Cohomological Properties of Toric Degenerations of Calabi-Yau Pairs

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Introduction.

This thesis can be regarded as a sequel to [15], [16] and [17], which together establish a paradigm studying mirror symmetry via logarithmic algebraic geometry. This approach can be viewed as an algebro-geometric version of the Strominger-Yau-Zaslow (SYZ) program [35]. This thesis concentrates nevertheless mainly on constructions on one side of mirror symmetry.

In [15], the construction begins with a compact integral affine manifold B without boundary containing singularities Δ of codimension ≥ 2. By means of the discrete Legendre transform of B (see [15, §1.4]), one gets another affine manifold $\check{B}$ with singularities $\check{\Delta}$. Concentrating on one side, say B, one then looks at a polyhedral decomposition $\mathscr{P}$ of B. With the decomposition $\mathscr{P}$, we are able to build a space $X_0 := X_0(B, \mathscr{P})$ from pieces of algebraic varieties, denoted by X_τ. Such pieces of varieties X_τ are called *toric strata*. They are acquired via methods in toric geometry, where every single X_τ is in fact a toric variety. The algebraic space X_0 is thus obtained by gluing together different X_τ torically, whereby toric prime divisors of X_τ are identified.

A log structure is then put on X_0 to get a *log Calabi-Yau space* $X_0^\dagger$, which is treated as a central fibre of a degeneration. The log structure is important as it carries algebraic information about the degeneration and the central fibre. With the help of log geometry, it is described in [17, Thm. 1.29] that, under certain assumptions, there exists a *toric degeneration* $\pi\colon \mathfrak{X} \to T$ with X_0 and X_η as the central fibre and the generic fibre respectively. Here T is the spectrum of a discrete valuation $\Bbbk$-algebra with closed point $O \in T$ and $X_0 := \pi^{-1}(O)$ (see Definition 1.3).

$$\begin{tikzcd}
X_\eta \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d] & X_0 \arrow[l, hook'] \arrow[d] \\
\{\eta\} \arrow[r, hook] & T & \{O\} \arrow[l, hook']
\end{tikzcd}$$

Given a smooth Calabi-Yau variety X_η, mirror symmetry studies the properties of X_η and its mirror $\check{X}_\eta$ (which is also a Calabi-Yau variety), and how these properties are related. The notion "mirror symmetry" originated from the study of string theory in physics. It led to exploration and investigation of many interesting mathematical phenomena. Among them, the phenomenon most related to this thesis is the exchange and computation of Hodge numbers (see [3, 4, 12]). More precisely, one has $h^{1,1}(X_\eta) = h^{1,2}(\check{X}_\eta)$ and $h^{1,2}(X_\eta) = h^{1,1}(\check{X}_\eta)$ for a mirror pair X_η and $\check{X}_\eta$ of Calabi-Yau varieties with $\dim X_\eta = \dim \check{X}_\eta = 3$.

In the above described framework, the mirror varieties X_η and $\check{X}_\eta$ are taken to be the generic fibres of degenerations $\pi\colon \mathfrak{X} \to T$ and $\check{\pi}\colon \check{\mathfrak{X}} \to \check{T}$ respectively. It is illustrated in [16, Thm. 3.23] and in the proof of [16, Cor. 3.24] that the exchange of Hodge numbers can be viewed more elementarily due to the discrete Legendre transform between B and $\check{B}$.

Therefore, mirror symmetry can be investigated by looking at the data exchange between B and $\check{B}$ providing the degenerations exist. On each side of the mirror, we then also look at mathematical properties between B and X_0, as well as between X_0 and X_η, eventually establishing correspondence of properties between Calabi-Yau varieties X_η and $\check{X}_\eta$.

In [16], as a continuation of [15], various consequences of the above construction were further investigated, including the computation of the Hodge theory of the log Calabi-Yau space $X_0^\dagger$ on each side and a base change theorem for the smoothings of the log Calabi-Yau spaces. In particular, the Hodge theory of the degeneration can be controlled by the data of $(B, \mathscr{P})$ under some technical assumptions.

Let $\Bbbk$ denote an algebraically closed field of characteristic zero. Let Z be the singular set of the log structure; it is a closed subset of X_0 of codimension ≥ 2. Denote the inclusion of the complement by $j\colon X_0 \setminus Z \hookrightarrow X_0$. Let Δ be the set of singularities of the affine structure on B and denote the inclusion of the complement by $i\colon B \setminus \Delta \hookrightarrow B$. The exchange of Hodge numbers in mirror symmetry is obtained via formulae relating the *log Dolbeault cohomology groups* $H^p(X_0, j_*\Omega^q_{X_0^\dagger/\Bbbk^\dagger})$ and the *affine Hodge groups* $H^p(B, i_* \bigwedge^q \check{\Lambda}^B \otimes_{\mathbb{Z}} \Bbbk)$. Here $\check{\Lambda}^B$ is the local system on $B \setminus \Delta$ of flat integral cotangent vectors.

The Hodge theory of X_η is eventually expressed in terms of the affine Hodge groups by applying the base change theorem (see [16, Thm. 4.2]). In particular, the Hodge numbers of X_η can be computed from B:

$$h^{p,q}(X_\eta) = h^q\big(B, i_* \textstyle\bigwedge^p \check{\Lambda}^B \otimes_{\mathbb{Z}} \Bbbk\big),$$

provided that a smoothing of X_0 exists. The proof is in two steps:

1. Equate the affine Hodge groups with the *logarithmic Dolbeault groups* $H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger})$. (see [16, Cor. 3.24])

2. Show that the Dolbeault cohomology groups of a toric degeneration $\mathfrak{X} \to T$ fit together into a vector bundle over the base space T. (see [16, Thm. 4.2])

The mirror phenomenon between Calabi-Yau varieties is generalized, for example by [9, 10, 13, 19, 23, 24], to *Fano varieties* and *Landau-Ginzburg models* (LG models). Striving for a unified framework, the concept of a *log Calabi-Yau pair* (log CY-pair), denoted by $(X_0^\dagger, D)$, is introduced in [17].

On the "Fano side", the generic fibre is a Calabi-Yau pair (X_η, D_η), where X_η is a variety with an effective anticanonical divisor $D_\eta \subset X_\eta$. In most cases we are interested, X_η is a Fano variety.

$$\begin{tikzcd}
(X_\eta, D_\eta) \arrow[r, hook] \arrow[d] & (\mathfrak{X}, \mathfrak{D}) \arrow[d] & (X_0, D) \arrow[l, hook'] \arrow[d] \\
\{\eta\} \arrow[r, hook] & T & \{O\} \arrow[l, hook']
\end{tikzcd}$$

On the other hand, the LG model is a variety $\check{X}_\eta$ with a regular function $W\colon \check{X}_\eta \to \mathbb{A}^1_{k(\eta)}$, where $k(\eta)$ denotes the residue field of a point η in a scheme. The pair $(\check{X}_\eta, W)$ is taken to be the mirror of X_η.

This thesis aims to apply the methods and results in [16] in order to investigate the cohomological consequences concerning log CY-pairs and their smoothings. We shall concentrate mainly on the Fano side.

A log CY-pair $(X_0^\dagger, D)$ is also determined by the data $(B, \mathscr{P})$, with the same gluing process as for $X_0^\dagger$ in [15, 16]. The greatest difference is that B is now non-compact without boundary. The new term $D = \bigcup D_\nu$ is a union of toric prime divisors in X_0. These toric divisors correspond to unbounded 1-cells in the decomposition $\mathscr{P}$ of B. In the toric degeneration, the effective divisor D_η has to be the smoothing of D.

On the central fibre, there are two log spaces $X_0^\dagger$ and $\check{X}_0^\dagger$ to be considered in this thesis. The log structure on $X_0^\dagger$ is determined by open gluing data analogously as in [15, Def. 2.25]. Let $\mathfrak{D}$ denote a divisor on $\mathfrak{X}$ such that $\mathfrak{D}_\eta = \mathfrak{D} \cap X_\eta := D_\eta$ is a smooth irreducible divisor on the generic fibre X_η and $\mathfrak{D}_0 := D$ is a collection of toric boundary (prime) divisors in X_0. In the perspective of toric degeneration $\pi\colon \mathfrak{X} \to T$, the log structure on $X_0^\dagger$ can be acquired by restriction of the log structure on $\mathfrak{X}$ given by $\mathcal{M}_{(\mathfrak{X}, \mathfrak{D}\cup X_0)}$ to X_0. Here $\mathcal{M}_{(\mathfrak{X}, \mathfrak{D}\cup X_0)}$ denotes the sheaf of regular functions on $\mathfrak{X}$ with zeros contained in $\mathfrak{D} \cup X_0$ (cf. [15, Ex. 3.2]). This log structure on $X_0^\dagger$ is the divisorial log structure induced by the toric boundary divisor during the toric construction of étale neighbourhoods of points of X_0 in $\mathfrak{X}$.

On the other hand, we have another log structure on $\mathfrak{X}$ given by $\mathcal{M}_{(\mathfrak{X}, X_0)}$, the sheaf of regular functions on $\mathfrak{X}$ with zeros contained in X_0. One then restricts this log structure to X_0 to get the log structure on $\check{X}_0^\dagger$. In order to obtain the ordinary Dolbeault cohomology groups $H^q(X_\eta, \Omega^p_{X_\eta})$ of the generic fibre X_η, this log structure has to be considered. This log structure has the advantage that the allowed log poles of differential forms on $\mathfrak{X}$ are not located on the generic fibre X_η while it is not the case for the log structure on $X_0^\dagger$.

In the situation of a variety with an effective anticanonical divisor (X_η, D_η), there are two natural classes of cohomology groups of Dolbeault type, the *ordinary Dolbeault group* $H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}\big)$ and the one with logarithmic poles along D_η, that is, the *logarithmic Dolbeault group* $H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big)$. Note that we have an abuse of notation here (see Remarks 3.10 and 3.12). The first main result of this thesis is the following generalization of the results [16, Thm. 3.21 and 4.2].

Theorem 0.1. *Let $(X_0^\dagger, D)$ be a log Calabi-Yau pair associated to an integral affine manifold with singularities $(B, \mathscr{P})$. Assume that $(B, \mathscr{P})$ is positive and simple (see [15, §1.5]) and that the monodromy around every cell $\tau \in \mathscr{P}$ in B is unimodular. Suppose that a smoothing of $X_0^\dagger \to \operatorname{Spec} \Bbbk^\dagger$ in a toric degeneration (see Definition 1.3) exists. Then the following holds:*

$$\dim_{k(\eta)} H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big) = \dim_{\Bbbk} H^q\big(B, i_*\big(\bigwedge^p \check{\Lambda}^B \otimes_{\mathbb{Z}} \Bbbk\big)\big).$$

This is acquired by a base change result of the hypercohomology groups $\mathbb{H}^k(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}})$ with respect to the log structure $X_0^\dagger$. It can be regarded as an affine cohomological control of the log Dolbeault groups of the generic fibre X_η. Same to the situation in [16], the above result relies on the technical assumption that the monodromy polytope $\mathrm{Conv}(\bigcup_{i=1}^q \check{\Delta}_i \times \{e_i\})$ is a standard simplex for every cell $\tau \in \mathscr{P}$ (see [15, Def. 1.60] and [16, Thm. 3.21]), which we call the monodromy is *unimodular* around the cell τ in the above theorem. The relaxation of this assumption for Calabi-Yau varieties X_η is handled in [32].

To get a similar cohomological control for the log structure $\breve{X}_0^\dagger$, we first notice that there is also a base change result for the hypercohomology groups $\mathbb{H}^k(\mathfrak{X}, \breve{\Omega}^\bullet_{\mathfrak{X}})$ with respect to the log structure on $\breve{X}_0^\dagger$. Then we introduce the notions Λ and Λ_0 of local systems of flat integral vector fields on the cone picture $\check{B}$, so as to get an (integral) affine analogue of the Poincaré residue map in complex algebraic geometry. After establishing an affine control of the log Dolbeault groups $H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big)$ from the cone picture (the above theorem is an affine control from the fan picture), we are able to express the affine cohomological control of the ordinary Dolbeault groups of X_η as

$$\dim_{k(\eta)} H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}\big) = \dim_{\Bbbk} H^q(\check{B}, i_*(\bigwedge^p \Lambda_0^{\check{B}} \otimes_{\mathbb{Z}} \Bbbk))$$

provided that D_η is irreducible in X_η. This fact is obtained by writing down cohomology long exact sequences of $\check{B}$ and X_η and the consequent identification of cohomology groups.

The above two affine cohomological controls establish links between the Dolbeault cohomology theories on X_η induced by Kähler geometry and the cohomology theories on B under toric degeneration. Given an X_η, provided that it is the generic fibre of a toric degeneration and its ordinary Dolbeault and log Dolbeault cohomology groups are known, it is possible to recover the corresponding singularities on the affine manifold B by Čech cohomology calculations on B. These will be illustrated by the calculations of examples of low dimensions in §4.1. Besides, a relation between the birational geometry on X_η and singularities on B is expected in higher dimensions, which is inspired by the calculation in dimension 2 because blowing up a point in X_η is equivalent to adding a singularity on B. A more detailed discussion will be conducted in §4.2 (2).

Another immediate observation of these considerations is the *simultaneous degeneration* of the spectral sequences of the four complexes of sheaves $\Omega^\bullet_{X_\eta}(\log D_\eta)$, $\Omega^\bullet_{X_\eta}$, $i_* \bigwedge^\bullet \Lambda^{\check{B}} \otimes \mathbb{C}$ and $i_* \bigwedge^\bullet \Lambda_0^{\check{B}} \otimes \mathbb{C}$ at E_1 level (with respect to the trivial filtrations). It illustrates a good correspondence between the cohomology theory of affine geometry on $\check{B}$ (equivalent to the affine geometry of B via the discrete Legendre transform) and that of the "induced" Kähler geometry on X_η under the setting of toric degeneration. The degeneration result of $\Omega^\bullet_{X_\eta}(\log D_\eta)$ at E_1 is especially impressing; we recover a classical result of Deligne [8] on the closed smooth fibres X_s with $k(s) = \mathbb{C}$ provided that an algebraic family with central fibre X_0 (as an algebraic space over $\Bbbk = \mathbb{C}$) exists.

This thesis is organized as follows. §1 is an introductory section. §1.1 reviews the settings and some definitions in [15, 16, 17] and gives an overview of the main results of this thesis. §1.2 discusses possible singular behaviour of a toric degeneration in higher dimensions and the impact of coherency of log structures on cohomology theories. §1.3 and §1.4 are actually the CY-pair version of Construction 2.1 and Theorem 2.6 in [16].

§2 is the most technical part of this thesis. This section follows the lines of §1, §3.1 and §3.2 of [16], applying the arguments of which and stating results in the setting of CY-pair with respect to the two log structures considered.

The important results of this thesis are written down in §3. In §3.1, we are able to get isomorphisms between log Dolbeault groups on X_0 and the affine Hodge groups on B with the help of the log structure on $X_0^\dagger$, which is our first affine cohomological control. Besides, we also have the Hodge decomposition for the hypercohomology with respect to $X_0^\dagger$. §3.2 contains the base change result for the hypercohomology groups of both log structures. §3.3 will review the discrete Legendre transform of an affine integral manifold B and introduce the notions Λ and Λ_0 and consequently obtain an affine analogue of the Poincaré residue map and the second affine cohomological control. §3.4 will analyse the spectral sequences of the complexes of sheaves on X_η and B.

In §4.1, we calculate some examples in dimension 1 and dimension 2. We will discuss some undeveloped aspects of this thesis and possible outcomes of the results in §4.2.

§5 is the appendix. It proves some statements relating the log spaces $X_0^\dagger$ and $\breve{X}_0^\dagger$, complementing the local description Theorem 1.12.

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LAUS DEO.

Chapter 1

Toric Degenerations and their Local Descriptions

This first section serves to provide a review of the notion of toric degeneration and a panoramic view of the main results in this thesis. It mentions in particular how the results in cohomology theories in the generic fibre of a toric degeneration and the related affine manifold B (or $\check{B}$) are going to appear.

1.1 An overview

Let $\Bbbk$ always be an algebraically closed field of characteristic zero. Following the notations of [15, §3.1] and [16, §1], consider a morphism of logarithmic spaces $\pi : X^\dagger = (X, \mathcal{M}_X) \to S^\dagger = (S, \mathcal{M}_S)$. Here $\mathcal{M}_X$ is a sheaf of monoids on X and the dagger is always used to denote a logarithmic space. In particular, the notion $\Bbbk$ denotes the trivial log point and $\Bbbk^\dagger$ denotes the standard log point (see [15, Ex. 3.5]). Recall related definitions concerning toric degenerations (see [17, Def. 1.6 – 1.9]).

Definition 1.1. A *totally degenerate CY-pair* is a reduced variety X_0 together with a reduced divisor $D \subseteq X_0$ fulfilling the following conditions: Let $\nu : \tilde{X}_0 \to X_0$ be the normalization and $C \subseteq \tilde{X}_0$ its conductor locus. Then $\tilde{X}_0$ is a disjoint union of algebraically convex toric varieties, and C is a reduced divisor such that $C + \nu^* D$ is the sum of all toric prime divisors, $\nu|_C : C \to \nu(C)$ is unramified and generically two-to-one, and the square

$$\begin{array}{ccc} C & \longrightarrow & \tilde{X}_0 \\ \big\downarrow & & \big\downarrow \nu \\ \nu(C) & \longrightarrow & X_0 \end{array}$$

is cartesian and cocartesian. □

A central concept in log geometry is (log) smoothness, which runs analogously to formal smoothness for schemes. In the following, we use the characterization of log smooth morphisms due to Kato (cf. [20, Thm. 3.5]) as in [15, Def. 3.8] and [17, Def. 1.7].

Definition 1.2. Let T be the spectrum of a discrete valuation $\Bbbk$-algebra R with closed point O and uniformizing parameter $t \in \mathcal{O}(T)$. Let $\mathfrak{X}$ be a $\Bbbk$-scheme and $\mathfrak{D}, X \subseteq \mathfrak{X}$ reduced divisors. A *log smooth morphism* $\pi : (\mathfrak{X}, X; \mathfrak{D}) \to (T, O)$ is a morphism $\pi : (\mathfrak{X}, X) \to (T, O)$ of pairs of $\Bbbk$-schemes with the following properties: For any $x \in \mathfrak{X}$ there exists an étale neighbourhood $U \to \mathfrak{X}$ of x such that $\pi|_U$ fits into a commutative diagram of the following form.

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d, "\pi|_U"'] & \operatorname{Spec} \Bbbk[P] \arrow[d, "G"] \\ T \arrow[r, "\Psi"] & \operatorname{Spec} \Bbbk[\mathbb{N}] \end{tikzcd}$$

Here P is a toric monoid, Ψ and G are defined respectively by mapping the generator $z^1 \in \Bbbk[\mathbb{N}]$ to t and to a non-constant monomial $z^\rho \in \Bbbk[P]$, and Φ is étale with preimage of the toric boundary divisor equal to the pull-back to U of $X \cup \mathfrak{D}$. $\square$

Although the log structures do not seem to be involved in this definition at first glance, this definition is indeed a modified version of [15, Def. 3.8] for the setup of toric degeneration.

Definition 1.3. Let T be the spectrum of a discrete valuation $\Bbbk$-algebra R with closed point $O \in T$ and uniformizing parameter $t \in \mathcal{O}(T)$. A *toric degeneration of CY-pairs over* T is a flat morphism $\pi : \mathfrak{X} \to T$ together with a reduced divisor $\mathfrak{D} \subseteq \mathfrak{X}$, with the following properties:

(i) $\mathfrak{X}$ is normal.

(ii) The central fibre $X_0 := \pi^{-1}(O)$ together with $D = \mathfrak{D} \cap X_0$ is a totally degenerate CY-pair.

(iii) Away from a closed subset $\mathfrak{Z} \subseteq \mathfrak{X}$ of relative codimension ≥ 2 not containing any toric stratum of X_0, the map $\pi : (\mathfrak{X}, X_0; \mathfrak{D}) \to (T, O)$ is log smooth. $\square$

The above definitions are taken from [17] for a CY-pair. In fact, these are generalizations of [15, Def. 4.1] and [15, Def. 4.3].

In [16], we have Figure 1.1 provided that a smoothing of a log Calabi-Yau space $X_0^\dagger$ exists. In the sense of above definitions, this log Calabi-Yau space is indeed a log CY-pair $(X_0^\dagger, D)$ with $D = \emptyset$. The generic fibre X_η is a Calabi-Yau variety of dimension n and hence in this thesis this type of degeneration is called the Calabi-Yau case.

In this thesis, we consider the situation of Figure 1.2 in which the generic fibre X_η has an effective anticanonical divisor $-K_{X_\eta}$. Now X_η is the smoothing of a log Calabi-Yau pair $(X_0^\dagger, D)$. We call this the Fano case since X_η is a Fano variety in most cases. We have an extra divisor $\mathfrak{D} \subset \mathfrak{X}$ such that $\mathfrak{D} \cap X_\eta = D_\eta$ is an effective divisor in X_η and $\mathfrak{D} \cap X_0 = D$.

In each of the above cases, a log space $X_0^\dagger$ is taken as the central fibre. In contrast to the Calabi-Yau case, we view however the central fibre as two log spaces $X_0^\dagger$ and $\breve{X}_0^\dagger$ with the same underlying scheme X_0. The log space $\breve{X}_0^\dagger$ is constructed locally using a relative chart $\breve{P}'$ of the chart P', where P' is a local chart for $X_0^\dagger$ (see Remark 1.13 (2)).

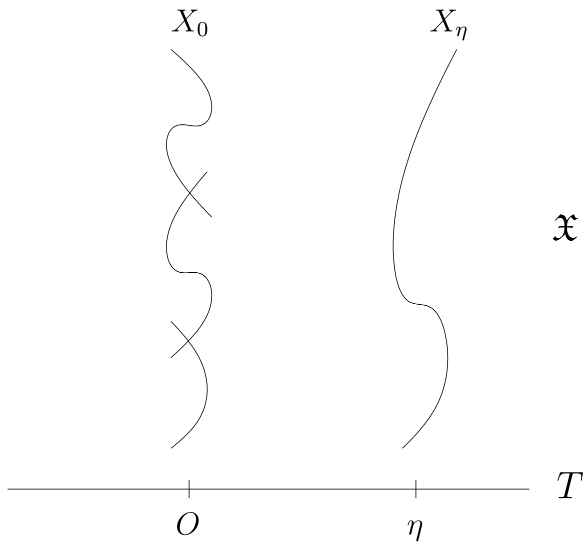


Figure 1.1: Calabi-Yau case

To calculate cohomology groups on X_0 and eventually on X_η, we note that there is a notion of fine log structure in logarithmic geometry (see [28]). The log structures we consider now have a nonempty locus Z on X_0 (eventually $\mathfrak{Z}$ on $\mathfrak{X}$) where the log structures fail to be fine. As seen in [16, Ex. 1.11], the sheaf of log differentials behaves poorly at points where the log structure is not fine. We will hence use the push-forward of the sheaf of log differentials in the log smooth part of X by $j\colon X \setminus Z \hookrightarrow X$ (similarly $j\colon \mathfrak{X} \setminus \mathfrak{Z} \hookrightarrow \mathfrak{X}$).

Definition 1.4. A *log derivation* on $X^\dagger$ over $S^\dagger$ with values in an $\mathcal{O}_X$-modules $\mathcal{E}$ is a pair $(\mathrm{D}, \mathrm{Dlog})$, where $\mathrm{D} : \mathcal{O}_X \to \mathcal{E}$ is an ordinary derivation of X/S and $\mathrm{Dlog} : \mathcal{M}_X^{\mathrm{gp}} \to \mathcal{E}$ is a homomorphism of abelian sheaves with $\mathrm{Dlog} \circ \pi^{\#} = 0$; these fulfill the following compatibility condition

$$\mathrm{D}\big(\alpha_X(m)\big) = \alpha_X(m) \cdot \mathrm{Dlog}(m), \tag{1.1}$$

for all $m \in \mathcal{M}_X$, where $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ is the log structure.

Denote the sheaf of log derivations of $X^\dagger$ over $S^\dagger$ with values in $\mathcal{O}_X$ by $\Theta_{X^\dagger/S^\dagger}$. □

Lemma 1.5 (Lem. 1.9 in [16])**.** *Given a morphism $\pi : X^\dagger \to S^\dagger$ of log schemes, let*

$$\Omega^1_{X^\dagger/S^\dagger} = \big(\Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{gp}})\big)/\mathcal{K},$$

with $\mathcal{K}$ the $\mathcal{O}_X$-module generated by

$$(\mathrm{d}\alpha_X(m), -\alpha_X(m) \otimes m), \quad \textit{and} \quad (0, 1 \otimes \pi^*(n)),$$

for $m \in \mathcal{M}_X$, $n \in \mathcal{M}_S$. Then the pair $(\mathrm{d}, \mathrm{dlog})$ of natural maps

$$\mathrm{d} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \longrightarrow \Omega^1_{X^\dagger/S^\dagger}, \quad \mathrm{dlog} : \mathcal{M}_X^{\mathrm{gp}} \xrightarrow{1\otimes;} \mathcal{O}_X \otimes \mathcal{M}_X^{\mathrm{gp}} \longrightarrow \Omega^1_{X^\dagger/S^\dagger},$$

is a universal log derivation.

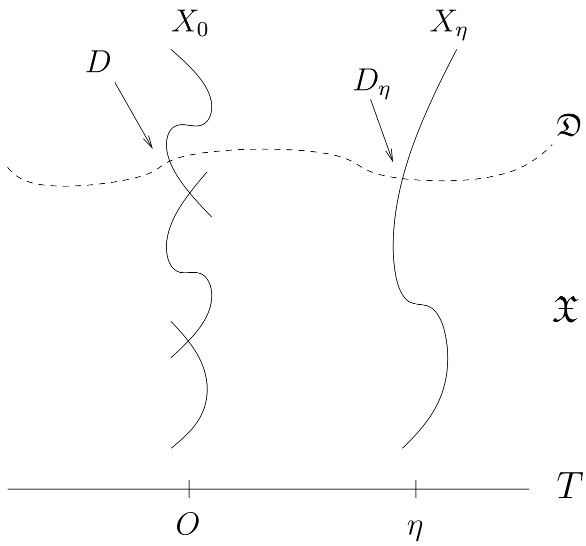


Figure 1.2: Fano case

The $\mathcal{O}_X$-module $\Omega^1_{X^\dagger/S^\dagger}$ is the module of *log differentials*. If π is log smooth then $\Omega^1_{X^\dagger/S^\dagger}$ is locally free (see [20, Prop. 3.10]). Use the convention $\Omega^0_{X^\dagger/S^\dagger} = \mathcal{O}_X$ and denote

$$\Omega^r_{X^\dagger/S^\dagger} = \bigwedge^r \Omega^1_{X^\dagger/S^\dagger}.$$

In the perspective of degeneration, the log structure on $X_0^\dagger$ is actually the restriction of the log structure given by $\mathcal{M}_{(\mathfrak{X},X_0\cup\mathfrak{D})}$ in $\mathfrak{X}$ to X_0, where $\mathcal{M}_{(\mathfrak{X},X_0\cup\mathfrak{D})}$ denotes the sheaf of regular functions on $\mathfrak{X}$ with zeros contained in $X_0 \cup \mathfrak{D}$ (cf. [15, Ex. 3.2]). This log structure on $X_0^\dagger$ is the divisorial log structure induced by the toric boundary divisor during the toric construction of étale neighbourhoods of points on X_0 in $\mathfrak{X}$ (cf. Remark 2.25).

On the other hand, we have another log structure on $\mathfrak{X}$ given by $\mathcal{M}_{(\mathfrak{X},X_0)}$. The log structure on the space $\breve{X}_0^\dagger$ is indeed the restriction of this log structure on $\mathfrak{X}$ to X_0. Under toric degeneration, this log structure on the space $\breve{X}_0^\dagger$ is also considered under the framework of toric degeneration with the following insight.

Speaking locally on $\mathfrak{X}$ with the use of a system of local coordinates $(z_1, \ldots, z_{n+1})$ such that $\mathfrak{D} = \{z_1 = 0\}$, the sheaf of monoids $\mathcal{M}_{(\mathfrak{X},X_0\cup\mathfrak{D})}$ will lead to the differential $\operatorname{dlog} z_1 = \frac{dz_1}{z_1}$ while the sheaf $\mathcal{M}_{(\mathfrak{X},X_0)}$ will not. Since $D_\eta = \mathfrak{D} \cap X_\eta$, the differential $\operatorname{dlog} z_1$ is restricted to an element of $\Omega^1_{X_\eta/k(\eta)}(\log D_\eta)$ and this element has poles along D_η. On the other hand, it is easy to see that the allowed log poles of differential forms on $\mathfrak{X}$ are not located on the generic fibre X_η by using the log structure on $\breve{X}_0^\dagger$.

The essence of [16] is to compute the ordinary Dolbeault groups on X_η with the help of the log structure on $X_0^\dagger$ under toric degeneration, which cannot be achieved by considering $X_0^\dagger$ in the Fano case. More precisely, one only recovers the cohomology group $H^q(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta))$ from $H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger})$ using the log structure on $X_0^\dagger$ via base change by Theorem 3.9, which is not the usual Dolbeault group in Kähler geometry. However, we do get the Dolbeault group $H^q(X_\eta, \Omega^p_{X_\eta/k(\eta)})$ with the consideration of the log structure on $\check{X}_0^\dagger$.

In both the Calabi-Yau and Fano case, the central fibre $X_0^\dagger$ is first constructed by toric geometry from an integral affine manifold B with singularities Δ and a polyhedral decomposition $\mathscr{P}$. The correspondence is summarized in the following table:

	Fano variety	LG model	Calabi-Yau variety
Fan Picture (Dual Intersection Complex)	B non-compact $\partial B = \emptyset$	B compact $\partial B \neq \emptyset$	B compact $\partial B = \emptyset$
Cone Picture (Intersection Complex)	$\check{B}$ compact $\partial \check{B} \neq \emptyset$	$\check{B}$ non-compact $\partial \check{B} = \emptyset$	$\check{B}$ compact $\partial \check{B} = \emptyset$

We briefly recall the construction of fan picture and cone picture here.

Given a polyhedral decomposition $\mathscr{P}$ of B in the fan picture, a fan structure is always specified at each vertex $v \in \mathscr{P}$. More explicitly, a complete rational polyhedral fan Σ_v is defined at v. We then get a toric variety $X_v := X(\Sigma_v)$ from this fan, which is an irreducible component X_v of X_0. Similarly, this construction is performed for faces τ in $\mathscr{P}$ of arbitrary dimensions (cf. [15, Def. 2.7]).

The cone picture $\check{B}$ is related to B by the discrete Legendre transform. Given $(\check{B}, \check{\mathscr{P}})$ and a cell $\tau \in \check{\mathscr{P}}$, we can define

$$\check{X}_\tau := \operatorname{Proj} \Bbbk[\check{P}_\tau],$$

where $\check{P}_\tau := C(\tau) \cap (\Lambda_\tau \oplus \mathbb{Z})$ and $C(\tau) := \{(rm, r) | r \geq 0, m \in \tau\}$ (cf. [15, Def. 2.1]). Suppose that $\dim B = n$. In the fan picture, $\dim X_\tau = n - p$ if $\dim \tau = p$. In the cone picture, on the other hand, $\dim \check{X}_\tau = p$ if $\dim \tau = p$.

We use the fan picture until §3.1 in this thesis because the cone picture provides the additional data of an ample line bundle, which are inessential for the Hodge-theoretical results here. Following the methods and arguments in [16], the first isomorphisms

$$H^q\big(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger}\big) \cong H^q(B, i_*(\bigwedge^p \check{\Lambda}^B \otimes_{\mathbb{Z}} \Bbbk)),$$

are obtained firstly in terms of the fan picture after

1. construction of an acyclic resolution $\mathscr{C}^\bullet(\Omega^p)$ of the sheaf Ω^p on X_0 in §2.4,

2. proving related cohomology vanishing theorems (Lemma 3.1 and Lemma 3.4),

3. identification of global sections on open covers of X_0 and B (Lemma 3.4),

which are summarized in Theorem 3.5. An application of base change (Theorem 3.9) yields

$$H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big) \cong H^q\big(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger}\big) \otimes_{\Bbbk} k(\eta).$$

So we get the first type of isomorphisms

$$H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big) \cong H^q(B, i_*(\bigwedge^p \check{\Lambda}^B \otimes_{\mathbb{Z}} k(\eta))). \tag{1.2}$$

We call this type of isomorphisms (1.2) the *first affine cohomological control.* This is one of the main results of this thesis. In terms of the data in the cone picture, this first control can be written as

$$H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)\big) \cong H^q(\check{B}, i_*(\bigwedge^p \Lambda^{\check{B}} \otimes_{\mathbb{Z}} k(\eta))). \tag{1.3}$$

With the consideration of the Poincaré residue map

$$0 \to \Omega^p_{X_\eta} \to \Omega^p_{X_\eta}(\log D_\eta) \to \Omega^{p-1}_{D_\eta} \to 0$$

and some new definitions Λ and Λ_0 in the cone picture $\check{B}$ (see Construction 3.14), we are able to get an (integral) affine analogue of the Poincaré residue map

$$0 \to \bigwedge^p \Lambda_0^{\check{B}} \to \bigwedge^p \Lambda^{\check{B}} \to \bigwedge^{p-1} \Lambda^{\partial\check{B}} \to 0.$$

By the comparison of the cohomology long exact sequences induced by the above two short exact sequences, we are able to get the second type of isomorphisms

$$H^q\big(X_\eta, \Omega^p_{X_\eta/k(\eta)}\big) \cong H^q(\check{B}, i_*(\bigwedge^p \Lambda_0^{\check{B}} \otimes_{\mathbb{Z}} k(\eta))). \tag{1.4}$$

This is another main result of this thesis. We shall call this type of isomorphisms the *second affine cohomological control.*

We call (1.3) (which is equivalent to (1.2)) and (1.4) the *affine cohomological controls* because the logarithmic and the ordinary Dolbeault cohomology groups on the variety X_η are controlled by the cohomology groups on $\check{B}$ (and vice versa) in the framework of toric degeneration.

We need to switch to the cone picture since the *second affine cohomological control* (1.4) is more natural to be expressed in the latter one. For the proof of the second cohomological control, we make use of

$$H^q\big(D_\eta, \Omega^p_{D_\eta/k(\eta)}\big) \cong H^q(\partial\check{B}, i_*(\bigwedge^p \Lambda^{\partial\check{B}} \otimes_{\mathbb{Z}} k(\eta)))$$

which is the result in [16], together with the fact that the affine manifold $\partial\check{B}$ is compact without boundary and is flat with respect to the affine structure. Indeed, the divisor D_η is then a Calabi-Yau variety and corresponds to $\partial\check{B}$. We lack however a good description in the fan picture, partly due to the fact that the affine manifold B is unbounded without boundary so that there is no corresponding analogue for the sheaf $\Lambda_0^{\check{B}}$ (defined on the cone picture $\check{B}$) in the fan picture B, which provides the necessary (integral) affine analogue of Poincaré residue map in the fan picture.

1.2 Issue of singularities and coherence of log structures

The toric degenerations do not necessarily always behave very nicely. In fact, singularities may occur on X_η, but we have the following Proposition 1.6 to classify them in terms of the local toric data.

These toric data are given by toric monoids P. A toric monoid is a finitely generated, saturated and integral monoid. Such a monoid P is precisely of the form $\sigma^\vee \cap \mathbb{Z}^n$ for $\sigma \subseteq \mathbb{R}^n$ a strictly convex, rational polyhedral cone. As we shall see later in detail in §1.4, each toric monoid P encodes the information about X_τ étale locally for a cell $\tau \in \mathscr{P}$ (e.g. its codimension q in B) as well as the monodromy behaviour of the affine structure around τ in terms of Newton polytopes Δ_i for $1 \leq i \leq q$, provided that $(B, \mathscr{P})$ is positive and simple ([15, Def. 1.54 and Def. 1.60]).

