega B \coprod_{n\geq 0} BGL_n(R).$$

Note that this definition gives a different $K_0R$ than the one we defined in the beginning. Here, $K_0R = Gr\pi_0(\coprod_{n\geq 0} BGL_n(R)) = \mathbb{Z}$ for all rings $R$. The higher homotopy groups agree with those defined above.

As explained in the introduction, this concept of taking a symmetric monoidal category $\mathcal{C}$ and associating to it another category $\mathcal{C}'$ such that $B\mathcal{C}'$ is the group completion of $B\mathcal{C}$ developed into what we now call algebraic $K$-theory of symmetric monoidal categories. There are different constructions of the category $\mathcal{C}'$. The one we present was written down by Daniel Grayson and inspired by Daniel Quillen (cf. [Gra76]). Hence, we refer to it as the Grayson-Quillen model.

**Definition 2.5.** Let $(\mathcal{C}, \oplus, 0)$ be a symmetric monoidal category. Then $(-\mathcal{C})\mathcal{C}$ is defined to be the category whose objects are pairs $(a, b)$ of objects in $\mathcal{C}$ and whose morphisms $(a, b) \to (c, d)$ are equivalence classes $(s, \alpha, \beta)$ consisting of an object $s \in \mathcal{C}$ and morphisms

$$\alpha\colon a \oplus s \longrightarrow c,\ \ \beta\colon b \oplus s \longrightarrow d.$$

Two morphisms $(s, \alpha, \beta), (s', \alpha', \beta')$ are equivalent if and only if there is an isomorphism $\gamma\colon s \to s'$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} a \oplus s \arrow[d, "\mathrm{id}_a \oplus \gamma"'] \arrow[dr, "\alpha"] & \\ a \oplus s' \arrow[r, "\alpha'"'] & c, \end{tikzcd} \qquad \begin{tikzcd} b \oplus s \arrow[d, "\mathrm{id}_b \oplus \gamma"'] \arrow[dr, "\beta"] & \\ b \oplus s' \arrow[r, "\beta'"'] & d. \end{tikzcd}$$

The composition of $(s, \alpha, \beta)\colon (a, b) \to (c, d)$ and $(t, \gamma, \delta)\colon (c, d) \to (e, f)$ is given by

$$(s \oplus t,\ \gamma \circ (\alpha \oplus \mathrm{id}_t),\ \delta \circ (\beta \oplus \mathrm{id}_t)).$$

There is a functor $i\colon \mathcal{C} \to (-\mathcal{C})\mathcal{C}$ given by $i(a) = (0, a)$ on objects and $i(\alpha) = (0, \rho_0, \alpha \circ \rho_a)$ on morphisms.

**Proposition 2.6.** *Let $\mathcal{C}$ be a symmetric monoidal category such that every morphism is an isomorphism and the translation functor $x \oplus \_\colon \mathcal{C} \to \mathcal{C}$ is faithful for every object $x \in \mathcal{C}$. Then $i\colon \mathcal{C} \to (-\mathcal{C})\mathcal{C}$ induces a group completion $B\mathcal{C} \to B(-\mathcal{C})\mathcal{C}$.*

We now proceed with algebraic $K$-theory of strictly bimonoidal categories.

## 2.2 $K$-theory of strictly bimonoidal categories

In [BDR04], Baas, Dundas and Rognes give a definition of the K-theory of a strictly bimonoidal category. The following presentation is taken from [BDRRb].

Let $\mathcal{R}$ be a strictly bimonoidal category.

**Definition 2.7.** The category of $n \times n$-matrices over $\mathcal{R}$, $M_n(\mathcal{R})$, is defined as follows. The objects of $M_n(\mathcal{R})$ are matrices $X = (X_{i,j})_{i,j=1}^n$ of objects of $\mathcal{R}$ and morphisms from $X = (X_{i,j})_{i,j=1}^n$ to $Y = (Y_{i,j})_{i,j=1}^n$ are matrices $f = (f_{i,j})_{i,j=1}^n$ where each $f_{i,j}$ is a morphism in $\mathcal{R}$ from $X_{i,j}$ to $Y_{i,j}$.

**Lemma 2.8.** *For a strictly bimonoidal category $(\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus, \otimes, 1_\mathcal{R})$ the category $M_n(\mathcal{R})$ is a monoidal category with respect to the matrix multiplication bifunctor*

$$M_n(\mathcal{R}) \times M_n(\mathcal{R}) \xrightarrow{\cdot} M_n(\mathcal{R}),\ (X_{i,j})_{i,j=1}^n \cdot (Y_{i,j})_{i,j=1}^n = (Z_{i,j})_{i,j=1}^n$$

*with*

$$Z_{i,j} = \bigoplus_{k=1}^n X_{i,j} \otimes Y_{k,j}.$$

*The unit of this structure is given by the unit matrix object $E_n$ which has $1_\mathcal{R} \in \mathcal{R}$ as diagonal entries and $0_\mathcal{R}$ in all other places.*

The property of $\mathcal{R}$ being bimonoidal gives $\pi_0\mathcal{R}$ the structure of a ring without negative elements and its additive group completion $Gr(\pi_0\mathcal{R})$ is a ring.

**Definition 2.9.** The weakly invertible $n \times n$-matrices over $\pi_0\mathcal{R}$, $GL_n(\pi_0\mathcal{R})$, are defined as the $n \times n$-matrices over $\pi_0\mathcal{R}$ that are invertible as matrices over $Gr(\pi_0\mathcal{R})$. In other words,

$GL_n(\pi_0\mathcal{R})$ can be defined by the pullback square

$$\begin{tikzcd} GL_n(\pi_0\mathcal{R}) \arrow[r] \arrow[d] & GL_n(Gr(\pi_0\mathcal{R})) \arrow[d] \\ M_n(\pi_0\mathcal{R}) \arrow[r] & M_n(Gr(\pi_0\mathcal{R})). \end{tikzcd}$$

**Definition 2.10.** The category of weakly invertible $n \times n$-matrices over $\mathcal{R}$, $GL_n(\mathcal{R})$, is the full subcategory of $M_n(\mathcal{R})$ with objects all matrices $X = (X_{i,j})_{i,j=1}^n \in M_n(\mathcal{R})$ whose matrix of $\pi_0$-classes $|X| = (|X_{i,j}|)_{i,j=1}^n$ is contained in $GL_n(\pi_0\mathcal{R})$.

The category $GL_n(\mathcal{R})$ inherits a monoidal structure from $M_n(\mathcal{R})$ since matrix multiplication is compatible with the property of being weakly invertible.

There is a canonical stabilization functor $GL_n(\mathcal{R}) \to GL_{n+1}(\mathcal{R})$ which is induced by taking the block sum with $E_1 \in GL_1(\mathcal{R})$. Let $GL(\mathcal{R})$ be the sequential colimit of the categories $GL_n(\mathcal{R})$.

**Definition 2.11.** Let $(\mathcal{C}, \cdot, 1)$ be a monoidal category. The bar construction $B_\bullet\mathcal{C}$ is a simplicial category $[q] \to B_q\mathcal{C}$. The category $B_q\mathcal{C}$ has objects consisting of

- an object $c^{ij} \in \mathcal{C}$ for each $0 \leq i < j \leq q$,
- an isomorphism $\phi^{ijk}\colon c^{ij} \cdot c^{jk} \to c^{ik}$ in $\mathcal{C}$ for each $0 \leq i < j < k \leq q$, such that for $\alpha\colon (c^{ij} \cdot c^{jk}) \cdot c^{kl} \xrightarrow{\cong} c^{ij} \cdot (c^{jk} \cdot c^{kl})$ the diagram

$$\begin{tikzcd} (c^{ij} \cdot c^{jk}) \cdot c^{kl} \arrow[rr, "\alpha"] \arrow[d, "\phi^{ijk}\cdot\mathrm{id}"'] & & c^{ij} \cdot (c^{jk} \cdot c^{kl}) \arrow[d, "\mathrm{id}\cdot\phi^{jkl}"] \\ c^{ik} \cdot c^{kl} \arrow[r, "\phi^{ikl}"] & c^{il} & c^{ij} \cdot c^{jl} \arrow[l, "\phi^{ijl}"'] \end{tikzcd}$$

  commutes for all $i < j < k < l$.

A morphism $f\colon \{c, \phi\} \to \{d, \psi\}$ of $B_q\mathcal{C}$ consists of morphisms $f^{ij}\colon c^{ij} \to d^{ij}$ in $\mathcal{C}$ for $0 \leq i < j \leq q$ such that $\psi^{ijk} \circ (f^{ij} \cdot f^{jk}) = f^{ik} \circ \phi^{ijk}$.
For $\varphi\colon [q] \to [p] \in \Delta$, the functor $B_p\mathcal{C} \to B_q\mathcal{C}$ is obtained by precomposing with $\varphi$. For instance, $d_1(c)$ is gotten by deleting all entries with indices containing 1 from the data giving $c$. In order to allow for degenary maps $s_i$, we use the convention that all objects of the form $c_{ii}$ are the unit of the monoidal structure and all isomorphisms of the form $\phi^{iik}, \phi^{ikk}$ are identities.

The following is a corollary of Lemma 3.3 in [BDRRb].

**Lemma 2.12.** *For each $q$ there is an equivalence of categories between $B_q\mathcal{C}$ and the product category $\mathcal{C}^q$ where the map $B_q\mathcal{C} \to \mathcal{C}^q$ is given by sending an object $c \in B_q\mathcal{C}$ to the diagonal $(c^{0,1}, c^{1,2}, \ldots, c^{q-1,q})$.*

**Definition 2.13.** Let $\mathcal{R}$ be a strictly bimonoidal category. The algebraic $K$-theory space of $\mathcal{R}$ is defined as

$$K(\mathcal{R}) = \Omega B(\coprod_{n\geq 0} |B_\bullet GL_n(\mathcal{R})|).$$

The coproduct $\coprod_{n\geq 0} |B_\bullet GL_n(\mathcal{R})|$ is a topological monoid, where the monoidal structure is induced by the block sum of matrices $GL_n(\mathcal{R}) \times GL_m(\mathcal{R}) \to GL_{n+m}(\mathcal{R})$. The looped bar construction provides a group completion of this topological monoid.

Given a strictly bimonoidal category $\mathcal{R}$, the associated spectrum $Spt(\mathcal{R})$ is in fact a strict ring spectrum. To be precise, there is a model of the functor $Spt$ such that this statement is true. Main references are [EM06], [May77] and [May09].

**Theorem 2.14** (Theorem 1.1 in [BDRRb])**.** *The $K$-theory space of a small topological strictly bimonoidal category $\mathcal{R}$, defined as above, is equivalent to the $K$-theory space of the ring spectrum $Spt(\mathcal{R})$, i.e.*

$$K(\mathcal{R}) \simeq K(Spt(\mathcal{R}))$$

*provided that $\mathcal{R}$ is a groupoid and the translation functor $X \oplus (\ )$ is faithful for every object $X \in \mathcal{R}$.*

This theorem provides possible geometric interpretations for cohomology theories which are highly desired.

A key point in the proof of this theorem is the notion of a multiplicative group completion. For a long time, it has been an open problem if there was a group completion that does not destroy an existing multiplication. Supposed solutions based on the Grayson-Quillen model turned out to be wrong (cf. [Tho80]).
In [BDRRa], Baas, Dundas, Richter and Rognes give a solution to this problem. The main idea is to consider a category that is graded in a certain sense. In the next chapter, we will explain this concept and give an example of a multiplicative group completion.

# 3 A multiplicative group completion

## 3.1 Graded categories

The following definitions are taken from [BDRRa], section 2. We suggest to think of a graded ring if one has not seen these definitions of graded categories before.

**Definition 3.1.** Let $(\mathbf{J}, +, 0, \chi)$ be a small permutative category. A $\mathbf{J}$-graded bipermutative category is a functor $X$ from $\mathbf{J}$ to the category **Strict** of small permutative categories and strict symmetric monoidal functors, together with the following data and subject to the following conditions where we denote the permutative structure of $X(i)$ by $(X(i), \oplus, 0_i, \gamma_\oplus)$:

1. A functor $\otimes\colon X(i) \times X(j) \to X(i+j)$ such that for any pair of morphisms $\phi\colon i \to k,\ \psi\colon j \to l$ in $\mathbf{J}$ the following diagram commutes:

$$\begin{tikzcd} X(i) \times X(j) \arrow[r, "\otimes"] \arrow[d, "X(\phi)\times X(\psi)"'] & X(i+j) \arrow[d, "X(\phi+\psi)"] \\ X(k) \times X(l) \arrow[r, "\otimes"] & X(k+l). \end{tikzcd}$$

2. An object 1 of $X(0)$ such that composition of the inclusion $\{1\} \times X(j) \to X(0) \times X(j)$ followed by $\otimes\colon X(0) \times X(j) \to X(0+j) = X(j)$ equals the projection isomorphism $\{1\} \times X(j) \cong X(j)$. Likewise for $X(j) \times \{1\}$.

3. Natural isomorphisms
$$\gamma_\otimes^{a,b}\colon a \otimes b \to X(\chi^{j,i})(b \otimes a)$$
in $X(i+j)$ for all $a \in X(i)$ and $b \in X(j)$ such that
$$X(\phi+\psi)(\gamma_\otimes^{a,b}) = \gamma_\otimes^{X(\phi)(a), X(\psi)(b)}$$
for any pair $\phi, \psi$ as above. We require that $X(\chi^{j,i})(\gamma_\otimes^{b,a}) \circ \gamma_\otimes^{a,b}$ is equal to the identity on $a \otimes b$ and $\gamma_\otimes^{a,1}$ and $\gamma_\otimes^{1,a}$ agree with the identity morphism on $a$ for all objects $a$.

4. The composition $\otimes$ is strictly associative and the diagram

$$\begin{tikzcd}
a \otimes b \otimes c \arrow[r, "\gamma_\otimes^{a\otimes b,c}"] \arrow[d, "\mathrm{id}\otimes\gamma_\otimes^{b,c}"'] & X(\chi^{k,i+j})(c \otimes a \otimes b) \arrow[d, "X(\chi^{k,i+j})(\gamma_\otimes^{c,a}\otimes\mathrm{id})"] \\
X(\mathrm{id}+\chi^{k,j})(a \otimes c \otimes b) \arrow[r, equal] & X(\chi^{k,i+j})X(\chi^{i,k}+\mathrm{id})(a \otimes c \otimes b)
\end{tikzcd}$$

commutes for all objects $a, b, c$.

5. For each $i \in \mathbf{J}$ the zero object $0_i$ annihilates everything multiplicatively, i.e., $\{0_i\} \times X(j) \to X(i) \times X(j) \to X(i+j)$ is the constant map to $0_{i+j}$. Likewise for the composite functor from $X(i) \times \{0_j\}$.

6. Right distributivity holds strictly, that is the diagram

$$\begin{tikzcd}
(X(i) \times X(i)) \times X(j) \arrow[r, "\oplus\times\mathrm{id}"] \arrow[d, "\Delta"'] & X(i) \times X(j) \arrow[dd, "\otimes"] \\
(X(i) \times X(j)) \times (X(i) \times X(j)) \arrow[d, "\otimes\times\otimes"'] & \\
X(i+j) \times X(i+j) \arrow[r, "\oplus"] & X(i+j)
\end{tikzcd}$$

commutes, where $\oplus$ is the monoidal structure in $X(i)$ and $X(i+j)$ respectively and $\Delta$ is the diagonal on $X(j)$ combined with the identity on $X(i) \times X(i)$, followed by a twist. We denote these instances of identities by $d_r$.

7. The left distributivity transformation, $d_l$, is given in terms of $d_r$ and $\gamma_\otimes$ as

$$d_l = \gamma_\otimes \circ d_r \circ (\gamma_\otimes \oplus \gamma_\otimes)\,.$$

(Here we suppress the twist $X(\chi)$ from the notation.) More explicitly, for all $i, j \in \mathbf{J}$

and $a \in X(i)$, $b, b' \in X(j)$ the following diagram defines $d_l$:

$$\begin{tikzcd}
a \otimes b \oplus a \otimes b' \arrow[rr, "\gamma_\otimes^{a,b} \oplus \gamma_\otimes^{a,b'}"] \arrow[d, "d_l"'] & & X(\chi^{j,i})(b \otimes a) \oplus X(\chi^{j,i})(b' \otimes a) \arrow[d, equal] \\
a \otimes (b \oplus b') \arrow[d, equal] & & X(\chi^{j,i})(b \otimes a \oplus b' \otimes a) \arrow[d, "X(\chi^{j,i})(d_r)=\mathrm{id}"] \\
X(\chi^{j,i})X(\chi^{i,j})(a \otimes (b \oplus b')) & & X(\chi^{j,i})((b \oplus b') \otimes a). \arrow[ll, "X(\chi^{j,i})(\gamma_\otimes^{b\oplus b',a})"']
\end{tikzcd}$$

8. The diagram

$$\begin{tikzcd}
(a \otimes b) \oplus (a \otimes b') \arrow[r, "d_l"] \arrow[d, "\gamma_\oplus"'] & a \otimes (b \oplus b') \arrow[d, "\mathrm{id}\otimes\gamma_\oplus"] \\
(a \otimes b') \oplus (a \otimes b) \arrow[r, "d_l"] & a \otimes (b' \oplus b)
\end{tikzcd}$$

commutes for all objects. The analogous diagram for $d_r$ also commutes. Due to the definition of $d_l$ in terms of $\gamma_\otimes$ and the identity $d_r$, it suffices to demand that $\gamma_\oplus \circ (\gamma_\otimes \oplus \gamma_\otimes) = (\gamma_\otimes \oplus \gamma_\otimes) \circ \gamma_\oplus$ and $(\gamma_\oplus \otimes \mathrm{id}) \circ \gamma_\otimes = \gamma_\otimes \circ (\mathrm{id} \otimes \gamma_\oplus)$.

9. The distributivity transformations are associative, i.e., the diagram

$$\begin{tikzcd}
(a \otimes b \otimes c) \oplus (a \otimes b \otimes c') \arrow[d, "d_l"'] \arrow[dr, "d_l"] & \\
a \otimes ((b \otimes c) \oplus (b \otimes c')) \arrow[r, "\mathrm{id}\otimes d_l"] & a \otimes b \otimes (c \oplus c')
\end{tikzcd}$$

commutes for all objects.

10. The following pentagon diagram commutes

$$\begin{tikzcd}[column sep=small]
 & (a \otimes (b \oplus b')) \oplus (a' \otimes (b \oplus b')) \arrow[dr, "d_r"] & \\
(a \otimes b) \oplus (a \otimes b') \oplus (a' \otimes b) \oplus (a' \otimes b') \arrow[ur, "d_l \oplus d_l"] \arrow[d, "\mathrm{id} \oplus \gamma_{\oplus} \oplus \mathrm{id}"'] & & (a \oplus a') \otimes (b \oplus b') \\
(a \otimes b) \oplus (a' \otimes b) \oplus (a \otimes b') \oplus (a' \otimes b') \arrow[dr, "d_r \oplus d_r"'] & & \\
 & ((a \oplus a') \otimes b) \oplus ((a \oplus a') \otimes b') \arrow[uur, "d_l"'] &
\end{tikzcd}$$

for all objects $a, a' \in X(i)$ and $b, b' \in X(j)$.

**Definition 3.2.** A **J**-graded strictly bimonoidal category is a functor $X \colon \mathbf{J} \to \mathbf{Strict}$ to the category of permutative categories and strict symmetric monoidal functors satisfying the same conditions as a **J**-graded bipermutative category except for the existence of a natural isomorphism $\gamma_{\otimes}$. The left distributivity isomorphism $d_l$ is thus not given in terms of $d_r$. Instead, the distributivity isomorphisms $d_l$ and $d_r$ are subject to the condition that the diagram

$$\begin{tikzcd}
a \otimes b \otimes c \oplus a \otimes b' \otimes c \arrow[r, "d_r"] \arrow[d, "d_l"'] & (a \otimes b \oplus a \otimes b') \otimes c \arrow[d, "d_l \otimes \mathrm{id}"] \\
a \otimes (b \otimes c \oplus b' \otimes c) \arrow[r, "\mathrm{id} \otimes d_r"] & a \otimes (b \oplus b') \otimes c
\end{tikzcd}$$

commutes for all objects.
In the case of a **J**-graded bipermutative category, this condition follows from the other axioms.

Let $\mathbf{Perm}^{nz}$ denote the category of permutative categories without zero object and lax symmetric monoidal functors. The adjoint pair of functors

$$U \colon \mathbf{Perm} \longrightarrow \mathbf{Perm}^{nz}, \quad F \colon \mathbf{Perm}^{nz} \longrightarrow \mathbf{Perm}$$

where $U$ is the forgetful functor and $F$ is given by $F(\mathcal{C}) = \mathcal{C}_+$, i.e., adding a disjoint zero object, defines a simplicial resolution $Z$ (cf. [Wei94], section 8.6, for further information on

simplicial resolutions).
For a definition of the homotopy colimit see [Tho82], section 3. The derived version of the homotopy colimit is necessary to fix the "zero problem" (see section 4.2 of [BDRRa]). It makes use of the simplicial resolution $Z$.

