The excluded minor structure theorem, and linkages in large graphs of bounded tree-width

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On the excluded minor structure theorem for graphs of large treewidth

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Abstract

At the core of the Robertson-Seymour theory of graph minors lies a powerful structure theorem which captures, for any fixed graph H, the common structural features of all the graphs not containing H as a minor. Robertson and Seymour prove several versions of this theorem, each stressing some particular aspects needed at a corresponding stage of the proof of the main result of their theory, the graph minor theorem.

We prove a new version of this structure theorem: one that seeks to combine maximum applicability with a minimum of technical ado, and which might serve as a canonical version for future applications in the broader field of graph minor theory. Our proof departs from a simpler version proved explicitly by Robertson and Seymour. It then uses a combination of traditional methods and new techniques to derive some of the more subtle features of other versions as well as further useful properties, with substantially simplified proofs.

1 Introduction

Graphs in this paper are finite and may have loops and multiple edges. Otherwise we use the terminology of [6]. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges.

The theory of graph minors was developed by Robertson and Seymour, in a series of 23 papers published over more than twenty years, with the aim of proving a single result: the *graph minor theorem*, which says that in any infinite collection of finite graphs there is one that is a minor of another. As with other deep results in mathematics, the body of theory developed for the

proof of the graph minor theorem has also found applications elsewhere, both within graph theory and computer science. Yet many of these applications rely not only on the general techniques developed by Robertson and Seymour to handle graph minors, but also on one particular auxiliary result that is also central to the proof of the graph minor theorem: a result describing the structure of all graphs G not containing some fixed other graph H as a minor.

This structure theorem has many facets. It roughly says that every graph G as above can be decomposed into parts that can each be 'almost' embedded in a surface of bounded genus (the bound depending on H only), and which fit together in a tree structure [6, Thm. 12.4.11]. Although later dubbed a 'red herring' (in the search for the proof of the graph minor theorm) by Robertson and Seymor themselves [15], this simplest version of the structure theorem is the one that appears now to be best known, and which has also found the most algorithmic applications [2, 3, 4, 11].

A particularly simple form of this structure theorem applies when the excluded minor H is planar: in that case, the said parts of G—the parts that fit together in a tree-structure and together make up all of G—have bounded size, i.e., G has bounded *tree-width*. If H is not planar, the graphs G not containing H as a minor have unbounded tree-width, and therefore contain arbitrarily large grids as minors and arbitrarily large walls as topological minors [6]. Such a large grid or wall identifies, for every low-order separation of G, one side in which most of that grid or wall lies. This is formalized by the notion of a *tangle*: the larger the tree-width of G, the larger the grid or wall, the order of the separations for which this works, and (thus) the *order* of the tangle. Since adjacent parts in our tree-decomposition of G meet in only a bounded number of vertices and thus define low-order separations, our large-order tangle 'points to' one of the parts, the part G' that contains most of its defining grid or wall.

The more subtle versions of the structure theorem, such as Theorem (13.4) from Graph Minors XVII [16], now focus just on this part G' of G. Like every part in our decomposition, it intersects every other part in a controlled way. Every such intersection consists of a bounded number of vertices, of which some lie in a fixed *apex set* $A \subseteq V(G')$ of bounded size, while the others are either at most 3 vertices lying on a face boundary of the portion G_0 of G' embedded in the surface, or else lie in (a common bag of) a so-called *vortex*, a ring-like subgraph of G' that is not embedded in the surface and meets G_0 only in (possibly many) vertices of a face boundary of G_0. The precise structure of these vortices, of which G' has only boundedly many, will be the focus of our attention for much of the paper. Our theorem describes in detail both the inner structure of the vortices and the way in which they are

linked to each other and to the large wall, by disjoint paths in the surface. These are the properties that have been used in applications of the structure theorems such as [1, 7], and which will doubtless be important also in future applications. An important part of the proofs is a new technique for analyzing vortices. We note that these techniques have also been independently developed by Geelen and Huynh [9].

The basis for this paper is Theorem (3.1) from Graph Minors XVI [15], which we shall restate as Theorem 1. Together with the 'grid-theorem' that large enough tree-width forces arbitrarily large grid minors (see [6]), and a simple fact about tangles from Graph Minors X [14], these are all the results we require from the Graph Minor series.

This paper is organized as follows. In Section 2 we introduce the terminology we need to state our results, as well as the theorem from Graph Minors XVI [15] on which we shall base our proof. Section 3 explains how we can find the tree-decomposition indicated earlier, with some additional information on how the parts of the tree-decomposition overlap. Section 4 collects some lemmas about graphs embedded in a surface, partly from the literature and partly new. In Section 5 we show how a given near-embedding of a graph can be simplified in various ways if we allow ourselves to remove a bounded number of vertices (which, in applications of these tools, will be added to the apex set). Section 6 contains lemmas showing how to obtain path systems with nice properties. Section 7 contains the proof of our structure theorem. In the last section, we give an alternative definition of vortex decompositions and show that our result works with these 'circular' decompositions as well.

2 Structure Theorems

A *vortex* is a pair $V = (G, \Omega)$, where G is a graph and $\Omega =: \Omega(V)$ is a linearly ordered set $(w_1, \ldots, w_n)$ of vertices in G. These vertices are the *society vertices* of the vortex; their number n is its *length*. We do not always distinguish notationally between a vortex and its underlying graph or the vertex set of that graph; for example, a *subgraph of* V is just a subgraph of G, a *subset of* V is a subset of $V(G)$, and so on. Also, we will often use Ω to refer to the linear order of the vertices $w_1, \ldots, w_n$ as well as the set of vertices $\{w_1, \ldots, w_n\}$.

A path-decomposition $\mathcal{D} = (X_1, \ldots, X_m)$ of G is a *decomposition* of our vortex V if $m = n$ and $w_i \in X_i$ for all i. The *depth* of the vortex V is the minimum width of a path-decomposition of G that is a decomposition of V.

When $n > 1$, the *adhesion* of our decomposition $\mathcal{D}$ of V is the maximum value of $|X_i \cap X_{i+1}|$, taken over all $1 \leq i < n$. We define the *adhesion* of a

vortex V as the minimum adhesion of a decomposition of that vortex.

When $\mathcal{D}$ is a decomposition of a vortex V as above, we write $Z_i := (X_i \cap X_{i+1}) \setminus \Omega$, for all $1 \leq i < n$. These Z_i are the *adhesion sets* of $\mathcal{D}$. We call $\mathcal{D}$ *linked* if

- all these Z_i have the same size;
- there are $|Z_i|$ disjoint Z_{i-1}–Z_i paths in $G[X_i] - \Omega$, for all $1 < i < n$;
- $X_i \cap \Omega = \{w_{i-1}, w_i\}$ for all $1 \leq i \leq n$, where $w_0 := w_1$.

Note that $X_i \cap X_{i+1} = Z_i \cup \{w_i\}$, for all $1 \leq i < n$ (Fig. 1).

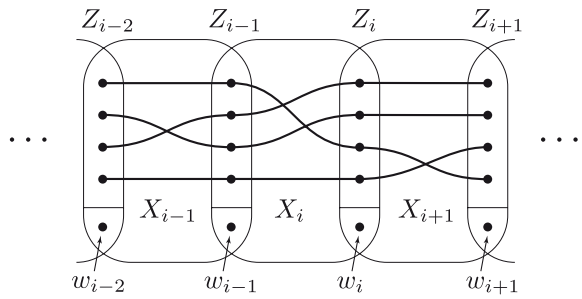

Figure 1: A linked vortex decomposition

The union over all $1 < i < n$ of the Z_{i-1}–Z_i paths in a linked decomposition of V is a disjoint union of X_1–X_n paths in G; we call the set of these paths a *linkage* of V with respect to $(X_1, \ldots, X_m)$.

Clearly, if V has a linked decomposition as above, then G has no edges between non-consecutive society vertices, since none of the X_i could contain both ends of such an edge. Conversely, if G has no such edges then V does have a linked decomposition: just let X_i consist of all the vertices of $G - \Omega$ plus w_{i-1} and w_i. We shall be interested in linked vortex decompositions whose adhesion is small, unlike in this example.

Let $V = (G, \Omega)$ be a vortex, and v a vertex of some supergraph of V. Clearly, $(G - v, \Omega \setminus \{v\})$ is a vortex, too, which we denote by $V - v$. If the length of V is greater than 2, this operation cannot increase the adhesion q of V: This is clear for $v \notin \Omega$, so suppose $\Omega = (w_1, \ldots, w_n)$ with $v = w_k$ for some $1 \leq k \leq n$. We may assume without loss of generality that $k \neq n$. Take a decomposition $(X_1, \ldots, X_n)$ of V of adhesion q. Then, it is easy to see that

$$(X_1, \ldots, X_{k-1}, (X_k \cup X_{k+1}) \setminus \{w_k\}, X_{k+1}, \ldots, X_n)$$

is a decomposition of $V - v$ of adhesion at most q. We shall not be interested in the adhesion of vortices of length at most 2. For a vertex set $A \subseteq V$ we denote by $V - A$ the vortex we obtain by deleting the vertices of A in turn. For a set of vortices $\mathcal{V}$ we define $\mathcal{V} - A := \{V - A : V \in \mathcal{V}, V - A \neq \emptyset\}$.

A (directed) *separation* of a graph G is an ordered pair (A, B) of non-empty subsets of $V(G)$ such that $G[A] \cup G[B] = G$. The number $|A \cap B|$ is the *order* of (A, B). Whenever we speak of separations in this paper, we shall mean such directed separations.

A set $\mathcal{T}$ of separations of G, all of order less than some integer θ, is a *tangle of order* θ if the following holds:

(1) For every separation (A, B) of G of order less than θ, either (A, B) or (B, A) lies in $\mathcal{T}$.

(2) If $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$, then $G[A_1] \cup G[A_2] \cup G[A_3] \neq G$.

Note that if $(A, B) \in \mathcal{T}$ then $(B, A) \notin \mathcal{T}$; we think of A as the 'small side' of the separation (A, B), with respect to this tangle.

Given a tangle $\mathcal{T}$ of order θ in a graph G, and a set $Z \subseteq V(G)$ of fewer than θ vertices, let $\mathcal{T} - Z$ denote the set of all separations (A', B') of $G - Z$ of order less than $\theta - |Z|$ such that there exists a separation $(A, B) \in \mathcal{T}$ with $Z \subseteq A \cap B$, $A - Z = A'$ and $B - Z = B'$. It is shown in [14, Theorem (6.2)] that $\mathcal{T} - Z$ is a tangle of order $\theta - |Z|$ in $G - Z$.

Given a subset D of a surface Σ, we write $\mathring{D}$, ∂D, and $\overline{D}$ for the topological interior, boundary, and closure, of D in Σ, respectively. For positive integers $\alpha_0, \alpha_1, \alpha_2$ and $\alpha := (\alpha_0, \alpha_1, \alpha_2)$, a graph G is *α-nearly embeddable* in Σ if there is a subset $A \subseteq V(G)$ with $|A| \leq \alpha_0$ such that there are integers $\alpha' \leq \alpha_1$ and $n \geq \alpha'$ for which $G - A$ can be written as the union of $n + 1$ edge-disjoint graphs $G_0, \ldots, G_n$ with the following properties:

(i) For all $1 \leq i \leq j \leq n$ and $\Omega_i := V(G_i \cap G_0)$, the pairs $(G_i, \Omega_i) =: V_i$ are vortices, and $G_i \cap G_j \subseteq G_0$ when $i \neq j$.

(ii) The vortices $V_1, \ldots, V_{\alpha'}$ are disjoint and have adhesion at most α_2; we denote the set of these vortices by $\mathcal{V}$. We will sometimes refer to these vortices as *large* vortices.

(iii) The vortices $V_{\alpha'+1}, \ldots, V_n$ have length at most 3; we denote the set of these vortices by $\mathcal{W}$. These are the *small* vortices of the near-embedding.

(iv) There are closed discs in Σ, with disjoint interiors $D_1, \ldots, D_n$, and an embedding $\sigma : G_0 \hookrightarrow \Sigma - \bigcup_{i=1}^{n} D_i$ such that $\sigma(G_0) \cap \partial D_i = \sigma(\Omega_i)$ for

all i and the generic linear ordering of Ω_i is compatible with the natural cyclic ordering of its image (i.e., coincides with the linear ordering of $\sigma(\Omega_i)$ induced by $[0,1)$ when ∂D_i is viewed as a suitable homeomorphic copy of $[0,1]/\{0,1\}$). For $i = 1, \ldots, n$ we think of the disc D_i as *accommodating* the (unembedded) vortex V_i, and denote D_i as $D(V_i)$.

We call $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ an *α-near embedding* of G in Σ, or just a *near-embedding*, with *apex set* A. For an integer α' larger than all the α_i we call $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ an *α'-near embedding*. It *captures* a tangle $\mathcal{T}$ of G if the 'large side' B' of an element $(A', B') \in \mathcal{T} - A$ is never contained in a vortex.

A direct implication of Theorem (3.1) from [15], stated in this terminology, reads as follows:

Theorem 1. *For every non-planar graph R there exist integers $\theta, \alpha \geq 0$ such that the following holds: Let G be a graph that does not contain R as a minor, and let $\mathcal{T}$ be a tangle in G of order at least θ. Then G has an α-near embedding, with apex set A say, in a surface Σ in which R cannot be drawn, and this embedding captures $\mathcal{T} - A$.*

We shall use Theorem 1 as the basis of our proofs in this paper.

Given a near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of G, let G_0' be the graph resulting from G_0 by joining any two nonadjacent vertices $u, v \in G_0$ that lie in a common small vortex $V \in \mathcal{W}$; the new edge uv of G_0' will be called a *virtual edge*. By embedding these virtual edges disjointly in the discs $D(V)$ accommodating their vortex V, we extend our embedding $\sigma\colon G_0 \hookrightarrow \Sigma$ to an embedding $\sigma'\colon G_0' \hookrightarrow \Sigma$. We shall not normally distinguish G_0' from its image in Σ under σ'.

A vortex (G_i, Ω_i) is *properly attached* if $|\Omega_i| \leq 3$ and it satisfies the following two requirements. First, for every pair of distinct vertices $u, v \in \Omega_i$ the graph G_i must contain an Ω_i-path (one with no inner vertices in Ω_i) from u to v. Second, whenever $u, v, w \in \Omega_i$ are distinct vertices (not necessarily in this order), there are two internally disjoint Ω_i-paths in G_i linking u to v and v to w, respectively.

Clearly, if (G_i, Ω_i) is properly attached to G_0, the vortex $(G_i - v, \Omega_i \setminus \{v\})$ is properly attached to $G_0 - v$ for any vertex $v \in \Omega_i$.

Given a graph H embedded in our surface Σ, a curve C in Σ is *H-normal* if it hits H in vertices only. The *distance in* Σ of two points $x, y \in \Sigma$ is the minimal value of $|G_0' \cap C|$ taken over all G_0'-normal curves C in the surface that link x to y. The *distance in* Σ of two vortices V and W is the minimum distance in Σ of a vertex in $\Omega(V)$ from a vertex in $\Omega(W)$. Similar, the *distance in* Σ of two subgraphs H and H' of G_0' is the minimum distance in Σ of a vertex in H from a vertex in H'.

A cycle C in Σ is *flat* if C bounds an open disc $D(C)$ in Σ. Disjoint cycles $C_1, \ldots, C_n$ in Σ are *concentric* if they bound open discs $D(C_1) \supseteq \ldots \supseteq D(C_n)$ in Σ. A set $\mathcal{P}$ of paths intersects $C_1, \ldots, C_n$ *orthogonally*, and is *orthogonal to* $C_1, \ldots, C_n$, if every path P in $\mathcal{P}$ intersects each of the cycles in a (possibly trivial but non-empty) subpath of P.

Let G be a graph embedded in a surface Σ, and Ω a subset of its vertices. Let $C_1, \ldots, C_n$ be cycles in G that are concentric in Σ. The cycles $C_1, \ldots, C_n$ *enclose* Ω if $\Omega \subseteq D(C_n)$. They *tightly enclose* Ω if, in addition, the following holds:

> *For all $1 \leq k \leq n$ and every point $v \in \partial D(C_k)$, there is a vertex $w \in \Omega$ whose distance from v in Σ is at most $n-k+2$.*

For a near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in a surface Σ and concentric cycles $C_1, \ldots, C_n$ in G_0', a vortex $V \in \mathcal{V}$ is *(tightly) enclosed* by these cycles if they (tightly) enclose $\Omega(V)$.

A flat triangle in G_0' is a *boundary triangle* if it bounds a disc that is a face of G_0' in Σ.

For positive integers $r \geq 3$, define a graph H_r as follows (Fig. 2). Let $P_1, \ldots, P_r$ be r disjoint ('horizontal') paths of length $r-1$, say $P_i = v_1^i \ldots v_r^i$. Let $V(H_r) = \bigcup_{i=1}^r V(P_i)$, and let

$$E(H_r) = \bigcup_{i=1}^{r} E(P_i) \cup \left\{ v_j^i v_j^{i+1} \mid \ i, j \text{ odd}; \ 1 \leq i < r; \ 1 \leq j \leq r \right\}$$
$$\cup \left\{ v_j^i v_j^{i+1} \mid \ i, j \text{ even}; \ 1 \leq i < r; \ 1 \leq j \leq r \right\}.$$

We call the paths P_i the *rows* of H_r; the paths induced by the vertices $\{v_j^i, v_{j+1}^i : 1 \leq i \leq r\}$ for an odd index i are its *columns*.

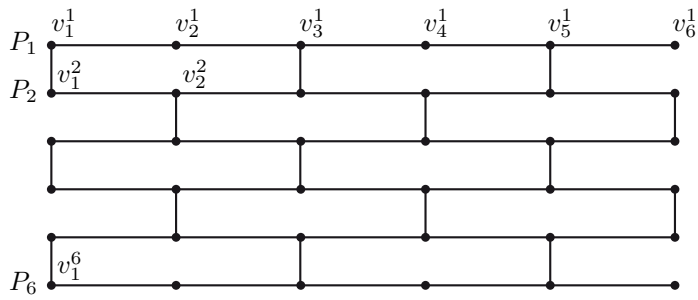

Figure 2: The graph H_6

The 6-cycles in H_r are its *bricks*. In the natural plane embedding of H_r, these bound faces of H. The outer cycle of the unique maximal 2-connected subgraph of H_r is the *boundary cycle* of H_r.

Any subdivision $H = TH_r$ of H_r will be called an *r-wall*, or a *wall of size r*. The *bricks* and the *boundary cycle* of H are its subgraphs that form subdivisions of the bricks and the boundary cycle of H_r, respectively. An embedding of H in a surface Σ is a *flat* embedding, and H is *flat* in Σ, if the boundary cycle C of H bounds an open disc $D(H)$ in Σ such that all its bricks B_i bound disjoint, open discs $D(B_i)$ in Σ with $D(B_i) \subseteq D(H)$ for all i.

For topological concepts used but not defined in this paper we refer to [6, Appendix B]. When we speak of the *genus* of a surface Σ we always mean its Euler genus, the number $2 - \chi(\Sigma)$.

A closed curve C in Σ is *genus-reducing* if the (one or two) surfaces obtained by 'capping the holes' of the components of $\Sigma \setminus C$ have smaller genus than Σ. Note that if C separates Σ and one of the two resulting surfaces is homeomorphic to S^2, the other is homeomorphic to Σ. Hence in this case C was not genus-reducing.

The *representativity* of an embedding $G \hookrightarrow \Sigma \not\simeq S^2$ is the smallest integer k such that every genus-reducing curve C in Σ that meets G only in vertices meets it in at least k vertices. We remark that, by [6, Lemmas B.5 and B.6], all faces of an embedded graph are discs if the representativity of the embedding is positive.

An $(\alpha_0, \alpha_1, \alpha_2)$-near embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in some surface Σ is *(β, r)-rich* for integers $3 \leq \beta \leq r$ if the following statements hold:

(i) G_0' contains a flat r-wall H.

(ii) If $\Sigma \not\simeq S^2$, the representativity of G_0' in Σ is at least β.

(iii) For every vortex $V \in \mathcal{V}$ there are β concentric cycles $C_1(V), \ldots, C_\beta(V)$ in G_0' tightly enclosing V and bounding open discs $D_1(V) \supseteq \ldots \supseteq D_\beta(V)$, such that $D_\beta(V)$ contains $\Omega(V)$ and $\overline{D(H)}$ does not meet $\overline{D_1(V)}$. For distinct, large vortices $V, W \in \mathcal{V}$, the discs $\overline{D_1(V)}$ and $\overline{D_1(W)}$ are disjoint. In particular, every two vortices in $\mathcal{V}$ have distance greater than β in Σ.

(iv) Let $V \in \mathcal{V}$ with $\Omega(V) = (w_1, \ldots, w_n)$. Then there is a linked decomposition of V of adhesion at most α_2 and a path P in $V \cup \bigcup \mathcal{W}$ with $V(P \cap G_0) = \Omega(V)$ that avoids all the paths of the linkage of V, and traverses $w_1, \ldots, w_n$ in this order.

(v) For every vortex $V \in \mathcal{V}$, its set of society vertices $\Omega(V)$ is linked in G'_0 to branch vertices of H by a set $\mathcal{P}(V)$ of β disjoint paths having no inner vertices in H.

(vi) For every vortex $V \in \mathcal{V}$, the paths in $\mathcal{P}(V)$ intersect the cycles $C_1(V), \ldots, C_\beta(V)$ orthogonally.

(vii) All vortices in $\mathcal{W}$ are properly attached.

Using this terminology, we can now state the main result of our paper:

Theorem 2. *For every non-planar graph R and integers $3 \leq \beta \leq r$ there exist integers $\alpha_0 = \alpha_0(R, \beta)$, $\alpha_1 = \alpha_1(R)$ and $w = w(\alpha_0, R, \beta, r)$ such that the following holds with $\alpha = (\alpha_0, \alpha_1, \alpha_1)$. Every graph G of tree-width $tw(G) \geq w$ that does not contain R as a minor has an α-near, (β, r)-rich embedding in some surface Σ in which R cannot be embedded.*

For our proof of Theorem 2 we shall use Theorem 1, but not directly. Instead, we use Theorem 1 in the next section to prove Theorem 4, stated below, which is a strengthening of Theorem (1.3) of [15]. Our proof of Theorem 2 will then be based on Theorem 4.

3 Finding a tree-decomposition

The following lemma shows that we can slightly modify a given α-near embedding by embedding some more vertices of the graph in the surface, so that all the small vortices are properly attached to G_0.

Lemma 3. *Given an integer α and an α-near embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in a surface Σ, there exists an α-near embedding $(\hat{\sigma}, \hat{G}_0, A, \mathcal{V}, \hat{\mathcal{W}})$ of G in Σ such that $G_0 \subseteq \hat{G}_0$ and $\hat{\sigma}|_{G_0} = \sigma$, each vortex in $\hat{\mathcal{W}}$ is properly attached to $\hat{G}_0$, and consecutive society vertices of vortices $V \in \mathcal{V}$ are never adjacent in V.*

Proof. Let us consider the following modifications of our near-embedding, each resulting in another α-near embedding.

(1) By embedding edges between society vertices of a small vortex V in $D(V)$, we may assume that no vortex in $\mathcal{W}$ contains an edge between two of its society vertices.

(2) By first performing (1) and then splitting a small vortex (G_i, Ω_i) into several small vortices each consisting of a component of $G_i - \Omega_i$ together with only their neighbours in Ω_i as society vertices, we may assume that every $(G_i, \Omega_i) \in \mathcal{W}$ satisfies

The graph $G_i - \Omega_i$ is connected and receives an edge from every vertex in Ω_i. $(\star)$

(3) If a society vertex w of a small vortex $V = (G_i, \Omega_i)$ has only one neighbour v in G_i, we can embed v and all $v - \Omega_i$ edges in $D(V)$ and replace w with v in Ω_i. Thus, we may assume that for every vortex $(G_i, \Omega_i) \in \mathcal{W}$ every society vertex has at least two neighbours in $G_i - \Omega_i$.

(4) Let $V := (G_i, \Omega_i) \in \mathcal{W}$ be a vortex of length 3. If there is a vertex $z \in V(G_i) \setminus \Omega_i$, that separates one society vertex $w \in \Omega_i$ from the other two society vertices w', w'', we can write (G_i, Ω_i) as the union of two small vortices $V^1 := (G_i^1, \{z, w\})$ and $V^2 := (G_i^2, \{z, w', w''\})$. Let G_0^+ denote the graph we obtain from G_0 by adding z to its vertex set and extending σ to an embedding σ^+ of G_0^+ in Σ by mapping z to a point in $D(V)$. It is easy to see that $(\sigma^+, G_0^+, A, \mathcal{V}, (\mathcal{W} \setminus \{V\}) \cup \{V^1, V^2\})$ is an α-near embedding of G in Σ.

We can iterate these two modifications only finitely often: Every application of (1), (3) or (4) increases either the number of embedded vertices or the number of embedded edges of the graph G while an application of (2) reduces the number of small vortices not satisfying $(\star)$.

Let $(\hat{\sigma}, \hat{G}_0, A, \mathcal{V}, \hat{\mathcal{W}})$ be the α-near embedding obtained by applying the two modifications as often as possible. Then, for every vortex $(G_i, \Omega_i) \in \mathcal{W}$ every $w \in \Omega_i$ has at least two neighbours in $G_i - \Omega_i$, which is connected. In particular every two vertices in Ω_i are linked by an Ω_i-path in G_i.

Suppose now that $\Omega_i = \{u, v, w\}$, and let us find paths $P = u \dots v$ and $Q = v \dots w$ in G_i that meet only in v. Let v', v'' be distinct neighbours of v in $G_i - \Omega_i$. We can find P and Q as desired unless the sets $\{v', v''\}$ and $\{u, w\}$ are separated in G_i by one vertex z. Then $z \neq v$, since $G_i - \Omega_i$ is connected and u, w send edges there. So z also separates v from $\{u, w\}$ in G_i, contrary to (4).

Thus, (G_i, Ω_i) is properly attached. Clearly, $G_0 \subseteq \hat{G}_0$ and $\hat{\sigma}|_{G_0} = \sigma$. Embedding any vortex edges between adjacent society vertices of large vortices V in the surface instead, we may assume that V contains no such edges, as desired. $\square$

Given two graphs G and H, we say that H is *properly attached to* G if the vortex $(H, V(H) \cap V(G))$ is properly attached to G.

Theorem 4. *For every non-planar graph R and for every integer m there exists an integer α such that for every graph G that does not contain R as a minor and every $Z \subseteq V(G)$ with $|Z| \leq m$ there exist a tree-decomposition $(V_t)_{t\in T}$ of G and a choice $r \in V(T)$ of a* root *of T such that, for every $t \in T$, there is a surface Σ_t in which R cannot be embedded, and the torso G_t of V_t has an α-near embedding $(\sigma_t, G_{t0}, A_t, \mathcal{V}_t, \emptyset)$ in Σ_t with the following properties:*

(i) *The vortices $V \in \mathcal{V}_t$ have decompositions of width at most α satisfying* (ii) *below.*

(ii) *For every $t' \in T$ with $tt' \in E(T)$ and $t \in rTt'$ the overlap $V_t \cap V_{t'}$ is contained in $A_{t'}$, and $(V_t \cap V_{t'}) \setminus A_t$ is contained either in a part $X_{tt'}$ of a vortex decomposition from* (i) *or in a subset $X_{tt'}$ of $V(G_{t0})$ that spans in G_{t0} either a K_1 or a K_2 or a K_3 bounding a face of G_{t0} in Σ_t. In the latter case, $G_{t'} - A_t$ is properly attached to G_{t0}.*

(iii) *If $t = r$, then $Z \subseteq A_r$. We say that the part V_r (with the chosen near-embedding of G_r)* accomodates *Z.*

We remark that the statement about Z in Theorem 4 only serves a technical purpose, to facilitate induction. The main difference between Theorem 4 and Theorem 1 is that the vortex decompositions required in Theorem 4 have bounded width, while those in Theorem 1 are only required to have bounded adhesion. It is this difference that requires the extra work when we deduce Theorem 4 from Theorem 1: Starting from the α-near embedding of G provided by Theorem 1, we have to split off small vortices of large width, and large parts to decompose those parts of G inductively.

Proof of Theorem 4. Applying Theorem 1 with the given graph R yields two constants $\hat{\alpha}$ and $\hat{\theta}$. We may assume that m is large enough that $\theta := (m + 2)/3 \geq \max(\hat{\theta}, 3\hat{\alpha} + 3)$ and θ is integral and let $\alpha := 4\theta - 2$.

The proof proceeds by induction on $|G|$, for these (now fixed) R, m and α. We may assume that $|Z| = m(= 3\theta - 2)$, since if it is smaller we can add arbitrary vertices to Z. (We may assume that such vertices exist, as the theorem is trivial for $|G| < \alpha$.)

We may assume that

There is no separation (A, B) of G of order at most θ such that both $|Z \cap A|$ and $|Z \cap B|$ are of size at least $|A \cap B|$. (1)

Otherwise, let $Z_A := (Z \setminus B) \cup (A \cap B)$. By assumption, $|A \cap B| \leq |Z \cap B|$, so $|Z_A| = |Z \setminus B| + |A \cap B| \leq |Z| = m$. We apply our theorem inductively to

$G[A]$ and Z_A, which yields a tree-decomposition of $G[A]$ such that the torso of its root part G_A has its apex set in a suitable near-embedding contain Z_A. Similarly, we apply the theorem to $G[B]$ and $Z_B := (Z \setminus A) \cup (A \cap B)$. We combine these two tree-decompositions by joining a new part $Z \cup (A \cap B)$ to both G_A and G_B and obtain a tree-decomposition of G with the desired properties of the theorem: The new part, which we make the root, contains at most $|Z| + |A \cap B| \leq 4\theta - 2$ vertices, so all these can be put in the apex set of an α-near embedding. Finally, the new part contains Z, and the new decomposition inherits all the remaining desired properties from the decomposition of $G[A]$ and $G[B]$. This proves (1).

Let $\mathcal{T}$ be the set of separations (A, B) of G of order less than θ such that $|Z \cap B| > |Z \cap A|$. Let us show that

$$\mathcal{T} \textit{ is a tangle of } G \textit{ of order } \theta. \tag{2}$$

By (1) and our assumption that $|Z| = m = 3\theta - 2$, for every separation (A, B) of G of order less than θ exactly one of the sets $Z \cap B$ and $Z \cap A$ has size less than θ. This implies both conditions from the definition of a tangle.

From (1) and the definition of $\mathcal{T}$ we conclude

$$|Z \cap A| < |A \cap B| \textit{ for every } (A, B) \in \mathcal{T}. \tag{3}$$

As $\theta \geq \hat{\theta}$, Theorem 1 gives us an $\hat{\alpha}$-near embedding $(\sigma, G_0, \hat{A}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$ of G in some surface Σ that captures $\mathcal{T}$. Our plan now is to split G at separators consisting of apex vertices, of society vertices of vortices in $\hat{\mathcal{W}}$, or single parts of vortex decompositions of vortices in $\hat{\mathcal{V}}$. We shall retain intact a part of G that contains G_0, and which we know how to embed α-nearly; this part is going to be a part of a new tree-decomposition. For the subgraphs of G that we split off we shall find tree-decompositions inductively, and eventually we shall combine all these tree-decompositions to one tree-decomposition of G that satisfies our theorem.

By Lemma 3, we may assume that large vortices contain no edges between consecutive society vertices, and that all small vortices are properly attached to G_0. Let us consider such a vortex $(G_i, \Omega_i) \in \hat{\mathcal{W}}$. Since our embedding captures $\mathcal{T}$, the separation $(V(G_i) \cup \hat{A}, V(G \setminus (G_i \setminus \Omega_i)) \cup \hat{A})$, whose order is at most $3 + |\hat{A}| < \theta$, lies in $\mathcal{T}$. By (3), G_i contains at most $2 + |\hat{A}|$ vertices of Z. Thus, $Z'_i := \Omega_i \cup \hat{A} \cup (Z \cap G_i)$ has size at most $5 + 2\hat{\alpha} \leq m$. We apply our theorem inductively to the smaller graph $G[V(G_i) \cup \hat{A}]$ with Z'_i. Let H^i be the torso of the root part of the resulting tree-decomposition $(T^i, \mathcal{H}^i)$, the one that accomodates Z'_i. Recall that G_i was properly attached to G_0. The Ω_i-paths witnessing this have no vertices in $\hat{A}$, and by replacing any H^i-subpaths they contain with torso-edges of H^i, we can turn them into paths witnessing that also $H^i - \hat{A}$ is properly attached to G_0.

For every vortex $(G_i, \Omega_i) \in \hat{\mathcal{V}}$, with $\Omega_i = \{w^i_1, \ldots, w^i_{n(i)}\}$ say, let us choose a fixed decomposition $(\hat{X}^i_1, \ldots, \hat{X}^i_{n(i)})$ of adhesion at most $\hat{\alpha}$. We define

$$X^i_j := \begin{cases} \big(\hat{X}^i_1 \cap \hat{X}^i_2\big) & \text{for } j = 1 \\ \big(\hat{X}^i_j \cap (\hat{X}^i_{j-1} \cup \hat{X}^i_{j+1})\big) & \text{for } 1 < j < n(i) \\ \big(\hat{X}^i_{n(i)} \cap \hat{X}^i_{n(i)-1}\big) & \text{for } j = n(i) \end{cases}$$

By G^-_i we denote the graph on $X^i_1 \cup \ldots \cup X^i_{n(i)}$ in which every X^i_j induces a complete graph but no further edges are present. Now, as the adhesion of (G_i, Ω_i) is at most $\hat{\alpha}$, every X^i_j contains at most $2\hat{\alpha}$ vertices and thus, $(X^i_1, \ldots, X^i_{n(i)})$ is a decomposition of the vortex $V^-_i := (G^-_i, \Omega_i)$ of depth at most $2\hat{\alpha} \leq \alpha$. Let $\mathcal{V}$ denote the set of these new vortices.

For every $j = 1, \ldots, n(i)$, the pair

$$\big(\hat{X}^i_j \cup \hat{A}, (V(G) \setminus (\hat{X}^i_j \setminus X^i_j)) \cup \hat{A}\big)$$

is a separation of order at most $|X^i_j \cup \hat{A}| \leq 2\hat{\alpha} + \hat{\alpha} \leq \theta$. As before, our embedding captures $\mathcal{T}$, so the separation lies in $\mathcal{T}$. By (3), at most $\theta - 1$ vertices from Z lie in $\hat{X}^i_j \cup \hat{A}$. Let $Z'_{ij} := X^i_j \cup \hat{A} \cup (Z \cap \hat{X}^i_j)$. This set contains at most $2\theta - 1 \leq m$ vertices and, as before, we can apply our theorem inductively to the smaller graph $G[\hat{X}^i_j \cup \hat{A}]$ with Z'_{ij}. We obtain a tree-decomposition $(T^i_j, \mathcal{H}^i_j)$ of this graph, with the root torso H^i_j accomodating Z'_{ij}.

Now, with $V_0 := V(G_0) \cup \hat{A}$, we can write

$$G = G[V_0] \cup \big(\bigcup \mathcal{W}\big) \cup \big(\bigcup \{G[\hat{X}^i_j] : V_i \in \mathcal{V}, 1 \leq j \leq n(i)\}\big).$$

Let us now combine our tree-decompositions of the vortices in $\mathcal{W}$ and the graphs $G[\hat{X}^i_j]$ to a tree-decomposition of G: We just add a new tree vertex v_0 representing V_0 to the union of all the trees T^i and T^i_j, and add edges from v_0 to every vertex representing an H^i or an H^i_j we found in our proof.

We still have to check that the torso of the new part V_0 can be α-nearly embedded as desired. But this is easy: Let G'_0 be the graph obtained from G_0 by adding an edge xy for every two nonadjacent vertices x and y that lie in a common vortex $V \in \mathcal{W}$. We can extend the embedding $\sigma : G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' : G'_0 \hookrightarrow \Sigma$ by mapping the new edges disjointly to the discs $D(V)$. Then $G' := G'_0 \cup \bigcup G^-_i$ is the torso of V_0 in our new tree-decomposition, and $(\sigma', G'_0, \hat{A} \cup Z, \mathcal{V}, \emptyset)$ is an α-near embedding of G' in Σ whose apex set contains Z. $\square$

As noted, Theorem (1.3) of [15] is a direct result from Theorem 4:

Corollary 5. *For every nonplanar graph R there exists an integer α such that every graph with no R-minor has a tree-decomposition $(V_t)_{t\in T}$ such that for every $t \in T$ there is a surface Σ_t in which R cannot be embedded but in which the torso G_t corresponding to t has an α-near embedding $(\sigma_t, G_{t0}, A_t, \mathcal{V}_t, \emptyset)$.*

4 Graphs on Surfaces

In this section, we collect results about graphs embedded in surfaces. Except for the last one, these results are not directly related to near-embeddings. Our first tool is the *grid-theorem* from [13]; see [6] for a short proof.

Theorem 6. *For every integer k there exists an integer $f(k)$ such that every graph of tree-width at least $f(k)$ contains a wall of size at least k.*

Every large enough wall embedded in a surface contains a large flat subwall:

Lemma 7. *For all integers k, g there is an integer $\ell = \ell(k, g)$ such that any wall of size ℓ embedded in a surface of genus at most g contains a flat wall of size k.*

Proof. Let ℓ be chosen large enough that every ℓ-wall contains $g+1$ disjoint $k+1$-walls. By [6, Lemma B.6], any ℓ-wall H in a surface Σ of genus g contains a k-wall H' each of whose bricks bounds an open disc in Σ. If none of these open discs contains a point of H, the wall H' is flat. Otherwise, the disc containing a point of H contains all the other $k+1$-walls we considered, and thus, all these are flat. □

Lemma 8. *Let G be a graph of tree-width at least w. Then in every tree-decomposition of G the torso of at least one part also has tree-width at least w.*

Proof. If every torso has a tree-decomposition of width at most $w-1$, we can use [6, Lemma 12.3.5] to combine these into a tree-decomposition of G of width at most $w-1$. □

The following lemma is a direct corollary of Lemmas B.4 and B.5 from [6].

Lemma 9. *For every surface Σ, every closed curve $C \subset \Sigma$ that does not bound a disc in Σ is genus-reducing.*

Let Σ be a (closed) surface and G be a graph embedded in Σ. For a face f of G, let S be the set of vertices that lie on ∂f. If we delete S and add a

new vertex v to G with neighbours $N(S)$, we obtain a graph G'. It is easy to see that we can extend the induced embedding of $G - S$ to an embedding of G'. We say that the graph G' embedded in Σ was obtained from G by *contracting f to v*.

The following lemma is from Demaine and Hajiaghayi [5].

Lemma 10. *For every two integers t and g there exists an integer $s = s(t, g)$ such that the following holds. Let G be a graph of tree-width at least s embedded in some surface Σ of genus g. If G' is obtained from G by contracting a face to a vertex, then G' has tree-width at least t.*

Our next lemma is due to Mohar and Thomassen [12]:

Lemma 11. *Let $G \hookrightarrow \Sigma \neq S^2$ be an embedding of representativity at least $2k+2$ for some $k \in \mathbb{N}$. Then, for every face f of G in Σ there are k concentric cycles $(C_1, \ldots, C_k)$ in G such that $f \subseteq \mathring{D}(C_k)$.*

For an oriented curve C and points $x, y \in C$ we denote by xCy the subcurve of C with endpoints x, y that is oriented from x to y. For a graph G embedded in a surface Σ, a face f of G, and a closed curve C in Σ, let $\mathcal{C}(C, f)$ denote the number of components of $C \cap f$.

Lemma 12. *Let G be a graph embedded in a surface Σ, and let F be the set of faces. For an integer $r > 0$, consider all G-normal, genus-reducing curves C in Σ that satisfy $|C \cap G| < r$. Let C be chosen so that $\sum_{f \in F} \mathcal{C}(C, f)$ is minimal. Then, $\mathcal{C}(C, f) \leq 1$ for all $f \in F$.*

Proof. Suppose there is an $f \in F$ with $\mathcal{C}(C, f) > 1$. Then there is a component D of $f - C$ whose boundary ∂D contains two distinct components of $f \cap C$, say the interiors of disjoint arcs xCy and zCw following some fixed orientation of C. Then x, y, z, w appear in this (cyclic) order on C.

Join x to z by an arc A through D. Then $C_w := xAzCx$ and $C_y := zAxCz$ are closed curves in Σ meeting precisely in A. Each of them meets G in fewer vertices than C does, so neither C_w nor C_y are genus reducing. This implies by Lemma 9 that C_w and C_y bound discs D_w and D_y in Σ. If D_w contains a point of C_y then $C_y \setminus A \subseteq D_w$ and hence $C \subseteq D_w$. But then C bounds a disc contained in $D_w \subseteq \Sigma$ (by the Jordan curve theorem), which contradicts our assumption that C is genus-reducing in Σ. Hence $D_w \cap C_y = \emptyset$, and similarly $D_y \cap C_w = \emptyset$. This implies that $\overline{D_w} \cap \overline{D_y} = A$. But then $\overline{D_w} \cup \overline{D_y}$ is a closed disc in Σ bounded by C, a contradiction as earlier. $\square$

Whenever there are cycles enclosing a vortex V, we can find cycles tightly enclosing V:

Lemma 13. *For an integer $\alpha > 0$, let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an α-near embedding of some graph G in a surface Σ and let $C_1, \ldots, C_n$ be cycles enclosing a vortex $V \in \mathcal{V}$. Then, there are n cycles $C'_1, \ldots, C'_n$ in G_0 that enclose V tightly, such that $D(C'_1) \subseteq D(C_1)$.*

Proof. Let us write $D_k := D(C_k)$ for $1 \leq k \leq n$ and $D_{n+1} := D(V)$ and $V(C_{n+1}) := \Omega(V)$. Suppose there is a cycle $C \subseteq G'_0 \cap \overline{D_k} \setminus \overline{D_{k+1}}$ for some $1 \leq k \leq n$ such that $C \neq C_k$. Then, we can replace C_k by C and obtain a new set of cycles in G_0 enclosing V. By this replacement, we reduce the number of vertices and edges of G'_0 in $\overline{D_k}$, so we can repeate this step only finitely often. We may assume that $C_1, \ldots, C_n$ were chosen so that such that a replacement is not possible.