Given these monoids P, we can consider a collection of schemes $\operatorname{Spec} \Bbbk[P]$, which then constitutes an open cover of the total space $\mathfrak{X}$ of the degeneration. In particular, every closed geometric point $\bar{x} \in Z$ (Z is where the log structure on $X_0^\dagger$ fails to be fine) is covered by such an étale neighbourhood $\operatorname{Spec} \Bbbk[P]$.

Proposition 1.6. *Let P be a toric monoid as mentioned above (see Construction 1.8), which is determined by $\tau \in \mathscr{P}$ and Newton polytopes $\Delta_1, \ldots, \Delta_q$ capturing the local monodromy behaviour around τ. Then the generic fibre of $f : \operatorname{Spec} \Bbbk[P] \to \operatorname{Spec} \Bbbk[\mathbb{N}]$ induced by $\rho = e_0^*$ is non-singular if and only if*

$$\operatorname{Conv}\left(\bigcup_{i=1}^{q} \Delta_i \times \{e_i\}\right)$$

is a standard simplex.

If

$$\operatorname{Conv}\left(\bigcup_{i=1}^{q} \Delta_i \times \{e_i\}\right)$$

is an elementary simplex (i.e. its only integral points are its vertices) then the generic fibre of f has codimension at least four Gorenstein quotient singularities.

Proof. The proof is the same as [16, Prop. 2.2], since the generic fibre of f is defined by the cone

$$K \cap \rho^\perp = \operatorname{Cone}\left(\bigcup_{i=1}^{q} \Delta_i \times \{e_i\}\right).$$

The only difference between the Calabi-Yau case and Fano case is that we allow $\Delta_0 := \tau$ to be unbounded. In particular, the cone $K \cap \rho^\perp$ behaves the same as in [16, Prop. 2.2].

□

Remark 1.7. Assuming that the simplex

$$\operatorname{Conv}\left(\bigcup_{i=1}^{q} \Delta_i \times \{e_i\}\right)$$

satisfies corresponding properties for every toric monoid P associated to $\tau \in \mathscr{P}$, then we obtain the desired properties of generic fibre of a toric degeneration accordingly. We thus see that the properties of generic fibre depend on the local monodromy around every cell $\tau \in \mathscr{P}$.

In [31, §3.3], there is a calculation of a 4-dimensional Fermat Calabi-Yau hypersurface X. The corresponding simplex is elementary and this X has four terminal Gorenstein singularities so that it is a case where the general fibre of a toric degeneration fails to be smooth. $\square$

An intermediate consequence of the above proposition is that, the generic fibre of π only has singularities with codimension ≥ 4. Thus, the generic fibre X_η is always smooth for $\dim X_\eta \leq 3$.

On the other hand, the log singular locus $\mathfrak{Z}$ also prevents us to get the isomorphisms between $H^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta))$ and $H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger})$ as well as between $H^q(X_\eta, \Omega^p_{X_\eta})$ and $H^q(X_0, j_*\Omega^p_{\check{X}_0^\dagger/\Bbbk^\dagger})$ because *a priori* we only get $H^q(X_\eta, (j_*\Omega^p_{\mathfrak{X}^\dagger/R^\dagger})_\eta)$ and $H^q(X_\eta, (j_*\Omega^p_{\check{\mathfrak{X}}^\dagger/R^\dagger})_\eta)$ by base change. The locus $\mathfrak{Z} \subseteq \mathfrak{X}$ where the log structures fail to be fine is of relative codimension ≥ 2 by the definition of toric degeneration. However, $\mathfrak{Z}_\eta = \mathfrak{Z} \cap X_\eta$ is empty as long as the generic fibre X_η is smooth.

Otherwise $\mathfrak{Z}_\eta$ is of codimension at least 4 in X_η (when there are singularities on X_η). In this case, there is no difference between the Čech cohomology groups $\check{H}^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta))$ and $\check{H}^q(X_\eta, (j_*\Omega^p_{\mathfrak{X}^\dagger/R^\dagger})_\eta)$. When the sheaf $j_*\Omega^p_{\mathfrak{X}^\dagger/R^\dagger}$ is locally free, both Čech cohomology computations are the same because a regular function on $X_\eta \setminus \mathfrak{Z}_\eta$ extends uniquely to a regular function on X_η. Similarly, the situation is also the same for the groups $\check{H}^q(X_\eta, \Omega^p_{X_\eta})$ and $\check{H}^q(X_\eta, (j_*\Omega^p_{\check{\mathfrak{X}}^\dagger/R^\dagger})_\eta)$.

1.3 The setting for a Calabi-Yau pair

Let B be an integral affine manifold with singularities, which is non-compact and without boundary (cf. [17, §1.1]). It carries a toric polyhedral decomposition $\mathscr{P}$, and we suppose $(B, \mathscr{P})$ is positive and simple, which is a condition on the local affine monodromy around $\Delta \subseteq B$. Then a choice of open gluing data $s = (s_e)_{e \in \mathrm{Hom}\,\mathscr{P}}$ (cf. [17, Def. 1.17]) determines a CY-pair (X_0, D). We require the CY-pair (X_0, D) with $D = \bigcup D_\mu$ to fulfill the following condition:

$$D_\mu \subset X_0 \text{ are toric } (n-1)\text{-strata in } X_0 \text{ satisfying } D_\mu \nsubseteq (X_0)_{\mathrm{sing}}, \tag{1.5}$$

where $(X_0)_{\mathrm{sing}}$ is the union of all toric $(n-1)$-strata besides all D_μ. Equivalently, $D \cup (X_0)_{\mathrm{sing}}$ contains all toric $(n-1)$-strata in X_0. The log schemes X_0 are S_2 as in [16]; this is a result of the construction of X_0.

At the same time, the open gluing data s mentioned above determine a log structure $X_0^\dagger$ on (X_0, D) (cf. [17, §1.2]), so that a log Calabi-Yau pair $(X_0^\dagger, D) = X_0(B, \mathscr{P}, s)^\dagger$ is obtained (cf. [15, Def. 4.3] and [17, Def. 1.22]). It is equipped with two log structures, along with two morphisms of log schemes

$$X_0^\dagger \to \operatorname{Spec} \Bbbk^\dagger$$

and

$$\breve{X}_0^\dagger \to \operatorname{Spec} \Bbbk^\dagger,$$

which are log smooth off of a codimension two set Z. The latter log structure is locally given by a natural relative chart $\breve{P}'$ of the chart P' for the log structure on $X_0^\dagger$ in the standard étale covering of X_0 (see Remark 1.13 (2) and Lemma 5.3). We will fix $(B, \mathscr{P}, s)$ now and write $(X_0^\dagger, D)$ instead of $X_0(B, \mathscr{P}, s)^\dagger$.

1.4 Local description

Consider a closed geometric point $\bar{x}$ in an irreducible component X_v of X_0, where $v \in \mathscr{P}$ is a vertex in fan picture B with polyhedral decomposition $\mathscr{P}$.

With reference to Definition 1.3, if π is log smooth at $\bar{x}$, then π is étale locally described by a usual toric degeneration in the sense of [18, §1.2], in which $\tilde{\sigma}$ there corresponds to the toric monoid P later in this section, yet without local monodromy data. Note further that the construction in [18] is done primarily using the cone picture, where codimension one faces correspond to the toric prime divisors (see comments before [18, Ex. 1.2]) while we are using the fan picture for the construction in this section.

As remarked in §1.2, the collection of schemes $\operatorname{Spec} \Bbbk[P]$ forms an open cover for the total space $\mathfrak{X}$ of the degeneration and it enables us to study global cohomology objects based on local computations on X_0 and its local models.

Construction 1.8 (cf. Constr. 2.1 in [16])**.** Let M' be a lattice, N' the dual lattice, and set $M = M' \oplus \mathbb{Z}^{q+1}$, N the dual lattice of M. We write $e_0, \dots, e_q$ for the standard basis of $\mathbb{Z}^{q+1}$, and we identify these with $(0, e_0), \dots, (0, e_q)$ in M. Thus we can write a general element of M as $m + \sum a_i e_i$ for $m \in M'$. Similarly, we write $e_0^*, \dots, e_q^*$ for the dual basis, which we view as elements of N.

Fix a convex lattice polytope $\tau \subseteq M'_{\mathbb{R}}$ where $\dim \tau = \dim M'_{\mathbb{R}}$, with normal fan $\check{\Sigma}_\tau$ living in $N'_{\mathbb{R}}$ (see [15, Def. 1.38] for the convention concerning the normal fan). We obtain a cone $C'(\tau) \subseteq M'_{\mathbb{R}} \oplus \mathbb{R}$, $C'(\tau) = \{(rm, r) | r \geq 0, m \in \tau\}$, and a monoid $P' = C'(\tau)^\vee \cap (N' \oplus \mathbb{Z})$. Define $\rho' \in P'$ to be given by the projection $M' \oplus \mathbb{Z} \to \mathbb{Z}$. We set

$$V'(\tau) = \operatorname{Spec} \Bbbk[P']/(z^{\rho'}) = \operatorname{Spec} \Bbbk[\partial P']$$

(cf. [15, Def. 2.13]). Here $\partial P'$ is the monoid consisting of elements of the boundary of P' and ∞, with $p + p'$ defined to be $p + p'$ if $p + p'$ lies in the boundary of P' and ∞ otherwise. As in [15], we identify $\partial P'$ as a set with $N' \cup \{\infty\}$ via projection to N'. We always use the convention that $z^\infty = 0$.

Let $\check{\psi}_1, \ldots, \check{\psi}_q$ be integral piecewise linear functions on $\check{\Sigma}_\tau$ whose Newton polytopes are $\Delta_1, \ldots, \Delta_q \subseteq M'_{\mathbb{R}}$, i.e.

$$\check{\psi}_i(n) = -\inf\{\langle n, m\rangle | m \in \Delta_i\}.$$

Similarly, let $\check{\psi}_0$ have Newton polytope τ, i.e.

$$\check{\psi}_0(n) = -\inf\{\langle n, m\rangle | m \in \tau\}.$$

Here, the function $\check{\psi}_0$ is allowed to take the value ∞, which is the case whenever τ is unbounded. For convenience of notation, we set $\Delta_0 := \tau$.

Given these data, we can define a monoid $P \subseteq N$ given by

$$\begin{aligned} P &= \left\{ n + \sum_{i=0}^{q} a_i e_i^* \,\middle|\, n \in N' \text{ such that } \check{\psi}_0(n) \neq \infty \text{ and } a_i \geq \check{\psi}_i(n) \text{ for } 0 \leq i \leq q \right\} \\ &= \left\{ (n, a_0, a_1, \ldots, a_q) \middle| \check{\psi}_0(n) \neq \infty \text{ and } a_i \geq \check{\psi}_i(n), 0 \leq i \leq q \right\}. \end{aligned}$$

Set $Y = \operatorname{Spec} \Bbbk[P]$. Note that $P = K^\vee \cap N$ where K is the cone in $M_{\mathbb{R}}$ generated by

$$\bigcup_{i=0}^{q} (\Delta_i \times \{e_i\}).$$

In particular, we see Y is Gorenstein because $\rho_K = \sum_{i=0}^{q} e_i^*$ takes the value 1 on each primitive integral generator of an extremal ray of K. Letting $X = \operatorname{Spec} \Bbbk[P]/(z^\rho)$ as usual with $\rho := e_0^*$, we describe X explicitly by defining

$$\begin{aligned} Q &= \left\{ n + \sum_{i=0}^{q} a_i e_i^* \in P \,\middle|\, a_0 = \check{\psi}_0(n) \right\} \cup \{\infty\} \\ &= \left\{ (n, \check{\psi}_0(n), a_1, \ldots, a_q) \in P \middle| a_i \geq \check{\psi}_i(n), 1 \leq i \leq q \right\} \cup \{\infty\} \end{aligned}$$

with addition on Q defined by

$$q_1 + q_2 = \begin{cases} q_1 + q_2 & \text{if } q_1 + q_2 \in Q \\ \infty & \text{otherwise.} \end{cases}$$

Then $Q \setminus \{\infty\}$ is, as a set, $P \setminus (\rho + P)$, so it is clear that $X = \operatorname{Spec} \Bbbk[Q]$. Note that $Q \cong \partial P' \oplus \mathbb{N}^q$, via

$$(n, a_0, \ldots, a_q) \mapsto (n, 0, a_1 - \check{\psi}_1(n), \ldots, a_q - \check{\psi}_q(n)).$$

Thus $X \cong V'(\tau) \times \mathbb{A}^q$.

We define subschemes Z_i of X by their ideals, for $1 \leq i \leq q$, with $I_{Z_i/X}$ generated by the set of monomials

$$\{z^{e_i^*}\} \cup \left\{ z^p \,\middle|\, \begin{array}{c} p = n + \sum a_j e_j^* \text{ such that there exists a unique} \\ \text{vertex } w \text{ of } \Delta_i \text{ such that } \langle n, w\rangle = -\check{\psi}_i(n) \end{array} \right\}.$$

The effect of the right-hand set is to select those irreducible components of the singular locus of X corresponding to edges of Δ_i, and $z^{e_i^*}$ defines a closed subscheme of this set of components. Set

$$Z = \bigcup_{i=1}^{q} Z_i.$$

This will be the locus where the log structure on X fails to be coherent. Let

$$u_i := z^{e_i^*}$$

for $1 \leq i \leq q$. For any vertex v of τ, denote by $\mathrm{Vert}_i(v)$ the vertex of Δ_i which represents the function $-\check{\psi}_i$ restricted to the maximal cone $\check{v}$ of $\check{\Sigma}_\tau$ corresponding to v. For every edge $\omega \subseteq \tau$, choose a primitive generator d_ω of the tangent space of ω, and let $v_\omega^{\pm}$ be the two vertices of ω, labelled so that d_ω points from v_ω^+ to v_ω^- as in [15, §1.5]. Set

$$\Omega_i = \{\omega \subseteq \tau | \dim \omega = 1 \text{ and } \mathrm{Vert}_i(v_\omega^+) \neq \mathrm{Vert}_i(v_\omega^-)\}.$$

(This notation is compatible with that in the definition of simplicity, cf. [15, Def. 1.60].) It is then easy to see that

$$Z_i = \{u_i = 0\} \cap \bigcup_{\omega \in \Omega_i} V_\omega.$$

Here for $\omega \subseteq \tau$ any face, we define $V_\omega \subseteq X$ to be the closed toric stratum of Y defined by the face of K generated by $\omega \times \{e_0\}$.

Similarly we define V'_ω, for any face $\omega \subseteq \tau$, to be the closed stratum of $V'(\tau)$ corresponding to $C'(\omega) \subseteq C'(\tau)$. □

Remark 1.9 (Rem. 2.5 in [16]). We shall always assume $(B, \mathscr{P})$ is positive and simple in this thesis (see [15, Def. 1.60]). Thus for a cell $\tau \in \mathscr{P}$ with $0 < \dim \tau < \dim B$, we always obtain associated to τ the data $\Omega_1, \ldots, \Omega_p$, $R_1, \ldots, R_p$, $\Delta_1, \ldots, \Delta_p$ and $\check{\Delta}_1, \ldots, \check{\Delta}_p$, with $\Delta_i \subseteq \Lambda_{\tau,\mathbb{R}}$ and $\check{\Delta}_i \subseteq \Lambda_{\tau,\mathbb{R}}^{\perp}$ elementary simplices ($\Lambda_{\tau,\mathbb{R}}$ is the tangent space to τ in B: see [15, Def. 1.31]). We call these data *simplicity data* for τ.

Recall now the definition of strict étale morphism in order to consider certain sorts of étale neighbourhoods of log schemes:

Definition 1.10 (Def. 2.3 in [16]). A morphism $\phi : X^\dagger \to Y^\dagger$ is *strict étale* if it is étale as a morphism of schemes and is strict, i.e. the log structure on $X^\dagger$ is the same as the pull-back of the log structure on $Y^\dagger$.

Remark 1.11 (Rem. 2.4 in [16]). Strict étale morphisms have the following standard property of étale morphisms: If $Y^\dagger$ is a log scheme, and $Y_0^\dagger$ is a closed subscheme of $Y^\dagger$ defined by a nilpotent sheaf of ideals with the induced log structure on Y_0, then there is an equivalence between the categories of strict étale $Y^\dagger$-schemes and strict étale $Y_0^\dagger$-schemes. Indeed, $X \mapsto$

$X_0 = X \times_Y Y_0$ gives an equivalence of categories between étale Y-schemes and étale Y_0-schemes (cf. [26, Chap. I, Thm. 3.23]), and to obtain the log structures on X or X_0, one just pulls back the log structure on Y or Y_0.

In particular, if we have a strict étale morphism $X_0^\dagger \to Y_0^\dagger$ and a thickening $Y^\dagger$ of $Y_0^\dagger$, we can talk about pulling back this thickening to $X_0^\dagger$ giving $X^\dagger$. Note also that if $f : X^\dagger \to Y^\dagger$ is a strict étale morphism over $\operatorname{Spec} \Bbbk^\dagger$, then $\Theta_{X^\dagger/\Bbbk} = f^*\Theta_{Y^\dagger/\Bbbk}$ and $\Theta_{X^\dagger/\Bbbk^\dagger} = f^*\Theta_{Y^\dagger/\Bbbk^\dagger}$, as is easily checked. □

Similar to the situation in [16, Thm. 2.6], we wish to describe the local models X for X_0 at points of Z. The singularities of the log structure will be well-behaved due to simplicity of $(B, \mathscr{P})$.

Theorem 1.12 (cf. Thm. 2.6 in [16])**.** *Given $(B, \mathscr{P})$ positive and simple and s lifted open gluing data and suppose that a CY-pair $(X_0^\dagger, D) = X_0(B, \mathscr{P}, s)^\dagger$ determined by these data exists. Let $\bar{x} \to Z \subseteq X_0$ be a closed geometric point. Then there exists data τ, $\check{\psi}_1, \ldots, \check{\psi}_q$ as in Construction 1.8 defining a monoid P, and an element $\rho \in P$, hence log spaces $Y^\dagger$, $X^\dagger \to \operatorname{Spec} \Bbbk^\dagger$ as in §1.1, such that there is a diagram over* $\operatorname{Spec} \Bbbk^\dagger$

$$\begin{tikzcd} & (V^\dagger, D') \arrow[dl] \arrow[dr, "\phi"] & \\ (X_0^\dagger, D) & & (X^\dagger, \tilde{D} \cap X) \end{tikzcd} \tag{1.6}$$

with both maps strict étale and $V^\dagger$ an étale neighbourhood of $\bar{x}$.

Proof. The proof is basically the same as that of [16, Thm. 2.6], in which one relates different information in a toric stratum of X_0 to that of a toric monoid étale locally.

As in [16, notational summary], for every $\tau \in \mathscr{P}$ there is an inclusion map

$$q_\tau \colon X_\tau \to X_0, \tag{1.7}$$

where every toric stratum X_τ is defined by $X_\tau := X(\Sigma_\tau)$ ([15, Def. 2.7]), in which the boundedness assumption of τ is not involved. This is the normalization of the stratum of X_0 corresponding to τ (see also Definition 1.1 and the remark after [15, Def. 4.1]). Consequently, the arguments of [15, Cor. 5.8] apply also for each unbounded cell τ (with open gluing data s for $(B, \mathscr{P})$), so that $q_\tau^{-1}(Z) = Z_1^\tau \cup \cdots \cup Z_q^\tau \cup Z'$ where $Z' \subseteq D_\tau$ is of codimension at least two in X_τ and Z_i^τ is a hypersurface in X_τ with Newton polytope $\check{\Delta}_i$.

Therefore, there exists a unique $\tau \in \mathscr{P}$ with $\bar{x} \in q_\tau(X_\tau \setminus \partial X_\tau)$ (see (1.7)) for a given $\bar{x} \in Z$, such that $0 < \dim \tau < \dim B$ since $\bar{x} \in Z$ (where Z is of codimension 2 in X_0, see §1.3). By [15, Cor. 5.8], we thus obtain simplicity data associated to τ as in Remark 1.9 and also other data in Construction 1.8. According to Construction 1.8, we are able to obtain X. The term $\tilde{D}$ (see Construction 2.1) occurs in Y whenever the face $\tau \in \mathscr{P}$ is unbounded.

Then pick some $g : \tau \to \sigma \in \mathscr{P}_{\max}$ so that we obtain an open set $V(\tau) \subseteq V(\sigma)$ (cf. [17, Constr. 1.16], note that this thesis uses the fan picture mainly). If σ is bounded, it is just the case considered in [16].

Hence, consider σ unbounded. By [17, §1.2] and careful examination, the boundedness assumption of τ and σ is not needed in the proof of [16, Thm. 2.6]. Therefore we can apply the result of [16, Thm. 2.6] to obtain first the diagram (1.6) without D', D and $\tilde{D} \cap X$.

Consider now the correspondence between D', D and $\tilde{D} \cap X$. In fact, $\tilde{D} \cap X$ is constructed from D via the underlying affine geometry in $(B, \mathscr{P})$. Recall in §1.3, we have $D = \bigcup D_\mu$ such that each D_μ is a toric $(n-1)$-stratum in X_0.

In other words, each component D_μ of D corresponds to a toric stratum $X_{\tau'}$, where τ' is an unbounded 1-cell. By Definition 1.3, we have $D \subset X_0$, then for a cell $\tau \in \mathscr{P}$, either the components of D have a nonempty intersection with $q_\tau(X_\tau)$ in X_0 or they do not. A component D_μ of D intersects $q_\tau(X_\tau)$ if and only if there exists a cell τ_0 containing τ and an unbounded 1-cell τ'. There are two cases.

Case 1: If the cell τ is bounded, then $\tilde{D}$ and D' are absent by Construction 2.1. In this case, diagram (1.6) is still true because the the local model X and eventually the étale neighbourhood V of $\bar{x}$ do not "see" the divisor D in X_0 in this case.

Case 2: If the cell τ is unbounded, then the divisor $\tilde{D}$ will be present in the local model X of X_0 via construction (see Construction 2.1). The étale neighbourhood V of $\bar{x}$ then has the corresponding divisor D'.

Moreover, $q_{\tau_0}(X_{\tau_0})$ is contained in $q_\tau(X_\tau)$ and $q_{\tau'}(X_{\tau'})$ since the cell τ_0 contains τ and τ'. Such a cell τ_0 is of course unbounded as τ' is unbounded and actually we have

$$q_{\tau_0}(X_{\tau_0}) \subseteq D_\mu = q_{\tau'}(X_{\tau'}).$$

Therefore, we also have the correspondence of D', D and $\tilde{D} \cap X$ étale locally.

□

Remark 1.13. 1. The Calabi-Yau pair $(X_0^\dagger, D)$ with a new term D generalizes the setting in [16]. Whenever the divisor D is trivial, we recover the situation in [16]. From the perspective of the geometry of $(B, \mathscr{P})$, the term D corresponds to the unbounded rays in the polyhedral decomposition $\mathscr{P}$. The affine manifold B is now non-compact, unbounded and without boundary, while B is compact without boundary in [16].

For the sake of brevity of this section, we postpone the construction details of $\tilde{D}$ in the local model to Construction 2.1. The term $\tilde{D}$ occurs in Y whenever the face $\tau \in \mathscr{P}$ is unbounded. It is worth noting that we get agreeing log structures $\mathcal{M}_{\check{Y}} := \mathcal{M}_{(Y,X)} = \mathcal{M}_{(Y,X\cup\tilde{D})} := \mathcal{M}_Y$ when $\tilde{D} = 0$.

2. In the above theorem, only the correspondence between the log spaces $X^\dagger$ and $X_0^\dagger$ (induced by $\mathcal{M}_{(Y,X\cup\tilde{D})}$ and $\mathcal{M}_{(\mathfrak{X},\mathfrak{D}\cup X_0)}$ respectively) is handled. The correspondence

between $\breve{X}^\dagger$ (on $\breve{Y}^\dagger$) and $\breve{X}_0^\dagger$ (induced by $\mathcal{M}_{(Y,X)}$ and $\mathcal{M}_{(\mathfrak{X},X_0)}$ respectively) follows also from the arguments similarly.

Indeed, both log spaces have the same underlying topological space X_0; the difference is the charts for the log structures. We use the chart P' for the log structure on $X_0^\dagger$. For $\breve{X}_0^\dagger$, one uses a subset $\breve{P}'$ of P' for the local chart. For $g : \tau \to \sigma \in P_{\max}$, where σ is unbounded, one has a natural inclusion P' in P_σ (see [15, Def. 2.12]). The chart $\breve{P}'$ is obtained simply by restricting the chart $\breve{P}_\sigma \subseteq P_\sigma$ onto P' via the above natural inclusion. A more explicit description of $\breve{P}_\sigma$ will be given in the proof of Lemma 5.3.

Chapter 2

Cohomology of log Calabi-Yau pairs

The goal of this section is to prepare for the calculation of the logarithmic cohomology theories on X_0. We first find out the global sections of log differentials on open subsets of X_0. Then we lift the local descriptions on $X^\dagger$ to the global situation on $X_0^\dagger$ (and do not handle the global situation for $\breve{X}_0^\dagger$ for the time being) in order to consider resolutions of the sheaves of log differentials Ω^r on $X_0^\dagger$ and investigate the properties of such resolutions.

In §2.1 and §2.2, we examine the étale neighbourhoods X of X_0 and use the methods and arguments in [16, §1] and formulate the results concerning log derivations and log differentials with respect to the log spaces $X^\dagger$ and $\breve{X}^\dagger$ in our new setting. In §2.3, we will calculate global sections of the sheaves of log differentials in X and look at some examples. In §2.4, we lift the local descriptions to the global situation of the log space $X_0^\dagger$ and give an account for the resolutions $\mathscr{C}^\bullet(\Omega^r)$.

2.1 Derivations

Below is the simplified version of Construction 1.8, in the sense that the simplicity data related to a cell τ are not used explicitly in the construction. This construction follows [16, §1], which emphasizes the correspondence between toric divisors of Y and extremal rays of the toric monoid P. The results about log derivations and log differentials are readily expressed in terms of the data by this construction.

Construction 2.1 (cf. §1 in [16])**.** Let $M' = \mathbb{Z}^n$, $M'_{\mathbb{R}} = M' \otimes_{\mathbb{Z}} \mathbb{R}$, $N' = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Z})$ and $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. Fix convex lattice polytopes $\Delta_0, \ldots, \Delta_q \subseteq M'_{\mathbb{R}}$ with $\dim \Delta_0 = \dim M'_{\mathbb{R}}$, where $\Delta_1, \ldots, \Delta_q$ are bounded but Δ_0 can be either bounded or unbounded. Set $M = M' \oplus \mathbb{Z}^{q+1}$ and N the dual lattice of M. From these, we obtain a strictly convex rational polyhedral cone $\sigma \subseteq M'_{\mathbb{R}} \oplus \mathbb{R}^{q+1} = M_{\mathbb{R}}$ where σ is of the form

$$\mathrm{cl}\left(\mathbb{R}_{\geq 0} \cdot \bigcup_{i=0}^{q} (\Delta_i \times \{e_i\})\right) = \mathrm{cl}\left(\sum_{i=0}^{q} \mathbb{R}_{\geq 0} \cdot (\Delta_i \times \{e_i\})\right),$$

in which cl denotes the set-theoretical closure. Then we define the toric monoid P to be $P = \sigma^\vee \cap (N' \oplus \mathbb{Z}^{q+1}) = \sigma^\vee \cap N$.

Set $Y = \mathrm{Spec}\, \Bbbk[P]$. Let $\rho = e_0^*$ and let $P_1, \ldots, P_s$ be the facets of P not containing ρ.

Further let $X := Y \setminus U$, thus we have

$$X = Y \setminus U = \bigcup_{i=1}^{s} X_i = \bigcup_{i=1}^{s} \operatorname{Spec} \Bbbk[P_i],$$

where the X_i's are the toric divisors of Y contained in X corresponding to P_i's. As a subset of X of codimension one, we denote

$$X_{\text{sing}} = \bigcup_{i\neq j, 1\leq i,j\leq s} (X_i \cap X_j).$$

We are going to define the divisors

$$\tilde{D} = \bigcup_{j=1}^{r} \tilde{D}_j$$

and

$$E = \bigcup_{j=1}^{t} E_j,$$

where $\tilde{D} \cup E$ is the union of toric divisors of Y not contained in X (The divisors $E_1, \cdots, E_t$ are actually the divisors $D_1, \cdots, D_t$ in [16, §1]). The divisors $\tilde{D}_1, \ldots, \tilde{D}_r$ exist only if the cell Δ_0 is unbounded.

Let $v_1, \ldots, v_{s+r+t}$ be the primitive generators of the extremal rays of σ, where $v_1, \ldots, v_s$ correspond to $X_1, \ldots, X_s$; $v_{s+1}, \ldots, v_{s+r}$ correspond to $\tilde{D}_1, \ldots, \tilde{D}_r$ and $v_{s+r+1}, \ldots, v_{s+r+t}$ correspond to $E_1, \ldots, E_t$. More precisely, $v_1, \ldots, v_s$ are taken to be the vertices of Δ_0 and $v_{s+r+1}, \ldots, v_{s+r+t}$ are taken to be the vertices of Δ_j for $j \geq 1$ (repetition of vertices among different Δ_j's is allowed).

The extremal rays generated by $v_{s+1}, \ldots, v_{s+r}$ exist when Δ_0 is unbounded. These extremal rays lies actually inside the subspace $(M'_{\mathbb{R}}, 0, \ldots, 0)$. In other words, these extremal rays are "horizontal" in the sense that they are not pointing in the $\mathbb{Z}^{q+1}$ direction. Moreover, we can express them in terms of the data from Δ_0.

Each of these extremal rays is of the form $\mathbb{R}_{\geq 0} \cdot (\hat{v}_{s+j} - v_{k_j})$ for $1 \leq j \leq r$, where v_{k_j} is the vertex of an unbounded edge and $\hat{v}_{s+j}$ is the integral point (thus an element of M') on the unbounded edge of Δ_0 nearest to v_{k_j}. These rays $\mathbb{R}_{\geq 0} \cdot (\hat{v}_{s+j} - v_{k_j})$ are *a priori* elements of $(M'_{\mathbb{R}}, 1, 0, \ldots, 0)$; however, we can vertically translate these rays to $(M'_{\mathbb{R}}, 0, 0, \ldots, 0)$ and identify v_{k_j} with the origin of $M_{\mathbb{R}}$ by a horizontal translation afterwards. Therefore, we have $v_{s+j} = \hat{v}_{s+j} - v_{k_j}$ (as an element in M) and every $\mathbb{R}_{\geq 0} \cdot (\hat{v}_{s+j} - v_{k_j})$ represents an actual extremal ray (not necessarily uniquely) of P of this type.

Note that we have $v_i \in M'$ for all i above. For ease of notation, we write

$$v_j^{\delta} = v_{s+j} \text{ for } 1 \leq j \leq r,$$

and

$$w_j = v_{s+r+j} \text{ for } 1 \leq j \leq t.$$

Let $\{P_1^\delta, \ldots, P_r^\delta\}$ and $\{Q_1, \ldots, Q_t\}$ be the facets (maximal proper faces) of P corresponding to $\{v_1^\delta, \ldots, v_r^\delta\}$ and $\{w_1, \ldots, w_t\}$ respectively. And the $\{P_1, \ldots, P_s\}$ previously defined are the facets related to $\{v_1, \ldots, v_s\}$. Note that $X_i = \operatorname{Spec} \Bbbk[P_i]$ and $\tilde{D}_j = \operatorname{Spec} \Bbbk[P_j^\delta]$. Since the P_i's are the facets of P not containing ρ, we have indeed $\rho \in P_1^\delta \cap \ldots \cap P_r^\delta \cap Q_1 \cap \ldots \cap Q_t$.

The various correspondences in above can be summarized by the following table.

Toric Divisors in $Y = \operatorname{Spec} \Bbbk[P]$	Irreducible Components	Facets in P	Generators of Extremal Rays of σ	Appearance in [16]
X_0	$X_1, \ldots, X_s$	$P_1, \ldots, P_s$	$(v_1, e_0), \ldots, (v_s, e_0)$	Yes
$\tilde{D}$	$\tilde{D}_1, \ldots, \tilde{D}_r$	$P_1^\delta, \ldots, P_r^\delta$	$(v_1^\delta, 0), \ldots, (v_r^\delta, 0)$	No
Remaining Toric Divisors	$E_1, \ldots, E_t$	$Q_1, \ldots, Q_t$	$(w_1, e_{k_1}), \ldots, (w_t, e_{k_t})$	Yes

□

In contrast to [16], we consider now three log structures on Y. The first one is given by

$$\mathcal{M}_{\check{Y}} = \mathcal{M}_{(Y,X)} = j_*(\mathcal{O}_U^\times) \cap \mathcal{O}_Y.$$

The second one is given by

$$\mathcal{M}_Y = \mathcal{M}_{(Y, X \cup \tilde{D})}.$$

The third one is induced by the chart $P \to \Bbbk[P]$, which is a fine log structure, and we denote the log space as $\mathcal{M}_{\tilde{Y}}$ (which means exactly the same as in [16]). There is an obvious inclusion $\mathcal{M}_{\check{Y}} \subseteq \mathcal{M}_Y \subseteq \mathcal{M}_{\tilde{Y}}$. We write $\check{Y}^\dagger$, $Y^\dagger$ and $\tilde{Y}^\dagger$ for the three log structures respectively.

Remark 2.2. If the polytope Δ_0 is bounded, we recover the case for σ and P in the construction before [16, Prop. 1.5] and we also do not have the terms $\tilde{D}_j$.

The new term $\tilde{D}$, which does not exist in [16], corresponds étale locally to the term $\mathfrak{D}$ in Definition 1.2. Yet the toric boundary divisor now is not only $X \cup \mathfrak{D}$, but with the extra term E. The toric divisor E in this étale local picture depends on the monodromy around $\Delta_0 := \tau$, which is also the case in [16]. The monodromy around τ could be trivial, then we have $E = \emptyset$. The readers can refer back to Construction 1.8 in order to have more insight concerning the terms $\tilde{D}$ and E in association with (X_0, D). □

Recall that we have defined what a log derivation is in Definition 1.4. We now restate [16, Prop. 1.5], [16, Cor. 1.7] and [16, Prop. 1.8] in the above situation, taking into account the newly introduced $\tilde{D}$ term and the new log structure $\mathcal{M}_{\check{Y}}$.

