Two constructions in monoidal categories

Equivariantly extended Drinfel'd Centers and Partially dualized Hopf Algebras

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Introduction

The fruitful interplay between topology and algebra has a long tradition. On one hand, invariants of topological spaces, such as the homotopy groups, homology groups, etc. carry algebraic structures and the goal of a systematic understanding of homotopy and homology led to definitions of categories, functors and natural transformations.

On the other hand, topological methods and principles inspire algebraic structures; the cobordism hypothesis for example has motivated higher categorical structures like (∞, n)-categories. Further, topology provides methods to prove algebraic theorems and explains the appearance of certain other phenomena; a particular nice example is that the quadruple dual functor $(\_)^{****}\colon \mathcal{C} \to \mathcal{C}$ of a finite tensor category $\mathcal{C}$ is monoidally isomorphic to the functor $D \otimes \_ \otimes D^{-1}$ for a distinguished invertible object D in $\mathcal{C}$ which was proven first by algebraical means in [ENO04] and reexamined with topological methods in [DSPS13].

The transition from the realm of categories to higher categories poses the problem of an algebraic description leading to something one might call 'higher algebraic structures'. For example tensor categories are a categorical generalization of algebras and module categories over tensor categories generalize modules over an algebra.

In this thesis we are concerned with the algebraic understanding of such higher algebraic structures coming from topological field theories, which were inspired by mathematical physics.

Topological field theories and generalizations

The topological invariants we are interested in are topological (quantum) field theories (TFTs). The basic idea of a d-dimensional TFT is that we assign to a d-dimensional manifold M a number which we can compute by subdividing M along $(d-1)$-dimensional submanifolds into smaller pieces. An abstraction from the Atiyah-Segal axioms for an d-dimensional TFT is expressed by saying that a TFT is a symmetric monoidal functor Z from a symmetric monoidal category $\mathcal{B}$ of d-dimensional cobordisms, which is of topological or geometrical nature, to another symmetric monoidal category $\mathcal{C}$, which is of algebraic nature. Informally speaking, a d-dimensional cobordism category has as objects manifolds of a fixed dimension $d-1$ and morphisms between those are given by equivalence classes of d-dimensional cobordisms. The monoidal structure of these cobordism categories is given by the disjoint union of manifolds and the unit object is the empty set considered as a $(d-1)$-dimensional manifold. Manifolds without boundary, seen as cobordisms from the empty set to the empty set, are sent by the TFT to endomorphisms of the unit object of the target category $\mathcal{C}$.

If $\mathcal{C} = \mathbf{vect}_{\Bbbk}$ is the category of vector spaces over a field $\Bbbk$ and $\mathcal{B}$ is one of the following cobordism categories, TFTs are characterized by well-understood algebraic structures, see [Koc04, LP08]

- Let $\mathcal{B}$ be the category with finite, ordered sets of points signed by $+$ or $-$ as objects and diffeomorphism classes of one dimensional, oriented, compact, smooth manifolds as morphisms. Then a TFT $Z\colon \mathcal{B} \to \mathbf{vect}_{\Bbbk}$ is already determined by the vector space $Z(+)$, which has to be finite dimensional.

- Let $\mathcal{B}$ be the category with objects: one dimensional, compact, oriented, smooth manifolds without boundary and morphisms diffeomorphism classes of two dimensional, compact, oriented, smooth manifolds with boundary. In this case a TFT $Z\colon \mathcal{B} \to \mathbf{vect}_{\Bbbk}$ is determined by a finite dimensional, commutative Frobenius algebra with underlying vector space $A = Z(S^1)$. The Frobenius algebra structure comes from the images of certain cobordisms under Z.

- Let $\mathcal{B}$ be the category with objects: one dimensional, compact, oriented, smooth manifolds (possibly with boundary) and morphisms given by diffeomorphism classes of two dimensional, compact, oriented, smooth manifolds with corners. In this case a TFT $Z\colon \mathcal{B} \to \mathbf{vect}_{\Bbbk}$ is determined by a knowledgeable Frobenius algebra, i.e. a pair of Frobenius algebras $A = Z(S^1)$ and $B = Z([0,1])$ with certain compatibility conditions, cf. [LP08].

The algebraic structures investigated in this thesis appear naturally in the context of two possible generalizations of TFTs; these are extended TFTs on the one hand and homotopy (quantum) field theories on the other.

Extended topological field theories A d-dimensional TFT $Z\colon \mathcal{B} \to \mathbf{vect}_{\Bbbk}$ assigns to every d-dimensional manifold without boundary an element of

the endomorphism ring of $\Bbbk$, which is isomorphic to $\Bbbk$ itself. Since Z is a functor, one can compute the invariants assigned to a d-manifold by cutting along $(d-1)$-dimensional submanifolds. **E**xtended TFTs are designed to give the possibility to compute invariants assigned to d-manifolds by subdividing with submanifolds of higher codimensions than 1. This gives a hierarchy of notions for locality of an extended TFT: the higher the allowed codimension of allowed submanifolds is, the higher is the knowledge of the extended TFT about local properties of closed manifolds.
For precise statements about extended TFTs one needs a notion of symmetric monoidal n-category. We do not define what an n-category should be, since already for $n = 3$ the axioms for a non-strict 3-category are rather involved [GPS95] and a strictification to a strict 3-category does not exist in general. Nevertheless, we want to give the reader an intuition of what features n-categories should provide.
Categories consist of objects, morphisms between objects and an associative composition of morphisms. An n-category should consist of objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, ..., n-morphisms between $(n-1)$-morphisms and k ways of composing k-morphisms for each $1 \leq k \leq n$ in an *associative manner*. The vague term associative manner means that differently composed triples of k-morphism may differ up to a *weakly invertible* $(k+1)$-morphism fulfilling coherence conditions; the attribute weakly invertible means invertible up to a weakly invertible $(k+2)$-morphism.
One of the easiest examples for non-strictly associative compositions is the bicategory $\mathbf{Alg}_{\Bbbk}$ with objects given by $\Bbbk$-algebras A, B, C, etc. 1-morphisms given by bimodules $M = {}_AM_B, N = {}_BN_C, P = {}_CP_D$, etc. and 2-morphisms given by bimodule homomorphisms $f\colon {}_AM_B \to {}_AM'_B$, etc.
Composition of 1-morphisms is then given by the tensor product over the middle algebra: $M \circ N := M \otimes_B N$. This composition of 1-morphisms is not associative, but there is always an invertible 2-morphism between $(M \circ N) \circ P$ and $M \circ (N \circ P)$ fulfilling a pentagon identity for the 4-fold compositions. Bimodule homomorphisms can be composed either by composition of morphisms or by tensor product.

We resume our description of extended TFTs: allowing decompositions of d-dimensional cobordisms by submanifolds up to codimension n leads to a symmetric monoidal n-category $\mathcal{B}_n$ of cobordisms. Objects are $(d-n)$-dimensional manifolds and k-morphisms $(1 \leq k \leq n)$ are given by $(d+k-n)$-dimensional cobordisms with corners; only the n-morphisms, i.e.

the d-cobordisms are considered up to diffeomorphism.
An *n-extended d-dimensional TFT* is then a symmetric monoidal n-functor from such a symmetric monoidal n-category $\mathcal{B}_n$ of topological nature into a symmetrical monoidal n-category of algebraic nature.
Baez and Dolan [BD95] conjectured a statement about fully extended TFTs, i.e. allowing submanifolds of codimension d. This statement is known under the name *cobordisms hypothesis* and was proven by Lurie [Lur09].

We observed above that ordinary TFTs are related to algebraic structures; for example a commutative Frobenius algebra is the image of a generating set of a cobordism category. If we have a set of generators and relations for our cobordism category, extended TFTs yield higher 'algebraic structures'. It is commonly believed that a 2-extended $3d$-TFT Z with values in the bicategory of 2-vector spaces gives a modular category $\mathcal{C} = Z(S^1)$, see for example the discussion [BK01, Chapter 5]. Modular categories are a certain class of braided categories and the first construction of this thesis is related to the existence of a minimal extension of braided categories to modular ones. Below we will discuss this extension problem in more detail. A precise definition of modular category is given in Appendix A.4, for the following discussion it suffices to know that the attribute modular is a non-degeneracy condition on the braiding.
This non-degeneracy of the braiding is needed in the construction of the famous Reshetikhin-Turaev invariants of 3-manifolds [RT91] which provide a (non-extended) $3d$-TFT [BK01].

Homotopy field theories Homotopy field theories (HFTs) were introduced in [Tur00], a recent survey on the topic is provided by [Tur10]. While TFTs give invariants of manifolds, HFTs produce invariants of maps $f\colon M \to X$, where M is a manifold and X a CW-complex. A particularly interesting class of CW-complexes are the Eilenberg-MacLane spaces of type $K(\Gamma, 1)$ for a discrete group Γ. These are CW-complexes X whose fundamental group $\pi_1(X, x)$ is isomorphic to the group Γ and all higher homotopy groups $\pi_k(X, x)$ with $k \geq 2$ are trivial. Such an Eilenberg-MacLane space is unique up to homotopy. If X is a $K(\Gamma, 1)$, an HFT is even an invariant of the homotopy class of the map $f\colon M \to X$.
In analogy to TFTs, an HFT is defined as a symmetric monoidal functor; one fixes a CW-complex X and an object in the source category is a cobordism together with a map to X. It is not too surprising that for the investigation of HFTs one is led to algebraic structures again. Describing HFTs for an Eilenberg-MacLane space of type $K(\Gamma, 1)$ yielded Turaev's

definition of Γ-Frobenius algebras (characterizing HFTs in dimension two) and Γ-braided modular categories (used to construct three dimensional HFTs of Reshetikhin-Turaev type) [Tur10].

In this thesis we are mainly concerned with Γ-braided categories. The word 'modular' above is again a non-degeneracy condition on the Γ-braiding and the Γ-grading of such a category. Müger showed [Müg04] that Galois extensions of braided categories obtained from a construction called de-equivariantization [DGNO10] are Γ-braided categories.

We deal with a natural extension problem related to Γ-braided categories. This is the problem of *extending a braided category with a group action* of the group Γ to a Γ-braided category. The extension problem of braided categories with a group action is equivalent to the *minimal extension problem* of braided categories to modular ones posed in [Müg03b].

Next we will, based on the discussion in [Tur10, Appendix 5], describe both of these extension problems in more detail and explain which constructions relate them.

Extending braided categories

Equivariant extensions A Γ-braided category is a monoidal, $\Bbbk$-linear, abelian category $\mathcal{C} = \bigoplus_{\alpha\in\Gamma} \mathcal{C}_\alpha$ graded by a discrete group Γ together with an action $\Phi = \{\Phi^\alpha : \mathcal{C} \to \mathcal{C}\}_{\alpha\in\Gamma}$ by monoidal autoequivalences and a Γ-braiding, i.e. natural isomorphisms $c_{X,Y} : X \otimes Y \to \bigoplus_{\alpha\in\Gamma} \Phi^\alpha(Y) \otimes X$ fulfilling several coherence conditions, see Definition 3.3.1 for details.

The axioms of a Γ-braided category imply that the neutral component $\mathcal{C}_1 \subset \mathcal{C}$ is an ordinary braided tensor category together with an action of Γ by braided autoequivalences. This naturally leads to the following extension problem: given a braided tensor category $\mathcal{D}$ together with an action Ψ of a group Γ by braided autoequivalences. Does there exist a Γ-braided category $\mathcal{C} = \bigoplus_{\alpha\in\Gamma} \mathcal{C}_\alpha$ such that $\mathcal{C}_\alpha \neq 0$ for all $\alpha \in \Gamma$, the neutral component $\mathcal{C}_1$ is equivalent to $\mathcal{D}$ and the action Φ on $\mathcal{C}$ restricts to the action Ψ on $\mathcal{D}$? We call $\mathcal{C}$ a Γ-braided extension of $\mathcal{D}$ and say that $\mathcal{C}$ solves the Γ-*braided extension problem.*

In this thesis we answer the existence question of such an extension, in the special case that $\mathcal{D} = \mathcal{Z}(\mathcal{E})$ is the Drinfel'd center of a monoidal category $\mathcal{E}$ together with an action of Γ on $\mathcal{D}$ that is a distinguished lift of an action on $\mathcal{E}$.

Minimal modular extensions It was shown by Müger [Tur10, Appendix 5, Thm. 5.4] that the Γ-braided extension problem for modular categories is equivalent to the minimal extension problem of premodular categories which we explain now: premodular categories are $\Bbbk$-linear, abelian, braided categories with finitely many simple objects and a spherical structure. This allows to speak of the dimension of an object and the dimension of the category. Let $\mathcal{C}$ be a braided category with braiding isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$. Denote by $\mathcal{Z}_{\mathrm{sym}}(\mathcal{C})$ the full subcategory of transparent objects in $\mathcal{C}$, i.e. objects X in $\mathcal{C}$ fulfilling $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all Y in $\mathcal{C}$. If $\mathcal{C}$ is a premodular category, one associates an element $\dim \mathcal{C}$ of the field $\Bbbk$, called the dimension of $\mathcal{C}$. Let Γ be a finite group, $\mathcal{C}$ a premodular category over $\mathbb{C}$ and $\mathcal{M}$ a modular category over $\mathbb{C}$ containing $\mathcal{C}$ as a full tensor subcategory. If now $\mathcal{Z}_{\mathrm{sym}}(\mathcal{C}) \subset \mathcal{C}$ is equivalent to the symmetric category $\mathbb{C}[\Gamma]$-**mod** of finite dimensional representations of Γ, one can show that the inequality

$$\dim \mathcal{M} \geq |\Gamma| \cdot \dim \mathcal{C} \tag{0.1}$$

holds, cf. Proposition 5.1 in [Müg03b]. Call $\mathcal{M}$ a *minimal modular extension of* $\mathcal{C}$, if (0.1) is an equality.

It is not clear whether this minimal extension problem has a solution. Even if there is a minimal extension of $\mathcal{C}$, in general there might exist inequivalent categories providing a solution to the problem: it is demonstrated in [Müg03b, Remark 5.3] that in the case $\mathcal{C} = \mathbb{C}[\Gamma]$-**mod** the representations of the twisted Drinfel'd doubles $D^{\omega}(\Gamma)$ provide solutions of the minimal extension problem. Here ω is any 3-cocycle $\omega\colon G^{\times 3} \to \mathbb{C}^{\times}$ and the representation categories of $D^{\omega}(\Gamma)$ and $D^{\omega'}(\Gamma)$ might be inequivalent braided categories. Theorem 9.4 in [MN01] gives a precise statement, when the braided categories $D^{\omega}(\Gamma)$-**mod** and $D^{\omega'}(\Gamma)$-**mod** are equivalent.

Equivalence of the extension problems As mentioned above, the minimal extension problem is equivalent to the problem of extending modular categories to Γ-braided categories. More concretely, the equivalence of the problems is provided via the de-equivariantization procedure and taking the orbifold category. The latter is inverse to the procedure of forming the de-equivariantization. We sketch this correspondence proven by Müger with the help of the diagram:

$$\begin{tikzcd}
\text{pre-modular} & \mathcal{C} \arrow[rr, "\text{minimal extension}"] \arrow[d, shift right, "\text{mod.}"'] & & \mathcal{M} \arrow[d, shift right, "\text{de-equi.}"'] & \text{modular} \\
\text{modular} & \mathcal{D} \arrow[rr, "\Gamma\text{-braided extension}"'] \arrow[u, shift right, "\text{orb.}"'] \arrow[loop below, "\Gamma"] & & \mathcal{N} \arrow[u, shift right, "\text{orb.}"'] \arrow[loop below, "\Gamma"] & \Gamma\text{-braided}
\end{tikzcd}$$

Here the left-most arrow stands for de-equivariantization, but in the particular situation it is better known as modularization.
Chapter 4 of this thesis deals with a construction for the arrow from $\mathcal{D}$ to $\mathcal{N}$, for the case that $\mathcal{D}$ is a Drinfel'd center and the action of Γ is of a certain form.

We will now explain the diagram: let $\mathcal{C}$ be a finite premodular category with $\mathcal{Z}_{\text{sym}}(\mathcal{C}) \simeq \mathbb{C}[\Gamma]$-**mod** for some finite group Γ. The category $\mathcal{C}$ is modular, iff Γ is the trivial group. De-equivariantization in the case $\mathcal{Z}_{\text{sym}}(\mathcal{C}) \simeq \mathbb{C}[\Gamma]$-**mod** was described under the name modularization [Bru00] and yields a modular category $\mathcal{D}$ with $|\Gamma| \cdot \dim \mathcal{D} = \dim \mathcal{C}$ and an action of the group Γ. If we solve the Γ-braided extension problem for $\mathcal{D}$, we get a Γ-braided category $\mathcal{N}$ of dimension $\dim \mathcal{N} = |\Gamma| \cdot \dim \mathcal{D}$ whose orbifold category $\mathcal{M}$ is modular and has dimension $\dim \mathcal{M} = |\Gamma| \cdot \dim \mathcal{N}$. So $\mathcal{M}$ is a modular category with $\dim \mathcal{M} = |\Gamma| \cdot \dim \mathcal{C}$; that $\mathcal{C}$ embeds into $\mathcal{M}$ is shown in [Tur10, Appendix 5].

Conversely, let $\mathcal{D}$ be a modular category with an action of a finite group Γ. Its orbifold category $\mathcal{C}$, also known as equivariantization [DGNO10], fulfills $\mathcal{Z}_{\text{sym}}(\mathcal{C}) \simeq \mathbb{C}[\Gamma]$-**mod**. If we find a minimal extension $\mathcal{M}$ of $\mathcal{C}$, we can see $\mathbb{C}[\Gamma]$-**mod** as a full subcategory of $\mathcal{M}$ and apply the de-equivariantization procedure. The results in [Tur10, Appendix 5] or [KJ04] ensure that the category $\mathcal{N}$ obtained in this manner, is a Γ-braided category with components $\mathcal{N}_\alpha \neq 0$ for all $\alpha \in \Gamma$. Further, the neutral sector $\mathcal{N}_1 \subset \mathcal{N}$ is equivalent to the initial category $\mathcal{D}$ as braided category with Γ-action.

Algebraic structures and monoidal categories

In the search for extensions of monoidal categories one is led to the investigation of algebraic structures in monoidal categories. More concretely, the homogeneous component $\mathcal{C}_\alpha$ of a Γ-graded category $\mathcal{C}$ is a bimodule category over the monoidal subcategory $\mathcal{C}_1 \subset \mathcal{C}$. The notion of bimodule category over a monoidal category categorifies the notion of a bimodule over an algebra. Any bimodule category over $\mathcal{C}_1$ can be realized as the category of modules over an algebra in a certain monoidal category, namely $\mathcal{C}_1 \boxtimes \mathcal{C}_1^{\otimes \mathrm{op}}$. We will not follow this line of thoughts, but use the notion of module category to explain an equivalence relation of monoidal categories.

Recall that two algebras over a field $\Bbbk$ are called Morita equivalent, if their categories of modules are equivalent. Seeing monoidal categories as a categorified version of an algebra one might call two monoidal categories 2-Morita equivalent, if their bicategories of module categories are equivalent. Let $\mathcal{C}$ and $\mathcal{D}$ be semi-simple tensor categories. It was shown that $\mathcal{C}$ and $\mathcal{D}$ have equivalent bicategories of module categories, iff their Drinfel'd centers $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{D})$ are equivalent as braided categories [ENO11, Thm 3.1],[Müg03a, Rem. 3.18]. This is analogous to the fact that two semi-simple algebras A and B have equivalent categories of modules, iff their centers $Z(A)$ and $Z(B)$ are isomorphic algebras.

It is desirable to get a better understanding of this equivalence relation for monoidal categories. Let A and B be not necessarily semi-simple Hopf algebras over the same field. In this thesis we provide a construction that relates the monoidal categories A-**Mod** and B-**Mod** in the sense that their Drinfel'd centers are equivalent as braided categories.

Outline

The results of this thesis split into two main parts.

Group-braided categories from non-braided monoidal categories with group action

Chapter 4, based on [Bar13], describes a construction of Γ-braided categories from the following input data: a monoidal category $\mathcal{C}$ together with an action of a discrete group Γ by monoidal autoequivalences $\Phi^\alpha \colon \mathcal{C} \to \mathcal{C}$ for every $\alpha \in \Gamma$. The Γ-braided category $\mathcal{Z}^\Gamma(\mathcal{C})$ constructed from these data has the neutral component $\mathcal{Z}(\mathcal{C})$, the Drinfel'd center of $\mathcal{C}$. The Γ-action on $\mathcal{Z}(\mathcal{C})$ is a distinguished lift of the action Φ on $\mathcal{C}$. In Remark 4.2.6 and Section 4.3 we compare our category $\mathcal{Z}^\Gamma(\mathcal{C})$ to other constructions of Γ-braided categories by Zunino [Zun04] and Virelizier [Vir05].

In Zunino's work the initial datum is a Γ-graded category with an action by monoidal autoequivalences by the same group Γ compatible with the grading. We discuss in Remark 4.2.6 why our construction is not a special case of Zunino's construction.

Another Γ-braided category is obtained from a Hopf-algebraic construction of Virelizier. In [Vir05] the author starts with a group Γ acting by Hopf algebra automorphisms on a Hopf algebra H. He defines a Γ-Hopf coalgebra, whose representation category is a Γ-braided category with neutral component equivalent to the representations of the Drinfel'd double $D(H)$ of H.
Since the representations of $D(H)$ form a category which is equivalent to the Drinfel'd center $\mathcal{Z}(H\text{-}\mathbf{mod})$ of the representation category of H, it is natural to ask, whether our category $\mathcal{Z}^\Gamma(\mathcal{C})$ generalizes Virelizier's Hopf-algebraic construction. We give a positive answer to this question in Proposition 4.3.10. In the investigation of this question we encounter an algebraic structure, which we call *twisted Yetter-Drinfel'd modules*. It is similar to *generalized Yetter-Drinfel'd modules*, introduced in [PS07].

Partial dualization of Hopf algebras

Chapter 5 is based on [BLS14]. We present a construction called *partial dualization*, inspired by the work of Heckenberger and Schneider [HS13] on reflections of Nichols algebras.

Heckenberger and Schneider start with two Hopf algebras R and $R^\vee$ in the braided category ${}^E_E\mathcal{YD}$ of Yetter-Drinfel'd modules over an ordinary Hopf algebra E. The (monoidal) category of modules over R in ${}^E_E\mathcal{YD}$ is equivalent to the category of modules over an ordinary Hopf algebra $R \rtimes E$, the bosonization of R. Assume that R and $R^\vee$ from above are connected by a non-degenerate Hopf pairing. Heckenberger and Schneider prove under the use of the Hopf algebra structures of $R \rtimes E$ and $R^\vee \rtimes E$ that the categories ${}^{R\rtimes E}_{R\rtimes E}\mathcal{YD}_{\mathrm{rat}}$ and ${}^{R^\vee\rtimes E}_{R^\vee\rtimes E}\mathcal{YD}_{\mathrm{rat}}$ of rational Yetter-Drinfel'd modules are equivalent as braided categories, cf. Sections 5-7 in [HS13].

We put this equivalence to the following abstract setting: let A and B be Hopf algebras in a braided category $\mathcal{C}$ together with a non-degenerate Hopf pairing. We show (directly in the category $\mathcal{C}$) that the braided categories ${}^A_A\mathcal{YD}(\mathcal{C})$ and ${}^B_B\mathcal{YD}(\mathcal{C})$ of Yetter-Drinfel'd modules in $\mathcal{C}$ over A resp. B are equivalent as braided categories via an equivalence $\Omega\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C})$. Assuming that H is a finite dimensional Hopf algebra and setting $\mathcal{C} = {}^E_E\mathcal{YD}$ we get back the result of Heckenberger and Schneider for finite dimensional Yetter-Drinfel'd modules, see Remark 2.2.9 for more details.

Now we start to explain the partial dualization of a Hopf algebra: let H be a Hopf algebra in a braided category $\mathcal{C}$ and let A be a Hopf subalgebra of H together with a Hopf algebra projection $\pi\colon H \to A$. These data allow to decompose H into a biproduct $K \rtimes A$, where K is a Hopf algebra in the category ${}^A_A\mathcal{YD}(\mathcal{C})$. Let $\Omega\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C})$ be the braided equivalence mentioned above. Then $\Omega(K)$ is a Hopf algebra in $\mathcal{C}$ and the partial dual of H with respect to A is defined as the biproduct $H' := \Omega(K) \rtimes B$, which is a Hopf algebra in $\mathcal{C}$.

Our partial dualization abstracts the following setting of Hopf algebras considered in Section 8 of [HS13]: let $\mathcal{C}$ be the category ${}^E_E\mathcal{YD}$ of Yetter-Drinfel'd modules over a Hopf algebra E and let M and N be objects in $\mathcal{C}$. To each Yetter-Drinfel'd module X over E one associates a Hopf algebra in $\mathcal{C}$, the Nichols algebra $\mathcal{B}(X)$. The projection and injection of the direct sum $M \oplus N$ induce Hopf algebra homomorphisms $H := \mathcal{B}(M \oplus N) \to \mathcal{B}(N) =: A$ and $\mathcal{B}(N) \to \mathcal{B}(M \oplus N)$ and the evaluation $N \otimes N^* \to \Bbbk$ induces a non-degenerate Hopf pairing between $A := \mathcal{B}(N)$ and $B := \mathcal{B}(N^*)$.

The benefits of our abstract approach are

- A simplification of the construction considered by Heckenberger and Schneider in the case of a finite dimensional Nichols algebra.
- Generalizing the construction of Heckenberger and Schneider to braided categories that are not given by Yetter-Drinfel'd modules over some Hopf algebra.

Moreover, our considerations allow to deduce the following connections between H and its partial dual H'

- The partial dual of H' with respect to the Hopf subalgebra B, is canonically isomorphic to H.
- The categories ${}^H_H\mathcal{YD}(\mathcal{C})$ and ${}^{H'}_{H'}\mathcal{YD}(\mathcal{C})$ are equivalent as braided categories.

This thesis is based on the following publications:

[Bar13] A. Barvels. Equivariant categories from categorical group actions on monoidal categories. arXiv preprint arXiv:1305.0679, 2013.

[BLS14] A. Barvels, S. Lentner, and C. Schweigert. Partially dualized Hopf algebras have equivalent Yetter-Drinfel'd modules. arXiv preprint arXiv:1402.2214, 2014.

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1. Algebraic structures in monoidal categories

In this preliminary chapter we give the definitions of common algebraic structures, such as algebras and modules in a monoidal category. This allows to express many concepts in a unified language. We will give these definitions explicitly only for strict monoidal categories. Suitable insertion of associativity and unit isomorphism is not difficult and can be done by the skeptical reader.

1.1. Conventions and notations

Before starting, we list our conventions for monoidal categories and functors between them without definitions. For precise definitions we refer to the appendix or standard literature like [Kas95] and [Mac98].

Monoidal categories and braiding Unless stated otherwise we will assume, without loss of generality, that our monoidal categories are strict, cf. Remark A.3.6. If $\mathcal{C}$ is a (strict) monoidal category, the symbol $\otimes$ shall always denote the tensor product functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and the symbol $\mathbf{1}$ always denotes the unit object of $\mathcal{C}$.

The functor $\otimes^{\mathrm{op}}\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given on objects by $(X, Y) \mapsto Y \otimes X$. Recursively we define the functors $\otimes^2 := \otimes$ and $\otimes^{n+1} := \otimes \circ (\otimes^n \times \mathrm{Id})$ for $n \geq 2$.

Let $\mathcal{C}$ be a braided category. We will denote the braiding isomorphism always by $c\colon \otimes \to \otimes^{\mathrm{op}}$. By $\overline{\mathcal{C}}$ we mean the mirror category, i.e. the monoidal category $\mathcal{C}$ with inverse braiding $\overline{c}_{X,Y} := c^{-1}_{Y,X}$.

Monoidal functors Given two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$, we write $F \otimes G$ instead of $\otimes \circ (F \times G)$.
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax monoidal functor (Definition A.3.2); we denote its monoidal structure by $F_2(X, Y)\colon FX \otimes FY \to F(X \otimes Y)$ and $F_0\colon \mathbf{1} \to F\mathbf{1}$.
Let $F\colon \mathcal{C} \to \mathcal{D}$ be an oplax monoidal functor (Definition A.3.2); we denote

its monoidal structure by $F^2(X,Y)\colon F(X\otimes Y) \to FX\otimes FY$ and $F^0\colon F\mathbf{1} \to \mathbf{1}$. If F^0 and all $F^2(X,Y)$ are isomorphisms, we consider F as a strong monoidal functor with monoidal structure $F_2(X,Y) := \big(F^2(X,Y)\big)^{-1}$ and $F_0 := \big(F^0\big)^{-1}$.

Given a lax or oplax monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, an integer $n \geq 3$ and objects $X_1, X_2, \ldots, X_n$ in $\mathcal{C}$, we define recursively the natural transformation $F_n\colon F \otimes F \otimes \ldots \otimes F \to F \circ \otimes^n$ by

$$\begin{aligned} F_n(X_1,\ldots,X_n) := F_2(X_1 \otimes \ldots \otimes X_{n-1}, X_n) \\ \circ\, (F_{n-1}(X_1,\ldots,X_{n-1}) \otimes \mathrm{id}_{FX_n}) \end{aligned} \tag{1.1}$$

and the transformation $F^n\colon F \circ \otimes^n \to F \otimes F \otimes \ldots \otimes F$ by

$$\begin{aligned} F^n(X_1,\ldots,X_n) := (F^{n-1}(X_1,\ldots,X_{n-1}) \otimes \mathrm{id}_{FX_n}) \\ \circ\, F^2(X_1 \otimes \ldots \otimes X_{n-1}, X_n)\ . \end{aligned} \tag{1.2}$$

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor and $\varphi\colon X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$ a morphism in $\mathcal{C}$ $(n,m \geq 1)$. We define the morphism $F.\varphi\colon FX_1 \otimes \ldots \otimes FX_n \to FY_1 \otimes \ldots \otimes FY_m$ as

$$F.\varphi := F_m^{-1}(Y_1,\ldots,Y_m) \circ F(\varphi) \circ F_n(X_1,\ldots,X_n)\ , \tag{1.3}$$

for $n = 1$ interpret $F_1(X_1)$ as the identity morphism id_{FX_1}.
Given morphisms $\psi\colon \mathbf{1} \to Y_1 \otimes \ldots \otimes Y_n$ and $\psi'\colon X_1 \otimes \ldots \otimes X_n \to \mathbf{1}$ in $\mathcal{C}$, define

$$F.\psi := F_n^{-1}(Y_1,\ldots,Y_n) \circ F(\psi) \circ F_0 \quad \text{and} \tag{1.4}$$

$$F.\psi := F_0^{-1} \circ F(\psi) \circ F_n(X_1,\ldots,X_n)\ . \tag{1.5}$$

Rigid categories Let $\mathcal{C}$ be a rigid monoidal category and X an object in $\mathcal{C}$. We write ${}^\vee X$ for the left dual of X, $\mathrm{ev}_X\colon {}^\vee X \otimes X \to \mathbf{1}$ denotes the evaluation and $\mathrm{coev}_X\colon \mathbf{1} \to X \otimes {}^\vee X$ the coevaluation. The right dual of X is denoted by $X^\vee$, it comes with evaluation $\widetilde{\mathrm{ev}}_X\colon X \otimes X^\vee \to \mathbf{1}$ and coevaluation $\widetilde{\mathrm{coev}}_X\colon \mathbf{1} \to X^\vee \otimes X$.

Graphical calculus Let $\mathcal{C}$ be a monoidal category. In Figure 1.1 we present our use of a graphical calculus; diagrams are read from bottom to top. We represent a morphism $f\colon X \to Y$ as a rectangular coupon labeled by f with a string attached to the bottom, labeled by X, and a string attached to the top, labeled by Y. If X or Y is the unit object, we either omit the string or draw a dashed line labeled by $\mathbf{1}$.

We depict the identity morphism of an object X by a straight line labeled by X. The composition of $f\colon X \to Y$ and $g\colon Y \to Z$ is given by putting g on top of f. A morphism $h\colon X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$ is depicted by a coupon with label h and n lines on the bottom and m on the top labeled by the objects $X_1, \ldots, X_n$ resp. $Y_1, \ldots, Y_m$. The tensor product of $f\colon X \to Y$ and $f'\colon X' \to Y'$ is drawn by juxtaposition.

Let $f\colon X \to Y$ be morphism in $\mathcal{C}$ and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We depict the morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$ as the morphism f in a tube labeled by F. If F is lax or oplax monoidal, we depict the components of the monoidal structure as in Figure 1.2. For another functor $G\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha\colon F \to G$ we sometimes depict the component α_X as a coupon labeled by α only rather than α_X.

For rigid and braided categories we denote the evaluations and coevaluations resp. the braiding isomorphisms and their inverses as shown in Figure 1.3.

1.2. Categories of modules

1.2.1. Algebras and coalgebras

Definition 1.2.1 Let $\mathcal{C}$ be monoidal category. An *associative algebra in* $\mathcal{C}$ is an object A together with a morphism $\mu = \mu_A\colon A \otimes A \to A$, such that

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu). \tag{1.6}$$

The algebra (A, μ) is called *unital*, if there is a morphism $\eta = \eta_A\colon \mathbf{1} \to A$ obeying

$$\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta). \tag{1.7}$$

The morphism μ is called *multiplication* and η is called unit of A.

Remark 1.2.2 Although the multiplication $\mu\colon A \otimes A \to A$ is an essential and usually non-unique part of the structure of an algebra, we will omit it often and only speak about the algebra A. The multiplication will always be denoted by μ or μ_A. Further we define recursively the morphisms $\mu^2 := \mu$ and $\mu^n := \mu \circ (\mu^{n-1} \otimes \mathrm{id}_A)$ for $n \geq 3$.
As one knows from the theory of ordinary algebras, the unit is unique, if it exists: let η and η' be units of an algebra A in a monoidal category $\mathcal{C}$.

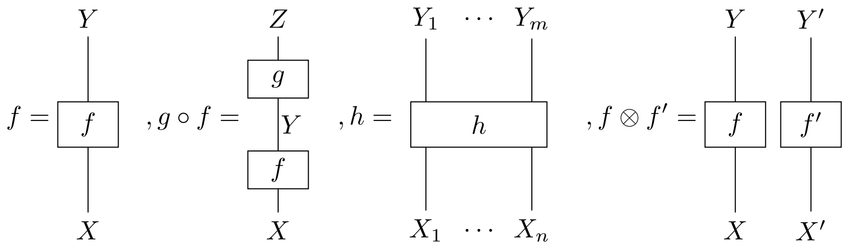


Figure 1.1.: Graphical notation of morphisms

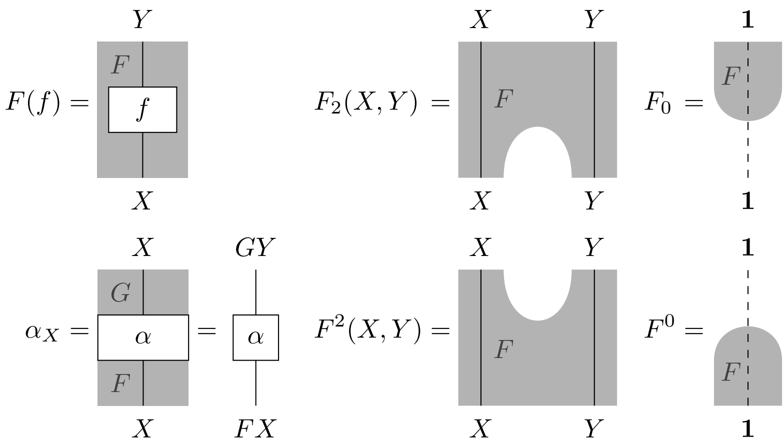


Figure 1.2.: Graphical notation involving functors and transformations

Then by using first that η' is a unit and second that η is one, we see

$$\eta = \mu \circ (\eta \otimes \eta') = \eta'.$$

We hid in the above computation that the tensor product is functorial, which is of course essential for the proof.
The uniqueness of the unit makes it unnecessary to speak of an algebra as a triple (A, μ, η).

To illustrate the strength of Definition 1.2.1 we present here a list of important examples:

The unit object For any monoidal category $\mathcal{C}$ the unit object $\mathbf{1}$ is always an algebra with multiplication given by the unit constraints $\mathsf{l}_{\mathbf{1}} = \mathsf{r}_{\mathbf{1}} \colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$.

Monoids The category **Sets** consisting of sets and maps is a monoidal category together with the cartesian product $\times$ of sets. Every one-element set $\{*\}$ is a unit object in **Sets**. A unital, associative algebra in **Sets** is the same thing as a monoid M. The usual axioms for a monoid express the unit e of M as an element in the set M. The formulation in terms of morphisms gives back the element formulation by defining $e := \eta(*)$.

Algebras over a ring If R is a commutative ring, the category R-**Mod** of R-modules is a monoidal category with tensor product $\otimes_R$ and unit object R. The axioms of an algebra A in R-**Mod** in the sense of Definition 1.2.1 are easily translated into the common textbook definition of an R-algebra as a ring A together with a unit preserving ring homomorphism $R \to Z(A)$. In particular for $R = \mathbb{Z}$ we get back the definition of a ring.