We claim that these cycles enclose V tightly. To see this, consider for a vertex $v \in V(C_k)$ the set F of all faces f of G'_0 with $f \subseteq D(C_k)$ and $v \in \partial f$. For every two neighbours x, y of v that lie on the boundary of the same face $f \in F$, there is a path in $G'_0 \cap \partial f$ linking x, y and avoiding v. Therefore, there is a face $f_v \in F$ such that ∂f_v contains a vertex $v' \in V(C_{k+1})$[1]: otherwise, $\bigcup\{\partial f : f \in F\}$ would contain a path between the two neighbours v^-, v^+ of v in C_k, that avoids v. Substituting this path for the path $v^- v v^+$ in C_k then turns C_k into a cycle in $G'_0 \cap (\overline{D}_k \setminus \overline{D}_{k+1})$ avoiding v, contradicting the choice of the C_i. Similarly, every edge e of C_k lies on the boundary of a face f of G'_0 that also contains a vertex v' of C_{k+1}. Then every inner point x of e can be linked to v' by a curve through f. By induction on $n - k$, we may assume that, unless $k = n$ and $v' \in \Omega(V)$, there is a curve C linking $v' \in V(C_{k+1})$ to some $w \in \Omega(V)$, with $|C \cap G'_0| \leq n - (k+1) + 2$. We extend this curve by a curve in f from v or x to v' which gives us a curve as desired. □

5 Taming a Vortex

In this section we describe how to obtain a new (and simpler) near-embedding from an old one if we are allowed to move a bounded number of vertices from the embedded part of the graph to the apex set. For example, we might reduce the number of large vortices by combining two of them, or reduce the genus of the surface by cutting along a genus-reducing curve.

Lemma 14. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ. If there are two vortices $V, W \in \mathcal{V}$ of length at least 4 and a G-normal curve C in Σ from $D(V)$ to $D(W)$ that meets G in at most d*

[1]Indeed, thickening G'_0 in Σ turns $\overline{f}$ into a compact surface with boundary. By the classification of these surfaces, the component of ∂f meeting the thickened vertex v is a circle, which defines a closed walk in G'_0. This walk contains the desired path.

vertices, then there is a vertex set $A' \subseteq V(G_0)$ of size $|A'| \leq 2\alpha_2 + d$ and a vortex $V' \subseteq G - A'$ such that G has an $(\alpha_0 + 2\alpha_2 + 2 + d, \alpha_1 - 1, \alpha_2)$-near embedding

$$(\sigma|_{G_0 - A'}, G_0, A \cup A', \mathcal{V}', \mathcal{W}')$$

in Σ with $\mathcal{V}' \subseteq (\mathcal{V} \setminus \{V, W\}) - A' \cup \{V'\}$ and $\mathcal{W}' = \mathcal{W} - A' \cup \{V - A' : V \in \mathcal{V}, |\Omega(V) \setminus A'| \leq 3\}$.

Proof. Let us choose decompositions $(X_1, \ldots, X_n)$ of V and $(Y_1, \ldots, Y_m)$ of W of adhesion at most α_2, where $\Omega(V) = (v_1, \ldots, v_n)$ and $\Omega(W) = (w_1, \ldots, w_m)$. By slightly adjusting C we may assume that the endpoints of C are vertices v_k and w_ℓ, so that $C \cap D(V) = \{v_k\}$ and $C \cap D(W) = \{w_\ell\}$ for some indices $1 \leq k \leq n$ and $1 \leq \ell \leq m$. Let S be the set of vertices in Σ on C. By fattening C to a disc D we obtain a closed disc $D' := \overline{D \cup D(V) \cup D(W)}$ such that $D' \cap (G_0 - S) = (\Omega(V) \cup \Omega(W)) \setminus \{v_k, w_\ell\}$. By reindexing if neccesary we may assume that the orientations of $\partial D'$ induced by $\Omega(V) \setminus \{v_k\}$ and $\Omega(W) \setminus \{w_\ell\}$ agree.

Let $X := (X_k \cap X_{k+1})$ if $k < n$ and $X := \{v_k\}$ if $k = n$ and let $Y := (Y_\ell \cap Y_{\ell+1})$ if $\ell < m$ and $Y := \{w_\ell\}$ if $\ell = m$. Note that $|X| \leq \alpha_2$ and $|Y| \leq \alpha_2$. If $k \neq 1$, let $X'_{k-1} := (X_{k-1} \cup X_k) \setminus X$ and $X'_i := X_i \setminus X$ for all $i \notin \{k-1, k\}$. If $k = 1$, let $X'_2 := (X_1 \cup X_2) \setminus X$ and $X'_i := X_i \setminus X$ for all $i \geq 3$. Define sets Y'_j analoguosly with ℓ and m replacing k and n. Finally, let $A' := S \cup X \cup Y$ and $G' := (V \cup W) - A'$. Then for

$$\Omega' := (v_{k+1}, \ldots, v_n, v_1, \ldots, v_{k-1}, w_{\ell+1}, \ldots, w_m, w_1, \ldots, w_{\ell-1})$$

the tuple $V' := (G', \Omega')$ is a vortex with a decomposition

$$(X'_{k+1}, \ldots, X'_n, X'_1, \ldots, X'_{k-1}, Y'_{\ell+1}, \ldots, Y'_m, Y'_1, \ldots, Y'_{\ell-1})$$

of adhesion at most α_2. Now it is easy to see that A' satisfies the conditions as desired. □

Our next lemma shows how to make vortices linked. The techniques used in its proof originate from [8] and were extended by Geelen and Huynh [9].

Lemma 15. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ such that every small vortex $W \in \mathcal{W}$ is properly attached. Moreover, assume that*

(i) *For every vortex $V \in \mathcal{V}$ there are $\alpha_2 + 1$ concentric cycles $C_0(V), \ldots, C_{\alpha_2}(V)$ in G'_0 tightly enclosing V.*

(ii) *For distinct vortices $V, W \in \mathcal{V}$, the discs $\overline{D(C_0(V))}$ and $\overline{D(C_0(W))}$ are disjoint.*

Then there is a graph $\tilde{G}_0 \subseteq G_0$ containing $G_0 \setminus \left(\bigcup_{V\in\mathcal{V}} D(C_0(V))\right)$, a set $\tilde{A} \subseteq V(G) \setminus V(\tilde{G}_0)$ of size $|\tilde{A}| \leq \tilde{\alpha} := \alpha_0 + \alpha_1(2\alpha_2 + 2)$, and sets $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}} \subseteq \mathcal{W}$ of vortices such that, with $\tilde{\sigma} := \sigma|_{\tilde{G}'_0}$, the tuple $(\tilde{\sigma}, \tilde{G}_0, A \cup \tilde{A}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ is an $(\tilde{\alpha}, \alpha_1, \alpha_2 + 1)$-near embedding of G in Σ such that every vortex $\tilde{V} \in \tilde{\mathcal{V}}$ satisfies condition (iv) *of the definition of (β, r)-rich, and $D(\tilde{V}) \supseteq D(V)$ for some $V \in \mathcal{V}$.*

Proof. We will convert the vortices in $\mathcal{V}$ into linked vortices one by one, so let us focus on one vortex $V \in \mathcal{V}$. The idea is as follows: we delete one vertex from each of the enclosing cycles, which gives us a set of α_2+1 disjoint paths. If necessary, we also delete an adhesion set of V which allows us to assume that the paths are 'aligned' to the vortex. Then, we 'push' these paths as far into the vortex as possible. As the adhesion of the vortex is bounded by α_2, at least one of the paths remains entirely in the surface. The vertices of the innermost such path, later denoted by P_0, become the society vertices of our new vortex, and the shifted path system shows that this new vortex is linked.

By assumption, V has a decomposition $(X'_1, \ldots, X'_{n'})$ with adhesion sets $Z'_i := X'_i \cap X'_{i+1}$ of size at most α_2, for all $i < n'$. Pick a vertex $v \in C_0(V)$. As $C_0(V), \ldots, C_{\alpha_2}(V)$ enclose V tightly, there is a curve C from v to $\Omega(V) := \{w'_1, \ldots, w'_{n'}\}$ that contains at most $\alpha_2 + 2$ vertices of G'_0. Let S denote the set of these vertices. Clearly, S consists of exactly one vertex from each $C_i(V)$, $0 \leq i \leq \alpha_2$ and one society vertex w'_j of V.

Put $n := n' - 1$ and $Z'_{n'} := \emptyset$ and let $Z := Z'_j \cup \{w'_j\}$. If $j = 1$ let

$$(X_1, \ldots, X_n) := ((X'_1 \cup X'_2) \setminus Z, X'_3 \setminus Z, \ldots, X'_{n'} \setminus Z)\,.$$

If $j > 1$, let

$$(X_1, \ldots, X_n) := (X'_{j+1} \setminus Z, X'_{j+2} \setminus Z, \ldots, X'_{n'} \setminus Z, X'_1 \setminus Z, \ldots, (X'_{j-1} \cup X'_j) \setminus Z).$$

Then $(X_1, \ldots, X_n)$ is a decomposition of adhesion at most α_2 of the vortex $V - Z$ taken with respect to the society $(w_1, \ldots, w_n)$ defined by $w_i := X_i \cap \Omega(V)$ for all i. For $i < n$, let $Z_i := X_i \cap X_{i+1}$.

Recall that the linear ordering of Ω is induced by an orientation of the disc $\overline{D(V)}$. The extension of this orientation to $\overline{D}(C_i)$ induces a cyclic ordering on $V(C_i)$, for each $0 \leq i \leq \alpha_2$, in which we let x_i denote the successor, and y_i the predecessor of the unique vertex in $S \cap V(C_i)$. Let $X := \{x_0, \ldots, x_{\alpha_2}\}$ and $Y := \{y_0, \ldots, y_{\alpha_2}\}$. Now we delete $S \cup Z$, a set of at most $2\alpha_2 + 1$ vertices, and put

$$G' := \left(\left(G'_0 \cap \overline{D(C_0(V))}\right) \cup V\right) - (S \cup Z).$$

Clearly, the graph G' still contains a set of $\alpha_2 + 1$ disjoint X–Y paths. Let us show that

$$\textit{For every set } \mathcal{P} \textit{ of } \alpha_2 + 1 \textit{ disjoint } X\text{–}Y\textit{-paths in } G', \textit{ the path } P_0 \textit{ starting in } x_0 \textit{ lies in } G'_0. \tag{4}$$

Otherwise, let w_q be the vertex of P_0 preceding its first vertex in $V - \Omega(V)$. As the subpath $P_0 w_q$ of P_0 lives entirely in the plane graph $G'_0 \cap \overline{D(C_0(V))}$, the set $V(P_0 w_q) \cup Z_q$ separates X from Y. Thus, all $\alpha_2 + 1$ paths in $\mathcal{P}$ have to pass through Z_q, a set of at most α_2 vertices, a contradiction. This proves (4).

By planarity, (4) implies that the paths in $\mathcal{P} \setminus \{P_0\}$ cannot cross P_0, so P_0 ends in y_0. Together with v and the edges $x_0 v$ and $v y_0$ the path P_0 forms a cycle in G'_0; for our original set $\mathcal{P}$, this is the cycle C_0, but for every $\mathcal{P}$ satisfying (4) its x_0–y_0 path P_0 defines such a cycle in G'_0. This cycle bounds a disc $D(\mathcal{P})$ in Σ containing $\Omega(V)$, and we define

$$G(\mathcal{P}) := \Big(\big(G'_0 \cap D(\mathcal{P}) \big) \cup V \Big) - (S \cup Z).$$

Clearly, $G(\mathcal{P})$ contains the paths from $\mathcal{P}$.

Let us choose $\mathcal{P}$ as in (4) with $G(\mathcal{P})$ minimal, and let the vertices of P_0 be labeled $p_0, \ldots, p_r$. Then we have the following:

$$\textit{For every vertex } p_i \in V(P_0) \textit{ there is a set } T \textit{ of } \alpha_2 + 1 \textit{ vertices of } G(\mathcal{P}) \textit{ that contains } p_i \textit{ and separates } X \textit{ from } Y \textit{ in } G'. \tag{5}$$

Indeed, if $i \in \{0, r\}$ then $T \in \{X, Y\}$ will do so assume that $0 < i < r$. By the minimality of $G(\mathcal{P})$, there is no set of $\alpha_2 + 1$ disjoint X–Y paths in $G' - p_i$. Hence by Menger's theorem, an X–Y separator T' of size at most α_2 exists in $G' - p_i$. Since $G(\mathcal{P}) - p_i \subseteq G' - p_i$ contains the α_2 paths of $\mathcal{P} \setminus \{P_0\}$, we have $T' \subseteq V(G(\mathcal{P}))$ and $|T'| = \alpha_2$. Now $T := T' \cup \{p_i\}$ is as desired. This completes the proof of (5).

Let us pick for each $i = 0, \ldots, r$ a separation (A_i, B_i) of $G(\mathcal{P})$ as in (5), with $p_i \in T_i := A_i \cap B_i$ and $|B_i|$ minimal such that $X \subseteq A_i$ and $Y \subseteq B_i$. Clearly, each T_i contains exactly one vertex from each path in $\mathcal{P}$. Let us show the following:

$$B_i \supsetneq B_j \textit{ for all } 0 \leq i < j \leq r. \tag{6}$$

Note first that $B_i \ni p_i \in P_0 p_i \subseteq A_j \setminus B_j$, so it suffices to show that $B_i \supseteq B_j$. Suppose this fails. Then $|B_j| \cap B_i| < |B_j$, which will contradict our choice of (A_j, B_j) if we can show that we could have chosen the separation $(A_i \cup A_j, B_i \cap B_j)$ instead of (A_j, B_j). Clearly, $p_j \in p_j P_0 \subseteq B_i \cap B_j$, since

$i < j$. Moreover, the new separator $T_B := (A_i \cup A_j) \cap (B_i \cap B_j)$ contains at least one vertex from each path of $\mathcal{P}$, so $|T_B| \geq |\mathcal{P}|$. Likewise, the X–Y separator $T_A := (A_i \cap A_j) \cap (B_i \cup B_j)$ meets every path in $\mathcal{P}$, so $T_A \geq |\mathcal{P}|$. But $|T_A| + |T_B| = |T_i| + |T_j| = 2|\mathcal{P}|$. Hence both inequalities hold with equality; in particular, $|T_B| = |\mathcal{P}| = \alpha_2 + 1$ as desired. This proves (6).

We set $\Omega := V(P_0)$ and $X_0 := A_0$ and $X_i := A_i \cap B_{i-1}$ for $1 < i < r$ and $X_r := B_{r-1}$. It is easy to check that $(X_0, \ldots, X_r)$ is a linked path decomposition of the vortex $(G(\mathcal{P}), \Omega)$. Finally, consider all $W \in \mathcal{W}$ such that $\Omega(W)$ intersects $G(\mathcal{P})$. If $\Omega(W) \subseteq G(\mathcal{P})$, we add W to $G(\mathcal{P})$ and delete it from $\mathcal{W}$. Otherwise, any vertices of $\Omega(W)$ in $G(\mathcal{P})$, and any edges of G'_0 between them, are vertices and edges of P_0. We then delete any such edges from $G(\mathcal{P})$ and dent $D((G(\mathcal{P}), \Omega))$ a little, so that its boundary no longer meets the interior of such edges. Since these W were properly attached, such edges can be replaced on P_0 by paths through W. Hence, property (iv) from (β, r)-rich follows for the new vortex $G(\mathcal{P})$. □

Lemma 16. *Let $z > 0$ be an integer, and $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ such that every two vertices in $\mathcal{V}$ have distance at least z in Σ. If the representativity of G'_0 in Σ is less than z, then there is a vertex set $A' \subseteq V(G_0)$ with $|A'| < a := 2\alpha_2 + 2 + z$, such that one of the following statements holds:*

a) There exists a set $\mathcal{V}'$ of vertices in G, a surface Σ' with $g(\Sigma') < g(\Sigma)$ and an $(\alpha_0 + a, \alpha_1 + 1, \alpha_2)$-near embedding

$$(\sigma', G_0 - A', A \cup A', \mathcal{V}', \mathcal{W} - A')$$

of G in Σ'.

b) There exists a separation (A_1, A_2) of G with $A_1 \cap A_2 = A'$ such that for $i = 1, 2$ there are surfaces Σ_i and $(\alpha_0 + a, \alpha_1, \alpha_2)$-near embeddings $(\sigma^i, G^i_0, A^i, \mathcal{V}^i, \mathcal{W}^i)$ of $G[A_i]$ into Σ_i such that $g(\Sigma_i) < g(\Sigma)$.

Proof. Let C be a genus-reducing curve in Σ that hits less than z vertices of G'_0. Let us assume that C meets the open disc $D(V)$ of a large vortex V, the case when it does not is even easier. Note that C cannot meet another large vortex, since the distance in Σ of two large vortices is at least z. By Lemma 12 we may assume that C meets the face f of G'_0 containing $D(V)$ in at most one component (of $C \cap f$). We can therefore modify C so that there are society vertices $x, y \in \Omega(V)$ with $\partial D(V) \cap C = \{x, y\}$. If C enters $D(V)$ from a disc $D(W)$ of a small vortex W with $x \in \Omega(W) = \{w_{i-1}, w_i, w_{i+1}\} \subseteq \Omega(V)$, where w_{i-1}, w_i, w_{i+1} are enumerated as in $\Omega(V)$, we choose C so that $x = w_{i+1}$. After this modification, C hits at most $z+2$ vertices. Deleting the

two appropriate adhesion sets splits V into two vortices V', V'', and using facts from elementary point-set topology such as in [6, Chapter 4.1] we can partition $D(V)$ into two discs $D(V'), D(V'')$ to accomodate them. Let A' be the union of the deleted adhesion sets and the vertices hit by C. This set contains up to a vertices.

Now, A' is a separator of G. We delete C from Σ and cap the holes of the resulting components; cf. [6, Appendix B]. If $\Sigma \setminus C$ has one component, statement (a) follows, otherwise (b) is true. $\square$

Lemma 17. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ. Let $V \in \mathcal{V}$ be a vortex, and $(C_1, \ldots, C_\ell)$ cycles tightly enclosing V. Then there is a set $\mathcal{X}$ of disjoint open discs with $\bigcup_{\Delta \in \mathcal{X}} \overline{\Delta} = \overline{D(C_1)}$ such that the following holds: For every disc $\Delta \in \mathcal{X}$ there are sets $S \subseteq V(G_0')$ of size $|S| \leq 2\ell + 2$ and $S' \subseteq V$ of size $|S'| \leq 2\alpha_2 - 2$ such that*

- $S = G_0' \cap \partial\Delta \subseteq V(C_1) \cup \ldots \cup V(C_\ell) \cup \Omega(V)$
- $|S \cap C_i| \leq 2$ *for each* $i = 1, \ldots, \ell$
- $|S \cap \Omega(V)| \leq 2$
- *there is a separation (X_1, X_2) of G with $X_1 \cap X_2 = S \cup S'$ and $G_0 \cap X_1 = G_0 \cap \Delta$.*

Proof. Pick a point $p \in D(V)$. Let $(x_0, \ldots, x_n)$ denote the vertices of $C_1(V)$ ordered linearly in a way compatible with a cyclic orientation of $C_1(V)$. For $1 \leq i \leq n$, denote the edges $x_{i-1}x_i$ by e_i, and put $e_{n+1} := x_n x_0$ and $e_0 := \emptyset$. We will inductively define curves linking $x_0, \ldots, x_n$ to p. First, let us choose for all $i = 0, \ldots, n$ a curve L_i' linking x_i to p so that $L_i' \cap G_0'$ consists of exactly one vertex from each of the cycles $C_1, \ldots, C_\ell$ and one society vertex $w_i \in \Omega(V)$ and so that $L_i' w_i \cap D(V) = \emptyset$ and $w_i L_i' \subseteq \overline{D(V)}$. Then put $L_0 := L_0'$, and for $i = 1, \ldots, n$ define L_i inductively as follows. Let z be the first point of L_i' on $L_0 \cup L_{i-1}$. If $z \in L_0$, let $L_i := L_i' z$; otherwise let $L_i := L_i' z L_{i-1}$. Note that, for $1 \leq i \leq n$ and every point $z \in L_i'$, $|L_i' z \cap (C_1 \cup \ldots \cup C_\ell)| = k$ if and only if $z \in \overline{D(C_k)} \setminus \overline{D(C_{k+1})}$ where $D(C_{\ell+1}) := \emptyset$.

Elementary topology implies (as in the proof of Lemma 13) that $D(C_1) \setminus L_0$ has a unique component homeomorphic to an open disc Δ_0'. Assume inductively that, for some $1 \leq i \leq n$, we have defined an open disc $\Delta_{i-1}' \subseteq D(C_1)$ whose boundary is contained in $L_0 \cup L_{i-1} \cup C_1$, contains L_{i-1}, and meets C_1 in exactly $C_1 \setminus (e_0 \cup \ldots \cup e_{i-1})$. Then Δ_{i-1}' contains the interior of L_i, which joins two points of $\partial\Delta_{i-1}'$ and thus divides Δ_{i-1}' into two open discs Δ_i and Δ_i'. We let Δ_i' be the disc whose boundary satisfies for i the requirements

analogous to those made earlier for $i-1$ on Δ'_{i-1}, and let $\Delta_i := \Delta'_{i-1} \setminus \partial\overline{\Delta'_i}$ be the other disc. Finally, put $\Delta_{n+1} := \Delta'_n$, set $\mathcal{X} := \{\Delta_1, \ldots, \Delta_{n+1}\}$, and let $\partial\Delta_i \cap G_0 =: S_i$ for all i.

Induction on i shows that S_i has exactly one edge on C_1 and otherwise lies in $L_0 \cup L_i$, so $|S_i| \leq 2\ell + 2$. If $\Delta_i \cap D(V) = \emptyset$, then S_i and $S'_i := \emptyset$ are as desired. Otherwise, $\partial\Delta_i$ meets $\overline{D(V)}$ in an arc linking distinct vertices w_i, w_j. Let S'_i be the union of the corresponding adhesion sets Z_i, Z_j in a vortex decomposition of V of adhesion $\leq \alpha_2$. Again, S_i and S'_i are as desired. □

6 Streamlining Path Systems

In this section we provide tools that allow us to find path systems in near-embeddings that satisfy conditions (v) and (vi) in the definition of (β, r)-rich.

Lemma 18. *Let G be a graph and A, B, C subsets of $V(G)$ with $|B| = 2k-1$ for some integer k. If G contains a set $\mathcal{P}$ of $2k-1$ disjoint A–B paths, and a set $\mathcal{Q}$ of $2k-1$ disjoint B–C paths, then there are k disjoint A–C paths in G.*

Proof. Let $\mathcal{P}$ be a set of $2k-1$ disjoint A–B paths, and let $\mathcal{Q}$ a set of $2k-1$ disjoint B–C paths in G. For every set $S \subseteq V(G)$ with $|S| < k$, at least k paths in $\mathcal{P}$ and at least k paths in $\mathcal{Q}$ avoid S. Two of these paths contain a common vertex of B, so $G - S$ contains a path from A to C. The existence of k disjoint A–C paths now follows by Menger's theorem. □

Lemma 19. *Let G be a graph embedded in a surface, let H be a flat wall in G of size $32k^2+r$ in G for integers $k < r$, and let Ω be a subset of $V(G)$ avoiding $D(H)$ such that there are $16k^2$ disjoint paths from Ω to branch vertices of H. Then H contains a wall H_0 of size r and k disjoint paths from Ω to branch vertices of H_0 that lie on the boundary cycle of H_0.*

Proof. Let H_0 be an r-wall in H with $8k^2$ concentric cycles in H enclosing H_0. Choose a set $\mathcal{P}$ of $16k^2$ disjoint paths from Ω to branch vertices of H such that $|E(\mathcal{P}) \setminus E(H)|$ is minimal, and among these so that $\sum_{P \in \mathcal{P}} |P|$ is minimal.

We claim that no path $P \in \mathcal{P}$ meets H_0. Otherwise P would meet each of our $8k^2$ concentric cycles $C \subseteq H$ without ending on C. By the choice of $\mathcal{P}$, this means that P contains no branch vertex of H, but meets C only inside one subdivided edge of C before leaving it again. By our first condition for the choice of $\mathcal{P}$, the branch vertices of H that are the ends of this subdivided edge must each lie on another path from $\mathcal{P}$, which must end there. So we

have at least $16k^2$ paths from $\mathcal{P}$ other than P ending at such branch vertices, which contradicts our assumption that $|\mathcal{P}| = 16k^2$.

As $|\mathcal{P}| = 16k^2$, either at least $4k$ rows or at least $4k$ columns contain terminal vertices of paths in $\mathcal{P}$. In either case it is easy to see that at least half of these branch vertices (i.e. $\geq 2k$) can be linked disjointly to branch vertices on the boundary cycle of H_0. Lemma 18 completes the proof. $\square$

Let G be a graph and $X, Y \subseteq V(G)$ with $|X| = |Y| =: k$. An X–Y *linkage* in G is a set of k disjoint paths in G such that each of these paths has one end in X and the other end in Y.

An X–Y linkage $\mathcal{P}$ in G is *singular* if $V(\bigcup \mathcal{P}) = V(G)$ and G does not contain any other X–Y linkage. The next lemma will be used in the proof of Lemma 21.

Lemma 20. *If a graph G contains a singular X–Y linkage $\mathcal{P}$ for vertex sets $X, Y \subseteq V(G)$, then G has path-width at most $|\mathcal{P}|$.*

Proof. Let $\mathcal{P}$ be a singular X–Y linkage in G. Applying induction on $|G|$, we show that G has a path-decomposition $(X_0, \ldots, X_n)$ of width at most $|\mathcal{P}|$ such that $X \subseteq X_0$. For every $x \in V(G)$ let $P(x)$ denote the path $P \in \mathcal{P}$ that contains x. Suppose first that every $x \in X$ has a neighbour $y(x)$ in G that is not its neighbour on $P(x)$. Then $y(x) \notin P(x)$ by the uniqueness of $\mathcal{P}$. The digraph on $\mathcal{P}$ obtained by joining for every $x \in X$ the 'vertex' $P(x)$ to the 'vertex' $P(y(x))$ contains a directed cycle D. Let us replace in $\mathcal{P}$ for each $x \in X$ with $P(x) \in D$ the path $P(x)$ by the X–Y path that starts in x, jumps to $y(x)$, and then continues along $P(y(x))$. Since every 'vertex' of D has in- and outdegree 1 in D, this yields an X–Y linkage with the same endpoints as $\mathcal{P}$ but different from $\mathcal{P}$. This contradicts our assumption that $\mathcal{P}$ is singular. Thus, there exists an $x \in X$ without any neighbours in G other than (possibly) its neighbour on $P(x)$. Consider this x.

If $P(x)$ is trivial, then x is isolated in G and $x \in X \cap Y$. By induction, $G - x$ has a path-decomposition $(X_1, \ldots, X_n)$ of width at most $|\mathcal{P}| - 1$ with $X \setminus \{x\} \subseteq X_1$. Add $X_0 := X$ to obtain the desired path-decomposition of G. If $P(x)$ is not trivial, let x' be its second vertex, and replace x in X by x' to obtain X'. By induction, $G - x$ has a path-decomposition $(X_1, \ldots, X_n)$ of width at most $|\mathcal{P}|$ with $X' \subseteq X_1$. Add $X_0 := X \cup \{x'\}$ to obtain the desired path-decomposition of G. $\square$

Our next lemma is a weaker version of Theorem 10.1 of [10].

Lemma 21. *Let s, and t be positive integers with $s \geq t$. Let G' be a graph embedded in the plane, and let $X \subseteq V(G')$ be a set of t vertices on a common*

face boundary of G'. Let $(C_1, \ldots, C_s)$ be concentric cycles in G', tightly enclosing X. Let G'' be another graph, with $V(G') \cap V(G'') \subseteq V(C_1)$. Assume that $G' \cup G''$ contains an X–Y linkage $\mathcal{P}$ with $Y \subseteq V(C_1)$. Then there exists an X–Y linkage $\mathcal{P}'$ in $G' \cup G''$ such that $\mathcal{P}'$ is orthogonal to $C_{t+1}, \ldots, C_s$.

Proof. Assume the lemma is false, and let G', G'', $\mathcal{P}$, and $(C_1, \ldots, C_s)$ form a counterexample with the minimum number of edges. Then $G := G' \cup G'' = \bigcup_{i=1}^{s} C_i \cup \mathcal{P}$, and $V(G) = V(\bigcup \mathcal{P})$. (Delete isolated vertices if necessary). Let us show that, for all $P \in \mathcal{P}$ and for all $1 \leq i \leq s$, every component of $P \cap C_i$ is a single vertex. Indeed, if $P \cap C_i$ had a component containing an edge e, then G'/e would form a counterexample with fewer edges, since any path using the contracted vertex v_e would still form, or could be expanded to a path for our desired path system $\mathcal{P}'$.

Next, let us show that

$$\mathcal{P} \text{ is singular.} \tag{7}$$

If there exists an X–Y linkage $\overline{\mathcal{P}}$ distinct from $\mathcal{P}$, then at least one of the edges of $\mathcal{P}$ is not contained in $\overline{\mathcal{P}}$. Since, as noted above, the paths in $\mathcal{P}$ have no edges on $C_1, \ldots, C_s$, the subgraph $\bigcup_{i=1}^{s} C_i \cup \bigcup \overline{\mathcal{P}}$ forms a counterexample with fewer edges, a contradiction. This proves (7).

Our choice of the cycles C_i as tightly enclosing X implies at once:

$$\text{There is no subpath } Q \text{ of some path } P \in \mathcal{P} \text{ in } D(C_j) \text{ with both endpoints in } C_j \text{ for some } j \text{ and otherwise disjoint from } \bigcup_i V(C_i). \tag{8}$$

A *local peak* of $\mathcal{P}$ is a subpath Q of a path $P \in \mathcal{P}$ such that Q has both endpoints on C_j for some $j > 1$ and every internal vertex of Q in $(\bigcup_i V(C_i))$ lies in $V(C_{j-1})$.

Let us show the following:

$$\mathcal{P} \text{ has no local peak} \tag{9}$$

Suppose $Q = x \ldots y \subseteq P \in \mathcal{P}$ is a local peak, with endpoints in C_j say, chosen so that j is maximal.

Let xC_jy denote the subpath of C_j such that the cycle $xC_jy \cup Q$ bounds a disc $D \subseteq D(C_{j-1}) \setminus D(C_j)$. If no interior vertex of xC_jy lies on a path from $\mathcal{P}$, we can replace Q by xC_jy on P and then contract this subpath of P, to obtain a counterexample with fewer edges. Hence xC_jy does have an interior vertex z on a path $P' \in \mathcal{P}$. Let $zP'z'$ be a minimal non-trivial subpath of P' such that $z' \in \bigcup_i C_i$ (This exists, as $j > 1$.) If $z' \in C_j$, then $zP'z' \subseteq D$ by (8). We then repeat the argument, with the local peak $zP'z'$ instead of

Q. This can happen only finitely often, and will eventually contradict the minimality of our counterexample. We may thus assume that z' cannot be chosen in C_j. Then $zP'z' \cap D = \emptyset$ and $z' \in C_{j+1}$, and $zP'z'$ extends to a subpath $z''P'z'$ of P' with $z'' \in C_{j+1}$ and no vertex other than z, z', z'' in $\bigcup_i C_i$. This path is a local peak of $\mathcal{P}$ that contradicts our choice of Q with j minimal, completing the proof of (9).

An immediate consequence of (8) and (9) is the following. For every $P \in \mathcal{P}$, let x be the endpoint of P in X and let y be the vertex of $V(C_1) \cap V(P)$ closest to x on P. Then the subpath $\overline{P} := xPy$ of P is orthogonal to the cycles $C_1, \ldots, C_s$. In fact, $\overline{P} \cap C_i$ is a single vertex, for each $1 \leq i \leq s$.

The final claim will complete our proof:

For every $P \in \mathcal{P}$, the path $P - \overline{P}$ does not meet C_{t+1}. (10)

To prove (10), suppose there exists $P \in \mathcal{P}$ such that $(P - \overline{P}) \cap C_{t+1} \neq \emptyset$. As before, it follows now from (8) and (9) that $P - \overline{P}$ contains a subpath Q from C_{t+1} to C_1 that is orthogonal to the cycles $C_{t+1}, C_t \ldots, C_1$. Together with final segments of our paths $\overline{P}$ and the cycles $C_1, \ldots, C_{t+1}$, this path Q' forms a subdivision of the $(t+1) \times (t+1)$ grid, which is well known to have path-width $t+1$. This contradicts (7) and Lemma 20, proving (10). □

7 Proof of the Main Result

Before proceeding with the proof of Theorem 2, we will need one more lemma. A similar result can be found in [5].

Lemma 22. *For every integer t and all integers $\alpha, g > 0$ there is an integer $s > 0$ such that the following holds. Let G be a graph of tree-width at least s and $(\sigma, G_0, A, \mathcal{V}, \emptyset)$ an α-near embedding of G in a surface Σ of genus g such that all vortices $V \in \mathcal{V}$ have depth at most α. Then G_0 has tree-width at least t.*

Proof. Let t, α, g be given. By Lemma 10, there is an integer r such that for every graph H of tree-width at least r embedded in a surface Σ of genus g the contraction of α disjoint faces of H to vertices leaves a graph of tree-width at least $t + \alpha$.

Let G be a graph as stated in the Lemma, of tree-width $s > \alpha r$. Let G_0^+ be the graph we obtain if for every vortex $V \in \mathcal{V}$ with $\Omega(V) = (w_1, \ldots, w_n)$ say, we add to G_0 all edges $w_j w_{j+1}$ for $1 \leq j \leq n$, where $n + 1 := 1$ if not already in G_0. Clearly, σ can be extended to an embedding of G_0^+ by embedding the new edges in the corresponding discs $D(V)$.

The tree-width of G_0^+ is at least r. (11)

Otherwise, choose a tree-decomposition $(V_t)_{t\in T}$ of G_0^+ of width less than r. For every vortex $V \in \mathcal{V}$ choose a fixed decomposition $(X_1, \dots, X_n)$ of depth at most α. For every $t \in T$ define

$$V_t' := V_t \cup \bigcup_{V\in\mathcal{V}} \{X_j : w_j \in V_t\}$$

Note that, as all vortices are disjoint and thus every vertex in V_t can be a society vertex of at most one vortex, we have $|V_t'| \le \alpha|V_t| < \alpha r$. We claim that $(V_t')_{t\in T}$ is a tree-decomposition of $G \cup G_0^+$. To see this, pick a vertex $v \in V_{t_1} \cap V_{t_3}$ for distinct $t_1, t_3 \in T$. We have to show that $v \in V_{t_2}'$ for all $t_2 \in t_1Tt_3$. Let us assume that $v \notin V(G_0^+)$ as the other case is easy. By construction, there is a vortex V with $\Omega(V) = (w_1, \dots, w_n)$ such that for some $w_j, w_k \in \Omega(V)$, we have $v \in X_j \cap X_k$ and $w_j \in V_{t_1}$ and $w_k \in V_{t_3}$. We may assume without loss of generality that $j < k$. By construction, G_0^+ contains path $w_jw_{j+1}\dots w_k$. As V_{t_2} separates V_{t_1} from V_{t_3} in $G \cup G_0^+$, there is a vertex $w_\ell \in V_{t_2}$ for some $j \le \ell \le k$. Then $v \in X_\ell$, since $(X_1, \dots, X_n)$ is a path-decomposition, so $v \in V_{t_2}$ as desired.

Clearly, $(V_t)_{t\in T}$ is a tree-decomposition of G as well, but it has width at most αr, a contradiction to our choice of G. This proves (11).

For every vortex $V \in \mathcal{V}$ there is a face $f \subseteq D(V)$ of G_0^+ with $\Omega(V) = \partial f \cap G_0^+$. By the choice of r, contracting all these faces to vertices yields a graph of tree-width at least $t + \alpha$. Removing the new vertices, of which we have at most α, results in the graph $G_0 \setminus \bigcup \mathcal{V}$ with tree-width at least t: note that the new edges of G_0^+ disappear in these two steps. Thus, the graph G_0 has tree-width at least t as well, proving the lemma. □

Proof of Theorem 2. Let $\hat\alpha$ be the integer α provided for R and $m = 0$ by Theorem 4, and let $\hat\gamma$ be an integer such that R embeds in every surface Σ with $g(\Sigma) > \hat\gamma$. By Theorems 4 and 6 and Lemmas 22, 7 and 8, there is an integer w such that if the tree-width of our graph G is larger than w, the following holds: There is a tree-decomposition $(V_t)_{t\in T}$ of G such that the torso $\hat G$ of one part V_{t_0} has an $\hat\alpha$-near embedding $(\hat\sigma, \hat G_0, \hat A, \hat{\mathcal{V}}', \emptyset)$ in a surface $\hat\Sigma$ in which R cannot be embedded such that $\hat G_0'$ contains a flat wall of size at least

$$6^{\hat\alpha+2\hat\gamma+1}(r + \hat\alpha(\beta + \hat\alpha + 3) + p),$$

where $p := 2\hat\alpha(\beta + 2\hat\alpha + 2\hat\gamma + 4) + 4$. We will show that, with these constants, we find an α-near embedding of G for $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ defined as

$$\begin{aligned}
\alpha_0 &:= \hat\alpha + p(2\hat\gamma + \hat\alpha) + 2\hat\alpha^2 + 2\hat\alpha \\
\alpha_1 &:= \hat\alpha + \hat\gamma \\
\alpha_2 &:= 2\hat\alpha + \hat\gamma
\end{aligned}$$

that is almost (β, r)-rich: The near embedding satisfies all the desired properties except for (v) and (vi). Instead, we only find paths linking the societies of large vortices to arbitrary branch vertices of a large wall. But this can be remedied: We apply the result for $32\beta^2 + r$ and $16\beta^2$ instead for r and β, respectively, and with Lemmas 19 and 21 we obtain a (β, r)-rich near-embedding as desired.

First, we will convert the near-embedding of the torso $\hat{G}$ into a near-embedding of the whole graph G by accomodating the rest in its vortices. To accomplish this, we will use property (ii) from Theorem 4 of our tree-decomposition. Pick a component T' of $T - t_0$ and let t' be the vertex in this component adjacent to t_0 in T. Let $Y := (\bigcup_{t \in T'} V_t) \setminus \hat{A}$. If t' is the parent of t_0 in T, i.e., if $t' \in rTt_0$, then $V_{t'} \cap V_{t_0} \subseteq \hat{A}$ and hence $Y \cap V_{t_0} = \emptyset$. We then add $G[Y]$ to $\hat{G}$ as a small vortex. Suppose now that t' is a child of t_0, i.e., that $t_0 \in rTt'$. Then, by (ii) of Theorem 4, we can either add $G[Y]$ to a part $X_{tt'}$ of a vortex V of $\hat{G}$ without increasing the adhesion of V, or we can add $G[Y]$ as a small vortex that is properly attached.

We perform this modification for all components of $T - t_0$. Let us collect in a set $\hat{\mathcal{W}}$ the new small vortices defined, and let $\hat{\mathcal{V}}$ denote the set of the new possibly modified, large vortices. By merging vortices if necessary, we may assume that there are no two vortices $W, W' \in \hat{\mathcal{W}}$ with $\Omega(W) \subseteq \Omega(W')$. Note that $(\hat{\sigma}, \hat{G}_0, \hat{A}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$ is an $\hat{\alpha}$-near-embedding of all of G, and thus also an α-near embedding.

Let us, more generally, consider α-near-embeddings $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of G in surfaces Σ such that

$$\left.\begin{array}{ll}
\bullet\ \textit{All vortices in } \mathcal{W} \textit{ are properly attached} & \\
\bullet\ \textit{All vortices in } \mathcal{V} \textit{ have adhesion at most} & \\
\quad \hat{\alpha} + g(\hat{\Sigma}) - g(\Sigma) + |\hat{\mathcal{V}}| - |\mathcal{V}| & \\
\bullet\ g(\Sigma) \leq g(\hat{\Sigma}) & \\
\bullet\ |\mathcal{V}| \leq |\hat{\mathcal{V}}| + (g(\hat{\Sigma}) - g(\Sigma)) \qquad (\leq \alpha_1) & \\
\bullet\ |A| \leq |\hat{A}| + p\Big(2\big(g(\hat{\Sigma}) - g(\Sigma)\big) + |\hat{\mathcal{V}}| - |\mathcal{V}|\Big) \qquad (\leq \alpha_0) &
\end{array}\right\} \quad (\star)$$

and further

$$G_0' \textit{ contains a flat wall } H_0 \textit{ of size at least } 6^q\mu. \qquad (\star\star)$$

where

$$q := |\mathcal{V}| + 2g(\Sigma) + 1$$
$$\mu := \mu(\sigma, G_0, A, \mathcal{V}, \mathcal{W}) := r + |\mathcal{V}|(\beta + 2\hat{\alpha} + \hat{\gamma} + 3) + p$$

Such near-embeddings exist, since $(\hat{\sigma}, \hat{G}_0, \hat{A}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$ satisfies $(\star)$ and $(\star\star)$.

Our next task is to find, among all such near embeddings, one with the following additional properties (P1)–(P4):

(P1) Every two vortices have distance at least $2\lambda + 3$ in Σ.

(P2) For every vortex $V \in \mathcal{V}$ there exist λ cycles $(C_1, \ldots, C_\lambda)$ tightly enclosing V. If $\Sigma \not\simeq S^2$, the representativity of G_0' in Σ is at least λ.

(P3) For all distinct vortices $V, W \in \mathcal{V}$, the discs $\overline{D(C_1(V))}$ and $\overline{D(C_1(W))}$ are disjoint.