Proposition 2.3 (cf. Prop. 1.5 in [16]). *In the above situation,* $\Gamma(Y, \Theta_{Y^\dagger/\Bbbk})$ *splits into* P^{gp}*-homogeneous pieces*

$$\bigoplus_{p \in P^{\mathrm{gp}}} z^p (\Theta_{Y^\dagger/\Bbbk})_p,$$

where

$$(\Theta_{Y^\dagger/\Bbbk})_p = \begin{cases} M \otimes_{\mathbb{Z}} \Bbbk & \text{if } p \in P, \\ \mathbb{Z}v_i \otimes_{\mathbb{Z}} \Bbbk & \text{if there exists an } i,\ s+r+1 \leq i \leq s+r+t, \\ & \text{with } \langle v_i, p\rangle = -1,\ \langle v_j, p\rangle \geq 0 \text{ for } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\Gamma(Y, \Theta_{\breve{Y}^\dagger/\Bbbk})$ *splits into* P^{gp}*-homogeneous pieces*

$$\bigoplus_{p \in P^{\mathrm{gp}}} z^p (\Theta_{\breve{Y}^\dagger/\Bbbk})_p,$$

where

$$(\Theta_{\breve{Y}^\dagger/\Bbbk})_p = \begin{cases} M \otimes_{\mathbb{Z}} \Bbbk & \text{if } p \in P, \\ \mathbb{Z}v_i \otimes_{\mathbb{Z}} \Bbbk & \text{if there exists an } i,\ s+1 \leq i \leq s+r+t, \\ & \text{with } \langle v_i, p\rangle = -1,\ \langle v_j, p\rangle \geq 0 \text{ for } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

In both cases, an element $m \in (\Theta_{\breve{Y}^\dagger/\Bbbk})_p$ *or* $m \in (\Theta_{Y^\dagger/\Bbbk})_p$ *is written as* ∂_m*. The term* $z^p\partial_m$ *acts on the monomial* z^q *by*

$$z^p \partial_m z^q = \langle m, q\rangle z^{p+q}.$$

Proof. Observe that the ideals of $X \cup \tilde{D}$ and X are generated by $P \setminus (P_1 \cup \cdots \cup P_s \cup P_1^\delta \cup \cdots \cup P_r^\delta)$ and $P \setminus (P_1 \cup \cdots \cup P_s)$ respectively. Then the proof is essentially the same as that of [16, Prop. 1.5], using the fact that $\Theta_{Y^\dagger/\Bbbk}$ and $\Theta_{\breve{Y}^\dagger/\Bbbk}$ consist of usual derivations of Y which preserve the ideals of $X \cup \tilde{D}$ and X respectively. □

Corollary 2.4 (cf. Cor. 1.7 in [16])**.** *In the situation of Proposition 2.3, let* $S = \operatorname{Spec} \Bbbk[\mathbb{N}]$ *with the log structure defined by the obvious chart* $\mathbb{N} \to \Bbbk[\mathbb{N}]$*. Then* $z^\rho = z^{e_0^*}$ *induces the log morphisms* $Y^\dagger \to S^\dagger$ *and* $\breve{Y}^\dagger \to S^\dagger$*. Furthermore,*

$$\Gamma(Y, \Theta_{Y^\dagger/S^\dagger}) = \bigoplus_{p \in P^{\mathrm{gp}}} z^p (\Theta_{Y^\dagger/S^\dagger})_p,$$

where

$$(\Theta_{Y^\dagger/S^\dagger})_p = \begin{cases} \rho^\perp \otimes_{\mathbb{Z}} \Bbbk & \text{if } p \in P, \\ \mathbb{Z}v_i \otimes_{\mathbb{Z}} \Bbbk & \text{if there exists an } i,\ s+r+1 \leq i \leq s+r+t, \\ & \text{with } \langle v_i, p\rangle = -1,\ \langle v_j, p\rangle \geq 0 \text{ for } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\Gamma(Y, \Theta_{\breve{Y}^\dagger/S^\dagger}) = \bigoplus_{p \in P^{\mathrm{gp}}} z^p (\Theta_{\breve{Y}^\dagger/S^\dagger})_p,$$

where

$$(\Theta_{\breve{Y}^\dagger/S^\dagger})_p = \begin{cases} \rho^\perp \otimes_{\mathbb{Z}} \Bbbk & \text{if } p \in P, \\ \mathbb{Z}v_i \otimes_{\mathbb{Z}} \Bbbk & \text{if there exists an } i,\ s+1 \le i \le s+r+t, \\ & \text{with } \langle v_i, p\rangle = -1,\ \langle v_j, p\rangle \ge 0 \text{ for } j \ne i, \\ 0 & \text{otherwise,} \end{cases}$$

Proof. Both cases follow by observing that elements of $\Theta_{\breve{Y}^\dagger/S^\dagger}$ and $\Theta_{Y^\dagger/S^\dagger}$ must annihilate z^ρ. □

Proposition 2.5 (cf. Prop. 1.8 in [16])**.** *Let $A_k = \Bbbk[t]/(t^{k+1})$, with natural map $\operatorname{Spec} A_k \to S = \operatorname{Spec} \Bbbk[\mathbb{N}]$. Pull back the log structure $S^\dagger$ on S, which is defined by the chart $\mathbb{N} \to \Bbbk[\mathbb{N}]$, to $\operatorname{Spec} A_k$ to yield the log scheme $\operatorname{Spec} A_k^\dagger$. Consider the scheme $\mathfrak{X}_k = \operatorname{Spec} \Bbbk[P]/(z^{(k+1)\rho})$.*

1. *Consider the log scheme $\mathfrak{X}_k^\dagger$ with the log structure induced from $Y^\dagger$ by the canonical map $\Bbbk[P] \to \Bbbk[P]/(z^{(k+1)\rho})$. Then $\Gamma(\mathfrak{X}_k, \Theta_{\mathfrak{X}_k^\dagger/\Bbbk})$ and $\Gamma(\mathfrak{X}_k, \Theta_{\mathfrak{X}_k^\dagger/A_k^\dagger})$ split into P^{gp}-homogeneous pieces*

   $$\bigoplus_{p \in P^{\mathrm{gp}}} z^p \left(\Theta_{\mathfrak{X}_k^\dagger/\Bbbk}\right)_p \quad \text{and} \quad \bigoplus_{p \in P^{\mathrm{gp}}} z^p \left(\Theta_{\mathfrak{X}_k^\dagger/A_k^\dagger}\right)_p$$

   respectively, where

   $$\left(\Theta_{\mathfrak{X}_k^\dagger/\Bbbk}\right)_p = \left(\Theta_{\mathfrak{X}_k^\dagger/A_k^\dagger}\right)_p = 0$$

   if there does not exist an i, $1 \le i \le s+r$, such that $0 \le \langle v_i, p\rangle \le k$; otherwise

   $$\left(\Theta_{\mathfrak{X}_k^\dagger/\Bbbk}\right)_p = \left(\Theta_{Y^\dagger/\Bbbk}\right)_p \quad \text{and} \quad \left(\Theta_{\mathfrak{X}_k^\dagger/A_k^\dagger}\right)_p = \left(\Theta_{Y^\dagger/S^\dagger}\right)_p.$$

2. *Consider the log scheme $\breve{\mathfrak{X}}_k^\dagger$ with the log structure induced from $\breve{Y}^\dagger$ by the canonical map $\Bbbk[P] \to \Bbbk[P]/(z^{(k+1)\rho})$. Then $\Gamma(\mathfrak{X}_k, \Theta_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk})$ and $\Gamma(\mathfrak{X}_k, \Theta_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger})$ split into P^{gp}-homogeneous pieces*

   $$\bigoplus_{p \in P^{\mathrm{gp}}} z^p \left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}\right)_p \quad \text{and} \quad \bigoplus_{p \in P^{\mathrm{gp}}} z^p \left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger}\right)_p$$

   respectively, where

   $$\left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}\right)_p = \left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger}\right)_p = 0$$

   if there does not exist an i, $1 \le i \le s$, such that $0 \le \langle v_i, p\rangle \le k$; otherwise

   $$\left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}\right)_p = \left(\Theta_{\breve{Y}^\dagger/\Bbbk}\right)_p \quad \text{and} \quad \left(\Theta_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger}\right)_p = \left(\Theta_{\breve{Y}^\dagger/S^\dagger}\right)_p.$$

Proof. Consider the restriction maps $\Theta_{Y^\dagger/\Bbbk} \to \Theta_{\mathfrak{X}_k^\dagger/\Bbbk}$ and $\Theta_{\breve{Y}^\dagger/\Bbbk} \to \Theta_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}$ and use the arguments in the proof of [16, Prop. 1.8]. Then the results follow with respect to the new notations and log structures. □

2.2 Differentials

Recall we have defined the notion of log differentials in Lemma 1.5. We now restate [16, Prop. 1.12] and [16, Cor. 1.13], which describe the various sheaves of log differentials on the log smooth part of X. The pushforward of the sheaf of log differentials on the log smooth part of X is considered because the log differentials behave poorly at points where the log structures are not fine, as demonstrated in [16, Ex. 1.11].

Proposition 2.6 (cf. Prop. 1.12 in [16]). *In the situation of Proposition 2.5, let $Z := E \cap X_{\text{sing}} \subseteq |\mathfrak{X}_k| = |X|$ be the locus where the log structures on X fail to be fine. (Here $|\mathfrak{X}_k|$ denotes the underlying topological space.) Then*

1. *$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk})$ is naturally a P-module with a decomposition into P-homogeneous pieces given as follows:*

$$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk}) = \bigoplus_{p \in P \setminus ((k+1)\rho + P)} \bigwedge^r \left(\bigcap_{\{j | p \in Q_j\}} Q_j^{\text{gp}} \right) \otimes_{\mathbb{Z}} \Bbbk.$$

   For $a \in \Bbbk$ and $n_i \in P^{\text{gp}}$, $an_1 \wedge \cdots \wedge n_r$ in the summand of degree p corresponds to the restriction of $az^p \operatorname{dlog} n_1 \wedge \cdots \wedge \operatorname{dlog} n_r \in \Gamma(Y \setminus Z, \Omega^r_{Y^\dagger/\Bbbk})$ to $\mathfrak{X}_k$.

2. *$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\check{\mathfrak{X}}_k^\dagger/\Bbbk})$ is naturally a P-module with a decomposition into P-homogeneous pieces given as follows:*

$$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\check{\mathfrak{X}}_k^\dagger/\Bbbk}) = \bigoplus_{p \in P \setminus ((k+1)\rho + P)} \bigwedge^r \left(\bigcap_{\{j | p \in P_j^\delta\}} (P_j^\delta)^{\text{gp}} \cap \bigcap_{\{j | p \in Q_j\}} Q_j^{\text{gp}} \right) \otimes_{\mathbb{Z}} \Bbbk.$$

   For $a \in \Bbbk$ and $n_i \in P^{\text{gp}}$, $an_1 \wedge \cdots \wedge n_r$ in the summand of degree p corresponds to the restriction of $az^p \operatorname{dlog} n_1 \wedge \cdots \wedge \operatorname{dlog} n_r \in \Gamma(Y \setminus Z, \Omega^r_{\check{Y}^\dagger/\Bbbk})$ to $\mathfrak{X}_k$.

Remark 2.7. 1. In the above statement as well as in Proposition 1.12 and in Corollary 1.13 in [16], there is an abuse of notation concerning the sheaf of log differentials. Consider the inclusion

$$j \colon \mathfrak{X}_k \setminus Z \hookrightarrow \mathfrak{X}_k,$$

where $Z = E \cap X_{\text{sing}}$ in this article while $Z = D \cap X_{\text{sing}}$ in [16]. With reference to the discussion just before Proposition 2.6, we seek to consider the pushforward of the sheaf of log differentials on $\mathfrak{X}_k \setminus Z$; it is the log smooth part of $\mathfrak{X}_k$. Actually, our purpose is to compute $\Gamma(\mathfrak{X}_k, j_*\Omega^r_{(\mathfrak{X}_k \setminus Z)^\dagger/\Bbbk})$ instead of $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk})$, in which we just define

$$\Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk} := j_*\Omega^r_{(\mathfrak{X}_k \setminus Z)^\dagger/\Bbbk}.$$

by the fact that $\Gamma(X, j_*\mathcal{F}) = \Gamma(U, \mathcal{F})$ for $j\colon U \hookrightarrow X$ and $\mathcal{F}$ a sheaf on U.

For sake of convenience and clarity, we will adopt a similar abuse of notation as in [16], for instance $\Omega^r_{X_0^\dagger/\Bbbk}$ instead of $j_*\Omega^r_{(X_0\setminus Z)^\dagger/\Bbbk}$ (and similarly for the $A_k^\dagger$ case) in the rest of this article.

2. In the above Proposition, suppose that the restriction of $z^p \operatorname{dlog} n_1 \wedge \ldots \wedge \operatorname{dlog} n_r$ to $\mathfrak{X}_k$ is nonzero in $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk})$. Then for $p' \in P \setminus ((k+1)\rho + P)$, the above notation means that

$$z^{p'} \cdot z^p \operatorname{dlog} n_1 \wedge \ldots \wedge \operatorname{dlog} n_r = z^{p'+p} \operatorname{dlog} n_1 \wedge \ldots \wedge \operatorname{dlog} n_r$$

is nonzero after restriction to an element in $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk})$ only if $n_k \in \bigcap_{\{j | p'+p \in Q_j\}} Q_j^{\mathrm{gp}}$ for all k. This fact also holds similarly for $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk})$.

Proof. First note the fact that $\Omega^1_{\breve{Y}^\dagger/\Bbbk}|_{Y\setminus Z} \subseteq \Omega^1_{Y^\dagger/\Bbbk}|_{Y\setminus Z} \subseteq \Omega^1_{\tilde{Y}^\dagger/\Bbbk}|_{Y\setminus Z}$. Consequently,

$$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}) \subseteq \Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk}) \subseteq \Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\tilde{\mathfrak{X}}_k^\dagger/\Bbbk}).$$

On the other hand, we know that

$$\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\tilde{\mathfrak{X}}_k^\dagger/\Bbbk}) = \bigoplus_{p \in P\setminus((k+1)\rho+P)} z^p \left(\bigwedge\nolimits^r P^{\mathrm{gp}}\right) \otimes_{\mathbb{Z}} \Bbbk.$$

The action of the algebraic torus $\operatorname{Spec} \Bbbk[P^{\mathrm{gp}}]$ respects the inclusions $X \subseteq Y$ and $\tilde{D} \cup E \subseteq Y$, $X \cup \tilde{D} \subseteq Y$ and $E \subseteq Y$; so it induces an action on $\Gamma(Y, \Omega^r_{\breve{Y}^\dagger/\Bbbk}) \subseteq \Gamma(Y, \Omega^r_{Y^\dagger/\Bbbk}) \subseteq \Gamma(Y, \Omega^r_{\tilde{Y}^\dagger/\Bbbk})$.

Therefore, for each $p \in P$ there exist $\Bbbk$-vector subspaces $\breve{V}_p^r, V_p^r \subseteq \bigwedge^r P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \Bbbk$ such that

$$\begin{aligned} \Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}) &= \bigoplus_{p \in P\setminus((k+1)\rho+P)} z^p \breve{V}_p^r, \\ \text{and} \quad \Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk}) &= \bigoplus_{p \in P\setminus((k+1)\rho+P)} z^p V_p^r. \end{aligned}$$

To finish the proof it remains to describe $\breve{V}_p^r$ and V_p^r for $p \in P \setminus ((k+1)\rho + P)$ because all monomials in $(k+1)\rho + P$ restrict to zero on $\mathfrak{X}_k$.

As in the proof of [16, Prop. 1.12], we can compute V_p^r by induction on r. For $r = 0$, $\Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk} = \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk} = \mathcal{O}_{\mathfrak{X}_k}$ and

$$\Gamma(\mathfrak{X}_k \setminus Z, \mathcal{O}_{\mathfrak{X}_k}) = \bigoplus_{p \in P\setminus((k+1)\rho+P)} z^p \otimes_{\mathbb{Z}} \Bbbk.$$

Then we apply the fact that an element of $z^p(\bigwedge^r P^{\mathrm{gp}})$ is in $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk})$ if and only if the contraction of it with any element of $\Gamma(\mathfrak{X}_k \setminus Z, \Theta_{\mathfrak{X}_k^\dagger/\Bbbk}) = \Gamma(\mathfrak{X}_k, \Theta_{\mathfrak{X}_k^\dagger/\Bbbk})$ is in $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^{r-1}_{\mathfrak{X}_k^\dagger/\Bbbk})$ via similar arguments as in the proof of [16, Prop. 1.12].

The computation of $\breve{V}_p^r$ is similar. We note that the new terms P_j^δ (together with Q_j) now take the role of Q_j in the case of V_p^r. In the same manner, we also arrive at the result for $\Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}$. □

Corollary 2.8. *In the situation of Proposition 2.6,* $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/A_k^\dagger})$ *and* $\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger})$ *are naturally P-modules with decompositions into P^{gp}-homogeneous pieces as follows:*

$$\begin{aligned}\Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\mathfrak{X}_k^\dagger/A_k^\dagger}) &= \bigoplus_{p\in P\setminus((k+1)\rho+P)} \bigwedge^r \Big(\bigcap_{\{j|p\in Q_j\}} Q_j^{\mathrm{gp}}/\mathbb{Z}\rho \Big) \otimes_{\mathbb{Z}} \Bbbk,\\ \Gamma(\mathfrak{X}_k \setminus Z, \Omega^r_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger}) &= \bigoplus_{p\in P\setminus((k+1)\rho+P)} \bigwedge^r \Big(\bigcap_{\{j|p\in P_j^\delta\}} (P_j^\delta)^{\mathrm{gp}}/\mathbb{Z}\rho \cap \bigcap_{\{j|p\in Q_j\}} Q_j^{\mathrm{gp}}/\mathbb{Z}\rho \Big) \otimes_{\mathbb{Z}} \Bbbk.\end{aligned}$$

Proof. The corollary is an immediate consequence of Proposition 2.6. □

2.3 Local calculations

We have proved Theorem 1.12 in §1.4, which is the analogue of [16, Thm. 2.6]. We now begin with the calculations for our local models as in [16, §3.1].

Construction 2.9 (cf. the paragraph before Lem. 3.2 in [16])**.** Suppose we are given data $\tau \subseteq M'_{\mathbb{R}}$, $\Delta_1, \dots, \Delta_q$ as in Construction 1.8, yielding a cone $K \subseteq M_{\mathbb{R}}$, $P = K^\vee \cap N$, $\rho \in P$, $Y = \operatorname{Spec} \Bbbk[P]$, $X = \operatorname{Spec} \Bbbk[P]/(z^\rho)$, $\mathfrak{X}_k = \operatorname{Spec} \Bbbk[P]/(z^{(k+1)\rho})$, where $\dim_{\Bbbk} X = n$.

For every face ω of τ, we have a stratum $V_\omega \subseteq X$, with $V_\omega = \operatorname{Spec} \Bbbk[P_\omega]$ where P_ω is the face of P given by $P \cap (\omega + e_0)^\perp$. For every $k \geq 0$, consider the monoid ideal

$$I^k_\omega = \{p \in P | \langle p, m\rangle > k \text{ for some } m \in \omega + e_0\}.$$

This defines a thickening

$$V^k_\omega = \operatorname{Spec} \Bbbk[P]/I^k_\omega.$$

Note that V^k_ω is a closed subscheme of $\mathfrak{X}_k$. Let $q_\omega : V^k_\omega \to \mathfrak{X}_k$ be the embedding.

Let $Z = \bigcup_i Z_i$ be the subscheme of X defined in Construction 1.8, with $j : X \setminus Z \hookrightarrow X$ the inclusion. Set $D_\omega = \bigcup_{\omega \subsetneq \omega' \subseteq \tau} V_{\omega'}$. This is a subset of the toric boundary of V_ω consisting of proper intersections of the stratum V_ω with other strata of X. Let

$$\kappa_\omega : V^k_\omega \setminus (D_\omega \cap Z) \to V^k_\omega$$

be the inclusion. With reference to Remark 2.7, we utilize the notation for the log structure $\mathfrak{X}_k^\dagger$ with $\Omega^r_k = j_*\Omega^r_{\mathfrak{X}_k^\dagger/\Bbbk}$ or $\Omega^r_k = j_*\Omega^r_{\mathfrak{X}_k^\dagger/A_k^\dagger}$ and let

$$\Omega^r_{\omega,k} = \kappa_{\omega*}\kappa^*_\omega(q^*_\omega\Omega^r_k/\operatorname{Tors}).$$

Meanwhile, for the log structure $\breve{\mathfrak{X}}_k^\dagger$, we denote $\breve{\Omega}^r_k = j_*\Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}$ or $\breve{\Omega}^r_k = j_*\Omega^r_{\breve{\mathfrak{X}}_k^\dagger/A_k^\dagger}$ and let

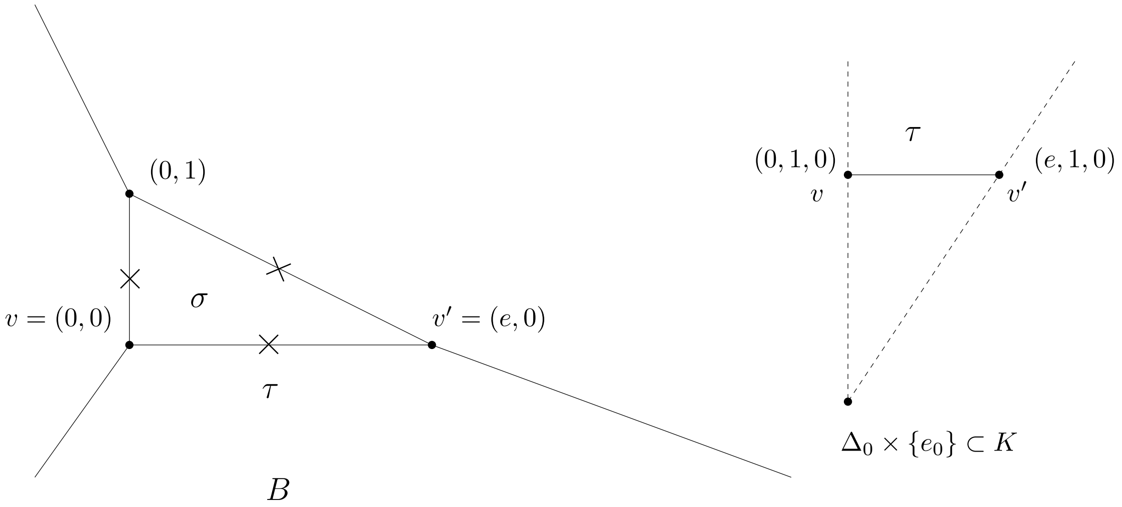
$$\breve{\Omega}^r_{\omega,k} = \kappa_{\omega*}\kappa^*_\omega(q^*_\omega\breve{\Omega}^r_k/\operatorname{Tors}).$$

□

As in [16, §3], we have the Tors term in the definition of sheaf of log differentials. This term denotes the submodule of torsion elements, which have supports on proper closed subsets. We compute now an example to explain the existence of this term.

Example 2.10. Recall from Remark 2.7 that we consider the sheaf $j_*\Omega^r_{(\mathfrak{X}_k\setminus Z)^\dagger/\mathbb{k}}$ because the sheaf of log differentials is not fine at points of Z. In the following, we demonstrate its relation to Tors in Construction 2.9 and forthcoming Lemma 2.11 ([16, Lem. 3.1]).

It is an elaboration of [17, Ex. 1.12]. Consider τ and σ as in the diagram, where σ is a maximal cell in the fan picture B and the crosses denote the focus-focus singularities on B (see [18, §2.2]).



Consider the cell τ on B and use the notations as in Construction 1.8 to compute the local model near the point $\bar{x} \in X_\tau$, where the log structure fails to be fine. Then we have $M' = \mathbb{Z}$ and $M = M' \oplus \mathbb{Z}^2$ and for $n \in N'_{\mathbb{R}}$,

$$\check{\psi}_0(n) = \begin{cases} 0 & \text{for } n \geq 0 \\ -en & \text{for } n < 0 \end{cases} \quad \text{and} \quad \check{\psi}_1(n) = \begin{cases} 0 & \text{for } n \geq 0 \\ -n & \text{for } n < 0 \end{cases}$$

so that $\Delta_0 = \tau$ and $\Delta_1 = [0,1]$.

P is generated by $\{(1,0,0),(0,1,0),(0,0,1),(-1,e,1)\}$, which correspond to variables $\{z^{(1,0,0)}, z^{(0,1,0)}, z^{(0,0,1)}, z^{(-1,e,1)}\}$. Q is generated by $\{(1,0,0),(0,0,1),(-1,e,1)\}$.

$$\begin{aligned} Y &= \operatorname{Spec} \mathbb{k}[P] = \operatorname{Spec} \mathbb{k}[x,t,w,x^{-1}t^e w] \\ &= \operatorname{Spec} \frac{\mathbb{k}[x,t,w,y]}{(xy - wt^e)}, \end{aligned}$$

in which $\mathbb{k}[x,t,w,x^{-1}t^e w]$ is a subring of $\mathbb{k}[x^\pm, t^\pm, w^\pm]$ and we write $x = z^{(1,0,0)}$, $t = z^{(0,1,0)}$ and $w = z^{(0,0,1)}$. As a subscheme of Y,

$$X = \operatorname{Spec} \mathbb{k}[P]/(z^{(0,1,0)}) = \operatorname{Spec} \frac{\mathbb{k}[x,w,y]}{(xy)}$$

By evaluation of $\check{\psi}_1$, we conclude that $Z = Z_1 = V(x, x^{-1}t^e w, w) = V(x, y, w)$.

For simplicity, we consider $e = 1$ in the rest of this example. Firstly, using notations in Construction 2.9, we have

$$V_\tau = \operatorname{Spec} \Bbbk[P_\tau] = \operatorname{Spec} \Bbbk[P \cap (\tau + e_0)^\perp] = \operatorname{Spec} \Bbbk[w] \subset X \cong \operatorname{Spec} \frac{\Bbbk[x, w, y]}{(xy)}.$$

As faces of τ, the vertices v and v' yield

$$V_v = \operatorname{Spec} \Bbbk[P_v] = \operatorname{Spec} \Bbbk[x, w] = V(y)$$

and

$$V_{v'} = \operatorname{Spec} \Bbbk[w, y] = V(x)$$

as subschemes of X. Furthermore, $D_v = D_{v'} = V_\tau$ and $D_v \cap Z = V(x, y, w)$. So we have the embedding

$$q_v \colon \operatorname{Spec} \Bbbk[x, w] \longrightarrow \operatorname{Spec} \frac{\Bbbk[x, w, y]}{(xy)} \cong X.$$

Consider the sheaf $\Omega_0^r = j_* \Omega^r_{\mathfrak{X}_0^\dagger / k^\dagger}$ on $X = \mathfrak{X}_0$. Consequently, $q_v^* \Omega_0^r$ and $q_v^* \Omega_0^r / \operatorname{Tors}$ are sheaves on V_v. We now compute the term Tors in $q_v^* \Omega_0^r$ and its relation with V_τ. As a subscheme of $Y \cong \operatorname{Spec} \Bbbk[x, t, w, y]/(xy - wt)$, observe that the relation

$$\operatorname{dlog} x + \operatorname{dlog} y = \operatorname{dlog} w + \operatorname{dlog} t$$

holds on X. Since we are working $/\Bbbk^\dagger$, i.e. modulo $\operatorname{dlog} t$, this relation is reduced to

$$\operatorname{dlog} x + \operatorname{dlog} y = \operatorname{dlog} w. \tag{2.1}$$

On X, the sheaf Ω_0^r has stalks generated by $\{\operatorname{dlog} x, \operatorname{dlog} y, \operatorname{dlog} w\}$ (depending on where the points lie) while $q_v^* \Omega_0^r$ on V_v and $q_{v'}^* \Omega_0^r$ on $V_{v'}$ have stalks generated by $\{\operatorname{dlog} x, dy, \operatorname{dlog} w\}$ and $\{dx, \operatorname{dlog} y, \operatorname{dlog} w\}$ respectively.

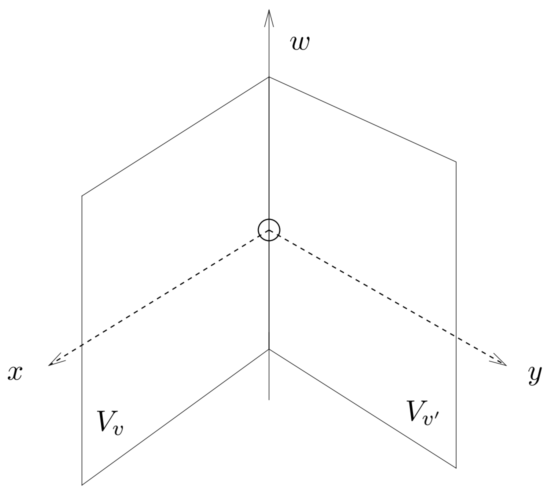
Nevertheless, there are some problematic terms on V_v and $V_{v'}$ so that we have to use the sheaves $q_v^* \Omega_0^r / \operatorname{Tors}$ and $q_{v'}^* \Omega_0^r / \operatorname{Tors}$. Examine the "mixed term" $y \operatorname{dlog} x$, which is *a priori* zero on $V_v \setminus V_\tau = \{(x, y, w) \in X \mid y = 0 \text{ and } x \neq 0\}$. By equation (2.1) and the observation that the term $dy = y \operatorname{dlog} y$ is well-defined on V_v,

$$\begin{aligned} y \operatorname{dlog} x &= y(\operatorname{dlog} w - \operatorname{dlog} y) \\ &= y \operatorname{dlog} w - y \operatorname{dlog} y \\ &= y \operatorname{dlog} w - dy \\ &= -dy, \end{aligned}$$

which is not a trivial term.

On the other hand, the term $y \operatorname{dlog} x$ is well-defined and not zero on $V_\tau = \{(x, y, w) \in X \mid x = y = 0\}$, which is a proper closed set of V_v. More precisely, we have

$$dx = y \operatorname{dlog} x = -dy \neq 0$$



because $xy = 0$ on V_τ so that $ydx + xdy = 0$ and consequently

$$dx + dy = \frac{y}{x} dx + dy = 0.$$

Therefore $y \operatorname{dlog} x$ is a term with support on a proper closed set on V_v and thus lies in Tors. In addition, the term $y^p \operatorname{dlog} x = 0$ for any $p > 1$ is not a cause of worry.

Similarly, the above phenomenon also occurs for the "mixed term" $x \operatorname{dlog} y$ on $V_{v'}$. Since these "mixed terms" are ill-behaved, it is natural to take the quotient by the module Tors to define the sheaf of log differentials. The nonexistence of these "mixed terms" is also the reason why we can have an obvious Hodge decomposition later in Theorem 3.6 (cf. [16, Thm. 3.26]).

□

In order to study the sheaves of differentials for our local models in Proposition 2.6 and Corollary 2.8, we recall a technical lemma here for easy reference.

Lemma 2.11 (Lem. 3.1 in [16]). *Let P be a toric monoid, $Q \subseteq P$ a face, $Y = \operatorname{Spec} \Bbbk[P]$, $I \subseteq P$ a monoid ideal with radical $P \setminus Q$, $X = \operatorname{Spec} \Bbbk[P]/I$. Suppose furthermore that $p \in P$, $q \in Q$, $p + q \in I$ implies $p \in I$. Consider a $\Bbbk[P]$-module F of the form*

$$F = \bigoplus_{p \in P} z^p F_{\langle p \rangle}$$

where $\langle p \rangle$ denotes the minimal face of P containing p, $F_{\langle p \rangle}$ a $\Bbbk$-vector space in $N \otimes_{\mathbb{Z}} \Bbbk$ containing p, and $F_{P_1} \subseteq F_{P_2}$ whenever $P_1 \subseteq P_2$. Then

1.

$$F \otimes_{\Bbbk[P]} \Bbbk[P]/I = \bigoplus_{p \in P} z^p \left(\frac{F_{\langle p \rangle}}{\sum_{\substack{p' \in P, q \in I \\ p' + q = p}} F_{\langle p' \rangle}} \right).$$

2. If $F_Q = F_P$ then

$$F_X := (F \otimes_{\Bbbk[P]} \Bbbk[P]/I)/\operatorname{Tors} = \bigoplus_{p \in P \setminus I} z^p F_{\langle p \rangle},$$

where Tors *denotes the submodule of elements of $F \otimes_{\Bbbk[P]} \Bbbk[P]/I$ with support on a proper closed subset of X.*

3. Let $J \subseteq P$ be a monoid ideal such that if $Z \subseteq X$ is the closed subscheme defined by $(I+J)/I \subseteq \Bbbk[P]/I$, then Z is codimension ≥ 2 in X. Let $\kappa\colon X \setminus Z \hookrightarrow X$ be the inclusion. If $\mathcal{F}_X$ is the sheaf on X associated to F_X, then

$$\Gamma(X, \kappa_* \kappa^* \mathcal{F}_X) = \bigoplus_{p \in P^{\mathrm{gp}}} z^p \bigcap_{q \in J \cap Q} \bigcup_{\substack{n \geq 0 \\ p+nq \in P \setminus I}} F_{\langle p+nq \rangle}$$

Using the notations in Construction 2.9, we are ready to calculate the global sections of the sheaves $\Omega^r_{w,k}$ and $\breve{\Omega}^r_{w,k}$, by putting $I = I_\omega$, $Q = P_\omega$ and $J = P \setminus P_{D_\omega \cap Z}$ in the above Lemma 2.11 (the notation $P_{D_\omega \cap Z}$ will appear in the following Lemma). We make several remarks first before we proceed.

Remark 2.12. 1. Under the new setting, Lemma 2.11 still holds true when both ω and τ are unbounded, or just τ is unbounded. By careful examinations, one notes that the unboundedness of τ or ω does not affect the algebraic arguments applied on the toric monoid P.

2. In the statement of [16, Lem. 3.2], for $i > 0$ and given $\omega \subseteq \tau$, ω_i is defined to be the largest face of Δ_i with respect to ω such that

$$\langle n, m \rangle = -\check{\psi}_i(n) \quad \text{for any } n \in \check{\omega} \text{ and } m \in \omega_i.$$

Hence, ω_i is associated with Δ_i or $\check{\psi}_i$. In fact, the functions $\check{\psi}_i$ build up a correspondence between $\{\omega_i\} \subseteq \Delta_i$ and $\{\omega\} \subseteq \Delta_0 = \tau$ as follows.

Given $\rho_i \in R_i$, recall that Δ_i is defined to be the convex hull of $\{m^{\rho_i}_{v_0 v} | v \in \tau\} \subseteq \Lambda_{\tau,\mathbb{R}}$ after fixing a vertex $v_0 \in \tau$ as a reference vertex. Moreover, for each vertex v of τ, $\mathrm{Vert}_i(v)$ denotes the vertex of Δ_i which represents the function $-\check{\psi}_i$ restricted to the maximal cone $\check{v}$ of $\check{\Sigma}_\tau$. Without loss of generality, we assume $v_0 = 0 \in M'$ and we have by definition $m^{\rho_i}_{v_0 v_0} = 0$. By positivity of $(B, \mathscr{P})$ and the definition $\check{\psi}_i(n) := -\inf\{\langle n, m \rangle | m \in \Delta_i\}$, we can see that

$$\mathrm{Vert}_i(v_0) = -\check{\psi}_i|_{\check{K}_{v_0}} = 0 = v_0 \in \Delta_i \subseteq M'$$

after identification of $\Delta_i \subseteq \Lambda_{\tau,\mathbb{R}}$ in $M'_{\mathbb{R}}$ (cf. [15, Rem. 1.56]; using the notation $e : \tau \to \rho_i$, $\Delta_e(\tau)$ is a face of $\Delta(\rho_i)$ by [15, Def. 1.58]). Consequently, $(v_0)_i = v_0$.

Consider $\dim \omega = 1$ with $v_0 \subseteq \omega \subseteq \tau$ and let v_1 be the other vertex of ω. Since $\check{\omega} \subseteq \check{v}_0$, it follows that for $m = 0 = (v_0)_i$,

$$0 = \langle n, m\rangle = -\check{\psi}_i(n) \quad \text{for any } n \in \check{v} \supseteq \check{\omega},$$

and therefore $(v_0)_i \subseteq \omega_i$. To determine ω_i, it suffices to find $m \in \Delta_i$ such that $\langle n, m\rangle = 0$ for any $n \in \check{\omega}$. Hence, ω_i can either be of dimension 0 or 1.

Assume $\dim \omega_i = 0$ and thus $(v_0)_i = \omega_i$, then

$$\check{\psi}_i|_{\check{K}_{v_1}} = 0$$

in the sense of [15, Rem. 1.56]. In other words, $\mathrm{Vert}_i(v_1) = \mathrm{Vert}_i(v_0)$ and thus we have $(v_1)_i = (v_0)_i$. This occurs when $m^{\rho_i}_{v_0 v_1} = 0$, exactly when there is no change of inner monodromy between v_0 and v_1 across ω. From another viewpoint, the function $\check{\psi}_i$ remains constant moving from $\check{v}_0$ to $\check{v}_1$ across $\check{\omega}$.

