Attraction properties and non-asymptotic stability of simple heteroclinic cycles and networks in $\mathbb{R}^4$

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Even when I walk through the dark valley of death, I will fear no evil, for you are close beside me.

Psalm 23,4

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Introduction

In dynamical systems heteroclinic cycles are invariant objects consisting of finitely many equilibria and trajectories connecting them in a circular fashion. They occur in various applications and are particularly useful to model stop-and-go dynamics: a trajectory near a heteroclinic cycle will spend a long time in a neighbourhood of an equilibrium, close to a steady state, before rapidly switching along a connecting trajectory towards another equilibrium, where it lingers for a long time again. Such behaviour is displayed, for instance, by the magnetic field of the Earth: its quick, aperiodic reversals of polarity are followed by long periods in an almost stationary state. This has been attributed by many authors to the possible presence of a heteroclinic cycle in the equations governing the geodynamic processes, e.g. Melbourne et al. [31] or Chossat et al. [10]. Another example occurs in population dynamics and is treated extensively by Hofbauer and Sigmund [20]: as a model for competition between three or more species, Lotka-Volterra equations can possess attracting heteroclinic cycles between steady states in which there is only one species. To an observer of the system it may look for some time as if a single species wins the competition and all others become extinct, before its density suddenly drops and its dominant position is taken by a different species. Other applications of dynamical systems theory in which heteroclinic cycles play a role include game theory, see also [20] or Aguiar and Castro [1], and neurodynamics, where Ashwin et al. [4] study cycles again for Lotka-Volterra equations, as well as in coupled-cell systems.

In arbitrary smooth dynamical systems, however, heteroclinic cycles are generally of high codimension, the saddle-saddle connections being destroyed by small perturbations of the vector field. Not until the 1980s and the famous works of dos Reis [12] and Guckenheimer and Holmes [18] was it discovered that they are structurally stable in an equivariant setting, where the vector field on the right hand side of the differential equation commutes with the action of a symmetry group. Subsequently, their study gained importance. Placing the connections in flow-invariant subspaces, where they are of saddle-sink type, and restricting to perturbations that respect these, turns heteroclinic cycles into a robust phenomenon: if the invariant subspaces persist, the heteroclinic cycle is also maintained. The existence of flow-invariant subspaces is natural in the context of symmetry. In [25] Krupa provides a comprehensive overview of early results on robust heteroclinic cycles along with plenty of examples.

This discovery of robustness sparked a huge interest in heteroclinic cycles in the 1990s. In [29] Lauterbach and Roberts discussed various cases of how spontaneous symmetry breaking may lead to the existence of heteroclinic cycles. Necessary and sufficient conditions for asymptotic stability of different types of cycles were derived by Krupa and Melbourne in [26] and [28]. However, being more complex in structure than a single hyperbolic equilibrium, heteroclinic cycles exhibit more intricate stability properties than the dichotomy between asymptotic stability and instability in the sense that everything except for a set of measure zero leaves a neighbourhood. In [30] Melbourne gives an example of a cycle that is a non-asymptotically stable attractor: it attracts a set of positive measure which is not a full neighbourhood of the cycle. This observation becomes even more important when several cycles are joined together to form a heteroclinic network. Then it is impossible for one of them to be asymptotically stable due to the connection to another cycle in a transverse direction. Nevertheless, in many cases there is a dominant cycle that is more stable than the others in the sense that it is the ω-limit set of a large measure set of points in a neighbourhood of the network – even though the other cycles may also attract sets of positive (but smaller) measure. Kirk and Silber address this question of competition between cycles in a heteroclinic network in [23]. In order to gain a broader understanding of the dynamics associated with such a network, it is crucial to accurately distinguish between various forms of non-asymptotic stability. An important one is *predominant asymptotic stability*, a term coined by Podvigina and Ashwin in [33]. A predominantly asymptotically stable (p.a.s.) set attracts everything in its neighbourhood except for a thin cusp-shaped region of points. The same concept had been used before as *essentially asymptotically stable (e.a.s.)*, however, there exist contradicting definitions in the literature: while that of Melbourne [30] is equivalent to simply attracting any set of positive measure, in [6] Brannath defines e.a.s. in the same way as p.a.s. is defined in [33]. In all previous work known to this author, when a set is claimed to be e.a.s. it is indeed shown to be p.a.s. That is why in order to avoid further confusion, we stop using the term e.a.s. after proving the equivalence mentioned above.

Recently, significant progress in the quantitative study of stability of heteroclinic cycles was made in [33]: Podvigina and Ashwin introduced a stability index that provides a measure of the basin of attraction for any compact invariant set. Since it is constant along trajectories of the flow, a finite number of these indices is sufficient to characterize the stability of a heteroclinic cycle. One central aim of this work is to establish relations between the sign of this index and the different types of non-asymptotic stability mentioned above. The most important results in this regard are theorems 1.33 and 1.34.

This thesis is divided into two major chapters, focusing on heteroclinic cycles and networks, respectively. The first section of chapter 1 sets the scene by reviewing relevant results from the literature: we give the precise definition of a heteroclinic cycle and introduce the equivariant setting in which it is a structurally stable phenomenon. In particular, we recall the division of *very simple heteroclinic cycles* into types A, B and C and discuss their characteristics in $\mathbb{R}^4$. Next, we provide an overview of the well-known results on asymptotic stability of heteroclinic cycles obtained to a large extent by Krupa and Melbourne in [26] and [28]. We also briefly look at the famous homoclinic cycle studied by Busse and Heikes in [7] as well as by Guckenheimer and Holmes in [18]. In section 1.2, the stability index from [33] is introduced along with various types of non-asymptotic stability. We then have all tools at hand to build on the previously known results: our main theorems 1.33 and 1.34 establish important relations between the stability index and non-asymptotic stability properties – under appropriate assumptions a heteroclinic cycle is predominantly asymptotically stable if and only if all local stability indices along its connecting trajectories are positive. A similar equivalence (with some restrictions) holds for negative indices and predominant instability. Along the way, we prove the equivalence of fragmentary and essential asymptotic stability in proposition 1.24, making the latter, ambiguous term superfluous. Moreover, in corollary 1.27, we confirm a conjecture made by Ashwin and Timme in [5] regarding the existence of unstable attractors: in a smooth system there are none. The third section is devoted to the explicit study of very simple heteroclinic cycles in $\mathbb{R}^4$. Combining the results from [28] and [33] mentioned above with our own observations from section 1.2, we derive necessary and sufficient conditions for predominant asymptotic stability of cycles of types B and C in terms of the eigenvalues of the linearization at the equilibria. In particular, we prove that in $\mathbb{R}^4$ an asymptotically stable cycle of type B or C generically becomes p.a.s. when a transverse eigenvalue turns positive (corollaries 1.58 and 1.64).

In chapter 2 we transfer our results to heteroclinic networks. Section 2.1 explores the general use of the stability index in this context: for each trajectory it can be computed either with respect to the whole network or with respect to any subcycle. Both quantities are crucial for an understanding of relative stability in a network and our results in lemmas 2.3 and 2.4 simplify index calculations in certain cases. Finally, in section 2.2, we classify very simple heteroclinic networks in $\mathbb{R}^4$ and investigate the competition between their subcycles in detail. We establish that only three such networks exist (proposition 2.8), two of which, to our knowledge, have not been studied before. This can be viewed as an extension and generalization of the Kirk and Silber study in [23]. We encounter several ways in which a network can be p.a.s. – with very different stability properties for the respective subcycles. Substantial parts of this section have been obtained in collaboration with Sofia Castro and are published in [8].

Our work is completed by a few comments on what we have done and some questions that may guide future work in different directions concerning the topics at hand. In a short appendix we discuss generalizations that are necessary to understand the results from [28] on asymptotic stability, but are otherwise of little relevance for this work.

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1. Stability of heteroclinic cycles

The first chapter of this work concerns stability properties of heteroclinic cycles. It is divided into three sections: In 1.1 we define heteroclinic cycles and introduce the precise setting in which they occur naturally, namely when the connecting trajectories lie in invariant subspaces that are created by equivariance of the system. We also recall classic results by Krupa and Melbourne on asymptotic stability, first proved in [26], [28] and [27]. The analysis by Guckenheimer and Holmes of the well-known cycle in [18] is discussed briefly as an introductory example.

Section 1.2 deals with the stability index defined in [33] by Podvigina and Ashwin. Much attention is devoted to relating it to various forms of non-asymptotic stability. Important aspects are illustrated by several examples.

Finally, in section 1.3 we focus on very simple heteroclinic cycles in $\mathbb{R}^4$, recalling the enumeration of type B and C cycles in [28]. Combining it with results from [33] and our theorems 1.33 and 1.34 we characterize the stability properties of type A, B and C cycles in $\mathbb{R}^4$ exhaustively.

1.1. Heteroclinic cycles

The aim of this section is to introduce the general setting and definitions that provide the framework for the rest of this thesis. In doing so we trim things in a way that best suits $\mathbb{R}^4$ and finite symmetry groups, the context in which we mainly work later. Nevertheless, important results on the asymptotic stability of heteroclinic cycles are stated in their full generality. We refer the interested reader to the appendix of this work for technical information on generalizing definitions and techniques to $\mathbb{R}^n$ and continuous symmetry groups.

The last subsection is devoted to one of the best-known heteroclinic cycles in the literature. It arises in the context of convection in a rotating layer of fluid and was first mentioned in [7] by Busse and Heikes. The authors of [18] used this example to help establish heteroclinic cycles as structurally stable objects in the context of symmetry.

1.1.1. Definitions and Motivation

We consider a smooth dynamical system in $\mathbb{R}^n$ given by the ordinary differential equation

$$\dot{x} = f(x), \tag{1.1}$$

and we denote by $\phi_t(x)$ the associated flow.

Definition 1.1 (e.g. [33])**.** Let $\xi_1, \ldots, \xi_m$ be mutually distinct hyperbolic equilibria for system (1.1). Let $s_j \subset W^u(\xi_{j-1}) \cap W^s(\xi_j)$ be connecting trajectories, where we set $\xi_0 := \xi_m$. Then the set of connecting trajectories and equilibria is called a *heteroclinic cycle*.

In general, a heteroclinic cycle disappears when f is only slightly perturbed, the saddle-saddle-connections being structurally unstable. Suppose, however, that within some flow-invariant subspace the connecting trajectories are of saddle-sink type. Then these persist under perturbations which respect the invariance of the subspaces, hence the cycle is preserved as well. This turns heteroclinic cycles from a phenomenon of high codimension into a possibly structurally stable one, reinforcing the relevance of understanding the dynamics associated with them.

It is well-known that equivariance of (1.1) naturally leads to the existence of invariant subspaces. So assume that f is equivariant under the action of a finite group $\Gamma \subset O(n)$, i.e. $\gamma f(x) = f(\gamma x)$ for all $x \in \mathbb{R}^n$ and all $\gamma \in \Gamma$. In this situation we recall the following definitions.

Definition 1.2 (e.g. [11], chapter 2.1)**.** For a group $\Gamma \subset O(n)$ and $x \in \mathbb{R}^n$ the subgroup $\Sigma_x := \{\gamma \in \Gamma \mid \gamma x = x\} \subset \Gamma$ is called the *isotropy subgroup* of x.

If $\Sigma \subset \Gamma$ is a subgroup, then Fix$(\Sigma) := \{x \in \mathbb{R}^n \mid \forall \sigma \in \Sigma \quad \sigma x = x\}$ is called the *fixed-point subspace* of Σ.

Clearly, $\Sigma_{\gamma x} = \gamma^{-1}\Sigma_x\gamma$ for $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$, establishing that isotropy subgroups are conjugated along group orbits. Moreover, $\Sigma_{\phi_t(x)} = \Sigma_x$, so they are constant along trajectories of the flow.

Now assume that for each j there is a subgroup $\Sigma_j \subset \Gamma$ such that $s_{j+1} \subset P_j = \text{Fix}(\Sigma_j)$ and ξ_{j+1} is a sink in P_j. Set $L_j := P_{j-1} \cap P_j$. These fixed-point subspaces persist under Γ-equivariant perturbations of f and within them the s_j are structurally stable saddle-sink-connections, so the cycle is preserved, too. This is why such a cycle is called *robust*.

Remark 1.3. In the symmetric context one usually identifies equilibria that lie on the same group orbit. This means in definition 1.1 the group orbits $\Gamma\xi_j$ are assumed to be distinct. If there is a connection $[\xi_0 \to \gamma\xi_0]$ for some $\gamma \in \Gamma$, the cycle $[\xi_0 \to \gamma\xi_0 \to \cdots \to \xi_0]$ is called *homoclinic*.

As we will see later, stability of a heteroclinic cycle generically depends only on the eigenvalues of the linearization of f at the equilibria making up the cycle. We therefore follow the well-established convention of labeling them as follows.

Definition 1.4 (e.g. [26])**.** Depending on the geometry of their eigenspaces, the eigenvalues of $df(\xi_j)$ are divided into four classes as listed in table 1.1.

Note that for an equilibrium in a heteroclinic cycle the sets of radial, contracting and expanding eigenvalues are never empty. If there are no transverse eigenvalues, we set $t_j = -\infty$. By $-c_j$, e_j and t_j the weakest contracting, strongest expanding and most unstable transverse eigenvalue are selected, respectively. The radial eigenvalues do not play a role for the

Table 1.1.: Types of eigenvalues

Eigenvalues	Description	max. real part
radial	eigenspace in L_j	$-r_j < 0$
contracting	nonradial, eigenspace in P_{j-1}	$-c_j < 0$
expanding	nonradial, eigenspace in P_j	$e_j > 0$
transverse	eigenspace not in $P_{j-1} \cup P_j$	$t_j \lesseqgtr 0$

(asymptotic and non-asymptotic) stability of the cycle, as we will see in subsection 1.1.2 and later in section 1.3.

We thus consider as the basic equation that underlies our investigations

$$\dot{x} = f(x), \text{ with } \gamma f(x) = f(\gamma x) \quad \forall x \in \mathbb{R}^n \quad \forall \gamma \in \Gamma, \tag{1.2}$$

where $\Gamma \subset O(n)$ is a finite group.

During most of this work we consider heteroclinic cycles in $\mathbb{R}^4$ with a particularly intuitive structure. Note that in [32] Podvigina extends the following definition of simple cycles to $\mathbb{R}^n$, relaxing the first condition by demanding merely that there is only one contracting direction at each equilibrium. We stick with the classical definition for $\mathbb{R}^4$ from [28].

Definition 1.5 ([28])**.** In a setting as above, a robust heteroclinic cycle $X \subset \mathbb{R}^4 \setminus \{0\}$ is called *simple* if for all j

- $\dim(P_j) = 2$,
- X intersects each connected component of $L_j \setminus \{0\}$ in at most one point.

It is called *very simple* if, in addition, for all j the eigenvalues of $df(\xi_j)$ are distinct.

Note that in [28] there is no distinction between simple and very simple cycles. As Podvigina and Chossat point out in [34], the authors seem to silently assume that there are no double eigenvalues.

In the case of a very simple heteroclinic cycle in $\mathbb{R}^4$ the fixed-point space L_j is one-dimensional for every j and there is a unique (real) eigenvalue of each type. Denote the contracting eigenspace (the orthogonal complement of L_j in P_{j-1}) by $V_j(c) := P_{j-1} \ominus L_j$, the expanding one by $V_j(e) := P_j \ominus L_j$ and the transverse one by $V_j(t) := (P_{j-1} + P_j)^\perp$. Then

$$\mathbb{R}^4 = L_j \oplus V_j(c) \oplus V_j(e) \oplus V_j(t)$$

is the isotypic decomposition of $\mathbb{R}^4$ under the isotropy group T_j of points in $L_j \setminus \{0\}$. Elements of T_j map isotypic components into themselves, so they are diagonal matrices. Since $L_j = \text{Fix}(T_j)$ and $T_j \subset O(4)$ this implies $T_j \subset \mathbb{Z}_2^3$, where $\mathbb{Z}_2^3$ consists of diagonal matrices with entries $\{1, \pm 1, \pm 1, \pm 1\}$. Since $\Sigma_j \subset T_j$ this leaves us with only two possibilities that are compatible with the requirements that $\dim(\text{Fix}(T_j)) = \dim(L_j) = 1$ and

$\dim(\text{Fix}(\Sigma_j)) = \dim(P_j) = 2$, namely

$$T_j \cong \mathbb{Z}_2^3 \text{ and } \Sigma_j \cong \mathbb{Z}_2^2, \tag{1.3}$$

$$T_j \cong \mathbb{Z}_2^2 \text{ and } \Sigma_j \cong \mathbb{Z}_2. \tag{1.4}$$

The same goes for $\Sigma_{j-1} \subset T_j$ and thus we have either $\Sigma_j \cong \mathbb{Z}_2^2$ or $\Sigma_j \cong \mathbb{Z}_2$ along all connecting trajectories. This observation prompts us to divide the set of very simple heteroclinic cycles in $\mathbb{R}^4$ into three types as was first done by [9] for homoclinic cycles. The definition was reformulated and extended to a more general framework in [28].

Definition 1.6 ([28]). A heteroclinic cycle $X \subset \mathbb{R}^4 \setminus \{0\}$ is of

- *type* A if and only if $\Sigma_j \cong \mathbb{Z}_2$ for all j,
- *type* B if and only if X lies in a three-dimensional fixed-point subspace $Q \subset \mathbb{R}^4$,
- *type* C if and only if it is not of type A or B.

Clearly, every very simple heteroclinic cycle in $\mathbb{R}^4$ belongs to one of those types. Moreover, they are mutually exclusive: X being of type B implies the existence of a subgroup $\Sigma^* \subset \Gamma$ such that

$$\text{Fix}(\Sigma_j) = P_j \subsetneq Q = \text{Fix}(\Sigma^*),$$

which means that $\Sigma^* \subsetneq \Sigma_j$ is a non-trivial subgroup. But that is not compatible with $\Sigma_j \cong \mathbb{Z}_2$, so the cycle is not of type A.

In [28] there is an alternative way of characterizing type A cycles in $\mathbb{R}^4$.

Lemma 1.7 ([28]). *A very simple heteroclinic cycle $X \subset \mathbb{R}^4 \setminus \{0\}$ is of type A if and only if there is no element $\gamma \in \Gamma$ that acts as a reflection on $\mathbb{R}^4$.*

Proof. See corollary 3.5 in [28], we briefly recall the important steps. First, note that the subgroups $T_i \subset \Gamma$ contain reflections if and only if the cycle is not of type A.

Now let τ be a reflection and look at $E := \text{Fix}(\tau) \cap P_1$. This is either a line or a plane. If it is a plane, then τ fixes all points in P_1, so $\tau \in \Sigma_1 \subset T_1$ and thus T_1 contains a reflection. In [28] it is shown that, if E is a line, it is conjugate to L_1 or L_2, since up to conjugation these are the only invariant lines in P_1. But then $\tau \in T_1$ or $\tau \in T_2$, and thus one of the T_i contains the reflection τ. So in either case, the cycle is not of type A. $\square$

1.1.2. Asymptotic stability of cycles

Since the mid 1990s the question of asymptotic stability of heteroclinic cycles has been studied extensively by several authors, most notably Ian Melbourne and Martin Krupa. In this subsection we review their most prominent results in this field – references are given below, where the theorems are stated. These are not restricted to simple cycles in $\mathbb{R}^4$, but hold for general robust heteroclinic cycles in $\mathbb{R}^n$. The classification into types A, B and C can be extended to this setting as discussed in appendix A.1. The same goes for the generalization from finite to continuous symmetry groups, we briefly comment on this in appendix A.2. For all following theorems we place the heteroclinic cycle within a system equivariant under the action of some compact Lie group Γ. By $N(\Sigma_j)$ we denote the normalizer of Σ_j in Γ.

Throughout this subsection we assume all transverse eigenvalues to be negative unless explicitly stated otherwise. This condition is clearly necessary for asymptotic stability and we only loosen it when turning our attention to non-asymptotic stability.

Proofs of the theorems we state can be found in the works of Krupa and Melbourne specified below. We do not reproduce them here, but briefly explain the general idea of how they are accomplished, because the concepts are important throughout this work. A basic tool to investigate stability of a heteroclinic cycle are *return maps* (also called *Poincaré maps*) defined on local cross sections transverse to the flow near the equilibria. For ξ_j we denote a section across the incoming trajectory from ξ_{j-1} by $H_j^{\text{in},j-1}$ and one across the outgoing trajectory to ξ_{j+1} by $H_j^{\text{out},j+1}$. Without loss of generality the linearized equation in local coordinates near ξ_j for a cycle in $\mathbb{R}^4$ reads:

$$\begin{cases} \dot{x}_1 = -r_j x_1 \\ \dot{x}_2 = -c_j x_2 \\ \dot{x}_3 = \quad e_j x_3 \\ \dot{x}_4 = t_j x_4 \end{cases}$$

Then the *local map* approximating the flow in a neighbourhood of ξ_j is given by:

$$\phi_j : H_j^{\text{in},j-1} \to H_j^{\text{out},j+1}, \quad (x_1,1,x_3,x_4) \mapsto \left(x_1 x_3^{\frac{r_j}{e_j}}, x_3^{\frac{c_j}{e_j}}, 1, x_4 x_3^{-\frac{t_j}{e_j}} \right)$$

Furthermore, *connecting diffeomorphisms* or *global maps* $\psi_j : H_j^{\text{out},j+1} \to H_{j+1}^{\text{in},j}$ are constructed. Due to the equivariance of the vector field these map the radial and expanding directions near ξ_j to the radial and contracting directions near ξ_{j+1}. Then compose $g^j := \psi_j \circ \phi_j$ to finally obtain return maps

$$g_j : H_j^{\text{in},j-1} \to H_j^{\text{in},j-1}, \quad g_j := g^{j-1} \circ g^{j-1} \circ \cdots \circ g^1 \circ g^m \circ \cdots \circ g^{j+1} \circ g^j$$

for each j. Their domains of definition may be restricted through cusps shaped by ratios of the respective eigenvalues. This is a standard construction and more details can be found in [26] and [28], for instance. There it is also shown that under certain circumstances only two components of the return maps are relevant for the stability of a cycle, even in $\mathbb{R}^n$ (where the construction of the return maps works analogously). With a logarithmic coordinate change the local maps ϕ_j reduce to two-dimensional linear maps represented by *transition matrices* M_j, the entries of which depend on the eigenvalues of $df(\xi_j)$. The return map g_j is then given by the product of the transition matrices, which we may refer to as a transition matrix as well, when no confusion is possible. This concept was first used in [16].

The conditions in the theorems below can be derived by studying transition matrices for the different types of cycles. We begin with necessary and sufficient conditions for cycles of type A.

Theorem 1.8 ([26]). *Let $X \subset \mathbb{R}^n$ be a robust heteroclinic cycle of type A. Suppose that* $\dim(W^u(\xi_j)) = \dim(N(\Sigma_j)/\Sigma_j) + 1$ *for all j. Then generically X is asymptotically stable if and only if*

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j. \tag{1.5}$$

Proof. See [26], theorem 3.1. □

Note that for finite symmetry groups the condition on the dimension of the unstable manifold of ξ_j reduces to $\dim(W^u(\xi_j)) = 1$, just as we demand for simple cycles in $\mathbb{R}^4$ with negative transverse eigenvalues.

There is a similar result for cycles of type B.

Theorem 1.9 ([28]). *For a robust heteroclinic cycle $X \subset \mathbb{R}^n$ of type B a sufficient condition for asymptotic stability is given by*

$$\prod_{j=1}^{m} c_j > \prod_{j=1}^{m} e_j. \tag{1.6}$$

If in addition $\dim(W^u(\xi_j)) = \dim(N(\Sigma_j)/\Sigma_j) + 1$ *for all j and there exists a Σ_j-isotypic component $\widetilde{Q}_j$ such that the eigenpaces corresponding to c_j and e_{j+1} lie in $\widetilde{Q}_j$, then, generically, condition (1.6) is necessary and sufficient for asymptotic stability of X.*

Proof. See [28], theorems 5.2 and 5.3. □

This result is very intuitive when restricting it to $\mathbb{R}^4$. There, a cycle of type B is contained in a three-dimensional fixed-point space $Q \subset \mathbb{R}^4$. Since all transverse eigenvalues are negative, stability of the cycle is determined by its stability within Q. Restricting to Q, however, we have $\Sigma_j \cong \mathbb{Z}_2$ and the cycle is therefore of type A (within Q). So the previous theorem applies, and without transverse eigenvalues condition (1.5) becomes (1.6).

The condition for type C cycles is of a slightly different nature and is formulated in terms of the transition matrices introduced above. Here we have to distinguish between connections of different types: a C-cycle may have some connections of type B, however, there must be at least one type C connection. Details on this can be found in appendix A.1. Despite this lack of precision in terminology, the essence of the following theorem becomes clear at this point.

Theorem 1.10 ([28]). *For a robust heteroclinic type C cycle $X \subset \mathbb{R}^n$ a sufficient condition for asymptotic stability is given by*

$$\mathrm{tr}(M) > \min(2,\ 1 + \det(M)). \tag{1.7}$$

Here $M = M_m \dots M_2 M_1$ is the product of the transition matrices, i.e. depending on whether the j-th connection is of type B or C set

$$M_j := \begin{pmatrix} c_j/t_j & 0 \\ -t_j/e_j & 1 \end{pmatrix} \qquad \textit{or} \qquad M_j := \begin{pmatrix} -t_j/e_j & 1 \\ c_j/e_j & 0 \end{pmatrix}.$$

Suppose in addition that for each j we have $\dim(W^u(\xi_j)) = \dim(N(\Sigma_j)/\Sigma_j) + 1$. *Moreover, assume that for all j there exist Σ_j-isotypic components $\widetilde{Q}_j \subset Q_j := P_j \oplus V_j(c)$ and $\widetilde{R}_j \subset R_j := P_j \oplus V_j(t)$ such that*

(i) for a type B connection the eigenvectors corresponding to c_j, e_{j+1} lie in $\widetilde{Q}_j$ and those for t_j, t_{j+1} in $\widetilde{R}_j$

(ii) for a type C connection the eigenvectors corresponding to c_j, t_{j+1} lie in $\widetilde{Q}_j$ and those for t_j, e_{j+1} in $\widetilde{R}_j$.

Then, generically, condition (1.7) is necessary and sufficient for asymptotic stability of X.

Proof. See [28], theorems 5.7 and 5.8. □

With this we have a comprehensive overview of asymptotic stability for robust heteroclinic cycles and see that it is understood to a fairly satisfactory extent. We conclude this subsection with a short excursion into the realm of non-asymptotic stability, even though not all of the necessary terminology has been introduced yet.

The following theorem is also by Krupa and Melbourne ([27]) and gives information on non-asymptotic stability for a subset of type A cycles which we label type A^*. In $\mathbb{R}^4$ this emcompasses all type A cycles. The reader is again referred to appendix A.1 for a definition of type A^*. The terms *predominantly (un)stable* have also not been defined at this point. For now they may simply be understood intuitively, which is to say that predominantly stable means “a bit less stable than asymptotically stable”, while predominantly unstable means “a bit more stable than completely unstable”. This is made precise in definitions 1.14 and 1.15.

Following notation in [27] we set

$$\rho := \rho_1 \cdot ... \cdot \rho_m, \quad \text{where} \quad \rho_j := \min\left(\frac{c_j}{e_j},\ 1 - \frac{t_j}{e_j}\right).$$

Theorem 1.11 ([27])**.** *Suppose X is a heteroclinic cycle of type A^*. Then generically one of the following holds.*

(a) X is asymptotically stable ($\rho > 1$, $t_j < 0$ for all j).

(b) X is predominantly asymptotically stable ($\rho > 1$, $t_j < e_j$ for all j, $t_j > 0$ for some j).

(c) X is predominantly unstable ($\rho > 1$, $t_j > e_j$ for some j).

(d) X is completely unstable ($\rho < 1$).

Proof. See [27], theorem 2.4. □

To this author’s knowledge there are no similar results for cycles of type B and C. In section 1.3 we fill this gap at least for cycles in $\mathbb{R}^4$.

1.1.3. The Busse-Heikes-Guckenheimer-Holmes cycle

In this subsection we briefly discuss one of the most famous heteroclinic cycles. The system in which it arises was first discussed by Busse and Heikes in [7], in the context of convection in a rotating layer of fluid heated from below. Later, in [18], Guckenheimer and Holmes investigated existence and stability of the cycle in detail. Moreover, this was the first time a heteroclinic cycle was shown to be structurally stable, existing for an open set of equivariant vector fields.

We denote the cycle by $X \subset \mathbb{R}^3$. It consists of six equilibria on the coordinate axes and there are eight subcycles X_i, $i = 1,...,8$, one in each (closed) octant of $\mathbb{R}^3$. All equilibria are related by symmetry, which means according to remark 1.3 X is a homoclinic cycle (and not a proper heteroclinic network). The equilibria are connected by a total of twelve trajectories lying in the coordinate planes. Restricted to an octant the cycle looks basically like the one in figure 1.1.

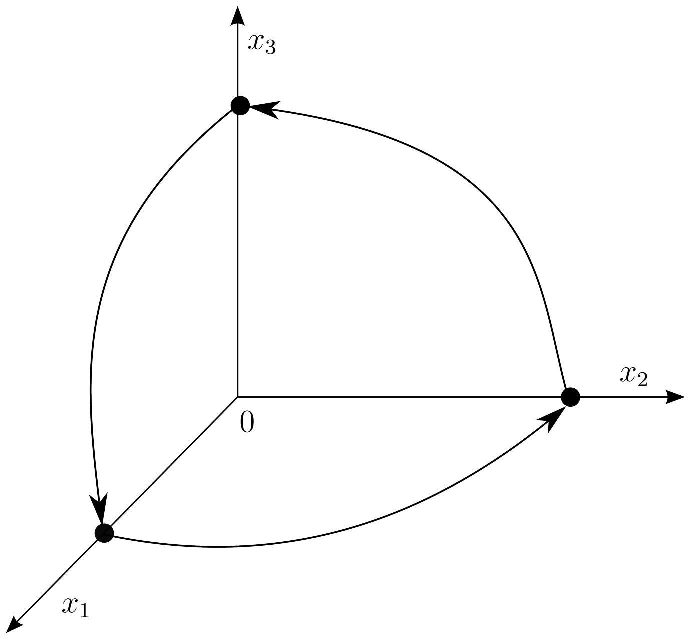


Figure 1.1.: A heteroclinic cycle with three equilibria, the P_i are the coordinate planes

We take a closer look at the setting in which X arises, following [18]. The authors consider a vector field f given by the right hand side of:

$$\begin{cases} \dot{x} = x(l + ax^2 + by^2 + cz^2) \\ \dot{y} = y(l + ay^2 + bz^2 + cx^2) \\ \dot{z} = z(l + az^2 + bx^2 + cy^2) \end{cases}$$

Rescaling the coordinates allows us to assume $|l| = 1$ and $|a + b + c| = 1$, without loss of generality. All coordinate axes and planes are flow-invariant, the same goes for the lines $x = \pm y = \pm z$. The vector field is equivariant under the action of the 24-element group $G \subset O(3)$ that is generated by

$$r_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The authors of [18] focus on the case where

$$(1)\ l = 1, \quad (2)\ a + b + c = -1, \quad (3)\ c < a < b < 0, \quad (4)\ -\frac{1}{3} < a. \qquad (1.8)$$

Besides the origin there are two equilibria on each coordinate axis, located at $\pm a^* := \pm 1/\sqrt{-a}$. They all belong to the same G-orbit. Condition (3) guarantees that there are no other equilibria in any of the coordinate planes. Due to condition (2) there are equilibria on the diagonals at $(\pm 1, \pm 1, \pm 1)$. The origin is unstable by condition (1), since $df(0) = \mathbb{1}$. For stability of the nontrivial equilibria on the axes consider

$$df(\pm a^*,0,0) = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 - \frac{c}{a} & 0 \\ 0 & 0 & 1 - \frac{b}{a} \end{pmatrix}.$$

Also by condition (3), this implies that $(\pm a^*,0,0)$ is locally attracting in the x-y-plane and repelling in the z-direction. The calculations for the other equilibria are similar, eventually leading to the existence of an asymptotically stable cycle

$$[(a^*,0,0) \to (0,0,a^*) \to (0,a^*,0) \to (a^*,0,0)].$$

Still following the reasoning in [18] we claim that in each octant of $\mathbb{R}^3$ all trajectories off the lines $x = \pm y = \pm z$ converge to the subcycle lying in the respective parts of the coordinate planes bounding the octant. To prove this, consider the functions

$$G(x,y,z) = x^2 + y^2 + z^2 - 3 \quad \text{and} \quad F(x,y,z) = xyz.$$

Using the Cauchy-Schwarz inequality in the second to last step, it is easy to verify that

$$\begin{aligned} \tfrac{1}{2}\langle \nabla G, f\rangle &= x^2(1 + ax^2 + by^2 + cz^2) + y^2(1 + ay^2 + bz^2 + cx^2) \\ &\quad + z^2(1 + az^2 + bx^2 + cy^2) \\ &= (x^2 + y^2 + z^2) + a(x^4 + y^4 + z^4) + (b + c)(x^2y^2 + x^2z^2 + y^2z^2) \\ &= (G + 3) + a(G + 3)^2 - (1 + 3a)(x^2y^2 + x^2z^2 + y^2z^2) \\ &\geq (G + 3)(1 + a(G + 3)) - \left(\tfrac{1}{3} + a\right)(G + 3)^2 \\ &= -\tfrac{1}{3}G(G + 3). \end{aligned}$$

Thus, G increases along non-zero trajectories $\mathbf{x}(t) = (x(t),y(t),z(t))$ inside the ball bounded by $G \equiv 0$, where $G(x,y,z) \in (-3,0)$, since

$$\frac{d}{dt}G(\mathbf{x}(t)) = \langle \nabla G, \dot{\mathbf{x}}\rangle = \langle \nabla G, f\rangle = -\tfrac{2}{3}G(G + 3) > 0.$$

Trajectories intersect the boundary of this ball transversely, with the obvious exception of the equilibria $(\pm 1, \pm 1, \pm 1)$, because the Cauchy-Schwarz inequality is strict for linearly independent vectors. Also, observe that

$$(G + 3)(1 + a(G + 3)) < 0 \quad \text{if} \quad G + 3 > -\frac{1}{a},$$

so G decreases along trajectories outside the ball with radius a^*. Thus, all trajectories are attracted to the spherical shell A that is bounded by the spheres of radius $\sqrt{3}$ and a^*, respectively. Within A, we have

$$\frac{d}{dt}F(\mathbf{x}(t)) = \langle \nabla F, f \rangle = xyz\left(3 + (a+b+c)(x^2+y^2+z^2)\right) = -FG,$$

which has the opposite sign of F and equals zero if and only if $F = 0$ or $G = 0$. Thus, all trajectories within A, except for the invariant lines $x = \pm y = \pm z$, approach the surface $F = 0$, which is the union of the coordinate planes.

Within the planes there are no other equilibria and also no periodic orbits, so by a standard Poincaré-Bendixson argument all trajectories converge to the respective equilibrium on the axes. Existence of the cycle follows from the fact that in each plane one of the two equilibria is a saddle and the other is a sink. Embedded canonically in $\mathbb{R}^4$ (extending the symmetry of the system to include a reflection in the additional space dimension) it is of type B since $X \subset \mathbb{R}^3 = \text{Fix}(G)$.

Note that the sufficient condition for asymptotic stability of type B cycles in theorem 1.9 is satisfied since with $m = 1$ we get

$$\prod_{j=1}^{m} c_j > \prod_{j=1}^{m} e_j \quad \Leftrightarrow \quad -\left(1 - \frac{c}{a}\right) > 1 - \frac{b}{a} \quad \Leftrightarrow \quad 0 < 2a - b - c = 1 + 3a.$$

Here we used condition (2) and the inequality $0 < 1+3a$ is satisfied due to (4) from equations (1.8) above. We revisit this cycle in example 1.38 to illustrate the calculation of stability indices in a simple case. For this, keep in mind that we have shown the following: in a transverse section across any connecting trajectory of the cycle all points on one side of the coordinate plane in which the trajectory is contained are attracted to a different subcycle than the ones on the other side.

1.2. Non-asymptotic stability and attraction

We have seen that asymptotic stability is well understood for a large class of heteroclinic cycles. However, in many cases this is not satisfactory, because a heteroclinic cycle may attract a set of large measure, that is not a full neighbourhood, and therefore dominate a dynamical system even though it is not asymptotically stable. Ian Melbourne, see [30], was the first to give an explicit example of such an attractor, prompting him to establish the notion of essential asymptotic stability. In this section we discuss various forms of non-asymptotic stability that have subsequently been developed and introduce the stability index from [33] as a means of quantifying them.

There are three main achievements in this section: we begin by reviewing the existing notions of stability for general compact sets $X \subset \mathbb{R}^n$ and clean up the list of definitions by proving the equivalence of fragmentary and essential asymptotic stability in proposition

1.24. Then, most importantly, theorems 1.33 and 1.34 connect these attraction properties to the stability index. The third point worth noting is that along the way to these results we confirm a conjecture made in [5] regarding the existence of unstable attractors in smooth systems.

1.2.1. Notions of stability

Consider a compact subset $X \subset \mathbb{R}^n$ that is invariant under the flow $\phi_t(.)$ generated by equation (1.2). Following definitions in [33] let $\mathcal{B}(X)$ and $\mathcal{B}_\varepsilon(X)$ denote its (ε-local) basin of attraction, i.e.

$$\mathcal{B}(X) := \{x \in \mathbb{R}^n \mid \omega(x) \subset X\}$$

and for $\varepsilon > 0$

$$\mathcal{B}_\varepsilon(X) := \{x \in B_\varepsilon(X) \mid \omega(x) \subset X \ \wedge \ \phi_t(x) \in B_\varepsilon(X) \ \forall t > 0\}.$$

Here $B_\varepsilon(X)$ is an open ε-neighbourhood of X. In the following, a neighbourhood is always considered to be open unless stated otherwise. $\ell(.)$ denotes n-dimensional Lebesgue measure. When referring to m-dimensional Lebesgue measure with $m \neq n$ we denote this explicitly by $\ell_m(.)$. We say that X is an *attractor* if it attracts a set of positive measure, i.e. if $\ell(\mathcal{B}(X)) > 0$.