Monads Let $\mathcal{C}$ be a small category and $\mathcal{E}\mathrm{nd}(\mathcal{C})$ the category of endofunctors as objects and natural transformations as morphisms. The composition of morphisms is given by vertical composition of natural transformations. $\mathcal{E}\mathrm{nd}(\mathcal{C})$ is a monoidal category with tensor product given on objects by the composition of functors and on morphisms by horizontal composition of natural transformations.
Let now (T, μ, η) be a unital, associative algebra in $\mathcal{E}\mathrm{nd}(\mathcal{C})$. This means that $T \colon \mathcal{C} \to \mathcal{C}$ is a functor and $\mu \colon T^2 \to T$ and $\eta \colon \mathrm{Id} \to T$ are natural

transformations such that the equalities

$$\mu_X \circ T(\mu_X) = \mu_X \circ \mu_{TX}$$
$$\mu_X \circ \eta_{TX} = \mathrm{id}_{TX} = \mu_X \circ T(\eta_X)$$

hold for all objects X in $\mathcal{C}$. These are the usual axioms of a *monad on* $\mathcal{C}$, cf. Section A.2.

Coalgebras Let $\mathcal{C}$ be a monoidal category, the opposite category $\mathcal{C}^{\mathrm{op}}$ is a monoidal category as well. The algebras in $\mathcal{C}^{\mathrm{op}}$ are known under the name coalgebras:

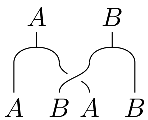
Definition 1.2.3 An object C in $\mathcal{C}$ together with morphisms $\Delta\colon C \to C \otimes C$ (comultiplication) and $\varepsilon\colon C \to \mathbf{1}$ (counit) is called *coalgebra*, if

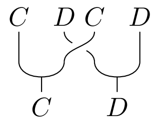
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{1.8}$$
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta\ . \tag{1.9}$$

We introduce in Figure 1.4 standard notations for (co)multiplication and (co)unit of a (co)algebra in a monoidal category $\mathcal{C}$.

We continue or list of examples for (co)algebras:

Tensor products of algebras and coalgebras Let $\mathcal{C}$ be a braided category, let A and B be algebras in $\mathcal{C}$ and let C and D be coalgebras in $\mathcal{C}$. We can define an algebra structure on the object $A \otimes B$ and a coalgebra structure on $C \otimes D$ by

$\mu_{A\otimes B} :=$ 

and $\Delta_{C\otimes D} :=$ 

The unit of $A \otimes B$ is $\eta_A \otimes \eta_B$, the counit of $C \otimes D$ is $\varepsilon_C \otimes \varepsilon_D$.

Coalgebras in cartesian categories Let $\mathcal{C}$ be a category which has all two-fold products and a terminal object, e.g. **Sets**.

The *cartesian category* associated to $\mathcal{C}$ is the monoidal category which has the product functor $\sqcap\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as tensor product, every terminal object $*$ is a unit object. In a cartesian category $\mathcal{C}$ every object has a unique structure as a counital coalgebra given by the unique morphism $\Delta\colon X \to X \sqcap X$ fulfilling $p_1 \circ \Delta = \mathrm{id}_X = p_2 \circ \Delta$. In the case of $\mathcal{C} = $ **Sets** the map Δ is the

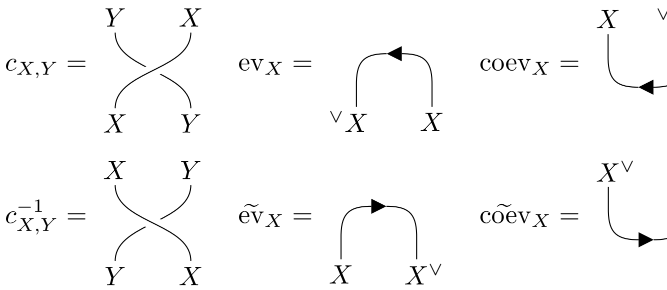


Figure 1.3.: Braidings, evaluations and coevaluations

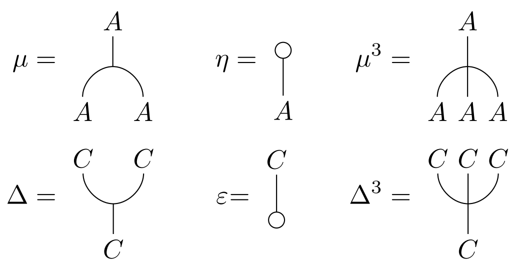


Figure 1.4.: Multiplication and comultiplication

diagonal map $x \mapsto (x, x)$.

We end this subsection with the following definition.

Definition 1.2.4 Let A and B be associative algebras in $\mathcal{C}$. A morphism $\varphi\colon A \to B$ is called *algebra homomorphism*, if

$$\varphi \circ \mu_A = \mu_B \circ (\varphi \otimes \varphi) \ .$$

Let A and B be unital algebras. We call φ *unital* or *unit preserving*, if

$$\varphi \circ \eta_A = \eta_B \ .$$

1.2.2. Modules and comodules

Definition 1.2.5 Let A be an associative algebra in a monoidal category $\mathcal{C}$. A *(left) A-module* or *module over A* is an object X in $\mathcal{C}$ together with a morphism $\rho = \rho_X : A \otimes X \to X$ such that

$$\rho \circ (\mu \otimes \mathrm{id}) = \rho \circ (\mathrm{id} \otimes \rho) \ . \tag{1.10}$$

If A is unital, we also require

$$\rho \circ (\eta \otimes \mathrm{id}) = \mathrm{id} \ . \tag{1.11}$$

Let X and Y be modules over an algebra A. A morphism $f\colon X \to Y$ is called *A-module homomorphism* or *A-linear morphism*, if

$$f \circ \rho_X = \rho_Y \circ (\mathrm{id} \otimes f) \ . \tag{1.12}$$

Modules over A and A-linear morphisms in $\mathcal{C}$ form a category A-$\mathbf{Mod}(\mathcal{C})$.

One defines right A-modules as pairs $(X, \rho^r : X \otimes A \to X)$ fulfilling conditions analogous to (1.10) and (1.11). We denote the category of right modules over A by $\mathbf{Mod}(\mathcal{C})$-$A$.

Dual to the notion of a module over an algebra is the notion of a comodule over a coalgebra C. These are pairs $(X, \delta\colon X \to C \otimes X)$ (left comodule) or $(X, \delta^r : X \to X \otimes C)$ (right comodule) fulfilling the axioms analogous to (1.10) and (1.11).

A morphism $f\colon X \to Y$ between C-comodules is called *C-comodule homomorphism* or *C-colinear morphism*, if the suitably changed condition (1.12) holds.

The category of left C-comodules is denoted by C-$\mathbf{Cmd}(\mathcal{C})$ and the category of right C-comodules by $\mathbf{Cmd}(\mathcal{C})$-$C$.

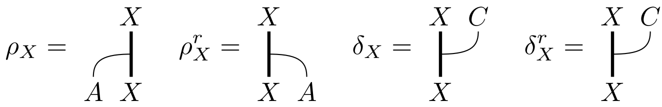


Figure 1.5.: Graphical notation of actions and coactions

In the graphical notation for morphisms we denote the action resp. coaction of a module resp. comodule as in Figure 1.5. To improve readability of the diagrams we will use different colors or thicker lines to distinguish modules and algebras.

We will now list examples for modules and comodules.

Modules over the unit object In a monoidal category $\mathcal{C}$ the unit object $\mathbf{1}$ is always an algebra and a coalgebra. Every object X is a left module over $\mathbf{1}$ via the unit isomorphism $\mathsf{l}_X \colon \mathbf{1} \otimes X \to X$ and a comodule over $\mathbf{1}$ via its inverse $\mathsf{l}_X^{-1} \colon X \to \mathbf{1} \otimes X$. The category of modules over $\mathbf{1}$ and the category $\mathcal{C}$ are obviously isomorphic as ordinary categories.

Sets and maps We have seen before that an algebra in the category **Sets** of sets and maps is the same thing as a monoid. A module over a monoid M is the same thing as a set X with an associative action $M \times X \to X$ of M. A module homomorphism is an equivariant map.

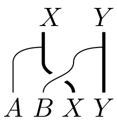
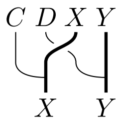
We also observed that every set X is a coalgebra in **Sets** by the diagonal map. Now let $(Y, \delta \colon Y \to X \times Y)$ be a comodule over X. Counitality implies that $\delta(y) = (f(y), y)$ for some $f(y) \in X$, hence δ defines a map $f \colon Y \to X$. Conversely, every map $f \colon Y \to X$ defines a comodule structure on Y by $\delta(y) := (f(y), y)$.
Now let Y be an X-comodule via $f \colon Y \to X$ and let Z be an X-comodule via $g \colon Z \to X$. An X-morphisms from Y to Z is a map $\varphi \colon Y \to Z$ with $g \circ \varphi = f$.

Regular modules and comodules Every algebra A in a monoidal category $\mathcal{C}$ is a left, as-well as a right A-module via its multiplication. This module is called the *regular (left/right) module*. Similar, every coalgebra

C is a left and right comodule over C via the comultiplication. It is called the *regular (left/right) comodule.*

Modules over a monad Let T be a monad on $\mathcal{C}$. A module over T in the sense of Definition 1.2.5 is an endofunctor of $\mathcal{C}$ together with a natural transformation plus the suitable axioms. Despite this notion of a module over T, in this thesis we mean by a T-module an object of the Eilenberg-Moore category T. These are pairs (X, r) where X is an object in $\mathcal{C}$ and $r\colon TX \to X$ is a morphism compatible with the multiplication μ and the unit η of T, see Appendix A.2 for a precise definition.
Note that in the literature T-modules are also called T-algebras, which we will not to since our interest is in the following type of monads:
Given an algebra A in a monoidal category $\mathcal{C}$ we get a monad $T = A \otimes \_$ with multiplication $\mu_X := \mu_A \otimes \mathrm{id}_X$ and unit $\eta_X := \eta_A \otimes \mathrm{id}_X$. A T-module is the same thing as a left A-module. Similarly, an algebra A defines a monad $T' := \_ \otimes A$ whose modules are right A-modules.

Tensor products Let $\mathcal{C}$ be a braided category, let A and B be algebras in $\mathcal{C}$ and let C and D be coalgebras in $\mathcal{C}$. Given an A-module resp. C-comodule X and a B-module resp. D-comodule Y in $\mathcal{C}$. The object $X \otimes Y$ is an $A \otimes B$-module resp. a $C \otimes D$-comodule with

$\rho :=$  resp. $\delta :=$  (1.13)

1.2.3. Functors related to (co)modules

Lemma 1.2.6 Let A and B be algebras and C and D be coalgebras in a monoidal category $\mathcal{C}$.

1. If $\varphi\colon A \to B$ is an algebra homomorphism, we get a *pull-back* or *restriction functor* $\varphi^*\colon B\text{-}\mathbf{Mod}(\mathcal{C}) \to A\text{-}\mathbf{Mod}(\mathcal{C})$ by

$$\varphi^* := \begin{cases} (X, \rho) & \mapsto (X, \rho \circ (\varphi \otimes \mathrm{id})) \\ (X \xrightarrow{f} Y) & \mapsto (X \xrightarrow{f} Y)\,. \end{cases}$$

2. If $\psi\colon C \to D$ is a coalgebra homomorphism, we get a *push-forward* or *corestriction functor* $\psi_*\colon C\text{-}\mathbf{Cmd}(\mathcal{C}) \to D\text{-}\mathbf{Cmd}(\mathcal{C})$ with

$$\psi_* := \begin{cases} (X,\delta) & \mapsto (X,(\psi\otimes \mathrm{id})\circ\delta) \\ (X \xrightarrow{f} Y) & \mapsto (X \xrightarrow{f} Y)\,. \end{cases}$$

Proof. Follows directly from the (co)algebra homomorphism axioms. $\square$

Let (X,ρ^r_X) be a right A-module and let (Y,ρ_Y) be a left A-module in a monoidal category $\mathcal{C}$. Define the object $X\otimes_A Y$ as the coequalizer of the two morphisms $\rho^r_X\otimes \mathrm{id}_Y$ and $\mathrm{id}\otimes\rho_Y$; if it exists, the object $X\otimes_A Y$ is called *tensor product of* X *and* Y *over* A.
Let (X,δ^r_X) be a right C-comodule and let (Y,δ_Y) be a left C-comodule in $\mathcal{C}$. Define the object $X\Box_C Y$ as the equalizer of $\delta^r_X\otimes\mathrm{id}_Y$ and $\mathrm{id}_X\otimes\delta_Y$; if it exists, the object $X\Box_C Y$ is called *cotensor product of* X *and* Y *over* C.

Definition 1.2.7 Let A and B be algebras in a monoidal category $\mathcal{C}$. An *A-B-bimodule* is an object X in $\mathcal{C}$ that is a left A-module and right B-module, such that the left and right actions commute:

$$\rho\circ(\mathrm{id}_A\otimes\rho^r) = \rho^r\circ(\rho\otimes\mathrm{id}_B)\,.$$

A *morphism of A-B-bimodules* is a morphism that is A-linear and B-linear. The category of A-B-bimodules and bimodule morphisms is denoted by the symbol A-B-$\mathbf{Bimod}(\mathcal{C})$. Analogously one defines the category of C-D-bicomodules and bicomodule morphisms which we denote by C-D-$\mathbf{Bicom}(\mathcal{C})$.

Lemma 1.2.8 Let $\mathcal{C}$ be a monoidal category with all coequalizers and let A, B and C be algebras in $\mathcal{C}$. Every A-B-bimodule X defines functors

$$\begin{aligned} X\otimes_B \_ &\colon B\text{-}C\text{-}\mathbf{Bimod}(\mathcal{C}) \to A\text{-}C\text{-}\mathbf{Bimod}(\mathcal{C}) \\ \_\otimes_A X &\colon C\text{-}A\text{-}\mathbf{Bimod}(\mathcal{C}) \to C\text{-}B\text{-}\mathbf{Bimod}(\mathcal{C})\,. \end{aligned}$$

If the monoidal product $\otimes$ of $\mathcal{C}$ preserves coequalizers in both variables and Y is a B-C-bimodule, the composed functor $(X\otimes_B \_)\circ(Y\otimes_C \_)$ is isomorphic to $(X\otimes_B Y)\otimes_C \_$.

Proof. Follows by standard diagram chases. $\square$

Corollary 1.2.9 Let $\mathcal{C}$ be a monoidal category with all coequalizers and assume that the product $\otimes$ preserves coequalizers in both variables. The category A-$\mathbf{Bimod}(\mathcal{C})$ of A-A-bimodules is a monoidal category with tensor product $\otimes_A$ and unit object A.

Of course, there is also a bicomodule version of Lemma 1.2.8 and Corollary 1.2.9, which we do not spell out.

Lemma 1.2.10 Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors and $\alpha\colon F \to G$ be a transformation.

1. Let A be an algebra in $\mathcal{C}$ and let F be a lax monoidal functor with monoidal structure (F_2, F_0). The object FA is an algebra in $\mathcal{D}$ with multiplication $F(\mu) \circ F_2(A, A)\colon FA \otimes FA \to FA$ and unit $F(\eta) \circ F_0\colon \mathbf{1} \to FA$.

2. If X is an A-module with action $\rho\colon A \otimes X \to X$, the object FX is an FA-module with action $F(\rho) \circ F_2(A, X)\colon FA \otimes FX \to FX$.

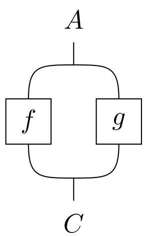
3. Let F and G be lax monoidal functors $\mathcal{C} \to \mathcal{D}$ and let $\alpha\colon F \to G$ be a monoidal transformation. The morphism $\alpha_A\colon FA \to GA$ is an algebra homomorphism.

4. Analogous statements to those above hold for coalgebras, comodules, oplax monoidal functors and monoidal transformations between oplax functors.

1.3. Bialgebras and Hopf algebras

1.3.1. Definitions

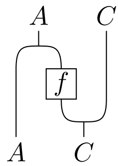
Let $\mathcal{C}$ be a monoidal category, C a coassociative coalgebra and A an associative algebra in $\mathcal{C}$. The morphisms $\mathcal{C}(C, A)$ form a monoid called the *convolution algebra*. If $\mathcal{C}$ is a $\Bbbk$-linear category, the convolution algebra is even a $\Bbbk$-algebra. The product is the *convolution product*: Let $f, g\colon C \to A$ be morphisms in $\mathcal{C}$ and define

$f * g :=$  .

The associativity of $*$ follows from coassociativity of Δ and associativity of μ. If C is counital and A is unital, the algebra $\mathcal{C}(C, A)$ is unital with

unit $\eta \circ \varepsilon \colon C \to A$.
Note that the object $A \otimes C$ is a left A-module via $\mu_A \otimes \mathrm{id}_C$ and a right C-comodule via $\mathrm{id}_A \otimes \Delta$. We can look at the set resp. $\Bbbk$-vector space ${}_A\operatorname{End}^C(A \otimes C)$ of A-linear and C-colinear morphisms $A \otimes C \to A \otimes C$. The set ${}_A\operatorname{End}^C(A \otimes C)$ is also a monoid with multiplication given by composition of morphisms. For $f \colon C \to A$ define

$$F_C^A(f) :=$$



One immediately verifies

Lemma 1.3.1 The assignment F_C^A defines an isomorphism of sets/vector spaces

$$\mathcal{C}(C, A) \to {}_A\operatorname{End}^C(A \otimes C) .$$

The inverse of F_C^A is given on $\varphi \colon A \otimes C \to A \otimes C$ by

$$\varphi \mapsto (\mathrm{id}_A \otimes \varepsilon) \circ \varphi \circ (\eta \otimes \mathrm{id}_C) .$$

Further, F_C^A is an anti-monoid homomorphism, i.e.

$$F_C^A(f * g) = F_C^A(g) \circ F_C^A(f)$$

for all $f, g \colon C \to A$.

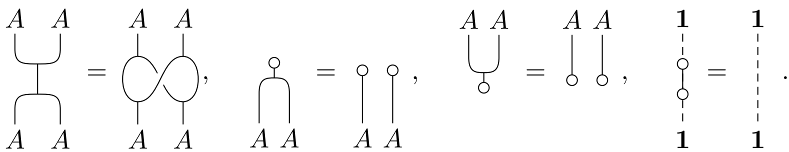
Definition 1.3.2 Let $\mathcal{C}$ be a braided category. A *bialgebra in $\mathcal{C}$* is an object A which is a unital, associative algebra and a counital, coassociative coalgebra in $\mathcal{C}$, such that one of the following equivalent conditions holds:

- The comultiplication $\Delta \colon A \to A \otimes A$ and the counit $\varepsilon \colon A \to \mathbf{1}$ are unital algebra homomorphisms.

- The multiplication $\mu \colon A \otimes A \to A$ and the unit morphism $\eta \colon \mathbf{1} \to A$ are counital coalgebra homomorphisms.

Let A and B be bialgebras in $\mathcal{C}$. A morphism $\varphi \colon A \to B$ is called *bialgebra homomorphism*, if φ is a unital algebra and counital coalgebra homomorphism.

We spell out the above definition: a bialgebra is an algebra and a coalgebra in $\mathcal{C}$ with the following compatibilities of algebra and coalgebra structure



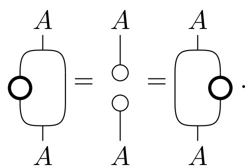
(1.14)

A special class of bialgebras is related to the convolution algebra $\mathcal{C}(A, A)$ obtained from the underlying coalgebra resp. algebra of A.

Definition 1.3.3 A bialgebra A is called *Hopf algebra*, if one of the following equivalent conditions holds:

- There is a morphism $S = S_A : A \to A$ which is convolution inverse to id_A.
- The morphism $\mathsf{H} = \mathsf{H}_A := (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) : A \otimes A \to A \otimes A$ is composition invertible.

From Lemma 1.3.1 one sees that S and H are related by $F_A^A(S) = \mathsf{H}^{-1}$. From the definition it is clear, that the antipode of a Hopf algebra is unique. We will depict S by a circle as in (1.15). The first condition of Definition 1.3.3 characterizes $S : A \to A$ as the unique morphism fulfilling



(1.15)

Many properties of the antipode of a Hopf algebra in a braided category are analogous to the properties of the antipode of a Hopf algebra over a field. Proofs can be given in terms of convolution algebras. We list some examples of such properties:

Lemma 1.3.4 Let A be a Hopf algebra in a braided category $\mathcal{C}$. The following holds:

1. $S \circ \eta = \eta$ and $\varepsilon \circ S = \varepsilon$.

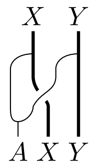
2. $S \circ \mu = \mu \circ c_{A,A} \circ (S \otimes S)$ and $\Delta \circ S = (S \otimes S) \circ c_{A,A} \circ \Delta$.
3. If $\varphi\colon A \to B$ is a bialgebra homomorphism, then $f \circ S_A = S_B \circ f$.

Proof. That S preserves the unit follows by

$$S \circ \eta = \mu \circ ((S \circ \eta) \otimes \eta) = \mu \circ (S \otimes \mathrm{id}) \circ \Delta \circ \eta = \eta \circ \varepsilon \circ \eta = \eta.$$

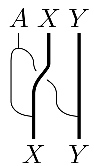
Analogously, we conclude $\varepsilon \circ S = \varepsilon$. The proof of $S \circ \mu = \mu \circ c_{A,A} \circ (S \otimes S)$ follows by showing that both sides are convolution inverse to $\mu \in \mathcal{C}(A \otimes A, A)$ and thus have to be equal. Similarly one shows that $\Delta \circ S$ and $(S \otimes S) \circ c_{A,A} \circ \Delta$ are convolution inverse to $\Delta \in \mathcal{C}(A, A \otimes A)$ and $f \circ S_A = S_B \circ f$ follows, since both are convolution inverse to $f \in \mathcal{C}(A, B)$. $\square$

If A is a bialgebra in $\mathcal{C}$, we can equip the categories of left modules or comodules over A with a monoidal structure: let X and Y be left A-modules. Recall that $X \otimes Y$ is an $A \otimes A$-module with the action defined in (1.13). Using that $\Delta\colon A \to A \otimes A$ is an algebra homomorphism gives, that

$\rho_{X \otimes Y} :=$ 

(1.16)

is a left A-action on $X \otimes Y$; it is called the *diagonal action*. Due to the counit $\varepsilon\colon A \to \mathbf{1}$, every object X in $\mathcal{C}$ becomes an A-module via the *trivial action* $\rho_X^{\mathrm{triv}} := \varepsilon \otimes \mathrm{id}_X$.
Analogously, for A-comodules X and Y the morphism

$\delta_{X,Y} :=$ 

(1.17)

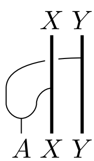
is an A-comodule structure on $X \otimes Y$, called the *diagonal coaction* and the unit $\eta\colon \mathbf{1} \to A$ defines on every object X a comodule structure $\delta_X^{\mathrm{triv}} := \eta \otimes \mathrm{id}_X$.

Proposition 1.3.5 The categories A-$\mathbf{Mod}(\mathcal{C})$ and A-$\mathbf{Cmd}(\mathcal{C})$ become monoidal categories with the tensor product of (co)modules defined by the diagonal (co)action and the unit object given by the unit object $\mathbf{1}$ of $\mathcal{C}$ seen as the trivial (co)module.

Proof. Due to coassociativity resp. counitality of Δ the associator and unit isomorphisms of $\mathcal{C}$ are A-linear. Thus the category A-$\mathbf{Mod}(\mathcal{C})$ inherits the associator resp. the unitators of $\mathcal{C}$. Hence A-$\mathbf{Mod}(\mathcal{C})$ is strict, if $\mathcal{C}$ is.
The category A-$\mathbf{Cmd}(\mathcal{C})$ is monoidal due to associativity and unitality of of the multiplication μ. $\square$

Note that the categories A-$\mathbf{mod}(\mathcal{C})$ and A-$\mathbf{cmd}(\mathcal{C})$ are in general not braided although $\mathcal{C}$ is braided. Further, A-$\mathbf{mod}(\mathcal{C})$ and A-$\mathbf{cmd}(\mathcal{C})$ are rather different categories.

Remark 1.3.6 Of course one can define a diagonal (co)action on two right modules, as well as a trivial right (co)action on each object of $\mathcal{C}$. As in the case of left (co)modules, we can see the categories $\mathbf{Mod}(\mathcal{C})$-$A$ and $\mathbf{Cmd}(\mathcal{C})$-$A$ as monoidal categories, which we will always do, if A is a bialgebra.
Note that in the definition of $\rho_{X\otimes Y}$ in (1.16) the choice of the braiding is not arbitrary: if we use the inverse braiding to define $\rho_{X\otimes Y}$, the pair $(X\otimes Y, \rho_{X\otimes Y})$ will not be an A-module in general. This comes from defining the multiplication on $A\otimes A$ with the help of $c_{A,A}$.
However, there is always a second A-module structure on $X\otimes Y$ given by the family

$$\rho'_{X\otimes Y} :=$$


(1.18)

Why we prefer ρ and not ρ' is merely a question of convention; in Remark 1.3.8 we will see that ρ' is the diagonal action of another bialgebra in another braided category.

The first subtle difficulty, in comparison to the theory of bialgebras over a field, turns out to be the definition of the opposed and coopposed bialgebra. Naively, one can try to define the opposed algebra of (A,μ,Δ) as the triple $(A, \mu^+ := \mu\circ c_{A,A}, \Delta)$. All axioms of a bialgebra in $\mathcal{C}$ hold, except for the first equality in (1.14). We can repair this flaw by assuming that A is a bialgebra not in $\mathcal{C}$, but rather in the mirror category $\overline{\mathcal{C}}$, i.e. $\mathcal{C}$ with the inverse braiding $\bar{c}_{X,Y} := c^{-1}_{Y,X}$. Note that $\overline{\overline{\mathcal{C}}}$ and $\mathcal{C}$ are equal as braided categories. We arrive at

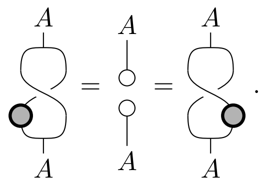
Definition 1.3.7 Let A be a bialgebra in a braided category $\mathcal{C}$. The *opposed bialgebra* A^{op} of A is the triple $(A, \mu^- := \mu\circ c^{-1}_{A,A}, \Delta)$, which is a bialgebra in $\overline{\mathcal{C}}$. The *coopposed bialgebra* A^{cop} of A is the triple $(A, \mu, \Delta^- := c_{A,A}^{-1} \circ \Delta)$, which is a bialgebra in $\overline{\mathcal{C}}$.

Remark 1.3.8 We said in Remark 1.3.6 that the morphisms (1.16) and (1.18) both define a left module structure on $X \otimes Y$. From the picture of ρ' one immediately verifies that this is the diagonal action of A^{cop} in $\overline{\mathcal{C}}$ on $X \otimes Y$. So we loose nothing, when we prefer one of the actions on $X \otimes Y$, since we can see $X \otimes Y$ either as an A-module in $\mathcal{C}$ or as an A^{cop}-module in $\overline{\mathcal{C}}$.

Remark 1.3.9 Let A be a bialgebra in a braided category $\mathcal{C}$. The opposed bialgebra of the opposed bialgebra of A, i.e. $(A^{\mathrm{op}})^{\mathrm{op}}$ in $\overline{\overline{\mathcal{C}}} = \mathcal{C}$, is equal to A itself. Also $(A^{\mathrm{cop}})^{\mathrm{cop}}$ is equal to A as a bialgebra in $\mathcal{C}$. In contrast to that, the bialgebras $(A^{\mathrm{op}})^{\mathrm{cop}} = (A, \mu^-, \Delta^+)$ and $(A^{\mathrm{cop}})^{\mathrm{op}} = (A, \mu^+, \Delta^-)$ are usually different as bialgebras in $\mathcal{C}$. Nevertheless, if A is a Hopf algebra, the antipode provides bialgebra homomorphisms

$$S\colon (A^{\mathrm{op}})^{\mathrm{cop}} \to A \quad \text{and} \quad S\colon A \to (A^{\mathrm{cop}})^{\mathrm{op}}$$

what we see from Lemma 1.3.4.
Already in classical Hopf algebra theory, the (co)opposite bialgebra of a Hopf algebra is not necessarily a Hopf algebra; this is precisely the case when the antipode is not invertible, cf. Lemma 1.3.10. From (1.15) one sees that the problem, whether A^{op} is a Hopf algebra depends on the existence of a morphism $T\colon A \to A$, which we depict by a gray circle, obeying



(1.19)

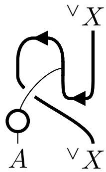
Such a morphism is also called a *skew-antipode* for the bialgebra A; note that $c_{A,A}^{-1}$, and not $c_{A,A}$, appears in (1.19).

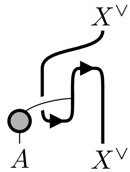
Parallel to classic Hopf algebra theory one has the following result:

Lemma 1.3.10 Let A be a bialgebra in a braided category $\mathcal{C}$, let $S\colon A \to A$ be an antipode for A and let $T\colon A \to A$ be a skew-antipode for A. Then $S \circ T = \mathrm{id}_A = T \circ S$.
In particular, A is a Hopf algebra with invertible antipode, iff A^{op} and A^{cop} are Hopf algebras with invertible antipode.

If A is a Hopf algebra in $\mathcal{C}$ with invertible antipode, and $\mathcal{C}$ is rigid then the modules and comodules over A are also rigid categories: let X be an object in $\mathcal{C}$ with left-dual object ${}^{\vee}X$ and right-dual object $X^{\vee}$.

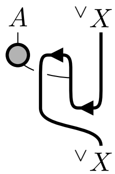
1. Given a left module $(X, \rho\colon A \otimes X \to X)$, define ${}^{\vee}(X, \rho)$ as the pair $({}^{\vee}X, {}^{\wedge}\rho\colon A \otimes {}^{\vee}X \to {}^{\vee}X)$ and define $(X, \rho)^{\vee}$ as the pair $(X^{\vee}, \rho^{\wedge}\colon A \otimes X^{\vee} \to X^{\vee})$ with

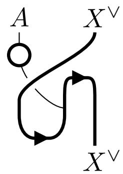
   ${}^{\wedge}\rho :=$ 

   and $\rho^{\wedge} :=$ 

   .

2. Given a left comodule $(X, \delta\colon X \to A \otimes X)$, define ${}^{\vee}(X, \delta)$ as the pair $({}^{\vee}X, {}^{\wedge}\delta\colon {}^{\vee}X \to A \otimes {}^{\wedge}X)$ and further define $(X, \delta)^{\vee}$ as the pair $(X^{\vee}, \delta^{\wedge}\colon X^{\vee} \to A \otimes X^{\vee})$ with

   ${}^{\wedge}\delta :=$ 

   and $\delta^{\wedge} :=$ 

   .

Lemma 1.3.11 Let $\mathcal{C}$ be a rigid, braided category and A a Hopf algebra in $\mathcal{C}$ with invertible antipode $S\colon A \to A$. Then also the categories of left A-modules resp. left A-comodules over A are rigid.

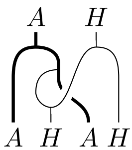
Proof. Let (X, ρ) be a left A-modules. One checks that ${}^{\vee}(X, \rho)$ is a left A-module and that ev_X and coev_X are A-linear morphisms fulfilling the rigidity axioms. Thus ${}^{\vee}(X, \rho)$ is a left dual object for (X, ρ). Similar arguments show that $(X, \rho)^{\vee}$ is a right dual object for (X, ρ) and that ${}^{\vee}(X, \delta)$ and $(X, \delta)^{\vee}$ are left resp. right dual to an A-comodule (X, δ). □

1.3.2. Smash products

In this subsection we give a short application of the viewpoint of algebras in monoidal categories.

Definition 1.3.12 Let H be a bialgebra in a braided category $\mathcal{C}$. A *module algebra over H* is an algebra in the monoidal category H-$\mathbf{Mod}(\mathcal{C})$ and a *comodule algebra over H* is an algebra in the monoidal category H-$\mathbf{Cmd}(\mathcal{C})$.
The term *module coalgebra over H* stands for a coalgebra in the category H-$\mathbf{Mod}(\mathcal{C})$ and the term *comodule coalgebra over H* for a coalgebra in the category H-$\mathbf{Cmd}(\mathcal{C})$.

If A is a module algebra over a bialgebra H in $\mathcal{C}$, we can define a multiplication morphism on $A \otimes H$

$\mu :=$ 

(1.20)

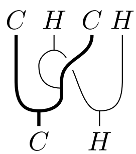
The pair $(A \otimes H, \mu)$ is called *the smash product of A over H* and we will denote it by $A \rtimes H$.

Lemma 1.3.13 Let H be a bialgebra in a braided category $\mathcal{C}$ and let A be a module algebra over H. The smash product $A \rtimes H$ is an associative algebra in $\mathcal{C}$ and the categories $(A \rtimes H)$-$\mathbf{Mod}(\mathcal{C})$ and A-$\mathbf{Mod}(H\text{-}\mathbf{Mod}(\mathcal{C}))$ are isomorphic categories.

Sketch of proof. An A-module X in H-$\mathbf{Mod}(\mathcal{C})$ with A-action ρ^A and H-action ρ^H becomes an $A \rtimes H$-module with action $\rho^A \circ (\mathrm{id}_A \otimes \rho^H)$.

Conversely, an $A \rtimes H$-module X becomes an H-module by composing the $A \rtimes H$-action with $\eta_A \otimes \mathrm{id}$ and X becomes an A-module in H-$\mathbf{Mod}(\mathcal{C})$ by composing the $A \rtimes H$-action with $\mathrm{id} \otimes \eta_H$ $\square$

Remark 1.3.14 Given a comodule coalgebra C over a bialgebra H in $\mathcal{C}$, one defines the *cosmash product* $C \rtimes H$ as the pair $(C \otimes H, \Delta)$ with

$\Delta =$ 

. (1.21)

Analogous to Lemma 1.3.13, we have that $C \rtimes H$ is a coalgebra in $\mathcal{C}$ whose category of comodules is isomorphic to the category of C-comodules in H-$\mathbf{Cmd}(\mathcal{C})$.

1.3.3. Functors between modules and comodules

From elementary algebra one knows that for a ring R the categories of left R-modules and right R^{op}-modules are isomorphic. For an algebra A in a braided category we have a similar equivalence, which turns out to be strict monoidal, if A is even a bialgebra.

Lemma 1.3.15 Let A be a bialgebra in a braided category $\mathcal{C}$ and (X, ρ) a left A-module. The assignment

$$
{}_A\mathsf{T} = \begin{cases} (X, \rho) & \mapsto (X, \rho^- := \rho \circ c^{-1}_{A,X}) \\ \left(X \xrightarrow{f} Y\right) & \mapsto \left(X \xrightarrow{f} Y\right) \end{cases}
$$

is a strict monoidal functor ${}_A\mathsf{T} \colon A\text{-}\mathbf{Mod}(\mathcal{C}) \to \mathbf{Mod}(\overline{\mathcal{C}})\text{-}A^{\mathrm{op}}$.

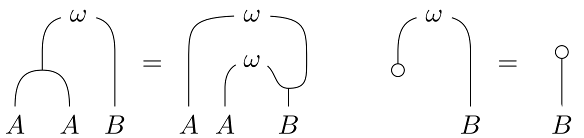
Remark 1.3.16 We call the functor ${}_A\mathsf{T}$ *side switch functor*. Similarly, we have side switch functors for right modules and left and right comodules, which are also strict monoidal:

$$
\begin{aligned}
& {}^A\mathsf{T} \colon A\text{-}\mathbf{Cmd}(\mathcal{C}) \to \mathbf{Cmd}(\overline{\mathcal{C}})\text{-}A^{\mathrm{cop}}, \\
& \mathsf{T}_A \colon \mathbf{Mod}(\mathcal{C})\text{-}A \to A^{\mathrm{op}}\text{-}\mathbf{Mod}(\overline{\mathcal{C}}) \\
\text{and} \qquad & \mathsf{T}^A \colon \mathbf{Cmd}(\mathcal{C})\text{-}A \to A^{\mathrm{cop}}\text{-}\mathbf{Cmd}(\overline{\mathcal{C}}) \ .
\end{aligned}
$$

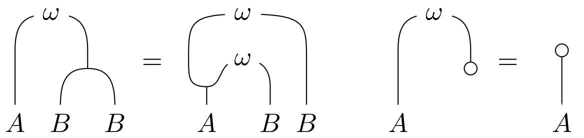
They are given on objects by composing the (co)action of a (co)module with suitable instances of the inverse braiding.
Note that $\mathsf{T}_{A^{\mathrm{op}}} \colon \mathbf{Mod}(\overline{\mathcal{C}})\text{-}A^{\mathrm{op}} \to (A^{\mathrm{op}})^{\mathrm{op}}\text{-}\mathbf{Mod}(\overline{\overline{\mathcal{C}}}) = A\text{-}\mathbf{Mod}(\mathcal{C})$ is inverse to the functor ${}_A\mathsf{T}$, thus we have an isomorphism of monoidal categories.

A Hopf pairing between two bialgebras allows us to define functors between the categories of modules and comodules.

Definition 1.3.17 Let A and B be bialgebras in a braided category $\mathcal{C}$. A morphism $\omega \colon A \otimes B \to \mathbf{1}$ is called *Hopf pairing between A and B*, if the following equations hold



(1.22)



(1.23)

A Hopf pairing is called *non-degenerate*, if there is an *inverse copairing*, i.e. is a morphism $\omega'\colon \mathbf{1} \to B \otimes A$, such that

$$(\omega \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \omega') = \mathrm{id}_A \quad \text{and} \quad (\mathrm{id}_B \otimes \omega) \circ (\omega' \otimes \mathrm{id}_B) = \mathrm{id}_B \ .$$

Lemma 1.3.18 Let A and B be bialgebras in a braided category $\mathcal{C}$ and $\omega\colon A \otimes B \to \mathbf{1}$ a Hopf pairing. The following holds

1. The morphism ω is a Hopf pairing between A^{op} and B^{cop} resp. between A^{cop} and B^{op}.

2. If ω is non-degenerate, the inverse copairing $\omega'\colon \mathbf{1} \to B \otimes A$ is unique and a *Hopf copairing*, i.e. ω' obeys axioms analogous to (1.22) and (1.23).