Lemma 2.13 (cf. Lem. 3.2 in [16])**.** *Consider the log space $\mathfrak{X}^\dagger_k$ (see Construction 2.9), which induces a log structure on V^k_ω. Given $\omega \subseteq \tau$, let $\omega_i \subseteq \Delta_i$ be the largest face of Δ_i such that $\langle n, m\rangle = -\check{\psi}_i(n)$ for all $n \in \check{\omega}$, $m \in \omega_i$ (Here $\check{\omega}$ is the cone in the normal fan $\check{\Sigma}_\tau$ of τ corresponding to ω). Then*

$$\Gamma(V^k_\omega, \Omega^r_{\omega,k}) = \bigoplus_{p \in P_{\omega,k}} z^p \left(\bigwedge^r \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^\perp\}} ((v+e_j)^\perp \cap N) \otimes_{\mathbb{Z}} \Bbbk \right)$$

or

$$\bigoplus_{p \in P_{\omega,k}} z^p \left(\bigwedge^r \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^\perp\}} (((v+e_j)^\perp \cap N)/\mathbb{Z}\rho) \otimes_{\mathbb{Z}} \Bbbk \right)$$

in the $/\Bbbk$ or $/A^\dagger_k$ cases respectively, where v runs over vertices of ω_j for any j and

$$P_{\omega,k} := \left\{ p \in P^{\mathrm{gp}} \;\middle|\; \begin{array}{l} \langle p, v\rangle \geq 0 \text{ for all } v \in \omega_i + e_i,\ 1 \leq i \leq q \\ \langle p, v\rangle \leq k \text{ for all } v \in \omega + e_0 \\ \langle p, v\rangle \geq 0 \text{ for all } v \in \tau + e_0 \end{array} \right\}.$$

Proof. We will do the $/\Bbbk$ case; the other case is identical.

With reference to Construction 1.8, let $P^\delta_1, \ldots P^\delta_r$, $Q_1, \ldots, Q_t$ be the maximal proper faces of P containing ρ. We remark that r can be zero, i.e., there could be no P^δ_i term; and then we are in the situation of [16, Lem. 3.2]. The collection of all $\operatorname{Spec} \Bbbk[P^\delta_j]$ and $\operatorname{Spec} \Bbbk[Q_j]$ is the collection of all the toric divisors of Y not contained in X. Set, for $p \in P$,

$$\Omega^r_p = \bigwedge^r \bigcap_{\{j | p \in Q_j\}} Q^{\mathrm{gp}}_j \otimes_{\mathbb{Z}} \Bbbk, \tag{2.2}$$

so that $\bigoplus_{p\in P} z^p\Omega^r_p$ defines a sheaf Ω^r_Y on Y. Note that Ω^r_p only depends on $\langle p\rangle$, so set $\Omega^r_{\langle p\rangle} := \Omega^r_p$. One checks easily that Proposition 2.6 implies $\Omega^r_{Y^\dagger/\Bbbk}|_{\mathfrak{X}_k} \cong j_*\Omega^r_{\mathfrak{X}^\dagger_k/\Bbbk}$. Then by Lemma 2.11, (2) applied with $I = I^k_\omega$ and $F_{\langle p\rangle} = \Omega^r_{\langle p\rangle}$, we have

$$\Gamma(V^k_\omega, (q^*_\omega j_*\Omega^r_{\mathfrak{X}^\dagger_k/\Bbbk})/\operatorname{Tors}) = \bigoplus_{p\in P\setminus I^k_\omega} z^p\Omega^r_p.$$

Denote the degree p piece of $\Gamma(V^k_\omega, \Omega^r_{\omega,k})$ by $\Gamma(V^k_\omega, \Omega^r_{\omega,k})_p$. Let

$$J := P \setminus P_{D_\omega\cap Z}$$

be the monoid ideal defining $D_\omega \cap Z$.

With reference to Remark 2.12, (1) in our new setting, we apply Lemma 2.11, (3) with $F_{\langle p+nq\rangle} = \Omega^r_{\langle p+nq\rangle}$, $Q = P_w$ and thus have the expression

$$\Gamma(V^k_\omega, \Omega^r_{\omega,k})_p = \bigcap_{q\in J\cap P_\omega} \bigcup_{\substack{n\geq 0\\ p+nq\in P\setminus I^k_\omega}} \Omega^r_{\langle p+nq\rangle}. \tag{2.3}$$

In the same manner as in [16, Lem. 3.2], we investigate the one-to-one inclusion reversing correspondence between faces P' of P_ω and cones K' with $K_\omega \subseteq K' \subseteq K$, where $K_\omega = C(\omega + e_0)$, in order to obtain a more explicit form of (2.3).

Now a stratum corresponding to K' is in $D_\omega \cap Z$ if it is contained in D_ω and Z_i for some i. The stratum is contained in D_ω provided $C(\omega' + e_0) \subseteq K'$ for some ω' with $\omega \subsetneq \omega' \subseteq \tau$. On the other hand, it is contained in Z_i if, first, $u_i = 0$ on the stratum, i.e. $K' \cap C(\Delta_i + e_i) \neq 0$ (otherwise $K' \subseteq (e^*_i)^\perp$); second, the stratum is contained in $V_{\omega''}$ for some $\omega'' \in \Omega_i$, this being equivalent to $\dim \omega'_i > 0$.

Thus, let $P_{D_\omega\cap Z}$ be the union of faces of P_ω corresponding to cones K' satisfying

1. $K' \cap C(\Delta_0 + e_0) = C(\omega' + e_0)$ for some $\omega' \supsetneq \omega$;
2. $K' \cap C(\Delta_i + e_i) \neq 0$ and $\dim \omega'_i > 0$ for some $1 \leq i \leq q$.

Let $q \in J \cap P_\omega$, and we consider the union in the above expression for this q. Then $Q := \langle q\rangle \subseteq P_\omega$ corresponds to some K' with $K_\omega \subseteq K' \subseteq K$ such that K' fails to satisfy either property (1) or property (2) above. We consider similarly three cases as in [16, Lem. 3.2].

After careful examinations, we see that the arguments in [16, Lem. 3.2] still work in the current situation, independent of the boundedness of ω, ω' and τ.

□

Lemma 2.14. *Consider the log space $\breve{\mathfrak{X}}^\dagger_k$ (see Construction 2.9), which induces a log structure on V^k_ω. Given $\omega \subseteq \tau$, let $\omega_i \subseteq \Delta_i$ be the largest face of Δ_i such that $\langle n, m\rangle = -\check{\psi}_i(n)$ for all $n \in \check{\omega}$, $m \in \omega_i$. (Here $\check{\omega}$ is the cone in the normal fan $\check{\Sigma}_\tau$ of τ corresponding to ω). Then the* *set of global sections* $\Gamma(V^k_\omega, \breve{\Omega}^r_{\omega,k})$ *is of the following form*

$$\bigoplus_{p\in P_{\omega,k}} z^p \left(\bigwedge^r \bigcap_{\{l|p\in(v^\delta_l)^\perp\}} ((v^\delta_l)^\perp \cap N) \cap \bigcap_{\{(v,j)|v\in\omega_j, p\in(v+e_j)^\perp\}} ((v+e_j)^\perp \cap N) \right) \otimes_{\mathbb{Z}} \Bbbk$$

or

$$\bigoplus_{p\in P_{\omega,k}} z^p \bigwedge^r \left(\bigcap_{\{l|p\in(v^\delta_l)^\perp\}} ((v^\delta_l)^\perp \cap N/\mathbb{Z}\rho) \cap \bigcap_{\{(v,j)|v\in\omega_j, p\in(v+e_j)^\perp\}} ((v+e_j)^\perp \cap N/\mathbb{Z}\rho) \right) \otimes_{\mathbb{Z}} \Bbbk$$

in the $/\Bbbk$ *or* $/A^\dagger_k$ *cases respectively, where* v *runs over vertices of* ω_j *for any* j *and*

$$P_{\omega,k} := \left\{ p \in P^{\mathrm{gp}} \;\middle|\; \begin{array}{l} \langle p, v\rangle \geq 0 \text{ for all } v \in \omega_i + e_i,\ 1 \leq i \leq q \\ \langle p, v\rangle \leq k \text{ for all } v \in \omega + e_0 \\ \langle p, v\rangle \geq 0 \text{ for all } v \in \tau + e_0 \end{array} \right\}.$$

Proof. We only take care of the $/\Bbbk$ case, as the remaining case follows easily. The proof is similar to that of Lemma 2.13.

Set, for $p \in P$,

$$\breve{\Omega}^r_p = \bigwedge^r \left(\bigcap_{\{j|p\in P^\delta_j\}} (P^\delta_j)^{\mathrm{gp}} \cap \bigcap_{\{j|p\in Q_j\}} Q^{\mathrm{gp}}_j \right) \otimes_{\mathbb{Z}} \Bbbk, \tag{2.4}$$

so that $\bigoplus_{p\in P} z^p \breve{\Omega}^r_p$ defines a sheaf $\breve{\Omega}^r_Y$ on Y. Note that $\breve{\Omega}^r_p$ only depends on $\langle p\rangle$, so set $\breve{\Omega}^r_{\langle p\rangle} := \breve{\Omega}^r_p$. One checks that Proposition 2.6 implies $\Omega^r_{\breve{Y}^\dagger/\Bbbk}|_{\mathfrak{X}_k} \cong j_*\Omega^r_{\breve{\mathfrak{X}}^\dagger_k/\Bbbk}$. Then by Lemma 2.11, (2) applied with $I = I^k_\omega$ and $F_{\langle p\rangle} = \breve{\Omega}^r_{\langle p\rangle}$,

$$\Gamma(V^k_\omega, (q^*_\omega j_*\Omega^r_{\breve{\mathfrak{X}}^\dagger_k/\Bbbk})/\operatorname{Tors}) = \bigoplus_{p\in P\setminus I^k_\omega} z^p \breve{\Omega}^r_p.$$

Denote the degree p piece of $\Gamma(V^k_\omega, \breve{\Omega}^r_{\omega,k})$ by $\Gamma(V^k_\omega, \breve{\Omega}^r_{\omega,k})_p$.

Let

$$J := P \setminus P_{D_\omega\cap Z}$$

be the monoid ideal defining $D_\omega \cap Z$.

With reference to Remark 2.12, (1) in our new setting, we apply Lemma 2.11, (3) and thus have the expression

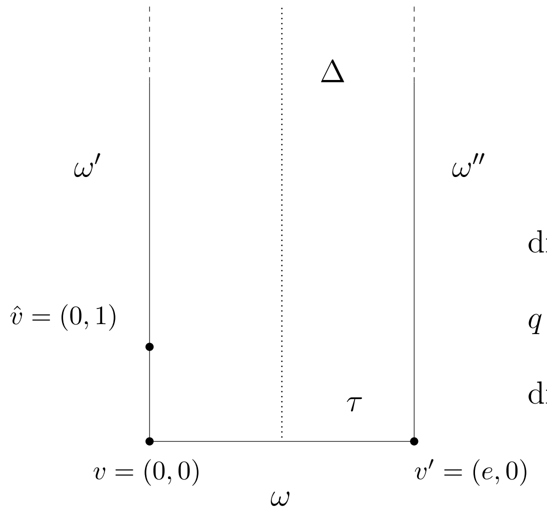
$$\Gamma(V^k_\omega, \breve{\Omega}^r_{\omega,k})_p = \bigcap_{q\in J\cap P_\omega} \bigcup_{\substack{n\geq 0 \\ p+nq\in P\setminus I^k_\omega}} \breve{\Omega}^r_{\langle p+nq\rangle}. \tag{2.5}$$

The only difference now is only the term $\breve{\Omega}^r_{\langle p+nq\rangle}$, which does not affect the process of taking intersection and union with respect to p and q in P. By the table at the end of

Construction 2.1, the term P_l^δ corresponds to the term $v_l^\delta = \hat{v}_{s+l} - v_{k_l}$; equivalently, $(P_l^\delta)^{\mathrm{gp}}$ corresponds to $(v_l^\delta)^\perp$ in this lemma, hence we can conclude that $\Gamma(V_\omega^k, \breve{\Omega}^r_{\omega,k})$ is in the form as written in the statement.

□

Example 2.15. Consider an unbounded cell τ of dimension 2 inside a maximal cell σ in B of dimension 3.



$\dim \tau = 2, \dim \sigma = 3$

$q = \operatorname{codim} \tau = 1$

$\dim \Delta_1 = 1$

Use the notations as in Construction 1.8 to compute the local model near the point $\bar{x} \in X_\tau$, where the log structure fails to be fine. Thus $M' = \mathbb{Z}^2$ and $M = M' \oplus \mathbb{Z}^2$ and for $n \in N'_{\mathbb{R}}$,

$$\check{\psi}_0(n_1,n_2) = \begin{cases} 0 & \text{for } n_1 \geq 0, n_2 \geq 0 \\ -n_1 e & \text{for } n_1 < 0, n_2 \geq 0 \\ \infty & \text{for } n_2 < 0 \end{cases} \quad \text{and} \quad \check{\psi}_1(n_1,n_2) = \begin{cases} 0 & \text{for } n_1 \geq 0, n_2 \geq 0 \\ -n_1 & \text{for } n_1 < 0, n_2 \geq 0, \end{cases}$$

with $\Delta_0 = \tau$ and $\Delta_1 = \{(m_1, m_2) \mid m_1 \in [0,1] \text{ and } m_2 = 0\}$.

The toric monoid P is generated by $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(-1,0,e,1)\}$, which correspond to variables $\{z^{(1,0,0,0)}, z^{(0,1,0,0)}, z^{(0,0,1,0)}, z^{(0,0,0,1)}, z^{(-1,0,e,1)}\}$. Q is generated by $\{(1,0,0,0),(0,1,0,0),(0,0,0,1),(-1,0,e,1)\}$. Hence,

$$\begin{aligned} Y &= \operatorname{Spec} \Bbbk[P] = \operatorname{Spec} \Bbbk[x_1, x_2, t, u, x_1^{-1} t^e u] \\ &= \operatorname{Spec} \frac{\Bbbk[x_1, x_2, t, u, y]}{(x_1 y - u t^e)}. \end{aligned}$$

As a subscheme of Y,

$$X = \operatorname{Spec} \frac{\Bbbk[x_1, x_2, u, y]}{(x_1 y)}$$

Again for simplicity, we consider $e = 1$ in the rest of this example. By evaluation of $\check{\psi}_1$, we conclude that $Z = V(u, x_1, y) \cong \operatorname{Spec} \Bbbk[x_2] \subset X$. Consequently, we have

$$V_\tau = \operatorname{Spec} \Bbbk[P_\tau] = \operatorname{Spec} \Bbbk[P \cap (\tau + e_0)^\perp] = \operatorname{Spec} \Bbbk[u] \subset X \cong \operatorname{Spec} \frac{\Bbbk[x_1, x_2, u, y]}{(x_1 y)}.$$

As faces of τ, the vertices v and v' yield

$$V_v = \operatorname{Spec} \Bbbk[P_v] = \operatorname{Spec} \Bbbk[x_1, x_2, u] = V(y)$$

and

$$V_{v'} = \operatorname{Spec} \Bbbk[x_2, u, y] = V(x_1)$$

as subschemes of X. For the 1-cells ω and ω',

$$V_\omega = \operatorname{Spec} \Bbbk[x_2, u] \quad \text{and} \quad V_{\omega'} = \operatorname{Spec} \Bbbk[x_1, u].$$

In particular, we have the new term $\tilde{D}$ of the form

$$\tilde{D} = \operatorname{Spec} \Bbbk[P \cap (v^\delta)^\perp] = \operatorname{Spec} \frac{\Bbbk[x_1, t, u, y]}{(x_1 y - tu)} = V(x_2) \subseteq Y,$$

where $v^\delta = \hat{v} - v$ is the generator of the extremal ray of K, which exists as τ is unbounded.

Observe that

1. $\tilde{D} \not\subseteq X$,
2. $\tilde{D} \cap V_\tau = V_\tau$,
3. $\tilde{D} \cap V_v = V_{\omega'}$,
4. $\tilde{D} \cap X = \operatorname{Spec} \Bbbk[x_1, u, y]/(x_1 y)$,

where $\operatorname{Spec} \Bbbk[x_1, u, y]/(x_1 y)$ is the fibred coproduct of 2 copies of the scheme $\operatorname{Spec} \Bbbk[u] \times \mathbb{A}^1_\Bbbk$.

To proceed, we summarize first the relation between rays of P and variables above:

Variable	Ray in P
x_1	$(1, 0, 0, 0)$
x_2	$(0, 1, 0, 0)$
t	$(0, 0, 1, 0)$
u	$(0, 0, 0, 1)$
y	$(-1, 0, e, 1)$

In particular, we expect *a priori* that $\operatorname{dlog} x_2 \in \Omega^1_{v,k}$ while $\operatorname{dlog} x_2 \notin \breve{\Omega}^1_{v,k}$ due to the definitions of the log structures and the fact $\tilde{D} = V(x_2)$. Actually, we have the following table, using the notations in Lemma 2.13 and Lemma 2.14:

Facet of P	Generator of Ray of $K \subseteq M_{\mathbb{R}}$	Related Point(s) in M'	Relation with Δ_i
$P_1 = (0,0,1,0)^{\perp}$ $P_2 = (e,0,1,0)^{\perp}$	$(0,0,1,0) = v + e_0$ $(e,0,1,0) = v' + e_0$	$v = (0,0)$ $v' = (e,0)$	vertex of $\Delta_0 = \tau$ vertex of $\Delta_0 = \tau$
$P^{\delta} = (0,1,0,0)^{\perp}$	$(0,1,0,0) = v^{\delta}$	$\hat{v} = (0,1)$, $v = (0,0)$	$v^{\delta} = \hat{v} - v$ and $\hat{v}$ are not vertices of Δ_0
$Q_1 = (0,0,0,1)^{\perp}$ $Q_2 = (1,0,0,1)^{\perp}$	$(0,0,0,1) = v + e_1$ $(1,0,0,1) = v_1 + e_1$	$v = (0,0)$ $v_1 = (1,0)$	vertex of Δ_1 vertex of Δ_1

Consequently, we observe that $x_2, t \in (Q_1 \cap Q_2)$ and $t \in (P^{\delta} \cap Q_1 \cap Q_2)$. Also,

$$x_1, t \in (v^{\delta})^{\perp} \cap (v + e_1)^{\perp},$$

which are the generators $\{\operatorname{dlog} x_1, \operatorname{dlog} t\}$ for $\Gamma(V_v^1, \breve{\Omega}^1_{v,1})$ and

$$x_1, x_2, t \in (v + e_1)^{\perp},$$

which are the generators $\{\operatorname{dlog} x_1, \operatorname{dlog} x_2, \operatorname{dlog} t\}$ for $\Gamma(V_v^1, \Omega^1_{v,1})$. Note that $\operatorname{dlog} t$ is not an element of $\Gamma(V_v^0, \Omega^1_{v,0})$ nor $\Gamma(V_v^0, \breve{\Omega}^1_{v,0})$. $\square$

Proposition 2.16 (cf. Prop. 3.3 in [16])**.** *Given faces $\omega \subseteq \omega' \subseteq \tau$, we have $I_\omega^k \subseteq I_{\omega'}^k$, and hence a closed embedding $V_{\omega'}^k \to V_\omega^k$. Then the set of global sections $\Gamma(V_{\omega'}^k, \Omega^r_{\omega,k}|_{V_{\omega'}^k} / \operatorname{Tors})$ is*

$$\Gamma(V_{\omega'}^k, \Omega^r_{\omega,k}|_{V_{\omega'}^k} / \operatorname{Tors}) = \bigoplus_{p \in P_{\omega,\omega',k}} z^p \left(\bigwedge^r \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^{\perp}\}} ((v + e_j)^{\perp} \cap N) \otimes_{\mathbb{Z}} \Bbbk \right)$$

or

$$\bigoplus_{p \in P_{\omega,\omega',k}} z^p \left(\bigwedge^r \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^{\perp}\}} (((v + e_j)^{\perp} \cap N)/\mathbb{Z}\rho) \otimes_{\mathbb{Z}} \Bbbk \right)$$

and the set of global sections $\Gamma(V_{\omega'}^k, \breve{\Omega}^r_{\omega,k}|_{V_{\omega'}^k} / \operatorname{Tors})$ is

$$\bigoplus_{p \in P_{\omega,\omega',k}} z^p \left(\bigwedge^r \bigcap_{\{l | p \in (v_l^{\delta})^{\perp}\}} ((v_l^{\delta})^{\perp} \cap N) \cap \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^{\perp}\}} ((v + e_j)^{\perp} \cap N) \right) \otimes_{\mathbb{Z}} \Bbbk$$

or

$$\bigoplus_{p \in P_{\omega,\omega',k}} z^p \bigwedge^r \left(\bigcap_{\{l | p \in (v_l^{\delta})^{\perp}\}} ((v_l^{\delta})^{\perp} \cap N/\mathbb{Z}\rho) \cap \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j)^{\perp}\}} ((v + e_j)^{\perp} \cap N/\mathbb{Z}\rho) \right) \otimes_{\mathbb{Z}} \Bbbk$$

in the $/\Bbbk$ or $/A_k^{\dagger}$ cases respectively, where v runs over vertices of ω_j and

$$P_{\omega,\omega',k} := \left\{ p \in P^{\mathrm{gp}} \;\middle|\; \begin{array}{l} \langle p, v \rangle \geq 0 \text{ for all } v \in \omega_i + e_i,\ 1 \leq i \leq q \\ \langle p, v \rangle \leq k \text{ for all } v \in \omega' + e_0 \\ \langle p, v \rangle \geq 0 \text{ for all } v \in \tau + e_0 \end{array} \right\}.$$

Note the only difference between this set and $P_{\omega,k}$ defined in Lemma 2.13 is that in the latter, $\langle p, v\rangle \leq k$ for all $v \in \omega + e_0$ instead of for all $v \in \omega' + e_0$.

Proof. We also only consider the $/\Bbbk$ case as in Lemma 2.13 and Lemma 2.14. Let $\tilde{P}$ be the monoid

$$\tilde{P} := \left\{ p \in P^{\mathrm{gp}} \middle| \begin{array}{l} \langle p, v\rangle \geq 0 \text{ for all } v \in \omega_i + e_i,\ 1 \leq i \leq q \\ \langle p, v\rangle \geq 0 \text{ for all } v \in \tau + e_0 \end{array} \right\}$$

with ideals

$$\tilde{I}^k_\omega := \{p \in \tilde{P} | \langle p, v\rangle > k \text{ for some } v \in \omega + e_0\}$$

and

$$\tilde{I}^k_{\omega'} := \{p \in \tilde{P} | \langle p, v\rangle > k \text{ for some } v \in \omega' + e_0\}.$$

Note $P \subseteq \tilde{P}$, $I^k_\omega = P \cap \tilde{I}^k_\omega$, $I^k_{\omega'} = P \cap \tilde{I}^k_{\omega'}$. Let F and $\breve{F}$ be the $\Bbbk[\tilde{P}]$-modules defined by

$$F = \bigoplus_{p\in\tilde{P}} z^p \left(\bigwedge^r \bigcap_{\{(v,j)|v\in\omega_j, p\in(v+e_j)^\perp\}} ((v + e_j)^\perp \cap N) \otimes_{\mathbb{Z}} \Bbbk \right)$$

and

$$\breve{F} = \bigoplus_{p\in\tilde{P}} z^p \left(\bigwedge^r \bigcap_{\{l|p\in(v^\delta_l)^\perp\}} ((v^\delta_l)^\perp \cap N) \cap \bigcap_{\{(v,j)|v\in\omega_j, p\in(v+e_j)^\perp\}} ((v + e_j)^\perp \cap N) \right) \otimes_{\mathbb{Z}} \Bbbk,$$

then from Lemma 2.11, (2) and Lemma 2.13, we see that

$$\Gamma(V^k_\omega, \Omega^r_{\omega,k}) \cong (F \otimes_{\Bbbk[\tilde{P}]} \Bbbk[\tilde{P}]/\tilde{I}^k_\omega)/\operatorname{Tors}$$

and

$$\Gamma(V^k_\omega, \breve{\Omega}^r_{\omega,k}) \cong (\breve{F} \otimes_{\Bbbk[\tilde{P}]} \Bbbk[\tilde{P}]/\tilde{I}^k_\omega)/\operatorname{Tors}.$$

Hence we can arrive at the conclusions that

$$\Gamma(V^k_{\omega'}, \Omega^r_{\omega,k}|_{V^k_{\omega'}}/\operatorname{Tors}) \cong (F \otimes_{\Bbbk[\tilde{P}]} \Bbbk[\tilde{P}]/\tilde{I}^k_{\omega'})/\operatorname{Tors},$$

and

$$\Gamma(V^k_{\omega'}, \breve{\Omega}^r_{\omega,k}|_{V^k_{\omega'}}/\operatorname{Tors}) \cong (\breve{F} \otimes_{\Bbbk[\tilde{P}]} \Bbbk[\tilde{P}]/\tilde{I}^k_{\omega'})/\operatorname{Tors}$$

with the use of Lemma 2.11, (2) again. For both cases, namely $\square = \Omega^r_{\omega,k}$ or $\square = \breve{\Omega}^r_{\omega,k}$, consider

$$\tilde{M} = (\Gamma(V^k_\omega, \square) \otimes_{\Bbbk[\tilde{P}]/\tilde{I}^k_\omega} \Bbbk[\tilde{P}]/\tilde{I}^k_{\omega'})/\operatorname{Tors}$$

with

$$M = (\Gamma(V^k_\omega, \square) \otimes_{\Bbbk[P]/I^k_\omega} \Bbbk[P]/I^k_{\omega'})/\operatorname{Tors}.$$

The proof for both cases follows the same argument for the isomorphism between M and $\tilde{M}$ in [16, Prop. 3.3].

$\square$

Corollary 2.17 (cf. Cor. 3.4 in [16]). *1. Given faces $\omega_1 \subseteq \omega_2 \subseteq \omega_3$ of τ, we have the inclusions*

$$(\Omega^r_{\omega_2,k}|_{V^k_{\omega_3}})/\operatorname{Tors} \subseteq (\Omega^r_{\omega_1,k}|_{V^k_{\omega_3}})/\operatorname{Tors}$$

and

$$(\breve{\Omega}^r_{\omega_2,k}|_{V^k_{\omega_3}})/\operatorname{Tors} \subseteq (\breve{\Omega}^r_{\omega_1,k}|_{V^k_{\omega_3}})/\operatorname{Tors}.$$

2. Given $\omega_1 \subseteq \omega_2$ faces of τ,

$$(\Omega^r_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors} = \bigcap_{v\in\omega_1} (\Omega^r_{v,k}|_{V^k_{\omega_2}})/\operatorname{Tors}$$

and

$$(\breve{\Omega}^r_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors} = \bigcap_{v\in\omega_1} (\breve{\Omega}^r_{v,k}|_{V^k_{\omega_2}})/\operatorname{Tors},$$

where v runs over vertices of ω_1, and the intersection can be viewed as taking place in $j_*(\Omega^r_{v,k}|_{V^k_{\omega_2}\setminus Z})$ and $j_*(\breve{\Omega}^r_{v,k}|_{V^k_{\omega_2}\setminus Z})$, which is independent of v since Ω^r_k and $\breve{\Omega}^r_k$ are locally free away from Z.

Proof. These statements follow immediately from the explicit formulae of the previous corollary. □

We can now define resolutions of Ω^r_k and $\breve{\Omega}^r_k$. For Ω^r_k, define a barycentric complex by

$$\mathscr{C}^p(\Omega^r_k) = \bigoplus_{\omega_0\subsetneq\cdots\subsetneq\omega_p\subseteq\tau} (\Omega^r_{\omega_0,k}|_{V^k_{\omega_p}})/\operatorname{Tors}$$

and a differential

$$d_{\mathrm{bct}}\colon \mathscr{C}^p(\Omega^r_k) \to \mathscr{C}^{p+1}(\Omega^r_k)$$

given by

$$(d_{\mathrm{bct}}(\alpha))_{\omega_0\subsetneq\cdots\subsetneq\omega_{p+1}} = \sum_{i=0}^{p}(-1)^i \alpha_{\omega_0\subsetneq\cdots\subsetneq\hat{\omega}_i\subsetneq\cdots\subsetneq\omega_{p+1}} + (-1)^{p+1}\alpha_{\omega_0\subsetneq\cdots\subsetneq\omega_p}|_{V^k_{\omega_{p+1}}}. \tag{2.6}$$

For $\breve{\Omega}^r_k$, we similarly define a complex

$$\mathscr{C}^p(\breve{\Omega}^r_k) = \bigoplus_{\omega_0\subsetneq\cdots\subsetneq\omega_p\subseteq\tau} (\breve{\Omega}^r_{\omega_0,k}|_{V^k_{\omega_p}})/\operatorname{Tors}$$

and a differential

$$\breve{d}_{\mathrm{bct}}\colon \mathscr{C}^p(\breve{\Omega}^r_k) \to \mathscr{C}^{p+1}(\breve{\Omega}^r_k),$$

which is also in the form as (2.6) above, hence we have

$$(\breve{d}_{\mathrm{bct}}(\alpha))_{\omega_0\subsetneq\cdots\subsetneq\omega_{p+1}} = \sum_{i=0}^{p}(-1)^i \alpha_{\omega_0\subsetneq\cdots\subsetneq\hat{\omega}_i\subsetneq\cdots\subsetneq\omega_{p+1}} + (-1)^{p+1}\alpha_{\omega_0\subsetneq\cdots\subsetneq\omega_p}|_{V^k_{\omega_{p+1}}}.$$

In both cases, the differentials are well-defined because the inclusions of Corollary 2.17, (1) enable us to identify the elements of $\mathscr{C}^p(\Omega^r_k)$ and $\mathscr{C}^p(\breve{\Omega}^r_k)$ with elements of $\mathscr{C}^{p+1}(\Omega^r_k)$ and $\mathscr{C}^{p+1}(\breve{\Omega}^r_k)$ respectively.

Theorem 2.18 (cf. Thm. 3.5 in [16])**.** *Consider the barycentric complexes* $\mathscr{C}^p(\Omega^r_k)$ *and* $\mathscr{C}^p(\breve{\Omega}^r_k)$ *with differentials* d_{bct} *and* $\breve{d}_{\mathrm{bct}}$ *respectively. Then*

1. $\mathscr{C}^\bullet(\Omega^r_k)$ *is a resolution of* Ω^r_k.

2. $\mathscr{C}^\bullet(\breve{\Omega}^r_k)$ *is a resolution of* $\breve{\Omega}^r_k$.

Proof. The arguments for the both claims are the same as that of [16, Thm. 3.5]. □

Proposition 2.19 (cf. Prop. 3.6 in [16])**.** *Consider the differential* d *and the log differential* dlog.

1. *The differential* $d\colon j_*\Omega^r_{\mathfrak{X}^\dagger_k/\Bbbk} \to j_*\Omega^{r+1}_{\mathfrak{X}^\dagger_k/\Bbbk}$ *(or* $d: j_*\Omega^r_{\mathfrak{X}^\dagger_k/A^\dagger_k} \to j_*\Omega^{r+1}_{\mathfrak{X}^\dagger_k/A^\dagger_k}$*) is given on the degree* p *piece of* $\Gamma(\mathfrak{X}_k, j_*\Omega^r_{\mathfrak{X}^\dagger_k/\Bbbk})$ *by* $z^p n \mapsto z^p \cdot p \wedge n$*. For any pair of faces* $\omega_1 \subseteq \omega_2 \subseteq \tau$*, this induces a map* $d\colon (\Omega^r_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors} \to (\Omega^{r+1}_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors}$.

2. *The differential* $d\colon j_*\Omega^r_{\breve{\mathfrak{X}}^\dagger_k/\Bbbk} \to j_*\Omega^{r+1}_{\breve{\mathfrak{X}}^\dagger_k/\Bbbk}$ *(or* $d: j_*\Omega^r_{\breve{\mathfrak{X}}^\dagger_k/A^\dagger_k} \to j_*\Omega^{r+1}_{\breve{\mathfrak{X}}^\dagger_k/A^\dagger_k}$*) is given on the degree* p *piece of* $\Gamma(\mathfrak{X}_k, j_*\Omega^r_{\breve{\mathfrak{X}}^\dagger_k/\Bbbk})$ *by* $z^p n \mapsto z^p \cdot p \wedge n$*. For any pair of faces* $\omega_1 \subseteq \omega_2 \subseteq \tau$*, this induces a map* $d\colon (\breve{\Omega}^r_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors} \to (\breve{\Omega}^{r+1}_{\omega_1,k}|_{V^k_{\omega_2}})/\operatorname{Tors}$.

Proof. The same argument as in [16, Prop. 3.6]. □

2.4 Global calculations

Let $(B,\mathscr{P})$ be a positive and simple integral affine manifold with singularities and a polyhedral decomposition $\mathscr{P}$. Let s be open gluing data for $(B,\mathscr{P})$, yielding $X_0 := X_0(B,\mathscr{P},s)$. This s together with the condition (LC) (see [15, Prop. 4.25]) also determines the log structure $X_0^\dagger$ on X_0 over $\operatorname{Spec}\Bbbk^\dagger$ with singular set $Z \subseteq X_0$. Take Ω^r to be the sheaf on X_0 which is either $j_*\Omega^r_{X_0^\dagger/\Bbbk}$ or $j_*\Omega^r_{X_0^\dagger/\Bbbk^\dagger}$, where $j\colon X_0 \setminus Z \to X_0$ is the inclusion. We refer to these as the $/\Bbbk$ and $/\Bbbk^\dagger$ cases respectively. On the other hand, take $\breve{\Omega}^r$ to be the sheaf on X_0 which is $j_*\Omega^r_{\breve{X}_0^\dagger/\Bbbk}$ and $j_*\Omega^r_{\breve{X}_0^\dagger/\Bbbk^\dagger}$ in the $/\Bbbk$ and $/\Bbbk^\dagger$ cases respectively. We will not handle this latter sheaf for the time being and leave the descriptions of this sheaf to §5.1.

Our goal is to calculate $H^p(X_0,\Omega^r)$. This section will be devoted to technical results which essentially lift the local descriptions of §2.3 to the global situation. The first goal is to obtain a nice resolution for Ω^r by defining a complex $\mathscr{C}^k(\Omega^r)$. The local form of this resolution has been studied in §2.3.

Let $q_\tau\colon X_\tau \to X_0$ be the usual inclusion of strata maps (cf. [15, Lem. 2.29]), D_τ the toric boundary of X_τ (the complement of the big torus orbit of X_τ) and let

$$\kappa_\tau\colon X_\tau \setminus (D_\tau \cap q_\tau^{-1}(Z)) \to X_\tau$$

be the inclusions. In analogy with the local case in §2.3, we define

$$\Omega^r_\tau := \kappa_{\tau*}\kappa^*_\tau(q^*_\tau\Omega^r/\operatorname{Tors}),$$

where Tors denotes the torsion subsheaf of $q^*_\tau\Omega^r$. In a similar fashion, define the sheaf $\breve{\Omega}^r_\tau$ on X_τ with respect to the log structure $\breve{X}^\dagger_0$.