We now introduce different notions of stability. The classic distinction between only asymptotic stability and (complete) instability is not sophisticated enough to describe the dynamics associated with heteroclinic cycles. This is particularly true for cycles that are part of a network: within a network no single cycle can be asymptotically stable, as there is always an equilibrium with an unstable direction belonging to another cycle. Yet there may still be a dominant cycle that is observed for a large proportion of initial states, making it "more stable" than the other cycles. Here we lay the foundations for investigating such structures in chapter 2.

As mentioned above Melbourne came up with a definition for an intermediate type of stability and an example of a heteroclinic cycle that possesses this property in [30]. We recall his definition here.

Definition 1.12 ([30])**.** X is called *essentially asymptotically stable (e.a.s.)* if there is a set $D \subset \mathbb{R}^n$, so that for any neighbourhood U of X and any $\varepsilon > 0$ there is a neighbourhood V of X such that

(a) for $x \in V \setminus D$ we have $\phi_t(x) \in U$ for all $t > 0$, as well as $\omega(x) \subset X$, and

(b) $\dfrac{\ell(V \setminus D)}{\ell(V)} > 1 - \varepsilon.$

An alternative approach to describing a form of weak attractiveness is made by Podvigina in [32].

Definition 1.13 ([32])**.** If $\ell(\mathcal{B}_\varepsilon(X)) > 0$ for all $\varepsilon > 0$, then X is called *fragmentarily asymptotically stable (f.a.s.)*.

Clearly, if X is e.a.s., then it is f.a.s., and X being f.a.s. implies that it is an attractor. In propositions 1.22 and 1.24 we prove that these three properties are in fact equivalent. To include examples where the basin of X is non-measurable but X is still an attractor we may widen definition 1.13 to the case where $\mathcal{B}(X)$ simply contains a set of positive measure. However, in the common examples $\mathcal{B}(X)$ is measurable anyway.

The term e.a.s. has led to some confusion since various authors have used it in slightly different interpretations. Already in [30] the cycle is actually shown to be predominantly asymptotically stable, a much stronger attraction property than given by definition 1.12, which we introduce below. The same is true for [23]. After showing that e.a.s. and f.a.s. are indeed equivalent we rather use the latter in the rest of this work. In order to precisely distinguish between different levels of (in)stability we recall/introduce the following definitions.

Definition 1.14. (Stability)

- (e.g. [19]). X is called *asymptotically stable* if for any neighbourhood U of X there is a neighbourhood V of X such that for all $x \in V$ we have $\phi_t(x) \in U$ for all $t > 0$ and $\omega(x) \subset X$.

- ([33]). X is called *asymptotically stable relative to (a.s.r.t.)* a set $N \subset \mathbb{R}^n$ if $X \subset \overline{N}$ and for any neighbourhood U of X there is a neighbourhood V such that for all $x \in V \cap N$ we have $\phi_t(x) \in U$ for all $t > 0$ and $\omega(x) \subset X$.

- ([33]). X is called *predominantly asymptotically stable (p.a.s.)* if it is asymptotically stable relative to some $N \subset \mathbb{R}^n$ with the property that

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1.$$

We usually use the strongest property a set possesses when talking about its stability, i.e. when we say that X is p.a.s. we implicitly mean that it is not asymptotically stable.

Definition 1.15. (Instability)

- ([27]). X is called *completely unstable (c.u.)* if there is a neighbourhood U of X and a set D with $\ell(D) = 0$, such that for all $x \in U \setminus D$ there is $t_0 > 0$ such that $\phi_{t_0}(x) \notin U$.

- X is called *completely unstable relative to (c.u.r.t.)* a set $N \subset \mathbb{R}^n$ if $X \subset \overline{N}$ and there is a neighbourhood U such that for all $x \in U \cap N$ there is $t_0 > 0$ such that $\phi_{t_0}(x) \notin U$.

- X is called *predominantly unstable (p.u.)* if it is completely unstable relative to some $N \subset \mathbb{R}^n$ with the property that

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1.$$

A fragmentarily asymptotically stable X can satisfy any of the definitions above except for complete instability (we prove that this is indeed impossible in corollary 1.27). To distinguish between truly fragmentarily asymptotically stable sets and those for which we can actually make a stronger statement we make the following definition.

Definition 1.16. X is called *properly fragmentarily asymptotically stable (p.f.a.s.)* if it is f.a.s. but neither p.a.s. nor p.u.

This provides us with the terminology to accurately describe stability properties of heteroclinic cycles and networks, in particular in the context of competition between cycles within a network in section 2.2. The most important type of stability we will be looking for is predominant asymptotic stability. A cycle with this property is likely to be visible in numerical simulations of the system. In contrast to that, for a predominantly unstable cycle usually no connection is visible.

Remark 1.17. For isolated invariant sets X the following three are equivalent:

(a) X is completely unstable

(b) $\ell(\mathcal{B}(X)) = 0$

(c) $\exists \delta > 0$ such that $\ell(\mathcal{B}_\delta(X)) = 0$

A set satisfying condition (c) is called *unstable* in [32] – as opposed to f.a.s. We have formulated the definition of completely unstable as a counterpart to asymptotic stability. The equivalence of (a), (b) and (c) follows from proposition 1.22. Isolation of X is required for (b) $\Rightarrow$ (a).

In [27] yet another concept of stability is introduced. We list it for the sake of completeness and to explain why the definitions above are better suited to our purposes.

Definition 1.18 ([27])**.** X is called *almost completely unstable (a.c.u.)* if there is a set D and a neighbourhood U of X such that for all $\varepsilon > 0$ there is a neighbourhood V of X such that

(a) for all $x \in V \setminus D$ there is $t_0 > 0$ such that $\phi_{t_0}(x) \notin U$ and

(b) $\dfrac{\ell(V \setminus D)}{\ell(V)} > 1 - \varepsilon$.

However, this does not give an adequate description of the stability properties of X. Any set X with a local basin of attraction that – for any $\varepsilon > 0$ – has positive measure but is not a full neighbourhood, not even up to a set of measure zero, is e.a.s. and a.c.u. at the same time. There are many such sets in section 2.2. To see why they are e.a.s., take the complement of $\mathcal{B}(X)$ as D in definition 1.12. Then for a given U and $\varepsilon > 0$ we may choose a sufficiently small asymmetric neighbourhood V that contains a subset of $\mathcal{B}(X)$ so large that condition (b) holds. Thus, X is e.a.s. It follows in the same way that it is also a.c.u.

In [27] the authors actually prove predominant instability when they talk about almost complete instability. So the former is what they seem to have in mind, which is why we abandon the concept of e.a.s./a.c.u. and replace it by the mutually exclusive p.a.s./p.u.

We conclude this subsection by proving the equivalence of e.a.s. and f.a.s., starting with some preparatory results.

Remark 1.19. Suppose X is a.s.r.t. some $N \subset \mathbb{R}^n$. Then by definition it is clear that for all $\delta > 0$ there is $\varepsilon > 0$ such that for all $x \in B_\varepsilon(X) \cap N$ we have $\phi_t(x) \in B_\delta(X)$ for all $t > 0$ and $\omega(x) \subset X$. In short this means that

$$\forall \delta > 0 \ \exists \varepsilon > 0 : \quad B_\varepsilon(X) \cap N \subset \mathcal{B}_\delta(X).$$

Lemma 1.20. *X is asymptotically stable for ϕ if and only if it is completely unstable for ϕ^{-1} with $D = X$.*

Proof. First, let X be asymptotically stable for ϕ and suppose the claim is not true. This means that for any neighbourhood U of X there exists an $x \in U \setminus X$ such that for all $t > 0$ we have $\phi_t^{-1}(x) = \phi_{-t}(x) \in U$. Thus, this trajectory stays in U forever, backwards and without loss of generality also forwards in time. Since X is compact, this means $\alpha(x) \neq \emptyset$. We have $\alpha(x) \cap X = \emptyset$ because X is asymptotically stable. But clearly, $\alpha(x) \subset U$, and this reasoning applies for any neighbourhood U. This means there is a ϕ-invariant set other than X itself in any neighbourhood of X and that contradicts the asymptotic stability of X.

Now we prove the converse. Assume X is c.u. for ϕ^{-1} with $D = X$ and call the respective neighbourhood U_1. Suppose it is not asymptotically stable for ϕ. Then there exists another neighbourhood U_2 of X such that for any neighbourhood W there is an $x \in W$ such that $\omega(x) \not\subset X$ or there is $t_0 > 0$ with $\phi_{t_0}(x) \notin U_2$. Set $U := U_1 \cap U_2$. Then any trajectory starting in $U \setminus X$ leaves U backwards in time by complete instability of X. This means U does not contain invariant sets other than X. Because X is not asymptotically stable, in any neighbourhood W we find $x \in W$ that also leaves U forwards in time. If this was not the case there would be $x \in W$ with $\omega(x) \not\subset X$ but $\omega(x) \subset U$ which contradicts the complete instability of X.

Now consider another neighbourhood $V \subsetneq U$ of X such that $\overline{V} \cap B_\varepsilon(\partial U) = \emptyset$ for some $\varepsilon > 0$. By the above, for all $n \in \mathbb{N}$ there is $x_n \in B_{\frac{1}{n}}(X) \cap V \setminus X$ that leaves U forwards and backwards in time. Let t_n be the smallest positive real number such that $\phi_{t_n}(x_n) \in \partial V$ and the positive trajectory through $\phi_{t_n}(x_n)$ does not return into V before it reaches $B_\varepsilon(\partial U)$. We can assume that the sequence $\phi_{t_n}(x_n)$ converges to some $x^+ \in \partial V$. The trajectory through x^+ leaves U backwards in finite time, so there exists $s > 0$ such that $\phi_{-s}(x^+) \in \partial U$. In particular, for all n sufficiently large, $\phi_{t_n - s}(x_n) \in B_\varepsilon(\partial U)$ by continuity of the flow. Assume that $t_n - s > 0$. This means there must be $r_n < t_n - s$ such that $\phi_{r_n}(x_n) \in \partial V$ and the trajectory through $\phi_{r_n}(x_n)$ does not reenter into V before reaching ∂U. This contradicts the definition of t_n, since $s > 0$. Therefore, we must have $t_n - s < 0$. But this is clearly impossible for n sufficiently large, since $t_n \to \infty$ as $n \to \infty$, because the trajectories come arbitrarily close to X. Therefore, X is asymptotically stable for ϕ. □

This result cannot be extended to the case of predominant (in)stability. A hyperbolic saddle point illustrates that, as it is p.u. for both flow directions. This is not surprising since the compactness arguments in the proof above fail if X is (un)stable only relative to some arbitrary set N which is not a full neighbourhood.

Next we show that the trajectory through a point $x \in \mathcal{B}(X)$ eventually enters any local basin $\mathcal{B}_\varepsilon(X)$.

Lemma 1.21. *For all $x \in \mathcal{B}(X)$ and $\varepsilon > 0$ there exists $t_x \in \mathbb{N}$ such that $\phi_t(x) \in B_\varepsilon(X)$ for all $t > t_x$. Moreover, for every $x \in \mathcal{B}(X)$ there exists $n \in \mathbb{N}$ such that $x \in \mathcal{B}_n(X)$.*

Proof. Suppose the first claim is not true. Since $\omega(x) \subset X$ for all $x \in \mathcal{B}(X)$ this means that there is $\varepsilon > 0$ and $x \in \mathcal{B}(X)$ such that the trajectory through x leaves (and enters) $B_\varepsilon(X)$ infinitely many times. The boundary $\partial B_\varepsilon(X)$ is compact and thus $\{\phi_t(x) \mid t \in \mathbb{R}\} \cap \partial B_\varepsilon(X)$ has an accumulation point x_0. But then $x_0 \in \omega(x)$, in contradiction with $\omega(x) \subset X$.

Now we prove the second statement. Let $x \in \mathcal{B}(X)$ and pick $\varepsilon > 0$, $t_0 \in \mathbb{R}$ such that $\phi_{t_0}(x) \in \mathcal{B}_\varepsilon(X)$. Then $T := \max_{t\in[0,t_0]}(\mathrm{d}(X,\phi_t(x)))$ exists and $x \in \mathcal{B}_n(X)$ for any $n > T$, where d(.,.) denotes the usual Euclidean distance (in this case between a set and a point). □

Not surprisingly, this implies that a set X with a positive measure basin of attraction $\mathcal{B}(X)$ must have positive measure local basins $\mathcal{B}_\varepsilon(X)$, as well. This is what we show next.

Proposition 1.22. *Suppose $\mathcal{B}(X)$ contains a set of positive measure. Then for all $\varepsilon > 0$ there is $M_\varepsilon \subset \mathcal{B}_\varepsilon(X)$ with $\ell(M_\varepsilon) > 0$. In other words, every attractor is f.a.s.*

Proof. Let $\varepsilon > 0$. Since the flow is smooth, the basin of attraction $\mathcal{B}(X)$ contains a set M with $\ell(M) > 0$, for which $M \subset B_\varepsilon(X)$ holds. For $T \in \mathbb{N}$ we set

$$M_T := \{x \in M \mid \exists t > T : \phi_t(x) \notin B_\varepsilon(X)\}.$$

Then M_T contains all points from M that still leave $B_\varepsilon(X)$ after time T. Now there are two cases:

- $\exists T \in \mathbb{N} : \ell(M_T) < \ell(M)$
  In this case we have $\ell(M \setminus M_T) > 0$ and $\phi_t(x) \in B_\varepsilon(X)$ for all $t > T$ and $x \in M \setminus M_T$, so the set $M_\varepsilon := \phi_T(M \setminus M_T)$ has the required properties.

- $\forall T \in \mathbb{N} : \ell(M_T) = \ell(M)$
  In this case $\ell(M \setminus M_T) = 0$ for all $T \in \mathbb{N}$. However, by lemma 1.21 for all $x \in M$ there exists $T \in \mathbb{N}$ such that $x \notin M_T$, so $M = \bigcup_{T\in\mathbb{N}} M \setminus M_T$. It follows that

$$0 < \ell(M) = \ell\left(\bigcup_{T\in\mathbb{N}} M \setminus M_T\right) \leq \sum_{T\in\mathbb{N}} \ell(M \setminus M_T) = 0.$$

  So this case cannot occur and the proof is complete. □

With this we have shown that being an attractor and fragmentary asymptotic stability are the same thing. The last missing ingredient to prove that f.a.s. and e.a.s. are also equivalent is the following theorem 2.1 (c) from [33], which we include for the sake of completeness.

Lemma 1.23 ([33]). *Let X be a compact invariant set for a continuous flow ϕ. Then X is e.a.s. if and only if there is a set $N \subset \mathbb{R}^n$, with $\ell(N \cap A) > 0$ for any neighbourhood A of X, such that X is asymptotically stable relative to N.*

Proof. First, let X be e.a.s. Then by property (a) from definition 1.12 it is asymptotically stable relative to D^c, which by (b) has positive measure intersection with any given neighbourhood of X.

Now assume there exists $N \subset \mathbb{R}^n$ as in the claim above. We need to find a set D with the properties in definition 1.12. Set $D := N^c$. Now let a neighbourhood U and $\varepsilon > 0$ be given. We construct a suitable neighbourhood V as follows. Since X is a.s.r.t. N there is a neighbourhood $\tilde{V}$ satisfying (a). Since $\ell(N \cap \tilde{V}) > 0$, we find at least one point of Lebesgue density for $N \cap \tilde{V}$, i.e. there is $x \in N \cap \tilde{V}$ such that

$$\lim_{\delta \to 0} \frac{\ell(B_\delta(x) \cap N \cap \tilde{V})}{\ell(B_\delta(x))} = 1.$$

Thus, we take $\delta > 0$ small enough that

$$\frac{\ell(B_\delta(x) \cap N \cap \tilde{V})}{\ell(B_\delta(x))} > 1 - \frac{\varepsilon}{2}.$$

Now we set $V := B_\delta(x) \cup B_\eta(X)$, where we choose $\eta > 0$ small enough that $V \subset \tilde{V}$ and

$$\frac{\ell(V \cap N)}{\ell(V)} > 1 - \varepsilon.$$

Therefore, condition (b) is fulfilled and X is e.a.s. □

As promised earlier, in the following proposition we finally state that e.a.s and f.a.s. are indeed the same, thus rendering the complicated definition of e.a.s. superfluous.

Proposition 1.24. *If X is f.a.s., then it is e.a.s. In particular, this means that the two terms are equivalent.*

Proof. If X is f.a.s., then for any $\varepsilon > 0$ the local basin $\mathcal{B}_\varepsilon(X)$ contains a set of positive measure. Suppose for the sake of simplicity that $\mathcal{B}_\varepsilon(X)$ is measurable for all $\varepsilon > 0$ (if not, take a measurable subset with the desired properties). We want to show that X is e.a.s. by lemma 1.23. So we have to construct a set N such that X is asymptotically stable relative to N and $\ell(N \cap A) > 0$ for any neighbourhood A of X.

For any monotonically decreasing sequence $\delta_j > 0$ with $\lim_{j\to\infty} \delta_j = 0$ we find another sequence $\alpha_j > 0$ with $\lim_{j\to\infty} \alpha_j = 0$ such that

$$\ell(N_j) > 0 \quad \text{with} \quad N_j = \mathcal{B}_{\delta_j}(X) \cap B_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X).$$

We set $N := \bigcup_{j\in\mathbb{N}} N_j \cup X$. By construction $\ell(N \cap A) > 0$ for any neighbourhood A of X. It remains to show that X is asymptotically stable relative to N. Let a neighbourhood U of X and $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that $B_{\delta_k}(X) \subset U$ holds and set $V := B_{\alpha_{k-1}}(X)$. Then for $x \in V \cap N$ we have

- $\phi_t(x) \in U$ for all $t > 0$, since

$$x \in V \cap N = B_{\alpha_{k-1}}(X) \cap N = \bigcup_{j \geq k} N_j \cup X \subset \bigcup_{j \geq k} \mathcal{B}_{\delta_j}(X) \subset \mathcal{B}_{\delta_k}(X),$$

  so $\phi_t(x) \in B_{\delta_k}(X) \subset U$ for all $t > 0$.

- $\omega(x) \subset X$, since $x \in V \cap N \subset N \subset \mathcal{B}(X)$.

Therefore, X is asymptotically stable relative to N and by lemma 1.23 X is e.a.s. □

We have now seen that the definitions in the beginning are indeed capable of adequately replacing e.a.s. and a.c.u. While the main aim of this subsection is therefore achieved, reformulating our results leads to an additional insight. In [5] Peter Ashwin and Marc Timme investigate unstable attractors. Corollary 1.27 confirms a conjecture they made. In order to state it, we recall two of their definitions.

Definition 1.25 ([5])**.** For an arbitrary subset $U \subset \mathbb{R}^n$ define the *lingering subset* of U to be

$$\mathcal{A}(U) := \{x \in U \mid \phi_t(x) \in U \quad \forall t \geq 0\}.$$

Definition 1.26 ([5])**.** An attractor X is called an *unstable attractor* if there is a neighbourhood U of X such that

$$\ell(\mathcal{A}(U)) = 0.$$

We rephrase what we have shown so far, extending proposition 1 in [5].

Corollary 1.27. *For a smooth flow there exist no unstable attractors.*

Proof. If X is an attractor, then $\ell(\mathcal{B}(X)) > 0$. So by proposition 1.22 we have $\ell(\mathcal{B}_\varepsilon(X)) > 0$ for any $\varepsilon > 0$ and therefore $\ell(\mathcal{A}(U)) > 0$ for any neighbourhood U of X. □

Within the scope of this work we are mainly interested in smooth systems, which is why we do not venture further into investigating unstable attractors.

1.2.2. The stability index

In this subsection we provide the main tool to quantitatively investigate stability – not only of heteroclinic cycles but of arbitrary compact, invariant sets $X \subset \mathbb{R}^n$. The stability index, introduced by Podvigina and Ashwin in [33], assigns to each point $x \in X$ a real number $\sigma(x) \in [-\infty,\infty]$. The fact that it is constant along trajectories turns it into a handy means of describing stability properties. The greater $\sigma(x)$, the larger the measure of $\mathcal{B}(X)$ locally near x. A positive stability index means that for smaller and smaller neighbourhoods $B_\varepsilon(x)$ the basin of attraction covers more and more of the neighbourhood, approaching full measure as $\varepsilon \to 0$. This prompts us to sometimes refer to such trajectories as "visible" since trajectories through points near it are likely to stay close to X. Note, however, that this intuitive appellation may be misleading: one cannot conclude that a trajectory with negative stability index with respect to some set X_1 is "invisible" in the sense that it is not part of the ω-limit set of nearly all points in its neighbourhood. This is because it may also belong to another invariant set X_2 with respect to which the stability index is positive – a common situation in heteroclinic networks. Therefore, one must be careful not to put too much weight on the phrase "visible".

The main results in this subsection are theorems 1.33 and 1.34, where, roughly speaking, we show that predominant asymptotic stability of a heteroclinic cycle is equivalent to positive local stability indices along all its connecting trajectories. A similar statement, with some additional restrictions, holds for predominant instability and negative local indices.

The stability index has recently been employed to quantify attraction properties in various kinds of dynamical systems, e.g. in [22] for chaotically driven concave maps. There is work in progress regarding its use in the context of attractors with riddled or intermingled basins of attraction.

Definition 1.28 ([33]). For $x \in X$ and $\varepsilon, \delta > 0$ define

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}, \qquad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Then set the *stability index* of X at x to be

$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$

where

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)}\right], \qquad \sigma_+(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(1-\Sigma_\varepsilon(x))}{\ln(\varepsilon)}\right].$$

We use the convention that $\sigma_-(x) = \infty$ if $\Sigma_\varepsilon(x) = 0$ for some $\varepsilon > 0$ and $\sigma_+(x) = \infty$ if $\Sigma_\varepsilon(x) = 1$. Then $\sigma(x) \in [-\infty, \infty]$. In the same way the *local stability index* is defined to be

$$\sigma_{\text{loc}}(x) := \sigma_{\text{loc},+}(x) - \sigma_{\text{loc},-}(x),$$

with

$$\sigma_{\text{loc},-}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}\right], \qquad \sigma_{\text{loc},+}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(1-\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}\right].$$

Near a point $x \in X$ the stability index $\sigma(x)$ quantifies the local extent of $\mathcal{B}(X)$, the basin of attraction of X. If $\sigma(x) > 0$, then in a small neighbourhood of x an increasingly large portion of points is attracted to X. If, on the other hand, $\sigma(x) < 0$, then the portion of such points goes to zero as the neighbourhood shrinks. Figure 1.2 illustrates this schematically. In the same way, the local index $\sigma_{\text{loc}}(x)$ quantifies the local basin $\mathcal{B}_\varepsilon(x)$.

Definition 1.28 also works in the contexts of maps rather than flows, since the concept of ω-limit sets exists for maps as well, so we can define a basin of attraction. In remark

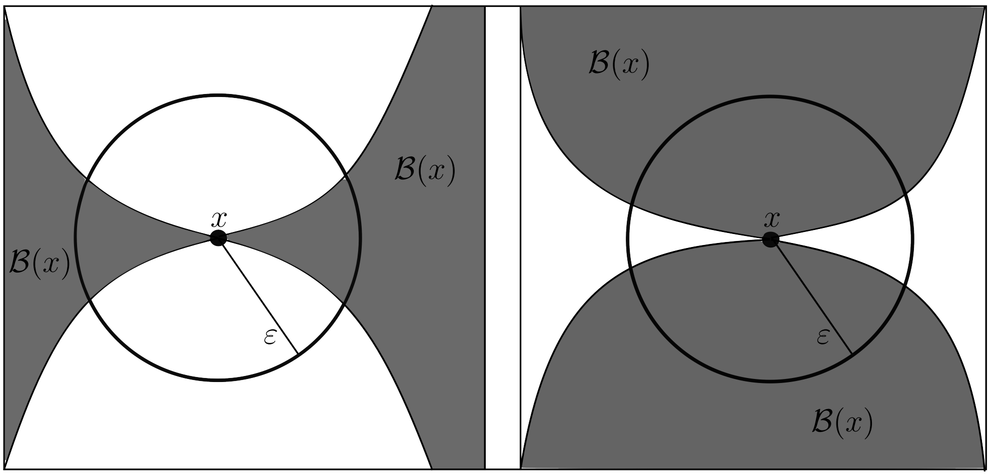


Figure 1.2.: Schematic illustration of the stability index: $\sigma(x) < 0$ left and $\sigma(x) > 0$ right

1.43 we see that the limits in the definition do not necessarily exist. The authors of [33] have explicitly calculated the stability indices for all very simple heteroclinic cycles in $\mathbb{R}^4$. We cite their results in subsection 1.3.2, then use them to derive necessary and sufficient conditions for non-asymptotic stability of different types of cycles in the rest of section 1.3.

We now recall two general properties of the stability index.

Theorem 1.29 ([33]). *Let X be a compact invariant set for a C^1-smooth flow and $x \in X$. Then the stability indices $\sigma(x)$ and $\sigma_{\text{loc}}(x)$ are constant along trajectories, whenever they exist.*

Proof. See [33], theorem 2.2. □

In fact, Podvigina and Ashwin show that both stability indices are invariant under C_1-conjugation of the flow. As mentioned before, theorem 1.29 enables us to characterize a heteroclinic cycle through a finite number of indices, namely the ones along the connecting trajectories. The indices at the equilibria turn out to be irrelevant since their signs do not influence predominant (in)stability of the cycle.

The following theorem simplifies the actual calculation of $\sigma(x)$ by reducing the dimension of the sets that need to be measured.

Theorem 1.30 ([33]). *Let X be a compact invariant set for a C^1-smooth flow. For $x \in X$ denote by S_x a codimension one section that is transverse to the flow at x. Then the stability indices $\sigma(x)$ and $\sigma_{\text{loc}}(x)$ can be computed by substituting $\Sigma_\varepsilon(x)$ and $\Sigma_{\varepsilon,\delta}(x)$ by*

$$\Sigma_{\varepsilon,S_x}(x) = \frac{\ell_{n-1}(B_\varepsilon(x) \cap \mathcal{B}(X) \cap S_x)}{\ell_{n-1}(B_\varepsilon(x) \cap S_x)} \quad \textit{and} \quad \Sigma_{\varepsilon,\delta,S_x}(x) = \frac{\ell_{n-1}(B_\varepsilon(x) \cap \mathcal{B}_\delta(X) \cap S_x)}{\ell_{n-1}(B_\varepsilon(x) \cap S_x)},$$

respectively.

Proof. See [33], theorem 2.4. □

Another consequence of proposition 1.24 is what [33] stated as theorem 2.3(a), but with an incomplete proof. Now that we know that e.a.s. and f.a.s. are equivalent, though, this statement follows immediately.

Corollary 1.31. *If there exists $x \in X$ with $\sigma(x) > -\infty$, then X is e.a.s.*

Proof. The basin $\mathcal{B}(X)$ contains a set of positive measure because otherwise we would have $\sigma(x) = -\infty$ everywhere. So X is e.a.s. by propositions 1.22 and 1.24. □

The next lemma characterizes the behaviour of $\Sigma_\varepsilon(x)$ for small $\varepsilon > 0$. Part of it was already established in [33] as lemma 2.2.

Lemma 1.32 (extends lemma 2.2 in [33])**.** *Suppose that the stability index $\sigma(x)$ exists for some $x \in X$ and let $c > 0$. Then the following is true.*

(a) If $\sigma_\pm(x) > 0$, then $\sigma_\mp(x) = 0$.

(b) $\Sigma_\varepsilon(x) = O(\varepsilon^c) \quad \Leftrightarrow \quad \sigma(x) \leq -c$

(c) $1 - \Sigma_\varepsilon(x) = O(\varepsilon^c) \quad \Leftrightarrow \quad \sigma(x) \geq c$

(d) $\Sigma_\varepsilon(x)$ is bounded away from 0 and $1 \quad \Rightarrow \quad \sigma(x) = 0$

Proof. (a). This is already shown in [33]: suppose $\sigma_-(x) > 0$. Then $\lim_{\varepsilon\to 0} \Sigma_\varepsilon(x) = 0$, so $\lim_{\varepsilon\to 0}(1 - \Sigma_\varepsilon(x)) = 1$ and therefore $\sigma_+(x) = 0$. The other implication follows similarly.

(b). If $\Sigma_\varepsilon(x) = O(\varepsilon^c)$, then there is $k > 0$ such that for small $\varepsilon > 0$ we have $\Sigma_\varepsilon(x) \leq k\varepsilon^c$. Then

$$\sigma_-(x) = \lim_{\varepsilon\to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \geq \lim_{\varepsilon\to 0} \frac{\ln(k\varepsilon^c)}{\ln(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\ln(k) + \ln(\varepsilon^c)}{\ln(\varepsilon)} = c > 0.$$

By (a) we have $\sigma_+(x) = 0$, and therefore

$$\sigma(x) = \sigma_+(x) - \sigma_-(x) \leq -c.$$

The converse is already shown in lemma 2.2 of [33]: suppose $\sigma(x) \leq -c$, then by (a) we have $\sigma_+(x) = 0$ and $\sigma_-(x) \geq c$. So

$$\lim_{\varepsilon\to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \geq c \quad \Rightarrow \quad \lim_{\varepsilon\to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon^c)} \geq 1.$$

Thus, for arbitrary $\delta > 0$ we can choose $\varepsilon > 0$ small enough such that

$$\ln(\Sigma_\varepsilon(x)) \leq \ln(\varepsilon^c) + \delta$$

and therefore

$$\Sigma_\varepsilon(x) \leq k\varepsilon^c$$

for some constant $k \in \mathbb{R}_+$. Statement (c) follows analogously.

For (d), note that if $\Sigma_\varepsilon(x)$ is bounded away from 0, then

$$\sigma_-(x) = \lim_{\varepsilon\to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \leq \lim_{\varepsilon\to 0} \frac{\text{const}}{\ln(\varepsilon)} = 0 \quad \Rightarrow \quad \sigma_-(x) = 0,$$

and analogously $\sigma_+(x) = 0$ if $\Sigma_\varepsilon(x)$ is bounded away from 1. So $\sigma(x) = 0$ as claimed. □

The converse of (d) does not hold, as can be seen from remark 1.42 (c). We are now in a position to prove that predominant (in)stability of a cycle is related to the signs of the indices along the connecting trajectories, the first step towards this is the following theorem.

Theorem 1.33. *Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle consisting of finitely many equilibria $\xi_1,\dots,\xi_m$ and connecting trajectories and suppose that $\ell_1(X) < \infty$. Assume that the local stability index $\sigma_{\text{loc}}(x)$ exists for all $x \in X$. Then the following holds.*

(a) X is p.a.s. $\Rightarrow$ $\ell_1(\{x \in X \mid \sigma_{\text{loc}}(x) < 0\}) = 0$

(b) X is p.u. $\Rightarrow$ $\ell_1(\{x \in X \mid \sigma_{\text{loc}}(x) > 0\}) = 0$

Proof. We begin with the first statement. Let X be p.a.s., in particular a.s.r.t. some $N \subset \mathbb{R}^n$. Assume that the implication is not true, so that $\ell_1(\hat{X}_a) > 0$, where

$$\hat{X}_a := \left\{ x \in X \setminus \bigcup_{j=1}^{m} \{\xi_j\} \mid \sigma_{\text{loc}}(x) < 0 \right\}.$$

Note that $\hat{X}_a$ is ϕ-invariant since the index is constant on trajectories. For $x \in \hat{X}_a$ we have $\sigma_{\text{loc},-}(x) > 0$ and therefore $\Sigma_{\varepsilon,\delta}(x) \to 0$ when $\varepsilon, \delta \to 0$. By theorem 1.30 the same is true for $\Sigma_{\varepsilon,\delta,S_x}(x)$ where S_x is a codimension one surface that is transverse to the flow at x. Thus, for all $x \in \hat{X}_a$ there exists $\gamma(x) > 0$ such that for $\delta, \varepsilon < \gamma(x)$ we have $\Sigma_{\varepsilon,\delta,S_x}(x) < \frac{1}{2}$. Since

$$0 < \ell_1(\hat{X}_a) = \ell_1 \left(\bigcup_{n\in\mathbb{N}} \left\{ x \in \hat{X}_a \,\middle|\, \gamma(x) \geq \frac{1}{n} \right\} \right) \leq \sum_{n\in\mathbb{N}} \ell_1 \left(\left\{ x \in \hat{X}_a \,\middle|\, \gamma(x) \geq \frac{1}{n} \right\} \right)$$

the measure of the sets in the sum cannot be zero for all $n \in \mathbb{N}$. So there is a set $Y_a \subset \hat{X}_a$ of positive one-dimensional measure where the bound is uniform, i.e. there is $\gamma > 0$ such that

$$\forall y \in Y_a \quad \forall \varepsilon, \delta < \gamma \quad \Sigma_{\varepsilon,\delta,S_y}(y) < \frac{1}{2}.$$

Without loss of generality we can assume that the transverse sections S_y are disjoint and of uniform size for all $y \in Y_a$, by excluding small neighbourhoods of the equilibria if necessary, without losing the property $\ell_1(Y_a) > 0$. We write $\ell_1(Y_a) = \alpha\ell_1(X)$ with $\alpha \in (0,1]$ and look at

$$W_\varepsilon(Y_a) := \bigcup_{y\in Y_a} (B_\varepsilon(y) \cap S_y) \subset B_\varepsilon(Y_a)$$

to discover

$$\frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} = \frac{\ell_{n-1}(B_\varepsilon)\ell_1(Y_a)}{\ell_{n-1}(B_\varepsilon)\ell_1(X) + O(\varepsilon^n)} \xrightarrow{\varepsilon\to 0} \frac{\ell_1(Y_a)}{\ell_1(X)} = \alpha,$$

because the volume of an $(n-1)$-dimensional ε-ball, $\ell_{n-1}(B_\varepsilon)$, is of order ε^{n-1}. Now for $\varepsilon, \delta < \gamma$ and small enough, Fubini's theorem gives

$$\begin{aligned}\ell(W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X)) &= \int\limits_{W_\varepsilon(Y_a)} \chi_{\mathcal{B}_\delta(X)}\, d\ell_n \\ &= \int\limits_{Y_a} \int\limits_{B_\varepsilon(y)\cap S_y} \chi_{\mathcal{B}_\delta(X)}\, d\ell_{n-1} d\ell_1 \\ &= \int\limits_{Y_a} \ell_{n-1}(B_\varepsilon(y) \cap S_y \cap \mathcal{B}_\delta(X))\, d\ell_1 \\ &< \frac{1}{2} \int\limits_{Y_a} \ell_{n-1}(B_\varepsilon(y) \cap S_y)\, d\ell_1 \\ &= \frac{1}{2} \ell(W_\varepsilon(Y_a)).\end{aligned}$$

X is asymptotically stable relative to N. So by remark 1.19, for $\delta < \gamma$ we find $\varepsilon < \gamma$ such that

$$\ell(B_\varepsilon(X) \cap N) \leq \ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))$$

Then by the above

$$\begin{aligned}\frac{\ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} &= \frac{\ell(W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \setminus W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} \\ &< \frac{1}{2}\frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \setminus W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} \\ &= 1 - \frac{1}{2}\frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))}.\end{aligned}$$

Since X is p.a.s., taking the limit $\varepsilon \to 0$ now gives

$$1 = \lim_{\varepsilon\to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} \leq \lim_{\varepsilon\to 0} \frac{\ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} \leq 1 - \frac{\alpha}{2}.$$

This is a contradiction, so $0 = \ell_1(\hat{X}_a) = \ell_1(\{x \in X \mid \sigma_{\text{loc}}(x) < 0\})$ as claimed.

Now let X be p.u., in particular let it be completely unstable relative to a set $M \subset \mathbb{R}^n$. Denote by U a neighbourhood such that all points in $U \cap M \setminus X$ leave U in finite positive time. In the same way as above, assume that $\ell_1(\hat{X}_b) > 0$, where

$$\hat{X}_b := \left\{ x \in X \setminus \bigcup_{j=1}^{m} \{\xi_j\} \mid \sigma_{\text{loc}}(x) > 0 \right\}.$$

We obtain a contradiction to this assumption in a similar way as before, so we just point out the steps where the reasoning is different. As before we find a set $Y_b \subset \hat{X}_b$ with $\ell_1(Y_b) = \beta \ell_1(X) > 0$ and $\gamma > 0$ such that

$$\forall y \in Y_b \quad \forall \varepsilon, \delta < \gamma \qquad \Sigma_{\varepsilon,\delta,S_y}(y) > \frac{1}{2}.$$

Again with Fubini's theorem and $W_\varepsilon(Y_b)$ as above we obtain for $\varepsilon, \delta > 0$ small enough

$$\ell(W_\varepsilon(Y_b) \cap \mathcal{B}_\delta(X)) > \frac{1}{2}\ell(W_\varepsilon(Y_b)),$$

and therefore

$$\ell(W_\varepsilon(Y_b) \cap (\mathcal{B}_\delta(X))^c) < \frac{1}{2}\ell(W_\varepsilon(Y_b)).$$

Since all points in $U \cap M \setminus X$ leave U, for ε, δ small enough we have

$$B_\varepsilon(X) \cap M \setminus X \subset B_\varepsilon(X) \cap (\mathcal{B}_\delta(X))^c.$$

This leads to a contradiction as follows

$$\begin{aligned}
1 &= \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap M)}{\ell(B_\varepsilon(X))} \\
&\leq \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(x) \cap (\mathcal{B}_\delta(X))^c)}{\ell(B_\varepsilon(X))} \\
&= \lim_{\varepsilon \to 0} \left[\frac{\ell(W_\varepsilon(Y_b) \cap (\mathcal{B}_\delta(X))^c)}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \cap (\mathcal{B}_\delta(X))^c \setminus W_\varepsilon(Y_b))}{\ell(B_\varepsilon(X))} \right] \\
&\leq \lim_{\varepsilon \to 0} \left[\frac{1}{2} \frac{\ell(W_\varepsilon(Y_b))}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \setminus W_\varepsilon(Y_b))}{\ell(B_\varepsilon(X))} \right] \\
&= 1 - \frac{\beta}{2},
\end{aligned}$$

thus completing the proof also for (b). □

We now take a look at the converse of this result.