3. If A and B are Hopf algebras, the Hopf pairing ω is compatible with the antipodes:

$$\omega \circ (S_A \otimes \mathrm{id}_B) = \omega \circ (\mathrm{id}_A \otimes S_B) \ . \tag{1.24}$$

Proof. The proof consists mostly on checking the claims directly. The last claim can be shown by proving that $\omega \circ (S_A \otimes \mathrm{id}_B)$ and $\omega \circ (\mathrm{id}_A \otimes S_B)$ are both convolution inverse to ω in the convolution algebra $\mathcal{C}(A' \otimes B, \mathbf{1})$, where A' is the coalgebra with comultiplication $c_{A,A} \circ \Delta$. □

Lemma 1.3.19 Let A and B be bialgebras in a braided category $\mathcal{C}$, let $\omega\colon A \otimes B \to \mathbf{1}$ be a Hopf pairing and let (X, δ) be a left B-comodule. The assignment

$${}^{\omega}\mathsf{D} := \begin{cases} (X, \delta) & \mapsto (X, (\omega \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \delta)) \\ (X \xrightarrow{f} Y) & \mapsto (X \xrightarrow{f} Y) \end{cases}$$

is a strict monoidal functor ${}^{\omega}\mathsf{D}\colon B\text{-}\mathbf{Cmd}(\mathcal{C}) \to A^{\mathrm{cop}}\text{-}\mathbf{Mod}(\overline{\mathcal{C}})$. If ω is non-degenerate the functor ${}^{\omega}\mathsf{D}$ is invertible.

Proof. Remember that A and A^{cop} have the same underlying algebra. That ${}^{\omega}\mathsf{D}(X)$ is an A-module (and thus an A^{cop}-module) follows directly from the equations (1.22).
Next one checks with the help of the first equality in (1.23) that for every pair of left B-comodules X and Y the A-module ${}^{\omega}\mathsf{D}(X \otimes Y)$ is equal to the A-module ${}^{\omega}\mathsf{D}(X) \otimes {}^{\omega}\mathsf{D}(Y)$ equipped with the diagonal action of A^{cop}.
Finally the second equation in (1.23) ensures the equality ${}^{\omega}\mathsf{D}(\mathbf{1}) \otimes {}^{\omega}\mathsf{D}(X) = {}^{\omega}\mathsf{D}(X) = {}^{\omega}\mathsf{D}(X) \otimes {}^{\omega}\mathsf{D}(\mathbf{1})$ of A-modules. Hence ${}^{\omega}\mathsf{D}$ is a strict monoidal functor.
The last thing to check is the invertibility in the case of a non-degenerate Hopf pairing: let ω' be the inverse copairing of ω and define the assignment ${}_{\omega'}\mathsf{D}\colon A^{\mathrm{cop}}\text{-}\mathbf{Mod}(\mathcal{C}) \to B\text{-}\mathbf{Cmd}(\mathcal{C})$ by

$$
{}_{\omega'}\mathsf{D} := \begin{cases} (X, \rho) & \mapsto (X, (\mathrm{id}_B \otimes \rho) \circ (\omega' \otimes \mathrm{id}_X)) \\ (X \xrightarrow{f} Y) & \mapsto (X \xrightarrow{f} Y)\,. \end{cases}
$$

That ${}_{\omega'}\mathsf{D}$ is indeed a functor follows from the Hopf copairing properties. Since ω' is the inverse copairing of ω, we see that ${}_{\omega'}\mathsf{D}$ is inverse to ${}^{\omega}\mathsf{D}$. $\square$

Example 1.3.20 Let A and B be Hopf algebras in a braided category $\mathcal{C}$.

1. Let $A^{\vee}$ be a right dual object of A. The triple $(A^{\vee}, \Delta^{\vee}, \mu^{\vee})$ is a Hopf algebra in $\mathcal{C}$ and the evaluation $\widetilde{\mathrm{ev}}\colon A \otimes A^{\vee} \to \mathbf{1}$ is a non-degenerate Hopf pairing with inverse copairing $\widetilde{\mathrm{coev}}\colon \mathbf{1} \to A^{\vee} \otimes A$.
   Let $\mathcal{C}$ be the category of vector spaces. A Hopf algebra A has a dual, iff it is a finite dimensional.

2. If $\omega\colon A \otimes B \to \mathbf{1}$ is a Hopf pairing, then $\omega^{+} := \omega \circ (S_A \otimes S_B) \circ c_{B,A}$ and $\omega^{-} := \omega \circ (S_A^{-1} \otimes S_B^{-1}) \circ c_{A,B}^{-1}$ are Hopf pairings $B \otimes A \to \mathbf{1}$.
   If ω is non-degenerate with inverse copairing ω', then ω^{+} and ω^{-} are non-degenerate as well. For instance, the inverse copairing of ω^{+} is given by $c_{A,B}^{-1} \circ (S_B^{-1} \otimes S_A^{-1}) \circ \omega'$.

Remark 1.3.21 Let A and B be bialgebras over a field $\Bbbk$. The definition of a Hopf pairing above reads in Sweedler notation as

$$
\omega(ab, x) = \omega(a, x_{(2)})\omega(b, x_{(1)}) \quad \text{and} \quad \omega(a, xy) = \omega(a_{(1)}, y)\omega(a_{(2)}, x)
$$

for all $a, b \in A$ and $x, y \in B$. The usual textbook definition of a Hopf pairing $\sigma\colon A \otimes B \to \Bbbk$ is

$$\sigma(ab, x) = \sigma(a, x_{(2)})\sigma(b, x_{(1)}) \quad \text{and} \quad \sigma(a, xy) = \sigma(a_{(1)}, x)\sigma(a_{(2)}, y)\ .$$

2. Yetter-Drinfel'd modules

Let $\mathcal{C}$ be a braided category and A a Hopf algebra in $\mathcal{C}$. We noticed before that neither the category of modules nor the category of comodules is a braided category. Nevertheless, there are always braided categories associated to A, namely the categories of Yetter-Drinfel'd modules over A. In this section we summarize facts about Yetter-Drinfel'd modules over a Hopf algebra A in a braided category $\mathcal{C}$ as defined by Bespalov [Bes97]. Bespalov's notion of Yetter-Drinfel'd module generalizes the classical notion of a Yetter-Drinfel'd module over a Hopf algebra over a field as given in [Mon93] also called crossed module [Kas95, Chapter IX.5].

2.1. Definitions

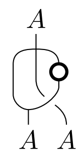
2.1.1. The category of Yetter-Drinfel'd modules

A Yetter-Drinfel'd module is a module and a comodule, subject to a compatibility condition. Actions and coactions can be on the left or right; thus there are four different types of Yetter-Drinfel'd modules.

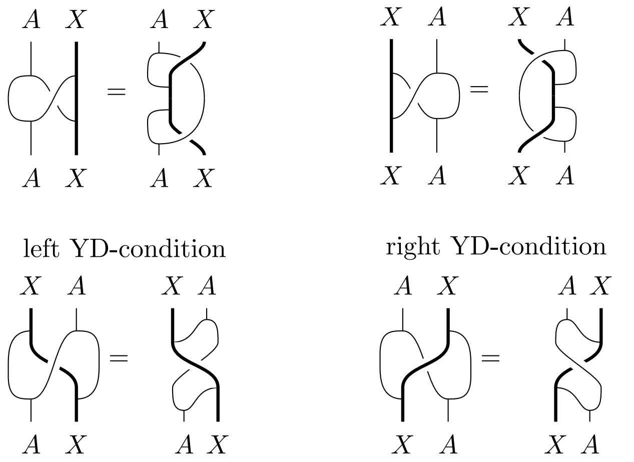
Definition 2.1.1 Let A be a Hopf algebra in a braided category $\mathcal{C}$; suppose that X is a module and comodule over A. We call X a *Yetter-Drinfel'd module over* A, if the suitable condition depicted in Figure 2.1 is fulfilled.

Example 2.1.2 Let A be a Hopf algebra in a braided category $\mathcal{C}$.

1. The (left) adjoint action $\mathrm{ad}_A\colon A \otimes A \to A$ of A on A is given by the morphism



   One can check that the triple $(A, \mathrm{ad}_A, \Delta)$ is a left Yetter-Drinfel'd module in $\mathcal{C}$.



left YD-condition right YD-condition

left-right YD-condition right-left YD-condition

Figure 2.1.: Yetter-Drinfel'd conditions

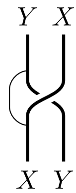
2. Let X be an arbitrary object in $\mathcal{C}$. It becomes a module and comodule with the trivial action by the counit and the trivial coaction by the unit of A. One sees that X is a Yetter-Drinfel'd module over A, iff $c_{X,A} \circ c_{A,X} = \mathrm{id}$.

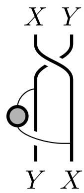
2.1.2. Monoidal structure of one-sided Yetter-Drinfel'd modules

The left Yetter-Drinfel'd modules over A are objects of a category ${}^A_A\mathcal{YD}(\mathcal{C})$; morphisms in ${}^A_A\mathcal{YD}(\mathcal{C})$ are morphisms in $\mathcal{C}$ that are A-linear and A-colinear. Moreover, the category ${}^A_A\mathcal{YD}(\mathcal{C})$ inherits a monoidal structure from $\mathcal{C}$ and there is a braiding on ${}^A_A\mathcal{YD}(\mathcal{C})$: the tensor product of a Yetter-Drinfel'd module X and a Yetter-Drinfel'd module Y is given by the object $X \otimes Y$ with the diagonal action and coaction of A. The unit object is the monoidal unit $\mathbf{1}$ of $\mathcal{C}$, together with trivial action given by the counit and trivial coaction given by the unit of A.
The braiding isomorphism

$$c^{\mathcal{YD}}_{X,Y} : X \otimes Y \to Y \otimes X$$

and its inverse are given by

$c^{\mathcal{YD}}_{X,Y} :=$ 

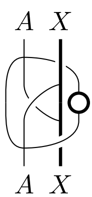
and $(c^{\mathcal{YD}}_{X,Y})^{-1} =$ 

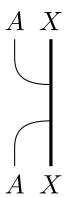
. (2.1)

We summarize this structure in the following proposition whose proof can be found in [Bes97].

Proposition 2.1.3 Let A be a Hopf algebra with invertible antipode in $\mathcal{C}$. The left Yetter-Drinfel'd modules over A in $\mathcal{C}$ have a natural structure of a braided monoidal category ${}^A_A\mathcal{YD}(\mathcal{C})$.

Remark 2.1.4 The definition of Yetter-Drinfel'd module does not require the existence of an antipode, so Yetter-Drinfel'd modules can be defined over a bialgebra as well. The braiding $c^{\mathcal{YD}}$ as defined above then fails to be an isomorphism, but still fulfills the hexagon axioms. So ${}^A_A\mathcal{YD}(\mathcal{C})$ is only a prebraided category.
If A is a Hopf algebra, the antipode allows us to reformulate the Yetter-Drinfel'd condition: a graphical calculation shows that a module and comodule X is a left Yetter-Drinfel'd module, iff



$=$ 

.

This reformulation is useful to prove the following lemma which is proven by straightforward calculations:

Lemma 2.1.5 Let A be a Hopf algebra with invertible antipode in $\mathcal{C}$. For a left Yetter-Drinfel'd module X consider the morphism

$$\theta_X := \rho_X \circ (S \otimes \mathrm{id}) \circ \delta_X \in \mathrm{End}_{\mathcal{C}}(X).$$

The following holds

1. $\theta_X \circ \rho_X = \rho_X \circ c_{X,A} \circ c_{A,X} \circ (S^2 \otimes \mathrm{id})$.

2. $\delta_X \circ \theta_X = (S^2 \otimes \mathrm{id}) \circ c_{X,A} \circ c_{A,X} \circ \delta_X$.
3. The inverse of θ_X is given by
$$\rho_X \circ c_{A,X}^{-1} \circ (\mathrm{id} \otimes S^{-2}) \circ c_{X,A}^{-1} \circ \delta_X \ .$$
4. If Y is another Yetter-Drinfel'd module, we have
$$c_{Y,X}^{\mathcal{YD}} \circ \theta_{Y\otimes X} \circ c_{X,Y}^{\mathcal{YD}} = c_{Y,X} \circ (\theta_Y \otimes \theta_X) \circ c_{X,Y} \ .$$

Remark 2.1.6 Let A be a Hopf algebra with invertible antipode in $\mathcal{C}$. The right Yetter-Drinfel'd modules also form a braided monoidal category, which is denoted by $\mathcal{YD}_A^A(\mathcal{C})$. The braiding is given by
$$c_{X,Y}^{\mathcal{YD}} := (\mathrm{id}_Y \otimes \rho_X^r) \circ (c_{X,Y} \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes \delta_Y^r).$$
If $\mathcal{C}$ is the category of vector spaces over a field $\Bbbk$, we also write ${}_A^A\mathcal{YD}_\Bbbk$ or ${}_A^A\mathcal{YD}$ for the category of Yetter-Drinfel'd modules.

Proposition 2.1.7 Let $\mathcal{C}$ be a rigid braided monoidal category and let A be a Hopf algebra with invertible antipode in $\mathcal{C}$. Then the category ${}_A^A\mathcal{YD}(\mathcal{C})$ is also rigid.

Proof. Let (X, ρ, δ) be a Yetter-Drinfel'd module over A. Recall that a braided category is rigid, if it is left or right rigid. So we only have to show the existence of a left dual Yetter-Drinfel'd module ${}^\vee(X, \rho, \delta)$. Choose as underlying object the left dual ${}^\vee X$ in $\mathcal{C}$, as left action ${}^\wedge\rho$ and as a left coaction ${}^\wedge\delta$ from Lemma 1.3.11. The object ${}^\vee X$ is obviously a module and comodule in such a way that the evaluation and coevaluation are A-linear and A-colinear morphisms fulfilling the rigidity axioms. The remaining check that ${}^\vee X$ is a Yetter-Drinfel'd module is a feasible calculation. □

2.1.3. Monoidal structure of two-sided Yetter-Drinfel'd modules

We have seen in the preceding subsection that we always have a braided monoidal structure on the category of left Yetter-Drinfel'd modules. Before we discuss, whether left-right Yetter-Drinfel'd modules form a braided category as well, we pause here and focus our attention on the braiding isomorphism
$$c_{X,Y}^{\mathcal{YD}} = (\rho_Y \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes c_{X,Y}) \circ (\delta_X \otimes \mathrm{id}_Y) \ .$$

The main idea of the braiding isomorphism $c^{\mathcal{YD}}_{X,Y}$ can be seen as repairing the flaw of $c_{X,Y}$ being neither A-linear nor A-colinear by involving the Hopf algebra A via connecting the objects X and Y by an action and coaction of A. Note that the morphism $c^{\mathcal{YD}}_{X,Y}$ is reasonably simple in the sense that we use a minimal amount of braiding isomorphisms from the underlying category $\mathcal{C}$ to form $c^{\mathcal{YD}}_{X,Y}$. Further, the hexagon-identity $c^{\mathcal{YD}}_{X\otimes Y,Z} = (c^{\mathcal{YD}}_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c^{\mathcal{YD}}_{Y\otimes Z})$ is due to associativity of the A-action of Z and the hexagon-identity $c^{\mathcal{YD}}_{X,Y\otimes Z} = (\mathrm{id}_X \otimes c^{\mathcal{YD}}_{X,Y\otimes Z}) \circ (c^{\mathcal{YD}}_{X,Y\otimes Z} \otimes \mathrm{id}_Z)$ is due to the coassociativity of the A-coaction of X. The last two observations also manifest in the following fact: denote ${}_A\mathcal{M} := A\text{-}\mathbf{Mod}(\mathcal{C})$ and ${}^A\mathcal{M} := A\text{-}\mathbf{Cmd}(\mathcal{C})$. The forgetful functors

$$ {}_A\mathcal{M} \longleftarrow {}^A_A\mathcal{YD}(\mathcal{C}) \longrightarrow {}^A\mathcal{M} $$

are strict monoidal.

We now turn to left-right Yetter-Drinfel'd modules. For any two Yetter-Drinfel'd modules over A the endomorphism $\gamma_{X,Y} := (\mathrm{id}_X \otimes \rho_Y) \circ (\delta_X \otimes \mathrm{id}_Y)$ of the object $X \otimes Y$ in $\mathcal{C}$ is defined. To turn this family of morphisms into a candidate for a braiding on the category ${}_A\mathcal{YD}^A(\mathcal{C})$ of left-right Yetter-Drinfel'd modules, we have to pre-compose or post-compose it with instances of the braiding c of $\mathcal{C}$. For reasons, that will become clear in a moment, we post-compose $\gamma_{X,Y}$ with $c^{-1}_{Y,X}$ rather than with $c_{X,Y}$

$$ c^{\mathcal{YD}}_{X,Y} := c^{-1}_{Y,X} \circ \gamma_{X,Y} \ . \tag{2.2} $$

As for left Yetter-Drinfel'd modules, the hexagon-identity for $c^{\mathcal{YD}}_{X\otimes Y,Z}$ holds, since the A-action of Z is associative. Despite the fact, that we do not have a monoidal structure on the category ${}_A\mathcal{YD}^A(\mathcal{C})$ yet, we can observe the following: the forgetful functor ${}_A\mathcal{YD}^A(\mathcal{C}) \to \mathcal{M}^A := \mathbf{Cmd}(\mathcal{C})\text{-}A$ would be strict monoidal, if the tensor product $X \otimes Y$ was equipped with the diagonal coaction of A.
Now look at the hexagon-identity for $c^{\mathcal{YD}}_{X,Y\otimes Z}$. If it holds, we have to equip $Y \otimes Z$ with the diagonal action of A^{cop} in $\overline{\mathcal{C}}$. Denote ${}_{A^{\mathrm{cop}}}\mathcal{M} := A^{\mathrm{cop}}\text{-}\mathbf{Mod}(\overline{\mathcal{C}})$. Again ignoring the fact that, so far, we have no monoidal structure on ${}_A\mathcal{YD}^A(\mathcal{C})$ we formulate: the forgetful functor ${}_A\mathcal{YD}^A(\mathcal{C}) \to {}_{A^{\mathrm{cop}}}\mathcal{M}$ would be strict monoidal, if the tensor product $X \otimes Y$ was equipped with the diagonal action of A^{cop}.

Post-composing $\gamma_{X,Y}$ with $c_{Y,X}^{-1}$ seems arbitrary in the sense that also

$$\tilde{c}_{X,Y}^{\mathcal{YD}} := \gamma_{Y,X} \circ c_{Y,X}^{-1} \tag{2.3}$$

seems to be good candidate for a braiding on ${}_A\mathcal{YD}^A(\mathcal{C})$. Following the discussions above now for $\tilde{c}_{X,Y}^{\mathcal{YD}}$ would allow strict monoidal forgetful functors

$${}_A\mathcal{M} \longleftarrow {}_A\mathcal{YD}^A(\mathcal{C}) \longrightarrow \mathcal{M}^{A^{\mathrm{op}}} := \mathbf{Cmd}(\overline{\mathcal{C}})\text{-}A^{\mathrm{op}} .$$

We end our observations with the next proposition, which in particular states that the above suspected monoidal products of Yetter-Drinfel'd modules are really Yetter-Drinfel'd modules.

Proposition 2.1.8 Let A be a Hopf algebra in a braided category $\mathcal{C}$. The category ${}_A\mathcal{YD}^A(\mathcal{C})$ admits two different structures of a braided monoidal category: let X and Y be Yetter-Drinfel'd modules over A.

1. The object $X \otimes Y$ becomes a Yetter-Drinfel'd module with diagonal action of A^{cop} and diagonal coaction of A. With this product of Yetter-Drinfel'd modules we get a braided monoidal category ${}_{A^{\mathrm{cop}}}\mathcal{YD}^A(\mathcal{C})$ with braiding isomorphisms (2.2).

2. The object $X \otimes Y$ becomes a Yetter-Drinfel'd module with diagonal action of A and diagonal coaction of A^{op}. With this product of Yetter-Drinfel'd modules we get a braided monoidal category ${}_A\mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$ with braiding isomorphisms (2.3).

Remark 2.1.9

1. The monoidal category ${}_{A^{\mathrm{cop}}}\mathcal{YD}^A(\mathcal{C})$ is not equal to the monoidal category ${}_A\mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$, nevertheless they are isomorphic.

2. For right-left Yetter-Drinfel'd modules one gets also two monoidal categories, which we will denote by ${}^{A^{\mathrm{op}}}\mathcal{YD}_A(\mathcal{C})$ and ${}^A\mathcal{YD}_{A^{\mathrm{cop}}}(\mathcal{C})$. These two categories come with strict monoidal forgetful functors

$$\mathcal{M}_A \longleftarrow {}^{A^{\mathrm{op}}}\mathcal{YD}_A(\mathcal{C}) \longrightarrow {}^{A^{\mathrm{op}}}\mathcal{M}$$
$$\mathcal{M}_{A^{\mathrm{cop}}} \longleftarrow {}^A\mathcal{YD}_{A^{\mathrm{cop}}}(\mathcal{C}) \longrightarrow {}^A\mathcal{M}$$

The category ${}^A\mathcal{YD}_{A^{\mathrm{cop}}}(\mathcal{C})$ is braided with braiding isomorphisms given by

$$c_{X,Y}^{\mathcal{YD}} := c_{Y,X}^{-1} \circ (\rho_X \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \delta_Y) . \tag{2.4}$$

The category ${}^{A^{\mathrm{op}}}\mathcal{YD}_A(\mathcal{C})$ is braided with braiding isomorphisms given by

$$c_{X,Y}^{\mathcal{YD}} := c_{Y,X}^{-1} \circ (\rho_X \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \delta_Y) . \tag{2.5}$$

2.2. Equivalences of Yetter-Drinfel'd categories

2.2.1. Left and right modules

In this subsection we discuss the side switch functor $\mathsf{T}\colon \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C})$ for Yetter-Drinfel'd modules. It turns out that, for our purposes, a non-trivial monoidal structure $\mathsf{T}_2\colon \mathsf{T}\otimes\mathsf{T} \to \mathsf{T}\circ\otimes$ has to be chosen for the switch functor, even in those cases (for $\mathcal{C}$ symmetric) where the identities provide a monoidal structure on T.

Lemma 2.2.1 The isomorphism ${}^A\mathsf{T}\colon \mathbf{Cmd}(\mathcal{C})\text{-}A \to \mathbf{Cmd}(\overline{\mathcal{C}})\text{-}A^{\mathrm{cop}}$ of categories from Remark 1.3.16 extends to an isomorphism of categories

$$ {}^A\mathsf{T}\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}). $$

The functor ${}^A\mathsf{T}$ is braided and strict monoidal, considered as a functor between the following monoidal categories:

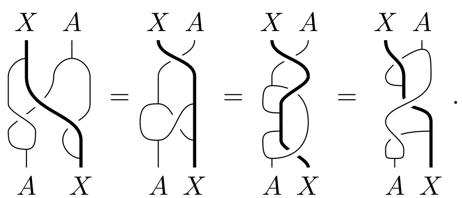
$$ {}^A\mathsf{T}\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}). $$

Remark 2.2.2 The equality $(A^{\mathrm{cop}})^{\mathrm{cop}} = A$ of Hopf algebras from Remark 1.3.9 might suggest the notation

$$ {}_A\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) := {}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) $$

which is not in conflict with other notation used in this thesis. To avoid confusion with the different monoidal category ${}_A\mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$, we refrain from using this notation.

Proof. Let $X = (X, \rho, \delta)$ be in ${}^A_A\mathcal{YD}(\mathcal{C})$. It follows from Remark 1.3.16 that ${}^A\mathsf{T}(X) = (X, \rho, c^{-1}_{X,A}\circ\delta)$ is an A^{cop}-comodule and A^{cop}-module in $\overline{\mathcal{C}}$. It remains to be shown that ${}^A\mathsf{T}(X)$ obeys the condition of a left-right A^{cop}-Yetter-Drinfel'd module in $\overline{\mathcal{C}}$:



One finally verifies that the braiding isomorphisms in the Yetter-Drinfel'd categories ${}^A_A\mathcal{YD}(\mathcal{C})$ and ${}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}})$ coincide as morphisms in the underlying category $\mathcal{C}$. $\square$

Remark 2.2.3 One can show by similar arguments that the isomorphisms ${}_A\mathsf{T}, \mathsf{T}^A$ and T_A extend to braided and strict monoidal functors

$$\begin{aligned}
{}_A\mathsf{T}\colon {}^A_A\mathcal{YD}(\mathcal{C}) &\to {}^{(A^{\mathrm{op}})^{\mathrm{op}}}\mathcal{YD}_{A^{\mathrm{op}}}(\overline{\mathcal{C}}),\\
\mathsf{T}^A\colon \mathcal{YD}^A_A(\mathcal{C}) &\to {}^{A^{\mathrm{cop}}}\mathcal{YD}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}(\overline{\mathcal{C}}),\\
\mathsf{T}_A\colon \mathcal{YD}^A_A(\mathcal{C}) &\to {}_{A^{\mathrm{op}}}\mathcal{YD}^{(A^{\mathrm{op}})^{\mathrm{op}}}(\overline{\mathcal{C}}).
\end{aligned}$$

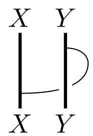
Theorem 2.2.4 Let A be a Hopf algebra in a braided category $\mathcal{C}$ and (X, ρ^r, δ^r) a right Yetter-Drinfel'd module over A. Consider

$$\mathsf{T}(X, \rho^r, \delta^r) = (X, \rho^- \circ (S^{-1} \otimes \mathrm{id}_X), (S \otimes \mathrm{id}_X) \circ \delta^+),$$

with $\rho^- := \rho^r \circ c^{-1}_{X,A}$ and $\delta^+ := c_{X,A} \circ \delta^r$. The functor

$$\mathsf{T} = (\mathsf{T}_A)^A\colon \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C})$$

has a monoidal structure $\mathsf{T}_2(X,Y)\colon \mathsf{T}(X) \otimes \mathsf{T}(Y) \to \mathsf{T}(X \otimes Y)$ given by

$$\mathsf{T}_2(X,Y) :=$$


$$= c^{\mathcal{YD}}_{Y,X} \circ c^{-1}_{Y,X}\ . \tag{2.6}$$

The monoidal functor $(\mathsf{T}, \mathsf{T}_2)$ is braided.

Proof. The functor $\mathsf{T}\colon \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C})$ is defined as the composition of the functors in the diagram

$$\begin{tikzcd}
\mathcal{YD}^A_A(\mathcal{C}) \arrow[r, dashed, "\mathsf{T}"] \arrow[d, "\mathsf{T}_A"'] & {}^A_A\mathcal{YD}(\mathcal{C}) \\
{}_{A^{\mathrm{op}}}\mathcal{YD}^{A^{\mathrm{op}}}(\overline{\mathcal{C}}) \arrow[r, "\mathsf{S}"] & {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) \arrow[u, "({}^A\mathsf{T})^{-1}"']
\end{tikzcd}$$

Here S denotes the functor of restriction along $S^{-1}\colon A^{\mathrm{cop}} \to A^{\mathrm{op}}$ and corestriction along $S\colon A^{\mathrm{op}} \to A^{\mathrm{cop}}$, thus T is a functor.
Since $c^{\mathcal{YD}}_{Y,X}$ is A-linear and A-colinear, we see from the right-hand side of (2.6), that the morphism $T_2(X,Y)$ is A-(co)linear, iff $c^{-1}_{Y,X}$ is A-(co)linear as a morphism $\mathsf{T}X \otimes \mathsf{T}Y \to \mathsf{T}(Y \otimes X)$; this is easily checked. Invertibility

of $\mathsf{T}_2(X,Y)$ is clear from (2.6), so it remains to check that T_2 is a monoidal structure, i.e. for all Yetter-Drinfel'd modules X, Y and Z we have

$$\begin{aligned}&\mathsf{T}_2(X\otimes Y, Z)\circ(\mathsf{T}_2(X,Y)\otimes \mathrm{id}_{\mathsf{T}(Z)})\\ =&\mathsf{T}_2(X, Y\otimes Z)\circ(\mathrm{id}_{\mathsf{T}(X)}\otimes \mathsf{T}_2(Y,Z))\ .\end{aligned}$$

This is a direct consequence of the Yetter-Drinfel'd condition. We conclude that $(\mathsf{T},\mathsf{T}_2)$ is a monoidal functor.
Finally we show that $(\mathsf{T},\mathsf{T}_2)$ is a braided monoidal functor, i.e. the equality

$$\mathsf{T}(c^{\mathcal{YD}}_{X,Y})\circ \mathsf{T}_2(X,Y) = \mathsf{T}_2(Y,X)\circ c^{\mathcal{YD}}_{\mathsf{T}X,\mathsf{T}Y}$$

holds. This is evident from drawing the morphisms. □

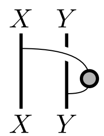
Remark 2.2.5 The functor $\mathsf{T} = (\mathsf{T}_A)^A$ in Theorem 2.2.4 is obtained by first turning the right action into a left action and then the right coaction into a left coaction, which shall be suggested by the notation $(\mathsf{T}_A)^A$.
There is another braided equivalence between the same braided categories of Yetter-Drinfel'd modules:

$$\mathsf{T}' = (\mathsf{T}^A)_A\colon \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C}).$$

The functor T' is given on objects by

$$\mathsf{T}'(X,\rho^r,\delta^r) = (X, \rho^+\circ(S\otimes \mathrm{id}_X), (S^{-1}\otimes\mathrm{id}_X)\circ\delta^-).$$

The monoidal structure $T_2'(X,Y)\colon \mathsf{T}'(X)\otimes\mathsf{T}'(Y)\to\mathsf{T}'(X\otimes Y)$ on T' is given by

$\mathsf{T}'_2(X,Y) :=$ 

$= \left(c^{\mathcal{YD}}_{X,Y}\right)^{-1}\circ c_{X,Y}\ .$

The two monoidal functors $\mathsf{T},\mathsf{T}'\colon \mathcal{YD}^A_A(\mathcal{C})\to {}^A_A\mathcal{YD}(\mathcal{C})$ are isomorphic as monoidal functors. An isomorphism $\mathsf{T}\to\mathsf{T}'$ is given by the family

$$\theta_{\mathsf{T}X} := \rho_{\mathsf{T}X}\circ(S\otimes\mathrm{id}_X)\circ\delta_{\mathsf{T}X} = \rho^r_X\circ(\mathrm{id}_X\otimes S)\circ\delta^r_X\ .$$

A right Yetter-Drinfel'd module version of Lemma 2.1.5 implies that θ is indeed a monoidal isomorphism.

2.2.2. Dually paired Hopf algebras

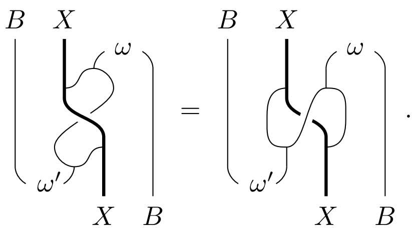
In this subsection, we prove that for Hopf algebras A and B that are related by a non-degenerate Hopf pairing, there is a braided monoidal equivalence between the categories ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}^{B}_{B}(\mathcal{C})$. This equivalence is a strict monoidal functor.

Lemma 2.2.6 Let $\omega\colon A \otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing with inverse copairing $\omega'\colon \mathbf{1} \to B \otimes A$. Then

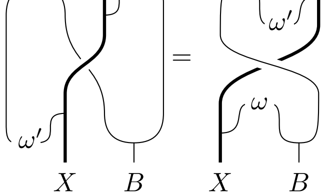
$$\begin{aligned} {}_{\omega'}\mathsf{D}^{\omega}\colon {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A}(\mathcal{C}) &\to {}^{B}\mathcal{YD}_{B^{\mathrm{cop}}}(\mathcal{C}) \\ (X,\rho,\delta) &\mapsto (X, (\mathrm{id}\otimes\omega)\circ(\delta\otimes\mathrm{id}), (\mathrm{id}\otimes\rho)\circ(\omega'\otimes\mathrm{id}))\ , \end{aligned}$$

is a strict monoidal braided functor. In particular, the two categories ${}_{A^{\mathrm{cop}}}\mathcal{YD}^{A}(\mathcal{C})$ and ${}^{B}\mathcal{YD}_{B^{\mathrm{cop}}}(\mathcal{C})$ are equivalent as braided monoidal categories.

Proof. Let (X,ρ,δ) be an A-Yetter-Drinfel'd module. From Lemma 1.3.19 it is clear that $\mathsf{D}(X,\rho,\delta)$ is a B-module and B-comodule. We have to check the Yetter-Drinfel'd condition. Since X is an A-Yetter-Drinfel'd module, we have the equality



Using that ω is a Hopf pairing, ω' is a Hopf copairing and $(\mathrm{id}_B\otimes\omega)\circ(\omega'\otimes\mathrm{id}_B) = \mathrm{id}_B$ we get the equality



which is the Yetter-Drinfel'd condition for the B-module and B-comodule structure on $\mathsf{D}(X)$. The functor D is strict monoidal, since the functors ${}_{\omega'}\mathsf{D}\colon A^{\mathrm{cop}}\text{-}\mathbf{Mod}(\mathcal{C}) \to B\text{-}\mathbf{Cmd}(\mathcal{C})$ and $\mathsf{D}^{\omega}\colon \mathbf{Cmd}(\mathcal{C})\text{-}A \to \mathbf{Mod}(\mathcal{C})\text{-}B^{\mathrm{cop}}$ are strict monoidal. Finally, the braiding is preserved

$$c^{\mathcal{YD}}_{\mathsf{D}(X),\mathsf{D}(Y)} = c^{\mathcal{YD}}_{X,Y} =: \mathsf{D}(c^{\mathcal{YD}}_{X,Y}) \ .$$

This follows from $(\omega \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \omega') = \mathrm{id}_A$. □

Corollary 2.2.7 Let $\omega\colon A \otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing with inverse copairing $\omega'\colon \mathbf{1} \to B \otimes A$. Then

$$\begin{aligned} {}^{\omega}_{\omega'}\mathsf{D}\colon {}^{A}_{A}\mathcal{YD}(\mathcal{C}) &\to \mathcal{YD}^{B}_{B}(\mathcal{C}) \\ (X, \rho_X, \delta_X) &\mapsto (X, \rho_{\mathsf{D}(X)}, \delta_{\mathsf{D}(Y)}) \end{aligned}$$

with

$$\begin{aligned} \rho_{\mathsf{D}(X)} &= (\mathrm{id} \otimes \omega) \circ (c^{-1}_{X,A} \otimes S^{-1}) \circ (\delta \otimes \mathrm{id}) \\ \delta_{\mathsf{D}(X)} &= c_{B,X} \circ (S \otimes \rho) \circ (\omega' \otimes \mathrm{id}) \end{aligned}$$

defines a strict monoidal braided functor.

In particular, the categories ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}^{B}_{B}(\mathcal{C})$ are equivalent as braided categories.

Proof. Note that $\omega\colon A \otimes B \to \mathbf{1}$ is a Hopf pairing of the two Hopf algebras A^{cop} and B^{op} in $\overline{\mathcal{C}}$. So we have the following composite of braided, strict monoidal functors

$$\begin{tikzcd} {}^{A}_{A}\mathcal{YD}(\mathcal{C}) \arrow[rr, dashed] \arrow[d, "{}^{A}\mathsf{T}"] & & \mathcal{YD}^{(B^{\mathrm{op}})^{\mathrm{cop}}}_{(B^{\mathrm{op}})^{\mathrm{cop}}}(\mathcal{C}) \arrow[r, "\mathsf{S}"] & \mathcal{YD}^{B}_{B}(\mathcal{C}) \\ {}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) \arrow[rr, "{}_{\omega'}\mathsf{D}^{\omega}"] & & {}^{(B^{\mathrm{op}})^{\mathrm{cop}}}\mathcal{YD}_{B^{\mathrm{op}}}(\overline{\mathcal{C}}) \arrow[u, "{}^{B^{\mathrm{op}}}\mathsf{T}"'] & \end{tikzcd}$$

Here S denotes the functor of restriction along $S^{-1}\colon B \to (B^{\mathrm{op}})^{\mathrm{cop}}$ and corestriction along $S\colon (B^{\mathrm{op}})^{\mathrm{cop}} \to B$. The top line of the above diagram is the functor ${}^{\omega}_{\omega'}\mathsf{D}$. □

Combining Theorem 2.2.4, Remark 2.2.5 and Corollary 2.2.7, we are now in a position to exhibit explicitly two braided equivalences

$$\Omega, \Omega'\colon {}^{A}_{A}\mathcal{YD}(\mathcal{C}) \to {}^{B}_{B}\mathcal{YD}(\mathcal{C}).$$

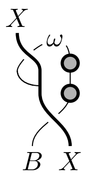
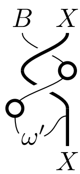
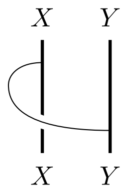
The first functor is the composition $\Omega := \mathsf{T} \circ \mathsf{D}$ with monoidal structure

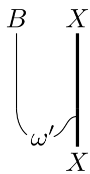
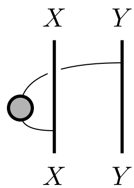
$$\begin{aligned}\Omega_2(X,Y) &= \mathsf{T}(\mathsf{D}_2(X,Y)) \circ \mathsf{T}_2(\mathsf{D}X,\mathsf{D}Y) = \mathrm{id}_{X\otimes Y} \circ (c^{\mathcal{YD}}_{\mathsf{D}Y,\mathsf{D}X} \circ c^{-1}_{Y,X}) \\ &= c^{\mathcal{YD}}_{Y,X} \circ c^{-1}_{Y,X}.\end{aligned}$$

The second to last equal sign uses that D is a strict braided functor. The other functor is $\Omega' := \mathsf{T}' \circ \mathsf{D}$ with monoidal structure

$$\Omega'_2(X,Y) = \mathsf{T}'(\mathsf{D}_2(X,Y)) \circ \mathsf{T}'_2(\mathsf{D}X,\mathsf{D}Y) = \left(c^{\mathcal{YD}}_{X,Y}\right)^{-1} \circ c_{X,Y}.$$

Graphically the functors and the monoidal structures are

$\Omega(X,\rho_X,\delta_X) = \Bigg(X,$  ,  $\Bigg)$, $\quad \Omega_2(X,Y) =$  ,

$\Omega'(X,\rho_X,\delta_X) = \Bigg(X,$  ,  $\Bigg)$, $\quad \Omega'_2(X,Y) =$  .