(P4) H_0 contains a flat wall H of size 6μ such that $D(H) \cap D(C_1(V)) = \emptyset$ for every $V \in \mathcal{V}$.

where

$$\lambda := \lambda(\sigma, G_0, A, \mathcal{V}, \mathcal{W}) := |\mathcal{V}|(\beta + 2\hat{\alpha} + \hat{\gamma} + 3).$$

From all α-near-embeddings satisfying $(\star)$ and $(\star\star)$ let us pick one minimizing $(g(\Sigma), |\mathcal{V}|)$ lexicographically. We will denote this near-embedding by ε. We will show that either ε itself has the properties (P1)–(P4) or we can find a disc in Σ such that, roughly said, the part of our graph nearly-embedded in this disc can be considered as a near-embedding in S^2 with these properties.

For the next steps in the proof, we will repeatedly make use of the following fact: for integers ℓ, r, consider a flat wall W of size $8\ell + 2r$ in G_0'. In W, we can find two subwalls W_1, W_2 of size r, together with ℓ concentric cycles $C_1(W_1), \ldots, C_\ell(W_1)$ around W_1 and ℓ concentric cycles $C_1(W_2), \ldots, C_\ell(W_2)$ around W_2 such that $\overline{D(C_1(W_1))}$ and $\overline{D(C_1(W_2))}$ are disjoint. In particular, W_1 and W_2 have distance at least $2\ell + 2$ in Σ. Further, if V is a vortex tightly enclosed by $k < \ell$ cycles $C_1(V), \ldots, C_k(V)$, then any two vertices picked from the cycles $C_1(V), \ldots, C_k(V)$ have distance at most $2k < 2\ell$ in Σ. Now, a comparison of the distances shows that one of the walls W_1, W_2 is disjoint from $\Omega(V)$ and all the cycles $C_1(V), \ldots, C_k(V)$.

Finally note that, if we delete a set X of k vertices from a wall H of size $\ell > k$, at most k rows and at most k columns of H are hit by X and thus, $H - X$ contains a wall of size at least $\ell - 4k$.

Let us show first that our near-embedding ε has property (P1). Otherwise we apply Lemma 14 with $d := 2\lambda$. This gives a vertex set A' of size at most $2(2\hat{\alpha} + \hat{\gamma}) + d \leq p$ and a near-embedding $\varepsilon' := (\sigma', G_0 - A', A \cup A', \mathcal{V}', \mathcal{W}')$ with $|\mathcal{V}'| \leq |\mathcal{V}| - 1$ of G in Σ. By Lemma 3, we may assume that its small vortices are properly attached. Then, ε' satisfies $(\star)$ and $(\star\star)$ but $(g(\Sigma), |\mathcal{V}'|) < (g(\Sigma), |\mathcal{V}|)$ lexicographically, which contradicts the choice of ε.

To show properties (P2)–(P4) we consider two cases: when $\Sigma \simeq S^2$ and when $\Sigma \not\simeq S^2$.

First, we assume that $\Sigma \simeq S^2$. Given a vortex $V \in \mathcal{V}$ it is easy to find in H_0 a subwall H half the size of H_0 such that $\overline{D(V)} \cap \overline{D(H)} = \emptyset$. Then H has size at least $3 \cdot 6^{q-1}\mu > 6^{q-1}\mu + 4\lambda$ (as $q - 1 \geq |\mathcal{V}| \geq 1$). Inside H there is a flat wall H_V of size at least $6^{q-1}\mu$ enclosed by λ cycles $C_1, \ldots, C_\lambda \subseteq H$. As $\Sigma \simeq S^2$, $C_\lambda, \ldots, C_1$ enclose V, and $\overline{D(C_\lambda)} \cap \overline{D(H_V)} = \emptyset$. Lemma 13 shows that there also exist λ cycles $C_1(V), \ldots, C_\lambda(V)$ tightly enclosing V such that $D(C_1(V))$ does not meet $D(H_V)$. Iterating this procedure for all $V \in \mathcal{V}$ establishes (P2), while replacing our original wall H_0 with a flat subwall H of size at least $6^{q-|\mathcal{V}|}\mu \geq 6\mu$ that satisfies $\overline{D(H)} \cap \overline{D(C_1(V))} = \emptyset \ \forall\ V \in \mathcal{V}$.

To prove (P3) suppose, that for two vortices $V, W \in \mathcal{V}$ the discs $\overline{D(C_1(V))}$ and $\overline{D(C_1(W))}$ intersect. By (P1), the cycles $C_1(V), \ldots, C_\lambda(V), C_1(W), \ldots, C_\lambda(W)$ are disjoint, so we may assume that $D(C_1(V)) \subseteq D(C_1(W))$. By Lemma 17, there is a disc $\Delta \subseteq D(C_1(W))$ containing $D(V)$ and a separation (X_1, X_2) of G of order at most $2\lambda + 2\hat{\alpha} \leq p$ such that $G_0 \cap X_1 = G_0 \cap \Delta$. Now let $\tilde{\mathcal{V}}$ be the set of all vortices of $\mathcal{V} - X_1$ with a society of size at least 4, and let $\tilde{\mathcal{W}}$ be the set of all vortices of $\mathcal{W} - X_1$, the vortex $(G[X_1], \emptyset)$, and the vortices of $\mathcal{V} - X_1$ with a society of at most 3 vertices. Clearly, $|\tilde{\mathcal{V}}| < |\mathcal{V}|$, as $\Omega(V) \subseteq X_1$. It is easy to see now that

$$(\sigma|_{G_0 - X_1}, G_0 - X_1, A \cup (X_1 \cap X_2), \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$$

is an $(\alpha_0, \alpha_1, \alpha_2)$ near-embedding of G in Σ, satisfying $(\star)$, and as H is a sufficiently large wall living in $G_0 - X_1$, condition $(\star\star)$ holds as well. This means that $(g(\Sigma), |\tilde{\mathcal{V}}|) < (g(\Sigma), |\mathcal{V}|)$, a contradiction to our choice of ε. This proves (P3) and (P4).

We now consider the case when $\Sigma \not\simeq S^2$. Our plan is to deduce (P2) from (P1) and Lemmas 11 and Lemmas 16. Thus, we must first show that the representativity of G_0' in Σ is at least $2\lambda + 2$. Suppose not, and apply Lemma 16 with $z := 2\lambda + 2$. If (a) of Lemma 16 holds, we have a near-embedding ε' of G in a surface Σ' with $g(\Sigma') < g(\Sigma)$ and $|\mathcal{V}'| \leq |\mathcal{V}| + 1$. The properties $(\star)$ and $(\star\star)$ are easy to verify; for $(\star\star)$, notice that $5 \cdot 6^{q-1}\mu \geq 8\lambda + 8$, so deleting up to z vertices from our wall H leaves a wall of size at least $6^{q-1}\mu$. Hence, the fact that $(g(\Sigma'), |\mathcal{V}'|) < (g(\Sigma), |\mathcal{V}|)$ contradicts our choice of ε. Similarly, if (b) of Lemma 16 holds, then one of the graphs G'^1_0, G'^2_0 contains a sufficiently large wall, so one of the near-embeddings $\varepsilon_1, \varepsilon_2$ satisfies $(\star)$ and $(\star\star)$, and $g(\Sigma_1), g(\Sigma_2) < g(\Sigma)$ yields the same contradiction as before. This shows that the representativity of G_0' in Σ is at least $2\lambda + 2$. We now apply Lemma 11 to each of the faces of G_0' that contain the disc $D(V)$ of a vortex $V \in \mathcal{V}$. Together with Lemma 13, this implies property (P2).

To show property (P3), assume that for two vortices $V, W \in \mathcal{V}$ the discs $\overline{D(C_1(V))}$ and $\overline{D(C_1(W))}$ intersect. As before we may assume that $D(C_1(V)) \subseteq D(C_1(W))$, and an application of Lemma 17 gives us a disc $\Delta \subseteq D(C_1(W))$ containing $D(V)$, and a separation (X_1, X_2) of G of order at most $2\lambda + 2\hat{\alpha} \leq p$ such that $G_0 \cap X_1 = G_0 \cap \Delta$. As noted earlier, there exists a flat subwall H of H_0 of size at least $6^{q-1}\mu$ that is disjoint from $\Omega(W)$ and all the cycles $(C_1(W), \ldots, C_\lambda(W))$, and hence from $X_1 \cap X_2$ (as $X_1 \cap X_2 \cap G_0' \subseteq \bigcup C_i(W) \cup \Omega(W)$ in Lemma 17). If $D(H) \cap \Delta = \emptyset$, then $H \subseteq G[X_2 \setminus X_1]$, so turning $G[X_1]$ into a small vortex attached (by an empty society) to $G_0' \cap G[X_2 \setminus X_1]$ we can reduce the number of large vortices of our near-embedding, leading to the same contradiction as earlier. Otherwise, $D(H) \subseteq \Delta$ and $H \subseteq G[X_1 \setminus X_2]$. We now turn $G[X_2]$ into a small vortex attached to $G_0' \cap G[X_1 \setminus X_2]$ and obtain a contradiction to the minimality of ε, since $g(\Delta) = 0 < g(\Sigma)$. This completes the proof of (P3) for the case of $\Sigma \not\simeq S^2$.

To show (P4), let us enumerate the vortices $\mathcal{V} =: \{V_1, \ldots, V_\ell\}$. We will prove by induction on k that, for $1 \leq k \leq \ell$, there is a flat wall $H_k \subseteq H_{k-1}$ of size $6^{q-k}\mu$ such that $D(H_k)$ avoids $D(C_1(V_1)), \ldots, D(C_1(V_k))$. For $k = 0$ this is precisely $(\star\star)$. Given $k \geq 1$, we have $q \geq k + 3$ and $\lambda \leq \mu/2$. Hence, as earlier, we can find a subwall $H_k \subseteq H_{k-1}$ of size $6^{q-(k-1)}\mu/6 \geq 6^{q-k}\mu \geq 6^3\mu$ from (P2) that avoids $\Omega(V_k)$ and all cycles $C_1(V_k), \ldots, C_\lambda(V_k)$. If $D(H_k) \cap D(C_1(V_k)) = \emptyset$, then this completes the induction step. Otherwise, Lemma 17 gives us a disc $\Delta \subseteq D(C_1(V_k))$ containing $D(H_k)$ and a separation (X_1, X_2) of G of order at most $2\lambda + 2\hat{\alpha} \leq p$ such that $G_0 \cap X_1 = G_0 \cap \Delta$ and $X_1 \cap X_2 \cap V(G_0) \subseteq \bigcup C_i(V_k) \cup \Omega(V_k)$. By (P3), this disc Δ does not contain $D(W)$ for any large vortex $W \neq V_k$. We now turn $G[X_2 \setminus X_1]$ into a small vortex attached by an empty society to $G_0' \cap G[X_1 \setminus X_2]$. We are now back in the case of $\Sigma = S^2$ treated before, and can find inside H_k (which we recall has size at least $6^{q-k} \geq 6^3\mu \geq 6\mu + 4\lambda$) a flat subwall H' of size $6\mu + 4\lambda$ with $\overline{D(H')} \cap \overline{D(V_k)} = \emptyset$. Inside H' there is a wall of size 6μ (which we note is large enough to satisfy $(\star\star)$ for our large vortex and genus 0) enclosed by λ cycles in H'. Re-interpreting these cycles as enclosing V_k, as earlier in our proof of (P2)–(P4) for $\Sigma \simeq S^2$, we once more obtain a contradiction to the minimality of ε. This completes the proof of (P4) for the case of $\Sigma \not\simeq S^2$.

From all α-near-embeddings $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of G into surfaces Σ satisfying $(\star)$ and $(\star\star)$ and (P1)–(P4) let us choose one minimizing $|\mathcal{V}|$. Let H be the wall from (P4).

An application of Lemma 15 now gives us a subgraph $\tilde{G}_0$ of G_0, a vertex set $\tilde{A} \subseteq V(G) \setminus V(\tilde{G}_0)$ with $|\tilde{A}| \leq 2\hat{\alpha}^2 + 2\hat{\alpha}$ disjoint from H and an α-near-embedding $\tilde{\varepsilon} := (\tilde{\sigma}, \tilde{G}_0, A \cup \tilde{A}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ of G such that every vortex $\tilde{V} \in \tilde{\mathcal{V}}$ has a linked decomposition of adhesion at most $\hat{\alpha}$, there are still at least

$\tilde{\lambda} := \lambda - (\hat{\alpha} + 1)$ cycles enclosing every $\tilde{V} \in \tilde{\mathcal{V}}$.

Let us show that there is no vertex set S in G'_0 of size less than β separating H from $\Omega(V)$ for some vortex $V \in \tilde{\mathcal{V}}$. Suppose there is and let us choose S minimal. We add to $\tilde{G}'_0$ a vertex v and edges from v to all society vertices $\Omega(V)$. Clearly, after adding v to X_2, the set S still separates H from $\Omega(V)$ and we can extend our embedding by mapping v and the new edges to $D(V)$. By the minimality of S, every vertex in S is adjacent to some vertex of the component T_0 of $G'_0 - S$ that contains v. Let T denote the (connected) graph T_0 together with S and all edges between T_0 and S. We note that T_0 contains $\Omega(V)$.

We fatten the embedded graph T to obtain a closed, connected set $D \subseteq \Sigma$ so that D contains T and further that that ∂D intersects with G'_0 only in edges incident with both S and $G'_0 \setminus T$.

Every component C of ∂D bounds a cycle in Σ. This is clear if $\Sigma \simeq S^2$ and if $\Sigma \not\simeq S^2$ we could otherwise slightly shift C in neighbourhoods of vertices in S to intersect with G'_0 only in S and obtain a genus reducing curve that hits G'_0 in less than β many vertices, contradicting (P2).

Let H' be a subwall of H of size at least $6\mu - 4\beta$ that avoids S and let Z be the component of $\Sigma - \partial D$ containing H'. We define $X_1 := (V(G'_0) \cap Z) \cup S$ and $X_2 := V(G'_0) \cap (\Sigma \setminus Z)$. This gives us a separation (X_1, X_2) with $S \subseteq X_1 \cap X_2$.

Further (X_1, X_2) can be modified so that, for every vortex $V' \in \tilde{\mathcal{V}} \setminus \{V\}$, its society $\Omega(V')$ and at least $\tilde{\lambda} - \beta$ many cycles enclosing V' are contained either in X_1 or in X_2. Indeed, as one of the cycles $C_1(V'), \ldots, C_\beta(V')$ enclosing V' is not hit by $X_1 \cap X_2$, it is contained in either X_1 or X_2. As this (plane) cycle C is a separator of G'_0, we can add all the vertices embedded in $D(C)$, in particular $\Omega(V')$ and the vertices of the cycles $C_{\beta+1}, \ldots, C_{\tilde{\lambda}}$, to the same X_i.

Let us consider the case that Z is a disc. As before, we add $X_1 \cap X_2$ to the apex set $A \cup \tilde{A}$ and add a new small vortex $W := (G[X_1 \setminus X_2], \emptyset)$ to $\tilde{\mathcal{W}}$. This new near-embedding of G in S^2 still satisfies $(\star)$, $(\star\star)$, and (P1)–(P4), but we have reduced $|\tilde{\mathcal{V}}|$ with this operation, a contradiction to our choice of the near-embedding.

Suppose Z is not a disc. Then, as each component of ∂Z bounds a disc and each component of $\Sigma \setminus Z$ contains a point of D, which is connected, $\Sigma \setminus Z$ has exactly one component $Z' \simeq S^2$. Again, we add $X_1 \cap X_2$ to the apex set and accomodate $X_2 \setminus X_1$ in a small vortex and obtain a near embedding of G with a reduced number of large vortices. Let us show that the representativity of our new embedded graph $\hat{G} := \tilde{G}'_0 - X_2$ is at least $\lambda - \beta$. Assume the opposite and pick a genus reducing $\hat{G}$-normal curve C in Σ that meets less than $\lambda - \beta$ vertices of $\hat{G}$. Then, we may assume by Lemma 12 that C intersects the face f of $\hat{G}$ containing Z' at most once. We reroute C

in f along $\partial Z'$. Now, C is also a $\tilde{G}'_0$-normal curve meeting at most β many additional vertices from S, which contradicts (P2) for $\tilde{G}'_0$. Now, (P1)–(P4) are easy to verify.

We conclude that, for every large vortex $V \in \tilde{\mathcal{V}}$, the society $\Omega(V)$ is connected to the branch vertices of H by β many disjoint paths.

Finally, as described in the beginning, Lemmas 19 and 21 finish the proof. □

8 Circular Vortex-Decompositions

In Graph Minors XVII [16], the structure theorem is stated with vortices having a circular instead of a linear structure. For most applications, the linear decompositions as discussed so far in this paper are sufficient, but sometimes the circular structure is necessary. In this section, we introduce circular vortex decompositions and point out how we can derive a new lemma from the proof of Lemma 15 that yields circular linkages for them. It is easy to see that we can apply this new lemma instead of Lemma 15 at the end of the proof of Theorem 2 and therefore, we can choose to have circular linkages for the large vortices when we apply the theorem.

For the remainder of this paper, we call decompositions of vortices as defined in Section 2 *linear decompositions* to distinguish them more clearly from the circular decompositions which we introduce now:

Let $V := (G, \Omega)$ be a vortex with $\Omega = (w_1, \ldots, w_n)$. Let us regard the ordering of Ω as a cyclic ordering. A tuple $\mathcal{D} := (X_1, \ldots, X_n)$ of subsets of $V(G)$ is a *circular decomposition* of V if the following properties are satisfied:

(i) $w_i \in X_i$ for all $1 \leq i \leq n$.

(ii) $X_1 \cup \ldots \cup X_n = V(G)$.

(iii) When $w_i < w_j < w_k < w_\ell$ are society vertices of V ordered with respect to the cyclic ordering Ω, then $X_i \cap X_k \subseteq X_j \cup X_\ell$

(iv) Every edge of G has both ends in X_i for some $1 \leq i \leq n$.

The *adhesion* of our circular decomposition $\mathcal{D}$ of V is the maximum value of $|X_{i-1} \cap X_i|$, taken over all $1 \leq i \leq n$. We define the *circular adhesion* of V as the minimum adhesion of a circular decomposition of that vortex.

When $\mathcal{D}$ is a circular decomposition of a vortex V as above, we write $Z_i := (X_i \cap X_{i+1}) \setminus \Omega$, for all $1 \leq i < n$. These Z_i are the *adhesion sets* of $\mathcal{D}$. We call $\mathcal{D}$ *linked* if

- all these Z_i have the same size;
- there are $|Z_i|$ disjoint Z_{i-1}–Z_i paths in $G[X_i] - \Omega$, for all $1 \leq i \leq n$;
- $X_i \cap \Omega = \{w_{i-1}, w_i\}$ for all $1 \leq i \leq n$.

Note that $X_i \cap X_{i+1} = Z_i \cup \{w_i\}$, for all $1 \leq i \leq n$ (Fig. 1).

The union of the Z_{i-1}–Z_i paths in a circular decomposition of V is a disjoint union of cycles in G each of which traverses the adhesion sets of $\mathcal{D}$ in cyclic order (possibly several times); We call the set of these cycles a *circular linkage* of V with respect to $\mathcal{D}$.

As described in Section 2 for linear decompositions, we see that we can delete a vertex from a circular decomposition of some vortex and obtain a new circular decomposition. This operation does not increase the adhesion but might decrease the number of society vertices.

Clearly, a linear decomposition of some vortex is a circular decomposition as well and it is easy to see that one can obtain a linear from a circular decomposition, if one deletes the overlap of two subsequent bags: Let $V := (G, \Omega)$ a vortex and $(X_1, \ldots, X_n)$ a circular decomposition of V. Delete the set $X_{i-1} \cap X_i$ from V for some index $1 \leq i \leq n$. We obtain a circular decomposition $\mathcal{D} := (X'_1, \ldots, X'_{n'})$ of $V - Z$ with $n' \leq n$. By shifting the indices if necessary we may assume that $X'_{n'} \cap X'_1$ is empty. $\mathcal{D}$ is a linear decomposition of $V - Z$: Pick a vertex $v \in X'_j \cap X'_\ell$ for indices $1 \leq j < \ell \leq n'$. This vertex avoids either X'_1 or $X'_{n'}$, let us assume the former. We apply property (iii) from the definition of a circular decomposition to w_1, w_j, w_k, w_ℓ for any k with $j < k < \ell$ and conclude that $v \in X'_k$.

To distinguish near-embeddings with linear decompositions from near-embeddings with circular decompositions, we will call the latter explicitly *near-embeddings with circular vortices*. Also, for a $(\alpha_0, \alpha_1, \alpha_2)$-near embedding with circular vortices let the third bound α_2 denote an upper bound for the circular adhesion of the large vortices.

We give a modified definition of β-rich to comply with the new concepts. For near-embeddings with circular decompositions we replace property (iv) by the following:

(iv') Let $V \in \mathcal{V}$ with $\Omega(V) = (w_1, \ldots, w_n)$. Then there is a circular, linked decomposition of V of adhesion at most α_2 and a cycle C in $V \cup \bigcup \mathcal{W}$ with $V(C \cap G_0) = \Omega(V)$ that avoids all the cycles of the circular linkage of V, and traverses $w_1, \ldots, w_n$ in this order.

Lemma 23. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ such that every small vortex $W \in \mathcal{W}$ is properly attached. Moreover, assume that*

(i) *For every vortex* $V \in \mathcal{V}$ *there are* $\alpha_2 + 1$ *concentric cycles* $C_0(V), \ldots, C_{\alpha_2}(V)$ *in* G'_0 *tightly enclosing* V.

(ii) *For distinct vortices* $V, W \in \mathcal{V}$, *the discs* $\overline{D(C_0(V))}$ *and* $\overline{D(C_0(W))}$ *are disjoint.*

Then there is a graph $\tilde{G}_0 \subseteq G_0$ *containing* $G_0 \setminus \left(\bigcup_{V \in \mathcal{V}} D(C_0(V))\right)$, *a set* $\tilde{A} \subseteq V(G) \setminus V(\tilde{G}_0)$ *of size* $|\tilde{A}| \leq \tilde{\alpha} := \alpha_0 + \alpha_1(2\alpha_2 + 2)$, *and sets* $\tilde{\mathcal{V}}$ *and* $\tilde{\mathcal{W}} \subseteq \mathcal{W}$ *of vortices such that, with* $\tilde{\sigma} := \sigma|_{\tilde{G}'_0}$, *the tuple* $(\tilde{\sigma}, \tilde{G}_0, A \cup \tilde{A}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ *is an* $(\tilde{\alpha}, \alpha_1, \alpha_2+1)$*-near embedding with circular vortices of* G *in* Σ *such that every vortex* $\tilde{V} \in \tilde{\mathcal{V}}$ *satisfies condition* (iv') *of the definition of* (β, r)*-rich, and* $D(\tilde{V}) \supseteq D(V)$ *for some* $V \in \mathcal{V}$.

Proof. This lemma can be proven almost exactly like Lemma 15. To avoid completely rewriting the proof, we just point out the differences.

The curve C in the surface hits the vertex set S which consists of exactly one vertex from each $C_i(V)$ and one society vertex w'_j of V. We split each vertex in $S \setminus \{w'_j\}$: For each $0 \leq i \leq \alpha_2$, we replace $v \in S \cap V(C_i(V))$ by two new vertices x_i, y_i and connect them with edges to the former neighbours of v such that C does not intersect any edges or vertices. The vertices $x_0, \ldots, x_{\alpha_2}$ and $y_0, \ldots, y_{\alpha_2}$ form the sets X and Y, respectively.

In the remainder of the proof we delete the set Z instead of $S \cup Z$. At the end, we identify the vertex pairs (x_i, y_i) for $0 \leq i \leq \alpha_2$ and obtain a linked, circular decomposition as desired. □

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Linkages in Large Graphs of Bounded Tree-Width

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Abstract

We show that all sufficiently large $(2k+3)$-connected graphs of bounded tree-width are k-linked. Thomassen has conjectured that all sufficiently large $(2k+2)$-connected graphs are k-linked.

1 Introduction

Given an integer $k \geq 1$, a graph G is *k-linked* if for any choice of $2k$ distinct vertices $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ of G there are disjoint paths $P_1, \ldots, P_k$ in G such that the end vertices of P_i are s_i and t_i for $i = 1, \ldots, k$. Menger's theorem implies that every k-linked graph is k-connected.

One can conversely ask how much connectivity (as a function of k) is required to conclude that a graph is k-linked. Larman and Mani [13] and Jung [8] gave the first proofs that a sufficiently highly connected graph is also k-linked. The bound was steadily improved until Bollobás and Thomason [3] gave the first linear bound on the necessary connectivity, showing that every

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$22k$-connected graph is k-linked. The current best bound shows that $10k$-connected graphs are also k-linked [20].

What is the best possible function $f(k)$ one could hope for which implies an $f(k)$-connected graph must also be k-linked? Thomassen [22] conjectured that $(2k+2)$-connected graphs are k-linked. However, this was quickly proven to not be the case by Jørgensen with the following example [23]. Consider the graph obtained from K_{3k-1} obtained by deleting the edges of a matching of size k. This graph is $(3k-3)$-connected but is not k-linked. Thus, the best possible function $f(k)$ one could hope for to imply k-linked would be $3k-2$. However, all known examples of graphs which are roughly $3k$-connected but not k-linked are similarly of bounded size, and it is possible that Thomassen's conjectured bound is correct if one assumes that the graph has sufficiently many vertices.

In this paper, we show Thomassen's conjectured bound is almost correct with the additional assumption that the graph is large and has bounded tree-width. This is the main result of this article.

Theorem 1.1. *For all integers k and w there exists an integer N such that a graph G is k-linked if*

$$\kappa(G) \geq 2k+3, \qquad \mathrm{tw}(G) < w, \quad \textit{and} \quad |G| \geq N.$$

where κ is the connectivity of the graph and tw *is the tree-width.*

The tree-width of the graph is a parameter commonly arising in the theory of graph minors; we will delay giving the definition until Section 2 where we give a more in depth discussion of how tree-width arises naturally in tackling the problem. The value $2k+2$ would be best possible; see Section 8 for examples of arbitrarily large graphs which are $(2k+1)$-connected but not k-linked.

Our work builds on the theory of graph minors in large, highly connected graphs begun by Böhme, Kawarabayashi, Maharry and Mohar [1]. Recall that a graph G contains K_t as a *minor* if there K_t can be obtained from a subgraph of G by repeatedly contracting edges. Böhme et al. showed that there exists an absolute constant c such that every sufficiently large ct-connected graph contains K_t as a minor. This statement is not true without the assumption that the graph be sufficiently large, as there are examples of small graphs which are $(t\sqrt{\log t})$-connected but still have no K_t minor [12, 21]. In the case where we restrict our attention to small values of t, one is able to get an explicit characterisation of the large t-connected graphs which do not contain K_t as a minor.

Theorem 1.2 (Kawarabayashi et al. [10]). *There exists a constant N such that every 6-connected graph G on N vertices either contains K_6 as a minor or there exists a vertex $v \in V(G)$ such that $G - v$ is planar.*

Jorgensen [7] conjectures that Theorem 1.2 holds for all graphs without the additional restriction to graphs on a large number of vertices. In 2010, Norine and Thomas [19] announced that Theorem 1.2 could be generalised to arbitrary values of t to either find a K_t minor in a sufficiently large t-connected graph or alternatively, find a small set of vertices whose deletion leaves the graph planar. They have indicated that their methodology could be used to show a similar bound of $2k+3$ on the connectivity which ensures a large graph is k-linked.

2 Outline

In this section, we motivate our choice to restrict our attention to graphs of bounded tree-width and give an outline of the proof of Theorem 1.1.

We first introduce the basic definitions of tree-width. A *tree-decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_t \subseteq V(G) : t \in V(T)\}$ is a collection of subsets of $V(G)$ indexed by the vertices of T. Moreover, $\mathcal{X}$ satisfies the following properties.

1. $\bigcup_{t \in V(T)} X_t = V(G)$,
2. for all $e \in E(G)$, there exists $t \in V(T)$ such that both ends of e are contained in X_t, and
3. for all $v \in V(G)$, the subset $\{t \in V(T) : v \in X_t\}$ induces a connected subtree of T.

The sets in $\mathcal{X}$ are sometimes called the *bags* of the decomposition. The *width* of the decomposition is $\max_{t \in V(T)} |X_t| - 1$, and the *tree-width* of G is the minimum width of a tree-decomposition.

Robertson and Seymour showed that if a $2k$-connected graph contains K_{3k} as a minor, then it is k-linked [16]. Thus, when one considers $(2k+3)$-connected graphs which are not k-linked, one can further restrict attention to graphs which exclude a fixed clique minor. This allows one to apply the excluded minor structure theorem of Robertson and Seymour [17]. The structure theorem can be further strengthened if one assumes the graph has large tree-width [5]. This motivates one to analyse separately the case when the tree-width is large or bounded. The proofs of the main results in [1] and

[10] similarly split the analysis into cases based on either large or bounded tree-width.

We continue with an outline of how the proof of Theorem 1.1 proceeds. Assume Theorem 1.1 is false, and let G be a $(2k+3)$-connected graph which is not k-linked. Fix a set $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ such that there do not exist disjoint paths $P_1, \ldots, P_k$ where the ends of P_i are s_i and t_i for all i. Fix a tree-decomposition $(T, \mathcal{X})$ of G of minimal width w.

We first exclude the possibility that T has a high degree vertex. Assume t is a vertex of T of large degree. By Property 3 in the definition of a tree-decomposition, if we delete the set X_t of vertices from G, the resulting graph must have at least $\deg_T(t)$ distinct connected components. By the connectivity of G, each component contains $2k+3$ internally disjoint paths from a vertex v to $2k+3$ distinct vertices in X_t. If the degree of t is sufficiently large, we conclude that the graph G contains a subdivision of $K_{a,2k+3}$ for some large value a. We now prove that that if a graph contains such a large complete bipartite subdivision and is $2k$-connected, then it must be k-linked (Lemma 7.1).

We conclude that the tree T does not have a high degree vertex, and consequently contains a long path. It follows that the graph G has a long *path decomposition*, that is, a tree-decomposition where the tree is a path. As the bags of the decomposition are linearly ordered by their position on the path, we simply give the path decomposition as a linearly ordered set of bags $(B_1, \ldots, B_t)$ for some large value t. At this point in the argument, the path-decomposition $(B_1, \ldots, B_t)$ may not have bounded width, but it will have the property that $|B_i \cap B_j|$ is bounded, and this will suffice for the argument to proceed. Section 3 examines this path decomposition in detail and presents a series of refinements allowing us to assume the path decomposition satisfies a set of desirable properties. For example, we are able to assume that $|B_i \cap B_{i+1}|$ is the same for all i, $1 \leq i < t$. Moreover, there exist a set $\mathcal{P}$ of $|B_1 \cap B_2|$ disjoint paths starting in B_1 and ending in B_t. We call these paths the *foundational linkage* and they play an important role in the proof. A further property of the path decomposition which we prove in Section 3 is that for each i, $1 < i < t$, if there is a bridge connecting two foundational paths in $\mathcal{P}$ in B_i, then for all j, $1 < j < t$, there exists a bridge connecting the same foundational paths in B_j. This allows us to define an auxiliary graph H with vertex set $\mathcal{P}$ and two vertices of $\mathcal{P}$ adjacent in H if there exists a bridge connecting them in some B_i $1 < i < t$.

Return to the linkage problem at hand; we have $2k$ terminals $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ which we would like to link appropriately, and $B_1, \ldots, B_t$ is our path decomposition with the foundational linkage running through it. Let the set $B_i \cap B_{i+1}$ be labeled S_i. As our path decomposition developed

in the previous paragraph is very long, we can assume there exists some long subsection $B_i, B_{i+1}, \ldots, B_{i+a}$ such that no vertex of $s_1, \ldots, s_k, t_1, \ldots, t_k$ is contained in $\bigcup_i^{i+a} B_i - (S_{i-1} \cup S_{i+a})$ for some large value a. By Menger's theorem, there exist $2k$ paths linking $s_1, \ldots, s_k, t_1, \ldots, t_k$ to the set $S_{i-1} \cup S_{i+a}$. We attempt to link the terminals by continuing these paths into the subgraph induced by the vertex set $B_i \cup \cdots \cup B_{i+a}$. More specifically, we extend the paths along the foundational paths and attempt to link up the terminals with the bridges joining the various foundational paths in each of the B_j. By construction, the connections between foundational paths are the same in B_j for all j, $1 < j < t$; thus we translate the problem into a token game played on the auxiliary graph H. There each terminal has a corresponding token, and the desired linkage in G will exist if it is possible to slide the tokens around H in such a way to match up the tokens of the corresponding pairs of terminals. The token game is rigorously defined in Section 4, and we present a characterisation of what properties on H will allow us to find the desired linkage in G.

The final step in the proof of Theorem 1.1 is to derive a contradiction when H doesn't have sufficient complexity to allow us to win the token game. In order to do so, we use the high degree in G and a theorem of Robertson and Seymour on crossing paths. We give a series of technical results in preparation in Section 5 and Section 6 and present the proof of Theorem 1.1 in Section 7.

3 Stable Decompositions

In this section we present a result which, roughly speaking, ensures that a highly connected, sufficiently large graph of bounded tree-width either contains a subdivision of a large complete bipartite graph or has a long path decomposition whose bags all have similar structure.

Such a theorem was first established by Böhme, Maharry, and Mohar in [2] and extended by Kawarabayashi, Norine, Thomas, and Wollan in [9], both using techniques from [14]. We shall prove a further extension based on the result by Kawarabayashi et al. from [9] so our terminology and methods will be close to theirs.

We begin this section with a general Lemma about nested separations. Let G be a graph. A *separation* of G is an ordered pair (A, B) of sets $A, B \subseteq V(G)$ such that $G[A] \cup G[B] = G$. If (A, B) is a separation of G, then $A \cap B$ is called its *separator* and $|A \cap B|$ its *order*. Two separations (A, B) and (A', B') of G are called *nested* if either $A \subseteq A'$ and $B \supseteq B'$ or $A \supseteq A'$ and $B \subseteq B'$. In the former case we write $(A, B) \leq (A', B')$ and in the

latter $(A, B) \geq (A', B')$. This defines a partial order $\leq$ on all separations of G. A set $\mathcal{S}$ of separations is called *nested* if the separations of $\mathcal{S}$ are pairwise nested, that is, $\leq$ is a linear order on $\mathcal{S}$. To avoid confusion about the order of the separations in $\mathcal{S}$ we do not use the usual terms like smaller, larger, maximal, and minimal when talking about this linear order but instead use *left*, *right*, *rightmost*, and *leftmost*, respectively (we still use *successor* and *predecessor* though). To distinguish $\leq$ from $<$ we say 'left' for the former and 'strictly left' for the latter (same for $\geq$ and right).

If (A, B) and (A', B') are both separations of G, then so are $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ and a simple calculation shows that the orders of $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ sum up to the same number as the orders of (A, B) and (A', B'). Clearly each of $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ is nested with both, (A, B) and (A', B').

For two sets $X, Y \subseteq V(G)$ we say that a separation (A, B) of G is an X–Y *separation* if $X \subseteq A$ and $Y \subseteq B$. If (A, B) and (A', B') are X–Y separations in G, then so are $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$. Furthermore, if (A, B) and (A', B') are X–Y separations of G of minimum order, say m, then so are $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ as none of the latter two can have order less than m but their orders sum up to $2m$.

Lemma 3.1. *Let G be a graph and $X, Y, Z \subseteq V(G)$. If for every $z \in Z$ there is an X–Y separation of G of minimal order with z in its separator, then there is a nested set $\mathcal{S}$ of X–Y separations of minimal order such that their separators cover Z.*

Proof. Let $\mathcal{S}$ be a maximal nested set of X–Y separations of minimal order in G (as $\mathcal{S}$ is finite the existence of a leftmost and a rightmost element in any subset of $\mathcal{S}$ is trivial). Suppose for a contradiction that some $z \in Z$ is not contained in any separator of the separations of $\mathcal{S}$.

Set $\mathcal{S}_L := \{(A, B) \in \mathcal{S} \mid z \in B\}$ and $\mathcal{S}_R := \{(A, B) \in \mathcal{S} \mid z \in A\}$. Clearly $\mathcal{S}_L \cup \mathcal{S}_R = \mathcal{S}$ and $\mathcal{S}_L \cap \mathcal{S}_R = \emptyset$. Moreover, if $\mathcal{S}_L$ and $\mathcal{S}_R$ are both non-empty, then the rightmost element (A_L, B_L) of $\mathcal{S}_L$ is the predecessor of the leftmost element (A_R, B_R) of $\mathcal{S}_R$ in $\mathcal{S}$. Loosely speaking, $\mathcal{S}_L$ and $\mathcal{S}_R$ contain the separations of $\mathcal{S}$ "on the left" and "on the right" of z, respectively, and (A_L, B_L) and (A_R, B_R) are the separations of $\mathcal{S}_L$ and $\mathcal{S}_R$ whose separators are "closest" to z.

By assumption there is an X–Y separation (A, B) of minimal order in G with $z \in A \cap B$. Set

$$(A', B') := (A \cup A_L, B \cap B_L) \quad \text{and} \quad (A'', B'') := (A' \cap A_R, B' \cup B_R)$$

(but $(A', B') := (A, B)$ if $\mathcal{S}_L = \emptyset$ and $(A'', B'') := (A', B')$ if $\mathcal{S}_R = \emptyset$). As (A_L, B_L), (A, B), and (A_R, B_R) are all X–Y separations of minimal order in

G so must be (A', B') and (A'', B''). Moreover, we have $z \in A'' \cap B''$ and thus $(A'', B'') \notin \mathcal{S}$.

By construction we have $(A_L, B_L) \leq (A', B')$ and $(A'', B'') \leq (A_R, B_R)$. To verify that $(A_L, B_L) \leq (A'', B'')$ we need to show $A_L \subseteq A' \cap A_R$ and $B_L \supseteq B' \cup B_R$. All required inclusions follow from $(A_L, B_L) \leq (A', B')$ and $(A_L, B_L) \leq (A_R, B_R)$. So by transitivity (A'', B'') is right of all elements of $\mathcal{S}_L$ and left of all elements of $\mathcal{S}_R$, in particular, it is nested with all elements of $\mathcal{S}$, contradicting the maximality of the latter. $\square$

We assume that every path comes with a fixed linear order of its vertices. If a path arises as an X–Y path, then we assume it is ordered from X to Y and if a path Q arises as a subpath of some path P, then we assume that Q is ordered in the same direction as P unless explicitly stated otherwise.

Given a vertex v on a path P we write Pv for the initial subpath of P with last vertex v and vP for the final subpath of P with first vertex v. If v and w are both vertices of P, then by vPw or wPv we mean the subpath of P that ends in v and w and is ordered from v to w or from w to v, respectively. By P^{-1} we denote the path P with inverse order.

Let $\mathcal{P}$ be a set of disjoint paths in some graph G. We do not distinguish between $\mathcal{P}$ and the graph $\bigcup \mathcal{P}$ formed by uniting these paths; both will be denoted by $\mathcal{P}$. By a *path of* $\mathcal{P}$ we always mean an element of $\mathcal{P}$, not an arbitrary path in $\bigcup \mathcal{P}$.

Let G be a graph. For a subgraph $S \subseteq G$ an *S-bridge in G* is a connected subgraph $B \subseteq G$ such that B is edge-disjoint from S and either B is a single edge with both ends in S or there is a component C of $G - S$ such that B consists of all edges that have at least one end in C. We call a bridge *trivial* in the former case and *non-trivial* in the latter. The vertices in $V(B) \cap V(S)$ and $V(B) \setminus V(S)$ are called the *attachments* and the *inner vertices* of B, respectively. Clearly an S-bridge has an inner vertex if and only if it is non-trivial. We say that an S-bridge B *attaches* to a subgraph $S' \subseteq S$ if B has an attachment in S'. Note that S-bridges are pairwise edge-disjoint and each common vertex of two S-bridges must be an attachment of both.

A *branch vertex* of S is a vertex of degree $\neq 2$ in S and a *segment* of S is a maximal path in S such that its ends are branch vertices of S but none of its inner vertices are. An S-bridge B in G is called *unstable* if some segment of S contains all attachments of B, and *stable* otherwise. If an unstable S-bridge B has at least two attachments on a segment P of S, then we call P a *host* of B and say that B is *hosted by* P. For a subgraph $H \subseteq G$ we say that two segments of S are *S-bridge adjacent* or just *bridge adjacent* in H if H contains an S-bridge that attaches to both.

If a graph is the union of its segments and no two of its segments have the

same end vertices, then it is called *unambiguous* and *ambiguous* otherwise. It is easy to see that a graph S is unambiguous if and only if all its cycles contain a least three branch vertices. In our application S will always be a union of disjoint paths so its segments are precisely these paths and S is trivially unambiguous.

Let $S \subseteq G$ be unambiguous. We say that $S' \subseteq G$ is a *rerouting* of S if there is a bijection φ from the segments of S to the segments of S' such that every segment P of S has the same end vertices as $\varphi(P)$ (and thus φ is unique by the unambiguity). If S' contains no edge of a stable S-bridge, then we call S' a *proper rerouting* of S. Clearly any rerouting of the unambiguous graph S has the same branch vertices as S and hence is again unambiguous.

The following Lemma states two observations about proper reroutings. The proofs are both easy and hence we omit them.

Lemma 3.2. *Let S' be a proper rerouting of an unambiguous graph $S \subseteq G$ and let φ be as in the definition. Both of the following statements hold.*

(i) Every hosted S-bridge has a unique host. For each segment P of S the segment $\varphi(P)$ of S' is contained in the union of P and all S-bridges hosted by P.

(ii) For every stable S-bridge B there is a stable S'-bridge B' with $B \subseteq B'$. Moreover, if B attaches to a segment P of S, then B' attaches to $\varphi(P)$.

Note that Lemma 3.2 (ii) implies that no unstable S'-bridge contains an edge of a stable S-bridge. Together with (i) this means that being a proper rerouting of an unambiguous graph is a transitive relation.

The next Lemma is attributed to Tutte; we refer to [9, Lemma 2.2] for a proof[1].