**Lemma 3.3** (Lemma 4.11 in [BDRRa])**.** *Let $[m]$ be an object of the category $\boldsymbol{I}$ of finite sets and injections. If $X\colon \boldsymbol{I} \to \boldsymbol{Perm}$ is a functor such that any $\varphi\colon [m] \to [n]$ in $\boldsymbol{I}$ is sent to an unstable (resp. stable) equivalence $X(\varphi)\colon X([m]) \to X([n])$, then the canonical chain*

$$X([m]) \xleftarrow{\sim} ZX([m]) \longrightarrow D\mathrm{hocolim}_{\boldsymbol{I}}X$$

*is an unstable (resp. stable) equivalence.*

**Proposition 3.4** (Proposition 5.1 in [BDRRa])**.** *Let $\boldsymbol{J}$ be a permutative category and $\mathcal{C}^\bullet$ a $\boldsymbol{J}$-graded bipermutative (or strictly bimonoidal) category. Then $D\mathrm{hocolim}_{\boldsymbol{J}}\mathcal{C}^\bullet$ is a simplicial bipermutative (strictly bimonoidal) category and*

$$\mathcal{C}^0 \xleftarrow{\sim} Z\mathcal{C}^0 \longrightarrow D\mathrm{hocolim}_{\boldsymbol{J}}\mathcal{C}^\bullet$$

*are maps of simplicial bipermutative (strictly bimonoidal) categories. Furthermore, for each $j \in \boldsymbol{J}$, the canonical maps*

$$\mathcal{C}^j \xleftarrow{\sim} Z\mathcal{C}^j \longrightarrow D\mathrm{hocolim}_{\boldsymbol{J}}\mathcal{C}^\bullet$$

*are maps of $Z\mathcal{C}^0$-modules.*

In the following, we present a model of a multiplicative group completion that might provide an alternative proof of Theorem 2.14.

## 3.2 Definition of the Segal model $K^\bullet\mathcal{R}$ and first properties

Given a permutative category $\mathcal{R}$, we define simplicial permutative categories $K^\bullet\mathcal{R}$. If $\mathcal{R}$ is strictly bimonoidal, these will provide an $\mathbf{I}$-graded strictly bimonoidal category, with $\mathbf{I}$ being the category of finite sets and injective maps.

The following construction is based on an idea of Graeme Segal ([Seg74], chapter 2) and was developed further by Nobuo Shimada and Kazuhisa Shimakawa ([SS79], chapter 2). Our presentation resembles the version of Elmendorf and Mandell ([EM06], chapter 4), only that we require the structure maps $\rho_C$ to be isomorphisms as in [SS79].

**Definition 3.5.** Let $(\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus)$ be a small permutative category.
For finite pointed sets $X^1_+, \dots, X^n_+$, $\bar{H}\mathcal{R}(X^1_+, \dots, X^n_+)$ is the category whose objects are the systems $\{C_{<S>}, \rho_C(< S >; i, T, U)\}$ with:

- $< S >= (S_1, \dots, S_n)$ runs through all $n$-tupels of basepoint-free subsets $S_i \subset X^i$ and
- the $C_{<S>}$ are objects of $\mathcal{R}$.
- Let $< S; i, T >$ denote $(S_1, \dots, S_{i-1}, T, S_{i+1}, \dots, S_n)$ for some subset $T \subset S_i$. Then the $\rho_C(< S >; i, T, U)$ are isomorphisms from $C_{<S;i,T>} \oplus C_{<S;i,U>}$ to $C_{<S>}$ for $i = 1, \dots, n$ and $T, U \subset S_i$ being disjoint subsets with $T \cup U = S_i$.

The objects $\{C_{<S>}, \rho_C(< S >; i, T, U)\}$ satisfy the following properties:

- If $S_i = \emptyset$ for $i \in \{1, \dots, n\}$, then $C_{<S>} = 0_\mathcal{R}$.
- If one of the $S_i, T, U$ is empty, then $\rho_C(< S >; i, T, U) = \mathrm{id}$.
- The isomorphisms $\rho_C(< S >; i, T, U)$ are compatible with the additive twist:

$$\rho_C(< S >; i, T, U) = \rho_C(< S >; i, U, T) \circ c_\oplus.$$

- The $\rho_C(< S >; i, T, U)$ are associative, that is for all $< S >, i$ and pairwise disjoint $T, U, V \subset S_i$ with $T \cup U \cup V = S_i$ the diagram

$$\begin{tikzcd} C_{<S;i,T>} \oplus C_{<S;i,U>} \oplus C_{<S;i,V>} \arrow[r, "\rho_C(<S;i,T\cup U>;i,T,U)\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\rho_C(<S;i,U\cup V>;i,U,V)"'] & C_{<S;i,T\cup U>} \oplus C_{<S;i,V>} \arrow[d, "\rho_C<S>;i,T\cup U,V)"] \\ C_{<S;i,T>} \oplus C_{<S;i,U\cup V>} \arrow[r, "\rho_C(<S>;i,T,U\cup V)"] & C_{<S>} \end{tikzcd}$$

commutes.

- The $\rho(< S >; i, T, U)$ satisfy the pentagon rule, that is for $i \neq j$ and $T, U \subseteq S_i, V, W \subseteq S_j$ with $T \cap U = \emptyset = V \cap W$ the following diagram commutes:

$$\begin{tikzcd}
& C_{<S;j,V>} \oplus C_{<S;j,W>} \arrow[dr, "\rho(<S>;j,V,W)"] & \\
C_{<S;i,T;j,V>} \oplus C_{<S;i,U;j,V>} \oplus C_{<S;i,T;j,W>} \oplus C_{<S;i,U;j,W>} \arrow[ur, "\rho(<S;j,V>;i,T,U)\oplus\rho(<S;j,W>;i,T,U)"] \arrow[d, "\mathrm{id}\oplus c_{\oplus}\oplus\mathrm{id}"'] & & C_{<S>} \\
C_{<S;i,T;j,V>} \oplus C_{<S;i,T;j,W>} \oplus C_{<S;i,U;j,V>} \oplus C_{<S;i,U;j,W>} \arrow[r, "\rho(<S;i,T>;j,V,W)\oplus\rho(<S;i,U>;j,V,W)"'] & C_{<S;i,T>} \oplus C_{<S;i,U>} \arrow[ur, "\rho(<S>;i,T,U)"'] &
\end{tikzcd}$$

Let us try to visualize the objects. If $X_+$ is a finite pointed set, an object $\{C_S, \rho_C(S;T,U)\}$ of $\bar{H}\mathcal{R}(X_+)$ is a collection of objects $C_S$ of $\mathcal{R}$ for each basepoint-free subset $S \subset X_+$ and isomorphisms $\rho_C(S;T,U)$ for each $T, U \subset S$ with $T \cup U = S$ and $T \cap U = \emptyset$. If, for example, $X_+ = [2]_+ = \{0,1,2\}$, where 0 denotes the basepoint, then an object of $\bar{H}\mathcal{R}(X_+)$ may be thought of as a square

$$\begin{tikzcd}
C_{\{1\}} \arrow[r, dash] \arrow[d, dash] & C_{\emptyset} = 0_{\mathcal{R}} \arrow[d, dash] \\
C_{\{1,2\}} \arrow[r, dash] & C_{\{2\}}
\end{tikzcd}$$

with $\rho(\{1,2\};\{1\},\{2\})\colon C_{\{1\}} \oplus C_{\{2\}} \xrightarrow{\cong} C_{\{1,2\}}$.
For $X'_+ = [3]_+ = \{0,1,2,3\}$, an object of $\bar{H}\mathcal{R}(X'_+)$ may be visualized as a cube

$$\begin{tikzcd}
& C_{\{1\}} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash, dashed] & & C_{\emptyset} = 0_{\mathcal{R}} \arrow[dl, dash] \arrow[dd, dash] \\
C_{\{1,2\}} \arrow[rr, dash] \arrow[dd, dash] & & C_{\{2\}} \arrow[dd, dash] & \\
& C_{\{1,3\}} \arrow[rr, dash, dashed] \arrow[dl, dash, dashed] & & C_{\{3\}} \arrow[dl, dash] \\
C_{\{1,2,3\}} \arrow[rr, dash] & & C_{\{2,3\}} &
\end{tikzcd} \tag{3.1}$$

with, amongst others,

$$\rho(\{1,2,3\};\{1\},\{2,3\})\colon C_{\{1\}} \oplus C_{\{2,3\}} \xrightarrow{\cong} C_{\{1,2,3\}},$$
$$\rho(\{1,2,3\};\{1,3\},\{2\})\colon C_{\{1,3\}} \oplus C_{\{2\}} \xrightarrow{\cong} C_{\{1,2,3\}},$$
$$\rho(\{1,2\};\{1\},\{2\})\colon C_{\{1\}} \oplus C_{\{2\}} \xrightarrow{\cong} C_{\{1,2\}}.$$

To shorten notation, we write $\rho_C(< S >)$ or simply $\rho_C$ instead of $\rho_C(< S >; i, T, U)$ if we do not need all the details.

Morphisms in this category are fairly easy. A morphism $f\colon \{C_{<S>}, \rho_C(< S >; i, T, U)\} \to \{D_{<S>}, \rho_D(< S >; i, T, U)\}$ consists of morphisms $f_{<S>}\colon C_{<S>} \to D_{<S>}$ in $\mathcal{R}$ which commute with the structure maps $\rho(< S >; i, T, U)$ and that are the identity if any of the $S_i$ is empty.

Let $\phi\colon X^1_+ \to X^2_+$ be a map of finite pointed sets. The induced functor $\phi_*\colon \bar{H}\mathcal{R}(X^1_+) \to \bar{H}\mathcal{R}(X^2_+)$ is defined such that an object $\{C_T, \rho_C(T; U_1, U_2)\}$ is sent to $\{\phi_* C_S, \phi_*\rho_C(S; U'_1, U'_2)\}$, $S \subset X^2_+$ basepoint-free, where $\phi_* C_S := C_{\phi^{-1}(S)}$ and the structure maps $\phi_*\rho_C$ are given by

$$\begin{tikzcd}
\phi_* C_{(S;U'_1)} \oplus \phi_* C_{(S;U'_2)} \arrow[r, equal] & C_{(\phi^{-1}(S);\phi^{-1}(U'_1))} \oplus C_{(\phi^{-1}(S);\phi^{-1}(U'_2))} \arrow[d, "\rho_C(\phi^{-1}(S);\phi^{-1}(U'_1),\phi^{-1}(U'_2))"] \\
 & C_{(\phi^{-1}(S);\phi^{-1}(U'_1)\cup\phi^{-1}(U'_2))} = \phi_* C_S.
\end{tikzcd}$$

A morphism $f = \{f_S\}$ is mapped to $\phi_* f := \{f_{\phi^{-1}(S)}\}$. This implies, that isomorphisms are mapped to isomorphisms.

Going back to the above example, consider the map $\phi\colon [3]_+ \to [2]_+$ with $\phi(0) = \phi(1) = 0, \phi(2) = 1$ and $\phi(3) = 2$. Then, for example, $\phi_* C_{\{1,2\}} = C_{\{2,3\}}$ and the cube in (3.1) would be mapped to

$$\begin{tikzcd}
C_{\{2\}} \arrow[r, dash] \arrow[d, dash] & C_{\emptyset} \arrow[d, dash] \\
C_{\{2,3\}} \arrow[r, dash] & C_{\{3\}}.
\end{tikzcd}$$

The category $\bar{H}\mathcal{R}(X^1_+,\dots,X^n_+)$ can be endowed with the structure of a permutative category which is induced by the permutative structure of $\mathcal{R}$. We define $\{C_{<S>},\rho_C\}\oplus\{D_{<S>},\rho_D\} := \{(C\oplus D)_{<S>},\rho_{C\oplus D}\}$ with $(C\oplus D)_{<S>} := C_{<S>}\oplus_{\mathcal{R}} D_{<S>}$ and $\rho_{C\oplus D} = (\rho_C\oplus_{\mathcal{R}}\rho_D)\circ(\mathrm{id}\oplus_{\mathcal{R}} c_{\oplus}\oplus_{\mathcal{R}}\mathrm{id})$ (we will omit the subscript $\mathcal{R}$ in the following). The unit is given by the zero cube, i.e., the object $\{C_{<S>},\rho_C(<S>;i,T,U)\}$ with $C_{<S>} = 0_{\mathcal{R}}$ and $\rho_C(<S>;i,T,U) = \mathrm{id}$ for all $<S>= (S_1,\dots,S_n)$ and $T,U\in S_i$ with the according properties. The structure maps $\alpha,\lambda,\rho$ (cf. Definition 1.7) and the additive twist are induced by the respective maps in $\mathcal{R}$ and the required diagrams commute because they commute in $\mathcal{R}$.

**Lemma 3.6.** *The functor $\bar{H}\mathcal{R}$ is a functor from the $n$-fold product of the category of finite pointed sets to the category of permutative categories and strict symmetric monoidal functors.*

*Proof.* We want to point out that $\alpha,\lambda$ and $\rho$ are indeed identity maps. In case of $\lambda,\rho$ this is clear due to the definition of the unit. Let $C_U$ denote $C_{<S;i,U>}$. We know that

$$\alpha\colon (C_U\oplus D_U)\oplus E_U \longrightarrow C_U\oplus(D_U\oplus E_U)$$

is an identity map for all $C_U,D_U,E_U\in\mathcal{R}$. To see that $\rho_{(C\oplus D)\oplus E} = \rho_{C\oplus(D\oplus E)}$ consider the following diagram:

$$\begin{tikzcd}
(C_U\oplus D_U)\oplus E_U\oplus(C_V\oplus D_V)\oplus E_V \arrow[r, equal] \arrow[d] & C_U\oplus(D_U\oplus E_U)\oplus C_V\oplus(D_V\oplus E_V) \arrow[d] \\
(C_U\oplus D_U)\oplus(C_V\oplus D_V)\oplus E_U\oplus E_V \arrow[d] & C_U\oplus C_V\oplus(D_U\oplus E_U)\oplus(D_V\oplus E_V) \arrow[d] \\
C_U\oplus(C_V\oplus D_U)\oplus D_V\oplus E_U\oplus E_V \arrow[r, equal] \arrow[d] & C_U\oplus C_V\oplus D_U\oplus(D_V\oplus E_U)\oplus E_V \arrow[d] \\
(C_{<S>}\oplus D_{<S>})\oplus E_{<S>} \arrow[r, equal] & C_{<S>}\oplus(D_{<S>}\oplus E_{<S>}),
\end{tikzcd}$$

where the horizontal identities are given by the associativity map in $\mathcal{R}$. The left hand side of the diagram is the map

$$\rho_{(C\oplus D)\oplus E}\colon [(C_U\oplus D_U)\oplus E_U]\oplus[(C_V\oplus D_V)\oplus E_V] \longrightarrow (C_{<S>}\oplus D_{<S>})\oplus E_{<S>}.$$

The right hand side is the map

$$\rho_{C\oplus(D\oplus E)}\colon [C_U \oplus (D_U \oplus E_U)] \oplus [C_V \oplus (D_V \oplus E_V)] \longrightarrow C_{<S>} \oplus (D_{<S>} \oplus E_{<S>}).$$

The diagram commutes since $\alpha$ and $c_\oplus$ are coherent and for this reason we have $\rho_{(C\oplus D)\oplus E} = \rho_{C\oplus(D\oplus E)}$. Thus,

$$\{(C_{<S>} \oplus D_{<S>}) \oplus E_{<S>}, \rho_{(C\oplus D)\oplus E}\} \xrightarrow{=} \{C_{<S>} \oplus (D_{<S>} \oplus E_{<S>}), \rho_{C\oplus(D\oplus E)}\}.$$

Nothing else remains to be done than to prove that the induced functors from above are in fact strict symmetric monoidal. Let $\phi_i\colon X^i_+ \to \tilde{X}^i_+$, $1 \leq i \leq n$, be maps of finite pointed sets and let $\phi^{-1}(< S >)$ denote $(\phi_1^{-1}(S_1), \ldots, \phi_n^{-1}(S_n))$. An easy calculation gives

$$\{\phi_* C_{<S>}, \phi_*\rho_C\} \oplus \{\phi_* D_{<S>}, \phi_*\rho_D\} = \{(\phi_* C \oplus \phi_* D)_{<S>}, (\phi_*\rho_C \oplus \phi_*\rho_D) \circ (\mathrm{id} \oplus c_\oplus \oplus \mathrm{id})\}$$

with

$$\begin{aligned}(\phi_* C \oplus \phi_* D)_{<S>} &= (\phi_* C)_{<S>} \oplus (\phi_* D)_{<S>} \\ &= C_{\phi^{-1}(<S>)} \oplus D_{\phi^{-1}(<S>)} = \phi_*(C \oplus D)_{<S>}\end{aligned}$$

and $(\phi_*\rho_C \oplus \phi_*\rho_D) \circ (\mathrm{id} \oplus c_\oplus \oplus \mathrm{id}) = \phi_*\rho_{C\oplus D}$. This last statement is true since $\phi_*\rho_{C\oplus D}$ is defined as

$$\begin{tikzcd} \phi_*(C \oplus D)_{<S;i,U>} \oplus \phi_*(C \oplus D)_{<S;i,V>} \arrow[d, equal] \\ (C_{\phi^{-1}(<S;i,U>)} \oplus D_{\phi^{-1}(<S;i,U>)}) \oplus (C_{\phi^{-1}(<S;i,V>)} \oplus D_{\phi^{-1}(<S;i,V>)}) \arrow[d, "\mathrm{id}\oplus c\oplus \mathrm{id}"] \\ (C_{\phi^{-1}(<S;i,U>)} \oplus C_{\phi^{-1}(<S;i,V>)}) \oplus (D_{\phi^{-1}(<S;i,U>)} \oplus D_{\phi^{-1}(<S;i,V>)}) \arrow[d, "\rho_C\oplus\rho_D"] \\ C_{\phi^{-1}(<S>)} \oplus D_{\phi^{-1}(<S>)} = \phi_*(C \oplus D)_{<S>} \end{tikzcd}$$

where $\rho_C \oplus \rho_D$ stands for

$$\rho_C(\phi^{-1} < S >; i, \phi^{-1}(U), \phi^{-1}(V)) \oplus \rho_D(\phi^{-1} < S >; i, \phi^{-1}(U), \phi^{-1}(V)).$$

Thus

$$\{\phi_* C_{<S>}, \phi_*\rho_C\} \oplus \{\phi_* D_{<S>}, \phi_*\rho_D\} = \{\phi_*(C \oplus D)_{<S>}, \phi_*\rho_{C\oplus D}\}.$$

The zero cube is of course mapped to the zero cube and all other conditions follow from the strictness in $\mathcal{R}$. □

Let $\Gamma^{op}$ be the skeleton of the following category of finite pointed sets and based set maps: There is one object $[n]_+ = \{0, 1, \ldots, n\}$ with basepoint 0 for each non-negative integer $n$ and morphisms are maps of sets that map 0 to 0. This category is equivalent to the opposite of Segal's category $\Gamma$ in [Seg74]. A $\Gamma$-category is a covariant functor $C$ from $\Gamma^{op}$ to **Cat** such that $C(\{0\})$ is equivalent to the category with a single morphism.
The functor $\bar{H}\mathcal{R}$ is a $\Gamma$-category. To see that $\bar{H}\mathcal{R}$ is indeed covariant, note that for maps of finite pointed sets $\phi_1\colon X^1_+ \to X^2_+$, $\phi_2\colon X^2_+ \to X^3_+$, an object $\phi_{2*}(\phi_{1*}C)_S$, $S \subset X^3_+$, is defined as $C_{\phi_1^{-1}(\phi_2^{-1}(S))}$. The structure maps $\rho_C$ and morphisms are defined analogously.
In particular, $\bar{H}\mathcal{R}$ is a very special $\Gamma$-category in the following sense:

**Lemma 3.7** (Lemma 2.2 in [SS79]). *For $k \in [m]_+$ let $i_k\colon [m]_+ \to [1]_+$ be the map*

$$i_k(j) = \begin{cases} 1 & j = k, \\ 0 & j \neq k. \end{cases}$$

*The canonical map*

$$\bar{H}\mathcal{R}([m]_+) \longrightarrow \bar{H}\mathcal{R}([1]_+) \times \cdots \times \bar{H}\mathcal{R}([1]_+),$$

*induced by the maps $i_k$ is an equivalence of categories.*

The functor $\bar{H}\mathcal{R}$ can be extended to a functor on the $n$-fold product of the category of pointed simplicial sets in the obvious way: Let $Y_1, \ldots, Y_n$ be pointed simplicial sets. We define $\bar{H}\mathcal{R}(Y_1, \ldots, Y_n)$ to be the $n$-simplicial permutative category with

$$\bar{H}\mathcal{R}(Y_1, \ldots, Y_n)_{(l_1,\ldots,l_n)} := \bar{H}\mathcal{R}((Y_1)_{l_1}, \ldots, (Y_n)_{l_n})$$

for $l_i \in \Delta$. This means that $\bar{H}\mathcal{R}(Y_1, \ldots, Y_n)$ is a functor from $(\Delta^{op})^{\times n}$ to the category of permutative categories.
In the extension to the simplicial setting, we want the $S_i$'s in $\{C_{<S>}, \rho_C(< S >; i, V, W)\}$ to be subfunctors of the $Y_i$'s. We remove the basepoint of $(S_i)_k$ for all $k \in \Delta$. Furthermore, we require $V, W$ to be subfunctors of $S_i$ with levelwise disjoint image and $V_{l_i} \cup W_{l_i} = S_{l_i}$.