Recall that X_0 can be viewed as the direct limit of a gluing functor $F_{S,s}$ defined in [15, Def. 2.11] and we take $S = \operatorname{Spec}\Bbbk$ as the base scheme. Since S and s are given, we shall write, for $\tau_1 \subseteq \tau_2$,

$$F_{\tau_1,\tau_2}\colon X_{\tau_2} \to X_{\tau_1}$$

for

$$F_{S,s}(\tau_1 \to \tau_2)\colon X_{\tau_2} \to X_{\tau_1}.$$

As noted in [17, §1.1], we restrict to the case where $(B, \mathscr{P})$ has no self-intersecting cells since the treatment of self-intersections is straightforward. Recall that

$$q_{\tau_2} = q_{\tau_1} \circ F_{\tau_1,\tau_2}.$$

Adapting the local results of §2.3 to the global situation, we have the following proposition.

Proposition 2.20. *If $\tau_1 \subseteq \tau_2$ with $\tau_1, \tau_2 \in \mathscr{P}$, then the functorial isomorphism on $X_{\tau_2} \setminus q^{-1}_{\tau_2}(Z)$*

$$\Omega^r_{\tau_2} = q^*_{\tau_2}\Omega^r \xrightarrow{\cong} F^*_{\tau_1,\tau_2}q^*_{\tau_1}\Omega^r = F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1}$$

extends to an inclusion

$$F^*_{\tau_1,\tau_2}\colon \Omega^r_{\tau_2} \to (F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors}.$$

Proof. This can be checked in an étale neighbourhood of a point $z \in Z$. By Theorem 1.12, this reduces to the case considered in Corollary 2.17, (1). □

We are now able to define our explicit resolution of the sheaf Ω^r. Define a barycentric complex

$$\mathscr{C}^k(\Omega^r) = \bigoplus_{\sigma_0 \subsetneq \cdots \subsetneq \sigma_k} q_{\sigma_k *}((F^*_{\sigma_0,\sigma_k}\Omega^r_{\sigma_0})/\operatorname{Tors})$$

with a differential $d_{\mathrm{bct}}\colon \mathscr{C}^k(\Omega^r) \to \mathscr{C}^{k+1}(\Omega^r)$ given by

$$\begin{aligned}(d_{\mathrm{bct}}(\alpha))_{\sigma_0,\cdots,\sigma_{k+1}} &= \alpha_{\sigma_1,\cdots,\sigma_{k+1}} + \sum_{i=1}^k (-1)^i \alpha_{\sigma_0,\cdots,\breve{\sigma}_i,\cdots,\sigma_{k+1}} \\ &\quad + (-1)^{k+1} F^*_{\sigma_k,\sigma_{k+1}}\alpha_{\sigma_0,\cdots,\sigma_k}.\end{aligned}$$

Here the term $\alpha_{\sigma_1,\cdots,\sigma_{k+1}} \in (F^*_{\sigma_1,\sigma_{k+1}}\Omega^r_{\sigma_1})/\operatorname{Tors}$ can be viewed, by Proposition 2.20, as an element of $(F^*_{\sigma_0,\sigma_{k+1}}\Omega^r_{\sigma_0})/\operatorname{Tors}$. Consider only this differential d_{bct}, we have first the following results:

Theorem 2.21. *$\mathscr{C}^\bullet(\Omega^r)$ is a resolution of Ω^r.*

Proof. This follows immediately from the local version, Theorem 2.18. □

Corollary 2.22.
$$H^p(X_0, \Omega^r) = \mathbb{H}^p(X_0, \mathscr{C}^\bullet(\Omega^r)).$$

Besides, there is the exterior differential $d\colon \Omega^r \to \Omega^{r+1}$, which is defined on $X_0 \setminus Z$, and hence on the pushforward, giving us a complex $(\Omega^\bullet, d)$, the *log de Rham complex* of X_0. By Proposition 2.19, d induces the map

$$d\colon (F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors} \to (F^*_{\tau_1,\tau_2}\Omega^{r+1}_{\tau_1})/\operatorname{Tors}$$

for $e\colon \tau_1 \to \tau_2$. Altogther, we have two maps of complexes

$$d_{\mathrm{bct}}\colon \mathscr{C}^k(\Omega^\bullet) \to \mathscr{C}^{k+1}(\Omega^\bullet)$$

and
$$d\colon \mathscr{C}^\bullet(\Omega^r) \to \mathscr{C}^\bullet(\Omega^{r+1}).$$

The two differentials d_{bct} and d together give us a double complex $\mathscr{C}^\bullet(\Omega^\bullet)$. We thus have the following immediate result:

Corollary 2.23.
$$\mathbb{H}^r(X_0, \Omega^\bullet) = \mathbb{H}^r(X_0, \operatorname{Tot}(\mathscr{C}^\bullet(\Omega^\bullet))),$$

where Tot *denotes the total complex of the double complex.*

In order to compute these cohomology groups explicitly, we need a useful global description for the sheaves Ω^r_ω. As in [16, §3.2], we first describe Ω^r_v for a vertex v of $\mathscr{P}$ in the new setting.

For a vertex v without unbounded rays, $\mathfrak{D}$ does not intersect X_v, or equivalently X_v does not contain any irreducible components of D. Then the properties of Ω^r_v are the same as before, which are described in [16, §3.2].

From now on in this section, consider a vertex v always with an unbounded ray. Pull back the log structure on $X_0^\dagger$ via q_v to obtain a log structure on $X_v \setminus q_v^{-1}(Z)$, with sheaf of monoids $\mathcal{M}_v$.

For a given X_v, it is true that $D \cap X_v$ is contained in the toric boundary D_v (cf. Construction 2.9 and Example 2.15). By [15, Lem. 5.13], we have a split exact sequence

$$0 \to \mathcal{M}^{\mathrm{gp}}_{(X_v,D_v)} \to \mathcal{M}^{\mathrm{gp}}_v \to \mathbb{Z}\rho \to 0, \tag{2.7}$$

where $\mathcal{M}_{(X_v,D_v)}$ is the sheaf of monoids associated to the divisorial log structure given by $D_v \subseteq X_v$, and ρ as usual is the image of $1 \in \mathbb{N}$ under the map of monoids induced by the log morphism $X_0^\dagger \to \operatorname{Spec} \Bbbk^\dagger$. Because $q_v^{-1}(Z) \subseteq X_v$ is codimension two, $j_*\mathcal{M}_v \to j_*\mathcal{O}_{X_v \setminus q_v^{-1}(Z)} =$

$\mathcal{O}_{X_v}$ determines a log structure on X_v, which we write as $X_v^\dagger$. Write $\mathcal{M}_v$ also for $j_*\mathcal{M}_v$. Then the exact sequence (2.7) still holds on X_v. From this exact sequence one sees that $\Omega^1_{X_v^\dagger/\Bbbk^\dagger}$ coincides with the ordinary sheaf of log derivations for the pair (X_v, D_v), which is canonically $\check{\Lambda}_v \otimes \mathcal{O}_{X_v}$ by [27, Prop. 3.1] while $\Omega^1_{X_v^\dagger/\Bbbk}$ is canonically $(\check{\Lambda}_v \oplus \mathbb{Z}\rho) \otimes \mathcal{O}_{X_v}$.

Lemma 2.24. *Let $v \in \mathscr{P}$ be a vertex. Then Ω^r_v is naturally isomorphic to $\Omega^r_{X_v^\dagger/\Bbbk}$ or $\Omega^r_{X_v^\dagger/\Bbbk^\dagger}$ in the $/\Bbbk$ and $/\Bbbk^\dagger$ cases respectively.*

Proof. First of all, $\Omega^1_{X_v^\dagger/\Bbbk^\dagger}$ coincides with the ordinary sheaf of log derivations for the pair (X_v, D_v). It is true in our new setting as well as in [16].

In [16, Lem. 3.12], the log structure of $(V(\sigma) \setminus Z) \cap X_v$ induced by $X_0^\dagger$ is given by the chart $P_e \to \mathcal{O}_{X_v}$ étale locally, where P_e is the maximal proper face of P_σ corresponding to $X_v \cap V(\sigma)$. Since the construction and consideration of P_e in [16] is unaltered for an unbounded maximal cell σ, this statement is true by the same argument of [16, Lem. 3.12] according to the definition of Ω^1_v and the log structure on $X_0^\dagger$. □

Remark 2.25. In the proof of the above lemma, we have the following observation. Let $e\colon v \to \sigma \in \mathscr{P}_{\max}$, where σ is unbounded. Then $D_\mu := D \cap X_v$ is nonempty. Since $D \cup (X_0)_{\text{sing}}$ is required to be the collection of all $(n-1)$-strata in X_0, so D_μ is contained in the toric boundary D_v. Hence, the "new term" D_μ is a part of the toric boundary of X_v. Furthermore, suppose D_μ is locally in the form $D_\mu = \{z_1 = 0\}$ with a choice of local coordinates $(z_1, \cdots, z_n)$. Then $\frac{dz_1}{z_1}$ is a local section of $\Omega^1_{X_v^\dagger/\Bbbk}$ and $\Omega^1_{X_v^\dagger/\Bbbk^\dagger}$.

In the same way as [16, Lem 3.12], Lemma 2.24 enables us to view Ω^r as being obtained by gluing together trivial vector bundles on the irreducible components of $X_0 \setminus Z$. Consider $\omega \in \mathscr{P}$ a bounded cell of dimension one, with vertices $e^\pm_\omega\colon v^\pm_\omega \to \omega$ arising from a choice of d_ω a primitive generator of Λ_ω. On $X_\omega \setminus q_\omega^{-1}(Z)$, there are the canonical identifications

$$F^*_{v^-_\omega,\omega}\Omega^r_{v^-_\omega} = F^*_{v^-_\omega,\omega} q^*_{v^-_\omega}\Omega^r = F^*_{v^+_\omega,\omega} q^*_{v^+_\omega}\Omega^r = F^*_{v^+_\omega,\omega}\Omega^r_{v^+_\omega}$$

On the other hand, using the isomorphism of Lemma 2.24 and on the left and right hand sides of the above identifications, we get on $X_\omega \setminus q_\omega^{-1}(Z)$ a map

$$\Gamma_\omega\colon F^*_{v^-_\omega,\omega}\Omega^r_{X^\dagger_{v^-_\omega}/\Bbbk} \xrightarrow{\cong} F^*_{v^+_\omega,\omega}\Omega^r_{X^\dagger_{v^+_\omega}/\Bbbk} \tag{2.8}$$

(or $/\Bbbk^\dagger$.) Let's describe Γ_ω explicitly.

Lemma 2.26 (Lem. 3.13 in [16])**.** *In the above situation, identify $\check{\Lambda}_{v^+_\omega}$ and $\check{\Lambda}_{v^-_\omega}$ via parallel transport through σ, and identify these with a lattice N. Then on $\operatorname{Sing}(V(\omega))$, in the $/\Bbbk$ case, Γ_ω is given by, for $n \in \bigwedge^\bullet(N \oplus \mathbb{Z}\rho)$,*

$$\Gamma_\omega(\operatorname{dlog} n) = -\left(\frac{df_\sigma}{f_\sigma} + l_\omega \operatorname{dlog}\rho\right) \wedge \operatorname{dlog}(\iota(d_\omega)n) + \operatorname{dlog} n,$$

where l_ω is a positive integer such that there is an integral affine isomorphism $[0, l_\omega] \to \omega$. The same formula holds modulo $\operatorname{dlog}\rho$ in the $/\Bbbk^\dagger$ case.

Remark 2.27. The formula only holds for bounded edge ω since unbounded edges do not have two vertices. In the proof of [16, Lem 3.13], the identification between the lifting of $\check{v}_\omega^\pm \cap N$ and $C(\omega)^\vee \cap (N \oplus \mathbb{Z})$ is concerned, which does not involve the boundedness of σ after fixing $\sigma \supseteq \omega$ (where $\sigma \in \mathscr{P}_{\max}$). $\square$

Next we describe Ω^r_τ for $\tau \in \mathscr{P}$ arbitrary. Pick a reference vertex $v \in \mathscr{P}$ with a morphism $g\colon v \to \tau$. We know by Proposition 2.20 that there is an inclusion of Ω^r_τ in $F^*_{v,\tau}\Omega^r_v$. We describe this subsheaf in the next step.

Recall that we assume $(B, \mathscr{P})$ is simple. Therefore, as in [15, Def. 1.60], for every $\tau \in \mathscr{P}$ with $\dim \tau \neq 0, n$, we have the following data:

$$\begin{aligned} \mathscr{P}_1(\tau) &= \{\omega \to \tau \mid \dim \omega = 1\} \\ \mathscr{P}_{n-1}(\tau) &= \{\tau \to \rho \mid \dim \rho = n-1\} \end{aligned}$$

Simplicity allows us to find disjoint sets

$$\begin{aligned} \Omega_1, \ldots, \Omega_q &\subseteq \mathscr{P}_1(\tau), \\ R_1, \ldots, R_q &\subseteq \mathscr{P}_{n-1}(\tau), \end{aligned}$$

and polytopes

$$\begin{aligned} \Delta_1, \ldots, \Delta_q &\subseteq \Lambda_{\tau,\mathbb{R}}, \\ \check{\Delta}_1, \ldots, \check{\Delta}_q &\subseteq \Lambda^\perp_{\tau,\mathbb{R}}. \end{aligned}$$

These have the property that if $\omega \in \Omega_i$, $e\colon \omega \to \tau$, then the monodromy polytope $\check{\Delta}_e(\tau) = \check{\Delta}_i$, and if $\rho \in R_i$, $f\colon \tau \to \rho$, then $\Delta_f(\tau) = \Delta_i$ (see [15, Def. 1.58]). The polytopes Δ_i are the Newton polytopes of the functions $\check{\psi}_i$ on $\check{\Sigma}_\tau$, the normal fan to τ (see [15, Rem. 1.59]; there is a typo: φ_ρ should be $\check{\psi}_\rho$). For any $g'\colon v' \to \tau$, we obtain vertices $\mathrm{Vert}_i(g')$ of Δ_i as in Construction 1.8. The reference vertex $g\colon v \to \tau$ then gives reference vertices $v_i := \mathrm{Vert}_i(g) \in \Delta_i$. The sets Ω_i are characterized by $\omega \in \Omega_i$ if and only if $\mathrm{Vert}_i(v_\omega^+) \neq \mathrm{Vert}_i(v_\omega^-)$.

In addition, simplicity includes the condition that the convex hulls of

$$\textstyle\bigcup_{i=1}^q \Delta_i \times \{e_i\} \text{ and } \bigcup_{i=1}^q \check{\Delta}_i \times \{e_i\}$$

in $\Lambda_{\tau,\mathbb{R}} \times \mathbb{R}^q$ and $\Lambda^\perp_{\tau,\mathbb{R}} \times \mathbb{R}^q$ respectively are elementary simplices. In particular, $\Delta_1, \ldots, \Delta_q$ and $\check{\Delta}_1, \ldots, \check{\Delta}_q$ are themselves elementary simplices, and their tangent spaces $T_{\Delta_1}, \ldots, T_{\Delta_q}$ give a direct sum decomposition of $\sum_{i=1}^q T_{\Delta_i} \subseteq \Lambda_{\tau,\mathbb{R}}$ and $T_{\check{\Delta}_1}, \ldots, T_{\check{\Delta}_q}$ give a direct sum decomposition of $\sum_{i=1}^q T_{\check{\Delta}_i} \subseteq \Lambda^\perp_{\tau,\mathbb{R}}$. Recall [16, Lem. 3.14] about the properties of Newton polytopes:

Lemma 2.28 (Lem. 3.14 in [16])**.** *If the convex hull of $\bigcup_{i=1}^q \Delta_i \times \{e_i\}$ is an elementary simplex, then there is a one-to-one correspondence between faces σ of $\Delta_\tau := \Delta_1 + \cdots + \Delta_q$*

(Minkowski sum) and q-tuples $(\sigma_1, \ldots, \sigma_q)$ *with* σ_i *a face of* Δ_i*, with* $\sigma = \sigma_1 + \cdots + \sigma_q$*. Furthermore,*

$$\dim \sigma = \sum_{i=1}^{q} \dim \sigma_i.$$

Remark 2.29. In [15, §1.5], the authors introduced the monodromy transformation around a loop and the concept of simplicity for the case where $(B, \mathscr{P})$ is compact and without boundary. The notion of monodromy transformation can be generalized in a straightforward way for unbounded cells of $\mathscr{P}$.

In particular, the operator $T_\omega^{e_1 e_2}$ is trivial for an unbounded 1-cell τ' ($T_\omega^{e_1 e_2}$ is first defined for the bounded case in the construction before [15, Def. 1.54]) because we now have only a vertex $v_{\tau'}$ for the 1-cell τ' (in contrast to the case of a bounded 1-cell ω with vertices $v_\omega^{\pm}$) so that the intersection $\Delta \cap \tau'$ of the 1-cell τ' and the discriminant locus Δ is empty. Then we can conclude that the monodromy polytope $\check{\Delta}(\tau') = 0$ (cf. [15, Def. 1.58]) for the unbounded 1-cell τ'.

We can then use [15, Def. 1.60] to define simplicity for $(B, \mathscr{P})$ also in the unbounded case by the observations above. □

Furthermore, we have the following proposition about the monodromy polytopes of an unbounded cell τ.

Proposition 2.30. *Let* τ' *be an unbounded 1-cell emerging from a vertex* v *in* B *and let* τ *be a cell containing* τ'*. Denote the monodromy polytopes of* τ *by* Δ_i *and* $\check{\Delta}_i$*. Then*

$$\sum_{i=1}^{q} T_{\check{\Delta}_i} \subseteq \Lambda_{\tau,\mathbb{R}}^{\perp} \cap \bigcap_{\substack{\tau' \neq \omega \subseteq \tau \\ \dim \omega = 1}} \Lambda_{\omega,\mathbb{R}}^{\perp}.$$

Besides, the nonzero elements of Δ_i *cannot occur in* $\Lambda_{\tau',\mathbb{R}}$ *for any unbounded 1-cell* τ' *contained in* τ*. In other words,*

$$\Delta_i \cap \Lambda_{\tau',\mathbb{R}} = 0.$$

As a result, it follows that

$$T_\sigma \cap \Lambda_{\tau',\mathbb{R}} = 0$$

for every unbounded 1-cell τ' *in* τ*, where* T_σ *denotes the tangent space to* σ *in* $\Lambda_{\tau,\mathbb{R}}$ *and* $\sigma = \sigma_1 + \cdots + \sigma_q$ *for* σ_i *a face of* Δ_i*, which is the Newton polyope of the function* $\check{\psi}_i$ *on* $\check{\Sigma}_\tau$*.*

Proof. Let τ' be an unbounded 1-cell emerging from a vertex v in B. By Remark 2.29, we have $\check{\Delta}(\tau') = 0$. By simplicity, it also implies that $\Delta(\tau') = 0$.

Consider now the Newton polytope $\check{\Delta}_i$ (which is the monodromy polytope of τ). The element of any Newton polytope $\check{\Delta}_i$ cannot occur in the direction of

$$\Lambda_{\tau',\mathbb{R}}^{\perp} \setminus \Big(\bigcup_{\substack{\tau' \neq \omega \subseteq \tau \\ \dim \omega = 1}} \Lambda_{\omega,\mathbb{R}}^{\perp} \Big).$$

Otherwise $\check{\Delta}_i$ would be a face of $\check{\Delta}(\tau')$ with $\dim \check{\Delta}_i \geq 1$, which is impossible. In other words, every Newton polytope $\check{\Delta}_i$ is a subset of

$$\bigcap_{\substack{\tau' \neq \omega \subseteq \tau \\ \dim \omega = 1}} \Lambda^{\perp}_{\omega,\mathbb{R}}.$$

Consequently, we have

$$\sum_{i=1}^{q} T_{\check{\Delta}_i} \subseteq \Lambda^{\perp}_{\tau,\mathbb{R}} \cap \bigcap_{\substack{\tau' \neq \omega \subseteq \tau \\ \dim \omega = 1}} \Lambda^{\perp}_{\omega,\mathbb{R}}.$$

To prove the second part of the proposition, we look at first the property of a polytope Δ_i with respect to $\tau \supseteq \tau'$. Let ρ be a cell containing τ and τ' with $\dim \rho = n-1$ and let $f' \colon \tau' \to \rho$. Firstly, we observe that $\Delta_{f'}(\tau')$ is a face of $\Delta(\rho)$ and $\Delta_{f'}(\tau') = \mathrm{Conv}\{m^{\rho}_{e\circ f, e\circ f'} | f' \colon v' \to \tau'\} = 0$ because v is the only vertex of τ'.

Let $f \colon \tau \to \rho$. Suppose there exists $m_0 \in \Lambda_{\tau',\mathbb{R}}$ such that $m_0 \in \Delta_i = \Delta_f(\tau)$. Hence $\mathrm{Conv}\{m_0, 0\} \subseteq \Lambda_{\tau',\mathbb{R}}$. As $\mathrm{Conv}\{m_0, 0\}$ is a face of $\Delta(\rho)$, thus by positivity and convexity of τ, it can only happen that $\mathrm{Conv}\{m_0, 0\} = \Delta_{f'}(\tau')$ or $\mathrm{Conv}\{m_0, 0\} = \Delta_{f''}(\tau'')$, where $f'' \colon \tau'' \to \rho$. In the former case, we arrive at the conclusion that $\mathrm{Conv}\{m_0, 0\} = \Delta_{f'}(\tau') = 0$ so that $m_0 = 0$. In the latter case, the 1-cell τ'' has to be unbounded due to convexity of τ; hence this case is reduced to the former case with $\Delta_{f''}(\tau'') = 0$. So the claim is proved.

Therefore, we can conclude that

$$T_{\sigma} \cap \Lambda_{\tau',\mathbb{R}} = 0$$

for every unbounded 1-cell τ' in τ.

□

As in [15], every toric stratum X_τ is defined by $X_\tau := X(\Sigma_\tau)$ (see [15, Def. 2.7]), in which the boundedness assumption of τ is not involved. Consequently, the arguments of [15, Cor. 5.8] apply, so that $q_\tau^{-1}(Z) = Z_1^\tau \cup \cdots \cup Z_q^\tau \cup Z'$ where $Z' \subseteq D_\tau$ is of codimension at least two in X_τ and Z_i^τ is a hypersurface in X_τ, with Newton polytope $\check{\Delta}_i$. Furthermore, from the proof of [15, Cor. 5.8], $Z_i^\tau = F_{\omega,\tau}^{-1}(Z_\omega)$ for any $\omega \in \Omega_i$, where Z_ω is the irreducible component of Z contained in the codimension one stratum X_ω of X_0.

For an index set $I \subseteq \{1, \ldots, q\}$, set $Z_I^\tau := \bigcap_{i \in I} Z_i^\tau$. For the log structure on $X_v^\dagger$, pull it back on X_v to X_τ via $F_{v,\tau}$, and then restrict it further to Z_I^τ, for any I. We write these structures as $X_\tau^\dagger$ and $(Z_I^\tau)^\dagger$, but keep in mind these are not intrinsic and depend on the choice of vertex $g \colon v \to \tau$. Note that these are all defined over $\mathrm{Spec}\, \Bbbk^\dagger$, by composing the inclusions into X_v with $X_v^\dagger \xrightarrow{q_v} X_0^\dagger \to \mathrm{Spec}\, \Bbbk^\dagger$. Viewing $Z_I^\tau \subseteq X_v$ via the inclusion $F_{v,\tau} \colon X_\tau \to X_v$, we have the following lemma.

Lemma 2.31 (cf. Lem. 3.15 in [16]). *1. There are exact sequences*

$$0 \to \bigoplus_{i \in I} \mathcal{O}_{Z_I^\tau}(-Z_i^\tau) \to \Omega^1_{X_v^\dagger/\Bbbk}|_{Z_I^\tau} \to \Omega^1_{(Z_I^\tau)^\dagger/\Bbbk} \to 0$$

and

$$0 \to \bigoplus_{i\in I} \mathcal{O}_{Z_I^\tau}(-Z_i^\tau) \to \Omega^1_{X_v^\dagger/\Bbbk^\dagger}|_{Z_I^\tau} \to \Omega^1_{(Z_I^\tau)^\dagger/\Bbbk^\dagger} \to 0.$$

Here $\mathcal{O}_{Z_I^\tau}(-Z_i^\tau)$ *denotes the restriction of the line bundle* $\mathcal{O}_{X_\tau}(-Z_i^\tau)$ *to* Z_I^τ. *In addition,* $\Omega^1_{(Z_I^\tau)^\dagger/\Bbbk}$ *and* $\Omega^1_{(Z_I^\tau)^\dagger/\Bbbk^\dagger}$ *are locally free* $\mathcal{O}_{Z_I^\tau}$*-modules.*

2. *If* $Y \subseteq X_\tau$ *is a toric stratum, then* $\Omega^r_{(Z_I^\tau)^\dagger/\Bbbk}|_Y = \Omega^r_{(Z_I^\tau\cap Y)^\dagger/\Bbbk}$ *and*

$$\mathrm{Tor}_j^{\mathcal{O}_{X_\tau}}(\Omega^r_{(Z_I^\tau)^\dagger/\Bbbk}, \mathcal{O}_Y) = 0$$

for $j > 0$. *Here the log structure on* $Z_I^\tau \cap Y$ *is the pull-back of the one on* Z_I^τ. *The same holds for the* $/\Bbbk^\dagger$ *case.*

Proof. Choose $h\colon \tau \to \sigma \in \mathscr{P}_{\max}$. The log structure $X_v^\dagger$ is given by the chart $P_\sigma \to \Bbbk[P_{h\circ g}]$, where $P_{h\circ g}$ is the maximal proper face of P_σ corresponding to $h\circ g\colon v \to \sigma$. This consideration and its consequence in [16, Lem. 3.15] is unaltered for an unbounded maximal cell σ.

Furthermore, Z_I^τ is defined by equations $\{f_i = 0 | i \in I\}$ in X_τ, in which f_i is given by the Newton polytope $\check{\Delta}_i$. By simplicity, the Newton polytopes $\check{\Delta}_i$ are elemetary simplices and thus are bounded like before. Hence, the arguments of [16, Lem. 3.15] apply also in the new situation and give the result. □

Proposition 2.32 (cf. Prop. 3.17 in [16]). *Given* $v \to \tau_1 \to \tau_2$, *the image of the inclusion* $(F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\,\mathrm{Tors}$ *in* $F^*_{v,\tau_2}\Omega^r_v$ *is*

$$\ker\left(F^*_{v,\tau_2}\Omega^r_v \xrightarrow{\delta_0} \bigoplus_{\substack{i=1,\ldots,q\\ w_i\neq v_i}} \Omega^{r-1}_{(Z_i^{\tau_2})^\dagger/\Bbbk}\right)$$

or

$$\ker\left(F^*_{v,\tau_2}\Omega^r_v \xrightarrow{\delta_0} \bigoplus_{\substack{i=1,\ldots,q\\ w_i\neq v_i}} \Omega^{r-1}_{(Z_i^{\tau_2})^\dagger/\Bbbk^\dagger}\right)$$

in the $/\Bbbk$ *and* $/\Bbbk^\dagger$ *cases respectively, where:*

1. *The direct sum is over all* i *and all vertices* w_i *of* Δ_i, $w_i \neq v_i$, *and* $\Delta_1, \ldots, \Delta_q$ *are parts of the simplicity data for* τ_1.

2. $Z_i^{\tau_2} = F^{-1}_{\tau_1,\tau_2}(Z_i^{\tau_1})$ *where* $Z_1^{\tau_1}, \ldots, Z_q^{\tau_1}$ *are as usual the codimension one irreducible components of* $q_{\tau_1}^{-1}(Z)$ *with Newton polytopes* $\check{\Delta}_1, \ldots, \check{\Delta}_q$.

3. *For* $\alpha \in F^*_{v,\tau_2}\Omega^r_v$, *the component of* $\delta_0(\alpha)$ *in the direct summand* $\Omega^{r-1}_{(Z_i^{\tau_2})^\dagger/\Bbbk}$ *or* $\Omega^{r-1}_{(Z_i^{\tau_2})^\dagger/\Bbbk^\dagger}$ *corresponding to some* w_i *is given by* $\iota(\partial_{w_i-v_i})\alpha|_{(Z_i^{\tau_2})^\dagger}$.

Proof. This proposition is the generalization of [16, Prop. 3.17] to the case where τ_1 and τ_2 are allowed to be unbounded. This proof actually follows the lines of [16, Prop. 3.17] with the application of Proposition 2.30.

Consider the $/\Bbbk$ case and the $/\Bbbk^\dagger$ case will follow. The first part of the proposition concerning $F^*_{\bullet,\tau_2}\Omega^r_\bullet$ follows by using the characterization of Corollary 2.17, (2) of $(F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors}$ as

$$\bigcap_{g'\colon v'\to\tau_1} F^*_{v',\tau_2}\Omega^r_{v'},$$

using Lemma 2.26 for the explicit identification of this intersection with a subsheaf of $F^*_{v,\tau_2}\Omega^r_v$.

Let α be a section of $F^*_{v,\tau_2}\Omega^r_v$. Then for any j and vertex $w_j \neq v_j$ of Δ_j, we can find a sequence of edges $h_i\colon \omega_i \to \tau_1$, $i = 1,\dots,m$ of τ_1, with d_{ω_i} chosen appropriately, so that

- $v^-_{\omega_1} = v$;
- $v^+_{\omega_i} = v^-_{\omega_{i+1}}$ for $i < m$;
- $\mathrm{Vert}_l(v^+_{\omega_i}) = v_l$ for $i < m$, for all l;
- $\mathrm{Vert}_l(v^+_{\omega_m}) = \begin{cases} v_l & l \neq j, \\ w_j & l = j. \end{cases}$

Choose a maximal cell σ containing τ_2 for reference, and let $f_1,\dots,f_q$ be the equations defining $Z^{\tau_2}_1,\dots,Z^{\tau_2}_q$ in the affine chart $V(\sigma)\cap X_{\tau_2}$. Note that by Proposition 2.30, Δ_i has no element in $\Lambda_{\tau',\mathbb{R}}$ for unbounded 1-cells τ' so it is impossible that $\omega_i = \tau'$ for $i = 1,\dots m$, where τ' is an unbounded 1-cell. Using Lemma 2.26, apply $\Gamma_{\omega_1},\dots,\Gamma_{\omega_m}$ successively to α by the same arguments as in [16, Prop. 3.17] by noting that ω_i are bounded for $1 \le i \le m$.

Conversely, if $\alpha \in \ker\delta_0$, let v' be any vertex of τ_1. Then we can find a sequence of edges $\omega_i \to \tau_1$, $i = 1,\dots,m$ of τ_1, with d_{ω_i} chosen appropriately, so that

- $v^-_{\omega_1} = v$;
- $v^+_{\omega_i} = v^-_{\omega_{i+1}}$ for $i < m$;
- $v^+_{\omega_m} = v'$;
- For each $1 \le l \le q$, there is at most one i such that $\mathrm{Vert}_l(v^-_{\omega_i}) \neq \mathrm{Vert}_l(v^+_{\omega_i})$, and for this i, $\mathrm{Vert}_l(v^-_{\omega_i}) = v_l$, $\mathrm{Vert}_l(v^+_{\omega_i}) = v'_l = \mathrm{Vert}_l(v')$.

Then again using Lemma 2.26 repeatedly along each ω_i as in the proof of [16, Prop. 3.17], we can identify α with a rational section of $F^*_{v',\tau_2}\Omega^r_{v'}$ and argue that the section is actually regular. Hence α is in $\bigcap_{g'\colon v'\to\tau_1} F^*_{v',\tau_2}\Omega^r_{v'}$.

□

For $e\colon \tau_1 \to \tau_2$, we will now calculate the cohomology of $(F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors}$ by building a convenient resolution. The first two terms of such a resolution are given by Proposition 2.32; we need to extend this two-term complex.

Let $V \subseteq \Lambda_{\tau,\mathbb{R}}$ be a subspace. We have a subsheaf $\Omega^r_v|_{X_\tau} \cap V^\perp$ of $\Omega^r_v|_{X_\tau}$ given by forms α with $\iota(\partial_m)\alpha = 0$ for all $m \in V$. We define $\Omega^r_{(Z^\tau_I)^\dagger/\Bbbk} \cap V^\perp$ or $\Omega^r_{(Z^\tau_I)^\dagger/\Bbbk^\dagger} \cap V^\perp$ to be the image of $\Omega^r_v|_{X_\tau} \cap V^\perp$ in $\Omega^r_{(Z^\tau_I)^\dagger/\Bbbk}$ (or $/\Bbbk^\dagger$).

For $m \in \Lambda_\tau$, note that

$$\iota(\partial_m)\left(\operatorname{im}\left(\bigoplus_{i\in I}\mathcal{O}_{Z^\tau_I}(-Z^\tau_i)\xrightarrow{d}\Omega^1_v|_{Z^\tau_I}\right)\right) = 0,$$

as all monomials occuring in the equations for Z^τ_i are in $\Lambda^\perp_\tau$. We thus in particular have from Lemma 2.31 an exact sequence

$$0 \to \bigoplus_{i\in I}\mathcal{O}_{Z^\tau_I}(-Z^\tau_i) \to \Omega^1_v|_{Z^\tau_I} \cap V^\perp \to \Omega^1_{(Z^\tau_I)^\dagger/\Bbbk} \cap V^\perp \to 0$$

and a similar exact sequence for the $/\Bbbk^\dagger$ case.

Given $g\colon v \to \tau_1$, we can define complexes $\mathcal{F}^{r,\bullet}_v$ by

$$\mathcal{F}^{r,p}_v = \bigoplus_{\substack{\sigma\subseteq\Delta_{\tau_1}:\bar{v}\in\sigma\\ \dim\sigma=p}} \Omega^{r-p}_{(Z^{\tau_1}_{I(\sigma)})^\dagger/\Bbbk} \cap T^\perp_\sigma,$$

in the $/\Bbbk$ case and

$$\mathcal{F}^{r,p}_v = \bigoplus_{\substack{\sigma\subseteq\Delta_{\tau_1}:\bar{v}\in\sigma\\ \dim\sigma=p}} \Omega^{r-p}_{(Z^{\tau_1}_{I(\sigma)})^\dagger/\Bbbk^\dagger} \cap T^\perp_\sigma,$$

in the $/\Bbbk^\dagger$ case, where the above sums are over all p-dimensional $\sigma = \sigma_1 + \cdots + \sigma_q$, in which σ_i is a face of Δ_i containing v_i and

$\Delta_{\tau_1} = \Delta_1 + \cdots + \Delta_q$;

$\bar{v} = v_1 + \ldots + v_q$;

T_σ is the tangent space to σ in $\Lambda_{\tau,\mathbb{R}}$;

$I(\sigma) = \{i|\sigma_i \neq \{v_i\}\}$.

We use the convention that if $I(\sigma) = \emptyset$ then $\Omega^r_{(Z^\tau_{I(\sigma)})^\dagger/\Bbbk}$ (or $/\Bbbk^\dagger$) is $\Omega^r_v|_{X_\tau}$.

We define differentials $\delta_p\colon \mathcal{F}^{r,p}_v \to \mathcal{F}^{r,p+1}_v$ by

$$(\delta_p\alpha)_{\sigma'} = \sum_{\substack{\sigma\subseteq\sigma':\bar{v}\in\sigma\\ \dim\sigma=p}} \iota(\partial_{w_j-v_j})\alpha_\sigma|_{Z^{\tau_1}_{I(\sigma')}}.$$

Here σ' is a face of Δ_{τ_1} of dimension $p+1$, and we sum over all faces σ of σ' of dimension p containing v. For each such σ', by Lemma 2.28 there is a unique j such that $\sigma'_j \neq \sigma_j$, and w_j is the unique vertex of σ'_j not contained in σ_j. By Proposition 2.32,

$$\Omega^r_{\tau_1} = \ker(\delta_0\colon \mathcal{F}^{r,0}_v \to \mathcal{F}^{r,1}_v).$$

The following lemma is a continuation of Proposition 2.32. It extends the resolution of the term $(F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors}$.