Lemma 3.3. *Let G be a graph and $S \subseteq G$ unambiguous. There exists a proper rerouting S' of S in G such that if B' is an S'-bridge hosted by some segment P' of S', then B' is non-trivial and there are vertices $v, w \in V(P')$ such that the component of $G - \{v, w\}$ that contains $B' - \{v, w\}$ is disjoint from $S' - vP'w$.*

This implies that the segments of S' are induced paths in G as trivial S'-bridges cannot be unstable and no two segments of S' have the same end vertices.

[1] To check that Lemma 2.2 in [9] implies our Lemma 3.3 note that if S' is obtained from S by "a sequence of proper reroutings" as defined in [9], then by transitivity S' is a proper rerouting of S according to our definition. And although not explicitly included in the statement, the given proof shows that no trivial S'-bridge can be unstable.

Let G be a graph. A set of disjoint paths in G is called a *linkage*. If $X, Y \subseteq V(G)$ with $k := |X| = |Y|$, then a set of k disjoint X–Y paths in G is called an X–Y *linkage* or a *linkage from X to Y*. Let $\mathcal{W} = (W_0, \ldots, W_l)$ be an ordered tuple of subsets of $V(G)$. Then l is the *length* of $\mathcal{W}$, the sets W_i with $0 \leq i \leq l$ are its *bags*, and the sets $W_{i-1} \cap W_i$ with $1 \leq i \leq l$ are its *adhesion sets*. We refer to the bags W_i with $1 \leq i \leq l-1$ as *inner* bags. When we say that a bag W of $\mathcal{W}$ *contains* some graph H, we mean $H \subseteq G[W]$. Given an inner bag W_i of $\mathcal{W}$, the sets $W_{i-1} \cap W_i$ and $W_i \cap W_{i+1}$ are called the *left* and *right* adhesion set of W_i, respectively. Whenever we introduce a tuple $\mathcal{W}$ as above without explicitly naming its elements, we shall denote them by $W_0, \ldots, W_l$ where l is the length of $\mathcal{W}$. For indices $0 \leq j \leq k \leq l$ we use the shortcut $W_{[j,k]} := \bigcup_{i=j}^{k} W_i$.

The tuple $\mathcal{W}$ with the following five properties is called a *slim decomposition* of G.

(L1) $\bigcup \mathcal{W} = V(G)$ and every edge of G is contained in some bag of $\mathcal{W}$.

(L2) If $0 \leq i \leq j \leq k \leq l$, then $W_i \cap W_k \subseteq W_j$.

(L3) All adhesion sets of $\mathcal{W}$ have the same size.

(L4) No bag of $\mathcal{W}$ contains another.

(L5) G contains a $(W_0 \cap W_1)$–$(W_{l-1} \cap W_l)$ linkage.

The unique size of the adhesion sets of a slim decomposition is called its *adhesion*. A linkage $\mathcal{P}$ as in (L5) together with an enumeration $P_1, \ldots, P_q$ of its paths is called a *foundational linkage* for $\mathcal{W}$ and its members are called *foundational paths*. Each path P_α contains a unique vertex of every adhesion set of $\mathcal{W}$ and we call this vertex the *α-vertex* of that adhesion set. For an inner bag W of $\mathcal{W}$ the α-vertex in the left and right adhesion set of W are called the *left* and *right* α-vertex of W, respectively. Note that $\mathcal{P}$ is allowed to contain trivial paths so $\bigcap \mathcal{W}$ may be non-empty.

The enumeration of a foundational linkage $\mathcal{P}$ for $\mathcal{W}$ is a formal tool to compare arbitrary linkages between adhesion sets of $\mathcal{W}$ to $\mathcal{P}$ by their 'induced permutation' as detailed below. When considering another foundational linkage $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ for $\mathcal{W}$ we shall thus always assume that it induces the same enumeration as $\mathcal{P}$ on $W_0 \cap W_1$, in other words, Q_α and P_α start on the same vertex.

Suppose that $\mathcal{W}$ is a slim decomposition of some graph G with foundational linkage $\mathcal{P}$. Then any $\mathcal{P}$-bridge B in G is contained in a bag of $\mathcal{W}$, and this bag is unique unless B is trivial and contained in one or more adhesion sets.

We say that a linkage $\mathcal{Q}$ in a graph H is *p-attached* if each path of $\mathcal{Q}$ is induced in H and if some non-trivial $\mathcal{Q}$-bridge B attaches to a non-trivial path P of $\mathcal{Q}$, then either B attaches to another non-trivial path of $\mathcal{Q}$ or there are at least $p-2$ trivial paths Q of $\mathcal{Q}$ such that H contains a $\mathcal{Q}$-bridge (which may be different from B) attaching to P and Q.

We call a pair $(\mathcal{W}, \mathcal{P})$ of a slim decomposition $\mathcal{W}$ of G and a foundational linkage $\mathcal{P}$ for $\mathcal{W}$ a *regular decomposition of attachedness p of G* if there is an integer p such that the axioms (L6), (L7), and (L8) hold.

(L6) $\mathcal{P}[W]$ is p-attached in $G[W]$ for all inner bags W of $\mathcal{W}$.

(L7) A path $P \in \mathcal{P}$ is trivial if $P[W]$ is trivial for some inner bag W of $\mathcal{W}$.

(L8) For every $P, Q \in \mathcal{P}$, if some inner bag of $\mathcal{W}$ contains a $\mathcal{P}$-bridge attaching to P and Q, then every inner bag of $\mathcal{W}$ contains such a $\mathcal{P}$-bridge.

The integer p is not unique: A regular decomposition of attachedness p has attachedness p' for all integers $p' \leq p$. Note that $\mathcal{P}$ satisfies (L7) if and only if every vertex of G either lies in at most two bags of $\mathcal{W}$ or in all bags. This means that either all foundational linkages for $\mathcal{W}$ satisfy (L7) or none.

The next Theorem follows[2] from the Lemmas 3.1, 3.2, and 3.5 in [9].

Theorem 3.4 (Kawarabayashi et al. [9])**.** *For all integers $a, l, p, w \geq 0$ there exists an integer N with the following property. If G is a p-connected graph of tree-width less than w with at least N vertices, then either G contains a subdivision of $K_{a,p}$, or G has a regular decomposition of length at least l, adhesion at most w, and attachedness p.*

Note that [9] features a stronger version of Theorem 3.4, namely Theorem 3.8, which includes an additional axiom (L9). We omit that axiom since our arguments do not rely on it.

Let $(\mathcal{W}, \mathcal{P})$ be a slim decomposition of adhesion q and length l for a graph G. Suppose that $\mathcal{Q}$ is a linkage from the left adhesion set of W_i to the right adhesion set of W_j for two indices i and j with $1 \leq i \leq j < l$. The enumeration $P_1, \ldots, P_q$ of $\mathcal{P}$ induces an enumeration $Q_1, \ldots, Q_q$ of $\mathcal{Q}$ where Q_α is the path of $\mathcal{Q}$ starting in the left α-vertex of W_i. The map $\pi : \{1, \ldots, q\} \to \{1, \ldots, q\}$ such that Q_α ends in the right $\pi(\alpha)$-vertex of W_j for $\alpha = 1, \ldots, q$ is a permutation because $\mathcal{Q}$ is a linkage. We call it the *induced permutation of $\mathcal{Q}$*. Clearly the induced permutation of $\mathcal{Q}$ is the composition of the induced permutations of $\mathcal{Q}[W_i]$, $\mathcal{Q}[W_{i+1}]$, $\ldots$, $\mathcal{Q}[W_j]$. For

[2] The statement of Lemma 3.1 in [9] only asserts the existence of a minor isomorphic to $K_{a,p}$ rather than a subdivision of $K_{a,p}$ like we do. But its proof refers to an argument in the proof of [14, Theorem 3.1] which actually gives a subdivision.

any permutation π of $\{1,\dots,q\}$ and any graph Γ on $\{1,\dots,q\}$ we write $\pi\Gamma$ to denote the graph $(\{\pi(\alpha) \mid \alpha \in V(\Gamma)\}, \{\pi(\alpha)\pi(\beta) \mid \alpha\beta \in E(\Gamma)\})$. For a subset $X \subseteq \{1,\dots,q\}$ we set $\mathcal{Q}_X := \{Q_\alpha \mid \alpha \in X\}$.

Keep in mind that the enumerations $\mathcal{P}$ induces on linkages $\mathcal{Q}$ as above always depend on the adhesion set where the considered linkage starts. For example let $\mathcal{Q}$ be as above and for some index i' with $i < i' \leq j$ set $\mathcal{Q}' := \mathcal{Q}[W_{[i',j]}]$. Then $Q_\alpha[W_{[i',j]}]$ need not be the same as Q'_α. More precisely, we have $Q_\alpha[W_{[i',j]}] = Q'_{\tau(\alpha)}$ where τ denotes the induced permutation of $\mathcal{Q}[W_{[i,i'-1]}]$.

For some subgraph H of G the *bridge graph of* $\mathcal{Q}$ *in* H, denoted $B(H,\mathcal{Q})$, is the graph with vertex set $\{1,\dots,q\}$ in which $\alpha\beta$ is an edge if and only if Q_α and Q_β are $\mathcal{Q}$-bridge adjacent in H. Any $\mathcal{Q}$-bridge B in H that attaches to Q_α and Q_β is said to *realise* the edge $\alpha\beta$. We shall sometimes think of induced permutations as maps between bridge graphs.

For a slim decomposition $\mathcal{W}$ of length l of G with foundational linkage $\mathcal{P}$ we define the *auxiliary graph* $\Gamma(\mathcal{W},\mathcal{P}) := B(G[W_{[1,\ l-1]}],\mathcal{P})$. Clearly $B(G[W],\mathcal{P}[W]) \subseteq \Gamma(\mathcal{W},\mathcal{P})$ for each inner bag W of $\mathcal{W}$ and if $(\mathcal{W},\mathcal{P})$ is regular, then by (L8) we have equality.

Set $\lambda := \{\alpha \mid P_\alpha \text{ is non-tivial}\}$ and $\theta := \{\alpha \mid P_\alpha \text{ is trivial}\}$. Given a subgraph $\Gamma \subseteq \Gamma(\mathcal{W},\mathcal{P})$ and some foundational linkage $\mathcal{Q}$ for $\mathcal{W}$, we write $G^{\mathcal{Q}}_\Gamma$ for the graph obtained by deleting $\mathcal{Q} \setminus \mathcal{Q}_{V(\Gamma)}$ from the union of $\mathcal{Q}$ and those $\mathcal{Q}$-bridges in inner bags of $\mathcal{W}$ that realise an edge of Γ or attach to $\mathcal{Q}_{V(\Gamma)\cap\lambda}$ but to no path of $\mathcal{Q}_{\lambda\setminus V(\Gamma)}$. For a subset $V \subseteq \{1,\dots,q\}$ we write $G^{\mathcal{Q}}_V$ instead of $G^{\mathcal{Q}}_{\Gamma(\mathcal{W},\mathcal{P})[V]}$. Note that $\mathcal{Q}_\theta = \mathcal{P}_\theta$. Hence $G^{\mathcal{P}}_\lambda$ and $G^{\mathcal{Q}}_\lambda$ are the same graph and we denote it by G_λ.

A regular decomposition $(\mathcal{W},\mathcal{P})$ of a graph G is called *stable* if it satisfies the following two axioms where $\lambda := \{\alpha \mid P_\alpha \text{ is non-trivial}\}$.

(L10) If $\mathcal{Q}$ is a linkage from the left to the right adhesion set of some inner bag of $\mathcal{W}$, then its induced permutation is an automorphism of $\Gamma(\mathcal{W},\mathcal{P})$.

(L11) If $\mathcal{Q}$ is a linkage from the left to the right adhesion set of some inner bag W of $\mathcal{W}$, then every edge of $B(G[W],\mathcal{Q})$ with one end in λ is also an edge of $\Gamma(\mathcal{W},\mathcal{P})$.

Given these definitions we can further expound our strategy to prove the main theorem: We will reduce the given linkage problem to a linkage problem with start and end vertices in $W_0 \cup W_l$ for some stable regular decomposition $(\mathcal{W},\mathcal{P})$ of length l. The stability ensures that we maximised the number of edges of $\Gamma(\mathcal{W},\mathcal{P})$, i.e. no rerouting of $\mathcal{P}$ will give rise to new bridge adjacencies. We will focus on a subset $\lambda_0 \subseteq \lambda$ and show that the minimum degree of G forces a high edge density in $G^{\mathcal{P}}_{\lambda_0}$, leading to a high

number of edges in $\Gamma(\mathcal{W},\mathcal{P})[\lambda_0]$. Using combinatoric arguments, which we elaborate in Section 4, we show that we can find linkages using segments of $\mathcal{P}$ and $\mathcal{P}$-bridges in $G^{\mathcal{P}}_{\lambda_0}$ to realise any matching of start and end vertices in $W_0 \cup W_l$, showing that G is in fact k-linked.

We strengthen Theorem 3.4 by the assertion that the regular decomposition can be chosen to be stable. We like to point out that, even with the left out axiom (L9) included in the definition of a regular decomposition, Theorem 3.5 would hold. By almost the same proof as in [9] one could also obtain a stronger version of (L8) stating that for every subset $\mathcal{R}$ of $\mathcal{P}$ if some inner bag of $\mathcal{W}$ contains a $\mathcal{P}$-bridge attaching every path of $\mathcal{R}$ but to no path of $\mathcal{P} \setminus \mathcal{R}$, then every inner bag does.

Theorem 3.5. *For all integers $a, l, p, w \geq 0$ there exists an integer N with the following property. If G is a p-connected graph of tree-width less than w with at least N vertices, then either G contains a subdivision of $K_{a,p}$, or G has a stable regular decomposition of length at least l, adhesion at most w, and attachedness p.*

Before we start with the formal proof let us introduce its central concepts: disturbances and contractions. Let $(\mathcal{W},\mathcal{P})$ be a regular decomposition of a graph G. A linkage $\mathcal{Q}$ is called a *twisting* $(\mathcal{W},\mathcal{P})$*-disturbance* if it violates (L10) and it is called a *bridging* $(\mathcal{W},\mathcal{P})$*-disturbance* if it violates (L11). By a $(\mathcal{W},\mathcal{P})$*-disturbance* we mean either of these two and a disturbance may be twisting and bridging at the same time. If the referred regular decomposition is clear from the context, then we shall not include it in the notation and just speak of a disturbance. Note that a disturbance is always a linkage from the left to the right adhesion set of an inner bag of $\mathcal{W}$.

Given a disturbance $\mathcal{Q}$ in some inner bag W of $\mathcal{W}$ which is neither the first nor the last inner bag of $\mathcal{W}$, it is not hard to see that replacing $\mathcal{P}[W]$ with $\mathcal{Q}$ yields a foundational linkage $\mathcal{P}'$ for $\mathcal{W}$ such that $\Gamma(\mathcal{W},\mathcal{P}')$ properly contains $\Gamma(\mathcal{W},\mathcal{P})$ and we shall make this precise in the proof. As the auxiliary graph can have at most $\binom{w}{2}$ edges, we can repeat this step until no disturbances (with respect to the current decomposition) are left and we should end up with a stable regular decomposition, given that we can somehow preserve the regularity.

This is done by "contracting" the decomposition in a certain way. The technique is the same as in [2] or [9]. Given a regular decomposition $(\mathcal{W},\mathcal{P})$ of length l of some graph G and a subsequence $i_1,\dots,i_n$ of $1,\dots,l$, the *contraction of* $(\mathcal{W},\mathcal{P})$ *along* $i_1,\dots,i_n$ is the pair $(\mathcal{W}',\mathcal{P}')$ defined as follows. We let $\mathcal{W}' := (W'_0, W'_1, \dots, W'_n)$ with $W'_0 := W_{[0,\ i_1-1]}$,

$$W'_j := W_{[i_j,\ i_{j+1}-1]} \quad \text{for} \quad j = 1,\dots,n-1,$$

$W_n := W_{[i_n,l]}$, and $\mathcal{P}' = \mathcal{P}[W'_{[1,\ n-1]}]$ (with the induced enumeration).

Lemma 3.6. *Let $(\mathcal{W}', \mathcal{P}')$ be the contraction of a regular decomposition $(\mathcal{W}, \mathcal{P})$ of some graph G of adhesion q and attachedness p along the sequence $i_1, \ldots, i_n$. Then the following two statements hold.*

(i) $(\mathcal{W}', \mathcal{P}')$ is a regular decomposition of length n of G of adhesion q and attachedness p, and $\Gamma(\mathcal{W}', \mathcal{P}') = \Gamma(\mathcal{W}, \mathcal{P})$.

(ii) The decomposition $(\mathcal{W}', \mathcal{P}')$ is stable if and only if none of the inner bags $W_{i_1}, W_{i_1+1}, \ldots, W_{i_n-1}$ of $\mathcal{W}$ contains a $(\mathcal{W}, \mathcal{P})$-disturbance.

Proof. The first statement is Lemma 3.3 of [9]. The second statement follows from the fact that an inner bag W'_j of $\mathcal{W}'$ contains a $(\mathcal{W}', \mathcal{P}')$-disturbance if and only if one of the bags W_i of $\mathcal{W}$ with $i_j \leq i < i_{j+1}$ contains a $(\mathcal{W}, \mathcal{P})$-disturbance (unless $\mathcal{W}'$ has no inner bag, that is, $n = 1$). The "if" direction is obvious and for the "only if" direction recall that the induced permutation of $\mathcal{P}'[W'_j]$ is the composition of the induced permutations of the $\mathcal{P}[W_i]$ with $i_j \leq i < i_{j+1}$ and every $\mathcal{P}'$-bridge in W'_j is also a $\mathcal{P}$-bridge and hence must be contained in some bag W_i with $i_j \leq i < i_{j+1}$. □

Let $\mathcal{Q}$ be a linkage in a graph H and denote the trivial paths of $\mathcal{Q}$ by Θ. Let $\mathcal{Q}'$ be the union of Θ with a proper rerouting of $\mathcal{Q} \setminus \Theta$ obtained from applying Lemma 3.3 to $\mathcal{Q} \setminus \Theta$ in $H - \Theta$. We call $\mathcal{Q}'$ a *bridge stabilisation of $\mathcal{Q}$ in H*. The next Lemma tailors Lemma 3.2 and Lemma 3.3 to our application.

Lemma 3.7. *Let $\mathcal{Q}$ be a linkage in a graph H. Denote by Θ the trivial paths of $\mathcal{Q}$ and let $\mathcal{Q}'$ be a bridge stabilisation of $\mathcal{Q}$ in H. Let P and Q be paths of $\mathcal{Q}$ and let P' and Q' be the unique paths of $\mathcal{Q}'$ with the same end vertices as P and Q, respectively. Then the following statements hold.*

(i) P' is contained in the union of P with all $\mathcal{Q}$-bridges in H that attach to P but to no other path of $\mathcal{Q} \setminus \Theta$.

(ii) If P and Q are $\mathcal{Q}$-bridge adjacent in H and one of them is non-trivial, then P' and Q' are $\mathcal{Q}'$-bridge adjacent in H.

(iii) Let Z be the set of end vertices of the paths of $\mathcal{Q}$. If p is an integer such that for every vertex x of $H - Z$ there is an x–Z fan of size p, then $\mathcal{Q}'$ is p-attached.

Proof.

(i) This is trivial if $P \in \Theta$ and follows easily from Lemma 3.2 (i) otherwise.

(ii) The statement follows directly from Lemma 3.2 (ii) if P and Q are both non-trivial so we may assume that $P = P' \in \Theta$ and Q is non-trivial. By assumption there is a P–Q path R in H. Clearly $R \cup Q$ contains the end vertices of Q'. On the other hand, by (i) it is clear that $Q \cap \mathcal{Q}' \subseteq Q'$. We claim that $R \cap \mathcal{Q}' \subseteq Q'$. Since R is internally disjoint from $\mathcal{Q}$ all its inner vertices are inner vertices of some $(\mathcal{Q} \setminus \Theta)$-bridge B. If B is stable or unstable but not hosted by any path of $\mathcal{Q}$ (that is, it has at most one attachment), then Lemma 3.2 implies that no path of $\mathcal{Q}'$ contains an inner vertex of B and that our claim follows. If B is hosted by a path of $\mathcal{Q}$, then this path must clearly be Q and thus by Lemma 3.2 (i) $R \cap \mathcal{Q}' \subseteq Q'$ as claimed. Hence $R \cup Q$ contains a P–Q' path that is internally disjoint from $\mathcal{Q}'$ as desired.

(iii) Clearly all paths of $\mathcal{Q}'$ are induced in H, either because they are trivial or by Lemma 3.3. Let B be a non-trivial hosted $\mathcal{Q}'$-bridge and let Q' be the non-trivial path of $\mathcal{Q}'$ to which it attaches. Then by Lemma 3.3 there are vertices v and w on Q' and a separation (X, Y) of H such that $V(B) \subseteq X$, $X \cap Y \subseteq \{v, w\} \cup V(\Theta)$, and apart from the inner vertices of $vQ'w$ all vertices of $\mathcal{Q}'$ are in Y, in particular, $Z \subseteq Y$. But B has an inner vertex x which must be in $X \setminus Y$. So by assumption there is an x–$\{v, w\} \cup V(\Theta)$ fan of size p in $G[X]$ and thus also an x–Θ fan of size $p - 2$. It is easy to see that this can gives rise to the desired $\mathcal{Q}'$-bridge adjacencies in H.

□

Proof of Theorem 3.5. We will trade off some length of a regular decomposition to gain edges in its auxiliary graph. To quantify this we define the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ by $f(m) := (zlw!)^m l$ where $z := 2^{\binom{w}{2}}$ and call a regular decomposition $(\mathcal{W}, \mathcal{P})$ of a graph G *valid* if it has adhesion at most w, attachedness p, and length at least $f(m)$ where m is the number of edges in the complement of $\Gamma(\mathcal{W}, \mathcal{P})$ that are incident with at least one non-trivial path of $\mathcal{P}$.

Set $\lambda := f\left(\binom{w}{2}\right)$ and let N be the integer returned by Theorem 3.4 when invoked with parameters a, λ, p, and w. We claim that the assertion of Theorem 3.5 is true for this choice of N. Let G be a p-connected graph of tree-width less than w with at least N vertices and suppose that G does not contain a subdivision of $K_{a,p}$. Then by the choices of N and λ the graph G has a valid decomposition (the foundational linkage has at most w paths so there can be at most $\binom{w}{2}$ non-edges in the auxiliary graph). Among all valid decompositions of G pick $(\mathcal{W}, \mathcal{P})$ such that the number of edges of $\Gamma(\mathcal{W}, \mathcal{P})$ is maximal and denote the length of $(\mathcal{W}, \mathcal{P})$ by n.

We may assume that for any integer k with $0 \leq k \leq n-l$ one of the $l-1$ consecutive inner bags $W_{k+1}, \ldots, W_{k+l-1}$ of $\mathcal{W}$ contains a disturbance. If not, then by Lemma 3.6, the contraction of $(\mathcal{W}, \mathcal{P})$ along the sequence $k+1, k+2, \ldots, k+l$ is a stable regular decomposition of G of length l, adhesion at most w, and attachedness p as desired.

Claim 3.5.1. *Let $1 \leq k \leq k' \leq n-1$ with $k'-k \geq lw!-1$. Then the graph $H := G[W_{[k,k']}]$ contains a linkage $\mathcal{Q}$ from the left adhesion set of W_k to the right adhesion set of $W_{k'}$ such that $B(H, \mathcal{Q})$ is a proper supergraph of $\Gamma(\mathcal{W}, \mathcal{P})$, the induced permutation π of $\mathcal{Q}$ is the identity, and $\mathcal{Q}$ is p-attached in H.*

Proof. There are indices $k_0 := k, k_1, \ldots, k_{w!} := k'+1$, such that for $j \in \{1, \ldots, w!\}$ we have $k_j - k_{j-1} \geq l$. For each $j \in \{0, \ldots, w!-1\}$ one of the at least $l-1$ consecutive inner bags $W_{k_j+1}, W_{k_j+2}, \ldots, W_{k_{j+1}-1}$ contains a disturbance $\mathcal{Q}_j$ by our assumption. Let W_{i_j} be the bag of $\mathcal{W}$ that contains $\mathcal{Q}_j$ and let $\mathcal{Q}'_j$ be the bridge stabilisation of $\mathcal{Q}_j$ in $G[W_{i_j}]$.

If $\mathcal{Q}_j$ is a twisting $(\mathcal{W}, \mathcal{P})$-disturbance, then so is $\mathcal{Q}'_j$ as they have the same induced permutation. If $\mathcal{Q}_j$ is a bridging $(\mathcal{W}, \mathcal{P})$-disturbance, then so is $\mathcal{Q}'_j$ by Lemma 3.7 (ii). The set Z of end vertices of $\mathcal{Q}_j$ is the union of both adhesion sets of W_{i_j} and clearly for every vertex $x \in W_{i_j} \setminus Z$ there is an x–Z fan of size p in $G[W_{i_j}]$ as G is p-connected. So by Lemma 3.7 (iii) the linkage $\mathcal{Q}'_j$ is p-attached in $G[W_{i_j}]$.

For every $j \in \{0, \ldots, w!-1\}$ denote the induced permutation of $\mathcal{Q}'_j$ by π_j. Since the symmetric group S_q has order at most $q! \leq w!$ we can pick[3] indices j_0 and j_1 with $0 \leq j_0 \leq j_1 \leq w!-1$ such that $\pi_{j_1} \circ \pi_{j_1-1} \ldots \circ \pi_{j_0} = \mathrm{id}$.

Let $\mathcal{Q}$ be the linkage from the left adhesion set of W_k to the right adhesion set of $W_{k'}$ in H obtained from $\mathcal{P}[W_{[k,k']}]$ by replacing $\mathcal{P}[W_{i_j}]$ with $\mathcal{Q}'_j$ for all $j \in \{j_0, \ldots, j_1\}$. Of all the restrictions of $\mathcal{Q}$ to the bags $W_k, \ldots, W_{k'}$ only $\mathcal{Q}[W_{i_j}] = \mathcal{Q}_j$ with $j_0 \leq j \leq j_1$ need not induce the identity permutation. However, the composition of their induced permutations is the identity by construction and therefore the induced permutation of $\mathcal{Q}$ is the identity.

To see that $B(H, \mathcal{Q})$ is a supergraph of $\Gamma(\mathcal{W}, \mathcal{P})$ note that $k < i_{j_0}$ so $\mathcal{Q}$ and $\mathcal{P}$ coincide on W_k and hence by (L8) we have

$$\Gamma(\mathcal{W}, \mathcal{P}) = B(G[W_k], \mathcal{P}[W_k]) \subseteq B(H, \mathcal{Q}).$$

It remains to show that $B(H, \mathcal{Q})$ contains an edge that is not in $\Gamma(\mathcal{W}, \mathcal{P})$. Set $W := W_{i_{j_0}}$, $W' := W_{i_{j_0}+1}$, and $\pi := \pi_{j_0}$. If $\mathcal{Q}'_{j_0}$ is a bridging disturbance,

[3] Let $(G, \cdot)$ be a group of order n and $g_1, \ldots, g_n \in G$. Then of the $n+1$ products $h_k := \prod_{i=1}^{k} g_i$ for $0 \leq k \leq n$, two must be equal by the pigeon hole principle, say $h_k = h_l$ with $k < l$. This means $\prod_{i=k+1}^{l} g_i = e$, where e is the neutral element of G.

then $B_0 := B(G[W], \mathcal{Q}[W])$ contains an edge that is not in $\Gamma(\mathcal{W}, \mathcal{P})$. Since $\mathcal{Q}$ and $\mathcal{P}$ coincide on all bags prior to W (down to W_k) we must have $B_0 \subseteq B(H, \mathcal{Q})$.

If $\mathcal{Q}'_{j_0}$ is a twisting disturbance, then $j_1 > j_0$, in particular, W' comes before $W_{i_{j_0+1}}$ (there is at least one bag between $W_{i_{j_0}}$ and $W_{i_{j_0+1}}$, namely $W_{k_{j_0+1}}$). This means $\mathcal{Q}[W'] = \mathcal{P}[W']$ and hence we have

$$B_1 := B(G[W'], \mathcal{Q}[W']) = B(G[W'], \mathcal{P}[W']) = \Gamma(\mathcal{W}, \mathcal{P}).$$

On the other hand, the induced permutation of the restriction of $\mathcal{Q}$ to all bags prior to W' is π and thus $\pi^{-1}B_1 \subseteq B(H, \mathcal{Q})$. But π is not an automorphism of $\Gamma(\mathcal{W}, \mathcal{P})$ and therefore $\pi^{-1}B_1 = \pi^{-1}\Gamma(\mathcal{W}, \mathcal{P})$ contains an edge that is not in $\Gamma(\mathcal{W}, \mathcal{P})$ as desired. This concludes the proof of Claim 3.5.1 $\square$

To exploit Claim 3.5.1 we now contract subsegments of $lw!$ consecutive inner bags of $\mathcal{W}$ into single bags. We assumed earlier that $(\mathcal{W}, \mathcal{P})$ is not stable so the number m of non-edges of $\Gamma(\mathcal{W}, \mathcal{P})$ is at least 1 (if $\Gamma(\mathcal{W}, \mathcal{P})$ is complete there can be no disturbances). Set $n' := zf(m-1)$. As $(\mathcal{W}, \mathcal{P})$ is valid, its length n is at least $f(m) = zlw!f(m-1) = n'lw!$. Let $(\mathcal{W}', \mathcal{P}')$ be the contraction of $(\mathcal{W}, \mathcal{P})$ along the sequence $i_1, \ldots, i_{n'}$ defined by $i_j := (j-1)lw! + 1$ for $j = 1, \ldots, n'$. Then by Lemma 3.6 the pair $(\mathcal{W}', \mathcal{P}')$ is a regular decomposition of G of length n', adhesion at most w, it is p-attached, and $\Gamma(\mathcal{W}', \mathcal{P}') = \Gamma(\mathcal{W}, \mathcal{P})$.

By construction every inner bag W'_i of $\mathcal{W}'$ consists of $lw!$ consecutive inner bags of $\mathcal{W}$ and hence by Claim 3.5.1 it contains a bridging disturbance $\mathcal{Q}'_i$ such $\mathcal{Q}'_i$ is p-attached in $G[W'_i]$, its induced permutation is the identity, and $B(G[W'_i], \mathcal{Q}'_i)$ is a proper supergraph of $\Gamma(\mathcal{W}', \mathcal{P}')$.

Clearly $\Gamma(\mathcal{W}', \mathcal{P}')$ has at most $z-1$ proper supergraphs on the same vertex set. On the other hand, $\mathcal{W}'$ has at least $n' - 1 = zf(m-1) - 1$ inner bags. By the pigeonhole principle there must be $f(m-1)$ indices $0 < i_1 < \ldots < i_{f(m-1)} < n'$ such that $B(G[W'_{i_j}], \mathcal{Q}'_{i_j})$ is the same graph Γ for $j = 1, \ldots, f(m-1)$.

Let $(\mathcal{W}'', \mathcal{P}'')$ be the contraction of $(\mathcal{W}', \mathcal{P}')$ along $i_1, \ldots, i_{f(m-1)}$. Obtain the foundational linkage $\mathcal{Q}''$ for $\mathcal{W}''$ from $\mathcal{P}''$ by replacing $\mathcal{P}'[W_{i_j}]$ with $\mathcal{Q}_{i_j}$ for $1 \leq j \leq f(m-1)$. By construction $\mathcal{W}''$ is a slim decomposition of G of length $f(m-1)$ and of adhesion at most w. $\mathcal{Q}''$ is a foundational linkage for $\mathcal{W}''$ that satisfies (L7) because $\mathcal{P}''$ does. By construction $\mathcal{Q}''$ is p-attached and $B(G[W''], \mathcal{Q}''[W'']) = \Gamma$ for all inner bags W'' of $\mathcal{W}''$. Hence $(\mathcal{W}'', \mathcal{P}'')$ is regular decomposition of G. But it is valid and its auxiliary graph Γ has more edges than $\Gamma(\mathcal{W}, \mathcal{P})$, contradicting our initial choice of $(\mathcal{W}, \mathcal{P})$. $\square$

4 Token Movements

Consider the following token game. We place distinguishable tokens on the vertices of a graph H, at most one per vertex. A move consists of sliding a token along the edges of H to a new vertex without passing through vertices which are occupied by other tokens. Which placements of tokens can be obtained from each other by a sequence of moves?

A rather well-known instance of this problem is the 15-puzzle where tokens $1, \ldots, 15$ are placed on the 4-by-4 grid. It has been observed as early as 1879 by Johnson [6] that in this case there are two placements of the tokens which cannot be obtained from each other by any number of moves.

Clearly the problem gets easier the more "unoccupied" vertices there are. The hardest case with $|H|-1$ tokens was tackled comprehensively by Wilson [25] in 1974 but before we turn to his solution we present a formal account of the token game and show how it helps with the linkage problem.

Throughout this section let H be a graph and let $\mathcal{X}$ always denote a sequence $\mathcal{X} = X_0, \ldots, X_n$ of vertex sets of H and $\mathcal{M}$ a non-empty sequence $\mathcal{M} = M_1, \ldots, M_n$ of non-trivial paths in H. In our model the sets X_i are "occupied vertices", the paths M_i are paths along which the tokens are moved, and i is the "move count".

Formally, a pair $(\mathcal{X}, \mathcal{M})$ is called a *movement on* H if for $i = 1, \ldots, n$

(M1) the set $X_{i-1} \triangle X_i$ contains precisely the two end vertices of M_i, and

(M2) M_i is disjoint from $X_{i-1} \cap X_i$.

Then n is the *length* of $(\mathcal{X}, \mathcal{M})$, the sets in $\mathcal{X}$ are its *intermediate configurations*, in particular, X_0 and X_n are its *first* and *last configuration*, respectively. The paths in $\mathcal{M}$ are the *moves* of $(\mathcal{X}, \mathcal{M})$. A movement with first configuration X and last configuration Y is called an X–Y *movement*. Note that our formal notion of token movements allows a move M_i to have both ends in X_{i-1} or both in X_i. In our intuitive account of the token game this corresponds to "destroying" or "creating" a pair of tokens on the end vertices of M_i.

Let us state some obvious facts about movements. If $\mathcal{M}$ is a non-empty sequence of non-trivial paths in H and one intermediate configuration X_i is given, then there is a unique sequence $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{M})$ satisfies (M1). A pair $(\mathcal{X}, \mathcal{M})$ is a movement if and only if $((X_{i-1}, X_i), (M_i))$ is a movement for $i = 1, \ldots, n$. This easily implies the following Lemma so we spare the proof.

Lemma 4.1. *Let* $(\mathcal{X}, \mathcal{M}) = ((X_0, \ldots, X_n), (M_1, \ldots, M_n))$ *and* $(\mathcal{Y}, \mathcal{N}) = ((Y_0, \ldots, Y_m), (N_1, \ldots, N_m))$ *be movements on* H *and let* $Z \subseteq V(H)$.

(i) If $X_n = Y_0$, then the pair

$$\big((X_0, \dots, X_n = Y_0, \dots, Y_m), (M_1, \dots M_n, N_1, \dots, N_m)\big)$$

is a movement. We denote it by $(\mathcal{X}, \mathcal{M}) \oplus (\mathcal{Y}, \mathcal{N})$ and call it the concatenation *of $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{Y}, \mathcal{N})$.*

(ii) If every move of $\mathcal{M}$ is disjoint from Z, then the pair

$$\big((X_0 \cup Z, \dots, X_n \cup Z), (M_1, \dots M_n)\big)$$

is a movement and we denote it by $(\mathcal{X} \cup Z, \mathcal{M})$.

Let $(\mathcal{X}, \mathcal{M})$ be a movement. For $i = 1, \dots, n$ let R_i be the graph with vertex set $(X_{i-1} \times \{i-1\}) \cup (X_i \times \{i\})$ and the following edges:

1. $(x, i-1)(x, i)$ for each $x \in X_{i-1} \cap X_i$, and
2. $(x, j)(y, k)$ where x, y are the end vertices of M_i and j, k the unique indices such that $(x, j), (y, k) \in V(R_i)$.

Define a multigraph $\mathcal{R}$ with vertex set $\bigcup_{i=0}^{n}(X_i \times \{i\})$ where the multiplicity of an edge is the number of graphs R_i containing it. Observe that two graphs R_i and R_j with $i < j$ are edge-disjoint unless $j = i+1$ and M_i and M_j both end in the same two vertices x, y of X_j, in which case they share one edge, namely $(x, j)(y, j)$. Our reason to prefer the above definition of $\mathcal{R}$ over just taking the simple graph $\bigcup_{i=1}^{n} R_i$ is to avoid a special case in the following argument.

Every graph R_i is 1-regular. Hence in $\mathcal{R}$ every vertex (x, i) with $0 < i < n$ has degree 2 as (x, i) is a vertex of R_j if an only if $j = i$ or $j = i+1$. Every vertex (x, i) with $i = 0$ or $i = n$ has degree 1 as it only lies in R_1 or in R_n. This implies that a component of $\mathcal{R}$ is either a cycle (possibly of length 2) avoiding $(X_0 \times \{0\}) \cup (X_n \times \{n\})$ or a non-trivial path with both end vertices in $(X_0 \times \{0\}) \cup (X_n \times \{n\})$. We denote the subgraph of $\mathcal{R}$ consisting of these paths by $\mathcal{R}(\mathcal{X}, \mathcal{M})$. Intuitively, each path of $\mathcal{R}(\mathcal{X}, \mathcal{M})$ traces the position of one token over the course of the token movement or of one pair of tokens which is destroyed or created during the movement.

For vertex sets X and Y we call any 1-regular graph on $(X \times \{0\}) \cup (Y \times \{\infty\})$ an (X, Y)*-pairing*. An (X, Y)-pairing is said to be *balanced* if its edges form a perfect matching from $X \times \{0\}$ to $Y \times \{\infty\}$, that is, each edge has one end vertex in $X \times \{0\}$ and the other in $Y \times \{\infty\}$.

The components of $\mathcal{R}(\mathcal{X}, \mathcal{M})$ induce a 1-regular graph on $(X_0 \times \{0\}) \cup (X_n \times \{n\})$ where two vertices form an edge if and only if they are in the

same component of $\mathcal{R}(\mathcal{X},\mathcal{M})$. To make this formally independent of the index n, we replace each vertex (x,n) by (x,∞). The obtained graph L is an (X_0,X_n)-pairing and we call it the *induced pairing* of the movement $(\mathcal{X},\mathcal{M})$. A movement $(\mathcal{X},\mathcal{M})$ with induced pairing L is called an *L-movement*. If a movement induces a balanced pairing, then we call the movement *balanced* as well.

Given two sets X and Y and a bijection $\varphi : X \to Y$ we denote by $L(\varphi)$ the balanced X–Y pairing where $(x,0)(y,\infty)$ is an edge of $L(\varphi)$ if and only if $y = \varphi(x)$. Clearly an X–Y pairing L is balanced if and only if there is a bijection $\varphi : X \to Y$ with $L = L(\varphi)$.

Given sets X, Y, and Z let L_X be an X–Y pairing and L_Z a Y–Z pairing. Denote by $L_X \oplus L_Z$ the graph on $(X \times \{0\}) \cup (Z \times \{\infty\})$ where two vertices are connected by an edge if and only if they lie in the same component of $L_X \cup L(\mathrm{id}_Y) \cup L_Z$. The components of $L_X \cup L(\mathrm{id}_Y) \cup L_Z$ are either paths with both ends in $(X \times \{0\}) \cup (Z \times \{\infty\})$ or cycles avoiding that set. So $L_X \oplus L_Z$ is an X–Z pairing end we call it the *concatenation* of L_X and L_Z. The next Lemma is an obvious consequence of this construction (and Lemma 4.1 (i)).

Lemma 4.2. *The induced pairing of the concatenation of two movements is the concatenation of their induced pairings.*

Let $(\mathcal{X},\mathcal{M})$ be a movement on H. A vertex x of H is called $(\mathcal{X},\mathcal{M})$-*singular* if no move of $\mathcal{M}$ contains x as an inner vertex and $I_x := \{i \mid x \in X_i\}$ is an *integer interval*, that is, a possibly empty sequence of consecutive integers. Furthermore, x is called *strongly* $(\mathcal{X},\mathcal{M})$*-singular* if it is $(\mathcal{X},\mathcal{M})$-singular and I_x is empty or contains one of 0 and n where n denotes the length of $(\mathcal{X},\mathcal{M})$. We say that a set $W \subseteq V(H)$ is $(\mathcal{X},\mathcal{M})$*-singular* or *strongly* $(\mathcal{X},\mathcal{M})$*-singular* if all its vertices are. If the referred movement is clear from the context, then we shall drop it from the notation and just write singular or strongly singular.

Note that any vertex v of H that is contained in at most one move of $\mathcal{M}$ is strongly $(\mathcal{X},\mathcal{M})$-singular. Furthermore, v is singular but not strongly singular if it is contained in precisely two moves but neither in the first nor in the last configuration.

The following Lemma shows how to obtain linkages in a graph G from movements on the auxiliary graph of a regular decomposition of G. It enables us to apply the results about token movements from this section to our linkage problem.

Lemma 4.3. *Let $(\mathcal{W},\mathcal{P})$ be a stable regular decomposition of some graph G and set $\lambda := \{\alpha \mid P_\alpha$ is non-trivial$\}$ and $\theta := \{\alpha \mid P_\alpha$ is trivial$\}$. Let $(\mathcal{X},\mathcal{M})$ be a movement of length n on a subgraph $\Gamma \subseteq \Gamma(\mathcal{W},\mathcal{P})$ and denote its induced pairing by L. If θ is $(\mathcal{X},\mathcal{M})$-singular and W_a and W_b are inner bags of $\mathcal{W}$*

with $b-a=2n-1$, then there is a linkage $\mathcal{Q} \subseteq G^{\mathcal{P}}_{\Gamma}[W_{[a,b]}]$ and a bijection $\varphi : E(L) \to \mathcal{Q}$ such that for each $e \in E(L)$ the path $\varphi(e)$ ends in the left α-vertex of W_a if and only if $(\alpha, 0) \in e$ and $\varphi(e)$ ends in the right α-vertex of W_b if and only if $(\alpha, \infty) \in e$.