We summarize our findings:

Theorem 2.2.8 Let $\omega\colon A\otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing. The categories ${}^A_A\mathcal{YD}(\mathcal{C})$ and ${}^B_B\mathcal{YD}(\mathcal{C})$ are braided equivalent via the monoidal functors Ω and Ω' above.

Remark 2.2.9 We end this subsection by relating the equivalence Ω to the equivalence Ω^{HS} of rational modules over $\Bbbk$-Hopf algebras discussed in [HS13].

1. Let $\Bbbk$ be a field and $\mathcal{L}_\Bbbk$ the category of linearly topologized vector spaces over $\Bbbk$. Fix a Hopf algebra E in $\mathcal{L}_\Bbbk$ and two Hopf algebras $(R, R^\vee)$ in ${}^E_E\mathcal{YD}(\mathcal{L}_\Bbbk)$ that are related by a non-degenerate Hopf pairing. It is then shown in [HS13] that the categories ${}^{R\rtimes E}_{R\rtimes E}\mathcal{YD}_{\mathrm{rat}}$ and ${}^{R^\vee\rtimes E}_{R^\vee\rtimes E}\mathcal{YD}_{\mathrm{rat}}$ are equivalent as braided categories. Here, the subscript rat denotes the subcategory of rational modules.

The non-degenerate pairing $\langle , \rangle \colon R^\vee \otimes R \to \Bbbk$ and the structural morphisms of the bosonized Hopf algebra $R \rtimes E$ are used in [HS13, Theorem 7.1] to construct a functor

$$(\Omega^{\mathrm{HS}}, \Omega_2^{\mathrm{HS}}) \colon {}^{R\rtimes E}_{R\rtimes E}\mathcal{YD}_{\mathrm{rat}} \to {}^{R^\vee \rtimes E}_{R^\vee \rtimes E}\mathcal{YD}_{\mathrm{rat}} \ .$$

In detail, the functor Ω^{HS} is constructed as follows: let M be a rational $(R \rtimes E)$-Yetter-Drinfel'd module and denote the left R-coaction by $\delta(m) = m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}$.
The $(R^\vee \rtimes E)$-Yetter-Drinfel'd module $\Omega^{\mathrm{HS}}(M)$ is equal to M as an E-Yetter-Drinfel'd module and has the following $R^\vee$-Yetter-Drinfel'd structure

$$\begin{aligned} &\text{action:} && \xi m = \langle \xi, m_{\langle -1 \rangle} \rangle m_{\langle 0 \rangle} \\ &\text{coaction:} && \delta_{\Omega^{\mathrm{HS}}(M)} = \left(c^{\mathcal{YD}}_{M,R^\vee} \circ c^{\mathcal{YD}}_{R^\vee,M} \right) (m_{[-1]} \otimes m_{[0]}), \end{aligned}$$

where $m_{[-1]} \otimes m_{[0]}$ is the unique element of $R^\vee \otimes M$ such that for all $r \in R$ and $m \in M$ we have

$$rm = \left\langle m_{[-1]}, \theta_R(r) \right\rangle m_{[0]}.$$

The monoidal structure of Ω^{HS} is given by the family of morphisms

$$\begin{aligned} \Omega_2^{\mathrm{HS}}(M,N) \colon \Omega^{\mathrm{HS}}(M) \otimes \Omega^{\mathrm{HS}}(N) &\to \Omega^{\mathrm{HS}}(M \otimes N) \\ m \otimes n &\mapsto S^{-1}_{R\rtimes E} S_R(n_{\langle -1 \rangle}) m \otimes n_{\langle 0 \rangle}. \end{aligned}$$

2. In this thesis, we started with a non-degenerate Hopf pairing $\omega \colon A \otimes B \to \mathbf{1}$ and constructed an equivalence

$$\Omega^\omega \colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C}).$$

Let $\mathcal{C}$ be the category of finite dimensional Yetter-Drinfel'd modules over the finite dimensional Hopf algebra E. Set $A = R$ and $B = R^\vee$ and $\omega \colon A \otimes B \to \Bbbk$, such that $\omega^-(b \otimes a) = \langle b, a \rangle$, cf. Example 1.3.20. One can show by straight-forward computations, that our functor Ω^{ω^-} coincides with the functor Ω^{HS} on the full subcategory ${}^{R\rtimes E}_{R\rtimes E}\mathcal{YD}_{\mathrm{fin}} \subset {}^{R\rtimes E}_{R\rtimes E}\mathcal{YD}_{\mathrm{rat}}$ of finite dimensional $(R \rtimes E)$-Yetter-Drinfel'd modules.

2.2.3. The square of Ω

From a non-degenerate Hopf pairing $\omega\colon A\otimes B\to \mathbf{1}$, we obtained an equivalence $\Omega^\omega\colon {}^A_A\mathcal{YD}(\mathcal{C})\to {}^B_B\mathcal{YD}(\mathcal{C})$. As noted in Example 1.3.20, we also have a non-degenerate Hopf pairing $\omega^-\colon B\otimes A\to \mathbf{1}$ from which we obtain an equivalence $\Omega^{\omega^-}\colon {}^B_B\mathcal{YD}(\mathcal{C})\to {}^A_A\mathcal{YD}(\mathcal{C})$.

Proposition 2.2.10 The braided monoidal functor

$$\Omega^{\omega^-}\circ\Omega^\omega\colon {}^A_A\mathcal{YD}(\mathcal{C})\to {}^A_A\mathcal{YD}(\mathcal{C})$$

is isomorphic to the identity functor.

Proof. A direct computation shows that the monoidal functors

$$(\Omega^\omega,\Omega^\omega_2)\circ\left((\Omega')^{\omega^-},(\Omega')^{\omega^-}_2\right)\quad\text{and}\quad\left((\Omega')^{\omega^-},(\Omega')^{\omega^-}_2\right)\circ(\Omega^\omega,\Omega^\omega_2)$$

are both equal to the identity functor with identity monoidal structure. Remark 2.2.5 implies that $(\Omega')^{\omega^-}$ is monoidally isomorphic to Ω^{ω^-}.

Alternatively, a concrete calculation shows that $\Omega^{\omega^-}\circ\Omega^\omega$ is equal to the monoidal functor that sends the Yetter-Drinfel'd module (X,ρ,δ) to the Yetter-Drinfel'd module

$$(X,\rho\circ(S^{-2}\otimes\mathrm{id}_X)\circ c^{-1}_{A,X}\circ c^{-1}_{X,A},c_{X,A}\circ c_{A,X}\circ(S^2\otimes\mathrm{id}_X)\circ\delta).$$

The monoidal structure of $\Omega^{\omega^-}\circ\Omega^\omega$ is given by the family of isomorphisms

$$c^{\mathcal{YD}}_{Y,X}\circ c^{\mathcal{YD}}_{X,Y}\circ c^{-1}_{X,Y}\circ c^{-1}_{Y,X}.$$

From this and Lemma 2.1.5 it is clear that $\theta_X:=\rho_X\circ(S\otimes\mathrm{id}_X)\circ\delta_X$ defines a monoidal isomorphism

$$\theta\colon\Omega^{\omega^-}\circ\Omega^\omega\to\mathrm{Id}.$$

□

3. Graded categories and group actions

3.1. Graded categories and (co)graded bialgebras

An easy example of a monoidal category we would like to consider as graded over a group Γ is given by the category Γ-$\mathbf{Vect}_{\Bbbk}$ of Γ-graded $\Bbbk$-vector spaces. An object of Γ-$\mathbf{Vect}$ is a vector space X together with a family $\{X_\alpha \subset X\}_{\alpha\in\Gamma}$ of vector subspaces fulfilling $X = \bigoplus_{\alpha\in\Gamma} X_\alpha$.
Let X and Y be two Γ-graded vector spaces. A morphisms of graded vector spaces is a $\Bbbk$-linear map $f\colon X \to Y$ preserving the degree, i.e. $f(X_\alpha) \subset Y_\alpha$. The vector space $X \otimes Y$ becomes graded vector space by defining

$$(X \otimes Y)_\alpha := \bigoplus_{\beta\gamma=\alpha} X_\beta \otimes Y_\gamma$$

for all $\alpha \in \Gamma$. Abstracting this example leads us to

Definition 3.1.1 Let Γ be group, let $\mathcal{C}$ be a monoidal category and let $\{\mathcal{C}_\alpha \subset \mathcal{C}\}_{\alpha\in\Gamma}$ be a family of full subcategories with $\mathcal{C} = \coprod_{\alpha\in\Gamma} \mathcal{C}_\alpha$.

1. The family of subcategories is called a Γ-*grading*, if for any $X \in \mathcal{O}b(\mathcal{C}_\alpha)$ and $Y \in \mathcal{O}b(\mathcal{C}_\beta)$ we have $X \otimes Y \in \mathcal{O}b(\mathcal{C}_{\alpha\beta})$.

2. The subcategory $\mathcal{C}_\alpha$ is called the α-*component* resp. the neutral component for $\alpha = 1$. An object X in $\mathcal{C}_\alpha$ is called *homogeneous* of degree α. We also denote $\alpha =: |X|$.

3. The *support of* $\mathcal{C}$ is the set $\operatorname{supp}\mathcal{C} = \{\alpha \in \Gamma \mid \mathcal{C}_\alpha \neq \emptyset\}$. The grading is called *trivial*, if $\operatorname{supp}\mathcal{C} = 1$; it is called *full*, if $\operatorname{supp}\mathcal{C} = \Gamma$.

4. If $\mathcal{C}$ is $\Bbbk$-linear we require $\mathcal{C} = \bigoplus_{\alpha\in\Gamma} \mathcal{C}_\alpha$ instead of $\mathcal{C} = \coprod_{\alpha\in\Gamma} \mathcal{C}_\alpha$.

One easily sees that the unit object $\mathbf{1}$ of a Γ-graded monoidal category $\mathcal{C}$ is homogeneous of degree $1 \in \Gamma$. If $\mathcal{C}$ is rigid and X a homogeneous object of degree $\alpha \in \Gamma$, then ${}^\vee X$ and $X^\vee$ are homogeneous of degree α^{-1}. This and compatibility of $\otimes$ and group multiplication shows that $\operatorname{supp}(\mathcal{C}) \subset \Gamma$ is a subgroup, if $\mathcal{C}$ is left or right rigid.

Algebraic examples of graded categories arise as the categories of modules over a *Hopf* Γ*-coalgebra* or comodules over a *Hopf* Γ*-algebra*. These algebraic structures were introduced by Turaev in [Tur00]; an understanding in terms of bialgebras in certain monoidal categories is presented in [CDL06].

Definition 3.1.2 Let $\Bbbk$ be a field and Γ a group. A *semi-Hopf* Γ*-coalgebra* consists of a family $A = \{A_\alpha\}_{\alpha\in\Gamma}$ of unital, associative $\Bbbk$-algebras, a family $\Delta = \{\Delta_{\alpha,\beta} \colon A_{\alpha\beta} \to A_\alpha \otimes A_\beta\}$ of unital algebra homomorphisms and a unital algebra homomorphism $\varepsilon = \varepsilon_1 \colon A_1 \to \Bbbk$ fulfilling

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}) \circ \Delta_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma} \tag{3.1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta_{1,\alpha} = \mathrm{id}_{A_\alpha} = (\mathrm{id} \otimes \varepsilon) \circ \Delta_{\alpha,1} \tag{3.2}$$

for all $\alpha, \beta, \gamma \in \Gamma$. We call Δ the *comultiplication* and ε the *counit* of A. A semi-Hopf Γ-coalgebra is called a Hopf Γ-coalgebra, if it has an antipode, i.e. a family $S = \{S_\alpha \colon A_\alpha \to A_{\alpha^{-1}}\}_{\alpha\in\Gamma}$ of $\Bbbk$-linear maps obeying

$$\mu_{\alpha^{-1}} \circ (S_\alpha \otimes \mathrm{id}) \circ \Delta_{\alpha,\alpha^{-1}} = \eta_{\alpha^{-1}} \circ \varepsilon = \mu_{\alpha^{-1}} \circ (\mathrm{id} \otimes S_\alpha) \circ \Delta_{\alpha^{-1},\alpha} \, . \tag{3.3}$$

Remark 3.1.3

1. To every semi-Hopf Γ-coalgebra, we can associate the algebra

   $$\tilde{A} := \bigoplus_{\alpha\in\Gamma} A_\alpha$$

   which is unital, iff $\operatorname{supp}(A) = \{\alpha \in \Gamma \mid A_\alpha \neq 0\}$ is finite. The algebra $\tilde{A}$ becomes a bialgebra with comultiplication

   $$\tilde{\Delta} := \sum_{\alpha\in\Gamma} \Delta_\alpha$$

   where $\Delta_\alpha := \sum_{\beta\gamma=\alpha} \Delta_{\beta,\gamma}$, the counit of $\tilde{A}$ is ε_1. If A is even a Hopf Γ-coalgebra the linear map $\tilde{S} := \sum_{\alpha\in\Gamma} S_\alpha$ is an antipode.
   Note that the bialgebra $\tilde{A}$ comes with unital algebra homomorphisms $\pi_\alpha \colon \tilde{A} \to A_\alpha$ and that the inclusion $\iota_\alpha \colon A_\alpha \to \tilde{A}$ is a unital algebra homomorphism, iff $\alpha = 1$ and $\operatorname{supp}(A) = 1$. Further the underlying coalgebra of $\tilde{A}$ is Γ-graded, i.e.

   $$\tilde{\Delta}(A_\alpha) \subset \bigoplus_{\beta\gamma=\alpha} A_\beta \otimes A_\gamma \, . \tag{3.4}$$

2. Conversely, given a bialgebra $\tilde{A}$ that is Γ-*cograded*, i.e. there are unital algebras A_α such that $\tilde{A} = \bigoplus_{\alpha \in \Gamma} A_\alpha$ as unital algebra (this implies $|\Gamma| < \infty$) and the comultiplication $\tilde{\Delta}\colon \tilde{A} \to \tilde{A} \otimes \tilde{A}$ fulfills condition (3.4).
One can show that the restriction of the counit $\varepsilon\colon \tilde{A} \to \Bbbk$ to A_α is zero for $\alpha \neq 1$ and thus the the family of $A = \{A_\alpha\}_{\alpha \in \Gamma}$ is a semi-Hopf Γ-coalgebra with comultiplication given by the maps $(\pi_\alpha \otimes \pi_\beta) \circ \tilde{\Delta}_{|A_{\alpha\beta}}$ and counit given by $\varepsilon = \varepsilon_{|A_1}$.
If $\tilde{A}$ is even a Hopf algebra the associated semi-Hopf Γ-coalgebra is of course a Hopf Γ-coalgebra.

Now let Γ be a finite group and consider the (commutative) Hopf algebra $\Bbbk^\Gamma$ of $\Bbbk$-valued functions on Γ. If A is Γ-cograded Hopf algebra with $\mathrm{supp}(A) = \Gamma$, we can see $\Bbbk^\Gamma$ as a Hopf subalgebra of A: the unit of A is of the form $1 = \sum_\alpha 1_\alpha$, denote by $\{e_\alpha\}_{\alpha \in \Gamma}$ the standard basis of $\Bbbk^\Gamma$ and identify 1_α with e_α to obtain an injective bialgebra homomorphism $\iota\colon \Bbbk^\Gamma \to A$.
In the following we will always assume that Γ is a finite group, so we can identify a semi-Hopf Γ-coalgebra A with its associated Γ-cograded bialgebra $\tilde{A}$.
This identification helps to see that the category A-**Mod** of (say left) A-modules of a Γ-cograded bialgebra is a Γ-graded category: since A is the direct sum of of the A_α we have

$$A\text{-}\mathbf{Mod} = \bigoplus_{\alpha \in \Gamma} (A_\alpha\text{-}\mathbf{Mod})$$

as abelian categories. The tensor product $X \otimes Y$ of an A_α-module X and an A_β-module Y becomes a $A_{\alpha\beta}$-module by pulling back the $A_\alpha \otimes A_\beta$-module structure along the unital algebra homomorphism $\Delta_{\alpha,\beta}$.
If A is even a Hopf algebra, the category A-**mod** of finite dimensional A-modules becomes a rigid category: given a left A_α-module, the dual space $X^* = \mathrm{Hom}(X, \Bbbk)$ is a right A_α and becomes a left $A_{\alpha^{-1}}$-module by the antipode $S_{\alpha^{-1}}\colon A_{\alpha^{-1}} \to A_\alpha$, cf. Lemma 1.3.11.

3.2. Weak group actions

3.2.1. Definition

Let Γ be a group and denote by $\underline{\Gamma}$ the discrete monoidal category with the elements of Γ as objects and the multiplication of Γ as tensor product.

Further denote for a monoidal category $\mathcal{C}$ by $\mathcal{A}\mathrm{ut}_\otimes(\mathcal{C})$ the (strict) monoidal category of monoidal autoequivalences and monoidal isomorphisms with tensor product given by the composition of monoidal functors.

Definition 3.2.1 A *(weak) action of a group* Γ *on a monoidal category* $\mathcal{C}$ is defined to be a monoidal functor

$$\Phi\colon \underline{\Gamma} \to \mathcal{A}\mathrm{ut}_\otimes(\mathcal{C}) \;.$$

An action Φ is called *strict*, if Φ is a strict monoidal functor. For a braided category $\mathcal{C}$ a weak action Φ of Γ on $\mathcal{C}$ is called *braided*, if each Φ^α is a braided monoidal functor.

Remark 3.2.2 We unravel the definition of a weak action: every group element $\alpha \in \Gamma$ defines a monoidal autoequivalence $(\Phi^\alpha, \Phi_2^\alpha)\colon \mathcal{C} \to \mathcal{C}$, we will also write ${}^\alpha X$ instead of $\Phi^\alpha(X)$.
Since $\Phi\colon \underline{\Gamma} \to \mathcal{A}\mathrm{ut}_\otimes(\mathcal{C})$ is a monoidal functor itself, we have monoidal isomorphisms $\Phi_{\alpha,\beta}\colon \Phi^\alpha \circ \Phi^\beta \to \Phi^{\alpha\beta}$. That $\Phi_{\alpha,\beta}$ is a monoidal isomorphism means that for all objects X and Y in $\mathcal{C}$ the equality

$$\Phi_2^{\alpha\beta}(X,Y) \circ (\Phi_{\alpha,\beta,X} \otimes \Phi_{\alpha,\beta,Y}) = \Phi_{\alpha,\beta,X\otimes Y} \circ (\Phi^\alpha \circ \Phi^\beta)_2(X,Y)$$

holds, where $(\Phi^\alpha \circ \Phi^\beta)_2(X,Y)$ is the composition

$${}^\alpha(\Phi_2^\beta(X,Y)) \circ \Phi_2^\alpha({}^\beta X, {}^\beta Y) \;.$$

Moreover, the isomorphisms $\Phi_{\alpha,\beta}$ are compatible in the way that the diagram

$$\begin{tikzcd}
{}^\alpha({}^\beta({}^\gamma X)) \arrow[r, "{}^\alpha(\Phi_{\beta,\gamma,X})"] \arrow[d, "\Phi_{\alpha,\beta,{}^\gamma X}"'] & {}^\alpha({}^{\beta\gamma} X) \arrow[d, "\Phi_{\alpha,\beta\gamma,X}"] \\
{}^{\alpha\beta}({}^\gamma X) \arrow[r, "\Phi_{\alpha\beta,\gamma,X}"] & {}^{\alpha\beta\gamma} X
\end{tikzcd} \tag{3.5}$$

commutes for all objects X in $\mathcal{C}$ and all elements $\alpha, \beta, \gamma \in \Gamma$.

The above considerations show that we can weaken the definition of a (weak) action by saying that Φ is a strong monoidal functor with target $\mathcal{E}\mathrm{nd}_\otimes(\mathcal{C})$: by the monoidal isomorphisms $\Phi_{\alpha,\alpha^{-1}}$ and $\Phi_{\alpha^{-1},\alpha}$ the compositions $\Phi^\alpha \circ \Phi^{\alpha^{-1}}$ and $\Phi^{\alpha^{-1}} \circ \Phi^\alpha$ are both isomorphic to the functor Φ^1 which is isomorphic to $\mathrm{Id}_\mathcal{C}$. Thus every Φ^α is an autoequivalence of $\mathcal{C}$. More precisely, we have the following lemma.

Lemma 3.2.3 Let Φ be a weak action of Γ on $\mathcal{C}$. If Φ^1 is the identity functor $\mathrm{Id}_{\mathcal{C}}$, then, for all $\alpha \in \Gamma$, the 4-tuples $(\Phi^\alpha, \Phi^{\alpha^{-1}}, \eta_\alpha, \varepsilon_\alpha)$ are monoidal adjunctions with unit $\eta_\alpha = \Phi^{-1}_{\alpha^{-1},\alpha} \colon \mathrm{Id} \to \Phi^{\alpha^{-1}} \circ \Phi^\alpha$ and counit $\varepsilon_\alpha = \Phi_{\alpha,\alpha^{-1}} \colon \Phi^\alpha \circ \Phi^{\alpha^{-1}} \to \mathrm{Id}$.

3.2.2. The orbifold category

Definition 3.2.4 Let $\mathcal{C}$ be a $\Bbbk$-linear monoidal category with a weak action of a group Γ. The *orbifold category* or *category of fixed objects* of $\mathcal{C}$ is the category $(\mathrm{orb}\,\mathcal{C})^\Gamma$ given as follows: objects are pairs (X, u_X) consisting of an object in X in $\mathcal{C}$ with ${}^\alpha X \cong X$ for all $\alpha \in \Gamma$ and a family $u_X = \{u_X^\alpha \colon {}^\alpha X \xrightarrow{\cong} X\}_{\alpha\in\Gamma}$ of isomorphisms such that

$$u_X^{\alpha\beta} \circ \Phi_{\alpha,\beta,X} = u_X^\alpha \circ {}^\alpha(u_X^\beta) \quad \text{for all } \alpha, \beta \in \Gamma. \tag{3.6}$$

A morphism $(X, u_X) \xrightarrow{f} (Y, u_Y)$ in $(\mathrm{orb}\,\mathcal{C})^\Gamma$ is a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ obeying

$$f \circ u_X^\alpha = u_Y^\alpha \circ {}^\alpha f \quad \text{for all } \alpha \in \Gamma. \tag{3.7}$$

Remark 3.2.5 The category $(\mathrm{orb}\,\mathcal{C})^\Gamma$ is also called equivariantization [DGNO10]. The category $(\mathrm{orb}\,\mathcal{C})^\Gamma$ is monoidal with tensor product given on objects by

$$(X, \{u_X^\alpha\}) \otimes (Y, \{u_Y^\alpha\}) = (X \otimes Y, \{u_{X\otimes Y}^\alpha\})$$

with $u_{X\otimes Y}^\alpha := (u_X^\alpha \otimes u_Y^\alpha) \circ (\Phi_2^\alpha)^{-1}(X,Y)$. If $\mathcal{C}$ is a fusion category and carries the structure of a Γ-braided category as defined in Definition 3.3.1, then $(\mathrm{orb}\,\mathcal{C})^\Gamma$ is a braided fusion category containing the symmetric category $\Bbbk[\Gamma]$-**mod** as braided, full subcategory, cf. [DGNO10, Section 4.2.2].

3.2.3. Weak actions on Hopf algebras

The content of this subsection follows [Dav07]. Let A and B be bialgebras over a field $\Bbbk$ and $\varphi \colon A \to B$ a homomorphism of the underlying algebras. We know from Lemma 1.2.6 that there is a pull-back functor $\varphi^* \colon B$-**Mod** $\to A$-**Mod**; if φ is even a bialgebra homomorphism, the functor φ^* is strict monoidal. We also want to describe non-strict monoidal structures on a pull-back functor φ^* for an algebra homomorphism $\varphi \colon A \to B$.

Definition 3.2.6 Let A and B be bialgebras over a field $\Bbbk$, $\varphi\colon A \to B$ a homomorphism of algebras and $F \in B \otimes B$, such that the following holds

$$F \cdot (\varphi \otimes \varphi)(\Delta(x)) = \Delta(\varphi(x)) \cdot F \in B \otimes B \quad \text{for all } x \in A, \tag{3.8}$$
$$(\Delta \otimes \mathrm{id})(F) \cdot (F \otimes 1_B) = (\mathrm{id} \otimes \Delta)(F) \cdot (1_B \otimes F) \ \in B \otimes B \otimes B, \tag{3.9}$$
$$(\varepsilon_B \otimes \mathrm{id})(F) = 1_B = (\mathrm{id} \otimes \varepsilon_B)(F)\ , \tag{3.10}$$
$$\varepsilon_B \circ \varphi = \varepsilon_A\ . \tag{3.11}$$

The pair (φ, F) is called a *twisted homomorphism of bialgebras.*
Let $(\psi, G)\colon B \to C$ be another twisted homomorphism of bialgebras. We define their composition $(\psi, G) \circ (\varphi, F)$ as the pair

$$(\psi \circ \varphi, G \cdot (\psi \otimes \psi)(F))\,. \tag{3.12}$$

Remark 3.2.7 Let $\varphi\colon A \to B$ be an algebra isomorphism and $F \in B \otimes B$ invertible, such that (φ, F) is a twisted bialgebra homomorphism. Note that in general (φ^{-1}, F^{-1}) is not a twisted bialgebra homomorphism, even if $A = B$. In fact F^{-1} fulfills

$$(F^{-1} \otimes 1) \cdot (\Delta \otimes \mathrm{id})(F^{-1}) = (1 \otimes F^{-1}) \cdot (\mathrm{id} \otimes \Delta)(F^{-1})$$

rather than (3.9). One could call (φ^{-1}, F^{-1}) a quasi-twisted bialgebra homomorphism. In [Dav07] the above notions of twisted bialgebra homomorphism and quasi-twisted bialgebra homomorphism are the other way around.

Remark 3.2.8 It is common to denote an element $F \in B \otimes B$ by the sum $\sum_{(F)} F^{(1)} \otimes F^{(2)}$ or just by $F^{(1)} \otimes F^{(2)}$ omitting the sum symbol. With this notation condition (3.9) reads

$$\Delta(F^{(1)}) \cdot F \otimes F^{(2)} = F^{(1)} \otimes \Delta(F^{(2)}) \cdot F\ .$$

To deal with several copies of F in one equation, we will introduce ad hoc notations like $F = F^{(1)} \otimes F^{(2)} = f^{(1)} \otimes f^{(2)}$. Combining this together with the Sweedler notation for the comultiplication, we can express (3.9) by

$$\begin{aligned} &(F^{(1)})_{(1)} \cdot f^{(1)} \otimes (F^{(1)})_{(2)} \cdot f^{(2)} \otimes F^{(2)} \\ =&F^{(1)} \otimes (F^{(2)})_{(1)} f^{(1)} \otimes (F^{(2)})_{(2)} f^{(2)}\ . \end{aligned}$$

Proposition 3.2.9 Let $(\varphi, F)\colon A \to B$ and $(\psi, G)\colon B \to C$ be twisted bialgebra homomorphisms.

1. The composition $(\psi, G) \circ (\varphi, F)$ is a twisted bialgebra homomorphism $A \to C$.

2. The twisted homomorphism (φ, F) defines a lax monoidal functor $(\Phi, \Phi_2) = (\varphi, F)^* \colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$. The functor Φ is given by the restriction functor $\Phi := \varphi^* \colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ and the lax monoidal structure Φ_2 is given by the following family of A-linear maps

$$\begin{aligned}\Phi_2(X, Y) \colon \Phi(X) \otimes \Phi(Y) &\to \Phi(X \otimes Y) \\ x \otimes y \qquad &\mapsto F.(x \otimes y) \ .\end{aligned}$$

3. We have the following equality of lax monoidal functors $C\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$

$$((\psi, G) \circ (\varphi, F))^* = (\varphi, F)^* \circ (\psi, G)^* \ .$$

Proof.

1. It is clear that $\psi \circ \phi$ is a unital algebra homomorphism preserving the counit. That $G \cdot (\psi \otimes \psi)(F)$ fulfills (3.10) follows since ε is an algebra homomorphism and ψ preserves the counit.
   We show that $\psi \circ \varphi$ and $G \cdot (\psi \otimes \psi)(F)$ fulfill (3.8)

$$\begin{aligned}&G^{(1)} \cdot \psi(F^{(1)}) \cdot (\psi\varphi)(x_{(1)}) \otimes G^{(2)} \cdot \psi(F^{(2)}) \cdot (\psi\varphi)(x_{(2)}) \\ =&G^{(1)} \cdot \psi(\varphi(x)_{(1)} \cdot F^{(1)}) \otimes G^{(2)} \cdot \psi(\varphi(x)_{(2)} \cdot F^{(2)}) \\ =&(\psi\varphi)(x)_{(1)} \cdot G^{(1)} \cdot \psi(F^{(1)}) \otimes (\psi\varphi)(x)_{(2)} \cdot G^{(2)} \cdot \psi(F^{(2)}) \ .\end{aligned}$$

   We used that ψ is an algebra homomorphism in both steps. Further we used (3.8) for F in the first step and for G in the second.
   The last thing to show is that $G \cdot (\psi \otimes \psi)(F)$ obeys (3.9).
   Let $F = F^{(1)} \otimes F^{(2)} = f^{(1)} \otimes f^{(2)}$ and $G = G^{(1)} \otimes G^{(2)} = g^{(1)} \otimes g^{(2)}$,

then we have

$$\begin{aligned}
&(G^{(1)}\psi(F^{(1)}))_{(1)}g^{(1)}\psi(f^{(1)}) \otimes \\
&(G^{(1)}\psi(F^{(1)}))_{(2)}g^{(2)}\psi(f^{(2)}) \ \otimes G^{(2)}\psi(F^{(2)}) \\
=\ &(G^{(1)})_{(1)}g^{(1)}\psi((F^{(1)})_{(1)}f^{(1)})\otimes \\
&(G^{(1)})_{(2)}g^{(2)}\psi((F^{(2)})_{(1)}f^{(2)}) \otimes G^{(2)}\psi(F^{(2)}) \\
=\ &G^{(1)}\psi(F^{(1)}) && \otimes (G^{(2)})_{(1)}g^{(1)}\psi((F^{(2)})_{(1)}f^{(1)}) \\
& && \otimes (G^{(2)})_{(2)}g^{(2)}\psi((F^{(2)})_{(2)}f^{(2)}) \\
=\ &G^{(1)}\psi(F^{(1)}) && \otimes (G^{(2)}\psi(F^{(2)}))_{(1)}g^{(1)}\psi(f^{(1)}) \\
& && \otimes (G^{(2)}\psi(F^{(2)}))_{(2)}g^{(2)}\psi(f^{(2)}) \ .
\end{aligned}$$

Here the first and last equal sign follow from Δ and ψ being algebra homomorphisms in combination with (3.8) for (ψ, G). The second equal sign uses (3.9) for (φ, F) and (ψ, G).

2. We know already from Lemma 1.2.6 that $\Phi = \varphi^*$ is a functor, so we only have to check that Φ_2 is indeed a monoidal structure: the equality (3.8) implies that each $\Phi_2(X, Y)$ is an A-linear map, and (3.9) implies

$$\begin{aligned}
&\Phi_2(X \otimes Y, Z) \circ (\Phi_2(X, Y) \otimes \mathrm{id}_{\Phi Z}) \\
&= \Phi_2(X, Y \otimes Z) \circ (\mathrm{id}_{\Phi X} \otimes \Phi_2(Y, Z))
\end{aligned}$$

for all B-modules X, Y and Z. Finally (3.11) gives the equality $\Phi(\Bbbk, \varepsilon_B) = (\Bbbk, \varepsilon_A)$ of A-modules and (3.10) the equality

$$\Phi_2(X, \Bbbk) = \mathrm{id}_{\Phi\Bbbk} = \Phi(\Bbbk, X) \ .$$

3. trivial.

□

Remark 3.2.10 Another interpretation of a twisted homomorphism is the following: let H a bialgebra and let $F \in H \otimes H$ be an invertible element obeying (3.9) and (3.10). Denote $F = F^{(1)} \otimes F^{(2)}$ and $F^{-1} = F^{(-1)} \otimes F^{(-2)}$. It is well known that there is a bialgebra H_F with the same multiplication as H and comultiplication

$$\Delta_F(x) := F^{(-1)}x_{(1)}F^{(1)} \otimes F^{(-2)}x_{(2)}F^{(2)} \ ,$$

see for example Chapter XV in [Kas95]. The statement that (φ, F) is a twisted bialgebra homomorphism, is equivalent to φ being a bialgebra homomorphism $\varphi\colon H \to H_F$.
If H is a Hopf algebra, then H_F is as well: the element $f := S(F^{(1)})F^{(2)}$ is invertible, its inverse is $f^{-1} = F^{(-1)}S(F^{(-2)})$ and the antipode of H_F is given by $S_F(x) := f^{-1}S(x)f$.

Definition 3.2.11 Let $(\varphi, F), (\varphi', F')\colon A \to B$ be twisted bialgebra homomorphisms. An element $\theta \in B$ is called *transformation from* (φ, F) *to* (φ', F'), if

$$\varphi'(x) \cdot \theta = \theta \cdot \varphi(x) \quad \text{for all } x \in A \tag{3.13}$$
$$F' \cdot (\theta \otimes \theta) = \Delta(\theta) \cdot F\,. \tag{3.14}$$

We also write $(\varphi, F) \xrightarrow{\theta} (\varphi', F')$ to say that θ is a transformation from (φ, F) to (φ', F').

Lemma 3.2.12 Let $\theta \in B$ be a transformation $(\varphi, F) \xrightarrow{\theta} (\varphi', F')$, then the family

$$\theta_X\colon X \to X,\ x \mapsto \theta.x\,.$$

indexed by B-modules X, is a monoidal transformation $(\varphi, F)^* \to (\varphi', F')^*$.

Proof. Each θ_X is A-linear due to (3.13) and compatible with the monoidal structures due to (3.14). □

Definition 3.2.13 Let A be a bialgebra and let Γ be a group. A *weak action by twisted bialgebra automorphisms* consists of

1. twisted automorphisms $(\varphi_\alpha, F_\alpha)\colon A \to A$ for every $\alpha \in \Gamma$,

2. invertible transformations $(\varphi_{\alpha\beta}, F_{\alpha\beta}) \xrightarrow{\theta_{\alpha,\beta}} (\varphi_\alpha, F_\alpha) \circ (\varphi_\beta, F_\beta)$,

such that the cocycle condition

$$\varphi_\alpha(\theta_{\beta,\gamma}) \cdot \theta_{\alpha,\beta\gamma} = \theta_{\alpha,\beta} \cdot \theta_{\alpha\beta,\gamma} \tag{3.15}$$

is fulfilled for all $\alpha, \beta, \gamma \in \Gamma$.

Proposition 3.2.14 Let A be a bialgebra and let Γ be a group. A weak action $(\varphi_\alpha, F_\alpha, \theta_{\alpha,\beta})$ of by twisted homomorphisms of Γ on A defines a weak action by monoidal functors on the category A-**Mod**: set $(\Phi^\alpha, \Phi_2^\alpha) := (\varphi_{\alpha^{-1}}, F_{\alpha^{-1}})^*$ and for an A-module X define the isomorphism

$$\begin{aligned} \Phi_{\alpha,\beta}(X)\colon (\Phi^\alpha\Phi^\beta)(X) &\to \Phi^{\alpha\beta}(X) \\ x &\mapsto \theta^{-1}_{\beta^{-1},\alpha^{-1}}.x\ . \end{aligned}$$

Proof. It is clear from Proposition 3.2.9 that Φ^α is a monoidal functor and Lemma 3.2.12 and the third part of Proposition 3.2.9 show that $\Phi_{\alpha,\beta}$ is a monoidal transformation $\Phi^\alpha \circ \Phi^\beta \to \Phi^{\alpha\beta}$, since $\theta^{-1}_{\beta^{-1},\alpha^{-1}}$ is a transformation from $\varphi_{\beta^{-1}} \circ \varphi_{\alpha^{-1}} \to \varphi_{(\alpha\beta)^{-1}}$. Condition (3.15) ensures that the corresponding diagram (3.5) commutes. □

3.3. Equivariant categories and braidings

Let $\mathcal{C}$ be a $\Bbbk$-linear category with a full Γ-grading. It is easy to argue that $\mathcal{C}$ cannot be equipped with a braiding, if Γ is non-commutative. Nevertheless, if we assume an action of Γ on $\mathcal{C}$, Turaev introduced a notion of Γ-braiding on $\mathcal{C}$ which we will present now.

Definition 3.3.1 Let Γ be a group.

1. A Γ-*equivariant category* is a monoidal category $\mathcal{C}$ together with a Γ-grading and a weak Γ-action Φ, such that for homogeneous object X of degree $\beta \in \Gamma$ the object ${}^\alpha X$ is homogeneous of degree $\alpha\beta\alpha^{-1}$ for all $\alpha, \beta \in \Gamma$.