Lemma 2.33 (cf. Lem. 3.18 in [16]). *Fix $g\colon v \to \tau_1$. For any $\tau_1 \subseteq \tau_2$,*

$$F^*_{\tau_1,\tau_2}\mathcal{F}^{r,\bullet}_v$$

is a resolution of $(F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1})/\operatorname{Tors}$.

Proof. It suffices to show this Lemma for $\tau_1 = \tau_2 = \tau$ because the complex remains a resolution under pull-back by Lemma 2.31, (2).

With reference to the proof of [16, Lem. 3.18], consider faces $v \in \omega \subseteq \omega' \subseteq \Delta_\tau$, and consider the complex $\mathcal{F}^\bullet_{\omega,\omega'}$ defined by

$$\mathcal{F}^p_{\omega,\omega'} = \bigoplus_{\substack{\omega\subseteq\sigma\subseteq\omega' \\ \dim\sigma=p}} \Omega^{r-p}_{(Z^\tau_{I(\sigma)})^\dagger/\Bbbk} \cap T^\perp_\sigma,$$

with differential δ_p. Recall that in the proof of [16, Lem. 3.18], it is proven that $H^i(\mathcal{F}^\bullet_{\omega,\omega'}) = 0$ for $i > \dim\omega$ by an induction on $\dim\omega' - \dim\omega$.

By Proposition 2.30, $\Delta_\tau \cap \Lambda_{\tau',\mathbb{R}} = 0$ so that $\omega \cap \Lambda_{\tau',\mathbb{R}} = \omega' \cap \Lambda_{\tau',\mathbb{R}} = \sigma \cap \Lambda_{\tau',\mathbb{R}} = 0$ for unbounded 1-cell τ', by setting the vertex $v = 0$ in $\Lambda_{\tau,\mathbb{R}}$. In other words, the faces ω, ω' and σ remain bounded.

Consequently, we note that the arguments used in the proof of [16, Lem. 3.18] are independent of the boundedness of τ. Hence, the result follows. □

Chapter 3

Cohomology of smoothings and affine cohomological controls

After the lengthy preparations in the previous sections, we finally arrive at many results in this section.

In §3.1 and §3.2, we follow the lines and methods of [16] to have the first affine cohomological control, the decomposition of the log Dolbeault groups and the base change theorem. Then in §3.3, we obtain an affine analogue of the Poincaré residue map in complex algebraic geometry and the second affine cohomological control with some new definitions in the cone picture $\check{B}$. In §3.4, we will look into some immediate consequences in various spectral sequences along our construction in toric degenerations.

3.1 The first affine cohomological control and a Hodge decomposition

We continue with the notations of the previous section §2.4 and proceed in a similar way as in [16, §3.3] with some new definitions (Definition 3.2) in the fan picture B. Under the "standard simplex" hypothesis on the polytopes describing the outer monodromy of the cells $\tau \in \mathscr{P}$ [1], the cohomology groups of $F^*_{\tau_1,\tau_2}\Omega^r_{\tau_1}/\,\mathrm{Tors}$ vanish in degree ≥ 1, and the global sections of these sheaves are easily expressed in terms of data on B. The investigation of the various phenomena with the "standard simplex" assumption lessened (see [32]) will not be performed in this thesis but we will have a related discussion in §4.2 (5).

Lemma 3.1 (cf. Lem. 3.19 in [16]). *Suppose that for the cell $\tau \in \mathscr{P}$, the polytope $\mathrm{Conv}(\bigcup_{i=1}^q \check{\Delta}_i \times \{e_i\})$ is a standard simplex. Then*

1. *For $\sigma \subseteq \Delta_\tau$ a face,*

$$\Gamma(X_\tau, \Omega^r_{(Z_I^\tau)^\dagger/\Bbbk^\dagger} \cap T_\sigma^\perp) = \frac{\bigwedge^r T_\sigma^\perp}{\mathrm{Top}(I)_r} \otimes \Bbbk,$$

for $T_\sigma^\perp \subseteq \check{\Lambda}_{v,\mathbb{R}}$, $\mathrm{Top}(I)_r$ the degree r part of the ideal in the exterior algebra of $T_\sigma^\perp$ generated by

$$\bigcup_{i\in I} \bigwedge\nolimits^{\mathrm{top}} T_{\check{\Delta}_i}.$$

[1] Or equivalently, the monodromy is unimodular around τ (see Theorem 0.1 and the remarks after it).

2. $H^j(X_\tau, \Omega^r_{(Z_I^\tau)^\dagger/\Bbbk^\dagger} \cap T_\sigma^\perp) = 0$ *for* $j > 0$.

Proof. Let W be a complementary subspace to $\sum_{i\in I} T_{\check{\Delta}_i} \subseteq T_\sigma^\perp$. Then we can split $\Omega_v^1|_{X_\tau} \cap T_\sigma^\perp$ as $(\mathcal{O}_{X_\tau} \otimes W) \oplus \bigoplus_{i\in I}(\mathcal{O}_{X_\tau} \otimes T_{\check{\Delta}_i})$, and in addition $d(\mathcal{O}(-Z_i^\tau)) \subseteq \mathcal{O}_{X_\tau} \otimes T_{\check{\Delta}_i}$, as the polynomial defining Z_i^τ only involves monomials in $\check{\Delta}_i$. Let $d_i = \dim \check{\Delta}_i$. Then we obtain a splitting of the exact sequence of Lemma 2.31

$$0 \to \bigoplus_{i\in I} \mathcal{O}_{Z_I^\tau}(-Z_i^\tau) \to \Omega_v^1|_{Z_I^\tau} \cap T_\sigma^\perp \to \Omega^1_{(Z_I^\tau)^\dagger/\Bbbk^\dagger} \cap T_\sigma^\perp \to 0$$

into exact sequences, for $i \in I$,

$$0 \to \mathcal{O}_{Z_I^\tau}(-Z_i^\tau) \to \mathcal{O}_{Z_I^\tau} \otimes T_{\check{\Delta}_i} \to \Omega_i^1 \to 0, \tag{3.1}$$

where each of these sequences defines a locally free sheaf Ω_i^1 of rank $d_i - 1$. In addition, we have one remaining direct summand of the original exact sequence,

$$0 \to 0 \to \mathcal{O}_{Z_I^\tau} \otimes W \to \mathcal{O}_{Z_I^\tau} \otimes W \to 0.$$

If we show that for $j > 0$

$$H^j\left(Z_I^\tau, \left(\bigotimes_{i\in I} \Omega_i^{r_i}\right)\left(-\sum_{i\in I} a_i Z_i^\tau\right)\right) = 0 \tag{3.2}$$

for $0 \le a_i \le d_i - 1 - r_i$, then (2) of the Lemma follows. To prove (1), it suffices to show that

$$H^0(Z_I^\tau, (\bigotimes_{i\in I} \Omega_i^{r_i})(-\sum_{i\in I} a_i Z_i^\tau)) = 0 \tag{3.3}$$

for $0 \le a_i \le d_i - 1 - r_i$ if at least one $a_i > 0$, and

$$H^0(Z_I^\tau, \bigotimes_{i\in I} \Omega_i^{r_i}) = \bigotimes_{i\in I} \bigwedge^{r_i} T_{\check{\Delta}_i}. \tag{3.4}$$

Since the toric stratum X_τ is defined by means of $X_\tau := X(\Sigma_\tau)$ (cf. [15, Def. 2.7]), it is independent of the boundedness of τ. Similarly, the object $\mathcal{Q}_{\tau,\mathbb{R}}$ (cf. [15, Def. 1.33]) and the application of [27, §2.2] are also independent of boundedness. The proof now proceeds in the same way as in [16, Lem. 3.19], so that $H^j(X_\tau, \mathcal{O}_{X_\tau}(-\sum_{i\in I} a_i Z_i^\tau)) = 0$ for $j > 0$, $0 \le a_i \le d_i$.

Tensoring the Koszul complex (3.5) in [16] with $\mathcal{O}_{X_\tau}(-\sum_{i\in I} a_i Z_i^\tau)$ for $0 \le a_i \le d_i - 1$ and employing the algebraic properties of the above exact sequences to perform an induction as in [16, Lem. 3.19], we obtain the above vanishings (3.2) and (3.3). Finally, the cohomology in (3.4) follows by the same argument as in the proof of [16, Lem. 3.19].

□

In [15, Def. 1.25 and Def. 2.9], notions W_{τ_i}, $\mathscr{W}$ and $W_e \subseteq B$ for $e\colon \tau_1 \to \tau_2$ are defined for bounded cells τ_1 and τ_2. They are not yet defined for unbounded cells in unbounded affine manifold $(B, \mathscr{P})$. Thus, we have the following definition.

Definition 3.2. Let B be an unbounded affine manifold with a polyhedral decomposition $\mathscr{P}$.

1. For an unbounded 1-cell τ' emerging from a vertex v, fix $a_{\tau'} \in \Lambda_{\tau',\mathbb{R}}$ such that $v + a_{\tau'} \in \mathrm{Int}(\tau')$. Take $v + a_{\tau'}$ to be $\mathrm{Bar}(\tau')$. For higher dimensional unbounded cells τ, define $\mathrm{Bar}(\tau)$ by taking $\mathrm{Bar}(\tau)$ as the average of vertices of τ and $\mathrm{Bar}(\tau')$ of all unbounded 1-cells τ' bounding τ. Therefore, one also obtains $\mathrm{Bar}(\mathscr{P})$ for B. Then take W_τ as defined in [15, Def. 1.25] and set $\mathscr{W} = \{W_\tau | \tau \in \mathscr{P}\}$.

2. With the above definition, we can then define W_e for B unbounded following [15, Lem. 2.9]. Consequently, the conclusions thereof still hold. Note that W_e is a bounded set and it makes sense to consider loops whose interiors are in W_e later. In particular, all loops in B can be identified with loops inside $\mathscr{W}$.

Remark 3.3. Note that $\mathscr{W}$ does not cover B completely. But it covers the "bounded part" of B (i.e. $\mathscr{W}$ covers a neighbourhood of the bounded cells in the polyhedral decomposition $\mathscr{P}$ of B). Let $B^\flat = \{b \in W_\sigma \text{ for some } \sigma \in \mathscr{P}\} \subseteq B$. By definition, this set $B^\flat$ is covered by $\mathscr{W}$. In particular, $B^\flat$ is compact and is a deformation retract of B.

Moreover, $B^\flat$ provides a natural retraction $B \to \bigcup_{\tau \in \mathscr{P}} W_\tau$ that respects the local system $\check{\Lambda}$, so that the cohomology $H^q(B, i_* \bigwedge^p \check{\Lambda} \otimes \Bbbk)$ can be computed using the cover $\bigcup W_\tau$.

Lemma 3.4 (cf. Lem. 3.20 in [16]). *With the same hypotheses as in Lemma 3.1, in the* $/\Bbbk^\dagger$ *case, we have for any morphism* $e\colon \tau_1 \to \tau_2$, $W_e \subseteq B$ *the open subset defined in Definition 3.2 (cf. [15, Lem. 2.9]),*

$$\Gamma(W_e, i_* \bigwedge^r \check{\Lambda} \otimes \Bbbk) \cong H^0(X_{\tau_2}, (F^*_{\tau_1,\tau_2} \Omega^r_{\tau_1}) / \operatorname{Tors})$$

and

$$H^j(X_{\tau_2}, (F_{\tau_1,\tau_2})^* \Omega^r_{\tau_1}) / \operatorname{Tors}) = 0$$

for $j > 0$.

Proof. The proof is essentially the same as that of [16, Lem. 3.20]. Pick a vertex $g\colon v \to \tau_1$. Then

$$\begin{aligned} H^j(X_{\tau_2}, (F^*_{\tau_1,\tau_2} \Omega^r_{\tau_1}) / \operatorname{Tors}) &\cong \mathbb{H}^j(X_{\tau_2}, F^*_{\tau_1,\tau_2} \mathcal{F}^{r,\bullet}_v) \\ &= H^j(\Gamma(X_{\tau_2}, F^*_{\tau_1,\tau_2} \mathcal{F}^{r,\bullet}_v)) \end{aligned}$$

by Lemma 2.33 and Lemma 3.1, (2). Besides, by Lemma 3.1 (1), the complex $\Gamma(X_{\tau_2}, F^*_{\tau_1,\tau_2} \mathcal{F}^{r,\bullet}_v)$ coincides with the complex of $\Bbbk$-vector spaces $F^\bullet$, where $F^\bullet$ is defined by, if $\Omega_i, R_i, \Delta_i, \check{\Delta}_i$ are the simplicity data for τ_1,

$$F^s = \bigoplus_{\substack{\sigma \subseteq \Delta_{\tau_1} : \bar{v} \in \sigma \\ \dim \sigma = s}} \left(\bigwedge^{r-s} T^\perp_\sigma \right) / \operatorname{Top}(e, I(\sigma))_{r-s},$$

where $\sigma = \sigma_1 + \cdots + \sigma_q$, $I(\sigma) = \{i|\sigma_i \neq \{v_i\}\}$ as before, and $\mathrm{Top}(e, I(\sigma))_{r-s}$ is the degree $r-s$ part of the ideal of the exterior algebra of $T_\sigma^\perp$ generated by

$$\bigcup_{i \in I(\sigma)} \bigwedge^{\mathrm{top}} (T_{\check{\Delta}_i} \cap \Lambda_{\tau_2}^\perp).$$

Furthermore, the differential $\delta_s \colon F^s \to F^{s+1}$ is defined by

$$(\delta_s \alpha)_{\sigma'} = \sum_{\substack{\sigma \subseteq \sigma' : \bar{v} \in \sigma \\ \dim \sigma = s}} \iota(\partial_{w_j - v_j}) \alpha_\sigma.$$

We can then show $H^j(F^\bullet) = 0$ for $j > 0$ by repeating the arguments of [16, Lem. 3.18] and [16, Lem. 3.20], defining analogous complexes $F^\bullet_{\omega,\omega'}$ and proceeding by induction.

Hence, we now calculate $H^0(F^\bullet)$, and compare this with $\Gamma(W_e, i_* \bigwedge^r \check{\Lambda} \otimes_{\mathbb{Z}} \Bbbk)$. We identify this with monodromy invariant elements of $i_* \bigwedge^r \check{\Lambda}_v \otimes_{\mathbb{Z}} \Bbbk$ for loops based at v whose interior is in W_e. The monodromy action is then generated by transformations of the form $T^\rho_{feg,feg'} \colon \Lambda_v \to \Lambda_v$, (cf. [15, §1.5]) where we have $f \colon \tau_2 \to \rho$ with ρ codimension one, and $g' \colon v' \to \tau_1$ a vertex. Then one has the action on $\bigwedge^r \check{\Lambda}_v \otimes_{\mathbb{Z}} \Bbbk$ in the form

$$T^\rho_{feg,feg'}(n) = n + \check{d}_\rho \wedge \iota(m^\rho_{feg,feg'})n.$$

Thus $n \in \bigwedge^r \check{\Lambda}_v \otimes_{\mathbb{Z}} \Bbbk$ is invariant under all such monodromy operations if and only if $\check{d}_\rho \wedge \iota(m^\rho_{feg,feg'})n = 0$ for all choices of f and g'. Note that as $f \circ e$ runs through elements of R_i which factor through e, $\check{d}_\rho$ runs through a generating set for $T_{\check{\Delta}_i} \cap \Lambda^\perp_{\tau_2}$, and for any given f with $f \circ e \in R_i$, as g' varies over all vertices of τ_1, $m^\rho_{feg,feg'}$ runs over $\{v'_i - v_i | v'_i := \mathrm{Vert}_i(g') \text{ a vertex of } \Delta_i\}$. From this description, it is then clear that $\check{d}_\rho \wedge \iota(m^\rho_{feg,feg'})n = 0$ for all f, g' if and only if $n \in H^0(F^\bullet)$. □

We can now prove the main theorem of this section: the identification of the logarithmic Dolbeault groups with the cohomology groups on B, which is the first type of the affine cohomological controls.

Theorem 3.5 (cf. Thm. 3.21 in [16]). *Let B be an integral affine manifold with singularities, with polyhedral decomposition $\mathscr{P}$, and suppose $(B, \mathscr{P})$ is positive and simple. Assume furthermore that for all $\tau \in \mathscr{P}$, $\mathrm{Conv}(\bigcup_{i=1}^q \check{\Delta}_i \times \{e_i\})$ is a standard simplex (equivalently, the monodromy is unimodular around every cell $\tau \in \mathscr{P}$, see Theorem 0.1). Let s be lifted gluing data, with $X_0 = X_0(B, \mathscr{P}, s)$. Then there is a canonical isomorphism*

$$H^p(X_0, j_* \Omega^r_{X_0^\dagger/\Bbbk^\dagger}) \cong H^p(B, i_* \bigwedge^r \check{\Lambda} \otimes \Bbbk).$$

Proof. Firstly, we have by Corollary 2.22

$$H^p(X_0, j_* \Omega^r_{X_0^\dagger/\Bbbk^\dagger}) = \mathbb{H}^p(X_0, \mathscr{C}^\bullet(\Omega^r)).$$

Besides, we have the vanishings $H^j(X_{\tau_2}, (F_{\tau_1,\tau_2})^*\Omega^r_{\tau_1})/\operatorname{Tors}) = 0$ for $j > 0$ and $\tau_1 \subseteq \tau_2$ by Lemma 3.4 with the assumption of standard monodromy simplices. Thus $\mathscr{C}^\bullet(\Omega^r)$ is an acyclic resolution. In other words,

$$\mathbb{H}^p(X_0, \mathscr{C}^\bullet(\Omega^r)) \cong H^p(\Gamma(X_0, \mathscr{C}^\bullet(\Omega^r))).$$

Moreover,

$$\Gamma(X_0, \mathscr{C}^p(\Omega^r)) = \bigoplus_{\sigma_0 \subsetneq \cdots \subsetneq \sigma_p} \Gamma(W_{\sigma_0 \to \sigma_p}, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k})$$

by Lemma 3.4. However, $\Gamma(W_{\sigma_0 \to \sigma_p}, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k}) = \Gamma(W_{\sigma_0 \to \cdots \to \sigma_p}, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k})$ (where $W_{\sigma_0 \to \cdots \to \sigma_p}$ is the connected component of $W_{\sigma_0} \cap \cdots \cap W_{\sigma_p}$ indexed by $\sigma_0 \to \cdots \to \sigma_p$; if $\mathscr{P}$ has no self-intersecting cells, then $W_{\sigma_0} \cap \cdots \cap W_{\sigma_p}$ only has one connected component anyway) because the relevant monodromy operators, as considered in the proof of Lemma 3.4, only depend on $\sigma_0 \to \sigma_p$. Under this identification, the complex $\Gamma(X_0, \mathscr{C}^\bullet(\Omega^r))$ with differential d_{bct} then agrees with the Čech complex for $i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k}$ with respect to the open covering $\mathscr{W} = \{W_\sigma | \sigma \in \mathscr{P}\}$. More explicitly,

$$\Gamma(X_0, \mathscr{C}^p(\Omega^r)) \cong \mathscr{C}^p(\mathscr{W}, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k})$$

Consider the set $B^\flat = \{b \in W_\sigma \text{ for some } \sigma \in \mathscr{P}\} \subseteq B$ (see Remark 3.3). Now $\mathscr{W}$ covers $B^\flat$ and $B^\flat$ is the deformation retract of B. As a result,

$$H^p(\mathscr{W}, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k}) = H^p(B^\flat, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k}) \cong H^p(B, i_* \bigwedge^r \check{\Lambda} \otimes \mathbb{k}).$$

This proves the theorem. □

We now obtain the Hodge decomposition:

Theorem 3.6 (cf. Thm. 3.26 in [16]). *With the hypotheses of Theorem 3.5, there is a canonical isomorphism*

$$\mathbb{H}^r(X_0, j_* \Omega^\bullet_{X_0^\dagger/\mathbb{k}^\dagger}) \cong \bigoplus_{p+q=r} H^p(X_0, j_* \Omega^q_{X_0^\dagger/\mathbb{k}^\dagger}).$$

Proof. By Corollary 2.23 and Lemma 3.4,

$$\mathbb{H}^r(X_0, \Omega^\bullet) = H^r(\Gamma(X_0, \operatorname{Tot}(\mathscr{C}^\bullet(\Omega^\bullet)))).$$

But as $\Gamma(X_0, (F^*_{\tau,\sigma}\Omega^\bullet_\tau)/\operatorname{Tors})$ consists entirely of differentials of the form $\operatorname{dlog} n$, d is in fact zero in $\Gamma(X_0, \mathscr{C}^\bullet(\Omega^\bullet))$, and thus the global sections of the total complex split as a direct sum $\bigoplus_q \Gamma(X_0, \mathscr{C}^\bullet(\Omega^q)[-q])$, hence the result.

□

3.2 Base change

To relate the (log-)Dolbeault cohomology groups of the central fibre and that of the smoothing in a toric degeneration, it is necessary to have corresponding base change theorems for the (hyper)cohomology groups of both log spaces.

The first result of this section, Theorem 3.9, does not depend on the constructions in §2.3 and §2.4. Indeed, we only need Proposition 2.6 to derive the base change result for the cohomology theories on the total spaces $\mathfrak{X}^\dagger$ and $\breve{\mathfrak{X}}^\dagger$. Assuming the existence of a smoothing, we immediately have several corollaries about the cohomology groups on a generic fibre X_η.

Definition 3.7 (Def. 2.7 in [16])**.** Let $\mathfrak{X}^\dagger_{\Bbbk}$ be a toric log Calabi-Yau space over $\operatorname{Spec}\Bbbk^\dagger$, with positive and simple dual intersection complex $(B, \mathscr{P})$, and let $A \in Ob(\mathscr{C}_R)$, where $Ob(\mathscr{C}_R)$ denotes the category of Artin local R-algebras with residue field $\Bbbk$ and $R = \Bbbk[\mathbb{N}]$ with log structure of $\operatorname{Spec} R^\dagger$ induced by $\mathbb{N} \to R$, $n \mapsto t^n$. Then a divisorial log deformation of $\mathfrak{X}^\dagger_{\Bbbk}$ over $\operatorname{Spec} A^\dagger$ is data $f_A : \mathfrak{X}^\dagger_A \to \operatorname{Spec} A^\dagger$ together with an isomorphism $\mathfrak{X}^\dagger_A \times_{\operatorname{Spec} A^\dagger} \operatorname{Spec}\Bbbk^\dagger \cong \mathfrak{X}^\dagger_{\Bbbk}$ over $\operatorname{Spec}\Bbbk^\dagger$ such that

1. f_A is flat as a morphism of schemes, and $f_A|_{\mathfrak{X}_A \setminus Z}$ is log smooth.

2. For every closed geometric point $\bar{x} \in Z$, let P, Y and X be the data of Theorem 1.12 giving a diagram (1.6) over $\operatorname{Spec}\Bbbk^\dagger$. Let $X^\dagger_A = Y^\dagger \times_{\operatorname{Spec}\Bbbk[\mathbb{N}]^\dagger} \operatorname{Spec} A^\dagger$. Then there exists a diagram over $\operatorname{Spec} A^\dagger$
$$\begin{tikzcd} & \mathcal{V}^\dagger_A \arrow[dl] \arrow[dr, "\phi'"] & \\ \mathfrak{X}^\dagger_A & & X^\dagger_A \end{tikzcd} \tag{3.5}$$
with both maps strict étale.

Remark 3.8. By [16, Cor. 2.18], the existence of a smoothing of a log Calabi-Yau pair $\mathfrak{X}_{\Bbbk} = (X^\dagger_0, D)$ in a toric degeneration implies the existence of a divisorial log deformation of $X^\dagger_0$ over $\operatorname{Spec} A^\dagger$. In fact, $\mathfrak{X}^\dagger_A$ is the fibre over the thickened point $\operatorname{Spec} A^\dagger$.

Theorem 3.9 (cf. Thm. 4.1 in [16])**.** *Let A be a local Artinian $\Bbbk[t]$-algebra with residue class field $\Bbbk$ and $\operatorname{Spec} A^\dagger$ the scheme $\operatorname{Spec} A$ with log structure induced by $\mathbb{N} \to A$, $1 \mapsto t$. Assume that*
$$\pi\colon \mathfrak{X}^\dagger = (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow \operatorname{Spec} A^\dagger$$
and
$$\pi\colon \breve{\mathfrak{X}}^\dagger = (\mathfrak{X}, \breve{\mathcal{M}}_{\mathfrak{X}}) \longrightarrow \operatorname{Spec} A^\dagger$$
are divisorial deformations of positive and simple toric log Calabi-Yau spaces $X^\dagger_0 \to \operatorname{Spec}\Bbbk^\dagger$ and $\breve{X}^\dagger_0 \to \operatorname{Spec}\Bbbk^\dagger$ respectively. Denote by $\mathcal{Z} \subseteq \mathfrak{X}$ the singular set of the log structure $\mathfrak{X}^\dagger$ of relative codimension two, $j\colon \mathfrak{X} \setminus \mathcal{Z} \to \mathfrak{X}$ the inclusion of the complement and write

$\Omega^\bullet_{\mathfrak{X}} := j_*\Omega^\bullet_{\mathfrak{X}^\dagger/A^\dagger}$ *and* $\breve{\Omega}^\bullet_{\mathfrak{X}} := j_*\Omega^\bullet_{\breve{\mathfrak{X}}^\dagger/A^\dagger}$. *Then* $\mathbb{H}^p(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}})$ *and* $\mathbb{H}^p(\mathfrak{X}, \breve{\Omega}^\bullet_{\mathfrak{X}})$ *are free A-modules and both commute with base change.*

Proof. Similar to the proof of [16, Thm. 4.1], we follow [22, 34]. By the cohomology and base change theorem, it suffices to prove the surjectivity of the restriction maps

$$\mathbb{H}^p(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}}) \longrightarrow \mathbb{H}^p(X_0, \Omega^\bullet_{X_0}).$$

and

$$\mathbb{H}^p(\mathfrak{X}, \breve{\Omega}^\bullet_{\mathfrak{X}}) \longrightarrow \mathbb{H}^p(X_0, \breve{\Omega}^\bullet_{X_0}).$$

Here $\Omega^\bullet_{X_0} = j_*\Omega^\bullet_{X_0^\dagger/\Bbbk^\dagger}$ and $\breve{\Omega}^\bullet_{X_0} = j_*\Omega^\bullet_{\breve{X}_0^\dagger/\Bbbk^\dagger}$. Following [22, p.404], it suffices to prove these for $A = \Bbbk[t]/(t^{k+1})$ with the obvious $\Bbbk[t]$-algebra structure. For structural clarity we keep the notation A for the base ring.

Consider the complexes of $\mathcal{O}_{\mathfrak{X}}$-modules

$$\mathcal{L}^\bullet = j_*\Omega^\bullet_{\mathfrak{X}^\dagger/\Bbbk}[u] = \bigoplus_{s=0}^{\infty} j_*\Omega^\bullet_{\mathfrak{X}^\dagger/\Bbbk} \cdot u^s$$

and

$$\breve{\mathcal{L}}^\bullet = j_*\Omega^\bullet_{\breve{\mathfrak{X}}^\dagger/\Bbbk}[u] = \bigoplus_{s=0}^{\infty} j_*\Omega^\bullet_{\breve{\mathfrak{X}}^\dagger/\Bbbk} \cdot u^s,$$

both equipped with the same differential d as in [16, Thm. 4.1] of the form

$$\begin{aligned} \mathrm{d}\Big(\sum_{s=0}^{N} \alpha_s u^s\Big) &= \sum_{s=0}^{N} \mathrm{d}\alpha_s \cdot u^s + s\,\mathrm{dlog}\,\rho \wedge \alpha_s \cdot u^{s-1} \\ &= \mathrm{d}\alpha_N \cdot u^N + \sum_{s=0}^{N-1} (\mathrm{d}\alpha_s + (s+1)\,\mathrm{dlog}\,\rho \wedge \alpha_{s+1}) \cdot u^s, \end{aligned} \tag{3.6}$$

where $\rho \in \Gamma(\mathcal{M}_{\mathfrak{X}})$ is the pull-back of the section of $\mathcal{M}_A$ induced by t. Note that these are differentials relative $\operatorname{Spec} \Bbbk$ rather than relative $\operatorname{Spec} A^\dagger$, so $\mathrm{dlog}\,\rho \neq 0$ unlike in $\Omega_{\mathfrak{X}}$. In these complexes, the dummy variable u formally behaves like $\log t$, and the use of considering these complexes is to trade powers of $\mathrm{dlog}\,\rho$ with powers of u.

Now the projection $\sum \alpha_s u^s \mapsto \alpha_0$ defines maps

$$\mathcal{L}^\bullet \longrightarrow \Omega^\bullet_{\mathfrak{X}}.$$

and

$$\breve{\mathcal{L}}^\bullet \longrightarrow \breve{\Omega}^\bullet_{\mathfrak{X}}.$$

To finish the proof, it suffices to show that the compositions

$$\varphi^\bullet \colon \mathcal{L}^\bullet \longrightarrow \Omega^\bullet_{\mathfrak{X}} \longrightarrow \Omega^\bullet_{X_0}$$

and

$$\breve{\varphi}^\bullet\colon \breve{\mathcal{L}}^\bullet \longrightarrow \breve{\Omega}^\bullet_{\mathfrak{X}} \longrightarrow \breve{\Omega}^\bullet_{X_0}$$

are quasi-isomorphisms, that is, they induce isomorphisms of cohomology sheaves $H^p(\mathcal{L}^\bullet) \to H^p(\Omega^\bullet_{X_0})$ and $H^p(\breve{\mathcal{L}}^\bullet) \to H^p(\breve{\Omega}^\bullet_{X_0})$, respectively. Consequently, the induced composed maps of hypercohomology groups

$$\mathbb{H}^p(\mathcal{L}^\bullet) \longrightarrow \mathbb{H}^p(\Omega^\bullet_{\mathfrak{X}}) \longrightarrow \mathbb{H}^p(\Omega^\bullet_{X_0})$$

and

$$\mathbb{H}^p(\breve{\mathcal{L}}^\bullet) \longrightarrow \mathbb{H}^p(\breve{\Omega}^\bullet_{\mathfrak{X}}) \longrightarrow \mathbb{H}^p(\breve{\Omega}^\bullet_{X_0})$$

are isomorphisms, hence the surjectivity of the second maps as needed.

By this argument and since $\mathfrak{X}^\dagger \to \operatorname{Spec} A^\dagger$ and $\breve{\mathfrak{X}}^\dagger \to \operatorname{Spec} A^\dagger$ are divisorial deformations of ${X_0}^\dagger \to \operatorname{Spec}\Bbbk^\dagger$ and $\breve{X}_0^\dagger \to \operatorname{Spec}\Bbbk^\dagger$ respectively, for the rest of the proof we consider the following local situation. For every étale neighbourhood X of X_0, there is a toric variety $Y = \operatorname{Spec}\Bbbk[P]$ containing X as a toric Cartier divisor $V(z^\rho)$ such that the deformations $\mathfrak{X}^\dagger \to \operatorname{Spec} A^\dagger$ and $\breve{\mathfrak{X}}^\dagger \to \operatorname{Spec} A^\dagger$ are given by

$$\pi\colon \operatorname{Spec}\Bbbk[P]/(z^{(k+1)\cdot\rho}) \longrightarrow \operatorname{Spec}\Bbbk[t]/(t^{k+1}), \quad \pi^*(t) = z^\rho.$$

Since $\varphi^r\colon \mathcal{L}^r \to \Omega^r_{X_0}$ is surjective for any r we obtain a short exact sequence

$$0 \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathcal{L}^\bullet \xrightarrow{\varphi^\bullet} \Omega^\bullet_{X_0} \longrightarrow 0$$

of complexes by defining $\mathcal{K}^\bullet = \ker\varphi^\bullet$. Now $\varphi^\bullet$ is a quasi-isomorphism if and only if $\mathcal{K}^\bullet$ is acyclic, and this is what we are going to show.

For an explicit description of $\mathcal{K}^r$ let $\sum_{s=0}^N \alpha_s u^s \in \mathcal{L}^r$, that is, $\alpha_s \in j_*\Omega^r_{\mathfrak{X}^\dagger/\Bbbk}$ for all s. Then $\sum_{s=0}^N \alpha_s u^s \in \ker\varphi^r$ if and only if $\alpha_0|_{X_0} = 0$. On the other hand, the closedness equation $\mathrm{d}(\sum \alpha_s u^s) = 0$ is equivalent to the system of equations

$$\begin{aligned} \mathrm{d}\alpha_N &= 0 \\ \mathrm{d}\alpha_s + (s+1)\,\mathrm{dlog}\,\rho \wedge \alpha_{s+1} &= 0, \quad s < N. \end{aligned} \tag{3.7}$$

It is easy to solve these equations after decomposing the coefficients α_s according to weights, that is, according to the P-grading. First, Proposition 2.6 gives a decomposition of $\Gamma(\mathfrak{X}, j_*\Omega^r_{\mathfrak{X}^\dagger/\Bbbk})$ into homogeneous pieces

$$\Gamma(\mathfrak{X}, j_*\Omega^r_{\mathfrak{X}^\dagger/\Bbbk}) = \bigoplus_{p\in P\setminus((k+1)\rho+P)} z^p \cdot \bigwedge^r\Big(\bigcap_{\{j\,|\,p\in Q_j\}} Q_j^{\mathrm{gp}}\Big) \otimes_{\mathbb{Z}} \Bbbk.$$

as well as a similar decomposition for $\Gamma(\mathfrak{X}_k, j_*\Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk})$

$$\Gamma(\mathfrak{X}_k, j_*\Omega^r_{\breve{\mathfrak{X}}_k^\dagger/\Bbbk}) = \bigoplus_{p\in P\setminus((k+1)\rho+P)} z^p \cdot \bigwedge^r\left(\bigcap_{\{j|p\in P_j^\delta\}} (P_j^\delta)^{\mathrm{gp}} \cap \bigcap_{\{j|p\in Q_j\}} Q_j^{\mathrm{gp}}\right) \otimes_{\mathbb{Z}} \Bbbk.$$

Thus we obtain P-gradings on $\mathcal{L}^\bullet$ and $\check{\mathcal{L}}^\bullet$ by imposing the P-grading on each direct summand $j_*\Omega^r_{\mathfrak{X}^\dagger/\Bbbk}\cdot u^s \subset \mathcal{L}^\bullet$ and $j_*\Omega^r_{\check{\mathfrak{X}}^\dagger/\Bbbk}\cdot u^s \subset \check{\mathcal{L}}^\bullet$ respectively. In particular, we may then assume that the α_s in (3.6) are of the form

$$\alpha_s = z^p \operatorname{dlog}\omega_s, \quad s = 0, \cdots, N, \tag{3.8}$$

with $\omega_s \in \bigwedge^r V_p \otimes_{\mathbb{Z}} \Bbbk$ or $\omega_s \in \bigwedge^r \check{V}_p \otimes_{\mathbb{Z}} \Bbbk$ where

$$V_p := \bigcap_{\{j|p\in Q_j\}} Q_j^{\mathrm{gp}}$$

and

$$\check{V}_p := \bigcap_{\{j|p\in P_j^\delta\}} (P_j^\delta)^{\mathrm{gp}} \cap \bigcap_{\{j|p\in Q_j\}} Q_j^{\mathrm{gp}}.$$

On each $\Omega^\bullet_{X_0}$ and $\check{\Omega}^\bullet_{X_0}$, the P-grading is obtained by plugging $k = 0$ into the formula above and dividing by $\mathbb{Z}\rho$. Second, the differentials on $\mathcal{L}^\bullet$, $\check{\mathcal{L}}^\bullet$ and on $\Omega^\bullet_{X_0}$, $\check{\Omega}^\bullet_{X_0}$ commute with the respective P-gradings, and so do $\varphi^\bullet$ and $\check{\varphi}^\bullet$. Third, all sheaves involved are pull-backs under the morphism $\mathrm{Et}\colon \mathfrak{X}^{\mathrm{et}} \to \mathfrak{X}^{\mathrm{Zar}}$ relating the Zariski site on $\mathfrak{X}$ to the étale site.