Proof. Let us start with the general observation that for every connected subgraph $\Gamma_0 \subseteq \Gamma(\mathcal{W}, \mathcal{P})$ and every inner bag W of $\mathcal{W}$ the graph $G^{\mathcal{P}}_{\Gamma_0}[W]$ is connected: If $\alpha\beta$ is an edge of Γ_0, then some inner bag of $\mathcal{W}$ contains a $\mathcal{P}$-bridge realising $\alpha\beta$ and so does W by (L8). In particular, $G^{\mathcal{P}}_{\Gamma_0}[W]$, contains a P_α–P_β path. So $\mathcal{P}_{V(\Gamma_0)}[W]$ must be contained in one component of $G^{\mathcal{P}}_{\Gamma_0}[W]$ as Γ_0 is connected. But any vertex of $G^{\mathcal{P}}_{\Gamma_0}[W]$ is in $\mathcal{P}_{V(\Gamma_0)}$ or in a $\mathcal{P}$-bridge attaching to it. Therefore $G^{\mathcal{P}}_{\Gamma_0}[W]$ is connected.

The proof is by induction on n. Denote the end vertices of M_1 by α and β, that is, $X_0 \triangle X_1 = \{\alpha, \beta\}$. By definition the induced pairing L_1 of $((X_0, X_1), (M_1))$ contains the edges $(\gamma, 0)(\gamma, \infty)$ with $\gamma \in X_0 \cap X_1$ and w.l.o.g. precisely one of $(\alpha, 0)(\beta, 0)$, $(\alpha, 0)(\beta, \infty)$, and $(\alpha, \infty)(\beta, \infty)$. The above observation implies that $G^{\mathcal{P}}_{M_1}[W_a]$ is connected. Hence $P_\alpha[W_{[a,a+1]}] \cup G^{\mathcal{P}}_{M_1}[W_a] \cup P_\beta[W_{[a,a+1]}]$ is connected and thus contains a path Q such that $\mathcal{Q}_1 := \{Q\} \cup \mathcal{P}_{X_0 \cap X_1}[W_{[a,a+1]}]$ satisfies the following. There is a bijection $\varphi_1 : E(L_1) \to \mathcal{Q}_1$ such that for each $e \in E(L_1)$ the path $\varphi_1(e)$ ends in the left γ-vertex of W_a if and only if $(\gamma, 0) \in e$ and $\varphi_1(e)$ ends in the right γ-vertex of W_{a+1} if and only if $(\gamma, \infty) \in e$. Moreover, the paths of $\mathcal{Q}_1$ are internally disjoint from $W_{a+1} \cap W_{a+2}$.

In the base case $n = 1$ the linkage $\mathcal{Q} := \mathcal{Q}_1$ is as desired. Suppose that $n \geq 2$. Then $((X_1, \ldots, X_n), (M_2, \ldots, M_n))$ is a movement and we denote its induced permutation by L_2. Lemma 4.2 implies $L = L_1 \oplus L_2$. By induction there is a linkage $\mathcal{Q}_2 \subseteq G^{\mathcal{P}}_{\Gamma}[W_{[a+2,b]}]$ and a bijection $\varphi_2 : E(L_2) \to \mathcal{Q}_2$ such that for any $e \in E(L_2)$ the path $\varphi_2(e)$ ends in the left α-vertex of W_{a+2} (which is the right α-vertex of W_{a+1}) if and only if $(\alpha, 0) \in e$ and in the right α-vertex of W_b if and only if $(\alpha, \infty) \in e$.

Clearly for every $\gamma \in X_1$ the γ-vertex of $W_{a+1} \cap W_{a+2}$ has degree at most 1 in $\mathcal{Q}_1$ and in $\mathcal{Q}_2$. If a path of $\mathcal{Q}_1$ contains the γ-vertex of $W_{a+1} \cap W_{a+2}$ and $\gamma \notin X_1$, then $\gamma \in \theta$ so by assumption $I_\gamma = \{i \mid \gamma \in X_i\}$ is an integer interval which contains 0 but not 1. This means that no path of $\mathcal{Q}_2$ contains the unique vertex of P_γ. If the union $\mathcal{Q}_1 \cup \mathcal{Q}_2$ of the two graphs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contains no cycle, then it is a linkage $\mathcal{Q}$ as desired. Otherwise it only contains such a linkage. □

In the rest of this section we shall construct suitable movements as input for Lemma 4.3. Our first tool to this end is the following powerful theorem[4]

[4]Wilson stated his theorem for graphs which are neither bipartite, nor a cycle, nor a

of Wilson.

Theorem 4.4 (Wilson 1974)**.** *Let k be a postive integer and let H be a graph on $n \geq k+1$ vertices. If H is 2-connected and contains a triangle, then for every bijection $\varphi : X \to Y$ of sets $X, Y \subseteq V(H)$ with $|X| = k = |Y|$ there is a $L(\varphi)$-movement of length $m \leq n!/(n-k)!$ on H.*

The given bound on m is not included in the original statement but not too hard to check: Suppose that $(\mathcal{X}, \mathcal{M})$ is a shortest $L(\varphi)$-movement and m is its length. Since L is balanced we may assume that no tokes are "created" or "destroyed" during the movement, that is, all intermediate configurations have the same size and for every i with $1 \leq i \leq m$ there is an injection $\varphi_i : X \to V(H)$ such that the induced pairing of $((X_0, \ldots, X_i), (M_1, \ldots, M_i))$ is $L(\varphi_i)$. If there were $i < j$ with $\varphi_i = \varphi_j$, then

$$\big((X_0, \ldots, X_i = X_j, X_{j+1}, \ldots, X_m), (M_1, \ldots, M_i, M_{j+1}, \ldots, M_m)\big)$$

was an $L(\varphi)$-movement of length $m - j + i < m$ contradicting our choice of $(\mathcal{X}, \mathcal{M})$. But there are at most $n!/(n-k)!$ injections from X to $V(H)$ so we must have $m \leq n!/(n-k)!$.

For our application we need to generate L-movements where L is not necessarily balanced. Furthermore, Lemma 4.3 requires the vertices of θ to be singular with respect to the generated movement. Lemma 4.8 and Lemma 4.9 give a direct construction of movements if some subgraph of H is a large star. Lemma 4.10 provides an interface to Theorem 4.4 that incorporates the above requirements. The proofs of these three Lemmas require a few tools: Lemma 4.5 simply states that for sets X and Y of equal size there is a short balanced X–Y movement. Lemma 4.6 exploits this to show that instead of generating movements for every choice of $X, Y \subseteq V(H)$ and any (X, Y)-pairing L it suffices to consider just one choice of X and Y. Lemma 4.7 allows us to move strongly singular vertices from X to Y and vice versa without spoiling the existence of the desired X–Y movement.

We call a set A of vertices in a graph H *marginal* if $H - A$ is connected and every vertex of A has a neighbour in $H - A$.

Lemma 4.5. *For any two distinct vertex sets X and Y of some size k in a connected graph H and any marginal set $A \subseteq V(H)$ there is a balanced X–Y movement of length at most k on H such that A is strongly singular.*

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certain graph θ_0. If H properly contains a triangle, then it satisfies all these conditions and if H itself is a triangle, then our theorem is obviously true.

Proof. We may assume that H is a tree and that all vertices of A are leaves of this tree. This already implies that vertices of A cannot be inner vertices of moves. Moreover, we may assume that $X \cap Y \cap A = \emptyset$.

We apply induction on $|H|$. The base case $|H| = 1$ is trivial. For $|H| > 1$ let e be an edge of H. If the two components H_1 and H_2 of $H - e$ each contain the same number of vertices from X as from Y, then for $i = 1, 2$ we set $X_i := X \cap V(H_i)$ and $Y_i := Y \cap V(H_i)$. By induction there is a balanced X_i–Y_i movement $(\mathcal{X}_i, \mathcal{M}_i)$ of length at most $|X_i|$ on H_i such that each vertex of A is strongly $(\mathcal{X}_i, \mathcal{M}_i)$-singular where $i = 1, 2$. By Lemma 4.1 $(\mathcal{X}, \mathcal{M}) := (\mathcal{X}_1 \cup X_2, \mathcal{M}_1) \oplus (\mathcal{X}_2 \cup Y_1, \mathcal{M}_2)$ is an X–Y movement of length at most $|X_1| + |X_2| = |X| = k$ as desired. Clearly $(\mathcal{X}, \mathcal{M})$ is balanced and A is strongly $(\mathcal{X}, \mathcal{M})$-singular as H_1 and H_2 are disjoint.

So we may assume that for every edge e of H one component of $H - e$ contains more vertices from Y than from X and direct e towards its end vertex lying in this component. As every directed tree has a sink, there is a vertex y of H such that every incident edge e is incoming, that is, the component of $H - e$ not containing y contains more vertices of X than of Y. As $|X| = |Y|$, this can only be if y is a leaf in H and $y \in Y \setminus X$.

Let M be any X–y path and denote its first vertex by x. At most one of $x \in Y$ and $x \in A$ can be true by assumption. Clearly $((\{x\}, \{y\}), (M))$ is an $\{x\}$–$\{y\}$ movement and since $H - y$ is connected, by induction there is a balanced $(X \setminus \{x\})$–$(Y \setminus \{y\})$ movement $(\mathcal{X}', \mathcal{M}')$ of length at most $k - 1$ on $H - y$ such that A is strongly singular w.r.t. both movements. As before, Lemma 4.1 implies that

$$(\mathcal{X}, \mathcal{M}) := \big((X, (X \setminus \{x\}) \cup \{y\}), (M)\big) \oplus (\mathcal{X}' \cup \{y\}, \mathcal{M}')$$

is an X–Y movement of length at most k. Clearly $(\mathcal{X}, \mathcal{M})$ is balanced and by construction A is strongly $(\mathcal{X}, \mathcal{M})$-singular. □

Lemma 4.6. *Let k be a positive integer and H a connected graph with a marginal set A. Suppose that $X, X', Y', Y \subseteq V(H)$ are sets with $|X| + |Y| = 2k$, $|X'| = |X|$, and $|Y'| = |Y|$ such that $(X \cup X') \cap (Y' \cup Y)$ does not intersect A. If for each (X', Y')-pairing L' there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most n' on H such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular, then for each (X, Y)-pairing L there is an L-movement $(\mathcal{X}, \mathcal{M})$ of length at most $n' + 2k$ such that A is $(\mathcal{X}, \mathcal{M})$-singular and all vertices of A that are not strongly $(\mathcal{X}, \mathcal{M})$-singular are in $(X' \cup Y') \setminus (X \cup Y)$.*

Proof. Let $(\mathcal{X}_X, \mathcal{M}_X)$ be a balanced X–X' movement of length at most $|X|$ and let $(\mathcal{X}_Y, \mathcal{M}_Y)$ be a balanced Y'–Y movement of length at most $|Y|$ such

that A is strongly singular w.r.t. both movements. These exist by Lemma 4.5. For any X'–Y' movement $(\mathcal{X}', \mathcal{M}')$ such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular,

$$(\mathcal{X}, \mathcal{M}) := (\mathcal{X}_X, \mathcal{M}_X) \oplus (\mathcal{X}', \mathcal{M}') \oplus (\mathcal{X}_Y, \mathcal{M}_Y)$$

is a movement of length at most $|X| + n' + |Y| = n' + 2k$ by Lemma 4.1.

In a slight abuse of the notation we shall write $a \in \mathcal{M}_X$, $a \in \mathcal{M}'$, and $a \in \mathcal{M}_Y$ for a vertex $a \in A$ if there is a move of $\mathcal{M}_X$, $\mathcal{M}'$, and $\mathcal{M}_Y$, respectively, that contains a. Consequently, we write $a \notin M_X$, etc. if there is no such move. The set A is strongly singular w.r.t. each of $(\mathcal{X}_X, \mathcal{M}_X)$, $(\mathcal{X}', \mathcal{M}')$, and $(\mathcal{X}_Y, \mathcal{M}_Y)$. Therefore all moves of $\mathcal{M}$ are internally disjoint from A and each $a \in A$ is contained in at most one move from each of $\mathcal{M}_X$, $\mathcal{M}'$, and $\mathcal{M}_Y$. Moreover, for each $a \in A$

1. $a \in \mathcal{M}_X$ if and only if precisely one of $a \in X$ and $a \in X'$ is true,
2. $a \in \mathcal{M}'$ if and only if precisely one of $a \in X'$ and $a \in Y'$ is true, and
3. $a \in \mathcal{M}_Y$ if and only if precisely one of $a \in Y'$ and $a \in Y$ is true.

Clearly $A \setminus (X \cup X' \cup Y' \cup Y)$ is strongly $(\mathcal{X}, \mathcal{M})$-singular as none of its vertices is contained in a path of $\mathcal{M}$.

Let $a \in X \cap A$. Then by assumption $a \notin Y \cup Y'$ and thus $a \notin \mathcal{M}_Y$. If $a \in X'$, then $a \in \mathcal{M}'$ and $a \notin \mathcal{M}_X$. Otherwise $a \notin X'$ and therefore $a \in \mathcal{M}_X$ and $a \notin \mathcal{M}'$. In either case a is in at most one move of $\mathcal{M}$ and hence $X \cap A$ is strongly $(\mathcal{X}, \mathcal{M})$-singular. A symmetric argument shows that $Y \cap A$ is strongly $(\mathcal{X}, \mathcal{M})$-singular.

Let $a \in (X' \cup Y') \cap A$ with $a \notin X \cup Y$. Then $a \in X' \triangle Y'$ so $a \in \mathcal{M}'$ and precisely one of $a \in \mathcal{M}_X$ and $a \in \mathcal{M}_Y$ is true.

We conclude that every vertex of $a \in A$ is $(\mathcal{X}, \mathcal{M})$-singular and it is even strongly $(\mathcal{X}, \mathcal{M})$-singular if and only if $a \notin (X' \cup Y') \setminus (X \cup Y)$.

The induced pairings L_X of $(\mathcal{X}_X, \mathcal{M}_X)$ and L_Y of $(\mathcal{X}_Y, \mathcal{M}_Y)$ are both balanced and it is not hard to see that for a suitable choice of L' the induced pairing $L_X \oplus L' \oplus L_Y$ of $(\mathcal{X}, \mathcal{M})$ equals L. □

Lemma 4.7. *Let H be a connected graph and let $X, Y \subseteq V(H)$. Suppose that L is an (X, Y)-pairing and $(\mathcal{X}, \mathcal{M})$ an L-movement of length n. If $x \in X \cup Y$ is strongly $(\mathcal{X}, \mathcal{M})$-singular, then the following statements hold.*

(i) $(\mathcal{X} \triangle x, \mathcal{M})$ is an $(L \triangle x)$-movement of length n where $\mathcal{X} \triangle x := (X_0 \triangle \{x\}, \ldots, X_n \triangle \{x\})$ and $L \triangle x$ denotes the graph obtained from L by replacing $(x, 0)$ with (x, ∞) or vice versa (at most one of these can be a vertex of L).

(ii) A vertex $y \in V(H)$ is (strongly) $(\mathcal{X}, \mathcal{M})$-singular if and only if it is (strongly) $(\mathcal{X} \triangle x, \mathcal{M})$-singular.

Proof. Clearly $(\mathcal{X} \triangle x, \mathcal{M})$ is an $(L \triangle x)$-movement of length n. As its intermediate configurations differ from those of $\mathcal{X}$ only in x, the last assertion is trivial for $y \neq x$. For $y = x$ note that $\{i \mid x \notin X_i\}$ is an integer interval containing precisely one of 0 and n because $\{i \mid x \in X_i\}$ is. □

In the final three Lemmas of this section we put our tools to use and construct movements under certain assumptions about the graph. Note that it is not hard to improve on the upper bounds given for the lengths of the generated movements with more complex proofs. However, in our main proof we have an arbitrarily long stable regular decomposition at our disposal, so the input movements for Lemma 4.3 can be arbitrarily long as well.

Lemma 4.8. *Let k be a positive integer and H a connected graph with a marginal set A. If one of*

a) $|A| \geq 2k - 1$ and

b) $|N_H(v) \cap N_H(w) \cap A| \geq 2k - 3$ for some edge vw of $H - A$

holds, then for any X–Y pairing L such that $X, Y \subseteq V(H)$ with $|X| + |Y| = 2k$ and $X \cap Y \cap A = \emptyset$ there is an L-movement $(\mathcal{X}, \mathcal{M})$ of length at most $3k$ on H such that A is $(\mathcal{X}, \mathcal{M})$-singular.

The basic argument of the proof is that that if we place tokens on the leaves of a star but not on its centre, then we can clearly "destroy" any given pair of tokens by moving one on top of the other through the centre of the star.

Proof. Suppose that a) holds. Let $N_A \subseteq A$ with $|N_A| = 2k - 1$. There are sets $X', Y' \subseteq V(H)$ such that

1. $|X'| = |X|$ and $|Y'| = |Y|$,
2. $X \cap N_A \subseteq X'$,
3. $Y \cap N_A \subseteq Y'$, and
4. $N_A \subseteq X' \cup Y'$ and $X' \cap Y' \cap A = \emptyset$.

By Lemma 4.6 it suffices to show that for each X'–Y' pairing L' there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most k on H such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular. Assume w.l.o.g. that the unique vertex of $(X' \cup Y') \setminus N_A$ is in X'. Repeated application of Lemma 4.7 implies that the desired L'-movement $(\mathcal{X}', \mathcal{M}')$ exists if and only if for every $(X' \cup Y')$–$\emptyset$ pairing L'' there is an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most k on H such that A is strongly $(\mathcal{X}'', \mathcal{M}'')$-singular.

Let L'' be any $(X' \cup Y')$–$\emptyset$ pairing. Then $E(L'') = \{(x_i, 0)(y_i, 0) \mid i = 1, \ldots, k\}$ where $(X' \cup Y') \cap N_A = \{x_1, \ldots, x_k, y_2, \ldots, y_k\}$ and $(X' \cup Y') \setminus N_A = \{y_1\}$. For $i = 0, \ldots, k$ set $X_i := \{x_j, y_j \mid j > i\}$. For $i = 1, \ldots, k$ let M_i be an x_i–y_i path in H that is internally disjoint from A. Then $(\mathcal{X}'', \mathcal{M}'') := ((X_0, \ldots, X_k), (M_1, \ldots, M_k))$ is an L''-movement of length k and obviously A is strongly $(\mathcal{X}'', \mathcal{M}'')$-singular.

Suppose that b) holds and let $N_A \subseteq N_H(v) \cap N_H(w) \cap A$ with $|N_A| = 2k-3$ and set $N_B := \{v, w\}$. There are sets $X', Y' \subseteq V(H)$ such that

1. $|X'| = |X|$ and $|Y'| = |Y|$,
2. $X \cap N_A \subseteq X'$ and $X' \cap A \subseteq X \cup N_A$,
3. $Y \cap N_A \subseteq Y'$ and $Y' \cap A \subseteq Y \cup N_A$,
4. $N_A \subseteq X' \cup Y'$ and $X' \cap Y' \cap A = \emptyset$,
5. $N_B \subseteq X'$ or $X' \subset N_A \cup N_B$, and
6. $N_B \subseteq Y'$ or $Y' \subset N_A \cup N_B$.

By Lemma 4.6 (see case a) for the details) it suffices to find an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most k on H such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular where L' is any X'–Y' pairing. Since $|(X' \cup Y') \setminus N_A| = 3$ we may asssume w.l.o.g. that $N_B \subseteq X'$ and $Y' \subseteq N_A \cup \{v\}$. So either there is $z \in X' \setminus (N_A \cup N_B)$ or $v \in Y'$. By repeated application of Lemma 4.7 we may assume that $N_A \subseteq X'$. This means that L' has the vertices $\bar{N}_A := N_A \times \{0\}$, $\bar{v} := (v, 0)$, $\bar{w} := (w, 0)$, and $\bar{z} := (z, 0)$ in the first case or $\bar{z} := (v, \infty)$ in the second case. So L' must satisfy one of the following.

1. No edge of L' has both ends in $\{\bar{v}, \bar{w}, \bar{z}\}$.
2. $\bar{v}\bar{w} \in E(L')$.
3. $\bar{v}\bar{z} \in E(L')$.
4. $\bar{w}\bar{z} \in E(L')$.

This leaves us with eight cases in total. Since construction is almost the same for all cases we provide the details for only one of them: We assume that $v \in Y'$ and $\bar{w}\bar{z} \in E(L')$. Then L' has edges $(w,0)(v,\infty)$ and $\{(x_i,0)(y_i,0) \mid i = 1,\dots,k-1\}$ where $x_1 := v$ and $X' \cap N_A = \{x_2,\dots,x_{k-1},y_1,\dots,y_{k-1}\}$. For $i = 0,\dots,k-1$ set $X_i := \{w\} \cup \bigcup_{j>i}\{x_j,y_j\}$ and let $X_k := \{v\}$. Set $M_1 := vy_1$ and $M_i := x_i v y_i$ for $i = 2,\dots,k-1$ and let M_k be a w–z path in H that is internally disjoint from A. Then $(\mathcal{X}',\mathcal{M}') := ((X_0,\dots,X_k),(M_1,\dots,M_k))$ is an L'-movement and A is strongly $(\mathcal{X}',\mathcal{M}')$-singular. □

Lemma 4.9. *Let k be a positive integer and H a connected graph with a marginal set A. Let $X,Y \subseteq V(H)$ with $|X|+|Y| = 2k$ and $X \cap Y \cap A = \emptyset$. Suppose that there is a vertex v of $H-(X \cup Y \cup A)$ such that*

$$2|N_H(v) \setminus A| + |N_H(v) \cap A| \geq 2k+1.$$

Then for any (X,Y)-pairing L there is an L-movement of length at most $k(k+2)$ on H such that A is singular.

Although the basic idea is still the same as in Lemma 4.8 it gets a little more complicated here as our star might not have enough leaves to hold all tokens at the same time. Hence we prefer an inductive argument over an explicit construction.

Proof. Set $N_A := N_H(v) \cap A$ and $N_B := N_H(v) \setminus A$. If $|N_A| \geq 2k-1$, then we are done by Lemma 4.8 as $3k \leq k(k+2)$. So we may assume that $|N_A| \leq 2k-2$. Under this additional assumption we prove a slightly stronger statement than that of Lemma 4.9 by induction on k: We not only require that A is singular but also that all vertices of A that are not strongly singular are in $N_A \setminus (X \cup Y)$.

The base case $k = 1$ is trivial. Suppose that $k \geq 2$. There are sets $X',Y' \subseteq V(H)$ such that

1. $|X'| = |X|$ and $|Y'| = |Y|$,
2. $X \cap N_A \subseteq X'$ and $X' \cap A \subseteq X \cup N_A$,
3. $Y \cap N_A \subseteq Y'$ and $Y' \cap A \subseteq Y \cup N_A$,
4. $N_A \subseteq X' \cup Y'$ and $X' \cap Y' \cap A = \emptyset$,
5. $N_B \subseteq X'$ or $X' \subset N_A \cup N_B$,
6. $N_B \subseteq Y'$ or $Y' \subset N_A \cup N_B$, and
7. $v \notin X'$ and $v \notin Y'$.

By Lemma 4.6 it suffices to find an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most k^2 such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular where L' is any X'–Y' pairing.

If there are $x, y \in X' \cap N_H(v)$ such that $(x,0)(y,0) \in E(L')$, then set $X'' := X' \setminus \{x,y\}$, $Y'' := Y'$, $H'' := H - (A \setminus (X'' \cup Y''))$, and $L'' := L' - \{(x,0),(y,0)\}$. We have $N_{H''}(v) \setminus A = N_B$ and $N_{H''}(v) \cap A = N_A \setminus \{x,y\}$ as $N_A \subseteq X' \cup Y'$. This means

$$2|N_{H''}(v) \setminus A| + |N_{H''}(v) \cap A| \geq 2|N_B| + |N_A| - 2 \geq 2k - 1.$$

Hence by induction there is an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most $(k+1)(k-1)$ on H'' such that A is singular and all vertices of A that are not strongly singular are in $N_A \setminus (X'' \cup Y'')$. Since $N_A \cap V(H'') \subseteq X'' \cup Y''$ the set A is strongly $(\mathcal{X}'', \mathcal{M}'')$-singular. Then by construction $(\mathcal{X}', \mathcal{M}') := ((X', X''), (xvy)) \oplus (\mathcal{X}'', \mathcal{M}'')$ is an L'-movement of length at most k^2 and A is strongly $(\mathcal{X}', \mathcal{M}')$-singular.

The case $x, y \in Y' \cap N_H(v)$ with $(x,\infty)(y,\infty) \in E(L')$ is symmetric. If there are $x \in X' \cap N_H(v)$ and $y \in Y' \cap N_H(v)$ such that $(x,0)(y,\infty) \in E(L')$ and at least one of x and y is in N_A, then the desired movement exists by Lemma 4.7 and one of the previous cases.

By assumption

$$2|N_H(v)| \geq 2|N_B| + |N_A| \geq 2k + 1$$

and thus $|N_H(v)| \geq k+1$. If $N_B \subseteq X'$, then $N_H(v) \subseteq X' \cup (Y' \cap A)$ and there is a pair as above by the pigeon hole principle. Hence we may assume that $X' \subset N_B$ and by symmetry also that $Y' \subset N_B$. This implies that $N_A = \emptyset$ and that L' is balanced.

So we have $|X'| = k = |Y'|$, $X', Y' \subseteq N_B$ and $|N_B| \geq k+1$. It is easy to see that there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $2k \leq k^2$ on $H[\{v\} \cup N_B]$ such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular. □

Lemma 4.10. *Let $n \in \mathbb{N}$ and let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be the map that is recursively defined by setting $f(0) := 0$ and $f(k) := 2k + 2n! + 4 + f(k-1)$ for $k > 0$. Let k be a positive integer and let H be a connected graph on at most n vertices with a marginal set A. Let $X, Y \subseteq V(H)$ with $|X| + |Y| = 2k$ and $X \cap Y \cap A = \emptyset$ such that neither X nor Y contains all vertices of $H - A$. Suppose that there is a block D of $H - A$ such that D contains a triangle and $2|D| + |N(D)| \geq 2k + 3$. Then for any (X,Y)-pairing L there is an L-movement of length at most $f(k)$ on H such that A is singular.*

Proof. Set $N_A := N(D) \cap A$ and $N_B := N(D) \setminus A$. If $|N_A| \geq 2k - 1$, then we are done by Lemma 4.8 as $3k \leq f(k)$. So we may assume that $|N_A| \leq 2k - 2$.

Under this additional assumption we prove a slightly stronger statement than that of Lemma 4.10 by induction on k: We not only require that A is singular but also that all vertices of A that are not strongly singular are in $N_A \setminus (X \cup Y)$. As always the base case $k = 1$ is trivial. Suppose that $k \geq 2$.

Claim 4.10.1. *Suppose that $|V(D) \setminus X| \geq 1$ and that there is an edge $(x, 0)(y, 0) \in E(L)$ with $x \in V(D)$ and $y \in V(D) \cup N(D)$. Let $A' \subseteq A \setminus (X \cup Y)$ with $|A'| \leq 1$. Then there is an L-movement of length at most $|D|! + 1 + f(k-1)$ on $H - A'$ such that A is singular and every vertex of A that is not strongly singular is in $N_A \setminus (X \cup Y)$.*

Proof. Let y' be a neighbour of y in D. Here is a sketch of the idea: Move the token from x to y' by a movement on D which we can generate with Wilsons's Theorem 4.4 and then add the move yy'. This "destroys" one pair of tokens and allows us to invoke induction.

We assume $x \neq y'$ (in the case $x = y'$ we can skip the construction of $(\mathcal{X}_\varphi, \mathcal{M}_\varphi)$ in this paragraph). Set $X' := (X \setminus \{x\}) \cup \{y'\}$ if $y' \notin X$ and $X' := X$ otherwise. The vertices x and y' are both in the 2-connected graph D which contains a triangle. By definition $|X \cap V(D)| = |X' \cap V(D)|$ and by assumption both sets are smaller than $|D|$. Let $\varphi : X \to X'$ any bijection with $\varphi|_{X \setminus V(D)} = \mathrm{id}\,|_{X \setminus V(D)}$ and $\varphi(x) = y'$. By Theorem 4.4 there is a balanced $L(\varphi|_{V(D)})$-movement of length at most $|D|!$ on D so by Lemma 4.1 (ii) there is a balanced $L(\varphi)$-movement $(\mathcal{X}_\varphi, \mathcal{M}_\varphi)$ of length at most $|D|!$ on H such that all its moves are contained in D.

Set $X'' := X' \setminus \{y, y'\}$ and let L' be the X'–X'' pairing with edge set $\{(z, 0)(z, \infty) \mid z \in X''\} \cup \{(y, 0)(y', 0)\}$. Clearly $((X', X''), (yy'))$ is an L'-movement. Let L'' be the X''–Y pairing obtained from L by deleting the edge $(x, 0)(y, 0)$ and substituting every vertex $(z, 0)$ with $(\varphi(z), 0)$. By definition we have $L = L(\varphi) \oplus L' \oplus L''$.

The set $A'' := A' \cup (A \cap \{y\})$ has at most 2 elements and thus $2|D| + |N(D) \setminus A''| \geq 2k + 1$. So by induction there is an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most $f(k-1)$ on $H - A''$ such that A is $(\mathcal{X}'', \mathcal{M}'')$-singular and every vertex of A that is not strongly $(\mathcal{X}'', \mathcal{M}'')$-singular is in $N_A \setminus (X \cup Y)$. Hence the movement

$$(\mathcal{X}, \mathcal{M}) := (\mathcal{X}_\varphi, \mathcal{M}_\varphi) \oplus \big((X', X''), (yy')\big) \oplus (\mathcal{X}'', \mathcal{M}'')$$

on $H - A'$ has induced pairing L by Lemma 4.2 and length at most $|D|! + 1 + f(k-1)$. Every move of $\mathcal{M}$ that contains a vertex of $A \setminus \{y\}$ is in $\mathcal{M}''$. Hence $A \setminus \{y\}$ is $(\mathcal{X}, \mathcal{M})$-singular and every vertex of $A \setminus \{y\}$ that is not strongly $(\mathcal{X}, \mathcal{M})$-singular is in $N_A \setminus (X \cup Y)$. If $y \notin A$, then we are done. But if $y \in A$, then our construction of $(\mathcal{X}'', \mathcal{M}'')$ ensures that no move of $\mathcal{M}''$ contains y. Therefore y is strongly $(\mathcal{X}, \mathcal{M})$-singular. $\square$

Claim 4.10.2. *Suppose that $|V(D)\backslash X| \geq 2$ and that L has an edge $(x,0)(y,0)$ with $x, y \in N(D)$. Then there is an L-movement of length at most $2|D|! + 2 + f(k-1)$ on H such that A is singular and every vertex of A that is not strongly singular is in $N_A \setminus (X \cup Y)$.*

Proof. The proof is very similar to that of Claim 4.10.1. Let y' be a neighbour of y in D. We assume $y' \in X$ (in the case $y' \notin X$ we can skip the construction of $(\mathcal{X}_\varphi, \mathcal{M}_\varphi)$ in this paragraph). Let $z \in V(D) \setminus X$ and let $X' := (X \setminus \{y'\}) \cup \{z\}$. Let $\varphi : X \to X'$ be any bijection with $\varphi|_{X \setminus V(D)} = \mathrm{id}_{X \setminus V(D)}$ and $\varphi(y') = z$. Applying Theorem 4.4 and Lemma 4.1 as in the proof of Claim 4.10.1 we obtain a balanced $L(\varphi)$-movement $(\mathcal{X}_\varphi, \mathcal{M}_\varphi)$ of length at most $|D|!$ such that its moves are contained in D (in fact, we could "free" the vertex y' with only $|D|$ moves by shifting each token on a y'–z path in D by one position towards z, but we stick with the proof of Claim 4.10.1 here for simplicity).

Set $X'' := (X' \setminus \{y\}) \cup \{y'\}$ and let $\varphi' : X' \to X''$ be the bijection that maps y to y' and every other element to itself. Clearly $((X', X''), (yy'))$ is an $L(\varphi')$-movement. Let L'' be the X''–Y pairing obtained from L by substituting every vertex $(z, 0)$ with $(\varphi' \circ \varphi(z), 0)$. It is not hard to see that this construction implies $L = L(\varphi) \oplus L(\varphi') \oplus L''$. Since $(0, x)(0, y')$ is an edge of L'' with $x \in V(D) \cup N(D)$ and $y' \in V(D)$ we can apply Claim 4.10.1 to obtain an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most $|D|! + 1 + f(k-1)$ on $H - (\{y\} \cap A)$ (note that $y \in A \cap X$ implies $y \notin Y$ by assumption) such that $A \setminus \{y\}$ is $(\mathcal{X}'', \mathcal{M}'')$-singular and every vertex of $A \setminus \{y\}$ that is not strongly $(\mathcal{X}'', \mathcal{M}'')$-singular is in $N_A \setminus (X \cup Y)$. Hence the movement

$$(\mathcal{X}, \mathcal{M}) := (\mathcal{X}_\varphi, \mathcal{M}_\varphi) \oplus \big((X', X''), (yy')\big) \oplus (\mathcal{X}'', \mathcal{M}'')$$

on H has induced pairing L by Lemma 4.2 and length at most $2|D|! + 2 + f(k-1)$. The argument that A is $(\mathcal{X}, \mathcal{M})$-singular and the only vertices of A that are not strongly $(\mathcal{X}, \mathcal{M})$-singular are in $N_A \setminus (X \cup Y)$ is the same as in the proof of Claim 4.10.1. □

Pick any vertex $v \in V(D)$. There are sets $X', Y' \subseteq V(H)$ such that

1. $|X'| = |X|$ and $|Y'| = |Y|$,
2. $X \cap N_A \subseteq X'$ and $X' \cap A \subseteq X \cup N_A$,
3. $Y \cap N_A \subseteq Y'$ and $Y' \cap A \subseteq Y \cup N_A$,
4. $N_A \subseteq X' \cup Y'$ and $X' \cap Y' \cap A = \emptyset$,
5. $N_B \subseteq X'$ or $X' \subset N_A \cup N_B$,

6. $N_B \subseteq Y'$ or $Y' \subset N_A \cup N_B$,

7. $v \notin X'$ and $v \notin Y'$,

8. $V(D) \cup N_B \subseteq X' \cup \{v\}$ or $X' \subset V(D) \cup N(D)$, and

9. $V(D) \cup N_B \subseteq Y' \cup \{v\}$ or $Y' \subset V(D) \cup N(D)$.

By Lemma 4.6 it suffices to find an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $f(k) - 2k$ on H such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular where L' is any X'–Y' pairing.

Since $n \geq |D|$ we have $f(k) - 2k \geq 2|D|! + 2 + f(k-1)$ and by assumption $v \in V(D) \setminus X'$. If L' has an edge $(0, x)(0, y)$ with $x \in V(D)$ and $y \in V(D) \cup N(D)$, then by Claim 4.10.1 there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $f(k) - 2k$ on H such that A is strongly $(\mathcal{X}', \mathcal{M}')$-singular (recall that $N_A \setminus (X' \cup Y')$ is empty by choice of X' and Y'). So we may assume that L' contains no such edge and by Lemma 4.7 we may also assume that it has no edge $(x, 0)(y, \infty)$ with $x \in V(D)$ and $y \in N_A$.

Counting the edges of L' that are incident with a vertex of $(V(D) \cup N(D)) \times \{0\}$ we obtain the lower bound

$$\|L'\| \geq |X' \cap V(D)| + |X' \cap N_B|/2 + |(X \cup Y) \cap N_A|/2.$$

If $V(D) \cup N_B \subseteq X' \cup \{v\}$, then $|X' \cap V(D)| = |D| - 1$ and $|X' \cap N_B| = |N_B|$. Since $|(X' \cup Y') \cap N_A| = |N_A|$ this means

$$2k = 2\|L'\| \geq 2(|D| - 1) + |N_B| + |N_A| \geq 2|D| + |N(D)| - 2 \geq 2k + 1,$$

a contradiction. So we must have $X' \subset V(D) \cup N(D)$ and $|V(D) \setminus X'| \geq 2$. Applying Claim 4.10.2 in the same way as Claim 4.10.1 above we deduce that no edge of L' has both ends in $X \times \{0\}$ or one end in $X \times \{0\}$ and the other in $N_A \times \{\infty\}$. By symmetry we can obtain statements like Claim 4.10.1 and Claim 4.10.2 for Y instead of X thus by the same argument as above we may also assume that $Y' \subset V(D) \cup N(D)$ and that no edge of L' has both ends in $Y \times \{\infty\}$ or one end in $N_A \times \{0\}$ and the other in $Y \times \{\infty\}$. Hence L' is balanced and $N_A = \emptyset$. Let $\varphi' : X' \to Y'$ be the bijection with $L' = L(\varphi')$.

In the rest of the proof we apply the same techniques that we have already used in the proof of Claim 4.10.1 and again in that of Claim 4.10.2 so from now on we only sketch how to construct the desired movements. Furthermore, all constructed movements use only vertices of $V(D) \cup N_B$ for their moves so A is trivially strongly singular w.r.t. them.

If $N_B \setminus X' \neq \emptyset$, then we have $X' \subset N_B$ by assumption, so $|N_B| \geq k + 1$ and thus also $Y' \subset N_B$. This is basically the same situation as at the end of the proof for Lemma 4.9 so we find an L'-movement of length at most $2k \leq f(k)$. We may therefore assume that $N_B \subseteq X' \cap Y'$.

Claim 4.10.3. *Suppose that L' has an edge $(x,0)(y,\infty)$ with $x \in V(D)$ and $y \in N_B$. Then there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $|D|! + 2 + f(k-1)$ such that the moves of $\mathcal{M}'$ are disjoint from A.*

Proof. Let y' be a neighbour of y in $V(D)$. Since D is 2-connected, y' has two distinct neighbours y_l and y_r in D. Using Theorem 4.4 we generate a balanced movement of length at most $|D|!$ on H such that all its moves are in D and its induced pairing has the edge $(x,0)(y_l,\infty)$ and its final configuration does not contain y' or y_r. Adding the two moves $yy'y_r$ and $y_l y' y$ then results in a movement $(\mathcal{X}_x, \mathcal{M}_x)$ of length at most $|D|! + 2$ whose induced pairing L_x contains the edge $(x,0)(y,0)$.

It is not hard to see that there is a pairing L'' such that $L_x \oplus L'' = L$ and this pairing must have the edge $(y,0)(y,\infty)$. By induction there is an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most $f(k-1)$ such that none if its moves contains y. So $(\mathcal{X}', \mathcal{M}') := (\mathcal{X}_x, \mathcal{M}_x) \oplus (\mathcal{X}'', \mathcal{M}'')$ is an L' movement of length at most $|D|! + 2 + f(k-1)$ as desired. □

Claim 4.10.4. *Suppose that L' has an edge $(x,0)(y,\infty)$ with $x, y \in N_B$. Then there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $2|D|! + 4 + f(k-1)$ such that the moves of $\mathcal{M}'$ are disjoint from A.*

Proof. Let x' be a neighbour of x in $V(D)$. Since D is 2-connected, x' has two distinct neighbours x_l and x_r in D. With the same construction as in Claim 4.10.3 we can generate a movement $(\mathcal{X}_x, \mathcal{M}_x)$ of length at most $|D|! + 2$ such that its induced pairing L_x contains the edge $(x,0)(x_r,\infty)$ and $(x'',0)(x,\infty)$ for some vertex $x'' \in V(D) \cap X'$. There is a pairing L'' such that $L = L_x \oplus L''$ and L'' contains the edge $(x_r,0)(y,\infty)$.

By Claim 4.10.3 there is an L''-movement $(\mathcal{X}'', \mathcal{M}'')$ of length at most $|D|! + 2 + f(k-1)$. So $(\mathcal{X}', \mathcal{M}') := (\mathcal{X}_x, \mathcal{M}_x) \oplus (\mathcal{X}'', \mathcal{M}'')$ is an L' movement of length at most $2|D|! + 4 + f(k-1)$ as desired. □

Since $f(k) - 2k \geq 2|D|! + 4 + f(k-1)$ we may assume that $N_B \cap Y' = \emptyset$ and thus $N_B = \emptyset$ by Claim 4.10.3 and Claim 4.10.4. This means $X', Y' \subseteq V(D)$ and therefore by Theorem 4.4 there is an L'-movement $(\mathcal{X}', \mathcal{M}')$ of length at most $|D|! \leq n! \leq f(k) - 2k$. This concludes the induction and thus also the proof of Lemma 4.10. □

5 Relinkages

This section collects several Lemma that compare different foundational linkages for the same stable regular decomposition of a graph. To avoid tedious repetitions we use the following convention throughout the section.

Convention. Let $(\mathcal{W}, \mathcal{P})$ be a stable regular decomposition of some length $l \geq 3$ and attachedness p of a p-connected graph G and set $\lambda := \{\alpha \mid P_\alpha$ is non-trivial$\}$ and $\theta := \{\alpha \mid P_\alpha$ is trivial$\}$. Let D be a block of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ and let κ be the set of all cut-vertices of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ that are in D.

Lemma 5.1. *Let $\mathcal{Q}$ be a foundational linkage. If $\alpha\beta$ is an edge of $\Gamma(\mathcal{W}, \mathcal{Q})$ with $\alpha \in \lambda$ or $\beta \in \lambda$, then $\alpha\beta$ is an edge of $\Gamma(\mathcal{W}, \mathcal{P})$.*

Proof. Some inner bag W_k of $\mathcal{W}$ contains a $\mathcal{Q}$-bridge B realising $\alpha\beta$, that is, B attaches to Q_α and Q_β. For $i = 1, \ldots, k-1$ the induced permutation π_i of $\mathcal{Q}[W_i]$ is an automorphism of $\Gamma(\mathcal{W}, \mathcal{P})$ by (L10) and hence so is the induced permutation $\pi = \prod_{i=1}^{k-1} \pi_i$ of $\mathcal{Q}[W_{[1,k-1]}]$.