2. A Γ-*braided category* is a Γ-equivariant category $\mathcal{C}$ together with a Γ-*braiding*, i.e. isomorphisms $c_{X,Y}\colon X \otimes Y \to {}^\alpha Y \otimes X$, where X is homogeneous of degree α, such that the following three diagrams

commute for all $\alpha, \beta, \gamma \in \Gamma$, X in $\mathcal{C}_\alpha$, Y in $\mathcal{C}_\beta$ and Z in $\mathcal{C}$

$$
\begin{tikzcd}
& (X\otimes Y)\otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & {}^{\alpha\beta}Z\otimes(X\otimes Y) \arrow[dr, "\mathsf{a}^{-1}"] & \\
X\otimes(Y\otimes Z) \arrow[ur, "\mathsf{a}^{-1}"] \arrow[dr, "\mathrm{id}_X\otimes c_{Y,Z}"'] & & & ({}^{\alpha\beta}Z\otimes X)\otimes Y \\
& X\otimes({}^{\beta}Z\otimes Y) \arrow[r, "\mathsf{a}^{-1}"] & (X\otimes{}^{\beta}Z)\otimes Y \arrow[r, "c_{X,{}^{\beta}Z}"] & ({}^{\alpha}({}^{\beta}Z)\otimes X)\otimes Y \arrow[u, "(\phi_{\alpha,\beta,Z}\otimes\mathrm{id}_X)\otimes\mathrm{id}_Y"']
\end{tikzcd}
\tag{3.16}
$$

$$
\begin{tikzcd}
& X\otimes(Y\otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & {}^{\alpha}(Y\otimes Z)\otimes X & \\
(X\otimes Y)\otimes Z \arrow[ur, "\mathsf{a}"] \arrow[dr, "c_{X,Y}\otimes\mathrm{id}"'] & & & ({}^{\alpha}Y\otimes{}^{\alpha}Z)\otimes X \arrow[ul, "\phi_2^{\alpha}(Y,Z)\otimes\mathrm{id}"'] \\
& ({}^{\alpha}Y\otimes X)\otimes Z \arrow[r, "\mathsf{a}"] & {}^{\alpha}Y\otimes(X\otimes Z) \arrow[r, "\mathrm{id}\otimes c_{X,Z}"] & {}^{\alpha}Y\otimes({}^{\alpha}Z\otimes X) \arrow[u, "\mathsf{a}^{-1}"']
\end{tikzcd}
\tag{3.17}
$$

$$
\begin{tikzcd}
& {}^{\gamma}X\otimes{}^{\gamma}Y \arrow[r, "c_{{}^{\gamma}X,{}^{\gamma}Y}"] & {}^{\gamma\alpha\gamma^{-1}}({}^{\gamma}Y)\otimes{}^{\gamma}X \arrow[dr, "\Phi_{\gamma\alpha\gamma^{-1},\gamma,Y}\otimes\mathrm{id}"] & \\
{}^{\gamma}(X\otimes Y) \arrow[ur, "(\Phi_2^{\gamma}(X,Y))^{-1}"] \arrow[dr, "{}^{\gamma}(c_{X,Y})"'] & & & {}^{\gamma\alpha}Y\otimes{}^{\gamma}X \\
& {}^{\gamma}({}^{\alpha}Y\otimes X) \arrow[r, "(\Phi_2^{\gamma}({}^{\alpha}Y,X))^{-1}"] & {}^{\gamma}({}^{\alpha}Y)\otimes{}^{\gamma}X \arrow[ur, "\Phi_{\gamma,\alpha,Y}\otimes\mathrm{id}"] &
\end{tikzcd}
\tag{3.18}
$$

Example 3.3.2 1. One particular class of Γ-equivariant categories is given by the representations of a crossed Hopf Γ-coalgebra as defined in [Tur10]. This is a Hopf Γ-coalgebra $A = \{A_\alpha, \Delta_{\alpha,\beta}\}$ in the sense of Definition 3.1.2 together with a *crossing*, i.e. a family of algebra homomorphisms $\varphi_\alpha^\gamma \colon A_\alpha \to A_{\gamma\alpha\gamma^{-1}}$ fulfilling

$$
\begin{aligned}
\varphi_\alpha^{\gamma\beta\gamma^{-1}} \circ \varphi_\beta^\gamma &= \varphi_{\alpha\beta}^\gamma \\
(\varphi_\alpha^\gamma \otimes \varphi_\beta^\gamma) \circ \Delta_{\alpha,\beta} &= \Delta_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} \circ \varphi_{\alpha\beta}^\gamma \\
\varepsilon_1 \circ \varphi_1^\gamma &= \varepsilon_1
\end{aligned}
$$

for all $\alpha, \beta, \gamma \in \Gamma$. Usually we omit the upper index of φ_α^γ.
Recall from the discussion after Remark 3.1.3 that the monoidal category A-**Mod** is given as the disjoint union of the categories A_α-**Mod** and the tensor product is by pulling back along $\Delta_{\alpha,\beta}$. By the axioms of a crossing we get a strict action by strict monoidal functors, if we set $\Psi_\beta := (\varphi_\beta^{-1})^*$.

2. It is well-known that braidings on the representation category of a Hopf algebra are in one-to-one correspondence with R-matrices. In more detail, an R-matrix for a Hopf algebra H is an invertible element $R \in H \otimes H$ such that the assignment

$$x \otimes y \mapsto R^{(2)} y \otimes R^{(1)} x$$

defines the braiding isomorphism $c_{X,Y} \colon X \otimes Y \to Y \otimes X$. One obtains the R-matrix from the braiding by evaluating the morphism $c_{H,H}$ on the element $1_H \otimes 1_H$, here H is the left regular H-module. The pair (H, R) is called a *quasi-triangular Hopf algebra.*

3. Generalizing R-matrices of Hopf algebras to R-matrices of crossed Hopf Γ-coalgebras gives the notion of *quasi-triangular* Hopf Γ-coalgebra. An R-matrix for a crossed Hopf Γ-coalgebra is a family

$$\{R_{\alpha,\beta} \in A_\alpha \otimes A_\beta\}_{\alpha,\beta\in\Gamma}$$

satisfying compatibility conditions, see [Tur10, Vir05] for details, which allow to express the Γ-braiding of the modules $X \in A_\alpha$-**Mod** and $Y \in A_\beta$-**Mod** by

$$c_{X,Y}(x \otimes y) = R_\beta . y \otimes R_\alpha . x$$

where $R_{\alpha,\beta} = R_\alpha \otimes R_\beta$ is a notation omitting a sum symbol. These compatibility conditions can be derived by the following observation: given a crossed Hopf Γ-coalgebra A, a Γ-braiding c on A-**Mod** is completely determined by the elements

$$c_{A_\alpha,A_\beta}(1_\alpha \otimes 1_\beta) =: R_{\alpha,\beta} \in A_\alpha \otimes A_\beta \ .$$

4. Equivariant extension of the Drinfel'd center

4.1. Half-braidings

To a monoidal category $\mathcal{C}$ one associates a braided category $\mathcal{Z}(\mathcal{C})$, the Drinfel'd center of $\mathcal{C}$. Its objects are pairs consisting of an object in $\mathcal{C}$ and a so called half-braiding on this object.
In this section we discuss a generalization of a half-braiding on an object X of a monoidal category $\mathcal{C}$. The categories obtained by collecting all objects of $\mathcal{C}$ with a certain kind of half-braiding will provide the homogeneous components of a Γ-equivariant category that we will obtain from a monoidal category $\mathcal{C}$ with a (weak) action of a group Γ.

Definition 4.1.1 Let $\mathcal{C}$ be a monoidal category, let $(F, F^2, F^0)\colon \mathcal{C} \to \mathcal{C}$ be an oplax monoidal endofunctor of $\mathcal{C}$ and let X be an object in $\mathcal{C}$. A *lax F-half-braiding* on X is a family $\gamma_X^F = \gamma_X$ of morphisms $\gamma_{X,U}\colon X \otimes U \to F(U) \otimes X$ that is natural in U and obeys

$$(F^2(U,V) \otimes \mathrm{id}_X) \circ (\gamma_{X,U\otimes V}) = (\mathrm{id}_{F(U)} \otimes \gamma_{X,V}) \circ (\gamma_{X,U} \otimes \mathrm{id}_V) \tag{4.1}$$

$$(F^0 \otimes \mathrm{id}_X) \circ \gamma_{X,\mathbf{1}} = \mathrm{id}_X \tag{4.2}$$

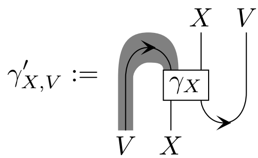
for all objects U and V in $\mathcal{C}$. We call γ_X a *strong F-half braiding*, if all $\gamma_{X,U}$ are isomorphisms.

Remark 4.1.2 Let $\mathcal{C}$ be a monoidal category and $F\colon \mathcal{C} \to \mathcal{C}$ an oplax monoidal functor.

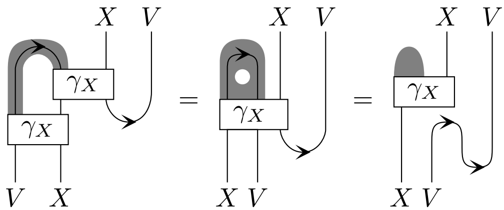
1. Let X be an object such that the functor $\_ \otimes X$ *reflects isomorphisms*, i.e. a morphism $f\colon U \to V$ in $\mathcal{C}$ is an isomorphism, if $f \otimes \mathrm{id}_X$ is an isomorphism. Under this assumption we derive from (4.2) that F^0 is an isomorphism. Further we see from (4.1) that also each $F^2(U,V)\colon F(U \otimes V) \to F(U) \otimes F(V)$ is an isomorphism, thus F is a (strong) monoidal functor.

2. Let $\mathcal{C}$ be a right rigid category and let F be strong monoidal. If γ_X is a F-half-braiding on X, then $\gamma_{X,V}$ is automatically strong. The

inverse of $\gamma_{X,V}$ is given by the morphism



We check that the composition $\gamma'_{X,V} \circ \gamma_{X,V}$ is equal to $\mathrm{id}_{X\otimes V}$:



The first equal sign is a consequence of (4.1). The second equal sign follows from $F_2(V, V^\vee) \circ F_2^{-1}(V, V^\vee) = \mathrm{id}_{F(V\otimes V^\vee)}$ and the naturality of γ_X. Finally the right-hand side is equal to $\mathrm{id}_{X\otimes V}$ by (4.2) and the rigidity axiom. We leave it to the reader to show that $\gamma_{X,V} \circ \gamma'_{X,V}$ is equal to $\mathrm{id}_{FV\otimes X}$.

For every oplax monoidal endofunctor F of a monoidal category $\mathcal{C}$ we get an ordinary category $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{C})$. The objects are pairs (X, γ_X) where X is an object of $\mathcal{C}$ and γ_X is an F-half-braiding on X. Morphisms in $\mathcal{Z}^F(\mathcal{C})$ are morphisms in $\mathcal{C}$ that *commute with half-braidings*: let (X_1, γ_{X_1}) and (X_2, γ_{X_2}) be objects in $\mathcal{Z}^F(\mathcal{C})$, a morphism $f\colon X_1 \to X_2$ is said to *commute with the half-braiding*, if

$$\gamma_{X_2,U} \circ (f \otimes \mathrm{id}_U) = (\mathrm{id}_{FU} \otimes f) \circ \gamma_{X_1,U} \tag{4.3}$$

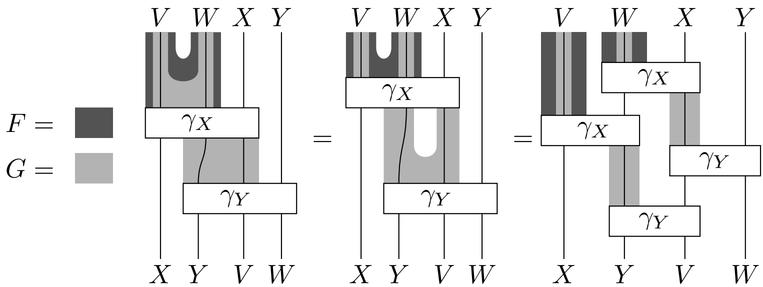
holds for all objects U in $\mathcal{C}$. We denote by $\mathcal{Z}^F(\mathcal{C})$ the full subcategory of $\mathcal{Z}_{\mathrm{lax}}(\mathcal{C})$ pairs (X, γ_X) where γ_X is a strong half-braiding. For $F = \mathrm{Id}$ the category $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{C})$ is the lax center $\mathcal{Z}_{\mathrm{lax}}(\mathcal{C})$ of $\mathcal{C}$ which is known to be a monoidal, and even prebraided, category. For generic F the category $\mathcal{Z}^F(\mathcal{C})$ is not monoidal. In special cases it can be equipped with the structure of a monoidal category, see Remark 4.1.4, which we will not discuss in detail in this thesis.

Lemma 4.1.3 Let $\mathcal{C}$ be a monoidal category and let $F, G\colon \mathcal{C} \to \mathcal{C}$ be oplax monoidal functors. Let also (X, γ_X) and (Y, γ_Y) be objects in $\mathcal{Z}^G_{\mathrm{lax}}(\mathcal{C})$ and $\mathcal{Z}^G_{\mathrm{lax}}(\mathcal{C})$, respectively.

1. The pair $(X \otimes Y, \gamma_{X\otimes Y})$ with
$$\gamma_{X\otimes Y,U} := (\gamma_{X,GU} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \gamma_{Y,U}) \tag{4.4}$$
is an object in $\mathcal{Z}^{FG}_{\mathrm{lax}}(\mathcal{C})$.

2. Given a monoidal transformation $\alpha\colon F \to G$, the family $\gamma'_{X,U} := (\alpha_U \otimes \mathrm{id}_X) \circ \gamma_{X,U}$ is a G-half-braiding on X.

Proof. We check condition (4.1) for $\gamma_{X\otimes Y,V\otimes W}$. By definition the morphism $((FG)^2(V,W) \otimes \mathrm{id}) \circ \gamma_{X\otimes Y,V\otimes W}$ is equal to the left-hand side of (4.5). Then use naturality of γ_X. We arrive at the right-hand side by using (4.1) for γ_X and γ_Y.



(4.5)

This is by definition equal to $(\mathrm{id} \otimes \gamma_{X\otimes Y,W})(\gamma_{X\otimes Y,V} \otimes \mathrm{id})$. We leave the easy proof of (4.2) to the reader.
The second part of the lemma follows in an analogous way, by using that α is a monoidal transformation. □

Remark 4.1.4 In [BV12] the category $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{C})$ is called the *F-center of $\mathcal{C}$*. Assume the existence of a monoidal transformation $\mu\colon F^2 \to F$.
From the Lemma 4.1.3 we see that for (X, γ_X) and (Y, γ_Y) in $\mathcal{Z}^F(\mathcal{C})$ we can equip the object $X \otimes Y$ with a lax F-half-braiding
$$(\mu_U \otimes \mathrm{id}_{X\otimes Y})(\gamma_{X,FU} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \gamma_{Y,U}) \, .$$
If there is a monoidal transformation $\eta\colon \mathrm{Id} \to F$, we can equip the unit object of $\mathcal{C}$ with an F-half-braiding $\gamma_{\mathbf{1},U} := \eta_U \otimes \mathrm{id}_{\mathbf{1}}$. One easily sees that $\mathcal{Z}^F(\mathcal{C})$ becomes a monoidal category in this way, if (F, μ, η) is a monad on $\mathcal{C}$, cf. Section A.2.
An oplax monoidal functor together with monoidal transformations μ and η as above is called *bimonad*. The Eilenberg-Moore category $\mathcal{C}^F$ becomes

a monoidal category, see [BV07].
Let $\mathcal{C}$ be left-rigid and assume the existence of the coend $\int^{U\in\mathcal{C}} {}^{\vee}(FU)\otimes X\otimes U$ for every object $X\in\mathcal{C}$. Then there is a unique functor $\mathsf{Z}_F\colon \mathcal{C}\to\mathcal{C}$ which is given on objects by

$$\mathsf{Z}_F(X) = \int^{U\in\mathcal{C}} {}^{\vee}(FU)\otimes X\otimes U\ .$$

If F is oplax monoidal, this functor Z_F is a monad and the Eilenberg-Moore category $\mathcal{C}^{\mathsf{Z}_F}$ is isomorphic to $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{C})$, see [BV12, Theorem 5.12]. Moreover, Bruguières and Virelizier proved that Z_F is a bimonad, if F is a bimonad. In this case categories $\mathcal{C}^{\mathsf{Z}_F}$ and $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{C})$ are isomorphic as monoidal categories.

Let $(\Psi,\Phi,\eta,\varepsilon)$ be a monoidal adjunction and $\Phi\colon\mathcal{D}\to\mathcal{C}$ and $\Psi\colon\mathcal{C}\to\mathcal{D}$ be strong monoidal functors. Let F be an oplax monoidal endofunctor of $\mathcal{D}$. Given (X,γ_X) in $\mathcal{Z}^F_{\mathrm{lax}}(\mathcal{D})$, we define γ_{Φ_*X} as the natural transformation given by the morphisms

$$\gamma_{\Phi_*X,U} := \Phi.(\gamma_{X,\Psi U})(\mathrm{id}_{\Phi X}\otimes\eta_U)\ , \tag{4.6}$$

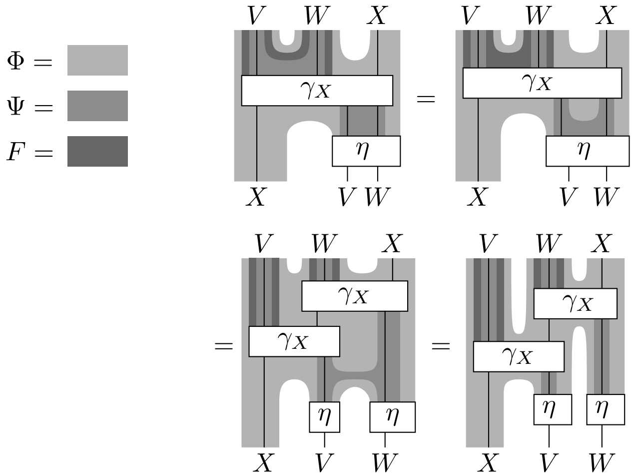
with $\Phi.(\gamma_{X,\Psi U}) := \Phi_2^{-1}(F\Psi U,X)\circ\Phi(\gamma_{X,\Psi U})\circ\Phi_2(X,\Psi U)$ as in (1.3).

Lemma 4.1.5 The family $\gamma_{\Phi_*X,\_}\colon \Phi X\otimes\_\to\Phi F\Psi(\_)\otimes\Phi X$ is a $\Phi F\Psi$-half-braiding on the object ΦX in $\mathcal{C}$. In particular the assignment

$$\Phi_* = \begin{cases}(X,\gamma_X) & \mapsto (\Phi X,\gamma_{\Phi_*X})\\ (f\colon X\to Y) & \mapsto (f\colon X\to Y)\end{cases}$$

is a functor $\Phi_*\colon \mathcal{Z}^F_{\mathrm{lax}}(\mathcal{D})\to\mathcal{Z}^{\Phi F\Psi}_{\mathrm{lax}}(\mathcal{C})$.

Proof. We prove that $\gamma_{\Phi_*,\_}$ fulfills (4.1) and leave the check of (4.2) to the reader. The following holds



The first term is merely the definition of the composition $((\Phi F\Psi)^2(V,W)\otimes \mathrm{id}_{\Phi X})\circ\gamma_{\Phi_* X, V\otimes W}$. The first equal sign follows from naturality of γ_X, the second follows by (4.1) and since $\eta\colon \mathrm{Id}\to\Phi\Psi$ is a monoidal transformation. The last equality is a consequence of $(\Phi\Psi)^2(V,W)$ being inverse to $(\Phi\Psi)_2(V,W)$. □

4.2. The main construction

Let $\mathcal{C}$ be a monoidal category and Γ a group. In this section we associate to every weak action of Γ on $\mathcal{C}$ a Γ-braided category $\mathcal{Z}^\Gamma(\mathcal{C})$. Recall that a weak action ϕ on $\mathcal{C}$ gives us monoidal autoequivalences $\phi^\alpha\colon \mathcal{C}\to\mathcal{C}$ and monoidal isomorphisms $\phi_{\alpha,\beta}\colon \phi^\alpha\circ\phi^\beta\to\phi^{\alpha\beta}$ for $\alpha,\beta\in\Gamma$; for simplicity assume $\phi^1=\mathrm{Id}_{\mathcal{C}}$.
Denote by $\mathcal{Z}_\alpha(\mathcal{C})$ the category $\mathcal{Z}^{\phi^\alpha}(\mathcal{C})$. In the following (X,γ^α_X) will always be an object in $\mathcal{Z}_\alpha(\mathcal{C})$. Define $(X,\gamma^\alpha_X)\odot(Y,\gamma^\beta_Y)$ as the pair $(X\otimes Y,\gamma_{X\otimes Y})$ where $\gamma_{X\otimes Y}$ is the natural isomorphism with components

$$\gamma_{X\otimes Y,U} := (\phi_{\alpha,\beta,U}\otimes \mathrm{id}_{X\otimes Y})(\gamma^\alpha_{X,{}^\beta U}\otimes \mathrm{id}_Y)(\mathrm{id}_X\otimes\gamma^\beta_{Y,U})\ . \tag{4.7}$$

By Lemma 4.1.3 this is a $\phi^{\alpha\beta}$-half-braiding on $X\otimes Y$.

Lemma 4.2.1 Let $\mathcal{C}$ be a strict monoidal category together with a weak action of the group Γ. The disjoint union $\mathcal{Z}^\Gamma(\mathcal{C}) := \coprod_{\alpha\in\Gamma}\mathcal{Z}_\alpha(\mathcal{C})$ is a Γ-

graded and strict monoidal category with tensor product $\odot$ and tensor unit $(\mathbf{1}, \mathrm{id})$.

Proof. The only thing left to check is, that $\odot$ is indeed associative: let $g, h, k \in \Gamma$ and let $(X, \gamma_X^g), (Y, \gamma_Y^h)$ and (Z, γ_Z^k) be in $\mathcal{Z}^\Gamma(\mathcal{C})$.
The object $(X \odot Y) \odot Z$ has the half-braiding

$$\begin{aligned}(\phi_{gh,k,U} \otimes \mathrm{id}_{X \otimes Y \otimes Z})(\phi_{g,h,{}^kU} \otimes \mathrm{id}_{X \otimes Y \otimes Z})(\gamma^g_{X,{}^h({}^kU)} \otimes \mathrm{id}_{Y \otimes Z}) \\ (\mathrm{id}_X \otimes \gamma^h_{Y,{}^kU} \otimes \mathrm{id}_Z)(\mathrm{id}_{X \otimes Y} \otimes \gamma^k_{Z,U})\end{aligned} \tag{4.8}$$

and the object $X \odot (Y \odot Z)$ has the half-braiding

$$\begin{aligned}(\phi_{g,hk,U} \otimes \mathrm{id}_{X \otimes Y \otimes Z})(\gamma^g_{X,{}^{hk}U} \otimes \mathrm{id}_{Y \otimes Z})(\mathrm{id}_X \circ \phi_{h,k,U} \circ \mathrm{id}_{Y \otimes Z}) \\ (\mathrm{id}_X \otimes \gamma^h_{Y,{}^kU} \otimes \mathrm{id}_Z)(\mathrm{id}_{X \otimes Y} \otimes \gamma^k_{Z,U})\end{aligned} \tag{4.9}$$

The term (4.9) is seen to equal (4.8) by using naturality of γ_X^g and the equality

$$\phi_{gh,k,U} \circ \phi_{g,h,{}^kU} = \phi_{g,hk,U} \circ {}^g\phi_{h,k,U} \ .$$

$\square$

Remark 4.2.2 If $\mathcal{C}$ is $\Bbbk$-linear, we will denote by $\mathcal{Z}^\Gamma(\mathcal{C})$ the category $\bigoplus_{\alpha \in \Gamma} \mathcal{Z}_\alpha(\mathcal{C})$ rather than $\coprod_{\alpha \in \Gamma} \mathcal{Z}_\alpha(\mathcal{C})$.

Remark 4.2.3 In the remainder of this section we will equip the Γ-graded category $\mathcal{Z}^\Gamma(\mathcal{C})$ with a compatible Γ-action Φ and a Γ-braiding.
During this construction we will encounter transformations that are built from the set $\left\{\phi^{\pm 1}_{\alpha,\beta}\right\}_{\alpha,\beta \in \Gamma}$ by horizontal and vertical compositions. In analogy to (1.1) we define for X in $\mathcal{C}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$ the morphism

$$\phi_{\alpha_1,\alpha_2,\ldots,\alpha_n,X} \colon {}^{\alpha_1}({}^{\alpha_2}(\cdots({}^{\alpha_n}X)\cdots)) \to {}^{\alpha_1\alpha_2\cdots\alpha_n}X$$

as $\phi_{\alpha_1,\alpha_2,X}$ for $n = 2$ and for $n \geq 3$ recursively by

$$\phi_{\alpha_1,\alpha_2,\ldots,\alpha_n,X} := \phi_{\alpha_1\alpha_2\cdots\alpha_{n-1},\alpha_n,X} \circ \phi_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},{}^{\alpha_n}X} \ . \tag{4.10}$$

Given $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \Gamma$ such that $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$, an induction argument and the coherence condition (3.5) show that any natural transformation $\xi \colon \phi^{\alpha_1} \cdots \phi^{\alpha_n} \to \phi^{\beta_1} \cdots \phi^{\beta_m}$ built from the transformations $\phi_{\alpha,\beta}$ resp. their inverses by successive vertical or horizontal composition, is of the form

$$\xi = \phi^{-1}_{\beta_1,\ldots,\beta_m} \circ \phi_{\alpha_1,\ldots,\alpha_n} \ .$$

Graphically we will denote $\phi_{\alpha_1,\dots,\alpha_n}$ by the symbol $\boxed{\alpha_1,\dots,\alpha_n}$ and the inverse $\phi^{-1}_{\alpha_1,\dots,\alpha_n}$ by the symbol $\boxed{\alpha_1,\dots,\alpha_n}$.

We want to define a Γ-action Φ on $\mathcal{Z}^\Gamma(\mathcal{C})$ in such a way that for all $\alpha,\beta\in\Gamma$ we have the inclusion $\Phi^\alpha(\mathcal{Z}_\beta(\mathcal{C}))\subset\mathcal{Z}_{\alpha\beta\alpha^{-1}}(\mathcal{C})$. Recall from Lemma 3.2.3 that the quadruple

$$(\phi^\alpha,\phi^{\alpha^{-1}},\phi^{-1}_{\alpha,\alpha^{-1}},\phi_{\alpha^{-1},\alpha})$$

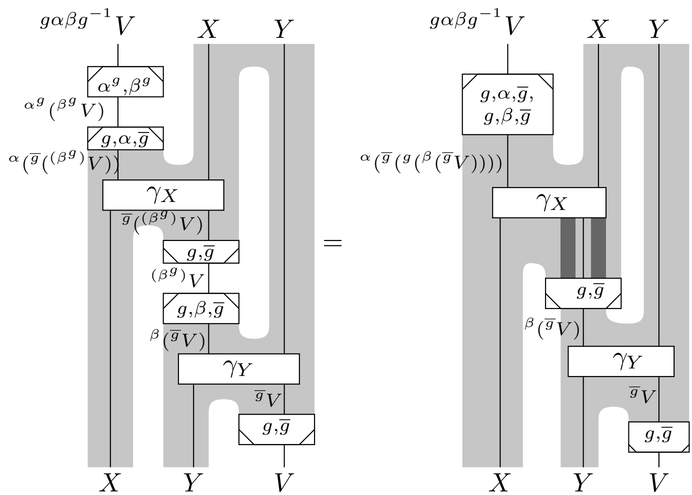
is always a monoidal adjunction. Hence, by Lemma 4.1.3 and Lemma 4.1.5, there is a functor $\Phi^\alpha_\beta\colon\mathcal{Z}_\beta(\mathcal{C})\to\mathcal{Z}_{\alpha\beta\alpha^{-1}}(\mathcal{C})$ for all $\alpha,\beta\in\Gamma$. The object (X,γ^β_X) is sent to $(\phi^\alpha(X),\gamma^{\alpha\beta\alpha^{-1}}_{{}^\alpha X})$ with

$$\gamma^{\alpha\beta\alpha^{-1}}_{{}^\alpha X,U}:=(\phi_{\alpha,\beta,\alpha^{-1},U}\otimes\mathrm{id}_{{}^\alpha X})\circ\phi^\alpha.(\gamma^\beta_{X,{}^{\alpha^{-1}}U})\circ(\mathrm{id}_{{}^\alpha X}\otimes\phi^{-1}_{\alpha,\alpha^{-1},U})\,. \tag{4.11}$$

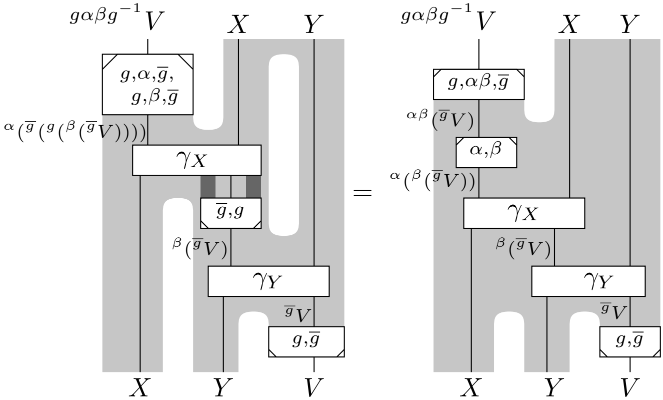
Here $\phi^\alpha.(\dots)$ is the notation from (1.3).

Lemma 4.2.4 Let $\alpha,\beta\in\Gamma$ and let (X,γ^α) and (Y,γ^β) be objects in $\mathcal{Z}^\Gamma(\mathcal{C})$. For any $g,h\in\Gamma$ the morphisms $\phi^g_2(X,Y)\colon{}^gX\otimes{}^gY\to{}^g(X\otimes Y)$, $\phi^g_0\colon\mathbf{1}\to{}^g\mathbf{1}$, $\phi_{g,h,X}\colon{}^g({}^hX)\to{}^{gh}X$ and $\gamma^\alpha_{X,Y}\colon X\otimes Y\to{}^\alpha Y\otimes X$ are morphisms in the category $\mathcal{Z}^\Gamma(\mathcal{C})$.

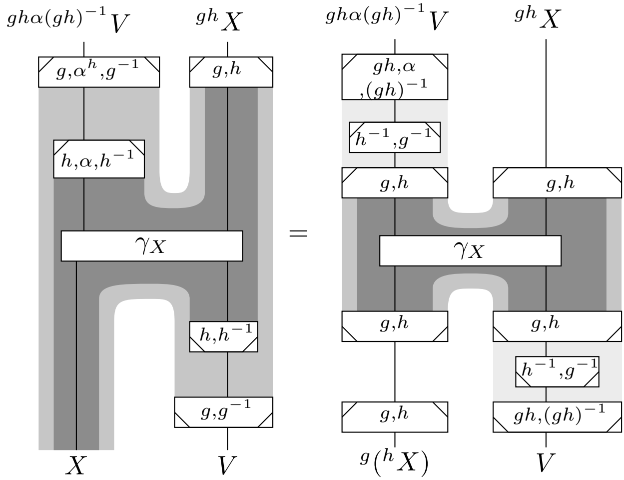
Proof. First we show the statement about $\phi^g_2(X,Y)$: by definition, the morphism $(\mathrm{id}\otimes\phi^g_2(X,Y))\circ\gamma_{\Phi^g(X)\otimes\Phi^g(Y),V}$ is equal to the left-hand side of the next line, where α^g denotes $g\alpha g^{-1}$ and $\overline{g}:=g^{-1}$:



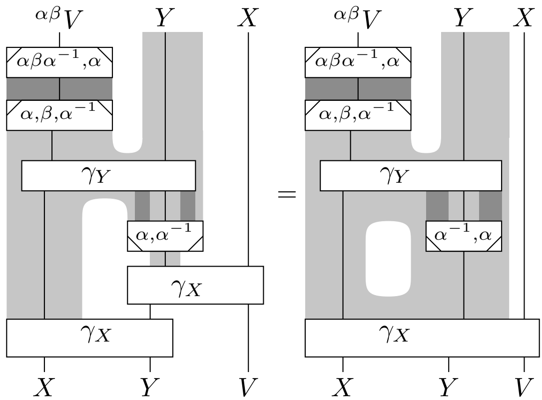
The above equality uses Remark 4.2.3: use naturality to slide the boxes labeled by g, α, g^{-1} and g, β, g^{-1} to the top left strand. Now we can use the equality $\phi_{g,g^{-1},{}^gU} = {}^g\phi_{g^{-1},g,U}$ with $U = {}^{\beta}({}^{g^{-1}}V)$ to arrive at.



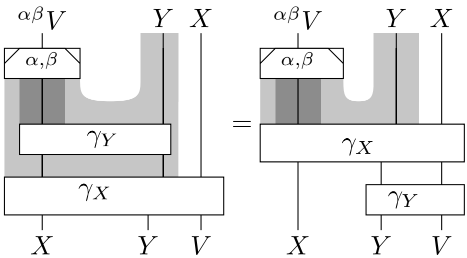
The equality follows from invertibility of $\phi_2(X, Y)$. Also the right-hand side is the morphism $\gamma_{\Phi^g(X\otimes Y),V} \circ (\phi_2^g(X, Y) \otimes \mathrm{id})$, hence $\phi_2^g(X, Y)$ is a morphism in $\mathcal{Z}^{\Gamma}(\mathcal{C})$. That ϕ_0^g is a morphism in $\mathcal{Z}^{\Gamma}(\mathcal{C})$ is trivial.
Next we come to $\phi_{g,h,X}$: spelling out the definition of $(\mathrm{id} \otimes \phi_{g,h,X}) \circ \gamma_{\Phi^g(\Phi^h X),V}$ gives the morphism



The right-hand side is obtained by a mere application of Remark 4.2.3. Using naturality, we can cancel out the boxes labeled by h^{-1}, g^{-1}. Using that the boxes labeled by g, h represent monoidal transformations all of them, except for the one bottom left, cancel out. The so obtained morphism equals $\gamma_{\Phi^{gh}(X),V} \circ (\phi_{g,h,X} \otimes \mathrm{id})$, thus $\phi_{g,h,X}$ is a morphism in $\mathcal{Z}^\Gamma(\mathcal{C})$. Finally we show that $\gamma_{X,Y}$ is a morphism in $\mathcal{Z}^\Gamma(\mathcal{C})$: the left-hand side of the next line is by definition the morphism $\gamma_{\Phi^\alpha(Y)\otimes X,V} \circ (\gamma_{X,Y} \otimes \mathrm{id})$.



The equal sign follows from (4.1) and $\phi_{\alpha,\alpha^{-1},{}^\alpha V} = {}^\alpha\phi_{\alpha^{-1},\alpha,V}$. Now use invertibility of $\phi_2^\alpha(Y,V)$ and Remark 4.2.3.



The last equality comes from naturality of γ_X. Now use (4.1) to see that the right-hand side is equal to $(\mathrm{id} \otimes \gamma_{X,Y}) \circ \gamma_{X\otimes Y,V}$. □

Theorem 4.2.5 Let $\mathcal{C}$ be a monoidal category with a Γ-action ϕ. The monoidal category $\mathcal{Z}^\Gamma(\mathcal{C}) = \coprod_{\alpha\in\Gamma} \mathcal{Z}_\alpha(\mathcal{C})$ from Lemma 4.2.1 and the functors $\Phi^\alpha_\beta \colon \mathcal{Z}_\beta(\mathcal{C}) \to \mathcal{Z}_{\alpha\beta\alpha^{-1}}(\mathcal{C})$ defined by (4.11) fulfill the following:

1. For every $\alpha \in \Gamma$ the endofunctor $\Phi^\alpha := \coprod_{\beta\in\Gamma} \Phi^\alpha_\beta$ of $\mathcal{Z}^\Gamma(\mathcal{C})$ is strong monoidal.

2. The assignment $\alpha \mapsto \Phi^\alpha$ extends to a Γ-equivariant action by monoidal functors on $\mathcal{Z}^\Gamma(\mathcal{C})$ in the sense of Definition 3.3.1.

3. The family $\left\{c_{(X,\gamma_X^\alpha),(Y,\gamma_Y^\beta)} := \gamma_{X,Y}^\alpha\right\}_{(X,\gamma_X^\alpha),(Y,\gamma_Y^\beta)\in\mathcal{Z}^\Gamma(\mathcal{C})}$ equips the category $\mathcal{Z}^\Gamma(\mathcal{C})$ with a Γ-braiding.

Proof. It is clear that Φ^α is a functor. Due to Lemma 4.2.4 the isomorphisms $\Phi_2^\alpha((X,\gamma_X),(Y,\gamma_Y)) := \phi_2^\alpha(X,Y)$ and $\Phi_0^\alpha := \phi_0^\alpha$ are morphisms in $\mathcal{Z}^\Gamma(\mathcal{C})$. Since the composition of morphisms is inherited from $\mathcal{C}$, the triple $(\Phi^\alpha, \Phi_2^\alpha, \Phi_0^\alpha)$ is a monoidal functor.
The morphisms $\Phi_{\alpha,\beta,(X,\gamma_X)} := \phi_{\alpha,\beta,X}$ are also in $\mathcal{Z}^\Gamma(\mathcal{C})$ and thus we see that we have a Γ-action on $\mathcal{Z}^\Gamma(\mathcal{C})$, which is Γ-equivariant by definition.
The conditions (3.16), (3.17) and (3.18) for the family $c_{(X,\gamma_X),(Y,\gamma_Y)}$ hold by the definitions of the respectively involved morphisms. □

Remark 4.2.6 Another construction of a Γ-braided category coming from a monoidal category with Γ-action is due to Zunino.