The proof now proceeds with the same argument for [16, Thm 4.1] to solve the above system of equations (3.7).

□

Remark 3.10. Provided that a smoothing of $(X_0^\dagger, D)$ exists, we want to deduce the cohomology groups $H^q(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta))$ and $H^q(X_\eta, \Omega^p_{X_\eta/k(\eta)})$ from data of the central fibre X_0 and eventually from data of B.

Assume that the generic fibre X_η is smooth over η so that it makes sense to talk about the above two types of cohomology groups (see §1.2). Thus, we concentrate on the situation when $\dim X_\eta \le 3$ or the generic fibre X_η is in addition smooth for $\dim X_\eta \ge 4$ (if and only if the monodromy around every cell of B is unimodular, see §1.2).

In the rest of this thesis, we write $H^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta))$ and $H^q(X_\eta, \Omega^p_{X_\eta})$ instead of $H^q(X_\eta, (j_*\Omega^p_{\mathfrak{X}^\dagger/R^\dagger})_\eta)$ and $H^q(X_\eta, (j_*\Omega^p_{\check{\mathfrak{X}}^\dagger/R^\dagger})_\eta)$.

□

Corollary 3.11 (cf. Thm. 4.2 in [16]). *Consider an integral affine manifold $(B, \mathscr{P})$ satisfying the hypotheses of Theorem 3.5. Suppose a smoothing X_η of the Calabi-Yau pair $(X_0^\dagger, D) = X_0(B, \mathscr{P}, s)^\dagger$ in a toric degeneration $\mathfrak{X} \to T = \operatorname{Spec} R$ exists, where the log space $T^\dagger$ is equipped with log structure induced by $\mathbb{N} \to R$, $1 \mapsto t$.*

Then $H^q(\mathfrak{X}, j_*\Omega^p_{\mathfrak{X}^\dagger/R^\dagger})$ is a locally free R-module, and it commutes with base change. In particular,

$$\dim_{k(\eta)} H^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta)) = \dim_{\Bbbk} H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger}).$$

Proof. This follows from Theorem 3.6 and Theorem 3.9 in a standard way, see [7, §5]. See also [16, Rem. 3.25 and 4.3].

□

Remark 3.12. The generic fibres X_η in our setting are indeed varieties over the field $k(\eta)$. It is quite often that one is interested in varieties over the field $\Bbbk$ (especially the case $\Bbbk = \mathbb{C}$) instead of over $k(\eta)$.

This ambiguity about the underlying field can be resolved if we restrict our attention only to those degenerations whose generic fibres X_η are projective. Nevertheless, we need to impose some extra assumptions on the cohomology of X_0 (e.g. $h^{2,0} = 0$) so that every deformation of X_0 is projective; and then we can actually talk about a general fibre X_s that is a variety over $\Bbbk$.

With such assumptions, we are able to have an algebraic family over a scheme of finite type over $\Bbbk$ with central fibre X_0, which is formally versal at the point 0. We have thus three types of fibres in such a family: the central fibre X_0, the generic fibre X_η and one further closed fibre X_s with $k(s) = \Bbbk$ (note that then X_s is smooth over s). Finally, we have the equality of the logarithmic Hodge number $\dim_{k(\eta)} H^q(X_\eta, \Omega^p_{X_\eta/k(\eta)}(\log D_\eta)) = \dim_{\Bbbk} H^q(X_s, \Omega^p_{X_s/\Bbbk}(\log D_s))$ by the above base change. The equality of the ordinary Hodge number $h^{p,q}(X_\eta) = h^{p,q}(X_s)$ follows from the equality $h^{p,q}(D_\eta) = h^{p,q}(D_s)$ and the second affine cohomological control (see §3.3).

□

Corollary 3.13. *Let $\Bbbk = \mathbb{C}$. Consider an integral affine manifold $(B, \mathscr{P})$ satisfying the hypotheses of Theorem 3.5. Suppose a smoothing X_η of the Calabi-Yau pair $(X_0^\dagger, D) = X_0(B, \mathscr{P}, s)^\dagger$ in a toric degeneration $\mathfrak{X} \to T$ exists. By Remark 3.12, assume X_s exists as a complex projective variety and assume that D_s is a simple normal crossing divisor in X_s. Then*

$$H^k(X_s \setminus D_s, \Bbbk) = \mathbb{H}^k(X_s, \Omega^\bullet_{X_s}(\log D_s)).$$

In particular,

$$\dim_{\Bbbk} H^k(X_s \setminus D_s, \Bbbk) = \dim_{k(\eta)} \mathbb{H}^k(X_\eta, \Omega^\bullet_{X_\eta}(\log D_\eta)) = \dim_{\Bbbk} \mathbb{H}^k(X_0, j_*\Omega^\bullet_{X_0^\dagger/\Bbbk^\dagger}).$$

Proof. It follows from Theorem 3.9 and the standard mixed Hodge theory for the compact complex manifold X_s (see [8], [30, Thm 4.2] and [36, §8.4.1]).

□

3.3 Second affine cohomological control

In this section, we consider $\Bbbk = \mathbb{C}$ and concentrate on the generic fibre X_η of the degeneration. Recall that in [15, §2.1] and [17, Ex. 1.13], the cone picture is defined (we are using the same notation as in [15, 16] in this thesis). The second affine cohomological control is better expressed in terms of the notation in the cone picture with affine manifold $\check{B}$.

Construction 3.14. Consider a compact affine manifold $\check{B}$ of dimension n with singularities $\check{\Delta}$ and boundary $\partial\check{B}$ such that $\partial\check{B}$ is a compact affine manifold of dimension $n-1$ without boundary, possibly with singularities inherited from $\check{B}$. Denote $\check{B}_0 = \check{B} \setminus \check{\Delta}$ and $i\colon \check{B}_0 \hookrightarrow \check{B}$.

Define $\Lambda^{\check{B}_0}$ and $\Lambda_0^{\check{B}_0}$ to be the local systems of flat integral vector fields on $\check{B}_0$ (cf. [15, Def. 1.9]) so that $\Lambda^{\check{B}_0}$ behaves at the boundary as if the boundary does not exist while $\Lambda_0^{\check{B}_0}$ restricted to the boundary is isomorphic to the usual local system of flat integral vector fields $\Lambda^{\partial\check{B}_0}$ on the boundary $\partial\check{B}_0$. In other words, we have

$$\begin{aligned} \Lambda^{\partial\check{B}_0} &\cong \Lambda_0^{\check{B}_0}|_{\partial\check{B}_0} \\ \operatorname{rank}\Lambda_0^{\check{B}_0}|_{\partial\check{B}_0} &= \operatorname{rank}\Lambda_0^{\check{B}_0\setminus\partial\check{B}_0} - 1 = n-1 \\ \operatorname{rank}\Lambda^{\check{B}_0}|_{\partial\check{B}_0} &= \operatorname{rank}\Lambda^{\check{B}_0}|_{\check{B}_0\setminus\partial\check{B}_0} = n \\ \operatorname{rank}\Lambda^{\check{B}_0}|_{\check{B}_0\setminus\partial\check{B}_0} &= \operatorname{rank}\Lambda_0^{\check{B}_0}|_{\check{B}_0\setminus\partial\check{B}_0} = n. \end{aligned}$$

For B unbounded without boundary, extend the discrete Legendre transform in [15, §1.4] by constructing the homeomorphism $B \to \check{B}\setminus\partial\check{B}$ via piecewise affine identification of barycentric subdivisons (see [17, Constr. 1.16]). Then we have $\Delta = \check{\Delta}$ and the isomorphism of the local systems $\check{\Lambda}^{B_0} \cong \Lambda^{\check{B}_0}$ (by the identification of the generization isomorphisms $\psi_{v\sigma}$ and $\psi_{\check{v}\check{\sigma}}^{-1}$). Furthermore, it is also true that $\check{B} \cong B^\flat \ncong B$ (homeomorphically) for B unbounded without boundary (see Definition 3.2 and proof of Theorem 3.5 for the definition of $B^\flat$).

To summarize the results in [16] and this thesis so far, we have the following proposition.

Proposition 3.15. *Suppose that (X_η, D_η) is the generic fibre of a toric degeneration of a log Calabi-Yau pair $(X_0^\dagger, D)$ with the dual intersection complex $(B, \mathscr{P})$ satisfying the hypotheses of Theorem 3.5 and the divisor D_η is smooth and irreducible on X_η.*

Then we have the following (non-canonical) isomorphisms:

$$H^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta)) \cong H^q(\check{B}, i_*\bigwedge^p \Lambda^{\check{B}} \otimes_{\mathbb{Z}} k(\eta)) \tag{3.9}$$

$$H^q(D_\eta, \Omega^p_{D_\eta}) \cong H^q(\partial\check{B}, i_*\bigwedge^p \Lambda^{\partial\check{B}} \otimes_{\mathbb{Z}} k(\eta)) \tag{3.10}$$

The first isomorphism follows from Theorem 3.5, the base change result Corollary 3.11 and the fact $\Lambda^{\check{B}_0} \cong \check{\Lambda}^{B_0}$ so that

$$H^q(\check{B}, i_*\bigwedge^p \Lambda^{\check{B}} \otimes \mathbb{C}) \cong H^q(B^\flat, i_*\bigwedge^p \check{\Lambda}^{B} \otimes \mathbb{C}) \cong H^q(B, i_*\bigwedge^p \check{\Lambda}^{B} \otimes \mathbb{C})$$

(for $B^\flat$, see Definition 3.2 and proof of Theorem 3.5).

The second isomorphism (3.10) follows actually from the insights of [5, §2]. Observe that the unbounded 1-cells in B correspond to $D = \bigcup_\mu D_\mu$ in X_0. Assuming that the unbounded 1-cells of the polyhedral decomposition $\mathscr{P}$ in the fan picture B are parallel (see [5, Prop. 2.1]), it follows that such a divisor D has a smoothing as an irreducible divisor D_η in the generic fibre X_η (see [5, Prop. 2.2] and Remark 5.2).

On the other hand, the unbounded 1-cells also correspond to $\partial\check{B}$ via the discrete Legendre transform. Therefore, $\partial\check{B}$ corresponds to the divisor D_η in the generic fibre. Viewing D_η as a generic fibre of dimension $n-1$ of a toric degeneration while regarding $\partial\check{B}$ as a real affine manifold of dimension $n-1$ associated to D_η, we thus get the isomorphism by [16, Thm. 3.22] and base change result [16, Thm. 4.2] because $\partial\check{B}$ is compact without boundary and D_η is a Calabi-Yau variety itself.

□

Remark 3.16. Let $\iota\colon \partial\check{B}_0 \hookrightarrow \check{B}_0$ denote the embedding of $\partial\check{B}_0$ into $\check{B}_0$. Consider $\iota^{-1}\check{\Lambda}^{\check{B}_0}$ on $\partial\check{B}_0$. There exists a global section $\alpha \in \Gamma(\partial\check{B}_0, \iota^{-1}\check{\Lambda}^{\check{B}_0})$ such that we have the restriction $\alpha|_{\check{\Lambda}^{\partial\check{B}_0}} = 0$ and the contraction $\alpha(\xi) = 1$ for any (integral) primitive normal vector field ξ of $\partial\check{B}_0$ with respect to B_0. In other words, α generates $\ker(\iota^{-1}\check{\Lambda}^{\check{B}_0} \to \check{\Lambda}^{\partial\check{B}_0})$ of the restriction map $\iota^{-1}\check{\Lambda}^{\check{B}_0} \to \check{\Lambda}^{\partial\check{B}_0}$, which is dual to the inclusion map $\Lambda^{\partial\check{B}_0} \hookrightarrow \Lambda^{\check{B}_0}$.

As a result, the contraction of α with $\gamma \in \bigwedge^p \Lambda^{\check{B}_0}$ induces a map

$$\bigwedge^{p} \Lambda^{\check{B}_0} \to \bigwedge^{p-1} \Lambda^{\partial\check{B}_0}$$

for every $p \geq 1$.

□

Recall that there is a notion of Poincaré residue map in algebraic geometry for a smooth algebraic variety X_η with a smooth irreducible divisor D_η (see e.g. [30, §4.2]). The above remark hence suggests an (integral) affine analogue of the Poincaré residue map.

Proposition 3.17. *Let (X_η, D_η) be the generic fibre of a toric degeneration of a log Calabi-Yau pair (X_0, D) with fan picture $(B, \mathscr{P})$. Suppose the cone picture $(\check{B}, \check{\mathscr{P}})$ is a compact affine manifold of dimension n with boundary $\partial\check{B}$ which is compact without boundary of dimension $n-1$.*

Now assume that the divisor D_η is irreducible on X_η. Using the notations in Remark 3.16, we then have a short exact sequence on $\check{B}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^{p} \Lambda_0^{\check{B}_0} & \longrightarrow & \bigwedge^{p} \Lambda^{\check{B}_0} & \longrightarrow & \bigwedge^{p-1} \Lambda^{\partial\check{B}_0} & \longrightarrow & 0 \\ & & & & \gamma & \longmapsto & \alpha(\gamma)\,|_{\partial\check{B}_0} & , & \end{array} \tag{3.11}$$

for $p \geq 1$, where $\alpha(\gamma)$ denotes the contraction of γ with the 1-form α, the generator of the kernel of the restriction map $\iota^{-1}\check{\Lambda}^{\check{B}_0} \to \check{\Lambda}^{\partial\check{B}_0}$ (see Remark 3.16).

Proof. The exactness of the sequence follows from the definition of $\Lambda_0^{\check{B}_0}$ and $\Lambda^{\check{B}_0}$ in Construction 3.14. □

Theorem 3.18. *Consider an integral affine manifold $(B, \mathscr{P})$ satisfying the hypotheses of Theorem 3.5. Suppose that a smoothing (X_η, D_η) of the Calabi-Yau pair $(X_0^\dagger, D) = X_0(B, \mathscr{P}, s)^\dagger$ in a toric degeneration $\mathfrak{X} \to T$ exists, where the divisor D_η is smooth and irreducible on X_η. Then we have an isomorphism*

$$H^q(X_\eta, \Omega^p_{X_\eta}) \cong H^q(\check{B}, i_* \bigwedge^p \Lambda_0^{\check{B}} \otimes k(\eta))$$

for any $p, q \geq 0$.

Proof. Since the divisor D_η is irreducible, one has the exact sequence

$$0 \to \Omega^p_{X_\eta} \to \Omega^p_{X_\eta}(\log D_\eta) \to \Omega^{p-1}_{D_\eta} \to 0,$$

which induces a cohomology long exact sequence (see e.g. [36, (8.8) in §8.4.2]). Together with the cohomology long exact sequence induced by (3.11) on $\check{B}$, we get a commutative diagram (with $K := k(\eta)$)

$$\begin{tikzcd} \cdots \arrow[r] & H^{q-1}(\partial\check{B}, i_* \bigwedge^{p-1} \Lambda^{\partial\check{B}} \otimes K) \arrow[r] \arrow[d, "\cong"] & H^q(\check{B}, i_* \bigwedge^{p} \Lambda_0^{\check{B}} \otimes K) \arrow[r] \arrow[d, dotted, "?"] & H^q(\check{B}, i_* \bigwedge^{p} \Lambda^{\check{B}} \otimes K) \arrow[r] \arrow[d, "\cong"] & H^q(\partial\check{B}, i_* \bigwedge^{p-1} \Lambda^{\partial\check{B}} \otimes K) \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & H^{q-1}(D_\eta, \Omega^{p-1}_{D_\eta}) \arrow[r] & H^q(X_\eta, \Omega^p_{X_\eta}) \arrow[r] & H^q(X_\eta, \Omega^p_{X_\eta}(\log D_\eta)) \arrow[r] & H^q(D_\eta, \Omega^{p-1}_{D_\eta}) \arrow[r] & \cdots \end{tikzcd}$$

with isomorphisms (3.9) and (3.10).

This diagram is indeed a commutative diagram of finite dimensional vector spaces with linear maps and exact rows:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"', "\cong"] & B \arrow[r, "j"] \arrow[d, "\beta"', "\cong"] & C \arrow[r, "k"] & D \arrow[r, "l"] \arrow[d, "\delta"', "\cong"] & E \arrow[d, "\epsilon"', "\cong"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

with the properties that

$$\begin{aligned} C &\cong \ker l \oplus \operatorname{coker} i \\ C' &\cong \ker l' \oplus \operatorname{coker} i'. \end{aligned}$$

By the commutativity of the diagram, we have then the isomorphism of the 2-term sequences $(A \to B) \cong (A' \to B')$ so that $\ker l \cong \ker l'$. Similarly, we have also the isomorphism $\operatorname{coker} i \cong \operatorname{coker} i'$ by the isomorphism of $(D \to E) \cong (D' \to E')$. As a result, $C \cong C'$ as vector spaces.

Therefore, the desired homomorphism exists and is an isomorphism. In other words,

$$H^q(X_\eta, \Omega^p_{X_\eta}) \cong H^q(\check{B}, i_* \bigwedge^p \Lambda_0^{\check{B}} \otimes k(\eta)).$$

□

$\mathscr{C}^n(\mathfrak{U}, \mathcal{O}_X)$	$\mathscr{C}^n(\mathfrak{U}, \Omega^1_X(\log D))$	$\cdots\cdots$	$\mathscr{C}^n(\mathfrak{U}, \Omega^n_X(\log D))$
$\vdots$			$\vdots$
$\mathscr{C}^1(\mathfrak{U}, \mathcal{O}_X)$	$\mathscr{C}^1(\mathfrak{U}, \Omega^1_X(\log D))$	$\cdots$	$\mathscr{C}^1(\mathfrak{U}, \Omega^n_X(\log D))$
$\mathscr{C}^0(\mathfrak{U}, \mathcal{O}_X)$	$\mathscr{C}^0(\mathfrak{U}, \Omega^1_X(\log D))$	$\cdots$	$\mathscr{C}^0(\mathfrak{U}, \Omega^n_X(\log D))$

Table 3.1: The E_0 page of $\Omega^\bullet_X(\log D)$

3.4 Analysis of spectral sequences

In this section, take $\Bbbk = \mathbb{C}$. By Remark 3.12, we consider the closed fibre X_s of the algebraic family, which is a variety over $\mathbb{C}$. For simplicity, write (X, D) for the pair (X_s, D_s). We first investigate some interesting phenomena of the spectral sequences of complexes of sheaves on both X and $\check{B}$, following the settings of previous sections.

Given a smooth closed fibre X of an algebraic family $\mathfrak{X}$, consider the spectral sequence of the complex of sheaves $\Omega^\bullet_X(\log D)$ on X. Let $\mathfrak{U}$ always be a Leray open cover of X with respect to the sheaves. Let $K_1^{p,q} = \mathscr{C}^q(\mathfrak{U}, \Omega^p_X(\log D))$ and consider the double complex $K_1^{\bullet,\bullet} = \bigoplus_{p,q\geq 0} K_1^{p,q}$ with differentials

$$d\colon K_1^{p,q} \to K_1^{p+1,q}$$
$$\delta\colon K_1^{p,q} \to K_1^{p,q+1},$$

where d is the standard De Rham differential (i.e. the exterior derivative) and δ is the standard Čech differential (i.e. the coboundary operator for the Čech complex). For $r \leq 2$, the terms $E_r^{p,q}$ are

$$\begin{aligned} E_0^{p,q} &= \mathscr{C}^q(\mathfrak{U}, \Omega^p_X(\log D)) \\ E_1^{p,q} &= H^q(X, \Omega^p_X(\log D)) \\ E_2^{p,q} &= H^p_d(H^q(X, \Omega^\bullet_X(\log D))). \end{aligned}$$

In diagrams, the $E_0^{p,q}$ and $E_1^{p,q}$ terms for (X, D) are given in Table 3.1 and Table 3.2 respectively.

Note that the differential d on X_0 (the exterior differential d before Corollary 2.23) as well as the differential d on $X(= X_s)$ are induced from the degeneration, which both inherit the differential d defined on $\mathfrak{X} \setminus \mathscr{Z}$ (see (3.6) and (3.8) in the proof of Theorem 3.9).

Now we determine when this spectral sequence degenerates. First, by Corollary 3.13 and Theorem 3.9, we have

$$H^k(X \setminus D, \Bbbk) = \mathbb{H}^k(X, \Omega^\bullet_X(\log D)) \cong \mathbb{H}^k(X_0, j_*\Omega^\bullet_{X_0^\dagger/\Bbbk^\dagger}).$$

$H^n(X,\mathcal{O}_X)$	$H^n(X,\Omega^1_X(\log D))$	$\cdots\cdots$	$H^n(X,\Omega^n_X(\log D))$
$\vdots$			$\vdots$
$H^1(X,\mathcal{O}_X)$	$H^1(X,\Omega^1_X(\log D))$	$\cdots$	$H^1(X,\Omega^n_X(\log D))$
$H^0(X,\mathcal{O}_X)$	$H^0(X,\Omega^1_X(\log D))$	$\cdots$	$H^0(X,\Omega^n_X(\log D))$

Table 3.2: The E_1 page of $\Omega^\bullet_X(\log D)$

Then Theorem 3.6 implies the direct sum decomposition of the cohomology group of the central fibre

$$\mathbb{H}^k(X_0, j_*\Omega^\bullet_{X_0^\dagger/\Bbbk^\dagger}) \cong \bigoplus_{p+q=k} H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger}).$$

Applying the (non-canonical) isomorphism $H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger}) \cong H^q(X,\Omega^p_X(\log D))$ from Corollary 3.11, we thus have the direct sum decomposition

$$H^k(X\setminus D,\Bbbk) \cong \bigoplus_{p+q=k} H^q(X,\Omega^p_X(\log D)). \tag{3.12}$$

It turns out that this direct sum decomposition (3.12) implies the degeneration of the spectral sequence at E_1. More explicitly, by the base change property in Corollary 3.11, one has

$$\dim E_1^{p,q} = \dim_\Bbbk H^q(X,\Omega^p_X(\log D)) = \dim_\Bbbk H^q(X_0, j_*\Omega^p_{X_0^\dagger/\Bbbk^\dagger})$$

while one has

$$\dim_\Bbbk H^k(X\setminus D,\Bbbk) = \dim_\Bbbk \mathbb{H}^k(X,\Omega^\bullet_X(\log D)) = \sum_{p+q=k}\dim E_\infty^{p,q}$$

by the definition of hypercohomology. Since every $E_\infty^{p,q}$ is a subquotient of $E_1^{p,q}$, we can conclude with the direct sum decomposition property (3.12)

$$\sum_{p+q=k}\dim E_\infty^{p,q} = \dim_\Bbbk \mathbb{H}^k(X,\Omega^\bullet_X(\log D)) = \sum_{p+q=k}\dim E_1^{p,q} \geq \sum_{p+q=k}\dim E_\infty^{p,q}$$

and therefore $E_1^{p,q} = E_\infty^{p,q}$ for any p,q.

Now consider the spectral sequence of the complex of sheaves $i_*\bigwedge^\bullet \Lambda^{\check{B}}\otimes\mathbb{C}$ on $\check{B}$. Let $K^{p,q}_{\mathrm{aff},1} = \mathscr{C}^q(\check{\mathfrak{U}}, i_*\bigwedge^p\Lambda^{\check{B}}\otimes\mathbb{C})$ (where $\check{U}$ denotes a Leray open cover of $\check{B}$). The double complex $K^{\bullet,\bullet}_{\mathrm{aff},1}$ is equipped with the Čech differential δ and the differential operator $d=0$.

Since $\check{\Lambda}$ is the discrete dual local system of flat sections of the dual connection on $\mathcal{T}_{B_0}$ (see [15, Def. 1.9]), the exterior differential d on the fan picture B is in fact a zero map. Therefore, the corresponding operator d on the cone picture $\check{B}$ is also the trivial zero map. As a result, the spectral sequence degenerates already at E_1, where the E_1 page is

$H^n(\check{B},\mathbb{C})$	$H^n(\check{B},i_*\bigwedge^1\Lambda\otimes\mathbb{C})$	$\cdots\cdots$	$H^n(\check{B},i_*\bigwedge^n\Lambda\otimes\mathbb{C})$
$\vdots$			$\vdots$
$H^1(\check{B},\mathbb{C})$	$H^1(\check{B},i_*\bigwedge^1\Lambda\otimes\mathbb{C})$	$\cdots$	$H^1(\check{B},i_*\bigwedge^n\Lambda\otimes\mathbb{C})$
$H^0(\check{B},\mathbb{C})$	$H^0(\check{B},i_*\bigwedge^1\Lambda\otimes\mathbb{C})$	$\cdots$	$H^0(\check{B},i_*\bigwedge^n\Lambda\otimes\mathbb{C})$

By Theorem 3.5 and Remark 3.12, we know that this E_1 page has all terms isomorphic to that of Table 3.2.

Now we consider the spectral sequence of the complex of sheaves $\Omega^\bullet_X$ on X with $K_2^{p,q} = \mathscr{C}^q(\mathfrak{U},\Omega^p_X)$. It is the Frölicher spectral sequence, which is well known for its degeneration at E_1 when X is a Kähler manifold, yielding

$H^n(X,\mathcal{O}_X)$	$H^n(X,\Omega^1_X)$	$\cdots\cdots$	$H^n(X,\Omega^n_X)$
$\vdots$			$\vdots$
$H^1(X,\mathcal{O}_X)$	$H^1(X,\Omega^1_X)$	$\cdots$	$H^1(X,\Omega^n_X)$
$H^0(X,\mathcal{O}_X)$	$H^0(X,\Omega^1_X)$	$\cdots$	$H^0(X,\Omega^n_X)$

which is exactly in the form of the Hodge diamond rotated clockwise by 45 degrees.

Consider the spectral sequence of $i_*\bigwedge^\bullet\check{\Lambda}_0^{\check{B}}\otimes\mathbb{C}$ on $\check{B}$. The double complex $K^{p,q}_{\mathrm{aff},2} = \mathscr{C}^q(\check{\mathfrak{U}},i_*\bigwedge^p\check{\Lambda}_0\otimes\mathbb{C})$ is also equipped with the differentials δ and d. Similar to the situation of $i_*\bigwedge^\bullet\check{\Lambda}^{\check{B}}\otimes\mathbb{C}$, the differential d is trivial, so the sequence degenerates at E_1. More explicitly, the E_1 page is

$H^n(B,\mathbb{C})$	$H^n(B,i_*\bigwedge^1\check{\Lambda}_0\otimes\mathbb{C})$	$\cdots\cdots$	$H^n(B,i_*\bigwedge^n\check{\Lambda}_0\otimes\mathbb{C})$
$\vdots$			$\vdots$
$H^1(B,\mathbb{C})$	$H^1(B,i_*\bigwedge^1\check{\Lambda}_0\otimes\mathbb{C})$	$\cdots$	$H^1(B,i_*\bigwedge^n\check{\Lambda}_0\otimes\mathbb{C})$
$H^0(B,\mathbb{C})$	$H^0(B,i_*\bigwedge^1\check{\Lambda}_0\otimes\mathbb{C})$	$\cdots$	$H^0(B,i_*\bigwedge^n\check{\Lambda}_0\otimes\mathbb{C})$

By Theorem 3.18 and Remark 3.12, the above two tables have isomorphic terms.

To conclude, we have the following theorem:

Theorem 3.19. *Let $\Bbbk = \mathbb{C}$. Consider a toric degeneration $\mathfrak{X} \to T$ satisfying the hypotheses of Theorem 3.18. Suppose this toric degeneration can be extended to an algebraic family with a smooth closed general fibre $X_s = X$. Then the spectral sequences of the four complexes of sheaves*

$$\Omega_X^\bullet(\log D),\ \Omega_X^\bullet,\ i_* \bigwedge^\bullet \check{\Lambda}^{\check{B}} \otimes \mathbb{C} \ \textit{ and } \ i_* \bigwedge^\bullet \check{\Lambda}_0^{\check{B}} \otimes \mathbb{C}$$

on X and $\check{B}$ degenerate simultaneously at E_1 level.

This theorem illustrates that the cohomology theory of affine geometry on $\check{B}$ (equivalent to the affine geometry of B via the discrete Legendre transform) is well related with that of the Kähler geometry on X_s under the setting of toric degeneration with unimodularity of the monodromy around every cell τ (i.e. the monodromy polytopes satisfy the "standard simplex" condition, see [16, Thm 3.21], [32, Thm 0.1] and Theorem 0.1 in this thesis).

Chapter 4

Examples, discussion and outlook

We conclude this article with this section. In §4.1, we look at some examples of toric degenerations in low dimensions and see how the affine cohomological controls work independently in the Kähler manifold X_s (which is the closed fibre in the algebraic family (see Remark 3.12)) and the corresponding affine manifold $\check{B}$. Then in §4.2, we will discuss some "undeveloped" insights of this article and their possible outcomes.

4.1 Examples

In this section, take $\Bbbk = \mathbb{C}$. We shall concentrate on the closed fibres X_s of the (not unique) algebraic family, which are varieties over $\mathbb{C}$ (see Remark 3.12). For simplicity, we write (X, D) for the pair (X_s, D_s).

First, consider examples arising from an affine manifold B without singularities. The general fibres are actually the same as the central fibres in the following examples. In §3.3, we require the irreducibility of the divisor D in order to prove the second affine cohomological control. As we will see in the following examples, the group $H^q(B, i_* \bigwedge^p \check{\Lambda} \otimes \mathbb{C})$ is isomorphic to $H^q(X, \Omega_X^p(\log D))$ in spite of the reducibility of the divisor D, because the irreducibility condition is not used in the proof of this isomorphism (see §2 and §3.1).

Example 4.1. Let $X = \mathbb{P}^1$ and $D = \{0, \infty\}$. We know that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)) = \mathbb{C}$$

while

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)) = 0$$

using the fact that $\Omega^1_{\mathbb{P}^1}(\log D) \cong \mathcal{O}$. Therefore E_1 of $\Omega_X^\bullet(\log D)$ reads

$$\begin{array}{|cc} 0 & 0 \\ \mathbb{C} & \mathbb{C} \\ \hline \end{array}$$

We also know that the spectral sequence of $\Omega_X^\bullet$ degenerates at E_1. In fact, the E_1 page in this case reads

$$\begin{array}{|cc} 0 & \mathbb{C} \\ \mathbb{C} & 0 \\ \hline \end{array}$$

Example 4.2. Let $X = \mathbb{P}^2$ and D be the union of the 3 coordinate lines of X. Using the facts $\Omega^1_{\mathbb{P}^2}(\log D) \cong \mathcal{O}^{\oplus 2}$ and $\Omega^2_{\mathbb{P}^2}(\log D) \cong \mathcal{O}$, we have

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \cong H^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}(\log D)) = \mathbb{C}$$

and

$$H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log D)) = \mathbb{C}^2$$

and $H^q(\mathbb{P}^2, \Omega^p_{\mathbb{P}^1}(\log D))$ vanishes for $q > 0$. Thus E_1 of $\Omega^\bullet_X(\log D)$ reads

$$\begin{array}{|ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\mathbb{C} & \mathbb{C}^2 & \mathbb{C} \\
\hline
\end{array}$$

Remark 4.3. The cohomology groups $H^q(B, i_* \bigwedge^p \check{\Lambda} \otimes \mathbb{C}) \cong H^q(\check{B}, i_* \bigwedge^p \Lambda \otimes \mathbb{C})$ in the above examples can be computed trivially since B possesses no singularities so that B is contractible as an affine manifold. We see that $H^q(B, i_* \bigwedge^p \check{\Lambda} \otimes \mathbb{C})$ is isomorphic to $H^q(X, \Omega^p_X(\log D))$ in above. In Example 4.1, the calculation for $H^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C})$ is easy (with the consideration of the boundary) and we see that it is isomorphic to $H^q(X, \Omega^p_X)$. In Example 4.2, we do not expect that $H^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C})$ is isomorphic to $H^q(X, \Omega^p_X)$ for any p, q since D is not locally irreducible.

In the following, consider the examples in [5, §6], which are toric degenerations of del Pezzo surfaces. In each of the examples, the affine manifold B possesses singularities and the divisor $D = D_s$ on the general fibre is irreducible.

Example 4.4. Let $X = \mathbb{P}^2$ and D is a smooth elliptic curve E on X. Consider the long exact sequence

$$\begin{array}{cccccc}
0 & \to & H^0(\Omega^1_X) & \to & H^0(\Omega^1_X(\log D)) & \to & H^0(\mathcal{O}_D) \\
 & \to & H^1(\Omega^1_X) & \to & H^1(\Omega^1_X(\log D)) & \to & H^1(\mathcal{O}_D) \\
 & \to & H^2(\Omega^1_X) & \to & H^2(\Omega^1_X(\log D)) & \to & 0
\end{array}$$

We know that $H^0(\Omega^1_X) \cong H^2(\Omega^1_X) = 0$ and $H^1(\Omega^1_X) \cong \mathbb{C}$. Besides, $H^0(\mathcal{O}_D) \cong H^1(\mathcal{O}_D) \cong \mathbb{C}$. Note that we have the isomorphisms

$$H^k(X \setminus D, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X(\log D)),$$

from (3.12). As $X \setminus D$ is a Stein space ([11, Kap. V, §1, Satz 5]), the cohomology group $H^k(X \setminus D, \mathbb{C})$ vanishes for $k > 2$ ([11, Kap. V, §5, Satz 9]). In this way, we have also the vanishing of $H^2(\Omega^1_X(\log D))$.

To calculate the remaining cohomology groups, we can make use of the affine geometry on B. Now the pair (X, D) corresponds to [17, Fig. 1.1] or the affine manifold with three singular points in [5, Fig. 6.2] (see also [5, Thm. 6.4]). By performing a Čech cohomology computation for the sheaf $i_*\check{\Lambda} \otimes \mathbb{C}$ on B with respect to the number of singular points as in the proof of [5, Prop. 6.11], we get $H^1(\Omega^1_X(\log D)) \cong H^1(B, i_*\check{\Lambda} \otimes \mathbb{C}) \cong \mathbb{C}$. From the same proof, we also obtain $H^0(\Omega^1_X(\log D)) \cong H^2(\Omega^1_X(\log D)) \cong 0$ using the affine geometry (see Remark 4.7).

Consider now the sheaf cohomology groups of $\Omega^2_X(\log D)$. Using the property of $X \setminus D$ as a Stein space again, it follows $H^q(\Omega^2(\log D))$ vanishes for $q > 0$. Eventually, the cohomology group $H^0(\Omega^2(\log D))$ can be obtained by a long exact sequence similar to above.

Hence E_1 of $\Omega^\bullet_X(\log D)$ reads

$$\begin{array}{|ccc} 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 \\ \mathbb{C} & 0 & \mathbb{C} \\ \hline \end{array}$$

Moreover, the E_1 page of $\Omega^\bullet_X$ is

$$\begin{array}{|ccc} 0 & 0 & \mathbb{C} \\ 0 & \mathbb{C} & 0 \\ \mathbb{C} & 0 & 0 \\ \hline \end{array}$$

which is well known.

Example 4.5. Consider the del Pezzo surface $X = dP_k$ and divisor $D = -K_{dP_k}$ with $k \geq 1$, where D is a smooth irreducible curve in X and k is the number of times of blowing-ups of $\mathbb{P}^2$ to get X. Let l be the number of focus-focus singularities of the corresponding affine manifold $\check{B}$. Then by [5, Thm. 6.4] and [5, Prop. 6.11], we have $l = k + 3$.