Clearly the restriction of any induced permutation to θ is always the identity, so $\pi(\alpha) \in \lambda$ or $\pi(\beta) \in \lambda$. Therefore $\pi(\alpha)\pi(\beta)$ must be an edge of $\Gamma(\mathcal{W}, \mathcal{P})$ by (L11) as B attaches to $\mathcal{Q}[W]_{\pi(\alpha)}$ and $\mathcal{Q}[W]_{\pi(\beta)}$. Since π is an automorphism this means that $\alpha\beta$ is an edge of $\Gamma(\mathcal{W}, \mathcal{P})$. □

The previous Lemma allows us to make statements about any foundational linkage $\mathcal{Q}$ just by looking at $\Gamma(\mathcal{W}, \mathcal{P})$, in particular, for every $\alpha \in \lambda$ the neighbourhood $N(\alpha)$ of α in $\Gamma(\mathcal{W}, \mathcal{P})$ contains all neighbours of α in $\Gamma(\mathcal{W}, \mathcal{Q})$. The following Lemma applies this argument.

Lemma 5.2. *Let $\mathcal{Q}$ be a foundational linkage such that $\mathcal{Q}[W]$ is p-attached in $G[W]$ for each inner bag W of $\mathcal{W}$. If λ_0 is a subset of λ such that $|N(\alpha) \cap \theta| \leq p - 3$ for each $\alpha \in \lambda_0$, then every non-trivial $\mathcal{Q}$-bridge in an inner bag of $\mathcal{W}$ that attaches to a path of $\mathcal{Q}_{\lambda_0}$ must attach to at least one other path of $\mathcal{Q}_\lambda$.*

Proof. Suppose for a contradiction that some inner bag W of $\mathcal{W}$ contains a $\mathcal{Q}$-bridge B that attaches to some path $Q_\alpha[W]$ with $\alpha \in \lambda_0$ but to no other path of $\mathcal{Q}_\lambda[W]$. Recall that either all foundational linkages for $\mathcal{W}$ satisfy (L7) or none does and $\mathcal{P}$ witnesses the former. Hence by (L7) a path of $\mathcal{Q}[W]$ is non-trivial if and only if it is in $\mathcal{Q}_\lambda[W]$. So by p-attachedness $Q_\alpha[W]$ is bridge adjacent to at least $p-2$ paths of $\mathcal{Q}_\theta$ in $G[W]$. Therefore in $\Gamma(\mathcal{W}, \mathcal{Q})$ the vertex α is adjacent to at least $p-2$ vertices of θ and by Lemma 5.1 so it must be in $\Gamma(\mathcal{W}, \mathcal{P})$, giving the desired contradiction. □

Lemma 5.3. *Let $\mathcal{Q}$ be a foundational linkage. Every $\mathcal{Q}$-bridge B that attaches to a path of $\mathcal{Q}_{\lambda \setminus V(D)}$ has no edge or inner vertex in $G_D^{\mathcal{Q}}$, in particular, it can attach to at most one path of $\mathcal{Q}_{V(D)}$.*

Proof. By assumption B attaches to some path Q_α with $\alpha \in \lambda \setminus V(D)$. This rules out the possibility that B attaches to only one path of $\mathcal{Q}_\lambda$ that happens to be in $\mathcal{Q}_{V(D)}$. So if B has an edge or inner vertex in $G_D^{\mathcal{Q}}$, then it must realise

an edge of D. Hence B attaches to paths Q_β and Q_γ with $\beta, \gamma \in V(D)$. This means that $\alpha\beta$ and $\alpha\gamma$ are both edges of $\Gamma(\mathcal{W}, \mathcal{Q})$ and thus of $\Gamma(\mathcal{W}, \mathcal{P})$ by Lemma 5.1. But D is a block of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ so no vertex of $\lambda \setminus V(D)$ can have two neighbours in D. $\square$

Given two foundational linkages $\mathcal{Q}$ and $\mathcal{Q}'$ and a set $\lambda_0 \subseteq \lambda$, we say that $\mathcal{Q}'$ is a $(\mathcal{Q}, \lambda_0)$-*relinkage* or a *relinkage of* $\mathcal{Q}$ *on* λ_0 if $Q'_\alpha = Q_\alpha$ for $\alpha \notin \lambda_0$ and $\mathcal{Q}'_{\lambda_0} \subseteq G^{\mathcal{Q}}_{\lambda_0}$.

Lemma 5.4. *If $\mathcal{Q}$ is a $(\mathcal{P}, V(D))$-relinkage and $\mathcal{Q}'$ a $(\mathcal{Q}, V(D))$-relinkage, then $G^{\mathcal{Q}'}_D \subseteq G^{\mathcal{Q}}_D$, in particular, $G^{\mathcal{Q}}_D \subseteq G^{\mathcal{P}}_D$.*

Proof. Clearly $G^{\mathcal{Q}}_D$ and $G^{\mathcal{Q}'}_D$ are induced subgraphs of G so it suffices to show $V(G^{\mathcal{Q}'}_D) \subseteq V(G^{\mathcal{Q}}_D)$. Suppose for a contradiction that there is a vertex $w \in V(G^{\mathcal{Q}'}_D) \setminus V(G^{\mathcal{Q}}_D)$. We have $G^{\mathcal{Q}'}_D \cap \mathcal{Q}' = \mathcal{Q}'_{V(D)} \subseteq G^{\mathcal{Q}}_D$ so w must be an inner vertex of a $\mathcal{Q}'$-bridge B'. But w is in $G_\lambda - G^{\mathcal{Q}}_D$ and thus in a $\mathcal{Q}$-bridge attaching to a path of $\mathcal{Q}_{\lambda \setminus V(D)}$, in particular, there is a w–$\mathcal{Q}_{\lambda \setminus V(D)}$ path R that avoids $G^{\mathcal{Q}}_D \supseteq \mathcal{Q}'_{V(D)}$. This means $R \subseteq B'$ and thus B' attaches to a path of $\mathcal{Q}'_{\lambda \setminus V(D)} = \mathcal{Q}_{\lambda \setminus V(D)}$, a contradiction to Lemma 5.3. Clearly $\mathcal{P}$ itself is a $(\mathcal{P}, V(D))$-relinkage so $G^{\mathcal{Q}}_D \subseteq G^{\mathcal{P}}_D$ follows from a special case of the statement we just proved. $\square$

Lemma 5.5. *Let $\mathcal{Q}$ be a $(\mathcal{P}, V(D))$-relinkage. If in $\Gamma(\mathcal{W}, \mathcal{P})$ we have $|N(\alpha) \cap \theta| \leq p - 3$ for all $\alpha \in \lambda \setminus V(D)$, then there is a $(\mathcal{Q}, V(D))$-relinkage $\mathcal{Q}'$ such that for every inner bag W of $\mathcal{W}$ the linkage $\mathcal{Q}'[W]$ is p-attached in $G[W]$ and has the same induced permutation as $\mathcal{Q}[W]$. Moreover, $\Gamma(\mathcal{W}, \mathcal{Q}')$ contains all edges of $\Gamma(\mathcal{W}, \mathcal{Q})$ that have at least one end in λ.*

Proof. Suppose that some non-trivial $\mathcal{Q}$-bridge B in an inner bag W of $\mathcal{W}$ attaches to a path $Q_\alpha = P_\alpha$ with $\alpha \in \lambda \setminus V(D)$ but to no other path of $\mathcal{Q}_\lambda$. Then B is also a $\mathcal{P}$-bridge and $\mathcal{P}[W]$ is p-attached in $G[W]$ by (L6) so $P_\alpha[W]$ must be bridge adjacent to at least $p - 2$ paths of $\mathcal{P}_\theta$ in $G[W]$ and thus α has at least $p - 2$ neighbours in θ, a contradiction. Hence every non-trivial $\mathcal{Q}$-bridge that attaches to a path of $\mathcal{Q}_{\lambda \setminus V(D)}$ must attach to at least one other path of $\mathcal{Q}_\lambda$.

For every inner bag W_i of $\mathcal{W}$ let $\mathcal{Q}'_i$ be the bridge stabilisation of $\mathcal{Q}[W_i]$ in $G[W_i]$. Then $\mathcal{Q}'_i$ has the same induced permutation as $\mathcal{Q}[W_i]$. Note that the set Z of all end vertices of the paths of $\mathcal{Q}[W_i]$ is the union of the left and right adhesion set of W_i. So by the p-connectivity of G for every vertex x of $G[W_i] - Z$ there is an x–Z fan of size p in $G[W_i]$. This means that $\mathcal{Q}'_i$ is p-attached in $G[W_i]$ by Lemma 3.7 (iii).

Hence $\mathcal{Q}' := \bigcup_{i=1}^{l-1} \mathcal{Q}'_i$ is a foundational linkage with $\mathcal{Q}'[W_i] = \mathcal{Q}'_i$ for $i = 1, \ldots, l-1$. Therefore $\mathcal{Q}'[W]$ is p-attached in $G[W]$ and $\mathcal{Q}'[W]$ has the same

induced permutations as $\mathcal{Q}[W]$ for every inner bag W of $\mathcal{W}$. There is no $\mathcal{Q}$-bridge that attaches to precisely one path of $\mathcal{Q}_{\lambda\setminus V(D)}$ but to no other path of $\mathcal{Q}_\lambda$ so we have $\mathcal{Q}'_{\lambda\setminus V(D)} = \mathcal{Q}_{\lambda\setminus V(D)}$ by Lemma 3.7 (i). The same result implies $\mathcal{Q}'_{V(D)} \subseteq G_D^{\mathcal{Q}}$ so $\mathcal{Q}'$ is indeed a relinkage of $\mathcal{Q}$ on $V(D)$.

Finally, Lemma 3.7 (ii) states that $\Gamma(\mathcal{W}, \mathcal{Q}')$ contains all those edges of $\Gamma(\mathcal{W}, \mathcal{Q})$ that have at least one end in λ. □

The "compressed" linkages presented next will allow us to fulfil the size requirement that Lemma 4.10 imposes on our block D as detailed in Lemma 5.7. Given a subset $\lambda_0 \subseteq \lambda$ and a foundational linkage $\mathcal{Q}$, we say that $\mathcal{Q}$ is *compressed to* λ_0 or λ_0*-compressed* if there is no vertex v of $G_{\lambda_0}^{\mathcal{Q}}$ such that $G_{\lambda_0}^{\mathcal{Q}} - v$ contains $|\lambda_0|$ disjoint paths from the first to the last adhesion set of $\mathcal{W}$ and v has a neighbour in $G_\lambda - G_{\lambda_0}^{\mathcal{Q}}$.

Lemma 5.6. *Suppose that in $\Gamma(\mathcal{W}, \mathcal{P})$ we have $|N(\alpha) \cap \theta| \leq p - 3$ for all $\alpha \in \lambda \setminus V(D)$ and let $\mathcal{Q}$ be a $(\mathcal{P}, V(D))$-relinkage. Then there is a $V(D)$-compressed $(\mathcal{Q}, V(D))$-relinkage $\mathcal{Q}'$ such that for every inner bag W of $\mathcal{W}$ the linkage $\mathcal{Q}'[W]$ is p-attached in $G[W]$.*

Proof. Clearly $\mathcal{Q}$ itself is a $(\mathcal{Q}, V(D))$-relinkage. Among all $(\mathcal{Q}, V(D))$-relinkages pick $\mathcal{Q}'$ such that $G_D^{\mathcal{Q}'}$ is minimal. By Lemma 5.4 and Lemma 5.5 we may assume that we picked $\mathcal{Q}'$ such that for every inner bag of W of $\mathcal{W}$ the linkage $\mathcal{Q}'[W]$ is p-attached in $G[W]$.

It remains to show that $\mathcal{Q}'$ is $V(D)$-compressed. Suppose not, that is, there is a vertex v of $G_D^{\mathcal{Q}'}$ such that v has a neighbour in $G_\lambda - G_D^{\mathcal{Q}'}$ and $G_D^{\mathcal{Q}'} - v$ contains an X–Y linkage $\mathcal{Q}''$ where X and Y denote the intersection of $V(G_D^{\mathcal{Q}'})$ with the first and last adhesion set of $\mathcal{W}$, respectively.

By Lemma 5.4 we have $G_D^{\mathcal{Q}''} \subseteq G_D^{\mathcal{Q}'} \subseteq G_D^{\mathcal{Q}}$ and thus $\mathcal{Q}''$ is a $(\mathcal{Q}, V(D))$-relinkage as well. This implies $G_D^{\mathcal{Q}''} = G_D^{\mathcal{Q}'}$ by the minimality of $G_D^{\mathcal{Q}'}$. The vertex v does not lie on a path of $\mathcal{Q}''$ by construction so it must be in a $\mathcal{Q}''$-bridge B''. But v has a neighbour w in $G_\lambda - G_D^{\mathcal{Q}'}$ and there is a w–$\mathcal{Q}'_{\lambda\setminus V(D)}$ path R that avoids $G_D^{\mathcal{Q}'}$. This means $R \subseteq B''$ and thus B'' attaches to a path of $\mathcal{Q}''_{\lambda\setminus V(D)}$, contradicting Lemma 5.3. □

Lemma 5.7. *Let $\mathcal{Q}$ be a $V(D)$-compressed foundational linkage. Let V be the set of all inner vertices of paths of $\mathcal{Q}_\kappa$ that have degree at least 3 in $G_D^{\mathcal{Q}}$. Then the following statements are true.*

(i) Either $2|D| + |N(D) \cap \theta| \geq p$ or $V(G_D^{\mathcal{Q}}) = V(\mathcal{Q}_{V(D)})$ and $\kappa \neq \emptyset$.

(ii) Either $2|D| + |N(D)| \geq p$ or there is $\alpha \in \kappa$ such that $|Q_\beta| \leq |V \cap V(Q_\alpha)| + 1$ for all $\beta \in V(D) \setminus \kappa$.

Note that $V(G_D^{\mathcal{Q}}) = V(\mathcal{Q}_{V(D)})$ implies that every $\mathcal{Q}$-bridge in an inner bag of $\mathcal{W}$ that realises an edge of D must be trivial.

Proof.

(i) Denote by X and Y the intersection of $G_D^{\mathcal{Q}}$ with the first and last adhesion set of $\mathcal{W}$, respectively. Let Z be the union of X, Y, and the set of all vertices of $G_D^{\mathcal{Q}}$ that have a neighbour in $G_\lambda - G_D^{\mathcal{Q}}$. Clearly $Z \subseteq V(\mathcal{Q}_\kappa) \cup X \cup Y$. Moreover, $G_D^{\mathcal{Q}} - z$ does not contain an X–Y linkage for any $z \in Z$: For $z \in X \cup Y$ this is trivial and for the remaining vertices of Z it holds by the assumption that $\mathcal{Q}$ is $V(D)$-compressed. Therefore for every $z \in Z$ there is an X–Y separation (A_z, B_z) of $G_D^{\mathcal{Q}}$ of order at most $|D|$ with $z \in A_z \cap B_z$. On the other hand, $\mathcal{Q}_{V(D)}$ is a set of $|D|$ disjoint X–Y paths in $G_D^{\mathcal{Q}}$ so every X–Y separation has order at least $|D|$. Hence by Lemma 3.1 there is a nested set $\mathcal{S}$ of X–Y separations of $G_D^{\mathcal{Q}}$, each of order $|D|$, such that $Z \subseteq Z_0$ where Z_0 denotes the set of all vertices that lie in a separator of a separation of $\mathcal{S}$.

We may assume that $(X, V(G_D^{\mathcal{Q}})) \in \mathcal{S}$ and $(V(G_D^{\mathcal{Q}}), Y) \in \mathcal{S}$ so for any vertex v of $G_D^{\mathcal{Q}} - (X \cup Y)$ there are $(A_L, B_L) \in \mathcal{S}$ and $(A_R, B_R) \in \mathcal{S}$ such that (A_L, B_L) is rightmost with $v \in B_L \setminus A_L$ and (A_R, B_R) is leftmost with $v \in A_R \setminus B_R$. Set $S_L := A_L \cap B_L$ and $S_R := A_R \cap B_R$.

Let z be any vertex of Z_0 "between" S_L and S_R, more precisely, $z \in (B_L \setminus A_L) \cap (A_R \setminus B_R)$. There is a separation $(A_M, B_M) \in \mathcal{S}$ such that its separator $S_M := A_M \cap B_M$ contains z. Then z witnesses that $A_M \nsubseteq A_L$ and $B_M \nsubseteq B_R$ and thus (A_M, B_M) is neither left of (A_L, B_L) nor right of (A_R, B_R). But $\mathcal{S}$ is nested and therefore (A_M, B_M) is strictly right of (A_L, B_L) and strictly left of (A_R, B_R). This means $v \in S_M$ otherwise (A_M, B_M) would be a better choice for (A_L, B_L) or for (A_R, B_R). So any separator of a separation of $\mathcal{S}$ that contains a vertex of $(B_L \setminus A_L) \cap (A_R \setminus B_R)$ must also contain v.

If $v \notin Z_0$, then $(B_L \setminus A_L) \cap (A_R \setminus B_R) \cap Z_0 = \emptyset$. This means that $S_L \cup S_R$ separates v from Z in $G_D^{\mathcal{Q}}$. So $S_L \cup S_R \cup V(\mathcal{Q}_{N(D) \cap \theta})$ separates v from $G - G_D^{\mathcal{Q}}$ in G. By the connectivity of G we therefore have

$$2|D| + |N(D) \cap \theta| \geq \left|S_L \cup S_R \cup V(\mathcal{Q}_{N(D) \cap \theta})\right| \geq p.$$

So we may assume that $V(G_D^{\mathcal{Q}}) = Z_0$ Since every separator of a separation of $\mathcal{S}$ consists of one vertex from each path of $\mathcal{Q}_{V(D)}$ this means $V(\mathcal{Q}_{V(D)}) \subseteq V(G_D^{\mathcal{Q}}) = Z_0 \subseteq V(\mathcal{Q}_{V(D)})$. If $\kappa = \emptyset$, then $X \cup Y \cup V(\mathcal{Q}_{N(D) \cap \theta})$ separates $G_D^{\mathcal{Q}} - (X \cup Y)$ from $G - G_D^{\mathcal{Q}}$ in G so this is just a special case of the above argument.

(ii) We may assume $\kappa \neq \emptyset$ by (i) and $\kappa \neq V(D)$ since the statement is trivially true in the case $\kappa = V(D)$. Pick $\alpha \in \kappa$ such that $|V \cap V(Q_\alpha)|$ is maximal and let $\beta \in V(D) \setminus \kappa$. For any inner vertex v of Q_β define (A_L, B_L) and (A_R, B_R) as in the proof of (i) and set $V_v := V \cap (B_L \setminus A_L) \cap (A_R \setminus B_R)$.

By (i) we have $V_v \subseteq Z_0$ and every separator of a separation of $\mathcal{S}$ that contains a vertex of V_v must also contain v. This means that $V_v \cap V_{v'} = \emptyset$ for distinct inner vertices v and v' of Q_β since no separator of a separation of $\mathcal{S}$ contains two vertices on the same path of $\mathcal{Q}_{V(D)}$.

Furthermore, $S_L \cup S_R \cup V_v$ separates v from $V(\mathcal{Q}_\kappa) \cup X \cup Y \supseteq Z$ in $G_D^{\mathcal{Q}}$ so by the same argument as in (i) we have $2|D| + |N(D) \cap \theta| + |V_v| \geq p$. Then $|N(D) \cap \lambda| \geq |V_v|$ would imply $2|D| + |N(D)| \geq p$ so we may assume that $|N(D) \cap \lambda| < |V_v|$ for all inner vertices v of Q_β. Clearly $N(D) \cap \lambda$ is a disjoint union of the sets $(N(\gamma) \cap \lambda) \setminus V(D)$ with $\gamma \in \kappa$ and these sets are all non-empty. Hence $|\kappa| \leq |N(D) \cap \lambda|$ and thus $|\kappa| + 1 \leq |V_v|$ for all inner vertices v of Q_β.

Write V for the inner vertices of $\mathcal{Q}_\beta$. Statement (ii) easily follows from

$$|V|(|\kappa| + 1) \leq \sum_{v \in V} |V_v| \leq |V| \leq |\kappa| \cdot |V \cap V(Q_\alpha)|. \qquad \square$$

6 Rural Societies

In this section we present the answer of Robertson and Seymour to the question whether or not a graph can be drawn in the plane with specified vertices on the boundary of the outer face in a prescribed order. We will apply their result to subgraphs of a graph with a stable decomposition.

A *society* is a pair (G, Ω) where G is a graph and Ω is a cyclic permutation of a subset of $V(G)$ which we denote by $\bar{\Omega}$. A society (G, Ω) is called *rural* if there is a drawing of G in a closed disc D such that $V(G) \cap \partial D = \bar{\Omega}$ and Ω coincides with a cyclic permutation of $\bar{\Omega}$ arising from traversing ∂D in one of its orientations. We say that a society (G, Ω) is *k-connected* for an integer k if there is no separation (A, B) of G with $|A \cap B| < k$ and $\bar{\Omega} \subseteq B \neq V(G)$. For any subset $X \subseteq \bar{\Omega}$ denote by $\Omega|X$ the map on X defined by $x \mapsto \Omega^k(x)$ where k is the smallest positive integer such that $\Omega^k(x) \in X$ (chosen for each x individually). Since Ω is a cyclic permutation so is $\Omega|X$.

Given two internally disjoint paths P and Q in G we write PQ for the cyclic permutation of $V(P \cup Q)$ that maps each vertex of P to its successor on P if there is one and to the first vertex of $Q - P$ otherwise and that maps

each vertex of $Q-P$ to its successor on $Q-P$ if there is one and to the first vertex of P otherwise.

Let R and S be disjoint $\bar{\Omega}$-paths in a society (G,Ω), with end vertices r_1, r_2 and s_1, s_2, respectively. We say that $\{R,S\}$ is a *cross in* (G,Ω), if $\Omega|\{r_1,r_2,s_1,s_2\} = (r_1s_1r_2s_2)$ or $\Omega|\{r_1,r_2,s_1,s_2\} = (s_2r_2s_1r_1)$.

The following is an easy consequence of Theorems 2.3 and 2.4 in [15].

Theorem 6.1 (Robertson & Seymour 1990)**.** *Any* 4*-connected society is rural or contains a cross.*

In our application we always want to find a cross. To prevent the society from being rural we force it to violate the implication given in following Lemma which is a simple consequence of Euler's formula.

Lemma 6.2. *Let (G,Ω) be a rural society. If the vertices in $V(G)\setminus\bar{\Omega}$ have degree at least* 6 *on average, then $\sum_{v\in\bar{\Omega}} d_G(v) \le 4|\bar{\Omega}|-6$.*

Proof. Since (G,Ω) is rural there is a drawing of G in a closed disc D with $V(G)\cap\partial D = \bar{\Omega}$. Let H be the graph obtained by adding one extra vertex w outside D and joining it by an edge to every vertex on ∂D. Writing $b := |\bar{\Omega}|$ and $i := |V(G)\setminus\bar{\Omega}|$, Euler's formula implies

$$\|G\| + b = \|H\| \le 3|H| - 6 = 3(i+b) - 3$$

and thus $\|G\| \le 3i + 2b - 3$. Our assertion then follows from

$$\sum_{v\in\bar{\Omega}} d_G(v) + 6i \le \sum_{v\in V(G)} d_G(v) = 2\|G\| \le 6i + 4b - 6 \qquad \square$$

In our main proof we will deal with societies where the permutation Ω is induced by paths (see Lemma 6.4 and Lemma 6.5). But every inner vertex on such a path that has degree 2 in G adds slack to the bound provided by Lemma 6.2 as it counts 2 on the left side but 4 on the right. This is remedied in the following Lemma which allows us to apply Lemma 6.2 to a "reduced" society where these vertices are suppressed.

Lemma 6.3. *Let (G,Ω) be a society and let P be a path in G such that all inner vertices of P have degree* 2 *in G. Denote by G' the graph obtained from G by suppressing all inner vertices of P and set $\Omega' := \Omega|V(G')$. Then (G',Ω') is rural if and only if (G,Ω) is.*

Proof. The graph G is a subdivision of G' so every drawing of G gives a drawing of G' and vice versa. Hence a drawing witnessing that (G,Ω) is rural can easily be modified to witness that (G',Ω') is rural and vice versa. $\square$

Two vertices a and b of some graph H are called *twins* if $N_H(a) \setminus \{b\} = N_H(b) \setminus \{a\}$. Clearly a and b are twins if and only if the transposition (ab) is an automorphism of H.

Lemma 6.4. *Let G be a p-connected graph and let $(\mathcal{W}, \mathcal{P})$ be a stable regular decomposition of G of length at least 3 and attachedness p. Set $\theta := \{\alpha \mid P_\alpha$ is trivial$\}$ and $\lambda := \{\alpha \mid P_\alpha$ is non-trivial$\}$. Let $\alpha\beta$ be an edge of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ such that $|N(\alpha) \cap \theta| \leq p-3$, $|N(\beta) \cap \theta| \leq p-3$, and for $N_{\alpha\beta} := N(\alpha) \cap N(\beta)$ we have $N_{\alpha\beta} \subseteq \theta$ and $|N_{\alpha\beta}| \leq p-5$. If α and β are not twins, then the society $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ is rural.*

Proof.

Claim 6.4.1. *Every $\mathcal{P}$-bridge with an edge in $G^{\mathcal{P}}_{\alpha\beta}$ must attach to P_α and P_β, in particular, $G^{\mathcal{P}}_{\alpha\beta} - P_\alpha$ and $G^{\mathcal{P}}_{\alpha\beta} - P_\beta$ are both connected.*

Proof. By Lemma 5.2 every non-trivial $\mathcal{P}$-bridge that attaches to P_α or P_β must attach to another path of $\mathcal{P}_\lambda$. Since P_α and P_β are induced this means that all $\mathcal{P}$-bridges with an edge in $G^{\mathcal{P}}_{\alpha\beta}$ must realise the edge $\alpha\beta$ and hence attach to P_α and P_β. □

Claim 6.4.2. *The set Z of all vertices of $G^{\mathcal{P}}_{\alpha\beta}$ that are end vertices of P_α or P_β or have a neighbour in $G - (G^{\mathcal{P}}_{\alpha\beta} \cup \mathcal{P}_{N_{\alpha\beta}})$ is contained in $V(P_\alpha \cup P_\beta)$.*

Proof. Any vertex v of $G^{\mathcal{P}}_{\alpha\beta} - (P_\alpha \cup P_\beta)$ is an inner vertex of some non-trivial $\mathcal{P}$-bridge B that attaches to P_α and P_β. Since $G^{\mathcal{P}}_{\alpha\beta}$ contains all inner vertices of B the neighbours of v in $G - G^{\mathcal{P}}_{\alpha\beta}$ must be attachments of B. But if B attaches to a path P_γ with $\gamma \neq \alpha, \beta$, then $\gamma \in N_{\alpha\beta}$ and therefore all neighbours of v are in $G^{\mathcal{P}}_{\alpha\beta} \cup \mathcal{P}_{N_{\alpha\beta}}$. □

Claim 6.4.3. *The society $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ is rural if and only if the society $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1}|Z)$ is.*

Proof. Clearly $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1}|Z)$ is rural if $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ is. For the converse suppose that $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1}|Z)$ is rural, that is, there is a drawing of $G^{\mathcal{P}}_{\alpha\beta}$ in a closed disc D such that $G^{\mathcal{P}}_{\alpha\beta} \cap \partial D = Z$ and one orientation of ∂D induces the cyclic permutation $P_\alpha P_\beta^{-1}|Z$ on Z.

For the rurality of $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ and $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1}|Z)$ it does not matter whether the first vertices of P_α and P_β are adjacent in $G^{\mathcal{P}}_{\alpha\beta}$ or not and the same is true for the last vertices of P_α and P_β. So we may assume that both edges exist and we denote the cycle that they form together with the paths P_α and P_β by C.

The closed disc D' bounded by C is contained in D. It is not hard to see that the interior of D' is the only region of $D - C$ that has vertices of both P_α and P_β on its boundary. But every edge of $G^{\mathcal{P}}_{\alpha\beta}$ lies on C or in a $\mathcal{P}$-bridge B with $B - (\mathcal{P} \setminus \{P_\alpha, P_\beta\}) \subseteq G^{\mathcal{P}}_{\alpha\beta}$. By Claim 6.4.1 such a bridge B must attach to P_α and P_β and in the considered drawing it must therefore be contained in D'. This means $G^{\mathcal{P}}_{\alpha\beta} \subseteq D'$ which implies that $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ is rural as desired. □

Claim 6.4.4. *For $H := G^{\mathcal{P}}_{\alpha\beta}$ and $\Omega := P_\alpha P_\beta^{-1}|Z$ the society (H, Ω) is 4-connected.*

Proof. Note that $\bar{\Omega} = Z$ since $Z \subseteq V(P_\alpha \cup P_\beta)$ by Claim 6.4.2. Set $T := V(\mathcal{P}_{N_{\alpha\beta}})$. Clearly $Z \cup T$ separates H from $G - H$ so for every vertex v of $H - Z$ there is a v–$T \cup Z$ fan of size at least p in G as G is p-connected. Since $|T| \leq p - 5$ this fan contains a v–Z fan of size at least 4 such that all its paths are contained in H. This means that (H, Ω) is 4-connected as desired. □

By the *off-road edges* of a cross $\{R, S\}$ in (H, Ω) we mean the edges in $E(R \cup S) \setminus E(P_\alpha \cup P_\beta)$. We call a component of $R \cap (P_\alpha \cup P_\beta)$ that contains an end vertex of R a *tail of* R. We define the *tails of* S similarly.

Claim 6.4.5. *If $\{R, S\}$ is a cross in (H, Ω) whose set E of off-road edges is minimal, then for every $z \in Z \setminus V(R \cup S)$ each z–$(R \cup S)$ path in $P_\alpha \cup P_\beta$ ends in a tail of R or S.*

Proof. Suppose not, that is, there is a Z–$(R \cup S)$ path T in $P_\alpha \cup P_\beta$ such that its last vertex t does not lie in a tail of R or S. W.l.o.g. we may assume that t is on R. Since t is not in a tail of R the paths Rt and tR must both contain an edge that is not in $P_\alpha \cup P_\beta$ so $E(T \cup Rt \cup S) \setminus E(P_\alpha \cup P_\beta)$ and $E(T \cup tR \cup S) \setminus E(P_\alpha \cup P_\beta)$ are both proper subsets of E. But one of $\{T \cup Rt, S\}$ and $\{T \cup tR, S\}$ is a cross in (H, Ω), a contradiction. □

Suppose now that α and β are not twins.

Claim 6.4.6. *(H, Ω) does not contain a cross.*

Proof. If (H, Ω) contains a cross, then we may pick a cross $\{R, S\}$ in (H, Ω) such that its set E of off-road edges is minimal. Since $Z \subseteq V(P_\alpha \cup P_\beta)$ we may assume w.l.o.g. that $\{R, S\}$ satisfies one of the following.

1. R and S both have their ends on P_α.

2. R has both ends on P_α. S has one end on P_α and one on P_β.

3. R and S both have one end on P_α and one on P_β.

We reduce the first case to the second. As P_β contains a vertex of Z but no end of R or S it must be disjoint from $R \cup S$ by Claim 6.4.5. But R and S both contain a vertex outside P_α (recall that P_α is induced by (L6)) so $R \cup S$ meets $H - P_\alpha$ which is connected by Claim 6.4.1.

Therefore there is a P_β–$(R \cup S)$ in $H - P_\alpha$, in particular, there is a Z–$(R \cup S)$ path T with its first vertex z in $Z \cap V(P_\beta)$ and we may assume that its last vertex t is on S. Denote by v the end of S that separates the ends of R in P_α.

Then $\{R, vSt \cup T\}$ is a cross in (H, Ω) and we may pick a cross $\{R', S'\}$ in (H, Ω) such that its set E' of off-road edges is minimal and contained in the set F of off-road edges of $\{R, vSt \cup T\}$. If $R' \cup S'$ contains no edge of T, then E' is a proper subset of E as it does not contain $E(S) \setminus E(vSt)$, a contradiction to the minimality of E. Hence $R' \cup S'$ contains an edge of T and hence must meet P_β. So by Claim 6.4.5 one of its paths, say S' ends in P_β as desired.

On the other hand, all off-road edges of $\{R', S'\}$ that are incident with P_β are in T and therefore the remaining three ends of R' and S' must all be on P_α. Hence $\{R', S'\}$ is a cross as in the second case.

In the second case we reroute P_α along R, more precisely, we obtain a foundational linkage $\mathcal{Q}$ from $\mathcal{P}$ by replacing the subpath of P_α between the two end vertices of R with R.

The first vertex of $R \cup S$ encountered when following P_β from either of its ends belongs to a tail of R or S by Claim 6.4.5. Obviously a tail contains precisely one end of R or S. Since R has no end on P_β and S only one, $(R \cup S) \cap P_\beta$ is a tail of S, in particular, R is disjoint from P_β and hence the paths of $\mathcal{Q}$ are indeed disjoint.

Clearly S must end in an inner vertex z of P_α. By the definition of Z there is a $\mathcal{P}$-bridge B in some inner bag W of $\mathcal{W}$ that attaches to z and to some path P_γ with $\gamma \in N(\alpha) \setminus N(\beta)$. But $B \cup S$ is contained in a $\mathcal{Q}$-bridge in $G[W]$ and therefore $\beta\gamma$ is an edge of $B(G[W], \mathcal{Q}[W])$ and thus of $\Gamma(\mathcal{W}, \mathcal{Q})$ but not of $\Gamma(\mathcal{W}, \mathcal{P})$. This contradicts Lemma 5.1.

In the third case Claim 6.4.5 ensures that the first and last vertex of P_α and of P_β in $R \cup S$ is always in a tail and clearly these tails must all be distinct. Hence by replacing the tails of R and S with suitable initial and final segments of P_α and P_β we obtain paths P'_α and P'_β such that the foundational linkage $\mathcal{Q} := (\mathcal{P} \setminus \{P_\alpha, P_\beta\}) \cup \{P'_\alpha, P'_\beta\}$ has the induced permutation $(\alpha\beta)$. Since $P_\gamma = Q_\gamma$ for all $\gamma \notin \{\alpha, \beta\}$ it is easy to see the there must be an inner bag W of $\mathcal{W}$ such that $\mathcal{Q}[W]$ has induced permutation $(\alpha\beta)$. But clearly $(\alpha\beta)$ is an automorphism of $\Gamma(\mathcal{W}, \mathcal{P})$ if and only if α and β are twins in

$\Gamma(\mathcal{W},\mathcal{P})$. Hence $\mathcal{Q}[W]$ is a twisting disturbance by the assumption that α and β are not twins. This contradicts the stability of $(\mathcal{W},\mathcal{P})$ and concludes the proof of Claim 6.4.6. $\square$

By Claim 6.4.4 and Theorem 6.1 the society (H,Ω) is rural or contains a cross. But Claim 6.4.6 rules out the latter so (H,Ω) is rural and by Claim 6.4.3 so is $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$. $\square$

In the previous Lemma we have shown how certain crosses in the graph $H = G^{\mathcal{P}}_{\alpha\beta}$ "between" two bridge-adjacent paths P_α and P_β of $\mathcal{P}$ give rise to disturbances. The next Lemma has a similar flavour; here the graph H will be the subgraph of G "between" P_α and Q_α where α is a cut-vertex of $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$ and $\mathcal{Q}$ a relinkage of $\mathcal{P}$.

Lemma 6.5. *Let G be a p-connected graph with a stable regular decomposition $(\mathcal{W},\mathcal{P})$ of attachedness p and set $\lambda := \{\alpha \mid P_\alpha$ is non-trivial$\}$ and $\theta := \{\alpha \mid P_\alpha$ is trivial$\}$. Let D be a block of $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$ and let κ be the set of cut-vertices of $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$ that are in D. If $|N(\alpha)\cap\theta| \leq p-4$ for all $\alpha \in \lambda$, then there is a $V(D)$-compressed $(\mathcal{P},V(D))$-relinkage $\mathcal{Q}$ such that $\mathcal{Q}[W]$ is p-attached in $G[W]$ for all inner bags W of $\mathcal{W}$ and for any $\alpha \in \kappa$ and any separation (λ_1,λ_2) of $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$ such that $\lambda_1 \cap \lambda_2 = \{\alpha\}$ and $N(\alpha)\cap\lambda_2 = N(\alpha)\cap V(D)$ the following statements hold where $H := G^{\mathcal{P}}_{\lambda_2} \cap G^{\mathcal{Q}}_{\lambda_1}$, q_1 and q_2 are the first and last vertex of Q_α, and Z_1 and Z_2 denote the vertices of $H - \{q_1,q_2\}$ that have a neighbour in $G_\lambda - G^{\mathcal{P}}_{\lambda_2}$ and $G_\lambda - G^{\mathcal{Q}}_{\lambda_1}$, respectively.*

(i) We have $Z_1 \subseteq V(P_\alpha)$ and $Z_2 \subseteq V(Q_\alpha)$. Furthermore, $Z := \{q_1,q_2\} \cup Z_1 \cup Z_2$ separates H from $G_\lambda - H$ in $G - \mathcal{P}_{N(\alpha)\cap\theta}$.

(ii) The graph H is connected and contains Q_α. The path P_α ends in q_2.

(iii) Every cut-vertex of H is an inner vertex of Q_α and is contained in precisely two blocks of H.

(iv) Every block H' of H that is not a single edge contains a vertex of $Z_1 \setminus V(Q_\alpha)$ and a vertex of $Z_2 \setminus V(P_\alpha)$ that is not a cut-vertex of H. Furthermore, $Q_\alpha[W]$ contains a vertex of Z_2 for every inner bag W of $\mathcal{W}$.

(v) There is $(\mathcal{P},V(D))$-relinkage $\mathcal{P}'$ with $\mathcal{P}' = (\mathcal{Q}\setminus\{Q_\alpha\}) \cup \{P'_\alpha\}$ and $P'_\alpha \subseteq H$ such that $Z_1 \subseteq V(P'_\alpha)$, $V(P'_\alpha \cap Q_\alpha)$ consists of q_1, q_2, and all cut-vertices of H, and $\mathcal{P}'[W]$ is p-attached in $G[W]$ for all inner bags W of $\mathcal{W}$.

(vi) Let H' be a block of H that is not a single edge. Then $P' := H' \cap P'_\alpha$ and $Q' := H' \cap Q_\alpha$ are internally disjoint paths with common first vertex q'_1 and common last vertex q'_2 and the society $(H', P'Q'^{-1})$ is rural.

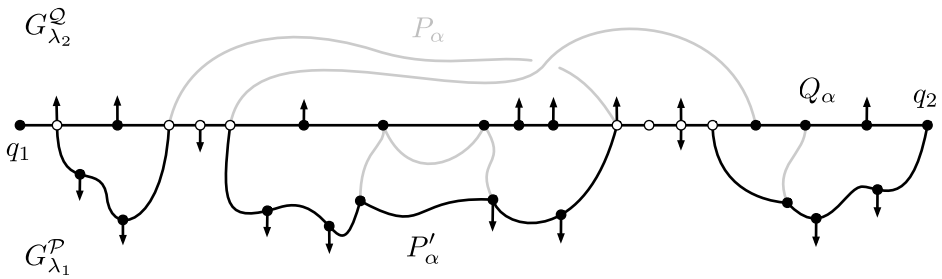

Figure 1: The graph $H = G^{\mathcal{P}}_{\lambda_2} \cap G^{\mathcal{Q}}_{\lambda_1}$.

Figure 1 gives an impression of H. The upper (straight) black q_1–q_2 path is Q_α and everything above it belongs to $G^{\mathcal{Q}}_{\lambda_2}$. The lower (curvy) black path is P'_α and everything below it belongs to $G^{\mathcal{P}}_{\lambda_1}$. The grey paths are subpaths of P_α and, as shown, P_α need not be contained in H and need not contain the vertices of $P_\alpha \cap P'_\alpha$ in the same order as P'_α. The white vertices are the cut-vertices of H. The vertices with an arrow up or down symbolise vertices of Z_2 and Z_1, respectively. The blocks of H that are not single edges are bounded by cycles in $P'_\alpha \cup Q_\alpha$ and Lemma 6.5 (vi) states that the part of H "inside" such a cycle forms a rural society.

Proof. For a $(\mathcal{P}, V(D))$-relinkage $\mathcal{Q}$ and $\beta \in \kappa$ any $G^{\mathcal{Q}}_D$-path $P \subseteq P_\beta$ such that some inner vertex of P has a neighbour in $G_\lambda - G^{\mathcal{P}}_D$ is called an *β-outlet of $\mathcal{Q}$*. By the *outlet graph of $\mathcal{Q}$* we mean the union of all components of $\mathcal{P}_\kappa - G^{\mathcal{Q}}_D$ that have a neighbour in $G_\lambda - G^{\mathcal{P}}_D$. In other words, the outlet graph of $\mathcal{Q}$ is obtained from the union of all β-outlets for all $\beta \in \kappa$ by deleting the vertices of $G^{\mathcal{Q}}_D$.

Clearly $\mathcal{P}$ itself is a $(\mathcal{P}, V(D))$-relinkage. Among all $(\mathcal{P}, V(D))$-relinkages pick $\mathcal{Q}'$ such that its outlet graph is maximal. By Lemma 5.6 there is a $V(D)$-compressed $(\mathcal{Q}', V(D))$-relinkage $\mathcal{Q}$ such that $\mathcal{Q}[W]$ is p-attached in $G[W]$ for all inner bags W of $\mathcal{W}$. Note that $G^{\mathcal{Q}}_D \subseteq G^{\mathcal{Q}'}_D$ by Lemma 5.4, so the outlet graph of $\mathcal{Q}$ is a supergraph of that of $\mathcal{Q}'$. Hence by choice of $\mathcal{Q}'$, they must be identical, in particular, the outlet graph of $\mathcal{Q}$ is maximal among the outlet graphs of all $(\mathcal{P}, V(D))$-relinkages.