Theorem 1.34. *Under the same assumptions as in the previous theorem the following holds.*

(a) $[\sigma_{\text{loc}}(x) > 0$ *along all connections*$] \Rightarrow X$ *is predominantly asymptotically stable.*

(b) If, in addition, we suppose that X is an isolated invariant set, then we also have $[\sigma_{\text{loc}}(x) < 0$ *along all connections*$] \Rightarrow X$ *is predominantly unstable.*

Proof. We start with (a). For all $x \in \hat{X} := X \setminus \bigcup_{j\in\mathbb{N}}\{\xi_j\}$ we have $\sigma_{\text{loc}}(x) > 0$, so it follows that $\lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0}\Sigma_{\varepsilon,\delta}(x) = 1$. Again the same is then true for $\Sigma_{\varepsilon,\delta,S_x}(x)$ with S_x as above. So for all $\rho_1 > 0$ and all $x \in \hat{X}$ there is $\varepsilon(x) > 0$ such that $\Sigma_{\varepsilon,\delta,S_x}(x) > 1 - \rho_1$ for $\varepsilon, \delta < \varepsilon(x)$. So, in particular

$$\forall x \in \hat{X} \quad \forall \varepsilon < \varepsilon(x) \quad \ell_{n-1}(\mathcal{B}_\varepsilon(X) \cap B_\varepsilon(x) \cap S_x) > (1-\rho_1)\ell_{n-1}(B_\varepsilon(x) \cap S_x).$$

We need a uniform lower bound for $\varepsilon(x)$. This can be found in the same way as in the previous lemma. Since

$$\hat{X} = \bigcup_{n\in\mathbb{N}} \left\{ x \in \hat{X} \,\middle|\, \varepsilon(x) \geq \frac{1}{n} \right\},$$

for any given $\rho_2 > 0$ we find $n \in \mathbb{N}$ and $Y \subset \hat{X}$ with

$$\ell_1(Y) > (1-\rho_2)\ell_1(X) \qquad \text{and} \qquad \forall y \in Y: \quad \varepsilon(y) \geq \frac{1}{n}.$$

Thus, we have $\ell(W_\varepsilon(Y)) > (1-\rho_2)\ell(B_\varepsilon(X))$ for $W_\varepsilon(Y)$ as in the previous lemma and ε small enough. If all sets involved are measurable we can now use Fubini's theorem to obtain

$$\begin{aligned}
\ell(\mathcal{B}_\varepsilon(X)) &= \int\limits_{B_\varepsilon(X)} \chi_{\mathcal{B}_\varepsilon(X)}\, d\ell_n \\
&\geq \int\limits_{W_\varepsilon(Y)} \chi_{\mathcal{B}_\varepsilon(X)}\, d\ell_n \\
&= \int\limits_Y \ell_{n-1}(B_\varepsilon(y) \cap S_y \cap \mathcal{B}_\varepsilon(X))\, d\ell_1 \\
&> (1-\rho_1)\int\limits_Y \ell_{n-1}(B_\varepsilon(y) \cap S_y)\, d\ell_1 \\
&= (1-\rho_1)\ell(W_\varepsilon) \\
&> (1-\rho_1)(1-\rho_2)\ell(B_\varepsilon(X)) \\
&> (1-\rho)\ell(B_\varepsilon(X))
\end{aligned}$$

for suitable choices of $\rho_1, \rho_2 > 0$ and a given $\rho > 0$. As in theorem 1.33 we exclude small neighbourhoods of the equilibria from Y to ensure uniform size of the transverse sections and also take ε small enough that the neighbourhoods do not overlap. So we have shown

$$\lim_{\varepsilon\to 0} \frac{\ell(\mathcal{B}_\varepsilon(X))}{\ell(B_\varepsilon(X))} = 1. \tag{1.9}$$

This is not yet sufficient for predominant asymptotic stability of X, we still have to construct a set N such that X is asymptotically stable relative to N and

$$\lim_{\varepsilon\to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1. \tag{1.10}$$

This can be done in a similar way as in the proof of proposition 1.24: we construct two monotonically decreasing sequences $\delta_j > 0$ with $\lim_{j\to\infty} \delta_j = 0$ and $\alpha_j > 0$ with $\lim_{j\to\infty} \alpha_j = 0$ and set

$$N := \bigcup_{j\in\mathbb{N}} N_j, \quad \text{where} \quad N_j := \mathcal{B}_{\delta_j}(X) \cap B_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X).$$

To make the choice of the sequences precise: for $j \in \mathbb{N}$ choose $\delta_j > 0$ such that for all $\delta \leq \delta_j$ we have

$$\ell(\mathcal{B}_\delta(X)) > \frac{j}{j+1}\ell(B_\delta(X)).$$

Then for $j \in \mathbb{N}$ pick $\alpha_j > 0$ in such a way that $\alpha_{j-1} \leq \delta_j$ and

$$\ell(B_{\alpha_j}(X)) < \frac{1}{j(j+1)}\ell(B_{\alpha_{j-1}}(X)).$$

This gives

$$\begin{aligned}
\ell(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)) &\geq \ell(\mathcal{B}_{\alpha_{j-1}}(X)) - \ell(B_{\alpha_j}(X)) \\
&> \frac{j}{j+1}\ell(B_{\alpha_{j-1}}(X)) - \frac{1}{j(j+1)}\ell(B_{\alpha_{j-1}}(X)) \\
&= \frac{j-1}{j}\ell(B_{\alpha_{j-1}}(X)).
\end{aligned}$$

With this we calculate for $\varepsilon > 0$ and $\alpha_k < \varepsilon \leq \alpha_{k-1}$

$$\begin{aligned}
\ell(B_\varepsilon(X) \cap N) &= \ell\left(B_\varepsilon(X) \cap \bigcup_{j\in\mathbb{N}} N_j\right) \\
&> \ell\left(B_\varepsilon(X) \cap \bigcup_{j\in\mathbb{N}} \mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&= \sum_{j\in\mathbb{N}} \ell\left(B_\varepsilon(X) \cap \mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&= \ell\left(B_\varepsilon(X) \cap \mathcal{B}_{\alpha_{k-1}}(X) \setminus B_{\alpha_k}(X)\right) + \sum_{j>k} \ell\left(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&\geq \ell\left(\mathcal{B}_\varepsilon(X) \setminus B_{\alpha_k}(X)\right) + \sum_{j>k} \ell\left(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&\geq \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \sum_{j>k} \ell\left(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&> \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \ell\left(\mathcal{B}_{\alpha_k}(X) \setminus B_{\alpha_{k+1}}(X)\right) \\
&> \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \frac{k}{k+1}\ell(B_{\alpha_k}(X)) \\
&= \ell(\mathcal{B}_\varepsilon(X)) - \frac{1}{k+1}\ell(B_{\alpha_k}(X)).
\end{aligned}$$

Now since $\alpha_k < \varepsilon$, we have $\ell(B_{\alpha_k}(X)) < \ell(B_\varepsilon(X))$, so

$$\frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} > \frac{\ell(\mathcal{B}_\varepsilon(X))}{\ell(B_\varepsilon(X))} - \frac{1}{k+1}\frac{\ell(B_{\alpha_k}(X))}{\ell(B_\varepsilon(X))} > \frac{\ell(\mathcal{B}_\varepsilon(X))}{\ell(B_\varepsilon(X))} - \frac{1}{k+1} \xrightarrow{\varepsilon\to 0} 1,$$

since $k = k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and the first term goes to 1 by (1.9). This shows that (1.10) is satisfied, so X is p.a.s.

Now we prove (b), so assume that X is an isolated invariant set and that $\sigma_{\text{loc}}(x)$ is negative along all connecting trajectories. This means that $\Sigma_{\varepsilon,\delta,S_x}(x) \to 0$ for $\delta,\varepsilon \to 0$, so the points in S_x converging directly to X form a thin cusp-shaped region at most. Since X is isolated, there is a neighbourhood U of X that contains no other invariant set. So all points in U, that are not in the thin part of the cusp-shaped set, leave U in finite positive time. If this was not the case, their ω-limit set would be contained in U, leading to a contradiction. Such points do not belong to $\mathcal{B}_\delta(X)$ for $\delta > 0$ so small that $B_\delta(X) \subset U$. With the same techniques as before, it follows that for fixed $\delta > 0$ small enough, the complement $\mathcal{B}_\delta(X)^c$ of the local basin of attraction satisfies

$$\lim_{\varepsilon\to 0} \frac{\ell(B_\delta(X)^c \cap B_\varepsilon(x))}{\ell(B_\varepsilon(X))} = 1,$$

proving predominant instability of X. □

Remark 1.35. Note that

$$[\sigma_{\text{loc}}(x) < 0 \textit{ along all trajectories}] \Rightarrow X \textit{ is predominantly unstable,}$$

does not hold without X being an isolated invariant set. To see why, imagine a cycle where along all trajectories the local basin of attraction takes the shape of a thin cusp in every transverse section (thus yielding a negative local index). At the same time, let all other points be stationary points of the return map, i.e. periodic orbits in the full system. Then X is not p.u. since there is no neighbourhood that all periodic orbits exit.

This asymmetry when reversing the implication from theorem 1.33 can be explained intuitively: the stability properties p.a.s. and p.u. both make statements about "many" trajectories in a neighbourhood of the invariant object – p.a.s. meaning most of them converge to it and p.u. meaning most of them leave the neighbourhood. The same is true for a positive stability index. It means that for most initial values the trajectories are eventually attracted to the cycle. Things are different for a negative stability index, though. Here, the only thing one knows is that very few trajectories converge to X. We do not know what happens to the others, so without further assumptions we cannot expect all (or even most) of them to leave the neighbourhood.

Note that heteroclinic cycles in a network are not isolated. However, this is not much of a problem for two reasons: first, when it comes to networks it is of greater interest to identify stable (p.a.s.) subcycles than unstable (p.u.) ones, so the p.a.s.-equivalence is the more important one. Second, as long as for each cycle the only other invariant set in its neighbourhood is another cycle, this allows us to control the behaviour of the remaining trajectories, so that the p.u.-equivalence holds, as well.

Remark 1.36. For heteroclinic cycles stability indices can be computed by iterating the return maps. We explain this in more detail in chapter 2. In most cases the basin of attraction is bounded by an exponential curve, which in turn means that generically $\sigma(x) \neq 0$ along connecting trajectories. This is the case for all very simple cycles in $\mathbb{R}^4$ as the computations of [33] show, see subsection 1.3.2. Then theorems 1.33 and 1.34 combine to prove that a heteroclinic cycle is p.a.s. if and only if $\sigma_{\text{loc}}(x) > 0$ along every trajectory. Otherwise, it is technically possible for a cycle to be p.a.s. even though there is a connection where $\sigma(x) = 0$. In remark 1.42 (c), however, we see that this requires a highly unusual geometry of the basin of attraction.

Since $\sigma(x) \geq \sigma_{\text{loc}}(x)$, positive non-local stability indices also imply predominant asymptotic stability of the cycle. The reversed implication fails, though, as can be seen from example 1.41. Moreover, note that in light of theorem 1.29 this result shows that being p.a.s. is an invariant property under C^1-conjugation of the flow.

These considerations have particularly drastic implications for heteroclinic cycles of type A^*.

Proposition 1.37. *Let X be a heteroclinic cycle of type A* and assume that the local stability index exists everywhere and is not equal to zero. Then generically either $\sigma_{\text{loc}}(x) > 0$ along all trajectories or $\sigma_{\text{loc}}(x) < 0$ along all trajectories. In $\mathbb{R}^4$ this holds for any cycle of type A.*

Proof. The first statement now follows immediately from the dichotomy for type A^* cycles in theorem 1.11. In $\mathbb{R}^4$ there is no difference between type A and A^* cycles, see definition A.5, which proves the second claim. □

In particular, this means that generically no cycle of type A* is properly fragmentarily asymptotically stable. If it is an attractor at all, it is typically either p.u. or p.a.s., i.e. in a small neighbourhood it attracts either almost everything or almost nothing. Therefore, for such a cycle either all trajectories are visible or none. This is not true for cycles of types B and C as we will see in subsections 1.3.4 and 1.3.5.

1.2.3. Examples

We now consider a few examples that illustrate important aspects about the stability index, that are not necessarily obvious at first sight. We begin with a quick look at the cycle from subsection 1.1.3.

Example 1.38. The stability index for the homoclinic type B cycle is easy to calculate and reveals nothing new about the cycle. We briefly discuss it anyway to get used to the concept of the index as a means of understanding stability.

Recall that since all connecting trajectories lie on the same G-orbit, there is only one stability index to calculate. Under the assumptions of (1.8) it is equal to $+\infty$ because the cycle is asymptotically stable and $\mathcal{B}(X)$ therefore covers an entire neighbourhood of any connecting trajectory. We can also determine the stability indices with respect to the subcycles X_i, temporarily ignoring the fact that the cycle is homoclinic. From the previous calculations it follows that each subcycle attracts all trajectories in its respective octant, except for the invariant diagonals. Therefore, when looking at a (symmetric) neighbourhood in a cross section transverse to a connection, exactly half of the points belong to $\mathcal{B}(X_i)$, independent of the size of the neighbourhood. So $\Sigma_\varepsilon(x)$ is constant with respect to X_i, thus along each connection the stability index (with respect to X_i) is equal to 0. Note that there is no difference between the local and non-local index here.

Now suppose, for instance, that $a < -\frac{1}{3}$, so that condition (4) in equations (1.8) is broken. Then no trajectory outside of the coordinate planes is attracted to the cycle anymore, it is completely unstable. The stability index is equal to $-\infty$.

We now move on to something a bit more involved. Clearly, a basin of attraction of measure zero leads to a stability index equal to $-\infty$. The converse is not true, however: a positive measure basin does not prevent the stability index from being equal to $-\infty$.

Remark 1.39. Consider an attractor $X \subset \mathbb{R}^n$ for which locally the basin of attraction $\mathcal{B}(X)$ has the form of a *superalgebraic cusp*. By this we mean that $\Sigma_\varepsilon(x) = O(\varepsilon^c)$ for all $c > 0$. For instance, let $\mathcal{B}(X)$ be shaped in such a way that for $x \in X$ we have $\Sigma_\varepsilon(x) = e^{-\frac{1}{\varepsilon}}$. Then for any $c > 0$

$$\begin{aligned}
\varepsilon^c e^{\frac{1}{\varepsilon}} &= \varepsilon^c \sum_{k=0}^{\infty} \frac{1}{k!\varepsilon^k} \\
&= \sum_{k=0}^{\infty} \frac{\varepsilon^c}{k!\varepsilon^k} \\
&= \varepsilon^c + \varepsilon^{c-1} + \frac{\varepsilon^{c-2}}{2} + \cdots + \frac{1}{c!} + \frac{1}{(c+1)!\varepsilon} + \cdots \\
&\xrightarrow{\varepsilon \to 0} \infty.
\end{aligned}$$

Indeed, this means that $\Sigma_\varepsilon(x) = O(\varepsilon^c)$ for all $c > 0$, because

$$\lim_{\varepsilon \to 0} \frac{\Sigma_\varepsilon(x)}{\varepsilon^c} = \lim_{\varepsilon \to 0} \frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon^c} = 0.$$

Thus, lemma 1.32 yields $\sigma(x) \leq -c$ for all $c > 0$. Therefore, $\sigma(x) = -\infty$, even though $\mathcal{B}(X)$ is of positive measure in any neighbourhood of x. Modifying the flow such that everything except for the cusp is contained in the basin of attraction, we get $\sigma(x) = +\infty$ even though not everything close to x is attracted to X.

We look at this in more detail: in $\mathbb{R}^2$, for instance, let X consist of the origin only. Then such a cusp is bounded by the x-axis and the graph of $f(x) = (2x+1)e^{-\frac{1}{x}}$. If the basin of attraction is shaped by f, then for small $\varepsilon > 0$ we have (up to a constant factor)

$$\Sigma_\varepsilon(0) = \frac{\ell(B_\varepsilon(0) \cap \mathcal{B}(0))}{\ell(B_\varepsilon(0))} = \frac{\int_0^\varepsilon (2x+1)e^{-\frac{1}{x}}dx}{\ell(B_\varepsilon(0))} \approx \frac{\int_0^\varepsilon (2x+1)e^{-\frac{1}{x}}dx}{\varepsilon^2} = \frac{\varepsilon^2 e^{-\frac{1}{\varepsilon}}}{\varepsilon^2} = e^{-\frac{1}{\varepsilon}}.$$

Remark 1.40. In many examples $\sigma(x)$ and $\sigma_{\text{loc}}(x)$ coincide. However, this is not always the case. The two indices are independent in the same way that the classical definitions of *stability* and *attractivity* do not go hand in hand. A simple example is given by a flow on S^1, as depicted in figure 1.3. Here X consists of a single equilibrium x and the whole space belongs to $\mathcal{B}(X)$, so $\sigma(x) = +\infty$. But only one side of x belongs to the local basin $\mathcal{B}_\delta(X)$ for $\delta > 0$ small enough. Thus, $\Sigma_{\varepsilon,\delta}(X)$ is constant and $\sigma_{\text{loc}}(x) = 0$. We now construct such an example in $\mathbb{R}^2$ (restricting notation to the upper half plane $\mathbb{H}^+ \subset \mathbb{R}^2$) and aim at the even more extreme case $\sigma(x) = +\infty$ while $\sigma_{\text{loc}}(x) = -\infty$. However, we only achieve this at the cost of smoothness.

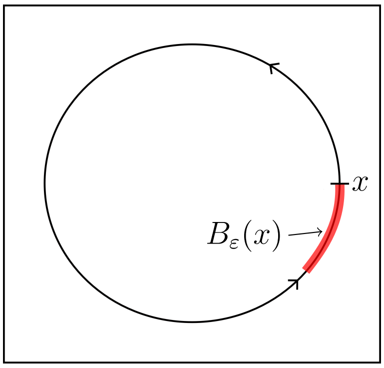


Figure 1.3.: $\sigma(x) = +\infty$ while $\sigma_{\text{loc}}(x) = 0$

Example 1.41. We modify a fairly well-known dynamical system that often serves as an example for an unstable but attractive invariant set. Consider the following ordinary differential equation on the upper half plane $\mathbb{H}^+ \subset \mathbb{R}^2$ that Hahn discusses in §40 of [19].

$$\dot{\xi}_1 = \frac{\xi_1^2(\xi_2 - \xi_1) + \xi_2^5}{(\xi_1^2 + \xi_2^2)(1 + (\xi_1^2 + \xi_2^2)^2)}, \qquad \dot{\xi}_2 = \frac{\xi_2^2(\xi_2 - 2\xi_1)}{(\xi_1^2 + \xi_2^2)(1 + (\xi_1^2 + \xi_2^2)^2)}. \tag{1.11}$$

Its phase portrait looks like figure 1.4. The invariant set $X = \{0\}$ consists of a single point,

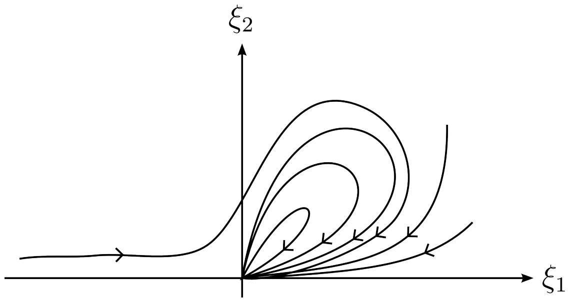


Figure 1.4.: Unstable but attractive equilibrium

the origin. It is unstable but attractive, the global basin $\mathcal{B}(X)$ is the entire space and thus has full measure in $B_\varepsilon(X) = B_\varepsilon(0)$ for any $\varepsilon > 0$, so $\sigma(0) = +\infty$. In the first quadrant there are infinitely many homoclinic orbits. For small $\delta > 0$ it contains the local basin of attraction $\mathcal{B}_\delta(X)$. Hahn also shows in [19] that the entire sector between the ξ_1-axis and the ray through the point $\left(\frac{25}{32},\frac{5}{8}\right)$ belongs to $\mathcal{B}_\delta(X)$. Denoting the opening angle of that sector by α this implies that for small $\varepsilon, \delta > 0$ we get

$$\frac{\alpha}{\pi} \leq \Sigma_{\varepsilon,\delta}(0) \leq \frac{1}{2},$$

and therefore $\sigma_{\text{loc}}(0) = 0$ by lemma 1.32. We modify this example so that $\sigma_{\text{loc}}(0) = -\infty$.

The idea is to transform the phase portrait into the one shown in figure 1.5 where the homoclinic orbits are all located below the line $\xi_2 = (2\xi_1 + 1)e^{-\frac{1}{\xi_1}}$, the boundary of a superalgebraic cusp as described above. The local basin of the origin is then contained in this cusp, so $\sigma_{\text{loc}}(0) = -\infty$, while the equilibrium is still globally attractive, hence $\sigma(0) = +\infty$.

In order to see how this is possible we make the following more general considerations.

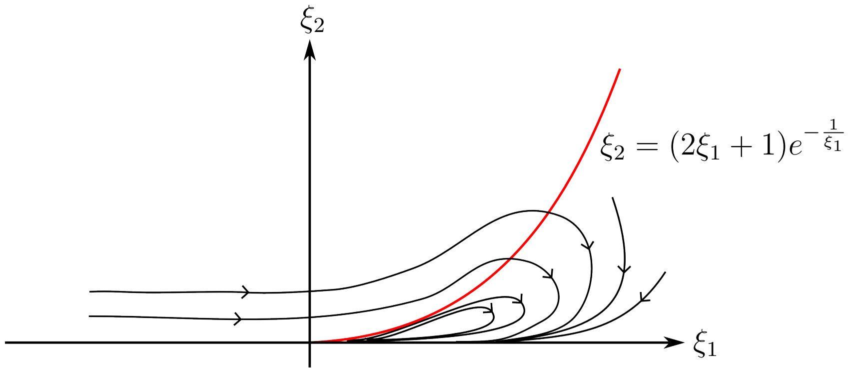


Figure 1.5.: Unstable but attractive equilibrium, its local basin contained in a superalgebraic cusp

Let $\dot{x} = f(x)$ be an ordinary differential equation generating a flow on $\mathbb{H}^+$ with some phase portrait (A), and let $x = x(t,x_0)$ be the solution for an inital value $x_0 \in \mathbb{H}^+$. Then consider a continuously differentiable homeomorphism $\Phi : \mathbb{H}^+ \to \mathbb{H}^+$ transforming the phase portrait in such a way that $y = y(t,y_0) := \Phi(x(t,x_0))$, with $y_0 := \Phi(x_0)$, are the trajectories of a desired phase portrait (B). Then $y(t,y_0)$ is the solution to an equation $\dot{y} = g(y)$ generating the desired flow (corresponding to (B)), where g is obtained by also transforming f with Φ in the following way:

$$\dot{y} = \frac{d}{dt}(\Phi(x(t))) = d\Phi(x(t)) \cdot \dot{x}(t) = [d\Phi \circ \Phi^{-1}](y(t)) \cdot [f \circ \Phi^{-1}](y(t)) =: g(y)$$

The right hand side g is continuous because Φ is a homeomorphism and $d\Phi$ is continuous. Now let us define Φ for our specific case:

$$\Phi : \mathbb{H}^+ \to \mathbb{H}^+, \quad (\xi_1,\xi_2) \mapsto \left(\xi_1 + \xi_2, (2\xi_2 + 1)e^{-\frac{1}{\xi_2}}\right)$$

It is simple to check that Φ is continuously differentiable. Moreover, it is bijective and its inverse Ψ is also continuous, since $\xi_2 \mapsto (2\xi_2 + 1)e^{-\frac{1}{\xi_2}}$ is strictly monotonically increasing and unbounded, thus open. However, note that Ψ is not differentiable along $\xi_2 = 0$, so neither is g. Nevertheless, this does exactly what we wanted: for $\xi_1 > 0$ we have $e^{-\frac{1}{\xi_2}} < e^{-\frac{1}{\xi_1+\xi_2}}$ and therefore $(2\xi_1+1)e^{-\frac{1}{\xi_2}} < (2(\xi_1+\xi_2)+1)e^{-\frac{1}{\xi_1+\xi_2}}$, so the first quadrant (and thus the entire local basin of attraction) is mapped below the boundary of a superalgebraic cusp. So we have $\sigma_{\text{loc}}(0) = -\infty$.

It is still unclear, however, if such an example can also be constructed with a smooth right hand side. The answer to this question cannot be expected from a construction like this. This is because by theorem 1.29 the stability index is invariant under topological equivalence. Therefore, we cannot use a diffeomorphism to alter the indices in the same way as above.

We come across a situation where $-\infty < \sigma_{\text{loc}}(x) < 0 < \sigma(x) < +\infty$ when studying heteroclinic networks in section 2.2, remark 2.18.

Remark 1.42. The above example serves to illustrate more interesting facts.

(a) In the second part of their theorem 2.3 the authors of [33] claim that if there is a $c > 0$ such that for all $x \in X$ we have $\sigma(x) > c$, then X is p.a.s. The original equations (1.11) show that this does not hold for arbitrary X. The origin is clearly not p.a.s. although $\sigma(0) = +\infty$. Note that the origin is not an isolated invariant set in this example.

(b) Again we see that lemma 1.20 cannot be extended to the case of predominant instability and predominant asymptotic stability instead of complete instability and asymptotic stability, respectively: in figure 1.5 the origin is p.u. for the original flow as well as for the time-reversed one (and thus not p.a.s.). A notable difference to the earlier example of a hyperbolic saddle point is that here the basin of attraction actually has positive measure. However, we emphasize once again that the present example has a right hand side that is continuous, but not smooth.

(c) A similar example can be constructed that shows that the converse of lemma 1.32 (d) does not hold: if $\Sigma_\varepsilon(x) = -\frac{1}{\ln(\varepsilon)}$, then there is no $c > 0$ such that $\Sigma_\varepsilon(x) = O(\varepsilon^c)$ which means $\sigma(x) > -c$ for all $c > 0$. Thus, $\sigma(x) = 0$, even though $\Sigma_\varepsilon(x) \to 0$. To verify this calculate

$$\lim_{\varepsilon\to 0} \frac{\Sigma_\varepsilon(x)}{\varepsilon^c} = \lim_{\varepsilon\to 0} -\frac{\varepsilon^{-c}}{\ln(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{c\varepsilon^{-c-1}}{\varepsilon^{-1}} = \lim_{\varepsilon\to 0} \frac{c}{\varepsilon^c} = \infty.$$

From the definition of the stability index it is by no means clear that the limits $\sigma_+(x)$ and $\sigma_-(x)$ always exist. In [33] Podvigina and Ashwin give an example where this is not the case. We now take their considerations a bit further. Note that here we consider the stability index for a map rather than a flow.

Remark 1.43. In [33] an example is given of a map for which 0 is a fragmentarily asymptotically stable fixed point such that the stability index does not exist. In fact, neither $\sigma(0)$ nor $\sigma_{\text{loc}}(0)$ exists since gobal and local basin coincide. We extend this example to see that it is possible that $\sigma(0)$ exists but $\sigma_{\text{loc}}(0)$ does not, or vice versa.

The map $M : [0,\infty) \to \mathbb{R}$ from [33] is defined as follows. For $k \in \mathbb{N}$ let $\epsilon_k := \exp(-2^k)$ and set

$$M(y) := \begin{cases} 0 & \text{if} \quad \epsilon_{2k+1} < |y| \leq \epsilon_{2k} \\ \dfrac{\epsilon_{2k-1}(y-\epsilon_{2k+2}) - \epsilon_{2k}(y-\epsilon_{2k+1})}{(\epsilon_{2k+1}-\epsilon_{2k+2})} & \text{if} \quad \epsilon_{2k+2} < |y| \leq \epsilon_{2k+1} \\ 0 & \text{if} \quad y = 0. \end{cases}$$

Figure 1.6 shows a qualitative graph of M. We consider the stability index for the dynamics

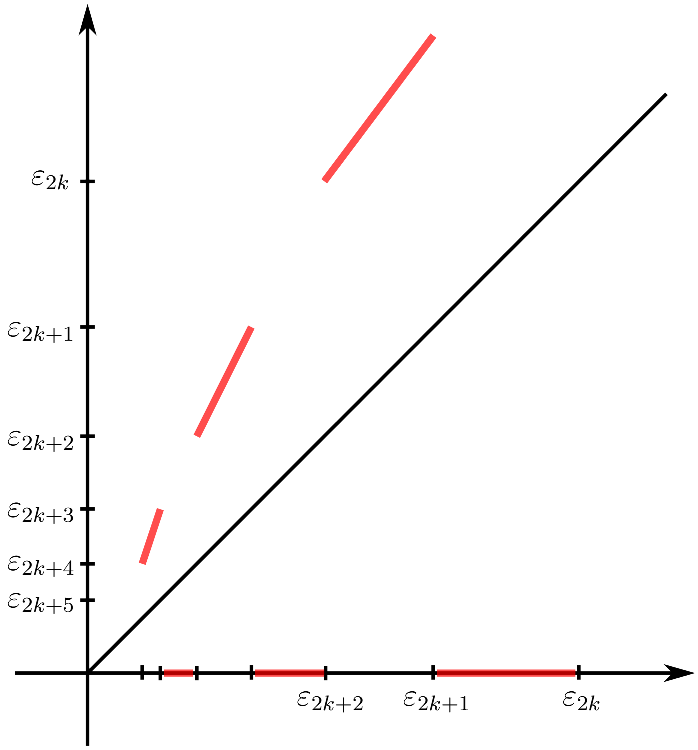


Figure 1.6.: $M : [0,\infty) \to [0,\infty)$

given by iterating M. For all $k \in \mathbb{N}$, points in $(\epsilon_{2k+2},\epsilon_{2k+1})$ move further and further away from 0, all other points are in the basin of attraction, they reach 0 after one iteration. Thus,

$$\Sigma_{\epsilon_{2k}}(0) > \frac{\epsilon_{2k}-\epsilon_{2k+1}}{\epsilon_{2k}} = 1-\epsilon_{2k} \quad \text{and} \quad \Sigma_{\epsilon_{2k+1}}(0) < \frac{\epsilon_{2k+2}}{\epsilon_{2+1}} = \epsilon_{2k+1},$$

and therefore

$$\frac{\ln(\Sigma_{\epsilon_{2k}}(0))}{\ln(\epsilon_{2k})} < \frac{\ln(1-\epsilon_{2k})}{\ln(\epsilon_{2k})} \overset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{\ln(\Sigma_{\epsilon_{2k+1}}(0))}{\ln(\epsilon_{2k+1})} > 1,$$

which shows that neither $\sigma(0)$ nor $\sigma_{\text{loc}}(0)$ exists. Now let us first modify M in a way that $\sigma(0)$ exists but $\sigma_{\text{loc}}(0)$ does not. We define a second map

$$M_1(y) := \begin{cases} M(y) & \text{for} \quad y \leq \frac{1}{2} \\ 0 & \text{for} \quad y > \frac{1}{2} \end{cases}$$

For this map the local basin is the same as for the original map, so $\sigma_{\text{loc}}(0)$ does not exist. However, all points in $[0,\infty)$ belong to the global basin of 0 because after initially leaving small neighbourhoods they reach the region where $y > \frac{1}{2}$ and they also are mapped to 0. Therefore, $\sigma(0) = +\infty$.

Now the other case. Set $M_2(y) := M(y)$ for $y \in (\epsilon_{2k},\epsilon_{2k-1}], k \in \mathbb{N}$, and instead of vanishing, let M_2 interpolate linearly between the existing values on the remaining intervals, except for $(\epsilon_1,\epsilon_0]$ where $M_2(y) = 0$ still. Then the local basin for small $\varepsilon > 0$ is empty, so $\sigma_{\text{loc}}(0) = -\infty$. However, the global basin now consists of precisely those points that are in the global basin for M. Thus, $\sigma(0)$ does not exist.

As in the original example of [33] these maps can be made continuous, but it is not clear whether similar examples exist for smooth maps or flows.

1.3. Non-asymptotic stability of very simple heteroclinic cycles in $\mathbb{R}^4$

In this section we take a detailed look at very simple heterolinic cycles in $\mathbb{R}^4$. This is the minimal dimension in which a cycle may have one eigenvalue of each type. While heteroclinic cycles do exist in $\mathbb{R}^3$, they have no transverse directions and we thus cannot expect the types of complex stability configurations that are associated with positive transverse eigenvalues. In particular, these allow us to combine cycles in a network (see chapter 2 of this thesis) and investigate different ways of stability loss.

In [28], Krupa and Melbourne enumerate all very simple cycles of types B and C in $\mathbb{R}^4$, we recall their results in subsection 1.3.1. Note again that they speak of simple cycles, but silently assume that $df(\xi_j)$ has no double eigenvalues. Combined with the computations in [33], which we revisit in subsection 1.3.2, we then deduce necessary and sufficient conditions for the various types of stability in the following subsections.

1.3.1. A classification of very simple heteroclinic cycles in $\mathbb{R}^4$

As mentioned above all cycles of types B and C in $\mathbb{R}^4$ are enumerated in [28]. We recall their result in this subsection, employing the usual notation $B_m^\pm$ and $C_m^\pm$, where m indicates the number of equilibria that make up the cycle. The superscript $\pm$ denotes whether $-\mathbb{1} \in \Gamma\,(-)$ or $-\mathbb{1} \notin \Gamma\,(+)$.

Lemma 1.44 ([28])**.** *There are seven distinct very simple heteroclinic cycles of type B and C in $\mathbb{R}^4$ and the only finite groups $\Gamma \subset O(n)$ that allow them are the ones denoted in parentheses:*

$$B_1^+(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^3),\ B_2^+(\mathbb{Z}_2^3),\ B_1^-(\mathbb{Z}_3 \ltimes \mathbb{Z}_2^4),\ B_3^-(\mathbb{Z}_2^4)$$
$$C_1^-(\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4),\ C_2^-(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4),\ C_4^-(\mathbb{Z}_2^4)$$

Proof. See [28], section 3 (b). □

As noted in [28], the enumeration of type A cycles is significantly more complicated, even in $\mathbb{R}^4$. There is a complete classification of homoclinic cycles of type A in $\mathbb{R}^4$ in [36]. In higher dimensions, an enumeration even of type B and C cycles, much less for type A, has not yet been done.

1.3.2. Stability results from [33]

In this subsection we recall the results that Podvigina and Ashwin obtained in [33] concerning the explicit calculation of stability indices for very simple cycles in $\mathbb{R}^4$. We transcribe their results below for ease of reference, following their notation by introducing the quantities $a_j = c_j/e_j$ and $b_j = -t_j/e_j$, depending on the eigenvalues of the linearization $df(\xi_j)$. These are used to define

$$\rho_j := \min(a_j,\ 1 + b_j) \quad \text{and} \quad \rho := \rho_1 \cdot \ldots \cdot \rho_m$$

as in subsection 1.1.2. We start by giving the stability indices for type A cycles and denote the index along the trajectory leading to ξ_j by σ_j. As in [33] we make no distinction between local and non-local indices here. Trajectories leaving a neighbourhood of the cycle are assumed to stay away from it for all positive times, so that $\sigma(x) = \sigma_{\text{loc}}(x)$. In the following subsections this allows us to use the sign of the indices to deduce attraction properties of the cycles by applying our results from section 1.2.

Theorem 1.45 ([33])**.** *Generically, for a simple robust heteroclinic cycle of type A in $\mathbb{R}^4$ the stability indices are as follows.*

(a) If $\rho > 1$ and $b_j > 0$ for all j, then $\sigma_j = +\infty$ for all j.
(b) If $\rho > 1$, $b_j > -1$ for all j and $b_j < 0$ for some j, then $\sigma_j > 0$ for all j.
(c) If $\rho < 1$ or there exists j such that $b_j < -1$, then $\sigma_j = -\infty$ for all j.

Proof. This is basically theorem 4.1 in [33]. In case (b) we do not give the full (complicated) expression for σ_j from [33], since we are only interested in the sign of the indices. In order to conclude that all indices are positive, one does not have to evaluate the expression for $\sigma_{j,+}$ by hand, since $\sigma_{j,+} \geq 0$ by construction and the case $\sigma_{j,+} = \sigma_{j,-} = 0$ is degenerate. □

In [33] finite stability indices are conveniently expressed through the function f^{index}, defined on $\mathbb{R}^2$ without the diagonal $\mathbb{D} := \{(x,y) \in \mathbb{R}^2 \mid x = -y\}$,

$$f^{\text{index}} : \mathbb{R}^2 \setminus \mathbb{D} \to [-\infty,\infty], \quad f^{\text{index}}(\alpha,\beta) := f^{+}(\alpha,\beta) - f^{-}(\alpha,\beta)$$

where $f^-(\alpha,\beta) := f^+(-\alpha,-\beta)$ and:

$$f^+(\alpha,\beta) := \begin{cases} +\infty, & \alpha,\beta \geq 0, \\ \ \ 0, & \alpha,\beta \leq 0, \\ -\frac{\beta}{\alpha} - 1, & \alpha < 0 < \beta,\ \frac{\beta}{\alpha} < -1 \\ \ \ 0, & \alpha < 0 < \beta,\ \frac{\beta}{\alpha} > -1 \\ -\frac{\alpha}{\beta} - 1, & \alpha > 0 > \beta,\ \frac{\alpha}{\beta} < -1 \\ \ \ 0, & \alpha > 0 > \beta,\ \frac{\alpha}{\beta} > -1 \end{cases}$$

We facilitate future use of f^{index} by the following result.