1. Let $\mathcal{C} = \coprod_{\alpha\in\Gamma} \mathcal{C}_\alpha$ be a Γ-equivariant category with a strict Γ-action by strict monoidal functors $\{\phi^\alpha\}_{\alpha\in\Gamma}$. In [Zun04, Section 4] the author defines for these data a Γ-braided category $\mathcal{Z} = \coprod_{\alpha\in\Gamma} \mathcal{Z}_\alpha$ as follows: objects of $\mathcal{Z}_\alpha$ are pairs (X,ξ) where X is an object in $\mathcal{C}_\alpha$ and ξ is a family of isomorphisms (called half-braiding) $\xi_U : X \otimes U \to {}^\alpha U \otimes X$ natural in $U \in \mathcal{C}$ and obeying

$$\xi_{U\otimes V} = (\mathrm{id}_{^\alpha U} \otimes \xi_V)(\xi_U \otimes \mathrm{id}_V)$$

for all objects U, V in $\mathcal{C}$. Morphisms in $\mathcal{Z}$ are morphisms in $\mathcal{C}$ that are compatible with half-braidings. The tensor product of (X,ξ) and (Y,ζ) is given by $(X\otimes Y, \eta)$ with $\eta_U := (\xi_{^\alpha U} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \zeta_U)$.
The action of Γ on $\mathcal{Z}$ is given by the functors Φ^α that send (X,ξ) in $\mathcal{Z}_\beta$ to $(\phi^\alpha(X), \xi^\alpha)$ in $\mathcal{Z}_{\alpha\beta\alpha^{-1}}$ with ξ^α being the natural isomorphism given by

$${}^\alpha(\xi_{\alpha^{-1}U})\colon {}^\alpha X \otimes \underbrace{{}^\alpha({}^{\alpha^{-1}}U)}_{=U} \to {}^{\alpha\beta\alpha^{-1}}U \otimes {}^\alpha X\ .$$

The Γ-braiding on $\mathcal{Z}$ is given by $c_{(X,\xi),(Y,\zeta)} := \xi_Y$. Note that the neutral component of $\mathcal{Z}$ is the Drinfel'd center of $\mathcal{C}_1 \subset \mathcal{C}$. This finishes our description of Zunino's category.

2. Any category $\mathcal{C}$ with Γ-action can be seen as a Γ-graded category with an equivariant Γ-action by choosing the trivial grading $\mathcal{C} = \mathcal{C}_1$. If we apply Zunino's construction in this setting, we obtain a Γ-braided category $\mathcal{Z}$ whose neutral component is the center of $\mathcal{C}$.
   In contrast to our category $\mathcal{Z}^\Gamma(\mathcal{C})$ the homogeneous components of $\mathcal{Z}$ are all trivial for $\alpha \neq 1$. Hence our category $\mathcal{Z}^\Gamma(\mathcal{C})$ does not reduce to Zunino's category, since our $\mathcal{Z}_\alpha(\mathcal{C})$ is non-trivial: under suitable finiteness conditions on $\mathcal{C}$ the coends from Remark 4.1.4 exist and $\mathcal{Z}_\alpha(\mathcal{C})$ is given by the category of modules over a monad. If $\mathcal{C}$ is the category of modules over a Hopf algebra see also Propositions 4.3.5 and 4.3.8 to see that our $\mathcal{Z}_\alpha(\mathcal{C})$ is non-trivial.

Remark 4.2.7 The following was pointed out to me by Sonia Natale. Let Γ be a finite group. In [GNN09] the authors construct for a Γ-graded fusion category $\mathcal{D} = \bigoplus_{\alpha\in\Gamma} \mathcal{D}_\alpha$ a Γ-braided category $\mathcal{Z}_{\mathcal{D}_1}(\mathcal{D})$. The support of $\mathcal{Z}_{\mathcal{D}_1}(\mathcal{D})$ equals the support of $\mathcal{D}$. The construction of the Γ-action on $\mathcal{Z}_{\mathcal{D}_1}(\mathcal{D})$ relies on $\mathcal{D}$ being a semi-simple category.

Let $\mathcal{C}$ be a tensor category together with an action by an arbitrary group Γ. In [Tam01] the author defines a Γ-graded category $\mathcal{C} \rtimes \Gamma$ where the objects of degree α are given by pairs (X, α) with an object X in $\mathcal{C}$. The tensor product of $\mathcal{C} \rtimes \Gamma$ is given by

$$(X, \alpha) \otimes (Y, \beta) := (X \otimes {}^\alpha Y, \alpha\beta)$$

and we consider $\mathcal{C}$ as the full monoidal subcategory with objects $(X, 1)$.

If $\mathcal{C}$ is a fusion category and Γ is finite, then $\mathcal{C} \rtimes \Gamma$ is a fusion category as well and we get by the results of Section 3A in [GNN09] that $\mathcal{Z}_{\mathcal{C}}(\mathcal{C} \rtimes \Gamma)$ is a Γ-braided category.

In Section 3D of [GNN09] the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{C} \rtimes \Gamma)$ is described as the category having as objects pairs (X, γ) where γ is a natural isomorphism $X \otimes \_ \xrightarrow{\cong} \_ \otimes {}^\alpha X$ subject to the compatibility condition

$$\gamma_{X, U\otimes V} = (\mathrm{id}_U \otimes \gamma)$$

This category appears to be closely related to the *mirror* of our category $\mathcal{Z}^\Gamma(\mathcal{C})$ from Theorem 4.2.5. The mirror of a Γ-braided category with a strict Γ-action is described in Section 2.5 of [Tur10, Chapter VI].

4.3. The Hopf algebra case

In the rest of this chapter H will always be a Hopf algebra with invertible antipode over a field $\Bbbk$. Let $\mathcal{C}$ be the monoidal category H-**Mod** of H-

modules together with a Γ-action coming from a weak action by twisted automorphisms of H as in Proposition 3.2.14. We will describe the category $\mathcal{Z}^\Gamma(\mathcal{C})$ as *twisted Yetter-Drinfel'd modules*, see Definition 4.3.1. For a finite dimensional Hopf algebra H and $\mathcal{C} = H$-**mod** the finite dimensional modules over H we can describe $\mathcal{Z}^\Gamma(\mathcal{C})$ also as the modules over a quasi-triangular Hopf Γ-coalgebra defined by Virelizier [Vir05].

4.3.1. Twisted Yetter-Drinfel'd modules

Let $(\varphi, F)\colon H \to H$ be a twisted automorphism and $\Phi = (\varphi, F)^*$ the monoidal pull-back functor as in 3.2.9. Set $\mathcal{C} = H$-**Mod** and consider H as the regular H-module. The comultiplication of H equips the module H with the structure of a coalgebra in H-**Mod**, and since $\Phi\colon \mathcal{C} \to \mathcal{C}$ is a strong monoidal functor we see that $\Delta^F\colon H \to H \otimes H$ given by $a \mapsto F^{-1}.(a_{(1)} \otimes a_{(2)})$ is a coassociative comultiplication on H. We will denote this coalgebra by H^F. Note that H^F is in general not a bialgebra, but a module coalgebra over H. We are now ready to define the algebraic structure which describes the category $\mathcal{Z}^\Phi(\mathcal{C})$.

Definition 4.3.1 Let H be a Hopf algebra over a field $\Bbbk$ and (φ, F) a twisted automorphism of H.

1. A vector space X together with a left H-action and a left H^F-coaction is called (φ, F)-*Yetter-Drinfel'd module* or simply φ-*Yetter-Drinfel'd module*, if the equality
$$\varphi(a_{(1)})x_{(-1)} \otimes a_{(2)}x_{(0)} = (a_{(1)}.x)_{(-1)}a_{(2)} \otimes (a_{(1)}.x)_{(0)} \tag{4.12}$$
holds for all $a \in H$ and $x \in X$.

2. A morphism of φ-twisted Yetter-Drinfel'd modules is a map $f\colon X \to Y$ that is H-linear and H^F-colinear. We denote the category of (φ, F)-Yetter-Drinfel'd modules by ${}^H_H\mathcal{YD}^F_\varphi$

Remark 4.3.2 1. For the twisted automorphism $(\mathrm{id}, 1\otimes 1)$ we get back the definition of a usual Yetter-Drinfel'd module.

2. Condition (4.12) is equivalent to
$$(a.x)_{(-1)} \otimes (a.x)_{(0)} = \varphi(a_{(1)})x_{(-1)}S(a_{(3)}) \otimes a_{(2)}x_{(0)} \tag{4.13}$$
holding for all $a \in H$ and $x \in X$.

Lemma 4.3.3 Let $\mathcal{C}$ be the category of H-modules and let Φ be the monoidal autoequivalence of $\mathcal{C}$ given by the pull-back functor associated to the twisted automorphism (φ, F) of H. Let X be a φ-Yetter-Drinfel'd module and U an H-module. Define the $\Bbbk$-linear map $\overline{\gamma}_U : X \otimes U \to \Phi(U) \otimes X$ by

$$\overline{\gamma}_U(x \otimes u) := x_{(-1)}.u \otimes x_{(0)} \ .$$

This defines a natural isomorphism $\overline{\gamma}\colon X \otimes \_ \to \Phi(\_) \otimes X$ and the pair $(X, \overline{\gamma})$ is an object in $\mathcal{Z}^{\Phi}(\mathcal{C})$.

Proof. For any H-module U the map $\overline{\gamma}_U$ is H-linear: for all $a \in H, x \in X$ and $u \in U$ we have

$$\begin{aligned} \overline{\gamma}(a.(x \otimes u)) = &\ (a_{(1)}.x)_{(-0)} a_{(2)} u \otimes (a_{(1)}.x)_{(0)} \\ \overset{(4.12)}{=} &\ \varphi(a_{(1)}) x_{(-1)} u \otimes a_{(2)} x_{(0)} \\ = &\ a.\overline{\gamma}_U(x \otimes u) \end{aligned}$$

Given an H-linear map $f\colon U \to V$ we have the equality

$$\overline{\gamma}_V \circ (\mathrm{id}_X \otimes f) = (\Phi(f) \otimes \mathrm{id}_X) \circ \overline{\gamma}_U$$

thus we have a natural transformation $\overline{\gamma}$. The inverse of $\overline{\gamma}_U$ is given by

$$\overline{\gamma}^{-1}\colon u \otimes x \mapsto x_{(0)} \otimes S^{-1}(F^{(2)} x_{(-1)}) F^{(1)} u$$

We see this as follows: write $F^{-1} = F^{(-1)} \otimes F^{(-2)}$. Since X is a comodule over H^F we have the equality

$$\begin{aligned} &x_{(-1)} \otimes (x_{(0)})_{(-1)} \otimes (x_{(0)})_{(0)} \\ =&F^{(-1)}(x_{(-1)})_{(1)} \otimes F^{(-2)}(x_{(-1)})_{(2)} \otimes x_{(0)} \ . \end{aligned} \tag{4.14}$$

We have for all $x \in X$ and $u \in U$ the equality

$$\begin{aligned} \overline{\gamma}^{-1} \circ \overline{\gamma}(x \otimes u) = &\ (x_{(0)})_{(0)} \otimes S^{-1}(F^{(2)}(x_{(0)})_{(-1)}) F^{(1)} x_{(-1)} u \\ \overset{(4.14)}{=} &\ x_{(0)} \otimes S^{-1}(F^{(2)} F^{(-2)} (x_{(-1)})_{(2)}) F^{(1)} F^{(-1)} (x_{(-1)})_{(1)} u \\ = &\ x_{(0)} \otimes S^{-1}((x_{(-1)})_{(2)})(x_{(-1)})_{(1)} u \\ = &\ x_{(0)} \otimes \varepsilon(x_{(-1)}) u = x \otimes u \ . \end{aligned}$$

Recall from Remark 3.2.10 that the elements $a := F^{(-1)}S(F^{(-2)}) \in H$ and $b := S(F^{(1)})F^{(2)} \in H$ are inverse to each other. In particular we have the equality

$$1 = S^{-1}(a)S^{-1}(b) = F^{(-2)}S^{-1}(F^{(-1)})S^{-1}(F^{(2)})F^{(1)} \tag{4.15}$$

and thus for all $x \in X$ and $u \in U$ the equality

$$\begin{aligned}
\overline{\gamma} \circ \overline{\gamma}^{-1}(u \otimes x) = {} & (x_{(0)})_{(-1)}S^{-1}(F^{(2)}x_{(-1)})F^{(1)}u \otimes (x_{(0)})_{(0)} \\
\overset{(4.14)}{=} {} & F^{(-2)}(x_{(-1)})_{(2)}S^{-1}(F^{(2)}F^{(-1)}(x_{(-1)})_{(1)})F^{(1)}u \otimes x_{(0)} \\
= {} & F^{(-2)}\varepsilon(x_{(-1)})S^{-1}(F^{(-1)})S^{-1}(F^{(2)})F^{(1)}u \otimes x_{(0)} \\
\overset{(4.15)}{=} {} & u \otimes x \; .
\end{aligned}$$

Now we show that $\overline{\gamma}$ is indeed a Φ-half-braiding, i.e. the equalities

$$\begin{aligned}
(\Phi_2^{-1}(U,V) \otimes \mathrm{id}_X) \circ \overline{\gamma}_{U \otimes V} &= (\mathrm{id}_{\Phi(X)} \otimes \overline{\gamma}_V) \circ (\overline{\gamma}_U \otimes \mathrm{id}_V) \\
\overline{\gamma_{\Bbbk}} &= \mathrm{id}_X
\end{aligned}$$

hold for all H-modules U and V. The second equality follows, since the H-action on $\Bbbk$ is given by ε, which is the counit of the coalgebra H^F. The first equality is equivalent to say that for all $x \in X, u \in U$ and $v \in V$ we have the equality

$$\begin{aligned}
&F^{(-1)}(x_{(-1)})_{(1)}u \otimes F^{(-2)}(x_{(-1)})_{(2)}v \otimes x_{(0)} \\
&= x_{(-1)}u \otimes (x_{(0)})_{(-1)}v \otimes (x_{(0)})_{(0)}
\end{aligned}$$

which follows by (4.14). □

The proof of the following Lemma is similar to the one of Lemma XIII.5.2 in [Kas95].

Lemma 4.3.4 Let $\mathcal{C}$ and Φ be as before and see H as H-module via left multiplication. For (X, γ) in $\mathcal{Z}^\Phi(\mathcal{C})$ define the $\Bbbk$-linear map

$$\begin{aligned}
\overline{\delta} \colon X &\to H \otimes X \\
x &\mapsto \gamma_H(x \otimes 1_H) \; .
\end{aligned}$$

The H-module X becomes a φ-Yetter-Drinfel'd module with coaction $\overline{\delta}$.

Proof. Note that the comultiplication $\Delta\colon H \to H \otimes H$ is a homomorphism of left H-modules. Thus for every $x \in X$ we have

$$\begin{aligned}&(F^{-1} \cdot \Delta \otimes \mathrm{id}_X) \circ \gamma_H(x \otimes 1_H)\\ =&(\Phi_2^{-1}(H,H) \otimes \mathrm{id}_X) \circ \gamma_{H\otimes H}(x \otimes 1_H \otimes 1_H)\\ =&(\mathrm{id}_H \otimes \gamma_H)(\gamma_H \otimes \mathrm{id}_H)(x \otimes 1_H \otimes 1_H)\end{aligned}$$

and so X is a comodule over H^F. For an H-module V and $v \in V$ denote by $\overline{v}$ the H-linear map $H \to V$ given by $1_H \to v$ and define write $x_{(-1)} \otimes x_{(0)} := \overline{\delta}(x)$. Since γ_V is H-linear we get for $x \in X$ and $v \in V$ the equality

$$\begin{aligned}\gamma_V(x \otimes v) &= (\gamma_V \circ (\mathrm{id} \otimes \overline{v}))(x \otimes 1_H)\\ &= (\Phi(\overline{v}) \otimes \mathrm{id}) \circ \gamma_H(x \otimes 1_H) = x_{(-1)}.v \otimes x_{(0)}\end{aligned} \tag{4.16}$$

Using H-linearity of γ_V we derive for all $a \in H$

$$\begin{aligned}&(a_{(1)}.x)_{(-1)}a_{(2)}v \otimes (a_{(1)}.x)_{(0)} = \gamma_V(a.(x \otimes v))\\ =&a.\gamma_V(x \otimes v) = \varphi(a_{(1)})x_{(-1)}.v \otimes a_{(2)}x_{(0)}\ .\end{aligned}$$

Specializing to $V = H$ and $v = 1_H$ we get condition (4.12). □

Proposition 4.3.5 Let H be a Hopf algebra with invertible antipode and let (φ, F) be a twisted bialgebra automorphism of H, set $\mathcal{C} := H$-**Mod** and $\Phi := (\varphi, F)^*$. The $\Bbbk$-linear categories ${}^H_H\mathcal{YD}^F_\varphi$ and $\mathcal{Z}^\Phi(\mathcal{C})$ are isomorphic.

Proof. The assignments from Lemma 4.3.3 and Lemma 4.3.4 define functors. We show that they are inverse to each other.
Let X be a φ-Yetter-Drinfel'd module. The half-braiding $\overline{\gamma_V}(x \otimes v) := x_{(-1)}v \otimes x_{(0)}$ from Lemma 4.3.3 specialized to $V = H$ and $v = 1_H$ is obviously equal to the coaction of X.
Conversely, let (X, γ) be an object in $\mathcal{Z}^\Phi(\mathcal{C})$. By (4.16) we see that the coaction $\gamma_H(x \otimes 1_H)$ determines the whole half-braiding. □

4.3.2. Virelizier's Hopf coalgebra

Let φ be a Hopf algebra automorphism. We want to compare the linear category ${}^H_H\mathcal{YD}_\varphi := {}^H_H\mathcal{YD}^{1\otimes 1}_\varphi$ with the category of representations of an algebra defined in [Vir05].
Let A and B be Hopf algebras and $\sigma\colon A \otimes B \to \Bbbk$ a Hopf pairing, i.e. the

equations

$$\sigma(ab, x) = \sigma(a, x_{(2)})\sigma(b, x_{(1)})$$
$$\sigma(a, xy) = \sigma(a_{(1)}, x)\sigma(a_{(2)}, y)$$
$$\sigma(1, x) = \varepsilon(x) \qquad \sigma(a, 1) = \varepsilon(a)$$

hold for all $a, b \in A$ and $x, y \in B$, cf. Remark 1.3.21. Virelizier showed for any bialgebra automorphism ψ of A, that $A \otimes B$ becomes an associative algebra with multiplication

$$(a \otimes x)(b \otimes y) := \sigma(\psi(b_{(1)}), S(x_{(1)}))\sigma(b_{(3)}, x_{(3)})ab_{(2)} \otimes x_{(2)}y \; .$$

Assume that H is finite dimensional. In this case H^* is also a Hopf algebra with $(f \cdot g)(a) = f(a_{(1)})g(a_{(2)})$ and $(f_{(1)} \otimes f_{(2)})(a \otimes b) = f(ab)$; the antipode is $\mathcal{S} := S^*$. Then, by definition, the linear map ev: $H \otimes (H^*)^{\mathrm{cop}} \to \Bbbk$ given by $\mathrm{ev}(a, f) := f(a)$ is a non-degenerate Hopf pairing:

$$\mathrm{ev}(ab, f) = f_{(1)} \otimes f_{(2)}(a \otimes b) = \mathrm{ev}(a, f_{(1)})\mathrm{ev}(b, f_{(2)}) \tag{4.17}$$
$$\mathrm{ev}(a, fg) = f(a_{(1)})g(a_{(2)}) = \mathrm{ev}(a_{(1)}, f)\mathrm{ev}(a_{(2)}g) \; . \tag{4.18}$$

Given a Hopf algebra automorphism φ of H. Virelizier's construction gives the algebra $D_\varphi(H)$ with the underlying vector space $H \otimes H^*$ and the multiplication of two elements in $H \otimes H^*$ is given by

$$\begin{aligned}(a \otimes f)(b \otimes g) :&= \mathrm{ev}(\varphi(b_{(1)}), \mathcal{S}(f_{(3)})\mathrm{ev}(b_{(3)}, f_{(1)})ab_{(2)} \otimes f_{(2)}g \\ &= f_{(3)}((S^{-1}\varphi)(b_{(1)})) \cdot f_{(1)}(b_{(3)}) \cdot ab_{(2)} \otimes f_{(2)}g \\ &= ab_{(2)} \otimes f(b_{(3)} \cdot ? \cdot (S^{-1}\varphi)(b_{(1)}))g \; .\end{aligned}$$

This explicit formula for the multiplication of $D_\varphi(H)$ helps to prove the following two lemmas by straightforward calculations

Lemma 4.3.6 Let X be a φ-Yetter-Drinfel'd module. The linear map $\rho\colon H \otimes H^* \otimes X \to X$ defined by

$$\rho(a \otimes f \otimes x) := (fS^{-1})(x_{(-1)})ax_{(0)}$$

is a $D_\varphi(H)$ action on the vector space X.

Lemma 4.3.7 Let X be a $D_\varphi(H)$ module and $\{a_i\} \subset H$ a basis of H with dual basis $\{a^i\} \subset H^*$. The vector space X becomes a φ-Yetter-Drinfel'd module over H with

$$\text{action:} \quad a.x : = (a \otimes \varepsilon).x$$
$$\text{coaction:} \quad \delta(x) : = \sum_i S(a_i) \otimes (1 \otimes a^i).x \; .$$

One checks that these two assignments define $\Bbbk$-linear functors and are inverse to each other. Hence we get

Proposition 4.3.8 Let H be a finite dimensional Hopf algebra and φ a Hopf algebra automorphism of H. The categories $D_\varphi(H)$-**Mod** and ${}^H_H\mathcal{YD}_\varphi$ are isomorphic $\Bbbk$-linear categories.

Corollary 4.3.9 The category ${}^H_H\mathcal{YD}_\varphi$ of φ-Yetter-Drinfel'd modules is abelian, if H is a finite dimensional.

Let Γ be a group and H a finite dimensional Hopf algebra. In [Vir05] the author starts with a group homomorphism $\phi\colon \Gamma \to \mathrm{Aut}_{\mathrm{Hopf}}(H)$. This gives a family $D_\phi(H) = \{D_{\phi_\alpha}\}_{\alpha\in\Gamma}$ of associative algebras. It can be equipped with the structure of a Hopf Γ-coalgebra, cf. Thm. 2.3 in [Vir05], by defining the comultiplication

$$\Delta_{\alpha,\beta}(a\otimes f) = (\phi_\beta(a_{(1)})\otimes f_{(2)})\otimes(a_{(2)}\otimes f_{(1)})\ .$$

Further, it is shown that $\varphi_\alpha(a\otimes f) := (\phi_\alpha(a)\otimes f\circ\phi_{\alpha^{-1}})$ defines a crossing on $D_\phi(H)$.
Let $A = (A_\alpha, \Delta_{\alpha,\beta}, \varphi_\beta)_{\alpha,\beta\in\Gamma}$ be a crossed Hopf Γ-coalgebra. The *mirror* $\overline{A}$ of A is the following crossed Hopf Γ-coalgebra, cf. Section VIII.1.6 in [Tur10]: set $\overline{A}_\alpha := A_{\alpha^{-1}}$ and $\overline{\varphi}_\beta := \varphi_\beta$ and define $\overline{\Delta}_{\alpha,\beta}$ as

$$\overline{\Delta}_{\alpha,\beta}(a) := (\varphi_\beta\otimes\mathrm{id})\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}} \in A_{\alpha^{-1}}\otimes A_{\beta^{-1}}\ .$$

We define the Γ-equivariant category $\mathcal{V} = \coprod_{\alpha\in\Gamma}\mathcal{V}_\alpha$ as the representation category of the mirror of Virelizier's crossed Hopf Γ-coalgebra. We spell out the structure of this category in more detail: the homogeneous component $\mathcal{V}_\alpha$ is the category $D_{\varphi_\alpha^{-1}}$-**Mod**. Let $X\in\mathcal{V}_\alpha$ and $Y\in\mathcal{V}_\beta$.
Since $\overline{\Delta}_{\alpha,\beta}(a\otimes f) = (a_{(1)}\otimes f_{(2)}\circ\phi_\beta^{-1})\otimes(a_{(2)}\otimes f_{(1)})$, the element $a\otimes f\in D_{\varphi_{\alpha\beta}^{-1}}$ acts on $X\otimes Y$ as follows

$$(a\otimes f).(x\otimes y) = (a_{(1)}\otimes f_{(2)}\circ\phi_\beta^{-1}).x\otimes(a_{(2)}\otimes f_{(1)}).y\ .$$

The action on $\mathcal{V}$ is given by the functors $\Psi_\beta := (\varphi_\beta^{-1})^*$, hence the element $a\otimes f\in D_{\phi_\alpha^{-1}}$ acts on the module $\Psi_\beta(X)$ as

$$(a\otimes f)._\beta x = (\phi_\beta^{-1}(a)\otimes f\circ\phi_\beta).x\ .$$

Now let $\mathcal{C}$ be the monoidal category H-**Mod** equipped with the Γ-action $\widetilde{\phi}_\alpha := \phi^*_{\alpha^{-1}}$ by pull-back functors. We want to show that in this case the category $\mathcal{Z}^\Gamma(\mathcal{C})$ is isomorphic to $\mathcal{V}$. To this end we first give a description of $\mathcal{Z} := \mathcal{Z}^\Gamma(\mathcal{C})$ in terms of ϕ_α-Yetter-Drinfel'd modules: the homogeneous component $\mathcal{Z}_\alpha$ is given by the category ${}^H_H\mathcal{YD}_{\phi_\alpha^{-1}}$. Given $X \in \mathcal{Z}_\alpha$ and $Y \in \mathcal{Z}_\beta$, their tensor product $X \otimes Y \in \mathcal{Z}_{\alpha\beta}$ has

$$\begin{aligned} \text{action:} \quad & a.(x \otimes y) = a_{(1)}x \otimes a_{(2)}y \\ \text{coaction:} \quad & \delta(x \otimes y) \ = \phi_\beta^{-1}(x_{(-1)})y_{(-1)} \otimes x_{(0)} \otimes y_{(0)} \end{aligned}$$

and the object $\Phi_\beta(X) \in \mathcal{Z}_{\beta\alpha\beta^{-1}}$ has

$$\begin{aligned} \text{action:} \quad & a._\beta x = \phi_\beta^{-1}(a).x \\ \text{coaction:} \quad & \delta(x) = \phi_\beta(x_{(-1)}) \otimes x_{(0)} \ . \end{aligned}$$

Now denote by $\mathsf{F}_\alpha \colon \mathcal{Z}_\alpha \to \mathcal{V}_\alpha$ the functor we get from Lemma 4.3.6, i.e. we map a ϕ_α^{-1}-Yetter-Drinfel'd module X to the $D_{\varphi_\alpha^{-1}}(H)$-module X with action given by

$$(a \otimes f).x = (fS^{-1})(x_{(-1)}) \otimes x_{(0)}$$

We will show that $\mathsf{F} := \coprod_\alpha \mathsf{F}_\alpha \colon \mathcal{Z} \to \mathcal{V}$ defines a strict monoidal functor, that commutes with the respective Γ-actions on $\mathcal{Z}$ and $\mathcal{V}$.
From the preceding discussion we easily see that the following diagrams of functors commute for every $\alpha, \beta \in \Gamma$

$$\begin{tikzcd} \mathcal{Z}_\alpha \times \mathcal{Z}_\beta \arrow[r, "\otimes"] \arrow[d, "\mathsf{F}_\alpha \times \mathsf{F}_\beta"'] & \mathcal{Z}_{\alpha\beta} \arrow[d, "\mathsf{F}_{\alpha\beta}"] \\ \mathcal{V}_\alpha \times \mathcal{V}_\beta \arrow[r, "\otimes"] & \mathcal{V}_{\alpha\beta} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{Z}_\alpha \arrow[r, "\Phi_\beta"] \arrow[d, "\mathsf{F}_\alpha"'] & \mathcal{Z}_{\beta\alpha\beta^{-1}} \arrow[d, "\mathsf{F}_{\beta\alpha\beta^{-1}}"] \\ \mathcal{V}_\alpha \arrow[r, "\Psi_\beta"] & \mathcal{V}_{\beta\alpha\beta^{-1}} \end{tikzcd}$$

This proves the claims about F. We summarize the content of this subsection in the following proposition:

Proposition 4.3.10 Let H be a finite dimensional Hopf algebra and let $\phi\colon \Gamma \to \mathrm{Aut}_{\mathrm{Hopf}}(H)$ be a group homomorphism. The category $\mathcal{Z}^\Gamma(\mathcal{C})$ associated to the Γ-action $\widetilde{\phi}_\alpha := \phi_\alpha^{-1}$ on $\mathcal{C} = H$-**Mod** is isomorphic, as a Γ-equivariant category, to the representations of the mirror of Virelizier's crossed Hopf Γ-coalgebra $D_\phi(H)$.

On the Γ-braiding of $\mathcal{V}$ and $\mathcal{Z}$ The crossed Hopf Γ-coalgebra $D_\phi(H)$ has also an R-matrix given by the family

$$R_{\alpha,\beta} = \sum_i (e_i \otimes \varepsilon) \otimes (1 \otimes e^i) \in D_{\phi_\alpha}(H) \otimes D_{\phi_\beta}(H)$$

where $\{e_i\}$ and $\{e^i\}$ are dual bases of H and H^*. Its inverse is given by

$$R^{-1}_{\alpha,\beta} = \sum_i (S(e_i) \otimes \varepsilon) \otimes (1 \otimes e^i) \in D_{\phi_\alpha}(H) \otimes D_{\phi_\beta}(H) \ .$$

Following Section VIII.1.6 in [Tur10], the R-matrix of the mirror $\overline{D_\phi}$ is given by the family

$$\overline{R}_{\alpha,\beta} = \sum_i (1 \otimes e^i) \otimes (S(e_i) \otimes \varepsilon) \in D_{\phi_\alpha^{-1}}(H) \otimes D_{\phi_\beta^{-1}}(H) \ .$$

Thus the braiding of the $\overline{D_\phi}$-modules $X \in \mathcal{V}_\alpha$ and $Y \in \mathcal{V}_\beta$ is the morphism

$$c^{\mathcal{V}}_{X,Y}(x \otimes y) = \sum_i (S(e_i) \otimes \varepsilon).y \otimes (1 \otimes e^i).x \ .$$

The image of X under F_α^{-1} is the vector space X together with

$$\begin{array}{rll} \text{action:} & (a \otimes f).x = (a \otimes \varepsilon).x & \\ \text{coaction:} & \delta(x) & = \sum_i S(x_i) \otimes (1 \otimes a^i).x \ . \end{array}$$

So the braiding of $\mathsf{F}_\alpha^{-1}(X)$ and $\mathsf{F}_\beta^{-1}(Y)$ is given by the morphism

$$\begin{aligned} c^{\mathcal{Z}}_{\mathsf{F}_\alpha^{-1}(X),\mathsf{F}_\beta^{-1}(Y)}(x \otimes y) &= x_{(-1)} y \otimes x_{(0)} \\ &= \sum_i \sum_i (S(x_i) \otimes \varepsilon).y \otimes (1 \otimes e^i).x \end{aligned}$$

which in particular means that F is an isomorphism of Γ-braided categories.

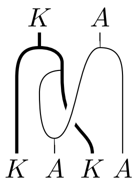
5. Partial dualization of Hopf algebras

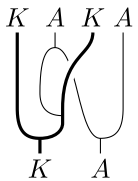
5.1. Radford biproduct and projection theorem

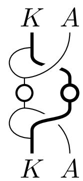
The following situation is standard: let A be a Hopf algebra over a field $\Bbbk$. Let K be a Hopf algebra in the braided category ${}^{A}_{A}\mathcal{YD}$ of A-Yetter-Drinfel'd modules.

The category of Yetter-Drinfel'd modules over K in ${}^{A}_{A}\mathcal{YD}$ can be described as the category of Yetter-Drinfel'd modules over a Hopf algebra $K \rtimes A$ over the field $\Bbbk$. The Hopf algebra $K \rtimes A$ is called *Majid bosonization* or *Radford's biproduct*. The definition of the biproduct $K \rtimes A$ directly generalizes to the description of Yetter-Drinfel'd modules over a Hopf algebra K in the braided category ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$, where $\mathcal{C}$ is now an arbitrary braided category. The biproduct of K and A is the smash product and the cosmash product of the underlying module algebra resp. comodule coalgebra. We collect in this subsection results from [Bes97] that will be needed in the construction of the partially dualized Hopf algebra in Section 5.2.

Definition 5.1.1 (Radford biproduct) Let $\mathcal{C}$ be a braided category and let $A \in \mathcal{C}$ and $K \in {}^{A}_{A}\mathcal{YD}(\mathcal{C})$ be Hopf algebras. The *Radford biproduct* $K \rtimes A$ is defined as the object $K \otimes A$ in $\mathcal{C}$ together with the following morphisms:

$\mu_{K \rtimes A} :=$ 

, $\Delta_{K \rtimes A} :=$ 

, $S_{K \rtimes A} :=$ 

.

Proposition 5.1.2 The Radford biproduct $K \rtimes A$ is a Hopf algebra in $\mathcal{C}$.

Definition 5.1.1 and a proof of Proposition 5.1.2 can be found in [Bes97, Subsection 4.1].

Remark 5.1.3 If K is a Hopf algebra in the category ${}^A_A\mathcal{YD}$ of Yetter-Drinfel'd modules over a $\Bbbk$-Hopf algebra A, the Radford biproduct is given by the following formulas for multiplication and comultiplication, cf. [Mon93, Section 10.6]:

$$\begin{aligned}(h \otimes a) \cdot (k \otimes b) &= h \cdot (a_{(1)}.k) \otimes a_{(2)} \cdot b \\ \Delta(h \otimes a) &= h_{(1)} \otimes (h_{(2)})_{(-1)} \cdot a_{(1)} \otimes (h_{(2)})_{(0)} \otimes a_{(2)}.\end{aligned}$$

This is a special case of the formulas which we expressed graphically in Definition 5.1.1.

Theorem 5.1.4 (Radford projection theorem) Let H and A be Hopf algebras in a braided category $\mathcal{C}$. Let $\pi\colon H \to A$ and $\iota\colon A \to H$ be Hopf algebra morphisms such that $\pi \circ \iota = \mathrm{id}_A$. If $\mathcal{C}$ has equalizers and $A \otimes \_$ preserves equalizers, there is a Hopf algebra K in the braided category ${}^A_A\mathcal{YD}(\mathcal{C})$, such that

$$H = K \rtimes A.$$

Proof. For a complete proof we refer to [AAFV00]. □

Remark 5.1.5 To illustrate the situation, we discuss the case when $\mathcal{C}$ is the braided category of $\Bbbk$-vector spaces and $\pi\colon H \to A$ is a projection onto a Hopf subalgebra $A \subset H$. The vector space underlying the Hopf algebra K in ${}^A_A\mathcal{YD}$ is then the space of coinvariants of H:

$$K := H^{\mathrm{coin}(\pi)} := \left\{ r \in H \mid r_{(1)} \otimes \pi(r_{(2)}) = r \otimes 1 \right\} .$$

One easily checks that K is a subalgebra of H and K is invariant under the left adjoint action of A on H.
The subspace K is also a left A-comodule with coaction $\delta_K(k) := \pi(k_{(1)}) \otimes k_{(2)}$. The fact that H is a left H-Yetter-Drinfel'd module with the adjoint action and regular coaction implies that K is even an A-Yetter-Drinfel'd module. The comultiplication of K is given by the formula

$$\Delta_K(k) := k_{(1)}\pi(S_H(k_{(2)})) \otimes k_{(3)}$$

and the antipode is $S_K(k) = \pi(k_{(1)})S_H(k_{(2)})$.

Theorem 5.1.6 (Bosonization theorem) Let A be a Hopf algebra in $\mathcal{C}$ and K a Hopf algebra in ${}^A_A\mathcal{YD}(\mathcal{C})$. There is an obvious isomorphism of braided categories

$${}^{K\rtimes A}_{K\rtimes A}\mathcal{YD}(\mathcal{C}) \cong {}^K_K\mathcal{YD}({}^A_A\mathcal{YD}(\mathcal{C})).$$

For a proof, we refer to [Bes97, Proposition 4.2.3].

5.2. The partial dual

We start with some definitions:

Definition 5.2.1 Let $\mathcal{C}$ be a braided monoidal category. A *partial dualization datum* $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$ for a Hopf algebra H in $\mathcal{C}$ consists of

- a Hopf algebra projection $\pi : H \to A$ onto a Hopf subalgebra $A \subset H$,
- a Hopf algebra B with a non-degenerate Hopf pairing $\omega\colon A \otimes B \to \mathbf{1}_{\mathcal{C}}$.

Given a partial dualization datum $\mathcal{A}$ for a Hopf algebra H in $\mathcal{C}$, the *partial dualization* $r_{\mathcal{A}}(H)$ is the following Hopf algebra in $\mathcal{C}$:

- By the Radford projection theorem 5.1.4, the projection $\pi\colon H \to A$ induces a Radford biproduct decomposition of H
$$H \cong K \rtimes A\ ,$$
where $K := H^{\mathrm{coin}(\pi)}$ is a Hopf algebra in the braided category ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$.
- The non-degenerate Hopf pairing $\omega\colon A \otimes B \to \mathbf{1}$ induces by Theorem 2.2.8 a braided equivalence:
$$\Omega\colon\ {}^{A}_{A}\mathcal{YD}\,(\mathcal{C}) \xrightarrow{\cong} {}^{B}_{B}\mathcal{YD}\,(\mathcal{C})\,.$$
Thus, the image of the Hopf algebra K in ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$ under the braided functor Ω is a Hopf algebra $L := \Omega\,(K)$ in the braided category ${}^{B}_{B}\mathcal{YD}\,(\mathcal{C})$.
- The Radford biproduct from Definition 5.1.1 of L over B allows us to introduce the partially dualized Hopf algebra,
$$r_{\mathcal{A}}(H) := L \rtimes B\ ,$$
which is a Hopf algebra in $\mathcal{C}$. As a Radford biproduct, it comes with a projection $\pi'\colon r_{\mathcal{A}}(H) \to B$.