Claim 6.5.1. *For any foundational linkage $\mathcal{R}$ of $\mathcal{W}$ we have $G^{\mathcal{R}}_{\lambda_1} \cup G^{\mathcal{R}}_{\lambda_2} = G_\lambda$ and $G^{\mathcal{R}}_{\lambda_1} \cap G^{\mathcal{R}}_{\lambda_2} = R_\alpha$.*

Proof. By Lemma 5.1 we have $\Gamma(\mathcal{W},\mathcal{R})[\lambda] \subseteq \Gamma(\mathcal{W},\mathcal{P})[\lambda]$, so (λ_1,λ_2) is also a separation of $\Gamma(\mathcal{W},\mathcal{R})[\lambda]$. Hence each $\mathcal{R}$-bridge in an inner bag of $\mathcal{W}$ has all its attachments in $\mathcal{R}_{\lambda_1\cup\theta}$ or all in $\mathcal{R}_{\lambda_2\cup\theta}$ and thus $G^{\mathcal{R}}_{\lambda_1} \cup G^{\mathcal{R}}_{\lambda_2} = G_\lambda$. The induced path R_α is contained in $G^{\mathcal{R}}_{\lambda_1} \cap G^{\mathcal{R}}_{\lambda_2}$ by definition. If $G^{\mathcal{R}}_{\lambda_1} \cap G^{\mathcal{R}}_{\lambda_2}$ contains a vertex that is not on R_α, then it must be in a non-trivial $\mathcal{R}$-bridge that attaches to R_α but to no other path of $\mathcal{R}_\lambda$. Such a bridge does not exist by Lemma 5.2 (applied to $\lambda_0 := \lambda$). □

Claim 6.5.2. *For every vertex v of $H - P_\alpha$ there is a v–Z_2 path in $H - P_\alpha$ and for every vertex v of $H - Q_\alpha$ there is a v–Z_1 path in $H - Q_\alpha$.*

Proof. Let v be a vertex of $H - P_\alpha \subseteq G^{\mathcal{P}}_{\lambda_2} - P_\alpha$. Then there is $\beta \in \lambda_2 \setminus \lambda_1$ such that v is on P_β or v is an inner vertex of some non-trivial $\mathcal{P}$-bridge attaching to P_β by Lemma 5.2 and the assumption that $|N(\alpha) \cap \theta| \leq p - 4$. In either case $G^{\mathcal{P}}_{\lambda_2} - P_\alpha$ contains a path R from v to the first vertex p of P_β. But p is also the first vertex of Q_β and therefore it is contained in $G_\lambda - G^{\mathcal{Q}}_{\lambda_1}$. Pick w on R such that Rw is a maximal initial subpath of R that is still contained in H. Then $w \neq p$ and the successor of w on R must be in $G_\lambda - G^{\mathcal{Q}}_{\lambda_1}$. This means $w \in Z_2$ as desired. If v is in $H - Q_\alpha$, then the argument is similar but slightly simpler as $Q_\beta = P_\beta$ for all $\beta \in \lambda_1 \setminus \lambda_2$. □

(i) Any vertex of $G^{\mathcal{P}}_{\lambda_2}$ that has a neighbour in $G^{\mathcal{P}}_{\lambda_1} - G^{\mathcal{P}}_{\lambda_2}$ must be on P_α by Claim 6.5.1. This shows $Z_1 \subseteq V(P_\alpha)$ and by a similar argument $Z_2 \subseteq V(Q_\alpha)$.

A neighbour v of H in G either is in no inner bag of $\mathcal{W}$, it is in G_λ, or it is in $\mathcal{P}_\theta$. In the first case v can only be adjacent to q_1 or q_2 as these are the only vertices of H in the first and last adhesion set of $\mathcal{W}$.

In the second case, note that $\mathcal{Q}$ is a $(\mathcal{P},\lambda_2)$-relinkage since $V(D) \subseteq \lambda_2$ and thus Lemma 5.4 yields $G^{\mathcal{Q}}_{\lambda_2} \subseteq G^{\mathcal{P}}_{\lambda_2}$ which together with Claim 6.5.1 implies

$$\begin{aligned} G_\lambda &= G^{\mathcal{P}}_{\lambda_1} \cup G^{\mathcal{P}}_{\lambda_2} = G^{\mathcal{P}}_{\lambda_1} \cup (G^{\mathcal{P}}_{\lambda_2} \cap G^{\mathcal{Q}}_{\lambda_1}) \cup (G^{\mathcal{P}}_{\lambda_2} \cap G^{\mathcal{Q}}_{\lambda_2}) \\ &= G^{\mathcal{P}}_{\lambda_1} \cup H \cup G^{\mathcal{Q}}_{\lambda_2}. \end{aligned}$$

Hence v is in $G_\lambda - G^{\mathcal{Q}}_{\lambda_1}$ or in $G_\lambda - G^{\mathcal{P}}_{\lambda_2}$ and thus all neighbours of v in H are in Z_2 or Z_1, respectively.

In the third case v is the unique vertex of some path P_β with $\beta \in \theta$. Let w be a neighbour of v in H. Either w is on P_α or there is a w–Z_2 path in H by Claim 6.5.2 which ends on Q_α as shown above. So $\alpha\beta$ is an edge of $\Gamma(\mathcal{W},\mathcal{P})$ or of $\Gamma(\mathcal{W},\mathcal{Q})$. The former implies $\beta \in N(\alpha)$ directly and the latter does with the help of Lemma 5.1. Hence we have

shown that $Z \cup V(\mathcal{P}_{N(\alpha)\cap\theta})$ separates H from the rest of G concluding the proof of (i).

(ii) We have $Q_\alpha \subseteq G^{\mathcal{Q}}_{\lambda_1}$ by definition and $Q_\alpha \subseteq G^{\mathcal{P}}_{\lambda_2}$ since $\mathcal{Q}$ is a $(\mathcal{P}, \lambda_2)$-relinkage. Hence $Q_\alpha \subseteq H$ and some component C of H contains Q_α. Suppose that v is a vertex of $H \cap P_\alpha$. Let w be the vertex of P_α such that $wP_\alpha v$ is a maximal subpath of P_α that is still contained in H. Since $P_\alpha \subseteq G^{\mathcal{P}}_{\lambda_2}$ we must have $w \in \{q_1\} \cup Z_2 \subseteq V(Q_\alpha)$ and hence v is in C. For any vertex v of $H - P_\alpha$ there is a v–Z_2 path in H by Claim 6.5.2 which ends on Q_α by (i). This means that v is in C and hence H is connected.

For every inner bag W of $\mathcal{W}$ the induced permutation π of $\mathcal{Q}[W]$ maps each element of $\lambda_1 \setminus \lambda_2$ to itself as $\mathcal{Q}$ is $(\mathcal{P}, \lambda_2)$-relinkage. Moreover, π is an automorphism of $\Gamma(\mathcal{W}, \mathcal{P})$ by (L10) and α is the unique vertex of λ_2 that has a neighbour in $\lambda_1 \setminus \lambda_2$. This shows $\pi(\alpha) = \alpha$. Hence Q_α and P_α must have the same end vertex, namely q_2.

(iii) Let v be a cut-vertex of H. By (ii) it suffices to show that all components of $H - v$ contain a vertex of Q_α. First note that every component of $H - v$ contains a vertex of Z: If a vertex w of $H - v$ is not in Z, then by (i) and the connectivity of G there is a w–Z fan of size at least $p - |N(\alpha) \cap \theta| \geq 2$ in H and at most one of its paths contains v. But any vertex $z \in Z \setminus V(Q_\alpha)$ is on P_α by (i) and the paths $q_1 P_\alpha z$ and $z P_\alpha q_2$ do both meet Q_α but at most one can contain v (given that $z \neq v$). So every component of $H - v$ must contain a vertex of Q_α as claimed.

Claim 6.5.3. *A Q_α-path $P \subseteq P_\alpha \cap H$ is an α-outlet if and only if some inner vertex of P is in Z_1, in particular, every vertex of $Z_1 \setminus V(Q_\alpha)$ lies in a unique α-outlet. Denoting the union of all α-outlets by U, no two components of $Q_\alpha - U$ lie in the same component of $H - U$.*

Proof. Clearly $Q_\alpha \subseteq G^{\mathcal{Q}}_D \cap H \subseteq G^{\mathcal{Q}}_{\lambda_2} \cap G^{\mathcal{Q}}_{\lambda_1} = Q_\alpha$ by Claim 6.5.1. Suppose that $P \subseteq P_\alpha \cap H$ has some inner vertex $z_1 \in Z_1$. Then P is a $G^{\mathcal{Q}}_D$-path and z_1 has a neighbour in $G_\lambda - G^{\mathcal{P}}_{\lambda_2} \subseteq G_\lambda - G^{\mathcal{P}}_D$ so P is an α-outlet.

Before we prove the converse implication let us show that $H \subseteq G^{\mathcal{P}}_D$. If some vertex v of $H \subseteq G^{\mathcal{P}}_{\lambda_2}$ is not in $G^{\mathcal{P}}_D$, then there is $\beta \in \lambda_2 \setminus V(D)$ such that v is on P_β or v is an inner vertex of a non-trivial $\mathcal{P}$-bridge attaching to P_β. But v is in $H - P_\alpha$ so by Claim 6.5.2 and (i) there is a v–Q_α path in H and hence $\alpha\beta$ is an edge of $\Gamma(\mathcal{W}, \mathcal{Q})[\lambda]$ and thus of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ by Lemma 5.1. But (λ_1, λ_2) is chosen such that $N(\alpha) \cap \lambda \subseteq \lambda_1 \cup V(D)$, a contradiction.

Suppose that P is an α-outlet. Then some inner vertex z of P has a neighbour in $G_\lambda - G_D^{\mathcal{P}} \subseteq G_\lambda - H$. So $z \in Z_1 \cup Z_2$ and therefore $z \in Z_1$ as $z \notin V(Q_\alpha) \supseteq Z_2$ by (i).

To conclude the proof of the claim we may assume for a contradiction that Q_α contains vertices r_1, r, and r_2 in this order such that $H - U$ contains an r_1–r_2 path R and r is the end vertex of an α-outlet. Let $\mathcal{Q}'$ be the foundational linkage obtained from $\mathcal{Q}$ by replacing the subpath $r_1 Q_\alpha r_2$ of Q_α with R. Clearly $\mathcal{Q}'$ is a $(\mathcal{P}, V(D))$-relinkage. It suffices to show that the outlet graph of $\mathcal{Q}'$ properly contains that of $\mathcal{Q}$ to derive a contradiction to our choice of $\mathcal{Q}$. By choice of R and the construction of $\mathcal{Q}'$ each β-outlet of $\mathcal{Q}$ for any $\beta \in \kappa$ is internally disjoint from $\mathcal{Q}'$ and hence is contained in a β-outlet of $\mathcal{Q}'$. But r is not on Q'_α so it is an inner vertex of some α-outlet of $\mathcal{Q}'$ so the outlet graph of $\mathcal{Q}'$ contains that of $\mathcal{Q}$ properly as desired. □

Claim 6.5.4. *Let r_1 and r_2 be the end vertices of an α-outlet P of $\mathcal{Q}$. Then $r_1 Q_\alpha r_2$ contains a vertex of $Z_2 \setminus V(P_\alpha)$.*

Proof. We assume that r_1 and r_2 occur on Q_α in this order. Set $Q := r_1 Q_\alpha r_2$. Clearly $P \cup Q$ is a cycle. Since P_α is induced in G, some inner vertex v of Q is not on P_α. By Claim 6.5.2 there is a v–Z_2 path R in $H - P_\alpha$ and its last vertex z_2 must be on Q_α (see (i)) but not on P_α. Finally, Claim 6.5.3 implies that v and z_2 must be in the same component of $Q_\alpha - P$ so both are on Q as desired. □

(iv) Clearly H' contains a cycle. Since Q_α is induced in G there must be a vertex v in $H' - Q_\alpha$ and the v–Z_1 path in $H - Q_\alpha$ that exists by Claim 6.5.2 avoids all cut-vertices of H by (iii) and thus lies in $H' - Q_\alpha$. So H' contains a vertex of $Z_1 - V(Q_\alpha)$ which lies on P_α by (i) and thus also an α-outlet by Claim 6.5.3. So by Claim 6.5.4 we must also have a vertex of $Z_2 \setminus V(P_\alpha)$ in H' that is neither the first nor the last vertex of Q_α in H'.

For any inner bag W of $\mathcal{W}$ the end vertices of $Q_\alpha[W]$ are cut-vertices of H. By (L8) $G[W]$ contains a $\mathcal{P}$-bridge realising some edge of D that is incident with α. So some vertex of P_α has a neighbour in $G_\lambda - G_{\lambda_1}^{\mathcal{P}}$. If $Q_\alpha[W] = P_\alpha[W]$, then $G_{\lambda_1}^{\mathcal{Q}}[W] = G_{\lambda_1}^{\mathcal{P}}[W]$ so this neighbour is also in $G_\lambda - G_{\lambda_1}^{\mathcal{Q}}$ and hence $Q_\alpha[W]$ contains a vertex of Z_2. If $Q_\alpha[W] \neq P_\alpha[W]$, then some block of H in $G[W]$ is not a single edge so by the previous paragraph $Q_\alpha[W]$ contains a vertex of Z_2.

Claim 6.5.5. *Every Z_1–Z_2 path in H is a q_1–q_2 separator in H.*

Proof. Suppose not, that is, H contains a q_1–q_2 path Q'_α and a Z_1–Z_2 path R such that R and Q'_α are disjoint. Clearly $H \cap \mathcal{Q} = Q_\alpha$ so $\mathcal{Q}' := (\mathcal{Q} \setminus \{Q_\alpha\}) \cup \{Q'_\alpha\}$ is a foundational linkage. The last vertex r_2 of R is in Z_2 and hence has a neighbour in $G_\lambda - G^{\mathcal{Q}}_{\lambda_1}$. So there is an r_2–$\mathcal{Q}'_{\lambda_2 \setminus \lambda_1}$ path R_2 that meets H only in r_2. Similarly, for the first vertex r_1 of R there is an r_1–$\mathcal{Q}'_{\lambda_1 \setminus \lambda_2}$ path R_1 that meets H only in r_1. Then $R_1 \cup R \cup R_2$ witness that $\Gamma(\mathcal{W}, \mathcal{Q}')$ has an edge with one end in $\lambda_1 \setminus \lambda_2$ and the other in $\lambda_2 \setminus \lambda_1$, contradicting Lemma 5.1. □

Claim 6.5.6. *Let H' be a block of H. Then $Q := H' \cap Q_\alpha$ is a path and its first vertex q'_1 equals q_1 or is a cut-vertex of H and its last vertex q'_2 equals q_2 or is a cut-vertex of H. Furthermore, there is a q'_1–q'_2 path $P \subseteq H'$ that is internally disjoint from Q such that $Z_1 \cap V(H') \subseteq V(P)$ and if a P-bridge B in H' has no inner vertex on Q_α, then for every $z_1 \in Z_1 \cap V(H')$ the attachments of B are either all on Pz_1 or all on z_1P.*

Proof. It follows easily from (iii) that Q is a path and q'_1 and q'_2 are as claimed. If H' is the single edge $q'_1q'_2$, then the statement is trivial with $P = Q$ so suppose not. Our first step is to show the existence of a q'_1–q'_2 path $R \subseteq H'$ that is internally disjoint from Q.

By (iv) some inner vertex z_2 of Q is in $Z_2 \setminus V(P_\alpha)$. Since H' is 2-connected there is a Q-path $R \subseteq H' - z_2$ with first vertex r_1 on Qz_2 and last vertex r_2 on z_2Q. Pick R such that r_1Qr_2 is maximal. We claim that $r_1 = q'_1$ and $r_2 = q'_2$.

Suppose for a contradiction that $r_2 \neq q'_2$. By the same argument as before there is Q-path $S \subseteq H' - r_2$ with first vertex s_1 on Qr_2 and last vertex s_2 on r_2Q. Note that s_1 must be an inner vertex of r_1Qr_2 by choice of R. Similarly, Q separates R from S in H' otherwise there was a Q-path from r_1 to s_2 again contradicting our choice of R.

But S has an inner vertex v as Q is induced and Claim 6.5.2 asserts the existence of a v–Z_1 path S' in $H - Q_\alpha$ which must be disjoint from R as Q separates S from R. So there is a Z_1–Z_2 path in $z_2Qs_1 \cup s_1Sv \cup S'$ which is disjoint from $Q_\alpha r_1Rr_2Q_\alpha$ by construction, a contradiction to Claim 6.5.5. This shows $r_2 = q'_2$ and by symmetry also $r_1 = q'_1$.

Among all q'_1–q'_2 paths in H' that are internally disjoint from Q pick P such that P contains as few edges outside P_α as possible. To show that P contains all vertices of $Z_1 \cap V(H')$ let $z_1 \in Z_1 \cap V(H')$. We may assume $z_1 \neq q'_1, q'_2$. If z_1 is an inner vertex of Q, then Q contains a Z_1–Z_2 path that is disjoint from P, a contradiction to Claim 6.5.5. So there is an α-outlet R which has z_1 as an inner vertex. Then $Rz_1 \cup Q$ and $z_1R \cup Q$ both contain a Z_1–Z_2 path and by Claim 6.5.5 P must intersect both paths. But P is

internally disjoint from Q so it contains a vertex t_1 of Rz_1 and a vertex t_2 of z_1R. If some edge of t_1Pt_2 is not on P_α, then $P' := q_1'Pt_1P_\alpha t_2Pq_2'$ is q_1'–q_2' path in H' that is internally disjoint from Q and has fewer edges outside P_α than P, contradicting our choice of P. This means $t_1Rt_2 \subseteq P$ and therefore z_1 is on P.

Finally, suppose that for some $z_1 \in Z_1$ there is a P-bridge B in H' with no inner vertex in Q_α and attachments $t_1, t_2 \neq z_1$ such that t_1 is on Pz_1 and t_2 is on z_1P (this implies $z_1 \neq q_1', q_2'$). Let R be the α-outlet containing z_1 and denote its end vertices by r_1 and r_2. By Claim 6.5.4 some inner vertex z_2 of r_1Qr_2 is in Z_2.

If B has an attachment in $z_1P - R$, then $z_1Rr_2 \cup z_2Qr_2$ contains a Z_1–Z_2 path that does not separate q_1' from q_2' in H' and therefore does not separate q_1 from q_2 in H, contradicting Claim 6.5.5. So B has no attachment in z_1P-R and a similar argument implies that B has no attachment in $Pz_1 - R$. So all attachments of B must be in $P \cap R \subseteq P_\alpha$. As $R \cup B$ contains a cycle and P_α is induced some vertex v of B is not on P_α. But then Claim 6.5.2 implies the existence of a v–Z_2 path that avoids P_α and hence uses only inner vertices of B, in particular, some inner vertex of B is in $Z_2 \subseteq V(Q_\alpha)$, contradicting our assumption and concluding the proof of this claim. □

(v) Applying Claim 6.5.6 to every block H' of H and uniting the obtained paths P gives a q_1–q_2 path $R \subseteq H$ such that $Z_1 \subseteq V(R)$ and $V(R \cap Q_\alpha)$ consists of q_1, q_2, and all cut-vertices of H. Moreover, for every $z_1 \in Z_1$ a P-bridge B in H that has no inner vertex in Q_α has all its attachments in Rz_1 or all in z_1R.

Set $\mathcal{Q}' := (\mathcal{Q} \setminus \{Q_\alpha\}) \cup \{R\})$. Let $\mathcal{P}'$ be the foundational linkage obtained by uniting the bridge stabilisation of $\mathcal{Q}'[W]$ in $G[W]$ for all inner bags W of $\mathcal{W}$. Then $\mathcal{P}'[W]$ is p-attached in $G[W]$ for all inner bags W of $\mathcal{W}$ by Lemma 3.7.

To show $P_\beta' = Q_\beta$ for all $\beta \in \lambda \setminus \{\alpha\}$ it suffices by Lemma 3.7 to check that every non-trivial $\mathcal{Q}'$-bridge B' that attaches to Q_β' attaches to at least one other path of $\mathcal{Q}_\lambda'$. If B' is disjoint from H it is also a $\mathcal{Q}$-bridge and thus attaches to some path $Q_\gamma = Q_\gamma'$ with $\gamma \in \lambda \setminus \{\alpha, \beta\}$ by Claim 6.5.1. If B' contains a vertex of H, then it attaches to $Q_\alpha' = R$ as H is connected (see (ii)) and $\mathcal{Q}' \cap H = R$.

To verify $P_\alpha' \subseteq H$ we need to show $B' \subseteq H$ for every $\mathcal{Q}'$-bridge B' that attaches to R but to no other path of $\mathcal{Q}_\lambda'$. Clearly for every vertex v of $G_{\lambda_1}^{\mathcal{P}} - P_\alpha$ there is a v–$\mathcal{P}_{\lambda_1 \setminus \{\alpha\}}$ path in $G_{\lambda_1}^{\mathcal{P}} - P_\alpha$. Similarly, for every vertex v of $G_{\lambda_2}^{\mathcal{Q}} - Q_\alpha$ there is a v–$\mathcal{Q}_{\lambda_2 \setminus \{\alpha\}}$ path in $G_{\lambda_2}^{\mathcal{Q}} - Q_\alpha$. But $Q_\beta' = P_\beta$ for all $\beta \in \lambda_1 \setminus \{\alpha\}$ and $Q_\beta' = Q_\beta$ for all $\beta \in \lambda_2 \setminus \{\alpha\}$ and

$G_\lambda - H = (G^{\mathcal{P}}_{\lambda_1} - P_\alpha) \cup (G^{\mathcal{Q}}_{\lambda_2} - Q_\alpha)$. This means that B' cannot contain a vertex of $G_\lambda - H$ and thus $B' \subseteq H$ as desired.

We have just shown that every bridge B' as above is an R-bridge in H. By construction and the properties (i) and (iv) every component of $Q_\alpha - R$ contains a vertex of Z_2 and hence lies in a $\mathcal{Q}'$-bridge attaching to some path Q'_β with $\beta \in \lambda_2 \setminus \{\alpha\}$. So B' is an R-bridge in H with no inner vertex in Q_α and therefore there must be $z_1, z'_1 \in Z_1 \cup \{q_1, q_2\}$ such that $z_1 R z'_1$ contains all attachments of B' and no inner vertex of $z_1 R z'_1$ is in Z_1. By Lemma 3.7 this implies that P'_α contains no vertex of $Q_\alpha - R$ and $Z_1 \subseteq V(P'_\alpha)$. On the other hand, P'_α must clearly contain the end vertices of R and all cut-vertices of H. This concludes the proof of (v).

(vi) We first show that (H', Ω) is rural where $\Omega := P'Q'^{-1}|Z$ where $Z' := Z \cap V(H')$. Since H is connected and $H \cap \mathcal{P}' = P_\alpha$ we must have $\beta \in N(\alpha) \cap \theta$ for each path P_β with $\beta \in \theta$ whose unique vertex has a neighbour in H. So the set T of all vertices of $\mathcal{P}_\theta$ that are adjacent to some vertex of H' has size at most $p-4$ by assumption. Clearly $Z' \cup T$ separates H' from the rest of G so for every vertex v of $H' - Z'$ there is a v–$(Z' \cup T)$ fan of size at least p and hence a v–Z fan of size at least 4. Hence (H', Ω) is 4-connected and hence it is rural or contains a cross by Theorem 6.1.

Suppose for a contradiction that (H', Ω) contains a cross. By the *off-road edges* of a cross $\{R, S\}$ in (H', Ω) we mean edge set $E(R \cup S) \setminus E(P' \cup Q')$. We call a component of $R \cap (P' \cup Q')$ that contains an end of R a *tail of* R and define the *tails of* S similarly.

Claim 6.5.7. *If $\{R, S\}$ is a cross in (H', Ω) such that its set of off-road edges is minimal, then for every $z \in Z$ that is not in $R \cup S$ the two z–$(R \cup S)$ paths in $P' \cup Q'$ both end in a tail of R or S.*

The proof is the same as for Claim 6.4.5 so we spare it.

Claim 6.5.8. *Every non-trivial $(P' \cup Q')$-bridge B in H' has an attachment in $P' - Q'$ and in $Q' - P'$.*

Proof. Let v be an inner vertex of B. Then $H - Q_\alpha$ contains a v–Z_1 path by Claim 6.5.2 so B must attach to P'. Note that v is in a non-trivial $\mathcal{P}'$-bridge B' and $B' \subseteq G^{\mathcal{P}}_{\lambda_2}$ since $Z_1 \subseteq V(P'_\alpha)$. Furthermore, B' must attach to a path $P'_\beta = Q_\beta$ with $\beta \in \lambda_2 \setminus \lambda_1$: This is clear if B' does not attach to P'_α and follows from Claim 6.5.1 if it does. So B'

contains a path R from v to $G^{\mathcal{Q}}_{\lambda_2} - Q_\alpha$ that avoids P'. But any such path contains a vertex of Z_2 (see (i)) and R does not contain q'_1 and q'_2 so some initial segment of R is a v–Z_2 path in $H' - P'$ as desired. □

Claim 6.5.9. *There is a cross $\{R', S'\}$ in (H', Ω) such that its set of off-road edges is minimal and neither P' nor Q' contains all ends of R' and S'.*

Proof. Pick a cross $\{R, S\}$ in (H', Ω) such that its set E of off-road edges is minimal. We may assume that P' contains all ends of R and S. By (iv) some inner vertex z_2 of Q' is in Z_2. So if $R \cup S$ contains an inner vertex of Q', then $Q' - P'$ contains a Z_2–$(R \cup S)$ path T whose last vertex t is an inner vertex of R say. Clearly one of $\{Rt \cup T, S\}$ and $\{tR \cup T, S\}$ is a cross in (H', Ω) whose set of off-road edges is contained in that of $\{R, S\}$ and hence is minimal as well. So either we find a cross $\{R', S'\}$ as desired or $Q' - P'$ is disjoint from $R \cup S$.

But $(R \cup S) - P'$ must be non-empty as P' is induced in G. So by Claim 6.5.8 there is a Q'–$(R \cup S)$ path in $H' - P'$, in particular, there is a Z_2–$(R \cup S)$ path T in $H' - P'$ and we may assume that its last vertex t is on R. Again one of $\{Rt \cup T, S\}$ and $\{tR \cup T, S\}$ is a cross in (H', Ω) and we denote its set of off-road edges by F. Pick a cross (R', S') in (H, Ω) such that its set E' of off-road edges minimal and $E' \subseteq F$.

Since t is not on P' each of Rt and tR contains an edge that is not in $P' \cup Q'$ so $F \setminus E(T)$ is a proper subset of E. This means that E' must contain an edge of T by minimality of E and hence it must contain $F \cap E(T)$ so $R' \cup S'$ contains a vertex of $Q' - P'$ and we have already seen that we are done in this case, concluding the proof of the claim. □

Claim 6.5.10. *For $i = 1, 2$ there is a q'_i–$(R' \cup S')$ path T_i in H' such that T_1 and T_2 end on one path of $\{R', S'\}$ and the other path has its ends in Z_1 and Z_2.*

Proof. It is easy to see that by construction one path of $\{R', S'\}$, say S', has one end in $Z_1 \setminus \{q'_1, q'_2\}$ and the other in $Z_2 \setminus \{q'_1, q'_2\}$. If for some i the vertex q'_i is in $R' \cup S'$, then it must be on R' and there is a trivial q'_i–R' path T_i. We may thus assume that neither of q'_1 and q'_2 is in $R' \cup S'$.

So $P' \cup Q'$ contains two q'_1–$(R' \cup S')$ paths T_1 and T'_1 that meet only in q'_1. By Claim 6.5.7 T_1 and T'_1 must both end in a tail of R' or S'. But (R', S') is a cross and no inner vertex of $T_1 \cup T'_1$ is an end of R' or S' so

we may assume that T_1 meets a tail of R'. By the same argument we find a q_2'–$(R' \cup S')$ path T_2 that end in a tail of R'. $\square$

To conclude the proof that (H', Ω) is rural note that Claim 6.5.10 implies the existence of a Z_1–Z_2 path in H that does not separate q_1 from q_2 in H and hence contradicts Claim 6.5.5. So (H', Ω) is rural and (vi) follows from this final claim:

Claim 6.5.11. *The society (H', Ω) is rural if and only if the society $(H', P'Q'^{-1})$ is.*

This holds by a simpler version of the proof of Claim 6.4.3 where Claim 6.5.8 takes the role of Claim 6.4.1.

$\square$

7 Constructing a Linkage

In our main theorem we want to construct the desired linkage in a long stable regular decomposition of the given graph. That decomposition is obtained by applying Theorem 3.5 which may instead give a subdivision of $K_{a,p}$. This outcome is even better for our purpose as stated by the following Lemma.

Lemma 7.1. *Every $2k$-connected graph containing a $TK_{2k,2k}$ is k-linked.*

Proof. Let G be a $2k$-connected graph and $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ two disjoint sets in $V(G)$ of size k each. We need to find a system of k disjoint S–T paths linking s_i to t_i for $i = 1, \ldots, k$.

By assumption G contains a subdivision of $K_{2k,2k}$, so there are disjoint sets $A, B \subseteq V(G)$ of size $2k$ each and a system $\mathcal{Q}$ of internally disjoint paths in G such that for every pair (a, b) with $a \in A$ and $b \in B$ there exists a unique a–b path in $\mathcal{Q}$ which we denote by Q_{ab}.

By the connectivity of G, there is a system $\mathcal{P}$ of $2k$ disjoint $(S \cup T)$–$(A \cup B)$ paths (with trivial members if $(S \cup T) \cap (A \cup B) \neq \emptyset$). Pick $\mathcal{P}$ such that it has as few edges outside of $\mathcal{Q}$ as possible. Our aim is to find suitable paths of $\mathcal{Q}$ to link up the paths of $\mathcal{P}$ as desired. We denote by A_1 and B_1 the vertices of A and B, respectively, in which a path of $\mathcal{P}$ ends, and let $A_0 := A \setminus A_1$ and $B_0 := B \setminus B_1$.

The paths of $\mathcal{P}$ use the system $\mathcal{Q}$ sparingly: Suppose that for some pair (a, b) with $a \in A_0$ and $b \in B$, the path Q_{ab} intersects a path of $\mathcal{P}$. Follow Q_{ab} from a to the first vertex v it shares with any path of $\mathcal{P}$, say P. Replacing

P by $Pv \cup Q_{ab}v$ in $\mathcal{P}$ does not give a system with fewer edges outside $\mathcal{Q}$ by our choice of $\mathcal{P}$. In particular, the final segment vP of P must have no edges outside $\mathcal{Q}$. This means $vP = vQ_{ab}$, that is, P is the only path of $\mathcal{P}$ meeting Q_{ab} and after doing so for the first time it just follows Q_{ab} to b. Clearly the symmetric argument works if $a \in A$ and $b \in B_0$. Hence

1. Q_{ab} with $a \in A_0$ and $b \in B_0$ is disjoint from all paths of $\mathcal{P}$,
2. Q_{ab} with $a \in A_1$ and $b \in B_0$ or with $a \in A_0$ and $b \in B_1$ is met by precisely one path of $\mathcal{P}$, and
3. Q_{ab} with $a \in A_1$ and $b \in B_1$ is met by at least two paths of $\mathcal{P}$.

In order to describe precisely how we link the paths of $\mathcal{P}$, we fix some notation. Since $|A_0| + |A_1| = |A| = 2k = |\mathcal{P}| = |A_1| + |B_1|$, we have $|A_0| = |B_1|$ and similarly $|A_1| = |B_0|$. Without loss of generality we may assume that $|B_0| \geq |A_0| = |B_1|$ and therefore $|B_0| \geq k$. So we can pick k distinct vertices $b_1, \ldots, b_k \in B_0$ and an arbitrary bijection $\varphi : B_1 \to A_0$. For $x \in S \cup T$ denote by P_x the unique path of $\mathcal{P}$ starting in x and by x' its end vertex in $A \cup B$.

For each i and $x = s_i$ or $x = t_i$ set

$$R_x := \begin{cases} Q_{x'b_i} & x' \in A_1 \\ Q_{\varphi(x')x'} \cup Q_{\varphi(x')b_i} & x' \in B_1 \end{cases}.$$

By construction R_x and R_y intersect if and only if $x, y \in \{s_i, t_i\}$ for some i, i.e. they are equal or meet exactly in b_i. The paths P_x and R_y intersect if and only if P_x ends in y', that is, if $x = y$. Thus for each $i = 1, \ldots, k$ the subgraph $C_i := P_{s_i} \cup R_{s_i'} \cup R_{t_i'} \cup P_{t_i}$ of G is a tree containing s_i and t_i. Furthermore, these trees are pairwise disjoint, finishing the proof. □

We now give the proof of the main theorem, Theorem 1.1. We restate the theorem before proceeding with the proof.

Theorem 1.1. For all integers k and w there exists an integer N such that a graph G is k-linked if

$$\kappa(G) \geq 2k + 3, \qquad \mathrm{tw}(G) < w, \quad \text{and} \quad |G| \geq N.$$

Proof. Let k and w be given and let f be the function from the statement of Lemma 4.10 with $n := w$. Set

$$n_0 := (2k+1)(n_1 - 1) + 1$$
$$n_1 := \max\{(2k-1)\binom{w}{2k}, 2k(k+3)+1, 12k+4, 2f(k)+1\}$$

We claim that the theorem is true for the integer N returned by Theorem 3.5 for parameters $a = 2k$, $l = n_0$, $p = 2k+3$, and w. Suppose that G is a $(2k+3)$-connected graph of tree-width less than w on at least N vertices. We want to show that G is k-linked. If G contains a subdivision of $K_{2k,2k}$, then this follows from Lemma 7.1. We may thus assume that G does not contain such a subdivision, in particular it does not contain a subdivision of $K_{a,p}$.

Let $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ be disjoint k-tuples of distinct vertices of G. Assume for a contradiction that G does not contain disjoint paths $P_1, \ldots, P_k$ such that the end vertices of P_i are s_i and t_i for $i = 1, \ldots, k$ (such paths will be called the *desired paths* in the rest of the proof).

By Theorem 3.5 there is a stable regular decomposition of G of length at least n_0, of adhesion $q \leq w$, and of attachedness at least $2k + 3$. Since this decomposition has at least $(2k + 1)(n_1 - 1)$ inner bags, there are $n_1 - 1$ consecutive inner bags which contain no vertex of $(S \cup T)$ apart from those coinciding with trivial paths. In other words, this decomposition has a contraction $(\mathcal{W}, \mathcal{P})$ of length n_1 such that $S \cup T \subseteq W_0 \cup W_{n_1}$. By Lemma 3.6 this contraction has the same attachedness and adhesion as the initial decomposition and the stability is preserved. Set $\theta := \{\alpha \mid P_\alpha \text{ is trivial}\}$ and $\lambda := \{\alpha \mid P_\alpha \text{ is non-trivial}\}$.

Claim 7.1.1. $\lambda \neq \emptyset$.

Proof. If $\lambda = \emptyset$, or equivalently, $\mathcal{P} = \mathcal{P}_\theta$, then all adhesion sets of $\mathcal{W}$ equal $V(\mathcal{P}_\theta)$. So by (L2) no vertex of $G - \mathcal{P}_\theta$ is contained in more than one bag of $\mathcal{W}$. On the other hand, (L4) implies that every bag W of $\mathcal{W}$ must contain a vertex $w \in W \setminus V(\mathcal{P}_\theta)$. Since $V(\mathcal{P}_\theta)$ separates W from the rest of G and G is $2k$-connected, there is a w–$\mathcal{P}_\theta$ fan of size $2k$ in $G[W]$. For different bags, these fans meet only in $\mathcal{P}_\theta$.

Since $\mathcal{W}$ has more than $(2k-1)\binom{q}{2k}$ bags, the pigeon hole principle implies that there are $2k$ such fans with the same $2k$ end vertices among the q vertices of $\mathcal{P}_\theta$. The union of these fans forms a $TK_{2k,2k}$ in G which may not exist by our earlier assumption. □

Claim 7.1.2. *Let Γ_0 be a component of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$. The following all hold.*

(i) $|N(\alpha) \cap \theta| \leq 2k - 2$ *for every vertex α of Γ_0.*

(ii) $|N(\alpha) \cap N(\beta) \cap \theta| \leq 2k - 4$ *for every edge $\alpha\beta$ of Γ_0.*

(iii) $2|N(\alpha) \cap \lambda| + |N(\alpha) \cap \theta| \leq 2k$ *for every vertex α of Γ_0.*

(iv) $2|D| + |N(D)| \leq 2k + 2$ *for every block D of Γ_0 that contains a triangle.*

Note that (iii) implies (i) unless Γ_0 is a single vertex and (iii) implies (ii) unless Γ_0 is a single edge. We need precisely these two cases in the proof of Claim 7.1.6.

Proof. The proof is almost identical for all cases so we do it only once and point out the differences as we go. Denote by Γ_1 the union of Γ_0 with all its incident edges of $\Gamma(\mathcal{W},\mathcal{P})$. Set $L := W_0 \cap W_1 \cap V(G_{\Gamma_1})$ and $R := W_{n_1-1} \cap W_{n_1} \cap V(G_{\Gamma_1})$. In case (iv) let α be any vertex of D. Let p and q be the first and last vertex of P_α. Then $(L \cup R) \setminus \{p,q\}$ separates $G^{\mathcal{P}}_{\Gamma_1} - \{p,q\}$ from $S \cup T$ in $G - \{p,q\}$. Hence by the connectivity of G there is a set $\mathcal{Q}$ of $2k$ disjoint $(S \cup T)$–$(L \cup R)$ paths in $G - \{p,q\}$, each meeting $G^{\mathcal{P}}_{\Gamma_1}$ only in its last vertex. For $i = 1,\dots,k$ denote by s'_i the end vertex of the path of $\mathcal{Q}$ that starts in s_i and by t'_i the end vertex of the path of $\mathcal{Q}$ that starts in t_i.

Our task is to find disjoint s'_i–t'_i paths for $i = 1,\dots,k$ in $G^{\mathcal{P}}_{\Gamma_1}$ and we shall now construct sets $X, Y \subseteq V(\Gamma_1)$ and an X–Y pairing L "encoding" this by repeating the following step for each $i \in \{1,\dots,k\}$. Let $\beta,\gamma \in V(\Gamma_1)$ such that s'_i lies on P_β and t'_i lies on P_γ. If $s'_i \in L$, then add β to X and set $\bar{s}_i := (\beta, 0)$. Otherwise $s'_i \in R \setminus L$ and we add β to Y and set $\bar{s}_i := (\beta, \infty)$. Note that $s'_i \in L \cap R$ if and only if $\beta \in \theta$. In this case our decision to add β to X is arbitrary and we could also add it to Y instead (and setting $\bar{s}_i$ accordingly) without any bearing on the proof. Handle γ and t'_i similarly. Then $\{\bar{s}_i\bar{t}_i \mid i = 1,\dots,k\}$ is the edge set of an (X,Y)-pairing which we denote by L.

We claim that there is an L-movement of length at most $(n_1-1)/2 \geq f(k)$ on $H := \Gamma_1$ such that the vertices of $A := V(\Gamma_1) \cap \theta$ are singular. Clearly $H - A = \Gamma_0$ is connected and every vertex of A has a neighbour in Γ_0 so A is marginal in H. The existence of the desired L-movement follows from Lemma 4.8 if (i) or (ii) are violated, from Lemma 4.9 if (iii) is violated, and from Lemma 4.10 if (iv) is violated (note that $|H| \leq w$). But then Lemma 4.3 applied to L implies the existence of disjoint s'_i–t'_i paths in $G^{\mathcal{P}}_{\Gamma_1}$ for $i = 1,\dots,k$ contradicting our assumption that G does not contain the desired paths. This shows that all conditions must hold. □

Claim 7.1.3. *We have* $2|\Gamma_0| + |N(\Gamma_0)| \geq 2k+3$ *(and necessarily* $N(\Gamma_0) \subseteq \theta$*) for every component* Γ_0 *of* $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$.

Proof. Let Γ_1 be the union of Γ_0 with all incident edges of $\Gamma(\mathcal{W},\mathcal{P})$. Set $L := W_0 \cap W_1 \cap V(G^{\mathcal{P}}_{\Gamma_1})$, $M := W_1 \cap W_2 \cap V(G^{\mathcal{P}}_{\Gamma_1})$, and $R := W_{n_1-1} \cap W_{n_1} \cap V(G^{\mathcal{P}}_{\Gamma_1})$. If $G - G^{\mathcal{P}}_{\Gamma_1}$ is non-empty, then $L \cup R$ separates it from M in G. Otherwise M separates L from R in $G = G^{\mathcal{P}}_{\Gamma_1}$. By the connectivity of G we have $2|\Gamma_0| + |N(\Gamma_0)| = |L \cup R| \geq 2k+3$ in the former case and $|M| = |\Gamma_0| + |N(\Gamma_0)| \geq 2k+3$ in the latter. □

We now want to apply Lemma 6.4 and Lemma 6.5. At the heart of both is the assertion that a certain society is rural and we already limited the number of their "ingoing" edges by Lemma 6.2. To obtain a contradiction we shall find societies exceeding this limit. Tracking these down is the purpose of the notion of "richness" which we introduce next.

Let $\Gamma \subseteq \Gamma(\mathcal{W}, \mathcal{P})[\lambda]$. We say that $\alpha \in V(\Gamma)$ is *rich in* Γ if the inner vertices of P_α that have a neighbour in both $G_\lambda - G_\Gamma^{\mathcal{P}}$ and $G_\Gamma^{\mathcal{P}} - P_\alpha$ have average degree at least $2 + |N_\Gamma(\alpha)|(2 + \varepsilon_\alpha)$ in $G_\Gamma^{\mathcal{P}}$ where $\varepsilon_\alpha := 1/|N(\alpha) \cap \lambda|$. A subgraph $\Gamma \subseteq \Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ is called *rich* if every vertex $\alpha \in V(\Gamma)$ is rich in Γ.

Claim 7.1.4. *For $\Gamma \subseteq \Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ and $\alpha \in V(\Gamma)$ the following is true.*

(i) If Γ contains all edges of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ that are incident with α, then α is rich in Γ.