Lemma 1.46. *For $\alpha,\beta \in \mathbb{R}$ we have*

(a) $f^{\text{index}}(\alpha,1) \in (0,+\infty)$ *if and only if* $\alpha \in (-1,0)$*,*
(b) $f^{\text{index}}(\alpha,1) \in (-\infty,0)$ *if and only if* $\alpha < -1$*,*
(c) $f^{\text{index}}(\beta,-1) \in (0,+\infty)$ *if and only if* $\beta > 1$*,*
(d) $f^{\text{index}}(\beta,-1) \in (-\infty,0)$ *if and only if* $\beta \in (0,1)$*.*

Proof. We calculate

$$f^+(\alpha,1) = \begin{cases} +\infty & \text{for } \alpha \geq 0 \\ -\frac{1}{\alpha} - 1 & \text{for } \alpha \in (-1,0) \\ \ \ 0 & \text{for } \alpha < -1 \end{cases}$$

and

$$f^-(\alpha,1) = f^+(-\alpha,-1) = \begin{cases} \ \ 0 & \text{for } \alpha \geq 0 \\ \ \ 0 & \text{for } \alpha \in (-1,0) \\ -\alpha - 1 & \text{for } \alpha < -1 \end{cases},$$

so

$$f^{\text{index}}(\alpha,1) = f^+(\alpha,1) - f^-(\alpha,1) = \begin{cases} +\infty & \text{for } \alpha \geq 0 \\ -\frac{1}{\alpha} - 1 > 0 & \text{for } \alpha \in (-1,0) \\ \ \ \alpha + 1 < 0 & \text{for } \alpha < -1 \end{cases}.$$

This proves (a) and (b), statements (c) and (d) follow in a similar manner. $\square$

The next lemmas give the stability indices for four non-homoclinic B- and C-cycles. In subsection 4.2.1 of [33] there are corresponding results for all such cycles in $\mathbb{R}^4$, we quote here only what we need later.

Lemma 1.47 ([33], p. 906)**.** *Generically, for a cycle of type B_2^+ in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) If $b_1 < 0$ and $b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(ii) Suppose $b_1 > 0$ and $b_2 > 0$.

(a) If $a_1a_2 < 1$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1a_2 > 1$, then the cycle is locally attracting and all stability indices are equal to $+\infty$.

(iii) Suppose $b_1 < 0$ and $b_2 > 0$.

(a) If $a_1a_2 < 1$ or $b_1a_2 + b_2 < 0$, then the cycle is not an attractor and all indices are equal to $-\infty$.

(b) If $a_1a_2 > 1$ and $b_1a_2 + b_2 > 0$, then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1,1), \quad \sigma_2 = +\infty.$$

Lemma 1.48 ([33], pp. 906–907)**.** *Generically, for a cycle of type B_3^- in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) If $b_1 < 0$, $b_2 < 0$ and $b_3 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(ii) Suppose $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$.

(a) If $a_1a_2a_3 < 1$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1a_2a_3 > 1$, then the cycle is locally attracting and all stability indices are equal to $+\infty$.

(iii) Suppose $b_1 < 0$, $b_2 > 0$ and $b_3 > 0$.

(a) If $a_1a_2a_3 < 1$ or $b_1a_2a_3 + b_3a_2 + b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1a_2a_3 > 1$ and $b_1a_2a_3 + b_3a_2 + b_2 > 0$, then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1,1), \quad \sigma_2 = +\infty, \quad \sigma_3 = f^{\text{index}}(b_3 + b_1a_3,1).$$

(iv) Suppose $b_1 < 0$, $b_2 < 0$ and $b_3 > 0$.

(a) If $a_1a_2a_3 < 1$ or $b_2a_1a_3 + b_1a_3 + b_3 < 0$ or $b_1a_2a_3 + b_3a_2 + b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) *If* $a_1a_2a_3 > 1$ *and* $b_2a_1a_3 + b_1a_3 + b_3 > 0$ *and* $b_1a_2a_3 + b_3a_2 + b_2 > 0$*, then the stability indices are*

$$\sigma_1 = \min\left(f^{\text{index}}(b_1,1),\ f^{\text{index}}(b_1 + b_2a_1,1)\right), \quad \sigma_2 = f^{\text{index}}(b_2,1), \quad \sigma_3 = +\infty.$$

Note that compared to the statement in [33], in lemma 1.48 (iv) (b) we have replaced $\sigma_3 = f^{\text{index}}(b_3 + b_1a_3,1)$ by $\sigma_3 = +\infty$. This is true since

$$b_2a_1a_3 + b_1a_3 + b_3 > 0 \quad \Rightarrow \quad b_1a_3 + b_3 > -b_2a_1a_3 > 0$$

and $f^{\text{index}}(\alpha,\beta) = +\infty$ for $\alpha,\beta > 0$.

Now we state the corresponding result for C_2^--cycles. Denote by λ_1, λ_2 the eigenvalues of the product of transition matrices

$$M := M_1M_2 = \begin{pmatrix} b_1b_2 + a_2 & b_1 \\ a_1b_2 & a_1 \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2$ if both are real. Note that tr$M = b_1b_2 + a_1 + a_2$ and $\det M = a_1a_2$.

Lemma 1.49 ([33], pp. 907–908)**.** *Generically, for a cycle of type C_2^- in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) *If* $b_1 < 0$ *and* $b_2 < 0$*, then the cycle is not an attractor and all stability indices are equal to* $-\infty$*.*

(ii) *Suppose* $b_1 > 0$ *and* $b_2 > 0$*.*

  (a) *If* $\max(\text{tr}M,\ 2(\text{tr}M - \det M)) < 2$*, then the cycle is not an attractor and all stability indices are equal to* $-\infty$*.*

  (b) *Otherwise the cycle is locally attracting and all stability indices are equal to* $+\infty$*.*

(iii) *Suppose* $b_1 < 0$ *and* $b_2 > 0$*.*

  (a) *If one of the following conditions is satisfied*

  - $(\text{tr}M)^2 - 4\det M < 0$,
  - $\max(\text{tr}M,\ 2(\text{tr}M - \det M)) < 2$,
  - $b_1b_2 - a_1 + a_2 < 0$,

  then the cycle is not an attractor and all stability indices are equal to $-\infty$*.*

  (b) *If none of the conditions above are satisfied, then the stability indices are*

$$\sigma_1 = f^{\text{index}}\left(\frac{b_1b_2 + a_1 - \lambda_2}{b_2},1\right), \quad \sigma_2 = f^{\text{index}}\left(\frac{\lambda_2 - b_1b_2 - a_2}{b_1}, -1\right).$$

Compared to the result in [33] we have replaced the first expression in (iii)(a):

$$\begin{aligned}(b_1b_2+a_2-a_1)^2+4a_1b_1b_2 &= (b_1b_2+a_2)^2-2a_1(b_1b_2+a_2)+a_1^2+4a_1b_1b_2\\ &= (b_1b_2+a_2)^2+2a_1(b_1b_2+a_2)+a_1^2-4a_1a_2\\ &= (b_1b_2+a_1+a_2)^2-4a_1a_2\\ &= (\text{tr}M)^2-4\det M\end{aligned}$$

The last result of this kind that we need is for C_4^--cycles. We list the indices only for the case we are interested in and do not reproduce the entire classification from [33]. Again some notation is required. The product of the transition matrices turns out to be:

$$M^{4,1} := M_4M_3M_2M_1 = \begin{pmatrix} (b_1b_2+a_1)(b_3b_4+a_3)+b_1a_2b_4 & (b_3b_4+a_3)b_2+b_4a_2 \\ a_4b_3(b_1b_2+a_1)+a_2a_4b_1 & a_4b_2b_3+a_2a_4 \end{pmatrix}$$

By $M^{j+3,j}$ we denote the matrix with cyclically permuted factors M_i. Here $j+3$ is to be understood mod 4 in the usual way. Again we denote the eigenvalues by $\lambda_1 \geq \lambda_2$ if both are real. They are independent of j since all $M^{j+3,j}$ are similar matrices, implying that they have equal eigenvalues, determinant and trace. The associated eigenvectors we write as

$$v_1^{j+3,j} = (v_{11}^{j+3,j}, v_{12}^{j+3,j}) \quad \text{and} \quad v_2^{j+3,j} = (v_{21}^{j+3,j}, v_{22}^{j+3,j}),$$

respectively. With this define

$$h^{j+3,j} := v_{11}^{j+3,j}v_{22}^{j+3,j} - v_{12}^{j+3,j}v_{21}^{j+3,j},$$

which puts us in a position to state the result. To reduce the number of sub- and superscripts set $M := M^{4,1}$.

Lemma 1.50 ([33], pp. 908–909)**.** *For a cycle of type C_4^-, suppose that $b_1 < 0$ and $b_j > 0$ for $j \neq 1$.*

(i) If one of the following holds, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(a) $\max(\text{tr}M,\ 2(\text{tr}M-\det M)) < 2$

(b) $(\text{tr}M)^2-4\det M < 0$

(c) $v_{11}^{1,2}v_{12}^{1,2} < 0$

(ii) Otherwise the stability indices along connecting trajectories are as follows:

$$\sigma_1 = \min\left(f^{\text{index}}(v_{22}^{4,1}/h^{4,1}, -v_{21}^{4,1}/h^{4,1}),\ f^{\text{index}}(b_1,1)\right)$$

$$\sigma_2 = \min\left(f^{\text{index}}(v_{22}^{1,2}/h^{1,2}, -v_{21}^{1,2}/h^{1,2}),\ f^{\text{index}}(b_1b_4+a_4,b_1)\right)$$

$$\sigma_3 = \min\left(f^{\text{index}}(v_{22}^{2,3}/h^{2,3}, -v_{21}^{2,3}/h^{2,3}),\ f^{\text{index}}(b_3(b_1b_4+a_4)+b_1a_3,b_1b_4+a_4)\right)$$

$$\sigma_4 = f^{\text{index}}(v_{22}^{3,4}/h^{3,4}, -v_{21}^{3,4}/h^{3,4})$$

Note that the expressions given for the stability indices in [33] differ from the ones above in this particular case. This has been noticed in private communication between Olga Podvigina, who subsequently provided the above correction, and this author.

We can use theorems 1.33 and 1.34 together with these lemmas to give necessary and sufficient criteria for predominant asymptotic stability of very simple robust heteroclinic cycles in $\mathbb{R}^4$ that depend only on the eigenvalues at the equilibria. We sum up how to do this in the following preparatory result.

Lemma 1.51. *Assume that for a very simple heteroclinic cycle X in $\mathbb{R}^4$ all stability indices exist. Then the following equivalences hold*

- $\sigma_i = +\infty$ *for all* $i \Leftrightarrow X$ *is asymptotically stable.*
- $\sigma_i = -\infty$ *for all* $i \Leftrightarrow X$ *is completely unstable.*
- $\sigma_i > 0$ *for all* $i \Leftrightarrow X$ *is predominantly asymptotically stable.*
- $\sigma_i < 0$ *for all* $i \Leftrightarrow X$ *is predominantly unstable.*

Proof. For the first two statements the implications from right to left are trivial. The other directions follow from results for the different types of cycles in subsections 4.2.1 and 4.2.2 of [33], most of which we have just listed above. The third and fourth statement follow from our considerations in subsection 1.2.2, keeping in mind that for very simple cycles in $\mathbb{R}^4$ the basin of attraction generically is an algebraic cusp shaped by ratios of the eigenvalues. □

There is one more result in [33], corollary 4.1, that we make use of later. It is a direct consequence of the calculation of stability indices for all very simple cycles in $\mathbb{R}^4$:

Corollary 1.52 ([33])**.** *For a very simple heteroclinic cycle in $\mathbb{R}^4$, $\sigma_j = -\infty$ for some j if and only if $\sigma_j = -\infty$ for all j.*

In the following subsections we investigate the different types of cycles one at a time.

1.3.3. Stability of type A cycles

For type A cycles in $\mathbb{R}^4$, theorem 1.11 simplifies to the following conditions for the different stability properties. This is basically a reformulation of theorem 2.4 in [27], with a slight refinement in case (c). Note that, as mentioned before, in $\mathbb{R}^4$ all type A cycles are of type A^*.

Theorem 1.53 ([27])**.** *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type A is generically*

(a) asymptotically stable if and only if $\rho > 1$ and $t_j < 0$ for all j,

(b) p.a.s. (but not a.s.) if and only if $\rho > 1$ and there is at least one positive transverse eigenvalue, but $t_j < e_j$ holds for all j,

(c) completely unstable if $\rho < 1$ or there is j with $t_j > e_j$.

Proof. This follows directly from combining theorem 1.45 with lemma 1.51. □

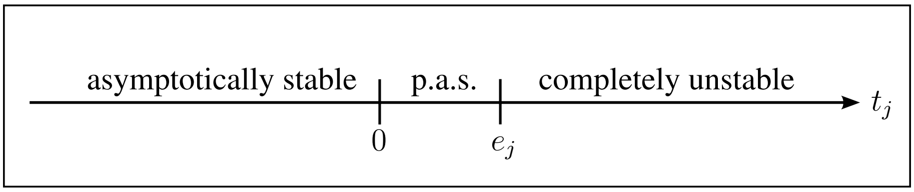


Figure 1.7.: Stability change for type A cycles when t_j turns positive

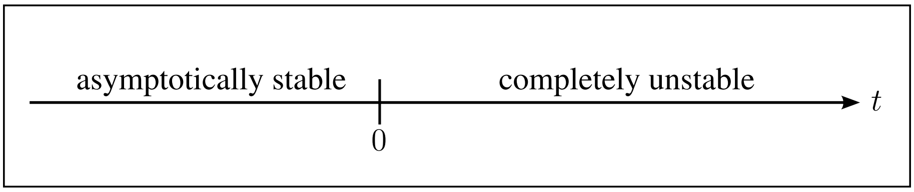


Figure 1.8.: Stability change for homoclinic type A and B cycles when t turns positive

We have hereby improved the dichotomy of theorem 1.11 for the special case $n = 4$. In $\mathbb{R}^4$ no type A cycle is genuinely predominantly unstable: either all indices are positive or they are all equal to $-\infty$. The change of stability for a type A cycle as one of the transverse eigenvalues becomes positive (while $\rho > 1$) is schematically depicted in figure 1.7. Note that if the cycle is homoclinic, then there is no p.a.s. region, see figure 1.8. This is because $\rho < 1$ if all transverse eigenvalues are positive (and for a homoclinic cycle there is only one), thus the cycle is completely unstable. The condition for asymptotic stability reduces to $c > e$ and $t < 0$ in the case of homoclinic cycles.

1.3.4. Stability of type B cycles

There are four cycles of type B in $\mathbb{R}^4$, two homoclinic and two heteroclinic ones, respectively. For these we obtain the following three theorems. The first one we only list for the sake of completeness as it has been proved in even greater generality in [28].

Theorem 1.54 ([28])**.** *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type B_1^+ or B_1^- is*

(a) asymptotically stable if and only if $c > e$ and $t < 0$,

(b) completely unstable otherwise.

Proof. This follows from the considerations in remark 3.3 of [28] or, alternatively, from subsection 4.2.1 in [33]. □

The stability change diagram is very simple for homoclinic cycles of type B. In fact, it is the same as that for homoclinic type A cycles and shown in figure 1.8, assuming $c > e$. Corresponding results for the two heteroclinic cycles of type B have not been known up to now. With the following two theorems we extend the results in subsection 1.1.2 to non-asymptotic stability, at least for $\mathbb{R}^4$.

Theorem 1.55. *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type B_2^+ is*

(a) asymptotically stable if and only if the following two conditions hold.

- $c_1c_2 > e_1e_2$
- $t_1,t_2 < 0$

(b) p.a.s. (but not a.s.) if and only if the following three conditions hold.

- $c_1c_2 > e_1e_2$
- $t_2 < 0$
- $0 < t_1 < \min\left(e_1, -\dfrac{e_1t_2}{c_2}\right)$

(c) p.f.a.s. if and only if the following three conditions hold.

- $c_1c_2 > e_1e_2$
- $t_2 < 0$
- $0 < e_1 < t_1 < -\dfrac{e_1t_2}{c_2}$

(d) completely unstable if and only if one of the following conditions holds.

- $c_1c_2 < e_1e_2$
- $t_1,t_2 > 0$
- $t_2 < 0 < t_1$ *and* $t_1 > -\dfrac{e_1t_2}{c_2}$.

Proof. By lemma 1.51 it suffices to calculate the conditions for the stability indices being

- all equal to $+\infty$ (a.s.),
- positive, but not all equal to $+\infty$ (p.a.s.),
- negative, but not all equal to $-\infty$ (p.u.),
- all equal to $-\infty$ (c.u.).

In the remaining cases the cycle is p.f.a.s. We calculate these conditions based on lemma 1.47.

(a) Both $\sigma_1 = \sigma_2 = +\infty$ if and only if $a_1a_2 > 1$ and $b_1,b_2 > 0$, i.e. if and only if $c_1c_2 > e_1e_2$ and $t_1, t_2 < 0$.

(b) We need $\sigma_1,\sigma_2 > 0$, but at least one of them not equal to infinity. This is only possible for $b_1 < 0 < b_2$ and $a_1a_2 > 1$, i.e. $t_2 < 0 < t_1$ and $c_1c_2 > e_1e_2$. Moreover, it is necessary that

$$b_1a_2 + b_2 > 0 \quad \Leftrightarrow \quad -\frac{t_1c_2}{e_1e_2} - \frac{t_2}{e_2} > 0 \quad \Leftrightarrow \quad t_1 < -\frac{e_1t_2}{c_2}.$$

Then we have $\sigma_2 = +\infty$ and $\sigma_1 = f^{\text{index}}(b_1,1)$. The latter expression has to be in $(0,\infty)$, which by lemma 1.46 is the case if and only if $b_1 \in (-1,0)$, which is the same as $0 < t_1 < e_1$.

(d) Both $\sigma_1 = \sigma_2 = -\infty$ if and only if the following holds

$$a_1a_2 < 1 \quad \vee \quad b_1,b_2 < 0 \quad \vee \quad (b_1 < 0 < b_2 \quad \wedge \quad b_1a_2 + b_2 < 0).$$

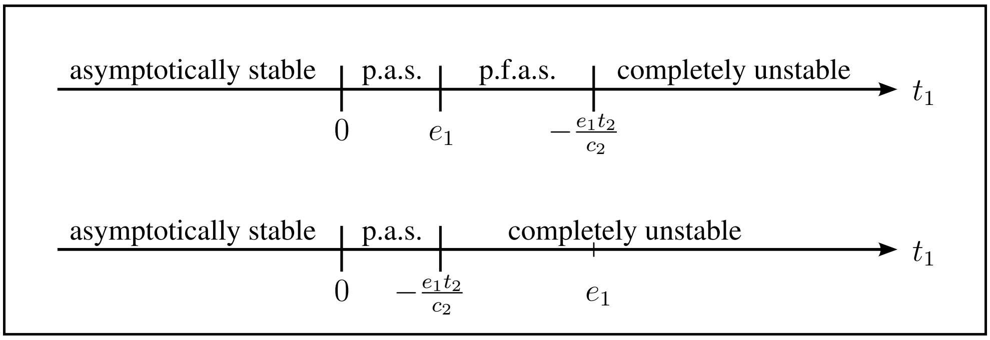


Figure 1.9.: Stability change for B_2^+-cycles when t_1 turns positive ($t_2 < 0$)

This is equivalent to

$$c_1 c_2 < e_1 e_2 \quad \vee \quad t_1{,}t_2 > 0 \quad \vee \quad \left(t_2 < 0 < t_1 \quad \wedge \quad t_1 > -\frac{e_1 t_2}{c_2} \right).$$

(c) The cycle is never predominantly unstable since in lemma 1.47 there is always either at least one index equal to $+\infty$ or all indices are equal to $-\infty$. Thus, in all remaining cases the cycle is p.f.a.s.

□

Theorem 1.55 gives us a clear description of the stability changes that the cycle undergoes when a transverse eigenvalue becomes positive, see figure 1.9. In these two diagrams $t_2 < 0$ is fixed and we consider t_1 as the bifurcation parameter all the while assuming that $c_1 c_2 > e_1 e_2$. Note that for such a cycle to be p.f.a.s it is necessary that $-c_2 > t_2$, i.e. the flow at ξ_2 has to be more strongly contracting in the transverse direction than in the direction of the cycle. In this case $\min\left(e_1, -\frac{e_1 t_2}{c_2}\right) = e_1$ and the p.a.s./p.f.a.s. question is decided by $t_1 \lessgtr e_1$, depending on which direction (transverse or expanding along the cycle) is more unstable.

In [30] a cycle of type B_2^+ is investigated and sufficient conditions for its predominant asymptotic stability are calculated. They obviously fall into (b) from above. In theorem 1.55 we have generalized those results, giving necessary and sufficient conditions for all stability types.

We now turn our attention to B_3^--cycles. There are three transverse eigenvalues to consider, resulting in a two-dimensional picture for the stability changes. Apart from that, the calculations are similar.

Theorem 1.56. *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type B_3^- is*

(a) asymptotically stable if and only if the following two conditions hold.

+ $c_1 c_2 c_3 > e_1 e_2 e_3$
+ $t_1{,}t_2{,}t_3 < 0$

(b) p.a.s. (but not a.s.) if and only if the following four conditions hold.

+ $c_1c_2c_3 > e_1e_2e_3$
+ $t_3 < 0$
+ $0 < t_1 < \min\left(e_1, \frac{e_1(e_3 - t_3)}{c_3}, \frac{e_1(-t_3c_2 - e_3t_2)}{c_2c_3}\right)$
+ $0 < t_2 < \min\left(e_2, \frac{e_2(e_1 - t_1)}{c_1}, \frac{e_2(-t_1c_3 - e_1t_3)}{c_1c_3}\right)$ *or* $t_2 < 0$

(c) completely unstable if and only if one of the following four conditions is satisfied.

- $c_1c_2c_3 < e_1e_2e_3$
- $t_1, t_2, t_3 > 0$
- $t_1 > 0 > t_3$ *and* $t_1 > -\frac{e_1(t_3c_2 + t_2e_3)}{c_2c_3}$
- $t_1, t_2 > 0 > t_3$ *and* $t_2 > -\frac{e_2(t_1c_3 + t_3e_1)}{c_1c_3}$.

(d) In all other cases the cycle is p.f.a.s. In particular, the cycle is never predominantly unstable.

Proof. Just as in the previous theorem it suffices to check, with lemma 1.48, when all stability indices are positive/negative or equal to $\pm\infty$.

(a) Clearly, $\sigma_1 = \sigma_2 = \sigma_3 = +\infty$ if and only if $a_1a_2a_3 > 1$ and $b_1, b_2, b_3 > 0$, i.e. if and only if $c_1c_2c_3 > e_1e_2e_3$ and $t_1, t_2, t_3 < 0$.

(b) For all stability indices to be positive but at least one of them not equal to $+\infty$, we need $a_1a_2a_3 > 1$, i.e. $c_1c_2c_3 > e_1e_2e_3$ and at least one positive and one negative transverse eigenvalue, say $t_1 > 0$ and $t_3 < 0$. Moreover, we must have

$$\begin{aligned} 0 < b_1a_2a_3 + a_2b_3 + b_2 \quad &\Leftrightarrow \quad 0 < -\frac{t_1c_2c_3}{e_1e_2e_3} - \frac{c_2t_3}{e_2e_3} - \frac{t_2}{e_2} \\ &\Leftrightarrow \quad t_1 < -\frac{e_1(t_3c_2 + e_3t_2)}{c_2c_3}. \end{aligned}$$

If then $t_2 < 0$, then we have $\sigma_2 = +\infty$ and for the other two indices to be positive we need

$$\sigma_1 = f^{\text{index}}(b_1, 1) > 0 \quad \Leftrightarrow \quad b_1 > -1 \quad \Leftrightarrow \quad t_1 < e_1,$$

and

$$\sigma_3 = f^{\text{index}}(b_3 + b_1a_3, 1) > 0 \quad \Leftrightarrow \quad b_3 + b_1a_3 > -1 \quad \Leftrightarrow \quad t_1 < \frac{e_1(e_3 - t_3)}{c_3}$$

Since $t_1 > 0$ we definitely have $\sigma_1 < +\infty$, so this is not a case of asymptotic stability.

If on the other hand $t_2 > 0$, then we also require

$$b_2a_1a_3 + a_3b_1 + b_3 > 0 \quad \Leftrightarrow \quad t_2 < -\frac{e_2(t_1c_3 + e_1t_3)}{c_1c_3}.$$

For positive stability indices we then need

$$\begin{aligned}
\sigma_1 &= \min\Big(f^{\text{index}}(b_1,1),\ f^{\text{index}}(b_1+b_2a_1,1)\Big) > 0 && \Leftrightarrow \quad t_1 < e_1 \ \wedge\ t_2 < \frac{e_2(e_1-t_1)}{c_1},\\
\sigma_2 &= f^{\text{index}}(b_2,1) > 0 && \Leftrightarrow \quad t_2 < e_2,\\
\sigma_3 &= f^{\text{index}}(b_3+b_1a_3,1) > 0 && \Leftrightarrow \quad t_1 < \frac{e_1(e_3-t_3)}{c_3}.
\end{aligned}$$

Again $t_1 > 0$, so $\sigma_1 < +\infty$ and this is also not a case of asymptotic stability.

(c) We have $\sigma_1 = \sigma_2 = \sigma_3 = -\infty$ if and only if one of the following holds:

- $a_1a_2a_3 < 1$
- $b_1,b_2,b_3 < 0$
- $b_1 < 0 < b_3$ and $b_1a_2a_3 + b_3a_2 + b_2 < 0$
- $b_1,b_2 < 0 < b_3$ and $b_2a_1a_3 + b_1a_3 + b_3 < 0$

These conditions are equivalent to:

- $c_1c_2c_3 < e_1e_2e_3$
- $t_1,t_2,t_3 > 0$
- $t_1 > 0 > t_3$ and $t_1 > -\frac{e_1(t_3c_2+e_3t_2)}{c_2c_3}$
- $t_1,t_2 > 0 > t_3$ and $t_2 > -\frac{e_2(t_1c_3+e_1t_3)}{c_1c_3}$

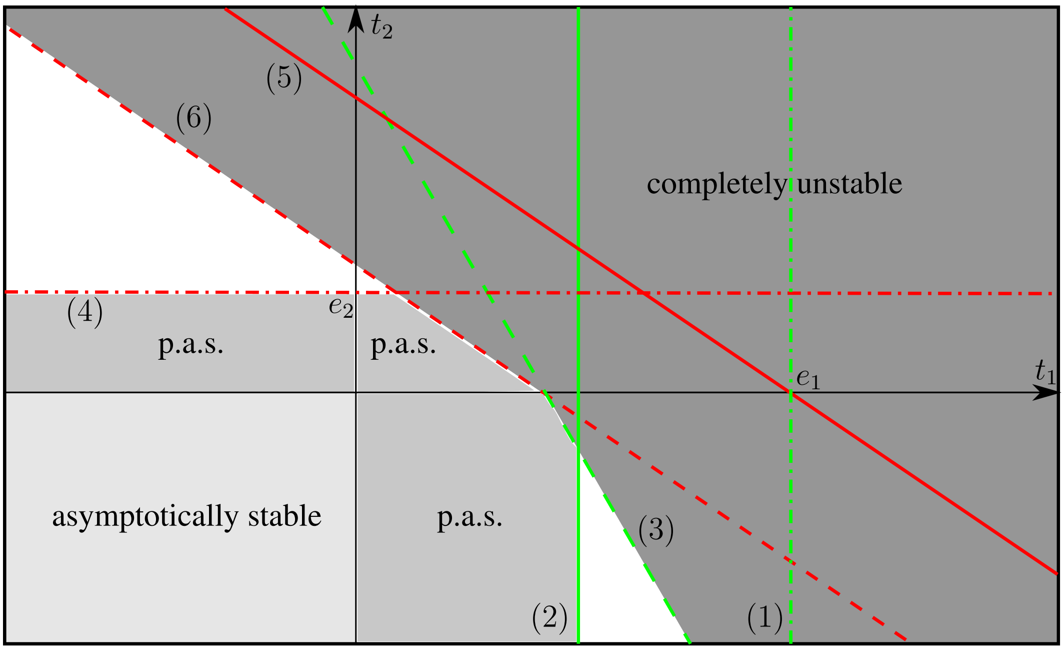
(d) Again it suffices to show that the cycle is never p.u. This follows from lemma 1.48 for the same reason as in the previous theorem.

□

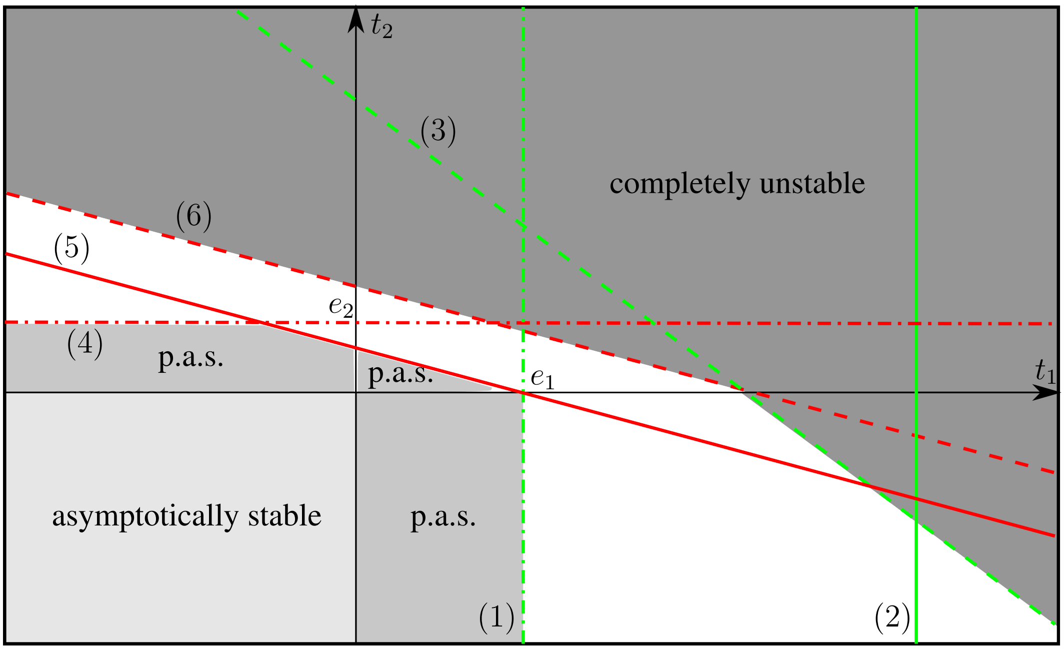
Stability Changes for B_3^--cycles

In the same way as for the B_2^+-cycles we now investigate what happens when transverse eigenvalues become positive for type B_3^--cycles. Here the situation is slightly more complex as we have not only t_1 but also t_2 as a varying parameter. We assume $c_1c_2c_3 > e_1e_2e_3$ and without loss of generality fix $t_3 < 0$ since three positive transverse eigenvalues immediately lead to complete instability. The diagram we obtain and the extent of the regions of different stability properties of course depend on the slopes of the boundary lines, and therefore on the eigenvalues at the equilibria. However, there are only two qualitatively different cases, depicted in figure 1.10.

From theorem 1.56 the stability properties for a parameter combination (t_1,t_2) are deter-



(a) $-c_3 < t_3$



(b) $-c_3 > t_3$

Figure 1.10.: Stability diagram for B_3^--cycles

mined by its relative position to the six lines given by:

$$t_1(t_2) = e_1 \tag{1}$$
$$t_1(t_2) = \frac{e_1(e_3 - t_3)}{c_3} \tag{2}$$
$$t_1(t_2) = \frac{e_1(-t_3c_2 - e_3t_2)}{c_2c_3} \tag{3}$$
$$t_2(t_1) = e_2 \tag{4}$$
$$t_2(t_1) = \frac{e_2(e_1 - t_1)}{c_1} \tag{5}$$
$$t_2(t_1) = \frac{e_2(-t_1c_3 - e_1t_3)}{c_1c_3} \tag{6}$$

The first three lines bound the region of predominant asymptotic stability whenever t_1 is positive, the other three whenever $t_2 > 0$. Lines (5) and (6) are parallel, they have the same slope $-\frac{e_2}{c_1}$. The intersections of all lines with the t_1- and t_2-axis are collected in table 1.2.

Table 1.2.: Intersections with the t_i-axes

	(1)	(2)	(3)	(4)	(5)	(6)
t_1-axis	e_1	$\frac{e_1(e_3-t_3)}{c_3}$	$-\frac{e_1t_3}{c_3}$	-	e_1	$-\frac{e_1t_3}{c_3}$
t_2-axis	-	-	$-\frac{c_2t_3}{e_3}$	e_2	$\frac{e_1e_2}{c_1}$	$-\frac{e_1e_2t_3}{c_1c_3}$

Since $e_i,c_i > 0$ and $c_1c_2c_3 > e_1e_2e_3$ we have

$$\frac{e_1(e_3 - t_3)}{c_3} > -\frac{e_1t_3}{c_3} \quad \text{and} \quad -\frac{e_1e_2t_3}{c_1c_3} = -\frac{e_1e_2e_3}{c_1c_2c_3}\frac{c_2t_3}{e_3} < -\frac{c_2t_3}{e_3},$$

restricting the number of possibilities for ordering the intersection points along the axes. Now consider two cases, depending on whether the contracting or transverse direction at ξ_3 dominates (in the sense that the contraction is stronger):

(a) $-c_3 < t_3$. Then $\frac{e_1e_2}{c_1} > -\frac{e_1e_2t_3}{c_1c_3}$ and $e_1 > -\frac{e_1t_3}{c_3}$.

(b) $-c_3 > t_3$. Then $\frac{e_1e_2}{c_1} < -\frac{e_1e_2t_3}{c_1c_3}$ and $e_1 < -\frac{e_1t_3}{c_3}$.

For both cases the (t_1,t_2)-plane is shown in figure 1.10. From the picture for (a) it is clear that the inequalities

$$e_1 \gtrless \frac{e_1(e_3 - t_3)}{c_3} \quad \text{and} \quad \frac{e_1e_2}{c_1} \gtrless -\frac{c_2t_3}{e_3}$$

as well as the relative position of the line $t_2(t_1) = e_2 > 0$ do not qualitatively affect the dynamics. In (b) all relative positions are fixed except for that of $t_2(t_1) = e_2 > 0$, which is qualitatively irrelevant in this case, too.

From figure 1.10 we deduce the following result.

Corollary 1.57. *Let X be a very simple heteroclinic cycle of type B_3^- in $\mathbb{R}^4$. Suppose $c_1c_2c_3 > e_1e_2e_3$ and $t_3 < 0$. In the (t_1,t_2)-plane consider paths leading from the region of asymptotic stability to that of complete instability.*

- *If $-c_3 < t_3$, then no such path that is sufficiently close to the origin leads through an open region where X is p.f.a.s.*
- *If $-c_3 > t_3$, then every such path leads through an open region where X is p.f.a.s.*

In other words, in the second case the p.f.a.s.-region in the (t_1,t_2)-plane (white in figure 1.10) is connected while in the first it is not. This corresponds to what we learned about cycles of type B_2^+ above: such a cycle can only be p.f.a.s if $-c_2 > t_2$. For B_3^--cycles, along a path that is sufficiently close to the origin proper fragmentary asymptotic stability only occurs if $-c_3 > t_3$.

In terms of the stability indices corollary 1.57 means that in the first case, along a path close to the origin in the (t_1,t_2)-plane, all indices along the cycle have the same sign, meaning either all trajectories of the cycle are visible or none. In particular, the cycle will go from all indices equal to $+\infty$, to all indices positive directly to all indices equal to $-\infty$. In the second case, for every path leading from asymptotic stability to complete instability, there is a region where there are indices along the cycle with opposite signs. From these considerations we can quickly deduce the following two statements about type B cycles in $\mathbb{R}^4$.

Corollary 1.58. *A very simple heteroclinic cycle of type B in $\mathbb{R}^4$ is generically p.a.s. after a transverse bifurcation.*

Corollary 1.59. *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type B is never predominantly unstable.*

1.3.5. Stability of type C cycles

There are three type C cycles in $\mathbb{R}^4$, one homoclinic and two heteroclinic ones. In the same way as before we derive conditions for the different forms of stability. Again the statement for homoclinic cycles is not new since there are no intermediate types of stability. It is a special case of the result for type C cycles in [28].

Theorem 1.60 ([28])**.** *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type C_1^- is*

(a) asymptotically stable if and only if $t < 0$ and $c - t > e$,

(b) completely unstable otherwise.

Proof. This follows from theorem 4.3 in [28] as well as from subsection 4.2.2 in [33]. □

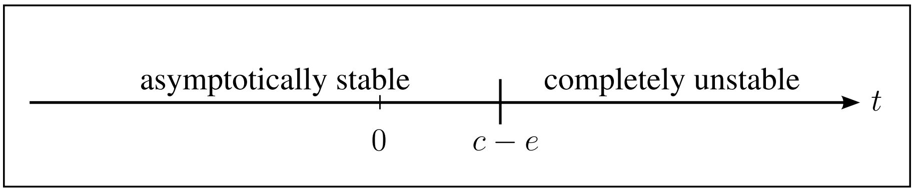


Figure 1.11.: Stability change for homoclinic type C cycles when t turns positive

The stability change diagram is shown in figure 1.11. In contrast to homoclinic A- and B-cycles the condition $c > e$ is not necessary to get anything but complete instability.

Now we move on to the C_2^--cycle. Recall that $\lambda_1 \geq \lambda_2$ denote the eigenvalues of its transition matrix product

$$M = \begin{pmatrix} b_1 b_2 + a_2 & b_1 \\ a_1 b_2 & a_1 \end{pmatrix}.$$

Then we have the following result.

Theorem 1.61. *In $\mathbb{R}^4$ a very simple heteroclinic cycle of type C_2^- is*

(a) asymptotically stable if and only if the following two conditions hold.

+ $t_1, t_2 < 0$
+ $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) > 2$

(b) completely unstable if and only if one of the following holds.