Using the same technique for $(X, D) = (\mathbb{P}^2, E)$, one also obtains the cohomology groups from the complex geometry as well as from the affine geometry. Thus E_1 of $\Omega^\bullet_X(\log D)$ reads

$$\begin{array}{|ccc} 0 & 0 & 0 \\ 0 & \mathbb{C}^{l-2} & 0 \\ \mathbb{C} & 0 & \mathbb{C} \\ \hline \end{array}$$

Moreover, E_1 of $\Omega^\bullet_X$ of course reads

$$\begin{array}{|ccc} 0 & 0 & \mathbb{C} \\ 0 & \mathbb{C}^{l-2} & 0 \\ \mathbb{C} & 0 & 0 \\ \hline \end{array}$$

Proposition 4.6. *Consider an affine manifold $\check{B}$ of dimension 2 with l focus-focus singularities and a nonempty boundary $\partial\check{B}$ as in Construction 3.14. Then the Čech cohomology groups have the following form*

$$\check{H}^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C}) \cong \begin{cases} \mathbb{C}^{l-2} & \text{for } (p,q) = (1,1), \\ \mathbb{C} & \text{for } (p,q) = (0,0) \text{ or } (2,2), \\ 0 & \text{for } (p,q) \text{ otherwise.} \end{cases}$$

In particular, the cohomology $\check{H}^1(\check{B}, i_*\Lambda_0 \otimes \mathbb{C})$ depends on the number of focus-focus singularities on $\check{B}$.

Proof. The above statement about $\check{H}^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C})$ is trivially true for $p = 0$ when $\check{B}$ is a disk. The proof is similar to that of [5, Prop. 6.11] but we take a different open cover so that the cover is a Leray cover with respect to $i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C}$.

In the following, assume that $\check{B}$ possesses l focus-focus singularities. Consider open sets U_0, $U_1, \ldots, U_l$, $V_1, \ldots, V_l$ which satisfy following conditions (i) - (v). We use the notations $\mathfrak{U} = \{U_0, U_1, \ldots, U_l, V_1, \ldots, V_l\}$, $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$ (similarly also for V_{ij}).

(i) Each of the open sets $U_1, U_2, \ldots, U_l$ contains exactly one singular point and $U_i \cap \partial B = \emptyset$ for all i.

(ii) The singular points lie on the boundary ∂U_0 of the open set U_0.

(iii) The union of V_1, $V_2, \ldots, V_l$ contains the boundary ∂B of the affine manifold while each of the sets U_1, $U_2, \ldots, U_l$ does not intersect ∂B.

(iv) Each of the open sets V_1, $V_2, \ldots, V_l$ does not contain any singular point.

(v) The open set U_{ijk} is empty for j, k pairwise different unless $i = 0$.

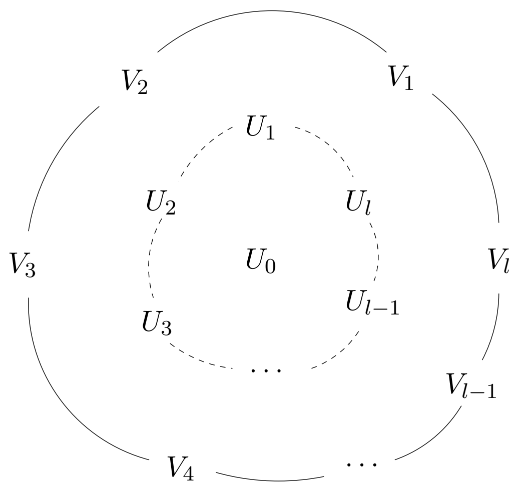


Figure 4.1: A diagram showing the relations between the open sets in the Leray cover for the general case.

First consider $i_*\Lambda_0 \otimes \mathbb{C}$. It is true that $\Gamma(U_0, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^2$ since there is no singular point for the affine structure in U_0; for each $1 \le r \le l$, we have $\Gamma(U_r, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}$ because there is an obstruction to one direction of the global sections of the vector fields due to the focus-focus singularity (the other direction is not obstructed, see [18, §2.2]); by the definition of Λ_0, one direction of the global sections of the vector fields is obstructed at the boundary $\partial\check{B}$, hence $\Gamma(V_r, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}$ for every $1 \le r \le l$. Therefore, we have

$$C^0(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^{2+2l}.$$

We see that U_{ij} is nonempty for $(i,j) = (0,r)$ for $1 \le r \le l$ and $(i,j) = (l,1)$ and $(r, r+1)$ for $1 \le r \le l-1$ with $\Gamma(U_{ij}, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^2$. Similarly, $U_r \cap V_r$ $(1 \le r \le l)$, $U_r \cap V_{r+1}$ $(1 \le r \le l-1)$ and $U_l \cap V_1$ are nonempty with $\Gamma(i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^2$. V_{ij} is also nonempty for $(i,j) = (l,1)$ or $(r, r+1)$ $(1 \le r \le l-1)$ yet with $\Gamma(i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}$ (due to the obstruction from the boundary). Thus, we have

$$C^1(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^{9l}.$$

Besides, for $V = U_{0,r,r+1}$, $U_{0,l,1}$, $U_{r,r+1} \cap V_{r+1}$, $U_{l1} \cap V_1$, $U_l \cap V_{l1}$ and $U_r \cap V_{r,r+1}$ (for $1 \le r \le l-1$), we have $\Gamma(V, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^2$ for these open sets. Hence,

$$C^2(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^{6l}.$$

It is easy to see that the map $C^0(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C}) \to C^1(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C})$ is injective while the map $C^1(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C}) \to C^2(\mathfrak{U}, i_*\Lambda_0 \otimes \mathbb{C})$ is surjective. Consequently, we have $\check{H}^1(\check{B}, i_*\Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}^{l-2}$ and $\check{H}^q(\check{B}, i_*\Lambda_0 \otimes \mathbb{C}) \cong 0$ for $q = 0, 2$.

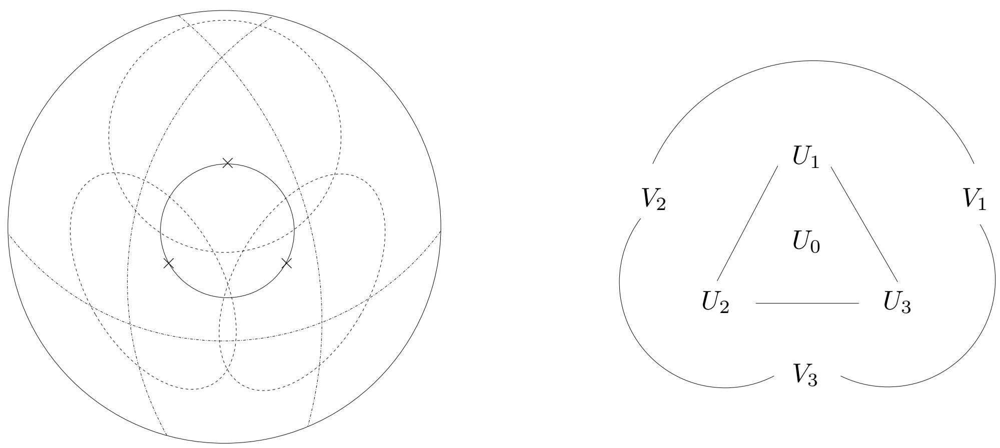


Figure 4.2: The Leray cover for the case $l = 3$ and a diagram showing the relations between the open sets.

Consider now $i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}$ using the same cover $\mathfrak{U}$. Consider the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

due to parallel transport counterclockwise around a focus-focus singularity given in [18, §2.2]. Then we have $\partial_{y'} = \partial_y$ and $\partial_{x'} = \partial_x - \partial_y$ so that $\partial_x \wedge \partial_y$ is invariant with respect to the parallel transport. Consequently one has $\Gamma(U_r, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}$ for $0 \leq r \leq l$. By the definition of Λ_0, the term $\partial_x \wedge \partial_y$ does not exist at the boundary, therefore $\Gamma(V_r, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \cong 0$ for $1 \leq r \leq l$. With such a cover, one is able to get

$$\begin{aligned} C^0(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) &\cong \mathbb{C}^{1+l}, \\ C^1(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) &\cong \mathbb{C}^{4l}, \\ C^2(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) &\cong \mathbb{C}^{3l}. \end{aligned}$$

The map $C^0(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \to C^1(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C})$ is injective. For the map $C^1(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \to C^2(\mathfrak{U}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C})$, the kernel is $\mathbb{C}^{1+l}$ and the image is $\mathbb{C}^{3l-1}$ so that the map is not surjective. Therefore, $\check{H}^2(\check{B}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \cong \mathbb{C}$ and $\check{H}^q(\check{B}, i_* \bigwedge^2 \Lambda_0 \otimes \mathbb{C}) \cong 0$ for $q = 0, 1$.

□

Remark 4.7. Some of the Čech cohomology groups for $i_* \bigwedge^\bullet \Lambda^{\check{B}} \otimes \mathbb{C}$ are actually computed in the fan picture B in [5, Prop. 6.11]. The computations are essentially the same in the cone picture $\check{B}$ by ignoring the boundary $\partial \check{B}$ due to the definition of the local system $\Lambda^{\check{B}}$ (see Construction 3.14). Indeed, we can use the open cover $\tilde{\mathfrak{U}} = \{U_0, U_1, \ldots, U_l\}$ to compute

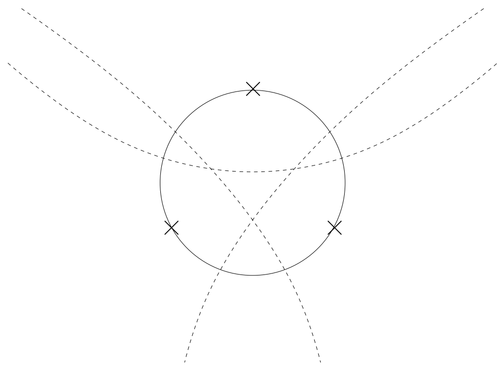


Figure 4.3: A Leray cover for computation of $\check{H}^q(\check{B}, i_* \bigwedge^p \Lambda \otimes \mathbb{C})$ for $l = 3$.

these cohomology groups for $\check{B}$, where we extend the open sets $U_1, U_2, \ldots, U_l$ (U_0 remains unchanged) in Proposition 4.6 to cover the boundary $\partial \check{B}$. With this observation and using similar methods as in the above proposition, it is not hard to verify that the Čech cohomology groups have the following form

$$\check{H}^q(\check{B}, i_* \bigwedge^p \Lambda \otimes \mathbb{C}) \cong \begin{cases} \mathbb{C}^{l-2} & \text{for } (p,q) = (1,1), \\ \mathbb{C} & \text{for } (p,q) = (2,0), \\ 0 & \text{for } (p,q) \text{ otherwise.} \end{cases}$$

Note that the map $\delta : C^0(\mathfrak{U}, i_* \bigwedge^2 \Lambda \otimes \mathbb{C}) \to C^1(\mathfrak{U}, i_* \bigwedge^2 \Lambda \otimes \mathbb{C})$ is not injective with

$$\ker \delta = \{(\omega_1, \omega_2, \dots, \omega_l) \mid \omega_1 = \cdots = \omega_l\} \cong \mathbb{C},$$

where $\omega_i = \partial_x \wedge \partial_y$ in the open sets U_i with local affine coordinates x, y.

Similar to the proof of [5, Prop 6.11] for $H^q(B, i_* \bigwedge^p \Lambda \otimes \mathbb{C})$, the open cover $\mathfrak{U}$ for the computation of the cohomology groups $H^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes \mathbb{C})$ is actually analogous to the acyclic cover $\check{\mathscr{W}}$ for $\check{B}$ in [15, Lem 5.5].

4.2 Discussion and outlook

1. One needs a polyhedral decomposition $\mathscr{P}$ on B in order to construct the central fibre X_0 in a toric degeneration. The isomorphisms

$$\begin{aligned} H^q(X_\eta, \Omega^p_{X_\eta}) &\cong H^q(\check{B}, i_* \bigwedge^p \Lambda_0 \otimes k(\eta)), \\ H^q(X_\eta, \Omega^p_{X_\eta}(\log D)) &\cong H^q(\check{B}, i_* \bigwedge^p \Lambda \otimes k(\eta)) \end{aligned}$$

   nonetheless forget the polyhedral decomposition on B. In other words, $(B, \mathscr{P})$ and $X_0^\dagger$ serve as a transition to get nice properties between B and X_η and the isomorphisms seem to be independent of $\mathscr{P}$. This is of course the philosophy of [15, 16] that the exchange of Hodge number between X_η and $\check{X}_\eta$ is independent of the polyhedral decomposition $\mathscr{P}$ on B, although the discrete Legendre transform between B and $\check{B}$ and isomorphisms between the cohomology groups of X_η, $X_0^\dagger$ and B (and similarly for the mirror $\check{X}_\eta$) depend on $(B, \mathscr{P})$.

2. The isomorphisms in (1) enable us to determine the cohomology of X_s by calculating the Čech cohomology on $\check{B}$. It is expected the computation of the Čech cohomology on an affine manifold $\check{B}$ has deeper applications in higher dimensions (dim ≥ 3) under the framework of toric degeneration, in which the tropical geometry related to the singularities Δ will play a more significant role.

   In dimension 3, for example, it would be essential first to search for criteria to sort out Fano varieties X_s that could be the smooth closed fibres of the algebraic families induced by toric degenerations. Once this is done, we can study the relation between the algebraic geometry on X_s and the affine geometry on B. In dimension 2, recall that blowing up a point on a del Pezzo surface (as a smooth closed fibre of a toric degeneration) is equivalent to adding one more focus-focus singularity on $\check{B}$. Analogously, in dimension 3, the change of the singularities $\check{\Delta}$ in $\check{B}$ (in the perspective of [6, Assump. 2.2]) when blowing up a point in a Fano variety could be studied.

   Together with the framework of toric degeneration, the investigation of the tropical geometry (e.g. Mikhalkin and Zharkov considered the curve case in [25]) is expected to help with the construction of new Fano varieties in higher dimensions. In particular, the Hodge numbers of such Fano varieties can be computed via the affine geometry on $\check{B}$.

3. In regard to mirror symmetry, one is interested in finding the mirror (the Landau-Ginzburg model) of a given Fano variety X_η (or some varieties with weaker conditions on their anticanonical divisors). The mirror is written as $(\check{X}_\eta, W)$, where $W\colon \check{X}_\eta \to \mathbb{C}$ is a regular function called Landau-Ginzburg potential. The classical consideration of the relevant cohomology theory on the mirror side (due to an unpublished work of Barannikov and Kontsevich and eventually [29, 33]) is to consider the *twisted de Rham complex* of $\check{X}_\eta$
$$(\Omega^\bullet_{\check{X}_\eta},\ d + dW\wedge)$$
with the help of the potential W (see [14, §2.2.1]).

   In this thesis, the degeneration X_0 of a Fano variety X_η is seen as two different log spaces $X_0^\dagger$ and $\check{X}_0^\dagger$. Each consideration contributes to the computation of one type of cohomology groups. Namely, considering $X_0^\dagger$ computes the log Dolbeault cohomology $H^q(X_\eta, \Omega^p(\log D_\eta))$ on X_η while considering $\check{X}_0^\dagger$ computes the usual Dolbeault cohomology on X_η.

   Under this viewpoint, it is expected that there exists another corresponding cohomology theory on the mirror side $\check{X}_\eta$ besides the one expected from the twisted de Rham complex. Moreover, the identification between the log structures and cohomology theories (if it exists) on $\check{X}_\eta$ would be the topic for later work.

4. The degeneration of the spectral sequences of the complexes of sheaves $\Omega^\bullet_{X_\eta}(\log D_\eta)$ on X_η and $\Omega^\bullet_{X_s}(\log D_s)$ on X_s at E_1 level is a trivial consequence along the construction of a toric degeneration (and in the sense of Remark 3.12). In general, when a toric degeneration does not exist for a compact Kähler manifold X_s, the degeneration of the spectral sequence is nevertheless true (see [8, Cor. 3.2.13] or [36, Thm. 8.35]). As a result, the construction of a toric degeneration should have some more fundamental meaning in homological algebra and the underlying geometry of the closed fibre X_s as well as the generic fibre X_η. It is believed that further constructions concerning the limiting mixed Hodge structure as in [21] by Katz and Stapledon of the two types of cohomology theories on $\mathfrak{X}$ would be possible.

5. Consider a cell $\tau \in \mathscr{P}$ with $1 \leq \dim\tau \leq n-1$. In [15, Def. 1.60], the authors consider the convex hull of the polytopes
$$\mathrm{Conv}\Big(\bigcup_{i=1}^{q} \check{\Delta}_i \times \{e_i\}\Big)$$

and denote it by $\check{\Delta}(\tau)$ so as to define when $(B, \mathscr{P})$ is simple. Geometrically, the Newton polytopes $\check{\Delta}_i$ capture the "outer monodromy" of a cell τ (see [16, notational summary]). In [16] and this article, it is assumed that $\check{\Delta}(\tau)$ is a *standard simplex* for every cell $\tau \in \mathscr{P}$ [1] so that we can obtain the required cohomology vanishing results for the resolution of Ω^r on X_0. Eventually, we could generalize the nice results [16, Thm. 3.21] and Theorem 3.5 in this article.

In [32], the author studies how the cohomology theories on X_0 behave under weaker assumptions on $\check{\Delta}(\tau)$. In fact, we have the following implications about the assumptions:

$$\text{standard} \Rightarrow \text{elementary} \underset{\neq}{\Rightarrow} \text{simplicial} \Rightarrow \text{general},$$

where a polytope is *elementary* if it does not contain any interior integral points and a polytope is *simplicial* if all of its proper faces are simplices. In particular, the first implication is in fact equivalent for $\dim B \leq 3$. Two classes of central fibre X_0, *hypersurface type* and *complete intersection type*, are defined and investigated; the former corresponds to $\check{\Delta}(\tau)$ being elementary ([32, Thm. 1.13]) and the latter corresponds to $\check{\Delta}(\tau)$ being simplicial ([32, Thm. 1.6]). In particular, the results concerning hypersurface type are especially beautiful and concise (e.g. Theorem 0.1, 1.13 and 1.15 in [32]). Also, a relation between the stringy Hodge numbers defined in [1, 2] and ordinary Hodge numbers is established (see [32, Thm. 1.15]).

It is expected the approach in [32] can be applied under the setting of this article. Especially, the author is unsure at this moment if there would be any relation between the cohomology theory $H^*_{\mathrm{st}}(X_\eta)$ (in the sense of [2]) and the cohomology theories in this article or rather a mix of the theories of this article and that of [32].

[1] Equivalently, the monodromy is unimodular around every cell $\tau \in \mathscr{P}$ (see Remark 0.1 and the remarks after it).

Chapter 5

Appendix

5.1 Statements about the log structure on $\breve{X}_0^\dagger$

This appendix serves to prove statements relating the log structures on the log spaces $X_0^\dagger$ and $\breve{X}_0^\dagger$. First, it supplies complementary information to §1.4 about the local description about the log structure on $\breve{X}_0^\dagger$. Second, we can also see why the irreducibility assumption of the divisor D_η on the general fibre of a toric degeneration is a reasonable one.

Analogous to §2.4, it is possible to calculate $H^p(X_0, \breve{\Omega}^r)$ by constructing an acyclic resolution for $\breve{\Omega}^r$ and proving related vanishing theorems. It is nevertheless much more lengthy to calculate $H^p(X_0, \breve{\Omega}^r)$ than $H^p(X_0, \Omega^r)$ and so we shall omit this calculation in this article.

Let $(B, \mathscr{P})$ be a positive and simple integral affine manifold with singularities and a polyhedral decomposition $\mathscr{P}$. Let s be open gluing data for $(B, \mathscr{P})$, yielding $X_0 := X_0(B, \mathscr{P}, s)$. This s together with the condition (LC) (see [15, Prop. 4.25]) also determines the log structure $X_0^\dagger$ on X_0 over $\operatorname{Spec} \Bbbk^\dagger$ with singular set $Z \subseteq X_0$. Take $\breve{\Omega}^r$ to be the sheaf on X_0 which is $j_*\Omega^r_{\breve{X}_0^\dagger/\Bbbk}$ and $j_*\Omega^r_{\breve{X}_0^\dagger/\Bbbk^\dagger}$ in the $/\Bbbk$ and $/\Bbbk^\dagger$ cases respectively, where $j \colon X_0 \setminus Z \to X_0$ is the inclusion. This sheaf is described étale locally in Remark 1.13 (2), which will be further elaborated in this section. Let $q_\tau \colon X_\tau \to X_0$ be the usual inclusion of strata maps, D_τ the toric boundary of X_τ and let

$$\kappa_\tau \colon X_\tau \setminus (D_\tau \cap q_\tau^{-1}(Z)) \to X_\tau$$

be the inclusions. We then define

$$\breve{\Omega}^r_\tau := \kappa_{\tau *}\kappa_\tau^*(q_\tau^*\breve{\Omega}^r / \operatorname{Tors}),$$

where Tors denotes the torsion subsheaf of $q_\tau^*\breve{\Omega}^r$. All these constructions is done in §2.4 for the sheaf Ω^r.

Consider an irreducible component X_v of X_0 for a vertex v of $\mathscr{P}$. For a vertex v without outgoing unbounded rays, $\mathfrak{D}$ does not intersect X_v, or equivalently X_v does not contain any irreducible components of D, one sees immediately that $\Omega^r_v = \breve{\Omega}^r_v$ in this case and the properties of Ω^r_v are the same as before, which is described in [16, §3.2].

From now on in this section, consider a vertex v always with an outgoing unbounded ray. Pull back the log structures $X_0^\dagger$ and $\breve{X}_0^\dagger$ on X_0 via q_v to obtain log structures on $X_v \setminus q_v^{-1}(Z)$. Denote the sheaves of monoids of both by $\mathcal{M}_v$ and $\breve{\mathcal{M}}_v$ respectively.

Recall that in §2.4 we consider $j_*\mathcal{M}_v \to j_*\mathcal{O}_{X_v \setminus q_v^{-1}(Z)} = \mathcal{O}_{X_v}$, which determines a log structure on X_v that is written as $X_v^\dagger$.

Now $j_*\breve{\mathcal{M}}_v \to j_*\mathcal{O}_{X_v \setminus q_v^{-1}(Z)} = \mathcal{O}_{X_v}$ determines another log structure on X_v, which is denoted as $\breve{X}_v^\dagger$. Write $\breve{\mathcal{M}}_v$ also for $j_*\breve{\mathcal{M}}_v$. In the following, we describe the log differential $\Omega^1_{\breve{X}_v^\dagger/\Bbbk}$ with the help of $\Omega^1_{X_v^\dagger/\Bbbk}$.

Proposition 5.1. *Suppose that a toric degeneration of $(X_0^\dagger, D)$ exists and the divisor D_η is irreducible in the general fibre X_η. Let $v \in \mathscr{P}$ be a vertex and D_μ be the unique component of D contained in the toric stratum X_v. Then we have the following short exact sequence*

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_{\breve{X}_v^\dagger/\Bbbk^\dagger} & \to & \Omega^1_{X_v^\dagger/\Bbbk^\dagger} & \to & \mathcal{O}_{D_\mu} & \to & 0 \\ & & & & \omega & \mapsto & \eta\,|_{D_\mu}, & & \end{array} \tag{5.1}$$

where $\omega = \eta \wedge \frac{dz_1}{z_1} + \eta'$ and η' does not contain dz_1 and $(z_1, \ldots, z_n)$ is a choice of local coordinates with $D_\mu = \{z_1 = 0\}$. Analogously, there is also a short exact sequence for the $/\Bbbk$ case

$$0 \to \Omega^1_{\breve{X}_v^\dagger/\Bbbk} \to \Omega^1_{X_v^\dagger/\Bbbk} \to \mathcal{O}_{D_\mu} \to 0\,. \tag{5.2}$$

Proof. Suffice to prove the statement étale locally. Fix $e\colon v \to \sigma \in \mathscr{P}_{\max}$ and we view $V(\sigma) = \operatorname{Spec} \Bbbk[P_\sigma]/(z^\rho)$ as an open subset of X_0. (See [15, Def. 2.12] for P_σ.) However, it can happen that the divisor D does not occur in the neighbourhood $X_v \cap V(\sigma)$ (under étale identification $p_\sigma : V(\sigma) \to X_0 \supseteq X_v$), and the sheaves $\Omega^1_{X_v^\dagger/\Bbbk^\dagger}$ and $\Omega^1_{\breve{X}_v^\dagger/\Bbbk^\dagger}$ are isomorphic in the neighbourhood in this case. The above happens when either σ is bounded or v has no outgoing unbounded rays in an unbounded maximal cell σ.

Hence we assume now the maximal cell σ is unbounded and there exists at least an unbounded 1-cell emerging from v. This unbounded 1-cell is then unique as D_η is irreducible (cf. Remark 5.2). Let D_μ be the unique component of D contained in X_v. Hence, there is only one unbounded 1-cell τ' emerging from v and we denote $e'\colon \tau' \to \sigma$.

Denote the maximal proper face of P_σ corresponding to $X_v \cap V(\sigma)$ (under étale identification) by P_e, i.e. $X_v \cap V(\sigma) \cong \operatorname{Spec} \Bbbk[P_e]$; similarly denote the proper face of P_σ corresponding to $(X_v \cap V(\sigma)) \cap \operatorname{Spec} \Bbbk[\check{\tau}' \cap (N \oplus \mathbb{Z})])$ by $P_{e'}$ (i.e. $\check{\tau}' \cap (N \oplus \mathbb{Z}) = P_{e'}$), following [15, Constr. 2.15] by considering $\tau' \subseteq \sigma$.

By the above definition, it is easy to see that $P_{e'} \otimes_{\mathbb{Z}} \mathbb{Q}$ is one dimension lower than $P_e \otimes_{\mathbb{Z}} \mathbb{Q}$ and $P_{e'}$ is contained in P_e as a proper face. Consequently, we can conclude $P_{e'}$ corresponds to $D \cap (X_v \cap V(\sigma))$ and $\mathcal{O}_D \cong ((P_{e'} \cap P_e) \otimes \mathcal{O}_{X_v}|_D)$ étale locally.

Therefore, in the open set $X_v \cap V(\sigma)$ of X_v, $D_\mu = D \cap X_v$ can be expressed locally as the zeros of the monomial z_1 with a choice of local coordinates $(z_1, \ldots, z_n)$. Therefore, we have the exact sequence by consideration of the residue map

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_{\check{X}_v^\dagger/\Bbbk^\dagger} & \to & \Omega^1_{X_v^\dagger/\Bbbk^\dagger} & \to & \mathcal{O}_{D_\mu} & \to & 0 \\ & & & & \omega & \mapsto & \eta|_{D_\mu}, & & \end{array} \tag{5.3}$$

where $\omega = \eta \wedge \frac{dz_1}{z_1} + \eta'$ and η' does not contain dz_1.

□

Remark 5.2. 1. According to [5, Prop. 2.1], the unbounded rays $\tau, \tau' \subseteq \sigma$ of any cell σ have to be parallel (i.e. $\Lambda_\tau = \Lambda_{\tau'}$) so that D_η is irreducible in the general fibre X_η and the Landau-Ginzburg mirror $\check{X}_0$ has a proper superpotential $W^0 : \check{X}_0 \to \Bbbk$. With the parallel assumption, maximal only one unbounded ray can emerge from a vertex v, and so we have the above exact sequence (5.1). If this condition is dropped, not only the open set $V(\sigma)$ would be nonreduced, but we would have also an exact sequence of the form

$$0 \to \Omega^m_{\check{X}_v^\dagger/\Bbbk^\dagger} \to \Omega^m_{X_v^\dagger/\Bbbk^\dagger} \to \mathcal{O}_{D_I} \to 0,$$

where m is number of unbounded rays emerging from the vertex (which is also the number of components of D_μ in X_v), $D_I = D_{i_1} \cap \cdots \cap D_{i_m}$ and $D_v = D_1 \cup \cdots \cup D_N$ is the toric boundary (cf. [30, Def. 4.5]). Note that the above proposition implies in general

$$0 \to \Omega^\bullet_{\check{X}_v^\dagger/\Bbbk^\dagger} \to \Omega^\bullet_{X_v^\dagger/\Bbbk^\dagger} \to \Omega^{\bullet-1}_D \to 0$$

(cf. (8.8) in [36, §8.4.2]). To summarize, we would not have a straightforward relation between $\Omega^\bullet_{X_v^\dagger/\Bbbk^\dagger}$, $\Omega^\bullet_{\check{X}_v^\dagger/\Bbbk^\dagger}$ and $\Omega^{\bullet-1}_D$ without the irreducibility of D_η.

2. We saw in the above proof that $P_{e'}$ corresponds étale locally to $D \cap (X_v \cap V(\sigma))$. Consider v_l^δ and $v_{k_l} = v$ such that τ' is generated by $l_v := v_l^\delta - v_{k_l}$. This ray $l_v \subseteq C(\sigma)$ corresponds étale locally to $\mathfrak{D}$ in $\mathfrak{X}$. We can also define V_{l_v} as $\operatorname{Spec} \Bbbk[P \cap l_v^\perp]$, which is exactly $\tilde{D} \subseteq Y$.

Lemma 5.3. *Let $v \in \mathscr{P}$ be a vertex. Then $\breve{\Omega}^r_v$ is naturally isomorphic to $\Omega^r_{\check{X}_v^\dagger/\Bbbk}$ or $\Omega^r_{\check{X}_v^\dagger/\Bbbk^\dagger}$ in the $/\Bbbk$ and $/\Bbbk^\dagger$ cases respectively.*

Proof. We'll do the $/\Bbbk$ case, the $/\Bbbk^\dagger$ case being similar. Functoriality of log differentials gives a map $q_v^* : q_v^* \breve{\Omega}^1 \to \Omega^1_{\check{X}_v^\dagger/\Bbbk}$ on $X_v \setminus q_v^{-1}(Z)$. This map is injective since it is generically injective and $q_v^* \breve{\Omega}^1$ is locally free on each affine étale neighbourhood $X_v \cap V(\sigma)$ of $X_v \setminus q_v^{-1}(Z)$ for $\sigma \in \mathscr{P}_{\max}$. Hence, suffice to prove the surjectivity of the map q_v^*.

As observed in the proof of Proposition 5.1, we only need to consider $e\colon v \to \sigma \in \mathscr{P}_{\max}$ with the maximal cell σ being unbounded. Let P_e be the maximal proper face of P_σ corresponding to $X_v \cap V(\sigma)$. Let τ' be the unbounded 1-cell emerging from v and denote $P_{e'} = \check{\tau}' \cap (N \oplus \mathbb{Z})$ where $e'\colon \tau' \to \sigma$ (cf. [15, Constr. 2.15]). View $V(\sigma) = \operatorname{Spec} \Bbbk[P_\sigma]/(z^\rho)$ as an open subset of X_0. Consider the neighbourhood $X_v \cap V(\sigma)$.

Different from the proof of [16, Lem. 3.12], the log structure on $V(\sigma)\setminus Z$ corresponding to $\mathcal{M}_X$ is not given by charts φ_i on an open cover $\{U_i\}$ of $V(\sigma)\setminus Z$, $\varphi_i\colon P_\sigma\to\mathcal{O}_{U_i}$ a monoid homomorphism.

For $1\le j\le k-1$, let $e_j\colon\tau_j\to\sigma$ be the remaining unbounded 1-cells in σ and denote $\breve{P}_\sigma := P_{e'}\cup P_{e_1}\cup\cdots\cup P_{e_{k-1}}$, which is a fine monoid. Hence, we have a monoid homomorphism $\alpha_i\colon\breve{P}_\sigma\to\mathcal{O}_{U_i}$ which is induced by restricting $\varphi_i\colon P_\sigma\to\mathcal{O}_{U_i}$ on $\breve{P}_\sigma$. By [15, Def. 3.4], we obtain a log structure on open cover $\{U_i\}$ of $V(\sigma)\setminus Z$ associated with the monoid $\breve{P}_\sigma$. This log structure possesses charts étale locally, so it is also a fine log structure. It is the log structure to be considered.

Restricting these charts α_i to $U_i\cap X_v$ gives charts $\alpha_i\colon\breve{P}_\sigma\to\mathcal{O}_{U_i\cap X_v}$, which can be easily shown to be of the form (following the proof of [15, Lem. 5.13])

$$p\mapsto\begin{cases}0 & p\notin P_{e'}\\ h_pz^p & p\in P_{e'},\end{cases}$$

where $P_{e'}\ni p\mapsto h_p\in\mathcal{O}^\times_{U_i\cap X_v}$ is a monoid homomorphism. Note that P_e contains $P_{e'}$ with a corank of 1. (cf. charts $\varphi_i : P_\sigma\to\mathcal{O}_{U_i\cap X_v}$ in the proof of [16, Lem. 3.12])

This chart lifts to a monoid homomorphism $\alpha_i\colon\breve{P}_\sigma\to\breve{\mathcal{M}}_{U_i}$, so for $p\in P_{e'}$, $\operatorname{dlog}(\alpha_i(p))\in\Gamma(U_i,\Omega^1)$ pulls back via q_v^* to $\operatorname{dlog}(h_pz^p)=\frac{d(h_p)}{h_p}+\operatorname{dlog}(z^p)$ in $\Omega^1_{\breve{X}_v^\dagger/\Bbbk}$. By extending h_p to U_i, we see dh_p is in the image of $q_v^*\breve{\Omega}^1\to\Omega^1_{\breve{X}_v^\dagger/\Bbbk}$, so $\operatorname{dlog} z^p$ is also for all $p\in P_{e'}^{\mathrm{gp}}$. On the other hand, $\operatorname{dlog}\rho$ clearly pulls back to $\operatorname{dlog}\rho\in\Omega^1_{\breve{X}_v^\dagger/\Bbbk}$. Thus q_v^* is surjective on each U_i, hence on $X_v\setminus q_v^{-1}(Z)$.

Now on $X_v\setminus q_v^{-1}(Z)=X_v\setminus(D_v\cap q_v^{-1}(Z))$, $\breve{\Omega}^1_v=\kappa_{v*}\kappa_v^*(q_v^*\breve{\Omega}^1/\operatorname{Tors})=\kappa_{v*}\kappa_v^*q_v^*\breve{\Omega}^1$, so we get an isomorphism on X_v

$$\breve{\Omega}^1_v=\kappa_{v*}\kappa_v^*q_v^*\breve{\Omega}^1\to\kappa_{v*}(\Omega^1_{\breve{X}_v^\dagger/\Bbbk})=\Omega^1_{\breve{X}_v^\dagger/\Bbbk},$$

the latter equality as X_v is S_2 and $q_v^{-1}(Z)$ is of codimension at least two in X_v. Similarly, we obtain $\breve{\Omega}^r_v\cong\Omega^r_{\breve{X}_v^\dagger/\Bbbk}$. □

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Zusammenfassung

Gross und Siebert haben ,,torische Entartungen" (toric degenerations) eingeführt, um ein besseres Verständnis der Spiegelsymmetrie von Calabi-Yau-Varietäten zu gewinnen. Eine der Hauptideen ist, die torische Entartung aus Daten der entsprechenden unterliegenden affinen Mannigfaltigkeit B aufzubauen. Mithilfe der logarithmischen algebraischen Geometrie war es Gross und Siebert möglich, die Isomorphie der Dolbeault-Kohomologiegruppen der Glättung und der affinen Hodgegruppen von B zu beweisen. Diese Dissertation ist der Versuch, die Entartungskonstruktion auch auf Varietäten mit effektiven antikanonischen Garben (zum Beispiel Fano-Varietäten) zu erweitern. Es wird bewiesen, dass Isomorphismen zwischen den zwei Typen von Dolbeault-Kohomologiegruppen der Glättung (gewöhnliche und logarithmische Dolbeaultgruppen) und den entsprechenden affinen Hodgegruppen von B unter bestimmten technischen Voraussetzungen existieren. Unter den gleichen Voraussetzungen entarten die vier zugehörigen Spektralfolgen gleichzeitig bei E_1.

Lebenslauf

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