(ii) If α is rich in Γ, then the inner vertices of P_α that have a neighbour in $G_\Gamma^{\mathcal{P}} - P_\alpha$ have average degree at least $2 + |N_\Gamma(\alpha)|(2 + \varepsilon_\alpha)$ in $G_\Gamma^{\mathcal{P}}$.

(iii) Suppose that Γ is induced in $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ and that there are subgraphs $\Gamma_1, \ldots, \Gamma_m \subseteq \Gamma$ such that α separates any two of them in $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ and $\bigcup_{i=1}^m \Gamma_i$ contains all edges of Γ that are incident with α. If α is rich in Γ, then there is $j \in \{1, \ldots, m\}$ such that α is rich in Γ_j.

Proof.

(i) The assumption implies that $G_\Gamma^{\mathcal{P}}$ contains every edge of G_λ that is incident with P_α so no vertex of P_α has a neighbour in $G_\lambda - G_D^{\mathcal{P}}$ and therefore the statement is trivially true.

(ii) The inner vertices of P_α that have a neighbour in $G_\lambda - G_\Gamma^{\mathcal{P}}$ and in $G_\Gamma^{\mathcal{P}} - P_\alpha$ have the desired average degree by assumption. We show that each inner vertex of P_α that has no neighbour in $G_\lambda - G_\Gamma^{\mathcal{P}}$ has at least the desired degree. Clearly we have $d_{G_\Gamma^{\mathcal{P}}}(v) = d_{G_\lambda}(v)$ for such a vertex v. Furthermore, $d_G(v) \geq 2k + 3$ since G is $(2k + 3)$-connected. Every neighbour of v in $\mathcal{P}_\theta$ gives rise to a neighbour of α in θ and by Claim 7.1.2 (iii) there can be at most $|N(\alpha) \cap \theta| \leq 2k - 2|N(\alpha) \cap \lambda|$ such neighbours. This means

$$d_{G_\Gamma^{\mathcal{P}}}(v) = d_{G_\lambda}(v) \geq 2k + 3 - |N(\alpha) \cap \theta| \geq 2|N(\alpha) \cap \lambda| + 3$$

so (ii) clearly holds.

(iii) We may assume that α is not isolated in Γ and that each of the graphs $\Gamma_1, \dots, \Gamma_m$ contains an edge of Γ that is incident with α by simply forgetting those graphs that do not.

For $i = 0, \dots, m$ denote by Z_i the inner vertices of P_α that have a neighbour in $G_\lambda - G^{\mathcal{P}}_{\Gamma_i}$ and in $G^{\mathcal{P}}_{\Gamma_i} - P_\alpha$ where $\Gamma_0 := \Gamma$ and set $Z := \bigcup_{i=1}^m Z_i$. Clearly $P_\alpha \subseteq G^{\mathcal{P}}_{\Gamma_i}$ for all i. Each edge e of $G^{\mathcal{P}}_\Gamma$ that is incident with an inner vertex of P_α but does not lie in P_α is in a $\mathcal{P}$-bridge that realises an edge of Γ by (L6) and Lemma 5.2 since Claim 7.1.2 (iii) implies that $|N(\alpha) \cap \theta| \leq 2k - 2$. So at least one of the graphs $G^{\mathcal{P}}_{\Gamma_i}$ contains e. On the other hand, we have $G^{\mathcal{P}}_{\Gamma_i} \subseteq G^{\mathcal{P}}_\Gamma$ for $i = 1, \dots, m$. This implies $Z_0 \subseteq Z$.

By the same argument as in the proof of (ii) the vertices of Z have average degree at least $2 + |N_\Gamma(\alpha)|(2 + \varepsilon_\alpha)$ in $G^{\mathcal{P}}_\Gamma$. In other words, $G^{\mathcal{P}}_\Gamma$ contains at least $|Z||N_\Gamma(\alpha)|(2 + \varepsilon_\alpha)$ edges with one end on P_α and the other in $G^{\mathcal{P}}_\Gamma - P_\alpha$.

By assumption we have $|N_\Gamma(\alpha)| = \sum_{i=1}^m |N_{\Gamma_i}(\alpha)|$ and so the pigeon hole principle implies that there is $j \in \{1, \dots, m\}$ such that $G^{\mathcal{P}}_{\Gamma_j}$ contains a set E of at least $|Z||N_{\Gamma_j}(\alpha)|(2 + \varepsilon_\alpha)$ edges with one end on P_α and the other in $G^{\mathcal{P}}_{\Gamma_j} - P_\alpha$.

By assumption and Claim 6.5.1 the path P_α separates $G^{\mathcal{P}}_{\Gamma_i}$ from $G^{\mathcal{P}}_{\Gamma_j}$ in G_λ for $i \neq j$. For any vertex $z \in Z \setminus Z_j$ there is $i \neq j$ with $z \in Z_i$, so z has a neighbour in $G^{\mathcal{P}}_{\Gamma_i} - P_\alpha \subseteq G_\lambda - G^{\mathcal{P}}_{\Gamma_j}$. Then the only reason that z is not also in Z_j is that it has no neighbour in $G^{\mathcal{P}}_{\Gamma_j} - P_\alpha$, in particular, it is not incident with an edge of E. So the vertices of Z_j have average degree at least $2 + \frac{|Z|}{|Z_j|}|N_{\Gamma_j}(\alpha)|(2 + \varepsilon_\alpha)$ in $G^{\mathcal{P}}_{\Gamma_j}$ which obviously implies the claimed bound.

□

Claim 7.1.5. *Every component of* $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ *contains a rich block.*

Proof. Let Γ_0 be a component of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$. Suppose that α is a cut-vertex of Γ_0 and let $D_1, \dots, D_m$ be the blocks of Γ_0 that contain α. Clearly $N(\alpha) \cap \lambda \subseteq V(\bigcup_{i=1}^m D_i)$ so Claim 7.1.4 implies that α is rich in $\bigcup_{i=1}^m D_i$ by (i) and hence there is $j \in \{1, \dots, m\}$ such that α is rich in D_j by (iii).

We define an oriented tree R on the set of blocks and cut-vertices of Γ_0 as follows. Suppose that D is a block of Γ_0 and α a cut-vertex of Γ_0 with $\alpha \in V(D)$. If α is rich in D, then we let (α, D) be an edge of R. Otherwise we let (D, α) be an edge of R. Note that the underlying graph of R is the block-cut-vertex tree of Γ_0 and by the previous paragraph every cut-vertex

is incident with an outgoing edge of R. But every directed tree has a sink, so there must be a block D of Γ_0 such that every $\alpha \in \kappa$ is rich in D where κ denotes the set of all cut-vertices of Γ_0 that lie in D.

But the only vertices of $G_D^{\mathcal{P}}$ that may have a neighbour in $G_\lambda - G_D^{\mathcal{P}}$ are on paths of $\mathcal{P}_{V(D)}$ by Lemma 5.3 and of these clearly only the paths of $\mathcal{P}_\kappa$ may have neighbours in $G_\lambda - G_D^{\mathcal{P}}$. So all vertices of $V(D) \setminus \kappa$ are trivially rich in D and hence D is a rich block. □

Claim 7.1.6. *Every rich block D of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ contains a triangle.*

Proof. Suppose that D does not contain a triangle. By Claim 7.1.3 and Claim 7.1.2 (i) we may assume D is not an isolated vertex of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$, that is, D contains an edge. We shall obtain contradicting upper and lower bounds for the number

$$x := \sum_{v \in V(\mathcal{P}_{V(D)})} (d_{G_D^{\mathcal{P}}}(v) - d_{\mathcal{P}_{V(D)}}(v)).$$

For every $\alpha \in V(D)$ denote by V_α the subset of $V(P_\alpha)$ that consists of the ends of P_α and all inner vertices of P_α that have a neighbour in $G_D^{\mathcal{P}} - P_\alpha$. Set $V := \bigcup_{\alpha \in V(D)} V_\alpha$.

For the upper bound let $\alpha\beta$ be an edge of D. Then $N_{\alpha\beta} := N(\alpha) \cap N(\beta) \subseteq \theta$ as a common neighbour of α and β in λ would give rise to a triangle in D. Furthermore, $|N_{\alpha\beta}| \leq 2k - 4$ by Claim 7.1.2 (ii). By Lemma 6.4 the society $(G_{\alpha\beta}^{\mathcal{P}}, P_\alpha P_\beta^{-1})$ is rural if α and β are not twins. But if they are, then $N(\alpha) \cup N(\beta) = N_{\alpha\beta} \cup \{\alpha, \beta\}$. This means that D is a component of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ that consists only of the single edge $\alpha\beta$. So by Claim 7.1.3 we have $|N_{\alpha\beta}| = |N(D)| \geq 2k - 1$, a contradiction. Hence $(G_{\alpha\beta}^{\mathcal{P}}, P_\alpha P_\beta^{-1})$ is rural.

The graph $G - \mathcal{P}_{N_{\alpha\beta}}$ contains $G_{\alpha\beta}^{\mathcal{P}}$ and has minimum degree at least $2k + 3 - |\mathcal{P}_{N_{\alpha\beta}}| \geq 6$ by the connectivity of G. By Claim 7.1.2 (i) we have $|N(\gamma) \cap \theta| \leq 2k - 2$ for every $\gamma \in \lambda$ so Lemma 5.2 implies that every non-trivial $\mathcal{P}$-bridge in an inner bag of $\mathcal{W}$ attaches to at least two paths of $\mathcal{P}_\lambda$ or to none. A vertex v of $G_{\alpha\beta}^{\mathcal{P}} - (P_\alpha \cup P_\beta)$ is therefore an inner vertex of some non-trivial $\mathcal{P}$-bridge B that attaches to P_α and P_β and has all its inner vertices in $G_{\alpha\beta}^{\mathcal{P}}$. This means that a neighbour of v outside $G_{\alpha\beta}^{\mathcal{P}}$ must be an attachment of B on some path P_γ and hence $\gamma \in N_{\alpha\beta} \subseteq \theta$. So all vertices of $G_{\alpha\beta}^{\mathcal{P}} - (P_\alpha \cup P_\beta)$ have the same degree in $G_{\alpha\beta}^{\mathcal{P}}$ as in $G - \mathcal{P}_{N_{\alpha\beta}}$, namely at least 6.

The vertices of $G_{\alpha\beta}^{\mathcal{P}} - (P_\alpha \cup P_\beta)$ retain their degree if we suppress all inner vertices of P_α and P_β that have degree 2 in $G_D^{\mathcal{P}}$. Since the paths of $\mathcal{P}$ are induced by (L6) an inner vertex of P_α has degree 2 in $G_D^{\mathcal{P}}$ if and only if it has no neighbour in $G_D^{\mathcal{P}} - P_\alpha$. So we suppressed precisely those inner vertices

of P_α and P_β that are not in V_α or V_β. By Lemma 6.3 the society obtained from $(G^{\mathcal{P}}_{\alpha\beta}, P_\alpha P_\beta^{-1})$ in this way is still rural so Lemma 6.2 implies

$$\sum_{v \in V_\alpha \cup V_\beta} d_{G^{\mathcal{P}}_{\alpha\beta}}(v) \leq 4|V_\alpha| + 4|V_\beta| - 6.$$

Clearly $G^{\mathcal{P}}_D = \bigcup_{\alpha\beta \in E(D)} G^{\mathcal{P}}_{\alpha\beta}$ and $P_\alpha \subseteq G^{\mathcal{P}}_{\alpha\beta}$ for all $\beta \in N_D(\alpha)$ and thus

$$\begin{aligned}
x &= \sum_{v \in V} \Big(d_{G^{\mathcal{P}}_D}(v) - d_{\mathcal{P}_{V(D)}}(v) \Big) \\
&\leq \sum_{\alpha \in V(D)} \sum_{\beta \in N_D(\alpha)} \sum_{v \in V_\alpha} (d_{G^{\mathcal{P}}_{\alpha\beta}}(v) - d_{P_\alpha}(v)) \\
&= \sum_{\alpha\beta \in E(D)} \sum_{v \in V_\alpha \cup V_\beta} d_{G^{\mathcal{P}}_{\alpha\beta}}(v) - \sum_{\alpha \in V(D)} |N_D(\alpha)| \cdot (2|V_\alpha| - 2) \\
&\leq \sum_{\alpha\beta \in E(D)} (4|V_\alpha| + 4|V_\beta| - 6) - \sum_{\alpha \in V(D)} |N_D(\alpha)| \cdot (2|V_\alpha| - 2) \\
&= \sum_{\alpha \in V(D)} |N_D(\alpha)| \, (4|V_\alpha| - 3) - \sum_{\alpha \in V(D)} |N_D(\alpha)| \cdot (2|V_\alpha| - 2) \\
&< \sum_{\alpha \in V(D)} 2|N_D(\alpha)| \cdot |V_\alpha|.
\end{aligned}$$

To obtain the lower bound for x note that Claim 7.1.4 (ii) says that for any $\alpha \in V(D)$ the vertices of V_α without the two end vertices of P_α have average degree $2 + |N_D(\alpha)|(2 + \varepsilon_\alpha)$ in $G^{\mathcal{P}}_D$ where $\varepsilon_\alpha \geq 1/k$ by Claim 7.1.2 (iii). Clearly every inner bag of $\mathcal{W}$ must contain a vertex of V_α as it contains a $\mathcal{P}$-bridge realising some edge $\alpha\beta \in E(D)$. This means $|V_\alpha| \geq n_1/2 \geq 4k + 2$ and thus

$$\begin{aligned}
x &= \sum_{\alpha \in V(D)} \sum_{v \in V_\alpha} \Big(d_{G^{\mathcal{P}}_D}(v) - d_{P_\alpha}(v) \Big) \\
&\geq \sum_{\alpha \in V(D)} (|V_\alpha| - 2) \cdot |N_D(\alpha)| \cdot (2 + \varepsilon_\alpha) \\
&\geq \sum_{\alpha \in V(D)} |N_D(\alpha)| \cdot (2|V_\alpha| - 4 + 4k\varepsilon_\alpha) \\
&\geq \sum_{\alpha \in V(D)} 2|N_D(\alpha)| \cdot |V_\alpha|. \qquad \square
\end{aligned}$$

Claim 7.1.7. *Every rich block D of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ satisfies $2|D| + |N(D)| \geq 2k + 3$.*

Proof. Suppose for a contradiction that $2|D| + |N(D)| \leq 2k + 2$. By Lemma 6.5 there is a $V(D)$-compressed $(\mathcal{P}, V(D))$-relinkage $\mathcal{Q}$ with properties as listed in the statement of Lemma 6.5. Let us first show that we are done if D is *rich w.r.t. to* $\mathcal{Q}$, that is, for every $\alpha \in V(D)$ the inner vertices of Q_α that have a neighbour in $G_\lambda - G_D^{\mathcal{Q}}$ and in $G_D^{\mathcal{Q}} - Q_\alpha$ have average degree at least $2 + |N_D(\alpha)|(2 + \varepsilon_\alpha)$ in $G_D^{\mathcal{Q}}$.

Denote the cut-vertices of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ that lie in D by κ. For $\alpha \in \kappa$ let V_α be the set consisting of the ends of Q_α and of all inner vertices of Q_α that have a neighbour in $G_D^{\mathcal{Q}} - Q_\alpha$ and set $V := \bigcup_{\alpha \in \kappa} V_\alpha$. Pick $\alpha \in \kappa$ such that $|V_\alpha|$ is maximal. By Lemma 5.7 (with $p = 2k + 3$) every vertex of $G_D^{\mathcal{Q}}$ lies on a path of $\mathcal{Q}_{V(D)}$ and we have $|Q_\beta| < |V_\alpha|$ for all $\beta \in V(D) \setminus \kappa$.

The paths of $\mathcal{Q}$ are induced in G as $\mathcal{Q}[W]$ is $(2k + 3)$-attached in $G[W]$ for every inner bag W of $\mathcal{W}$. Hence V_α contains precisely the vertices of Q_α that are not inner vertices of degree 2 in $G_D^{\mathcal{Q}}$. By the same argument as in the proof of Claim 7.1.4 (ii) the vertices of V_α that are not ends of Q_α have average degree at least $2 + |N_D(\alpha)|(2 + \varepsilon_\alpha)$ in $G_D^{\mathcal{Q}}$.

We want to show that the average degree in $G_D^{\mathcal{Q}}$ taken over all vertices of V_α is larger than $2 + 2|N_D(\alpha)|$. Clearly the end vertices of Q_α have degree at least 1 in $G_D^{\mathcal{Q}}$ so both lack at most $1 + 2|N_D(\alpha)| \leq 3|N_D(\alpha)|$ incident edges to the desired degree. On the other hand, the degree of every vertex of V_α that is not an end of Q_α is on average at least $|N_D(\alpha)| \cdot \varepsilon_\alpha$ larger than desired. But $\varepsilon_\alpha \geq 1/k$ by Claim 7.1.2 (iii) and by Lemma 6.5 (iv) the path $Q_\alpha[W]$ contains a vertex of V_α for every inner bag of $\mathcal{W}$, in particular, $|V_\alpha| \geq n_1/2 > 6k + 2$ and hence $(|V_\alpha| - 2)\varepsilon_\alpha > 6$.

This shows that there are more than $2|V_\alpha| \cdot |N_D(\alpha)|$ edges in $G_D^{\mathcal{Q}}$ that have one end on Q_α and the other on another path of $\mathcal{Q}_{V(D)}$. By Lemma 5.1 these edges can only end on paths of $\mathcal{Q}_{N_D(\alpha)}$ so by the pigeon hole principle there is $\beta \in N_D(\alpha)$ such that $G_D^{\mathcal{Q}}$ contains more than $2|V_\alpha|$ edges with one end on Q_α and the other on Q_β.

Hence the society (H, Ω) obtained from $(G_{\alpha\beta}^{\mathcal{Q}}, Q_\alpha Q_\beta^{-1})$ by suppressing all inner vertices of Q_α and Q_β that have degree 2 in $G_{\alpha\beta}^{\mathcal{Q}}$ has more than $2|V_\alpha| + 2|V_\beta| - 2$ edges and all its $|V_\alpha| + |V_\beta|$ vertices are in $\bar{\Omega}$. So by Lemma 6.2 (H, Ω) cannot be rural. But it is trivially 4-connected as all its vertices are in $\bar{\Omega}$ and must therefore contain a cross by Theorem 6.1. The paths of $\mathcal{Q}$ are induced so this cross consists of two edges which both have one end on Q_α and the other on Q_β. Such a cross gives rise to a linkage $\mathcal{Q}'$ from the left to the right adhesion set of some inner bag W of $\mathcal{W}$ such that the induced permutation of $\mathcal{Q}'$ maps some element of $V(D) \setminus \{\alpha\}$ (not necessarily β) to α and maps every $\gamma \notin V(D)$ to itself. Since α has a neighbour outside D this is not an automorphism of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ and therefore $\mathcal{Q}'$ is a twisting

disturbance contradicting the stability of $(\mathcal{W}, \mathcal{P})$.

It remains to show that D is rich w.r.t. $\mathcal{Q}$. Suppose that it is not. By the same argument as for Claim 7.1.4 (i) there must be $\alpha \in \kappa$ such that the inner vertices of Q_α that have a neighbour in $G_\lambda - G_D^{\mathcal{Q}}$ and in $G_D^{\mathcal{Q}} - Q_\alpha$ have average degree less than $2 + |N_D(\alpha)|(2 + \varepsilon_\alpha)$ in $G_D^{\mathcal{Q}}$. Let (λ_1, λ_2) be a separation of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ with $\lambda_1 \cap \lambda_2 = \{\alpha\}$ and $N(\alpha) \cap \lambda_2 = N(\alpha) \cap V(D)$. Let H, Z_1, Z_2, $\mathcal{P}'$, q_1, and q_2 be as in the statement of Lemma 6.5. We shall obtain contradicting upper and lower bounds for the number

$$x := \sum_{v \in V(P'_\alpha \cup Q_\alpha)} \left(d_H(v) - d_{P'_\alpha \cup Q_\alpha}(v) \right).$$

Denote by $H_1, \ldots, H_m$ the blocks of H that are not a single edge and for $i = 1, \ldots, m$ let V_i be the set of vertices of $C_i := H_i \cap (P'_\alpha \cup Q_\alpha)$ that are a cut-vertex of H or are incident with some edge of H_i that is not in $P'_\alpha \cup Q_\alpha$ and set $V := \bigcup_{i=1}^m V_i$. By definition we have $d_H(v) = d_{P'_\alpha \cup Q_\alpha}(v)$ for all vertices v of $(P'_\alpha \cup Q_\alpha) - V$.

Note that H is adjacent to at most $|N(\alpha) \cap \theta|$ vertices of $\mathcal{P}_\theta$ by Lemma 6.5 (ii) and Lemma 5.1. So Claim 7.1.2 (iii) and the connectivity of G imply that every vertex of H has degree at least $2k + 3 - |N(\alpha) \cap \theta| \geq 2|N(\alpha) \cap \lambda| + 3$ in G_λ.

To obtain an upper bound for x let $i \in \{1, \ldots, m\}$. By Lemma 6.5 (vi) C_i is a cycle and the society $(H_i, \Omega(C_i))$ is rural where $\Omega(C_i)$ denotes one of two cyclic permutations that C_i induces on its vertices. Since $|N(\alpha) \cap \lambda| \geq 2$ every vertex of $H_i - C_i$ has degree at least 6 in H_i by the previous paragraph. This remains true if we suppress all vertices of C_i that have degree 2 in H_i. The society obtained in this way is still rural by Lemma 6.3. Since we suppressed precisely those vertices of C_i that are not in V_i Lemma 6.2 implies $\sum_{v \in V_i} d_{H_i}(v) \leq 4|V_i| - 6$. By definition of V we have $d_H(v) = d_{P'_\alpha \cup Q_\alpha}(v)$ for all vertices v of $P'_\alpha \cup Q_\alpha$ that are not in V. Hence we have

$$x = \sum_{v \in V} \left(d_H(v) - d_{P'_\alpha \cup Q_\alpha}(v) \right) = \sum_{i=1}^m \sum_{v \in V_i} \left(d_{H_i}(v) - d_{C_i}(v) \right) \leq \sum_{i=1}^m \left(2|V_i| - 6 \right).$$

Let us now obtain a lower bound for x. Clearly $G_D^{\mathcal{P}} \subseteq G_{\lambda_2}^{\mathcal{P}}$ and $G_D^{\mathcal{Q}} \subseteq G_{\lambda_2}^{\mathcal{Q}}$. To show that $d_{G_D^{\mathcal{Q}}}(v) = d_{G_{\lambda_2}^{\mathcal{Q}}}(v)$ for all $v \in V(H)$ (we follow the general convention that a vertex has degree 0 in any graph not containing it) it remains to check that an edge of G_λ that has precisely one end in H but is not in $G_D^{\mathcal{Q}}$ cannot be in $G_{\lambda_2}^{\mathcal{Q}}$. Such an edge e must be in a $\mathcal{Q}$-bridge that attaches to Q_α and some Q_β with $\beta \in \lambda \setminus V(D)$. But $N(\alpha) \cap \lambda_2 = V(D)$ and hence $\beta \in \lambda_1$. So e is an edge of $G_{\lambda_1}^{\mathcal{Q}}$ but not on Q_α and therefore not in $G_{\lambda_2}^{\mathcal{Q}}$.

This already implies $d_{G_D^{\mathcal{P}}}(v) = d_{G_{\lambda_2}^{\mathcal{P}}}(v)$ for all $v \in V(H)$ since $G_D^{\mathcal{Q}} \subseteq G_D^{\mathcal{P}}$ and $G_{\lambda_2}^{\mathcal{P}} = H \cup G_{\lambda_2}^{\mathcal{Q}}$ (see the proof of Lemma 6.5 (i) for the latter identity). The next equality follows directly from the definition of H.

$$d_H(v) + d_{G_{\lambda_2}^{\mathcal{Q}}}(v) = d_{G_{\lambda_2}^{\mathcal{P}}}(v) + d_{Q_\alpha}(v) \quad \forall v \in V(H).$$

Denote by U_1 the set of inner vertices of P_α that have a neighbour in both $G_\lambda - G_D^{\mathcal{P}}$ and $G_D^{\mathcal{P}} - P_\alpha$ and by U_2 the set of inner vertices of Q_α that have a neighbour in both $G_\lambda - G_D^{\mathcal{Q}}$ and $G_D^{\mathcal{Q}} - Q_\alpha$. In other words, U_1 and U_2 are the sets of those vertices of P_α and Q_α, respectively, that are relevant for the richness of α in D. Set $V' := (V \setminus \{q_1, q_2\}) \cup (Z_1 \cap Z_2))$, $V_P := V' \cap V(P'_\alpha)$, and $V_Q := V' \cap V(Q_\alpha)$. Then $U_1 = (V \cap Z_1) \cup (Z_1 \cap Z_2) = V' \cap Z_1 \subseteq V_P$ and $U_2 = (V \cap Z_2) \cup (Z_1 \cap Z_2) \subseteq V_Q$.

By our earlier observation every vertex of H has degree at least $2|N(\alpha) \cap \lambda| + 3$ in G_λ and therefore every vertex of $V_P \setminus Z_1$ must have at least this degree in $G_{\lambda_2}^{\mathcal{P}}$. Since $U_1 \subseteq V_P$ and α is rich in D this means that

$$\sum_{v \in V_P} d_{G_D^{\mathcal{P}}}(v) \geq |V_P| \left(2 + |N_D(\alpha)| \cdot (2 + \varepsilon_\alpha)\right).$$

Similarly, we have $U_2 \subseteq V_Q \subseteq V(Q_\alpha)$ and every vertex $v \in V_Q \setminus Z_2$ satisfies $d_{G_D^{\mathcal{Q}}}(v) = 2 = d_{Q_\alpha}(v)$. So by the assumption that α is not rich in D w.r.t. $\mathcal{Q}$ we have

$$\sum_{v \in V_Q} \left(d_{G_D^{\mathcal{Q}}}(v) - d_{Q_\alpha}(v)\right) < |V_Q| \cdot |N_D(\alpha)| \cdot (2 + \varepsilon_\alpha).$$

Observe that

$$2|N(\alpha) \cap \lambda| + 3 = 2 + |N(\alpha) \cap \lambda_1| \cdot (2 + \varepsilon_\alpha) + |N(\alpha) \cap \lambda_2| \cdot (2 + \varepsilon_\alpha)$$

and recall that $N_D(\alpha) = N(\alpha) \cap \lambda_2$. Combining all of the above we get

$$
\begin{aligned}
x &\geq \sum_{v\in V'} \big(d_H(v) - d_{P'_\alpha\cup Q_\alpha}(v)\big) \\
&= \sum_{v\in V'} \Big(d_{G_D^{\mathcal{P}}}(v) - d_{G_D^{\mathcal{Q}}}(v) + d_{Q_\alpha}(v) - d_{P'_\alpha\cup Q_\alpha}(v)\Big) \\
&= \sum_{v\in V_P} d_{G_D^{\mathcal{P}}}(v) + \sum_{v\in V'\setminus V_P} d_{G_D^{\mathcal{P}}}(v) - \sum_{v\in V_Q} \Big(d_{G_D^{\mathcal{Q}}}(v) - d_{Q_\alpha}(v)\Big) - 2|V'| - 2m \\
&> |V_P|\cdot|N_D(\alpha)|\cdot(2+\varepsilon_\alpha) + 2|V_P| + |V'\setminus V_P|\cdot(2|N(\alpha)\cap\lambda| + 3) \\
&\quad - |V_Q|\cdot|N_D(\alpha)|\cdot(2+\varepsilon_\alpha) - 2|V'| - 2m \\
&= |V'\setminus V_Q|\cdot|N_D(\alpha)|\cdot(2+\varepsilon_\alpha) + |V'\setminus V_P|\cdot|N(\alpha)\cap\lambda_1|\cdot(2+\varepsilon_\alpha) - 2m \\
&> 2|V'\setminus V_Q| + 2|V'\setminus V_P| - 2m = \sum_{i=1}^{m}(2|V_i| - 6)
\end{aligned}
$$

This shows that D is rich w.r.t. $\mathcal{Q}$ as defined above. So Claim 7.1.7 holds. □

By Claim 7.1.1 the graph $\Gamma(\mathcal{W},\mathcal{P})[\lambda]$ has a component. This component has a rich block D by Claim 7.1.5. By Claim 7.1.6 and Claim 7.1.7 we have a triangle in D and $|D|+|N(D)| \geq 2k+3$. This contradicts Claim 7.1.2 (iv) and thus concludes the proof of Theorem 1.1. □

8 Discussion

In this section we first show that Theorem 1.1 is almost best possible (see Proposition 8.1 below) and then summarise where our proof uses the requirement that the graph G is $(2k+3)$-connected.

Proposition 8.1. *For all integers k and N with $k \geq 2$ there is a graph G which is not k-linked such that*

$$\kappa(G) \geq 2k+1, \qquad \mathrm{tw}(G) \leq 2k+10, \quad \textit{and} \quad |G| \geq N.$$

Proof. We reduce the assertion to the case $k = 2$, that is, to the claim that there is a graph H which is not 2-linked but satisfies

$$\kappa(H) = 5, \qquad \mathrm{tw}(H) \leq 14, \quad \text{and} \quad |H| \geq N.$$

For any $k \geq 3$ let K be the graph with $2k-4$ vertices and no edges. We claim that $G := H * K$ (the disjoint union of H and K where every vertex of H is joined to every vertex of K by an edge) satisfies the assertion for k.

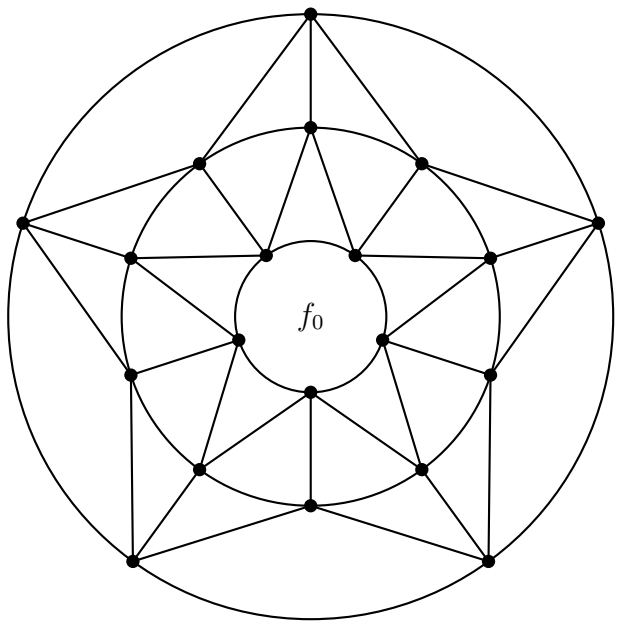

Figure 2: The 5-connected graph H_0 and its inner face f_0.

Clearly $|G| = |H| + 2k - 4 \geq N$. Taking a tree-decomposition of H of minimal width and adding $V(K)$ to every bag gives a tree-decomposition of G, so $\mathrm{tw}(G) \leq \mathrm{tw}(H) + 2k - 4 \leq 2k + 10$. To see that G is $(2k+1)$-connected, note that it contains the complete bipartite graph with partition classes $V(H)$ and $V(K)$, so any separator X of G must contain $V(H)$ or $V(K)$. In the former case we have $|X| \geq N$ and we may assume that this is larger than $2k$. In the latter case we know that $G - X \subseteq H$, in particular $X \cap V(H)$ is a separator of H and hence must have size at least 5, implying $|X| \geq |K| + 5 = 2k + 1$ as required.

Finally, G is not k-linked: By assumption there are vertices s_1, s_2, t_1, t_2 of H such that H does not contain disjoint paths P_1 and P_2 whereP_i ends in s_i and t_i for $i = 1, 2$. If G was k-linked, then for any enumeration $s_3, \ldots, s_k, t_3, \ldots, t_k$ of the $2k - 4$ vertices of $V(K)$ there were disjoint paths $P_1, \ldots, P_k$ in G such that P_i has end vertices s_i and t_i for $i = 1, \ldots, k$. In particular, P_1 and P_2 do not contain a vertex of K and are hence contained in H, a contradiction.

It remains to give a counterexample for $k = 2$. The planar graph H_0 in Figure 2 is 5-connected. Denote the 5-cycle bounding the outer face of H_0 by C_1 and the 5-cycle bounding f_0 by C_0. Then $(V(H_0 - C_0), V(H_0 - C_1))$ forms a separation of H_0 of order 10, in particular, H_0 has a tree-decomposition of

width 14 where the tree is K_2. Draw a copy H_1 of H_0 into f_0 such that the cycle C_0 of H_0 gets identified with the copy of C_1 in H_1. Since $H_0 \cap H_1$ has 5 vertices, the resulting graph is still 5-connected and has a tree-decomposition of width 14. We iteratively paste copies of H_0 into the face f_0 of the previously pasted copy as above until we end up with a planar graph H such that

$$\kappa(H) = 5, \qquad \mathrm{tw}(H) \leq 14, \quad \text{and} \quad |H| \geq N.$$

Still the outer face of H is bounded by a 5-cycle C_1, so we can pick vertices s_1, s_2, t_1, t_2 in this order on C_1 to witness that H is not 2-linked (any s_1–t_1 path must meet any s_2–t_2 path by planarity). $\square$

Where would our proof of Theorem 1.1 fail for a $(2k+2)$-connected graph G? There are several instances where we invoke $(2k+3)$-connectivity as a substitute for a minimum degree of at least $2k+3$. The only place where minimum degree $2k+2$ does not suffice is the proof of Claim 7.1.4. We need minimum degree $2k+3$ there to get the small "bonus" ε_α in our notion of richness. Richness only allows us to make a statement about the inner vertices of a path and the purpose of this bonus is to compensate for the end vertices. Therefore the arguments involving richness in the proofs of Claim 7.1.6 and Claim 7.1.7 would break down if we only had minimum degree $2k+2$.

But even if the suppose that G has minimum degree at least $2k+3$ there are still two places where our proof of Theorem 1.1 fails: The first is the proof of Claim 7.1.3 and the second is the application of Lemma 5.7 in the proof of Claim 7.1.7.

We use Claim 7.1.3 in the proof of Claim 7.1.6, to show that no component of $\Gamma(\mathcal{W}, \mathcal{P})[\lambda]$ can be a single vertex or a single edge. In both cases we do not use the full strength of Claim 7.1.3. So although we formally rely on $(2k+3)$-connectivity for Claim 7.1.3 we do not really need it here.

However, the application of Lemma 5.7 in the proof of Claim 7.1.7 does need $(2k+3)$-connectivity. Our aim there is to get a contradiction to Claim 7.1.2 (iv) which gets the bound $2k+3$ from the token game in Lemma 4.10. This bound is sharp: Let H be the union of a triangle $D = d_1d_2d_3$ and two edges d_1a_1 and d_2a_2 and set $A := \{a_1, a_2\}$. Clearly $H - A = D$ is connected and A is marginal in H. For $k = 3$ we have $2|D| + |N(D)| = 8 = 2k+2$. Let L be the pairing with edges $(a_1, 0)(a_2, 0)$ and $(d_i, 0)(d_i, \infty)$ for $i = 1, 2$. It is not hard to see that there is no L-movement on H as the two tokens from A can never meet.

So the best hope of tweaking our proof of Theorem 1.1 to work for $(2k+2)$-connected graphs is to provide a different proof for Claim 7.1.7. This would

also be a chance to avoid relinkages, that is, most of Section 5, and the very technical Lemma 6.5 altogether as they only serve to establish Claim 7.1.7.

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Entwicklung der Arbeit

Die Entwicklung der Arbeit "Linkages in Large Graphs of Bounded Tree-Width" gliedert sich zeitlich in drei Abschnitte.

1. Das Projekt wird zunächst von Ken-Ichi Kawarabayashi, Theodor Müller und Paul Wollan in Tokio im September 2009 bearbeitet. Hier wird die grundlegende Beweisstrategie entwickelt. Das allgemeine Linkage-Problem soll auf ein Linkage-Problem in einem Graphen mit einer langen Wegzerlegung zurückgeführt werden. Das Umleiten der Wege wird kombinatorisch über ein *Token-Game* beschrieben. Mit Hilfe der Resultate von Robertson und Seymour zu *rural societies* (cf. Theorem 6.1) sollen Kreuze in Brücken zwischen Fundamentalwegen gefunden werden, um die Brückenkonfiguration zur Anwendung des Token-Games zu verbessern. Es wird erwartet, dass das Token-Game nur in einfachen Fällen betrachtet werden muss (z.B. dass der Hilfsgraph ein Stern ist). Es gibt die Hoffnung, dass ein Zusammenhang von $2k + 2$ ausreicht. Technische Details werden nur oberflächlich diskutiert, da erwartet wird, dass die Umsetzung an vielen Stellen analog zu anderen Resultaten möglich sei (z.B. lange Wegesysteme wie in [1]; *Vortex* verdichten wie in [2]).

2. Von Oktober 2009 bis August 2011 arbeitet Theodor Müller in Hamburg allein weiter. In dieser Zeit fand eine tiefe technische Analyse der Beweisstrategie statt. Dabei offenbaren sich eine Reihe von größeren technischen und konzeptionellen Problemen. Ergebnis dieser Arbeit sind Lösungskriterien für das Token-Game. Dabei werden auch triviale Fundamentalwege berücksichtigt, die zuvor zu Schwierigkeiten geführt haben. Vorläufer der Begriffe *rich* und *rich block* werden entwickelt. Das Verdichten eines solchen Blocks um die Brückenkonfiguration zu verbessern wird für den Fall gelöst, dass der Block nur eine Artikulation des Hilfsgraphen enthält. Es zeigt sich, dass die Beweismethode einen Zusammenhang von $2k + 3$ erfordert.

3. Im August 2011 kommen Jan-Oliver Fröhlich und Julian Pott zum Projekt hinzu. Theodor Müller begleitet das Projekt ab diesem Zeitpunkt im Hintergrund und beteiligt sich dann, wenn Probleme auftreten. Der Begriff einer *stable regular decomposition* wird entwickelt. Die Verdichtung eines Blocks mit beliebig vielen Artikulationen wird gelöst. Das Token-Game wird in seiner endgültigen Fassung beschrieben. Die verschiedenen Beweisteile werden organisiert und formalisiert, die dazu benötigten Notationen werden entwickelt und vereinheitlicht. Die enge

Verstrickung zwischen den verschiedenen Beweisteilen führt dazu, dass selbst kleine notwendige technische Änderungen Auswirkungen auf den gesamten Beweis haben, sodass Definitionen und Notationen mehrfach vollständig überarbeitet werden müssen. So wird die endgültige Definition von *rich* erst im November 2013 gefunden. Die Ausarbeitung von technischen Details (z.B. Lemmas 6.4 und 6.5) nimmt viel Zeit in Anspruch. Sowohl konzeptionelle als auch technische Probleme werden häufig zu dritt gemeinsam an der Tafel bearbeitet. Die letztendliche Ausformulierung des Beweises wird von Jan-Oliver Fröhlich vorgenommen. Viele der dabei auftretenden technischen Problemen und Detailfragen werden gemeinsam geklärt. Die Ausarbeitung des Beweises kommt im Februar 2014 zum Abschluss. Im März 2014 verfasst Paul Wollan die Abschnitte 1 und 2 der Arbeit.

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Summary

A central part of the graph minor theory developed by Robertson and Seymour is the excluded minor structure theorem [5][6]. This theorem has found many applications elsewhere.

In the first part of this dissertation, we present a new version of this structure theorem with the goal to provide a broad feature set and an accessible terminology to allow for easier applications.

A graph G with $|G| \geq 2k$ is called *k-linked* if for every choice of distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ there are disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i. It has been shown by Larman and Mani [3] and Jung [2] that there exists a function $f(k)$ such that every $f(k)$-connected graph is k-linked. Bollobàs and Thomason [1] have given a linear bound of $22k$. This result has been improved by Thomas and Wollan [4] by proving that $10k$-connected graphs are k-linked.

In the second part, we show that for all integers k and w there is an integer N such that every $2k+3$-connected graph G of tree width less than w on at least N vertices is k-linked. A central part of the proof is the study of certain subgraphs of G. We analyze linear decompositions of these subgraphs with new techniques we derived from those techniques used in the first part.

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Zusammenfassung

Ein zentraler Bestandteil der Minorentheorie von Robertson und Seymour ist der Struktursatz über Graphen mit verbotenen Minoren [5][6]. Dieser Satz hat an vielen weiteren Stellen Anwendung gefunden.

Im ersten Teil dieser Dissertation stellen wir eine neue Version dieses Struktursatzes mit umfangreichen strukturellen Beschreibungen und einer zugänglichen Terminologie vor, um weitere Anwendungen zu vereinfachen.

Ein Graph G mit $|G| \geq 2k$ ist *k-verbunden*, wenn es für jede Wahl verschiedener Ecken $s_1, \ldots, s_k, t_1, \ldots, t_k$ disjunkte Wege $P_1, \ldots, P_k$ gibt, sodass jeder Weg P_i die Enden s_i und t_i hat.

Larman und Mani [3] und Jung [2] haben die Existenz einer Funktion $f(k)$ bewiesen, sodass jeder $f(k)$-zusammenhängende Graph k-verbunden ist. Bollobàs und Thomason [1] haben gezeigt, dass ein linearer Zusammenhang von $22k$ ausreicht. Dieses Ergebnis wurde von Thomas und Wollan [4] mit dem Beweis, dass $10k$-zusammenhängende Graphen k-verbunden sind, verbessert.

Im zweiten Teil dieser Dissertation zeigen wir, dass es für jede Wahl von natürlichen Zahlen k und w eine natürliche Zahl N gibt, sodass jeder $2k+3$-zusammenhängende Graph G mit Baumweite höchstens w und mindestens N Ecken k-verbunden ist. Ein zentraler Teil des Beweises ist die Untersuchung gewisser Teilgraphen von G. Wir analysieren lineare Zerlegungen dieser Teilgraphen mit weiterentwickelten Techniken aus dem ersten Teil.

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Lebenslauf

Entfällt aus datenschutzrechtlichen Gründen