We summarize:

Definition 5.2.2 For a partial dualization datum $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$, we call the Hopf algebra $r_{\mathcal{A}}(H)$ in $\mathcal{C}$ the *partial dual* of H with respect to the partial dualization datum $\mathcal{A}$.

Our construction is inspired by the calculations in [HS13] using smash-products. In Section 5.3.3, we explain the relation of these calculations to our general construction.

5.2.1. Involutiveness of partial dualizations

The Hopf algebra $r_{\mathcal{A}}(H)$ comes with a projection onto the subalgebra B. The two Hopf pairings $\omega^{\pm}\colon B \otimes A \to \mathbf{1}_{\mathcal{C}}$ from Example 1.3.20 yield two possible partial dualization data for $r_{\mathcal{A}}(H)$:

$$\mathcal{A}^+ = (r_{\mathcal{A}}(H) \xrightarrow{\pi'} B, A, \omega^+)$$
$$\mathcal{A}^- = (r_{\mathcal{A}}(H) \xrightarrow{\pi'} B, A, \omega^-)\ .$$

Recall from Subsection 2.2.3 the natural isomorphism

$$\theta\colon\ \Omega^{\omega^-} \circ \Omega^{\omega} \cong \mathrm{Id}_{{}^A_A\mathcal{YD}(\mathcal{C})}\ .$$

In a similar way, one has a natural isomorphism

$$\tilde{\theta}\colon\ \Omega^{\omega} \circ \Omega^{\omega^+} \cong \mathrm{Id}_{{}^B_B\mathcal{YD}(\mathcal{C})}.$$

Corollary 5.2.3 The two-fold partial dualization $r_{\mathcal{A}^-}(r_{\mathcal{A}}(H))$ is isomorphic to H, as Hopf algebra in the braided category $\mathcal{C}$. A non-trivial isomorphism of Hopf algebras is

$$r_{\mathcal{A}^-}(r_{\mathcal{A}}(H)) = \Omega^{\omega^-}(\Omega^{\omega}(K)) \rtimes A \xrightarrow{\theta_K \otimes id_A} K \rtimes A = H,$$

with $\theta_K = \rho_K \circ (S_A \otimes id_K) \circ \delta_K$ as in Lemma 2.1.5.

5.2.2. Relations between the representation categories

It is natural to look for relations between categories of representations of a Hopf algebra H in $\mathcal{C}$ and its partial dualization $r_{\mathcal{A}}(H)$:

Theorem 5.2.4 Let H be a Hopf algebra in a braided category $\mathcal{C}$, let $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$ be a partial dualization datum and $r_{\mathcal{A}}(H)$ the partially dualized Hopf algebra. Then the equivalence of braided categories

$$\Omega\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C})$$

from Theorem 2.2.8 induces an *equivalence of braided categories*:

$${}^H_H\mathcal{YD}(\mathcal{C}) \cong {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}(\mathcal{C})\right) \xrightarrow{\tilde{\Omega}} {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}(\mathcal{C})\right) \cong {}^{r_{\mathcal{A}}(H)}_{r_{\mathcal{A}}(H)}\mathcal{YD}(\mathcal{C})\,.$$

Proof. The Hopf algebra $L \in {}^B_B\mathcal{YD}(\mathcal{C})$ was defined as the image of $K \in {}^A_A\mathcal{YD}(\mathcal{C})$ under the functor Ω, i.e. $L = \Omega(K)$. The braided equivalence Ω induces an equivalence $\tilde{\Omega}$ of Yetter-Drinfel'd modules over the Hopf algebra K in the braided category ${}^A_A\mathcal{YD}(\mathcal{C})$ to Yetter-Drinfel'd modules over the Hopf algebra $L = \Omega(K)$ in ${}^B_B\mathcal{YD}(\mathcal{C})$

$$\begin{aligned} {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}(\mathcal{C})\right) \xrightarrow{\tilde{\Omega}} & {}^{\Omega(K)}_{\Omega(K)}\mathcal{YD}\left({}^B_B\mathcal{YD}(\mathcal{C})\right) \\ & =: {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}(\mathcal{C})\right). \end{aligned}$$

By Theorem 5.1.4, the source category of $\tilde{\Omega}$ is

$${}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}(\mathcal{C})\right) \cong {}^{K\rtimes A}_{K\rtimes A}\mathcal{YD}(\mathcal{C}) = {}^H_H\mathcal{YD}(\mathcal{C}) \ .$$

Similarly, we have for the target category of $\tilde{\Omega}$

$${}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}(\mathcal{C})\right) \cong {}^{L\rtimes B}_{L\rtimes B}\mathcal{YD}(\mathcal{C}) = {}^{r_{\mathcal{A}}(H)}_{r_{\mathcal{A}}(H)}\mathcal{YD}(\mathcal{C}) \ .$$

Altogether, we obtain a braided equivalence

$${}^H_H\mathcal{YD}(\mathcal{C}) \cong {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}(\mathcal{C})\right) \xrightarrow{\tilde{\Omega}} {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}(\mathcal{C})\right) \cong {}^{r_{\mathcal{A}}(H)}_{r_{\mathcal{A}}(H)}\mathcal{YD}(\mathcal{C}). \quad \square$$

If $\mathcal{C}$ is the category of vector spaces over a field $\Bbbk$, Yetter-Drinfel'd modules over a Hopf algebra H can be described as modules over the Drinfel'd double $\mathcal{D}(H)$. For a Hopf algebra H in a general braided category $\mathcal{C}$, a notion of a Drinfel'd double $\mathcal{D}(H)$ has been introduced in [BV13] in a way such that there exists a braided equivalence $\mathcal{D}(H)$-$\mathbf{Mod}(\mathcal{C}) \cong {}^H_H\mathcal{YD}(\mathcal{C})$. Hence Theorem 5.2.4 implies

Corollary 5.2.5 The categories of left modules over the Drinfel'd double $\mathcal{D}(H)$ of a Hopf algebra H and over the Drinfel'd double $\mathcal{D}(r_{\mathcal{A}}(H))$ of its partial dualization $r_{\mathcal{A}}(H)$ are braided equivalent.

5.3. Examples

We illustrate our general construction in three different cases:

5.3.1. The complex group algebra of a semi-direct product

For the complex Hopf algebra associated to a finite group G, we take

$$\mathcal{C} = \mathbf{vect}_{\mathbb{C}} \qquad H = \mathbb{C}[G].$$

To get a partial dualization datum for H, suppose that there is a split extension $N \to G \to Q$, which allows us to identify Q with a subgroup of G, i.e. $G = N \rtimes Q$. We then get a split Hopf algebra projection onto $A := \mathbb{C}[Q]$:

$$\pi\colon \mathbb{C}[G] \to \mathbb{C}[Q].$$

The coinvariants of H with respect to π, which by Theorem 5.1.4 have the structure of a Hopf algebra $K \in {}^A_A\mathcal{YD}(\mathcal{C})$, turn out to be

$$K := H^{\mathrm{coin}(\pi)} = \mathbb{C}[N] \ .$$

The A-coaction on the A-Yetter-Drinfel'd module K is trivial, since the Hopf algebra H is cocommutative. The A-action on K is non-trivial; it is given by the action of $Q \subset G$ on the normal subgroup N. Because of the trivial A-coaction, the self-braiding of K in ${}^A_A\mathcal{YD}$ is trivial; thus K is even a complex Hopf algebra. Writing H as in Theorem 5.1.4 as a Radford biproduct, we recover

$$H = K \rtimes A = \mathbb{C}[N] \rtimes \mathbb{C}[Q].$$

Since the A-coaction on K is trivial, the coalgebra structure is just given by the tensor product of the coalgebra structures on the group algebras.

As the dual of A, we take the commutative Hopf algebra of functions on Q, $B := \mathbb{C}^Q$; we denote its canonical basis by $(e_q)_{q\in Q}$; the Hopf pairing ω is the canonical evaluation. This gives the partial dualization datum

$$\mathcal{A} = (\mathbb{C}[G] \xrightarrow{\pi} \mathbb{C}[Q], \mathbb{C}^Q, \omega) \ .$$

Since the coaction of A on K is trivial, the morphism $\Omega_2(K, K)$ from the monoidal structure on Ω is trivial. Hence the functor Ω^ω maps K to the same complex Hopf algebra

$$L := \Omega^\omega(K) \cong \mathbb{C}[N] \ ,$$

which, however, has now to be seen as a Yetter-Drinfel'd module over $\mathbb{C}^Q$, i.e. $L \in {}^{\mathbb{C}^Q}_{\mathbb{C}^Q}\mathcal{YD}$: L has trivial action of $B = \mathbb{C}^Q$ and the coaction is given by the dualized action of Q on N

$$n \longmapsto \sum_{q\in Q} e_q \otimes q^{-1} n q.$$

The partial dualization $r_{\mathcal{A}}(H)$ is, by definition, the Radford biproduct

$$r_{\mathcal{A}}(H) = L \rtimes B = \mathbb{C}[N] \rtimes \mathbb{C}^Q.$$

In this biproduct, the algebra structure is given by the tensor product of algebras.

An H-module is a complex G-representation. To give an alternative description of the category $r_{\mathcal{A}}(H)$-**Mod**, we make the definition of $r_{\mathcal{A}}(H)$-modules explicit: an $r_{\mathcal{A}}(H)$-module V, with $r_{\mathcal{A}}(H) = \mathbb{C}[N] \rtimes \mathbb{C}^Q$ has the structure of a $\mathbb{C}^Q$-module and thus of a Q-graded vector space: $V = \bigoplus_{q\in Q} V_q$. Moreover, it comes with an action of N denoted by $n.v$ for $n \in N$ and $v \in V$. Since the algebra structure is given by the tensor product of algebras, the N-action preserves the Q-grading. The tensor product of two $r_{\mathcal{A}}(H)$-modules V and W is graded in the obvious way,

$$(V \otimes W)_q = \bigoplus_{q_1 q_2 = q} V_{q_1} \otimes W_{q_2}.$$

The non-trivial comultiplication

$$\Delta_{\mathbb{C}[N]\rtimes\mathbb{C}^Q}(n) = \sum_{q\in Q} (n \otimes e_q) \otimes (q^{-1}nq \otimes 1)$$

for the Radford biproduct implies a non-trivial N-action on the tensor product: on homogeneous components V_{q_1} and W_{q_2}, with $q_1, q_2 \in Q$, we have for $n \in N$

$$n.(V_{q_1} \otimes W_{q_2}) = (n.V_{q_1}) \otimes ((q_1^{-1} n q_1).V_{q_2}).$$

We are now in a position to give the alternative description of the category $r_{\mathcal{A}}(H)$-**Mod**. We denote by $\mathbf{vect}_G$ the monoidal category of G-graded finite-dimensional complex vector spaces, with the monoidal structure inherited from the category of vector spaces. Representatives of the isomorphism classes of simple objects are given by the one-dimensional vector spaces $\mathbb{C}_g$ in degree $g \in G$. Given a subgroup $N \leq G$, the object $\mathbb{C}[N] := \oplus_{n\in N}\mathbb{C}_n$ has a natural structure of an associative, unital algebra in $\mathbf{vect}_G$. It is thus possible to consider $\mathbb{C}[N]$-bimodules in the monoidal category $\mathbf{vect}_G$; together with the tensor product $\otimes_{\mathbb{C}[N]}$, these bimodules form a monoidal category $\mathbb{C}[N]$-**Bimod**($\mathbf{vect}_G$). In this setting, we have the following description of the category $r_{\mathcal{A}}(H)$-**Mod**:

Lemma 5.3.1 The monoidal category $r_{\mathcal{A}}(H)$-**Mod** is monoidally equivalent to the category of $\mathbb{C}[N]$-**Bimod**($\mathbf{vect}_G$).

The braided equivalence of Yetter-Drinfel'd modules over H and $r_{\mathcal{A}}(H)$ established in Theorem 5.2.4, more precisely the braided equivalence of the categories of modules over their Drinfel'd doubles from Corollary 5.2.5, implies the braided equivalence

$$\mathcal{Z}(\mathbb{C}[G]\text{-}\mathbf{Mod}) \cong \mathcal{Z}(\mathbf{vect}_G) \cong \mathcal{Z}(\mathbb{C}[N]\text{-}\mathbf{Bimod}(\mathbf{vect}_G))$$

which has been shown in [Sch01, Theorem 3.3] in a more general context.

Proof. It suffices to specify a monoidal functor

$$\Phi\colon \mathbb{C}[N]\text{-}\mathbf{Bimod}(\mathbf{vect}_G) \to r_{\mathcal{A}}(H)\text{-}\mathbf{Mod}$$

that is bijective on the spaces of morphisms and to give a preimage for every object $D \in r_{\mathcal{A}}(H)\text{-}\mathbf{Mod}$. Suppose that B is a $\mathbb{C}[N]$-bimodule in the category $\mathbf{vect}_G$, i.e. $B = \bigoplus_{g\in G} B_g$, with $\mathbb{C}[N]$-actions denoted by arrows $\rightharpoonup, \leftharpoonup$.

To define the functor Φ on objects, consider for a bimodule B the Q-graded vector space $\Phi(B) := \oplus_{q\in Q} B_q \subset B$, obtained by retaining only the homogeneous components with degree in $Q \subset G$. A left N-action is defined for any homogeneous vector $v_q \in \Phi(B)_q$ by

$$n.v_q := n \rightharpoonup v_q \leftharpoonup (q^{-1}n^{-1}q).$$

Moreover,

$$n.v_q = n \rightharpoonup v_q \leftharpoonup (q^{-1}n^{-1}q) \ \in \Phi(B)_{nq(q^{-1}n^{-1}q)} = \Phi(B)_q \ ,$$

since $\rightharpoonup, \leftharpoonup$ are morphisms in $\mathbf{vect}_G$. Thus the N-action preserves the Q-grading; we conclude that $\Phi(B)$ is an object in $r_{\mathcal{A}}(H)\text{-}\mathbf{Mod}$.

On the morphism spaces, the functor Φ acts by restriction to the vector subspace $\Phi(B) \subset B$. We show that this gives a bijection on morphisms: suppose $\Phi(f) = 0$, then $f(v_q) = 0$ for all v_q with grade $q \in Q$. For an arbitrary $v_g \in B$ with grade $g \in G$, we may write $g = nq$ with $n \cdot q \in N \rtimes Q$ and get an element $n^{-1} \rightharpoonup v_g$ of degree q. Using that f is a morphism of $\mathbb{C}[N]$-bimodules, we find $f(v_g) = n \rightharpoonup f(n^{-1} \rightharpoonup v_g) = 0$. Thus Φ is injective on morphisms. To show surjectivity, we take a morphism $f_\Phi\colon \Phi(B) \to \Phi(C)$; writing again $g = nq$, we define a linear map $f\colon B \to C$ on $v_g \in V_g$ by $f(v_g) := n \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g)$. This linear map is, by construction, a morphism of left $\mathbb{C}[N]$-modules in $\mathbf{vect}_G$. It remains to verify that f is also a morphism of right $\mathbb{C}[N]$-modules. We note that for $g \in G$, the decomposition $g = nq$ with $n \in N$ and $q \in Q$ implies $gm = (nqmq^{-1})q$ with $nqmq^{-1} \in N$ for all $m \in N$. We thus find:

$$\begin{aligned}
f(v_g \leftharpoondown m) &= (nqmq^{-1}) \rightharpoonup f_\Phi((nqmq^{-1})^{-1} \rightharpoonup v_g \leftharpoondown m) \\
&= (nqmq^{-1}) \rightharpoonup f_\Phi((qm^{-1}q^{-1}) \rightharpoonup (n^{-1} \rightharpoonup v_g) \leftharpoondown m) \\
&= (nqmq^{-1}) \rightharpoonup f_\Phi((qm^{-1}q^{-1}).(n^{-1} \rightharpoonup v_g)) \\
&= (nqmq^{-1}) \rightharpoonup (qm^{-1}q^{-1}).f_\Phi((n^{-1} \rightharpoonup v_g)) \\
&= (nqmq^{-1}) \rightharpoonup (qm^{-1}q^{-1}) \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g) \leftharpoondown m \\
&= n \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g) \leftharpoondown m \\
&= f(v_g) \leftharpoondown m
\end{aligned}$$

In the forth identity, we used that f_Φ is $r_{\mathcal{A}}(H)$-linear.

Next we show that Φ has a natural structure of a monoidal functor. Recall that the tensor product $V \otimes W$ in $\mathbf{vect}_G$ (resp. $\mathbf{vect}_Q$) is defined as the tensor product of vector spaces with diagonal grading $V_g \otimes W_h \subset (V \otimes W)_{gh}$. Furthermore, the tensor product in $\mathbb{C}[N]$-$\mathbf{Bimod}(\mathbf{vect}_G)$ is defined by $\otimes_{\mathbb{C}[N]}$. On the other side, the tensor product $\otimes$ in $r_{\mathcal{A}}(H)$-$\mathbf{Mod}$ is the tensor product of modules over the Hopf algebra $r_{\mathcal{A}}(H) = \mathbb{C}[N] \rtimes \mathbb{C}^Q$ with diagonal grading and action

$$n.(\Phi(B)_{q_1} \otimes \Phi(B)_{q_2}) = (n.\Phi(B)_{q_1}) \otimes ((qnq^{-1}).\Phi(B)_{q_2}).$$

We now show that the canonical projection of vector spaces $B \otimes C \to B \otimes_{\mathbb{C}[N]} C$ gives rise to a monoidal structure on Φ:

$$\Phi_2 \colon \Phi(B) \otimes \Phi(C) \to \Phi(B \otimes_{\mathbb{C}[N]} C).$$

It is clear that this map is compatible with the Q-grading. The compatibility with the N-action is calculated as follows: for $n \in N$, $b \in B_{q_1}$ $c \in C_{q_2}$:

$$\begin{aligned}
&(n.(b \otimes c)) = n.b \otimes (q_1^{-1}nq_1).c \\
&\xmapsto{\phi} (n.b) \otimes_{\mathbb{C}[N]} (q_1^{-1}nq_1).c \\
&= (n \rightharpoonup b \leftharpoondown (q_1^{-1}n^{-1}q_1)) \otimes_{\mathbb{C}[N]} ((q_1^{-1}nq_1) \rightharpoonup c \leftharpoondown (q_2^{-1}q_1^{-1}nq_1q_2)) \\
&= (n \rightharpoonup b \leftharpoondown (q_1^{-1}n^{-1}q_1q_1^{-1}nq_1)) \otimes_{\mathbb{C}[N]} (c \leftharpoondown (q_2^{-1}q_1^{-1}nq_1q_2)) \\
&= (n \rightharpoonup b) \otimes_{\mathbb{C}[N]} (c \leftharpoondown ((q_1q_2)^{-1}n(q_1q_2)) \\
&= n.\left(b \otimes_{\mathbb{C}[N]} c\right).
\end{aligned}$$

Moreover, Φ_2 is clearly compatible with the associativity constraint. We now show Φ_2 is bijective by giving an explicit inverse: consider an element

$v \otimes w \in B \otimes_{\mathbb{C}[N]} C$ which is in $\Phi(B \otimes_{\mathbb{C}[N]} C) \subset B \otimes_{\mathbb{C}[N]} C$. Restricting to homogeneous elements, we take $v \otimes w \in (B \otimes_{\mathbb{C}[N]} C)$ with v of degree $g \in G$ and w of degree $h \in G$. Since $v \otimes w$ is even in the subspace $\Phi(B \otimes_{\mathbb{C}[N]} C)$, we have $q := gh \in Q$. Writing $h = n'q'$ with $n' \in N$ and $q' \in Q$, have in the tensor product over $\mathbb{C}[N]$ the identity $v \otimes w = (v \leftharpoonup n') \otimes ((n')^{-1} \rightharpoonup w)$ with tensor factors both graded in Q, hence in $\Phi(B) \otimes \Phi(C)$. We may now define the inverse $\Phi_2^{-1}(v \otimes w) := (v \leftharpoonup n) \otimes (n^{-1} \rightharpoonup w)$, which is a left- and right-inverse of Φ_2. Finally the monoidal units in the categories are $\mathbb{C}_1,\ 1 \in Q$ resp. $\mathbb{C}[N]$; then $N \cap Q = \{1\}$ implies that there is an obvious isomorphism $\mathbb{C}_1 \cong \Phi(\mathbb{C}[N])$. Hence Φ is a monoidal functor.

To verify that Φ indeed defines an equivalence of tensor categories, it remains to construct an object $D \in \mathbb{C}[N]$-$\mathbf{Bimod}(\mathbf{vect}_G)$ for each module $V \in r_{\mathcal{A}}(H)$-$\mathbf{Mod}$, such that $\Phi(D)$ and V are isomorphic.

The following construction could be understood as an induced corepresentation via the cotensor product, but we prefer to keep the calculation explicit: for $V = \bigoplus V_q$ consider the vector space

$$D := \bigoplus_{q \in Q} \mathbb{C}[N] \otimes V_q.$$

Since $G = NQ$, the vector space D is naturally endowed with a G-grading. Left multiplication on $\mathbb{C}[N]$ gives a natural left N-action $\rightharpoonup$ via left-multiplication on $\mathbb{C}[N]$, which is clearly a morphism in $\mathbf{vect}_G$. We define a right N-action on D by

$$(n \otimes v_q) \leftharpoonup m \ := \ n(qmq^{-1}) \otimes (qn^{-1}q^{-1}).v_q.$$

Since the left action preserves the Q-grading, the vector $(n \otimes v_q) \leftharpoonup m$ has degree $n(qmq^{-1})q = (nq)m$; thus also the right action $\leftharpoonup$ is a morphism in $\mathbf{vect}_G$.

We finally verify that $\Phi(D) \cong V$: the homogeneous components of D with degree in the subgroup Q only are spanned by elements $1 \otimes v_q$, hence we can identify $\Phi(D)$ with V. We check that the N-action defined on $\Phi(D)$ coincides with the one on V we started with:

$$\begin{aligned} n.(1 \otimes v_q) &= n \rightharpoonup (1 \otimes v_q) \leftharpoonup (q^{-1}n^{-1}q) \\ &= n(q(q^{-1}n^{-1}q)q^{-1}) \otimes (q(q^{-1}n^{-1}q)^{-1}q^{-1}).v_q \\ &= 1 \otimes n.v_q. \end{aligned}$$

□

5.3.2. The Taft algebra

Fix a natural number d and let $\zeta \in \mathbb{C}$ be a primitive d-th root of unity. We consider the Taft algebra T_ζ which is a complex Hopf algebra. As an algebra, T_ζ is generated by two elements g and x modulo the relations

$$g^d = 1,\ x^d = 0 \quad \text{and} \quad gx = \zeta xg.$$

A coassociative comultiplication on T_ζ is defined by the unique algebra homomorphism $\Delta\colon T_\zeta \to T_\zeta \otimes T_\zeta$ with

$$\Delta(g) = g \otimes g \quad \text{and} \quad \Delta(x) = g \otimes x + x \otimes 1.$$

Lemma 5.3.2 Let ζ and ξ be primitive d-th roots of unity. If there exists an isomorphism $\psi\colon T_\zeta \to T_\xi$ of Hopf algebras, then $\zeta = \xi$.

Proof. The set $\{x^n g^m \mid 0 \leq n, m < d\}$ is a $\mathbb{C}$-basis of T_ξ consisting of eigenvectors for the automorphisms

$$\mathrm{ad}_h\colon T_\xi \to T_\xi, \quad a \mapsto hah^{-1},$$

with $h = g^c$ for $c \in \{1, 2, \dots, N-1\}$.

Suppose that $\psi\colon T_\zeta \to T_\xi$ is a Hopf algebra isomorphism. Then the image $h := \psi(g)$ of the generator g of T_ζ is equal to $g^c \in T_\xi$ for some $c \in \{1, 2, \dots, d-1\}$. The generator x of T_ζ is mapped by the algebra homomorphism ψ to an eigenvector $y := \psi(x)$ of ad_h to the eigenvalue ζ:

$$hyh^{-1} = \psi(gxg^{-1}) = \zeta y.$$

Since ξ is a primitive root of unit, we find $0 < n < d$ such that $\zeta = \xi^n$. Thus y is an element of the $\mathbb{C}$-linear subspace $\langle x^n g^m \mid 0 \leq m < d\rangle_{\mathbb{C}}$ of T_ξ. This implies that $y^k = 0$ for k the smallest number such that $kn \geq d$. Since ψ is an isomorphism, n has to be 1 and hence $\zeta = \xi$. $\square$

Denote by A the Hopf subalgebra of T_ζ generated by g. We will deduce from Proposition 5.3.5 that the partial dual of T_ζ with respect to A is isomorphic to T_ζ. Hence, to get a non-trivial behavior of the partial dual we have to look at a class of complex Hopf algebras which is more general than Taft algebras.

Let N be a natural number and let d be a divisor of N. Now let ζ be a primitive d-th root of unity and q a primitive N-th root of unity. Let $c + N\mathbb{Z}$ be the unique residue class such that $\zeta = q^c$. Define $\hat{T}_{\zeta,q}$ as the $\mathbb{C}$-algebra

$$\hat{T}_{\zeta,q} := \langle x, g \mid g^N = 1, x^d = 0, gx = \zeta xg\rangle$$

and define $\check{T}_{\zeta,q}$ as the $\mathbb{C}$-algebra

$$\check{T}_{\zeta,q} := \langle x, g \mid g^N = 1, x^d = 0, gx = qxg \rangle.$$

Both algebras are finite-dimensional of dimension Nd.

One checks the following

Lemma 5.3.3 Let $\hat{T}_{\zeta,q}$ and $\check{T}_{\zeta,q}$ be the algebras from above. The unique algebra homomorphisms $\hat{\Delta}\colon \hat{T}_{\zeta,q} \to \hat{T}_{\zeta,q} \otimes \hat{T}_{\zeta,q}$ and $\check{\Delta}\colon \check{T}_{\zeta,q} \to \check{T}_{\zeta,q} \otimes \check{T}_{\zeta,q}$ defined on the generators by

$$\begin{aligned} \hat{\Delta}(g) &:= g \otimes g & \qquad \hat{\Delta}(x) &:= g \otimes x + x \otimes 1 \\ \check{\Delta}(g) &:= g \otimes g & \qquad \check{\Delta}(x) &:= g^c \otimes x + x \otimes 1 \end{aligned}$$

give the structure of an coassociative counital Hopf algebra on $\hat{T}_{\zeta,q}$ and $\check{T}_{\zeta,q}$, respectively.

Furthermore, we have exact sequences of Hopf algebras, with $k := \frac{N}{d}$

$$\mathbb{C}[\mathbb{Z}_k] \longrightarrow \hat{T}_{\zeta,q} \longrightarrow T_\zeta$$

$$T_\zeta \longrightarrow \check{T}_{\zeta,q} \longrightarrow \mathbb{C}[\mathbb{Z}_k] \ .$$

The Hopf subalgebra $A \subset \hat{T}_{\zeta,q}$ generated by the grouplike element g and the Hopf subalgebra $B \subset \check{T}_{\zeta,q}$ generated by g are both isomorphic to the complex group Hopf algebra $\mathbb{C}[\mathbb{Z}_N]$. To apply a partial dualization, we need a Hopf pairing; it is given by the following lemma whose proof we leave to the reader:

Lemma 5.3.4 Let q be an N-th primitive root of unity and let $g \in \mathbb{C}[\mathbb{Z}_N]$ be a generator of the cyclic group $\mathbb{Z}_N$.

1. The bilinear form $\omega\colon \mathbb{C}[\mathbb{Z}_N] \times \mathbb{C}[\mathbb{Z}_N] \to \Bbbk$ given by $\omega(g^n, g^m) = q^{nm}$ is a Hopf pairing.

2. The linear map $\omega'\colon \Bbbk \to \mathbb{C}[\mathbb{Z}_N] \otimes \mathbb{C}[\mathbb{Z}_N]$ with

$$\omega'(1_\Bbbk) = \frac{1}{N} \sum_{k,\ell=1}^{N} q^{-k\ell} g^k \otimes g^\ell$$

is the inverse copairing of ω.

The partial dual of $\hat{T}_{\zeta,q}$ with respect to A and ω is isomorphic to $\check{T}_{\zeta,q}$:

Proposition 5.3.5 Let N be a natural number and d be a divisor of N. Let ζ be a primitive d-th root of unity and q a primitive N-th root of unity with $q^c = \zeta$. Let $A \subset \hat{T}_{\zeta,q}$ and $B \subset \check{T}_{\zeta,q}$ be as above and $\omega\colon A \otimes B \to \Bbbk$ the non-degenerate Hopf pairing from Lemma 5.3.4.

1. The algebra homomorphism $\pi\colon \hat{T}_{\zeta,q} \to A$ which sends g to g and x to 0 is a Hopf algebra projection onto A.
2. The partial dualization of $\hat{T}_{\zeta,q}$ with respect to the partial dualization datum $(\hat{T}_{\zeta,q} \xrightarrow{\pi} A, B, \omega)$ is isomorphic to $\check{T}_{\zeta,q}$.

In particular, for $N = d$, we have $\hat{T}_{\zeta,q} = \check{T}_{\zeta,q}$.

Proof. The space of coinvariants $K := \hat{T}_{\zeta,q}^{\,\mathrm{coin}(\pi)} = \{a \in \hat{T}_{\zeta,q} \mid \hat{\Delta}(a) = a \otimes 1\}$ equals the $\mathbb{C}$-linear span of $\{1, x, x^2, \dots, x^{d-1}\}$. Remark 5.1.5 implies that K is a Yetter-Drinfel'd module with A-action $\rho\colon A \otimes K \to K$ and A-coaction $\delta\colon K \to A \otimes K$ given by

$$\begin{aligned} \rho\colon g \otimes x &\mapsto gxg^{-1} = \zeta x, \\ \delta\colon x &\mapsto \pi(g) \otimes x = g \otimes x = x_{(-1)} \otimes x_{(0)}. \end{aligned}$$

Moreover, K has the structure of a Hopf algebra in ${}^A_A\mathcal{YD}$ with multiplication and comultiplication given by

$$\begin{aligned} \mu\colon x \otimes x &\mapsto x^2, \\ \Delta\colon x &\mapsto 1 \otimes x + x \otimes 1 \ . \end{aligned}$$

The dualization functor (Ω, Ω_2) from Section 2.2.2 for the Hopf pairing $\omega\colon A \otimes A \to \Bbbk$ yields the A-Yetter-Drinfel'd module $L = \langle 1, x, \dots, x^{d-1}\rangle_{\mathbb{C}}$ with action $\rho'\colon A \otimes L \to L$ and coaction $\delta'\colon L \to A \otimes L$ given by

$$\begin{aligned} \rho'\colon g \otimes x &\mapsto \omega(x_{(-1)}, g)x_{(0)} = qx, \\ \delta'\colon x &\mapsto \frac{1}{N}\sum_{k,\ell=1}^{N} q^{-k\cdot\ell} g^k \otimes \rho(g^\ell \otimes x) = \frac{1}{N}\sum_{k,\ell=1}^{N} (q^{-k}q^c)^\ell g^k \otimes x = g^c \otimes x. \end{aligned}$$

The Yetter-Drinfel'd module L has a natural structure of a Hopf algebra in ${}^A_A\mathcal{YD}$ with multiplication $\mu' = \mu \circ \Omega_2(K, K)$ and comultiplication $\Delta' = \Omega_2^{-1}(K, K) \circ \Delta$

$$\begin{aligned} \mu'\colon x \otimes x &\mapsto \zeta x^2, \\ \Delta'\colon x &\mapsto 1 \otimes x + x \otimes 1 \ . \end{aligned}$$

As an algebra, L is generated by x, so the biproduct $r_{\mathcal{A}}(\hat{T}_{\zeta,q}) = L \rtimes B$ is generated by $x \cong x \otimes 1$ and $g \cong 1 \otimes g$. In the biproduct $r_{\mathcal{A}}(\hat{T}_{\zeta,q})$, the relations

$$g^N = 1, x^d = 0 \text{ and } gx = \rho'(g \otimes x)g = qxg$$

hold. This gives a surjective algebra homomorphism $\psi\colon r_{\mathcal{A}}(\hat{T}_{\zeta,q}) \to \check{T}_{\zeta,q}$; since $\check{T}_{\zeta,q}$ and $r_{\mathcal{A}}(\hat{T}_{\zeta,q})$ have the same complex dimension, ψ is an isomorphism.

The map ψ also respects the coalgebra structures, since

$$\Delta_{r_{\mathcal{A}}(\hat{T}_{\zeta,q})}(x) = 1 \cdot x_{[-1]} \otimes x_{[0]} + x \otimes 1 = g^c \otimes x + x \otimes 1 \ . \qquad \square$$

5.3.3. Reflection on simple roots in a Nichols algebra

We finally discuss the example of Nichols algebras [HS13]. We take for $\mathcal{C}$ the category of finite-dimensional Yetter-Drinfel'd modules over a complex Hopf algebra h with bijective antipode, e.g. the complex group algebra of a finite group G. Let $M \in \mathcal{C}$ be a finite direct sum of simple objects $(M_i)_{i \in I}$,

$$M = \bigoplus_{i \in I} M_i \ .$$

Thus, M is a complex braided vector space. The Nichols algebra $\mathcal{B}(M)$ of M is defined as a quotient by the kernels of the quantum symmetrizer maps Q_n

$$\mathcal{B}(M) := \bigoplus_{n \geq 0} M^{\otimes n} / \ker(Q_n).$$

The Nichols algebra $\mathcal{B}(M)$ is a Hopf algebra in the braided category $\mathcal{C}$. If M is a direct sum of n simple objects in $\mathcal{C}$, the Nichols algebra is said to be of rank n.

Each simple subobject M_i of M provides a partial dualization datum: denote by M_i^* the braided vector space dual to M_i. Denote by $\mathcal{B}(M_i)$ the Nichols algebra for M_i. The fact that M_i is a subobject and a quotient of M implies that $\mathcal{B}(M_i)$ is a Hopf subalgebra of $\mathcal{B}(M)$ and that there is a natural projection $\mathcal{B}(M) \xrightarrow{\pi_i} \mathcal{B}(M_i)$ of Hopf algebras. Similarly, the evaluation and coevaluation for M induce a non-degenerate Hopf pairing $\omega_i\colon \mathcal{B}(M_i) \otimes \mathcal{B}(M_i^*) \to \mathbb{C}$ on the Nichols algebras. We thus have for each $i \in I$ a partial dualization datum

$$\mathcal{A}_i := (\mathcal{B}(M) \xrightarrow{\pi_i} \mathcal{B}(M_i), \mathcal{B}(M_i^*), \omega_i)$$

We denote by $r_i(\mathcal{B}(M)) := r_{\mathcal{A}_i}(\mathcal{B}(M))$ the partial dualization of $\mathcal{B}(M)$ with respect to $\mathcal{A}_i$. As usual, we denote by K_i the coinvariants for the the projection π_i; K_i is a Hopf algebra in the braided category of $\mathcal{B}(M_i)$-Yetter-Drinfel'd modules.

We summarize some results of [AHS10],[HS10] and [HS13]; for simplicity, we assume that the Nichols $\mathcal{B}(M)$ algebra is finite-dimensional. To make contact with our results, we note that the i-th partial dualization

$$r_i(\mathcal{B}(M)) := \Omega(K_i) \rtimes \mathcal{B}(M_i^*) \ ,$$

as introduced in the present paper, coincides with the i-th reflection of $\mathcal{B}(M)$ in the terminology of [AHS10].

Theorem 5.3.6 Let h be a complex Hopf algebra with bijective antipode . Let M_i be a finite collection of simple h-Yetter-Drinfel'd modules. Consider $M := \bigoplus_{i=1}^n M_i \in {}^h_h\mathcal{YD}$ and assume that the associated Nichols algebra $H := \mathcal{B}(M)$ is finite-dimensional. Then the following assertions hold:

- By construction, the Nichols algebras $\mathcal{B}(M), r_i(\mathcal{B}(M))$ have the same dimension as complex vector spaces.

- For $i \in I$, denote by $\hat{M}_i$ the braided subspace

  $$\hat{M}_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus M_{i+1} \oplus \ldots$$

  of M. Denote by $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$ the braided vector space obtained as the image of $\hat{M}_i \subset \mathcal{B}(M)$ under the adjoint action of the Hopf subalgebra $\mathcal{B}(M_i) \subset \mathcal{B}(M)$. Then, there is a unique isomorphism [HS13, Prop. 8.6] of Hopf algebras in the braided category ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}\left({}^h_h\mathcal{YD}\right)$:

  $$K_i \cong \mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i))$$

  which is the identity on $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$.

- Define, with the usual convention for the sign, $a_{ij} := -\max\{m \mid \mathrm{ad}^m_{M_i}(M_j) \neq 0\}$. Fix $i \in I$ and denote for $j \neq i$

  $$V_j := \mathrm{ad}_{M_i}^{-a_{ij}}(M_j) \subset \mathcal{B}(M) \ .$$

  The braided vector space

  $$R_i(M) = V_1 \oplus \cdots M_i^* \cdots \oplus V_n \in {}^h_h\mathcal{YD}$$

is called the the i-th reflection of the braided vector space M. Then there is a unique isomorphism [HS13, Thm. 8.9] of Hopf algebras in ${}^h_h\mathcal{YD}$

$$r_i(\mathcal{B}(M_1 \oplus \cdots \oplus M_n)) \cong \mathcal{B}(V_1 \oplus \cdots M_i^* \cdots \oplus V_n)$$

which is the identity on M.