∘ $t_1, t_2 > 0$
∘ $t_1, t_2 < 0$ *and* $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$
∘ $t_2 < 0 < t_1$ *and one of the following three:*
  * $(\mathrm{tr}M)^2 - 4\det M < 0$
  * $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$
  * $b_1 b_2 - a_1 + a_2 < 0$

(c) p.a.s. (but not a.s.) if and only if the following three conditions hold.

+ $t_2 < 0 < t_1$
+ *none of the conditions in (b) hold*
+ $\lambda_2 \in (b_1 b_2 + a_1,\ b_1 b_2 + \min(a_1 + b_2,\ a_2 + b_1))$

(d) p.u. if and only if the following three conditions hold.

+ $t_2 < 0 < t_1$
+ *none of the conditions in (b) hold*
+ $\lambda_2 \in (b_1 b_2 + \max(a_1 + b_2,\ a_2 + b_1),\ b_1 b_2 + a_2)$

(e) In all other cases the cycle is p.f.a.s.

Case (a) was already treated by [28]. In fact, it follows from their theorem 4.3 on asymptotic stability for type C cycles in $\mathbb{R}^4$. Theorem 1.61 extends this result.

Proof. Cases (a) and (b) follow directly from lemma 1.49. There is only one case in which the stability indices are not equal to $\pm\infty$. This is the one we need to investigate in order to prove (c) and (d). The first two conditions are the same in both cases and just the ones prohibiting complete instability. The stability indices then are

$$\sigma_1 = f^{\text{index}}(\alpha,1) \qquad \text{with} \quad \alpha = \frac{b_1 b_2 + a_1 - \lambda_2}{b_2}$$
$$\sigma_2 = f^{\text{index}}(\beta,-1) \qquad \text{with} \quad \beta = \frac{\lambda_2 - b_1 b_2 - a_2}{b_1}$$

As before we have to determine when $\sigma_1,\sigma_2 > 0$ for (c) and $\sigma_1,\sigma_2 < 0$ for (d), with at least one of them being finite in either case. Note that $\lambda_2 \in \mathbb{R}$ since

$$\lambda_2 = \frac{\text{tr}M}{2} - \sqrt{\frac{(\text{tr}M)^2}{4} - \det M},$$

and the expression under the root is positive because $(\text{tr}M)^2 - 4\det M > 0$. By lemma 1.46 we get

$$\sigma_1 > 0 \Leftrightarrow \alpha \in (-1,0); \qquad \sigma_2 > 0 \Leftrightarrow \beta > 1$$
$$\sigma_1 < 0 \Leftrightarrow \alpha < -1; \qquad \sigma_2 < 0 \Leftrightarrow \beta \in (0,1).$$

Solving this for λ_2 leads to

$$\sigma_1 > 0 \Leftrightarrow \lambda_2 \in (b_1b_2 + a_1,\ b_1b_2 + a_1 + b_2); \quad \sigma_2 > 0 \Leftrightarrow \lambda_2 < b_1b_2 + a_2 + b_1$$
$$\sigma_1 < 0 \Leftrightarrow \lambda_2 > b_1b_2 + a_1 + b_2; \qquad \sigma_2 < 0 \Leftrightarrow \lambda_2 \in (b_1b_2 + a_2 + b_1,\ b_1b_2 + a_2)$$

giving precisely the required conditions. □

Stability Changes for $\mathbf{C_2^-}$-cycles

The conditions for λ_2 in (c) and (d) cannot be interpreted in a straightforward way as for the cycles of type B, since $\lambda_2 = \lambda_2(t_1)$ and the boundaries of the intervals also depend on t_1. However, this can be overcome by looking at the nature of the dependency. We give a graphic interpretation in figure 1.12. The manner of visualisation is qualitatively different from before: we vary only one transverse eigenvalue, t_1, as for the B_2^+-cycle, yet the diagram is two-dimensional, as in the case of the B_3^--cycle. The second axis is not the t_2-axis, though. It is merely necessary since the stability changes can only be derived from following along the graph of $\lambda_2(t_1)$ and observing its position relative to other quantities depending on t_1. These are the boundaries of the intervals for λ_2 and they are given by the following linear functions of t_1:

$$\begin{aligned}
L_1(t_1) &= b_1 b_2 + a_1\\
L_2(t_1) &= b_1 b_2 + a_1 + b_2\\
L_3(t_1) &= b_1 b_2 + a_2 + b_1\\
L_4(t_1) &= b_1 b_2 + a_2
\end{aligned}$$

We have $L_1(0) = a_1 < a_1 + b_2 = L_2(0)$ and $L_3(0) = L_4(0) = a_2$. Since $b_1 b_2 - a_1 + a_2 > 0$ implies $a_2 > a_1 - b_1 b_2 > a_1$, we are left with two cases for the relative positions of the L_i:

(i) $a_1 < a_2 < a_1 + b_2$

(ii) $a_1 < a_1 + b_2 < a_2$

Note that L_1,L_2,L_4 all have the same slope $\dfrac{t_2}{e_1 e_2} < 0$, while that of L_3 is smaller:

$$\frac{t_2}{e_1 e_2} - \frac{1}{e_1} < \frac{t_2}{e_1 e_2}$$

In terms of the L_i, the conditions for predominant (in)stability can now be reformulated as

$$\begin{aligned}
\text{p.a.s.} \quad &\Leftrightarrow \quad \lambda_2 \in (L_1(t_1),\ \min(L_2(t_1), L_3(t_1)))\\
\text{p.u.} \quad &\Leftrightarrow \quad \lambda_2 \in (\max(L_2(t_1), L_3(t_1)),\ L_4(t_1)).
\end{aligned}$$

We have

$$\lambda_2(t_1) = \frac{b_1 b_2 + a_1 + a_2}{2} - \sqrt{\left(\frac{b_1 b_2 + a_1 + a_2}{2}\right)^2 - a_1 a_2},$$

which gives

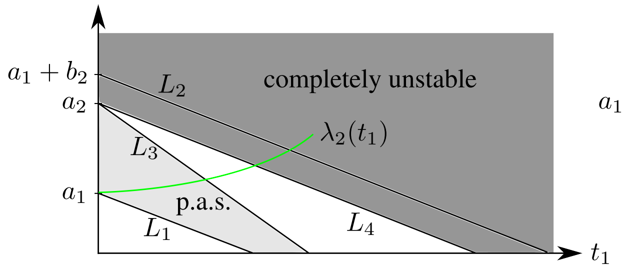
$$\lambda_2(0) = \frac{a_1 + a_2}{2} - \sqrt{\left(\frac{a_1 + a_2}{2}\right)^2 - a_1 a_2} = \frac{a_1 + a_2}{2} - \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2} = a_1.$$

Note that if $\lambda_2(t_1) > L_4(t_1)$, then the cycle is completely unstable, since

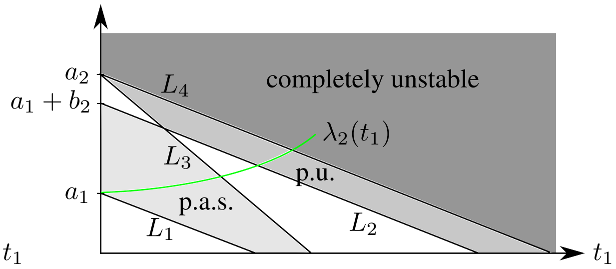
$$\begin{aligned}
\lambda_2(t_1) > b_1 b_2 + a_2 \quad &\Rightarrow \quad \frac{b_1 b_2 + a_1 + a_2}{2} > b_1 b_2 + a_2\\
&\Rightarrow \quad -b_1 b_2 + a_1 - a_2 > 0\\
&\Rightarrow \quad b_1 b_2 - a_1 + a_2 < 0,
\end{aligned}$$

which according to case (b) implies complete instability. Note that $\lambda_2(t_1) \in \mathbb{R}$ as long as $\mathrm{tr} M^2 - 4 \det M > 0$. The latter expression depends quadratically on t_1, its zeros are given through

$$b_1 = -\frac{\left(\sqrt{a_1} + \sqrt{a_2}\right)^2}{b_2} < 0 \quad \text{and} \quad b_1 = -\frac{\left(\sqrt{a_1} - \sqrt{a_2}\right)^2}{b_2} < 0,$$



(a) $a_2 < a_1 + b_2$



(b) $a_2 > a_1 + b_2$

Figure 1.12.: Stability change for a C_2^--cycle when t_1 turns positive

both corresponding to positive values of t_1:

$$t_1 = \frac{e_1}{b_2}\left(\sqrt{a_1} + \sqrt{a_2}\right)^2 \quad \text{and} \quad t_1 = \frac{e_1}{b_2}\left(\sqrt{a_1} - \sqrt{a_2}\right)^2$$

For small $t_1 > 0$ the eigenvalue is therefore real. Moreover, $\lambda_2(t_1)$ increases monotonically as long as $t_1 \in \left(0, \frac{e_1}{b_2}\left(\sqrt{a_1} + \sqrt{a_2}\right)^2\right)$. This completes the derivation of figure 1.12. Note that we have neglected the other conditions in (b) that lead to complete instability. Each of them constitutes an upper bound on t_1, above which the cycle is completely unstable. We have assumed all of these bounds to be sufficiently large so that they do not influence the picture. In case they are smaller, the dynamics are simplified in the sense that figure 1.12 is "cut off" at the respective value and the cycle is completely unstable for larger t_1.

From these considerations we conclude the following result.

Corollary 1.62. *A very simple heteroclinic cycle of type C_2^- in $\mathbb{R}^4$ is generically predominantly asymptotically stable after a transverse eigenvalue becomes positive. As the eigenvalue becomes larger, generically there exists an open interval where the cycle is p.f.a.s.*

Proof. The first statement is clear from figure 1.12. Concerning the second one: the only possibility for $\lambda_2(t_1)$ not to enter the region of proper fragmentary asymptotic stability (white in figure 1.12) is when it passes through the intersection point of $L_2(t_1)$ and $L_3(t_1)$. But that is a degenerate configuration. □

In contrast to type B cycles predominant instability is also possible for certain configurations of eigenvalues. In case $a_2 > a_1+b_2$ it is a generic state along each path from asymptotic stability to complete instability.

Stability of C_4^--cycles

The stability behaviour of the remaining type of very simple heteroclinic cycle in $\mathbb{R}^4$, type C_4^-, cannot be investigated in analogous fashion. In principle, it is possible, of course, to reformulate the classification in [33] as necessary and sufficient conditions for the different stability properties as we have done with the other types. However, for a C_4^--cycle there are four transverse eigenvalues, meaning we have to consider three of them becoming positive, making it impossible to illustrate the results graphically in the same way as before. Simply stating the conditions without graphical interpretation would be little more enlightening than the results given in [33]. There is, however, a useful conclusion that can be reached for the case where one transverse eigenvalue becomes positive.

Proposition 1.63. *Let X be a very simple heteroclinic cycle of type C_4^- in $\mathbb{R}^4$. Suppose that $b_1 < 0 < b_j$ for $j \neq 1$, and assume that none of the conditions (a), (b) and (c) in lemma 1.50 are satisfied. Then there is $\varepsilon_0 > 0$ such that for $0 < t_1 < \varepsilon_0$ (and all other parameters unchanged) the cycle is p.a.s.*

Proof. We use lemma 1.50 and show that for $t_1 > 0$ sufficiently small all stability indices are positive, but not all of them equal to $+\infty$. To this end, we first convince ourselves that

$$f^{\text{index}}(v_{22}^{j+3,j}/h^{j+3,j}, -v_{21}^{j+3,j}/h^{j+3,j}) = +\infty \tag{1.12}$$

for all j: by construction, for $j \neq 2$ the eigenvalues of $M^{j+3,j}$ can be determined from those of $M^{1,2}$ by multiplying with the matrices M_j, $j \neq 1$, in the correct fashion. These have positive entries only. Thus, it follows from the converse of condition (c) in lemma 1.50 that for all j the entries of $v_1^{j+3,j}$ have the same sign. For $t_1 > 0$ small enough all $M^{j+3,j}$ have only positive entries. Therefore, while eigenvectors corresponding to the greater one of their eigenvalues have same sign entries, those for the smaller one have opposite sign entries. So $v_{21}^{j+3,j}v_{22}^{j+3,j} < 0$, and thus the arguments of f^{index} above have the same sign. In fact, taking into account $h^{j+3,j}$, they are both positive. This proves (1.12), so $\sigma_4 = +\infty$ and the other stability indices are equal to the respective second expression in the minimum.

Now we choose $t_1 > 0$ small enough such that

$$(i)\ t_1 < e_1, \qquad (ii)\ b_1b_4 + a_4 > -b_1, \qquad (iii)\ b_1b_3b_4 + b_3a_4 + b_1a_3,\, b_1b_4 + a_4 > 0.$$

Then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1{,}1) = -\frac{1}{b_1} - 1 = \frac{e_1}{t_1} - 1 > 0,$$

$$\sigma_2 = f^{\text{index}}(b_1b_4 + a_4{,}b_1) = -\frac{b_1b_4 + a_4}{b_1} - 1 > 0,$$

$$\sigma_3 = f^{\text{index}}(b_1b_3b_4 + b_3a_4 + b_1a_3{,}b_1b_4 + a_4) = +\infty.$$

So all indices are positive for $t_1 > 0$ small enough, two of them even equal to $+\infty$, and thus the C_4^--cycle is predominantly asymptotically stable as claimed. $\square$

Together with the detailed study of the C_2^--cycle above, this allows us to close with the following general conclusion about (non-homoclinic) type C cycles in $\mathbb{R}^4$.

Corollary 1.64. *A very simple heteroclinic cycle of type C in $\mathbb{R}^4$ is generically p.a.s. after a transverse bifurcation.*

2. Stability of heteroclinic networks

It is possible for a dynamical system to contain more than one heteroclinic cycle. If two or more such cycles are joined by at least one common trajectory or equilibrium, it has become customary to speak of a heteroclinic network. This second chapter of the present work deals with such networks. As is the case with heteroclinic cycles, they occur in various applications. Among many others, examples can be found in the works of Aguiar, Castro and Labouriau [2] or Driesse and Homburg [13]. One of the best-known early studies of heteroclinic networks is [23], where Kirk and Silber investigate competition between two cycles forming a network. In section 2.2 we generalize their results and transfer their approach to networks formed by different types of cycles.

Heteroclinic networks give rise to many interesting and complex dynamical phenomena, such as *switching* (see [21]) and *cycling* (see [35]) to name just two keywords. Within this work, however, we restrict our study to pure stability questions. Let us now start by making the rough definition from above more precise.

Definition 2.1 (e.g. [23])**.** A *heteroclinic network* X is the union of finitely many heteroclinic cycles C_i such that $X = \bigcup_i C_i$ is connected.

In section 2.1 we discuss attraction and stability properties of heteroclinic networks as a whole. The stability index can be used to refine stability assessment by making out dominant cycles within the network that attract most trajectories within a sufficiently small neighbourhood. We derive general results on the index in the context of networks which allow us to explicitly calculate stability indices with respect to a network and investigate non-asymptotic stability properties. Within a network of non-homoclinic cycles, a single cycle cannot be asymptotically stable due to the expanding dimension in the direction of the next equilibrium. However, the network as a whole may have this property.

The main focus is on section 2.2, where we classify very simple heteroclinic networks in $\mathbb{R}^4$ and use the results from chapter 1 to discuss competition between their subcycles. This extends the study in [23] and places it in a broader context.

Note that throughout this chapter we assume that there are no invariant objects other than the heteroclinic network we consider. Trajectories leaving a neighbourhood of the network are assumed to stay away from it for all positive times. Just like in section 1.3 this allows us to calculate only local stability indices – as before we drop the subscript "loc". Then in most cases the (non-local) c-index coincides with the local n-index, see also remark 2.18.

2.1. Stability and attraction properties of networks

2.1.1. The stability index for networks

In subsection 1.2.2 we introduced the stability index $\sigma(x)$ from [33]. It is by definition tied to an underlying invariant set through the measure of its basin of attraction. When $X = \bigcup_i C_i$ is a heteroclinic network, it turns out to be insightful to calculate stability indices with respect to both the network X and its subcycles C_i. We refer to the former as *n-indices* and denote them by $\sigma^n(x)$. The latter we call *c-indices* and write $\sigma^i(x)$ (for the index with respect to cycle C_i) or simply $\sigma(x)$ unless stated otherwise. For a heteroclinic cycle we used subscripts to indicate that σ_j is the index along the trajectory leading to the equilibrium ξ_j. In a network there may be more than one such trajectory, so we usually write σ_{jk} for the stability index along the trajectory $[\xi_j \to \xi_k]$. We use analogous notation for the eigenvalues of the linearization when needed.

In this subsection we establish the intuitive relation $\sigma(x) \geq \sigma^i(x)$ and lay the foundations for explicitly calculating $\sigma(x)$ under certain circumstances. This is necessary since [33] only covers stability indices with respect to cycles, not networks, and the comparison of both quantities yields information on the relative stability of the cycles in a network. Our first result is a simple lemma that proves useful later on.

Lemma 2.2. *Let $a_k, b_k > 0$, $k \in \mathbb{N}$, and suppose that $\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k}$ converges for small $\varepsilon > 0$. Then we have*

$$\lim_{\varepsilon\to 0} \frac{\ln(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k})}{\ln(\varepsilon)} = \inf\{b_k \mid k \in \mathbb{N}\}.$$

Proof. Set $b := \inf\{b_k \mid k \in \mathbb{N}\}$ so $\varepsilon^{b_k - b}$ is bounded for $\varepsilon \to 0$. Now calculate

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\ln\left(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k}\right)}{\ln(\varepsilon)} &= \lim_{\varepsilon\to 0} \frac{\ln\left(\varepsilon^b \left(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k - b}\right)\right)}{\ln(\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \left(\frac{\ln(\varepsilon^b)}{\ln(\varepsilon)} + \frac{\ln\left(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k - b}\right)}{\ln(\varepsilon)} \right) \\
&= \lim_{\varepsilon\to 0} \left(b + \frac{\ln\left(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k - b}\right)}{\ln(\varepsilon)} \right) \\
&= b \\
&= \inf\{b_k \mid k \in \mathbb{N}\}.
\end{aligned}$$

To validate the second to last equality choose $N \in \mathbb{N}$, then set $K^* := \left\{k \in \mathbb{N} \mid b_k - b > \frac{1}{N}\right\}$ and $K_* := \left\{k \in \mathbb{N} \mid b_k - b \leq \frac{1}{N}\right\}$. Note that $K_* \neq \emptyset$ because b is the infimum of the b_k. For small $\varepsilon > 0$ this gives

$$\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k - b} \geq \sum_{k\in K^*} a_k \varepsilon^{b_k - b} + \sum_{k\in K_*} a_k \varepsilon^{\frac{1}{N}} \geq \varepsilon^{\frac{1}{N}} \sum_{k\in K_*} a_k, \tag{2.1}$$

and therefore

$$\lim_{\varepsilon\to 0} \frac{\ln\left(\sum_{k\in\mathbb{N}} a_k \varepsilon^{b_k - b}\right)}{\ln(\varepsilon)} \leq \lim_{\varepsilon\to 0} \frac{\ln(\varepsilon^{\frac{1}{N}} \sum_{k\in K_*} a_k)}{\ln(\varepsilon)} = \frac{1}{N}.$$

This holds for all $N \in \mathbb{N}$, so the proof is complete. Note that as long as the sum contains only finitely many terms, the statement is still true even if some a_k are negative (whereas the sum must remain positive). This is because for a finite sum we do not need (2.1), where the inequalities hold only for positive a_k. $\square$

When studying a heteroclinic network it is of central interest to identify its most stable subcycles, i.e. to understand the relative stability of the cycles forming the network. To this end it is useful to compute and compare both c- and n-indices, especially along trajectories belonging to more than one cycle. The next lemma is an intuitive step in this direction.

Lemma 2.3. *For a heteroclinic network $X \subset \mathbb{R}^n$ consisting of distinct heteroclinic cycles $C_1, C_2, \ldots, C_m$ let $\sigma(x)$ be the stability index at a point $x \in X$ with respect to the network $X = \bigcup_{i=1}^m C_i$ and $\sigma^i(x)$ the index w.r.t. a cycle C_i, $i = 1, \ldots, m$. Then for $x \in X$ the following is true:*

$$\sigma(x) \begin{cases} = +\infty & \textit{if} \quad \exists\varepsilon > 0 \quad \ell(\mathcal{B}(X) \cap B_\varepsilon(x)) = \ell(B_\varepsilon(x)) \\ \geq \max\{\sigma^i(x) \mid i = 1, \ldots, m\} & \textit{if} \quad \forall\varepsilon > 0 \quad \ell(\mathcal{B}(X) \cap B_\varepsilon(x)) < \ell(B_\varepsilon(x)) \end{cases}$$

An analogous statement holds for the local stability index.

Proof. The first case, where $\sigma(x) = +\infty$, is clear by the definition of the index, so we consider the second one. Because ω-limit sets of forward-bounded orbits are nonempty, compact, connected and flow-invariant, we note that for x with $\omega(x) \subset X$ we either have

- $\omega(x) = \{\xi_j\}$ for some equilibrium ξ_j, or
- $\omega(x) = C_i$, for some cycle C_i, or
- $\omega(x)$ is the union of several cycles.

Since the equilibria ξ_j are hyperbolic saddles, the dimension of their stable manifolds $W^s(\xi_j)$ is less than n, so $\ell(W^s(\xi_j)) = 0$ holds for all j. Therefore

$$\ell(\{x \in \mathbb{R}^n \mid \omega(x) = \{\xi_j\} \text{ for some equilibrium } \xi_j\}) = \ell\left(\bigcup\nolimits_j W^s(\xi_j)\right) = 0.$$

From this it follows that $\ell(\mathcal{B}(C_i) \cap \mathcal{B}(C_k)) = 0$ if $i \neq k$. Moreover, it means that in terms of calculating the stability index we are only interested in points that have more than a single point in their ω-limit sets. Focusing on those with $\omega(x) = C_i$ (a single cycle) for the moment yields the following inequality

$$\begin{aligned}
\Sigma_\varepsilon(x) &= \frac{\ell(\mathcal{B}(X) \cap B_\varepsilon(x))}{\ell(B_\varepsilon(x))} \\
&\geq \frac{\ell(\bigcup_{i=1}^m \mathcal{B}(C_i) \cap B_\varepsilon(x))}{\ell(B_\varepsilon(x))} \\
&= \sum_{i=1}^m \frac{\ell(\mathcal{B}(C_i) \cap B_\varepsilon(x))}{\ell(B_\varepsilon(x))} \\
&= \sum_{i=1}^m \Sigma_\varepsilon^i(x).
\end{aligned}$$

Together with $\Sigma_\varepsilon(x) \in [0,1]$ this makes it clear that we cannot have $\sigma^i(x) > 0$ and $\sigma^k(x) > 0$ simultaneously for $i \neq k$.

Furthermore, lemma 1.32 implies that for small $\varepsilon > 0$ we can write

$$\begin{cases} 1 - \Sigma_\varepsilon(x) = \alpha\varepsilon^c & \text{if} \quad \sigma(x) = c > 0 \\ \quad\;\; \Sigma_\varepsilon(x) = \beta\varepsilon^c & \text{if} \quad \sigma(x) = -c < 0 \end{cases}$$

with $\alpha,\beta > 0$, for simplicity of notation in our calculations. Now consider the following two cases:

(i) Let $\sigma^i(x) < 0$ for all $i = 1, \ldots, m$. If for some i we have $\sigma^i(x) = -\infty$, then it is obvious that $\sigma(x) \geq \sigma^i(x)$ holds. Otherwise we obtain

$$\Sigma_\varepsilon(x) \geq \sum_{i=1}^m \Sigma_\varepsilon^i(x) = \sum_{i=1}^m \beta_i \varepsilon^{-\sigma^i(x)},$$

and thus taking into account that $\ln(\varepsilon) < 0$ and using lemma 2.2

$$\sigma_-(x) \leq \lim_{\varepsilon \to 0} \frac{\ln\left(\sum\limits_{i=1}^m \beta_i \varepsilon^{-\sigma^i(x)}\right)}{\ln(\varepsilon)} = \min\{-\sigma^i(x) \mid i = 1, \ldots, m\} > 0.$$

Therefore $\sigma_+(x) = 0$ and we get

$$\sigma(x) = \sigma_+(x) - \sigma_-(x) \geq \max\{\sigma^i(x) \mid i = 1, \ldots, m\}.$$

(ii) Now let one of the indices be positive. Assume $\sigma^1(x) > 0 > \sigma^i(x)$ for $i = 2, \ldots, m$, without loss of generality. As above we may neglect the case that one of them is infinite. So

$$\Sigma_\varepsilon(x) \geq \sum_{i=1}^m \Sigma_\varepsilon^i(x) = 1 - \alpha_1 \varepsilon^{\sigma^1(x)} + \sum_{i=2}^m \beta_i \varepsilon^{-\sigma^i(x)}.$$

Since $\ell(\mathcal{B}(C_1) \cap \mathcal{B}(C_i)) = 0$ holds for the basins of attraction of the two cycles C_1 and C_i, $i = 2, \dots, m$, we find that $\sigma^1(x) \leq -\sigma^i(x)$ for all other i and therefore again by lemma 2.2

$$\begin{aligned}\sigma_+(x) &= \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)} \\ &\geq \lim_{\varepsilon \to 0} \frac{\ln\left(\alpha_1 \varepsilon^{\sigma^1(x)} - \sum_{i=2}^{m} \beta_i \varepsilon^{-\sigma^i(x)}\right)}{\ln(\varepsilon)} \\ &= \min\{\sigma^1(x), -\sigma^i(x) \mid i = 2, \dots, m\} \\ &= \sigma^1(x) \\ &= \max\{\sigma^i(x) \mid i = 1, \dots, m\},\end{aligned}$$

which proves the claim for this case as well.

The result for the local stability index follows in the same way. Note that this is only non-trivial when x lies on a trajectory that belongs to more than one of the cycles. □

It is not untypical for the inequality in lemma 2.3 to be strict. This is often, though not always, an indication of one cycle attracting trajectories from a neighbourhood of the other cycle(s), thus *winning the competition* between the cycles, to use the terminology of Kirk and Silber in [23]. In section 2.2 we come across several such instances when we look at very simple networks in $\mathbb{R}^4$.

The computations of Krupa and Melbourne in [26] and [28] show that, even in $\mathbb{R}^n$, stability of heteroclinic cycles under certain circumstances depends only on the behaviour of a two-dimensional map. In such a case the calculation of n-indices can be done as in the following lemma.

Lemma 2.4. *Let $X \subset \mathbb{R}^m$ be a heteroclinic cycle or network and $x \in X$ a point on a connecting trajectory. Suppose that for all points $y = (y_1,\dots,y_m) \in B_\varepsilon(x)$, stability with respect to X depends only on their (y_1,y_2)-components. Furthermore, assume that $\mathcal{B}(X) \cap B_\varepsilon(x) = B_\varepsilon(x) \setminus \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ where $\mathcal{E}_n$ are disjoint sets of the form*

$$\mathcal{E}_n = \{y \in B_\varepsilon(x) \mid k_n y_1^{\alpha_n} \leq y_2 \leq \tilde{k}_n y_1^{\alpha_n}\},$$

with constants $k_n, \tilde{k}_n > 0$. If the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is bounded away from 1 and not all α_n are negative, then

$$\sigma(x) = -1 + \min\left(\frac{1}{\alpha_{N_1}}, \alpha_{N_2}\right) > 0,$$

with $\alpha_{N_1} := \max\{\alpha_n \mid 0 < \alpha_n < 1\}$ and $\alpha_{N_2} := \min\{\alpha_n \mid \alpha_n > 1\}$.

If on the other hand $\alpha_n = 1$ for some $n \in \mathbb{N}$ or 1 is an accumulation point of the sequence $(\alpha_n)_{n\in\mathbb{N}}$, then $\sigma(x) = 0$.

Proof. First of all, note that the sets $\mathcal{E}_n$ where $\alpha_n < 0$ are irrelevant for the stability index since their intersection with a sufficiently small ε-neighbourhood is empty, see figure 2.1 (left).

Now consider the case where $(\alpha_n)_{n\in\mathbb{N}}$ is bounded away from 1. We show that $\sigma_+(x) > 0$ (and equal to the expression above) and thus $\sigma_-(x) = 0$ by lemma 1.32, which means $\sigma(x) = \sigma_+(x) > 0$. We start by calculating

$$\begin{aligned}\ell(\mathcal{B}(X) \cap B_\varepsilon(x)) &= \ell(B_\varepsilon(x)) - \ell\left(\bigcup_{n\in\mathbb{N}} \mathcal{E}_n \cap B_\varepsilon(x)\right) \\ &= \ell(B_\varepsilon(x)) - \sum_{n\in\mathbb{N}} \ell(\mathcal{E}_n \cap B_\varepsilon(x)).\end{aligned}$$

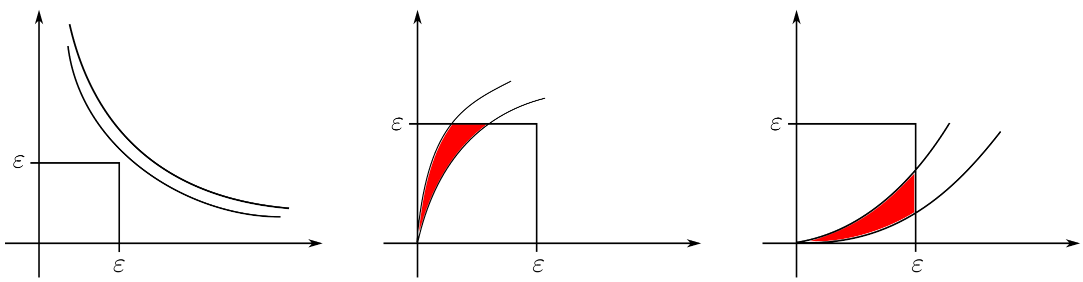
Grouping all constants terms in k_n and $\tilde{k}_n$ in each step, we determine

$$\ell(\mathcal{E}_n \cap B_\varepsilon(x)) = \begin{cases} k_n\varepsilon^{m-2}\int\limits_0^\varepsilon y_1^{\alpha_n}dy_1 = k_n\varepsilon^{m-1+\alpha_n} & \text{for} \quad \alpha_n > 1 \\ \tilde{k}_n\varepsilon^{m-2}\int\limits_0^\varepsilon y_2^{\frac{1}{\alpha_n}}dy_2 = \tilde{k}_n\varepsilon^{m-1+\frac{1}{\alpha_n}} & \text{for} \quad \alpha_n < 1. \end{cases}$$

Since $\ell(B_\varepsilon(x))$ is of order ε^m, we obtain

$$\begin{aligned}\Sigma_\varepsilon(x) = \frac{\ell(\mathcal{B}(X) \cap B_\varepsilon(x))}{\ell(B_\varepsilon(x))} &= 1 - \sum_{n\in\mathbb{N}} \frac{\ell(\mathcal{E}_n \cap B_\varepsilon(x))}{\ell(B_\varepsilon(x))} \\ &= 1 - \sum_{\alpha_n<1} \tilde{k}_n\varepsilon^{-1+\frac{1}{\alpha_n}} - \sum_{\alpha_n>1} k_n\varepsilon^{-1+\alpha_n}.\end{aligned}$$

Putting these pieces together and using lemma 2.2 for the third equality below, we get



(a) $\alpha_n < 0$ **(b)** $0 < \alpha_n < 1$ **(c)** $\alpha_n > 1$

Figure 2.1.: α_n and the ε-neighbourhood: $\mathcal{E}_n \cap B_\varepsilon(x)$ is coloured red.

$$
\begin{aligned}
\sigma_+(x) &= \lim_{\varepsilon\to 0} \frac{\ln(1-\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{\ln\left(\sum_{\alpha_n<1} \tilde{k}_n \varepsilon^{-1+\frac{1}{\alpha_n}} + \sum_{\alpha_n>1} k_n \varepsilon^{-1+\alpha_n}\right)}{\ln(\varepsilon)} \\
&= \inf\left\{-1+\frac{1}{\alpha_{n_1}}, -1+\alpha_{n_2} \;\middle|\; \alpha_{n_1}<1, \alpha_{n_2}>1\right\} \\
&= -1+\min\left(\frac{1}{\alpha_{N_1}}, \alpha_{N_2}\right) \\
&> 0.
\end{aligned}
$$

In case $\alpha_n \neq 1$ for all $n \in \mathbb{N}$ but 1 is an accumulation point of the α_n, the calculations work in the same way and the infimum above then yields 0. Since for $\varepsilon \to 0$ we still get $\Sigma_\varepsilon(x) \to 1$, we also have $\sigma_-(x) = 0$ and thus $\sigma(x) = 0$.

If $\alpha_{n_0} = 1$ for some $n_0 \in \mathbb{N}$, then for any $\varepsilon > 0$ the set $\mathcal{E}_{n_0}$ contributes a constant portion of $B_\varepsilon(x)$ to the complement of the basin $\mathcal{B}(X)$. So $\Sigma_\varepsilon(x)$ is bounded away from zero and one, therefore $\sigma(x) = 0$ by lemma 1.32. □

We finish this subsection with another lemma, giving information about the relation between c- and n-indices. Note that this does not hold for the local stability indices. It is important to clarify that whenever we call a map a *contraction*, we not only mean that it is contracting in the usual sense, but also assume that it maps its domain into itself. In doing so we follow the terminology of Kirk and Silber in [23].

Lemma 2.5 ([8]). *Let $[\xi_i \to \xi_j]$ be a common connecting trajectory between two non-homoclinic cycles constituting a very simple heteroclinic network $X \subset \mathbb{R}^n$. Suppose that for at least one of the cycles the return maps are contractions. Let σ_{ij} and $\tilde{\sigma}_{ij}$ be the global stability indices for each cycle and σ^n_{ij} the global stability index with respect to the whole network. We have*

$$\sigma^n_{ij} > 0 \quad \Leftrightarrow \quad \sigma_{ij} > 0 \;\vee\; \tilde{\sigma}_{ij} > 0.$$

Proof. The implication from right to left is simply lemma 2.3.

Assume $\sigma^n_{ij} > 0$. For a point x, that contributes to the index σ^n_{ij}, we have $\omega(x) \subset X$, so $\omega(x)$ is compact, non-empty and connected, leaving three possibilities:

(a) $\omega(x)$ is an equilibrium.
(b) $\omega(x)$ is one of the cycles.
(c) $\omega(x)$ is the whole network X.

The set of points for which (a) holds is the union of the stable manifolds of the equilibria and thus of measure zero. Case (c) does not occur: the trajectory through x would have to follow around both cycles infinitely many times, which is impossible since for at least one of the cycles the return maps are contractions. So almost all x with $\omega(x) \subset X$ fall into case (b). Therefore, one of the cycles has a large enough basin of attraction to make the index with respect to only this cycle positive. □

2.1.2. Non-asymptotic stability of networks

Non-asymptotic stability of heteroclinic networks can be understood to a large extent by studying stability of the single cycles and combining this information with the stability index with respect to the entire network. The proofs of theorems 1.33 and 1.34 generalize without difficulty to the context of heteroclinic networks, giving

Corollary 2.6. *Let $X \subset \mathbb{R}^n$ be a heteroclinic network consisting of finitely many equilibria and connecting trajectories and suppose that $\ell_1(X) < \infty$. Assume that the local stability index σ^n_{loc} exists for all connecting trajectories and is unequal to zero. Then X is predominantly asymptotically stable if and only if $\sigma^n_{\mathrm{loc}} > 0$ along all connecting trajectories.*

The result about predominant instability generalizes in the same way. It is of lesser interest though, since the more challenging task is to identify cycles winning the competition within a network. This again only seems worthwhile if the network is somewhat stable as a whole.

When looking at specific networks in $\mathbb{R}^4$ in more detail in the next section, we see that corollary 2.6 opens up various ways for a heteroclinic network to be p.a.s. The following list is not complete, but gives a good first impression of the diversity awaiting us.

1. Each cycle has positive c-indices along all its connecting trajectories, except for trajectories belonging to more than one cycle, where due to lemma 2.3 only one of them can have a c-index greater than zero. Such a situation occurs in propositions 2.14 and 2.15.

2. There is one dominant cycle, i.e. one cycle with only positive c-indices, while all other cycles have negative c-indices, but positive n-indices because most points in their neighbourhood are also attracted to the dominant cycle. This can be found in case (i) of propositions 2.16 and 2.17.

3. Even when no single cycle is p.a.s., there may still be a cycle that wins the competition in the sense that for most initial conditions near the network the trajectories end up converging to it. In case (ii) of propositions 2.16 and 2.17 there is a cycle which is not p.a.s. itself, since along the trajectory it shares with another cycle, most points make finitely many excursions around the other cycle first, before approaching the attracting one. This leads to only positive n-indices without having a cycle with positive c-indices. Such behaviour was called *e.a.s. in spirit* in [23], translating to *p.a.s. in spirit* in our terminology.

4. In proposition 2.22 we encounter yet another way in which predominant asymptotic stability may be obtained (and lost) by a network. Again no cycle is p.a.s., but this time there is no obviously dominant cycle – both have the same number of positive, negative and infinite indices.

Heteroclinic networks in dimensions higher than $n = 4$ are likely to display even more complex dynamics and yet qualitatively other possibilities to combine cycles into an attracting set, especially when more than two cycles are involved. In this work we restrict our attention to $\mathbb{R}^4$, though, where we have a complete enumeration of all type B and C cycles at hand, which helps us find possible networks as well. This is exactly what we do at the beginning of the next section.

2.2. Very simple heteroclinic networks in $\mathbb{R}^4$

We now turn our attention to $\mathbb{R}^4$ again, focusing on very simple heteroclinic networks of B- and C-cycles with at least one common trajectory. Note that in $\mathbb{R}^4$ type A cycles cannot exist simultaneously with B- or C-cycles due to symmetry restrictions. For the former, the symmetry group does not contain reflections, whereas for the latter it does, see lemma 1.7. Therefore, a network involving a type A cycle consists of type A cycles only. As a consequence, proposition 1.37 drastically limits the possibilities for competition in such networks: if there is an A-cycle with any positive c-indices at all, it has only positive c-indices. At the same time all other cycles sharing a connecting trajectory with it must have a negative c-index along the common connection, and therefore along all of their trajectories. So the existence of an A-cycle that attracts a set of positive measure immediately keeps all its neighbouring cycles from attracting anything at all.