For convenience, we will denote this again with $V, W \subset S_i$ and $V \cap W = \emptyset, V \cup W = S_i$.

Recall the small model of the simplicial one sphere, namely $\mathbb{S}^1_n = [n]_+ = \{0, 1, \ldots, n\}$ with 0 as basepoint and face and degeneracy maps $d_i \colon \mathbb{S}^1_n \to \mathbb{S}^1_{n-1},\ s_i \colon \mathbb{S}^1_n \to \mathbb{S}^1_{n+1}$ defined as

$$d_i(j) = \begin{cases} j & j < i, \\ i & j = i < n, \\ 0 & i = j = n, \\ j-1 & j > i, \end{cases} \quad \text{and} \quad s_i(k) = \begin{cases} k & k \leq i, \\ k+1 & k > i. \end{cases}$$

The simplicial path space of $\mathbb{S}^1_\bullet$ is defined as the simplicial set $(P\mathbb{S}^1)_\bullet$ with $(P\mathbb{S}^1)_n = \mathbb{S}^1_{n+1}$ and $d'_i = d_{i+1}, s'_i = s_{i+1}$. The renumbering of face and degeneracy maps leaves simplicial maps $d_0, s_0 \colon (P\mathbb{S}^1)_\bullet \to \mathbb{S}^1_\bullet$.
For the sake of readability, we omit the parentheses and write $P\mathbb{S}^1_n$ for $(P\mathbb{S}^1)_n$.

As a corollary of Lemma 3.6 we get that the simplicial map $d_0 \colon P\mathbb{S}^1_\bullet \to \mathbb{S}^1_\bullet$ induces a strict symmetric monoidal functor of simplicial permutative categories

$$d_{0*} \colon \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet) \longrightarrow \bar{H}\mathcal{R}(\mathbb{S}^1_\bullet).$$

By this we mean a simplicial functor that is a strict symmetric monoidal functor of permutative categories in each simplicial degree.

The following definition is due to Baas, Dundas, Richter and Rognes. However, since it is based on an idea of Graeme Segal we call it the Segal model. A similar construction can be found in [Tho79], section 4.

**Definition 3.8.** Let $\mathbf{J}$ be the category with objects $0, 1, 2$ and non-trivial morphisms $2 \to 0,\ 1 \to 0$. For a permutative category $\mathcal{R}$ we define $K^n_\bullet\mathcal{R}$ to be the $n$-simplicial permutative category that is the $n$-fold limit of the diagram $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n})$ in **sStrict** with $i_j \in \mathbf{J}$, $Y_0 = \mathbb{S}^1_\bullet, Y_1 = Y_2 = P\mathbb{S}^1_\bullet$ and $d_0 \colon P\mathbb{S}^1_\bullet \to \mathbb{S}^1_\bullet$.

**Example 3.9.** $K^1_\bullet\mathcal{R}$ is the pullback of

$$\begin{tikzcd} & \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet) \arrow[d, "d_{0*}"] \\ \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet) \arrow[r, "d_{0*}"] & \bar{H}\mathcal{R}(\mathbb{S}^1_\bullet) \end{tikzcd}$$

and $K^2_\bullet\mathcal{R}$ is the limit of

$$\begin{tikzcd} \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet) \arrow[r, "d_{0*}"] \arrow[d, "d_{0*}"] & \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, \mathbb{S}^1_\bullet) \arrow[d, "d_{0*}"] & \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet) \arrow[l, "d_{0*}"'] \arrow[d, "d_{0*}"] \\ \bar{H}\mathcal{R}(\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet) \arrow[r, "d_{0*}"] & \bar{H}\mathcal{R}(\mathbb{S}^1_\bullet, \mathbb{S}^1_\bullet) & \bar{H}\mathcal{R}(\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet) \arrow[l, "d_{0*}"'] \\ \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet) \arrow[r, "d_{0*}"] \arrow[u, "d_{0*}"] & \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, \mathbb{S}^1_\bullet) \arrow[u, "d_{0*}"] & \bar{H}\mathcal{R}(P\mathbb{S}^1_\bullet, P\mathbb{S}^1_\bullet). \arrow[l, "d_{0*}"'] \arrow[u, "d_{0*}"] \end{tikzcd}$$

Note that a limit in a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ exists if it exists for every object $c \in \mathcal{C}$ and is then computed pointwise. This means that $K^1_q\mathcal{R}$ is the pullback of

$$\begin{tikzcd} & \bar{H}\mathcal{R}(P\mathbb{S}^1_q) \arrow[d, "d_{0*}"] \\ \bar{H}\mathcal{R}(P\mathbb{S}^1_q) \arrow[r, "d_{0*}"] & \bar{H}\mathcal{R}(\mathbb{S}^1_q) \end{tikzcd}$$

(cf. [Bor94], 2.15). Furthermore, keep in mind that

$$\lim_{\mathbf{J}\times\mathbf{J}} \bar{H}\mathcal{R}(Y_{i_1}, Y_{i_2}) \cong \lim_{\mathbf{J}} \lim_{\mathbf{J}} \bar{H}\mathcal{R}(Y_{i_1}, Y_{i_2}).$$

This follows from the universal property of the limit by use of the isomorphism of functor categories $[\mathbf{J} \times \mathbf{J}, \mathbf{Strict}] \cong [\mathbf{J}, [\mathbf{J}, \mathbf{Strict}]]$.

We now turn to the question of group completeness and start with analyzing the category $K^1_q\mathcal{R}$: It consists of objects $(\{C_S, \rho_C\}, \{D_S, \rho_D\})$ and morphisms $(f, g)$ with $C_S = D_S$ and $f_S = g_S$ for all $S \subset d_0^{-1}([q])$. Since the preimage of $[q] = \{1, \ldots, q\}$ under $d_0\colon [q{+}1]_+ \to [q]_+$ is $\{2, \ldots, q+1\}$, $\{C_S, \rho_C\}$ and $\{D_S, \rho_D\}$ only differ in $S = \{1\}$ (and thus every $S \subset [q+1]$ containing $\{1\}$).
In degree zero, $K^1_\bullet\mathcal{R}$ is isomorphic to $\mathcal{R}\times\mathcal{R}$ as a permutative category: Since $P\mathbb{S}^1_0 = \{0, 1\}$, it is obvious that $\bar{H}\mathcal{R}(P\mathbb{S}^1_0) \cong \mathcal{R}$. Moreover, $\bar{H}\mathcal{R}(\mathbb{S}^1_0) = 0_\mathcal{R}$ which makes $d_{0*}$ the trivial

map on 0-simplices.
We denote objects of $\bar{H}\mathcal{R}(P\mathbb{S}^1_1)$ by $C_{\{1,2\}} \cong C_{\{1\}} \oplus C_{\{2\}}$ where the isomorphism is given by $\rho_C(\{1,2\};\{1\},\{2\})$. Elements of $K^1_1\mathcal{R}$ are pairs $(C_{\{1,2\}} \cong C_{\{1\}} \oplus C_{\{2\}}, D_{\{1,2\}} \cong D_{\{1\}} \oplus D_{\{2\}})$ with $C_{\{2\}} = D_{\{2\}}$.
There are two maps $d'_0, d'_1 \colon K^1_1\mathcal{R} \to K^1_0\mathcal{R}$, induced by the face maps $d'_0, d'_1 \colon P\mathbb{S}^1_1 \longrightarrow P\mathbb{S}^1_0$:

$$
\begin{aligned}
d'_0 &\colon (C_{\{1,2\}} \cong C_{\{1\}} \oplus C_{\{2\}}, D_{\{1,2\}} \cong D_{\{1\}} \oplus D_{\{2\}}) \longmapsto (C_{\{1,2\}}, D_{\{1,2\}}),\\
d'_1 &\colon (C_{\{1,2\}} \cong C_{\{1\}} \oplus C_{\{2\}}, D_{\{1,2\}} \cong D_{\{1\}} \oplus D_{\{2\}}) \longmapsto (C_{\{1\}}, D_{\{1\}}).
\end{aligned}
$$

Let $X = (C_{\{1,2\}} \cong C_{\{1\}} \oplus C_{\{2\}}, D_{\{1,2\}} \cong D_{\{1\}} \oplus D_{\{2\}}) \in K^1_1\mathcal{R}$. By use of the isomorphisms $\rho_C$ and $\rho_D$, we have $d'_0(X) \cong d'_1(X) \oplus (C_{\{2\}}, D_{\{2\}})$ as elements in $K^1_0\mathcal{R}$.

As usual, $\pi_0 K^1_\bullet\mathcal{R} = \pi_0|NK^1_\bullet\mathcal{R}| = \pi_0(\mathrm{diag}NK^1_\bullet\mathcal{R})$ where $N$ denotes the nerve functor. This means, $\pi_0 K^1_\bullet\mathcal{R} = N_0K^1_0\mathcal{R}/\sim = (\mathcal{R}\times\mathcal{R})/\sim$ with $\sim$ being induced by the face maps on the nerve and $K^1_\bullet\mathcal{R}$. More precisely, $(C,D) \sim (C',D')$ if and only if there exists a 1-simplex

$$(\{C_S, \rho_C\}, \{D_S, \rho_D\}) \longrightarrow (\{C'_S, \rho_{C'}\}, \{D'_S, \rho_{D'}\})$$

in $N_1K^1_1\mathcal{R}$ such that

$$d'_0(\{C_S, \rho_C\}, \{D_S, \rho_D\}) = (C_{\{1,2\}}, D_{\{1,2\}}) = (C, D)$$

and

$$d'_1(\{C'_S, \rho_{C'}\}, \{D'_S, \rho_{D'}\}) = (C'_{\{1\}}, D'_{\{1\}}) = (C', D')$$

with $d'_i \colon K^1_1\mathcal{R} \to K^1_0\mathcal{R}$ as above. Providing $\pi_0 K^1_\bullet\mathcal{R}$ with addition by components, we see

**Lemma 3.10.** *The set of path components $\pi_0 K^1\mathcal{R}$ is an abelian group. More precisely,*

$$\pi_0 K^1_\bullet\mathcal{R} = Gr\pi_0\mathcal{R}.$$

*Proof.* The basic idea is that the face maps on the nerve induce the restriction to path components in $\mathcal{R}$ and the face maps $d'_i$ on $K^1_\bullet\mathcal{R}$ induce the Grothendieck group structure. To understand the first statement, consider a map $(\{C_S, \rho_C\}, \{D_S, \rho_D\}) \longrightarrow (\{C'_S, \rho_{C'}\}, \{D'_S, \rho_{D'}\})$ with $(C_{\{2\}}, D_{\{2\}}) = (C'_{\{2\}}, D'_{\{2\}}) = (0_\mathcal{R}, 0_\mathcal{R})$. Then $(C_{\{1,2\}}, D_{\{1,2\}}) = (C_{\{1\}}, D_{\{1\}})$, $(C'_{\{1,2\}}, D'_{\{1,2\}}) = (C'_{\{1\}}, D'_{\{1\}})$ and the relation explained above translates into $(C,D) \sim (C',D')$ if there is map $(C,D) \to (C',D')$. We turn to the second statement. Obviously,

$(0_\mathcal{R}, 0_\mathcal{R})$ is the neutral element. We want to show: $(C, C) \sim (0_\mathcal{R}, 0_\mathcal{R})$ for all $C \in \mathcal{R}$. Since $\mathrm{diag}(\bar{H}\mathcal{R}(P\mathbb{S}^1_1) \times \bar{H}\mathcal{R}(P\mathbb{S}^1_1)) \subset K^1_1\mathcal{R}$, there is an object $(\{C_S, \rho_C\}, \{C_S, \rho_C\}) \in K^1_1\mathcal{R}$ with

$$d'_0(\{C_S, \rho_C\}, \{C_S, \rho_C\}) = (C_{\{1,2\}}, C_{\{1,2\}}) = (C, C)$$

and

$$d'_1(\{C_S, \rho_C\}, \{C_S, \rho_C\}) = (C'_{\{1\}}, C'_{\{1\}}) = (0_\mathcal{R}, 0_\mathcal{R})$$

for every $C \in \mathcal{R}$. Considering the identity map on this object gives $(C, C) \sim (0_\mathcal{R}, 0_\mathcal{R})$ and this implies that $(C_1, C_2)$ is the inverse of $(C_2, C_1)$. □

In [Tho79], section 4, Thomason describes functors $S = \bar{W}^1 S$ which are equivalent to our $\bar{H}\mathcal{R}$. Notably, Proposition 4.3.2 implies:

**Proposition 3.11.** *The canonical functor $\mathcal{R} \to K^1_0\mathcal{R}$ defined as*

$$c \longmapsto (c, 0), \;\; f \longmapsto (f, id_0).$$

*induces a group completion map $B\mathcal{R} \to BK^1_\bullet\mathcal{R}$.*

*Proof.* To see that Thomason's proof applies to $K^1_\bullet\mathcal{R}$, recall that

$$|N\Delta^{op} \textstyle\int K^1_\bullet\mathcal{R}| \simeq |\mathrm{hocolim} NK^1_\bullet\mathcal{R}| \simeq |\mathrm{diag} NK^1_\bullet\mathcal{R}|$$

where $\Delta^{op} \int K^1_\bullet\mathcal{R}$ is the Grothendieck construction on $K^1_\bullet\mathcal{R}$. The first equivalence holds because of Theorem 1.2 in [Tho79] and the second holds because the homotopy colimit in the category of bisimplicial sets is homotopy equivalent to the diagonal (cf. [BK72], XII, 4.3). □

## 3.3 $K^\bullet\mathcal{R}$ is an I-graded category

From now on, let $\mathcal{R} = (\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus, \otimes, 1_\mathcal{R}, \delta_r, \delta_l)$ be a strictly bimonoidal category unless stated otherwise.

### 3.3.1 Multiplicative structure

Let $X^i_+$ be finite pointed sets. The multiplication in $\mathcal{R}$ induces a pairing

$$\begin{array}{rccc} \mu\colon \bar{H}\mathcal{R}(X^1_+,\dots,X^n_+)\times\bar{H}\mathcal{R}(X^{n+1}_+,\dots,X^{n+m}_+) & \longrightarrow & \bar{H}\mathcal{R}(X^1_+,\dots,X^n_+,X^{n+1}_+,\dots,X^{n+m}_+) \\ (\{C_{<S>},\rho_C(<S>)\},\{D_{<T>},\rho_D(<T>)\}) & \longrightarrow & \{(C\otimes D)_{<S,T>},\rho_{C\otimes D}(<S,T>)\} \end{array}$$

with $< S,T >= (S_1,\dots,S_n,T_{n+1},\dots,T_{n+m})$ and

$$(C\otimes D)_{(S_1,\dots,S_n,T_{n+1},\dots,T_{n+m})} := C_{(S_1,\dots,S_n)}\otimes D_{(T_{n+1},\dots,T_{n+m})}.$$

For $1\le i\le n$ and $S_i = V\cup W,\ V\cap W=\emptyset$ and $n+1\le j\le n+m$ and $T_j = V'\cup W',\ V'\cap W'=\emptyset$ respectively, the corresponding isomorphisms are defined as

$$\begin{aligned} \rho_{C\otimes D}(< S,T >; i,V,W) &:= (\rho_C(< S >; i,V,W)\otimes \mathrm{id}_{D_{<T>}})\circ\delta_r \quad \text{and} \\ \rho_{C\otimes D}(< S,T >; j,V',W') &:= (\mathrm{id}_{C_{<S>}}\otimes\rho_D(< T >; j,V',W'))\circ\delta_l \end{aligned}$$

respectively.
To see how this works, consider the following example:
Let $U\cup V = S$ and $U'\cup V' = T$ with $U\cap V=\emptyset, U'\cap V'=\emptyset$. Then

$$\begin{tikzcd} \rho_{C\otimes D}((S,T);1,V,W)\colon\ (C\otimes D)_{(V,T)}\oplus(C\otimes D)_{(W,T)} \arrow[r, equal] & (C_V\otimes D_T)\oplus(C_W\otimes D_T) \arrow[d, "\delta_r"] \\ & (C_V\oplus C_W)\otimes D_T \arrow[d, "\rho_C\otimes\mathrm{id}_D"] \\ & C_S\otimes D_T \end{tikzcd}$$

and

$$\begin{tikzcd} \rho_{C\otimes D}((S,T);2,V',W')\colon\ (C\otimes D)_{(S,V')}\oplus(C\otimes D)_{(S,W')} \arrow[r, equal] & (C_S\otimes D_{V'})\oplus(C_S\otimes D_{W'}) \arrow[d, "\delta_l"] \\ & C_S\otimes(D_{V'}\oplus D_{W'}) \arrow[d, "\mathrm{id}_C\otimes\rho_D"] \\ & C_S\otimes D_T \end{tikzcd}$$

Recall that we defined $\delta_l$ in terms of $\delta_r$ and $c_\otimes$ and that it is usually not an identity morphism! These structure isomorphisms satisfy the requirements spelled out in Definition 3.5. In particular, they are associative: Let $n+1 \leq j \leq n+m$ and $U \cup V \cup W = T_j$. Then the diagram

$$\begin{tikzcd}
(C\otimes D)_{<S,T;j,W>}\oplus(C\otimes D)_{<S,T;j,U>}\oplus(C\otimes D)_{<S,T;j,V>} \arrow[r, "\mathrm{id}\oplus\delta_l"] \arrow[d, "\delta_l\oplus\mathrm{id}"'] & (C\otimes D)_{<S,T;j,W>}\oplus(C_{<S>}\otimes(D_{<T;j,U>}\oplus D_{<T;j,V>})) \arrow[d, "\mathrm{id}\oplus(\mathrm{id}_{C_{<S>}}\otimes\rho_D)"] \\
(C_{<S>}\otimes(D_{<T;j,W>}\oplus D_{<T;j,U>}))\oplus(C\otimes D)_{<S,T;j,V>} \arrow[d, "(\mathrm{id}_{C_{<S>}}\otimes\rho_D)\oplus\mathrm{id}"'] & (C\otimes D)_{<S,T;j,W>}\oplus(C\otimes D)_{<S,T;j,U\cup V>} \arrow[d, "\delta_l"] \\
(C\otimes D)_{<S,T;j,W\cap U>}\oplus(C\otimes D)_{<S,T;j,V>} \arrow[d, "\delta_l"'] & C_{<S>}\otimes(D_{<T;j,W>}\oplus D_{<T;j,U\cup V>}) \arrow[d, "\mathrm{id}_{C_{<S>}}\otimes\rho_D"] \\
C_{<S>}\otimes(D_{<T;j,W\cup U>}\oplus D_{<T;j,V>}) \arrow[r, "\mathrm{id}_{C_{<S>}}\otimes\rho_D"] & (C\otimes D)_{<S,T>}
\end{tikzcd}$$

commutes since $\delta_l$ is natural and associative and the additive structure isomorphisms $\rho_D$ are associative. In particular, $(\mathrm{id}\otimes(\rho_D\oplus\mathrm{id}))\circ\delta_l = \delta_l\circ(\mathrm{id}\otimes(\rho_D\oplus\mathrm{id}))$ since $\delta_l$ is natural. An analogous diagram for $1 \leq i \leq n$ and $S_i = U' \cup V' \cup W'$ commutes as well. Note that it makes use of the strict distributivity $\delta_r$ and the additive structure isomorphisms $\rho_C$. Furthermore, the structure isomorphisms satisfy the pentagon rule. Consult the next page for the diagram. We apologize for the missing details due to technical constraints.

In case of a bipermutative category $\mathcal{R} = (\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus, \otimes, 1_\mathcal{R}, c_\otimes, \delta_r, \delta_l)$, we define the multiplicative twist

$$\gamma_\otimes\colon \bar{H}\mathcal{R}(X_+^1, \ldots, X_+^n, X_+^{n+1}, \ldots, X_+^{n+m}) \longrightarrow \bar{H}\mathcal{R}(X_+^{n+1}, \ldots, X_+^{n+m}, X_+^1, \ldots, X_+^n)$$

as

$$\begin{aligned}
\gamma_\otimes^{C,D}\colon \{(C\otimes D)_{<S,T>}, \rho_{C\otimes D}\} &\longrightarrow \{(c_\otimes(C\otimes D))_{<T,S>}, \rho_{c_\otimes(C\otimes D)}(<S,T>;\chi^{m,n}(i), V, W)\} \\
&= \{(D\otimes C)_{<T,S>}, \rho_{D\otimes C}(<S,T>;\chi^{m,n}(i), V, W)\}
\end{aligned}$$

with $<S,T>= (S_1, \ldots, S_n, T_{n+1}, \ldots, T_{n+m})$ and $<T,S>= (T_{n+1}, \ldots, T_{n+m}, S_1, \ldots, S_n)$. This is a natural isomorphism for all $C, D \in \mathcal{R}$ since the twist $c_\otimes$ and the distributivity maps $\delta_r, \delta_l$ are natural in $\mathcal{R}$. Note that $\gamma_\otimes$ entails an exchange of the involved distributivity maps. Since $\delta_l\circ(c_\otimes\oplus c_\otimes) = c_\otimes\circ\delta_r$, this does not cause any problems.