- With the same definition for a_{ij} for $i \neq j$ and $a_{ii} := 2$, the matrix $(a_{ij})_{i,j=1,\ldots n}$ is a generalized Cartan matrix [AHS10, Thm. 3.12]. Moreover, one has $r_i^2(\mathcal{B}(M)) \cong \mathcal{B}(M)$, as a special instance of Corollary 5.2.3, and the Cartan matrices coincide, $a_{ij}^M = a_{ij}^{r_i(M)}$. In the terminology of [HS10, Thm. 6.10], one obtains a Cartan scheme.

- The maps r_i give rise to a Weyl groupoid which controls the structure of the Nichols algebra $\mathcal{B}(M)$. For details, we refer to [AHS10, Sect. 3.5] and [HS10, Sect. 5].

We finally give examples that illustrate the appearance of Nichols algebras as Borel algebras in quantum groups. We end with an example in which a reflected Nichols algebra is not isomorphic to the original Nichols algebra.

The first example serves to fix notation:

Example 5.3.7 Let $n > 1$ be a natural number and q be a primitive n-th root of unity in $\mathbb{C}$. Let M be the one-dimensional complex braided vector space with basis x_1 and braiding matrix $q_{11} = q$. As a quotient of the tensor algebra, the associated Nichols algebra $\mathcal{B}(M)$ inherits a grading, $\mathcal{B}(M) = \oplus_{k\in\mathbb{N}}\mathcal{B}(M)_{(k)}$. As a graded vector space, it is isomorphic to

$$\mathcal{B}(M) \cong \mathbb{C}[x_1]/(x_1^n)$$

and thus of complex dimension n. The Hilbert series is

$$\mathcal{H}(t) := \sum_{k\geq 0} t^k \dim\left(\mathcal{B}(M)_{(k)}\right) = 1 + t + \cdots t^{n-1}.$$

The next example exhibits the role of Nichols algebras as quantum Borel parts.

Example 5.3.8 Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra of rank n with Cartan matrix $(a_{ij})_{i,j=1\ldots n}$. Let $(\alpha_i)_{i=1,\ldots,n}$ be a set of simple roots for $\mathfrak{g}$ and let $d_i := \langle\alpha_i, \alpha_i\rangle/2$. We construct a braided vector space M with diagonal braiding as a Yetter-Drinfel'd module over an abelian group: fix a root $q \neq 1$ of unity and consider the braiding matrix

$$q_{ii} = q^{2d_i} \qquad q_{ij} = q^{d_i a_{ij}},\ i \neq j.$$

The associated Nichols algebra $\mathcal{B}(M)$ is then the quantum Borel part of the Frobenius-Lusztig kernel $u_q(\mathfrak{g})$. In this case, all Nichols algebras $r_i(\mathcal{B}(M))$ obtained by reflections are isomorphic as algebras. As Hopf algebras, they are isomorphic up to a Drinfel'd twist. The isomorphisms give rise to the Lusztig automorphisms T_{s_i} of the algebra $u_q(\mathfrak{g})$ for the simple root α_i. These automorphisms enter e.g. in the construction of a PBW-basis for $U(\mathfrak{g})$.

In the following example [Hec09], the two Nichols algebras describe two possible Borel parts of the Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2|1)$; they also appear in the description [ST13] of logarithmic conformal field theories. In this example, non-isomorphic Nichols algebras are related by reflections.

Example 5.3.9 Let $q \neq \pm 1$ be a primitive n-th root of unity. Find two two-dimensional diagonally braided vector spaces M, N, with bases $(x_1^{(M)}, x_2^{(M)})$ $(x_1^{(N)}, x_2^{(N)})$ respectively, such that

$$\begin{aligned} q_{11}^{(M)} = q_{22}^{(M)} &= -1 \qquad & q_{12}^{(M)} q_{21}^{(M)} &= q^{-1} \\ q_{11}^{(N)} = -1 \quad q_{22}^{(N)} &= q \qquad & q_{12}^{(N)} q_{21}^{(N)} &= q^{-1}. \end{aligned}$$

We describe a PBW-basis of the Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ by isomorphisms of graded vector spaces to symmetric algebras. To this end, denote for a basis element $x_i^{(M)}$ of M the corresponding Nichols subalgebra by $\mathcal{B}(x_i^{(M)})$, and similarly for N. (We will drop superscripts from now on, wherever they are evident.) A PBW-basis for the Nichols algebra $\mathcal{B}(x_i^{(M)})$ has been discussed in Example 5.3.7. Moreover, we need the shorthand $x_{12} := x_1 x_2 - q_{12} x_2 x_1$. One can show that the multiplication in the Nichols algebras leads to isomorphisms of graded vector spaces:

$$\begin{aligned} \mathcal{B}(M) &\xleftarrow{\sim} \mathcal{B}(x_1) \otimes \mathcal{B}(x_2) \otimes \mathcal{B}(x_1 x_2 - q_{12} x_2 x_1) \\ &\cong \mathbb{C}[x_1]/(x_1^2) \otimes \mathbb{C}[x_2]/(x_2^2) \otimes \mathbb{C}[x_{12}]/(x_{12}^n), \\ \mathcal{B}(N) &\xleftarrow{\sim} \mathcal{B}(x_1) \otimes \mathcal{B}(x_2) \otimes \mathcal{B}(x_1 x_2 - q_{12} x_2 x_1) \\ &\cong \mathbb{C}[x_1]/(x_1^2) \otimes \mathbb{C}[x_2]/(x_2^n) \otimes \mathbb{C}[x_{12}]/(x_{12}^2). \end{aligned}$$

Both Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are of dimension $4n$ and have a Cartan matrix of type A_2. Their Hilbert series can be read off from the

PBW-basis:

$$\mathcal{H}_{\mathcal{B}(M)}(t) = (1+t)(1+t)(1+t^2+t^4\cdots t^{2(n-1)}),$$
$$\mathcal{H}_{\mathcal{B}(N)}(t) = (1+t)(1+t+t^2+\cdots t^{n-1})(1+t^2).$$

The two Hilbert series are different; thus the two Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are *not* isomorphic. The Nichols algebras are, however, related by partial dualizations:

$$r_1(\mathcal{B}(M)) = \mathcal{B}(N) \qquad r_2(\mathcal{B}(M)) \cong \mathcal{B}(N)$$
$$r_1(\mathcal{B}(N)) = \mathcal{B}(M) \qquad r_2(\mathcal{B}(N)) = \mathcal{B}(N)$$

where r_i is the partial dualization with respect to the subalgebra $\mathcal{B}(\mathbb{C}x_i)$. For the isomorphism indicated by $\cong$, the generators x_1 and x_2 have to be interchanged.

A. Category theory

For the reader's convenience we list in this chapter the definitions of categorical notions, that appear throughout the text. For further details we refer the reader to [Kas95] and [Mac98].

A.1. Basic notions

Categories Recall that a *category* $\mathcal{C}$ consists of objects, morphisms and a unital, associative composition law. We say that a A *is an object in* $\mathcal{C}$ or simply write $A \in \mathcal{O}b(\mathcal{C})$ even when the objects of $\mathcal{C}$ form a proper class and not a set. If $\mathcal{O}b(\mathcal{C})$ is a set, the category $\mathcal{C}$ is said to be a *small category*. Denote the set of morphisms from an object A to an object B by $\mathcal{C}(A, B), \mathrm{Hom}_{\mathcal{C}}(A, B)$ or $\mathrm{Hom}(A, B)$ and write $\mathrm{id}_A, \mathrm{id}$ or A for the identity morphism of A. If $\mathcal{C}$ contains only identity morphisms, it is called a *discrete category*.
If we write $f\colon A \to B$ or $A \xrightarrow{f} B$, we mean $f \in \mathcal{C}(A, B)$, i.e. f is a morphism from A to B. By $g \circ f = gf$ or $\left(A \xrightarrow{f} B \xrightarrow{g} C\right)$ we mean the composition of $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$.
A morphism $f\colon A \to B$ is called *isomorphism*, if there is a morphism $g\colon B \to A$ with $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.
A morphism $f\colon A \to B$ is called *monomorphism*, if for any two morphisms $g_1, g_2\colon X \to A$ the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. A morphism $f\colon A \to B$ is called *epimorphism*, if for any two morphisms $h_1, h_2\colon B \to Y$ the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

For a category $\mathcal{C}$ we denote by $\mathcal{C}^{\mathrm{op}}$ the *opposite category* with the same objects as $\mathcal{C}$, morphism sets $\mathcal{C}^{\mathrm{op}}(A, B) := \mathcal{C}(B, A)$ and composition

$$\begin{aligned} \circ\colon \mathcal{C}^{\mathrm{op}}(B, C) \times \mathcal{C}^{\mathrm{op}}(A, B) &\to \mathcal{C}^{\mathrm{op}}(A, C)\ , \\ (g, f) &\mapsto g \circ_{\mathrm{op}} f := f \circ g\ . \end{aligned}$$

The *direct product* of a category $\mathcal{C}$ and a category $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$; it has as objects pairs of objects (C, D) with $C \in \mathcal{O}b(\mathcal{C})$ and

$D \in \mathcal{O}b(\mathcal{C})$ and a morphism form (C, D) to (C', D') is a pair $(f, g) = (f: C \to C', g: D \to D')$. The composition of morphisms (f, g) and (f', g') is given by $(f', g') \circ (f, g) := (f' \circ f, g' \circ g)$.

Let I be an index set and let $\{\mathcal{C}_i\}_{i \in I}$ be a family of categories. The *disjoint union* $\coprod_{i \in I} \mathcal{C}_i$ is the following category: objects are pairs (X_i, i) with X_i an object in $\mathcal{C}_i$ and $i \in I$, a morphisms are pairs (f_i, i) with f_i a morphism in $\mathcal{C}_i$ and $i \in I$, the composition of morphisms is given by composition in the first entry, i.e. $(f_i, i) \circ (g_i, i) := (f_i \circ g_i, i)$.

Functors Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of an object function, assigning an object $F(A)$ in $\mathcal{D}$ to each object A in $\mathcal{C}$, and a morphism function, assigning to each morphism $f: A \to B$ in $\mathcal{C}$ a morphism $F(f): F(A) \to F(B)$, such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{O}b(\mathcal{C})$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms in $\mathcal{C}$. Write $F: \mathcal{C} \to \mathcal{D}$ or $\mathcal{C} \xrightarrow{F} \mathcal{D}$ to express, that F is a functor from $\mathcal{C}$ to $\mathcal{D}$. The *identity functor* $\mathrm{Id}_{\mathcal{C}} = \mathrm{Id}: \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$ maps an object A to A and a morphism f to f.

Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories; the *composition* of a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and a functor $\mathcal{D} \xrightarrow{G} \mathcal{E}$ is the functor $G \circ F = GF: \mathcal{C} \to \mathcal{E}$ which assigns $G(F(A))$ to an object A in $\mathcal{C}$ and $G(F(f))$ to a morphism f in $\mathcal{C}$.

Transformations Let $F, G: \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* or simply a *transformation* α from F to G consists of a family $\alpha = \{\alpha_A : F(A) \to G(A)\}_{A \in \mathcal{O}b(\mathcal{C})}$ of morphisms in $\mathcal{D}$ that is *natural* in A, i.e. for all morphisms $f: A \to B$ in $\mathcal{C}$ the equality $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ holds. The morphisms α_A are called the *components of the transformation* α and we also say, by abuse of language, that $\alpha_A: F(A) \to G(A)$ is a natural transformation. Notations like $\alpha: F \to G$ or $F \xrightarrow{\alpha} G$ are used to express that α is a transformation from F to G.
A transformation $\alpha: F \to G$ is called *natural isomorphism*, if each component is an isomorphism.

Let $F, G, H: \mathcal{C} \to \mathcal{D}$ be functors and let $\alpha: F \to G$ and $\beta: G \to H$ be transformations. The *vertical composition* of α and β is the transformation $\beta \bullet \alpha: F \to H$ given by the components $(\beta \bullet \alpha)_A := \beta_A \circ \alpha_B$.

Let $F, G: \mathcal{C} \to \mathcal{D}$ and $H, K: \mathcal{D} \to \mathcal{E}$ be functors and let $\alpha: F \to G$ and $\beta: H \to K$ be transformations. The *horizontal composition* of α and

β is the transformation $\beta \circ \alpha\colon H \circ F \to K \circ G$ with the components $(\beta \circ \alpha)_A := K(\alpha_A) \circ \beta_{FA} = \beta_{GA} \circ H(\alpha_A)$.

Equivalences of categories We say that $F\colon \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories*, if there is an *inverse functor* $G\colon \mathcal{D} \to \mathcal{C}$, i.e. $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$.
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say F *is an equivalence of categories*, if there is a *quasi-inverse functor* $G\colon \mathcal{D} \to \mathcal{C}$, i.e. there are natural isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cong} GF$ and $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$. A quasi-inverse functor is unique, up to a natural isomorphism.

A.2. Adjunctions and monads

Adjoint functors Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors. The functor L is called *left-adjoint to* R resp. the functor R is called *right-adjoint to* L, if there are transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to RL$ and $\varepsilon\colon LR \to \mathrm{Id}_{\mathcal{D}}$, such that the compositions

$$L(C) \xrightarrow{L(\eta_C)} LRL(C) \xrightarrow{\varepsilon_{LC}} L(C) \qquad \text{and}$$
$$R(D) \xrightarrow{\eta_{R(D)}} RLR(C) \xrightarrow{R(\varepsilon_D)} R(D)$$

are identity morphisms. We write $L \dashv R$ and say that $(L, R, \eta, \varepsilon)$ is an *adjunction*, the transformation η is called *unit of the adjunction* and ε is called *counit of the adjunction*.
If R and R' are right-adjoint to a functor L, then there is a unique natural isomorphism $\varphi\colon R \to R'$ compatible with the units and counits of the corresponding adjunctions.

We say that $(L, R, \eta, \varepsilon)$ is an *adjoint equivalence*, if η and ε are natural isomorphisms. The functor R is then a quasi-inverse of L; every equivalence of categories is part of an adjoint equivalence.

Monads Let $T\colon \mathcal{C} \to \mathcal{C}$ be a functor and let $\mu\colon T \circ T \to T$ and $\eta\colon \mathrm{Id}_{\mathcal{C}} \to T$ be natural transformations. The triple (T, μ, η) is called a *monad on* $\mathcal{C}$, if the equalities

$$\mu_A \circ T(\mu_A) = \mu_A \circ \mu_{TA} \qquad \text{and}$$
$$\mu_A \circ \eta_{TA} = \mathrm{id}_{TA} = \mu_A \circ T(\eta_A)$$

hold for every object A in $\mathcal{C}$. We also say that T is a monad without mentioning the transformations μ and η, called *multiplication* and *unit* respectively.

A *module over* T or T*-module* is a pair (A, r) consisting of an object A in $\mathcal{C}$ and a morphism $r\colon TA \to A$, such that

$$r \circ \mu_A = r \circ T(r) \qquad \text{and}$$
$$r \circ \eta_A = \mathrm{id}_A \ .$$

If (A, r_A) and (B, r_B) are T-modules a morphism $f\colon A \to B$ is called T*-linear*, if

$$f \circ r_A = r_B \circ T(f) \ .$$

The *Eilenberg-Moore category* $\mathcal{C}^T$ of a monad T has as objects T-modules and as morphisms T-linear morphisms; the composition of morphisms is inherited from $\mathcal{C}$.

Monadic functors The *forgetful functor* $\mathsf{U}^T : \mathcal{C}^T \to \mathcal{C}$ sending (A, r_A) to A is part of an adjunction $(\mathsf{U}^T, \mathsf{F}^T, \eta^T, \varepsilon^T)$.
The right-adjoint functor of U^T is the *free functor* $\mathsf{F}_T : \mathcal{C} \to \mathcal{C}^T$, which sends an object A to the T-module (TA, μ_A) and a morphism f to the T-linear map $T(f)$. The unit of T defines the unit η^T of the adjunction and the counit η^T has the components $\varepsilon^T_{(A,r_A)} = r_A$.

The *monad of an adjunction* $(L, R, \eta, \varepsilon)$ is given by the functor $T := RL$ with multiplication $\mu_A := R(\varepsilon_{LA})\colon RLRL(A) \to RL(A)$ and unit $\eta_A\colon A \to RL(A)$.

Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor, that is part of an adjunction $(F, G, \eta, \varepsilon)$ and by T its associated monad on $\mathcal{C}$. The *comparison functor* $K : \mathcal{D} \to \mathcal{C}^T$ is given by $K(D) := (G(D), G(\varepsilon_D))$.
A functor $G\colon \mathcal{D} \to \mathcal{C}$ is called monodic, if it has a left-adjoint $F : \mathcal{C} \to \mathcal{D}$ and the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$ is an equivalence of categories.

A.3. Monoidal categories

Definition A.3.1 A monoidal category is a 6-tuple $(\mathcal{C}, \otimes, \mathbf{1}, \mathsf{a}, \mathsf{l}, \mathsf{r})$ consisting of a category $\mathcal{C}$, a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (tensor product), an object

$\mathbf{1}$ in $\mathcal{C}$, a natural isomorphism $\mathsf{a}\colon \otimes\circ(\otimes\times\mathrm{Id}_\mathcal{C})\to\otimes\circ(\mathrm{Id}_\mathcal{C}\times\otimes)$ (associator), a natural isomorphism $\mathsf{l}\colon \mathbf{1}\otimes\_\to\mathrm{Id}_\mathcal{C}$ (left unit isomorphism) and a natural isomorphism $\mathsf{r}\colon \_\otimes\mathbf{1}\to\mathrm{Id}_\mathcal{C}$ (right unit isomorphism), such that the following diagrams commute for all objects U, V, W and X in $\mathcal{C}$:

$$\begin{tikzcd}
 & (U\otimes V)\otimes(W\otimes X) \arrow[dr, "\mathsf{a}_{U,V,W\otimes X}"] & \\
((U\otimes V)\otimes W)\otimes X \arrow[ur, "\mathsf{a}_{U\otimes V,W,X}"] \arrow[d, "\mathsf{a}_{U,V,W}\otimes\mathrm{id}_X"'] & & U\otimes(V\otimes(W\otimes X)) \\
(U\otimes(V\otimes W))\otimes X \arrow[rr, "\mathsf{a}_{U,V\otimes W,X}"] & & U\otimes((V\otimes W)\otimes X) \arrow[u, "\mathrm{id}_U\otimes\mathsf{a}_{V,W,X}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(U\otimes\mathbf{1})\otimes V \arrow[rr, "\mathsf{a}_{U,\mathbf{1},V}"] \arrow[dr, "\mathsf{r}_U\otimes\mathrm{id}_V"'] & & U\otimes(\mathbf{1}\otimes V) \arrow[dl, "\mathrm{id}_U\otimes\mathsf{l}_V"] \\
 & (U\otimes\mathbf{1})\otimes V &
\end{tikzcd}$$

The category is called *strict*, if a, l and r are the identity transformations.

Definition A.3.2 Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $F\colon\mathcal{C}\to\mathcal{D}$ be a functor, let $F^0\colon F\mathbf{1}_\mathcal{C}\to\mathbf{1}_\mathcal{D}$ and $F_0\colon\mathbf{1}_\mathcal{D}\to F$ be morphisms and $F^2\colon F\circ\otimes\to F\otimes F$ and $F_2\colon F\otimes F\to F\circ\otimes$ transformations.

1. The triple (F, F_2, F_0) is called *lax monoidal functor* and F_2 and F_0 are called the *monoidal structure* of F, if for all objects U, V, W in $\mathcal{C}$ the diagrams

$$\begin{tikzcd}
(FU\otimes FV)\otimes FW \arrow[r, "\mathsf{a}_{FU,FV,FW}"] \arrow[d, "F_2(U,V)\otimes\mathrm{id}_{FW}"'] & FU\otimes(FV\otimes FW) \arrow[d, "\mathrm{id}_{FU}\otimes F_2(V,W)"] \\
F(U\otimes V)\otimes FW \arrow[d, "F_2(U\otimes V,W)"'] & FU\otimes F(V\otimes W) \arrow[d, "F_2(U,V\otimes W)"] \\
F((U\otimes V)\otimes W) \arrow[r, "F(\mathsf{a}_{U,V,W})"] & F(U\otimes(V\otimes W))
\end{tikzcd}$$

$$\begin{tikzcd}
F(\mathbf{1}\otimes U) \arrow[rr, "F(\mathsf{l}_U)"] & & FU & & F(U\otimes\mathbf{1}) \arrow[ll, "F(\mathsf{r}_U)"'] \\
F\mathbf{1}\otimes FU \arrow[u, "F_2(\mathbf{1},U)"] & \mathbf{1}\otimes FU \arrow[l, "F_0\otimes\mathrm{id}"'] \arrow[ur, "\mathsf{l}_{FU}"] & & FU\otimes\mathbf{1} \arrow[ul, "\mathsf{r}_{FU}"'] \arrow[r, "\mathrm{id}\otimes F_0"] & FU\otimes F\mathbf{1} \arrow[u, "F_2(U,\mathbf{1})"']
\end{tikzcd}$$

   commute. The monoidal functor is called *strong monoidal* or simply *monoidal*, if F_2 and F_0 are isomorphisms. It is called *strict*, if F_2 and F_0 are identities.

2. The triple (F, F^2, F^0) is called *oplax monoidal functor* and F_2 and F_0 are called the *(op)monoidal structure* of F, if the diagrams above

commute after changing F_2 to F^2 and F_0 to F^0 (and of course changing the direction of those arrows). If is called *strong (op)monoidal* resp. *strict*, if F^2 and F^0 are isomorphisms resp. identities.

3. Let F and G be lax monoidal resp. oplax monoidal functors. If defined, the composition $F \circ G$ is an (op)lax monoidal functor with monoidal structure

$$\begin{aligned}(FG)_2(U,V) &:= F(G_2(U,V)) \circ F_2(GU,GV) \quad \text{and} \\ (FG)_0 &:= F(G_0) \circ F_0\end{aligned}$$

resp. opmonoidal structure

$$\begin{aligned}(FG)^2(U,V) &:= F^2(GU,GV) \circ F(G^2(U,V)) \quad \text{and} \\ (FG)^0 &:= F^0 \circ F(G^0)\ .\end{aligned}$$

Definition A.3.3 Let F and G be (op)lax monoidal functors and let $\alpha\colon F \to G$ be a natural transformation. If the equalities

$$\alpha_{U\otimes V} \circ F_2(U,V) = G_2(U,V) \circ (\alpha_U \otimes \alpha_V) \quad \text{and} \quad \alpha_{\mathbf{1}} F_0 = G_0 \tag{A.1}$$

hold for all objects U and V in $\mathcal{C}$ we say that α is a *monoidal* transformation. We call α an *opmonoidal* transformation, if the equalities

$$(\alpha_U \otimes \alpha_V) \circ F^2(U,V) = G^2(U,V) \circ \alpha_{U\otimes V} \tag{A.2}$$

hold for all objects U and V in $\mathcal{C}$. By abuse of language we call opmonoidal transformations sometimes also monoidal.

Lemma A.3.4 Let $(F, G, \eta, \varepsilon)$ be an adjunction. The following statements hold, as well as the dual statements, i.e. exchange F and G and the words lax and oplax.

1. If G is a lax monoidal functor there is an oplax monoidal structure on F.

2. If F and G are lax monoidal functors and η and ε are monoidal transformations, then F is a strong monoidal functor.

3. If F is strong monoidal, there is a unique lax monoidal structure on G, such that η and ε are monoidal transformations.

Sketch of proof.

1. The morphism set $\mathrm{Hom}(X \otimes Y, G(FX \otimes FY))$ contains the morphism $f := G_2(FX, FY) \circ (\eta_X \otimes \eta_X)$. Define $F^2(X,Y)$ as the image of f under the isomorphism
$$\mathrm{Hom}(X \otimes Y, G(FX \otimes FY)) \cong \mathrm{Hom}(F(X \otimes Y), FX \otimes FY) ,$$
i.e. $F^2(X,Y) = \varepsilon_{FX \otimes FY} \circ F(f)$. Define $F^0 := \varepsilon_{\mathbf{1}} \circ F(G_0)$.

2. Under the additional assumptions on F, η and ε one proves that F^2 is inverse to F_2 and F^0 is inverse to F_0.

3. Assume that $G_2(X,Y)$ defines a monoidal structure on G. Since η and ε are monoidal transformations we conclude from 2. that $G_2(X,Y)$ defines an oplax monoidal structure $F^2(X,Y)$, whose components are inverse to $F_2(X,Y)$. Hence $F^2(X,Y) = F_2^{-1}(X,Y)$ and $G_2(X,Y)$ is the monoidal structure obtained from the dual statement of 1.

□

Corollary A.3.5 Let $(F, G, \eta, \varepsilon)$ be an adjoint equivalence. The functor F is strong monoidal, iff G is strong monoidal.

Remark A.3.6 Usually one can assume, without loss of generality, that one works in a strict monoidal category due to the fact that every monoidal category $\mathcal{C}$ is equivalent, as a monoidal category, to a strict monoidal category $\mathcal{C}^{\mathrm{str}}$. See for example Section $XI.5$ in [Kas95].

Definition A.3.7 Let $\mathcal{C}$ be a strict monoidal category and let X be an object in $\mathcal{C}$. A *left dual object* for X is an object ${}^\vee X$ together with morphisms $\mathrm{ev}_X : {}^\vee X \otimes X \to \mathbf{1}$ (evaluation) and $\mathrm{coev}_X : \mathbf{1} \to X \otimes {}^\vee X$ (coevaluation), such that
$$(\mathrm{ev}_X \otimes \mathrm{id}_{{}^\vee X})(\mathrm{id}_{{}^\vee X} \otimes \mathrm{coev}_X) = \mathrm{id}_{{}^\vee X} \quad \text{and}$$
$$(\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X \ .$$
Analogously one defines a *right dual object* as an object $X^\vee$ together with morphisms $\widetilde{\mathrm{ev}}_X : X \otimes X^\vee \to \mathbf{1}$ and $\widetilde{\mathrm{coev}}_X : \mathbf{1} \to X^\vee \otimes X$ fulfilling
$$(\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee} \quad \text{and}$$
$$(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}) = \mathrm{id}_X \ .$$
If every object X in $\mathcal{C}$ has a left resp. right dual, the category is called *left rigid* resp. *right rigid*. The category $\mathcal{C}$ is called *rigid*, if it is left and right rigid.

If X' and X'' are both left or right dual to X, there is a unique isomorphism $X' \to X''$ compatible with evaluation. Let $\mathcal{C}$ be (say) right rigid. If we choose exactly one dual object for each object X, we get a functor $(\_)^\vee \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ by defining for $f \colon X \to Y$ the morphism

$$f^\vee := (\mathrm{id} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id} \otimes f \otimes \mathrm{id})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}) \,.$$

Similarly, one defines a functor ${}^\vee(\_) \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. The functors ${}^\vee(\_)$ and $(\_)^\vee$ are strong monoidal $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\otimes\mathrm{op}}$ where $\mathcal{C}^{\otimes\mathrm{op}}$ denotes the monoidal category with opposed tensor product $X \otimes^{\mathrm{op}} Y := Y \otimes X$.

Definition A.3.8 Let $\mathcal{C}$ be a rigid category and $\alpha \colon {}^\vee(\_) \to (\_)^\vee$ a natural isomorphism. It is called *pivotal structure* on $\mathcal{C}$, if it is a monoidal isomorphism. For an endomorphism $f \colon X \to X$ in a pivotal category $\mathcal{C}$ one defines the *left trace* $\mathrm{tr}_\ell(f)$ and the *right trace* $\mathrm{tr}_r(f)$ as

$$\begin{aligned} \mathrm{tr}_\ell(f) &:= \mathrm{ev}_X \circ (\alpha_X^{-1} \otimes f) \circ \widetilde{\mathrm{coev}}_X \\ \mathrm{tr}_\ell(f) &:= \widetilde{\mathrm{ev}}_X \circ (f \otimes \alpha_X) \circ \mathrm{coev}_X \,. \end{aligned}$$

A pivotal structure is called *spherical*, if left and right trace coincide for all endomorphisms of $\mathcal{C}$.

Definition A.3.9 Let $\mathcal{C}$ be a monoidal category and $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ a family of morphisms, natural in $X, Y \in \mathcal{O}b(\mathcal{C})$. The family $c_{X,Y}$ is called *prebraiding*, if the following two hexagons commute for all $X, Y, Z \in \mathcal{O}b(\mathcal{C})$

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[rr, "c_{X \otimes Y, Z}"] & & Z \otimes (X \otimes Y) \arrow[dr, "\mathsf{a}^{-1}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\mathsf{a}^{-1}"] \arrow[dr, "\mathrm{id} \otimes c_{Y,Z}"'] & & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[rr, "\mathsf{a}^{-1}"] & & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[rr, "c_{X, Y \otimes Z}"] & & (Y \otimes Z) \otimes X \arrow[dr, "\mathsf{a}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\mathsf{a}"] \arrow[dr, "c_{X,Y} \otimes \mathrm{id}"'] & & & & Y \otimes (Z \otimes X) \\
 & (Y \otimes X) \otimes Z \arrow[rr, "\mathsf{a}"] & & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id} \otimes c_{X,Z}"'] &
\end{tikzcd}$$

If c is a natural isomorphism the category $\mathcal{C}$ is called a *braided* category with *braiding* c.

Let $\mathcal{C}$ and $\mathcal{D}$ be braided categories. A monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *braided*, if

$$F(c_{X,Y}) \circ F_2(X,Y) = F_2(Y,X) \circ c_{FX,FY} \ .$$

Definition A.3.10 Let $\mathcal{C}$ be a braided category. The *symmetric center* of $\mathcal{C}$ or the subcategory of *transparent objects* $\mathcal{Z}_{\mathrm{sym}}$ in $\mathcal{C}$ is defined as the full subcategory of $\mathcal{C}$ containing the objects X, such that $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all $Y \in \mathcal{O}b(\mathcal{C})$.

A.4. Modular categories

Let $\mathbb{k}$ be a field. We call a category $\mathcal{C}$ $\mathbb{k}$-*linear*, if $\mathcal{C}$ has a zero object, all finite sums and every Hom set is a $\mathbb{k}$-vector space such that the composition $\circ$ of morphisms is $\mathbb{k}$-linear in each variable.

Let I be an indexing set and $\{\mathcal{C}_i\}_{i\in I}$ a family of $\mathbb{k}$-linear categories. The *direct sum* $\bigoplus_{i\in I} \mathcal{C}_i$ is defined as the following category: objects are families $(X_i)_{i\in I}$ where X_i is an object in $\mathcal{C}_i$ and only finitely many X_i are not the zero object, the morphism set $\mathrm{Hom}((X_i)_{i\in I}, (Y_i)_{i\in I})$ is given by the vector space $\bigoplus_{i\in I} \mathrm{Hom}(X_i, Y_i)$.

An object X in a $\mathbb{k}$-linear category is called *simple*, if the Hom-space $\mathcal{C}(X,X)$ is one dimensional.

A $\mathbb{k}$-linear abelian category is called *finite*, if

- there are, up to isomorphism, only finitely many simple objects,
- every Hom space $\mathcal{C}(X,Y)$ is finite dimensional,
- every object has finite length,
- the category $\mathcal{C}$ has enough projectives.

A *tensor category* is a rigid, monoidal category $\mathcal{C}$ which is $\mathbb{k}$-linear, the unit object $\mathbf{1}$ is simple and the functor $\otimes$ is $\mathbb{k}$-linear in each variable.

A finite tensor category $\mathcal{C}$ is called *fusion category*, if every object is isomorphic to a finite sum of simple objects.

A *premodular category* is a braided fusion category together with a spherical structure.

A premodular category $\mathcal{C}$ is called *modular*, if $\mathcal{Z}_{\mathrm{sym}}(\mathcal{C})$ is equivalent to the category $\mathbf{vect}_{\mathbb{k}}$ of finite dimensional vector spaces.

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 08.05.2014 Unterschrift

Zusammenfassung

Monoidale Kategorien treten auf als Darstellungskategorien von Hopf-Algebren und anderen algebraischen Strukturen. Sie spielen unter anderem eine zentrale Rolle in Darstellungstheorie und topologischer Feldtheorie. In dieser Dissertation werden zwei verschiedene Konstruktionen durchgeführt, die aus diesen Zusammenhängen heraus motiviert sind. Im Besonderen spielen dabei algebraische und darstellungstheoretische Strukturen innerhalb dieser monoidalen Kategorien eine wichtige Rolle. Über diese gibt das erste Kapitel einen kurzen Überblick.

Kapitel 2 beschäftigt sich mit Yetter-Drinfel'd Moduln über Hopf-Algebren in verzopften Kategorien und stellt neue Resultate für die folgenden Kapitel bereit. Ein besonderer Schwerpunkt liegt dabei auf der Beschreibung eines Isomorphismus $\Omega\colon {}^{A}_{A}\mathcal{YD}(\mathcal{C}) \to {}^{A^\vee}_{A^\vee}\mathcal{YD}(\mathcal{C})$ von verzopften Kategorien, wobei, falls existent, $A^\vee$ die duale Hopf-Algebra von A bezeichnet.

Im dritten Kapitel erinnern wir, für eine diskrete Gruppe Γ, an die Definition einer Γ-verzopften Kategorie und einer quasi-triangulären Hopf Γ-Koalgebra.

Im vierten Kapitel wird die erste Konstruktion dieser Arbeit vorgestellt: Zu einer beliebigen monoidale Kategorie $\mathcal{C}$ mit schwacher Gruppenwirkung durch eine Gruppe Γ, konstruieren wir eine Γ-verzopfte Kategorie $\mathcal{Z}^\Gamma(\mathcal{C})$, deren neutrale Komponente das Drinfel'd Zentrum von $\mathcal{C}$ ist. Ist $\mathcal{C}$ die Kategorie von Moduln über einer Hopf-Algebra H, so lassen sich die homogenen Komponenten von $\mathcal{Z}^\Gamma(\mathcal{C})$ als getwistete Yetter-Drinfel'd Moduln über H beschreiben. Für eine endlich-dimensionale Hopf-Algebra H verallgemeinert unsere Kategorie $\mathcal{Z}^\Gamma(\mathcal{C})$ eine Hopf-algebraische Konstruktion von Virelizier.

Das letzte Kapitel behandelt die zweite Konstruktion dieser Arbeit. Motiviert durch Arbeiten von Heckenberger und Schneider führen wir den Begriff einer partiellen Dualisierung von Hopf-Algebren in verzopften Kategorien ein. Eine wichtige Idee in dieser Arbeit ist es, im Rahmen verzopfter Kategorien algebraisch aufwändige Rechnungen mit Smash-Produkten konzeptionell zu vereinfachen. Ausgehend von einer Hopf-Algebra H in einer verzopften Kategorie $\mathcal{C}$ mit einer Hopf-Unteralgebra A und einer Hopf-Algebraprojektion $\pi\colon H \to A$ konstruieren wir die partielle Dualisierung H' von H bezüglich A. Die Hopf-Algebren H und H' sind im Allgemeinen weder isomorph noch Morita-äquivalent. Wir zeigen aber, dass die Kategorien ${}^{H}_{H}\mathcal{YD}(\mathcal{C})$ und ${}^{H'}_{H'}\mathcal{YD}(\mathcal{C})$ immer isomorph als verzopfte Kategorien sind. Darüber hinaus ist die partielle Dualisierung von H' bzgl. der Unteralgebra $A^\vee$ wieder isomorph zu H.

Summary

Monoidal categories appear as representation categories of Hopf algebras and other algebraic structures. Among other things they play a central role in representation theory and topological field theory.

In this dissertation we consider two constructions which are motivated from these theories. Algebraic and representation-theoretical structures internal to monoidal categories play an important role for these constructions. The first chapter gives a short overview of them.

Chapter 2 deals with Yetter-Drinfel'd modules over Hopf algebras in a braided category and supplies new results used in the following chapters. The main goal is to describe an isomorphism $\Omega\colon {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^{A^\vee}_{A^\vee}\mathcal{YD}(\mathcal{C})$ of braided categories, where, if it exists, $A^\vee$ denotes the Hopf algebra dual to A.

In the third chapter we remind the reader of the definitions of Γ-braided categories and quasi-triangular Hopf Γ-coalgebras for a discrete group Γ.

Chapter 4 deals with the first construction of this thesis. Given a monoidal category $\mathcal{C}$ with a weak action by a group Γ, we construct a Γ-braided category $\mathcal{Z}^\Gamma(\mathcal{C})$ whose neutral component is the Drinfel'd center of $\mathcal{C}$. If $\mathcal{C}$ is the category of modules over a Hopf algebra H, we describe the homogeneous components of $\mathcal{Z}^\Gamma(\mathcal{C})$ as twisted Yetter-Drinfel'd modules over H. For a finite dimensional Hopf algebra H our category $\mathcal{Z}^\Gamma(\mathcal{C})$ generalizes a Hopf-algebraic construction of Virelizier.

Chapter 5 deals with the second construction in this thesis. Motivated by work of Heckenberger and Schneider, we introduce the notion of a partial dualization for Hopf algebras in a braided category. One of the basic ideas of this thesis is to give, using the framework of braided categories, a conceptual simplification of algebraically involved calculations with smash products. Starting with a Hopf algebra H in a braided category $\mathcal{C}$ together with a Hopf subalgebra A and a Hopf algebra projection $\pi\colon H \to A$ we construct a Hopf algebra H', the partial dual of H with respect to A. In general the Hopf algebras H and H' are neither isomorphic nor Morita equivalent, but we show that the categories ${}^H_H\mathcal{YD}(\mathcal{C})$ and ${}^{H'}_{H'}\mathcal{YD}(\mathcal{C})$ are always isomorphic as braided categories. Moreover, the partial dual of H' with respect to $A^\vee$ is isomorphic to H.