For B- and C-cycles the possibilities for competition in a network are much more diverse, as Kirk and Silber illustrate in [23]. That is why we concentrate on these types from now on. In order to make use of the earlier results on very simple heteroclinic cycles, we focus on networks made up of such cycles.

Definition 2.7. We call a heteroclinic network *very simple* if it consists of very simple heteroclinic cycles.

Furthermore, we restrict our investigation to very simple networks where the cycles have at least one connecting trajectory in common. This is particularly interesting since cycles can compete for the initial conditions close to the common trajectory. Then the stability index with respect to both cycles may be used to determine which cycle wins the competition. Nevertheless, networks with only equilibria in common may exhibit qualitatively different dynamics and shall be studied elsewhere.

While section 1.3 provides us with a convenient way of checking cycles in $\mathbb{R}^4$ for their stability properties, this is usually not sufficient for a thorough understanding of stability in a heteroclinic network. If each cycle in a network is either predominantly asymptotically stable or predominantly unstable, then generically only the p.a.s. cycle(s) will be observed and not the network itself. In this case, information on the stability indices with respect to both cycles suffices. However, the more interesting situations are those where none of the cycles is p.a.s. Then c-indices do not give us enough information on the network and its attraction properties. When looking at specific networks, we thus investigate all possible scenarios, but put an extra emphasis on parameter situations without p.a.s. cycles obviously dominating the competition.

As already mentioned in the introduction some results in this section have been obtained in collaboration with Sofia Castro and can be found in [8]. More precisely, this applies to subsection 2.2.2 and parts of subsection 2.2.3.

2.2.1. Classification of networks in $\mathbb{R}^4$

In this subsection we prove that only three very simple networks composed of type B and C cycles sharing a connecting trajectory exist in $\mathbb{R}^4$. One of them, the (B_3^-,B_3^-)-network, is the network in [23], the other two have not been studied before. In this sense we provide a generalization of [23].

The number of possible very simple networks is clearly limited by lemma 1.44, where all B- and C-cycles in $\mathbb{R}^4$ are listed. This enables us to proceed with the following result.

Proposition 2.8 ([8]). *Let X be a very simple heteroclinic network in $\mathbb{R}^4$ consisting of heteroclinic cycles of type B or C with at least one common connecting trajectory. Suppose that there are no critical elements other than the origin and the equilibria belonging to the cycles. Then X is of one of the following types*

- (B_2^+,B_2^+) – *with one common connecting trajectory*
- (B_3^-,B_3^-) – *with one common connecting trajectory*
- (B_3^-,C_4^-) – *with two common connecting trajectories*

Proof. According to lemma 1.44, there are four distinct non-homoclinic type B and C cycles and they can exist only under equivariance of the system $\dot{x} = f(x)$ with respect to the following symmetry groups:

$$B_2^+(\mathbb{Z}_2^3), \quad B_3^-(\mathbb{Z}_2^4), \quad C_2^-(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4), \quad C_4^-(\mathbb{Z}_2^4).$$

Therefore, the only cycles of different type that may exist simultaneously are B_3^- and C_4^-, under equivariance with respect to $\mathbb{Z}_2^4$. The proof proceeds in two steps.

Step 1: we show that the combinations (C_2^-,C_2^-) and (C_4^-,C_4^-) are not possible, starting with (C_4^-,C_4^-). Suppose we have a system with a C_4^--cycle joining equilibria

$$[\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1],$$

which, without loss of generality, lie on the respective coordinate axes. Now, also without loss of generality, there are three possibilities to introduce a second C_4^--cycle:

(a) Add a connection $[\xi_4 \to \xi_1]$.
(b) Add an equilibrium ξ_* and connections $[\xi_3 \to \xi_* \to \xi_1]$.
(c) Add two equilibria ξ_*, ξ_{**} and connections $[\xi_2 \to \xi_* \to \xi_{**} \to \xi_1]$.

In case (a), the new connection $[\xi_4 \to \xi_1]$ has to lie in P_{14} (the x_1-x_4-plane), so the phase portrait in this plane looks like figure 2.2. Applying the Poincaré-Bendixson Theorem within the invariant plane P_{14}, one of the following holds:

(a_1) ξ_4 is connected to ξ_1 by a two-dimensional set of trajectories.
(a_2) There exists another equilibrium or periodic orbit within P_{14}.

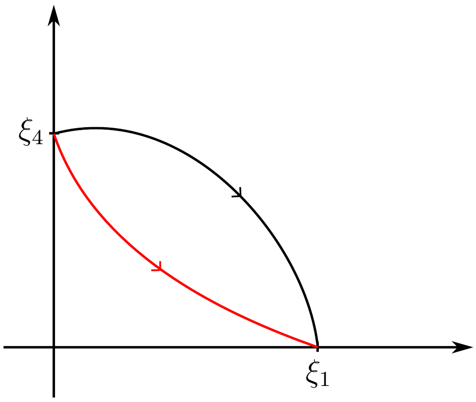


Figure 2.2.: Phase portrait in P_{14} for case (a).

Case (a_1) does not occur since for a very simple heteroclinic cycle the connection is one-dimensional in P_{14}. Case (a_2) is excluded by assumption, so an additional connection as in (a) is impossible.

In case (b), the new equilibrium ξ_* must lie in a one-dimensional fixed-point subspace. Under the standard action of $\mathbb{Z}_2^4$ (see section 3 (b) of [28]) the only such subspaces are the coordinate axes. So ξ_* must lie on the x_4-axis, since a C_4^--cycle is not contained in a three-dimensional subspace. Thus, the phase portrait in P_{14} looks like figure 2.3. By a Poincaré-Bendixson argument similar to the one above, case (b) is impossible as well.

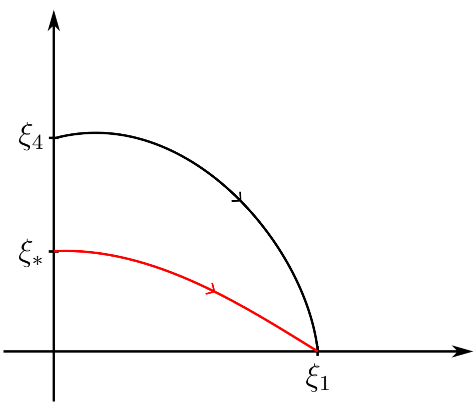


Figure 2.3.: Phase portrait in P_{14} for case (b).

In case (c), for reasons similar to the above, there are two subcases:

(c_1) ξ_* lies on the x_3-axis and ξ_{**} on the x_4-axis.
(c_2) ξ_* lies on the x_4-axis and ξ_{**} on the x_3-axis.

For (c_1), the phase portrait in P_{14} looks exactly like the one in case (b), replacing ξ_* with ξ_{**}. For (c_2), dynamics in P_{34} are shown in figure 2.4. Again, a Poincaré-Bendixson argument yields that case (c) is not possible.

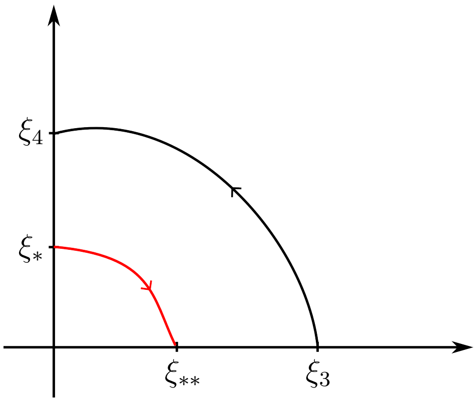


Figure 2.4.: Phase portrait P_{34} for case (c-ii).

The reasoning for a (C_2^-,C_2^-)-network is similar. A single cycle of type C_2^- occupies the whole of $\mathbb{R}^4$ in the same way that a C_4^--cycle does. There are also four equilibria, only now there are two pairs which are related by symmetry. Thus, analogous to the above, it follows that no additional C_2^--cycle can be introduced to the system by adding connections and/or equilibria.

Step 2: we have to convince ourselves that the remaining three networks do indeed exist.

A (B_3^-,C_4^-)-network may be put together as follows. Suppose we have a system $\dot{x} = f(x)$, equivariant under the action of $\mathbb{Z}_2^4$, with a heteroclinic cycle of type C_4^-. As above we assume it consists of four equilibria ξ_i on the x_i-axis, joined by connecting trajectories in the coordinate planes in the standard way:

$$[\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1]$$

It is impossible to introduce a B_3^--cycle to this system by adding an equilibrium (and two connections), for the same reasons as in step 1. However, the existence of the C_4^--cycle places no a priori restrictions on the dynamics in P_{13}. So we may assume that there is a connection $[\xi_3 \to \xi_1] \subset P_{13}$, making ξ_1 a sink in the three-dimensional coordinate space S_{134} and ξ_3 expanding in the x_1- and x_4-directions. Then we have a second cycle

$$[\xi_1 \to \xi_2 \to \xi_3 \to \xi_1],$$

contained in the three-dimensional fixed-point subspace S_{123} and thus of type B_3^-. It has two connections in common with the C_4^--cycle.

We construct a (B_2^+,B_2^+)-network in subsection 2.2.2. The existence of a (B_3^-,B_3^-)-network was already shown in [23] and is dealt with in subsection 2.2.3. □

We have thus established that B- and C-cycles allow only three types of very simple heteroclinic networks in $\mathbb{R}^4$: (B_2^+,B_2^+), (B_3^-,B_3^-) and (B_3^-,C_4^-), the latter being the only way to turn two cycles of different types into a network. There are two generic results that now follow immediately from the results in subsection 1.3.2.

Corollary 2.9 ([8]). *For the three very simple heteroclinic networks in $\mathbb{R}^4$, at least one connecting trajectory has stability index equal to $+\infty$, unless all indices are equal to $-\infty$.*

Let us finish this subsection with a clarifying remark on the coexistence of B- and C-cycles.

Remark 2.10. In appendix A.1 we show that in $\mathbb{R}^4$ simple heteroclinic cycles can be characterized by assigning to each connection a type A, B or C, given by definition A.1. Indeed, for such a cycle all connections are of the same type, and it is the same as that of the cycle.

For the (B_3^-,C_4^-)-network this gives rise to the question of which type the common connections are. The answer is that it depends on which cycle one considers. The connections $[\xi_1 \to \xi_2]$ and $[\xi_2 \to \xi_3]$ belong to both cycles. They are of type B when viewed as part of the B_3^--cycle and of type C when looking at the C_4^--cycle. To verify this denote by (x_1,x_2,x_3,x_4) the standard basis in $\mathbb{R}^4$ and by $\langle x_i \rangle$ the linear subspace spanned by x_i. Then we determine the invariant subspaces P_j and $Q_j = V_j(c) \oplus P_j$ for both cycles to be

$$\begin{aligned}
B_3^- : \quad & Q_1 = P_3 + P_1 = \langle x_3,x_1\rangle + \langle x_1,x_2\rangle = \langle x_1,x_2,x_3\rangle \Rightarrow P_2 = \langle x_2,x_3\rangle \subset Q_1 \\
& Q_2 = P_1 + P_2 = \langle x_1,x_2\rangle + \langle x_2,x_3\rangle = \langle x_1,x_2,x_3\rangle \Rightarrow P_3 = \langle x_3,x_1\rangle \subset Q_2 \\
& Q_3 = P_2 + P_3 = \langle x_2,x_3\rangle + \langle x_3,x_1\rangle = \langle x_1,x_2,x_3\rangle \Rightarrow P_1 = \langle x_1,x_2\rangle \subset Q_3 \\
C_4^- : \quad & Q_1 = P_4 + P_1 = \langle x_4,x_1\rangle + \langle x_1,x_2\rangle = \langle x_1,x_2,x_4\rangle \Rightarrow P_2 = \langle x_2,x_3\rangle \not\subset Q_1 \\
& Q_2 = P_1 + P_2 = \langle x_1,x_2\rangle + \langle x_2,x_3\rangle = \langle x_1,x_2,x_3\rangle \Rightarrow P_3 = \langle x_3,x_4\rangle \not\subset Q_2 \\
& Q_3 = P_2 + P_3 = \langle x_2,x_3\rangle + \langle x_3,x_4\rangle = \langle x_2,x_3,x_4\rangle \Rightarrow P_4 = \langle x_4,x_1\rangle \not\subset Q_3 \\
& Q_4 = P_3 + P_4 = \langle x_3,x_4\rangle + \langle x_4,x_1\rangle = \langle x_1,x_3,x_4\rangle \Rightarrow P_1 = \langle x_1,x_2\rangle \not\subset Q_4
\end{aligned}$$

Indeed, in both cases all the Q_j are reflection hyperplanes under the action of $\mathbb{Z}_2^4$. For the B-cycle we have $P_{j+1} \subset Q_j$, whereas for the C-cycle we have $P_{j+1} \not\subset Q_j$ for all j. The connections $[\xi_1 \to \xi_2]$ and $[\xi_2 \to \xi_3]$ adopt both types. The invariant planes P_j are different for the two cycles, and thus, so are the Q_j and also the connection types.

In the following subsections we investigate all three very simple heteroclinic networks in detail. For the (B_3^-,B_3^-)-network part of this has already been done in [23], we show some of their results in a different light (using the stability index) and complete their discussion of stability and competition.

2.2.2. The (B_2^+,B_2^+)-network

We start with the simplest of the three networks. Consider the group $\Gamma \cong \mathbb{Z}_2^3 \subset O(4)$ generated by the following elements of order 2:

$$\begin{aligned}\kappa_2.(x_1,x_2,x_3,x_4) &= (x_1,-x_2,x_3,x_4)\\ \kappa_3.(x_1,x_2,x_3,x_4) &= (x_1,x_2,-x_3,x_4)\\ \kappa_4.(x_1,x_2,x_3,x_4) &= (x_1,x_2,x_3,-x_4).\end{aligned}$$

For $i \neq j$ it is easily seen that Fix$(\langle\kappa_i,\kappa_j\rangle)$ is a two-dimensional space of the form

$$P_{1k} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid \ x_i = x_j = 0,\ k \neq i,j\}.$$

Therefore, we have Fix$(\langle\kappa_2,\kappa_3,\kappa_4\rangle) = L_1 = \{(x_1,0,0,0) \mid \ x_1 \in \mathbb{R}\}$. Let f be a vector field equivariant under this group action, given by the right hand side of:

$$\begin{cases}\dot{x}_1 = a_1x_1 + \sum_{i=1}^4 b_{1i}x_i^2 + c_1x_1^3\\ \dot{x}_2 = a_2x_2 + x_2\sum_{i\neq 2} b_{2i}x_i^2 + c_2x_2^3 + d_2x_1x_2\\ \dot{x}_3 = a_3x_3 + x_3\sum_{i\neq 3} b_{3i}x_i^2 + c_3x_3^3 + d_3x_1x_3\\ \dot{x}_4 = a_4x_4 + x_4\sum_{i\neq 4} b_{4i}x_i^2 + c_4x_4^3 + d_4x_1x_4\end{cases}$$

The origin is an equilibrium, we impose $a_j > 0$, so that it is repelling. Checking for more equilibria on the x_1-axis yields

$$0 \stackrel{!}{=} \dot{x}_1 = a_1 + b_{11}x_1 + c_1x_1^2 \quad \Leftrightarrow \quad x_1 = \frac{-b_{11} \pm \sqrt{b_{11}^2 - 4a_1c_1}}{2c_1}.$$

Assume that $b_{11}^2 - 4a_1c_1 > 0$ so that there are two equilibria, other than the origin, on the x_1-axis. Furthermore, set $c_1 < 0$ and label these ξ_a and ξ_b, such that the first coordinate of ξ_a is negative and the first coordinate of ξ_b is positive.

Checking for equilibria other than the origin on the x_j-axis, $j \neq 1$, yields

$$0 \stackrel{!}{=} \dot{x}_j = a_j + c_jx_j^2 \quad \Leftrightarrow \quad x_j = \pm\sqrt{-\frac{a_j}{c_j}}.$$

Assume $c_j > 0$ for $j \neq 1$, so that there are no other equilibria on the x_j-axis. The linearization of f at the equilibria $\xi_{a/b} = (x_{a/b},0,0,0)$ is

$$\begin{aligned}df(\xi_{a/b}) = \text{diag}\Big(&a_1 + 2b_{11}x_{a/b} + 3c_1x_{a/b}^2,\ a_2 + b_{21}x_{a/b}^2 + d_2x_{a/b},\\ &a_3 + b_{31}x_{a/b}^2 + d_3x_{a/b},\ a_4 + b_{41}x_{a/b}^2 + d_4x_{a/b}\Big)\\ = \text{diag}\Big(&b_{11}x_{a/b} + 2c_1x_{a/b}^2,\ a_2 + b_{21}x_{a/b}^2 + d_2x_{a/b},\ a_3 + b_{31}x_{a/b}^2 + d_3x_{a/b},\\ &a_4 + b_{41}x_{a/b}^2 + d_4x_{a/b}\Big).\end{aligned}$$

The remaining coefficients b_{j1} and d_j can be chosen so that

(a) ξ_a is a saddle and ξ_b a sink in P_{12},
(b) ξ_a is a sink and ξ_b a saddle in P_{13} and P_{14}.

We thus obtain a heteroclinic network made of three connections as follows:

$$[\xi_a \to \xi_b] \text{ in } P_{12}; \quad [\xi_b \to \xi_a] \text{ in } P_{13}; \quad [\xi_b \to \xi_a] \text{ in } P_{14}.$$

There are two cycles, each involving both equilibria:

$$C_3 = [\xi_a \to \xi_b \to \xi_a] \subset P_{12} \cup P_{13} \subset \text{Fix}(\langle\kappa_4\rangle)$$
$$C_4 = [\xi_a \to \xi_b \to \xi_a] \subset P_{12} \cup P_{14} \subset \text{Fix}(\langle\kappa_3\rangle)$$

They are schematically depicted in figure 2.5. Since both are contained in a three-dimensional fixed-point subspace and $-\mathbb{1} \notin \Gamma$, they are of type B_2^+.

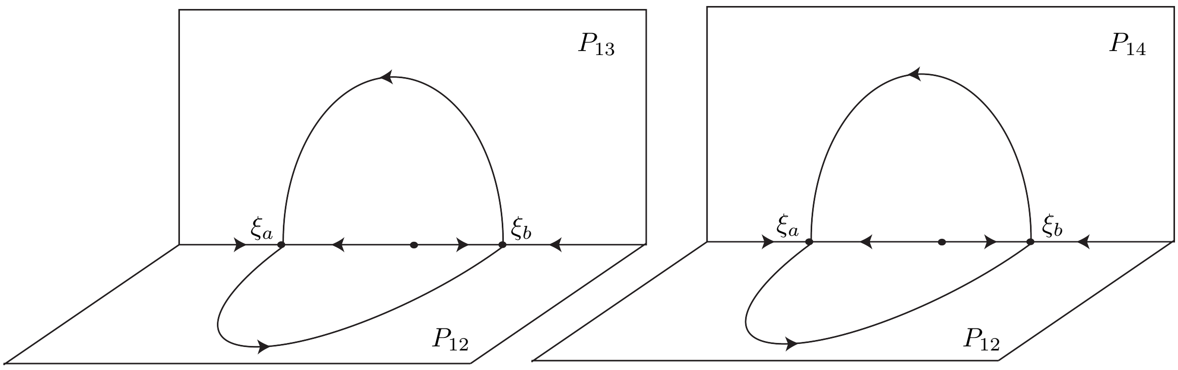


Figure 2.5.: The B_2^+-cycles in the network.

Dynamics near the (B_2^+,B_2^+)-network

In a way analogous to that by Kirk and Silber in [23] and as explained in subsection 1.1.2, we use the linearization at each equilibrium to derive local maps linearizing the flow near the equilibria. Global maps are defined as small perturbations of the identity, conditioning the domain of definition of the return maps around each cycle in the network.

Near ξ_a the local maps are defined for points in an incoming section to the flow approaching ξ_a, H_{ai}^{in} for $i = 3,4$, with image in an outgoing section to the flow leaving ξ_a, H_{a2}^{out}. Linearize the flow near ξ_a to obtain

$$\begin{cases} \dot{x}_1 = -r_a x_1 \\ \dot{x}_2 = \quad e_{a2} x_2 \\ \dot{x}_3 = -c_{a3} x_3 \\ \dot{x}_4 = -c_{a4} x_4, \end{cases}$$

where all the constants are positive.

Near ξ_b, the local maps are analogously defined but now we have two outgoing sections, H^{out}_{b3} and H^{out}_{b4}, and one incoming section, H^{in}_{b2}. Linearization of the flow near ξ_b provides

$$\begin{cases} \dot{x}_1 = -r_b x_1 \\ \dot{x}_2 = -c_{b2} x_2 \\ \dot{x}_3 = \quad e_{b3} x_3 \\ \dot{x}_4 = \quad e_{b4} x_4, \end{cases}$$

where again all the constants are positive. Assume from now on, and without loss of generality, that $e_{b3} > e_{b4}$.

The local coordinates for the sections to the flow are as follows:

$$\begin{aligned} H^{\text{out}}_{a2} = H^{\text{in}}_{b2} &= \{(x_1,1,x_3,x_4)\} \\ H^{\text{in}}_{a3} = H^{\text{out}}_{b3} &= \{(x_1,x_2,1,x_4)\} \\ H^{\text{in}}_{a4} = H^{\text{out}}_{b4} &= \{(x_1,x_2,x_3,1)\} \end{aligned}$$

Standard construction of the local maps using the linearized flow gives:

$$\begin{aligned} \varphi_{a3} : H^{\text{in}}_{a3} \to H^{\text{out}}_{a2}, \quad \varphi_{a3}(x_1,x_2,1,x_4) &= \left(x_1 x_2^{\frac{r_a}{e_{a2}}}, 1, x_2^{\frac{c_{a3}}{e_{a2}}}, x_4 x_2^{\frac{c_{a4}}{e_{a2}}} \right) \\ \varphi_{b3} : H^{\text{in}}_{b2} \to H^{\text{out}}_{b3}, \quad \varphi_{b3}(x_1,1,x_3,x_4) &= \left(x_1 x_3^{\frac{r_b}{e_{b3}}}, x_3^{\frac{c_{b2}}{e_{b3}}}, 1, x_4 x_3^{-\frac{e_{b4}}{e_{b3}}} \right) \\ \varphi_{a4} : H^{\text{in}}_{a4} \to H^{\text{out}}_{a2}, \quad \varphi_{a4}(x_1,x_2,x_3,1) &= \left(x_1 x_2^{\frac{r_a}{e_{a2}}}, 1, x_3 x_2^{\frac{c_{a3}}{e_{a2}}}, x_2^{\frac{c_{a4}}{e_{a2}}} \right) \\ \varphi_{b4} : H^{\text{in}}_{b2} \to H^{\text{out}}_{b3}, \quad \varphi_{b4}(x_1,1,x_3,x_4) &= \left(x_1 x_4^{\frac{r_b}{e_{b4}}}, x_4^{\frac{c_{b2}}{e_{b4}}}, x_3 x_4^{-\frac{e_{b3}}{e_{b4}}}, 1 \right) \end{aligned}$$

The domains of definition of the maps φ_{b3} and φ_{b4} are constrained by the inequalities

$$(1-\epsilon) x_3^{\frac{e_{b4}}{e_{b3}}} > x_4 \geq 0 \quad \text{and} \quad (1-\epsilon) x_4^{\frac{e_{b3}}{e_{b4}}} > x_3 \geq 0,$$

in the respective (local) coordinates. By composing these local maps with global maps analogous to those used in [23], without resorting to polar coordinates however, we obtain four return maps, one for each connection belonging to each cycle. These are, for C_3,

$$\begin{aligned} g_{3a} &: H^{\text{in}}_{a3} \to H^{\text{in}}_{a3} \\ g_{3b} &: H^{\text{in}}_{b2} \to H^{\text{in}}_{b2} \end{aligned}$$

and, for C_4,

$$\begin{aligned} g_{4a} &: H^{\text{in}}_{a4} \to H^{\text{in}}_{a4} \\ g_{4b} &: H^{\text{in}}_{b2} \to H^{\text{in}}_{b2}. \end{aligned}$$

The return maps are given by:

$$g_{3a}(x_1,x_2,1,x_4) = \left(A_1 x_1 x_2^{\frac{r_a}{e_{a2}} + \frac{c_{a3} r_b}{e_{a2} e_{b3}}},\ B_1 x_2^{\tilde{\rho}},\ 1,\ C_1 x_4 x_2^{\tilde{\delta}} \right),$$

with $0 \leq x_4 < k_1^{3a}(1-\epsilon)x_2^{\frac{c_{a3}}{e_{a2}}\left(\frac{e_{b4}}{e_{b3}}-\frac{c_{a4}}{c_{a3}}\right)}$;

$$g_{3b}(x_1,1,x_3,x_4) = \left(A_2 x_1 x_3^{\frac{r_b}{e_{b3}}+\frac{c_{b2}r_a}{e_{a2}e_{b3}}},\ 1,\ B_2 x_3^{\tilde{\rho}},\ C_2 x_4 x_3^{\frac{e_{b4}}{e_{b3}}(\rho-1)}\right),$$

with $0 \leq x_4 < k_2^{3a}(1-\epsilon)x_3^{\frac{e_{b4}}{e_{b3}}}$;

$$g_{4a}(x_1,x_2,x_3,1) = \left(D_1 x_1 x_2^{\frac{r_a}{e_{a2}}+\frac{c_{a4}r_b}{e_{a2}e_{b4}}},\ E_1 x_2^{\rho},\ x_3 x_2^{\delta},\ 1\right),$$

with $0 \leq x_3 < k_2^{3a}(1-\epsilon)x_2^{\frac{c_{a4}}{e_{a2}}\left(\frac{e_{b3}}{e_{b4}}-\frac{c_{a3}}{c_{a4}}\right)}$;

$$g_{4b}(x_1,1,x_3,x_4) = \left(D_2 x_1 x_4^{\frac{r_b}{e_{b4}}+\frac{c_{b2}r_a}{e_{b4}e_{a2}}},\ 1,\ E_2 x_3 x_4^{\frac{e_{b3}}{e_{b4}}(\tilde{\rho}-1)},\ F_2 x_4^{\rho}\right),$$

with $0 \leq x_3 < k_2^{3a}(1-\epsilon)x_4^{\frac{e_{b3}}{e_{b4}}}$, where

$$\rho := \frac{c_{a4}c_{b2}}{e_{a2}e_{b4}}, \quad \delta := \frac{c_{a3}}{e_{a2}} - \frac{e_{b3}c_{a4}}{e_{a2}e_{b4}}$$
$$\tilde{\rho} := \frac{c_{a3}c_{b2}}{e_{a2}e_{b2}}, \quad \tilde{\delta} := \frac{c_{a4}}{e_{a2}} - \frac{e_{b4}c_{a3}}{e_{a2}e_{b3}}.$$

The constants $A_i, B_i, C_i, D_i, E_i, F_i$ arise because the global maps are assumed to be perturbations of the identity. We denote the respective domains of definition, given through the inequalities above, by $\mathrm{dom}(g_j)$.

Stability indices for the (B_2^+,B_2^+)-network

In the terminology of lemma 1.47 we have $\tilde{\rho} = a_1a_2$ for C_3 and $\rho = a_1a_2$ for C_4. Since we want to avoid having all indices along one of the cycles equal to $-\infty$, from now on we assume $\rho,\tilde{\rho} > 1$. Note that precisely one of $\tilde{\delta}$ and δ is positive. We calculate the stability indices for two cases:

(i) $\delta < 0$ $(\Rightarrow \tilde{\delta} > 0)$
(ii) $\delta > 0$ $(\Rightarrow \tilde{\delta} < 0)$

We use the superscript n, writing σ_{ij}^n, when the invariant set for which the stability index is calculated is the whole network. Subscripts indicate the direction of the connection and the cycle: $\sigma_{ij,3}$ for the connection in C_3 and the stability index relative only to this cycle, whereas we write $\sigma_{ij,3}^n$ for the stability index along the same connection but now calculated for the network. Note that, when calculating the stability index of the common connection with respect to the network, we have $\sigma_{ab,3}^n = \sigma_{ab,4}^n$. In this case, we use σ_{ab}^n.

Theorem 2.11 ([8]). *Generically, the stability indices for connecting trajectories in the network are*

$$\textbf{Case (i):}\quad \sigma_{ab,4} = \sigma_{ba,4} = -\infty, \quad \sigma_{ab,3} > 0, \quad \sigma_{ba,3} = +\infty;$$
$$\sigma_{ab}^n, \sigma_{ba,4}^n > 0, \quad \sigma_{ba,3}^n = +\infty.$$

$$\textbf{Case (ii):}\quad \sigma_{ab,4} < 0, \quad \sigma_{ba,4} = +\infty, \quad \sigma_{ab,3} = \sigma_{ba,3} = -\infty;$$
$$\sigma^n_{ab}, \sigma^n_{ba,3} > 0, \quad \sigma^n_{ba,4} = +\infty.$$

Proof. The c-indices can be deduced from lemma 1.47, where $a_1 a_2 = \tilde{\rho}$ and $b_1 a_2 + b_2 = \tilde{\delta}$ for C_3, while $a_1 a_2 = \rho$ and $b_1 a_2 + b_2 = \delta$ for C_4. All n-indices that are equal to $+\infty$ follow from lemma 2.3. The same is true for $\sigma^n_{ab} > 0$ in (i). Of the remaining ones, we show how to calculate $\sigma^n_{ba,3}$ in case (ii), the others can be determined in a similar manner.

In case (ii) we have $\delta > 0$, which together with $\rho, \tilde{\rho} > 1$ implies that the return maps around C_4 are contractions. This allows us to determine all points in H^{in}_{a3}, that are not attracted to the network, in two steps: first we calculate the preimage $\mathcal{E}_0 \subset H^{\text{in}}_{a3}$ under φ_{a3} of the complement of $\text{dom}(g_{3b}) \cup \text{dom}(g_{4b}) \subset H^{\text{in}}_{b2}$. Then we take the union of preimages of $\mathcal{E}_0$ under the return map g_{3a}, $\mathcal{E}_n := g^{-n}_{3a}(\mathcal{E}_0)$. In the same way as Kirk and Silber do in [23], we restrict the calculations (and notation) to the two relevant components. Also, we adjust the constants $k, \hat{k}$ in every step.

$$\begin{aligned}
\mathcal{E}_0 &= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid \varphi_{a3}(x_2,x_4) \notin \text{dom}(g_{3b}) \cup \text{dom}(g_{4b})\Big\} \\
&= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid \Big(x_2^{\frac{c_{a3}}{e_{a2}}},\ x_4 x_2^{\frac{c_{a4}}{e_{a2}}}\Big) \notin \text{dom}(g_{3b}) \cup \text{dom}(g_{4b})\Big\} \\
&= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid k x_2^{\frac{e_{b4}}{e_{b3}}\frac{c_{a4}}{e_{a2}}} \le x_4 x_2^{\frac{c_{a4}}{e_{a2}}} \le \hat{k} x_2^{\frac{e_{b4}}{e_{b3}}\frac{c_{a4}}{e_{a2}}}\Big\} \\
&= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid k x_2^{-\tilde{\delta}} \le x_4 \le \hat{k} x_2^{-\tilde{\delta}}\Big\} \\
\Rightarrow \mathcal{E}_1 &= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid g_{3a}(x_2,x_4) \in \mathcal{E}_0\Big\} \\
&= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid \Big(B_1 x_2^{\tilde{\rho}},\ C_1 x_4 x_2^{\tilde{\delta}}\Big) \in \mathcal{E}_0\Big\} \\
&= \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid k x_2^{-\tilde{\delta}\tilde{\rho}-\tilde{\delta}} \le x_4 \le \hat{k} x_2^{-\tilde{\delta}\tilde{\rho}-\tilde{\delta}}\Big\}
\end{aligned}$$

Iteration leads to

$$\mathcal{E}_n = \Big\{(x_2,x_4) \in H^{\text{in}}_{a3} \mid k x_2^{\alpha_n} \le x_4 \le \hat{k} x_2^{\alpha_n}\Big\},$$
$$\text{where} \quad \alpha_n = -\tilde{\delta} \sum_{j=0}^{n} \tilde{\rho}^j.$$

The sequence of exponents $(\alpha_n)_{n\in\mathbb{N}}$ is monotonically increasing and unbounded, because $\alpha_{n+1} - \alpha_n = -\tilde{\delta}\tilde{\rho}^{n+1} > 0$. Therefore, in the generic case $\alpha_n \neq 1$ for all $n \in \mathbb{N}$, by lemma 2.4 we obtain $\sigma^n_{ba,3} > 0$.

For the calculation of σ^n_{ab} the sequence of exponents turns out to be

$$\beta_n = \frac{e_{b4}}{e_{b3}} \tilde{\rho}^n - (\rho - 1) \sum_{j=0}^{n-1} \tilde{\rho}^j.$$

This gives $\beta_{n+1} - \beta_n = \frac{e_{b4}}{e_{b3}} \tilde{\rho}^n (\tilde{\rho} - \rho)$ and since $\delta > 0$ if and only if $\tilde{\rho} > \rho$, this sequence is also monotonically increasing, yielding $\sigma^n_{ab} > 0$. □

Theorem 2.11 tells us everything about the generic stability configuration of the network. In case (i), C_3 is p.a.s., having only positive indices. The other cycle is completely unstable, with both indices equal to $-\infty$. By moving from the cycle to the network as the underlying set, these stability indices become positive, making the network as a whole p.a.s. However, from taking into account all of these quantities we know more than that: even though the whole network is p.a.s., one might say that C_3 wins the competition between the two cycles: most trajectories near the connection that belongs only to C_4 switch to C_3 to which they then converge. So starting with an arbitrary initial condition near the network, no matter which connection, we will most likely end up at C_3.

In case (ii) things are not entirely analogous. C_3 is completely unstable, just as C_4 was before. But now C_4 is not p.a.s. since it has a negative index along the common trajectory. This turns positive, however, when looking at the entire network, which is p.a.s. as before. The difference is that no individual cycle is p.a.s., even though C_4 is more stable when looking at c-indices only. However, almost all trajectories starting near the common trajectory make finitely many excursions around the C_3 before converging to C_4.

2.2.3. The (B_3^-, B_3^-)-network

This subsection covers the second very simple heteroclinic network, made up of two B_3^--cycles. We do not go into as much detail to explain its existence as we did for the (B_2^+, B_2^+)-network, since it was already studied in [23]. We do, however, proceed by recalling their general setting and some terminology.

Consider the group $\Gamma \cong \mathbb{Z}_2^4 \subset O(4)$ generated by the following elements of order 2:

$$
\begin{aligned}
\kappa_1.(x_1,x_2,x_3,x_4) &= (-x_1,x_2,x_3,x_4)\\
\kappa_2.(x_1,x_2,x_3,x_4) &= (x_1,-x_2,x_3,x_4)\\
\kappa_3.(x_1,x_2,x_3,x_4) &= (x_1,x_2,-x_3,x_4)\\
\kappa_4.(x_1,x_2,x_3,x_4) &= (x_1,x_2,x_3,-x_4).
\end{aligned}
$$

All coordinate axes and planes (denote the x_i-x_j-plane by P_{ij}) are fixed-point subspaces of appropriate subgroups. Assume that for a Γ-equivariant vector field f there is a non-trivial equilibrium ξ_j on each x_j-axis. Then coefficients can be chosen such that there is a very simple heteroclinic network made of five connections as follows:

$$[\xi_1 \to \xi_2] \text{ in } P_{12}; \quad [\xi_2 \to \xi_3] \text{ in } P_{23}; \quad [\xi_3 \to \xi_1] \text{ in } P_{13}; \quad [\xi_2 \to \xi_4] \text{ in } P_{24}; \quad [\xi_4 \to \xi_1] \text{ in } P_{14}.$$

We therefore have two cycles, as depicted in figure 2.6. Each involves three equilibria:

$$
\begin{aligned}
C_3 &= [\xi_1 \to \xi_2 \to \xi_3 \to \xi_1] \subset S_{123} = \mathrm{Fix}(\langle \kappa_4 \rangle)\\
C_4 &= [\xi_1 \to \xi_2 \to \xi_4 \to \xi_1] \subset S_{124} = \mathrm{Fix}(\langle \kappa_3 \rangle)
\end{aligned}
$$

Here, S_{ijk} denotes the three-dimensional coordinate space spanned by the x_i-, x_j- and x_k-axes. Again, both cycles are contained in a three-dimensional fixed-point subspace, but this time $-\mathbb{1} \in \Gamma$, so they are of type B_3^-. We also refer to C_3 and C_4 as the ξ_3- and ξ_4-cycle, respectively.

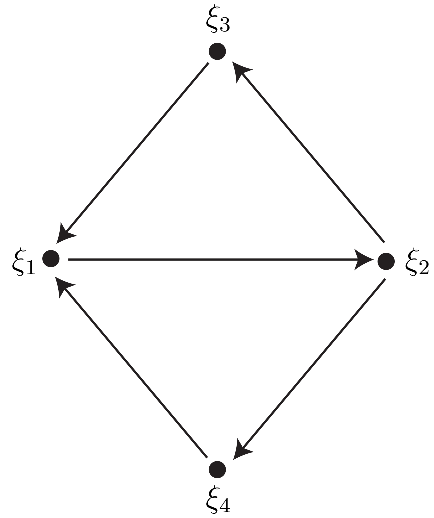


Figure 2.6.: The heteroclinic (B_3^-, B_3^-)-network

Dynamics near the (B_3^-, B_3^-)-network

In [23], Kirk and Silber introduce cross sections $H_i^{\mathrm{out},k}$ and $H_i^{\mathrm{in},k}$ along the connecting trajectories. Local and global maps are constructed just as we did in the previous subsection, with the sole difference that appropriate polar coordinates are introduced for convenience. Linearization of the flow at the equilibria in local coordinates, here at ξ_3, is assumed to be given by

$$\begin{cases} \dot{x}_1 = \quad e_{31}x_1 \\ \dot{x}_2 = -c_{32}x_2 \\ \dot{u}_3 = -r_3 u_3 \\ \dot{x}_4 = -c_{34}x_4, \end{cases}$$

where all constants are positive. This is done similarly for the other equilibria, labeling radial, contracting, expanding and transverse eigenvalues accordingly. For every equilibrium there is precisely one eigenvalue of each type. For ξ_1 and ξ_2 the type of the eigenvalues obviously depends on which cycle one considers. All transverse eigenvalues are collected in table 2.1, only ξ_2 possesses a positive one.