Let $U \cup V = S_i, U' \cup V' = T_j$.
We use the abbreviations $(U,U') :=< S,T;i,U;j,U' >$, $< S;U' >:=< S,T;j,U' >$ and $< T;U >=< S,T;i,U >$. Furthermore,

$$(1) = ((\rho_C(< S,T >;i,U,V) \otimes \mathrm{id}_D) \oplus (\rho_C(< S,T >;i,U,V) \otimes \mathrm{id}_D)) \circ (\delta_r \oplus \delta_r),$$
$$(2) = ((\mathrm{id}_C \otimes \rho_D(< S,T >;j,U',V')) \oplus (\mathrm{id}_C \otimes \rho_D(< S,T >;j,U',V'))) \circ (\delta_l \oplus \delta_l).$$

We get the following diagram:

$$\begin{tikzcd}
 & (C\otimes D)_{<S;U'>}\oplus(C\otimes D)_{<S;V'>} \arrow[ddr, "(\mathrm{id}_C\otimes\rho_D(<S,T>;j,U',V'))\circ\delta_l"] & \\
(C\otimes D)_{(U,U')}\oplus(C\otimes D)_{(V,U')}\oplus(C\otimes D)_{(U,V')}\oplus(C\otimes D)_{(V,V')} \arrow[ur, "(1)"] \arrow[dd, "\mathrm{id}\oplus c_\oplus\oplus\mathrm{id}"'] & & \\
 & & (C\otimes D)_{<S,T>} \\
(C\otimes D)_{(U,U')>}\oplus(C\otimes D)_{<(U,V')}\oplus(C\otimes D)_{(V,U')}\oplus(C\otimes D)_{(V,V')} \arrow[dr, "(2)"'] & & \\
 & (C\otimes D)_{<T;U>}\oplus(C\otimes D)_{<T;V>} \arrow[uur, "(\rho_C(<S,T>;i,U,V)\otimes\mathrm{id}_D)\circ\delta_r"'] &
\end{tikzcd}$$

The naturality of the distributivity maps implies

$$\delta_l \circ ((\rho_C(< S,T >;i,U,V) \otimes \mathrm{id}_D) \oplus (\rho_C(< S,T >;i,U,V) \otimes \mathrm{id}_D)) = ((\rho_C(< S,T >;i,U,V) \otimes \mathrm{id}_{D\oplus D}) \circ \delta_l,$$
$$\delta_r \circ ((\mathrm{id}_C \otimes \rho_D(< S,T >;j,U',V')) \oplus (\mathrm{id}_C \otimes \rho_D(< S,T >;j,U',V'))) = (\mathrm{id}_{C\oplus C} \otimes \rho_D(< S,T >;j,U',V')) \circ \delta_r.$$

That is, we can assort the distributivity maps at the "left hand side" of the diagram. Finally, the diagram commutes since $\delta_l \circ (\delta_r \oplus \delta_r) \circ (\mathrm{id} \oplus c_\oplus \oplus \mathrm{id}) = \delta_r \circ (\delta_l \oplus \delta_l)$ (cf. Definiton 3.1, condition 10, for $\mathbf{J} = *$) and the product in $\mathcal{R}$ is natural.

Now let $Y_1,\dots,Y_{n+m}$ be pointed simplicial sets. Similar to the one above, we get a pairing

$$\begin{aligned}\mu\colon \bar{H}\mathcal{R}(Y_1,\dots,Y_n)\times\bar{H}\mathcal{R}(Y_{n+1},\dots,Y_{n+m}) &\longrightarrow \bar{H}\mathcal{R}(Y_1,\dots,Y_n,Y_{n+1},\dots,Y_{n+m})\\ (\{C_{<S>},\rho_C(<S>)\},\{D_{<T>},\rho_D(<T>)\}) &\longrightarrow \{(C\otimes D)_{<S,T>},\rho_{C\otimes D}(<S,T>)\}\end{aligned}$$

where we define $(C\otimes D)_{<S,T>}$ to be the $(n+m)$-simplicial object

$$(C\otimes D)_{<S,T>_{(l_1,\dots,l_{n+m})}} := C_{((S_1)_{l_1},\dots,(S_n)_{l_n})}\otimes D_{((T_{n+1})_{l_{n+1}},\dots,(T_{n+m})_{l_{n+m}})}\in\mathcal{R}.$$

The isomorphisms $\rho_{C\otimes D}$ are defined analogously to the ones above and two morphisms $f_{<S>},g_{<T>}$ are mapped to $(f\otimes g)_{<S,T>} := f_{<S>}\otimes g_{<T>}$.
For a bipermutative category $\mathcal{R}$, the multiplicative twist is defined as above.

Note that $\mu$ is natural, meaning that the following diagram commutes:

$$\begin{tikzcd}(\{C_{<S>},\rho_C(<S>)\},\{D_{<T>},\rho_D(<T>)\}) \arrow[r, "\mu^{C,D}"] \arrow[d, "{f_{<S>},g_{<T>}}"'] & \{(C\otimes D)_{<S,T>},(\rho_C\otimes\rho_D)(<S,T>)\} \arrow[d, "(f\otimes g)_{<S,T>}"] \\ (\{C'_{<S>},\rho_{C'}(<S>)\},\{D'_{<T>},\rho_{D'}(<T>)\}) \arrow[r, "\mu^{C',D'}"] & \{(C'\otimes D')_{<S,T>},(\rho_{C'}\otimes\rho_{D'})(<S,T>)\}\end{tikzcd}.$$

The functor $\mu$ induces a functor $K(\mu)\colon K^n_\bullet\mathcal{R}\times K^m_\bullet\mathcal{R}\to K^{n+m}_\bullet\mathcal{R}$ and we will now explain how this works exactly. For simplicity, we restrict to the case $n=m=1$.
First of all, we want to point out what we mean with the product of diagrams $\bar{H}\mathcal{R}(Y_{i_1})\times\bar{H}\mathcal{R}(Y_{i_2})$. Recall that $\bar{H}\mathcal{R}(Y_{i_j})$ is a functor $\mathbf{J}\to\mathbf{sStrict}$. Its product is a functor $\mathbf{J}^2\longrightarrow\mathbf{sStrict}^2$ that maps an object $(i_1,i_2)\in\mathbf{J}^2$ to $(\bar{H}\mathcal{R}(Y_{i_1}),\bar{H}\mathcal{R}(Y_{i_2}))\in\mathbf{sStrict}^2$. On the other hand, $\bar{H}\mathcal{R}(Y_{i_1},Y_{i_2})$ is a functor $\mathbf{J}^2\to\mathbf{sStrict}$. Thus, the multiplication

$$\mu\colon \bar{H}\mathcal{R}(Y_1)\times\bar{H}\mathcal{R}(Y_2)\longrightarrow\bar{H}\mathcal{R}(Y_1,Y_2)$$

induces a map of diagrams

$$\bar{H}\mathcal{R}(Y_{i_1})\times\bar{H}\mathcal{R}(Y_{i_2})\longrightarrow\bar{H}\mathcal{R}(Y_{i_1},Y_{i_2})$$

by application on each vertex. Finally, the universal property of the limit provides the functor

$$K(\mu)\colon \lim_{\mathbf{J}} \lim_{\mathbf{J}} \big(\bar{H}\mathcal{R}(Y_{i_1}), \bar{H}\mathcal{R}(Y_{i_2})\big) \longrightarrow \lim_{\mathbf{J}} \lim_{\mathbf{J}} \bar{H}\mathcal{R}(Y_{i_1}, Y_{i_2}) \cong \lim_{\mathbf{J}\times\mathbf{J}} \bar{H}\mathcal{R}(Y_{i_1}, Y_{i_2}).$$

### 3.3.2 Induced functors $K^m\mathcal{R} \to K^n\mathcal{R}$

Let $\mathbf{I}$ be the skeleton of the category of finite sets and injective maps as in Definition 1.8. For each injective map $\phi\colon [m] \to [n]$ we want to define a functor $\phi_*\colon K^m\mathcal{R} \to K^n\mathcal{R}$. Our construction is inspired by the presentation in [EM06], section 4.

**Definition 3.12.** Let $\phi\colon [m] \to [n]$ be an arbitrary morphism in $\mathbf{I}$. We define

$$\phi_*\colon \bar{H}\mathcal{R}(Y_1, \ldots, Y_m) \longrightarrow \bar{H}\mathcal{R}(Y_1^\phi, \ldots, Y_n^\phi) = \bar{H}\mathcal{R}(Y_{\phi^{-1}(1)}, \ldots, Y_{\phi^{-1}(n)})$$

with $Y_{\phi^{-1}(i)} = [1]_+$, the constant simplicial set on $[1]_+$, if $\phi^{-1}(i) = \emptyset$, to be the following functor: An object $\{C_{<S>}, \rho_C(< S >; i, T, U)\}$ is mapped to $\{C^\phi_{\phi<S>}, \rho_{C^\phi}(\phi < S > ; \phi^{-1}(i), T, U)\}$ with $C^\phi_{\phi<S>} = C_{\phi<S>}$, $\phi < S >= (S_{\phi^{-1}(1)}, \ldots, S_{\phi^{-1}(n)})$ and $S_{\phi^{-1}(i)} \subset [1]$ if $\phi^{-1}(i) = \emptyset$. The structure isomorphisms $\rho_{C^\phi}$ are given by

$$\rho_{C^\phi}(\phi < S >; \phi^{-1}(i), T, U) = \begin{cases} \rho_C(\phi < S >; \phi^{-1}(i), T, U) & \phi^{-1}(i) \neq \emptyset, \\ \mathrm{id} & \phi^{-1}(i) = \emptyset \text{ and } S_{\phi^{-1}(i)} \subset [1]. \end{cases}$$

A morphism $f = \{f_{<S>}\}$ is mapped to $f^\phi = \{f^\phi_{\phi<S>}\} = \{f_{\phi<S>}\}$.

The superscript $^\phi$ only serves to indicate that we understand the respective object as an image under $\phi_*$
To make evident that the functor $\phi_*$ is well-defined, we discuss some special cases.

Let $\sigma \in \Sigma_n$. In particular, $\sigma$ is an injective map $[n] \to [n]$. By definition,

$$\sigma_*\colon \bar{H}\mathcal{R}(Y_1, \ldots, Y_n) \to \bar{H}\mathcal{R}(Y_{\sigma^{-1}(1)}, \ldots, Y_{\sigma^{-1}(n)})$$

sends an object $\{C_{<S>}, \rho_C(< S >; i, T, U)\}$ to $\{C^\sigma_{\sigma<S>}, \rho_{C^\sigma}(\sigma < S >; \sigma^{-1}(i), T, U)\}$ with $C^\sigma_{\sigma<S>} = C_{\sigma<S>}$, $\sigma < S >= (S_{\sigma^{-1}(1)}, \ldots, S_{\sigma^{-1}(n)})$ and

$$\rho_{C^\sigma}(\sigma < S >; \sigma^{-1}(i), T, U) = \rho_C(\sigma < S >; \sigma^{-1}(i), T, U).$$

On morphisms, $\sigma_*$ sends $\{f_{<S>}\}$ to $\{f^\sigma_{\sigma<S>}\}$ with $f^\sigma_{\sigma<S>} = f_{\sigma<S>}$.
That is, $\sigma_*$ only permutes the simplicial sets $Y_i$.

**Lemma 3.13.** *The functor $\sigma_*\colon \bar{H}\mathcal{R}(Y_1,\ldots,Y_n) \to \bar{H}\mathcal{R}(Y_{\sigma^{-1}(1)},\ldots,Y_{\sigma^{-1}(n)})$ is an isomorphism of permutative categories.*

*Proof.* Given an object $C = \{C_{<S>}, \rho_C\} \in \bar{H}\mathcal{R}(Y_1,\ldots,Y_n)$, the identity morphism on $C$ is given by $\{\mathrm{id}_{C_{<S>}}\}$. The functor $\sigma_*$ maps this morphism to $\{\mathrm{id}_{C^\sigma_{\sigma<S>}}\} = \{\mathrm{id}_{\sigma_*(C_{<S>})}\}$. Thus, $\sigma_*(\mathrm{id}_C) = \mathrm{id}_{\sigma_*(C)}$. Furthermore, $\sigma_*(f\circ g) = \sigma_*(f)\circ\sigma_*(g)$, since

$$\sigma_*(f\circ g) = \{(f\circ g)^\sigma_{\sigma<S>}\} = \{f^\sigma_{\sigma<S>}\circ g^\sigma_{\sigma<S>}\} = \{f^\sigma_{\sigma<S>}\}\circ\{g^\sigma_{\sigma<S>}\} = \sigma_*(f)\circ\sigma_*(g).$$

Evidently, $\sigma_*$ is a bijection on objects and morphisms and thus an isomorphism of categories. To see that $\sigma_*$ is in fact an isomorphism of permutative categories, consider

$$\{C^\sigma_{\sigma<S>}, \rho_{C^\sigma}\} \oplus \{D^\sigma_{\sigma<S>}, \rho_{D^\sigma}\} = \{(C^\sigma \oplus D^\sigma)_{\sigma<S>}, \rho_{C^\sigma\oplus D^\sigma}\}$$

since on both sides objects and structure maps are given by

$$C_{\sigma<S>} \oplus D_{\sigma<S>}$$

and

$$\begin{array}{c}
(C_{\sigma(<S;i,U>)} \oplus D_{\sigma(<S;i,U>)}) \oplus (C_{\sigma(<S;i,V>)} \oplus D_{\sigma(<S;i,V>)}) \\
\Big\downarrow{\scriptstyle \mathrm{id}\oplus c\oplus\mathrm{id}} \\
(C_{\sigma(<S;i,U>)} \oplus C_{\sigma(<S;i,V>)}) \oplus (D_{\sigma(<S;i,U>)} \oplus D_{\sigma(<S;i,V>)}) \\
\Big\downarrow{\scriptstyle \rho_C\oplus\rho_D} \\
C_{\sigma(<S>)} \oplus D_{\sigma(<S>)}
\end{array}$$

respectively. Thereby, $\sigma(<S;i,U>) = (S_{\sigma^{-1}(1)},\ldots,S_{\sigma^{-1}(n)};\sigma^{-1}(i),U)$. The zero cube is mapped to the zero cube and all other conditions follow from the strictness in $\mathcal{R}$. $\square$

**Example 3.14.** Let $\sigma\in\Sigma_3$ be given by $1\mapsto 2, 2\mapsto 3, 3\mapsto 1$. Then

$$\begin{aligned}
\sigma_*\colon \bar{H}\mathcal{R}(Y_1,Y_2,Y_3) &\longrightarrow \bar{H}\mathcal{R}(Y_3,Y_1,Y_2)\\
\{C_{(S_1,S_2,S_3)}, \rho_C\} &\longmapsto \{C^\sigma_{(S_3,S_1,S_2)}, \rho_{C^\sigma}\},\\
f_{(S_1,S_2,S_3)} &\longmapsto f^\sigma_{(S_3,S_1,S_2)}.
\end{aligned}$$

**Lemma 3.15.** *Let $\sigma, \tau \in \Sigma_n$. Then $(\sigma \circ \tau)_* = \sigma_* \circ \tau_*$.*

*Proof.* We observe:

$$\begin{aligned}(\sigma \circ \tau)_* \colon \bar{H}\mathcal{R}(Y_1, \ldots, Y_n) \longrightarrow & \bar{H}\mathcal{R}(Y_{(\sigma\circ\tau)^{-1}(1)}, \ldots, Y_{(\sigma\circ\tau)^{-1}(n)}) \\ & = \bar{H}\mathcal{R}(Y_{\tau^{-1}(\sigma^{-1}(1))}, \ldots, Y_{\tau^{-1}(\sigma^{-1}(n))})\end{aligned}$$

and the latter is the image of $\bar{H}\mathcal{R}(Y_1, \ldots, Y_n)$ under $\sigma_* \circ \tau_*$. □

Let $\iota\colon [n] \to [n+1]$ be the standard inclusion missing the last element $n+1$. We consider $\bar{H}\mathcal{R}(Y_1, \ldots, Y_n, [1]_+)$ as an $n$-simplicial category via

$$\bar{H}\mathcal{R}(Y_1, \ldots, Y_n, [1]_+)_{(l_1,\ldots,l_n)} = \bar{H}\mathcal{R}((Y_1)_{l_1}, \ldots, (Y_n)_{l_n}, [1]_+).$$

**Lemma 3.16.** *The map $\iota\colon [n] \to [n+1]$ induces an isomorphism of permutative categories $\iota_*\colon \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}) \to \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}, [1]_+)$.*

*Proof.* The object $\{C_{<S>}, \rho_C(< S >; i, T, U)\}$ is sent to $\{C^\iota_{\iota<S>}, \rho_{C^\iota}(\iota < S >; i, T, U)\}$ with

$$C^\iota_{(S_1,\ldots,S_n,1)} = C_{<S>},\ C^\iota_{(S_1,\ldots,S_n,\emptyset)} = 0_\mathcal{R}$$

and

$$\rho_{C^\iota}((S_1, \ldots, S_n, 1); i, T, U) = \rho_C(< S >; i, T, U) \text{ for } i < n+1,$$

$$\rho_{C^\iota}((S_1, \ldots, S_n, 1); n+1, T, U) = \rho_{C^\iota}((S_1, \ldots, S_n, \emptyset); i, T, U) = \mathrm{id}.$$

The morphism $f = \{f_{<S>}\}$ is sent to the morphism $f^\iota = \{f^\iota_{\iota<S>}\}$ where

$$f^\iota_{(S_1,\ldots,S_n,1)} = f_{<S>} \text{ and } f^\iota_{(S_1,\ldots,S_n,\emptyset)} = \mathrm{id}.$$

The inverse is induced by dropping the $\{1\}$ from $(S_1, \ldots, S_n, 1)$.
Obviously, $\iota_*$ respects the permutative structure. □

The isomorphism $\iota_*\colon \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}) \to \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}, [1]_+)$ induces a map from $K^n_\bullet\mathcal{R}$ to the limit of the system $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}, [1]_+)$ (via the universal property of $\lim \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}, [1]_+)$). The natural maps from $[1]_+$ to $P\mathbb{S}^1_0 = [1]_+$ and $\mathbb{S}^1_0 = \{0\}$ then induce a map from the limit of the system $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n}, [1]_+)$ to $K^{n+1}_\bullet\mathcal{R}$ (via the universal property of $K^{n+1}_\bullet\mathcal{R}$). Putting it all together, we get a functor of permutative categories $K^n_\bullet\mathcal{R} \to K^{n+1}_\bullet\mathcal{R}$.

Similarly, each inclusion $\iota_j\colon [n] \to [n+1]$ which misses the element $j$ induces an isomorphism $\bar{H}\mathcal{R}(Y_1,\dots,Y_n) \to \bar{H}\mathcal{R}(Y_1,\dots,Y_{n+1})$ with $Y_j = [1]_+$ and thus a functor $K^n_\bullet\mathcal{R} \to K^{n+1}_\bullet\mathcal{R}$.
Every order-preserving injective map $i\colon [m] \to [n]$ is a composition of standard inclusions $\iota_j$. This composition is not unique, but if $i = \iota_h \circ \iota_j = \iota_k \circ \iota_l$, then $\iota_{h*} \circ \iota_{j*} = \iota_{k*} \circ \iota_{l*}$ for obvious reasons. Thus, the above definition extends to the definiton of a functor $i_*\colon K^m_\bullet\mathcal{R} \to K^n_\bullet\mathcal{R}$.

The functor $\phi_*$ as in Definition 3.12 is well-defined because of these isomorphisms.

**Example 3.17.** Let $\phi\colon [2] \to [4]$ be the injective map $1 \mapsto 4,\ 2 \mapsto 2$. Then:

$$\begin{aligned}
\phi_*\colon \bar{H}\mathcal{R}(Y_1,Y_2) &\longrightarrow \bar{H}\mathcal{R}(Y_1^\phi,Y_2^\phi,Y_3^\phi,Y_4^\phi) = \bar{H}\mathcal{R}([1]_+,Y_2,[1]_+,Y_1)\\
(C_{(S_1,S_2)},\rho_C((S_1,S_2);i,T,U)) &\longmapsto (C^\phi_{(1,S_2,1,S_1)},\rho_{C^\phi}(1,S_2,1,S_1);\phi^{-1}(i),T,U))\\
f_{(S_1,S_2)} &\longmapsto f^\phi_{(1,S_2,1,S_1)}
\end{aligned}$$

where 1 is a subfunctor of the constant simplicial set [1].