Local and global maps are constructed in the standard way. The former we list for ease of reference, the latter are irrelevant for the stability indices and can be found in [23] along with details on the construction of local coordinates (x,y) as well as transverse sections $H_i^{\mathrm{out},j}$ and

Table 2.1.: Transverse eigenvalues

Equilibrium	ξ_3-cycle	ξ_4-cycle
ξ_1	$-c_{14}$	$-c_{13}$
ξ_2	e_{24}	e_{23}
ξ_3	$-c_{34}$	–
ξ_4	–	$-c_{43}$

$H_i^{\text{in},k}$. Note that in contrast to [23] we do not use polar coordinates for the maps ϕ_{ijk}.

$$\begin{aligned}
\phi_{123} &: H_2^{\text{in},1} \to H_2^{\text{out},3}, & \phi_{123}(1,x_2,x_3,x_4) &= \left(x_3^{\frac{c_{21}}{e_{23}}}, x_2 x_3^{\frac{r_2}{e_{23}}}, 1, x_4 x_3^{-\frac{e_{24}}{e_{23}}}\right)\\
\phi_{231} &: H_3^{\text{in},2} \to H_3^{\text{out},1}, & \phi_{231}(x_1,1,x_3,x_4) &= \left(1, x_1^{\frac{c_{32}}{e_{31}}}, x_3 x_1^{\frac{r_3}{e_{31}}}, x_4 x_1^{\frac{c_{34}}{e_{31}}}\right)\\
\phi_{312} &: H_1^{\text{in},3} \to H_1^{\text{out},2}, & \phi_{312}(x_1,x_2,1,x_4) &= \left(x_1 x_2^{\frac{r_1}{e_{12}}}, 1, x_2^{\frac{c_{13}}{e_{12}}}, x_4 x_2^{\frac{c_{14}}{e_{12}}}\right)\\
\phi_{124} &: H_2^{\text{in},1} \to H_2^{\text{out},4}, & \phi_{124}(1,x_2,x_3,x_4) &= \left(x_4^{\frac{c_{21}}{e_{24}}}, x_2 x_4^{\frac{r_2}{e_{24}}}, x_3 x_4^{-\frac{e_{23}}{e_{24}}}, 1\right)\\
\phi_{241} &: H_4^{\text{in},2} \to H_4^{\text{out},1}, & \phi_{241}(x_1,1,x_3,x_4) &= \left(1, x_1^{\frac{c_{42}}{e_{41}}}, x_3 x_1^{\frac{c_{43}}{e_{41}}}, x_4 x_1^{\frac{r_4}{e_{41}}}\right)\\
\phi_{412} &: H_1^{\text{in},4} \to H_1^{\text{out},2}, & \phi_{412}(x_1,x_2,x_3,1) &= \left(x_1 x_2^{\frac{r_1}{e_{12}}}, 1, x_3 x_2^{\frac{c_{13}}{e_{12}}}, x_2^{\frac{c_{14}}{e_{12}}}\right)
\end{aligned}$$

These are composed to form return maps $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{g}_3$ around C_3 and g_1, g_2, g_4 around C_4. which in turn reduce to the following two-dimensional maps.

$$\begin{aligned}
\tilde{h}_1 &: H_1^{\text{out},2} \to H_1^{\text{out},2}, & \tilde{h}_1(x,y) &= (x^{\tilde{\rho}}, y x^{\tilde{\nu}}) & &\text{for} \quad y < x^{\frac{e_{24}}{e_{23}}}\\
\tilde{h}_2 &: H_2^{\text{out},3} \to H_2^{\text{out},3}, & \tilde{h}_2(x,y) &= (x^{\tilde{\rho}}, y x^{\tilde{\delta}}) & &\text{for} \quad y < x^{-\tilde{\delta}}\\
\tilde{h}_3 &: H_3^{\text{out},1} \to H_3^{\text{out},1}, & \tilde{h}_3(x,y) &= (x^{\tilde{\rho}}, y x^{\tilde{\tau}}) & &\text{for} \quad y < x^{\tilde{\sigma}}\\
h_1 &: H_1^{\text{out},2} \to H_1^{\text{out},2}, & h_1(x,y) &= (x y^{\nu}, y^{\rho}) & &\text{for} \quad x < y^{\frac{e_{23}}{e_{24}}}\\
h_2 &: H_2^{\text{out},4} \to H_2^{\text{out},4}, & h_2(x,y) &= (x^{\rho}, y x^{\delta}) & &\text{for} \quad y < x^{-\delta}\\
h_4 &: H_4^{\text{out},1} \to H_4^{\text{out},1}, & h_4(x,y) &= (x^{\rho}, y x^{\tau}) & &\text{for} \quad y < x^{\sigma}
\end{aligned}$$

As before we denote domains of definition by $\text{dom}(h_j) \subset H_j^{\text{out},k}$. Constants arising from the global maps are ignored for the sake of readability and because they are irrelevant for the stability indices – as we saw in the proof of theorem 2.11. The exponents above depend on the eigenvalues in the following way:

$$\begin{aligned}
\rho &= \frac{c_{42}c_{14}c_{21}}{e_{24}e_{41}e_{12}} > 0 & \tilde{\rho} &= \frac{c_{32}c_{13}c_{21}}{e_{23}e_{31}e_{12}} > 0\\
\nu &= -\frac{e_{23}}{e_{24}} + \frac{c_{21}c_{43}}{e_{24}e_{41}} + \frac{c_{13}c_{42}c_{21}}{e_{41}e_{24}e_{12}} & \tilde{\nu} &= -\frac{e_{24}}{e_{23}} + \frac{c_{21}c_{34}}{e_{23}e_{31}} + \frac{c_{14}c_{32}c_{21}}{e_{31}e_{23}e_{12}}\\
\delta &= \frac{c_{43}}{e_{41}} + \frac{c_{13}c_{42}}{e_{12}e_{41}} - \frac{e_{23}c_{14}c_{42}}{e_{12}e_{41}e_{24}} & \tilde{\delta} &= \frac{c_{34}}{e_{31}} + \frac{c_{14}c_{32}}{e_{12}e_{31}} - \frac{e_{24}c_{13}c_{32}}{e_{12}e_{31}e_{23}}\\
\tau &= \frac{c_{13}}{e_{12}} - \frac{e_{23}c_{14}}{e_{12}e_{24}} + \frac{c_{14}c_{21}c_{43}}{e_{12}e_{41}e_{24}} & \tilde{\tau} &= \frac{c_{14}}{e_{12}} - \frac{e_{24}c_{13}}{e_{12}e_{23}} + \frac{c_{13}c_{21}c_{34}}{e_{12}e_{31}e_{23}}
\end{aligned}$$

Another quantity appearing later is

$$\sigma = \frac{c_{14}}{e_{12}}\left(\frac{e_{23}}{e_{24}} - \frac{c_{13}}{c_{14}}\right), \qquad \tilde{\sigma} = \frac{c_{13}}{e_{12}}\left(\frac{e_{24}}{e_{23}} - \frac{c_{14}}{c_{13}}\right).$$

Note that σ and $\tilde{\sigma}$ have opposite signs. Together with

$$\delta = \frac{c_{43}}{e_{41}} + \tilde{\sigma}\frac{c_{42}e_{23}}{e_{24}e_{41}}, \qquad \tilde{\delta} = \frac{c_{34}}{e_{31}} + \sigma\frac{c_{32}e_{24}}{e_{23}e_{31}}, \tag{2.2}$$

this leads to the observation that only one of δ and $\tilde{\delta}$ can be negative, as long as all the c_{ij} are positive. The eigenvalue ratios a_j and b_j, required for lemma 1.48, are given in table 2.2.

Table 2.2.: a_j and b_j

	ξ_3-cycle	ξ_4-cycle
values at ξ_1	$a_3 = c_{13}/e_{12}$; $b_3 = c_{14}/e_{12}$	$a_3 = c_{14}/e_{12}$; $b_3 = c_{13}/e_{12}$
values at ξ_2	$a_1 = c_{21}/e_{23}$; $b_1 = -e_{24}/e_{23}$	$a_1 = c_{21}/e_{24}$; $b_1 = -e_{23}/e_{24}$
values at ξ_3	$a_2 = c_{32}/e_{31}$; $b_2 = c_{34}/e_{31}$	–
values at ξ_4	–	$a_2 = c_{42}/e_{41}$; $b_2 = c_{43}/e_{41}$

Note that we swop the indices here in order to apply lemma 1.48 more conveniently. With the assumptions made in [23], namely that $c_{ij}, e_{ij} > 0$ for all i and j, we have $b_1 < 0 < b_2$,b_3 for both cycles. So we only have to look at case (iii) of lemma 1.48. Note that

$$\rho = a_1a_2a_3 = \tilde{\rho} \quad \text{and} \quad \delta = b_1a_2a_3 + b_3a_2 + b_2 = \tilde{\delta}, \tag{2.3}$$

where the equalities $\rho = \tilde{\rho}$ and $\delta = \tilde{\delta}$ are merely symbolic as an expression. In order to avoid complete instability we maintain the assumption that $\rho, \tilde{\rho} > 1$. Moreover, we henceforth assume without loss of generality that $e_{23} > e_{24}$.

When calculating stability indices, it is of importance to know whether or not the return maps are contractions. We summarize the corresponding results from [23] in the following lemma.

Lemma 2.12 ([23])**.** *If $\rho, \tilde{\rho} > 1$ and $e_{23} > e_{24}$, then the following holds.*

1. *The reduced maps h_j and $\tilde{h}_j$ are contractions if and only if the return maps g_j and $\tilde{g}_j$ are contractions.*
2. *If $\delta > 0$, then the maps g_i (around C_4) are contractions.*
3. *If $\tilde{\delta} > 0$, then the maps $\tilde{g}_i$ (around C_3) are contractions.*
4. *If $\tilde{\delta} > 0$ and $\tilde{\sigma} < 1$, then generically C_3 is p.a.s.*

Proof. See [23], lemmas 1 and 2. □

The fourth statement above is one of two main stability results in [23]. We recover it by means of the stability index and extend the study to all other parameter configurations.

The following basic observation is useful for future reference.

Lemma 2.13 ([8]). *All stability indices for the ξ_3- (respectively, ξ_4-) cycle are equal to $-\infty$ if and only if $\tilde{\delta} < 0$ (respectively, $\delta < 0$).*

Proof. Straightforward with lemma 1.48. In fact, since $\rho, \tilde{\rho} > 1$, and given (2.3), we have all stability indices equal to $-\infty$ for the cycle corresponding to δ or $\tilde{\delta}$ negative. □

Stability indices for the parameter situation in [23]

Under the assumptions specified above Kirk and Silber investigated two different situations in [23]:

- $\delta, \tilde{\delta} > 0$ – corresponding to figure 5 in [23]
- $\delta\tilde{\delta} < 0$ – corresponding to lemma 3 in [23]

We calculate stability indices for both of these configurations, confirming and refining their results about stability and competition. Consistent with the notation in [23] we distinguish between indices with respect to the ξ_3- and ξ_4-cycles by writing $\tilde{\sigma}_{ij}$ and σ_{ij}, respectively. Starting with the case $\delta, \tilde{\delta} > 0$, we have the following two propositions.

Proposition 2.14 ([8]). *The stability indices corresponding to the cases depicted in figure 5 of [23] are as follows:*

$$\tilde{\sigma}_{23} = \sigma_{24} = +\infty \qquad \tilde{\sigma}_{12} \in (0, +\infty) \qquad \sigma_{12} \in (-\infty, 0)$$

and either $\tilde{\sigma}_{31} = +\infty$ or $\sigma_{41} = +\infty$, but not both.

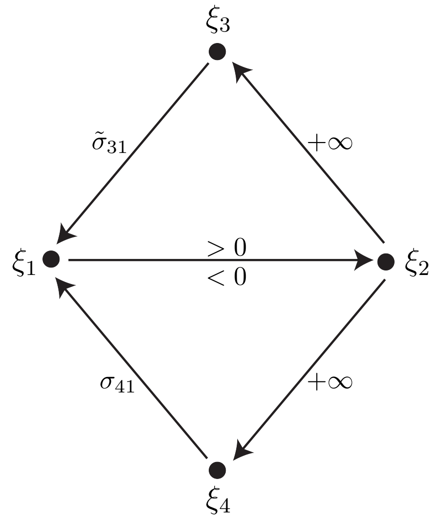


Figure 2.7.: The stability indices for the network in proposition 2.14. Exactly one of σ_{41} and $\tilde{\sigma}_{31}$ is equal to $+\infty$.

Proof. From case (iii)(b) in lemma 1.48, we obtain for the ξ_3-cycle

$$\begin{aligned}
\tilde{\sigma}_{12} &= f^{\text{index}}(b_1,1) = f^{\text{index}}\left(-\frac{e_{24}}{e_{23}},1\right) = f^{+}\left(-\frac{e_{24}}{e_{23}},1\right) - 0 = \frac{e_{23}}{e_{24}} - 1 > 0,\\
\tilde{\sigma}_{23} &= +\infty,\\
\tilde{\sigma}_{31} &= f^{\text{index}}(b_3 + b_1 a_3,1) = f^{\text{index}}(-\tilde{\sigma},1) = \begin{cases} +\infty & \text{if } \tilde{\sigma} \leq 0 \\ \frac{1}{\tilde{\sigma}} - 1 > 0 & \text{if } \tilde{\sigma} \in (0,1) \\ 1 - \tilde{\sigma} < 0 & \text{if } \tilde{\sigma} > 1 \end{cases},
\end{aligned}$$

and for the ξ_4-cycle

$$\begin{aligned}
\sigma_{12} &= f^{\text{index}}(b_1,1) = f^{\text{index}}\left(-\frac{e_{23}}{e_{24}},1\right) = -f^{+}\left(\frac{e_{23}}{e_{24}},-1\right) = 1 - \frac{e_{23}}{e_{24}} < 0,\\
\sigma_{24} &= +\infty,\\
\sigma_{41} &= f^{\text{index}}(b_3 + b_1 a_3,1) = f^{\text{index}}(-\sigma,1) = \begin{cases} +\infty & \text{if } \sigma \leq 0 \\ \frac{1}{\sigma} - 1 > 0 & \text{if } \sigma \in (0,1) \\ 1 - \sigma < 0 & \text{if } \sigma > 1 \end{cases}.
\end{aligned}$$

□

Now we determine the stability indices with respect to the entire network. This is done similarly to the proof of theorem 2.11, but is simpler, because the return maps around both cycles are contractions.

Proposition 2.15 ([8])**.** *The stability indices with respect to the network, corresponding to the cases depicted in figure* 5 *of [23], are all positive. Furthermore, no finite stability index becomes infinite when calculated with respect to the network.*

Proof. By lemma 2.3 we know that $\tilde{\sigma}^n_{23} = \sigma^n_{24} = +\infty$ and $\sigma^n_{12} > 0$. That σ^n_{12} is finite follows from the observation that the union of the domains of definition of the return maps around each cycle starting at this connection excludes a cusp-shaped region.

The proof proceeds by determining $\tilde{\sigma}^n_{31}$ and σ_{41}, case by case corresponding to each line of figure 5 in [23]. Because both δ and $\tilde{\delta}$ are positive, all return maps are contractions and we only have to calculate along each connection the set of points taken outside $\text{dom}(\tilde{h}_1) \cup \text{dom}(h_1)$ by the local maps.

Line 1: Because of $\sigma > 1$ and $\tilde{\sigma} < 0$ we have $\sigma_{41} < 0$ and $\tilde{\sigma}_{31} = +\infty$, which implies $\tilde{\sigma}^n_{31} = +\infty$. We calculate σ^n_{41} by looking at the set of points

$$\begin{aligned}
\mathcal{E}_0 &= \{(x,y) \in H_1^{\text{in},4} |\ \ \phi_{412}(x,y) \notin \text{dom}(\tilde{h}_1) \cup \text{dom}(h_1)\}\\
&= \{(x,y) \in H_1^{\text{in},4} |\ \ (yx^{c_{13}/e_{12}}, x^{c_{14}/e_{12}}) \notin \text{dom}(\tilde{h}_1) \cup \text{dom}(h_1)\}\\
&= \{(x,y) \in H_1^{\text{in},4} |\ \ x^{-\tilde{\sigma}} < y < x^{-\tilde{\sigma}}\}.
\end{aligned}$$

Since $-\tilde{\sigma} > 0$, the set $\mathcal{E}_0$ is a thin cusp and we have $\sigma^n_{41} > 0$ and finite.

Line 2: As in the previous case $\tilde{\sigma}^n_{31} = +\infty$. Moreover, $\sigma^n_{41} \geq \sigma_{41} > 0$ and finite because of $0 < \sigma < 1$.

Line 3: In this case and the next we have $\sigma_{41} = +\infty$ due to $\sigma < 0$, so $\sigma^n_{41} = +\infty$. Because of $0 < \tilde{\sigma} < 1$ it follows that $\tilde{\sigma}^n_{31} \geq \tilde{\sigma}_{31} > 0$.

Line 4: Since $\tilde{\sigma} > 1$ we have $\tilde{\sigma}_{31} < 0$ and thus determine $\tilde{\sigma}^n_{31}$ by calculating

$$\begin{aligned}\mathcal{E}_0 &= \{(x,y) \in H_1^{\text{in},3} |\ \phi_{312}(x,y) \notin \text{dom}(\tilde{h}_1) \cup \text{dom}(h_1)\} \\ &= \{(x,y) \in H_1^{\text{in},3} |\ (x^{c_{13}/e_{12}}, yx^{c_{14}/e_{12}}) \notin \text{dom}(\tilde{h}_1) \cup \text{dom}(h_1)\} \\ &= \{(x,y) \in H_1^{\text{in},3} |\ x^{\tilde{\sigma}} < y < x^{\tilde{\sigma}}\}.\end{aligned}$$

As in the first case this is a thin cusp and therefore we obtain $\tilde{\sigma}^n_{31} > 0$ and finite. □

Let us now consider the second case, where $\delta\tilde{\delta} < 0$, corresponding to lemma 3 in [23]. Again we calculate c- and n-indices.

Proposition 2.16 ([8]). *The stability indices in the conditions of lemma 3 of [23] are as follows.*

(i) For $\delta < 0:\quad \sigma_{12} = \sigma_{24} = \sigma_{41} = -\infty,\quad \tilde{\sigma}_{23} = \tilde{\sigma}_{31} = +\infty,\quad \tilde{\sigma}_{12} = \dfrac{e_{23}}{e_{24}} - 1 > 0$

(ii) For $\delta > 0:\quad \sigma_{12} = 1 - \dfrac{e_{23}}{e_{24}} < 0,\quad \sigma_{24} = \sigma_{41} = +\infty,\quad \tilde{\sigma}_{12} = \tilde{\sigma}_{23} = \tilde{\sigma}_{34} = -\infty$

Proof. The indices for the ξ_4-cycle in (i) and for the ξ_3-cycle in (ii) are clear by lemma 2.13. For the others we make use of proposition 2.14. Since σ and $\tilde{\sigma}$ have opposite signs, we have $\sigma < 0$ if and only if $\tilde{\delta} < 0$ and analogously for $\tilde{\sigma}$ and δ. This determines $\tilde{\sigma}_{31}$ and σ_{41}, yielding for case (i)

$$\tilde{\sigma}_{12} = \frac{e_{23}}{e_{24}} - 1 > 0, \qquad \tilde{\sigma}_{23} = +\infty, \qquad \tilde{\sigma}_{31} = +\infty,$$

and for case (ii)

$$\sigma_{12} = 1 - \frac{e_{23}}{e_{24}} < 0, \qquad \sigma_{24} = +\infty, \qquad \sigma_{41} = +\infty.$$

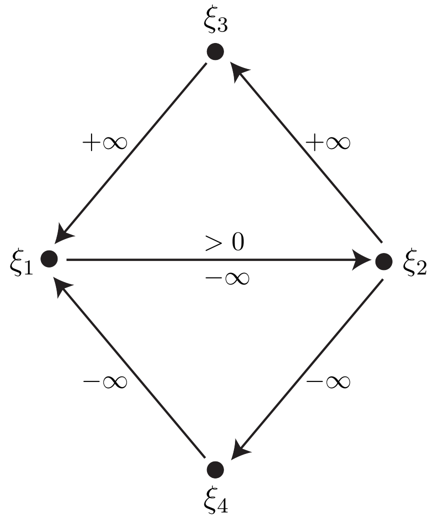
□

These c-indices are depicted in figure 2.8. Now we turn our attention to n-indices again. In both cases the return maps are contractions for only one of the cycles, so we must take into account preimages of the relevant sets again in order to find all points that leave the network. Some of the calculations can already be found in the proof of lemma 3 in [23], even though they are not related to the stability index there.

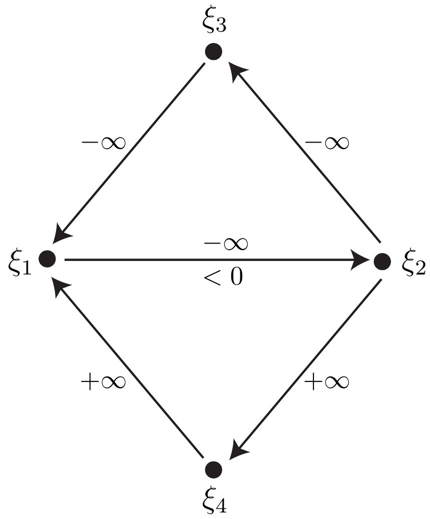
Proposition 2.17 ([8]). *Generically, the stability indices with respect to the network in lemma 3 of [23] are as follows.*

(i) For $\delta < 0:\quad \sigma^n_{12},\ \sigma^n_{24},\ \sigma^n_{41} > 0,\quad \tilde{\sigma}^n_{23} = \tilde{\sigma}^n_{31} = +\infty$

(ii) For $\delta > 0:\quad \sigma^n_{12},\ \tilde{\sigma}^n_{23},\ \tilde{\sigma}^n_{31} > 0,\quad \sigma^n_{24} = \sigma^n_{41} = +\infty$



(a) Case (i)



(b) Case (ii)

Figure 2.8.: The stability indices for the network in proposition 2.16.

Proof. We only show case (i), calculations for case (ii) are analogous. From lemma 2.3 and proposition 2.16 we deduce directly that $\tilde{\sigma}_{31}^{n} = \tilde{\sigma}_{31} = +\infty$ and $\tilde{\sigma}_{23}^{n} = +\infty$. The remaining indices can be determined in exactly the same way as in theorem 2.11. Therefore, we shorten the calculations accordingly.

For σ_{24}^{n} we investigate the section $H_2^{\text{out},4}$ across the trajectory $[\xi_2 \to \xi_4]$. We need the set $\mathcal{E}_0$ of points in $H_2^{\text{out},4}$ that do not land in $\text{dom}(h_1) \cup \text{dom}(\tilde{h}_1) \subset H_1^{\text{out},2}$ as they are transported around C_4 – they do not stay near the network any longer, while all others follow C_3 or C_4 again. To determine σ_{24}^{n} it is therefore necessary to calculate the measure of $\mathcal{E}_0$ and all its preimages under the (restricted) return map h_2 in an ε-neighbourhood since all other points belong to $\mathcal{B}(C_3 \cup C_4)$:

$$\mathcal{B}(C_3 \cup C_4) \cap B_\varepsilon = B_\varepsilon \setminus \bigcup_{n \in \mathbb{N}} h_2^{-n}(\mathcal{E}_0),$$

where B_ε is an ε-ball around 0 in $H_2^{\text{out},4}$. Note that the preimages $h_2^{-n}(\mathcal{E}_0)$ are disjoint. The above characterization of $\mathcal{E}_0$ yields

$$\begin{aligned} \mathcal{E}_0 &= \{(x,y) \in H_2^{\text{out},4} \mid \phi_{241}(x,y) \notin \text{dom}(h_1) \cup \text{dom}(\tilde{h}_1)\} \\ &= \{(x,y) \in H_2^{\text{out},4} \mid kx^{-\delta} \leq y \leq \tilde{k}x^{-\delta}\}, \end{aligned}$$

and therefore

$$\begin{aligned} h_2^{-1}(\mathcal{E}_0) &= \{(x,y) \in H_2^{\text{out},4} \mid h_2(x,y) = (x^\rho, yx^\delta) \in \mathcal{E}_0\} \\ &= \{(x,y) \in H_2^{\text{out},4} \mid kx^{-\delta(\rho+1)} \leq y \leq \tilde{k}x^{-\delta(\rho+1)}\}, \end{aligned}$$

where $k, \tilde{k} > 0$. Iterating this leads to

$$\mathcal{E}_n := h_2^{-n}(\mathcal{E}_0) = \{(x,y) \in H_2^{\text{out},4} \mid k_n x^{\alpha_n} \leq y \leq \tilde{k}_n x^{\alpha_n}\}$$

with suitable constants $k_n, \tilde{k}_n$ and the same monotonically increasing sequence of exponents $\alpha_n := -\delta \sum_{m=0}^{n} \rho^m > 0$ as in the proof of theorem 2.11. Thus, with $N \in \mathbb{N}$ such that $\alpha_N < 1 < \alpha_{N+1}$, lemma 2.4 yields

$$\sigma_{24}^n = -1 + \min\left(\frac{1}{-\delta \sum_{m=0}^{N} \rho^m}, -\delta \sum_{m=0}^{N+1} \rho^m\right) > 0.$$

In the same way, for σ_{41}^n we determine the set $\mathcal{F}_0 \subset H_4^{\text{out},1}$ of points that hit $H_1^{\text{out},2}$ in neither $\text{dom}(g_1)$ nor $\text{dom}(\tilde{g}_1)$, and its preimages (now under the appropriate return map h_4). We obtain

$$\mathcal{F}_n := h_4^{-n}(\mathcal{F}_0) = \left\{(x,y) \in H_4^{\text{out},1} \;\middle|\; k_n x^{\beta_n} \leq y \leq \tilde{k}_n x^{\beta_n}\right\}$$

where $k, \tilde{k}_n$ are again positive constants and $\beta_n = \sigma\rho^n - \tau \sum_{m=0}^{n-1} \rho^m$. The sequence $(\beta_n)_{n\in\mathbb{N}}$ is also monotonically increasing, since

$$\begin{aligned}
\beta_{n+1} - \beta_n &= \sigma\rho^{n+1} - \tau \sum_{m=0}^{n} \rho^m - \sigma\rho^n + \tau \sum_{m=0}^{n-1} \rho^m \\
&= \rho^n(\sigma(\rho - 1) - \tau) \\
&= -\frac{c_{14}c_{21}}{e_{12}e_{24}}\rho^n\delta \\
&> 0.
\end{aligned}$$

Also, $\beta_0 = \sigma > 0$. Therefore, choosing $M \in \mathbb{N}$ such that $\beta_M < 1 < \beta_{M+1}$, again by lemma 2.4, we obtain

$$\sigma_{41}^n = -1 + \min\left(\frac{1}{\sigma\rho^M - \tau \sum_{m=0}^{M-1} \rho^m},\ \sigma\rho^{M+1} - \tau \sum_{m=0}^{M} \rho^m\right) > 0.$$

The last index remaining for case (i) is σ_{12}^n. We know it is finite because $\text{dom}(h_1) \cup \text{dom}(\tilde{h}_1)$ never covers a full neighbourhood of the network in the section across $[\xi_1 \to \xi_2]$. In fact, we convince ourselves that $\sigma_{12}^n = \tilde{\sigma}_{12}$. Look at the set $\mathcal{G}_0 \subset H_1^{\text{out},2}$,

$$\mathcal{G}_0 = \left\{(x,y) \in H_1^{\text{out},2} \;\middle|\; k y^{\frac{e_{23}}{e_{24}}} < x < \tilde{k} y^{\frac{e_{23}}{e_{24}}}\right\}.$$

We have to consider its preimages under h_1, the return map around C_4, since C_3 is attracting, so nothing can "get lost" once it has hit $\text{dom}(\tilde{h}_1)$. Thus, with $h_1(x,y) = (xy^\nu, y^\rho)$ we get

$$h_1^{-n}(\mathcal{G}_0) = \left\{(x,y) \in H_1^{\text{out},2} \;\middle|\; k_n y^{\gamma_n} < x < \tilde{k}_n y^{\gamma_n}\right\},$$

$$\text{where} \quad \gamma_n = \rho^n \frac{e_{23}}{e_{24}} - \nu \sum_{m=0}^{n-1} \rho^m.$$

Now the sequence $(\gamma_n)_{n\in\mathbb{N}}$ increases monotonically, since

$$\begin{aligned}\gamma_{n+1}-\gamma_n &= \rho^{n+1}\frac{e_{23}}{e_{24}}-\rho^n\frac{e_{23}}{e_{24}}-\nu\rho^n\\ &= \rho^n\left(\frac{e_{23}}{e_{24}}(\rho-1)-\nu\right)\\ &= \rho^n\left(\frac{e_{23}}{e_{24}}(\rho-1)-\frac{e_{23}}{e_{24}}(\rho-1)-\frac{c_{21}}{e_{24}}\delta\right)\\ &= -\rho^n\frac{c_{21}}{e_{24}}\delta\\ &> 0,\end{aligned}$$

because $\delta<0$. Since $\gamma_0=\dfrac{e_{23}}{e_{24}}>1$, there is no $\gamma<1$, so

$$\sigma_{12}^n=-1+\min\{\gamma_n\mid n\in\mathbb{N}\}=-1+\gamma_0=\frac{e_{23}}{e_{24}}-1>0.$$

The genericity conditions mentioned in the statement of this proposition are $\alpha_n\neq 1$, $\beta_n\neq 1$ and $\gamma_n\neq 1$ for all $n\in\mathbb{N}$. □

Note that in case (i) the only indices that actually change when shifting from the cycle to the network are the ones along the trajectories belonging solely to C_4. For the common trajectory we get $\sigma_{12}^n=\tilde\sigma_{12}$. In case (ii), on the other hand, we have $\sigma_{12}^n>0>\max(\sigma_{12},\tilde\sigma_{12})$.

Remark 2.18. Case (ii) exhibits interesting behaviour along the trajectory $[\xi_1\to\xi_2]$ belonging to both cycles. Here we have $-\infty=\tilde\sigma_{12}<\sigma_{12}<0$, but $\sigma_{12}^n>0$. Only a small cusp-shaped region in the transverse section $H_1^{\text{out},2}$ is directly attracted to C_4, while everything else (except for another thin cusp that leaves the neighbourhood of the network) follows around C_3 finitely many times before eventually converging to C_4, too. Such points are in $\mathcal{B}(C_4)$, but not in $\mathcal{B}_\varepsilon(C_4)$ for $\varepsilon>0$ sufficiently small. Therefore, the n-index along this trajectory corresponds to the (non-local) c-index. Thus, in our original notation and with respect to C_4 we get for $x\in[\xi_1\to\xi_2]$

$$\sigma_{\text{loc}}(x)<0<\sigma(x).$$

This is related to example 1.41: here we have a smooth setting where for certain points the local stability index is negative while the (non-local) index is positive.

Stability indices for $c_{34}<0$ and $c_{43}<0$

In propositions 2.14 and 2.15, all return maps around each cycle are contractions. In the case studied in proposition 2.16, this is true for only one of the cycles and we have the competition described by Kirk and Silber in [23]. However, we note that in all cases there is one cycle with all stability indices equal to $-\infty$. That is, one of the cycles attracts hardly anything in its neighbourhood. It follows from corollary 2.9 that then the other cycle has at least one stability index equal to $+\infty$, so it attracts almost all points near at least one of its connections.

We now look for a more challenging case that has not been studied in [23]: no cycle in the network has all connections with stability index equal to $-\infty$. In order to do this, we admit that either c_{34} or c_{43} may be negative. This creates a positive transverse eigenvalue at ξ_3 or ξ_4, respectively. We assume that it is weaker than the expanding eigenvalue at the node: $|c_{34}| < e_{31}$ at ξ_3 and $|c_{43}| < e_{41}$ at ξ_4.

Having both $c_{34} < 0$ and $c_{43} < 0$ again leads to a cycle with all stability indices equal to $-\infty$. In fact, since $\sigma\tilde{\sigma} < 0$, we cannot have $\delta, \tilde{\delta} > 0$. Then by lemma 1.48 (iv)(a), the stability indices of the corresponding cycle are all equal to $-\infty$.

For the remainder of this section we assume that $\tau, \tilde{\tau}, \delta, \tilde{\delta} > 0$ to avoid stability indices equal to $-\infty$ and, as usual, $\rho, \tilde{\rho} > 1$.

Proposition 2.19 ([8]). *If $c_{34} < 0$, then $|\tilde{\sigma}_{12}|, |\sigma_{41}| < \infty$, $\tilde{\sigma}_{23} > 0$, $\tilde{\sigma}_{31}, \sigma_{24} = +\infty$ and $\sigma_{12} < 0$.*

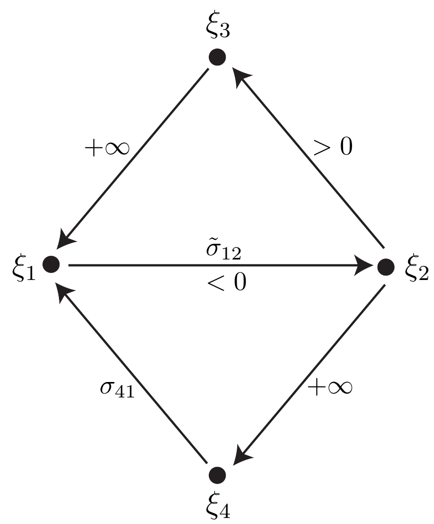


Figure 2.9.: The stability indices for the network in proposition 2.19. Stability indices $\tilde{\sigma}_{12}$ and σ_{41} are finite but have no predetermined sign.

Proof. The stability indices for the ξ_4-cycle are as in proposition 2.14. For the ξ_3-cycle, given the imposed signs of the parameters, we are in case (iv)(b) of lemma 1.48. Since b_1,$b_2 < 0$, we have $f^{\text{index}}(b_1 + b_2 a_1{,}1) < f^{\text{index}}(b_1{,}1)$ and

$$\tilde{\sigma}_{12} = f^{\text{index}}(b_1 + b_2 a_1{,}1) = \begin{cases} -\dfrac{1}{b_1 + b_2 a_1} - 1 > 0 & \text{if } b_1 + b_2 a_1 \in (-1{,}0) \\ b_1 + b_2 a_1 + 1 < 0 & \text{if } b_1 + b_2 a_1 < -1 \end{cases} .$$

The assumption that $|c_{34}| < e_{31}$ ensures that

$$\tilde{\sigma}_{23} = -\frac{e_{31}}{c_{34}} - 1 > 0.$$

Since we assume $\delta, \tilde{\delta} > 0$, their expressions as functions of $\tilde{\sigma}$ and σ, see equation (2.2), impose $\tilde{\sigma} < 0$ and $\sigma > 0$. Then $\tilde{\sigma}_{31} = +\infty$ and $|\sigma_{41}| < \infty$. □

An interesting case is that where $\tilde{\sigma}_{12} < 0$ and $\sigma_{41} > 0$. Then in terms of sign the cycles have the same collection of stability indices.

We now consider the analogous situation for $c_{43} < 0$ before moving on to n-indices in both cases.

Proposition 2.20 ([8]). *If $c_{43} < 0$, then $|\tilde{\sigma}_{31}| < \infty$, $\tilde{\sigma}_{12}, \sigma_{24} > 0$, $\tilde{\sigma}_{23}, \sigma_{41} = +\infty$ and $\sigma_{12} < 0$.*

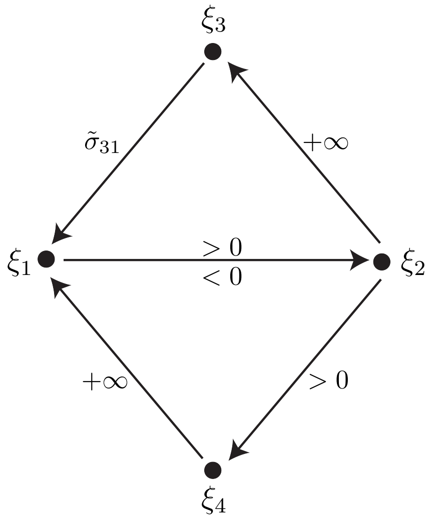


Figure 2.10.: The stability indices for proposition 2.20. Stability index $\tilde{\sigma}_{31}$ is finite but has no predetermined sign.

Proof. Here the stability indices for the ξ_3-cycle are as in proposition 2.14. For the ξ_4-cycle, given the imposed signs of the parameters, we are in case (iv)(b) of lemma 1.48. Since b_1,$b_2 < 0$, we have

$$\sigma_{12} = f^{\text{index}}(b_1 + b_2 a_1, 1) = \begin{cases} -\dfrac{1}{b_1 + b_2 a_1} - 1 > 0 & \text{if } b_1 + b_2 a_1 \in (-1,0) \\ b_1 + b_2 a_1 + 1 < 0 & \text{if } b_1 + b_2 a_1 < -1 \end{cases}.$$

The assumption that $|c_{43}| < e_{41}$ ensures that

$$\sigma_{24} = -\frac{e_{41}}{c_{43}} - 1 > 0.$$

Since we assume $\delta, \tilde{\delta} > 0$, again equation (2.2) imposes $\tilde{\sigma} > 0$ and $\sigma < 0$. Then $\tilde{\sigma}_{41} = +\infty$. □

Also for $c_{43} < 0$ it is possible to obtain the same collection of signs for the stability indices of both cycles: when $\tilde{\sigma}_{31} < 0$. We now investigate the n-indices, focusing in particular on the two situations where the signs correspond.