We now want to apply these results on diagrams of categories. For each injective map $\phi\colon [m] \to [n]$ the functor $\phi_*\colon \bar{H}\mathcal{R}(Y_1,\dots,Y_m) \to \bar{H}\mathcal{R}(Y_1^\phi,\dots,Y_n^\phi)$ induces a functor of diagrams $\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_m}) \to \bar{H}\mathcal{R}(Y_{i_1}^\phi,\dots,Y_{i_n}^\phi)$ in **sStrict** by applying $\phi_*$ on each vertex. We denote this functor of diagrams with $\phi_*$ as well.
One word on notation: $\{C_{(S_{i_1},\dots,S_{i_n})},\rho_C\} \in \bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_n})$ is a diagram of objects where $S_{i_j}$ is a subfunctor of $Y_{i_j}$ such that $(S_{i_j})_k$ does not contain the basepoint for all $k \in \Delta$. Likewise, a morphism $f$ in $\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_n})$ is a diagram of morphisms $f_{(S_{i_1},\dots,S_{i_n})}$. If we shorten notation we write again $< S >$ instead of $(S_{i_1},\dots,S_{i_n})$.

The functor $\phi_*$ on $\bar{H}\mathcal{R}$ induces a functor $K(\phi_*)\colon K^m\mathcal{R} \to K^n\mathcal{R}$ via the universal property:

$$\begin{tikzcd}
K^m\mathcal{R} \arrow[rr, "K(\phi_*)"] \arrow[d] & & K^n\mathcal{R} \arrow[d] \\
\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_m}) \arrow[rr, "\phi_*"] & & \bar{H}\mathcal{R}(Y_{i_1}^\phi,\dots,Y_{i_n}^\phi).
\end{tikzcd}$$

Since, $(\psi\circ\phi)^{-1} = \phi^{-1}\circ\psi^{-1}$, we have $\psi_*\circ\phi_* = (\psi\circ\phi)_*$. To sum up:

**Lemma 3.18.** *The assignment $[n] \to K^n\mathcal{R}$ defines a covariant functor $K^\bullet\mathcal{R}$ from $\boldsymbol{I}$ to the category of simplicial objects in **Strict**.*

### 3.3.3 Main theorem

The product and the functors induced by maps from $\mathbf{I}$ fit together in a very good way:

**Example 3.19.** Let $\phi\colon [1] \to [2],\ 1 \mapsto 2$ and $\psi : [2] \to [3],\ 1 \mapsto 3, 2 \mapsto 1$. Then:

$$\begin{tikzcd}
\bar{H}\mathcal{R}(Y_{i_1}) \times \bar{H}\mathcal{R}(Y_{j_1}, Y_{j_2}) \arrow[r, "\mu"] \arrow[d, "(\phi_*,\psi_*)"'] & \bar{H}\mathcal{R}(Y_{i_1}, Y_{j_1}, Y_{j_2}) \arrow[d, "\phi_*\otimes\psi_*"] \\
\bar{H}\mathcal{R}([1]_+, Y_{i_1}) \times \bar{H}\mathcal{R}(Y_{j_2}, [1]_+, Y_{j_1}) \arrow[r, "\mu"] & \bar{H}\mathcal{R}([1]_+, Y_{i_1}, Y_{j_2}, [1]_+, Y_{j_1})
\end{tikzcd}$$

On objects:

$$\begin{tikzcd}
(\{C_S, \rho_C(S)\}, \{D_{(T_1,T_2)}, \rho_D(T_1,T_2)\}) \arrow[r] \arrow[d, mapsto] & \{C_S \otimes D_{(T_1,T_2)}, \rho_{C\otimes D}(S, T_1, T_2)\} \arrow[d, mapsto] \\
(\{C^\phi_{(1,S)}, \rho^\phi_C(1,S)\}, \{D^\psi_{(T_2,1,T_1)}, \rho^\psi_D(T_2,1,T_1)\}) \arrow[r] & \{C^\phi_{(1,S)} \otimes D^\psi_{(T_2,1,T_1)}, \rho_{C^\phi\otimes D^\psi}(1,S,T_2,1,T_1)\}
\end{tikzcd}$$

where $\rho_{C^\phi\otimes D^\psi}(1,S,T_2,1,T_1)$ is defined as

$$\rho_{C^\phi\otimes D^\psi}((1,S,T_2,1,T_1); i, U, V) = (\rho_{C^\phi}((1,S); \phi^{-1}(i), U, V) \otimes \mathrm{id}_{D_{(T_2,1,T_1)}}) \circ \delta_r$$

for $i \in \{1,2\}$ and

$$\rho_{C^\phi\otimes D^\psi}((1,S,T_2,1,T_1); i, U', V') = (\mathrm{id}_{C_{(1,S)}} \otimes \rho_{D^\psi}((T_2,1,T_1); \psi^{-1}(i), U', V')) \circ \delta_l$$

for $i \in \{3,4,5\}$.

On morphisms:

$$\begin{tikzcd}
(f_S, g_{(T_1,T_2)}) \arrow[r] \arrow[d, mapsto] & (f \otimes g)_{(S,T_1,T_2)} \arrow[d, mapsto] \\
(f^\phi_{(1,S)}, g^\psi_{(T_2,1,T_1)}) \arrow[r] & (f^\phi \otimes g^\psi)_{(1,S,T_2,1,T_1)}
\end{tikzcd}$$

On limits:

$$\begin{tikzcd}
K^1\mathcal{R}\times K^1\mathcal{R} \arrow[rrr, "K(\mu)"] \arrow[ddd, "{(K(\phi_*),K(\psi_*))}"'] \arrow[dr] & & & K^2\mathcal{R} \arrow[dl] \arrow[ddd, "K(\phi_*)\otimes K(\psi_*)"] \\
& \bar{H}\mathcal{R}(Y_{i_1})\times\bar{H}\mathcal{R}(Y_{j_1},Y_{j_2}) \arrow[r, "\mu"] \arrow[d, "{(\phi_*,\psi_*)}"'] & \bar{H}\mathcal{R}(Y_{i_1},Y_{j_1},Y_{j_2}) \arrow[d, "\phi_*\otimes\psi_*"] & \\
& \bar{H}\mathcal{R}([1]_+,Y_{i_1})\times\bar{H}\mathcal{R}(Y_{j_2},[1]_+,Y_{j_1}) \arrow[r, "\mu"] & \bar{H}\mathcal{R}([1]_+,Y_{i_1},Y_{j_2},[1]_+,Y_{j_1}) & \\
K^2\mathcal{R}\times K^3\mathcal{R} \arrow[rrr, "K(\mu)"'] \arrow[ur] & & & K^5\mathcal{R} \arrow[ul]
\end{tikzcd}$$

The outer maps exist (and the diagram commutes) because of the universal property of the limit.

**Proposition 3.20.** *Let $\mathcal{R}$ be a bipermutative category and $(\boldsymbol{I},+,\emptyset,\chi)$ the skeleton of the category of finite sets and injective maps. The assignment $[n]\mapsto K^n\mathcal{R}$ with $K^0\mathcal{R}$ given by $\mathcal{R}$ turns $K^{\bullet}\mathcal{R}$ into an $\boldsymbol{I}$-graded bipermutative category.*

*Proof.* Basically, because of how the product and other structure maps are defined, all conditions hold since we require the underlying category $\mathcal{R}$ to be bipermutative. The numbering refers to Definition 3.1.

1. We have already discussed the product and given an example indicating that the product is natural in $\mathbf{I}$. In fact, for $\phi\colon [k]\to[m]$ and $\psi\colon [l]\to[n]$ the following diagrams commute for all objects $\{C_{<S>},\rho_C\}\in\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_k})$, $\{D_{<T>},\rho_D\}\in\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_l})$ and morphisms $f$ in $\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_k})$, $g$ in $\bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_l})$:

$$\begin{tikzcd}
\{C_{<S>},\rho_C\},\{D_{<T>},\rho_D\} \arrow[r, "\mu"] \arrow[d, mapsto, "{\phi_*,\psi_*}"'] & \{C_{<S>}\otimes D_{<T>},\rho_C\otimes\rho_D\} \arrow[d, mapsto, "\phi_*\otimes\psi_*"] \\
\{C^{\phi}_{\phi<S>},\rho_{C^{\phi}}\},\{D^{\psi}_{\psi<T>},\rho_{D^{\psi}}\} \arrow[r, "\mu"] & \{C^{\phi}_{\phi<S>}\otimes D^{\psi}_{\psi<T>},\rho_{C^{\phi}\otimes D^{\psi}}\},
\end{tikzcd}$$

$$\begin{tikzcd}
f_{<S>},g_{<T>} \arrow[r, "\mu"] \arrow[d, mapsto, "{\phi_*,\psi_*}"'] & f_{<S>}\otimes g_{<T>} \arrow[d, mapsto, "\phi_*\otimes\psi_*"] \\
f^{\phi}_{\phi<S>},g^{\psi}_{\psi<T>} \arrow[r, "\mu"] & f^{\phi}_{\phi<S>}\otimes g^{\psi}_{\psi<T>}.
\end{tikzcd}$$

The commutativity is guaranteed by the naturality of $\otimes$ in $\mathcal{R}$. Regarding the structure

maps $\rho$, recall that $\phi_*$ and $\psi_*$ only work on the simplicial sets $Y_{i_j}$, the structure maps themselves remain the same.
This implies that the respective diagram for limits commutes as well (based on the universal property of the limit) (cf. the diagram on the next page). We point out that all work is done on the level of diagrams $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n})$. Commutative diagrams on this level induce commutative diagramms on limits. We will therefore omit this last step in the following.

2. The unit 1 is given by $1_\mathcal{R} \in \mathcal{R}$ and the multiplication

$$\mathcal{R} \times \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_j}) \longrightarrow \bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_j})$$

by

$$(D, \{C_{<S>}, \rho_C(< S >; i, T, U)\}) \longrightarrow \{D \otimes C_{<S>}, (\mathrm{id}_D \otimes \rho_C(< S >; i, T, U)) \circ \delta_l\}.$$

Then

$$\begin{aligned}(1, \{C_{<S>}, \rho_C(< S >; i, T, U)\}) \xrightarrow{\otimes} &\{1_\mathcal{R} \otimes C_{<S>}, (\mathrm{id}_1 \otimes \rho_C(< S >; i, T, U)) \circ \delta_l\} \\ &= \{C_{<S>}, \rho_C(< S >; i, T, U)\}.\end{aligned}$$

3. We already defined the multiplicative twist. Evidently, $\gamma_\otimes$ is natural since it is induced by the twist $c_\otimes$ in $\mathcal{R}$. For the same reason, $\gamma_\otimes^{D,C} \circ \gamma_\otimes^{C,D}$ is the identity on $\{(C \otimes D)_{<S,T>}, \rho_{C\otimes D}\}$.
Moreover, the following diagram commutes for $\phi\colon [k] \to [n]$ and $\psi\colon [l] \to [m]$:

$$\begin{tikzcd}
C_{<S>} \otimes D_{<T>} \arrow[r, "c_\otimes^{C,D}"] \arrow[d, mapsto, "\phi_*\otimes\psi_*"'] & D_{<T>} \otimes C_{<S>} \arrow[d, mapsto, "\psi_*\otimes\phi_*"] \\
C^\phi_{\phi<S>} \otimes D^\psi_{\psi<T>} \arrow[r, "c_\otimes^{C^\phi,D^\psi}"] & D^\psi_{\psi<T>} \otimes C^\phi_{\phi<S>}
\end{tikzcd}$$

since it is a diagram in $\mathcal{R}$ and the twist $c_\otimes$ is natural in $\mathcal{R}$. Regarding the structure maps $\rho$, recall that all involved structure maps in $\mathcal{R}$ are natural and the maps $\rho$ remain unchanged.

4. Let $< S >= (S_1, \ldots, S_k)$, $< T >= (T_{k+1}, \ldots, T_{k+l})$ and $< U >= (U_{k+l+1}, \ldots, U_{k+l+m})$.

$$
\begin{tikzcd}
K^k\mathcal{R}\times K^l\mathcal{R} \arrow[rrr, "K(\mu)"] \arrow[dr] \arrow[ddd, "{(K(\phi_*),K(\psi_*))}"'] & & & K^{k+l} \arrow[dl] \arrow[ddd, "K(\phi_*)\otimes K(\psi_*)"] \\
& \bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_k})\times\bar{H}\mathcal{R}(Y_{j_1},\dots,Y_{j_l}) \arrow[r, "\mu"] \arrow[d, "{(\phi_*,\psi_*)}"'] & \bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_k},Y_{j_1},\dots,Y_{j_l}) \arrow[d, "\phi_*\otimes\psi_*"] & \\
& \bar{H}\mathcal{R}(Y^{\phi}_{i_1},\dots,Y^{\phi}_{i_m})\times\bar{H}\mathcal{R}(Y^{\psi}_{j_1},\dots,Y^{\psi}_{j_n}) \arrow[r, "\mu"] & \bar{H}\mathcal{R}(Y^{\phi}_{i_1},\dots,Y^{\phi}_{i_m},Y^{\psi}_{j_1},\dots,Y^{\psi}_{j_n}) & \\
K^m\mathcal{R}\times K^n\mathcal{R} \arrow[rrr, "K(\mu)"'] \arrow[ur] & & & K^{m+n} \arrow[ul]
\end{tikzcd}
$$

For all $\{C_{<S>}, \rho_C\} \in \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_k})$, $\{D_{<T>}, \rho_D\} \in \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_l})$ and $\{E_{<U>}, \rho_E\} \in \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_m})$, we know

$$\begin{aligned}(C \otimes (D \otimes E))_{<S,T,U>} &= C_{<S>} \otimes (D_{<T>} \otimes E_{<U>}) \\ &= (C_{<S>} \otimes D_{<T>}) \otimes E_{<U>} = ((C \otimes D) \otimes E)_{<S,T,U>}\end{aligned}$$

since the product in $\mathcal{R}$ is strictly associative (same argument for morphisms). Moreover, $\rho_{(C\otimes D)\otimes E}(< S, T, U >; i, V, W) = \rho_{C\otimes(D\otimes E)}(< S, T, U >; i, V, W)$ since both are defined as

$$\begin{aligned}&\rho_{C\otimes D\otimes E}(< S, T, U >; i, V, W) \\ =&\begin{cases}(\rho_C \otimes \mathrm{id}_{C\otimes E}) \circ \delta_r & 1 \le i \le k, \\ (\mathrm{id}_C \otimes \rho_D \otimes \mathrm{id}_E) \circ (\delta_l \otimes \mathrm{id}_E) \circ \delta_r & k+1 \le i \le k+l, \\ (\mathrm{id}_{C\otimes D} \otimes \rho_E) \circ \delta_l & k+l+1 \le i \le k+l+m.\end{cases}\end{aligned}$$

For $k+1 \le i \le k+l$, the map $\rho_{C\otimes D\otimes E}(< S, T, U >; i, V, W)$ could as well be defined as $(\mathrm{id}_C \otimes \rho_D \otimes \mathrm{id}_E) \circ (\mathrm{id}_C \otimes \delta_r) \circ \delta_l$. To see that both definitions agree recall that the diagram

$$\begin{tikzcd}
(C_{<S>}\otimes D_{<T;i,V>}\otimes E_{<U>})\oplus(C_{<S>}\otimes D_{<T;i,W>}\otimes E_{<U>}) \arrow[r, "\delta_l"] \arrow[d, "\delta_r"'] & C_{<S>}\otimes(D_{<T;i,V>}\otimes E_{<U>}\oplus D_{<T;i,W>}\otimes E_{<U>}) \arrow[d, "\mathrm{id}_{C_{<S>}}\otimes\delta_r"] \\
(C_{<S>}\otimes D_{<T;i,V>}\oplus C_{<S>}\otimes D_{<T;i,W>})\otimes E_{<U>} \arrow[r, "\delta_l\otimes\mathrm{id}_{E_{<U>}}"] & C_{<S>}\otimes(D_{<T;i,V>}\oplus D_{<T;i,W>})\otimes E_{<U>}
\end{tikzcd}$$

is a diagram in $\mathcal{R}$ and commutes since $\mathcal{R}$ is bipermutative (cf. Definitions 3.1, 3.2 for $\mathbf{J} = *$).

For $k+1 \le i \le k+l$ and $k+l+1 \le i \le k+l+m$, the structure maps make use of the left distributivity map. Recall that $\delta_l = c_\otimes \circ \delta_r \circ (c_\otimes \oplus c_\otimes)$. Thus, in all three cases, the structure map $\rho_{C\otimes D\otimes E}(< S, T, U >; i, V, W)$ is defined as: Shuffle the objects which you want to sum up ($C, D$ or $E$) to the left (if necessary). Use $\delta_r = \mathrm{id}$. Shuffle it back (if necessary). Apply the appopriate structure isomorphism ($\rho_C, \rho_D$ or $\rho_E$) together with the identity map on the other objects. Since the product is natural, the diagram

$$\begin{tikzcd}
(E_{<U;i,V>} \oplus E_{<U;i,W>}) \otimes C_{<S>} \otimes D_{<T>} \arrow[r, "c_\otimes"] \arrow[d, "\rho_E\otimes\mathrm{id}"'] & C_{<S>} \otimes D_{<T>} \otimes (E_{<U;i,V>} \oplus E_{<U;i,W>}) \arrow[d, "\mathrm{id}\otimes\rho_E"] \\
E_{<U>} \otimes C_{<S>} \otimes D_{<T>} \arrow[r, "c_\otimes"] & C_{<S>} \otimes D_{<T>} \otimes E_{<U>}
\end{tikzcd}$$

commutes. (So does the accordant one for $k+1 \leq i \leq k+l$.) Hence, we can say that in all cases, the structure maps are defined as: Shuffle the objects which you want to sum up to the left (if necessary). Use $\delta_r = \mathrm{id}$. Apply the appopriate structure isomorphism together with the identity map on the other objects. Shuffle it back (if necessary). This will be helpful when we now show that associativity is compatible with twists.
Let $\chi_1 = \chi(l,m), \chi_2 = \chi(k+l,m), \chi_3 = \chi(m,k)$ and consider the diagramm

$$\begin{tikzcd}
C_{<S>}\otimes D_{<T>}\otimes E_{<U>} \arrow[r] \arrow[d] & \chi_{2*}E_{<U>}\otimes\chi_{2*}C_{<S>}\otimes\chi_{2*}D_{<T>} \arrow[d] \\
C_{<S>}\otimes(\mathrm{id}_k\oplus\chi_1)_*E_{<U>}\otimes(\mathrm{id}_k\oplus\chi_1)_*D_{<T>} \arrow[r, "(*)"] & \chi_{2*}(\chi_{3*}C_{<S>})\otimes\chi_{2*}(\chi_{3*}E_{<U>})\otimes\chi_{2*}D_{<T>}
\end{tikzcd} \tag{3.2}$$

with

$$\begin{aligned}
\chi_{2*}E_{<U>} &:= E_{(U_{\chi_2^{-1}(1)},\dots,U_{\chi_2^{-1}(m)})},\\
\chi_{2*}C_{<S>} &:= C_{(S_{\chi_2^{-1}(m+1)},\dots,S_{\chi_2^{-1}(m+k)})},\\
\chi_{2*}D_{<T>} &:= D_{(T_{\chi_2^{-1}(m+k+1)},\dots,T_{\chi_2^{-1}(m+k+l)})},
\end{aligned}$$

$$\begin{aligned}
(\mathrm{id}_k\oplus\chi_1)_*E_{<U>} &:= E_{(U_{k+\chi_1^{-1}(1)},\dots,U_{k+\chi_1^{-1}(m)})},\\
(\mathrm{id}_k\oplus\chi_1)_*D_{<T>} &:= D_{(T_{k+\chi_1^{-1}(m+1)},\dots,T_{k+\chi_1^{-1}(m+l)})},
\end{aligned}$$

$$\begin{aligned}
\chi_{2*}(\chi_{3*}C_{<S>}) &:= C_{(S_{\chi_2^{-1}(\chi_3^{-1}(1))},\dots,S_{\chi_2^{-1}(\chi_3^{-1}(k))})},\\
\chi_{2*}(\chi_{3*}E_{<U>}) &:= E_{(U_{\chi_2^{-1}(\chi_3^{-1}(k+1))},\dots,U_{\chi_2^{-1}(\chi_3^{-1}(k+m))})},\\
\chi_{2*}D_{<T>} &= D_{(T_{\chi_2^{-1}(m+k+1)},\dots,T_{\chi_2^{-1}(m+k+l)})}.
\end{aligned}$$

The lower map $(*)$ is the identity:

$$\begin{aligned}
&C_{(S_{\chi_2^{-1}(\chi_3^{-1}(1))},\dots,S_{\chi_2^{-1}(\chi_3^{-1}(k))})}\otimes E_{(U_{\chi_2^{-1}(\chi_3^{-1}(k+1))},\dots,U_{\chi_2^{-1}(\chi_3^{-1}(k+m))})}\otimes D_{(T_{\chi_2^{-1}(m+k+1)},\dots,T_{\chi_2^{-1}(m+k+1)})}\\
&= C_{(S_{\chi_2^{-1}(m+1)},\dots,S_{\chi_2^{-1}(m+k)})}\otimes E_{(U_{\chi_2^{-1}(1)},\dots,U_{\chi_2^{-1}(m)})}\otimes D_{(T_{\chi_2^{-1}(m+k+1)},\dots,T_{\chi_2^{-1}(m+k+1)})}\\
&= C_{<S>}\otimes E_{(U_{k+l+1},\dots,U_{k+l+m})}\otimes D_{(T_{k+1},\dots,T_{k+l})}\\
&= C_{<S>}\otimes E_{(U_{k+\chi_1^{-1}(1)},\dots,U_{k+\chi_1^{-1}(m)})}\otimes D_{(T_{k+\chi_1^{-1}(m+1)},\dots,T_{k+\chi_1^{-1}(m+l)})}.
\end{aligned}$$

Diagram 3.2 commutes since it is a diagram in the bipermutative category $\mathcal{R}$. We get a commutative diagram of objects in $\bar{H}\mathcal{R}(Y_{i_1},\ldots,Y_{i_{k+l+m}})$, since the structure maps only differ in the order in which one shuffles the objects to be summed up to the left. However, in a permutative category (here, we refer to $(\mathcal{R},\otimes,c_\otimes)$), it does not matter in which order one permutes (cf. Definition 1.7).