Proposition 2.21 ([8]). *Let $c_{34} < 0$. The stability indices with respect to the network when $\tilde{\sigma}_{12} < 0$ and $\sigma_{41} > 0$ are as follows:*

$$\tilde{\sigma}^n_{31} = \sigma^n_{24} = +\infty; \quad \tilde{\sigma}^n_{23}, \sigma^n_{41} > 0; \quad \sigma^n_{12} < 0.$$

Proof. By lemma 2.3 and proposition 2.19 we have $\tilde{\sigma}^n_{31} = \sigma^n_{24} = +\infty$ and also $\tilde{\sigma}^n_{23}, \sigma^n_{41} > 0$.

Changing the sign of c_{34} does not affect the results on the return maps in lemma 2.12: since $\delta, \tilde{\delta} > 0$ all return maps are contractions. Their domains of definition, however, do not remain unchanged. Now that $c_{34} < 0$, the local map ϕ_{231} is defined only on a cusp shaped by $y < x^{-\frac{c_{34}}{e_{31}}}$. All other points near the trajectory from ξ_2 to ξ_3 leave the neighbourhood of the network in the transverse direction. This obviously affects dom$(\tilde{g}_2)$, restricting it by the same inequality.

All other domains of defintion of the local maps remain the same. However, the change in dom(ϕ_{231}) influences the domains of all return maps around the ξ_3-cycle. For $\tilde{g}_1$ we now need to make sure that $\phi_{123}(x,y)$ lands in dom(ϕ_{231}). This changes the restriction on dom$(\tilde{g}_1)$ from $y < x^{\frac{e_{24}}{e_{23}}}$ to $y < x^{\frac{e_{24}}{e_{23}} - \frac{c_{21}c_{34}}{e_{23}e_{31}}}$. The domain of $\tilde{g}_3$ has to be modified in the same way. Thus, the domains of the return maps around the ξ_3-cycle are restricted by the following inequalities:

$$\begin{aligned} \tilde{h}_1 : &\quad y < x^{\frac{e_{24}}{e_{23}} - \frac{c_{21}c_{34}}{e_{23}e_{31}}} = x^{-(b_1 + b_2 a_1)} \\ \tilde{h}_2 : &\quad y < x^{-\frac{c_{34}}{e_{31}}} \\ \tilde{h}_3 : &\quad y < x^{-\tilde{\tau}} \end{aligned}$$

The domains of the maps around the ξ_4-cycle are the same as before. Thus, $\tilde{\sigma}^n_{23}$ and σ^n_{41} remain finite.

Points in the complement of dom$(\tilde{g}_1) \cup$ dom(g_1) inside an ε-ball in $H_1^{\text{out},2}$ satisfy

$$x^{-(b_1+b_2a_1)} < y < x^{\frac{e_{24}}{e_{23}}}.$$

This is a thick cusp if $-(b_1 + b_2 a_1) > 1$. Then, from the definition of the stability index, we obtain $\sigma_-(x) > 0$ along this trajectory, so that $\sigma^n_{12} < 0$, when $-(b_1 + b_2 a_1) > 1$. This is precisely the case when $\tilde{\sigma}_{12} < 0$. □

This is the first instance where the common connection in the network has a negative n-index, meaning that many trajectories stop following the network at this point – it is not p.a.s.

In fact, we discover an interesting feature about the way in which the network may lose its predominant asymptotic stability. As seen above the sign of the stability index along the common trajectory is determined by

$$-(b_1 + b_2 a_1) \lesseqgtr 1 \quad \Leftrightarrow \quad c_{34} \gtreqless \frac{e_{31}(e_{24} - e_{23})}{c_{21}} < 0. \tag{2.4}$$

This means, as long as c_{34} is negative, but not too small, the network (and even the ξ_3-cycle) is still p.a.s., because then $\tilde{\sigma}_{12} > 0$. But once c_{34} becomes smaller than the fraction in (2.4), we have $\sigma^n_{12} < 0$ and neither a cycle nor the network is p.a.s. anymore. Thus, both lose predominant stability through an increasing transverse eigenvalue at ξ_3 – but in terms of stability indices the loss of trajectories is felt strongest along the connection from ξ_1 to ξ_2. All other indices remain greater than zero.

We now also give the indices with respect to the network for $c_{43} < 0$. Again we focus on the most competitive case where the cycles have qualitatively equal indices. In contrast to what we found for $c_{34} < 0$, we observe the existence of parameter values ensuring predominant asymptotic stability of the network while neither cycle is p.a.s..

Proposition 2.22 ([8]). *Let $c_{43} < 0$. The stability indices with respect to the network when $\tilde{\sigma}_{31} < 0$ are as follows:*

$$\tilde{\sigma}^n_{23} = \sigma^n_{41} = +\infty; \quad \sigma^n_{12}, \sigma^n_{24} > 0; \quad |\tilde{\sigma}^n_{31}| < \infty$$

Proof. $\tilde{\sigma}^n_{23} = \sigma^n_{41} = +\infty$ follows from lemma 2.3 and proposition 2.20, the same goes for $\sigma^n_{12}, \sigma^n_{24} > 0$. The latter two indices are not equal to $+\infty$ because the domains of the respective return maps again exclude a cusp shaped region of points that move away from the network. Determining the domains is analogous to the previous theorem, yielding the inequalities below.

$$\begin{aligned} h_1: &\quad x < y^{\frac{e_{23}}{e_{24}} - \frac{c_{21}c_{43}}{e_{24}e_{41}}} = y^{-(b_1+b_2a_1)} \\ h_2: &\quad y < x^{-\frac{c_{43}}{e_{41}}} \\ h_4: &\quad y < x^{-\tau} \end{aligned}$$

We need to investigate what happens along the connection from ξ_3 to ξ_1. A point $(x,y) \in H_3^{\text{out},1}$ belongs to the basin of attraction of the network if and only if

$$\phi_{312}(x,y) = \left(x^{\frac{c_{13}}{e_{12}}}, yx^{\frac{c_{14}}{e_{12}}}\right) \in \text{dom}(\tilde{g}_1) \cup \text{dom}(g_1),$$

which is equivalent to

$$yx^{\frac{c_{14}}{e_{12}}} < x^{\frac{c_{13}e_{24}}{e_{12}e_{23}}} \quad \vee \quad x^{\frac{c_{13}}{e_{12}}} < \left(yx^{\frac{c_{14}}{e_{12}}}\right)^{\frac{e_{23}}{e_{24}} - \frac{c_{21}c_{43}}{e_{24}e_{41}}},$$

and thus to

$$y < x^{\tilde{\sigma}} \quad \vee \quad y^{-\frac{e_{23}}{e_{24}} + \frac{c_{21}c_{43}}{e_{24}e_{41}}} < x^{-\tau}.$$

The first condition characterizes the thin side of a cusp, since $\tilde{\sigma} > 1$ was the condition for $\tilde{\sigma}_{31} < 0$. Whether the second condition describes the thin or the thick side of a cusp depends on

$$\alpha := -\frac{e_{23}}{e_{24}} + \frac{c_{21}c_{43}}{e_{24}e_{41}} \lesseqgtr -\tau.$$

For $\alpha < -\tau$, we have $\tilde{\sigma}^n_{31} < 0$. A short calculation yields that

$$\alpha < -\tau \quad \Leftrightarrow \quad \alpha < -\frac{c_{13}}{e_{12} + c_{14}}$$

and both this inequality and its reverse are compatible with $\tilde{\sigma} > 1$ which is equivalent to

$$-\frac{c_{13}}{e_{12}+c_{14}} < -\frac{e_{23}}{e_{24}}.$$

$\square$

This means that for $c_{43} < 0$, but not too small, the network may be p.a.s., but once c_{43} is so small that $\alpha < -\tau$, the network loses its stability along the trajectory $[\xi_3 \to \xi_1]$. In either case, none of the individual cycles is p.a.s..

It is straightforward to see from the calculations in the previous proof that, when $\alpha > -\tau$, most points in a neighbourhood of $[\xi_3 \to \xi_1]$ that follow the network end up in dom(g_1), that is, they switch from the ξ_3- to the ξ_4-cycle. Analogous calculations show that points $(x,y) \in H_4^{\text{out},1}$ will follow the network if $\phi_{412}(x,y) = \left(yx^{\frac{c_{13}}{e_{12}}}, x^{\frac{c_{14}}{e_{12}}}\right) \in$ dom$(\tilde{g}_1) \cup$ dom(g_1). We have $\phi_{412}(x,y) \in$ dom$(\tilde{g}_1)$ if

$$y > x^{\sigma}$$

and $\phi_{412}(x,y) \in$ dom(g_1) if

$$y < x^{-\tau}.$$

Since $\sigma < 0 < \tau$ to have the stability indices in proposition 2.22, almost all points in a sufficiently small neighbourhood of $[\xi_4 \to \xi_1]$ will remain close to the network by following the ξ_4-cycle. In this sense, the ξ_4-cycle wins the competition.

2.2.4. The (B_3^-,C_4^-)-network

In this subsection we briefly look at the (B_3^-,C_4^-)-network. Due to the huge number of cases arising from the classification in [33] we do not investigate existence and stability to the same level of detail as we did for the other two networks. We do show, however, that in the extreme cases of very small positive transverse eigenvalues for one of the cycles, the basic behaviour of the network is similar to that of (B_2^+,B_2^+)- and (B_3^-,B_3^-)-networks in the sense that one cycle is p.a.s. and the other one has all indices equal to $-\infty$.

Suppose we have a very simple (B_3^-,C_4^-)-network as sketched in the proof of lemma 2.8, meaning there are four equilibria ξ_j and two cycles

$$[\xi_1 \to \xi_2 \to \xi_3 \to \xi_1] \qquad \text{and} \qquad [\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1],$$

which are of types B_3^- and C_4^-, respectively. We denote the eigenvalues at ξ_j by positive quantities r_j, c_{ji}, e_{jk} in the standard way. Note that for both cycles there is a positive transverse eigenvalue at ξ_3. It is precisely the expanding eigenvalue of the other cycle, i.e. e_{34} for the B_3^--cycle and e_{31} for the C_4^--cycle. We assume all other transverse eigenvalues to be negative. Then we have the following result.

Proposition 2.23. *In the situation we have just described the following holds.*

(i) Suppose that all conditions for asymptotic stability of the C_4^--cycle (see proposition 1.63) are fulfilled with the exception that there is a positive transverse eigenvalue at ξ_3. Then, if $e_{31} > 0$ is sufficiently small ($e_{34} \gg e_{31}$), the C_4^--cycle is p.a.s. and the B_3^--cycle has all its stability indices equal to $-\infty$.

(ii) On the other hand, assume that all conditions for asymptotic stability of the B_3^--cycle (see theorem 1.56) are fulfilled with the exception that there is a positive transverse eigenvalue at ξ_3. Then, if $e_{34} > 0$ is small enough ($e_{31} \gg e_{34}$), the B_3^--cycle is p.a.s. and the C_4^--cycle has all its stability indices equal to $-\infty$.

Proof. We begin with the first statement and the stability indices for the B_3^--cycle. Due to the existence of a positive transverse eigenvalue at ξ_3 we are in case (iii) of lemma 1.48. With the appropriate permutation of subscripts a sufficient condition for all indices to be equal to $-\infty$ is given by

$$b_3 a_1 a_2 + b_2 a_1 + b_1 < 0, \quad \text{which is the same as} \quad b_3 = -\frac{e_{34}}{e_{31}} < -\frac{b_2 a_1 + b_1}{a_1 a_2}.$$

Since both sides of the second inequality are negative this can be achieved by choosing $e_{31} > 0$ small enough. By proposition 1.63 we get the C_4^- to be p.a.s. at the same time.

For the other statement we argue in a different manner. Suppose $e_{34} > 0$ is small enough for the B_3^--cycle to be p.a.s.. Then, again by case (iii) of lemma 1.48, the stability index σ_{12} for the trajectory $[\xi_1 \to \xi_2]$ with respect to the B_3^--cycle is

$$\sigma_{12} = f^{\text{index}}(b_2 + b_3 a_2, 1).$$

Choosing $e_{34} > 0$ small enough brings $b_3 = -\frac{e_{34}}{e_{31}} < 0$ (with respect to the B_3^--cycle) arbitrarily close to zero. Since $b_2 > 0$ we then have $b_2 + b_3 a_2 > 0$ and thus $\sigma_{12} = +\infty$ for the B_3^--cycle. This means that for the C_4^--cycle the index along $[\xi_1 \to \xi_2]$ must be equal to $-\infty$. But then all its stability indices are equal to $-\infty$ by corollary 1.52. $\square$

Note that an analogous statement holds for the other two very simple networks in $\mathbb{R}^4$. For the (B_2^+, B_2^+)-network it is a direct consequence of theorem 2.11. For the (B_3^-, B_3^-)-network we have just done the calculation in the first part of our proof above – it does not change if the other cycle is of type B_3^-, too.

This means that if a very simple heteroclinic network is created from a single heteroclinic cycle by turning a transverse eigenvalue t_j positive and establishing a new connection (or two), then there is $\varepsilon > 0$ such that for $t_j < \varepsilon$ the original cycle is predominantly asymptotically stable while the new cycle is completely unstable. With this conclusion we close our chapter on heteroclinic networks. A more detailed study of stability in the (B_3^-, C_4^-)-network is beyond the scope of this work.

3. Prospects and Comments

We close this work with a short summary and an overview of further questions that we consider worth investigating. Our primary aim was to relate the stability index to non-asymptotic attraction properties of heteroclinic cycles and subsequently apply our results to very simple cycles and networks of types B and C in $\mathbb{R}^4$. The former was achieved by proving that a heteroclinic cycle is predominantly asymptotically stable if and only if the indices along all its connecting trajectories are positive. The analogous result for predominant instability does not hold in quite the same generality, but is of lesser interest for reasons mentioned earlier. Along the way, an effort has been made to clarify the terminology for non-asymptotic stability. In particular, we proved the equivalence of fragmentary and essential asymptotic stability, rendering the latter unnecessary. This should simplify future communication on the topic, since e.a.s. had been used ambiguously by different authors before. Moreover, relying on results by Krupa and Melbourne [28] as well as by Ashwin and Podvigina [33] we classified very simple heteroclinic networks in $\mathbb{R}^4$ and discussed competition between cycles in the (B_2^+,B_2^+)- and (B_3^-,B_3^-)-networks. To a certain extent, yet not exhaustively, we also addressed this question for the (B_3^-,C_4^-)-network.

At several points, generalizations seem possible if not desirable. We list these below, together with some general comments.

1. We have not investigated the (B_3^-,C_4^-)-network to the same degree of detail as the other networks. In principle, one can apply our techniques to determine necessary and sufficient conditions for the various stability properties of C_4^--cycles. This is complex only due to the high number of cases involved and the more complicated expressions for the stability indices. The qualitative challenge is to interpret the results in a useful way, since two-dimensional visualization as for the other cycles is not possible. Such a study would allow a more elaborate investigation of stability and competition in the network.

2. The detailed study of non-asymptotic stability was done only for very simple cycles in $\mathbb{R}^4$. It is certainly conceivable to generalize these results in several directions:

   (a) from $\mathbb{R}^4$ to $\mathbb{R}^n$

   (b) from finite symmetry groups to compact Lie groups

   In particular, necessary and sufficient conditions for predominant asymptotic stability in the broader frameworks of (a) and (b) seem a worthwhile objective. However, for two reasons this is no small undertaking: firstly, though the type A, B, C classification generalizes to $\mathbb{R}^n$, there is no complete list of cycles of either type in $\mathbb{R}^n$. Our study of very simple cycles in $\mathbb{R}^4$ relied heavily on the achievements of [28] in this regard. Secondly, even if we had such a list, in order to apply the same methods as here, we still lack the explicit computation of the stability indices. This is where we benefited from the work in [33], where the calculations for $\mathbb{R}^4$ are done. Already these are rather cumbersome and the authors expect an extension to $\mathbb{R}^n$ to be so complex that little insight is likely be gained from it.

3. Even within $\mathbb{R}^4$, there is room for generalizations: it should be possible to obtain analogous results for all simple (not only very simple) heteroclinic cycles, or an even more general class. In [34] Podvigina and Chossat have determined all subgroups of $O(4)$ that allow very simple heteroclinic cycles in $\mathbb{R}^4$. In doing so, they have completed the list of [28], which encompassed only those of types B and C. Their techniques can be expected to work for simple cycles, as well. As above, one would still have to determine stability indices for the new list of cycles, though.

4. An alternative approach to the study of stability for heteroclinic cycles is pursued by Podvigina in [32]. She gives necessary and sufficient conditions for fragmentary asymptotic stability of what she calls type Z cycles – heteroclinic cycles in $\mathbb{R}^n$ for which all Σ_j-isotypic components of $P_j^\perp$ are one-dimensional. The outcome is, in principle, less than a full generalization of the results obtained for $\mathbb{R}^4$ here, but certainly a significant step towards a better understanding of non-asymptotic stability in $n > 4$ dimensions. It does not require tedious calculations for single types of cycles, but allows for more general results applicable to a broad class of cycles.

5. Another question that is not addressed in this work regards transverse bifurcations of heteroclinic cycles. When a transverse eigenvalue turns positive and an asymptotically stable heteroclinic cycle becomes p.a.s., then p.u. and finally completely unstable, it is of interest to know where the trajectories go once they do not converge to the cycle anymore. Transverse bifurcations of very simple homoclinic cycles in $\mathbb{R}^4$ have been studied extensively by Chossat et al. in [9]. This could be extended to non-homoclinic cycles, enhancing the stability loss analyses in section 1.3. Of course, heteroclinic cycles may also lose stability through other kinds of bifurcation, but that is something we have not touched at all in this work.

6. Similar possibilities for generalizations exist in the context of heteroclinic networks. Not only is it possible to form a network from more complex types of cycles than we did, it is also conceivable to construct networks where the cycles have only equilibria in common instead of at least one connecting trajectory. Competition in such networks may turn out to be different from what we have seen so far. Especially in higher dimensions the fact that equilibria have more than one transverse direction may lead to qualitatively new situations. Many authors have studied heteroclinic networks other

than those discussed in section 2.2, see [2] and [35], for instance. So an attempt at a systematic study of a more general class of networks seems justified.

7. Finally, heteroclinic trajectories may not only connect hyperbolic equilibria to form a cycle, but also periodic orbits – examples can be found in [3] and [15], for instance. How do the stability indices of the connecting trajectories and those of the invariant objects interact? In our case, stability indices at the equilibria turned out to be irrelevant for the stability of the whole cycle. This is not obvious for more complex constructions, where the equilibria are replaced by objects of positive dimension (possibly the same as that of the connections).

A. Background for asymptotic stability results

We provide a short appendix to our work in order to enable the reader to grasp the broader context of the results in subsection 1.1.2. As this is only a side note to what we have been concerned with, we keep it brief, quoting the deeper theorems in A.2 without proof. In A.1 we explain how the definitions of types A, B and C generalize to $\mathbb{R}^n$. This is followed by A.2, where we give an outline of how one may study heteroclinic cycles under the action of continuous symmetry groups.

A.1. Generalization of types A, B, C to $\mathbb{R}^n$

In this first part we show how the division of heteroclinic cycles into types A, B and C can be generalized to simple cycles in $\mathbb{R}^n$. This has been done by Krupa and Melbourne in [28], where all of the following can be found, and allows us to state the results in subsection 1.1.2 on asymptotic stability of heteroclinic cycles in their full generality. For the rest of this thesis, though, it is of little relevance, since we mostly discuss cycles and networks in $\mathbb{R}^4$.

In order to generalize types A, B and C to $\mathbb{R}^n$ we first stay in $\mathbb{R}^4$ and characterize them in a local manner, rather than in the global way we used in the main text. So in the usual setting in $\mathbb{R}^4$ consider the three-dimensional spaces $Q_j := P_{j-1} + P_j$. These may or may not be fixed-point subspaces, i.e. it is not clear whether or not there exist $\tau_j \in \Gamma$ such that $Q_j = \text{Fix}(\tau_j)$. We distinguish the following three cases.

Definition A.1 ([28])**.** Depending on the nature of Q_j the j-th connecting trajectory is said to be of type A, B or C:

(a) The j-th connection is of *type* A if Q_j is not a reflection hyperplane.

(b) The j-th connection is of *type* B if Q_j is a reflection hyperplane and $P_{j+1} \subset Q_j$.

(c) The j-th connection is of *type* C if Q_j is a reflection hyperplane and $P_{j+1} \not\subset Q_j$.

Obviously, for a simple heteroclinic cycle in $\mathbb{R}^4$ each connection is of precisely one of the types above. This works naturally with our type definition for the cycles, as the following result illustrates. In order to see how, we briefly recall its proof from [28].

Proposition A.2 ([28])**.** *A simple robust heteroclinic cycle $X \subset \mathbb{R}^4$ is of*

(a) type A if and only if each connection is of type A.

(b) type B if and only if each connection is of type B.

(c) type C if and only if each connection is of type C.

Proof. If X is of type A, then by lemma 1.7 there is no element that acts as a reflection on $\mathbb{R}^4$, so there is no three-dimensional fixed-point subspace. Thus, all connections are of type A. On the other hand, if X is not of type A, then due to (1.3) we have $\Sigma_{j-1} \cap \Sigma_j = \mathbb{Z}_2$ and therefore $Q_j = \mathrm{Fix}(\Sigma_{j-1} \cap \Sigma_j)$. So all connections are of type B or C.

For type B and C cycles the implication from left to right follows from going through the enumeration of cycles in [28], see lemma 1.44, and checking all connection types. In remark 2.10 we did this for B_3^-- and C_4^--cycles. Calculations for the other cycles are analogous, so we skip them.

The other implication for B and C cycles is seen rather easily: if all connections are of type B, then $\bigcup_i P_i \subset Q_j$, so the whole cycle is contained in the three-dimensional fixed-point space Q_j. Therefore, it is of type B. Similarly, if all connections are of type C, the cycle is not contained in a proper subspace of the four-dimensional space $Q_j + P_{j+1} = \mathbb{R}^4$, thus it is not of type B. It is also not of type A because the symmetry group contains reflections. $\square$

Through a similar local approach we can assign a type to a heteroclinic connection in $\mathbb{R}^n$. We need this for the definition of higher dimensional type C cycles, types A and B can be generalized without it. Before we proceed, recall the following notation for the generalized eigenspaces at an equilibrium ξ_j:

$$V_j(c) := P_{j-1} \ominus L_j, \quad V_j(e) := P_j \ominus L_j, \quad V_j(t) := (P_{j-1} + P_j)^\perp,$$

where $U \ominus V$ denotes the orthogonal complement of V in U. In the following we treat the three types of cycles one by one.

Type A cycles

In order to generalize the definition of type A cycles we briefly revisit the concept of isotypic decomposition. While the following definition and considerations are adapted from Krupa and Melbourne [28], background on this topic can be found in the books of Lauterbach and Chossat [11] or Golubitsky et al. [17].

Definition A.3 ([28])**.** Let $\Sigma \subset \Gamma$ be a subgroup and write $\mathbb{R}^n = U_0 \oplus U_1 \oplus \cdots \oplus U_p$ as a direct sum of Σ-irreducible subspaces. Grouping together U_i that carry isomorphic representations of Σ one obtains the unique *isotypic decomposition* $\mathbb{R}^n = W_0 \oplus W_1 \oplus \cdots \oplus W_q$. The spaces W_j are called *Σ-isotypic components*.

Note that $\mathrm{Fix}(\Sigma)$ is a Σ-isotypic component which makes $W_0 = \mathrm{Fix}(\Sigma)$ a common choice. Furthermore, two Σ-irreducible subspaces are in the same W_j if and only if they are isomorphic. Linear mappings commuting with the action of Σ map isotypic components into themselves. For $x \in \mathbb{R}^n$ and $\sigma \in \Sigma$ compute:

$$\sigma.df(x) = \frac{d}{dx}(\sigma.f(x)) = \frac{d}{dx}(f(\sigma.x)) = df(\sigma.x)\sigma$$

If $x \in \text{Fix}(\Sigma)$, then $\sigma.x = x$ and thus $df(x)$ commutes with Σ. Generically each generalized eigenspace (for a non-zero eigenvalue of $df(x)$) then lies in a single Σ-isotypic component.

Definition A.4 ([28]). A robust heteroclinic cycle $X \subset \mathbb{R}^n$ is said to be of *type* A if for all j the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in the same Σ_j-isotypic component.

Note that the eigenspaces of e_j and c_{j+1} belong to $\text{Fix}(\Sigma_j)$ by construction. Those corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in $\text{Fix}(\Sigma_j)^\perp$, which may or may not split into several Σ_j-isotypic components.

For $n = 4$, this coincides with the original definition where we demand that $\Sigma_j \cong \mathbb{Z}_2$ for all j: let us derive the Σ_j-isotypic decomposition of $\mathbb{R}^4$ for a cycle of type A. As in subsection 1.1.1 we view Σ_j as a subset of the diagonal matrices with entries $\{1, \pm 1, \pm 1, \pm 1\}$. The fixed-point space $P_j = \text{Fix}(\Sigma_j)$ is a Σ_j-isotypic component of dimension two, since it contains the radial and expanding directions at ξ_j. Since Σ_j consists of only one non-identity element, its actions on the remaining two one-dimensional spaces spanned by the contracting and transverse eigenvalues at ξ_j are isomorphic: the non-identity element must act as $-\mathbb{1}$ on these spaces since they do not belong to $\text{Fix}(\Sigma_j)$. Therefore, $V_j(c)$ and $V_j(t)$ belong to the same Σ_j-isotypic component, which means that Σ_j splits $\mathbb{R}^4$ into two components, both two-dimensional. Now the directions corresponding to c_j and t_j are not contained in P_j (since it contains those corresponding to r_j and e_j), so they are in the other component. The same goes for the directions corresponding to e_{j+1} and t_{j+1}, because P_j also contains those belonging to r_{j+1} and c_{j+1}. So all the required eigenspaces belong to the same component.

Note that this is not the case for type B or C cycles in $\mathbb{R}^4$: for these, $\Sigma_j \cong \mathbb{Z}_2^2$, thus there are three non-identity elements, one of them acting as $-\mathbb{1}$ on $V_j(c)$ and as $\mathbb{1}$ on $V_j(t)$, and a second one vice versa. Therefore, the representations of Σ_j on $V_j(c)$ and $V_j(t)$ are not isomorphic, meaning that they both form a one-dimensional Σ_j-isotypic component. A space containing the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1}, however, must be at least two-dimensional, so these do not lie in the same component.

There is a special subset of type A cycles that is worth distinguishing, since in [27] a powerful result regarding their non-asymptotic stability was proved, see theorem 1.11.

Definition A.5. We say that a robust heteroclinic cycle X is of *type* A^* if it is of type A and if for all equilibria ξ_j the generalized eigenspaces corresponding to all transverse eigenvalues of $df(\xi_j)$ with positive real part lie in the same Σ_j-isotypic component.

In $\mathbb{R}^4$ types A and A^* coincide since there is only one transverse direction at each equilibrium. This is generally not the case in $\mathbb{R}^n$ for $n > 4$.

Type B cycles

We proceed by generalizing type B cycles.

Definition A.6 ([28]). A robust heteroclinic cycle $X \subset \mathbb{R}^n$ is of *type* B if for each j there is a fixed-point space R_j such that

$$R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(t). \tag{A.1}$$

First, we convince ourselves that the new definitions of types A and B are mutually exclusive: the existence of the subspace $R_j \supset P_j$ means that there is a proper subgroup of Σ_j acting as the identity on $V_j(t)$ but not on $V_j(c)$. So its actions on these two spaces are not isomorphic. Therefore, the eigenspaces corresponding to c_j and t_j do not lie in the same Σ_j-isotypic component of $\mathbb{R}^n$ and the cycle is not of type A.

Condition (A.1) can roughly be thought of as ξ_j and ξ_{j+1} having the same transverse directions. In $\mathbb{R}^4$ this reduces to the original definition of type B: note that $\dim(R_j) = 3$, so it is a reflection hyperplane and the cycle is not of type A. Furthermore, the transverse direction is the same for all ξ_j (for each j there is only one). This means that for none of the equilibria this space dimension is contracting, expanding or radial. Thus we have $P_{j+1} \subset Q_j$ for all j and by definition A.1 above the cycle is of type B.

Type C cycles

In order to define type C cycles in $\mathbb{R}^n$, we generalize the local description of the connection types.

Definition A.7 ([28]). The j-th connection is of *type* B if there are fixed-point subspaces Q_j, R_j such that

$$Q_j = P_j \oplus V_j(c) = P_j \oplus V_{j+1}(e) \text{ and } R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(t).$$

The j-th connection is of *type* C if there are fixed-point subspaces Q_j, R_j such that

$$Q_j = P_j \oplus V_j(c) = P_j \oplus V_{j+1}(t) \text{ and } R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(e). \tag{A.2}$$

A connection cannot be of types B and C at the same time, since this would mean

$$P_j \oplus V_{j+1}(t) = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(e),$$

but transverse and expanding directions at ξ_{j+1} span different spaces together with P_j. However, if regarded with respect to two different cycles, a connection may have different types. We elaborated on this in remark 2.10. If all connections are of type B, then so is the cycle because (A.1) is satisfied. But definition A.6 is less restrictive, not requiring the existence of Q_j.

However, if one of the connections is of type C, then the cycle is not of type B: suppose it was, then

$$R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(t) = P_j \oplus V_j(c) = Q_j.$$

This is impossible because transverse and contracting directions at ξ_j do not span the same space together with P_j. Now we are able to define general type C cycles.

Definition A.8 ([28])**.** A robust heteroclinic cycle $X \subset \mathbb{R}^n$ is of *type* C if each connection is of type B or C and at least one connection is of type C.

Types C and A are mutually exclusive for the same reason as B and A: the existence of the subspaces Q_j and R_j in either case implies non-isomorphic actions of Σ_j on $V_j(c)$ and $V_j(t)$. Together with the considerations above this shows that also in $\mathbb{R}^n$ the three types are mutually exclusive. However, they are not exhaustive. Note, for instance, that (A.2) places restrictions on the dimensions of the generalized eigenspaces. There is no a priori reason why cycles that break these, and also the corresponding conditions for type B, should not exist.

Finally, we show that the type C definition also reduces to the original definition in $\mathbb{R}^4$: a cycle in $\mathbb{R}^4$ with a connection of type C as above does not fit into a three-dimensional space due to (A.2). So it is not of type B. It is also not of type A since Q_j and R_j are three-dimensional fixed-point spaces by construction, requiring the existence of reflections in Γ.

A.2. Continuous symmetry groups

We round off this appendix with a short remark on robust heteroclinic cycles generated by the action of continuous symmetry groups. We follow along the lines of [26] and [24], hereby providing the full theoretical framework for subsection 1.1.2. Let Γ be a compact Lie group acting on $\mathbb{R}^n$. Without loss of generality we assume this action to be orthogonal, i.e. $\Gamma \subset O(n)$. As before we consider a Γ-equivariant vector field f and the ordinary differential equation $\dot{x} = f(x)$ with the associated dynamics.

If for $x_0 \in \mathbb{R}^n$ the group orbit Γx_0 is invariant under the flow generated by f, then it is called a *relative equilibrium*. The simplest form of this is just a group orbit of equilibria, then the flow is trivial along the group orbit. We want to describe the dynamics near a relative equilibrium in a systematic way. In order to do so, we employ the concepts of tangent and normal vector fields: a vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is called *tangent* if $g(x)$ is tangent to the group orbit Γx for all $x \in \mathbb{R}^n$. Furthermore, g is called *normal* if for every $x \in \Gamma x_0$ the normal space N_x (containing all vectors normal to Γx_0 at x) is invariant under the flow of g. This does not have to be the case for any group orbits other than Γx_0.

In [24] Krupa proves two important results, that allow us to decompose the Γ-equivariant vector field f into a tangent and a normal field, both of which are also Γ-equivariant. Subsequently, the flow generated by f can be understood as the flow of the normal field coupled with a drift along group orbits. We state the two theorems here. Note that they hold regardless of whether or not the group orbit Γx_0 is a relative equilibrium.

Theorem A.9 ([24])**.** *In the situation described above, there is a* Γ*-invariant neighbourhood* $U \subset \mathbb{R}^n$ *of* Γx_0 *and there exist smooth* Γ*-equivariant vector fields* f_N *and* f_T*, where the* *former is normal and the latter is tangent, such that*

$$f(u) = f_N(u) + f_T(u)$$

for all $u \in U$.

Theorem A.10 ([24]). *Under the same assumptions, let* $u(t)$ *be a trajectory of the flow generated by* f*, which is contained in the neighbourhood* U *from above for all positive times. Then there is a smooth curve* $\gamma : \mathbb{R}_{\geq 0} \to \Gamma$ *and a trajectory* $y(t)$ *of the normal vector field* f_N *restricted to* N_{x_0}*, such that*

$$\gamma(t)y(t) = u(t)$$

for all $t \geq 0$.

The proofs in [24] rely on a rather technical lemma, which we do not reproduce here. Note that since Γ acts orthogonally, the normal space N_{x_0} is invariant under the action of the isotropy subgroup Σ_{x_0}. Therefore, the restriction of f_N to N_{x_0} is Σ_{x_0}-equivariant. Setting $k :=$ codim(Γx_0), the essential conclusion we draw from these results is that they allow us to study the dynamics of f in two steps: first, analyze the (k-dimensional) dynamics of the Σ_{x_0}-equivariant vector field $f_N \mid_{N_{x_0}}$. Then, find the *drift* $\gamma(t)$ to understand how f_N evolves along the group orbit.

The next crucial observation is that for a group orbit Γx_0 the real parts of the eigenvalues of $df_N(x)$ do not depend on the point $x \in \Gamma x_0$ at which they are calculated. This follows from proposition F and theorem G in [14]. Thus, it makes sense to call a relative equilibrium Γx_0 *hyperbolic* if (any) $x \in \Gamma x_0$ is a hyperbolic equilibrium of f_N. With this generalization we may consider heteroclinic cycles between hyperbolic relative equilibria ξ_j. The definition of radial, contracting, expanding and transverse eigenvalues at the (relative) equilibria generalizes naturally by considering the real parts of eigenvalues of $df_N(\xi_j)$.

The normalizer condition

$$\dim(W^u(\xi_j)) = \dim(N(\Sigma_j)/\Sigma_j) + 1$$

from subsection 1.1.2 intuitively makes sense, too, since $\dim(N(\Sigma_j)/\Sigma_j)$ is just the dimension of the group orbit Γx_0. With this, the earlier results by Krupa and Melbourne can be understood to a satisfactory extent in the general setting within which they hold. In fact, their proofs do not become significantly more complicated through the generalization – due to the above considerations they can be achieved via the same methods as those for the simpler context of finite groups, see [26] for details.

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Summary

This thesis deals with stability properties of compact invariant sets, aiming at a better understanding of non-asymptotic stability for heteroclinic cycles and networks.

In the first chapter we recall general aspects concerning the theory of heteroclinic cycles (such as the equivariant setting in which they are robust), and review well-known results by Krupa and Melbourne, e.g. from [28], on their asymptotic stability. Moreover, the stability index of Podvigina and Ashwin [33] is discussed together with various forms of non-asymptotic stability. The central result in this part is a connection between the signs of local stability indices and predominant asymptotic stability of a heteroclinic cycle. Building on results by all authors mentioned above, we use this relationship to characterize non-asymptotic stability of very simple heteroclinic cycles in $\mathbb{R}^4$, giving necessary and sufficient conditions for predominant asymptotic stability.

The second chapter is concerned with heteroclinic networks. Some results from before are transferred without much difficulty to this context. We derive a list of all very simple networks in $\mathbb{R}^4$, and stability indices with respect to subcycles and the whole network (c- and n-indices) enable us to investigate competition within these. Results in this chapter have partially been obtained in collaboration with Sofia Castro and may be viewed as an extension and generalization of the Kirk and Silber study [23] of competition in a (B_3^-, B_3^-)-network.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit Stabilitätseigenschaften kompakter invarianter Mengen. Hauptanliegen ist dabei, ein detaillierteres Verständnis der nicht-asymptotischen Stabilität von heteroklinen Zyklen und Netzwerken zu gewinnen.

Im ersten Kapitel werden grundlegende Begriffe aus der Theorie heterokliner Zyklen wiedergegeben (wie das äquivariante Setting, in dem sie robuste Objekte sind) sowie bekannte Resultate von Krupa und Melbourne, z.B. aus [28], zu asymptotischer Stabilität vorgestellt. Desweiteren wird der Stabilitätsindex von Podvigina und Ashwin [33] diskutiert und mit verschiedenen Formen nicht-asymptotischer Stabilität in Zusammenhang gestellt. Das wichtigste Resultat in diesem Teil verknüpft die Vorzeichen des lokalen Stabilitätsindex mit dem Stabilitätsbegriff *predominant asymptotic stability*. Aufbauend auf Ergebnissen der eben genannten Autoren beschreiben wir somit nicht-asymptotische Stabilität von sehr einfachen heteroklinen Zyklen im $\mathbb{R}^4$ vollständig, insbesondere leiten wir notwendige und hinreichende Bedingungen für *predominant asymptotic stability* her.

Das zweite Kapitel behandelt heterokline Netzwerke. Einige der Ergebnisse von vorher lassen sich ohne große Schwierigkeiten in diesen Kontext übertragen. Es wird eine vollständige Liste von sehr einfachen heteroklinen Netzwerken im $\mathbb{R}^4$ hergeleitet. Anschließend untersuchen wir die relative Stabilität von Zyklen in einem Netzwerk mithilfe des Stabilitätsindex – einerseits bezogen auf lediglich einen Unterzyklus, andererseits auf das gesamte Netzwerk. Einige dieser Ergebnisse sind in Zusammenarbeit mit Sofia Castro entstanden. Sie stellen eine Verallgemeinerung und Erweiterung der Untersuchung von Kirk und Silber [23] dar.

Lebenslauf

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