5. Recall that the zero object $0_j \in \bar{H}\mathcal{R}(Y_{i_1},\ldots,Y_{i_j})$ is the zero cube, that is the cube $\{C_{<S>},\rho_C(<S>;i,T,U)\}$ with $C_{<S>}=0_{\mathcal{R}}$ and $\rho_C(<S>;i,T,U)=\mathrm{id}$ for all $<S>=(S_1,\ldots,S_j)$ and $T,U\in S_i$ with the according properties. Then

$$\begin{aligned}(0_j,\{C_{<S'>},\rho_C(<S'>;i,T,U)\}) \xrightarrow{\otimes} &\{(0_{\mathcal{R}}\otimes C)_{<S,S'>},\mathrm{id}\otimes\rho_C(<S'>;i,T,U)\}\\ &=0_{j+m}\end{aligned}$$

for all objects $\{C_{<S'>},\rho_C(<S'>;i,T,U)\}\in\bar{H}\mathcal{R}(Y_{i_1},\ldots,Y_{i_j})$.

6. Let $\{C_{<S>},\rho_C\},\{D_{<S>},\rho_D\}\in\bar{H}\mathcal{R}(Y_{i_1},\ldots,Y_{i_n})$ and $\{E_{<T>},\rho_E\}\in\bar{H}\mathcal{R}(Y_{i_1},\ldots,Y_{i_m})$. We have to show that the right distributivity map $d_r$ is the identity:

$$\begin{gathered}\{((C\oplus D)\otimes E)_{<S,T>},(\rho_{(C\oplus D)\otimes E})(<S,T>;i,U,V)\}\\ =\\ \{(C\otimes E\oplus D\otimes E)_{<S,T>},\rho_{C\otimes E\oplus D\otimes E}(<S,T>;i,U,V)\}.\end{gathered}$$

Since right distributivity is strict in $\mathcal{R}$, we know that this is true on objects:

$$\begin{gathered}(C\oplus D)_{(S_{i_1},\ldots,S_{i_n})_{(l_1,\ldots,l_n)}}\otimes E_{(T_{i_1},\ldots,T_{i_m})_{(k_1,\ldots,k_m)}}\\ =\\ C_{(S_{i_1},\ldots,S_{i_n})_{(l_1,\ldots,l_n)}}\otimes E_{(T_{i_1},\ldots,T_{i_m})_{(k_1,\ldots,k_m)}}\oplus D_{(S_{i_1},\ldots,S_{i_n})_{(l_1,\ldots,l_n)}}\otimes E_{(T_{i_1},\ldots,T_{i_m})_{(k_1,\ldots,k_m)}}\end{gathered}$$

for all $(l_1,\ldots,l_n)\in\Delta^n,(k_1,\ldots,k_m)\in\Delta^m$.
Regarding the structure maps, we find

$$\rho_{(C\oplus D)\otimes E}(<S,T>;i,U,V)=\begin{cases}(\mathrm{id}_{C\oplus D}\otimes\rho_E)\circ\delta_l & i>n,\\ (\rho_{C\oplus D}\otimes\mathrm{id}_E)\circ\delta_r & i\leq n\end{cases}$$

and

$$\rho_{C\otimes E\oplus D\otimes E}(< S,T >; i,U,V) = \begin{cases} \delta_r \circ (\mathrm{id}_{C\oplus D} \otimes \rho_E) \circ \delta_l \circ (\delta_r \oplus \delta_r) & i > n, \\ \delta_r \circ (\rho_{C\oplus D}\mathrm{id}_E) \circ \delta_r \circ (\delta_r \oplus \delta_r) & i \leq n \end{cases}$$
$$= \begin{cases} (\mathrm{id}_{C\oplus D} \otimes \rho_E) \circ \delta_l & i > n, \\ (\rho_{C\oplus D} \otimes \mathrm{id}_E) \circ \delta_r & i \leq n \end{cases}$$

where the last equation is again valid since $\delta_r$ is an identity map in $\mathcal{R}$. To make this more palpable, we write out the case $i > n$:

$$((C_{<S>} \otimes E_{<T;i,U>}) \oplus (D_{<S>} \otimes E_{<T;i,U>})) \oplus ((C_{<S>} \otimes E_{<T;i,V>}) \oplus (D_{<S>} \otimes E_{<T;i,V>}))$$
$$\downarrow \delta_r \oplus \delta_r$$
$$((C_{<S>} \oplus D_{<S>}) \otimes E_{<T;i,U>}) \oplus ((C_{<S>} \oplus D_{<S>}) \otimes E_{<T;i,V>})$$
$$\downarrow \delta_l$$
$$(C_{<S>} \oplus D_{<S>}) \otimes (E_{<T;i,U>} \oplus E_{<T;i,V>})$$
$$\downarrow \mathrm{id}_{C\oplus D} \otimes \rho_E$$
$$(C_{<S>} \oplus D_{<S>}) \otimes E_{<T>}$$
$$\downarrow \delta_r$$
$$(C_{<S>} \otimes E_{<T>}) \oplus (D_{<S>} \otimes E_{<T>}).$$

Furthermore, the same argument as for objects applies to morphisms $f, g$ in $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n})$ and $h$ in $\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_m})$:

$$((f \oplus g) \otimes h)_{<S,T>} = ((f \otimes h) \oplus (g \otimes h))_{<S,T>}.$$

7. Left distributivity is defined as

$$d_l = \gamma_\otimes \circ d_r \circ (\gamma_\otimes \oplus \gamma_\otimes).$$

8. The conditions $\gamma_\oplus \circ (\gamma_\otimes \oplus \gamma_\otimes) = (\gamma_\otimes \oplus \gamma_\otimes) \circ \gamma_\oplus$ and $(\gamma_\oplus \otimes \mathrm{id}) \circ \gamma_\otimes = \gamma_\otimes \circ (\mathrm{id} \otimes \gamma_\oplus)$ hold, since the twists $\gamma$ are induced by the twists $c$ in $\mathcal{R}$ which satisfy these conditions. Moreover, the structure maps are defined summandwise and are hence not affected by the additive twists.

9. It is clear that the diagram

$$\begin{tikzcd}
(C \otimes D \otimes E)_{<S,T,U>} \oplus (C \otimes D \otimes E')_{<S,T,U>} \arrow[d, "\delta_l"'] \arrow[dr, "\delta_l"] & \\
C_{<S>} \otimes ((D \otimes E)_{<T,U>} \oplus (D \otimes E')_{<T,U>}) \arrow[r, "\mathrm{id}\otimes\delta_l"] & (C \otimes D)_{<S,T>} \otimes (E \oplus E')_{<U>}
\end{tikzcd}$$

commutes since it is a diagram of objects in the bipermutative category $\mathcal{R}$. Concerning the structure maps, recall that the product is strictly associative which in our case suffices to see that the distributivity maps are associative with respect to structure maps.

10. The pentagon diagram commutes because the underlying diagrams commute in $\mathcal{R}$. Note that in all cases, the structure maps are defined as a composition of distributivity maps and $(\rho_{C\oplus C'}) \otimes (\rho_{D\oplus D'})$. The respective diagram commutes since the according one for the distributivity maps in $\mathcal{R}$ commutes. □

**Corollary 3.21.** *Let $\mathcal{R}$ be a strictly bimonoidal category and $\boldsymbol{I}$ as above. Then $K^\bullet\mathcal{R}$ is an $\boldsymbol{I}$-graded strictly bimonoidal category.*

## 3.4 $K^\bullet\mathcal{R}$ defines a multiplicative group completion

**Proposition 3.22.** *Let $\mathcal{R}$ be a strictly bimonoidal category. The canonical inclusion $\iota\colon [n] \to [n+1]$ induces an unstable equivalence $K^n_\bullet\mathcal{R} \to K^{n+1}_\bullet\mathcal{R}$ for $n \geq 1$.*

The outline of the following proof is due to Baas, Dundas, Richter and Rognes.

*Proof.* We know that $K^n_\bullet\mathcal{R} \cong \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+)$ (induced by $\iota$, see Lemma 3.16). Since $n$ is at least one and the limit can be taken iteratively, $\pi_0(\lim\bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+))$ is a group by use of an anologous argument as for $K^1_\bullet\mathcal{R}$. Thus, the $H$-space $|\mathrm{diag}N \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+)|$ is group-like:

$$|\mathrm{diag}N \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+)| \simeq \Omega B|\mathrm{diag}N \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+)|.$$

By Proposition 2.1.3 in [Tho79],

$$\Omega B|\mathrm{diag}N \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, [1]_+) = \Omega|\mathrm{diag}N \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, \mathbb{S}^1_\bullet)|.$$

From [Tho79], Proposition 4.3.2 we deduce that

$$|\mathrm{diag}NK^1_\bullet\mathcal{R}| \simeq \Omega|\mathrm{diag}N\bar{H}\mathcal{R}(\mathbb{S}^1_\bullet)|.$$

Since $K^n_\bullet\mathcal{R} \cong \lim \bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_n},[1]_+)$ is a permutative category, we get as well

$$\Omega|\mathrm{diag}N\lim \bar{H}\mathcal{R}(Y_{i_1},\dots,Y_{i_n},\mathbb{S}^1_\bullet)| \simeq |\mathrm{diag}NK^{n+1}_\bullet\mathcal{R}|$$

which finishes the proof. □

We have proven so far that $K^\bullet\mathcal{R}$ is an $\mathbf{I}$-graded bipermutative category such that any $[m]\to[n]$ induces a stable equivalence $K^m_\bullet\mathcal{R}\to K^n_\bullet\mathcal{R}$ and that for $m,n\geq 1$ any such map induces an unstable equivalence $K^m_\bullet\mathcal{R}\to K^n_\bullet\mathcal{R}$. Hence, Lemma 3.3 and Proposition 3.4 yield that the canonical chain

$$K^m_\bullet\mathcal{R} \xleftarrow{\sim} ZK^m_\bullet\mathcal{R} \longrightarrow D\mathrm{hocolim}_{\mathbf{I}}K^\bullet\mathcal{R}$$

is a stable equivalence of $Z\mathcal{R}$-modules for all $[m]\in\mathbf{I}$ and an unstable equivalence for all $[m]\in\mathbf{I}, m\geq 1$. In particular, this gives an unstable equivalence

$$K^1_\bullet\mathcal{R} \xleftarrow{\sim} ZK^1_\bullet\mathcal{R} \longrightarrow D\mathrm{hocolim}_{\mathbf{I}}K^\bullet\mathcal{R}$$

and thus provides that $D\mathrm{hocolim}_{\mathbf{I}}K^\bullet\mathcal{R}$ is group complete. Furthermore,

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow D\mathrm{hocolim}_{\mathbf{I}}K^\bullet\mathcal{R}$$

are maps of simplicial bipermutative categories. Consequently, the Segal model defines a multiplicative group completion:

**Theorem 3.23.** *Let $\mathcal{R}$ be a strictly bimonoidal (bipermutative) category, then $\bar{\mathcal{R}} = D\mathrm{hocolim}_{\mathbf{I}}K^\bullet\mathcal{R}$ is a simplicial category such that $\pi_0\bar{\mathcal{R}}$ is a (commutative) ring and there are stable equivalences*

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}} \tag{3.3}$$

*of simplicial strictly bimonoidal (bipermutative) categories.*

The main motivation of Theorem 2.14 is the category of finite dimensional complex vector spaces which is in particular a topological category. Thus, before establishing an equivalence between the Segal model and the model used by Baas, Dundas, Richter and Rognes, we need to make sure that the above statements hold for topological categories as well. Originally, Segal's construction is a construction for topological categories. Furthermore, we point out that the limit, frequently used in our constructions, preserves an enrichment over **Top** since **Top** is complete. In addition, we need to check that the equivalences in 3.3 respect such an enrichment. In case of the equivalence $Z\mathcal{R} \to \mathcal{R}$ this is not difficult since the resolution is defined via adding disjoint zeros. The derived homotopy colimit makes use of the resolution $Z$ and the homotopy colimit of Thomason. Morphisms in Thomason's homotopy colimit $\mathrm{hocolim}_{\mathbf{I}} K^\bullet\mathcal{R}$ consist of triples of a surjection of sets, a morphism in $\mathbf{I}$ and a morphism in $K^i\mathcal{R}$. Thus, given a topology on the morphisms of $K^i\mathcal{R}$ it can be trivially extended to define a topology on the morphisms of $\mathrm{hocolim}_{\mathbf{I}} K^\bullet\mathcal{R}$ such that the equivalence $Z\mathcal{R} \to D\mathrm{hocolim}_{\mathbf{I}} K^\bullet\mathcal{R}$ is continuous in each degree.

Recall the Grayson-Quillen model $(-\mathcal{R})\mathcal{R}$ from Definition 2.5. Baas, Dundas, Richter and Rognes use a version of this model to define a multiplicative group completion and prove with it that – under mild restrictions on $\mathcal{R}$ – the $K$-theory space of a strictly bimonoidal category $\mathcal{R}$ is equivalent to the $K$-theory space of the associated ring spectrum $Spt(\mathcal{R})$.

**Proposition 3.24.** *Let $\mathcal{R}$ be a small symmetric monoidal category. If in addition $\mathcal{R}$ is a groupoid and the translation functor $x \oplus (\ )$ is faithful for every object $x \in \mathcal{R}$, then there are weak equivalences*

$$K^1_\bullet\mathcal{R} \xrightarrow{\sim} K^1_\bullet(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} (-\mathcal{R})\mathcal{R}.$$

*Proof.* The right map is a weak equivalence, since stable equivalences between group complete symmetric monoidal categories are weak equivalences. The left map being a weak equivalence is a consequence of [MT78], Lemma 2.3, since $\mathcal{R} \to (-\mathcal{R})\mathcal{R}$ is a group completion. □

This statement says that the Grayson-Quillen model and the Segal model are equivalent and thus the proof of [BDRRb] applies to the Segal model as well. Notably, this proof only works if $\mathcal{R}$ is a groupoid and the translation functor $x \oplus (\ )$ is faithful for every object $x \in \mathcal{R}$. This is due to the fact that in other cases the Grayson-Quillen model does not provide a group completion. However, the Segal model does. Hence, the obvious question is if theorem 2.14 holds in a more general context and if the Segal model could help proving it.

# 4 A tensor product of permutative categories

The main obstacle in giving a definition of Hochschild homology of a strictly bimonoidal category close to the original definition for rings is a definition of a suitable tensor product. Apart from that, the question if there is a tensor product of permutative categories is interesting by itself. Aiming for a tensor product in **Strict**, we give a construction that is based on the construction of the tensor product of abelian groups. The key point in the construction of the latter is the quotient group, thus we start with the definition and a discussion of quotient categories. However, the use of quotient categories or rather the fact that our tensor product has origin in a discrete setting already implies its main flaw: Many desirable applications of the tensor product require that certain twist maps are strict (cf. Prop. 4.12).

## 4.1 Quotient categories

Given a small category, we construct a quotient category with respect to equivalence relations on its objects and morphisms. Our construction is inspired by the one by Schubert in [Sch70], chapter 6.

Let $\mathcal{C}$ be a small category and $R$ an equivalence relation on its objects. Let further $K$ be a relation on the morphisms of $\mathcal{C}$ such that $s(f) \sim_R s(f')$ and $t(f) \sim_R t(f')$ for all $(f, f') \in K$. Given this data, we want to construct a quotient category, denoted $\mathcal{Q}(\mathcal{C})^R_{K'}$, with objects given by equivalence classes of $R$ and morphisms given by equivalence classes of a certain equivalence relation $K'$ associated to $K$. This equivalence relation $K'$ has to fulfill some requirements in order to ensure that $\mathcal{Q}(\mathcal{C})^R_{K'}$ is well-defined:
We need to make sure that we get decent identity morphisms. Therefore, we add to $K$ the pairs $(\mathrm{id}_c, \mathrm{id}_{c'})$ for all $c \sim_R c'$ and denote this extension by $K^0$. Furthermore, the

equivalence relation $K'$ has to be compatible with composition, i.e. $f \sim f'$ and $g \sim g'$ implies $f \circ g \sim f' \circ g'$ for composable morphisms $f, g$ and $f', g'$. This requirement is justified by the following observation:
Let $G$ be a group and let $\mathcal{C}$ be the category with one object and morphisms given by elements of $G$. An equivalence relation $K$ on mor$\mathcal{C}$ that is compatible with composition corresponds to a congruence on $G$. The main theorem on congruences (see [Jac85], pp. 55) then implies that [id] is a normal subgroup of $G$. On the other hand, the same theorem states that if $H$ is a normal subgroup and $K$ is the relation given by $a \sim b :\Leftrightarrow ab^{-1} \in H$, then $K$ is an equivalence relation on mor$\mathcal{C} = G$ that is compatible with composition.

However, we admit that the requirement that $f \sim f'$ and $g \sim g'$ implies $f \circ g \sim f' \circ g'$ is rather strong:

**Example 4.1.** Let $\mathcal{C}$ be the category with one object and mor$\mathcal{C} = \Sigma_3$. Consider the relation $K$ given by

$$a \sim b \Leftrightarrow ab^{-1} \in \{e, \sigma_1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

This is an equivalence relation on the morphisms since $\{e, \sigma_1\}$ is a group, but not compatible with composition. In particular, $[e]_K = \{e, \sigma_1\}$ is not a normal subgroup of $\Sigma_3$. However, if $K'$ denotes the smallest equivalence relation compatible with composition generated by $K$, then $[e]_{K'}$ has to be a normal subgroup and contain $[e]_K$. Since $A_3$ is the only non-trivial normal subgroup of $\Sigma_3$ but does not contain $[e]_K$, it is $K' = \Sigma_3 \times \Sigma_3$ and $[e]_{K'} = \Sigma_3$.

**Definition 4.2.** Let $\mathcal{C}$ be a small category and let $R, K$ and $K^0$ be as above. Let $K'$ be the smallest equivalence relation generated by $K^0$ that is compatible with composition. We define the quotient category $\mathcal{Q}(\mathcal{C})^R_{K'}$ to be the category with objects given by all equivalence classes of $R$. The morphisms of the quotient category are given by words $[f_n] \dots [f_1]$ of equivalence classes of $K'$ with $t(f_i) \sim_R s(f_{i+1})$. Composition is consequently defined as juxtaposition, i.e., $[f] \circ [g] = [f][g]$, and we require

$$[\mathrm{id}_b][f] = [f] = [f][\mathrm{id}_a] \tag{4.1}$$

for all morphisms $f \colon a \to b$ in $\mathcal{C}$.

To see that the use of words is necessary consider the following scenario: Let $c \neq c' \in \mathcal{C}$ such that $c \sim_R c'$ but there does not exist a morphism $c \to c'$. Then there might be morphisms $f, g$ in $\mathcal{C}$ such that $s[f] = c' \sim_R c = t[g]$. Thus, it makes sense to compose $[f]$ and $[g]$ in $\mathcal{Q}(\mathcal{C})$, though $f$ and $g$ are not composable in $\mathcal{C}$.

**Proposition 4.3.** *The quotient category $\mathcal{Q}(\mathcal{C})^R_{K'}$ from Definition 4.2 is indeed a category.*

*Proof.* Conveniently, the composition is associative by definition. We need to show that the source and target maps are well-defined maps of sets $\mathrm{mor}\mathcal{Q}(\mathcal{C}) \to \mathrm{ob}\mathcal{Q}(\mathcal{C})$, i.e. $s[f] = [s(f)]$ and $t[f] = [t(f)]$. We confine ourselves to discussing the source map, analogous statements hold for the target map.
Let $f \sim_{K'} f'$. This means there is a string of relations $f = f_1 \sim \cdots \sim f_i \sim \cdots \sim f_n = f'$ such that either

- $f_i \sim f_{i+1}$ is given by $f_i \sim_{K^0} f_{i+1}$ or
- $f_i = g_k \circ \cdots \circ g_1$ and $f_{i+1} = g'_k \circ \cdots \circ g'_1$ with $g_j \sim_{K^0} g'_j \; \forall j$.

In the first case, $f_i \sim_{K^0} f_{i+1}$ implies $s(f_i) \sim_R s(f_{i+1})$ and in the second case, it is $s(f_i) = s(g_1) \sim_R s(g'_1) = s(f_{i+1})$.
For any $[c] \in \mathcal{Q}(\mathcal{C})$, the identity morphism is given by $[\mathrm{id}_c]$ and we made sure that this morphism is well-defined and fulfills the required condition. Note that condition 4.1 is well-defined since we demanded that $\mathrm{id}_c \sim \mathrm{id}_{c'}$ if $c \sim c'$. □

**Proposition 4.4.** *The quotient category is equipped with a projection functor $\pi\colon \mathcal{C} \to \mathcal{Q}(\mathcal{C})^R_{K'}$, given by $c \mapsto [c]$, $f \mapsto [f]$. The functor $\pi$ satisfies a universal property: Let $T\colon \mathcal{C} \to \mathcal{D}$ be a functor and $R, K, K'$ relations on $\mathcal{C}$ as above. If $T(c) = T(c')$ for all $c \sim_R c'$ and $T(f) = T(f')$ for all $f \sim_{K'} f'$, then there exists a unique functor $S\colon \mathcal{Q}(\mathcal{C})^R_{K'} \to \mathcal{D}$ such that*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "T"] \arrow[dr, "\pi"'] & & \mathcal{D} \\ & \mathcal{Q}(\mathcal{C})^R_K \arrow[ur, "S"'] & \end{tikzcd}$$

*is a commutative diagram of categories and functors.*

*Proof.* The functor $S$ is defined as $S([c]) = T(c)$ for all objects $c \in \mathcal{C}$ and $S([f_k] \dots [f_1]) = T(f_k) \circ \cdots \circ T(f_1)$ for all morphisms $f_i$ in $\mathcal{C}$. Note that the latter is well defined since $T(s(f_{i+1})) = T(t(f_i))$ and we premised $T(f) = T(f')$ for all $f \sim_{K'} f'$. The fact that $S$ is unique is of course due to the commutativity condition: Let $S'$ be another functor with $T = \pi \circ S' = \pi \circ S$. Then $S'([c]) = T(c) = S([c])$ for all objects $c \in \mathcal{C}$ and $S'([f_k] \dots [f_1]) = T(f_k) \circ \cdots \circ T(f_1) = S([f_k] \dots [f_1])$ for all morphisms $f_i$ in $\mathcal{C}$. □

Let $(\mathcal{C}, \square, 0)$ be a monoidal category. We call an equivalence relation on the objects of $\mathcal{C}$ compatible with the monoidal structure if $\square$ preserves equivalences, i.e.

$$c \sim c' \text{ and } d \sim d' \text{ implies } c\square d \sim c'\square d'.$$

Note that as a consequence, $c \sim_R c'$ implies $c\square d \sim_R c'\square d$ and $d\square c \sim d\square c'$ for all $d \in \mathcal{C}$. Analogously, an equivalence relation on the morphisms of $\mathcal{C}$ is called compatible with the monoidal structure if

$$f \sim f' \text{ and } g \sim g' \text{ implies } f\square g \sim f'\square g'.$$

What we call compatible with a given monoidal structure is called congruence in the context of monoids and groups.

**Proposition 4.5.** *Let $(\mathcal{C}, \square, 0)$ be a monoidal category with relations $R, K$ and $K'$ as above. In addition, we require both equivalence relations to be compatible with the monoidal structure. Then $\mathcal{Q}(\mathcal{C})^R_{K'}$, defined as above, is a monoidal category.*

*Proof.* The monoidal structure on $\mathcal{Q}(\mathcal{C})^R_{K'}$ is induced by the one on $\mathcal{C}$. We define

$$[c]\square[c'] = [c\square c'],\ [f]\square[g] = [f\square g].$$

This is well-defined, since we required $R$ and $K'$ to be compatible with the monoidal structure. However, a priori we cannot make sense of $([f'][f])\square([g'][g])$. Therefore, we define

$$([f'] \circ [f])\square([g'] \circ [g]) := ([f']\square[g']) \circ ([f]\square[g]).$$

Now, $\square$ is defined on all morphisms. Words of different length are no problem:

$$[f]\square([g'][g]) = ([f] \circ [\mathrm{id}])\square([g'][g]) = ([f]\square[g']) \circ ([\mathrm{id}]\square[g]).$$

The unit is given by $[0]$. All structure isomorphisms are induced by the ones in $\mathcal{C}$. They are well defined since if $f\colon c \to c'$ is an isomorphism in $\mathcal{C}$, $[f]\colon [c] \to [c']$ is an isomorphism in $\mathcal{Q}(\mathcal{C})^R_{K'}$. Commutativity of all required diagrams is induced by commutativity of the respective diagrams in $\mathcal{C}$ since $[a]\square([b]\square([c]\square[d])) = [a\square(b\square(c\square d))]$ etc. $\square$

**Example 4.6.** Let $M$ be a monoid, considered as a discrete category. Let further $\equiv$ denote a congruence on $M$. Then the monoid $M/\equiv$, as in [Jac85], agrees with our definition of a quotient category.

**Example 4.7.** Let $\Sigma$ denote the bipermutative category of finite sets and permutations from Example 1.12. We construct the quotient category with respect to the relation $K$ on morphisms given by

$$\sigma \sim_K \sigma' :\Leftrightarrow \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma').$$

This relation is an equivalence relation and compatible with composition since $\operatorname{sgn}(\sigma \circ \sigma') = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\sigma')$. Moreover, $K$ is compatible with $\oplus$ as $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma'), \operatorname{sgn}(\tau) = \operatorname{sgn}(\tau')$ implies $\operatorname{sgn}(\sigma \oplus \tau) = \operatorname{sgn}(\sigma' \oplus \tau')$. Hence, the quotient category $\mathcal{Q}(\Sigma)_K$ has the same objects as $\Sigma$ and its morphisms are given by

$$\operatorname{mor}([m],[n]) = \begin{cases} \Sigma_n/A_n & m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{V}_{\mathbb{C}}$ be the category of complex vector spaces from Example 1.13 and $\mathcal{L}$ the category with objects $\mathbb{C}_n$ for all integers $n \in \mathbb{Z}$ and morphisms the linear automorphisms $\mathbb{C}_n^* = \mathbb{C}^*$, compare [Kra], Definition 2.3. Let $S\colon \Sigma \to \mathcal{V}_{\mathbb{C}}$ be the functor that takes $[n]$ to $\mathbb{C}^n$ and $\sigma \in \Sigma_n$ to $(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \in GL_n(\mathbb{C})$. Note that $S$ is covariant since $(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$ is the matrix $(a_{ij})$ with $a_{ij} = 1$ if $i = \sigma(j)$ and zero elsewhere. Thus, $S(\sigma_2) \cdot S(\sigma_1) = (c_{ij})$ with

$$c_{ij} = \begin{cases} 1 & i = \sigma_2(\sigma_1(j)), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\Lambda\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{L}$ be the functor that takes $\mathbb{C}^n$ to $\mathbb{C}_n$ and is the determinant on morphisms. Then $\Lambda \circ S =$sgn, i.e., on morphisms we get $(\Lambda \circ S)(\sigma) = 1$ if $\sigma \in A_n$ and $(\Lambda \circ S)(\sigma) = -1$ if $\sigma \in \Sigma_n \setminus A_n$. Thus, $\Lambda \circ S$ factors through $\mathcal{Q}(\Sigma)_K$ in the sense that we get a commutative diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "\Lambda\circ S"] \arrow[dr, "\pi"'] & & \mathcal{L} \\ & \mathcal{Q}(\Sigma)_K \arrow[ur] & \end{tikzcd}$$

of permutative categories.

## 4.2 The tensor product

We are now turning towards the definition of the tensor product of permutative categories. There are different requirements on a tensor product and we concentrate on two aspects:

Universality with respect to a certain kind of multilinear functors and a monoidal structure on the category **Strict** of permutative categories and strict symmetric monoidal functors. However, once we identify the correct version of multilinear functors and establish the appropriate universal property, it will be evident that our tensor product does not endow the category **Strict** with a monoidal structure.

Let $\mathcal{N}, \mathcal{M}$ be permutative categories and consider $P(\mathcal{N} \times \mathcal{M})$, the free permutative category on $\mathcal{N} \times \mathcal{M}$ (cf. Definition 1.14). We define $R$ to be the equivalence relation on objects generated by:

$$k[\dots,(n,m),(n,m'),\dots] \sim (k-1)[\dots,(n,m\Box m'),\dots],$$
$$k[\dots,(n,m),(n',m),\dots] \sim (k-1)[\dots,(n\Box n',m),\dots],$$

and

$$k[\dots,(e_{\mathcal{N}},m),\dots] \sim (k-1)[\dots,\widehat{(e_{\mathcal{N}},m)},\dots],$$
$$k[\dots,(n,e_{\mathcal{M}}),\dots] \sim (k-1)[\dots,\widehat{(n,e_{\mathcal{M}})},\dots]$$

for all $n \in \mathcal{N}, m \in \mathcal{M}$.
We can interpret a morphism

$$\sigma[f_1,\dots,f_k]\colon k[x_1,\dots,x_k] \to k[x'_1,\dots,x'_k]$$

as

$$\sigma[\mathrm{id},\dots,\mathrm{id}] \circ \mathrm{id}[f_1,\dots,f_k]$$

with $f_i\colon x_i \to x'_{\sigma(i)}$ and $\sigma[\mathrm{id},\dots,\mathrm{id}]$ is the morphism

$$k[x'_{\sigma(1)},\dots,x'_{\sigma(k)}] \longrightarrow k[x'_{\sigma^{-1}(\sigma(1))},\dots,x'_{\sigma^{-1}(\sigma(k))}] = k[x'_1,\dots,x'_k].$$

Based on this, we define $K$ to be the set of pairs of morphisms in $P(\mathcal{N} \times \mathcal{M})$ containing

$$(\mathrm{id}[\dots,(f,g),(f',g)\dots],\mathrm{id}[\dots,(f\Box f',g),\dots])$$
$$\text{and } (\mathrm{id}[\dots,(f,g),(f,g')\dots],\mathrm{id}[\dots,(f,g\Box g'),\dots])\,.$$

Furthermore, we require that $K$ contains the pairs

$$\Big(\sigma[\ldots,(\mathrm{id}_{e_\mathcal{N}},g),\ldots],\sigma'[\ldots,\widehat{(\mathrm{id}_{e_\mathcal{N}},g)},\ldots]\Big) \text{ and } \Big(\sigma[\ldots,(f,\mathrm{id}_{e_\mathcal{M}}),\ldots],\sigma'[\ldots,\widehat{(f,\mathrm{id}_{e_\mathcal{M}})},\ldots]\Big),$$

with

$$\sigma'(j) = \begin{cases} \sigma(j) & \sigma(j) < \sigma(i) \\ \sigma(j) - 1 & \sigma(j) > \sigma(i) \end{cases}, \ j \neq i,$$

if $(\mathrm{id}_{e_\mathcal{N}}, g)$ (resp. $(f, \mathrm{id}_{e_\mathcal{M}})$) is the $i^{\text{th}}$ entry.
Note that $R$ and $K$ are compatible with the monoidal structure on $P(\mathcal{N} \times \mathcal{M})$ and that $K = K^0$.

**Definition 4.8.** Let $\mathcal{N}, \mathcal{M}$ be permutative categories. We define the tensor product $\mathcal{N} \otimes \mathcal{M}$ to be the quotient category of $P(\mathcal{N} \times \mathcal{M})$ with regard to the equivalence relations $R$ and $K'$, where $K'$ is the smallest equivalence relation generated by $K$ that satisfies the conditions of Definition 4.2.

For the sake of lucidity, we denote objects of $\mathcal{N} \otimes \mathcal{M}$ by $k[(n_1, m_1), \ldots, (n_k, m_k)]$. Keep in mind that these are equivalence classes!

**Proposition 4.9.** *The tensor product $\mathcal{N} \otimes \mathcal{M}$ inherits the structure of a permutative category from $P(\mathcal{N} \times \mathcal{M})$.*

*Proof.* The neutral element is given by the equivalence class of $0[\,]$. We already know that $\mathcal{N} \otimes \mathcal{M}$ is a monoidal category. Since we defined the sum of equivalence classes to be the equivalence class of the sum (see 4.5), strictness is induced by the strictness in $P(\mathcal{N} \times \mathcal{M})$. The twist is induced by the twist in $P(\mathcal{N} \times \mathcal{M})$: $\gamma = \chi(n, k)[\mathrm{id}, \ldots, \mathrm{id}]$. □

Note that we need $\mathcal{N}, \mathcal{M}$ to be permutative so that $R$ and $K$ are well-defined. In particular, we need identities like

$$\begin{aligned} &(1[(n_1,m)] + 1[(n_2,m)]) + 1[(n_3,m)] \\ &= 3[(n_1 \Box_\mathcal{N} n_2) \Box_\mathcal{N} n_3, m)] \\ &= 3[(n_1 \Box_\mathcal{N} (n_2 \Box_\mathcal{N} n_3), m)] \\ &= 1[(n_1,m)] + (1[(n_2,m)] + 1[(n_3,m)]) \end{aligned}$$

and

$$1[(n,m)] + 1[(e_\mathcal{N},m)] = 1[(n \Box_\mathcal{N} e_\mathcal{N}, m)] = 1[(n,m)]$$

to hold in $\mathcal{N} \otimes \mathcal{M}$.

**Proposition 4.10.** *The tensor product is a bifunctor* $\otimes\colon \textbf{\textit{Strict}} \times \textbf{\textit{Strict}} \to \textbf{\textit{Strict}}$.

*Proof.* We have already proven that for permutative categories $\mathcal{N}, \mathcal{M}$ the tensor product $\mathcal{N} \otimes \mathcal{M}$ is a permutative category. Let now $F\colon \mathcal{N} \to \mathcal{N}', G\colon \mathcal{M} \to \mathcal{M}'$ be strict symmetric monoidal functors. Then $F \otimes G\colon \mathcal{N} \otimes \mathcal{M} \to \mathcal{N}' \otimes \mathcal{M}'$ is defined as

$$\begin{aligned} k[(n_1, m_1), \dots, (n_k, m_k)] &\longmapsto k[(F(n_1), G(m_1)), \dots, (F(n_k), G(m_k))], \\ \sigma[(f_1, g_1), \dots, (f_k, g_k)] &\longmapsto \sigma[(F(f_1), G(g_1)), \dots, (F(f_k), G(g_k))] \end{aligned}$$

on objects and morphisms respectively. Note that $F \otimes G$ is well-defined since

$$\begin{aligned} &(F \otimes G)(k[(n_1, m_1), \dots, (n_i, m_i), (n_{i+1}, m_i), \dots, (n_k, m_k)]) \\ &= k[(F(n_1), G(m_1)), \dots, (F(n_i), G(m_i)), (F(n_{i+1}), G(m_i)), \dots, (F(n_k), G(m_k))] \\ &= k[(F(n_1), G(m_1)), \dots, (F(n_i) \Box_{\mathcal{N}'} F(n_{i+1}), G(m_i)), \dots, (F(n_k), G(m_k))] \\ &= k[(F(n_1), G(m_1)), \dots, (F(n_i \Box_{\mathcal{N}} n_{i+1}), G(m_i)), \dots, (F(n_k), G(m_k))] \\ &= (F \otimes G)(k[(n_1, m_1), \dots, (n_i \Box_{\mathcal{N}} n_{i+1}, m_i), \dots, (n_k, m_k)]) \end{aligned}$$

and

$$\begin{aligned} &(F \otimes G)(k[(n_1, m_1), \dots, (e_{\mathcal{N}}, m_i), \dots, (n_k, m_k)]) \\ &= k[(F(n_1), G(m_1)), \dots, (F(e_{\mathcal{N}}), G(m_i)), \dots, (F(n_k), G(m_k))] \\ &= (k-1)[(F(n_1), G(m_1)), \dots, (F\widehat{(e_{\mathcal{N}}), G}(m_i)), \dots, (F(n_k), G(m_k))] \\ &= (F \otimes G)((k-1)[(n_1, m_1), \dots, \widehat{(e_{\mathcal{N}}, m_i)}, \dots, (n_k, m_k)]). \end{aligned}$$

There are analogous equations for $(F \otimes G)(k[(n_1, m_1), \dots, (n_i, m_i), (n_i, m_{i+1}), \dots, (n_k, m_k)])$ and $(F \otimes G)(k[(n_1, m_1), \dots, (n_i, e_{\mathcal{M}}), \dots, (n_k, m_k)])$. Be aware that we need $F$ and $G$ to be strict symmetric monoidal in order that all these equations hold.
From the definition, it is clear that $F \otimes G$ behaves well with composition of morphisms and $(F \otimes G)(\mathrm{id}) = \mathrm{id}$. Moreover, $F \otimes G$ is strict symmetric monoidal by definition. $\square$

**Proposition 4.11.** *Let $N, M$ be abelian groups, considered as discrete monoidal categories $\mathcal{C}_N, \mathcal{C}_M$. Then the set of path components of the permutative category $\mathcal{C}_N \otimes \mathcal{C}_M$ is isomorphic to $N \otimes M$ as a set.*

*Proof.* We recall the construction of $N \otimes M$. Consider the free abelian group with basis the set of tupels $(n, m)$ for $n \in N$ and $m \in M$. The tensor product $N \otimes M$ is the quotient of this group by the following relations:

$$(n, m) + (n, m') = (n, m + m'),$$
$$(n, m) + (n', m) = (n + n', m).$$

As usual, we denote an equivalence class by $n \otimes m$. Every morphism in $\mathcal{C}_N \otimes \mathcal{C}_M$ is of the form $\sigma[\mathrm{id}, \ldots, \mathrm{id}]$, i.e. a morphism only commutes summands. Since $N \otimes M$ is abelian and does not distinguish permuted sums,

$$N \otimes M \longrightarrow \pi_0(\mathcal{N} \otimes \mathcal{M}), \quad n \otimes m \longmapsto [1[(n, m)]]$$

is an isomorphism. Note that

$$1[(m, n)] + 1[(-m, n)] = 1[(m + (-m), n)] = 1[(0, n)] = 0[\,].$$

$\square$

Recall that a bifunctor is a functor whose domain is a product category (of two categories). That is, a bifunctor is a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that associates to each object $(c, d) \in \mathcal{C} \times \mathcal{D}$ an object $F((c, d)) \in \mathcal{E}$ and to each morphism $(f, g)\colon (c, d) \to (c', d')$ in $\mathcal{C} \times \mathcal{D}$ a morphism $F(f, g)\colon F((c, d)) \to F((c', d'))$ in $\mathcal{E}$ such that $F(\mathrm{id}_{(c,d)}) = \mathrm{id}_{F((c,d))}$ and $F((f', g') \circ (f, g)) = F(f', g') \circ F(f, g)$.

We call a bifunctor of permutative categories lax (strong, strict) monoidal in the first argument if $F(\ , d)\colon \mathcal{C} \to \mathcal{E}$ is a lax (strong, strict) monoidal functor for every $d \in \mathcal{D}$. In particular, the following diagram commutes for all $c_i \in \mathcal{C},\ d \in \mathcal{D}$ and morphisms $f_i\colon c_i \to c_i'$ in $\mathcal{C}$ and $g\colon d \to d'$ in $\mathcal{D}$:

$$\begin{tikzcd}
F(c_1, d)\Box F(c_2, d) \arrow[r] \arrow[d, "F(f_1,g)\Box F(f_2,g)"'] & F(c_1\Box c_2, d) \arrow[d, "F(f_1\Box f_2,g)"] \\
F(c_1', d')\Box F(c_2', d') \arrow[r] & F(c_1'\Box c_2', d').
\end{tikzcd}$$

We call a bifunctor of permutative categories lax (strong, strict) monoidal in the second argument if $F(c,\ )\colon \mathcal{D} \to \mathcal{E}$ satifies the corresponding properties. Finally, we call a bifunctor lax (strong, strict) bilinear if it is lax (strong, strict) monoidal in both arguments and the following diagram commutes:

$$\begin{tikzcd}
(F(c_1,d_1)\Box F(c_2,d_1))\Box(F(c_1,d_2)\Box F(c_2,d_2)) \arrow[r, "(*)"] \arrow[d] & (F(c_1,d_1)\Box F(c_1,d_2))\Box(F(c_2,d_1)\Box F(c_2,d_2)) \arrow[d] \\
F(c_1\Box c_2,d_1)\Box F(c_1\Box c_2,d_2) \arrow[dr] & F(c_1,d_1\Box d_2)\Box F(c_2,d_1\Box d_2) \arrow[d] \\
& F(c_1\Box c_2,d_1\Box d_2).
\end{tikzcd} \tag{4.2}$$

The map $(*)$ is given by $\mathrm{id}\Box c_\Box\Box\mathrm{id}$ and the appropriate associativity identities.
Note that in the case of strict bilinear bifunctors, the above condition implies that the map $(*)$ is an identity map. Be aware that there are very few categories that fulfill this condition, the main example being discrete categories.
The set of all bilinear bifunctors $\mathcal{C}\times\mathcal{D}\to\mathcal{E}$ is denoted by $Bilin(\mathcal{C}\times\mathcal{D},\mathcal{E})$.

The universal property of the quotient category provides the following

**Proposition 4.12.** *The tensor product $\mathcal{N}\otimes\mathcal{M}$ is equipped with a universal bilinear bifunctor $\pi\colon\mathcal{N}\times\mathcal{M}\to\mathcal{N}\otimes\mathcal{M}$. Here, universal means that any bilinear bifunctor $\mathcal{N}\times\mathcal{M}\to\mathcal{Z}$ factors uniquely through $\pi\colon\mathcal{N}\times\mathcal{M}\to\mathcal{N}\otimes\mathcal{M}$. In particular,*

$$\begin{aligned}\Phi\colon \boldsymbol{Strict}(\mathcal{N}\otimes\mathcal{M},\mathcal{Z}) &\longrightarrow StrictBilin(\mathcal{N}\times\mathcal{M},\mathcal{Z})\\ F &\longmapsto F\circ\pi\end{aligned}$$

*is a bijection.*

*Proof.* First of all, the functor $F\circ\pi$ is bilinear for all $F\in\mathbf{Strict}(\mathcal{N}\otimes\mathcal{M},\mathcal{Z})$ because of the tensor product relations. Note for example

$$\begin{aligned}(F\circ\pi)(n_1,m)\Box(F\circ\pi)(n_2,m) &= F(1[(n_1,m)])\Box F(1[(n_2,m)])\\ &\overset{(*)}{=} F(1[(n_1,m)]+1[(n_2,m)]) = F(1[(n_1\Box n_2,m)])\end{aligned}$$

where equation $(*)$ holds since $F$ is strict symmetric monoidal.

The inverse $\Psi\colon \mathit{StrictBilin}(\mathcal{N}\times\mathcal{M},\mathcal{Z}) \to \mathbf{Strict}(\mathcal{N}\otimes\mathcal{M},\mathcal{Z})$ is given by $G\mapsto \tilde{G}$ with

$$\begin{aligned}\tilde{G}(k[(n_1,m_1),\dots,(n_k,m_k)]) &= G(n_1,m_1)\Box\dots\Box G(n_k,m_k),\\ \tilde{G}(\sigma[(f_1,g_1),\dots(f_k,g_k)] &= \gamma\circ(G(f_1,g_1)\Box G(f_2,g_2)\Box\dots\Box G(f_k,g_k))\end{aligned}$$

where $\gamma$ denotes the appropriate composition of structure isomorphisms. By this we mean that a morphism $\sigma[\mathrm{id},\dots,\mathrm{id}]$ in $\mathcal{N}\otimes\mathcal{M}$ is mapped to a morphism in $\mathcal{Z}$ that permutes summands. However, permuting summands in a permutative category can be expressed by a composition of twists (and associativity identities). In detail, this works as follows: Let us start with a transposition $\tau\in\Sigma_k$ of the form $(i,i+1)$ and let $[\tau]$ abbreviate $\tau[\mathrm{id},\dots,\mathrm{id}]$. The image of $[\tau]$ is to be the composition of the appropriate structure isomorphisms $\gamma$ (and associativity identities). For example, let $\tau\in\Sigma_2$ denote the transposition that interchanges positions 1 and 2 and consider

$$[\tau]\colon 2[(n_1,m_1),(n_2,m_2)] \longrightarrow 2[(n_2,m_2),(n_1,m_1)].$$

Then $\tilde{G}([\tau])$ is defined as

$$\gamma_{G(n_1,m_1),G(n_2,m_2)}\colon G(n_1,m_1)\Box G(n_2,m_2) \longrightarrow G(n_2,m_2)\Box G(n_1,m_1).$$

We need to check that

$$\tilde{G}(\tau_2\circ\tau_1) = \tilde{G}(\tau_2)\circ\tilde{G}(\tau_1)$$

and

$$\tilde{G}([\tau]\circ\mathrm{id}[(f_1,g_1),\dots,(f_k,g_k)]) = \tilde{G}([\tau])\circ S(\mathrm{id}[(f_1,g_1),\dots,(f_k,g_k)]).$$

The first equation holds since we defined $\tilde{G}([\tau])$ as a composition of twists and the succession of these twists does not matter in the sense that in a permutative category all possible diagrams of the kind

$$\begin{tikzcd} a\Box(b\Box c) \arrow[r, equal] \arrow[d, "\mathrm{id}_a\Box\gamma"'] & (a\Box b)\Box c \arrow[r, "\gamma"] & c\Box(a\Box b) \arrow[d, equal] \\ a\Box(c\Box b) \arrow[r, equal] & (a\Box c)\Box b \arrow[r, "\gamma\Box\mathrm{id}_b"] & (c\Box a)\Box b \end{tikzcd}$$

commute. The second equation holds since $\tilde{G}(\mathrm{id}[(f_1,g_1),\dots,(f_k,g_k)])$ is well-defined and associativity is strict in the permutative category $\mathcal{Z}$.
We know that each permutation $\sigma\in\Sigma_k$ can be written as a product of transpositions $\tau_l\circ\dots\circ\tau_1$ but that there is no unique way to do this. However, since in a permutative

category it does not matter in which order one permutes this does not cause a problem. Hence, if $\tau_l \circ \dots \circ \tau_1$ is any decomposition of a given $\sigma \in \Sigma$, then $\tilde{G}([\sigma]) := \tilde{G}(\tau_l \circ \dots \circ \tau_1)$. Note that in case of morphisms, the definition of $\tilde{G}$ extends to words of equivalence classes via

$$\tilde{G}(\mathrm{id}[(f_k, g_k)] \dots \mathrm{id}[(f_1, g_1)]) = G(f_k, g_k) \circ \dots \circ G(f_1, g_k)$$

where $(f_i, g_i)$ denotes any morphism in $\mathcal{N} \times \mathcal{M}$.
The universal property of the quotient category (Prop. 4.4) yields that $\tilde{G}$ is unique and it is strict symmetric monoidal by definition. Note that we need commutativity of diagram 4.2 so that $\tilde{G}$ is well-defined. In $\mathcal{N} \otimes \mathcal{M}$, we have

$$\begin{tikzcd}
4[(n_1,m_1),(n_1,m_2),(n_2,m_1),(n_2,m_2)] \arrow[r, "\mathrm{id}+c_+ +\mathrm{id}"] \arrow[d, equal] & 4[(n_1,m_1),(n_2,m_1),(n_1,m_2),(n_2,m_2)] \arrow[d, equal] \\
2[(n_1, m_1\square m_2),(n_2, m_1\square m_2)] \arrow[dr, equal] & 2[(n_1\square n_2, m_1), n_1\square n_2, m_2)] \arrow[d, equal] \\
 & 1[(n_1\square n_2, m_1\square m_2)].
\end{tikzcd}$$

Hence, $\mathrm{id}_{(n_1,m_1)} + c_{(n_1,m_2),(n_2,m_1)} + \mathrm{id}_{(n_2,m_2)}$ is an identity map in $\mathcal{N} \otimes \mathcal{M}$. This does not imply that $2[(n_1,m_1),(n_2,m_2)] \to 2[(n_2,m_2),(n_1,m_1)]$ is the identity for all $n_i \in \mathcal{N}, m_i \in \mathcal{M}$!

□

In the following, when defining a functor $\mathcal{N} \otimes \mathcal{M} \to \mathcal{Z}$ of permutative categories, we will restrict to defining the functor morphisms of length one. This functor is then defined on all morphisms, i.e. words of any length, as long as it is well-defined.

Proposition 4.12 implies that there cannot be a unit object in **Strict** with respect to $\otimes$. Such a unit object is a permutative category $\mathcal{E}$ together with natural isomorphisms $\mathcal{N} \otimes \mathcal{E} \to \mathcal{N}$ and $\mathcal{E} \otimes \mathcal{M} \to \mathcal{M}$ for all permutative categories $\mathcal{N}, \mathcal{M}$. These natural isomorphisms have to be morphisms in **Strict**, that is strict symmetric monoidal functors. In other words, $\mathcal{N} \times \mathcal{E} \to \mathcal{N}$ has to be bilinear which requires diagram 4.2 to commute in $N$.

Given a strictly bimonoidal category $(\mathcal{R}, \oplus_{\mathcal{R}}, 0, c_{\oplus}, \otimes_{\mathcal{R}}, 1)$, Proposition 4.12 furthermore reveals that there is no well-defined functor $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$. To be precise:

**Proposition 4.13.** *Let $\mathcal{R}$ be a strictly bimonoidal category. The product $\otimes_{\mathcal{R}}$ in $\mathcal{R}$ does not induce a functor $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ where the tensor product $\mathcal{R} \otimes \mathcal{R}$ is taken with respect to*

$\oplus_{\mathcal{R}}$.

At first glance, it seems self-evident to ask for a functor $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ induced by $\otimes_{\mathcal{R}}$ since a reasonable functor $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ should be compatible with the multiplication on path components $\pi_0\mathcal{R} \times \pi_0\mathcal{R} \to \pi_0\mathcal{R}$ defined as $[c], [d] \mapsto [c \otimes_{\mathcal{R}} d]$.
Consider the following diagram:

$$\begin{tikzcd} \pi_0\mathcal{R} \times \pi_0\mathcal{R} \arrow[r] \arrow[d] & \pi_0\mathcal{R} \\ \pi_0(\mathcal{R} \times \mathcal{R}) \arrow[r] & \pi_0(\mathcal{R} \otimes \mathcal{R}), \end{tikzcd} \qquad \begin{tikzcd} {[c], [d]} \arrow[r, mapsto] \arrow[d, mapsto] & {[c \otimes_{\mathcal{R}} d]} \\ {[(c, d)]} \arrow[r, mapsto] & {[1[(c, d)]].} \end{tikzcd} \tag{4.3}$$

In order to induce a map on $\pi_0$ such that diagram 4.3 commutes, the most obvious ansatz is to define the functor $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ as

$$k[(r_1, s_1), \ldots, (r_k, s_k)] \longmapsto r_1 \otimes_{\mathcal{R}} s_1 \oplus_{\mathcal{R}} \cdots \oplus_{\mathcal{R}} r_k \otimes_{\mathcal{R}} s_k.$$

However, as we will see now, this functor is not well-defined.

*Proof of Prop. 4.13.* Proposition 4.12 gives $\mathbf{Strict}(\mathcal{R} \otimes \mathcal{R}, \mathcal{R}) \cong \mathit{StrictBilin}(\mathcal{R} \times \mathcal{R}, \mathcal{R})$. Thus, it suffices to show that $\otimes_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is not bilinear. In fact, $\otimes_{\mathcal{R}}$ fails to be strict monoidal in the second argument due to the non-strict left distributivity:

$$r \otimes_{\mathcal{R}} s \oplus_{\mathcal{R}} r \otimes_{\mathcal{R}} s' \neq r \otimes_{\mathcal{R}} (s \oplus_{\mathcal{R}} s').$$

□

We point out that this agrees with the a prediction of Elmendorf and Mandell in the introduction of [EM06]. There, they mention that ”permutative categories appear not to support a symmetric monoidal structure consistent with a reasonable notion of multiplicative structure”.
Of course, this is the reason why we were not able to define a Hochschild complex in this setting.

**Proposition 4.14.** *The tensor product is commutative in the sense that there are natural strict symmetric monoidal isomorphisms of categories $\gamma_{\mathcal{N},\mathcal{M}} : \mathcal{N} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{N}$ for all $\mathcal{N}, \mathcal{M}$ in* ***Strict****.*

*Proof.* Let $\gamma^{\otimes}_{\mathcal{N},\mathcal{M}} \colon \mathcal{N} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{N}$ be defined as

$$\begin{aligned} k[(n_1, m_1), \ldots, (n_k, m_k)] &\longmapsto k[(m_1, n_1), \ldots, (m_k, n_k)], \\ \sigma[(f_1, g_1), \ldots, (f_k, g_k)] &\longmapsto \sigma[(g_1, f_1), \ldots, (g_k, f_k)] \end{aligned}$$

on objects and morphisms respectively. That is, a morphism

$$\sigma[(f_1, g_1), \ldots, (f_k, g_k)] \colon k[(n_1, m_1), \ldots, (n_k, m_k)] \longrightarrow k[(n'_1, m'_1), \ldots, (n'_k, m'_k)]$$

is mapped to

$$\sigma[(g_1, f_1), \ldots, (g_k, f_k)] \colon k[(m_1, n_1), \ldots, (m_k, n_k)] \longrightarrow k[(m'_1, n'_1), \ldots, (m'_k, n'_k)]$$

with $f_i \colon n_i \to n'_{\sigma(i)}, g_i \colon m_i \to m'_{\sigma(i)}$ in both cases.
The functor $\gamma^{\otimes}_{\mathcal{N},\mathcal{M}}$ is well-defined since

$$\begin{aligned} \gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(2[(n, m_1), (n, m_2)]) &= 2[(m_1, n), (m_2, n)] \\ &= 1[(m_1 \square m_2, n)] = \gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(1[(n, m_1 \square m_2)] \end{aligned}$$

(analogously for $2[(n_1, m), (n_2, m)] = 1[(n_1 \square n_2, m)]$) and

$$\begin{aligned} &\gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(4[(n_1, m_1), (n_1, m_2), (n_2, m_1), (n_2, m_2)]) \\ &= 4[(m_1, n_1), (m_2, n_1), (m_1, n_2), (m_2, n_2)] \\ &= 1[(m_1 \square m_2, n_1 \square n_2)] \\ &= 4[(m_1, n_1), (m_1, n_2), (m_2, n_1), (m_2, n_2)] \\ &= \gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(4[(n_1, m_1), (n_2, m_1), (n_1, m_2), (n_2, m_2)]). \end{aligned}$$

Moreover,

$$\gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(k[\ldots, (e_{\mathcal{N}}, m), \ldots]) = \gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(k[\ldots, \widehat{(e_{\mathcal{N}}, m)}, \ldots])$$

and

$$\gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(k[\ldots, (n, e_{\mathcal{M}}), \ldots]) = \gamma^{\otimes}_{\mathcal{N},\mathcal{M}}(k[\ldots, \widehat{(n, e_{\mathcal{M}})}, \ldots]).$$

Obviously, $\gamma_{\otimes}$ is strict symmetric monoidal. Note that $\gamma^{\otimes}$ is natural: For a functor $\mathcal{A} \otimes \mathcal{B} \to$

$\mathcal{N} \otimes \mathcal{M}$ we get the following commutative diagram:

$$\begin{tikzcd}
k[(a_1,b_1),\dots,(a_k,b_k)] \arrow[r, mapsto] \arrow[d, mapsto, "\gamma^{\otimes}"'] & k[(n_1,m_1),\dots,(n_k,m_k)] \arrow[d, mapsto, "\gamma^{\otimes}"] \\
k[(b_1,a_1),\dots,(b_k,a_k)] \arrow[r, mapsto] & k[(m_1,n_1),\dots,(m_k,n_k)].
\end{tikzcd}$$

Two-fold application of $\gamma^{\otimes}$ gives the identity. $\square$

## 4.3 Comparison to existing constructions

In [Gra74], John Gray presents a tensor product for 2-categories that induces a closed monoidal category structure on the category of 2-categories. A strictly monoidal category $\mathcal{C}$ defines a 2-category with one 0-cell, the objects and morphisms of $\mathcal{C}$ being the 1-cells and 2-cells respectively and composition of 1-cells being defined by the monoidal structure in $\mathcal{C}$. That way, Gray's tensor product can be applied to permutative categories after forgetting the symmetric structure. Our construction is then similar to his' in the sense that we face similar problems, e.g. equivalence classes, words of morphisms etc. The main difference is that his construction gives a tensor product (as strictly monoidal category) with objects consisting of equivalence classes of strings of pairs $(a, b)$ with either $a$ or $b$ being the unit in its origin category.

In [EM09], Anthony Elmendorf and Michael Mandell construct a tensor product of multicategories and show that this tensor product equips the category of based multicategories $\mathbf{Mult}_*$ with a symmetric monoidal structure. In particular, $\mathbf{Mult}_*$ is closed. Every permutative category has an underlying based multicategory and the forgetful functor $\mathbf{Strict} \to \mathbf{Mult}_*$ is full and faithful. Conversely, the permutative structure can be recovered from the multicategory structure. However, the strict isomorphism class of a permutative category cannot be recovered from the isomorphism class of its underlying multicategory (cf. [EM09], Example 3.5). Hence, their construction defines a tensor product of permutative categories but it cannot distinguish between different permutative structures in a strict sense.

## 4.4 Conclusion

We have shown that the concept of bilinear functors in **Strict** – as we defined them – is incompatible with a monoidal structure on **Strict**. At least, if this monoidal structure should be induced by a bifunctor that is universal with respect to these bilinear functors. There is another aspect suggesting that one should, in fact, not work in **Strict**. A further criterion for a bifunctor that deserves the name 'tensor product' is that it should be adjoint to the hom functor. However, to our knowledge there is no internal morphism object in **Strict**. The following construction which reveals the obstruction in defining such a morphism object was kindly communicated to us by Bjørn Dundas.

**Lemma 4.15.** *Let $\mathcal{B}, \mathcal{C}$ be permutative categories. The categories $\boldsymbol{Strong}(\mathcal{B}, \mathcal{C}) \subseteq \boldsymbol{Perm}(\mathcal{B}, \mathcal{C})$ of strong/lax symmetric monoidal functors and transformations are themselves permutative categories.*

*Proof.* Given two symmetric monoidal functors $G$ and $G'$, their sum is the functor $G + G'$ which on a morphism $f: b \to b'$ is given by

$$G(f) + G'(f): G(b) \oplus G'(b) \longrightarrow G(b') \oplus G'(b').$$

Furthermore, this functor is equipped with natural transformations

$$(G + G')(b, b'): (G + G')(b) \oplus (G + G')(b') \longrightarrow (G + G')(b \oplus b')$$

given by

$$\begin{tikzcd}
(G+G')(b) \oplus (G+G')(b') \arrow[r, equal] & G(b) \oplus G'(b) \oplus G(b') \oplus G'(b') \arrow[d, "\mathrm{id}+\gamma^{\oplus}+\mathrm{id}"] \\
& G(b) \oplus G(b') \oplus G'(b) \oplus G'(b') \arrow[d, "G(b,b')+G'(b,b')"] \\
(G+G')(b \oplus b') \arrow[r, equal] & G(b+b') \oplus G'(b+b').
\end{tikzcd}$$

Extending this to symmetric monoidal transformations and checking the relevant diagrams gives the result. □

This implies that unless the twist $\gamma^{\oplus}$ in $\mathcal{C}$ is the identity,

$$(G + G')(b \oplus b') = (G + G')(b) \oplus (G + G')(b')$$

is never true. Thus, this construction does not apply to $\mathbf{Strict}(\mathcal{B}, \mathcal{C})$.

Hence, we promote looking for a tensor product in **Strong**. It should be defined as the left adjoint of the hom functor – if this left adjoint actually exists.

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**Zusammenfassung**
Diese Arbeit ordnet sich in den Kontext algebraischer $K$-Theorie ein. Zentral in vielen Konstruktionen algebraischer $K$-Theorie ist der Begriff der Gruppenvervollständigung. Im Zuge der Verallgemeinerung von $K$-Theorie von Ringen auf "ringähnliche" Objekte wie Ringspektren und bimonoidale Kategorien stellte sich die Frage nach einer multiplikativen Gruppenvervollständigung. Damit ist eine Vervollständigung bezüglich einer monoidalen Struktur gemeint, die eine existierende zweite monoidale Struktur respektiert. Diese Frage ist offen, seit Thomason 1980 auf Fehler in vermeintlichen Lösungen hinwies. Nils Baas, Bjørn Dundas, Birgit Richter und John Rognes präsentieren in ihrem Artikel "Ring completion of rig categories" (wird demnächst erscheinen) eine umfassende Lösung für dieses Problem. Neben allgemeingültigen theoretischen Betrachtungen konstruieren sie eine konkrete multiplikative Gruppenvervollständigung. Allerdings existiert ihr Modell nur für Gruppoide mit treuer Translation. In der vorliegenden Arbeit nutzen wir eine Idee von Graeme Segal aus dem Jahre 1974 um eine multiplikative Gruppenvervollständigung zu konstruieren, die diese Anforderungen nicht stellt, sondern auf beliebige strikte bimonoidale Kategorien anwendbar ist.

Des Weiteren konstruieren wir ein Tensorprodukt von permutativen Kategorien. Dies war motiviert durch das Streben nach einer Spurabbildung, welche die Arbeit mit $K$-Theorie erleichtern könnte. Über ein Tensorprodukt permutativer Kategorien existieren verschiedenste Mutmaßungen in der Literatur. Viele sind der Auffassung, dass es ein solches Tensorprodukt nicht gibt, zumindest nicht mit guten multiplikativen Eigenschaften. Wir geben eine konkrete Konstruktion und legen dar, wo Probleme entstehen und wo mögliche Auswege ansetzen könnten.

# Lebenslauf

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