E_n-cohomology as functor cohomology and additional structures

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1 Introduction

The little n-cubes PROP and the corresponding operad were introduced around 1970 in order to study n-fold loop spaces. In [9] Michael Boardman and Rainer Vogt define a tensor product and a bar construction for Lawvere theories and and use this to prove iteratively that a space on which the little n-cubes operad PROP acts can be delooped n times and that hence the little n-cubes PROP recognizes n-fold loop spaces. Peter May compares the monad associated to the little n-cubes operad with the monad $\Omega^n\Sigma^n$ and shows that applied to connected spaces these monads yield weakly equivalent results. He uses the bar construction for monads to also derive a recognition principle for n-fold loop spaces.
Since then, operads have received a lot of attention in topology, algebra and physics as a means to encode algebraic structures abstractly. In particular, one can study algebraic variants of the little n-cubes operad, so called E_n-operads, in the world of differential graded k-modules over a commutative unital ring k. Algebras over such an E_n-operad, called E_n-algebras, hence can be thought of as algebraic analogues of n-fold loop spaces. An E_n-algebra is a differential graded k-module endowed with a product which is associative up to a coherent set of all possible higher homotopies for associativity, but commutative only up to higher homotopies of a certain level, depending on n.
Important examples arise for $n = 1$ and $n = \infty$: An E_1-algebra is exactly an A_∞-algebra, an algebraic analogue of a space with A_∞-structure as introduced by Stasheff in his study of H-spaces [64]. A result by Kadeishvili (see [39]) shows that A_∞-algebras classify quasi-isomorphism types of differential graded k-algebras over a field k. In [60] Steffen Sagave proves that if one is willing to work with derived A_∞-algebras, which combine projective resolutions with A_∞-structures, similar results can be obtained over any commutative ring. Derived A_∞-algebras have been studied in an operadic context in [42] and [2]. On the other hand, E_∞-algebras are the right notion of commutative algebras up to coherent homotopy. An important example is given by the singular cochains on a space, with multiplication giving rise to the cup product on singular cohomology. Michael Mandell showed in [48] that the E_∞-algebra structure on the cochains of a nilpotent space of finite type determines this space up to weak equivalence. In the world of spectra E_∞-algebras, called E_∞ ring spectra, also play an important role.
In the differential graded setting, every operad which fulfills certain cofibrancy conditions automatically comes with a notion of homology and cohomology specifically suited to algebras over this operad. In particular, we can say what E_n-homology and E_n-cohomology of an E_n-algebra A with coefficients in a so-called representation of A are. Again, one finds that the cases $n = 1$ and $n = \infty$ yield familiar notions: For $n = 1$, one can show that E_1-homology coincides with Hochschild homology of A_∞-algebras as defined in [25]. If $n = \infty$, we retrieve the notion of Γ-homology (see [58]). Note that any commutative algebra is in particular an E_n-algebra for any n. For commutative algebras and trivial coefficients, E_n-homology is known to coincide with Teimuraz Pirashvili's higher order Hochschild homology

as defined in [52].
The operadic definition of E_n-homology and cohomology is given in terms of derived functors and is a priori difficult to compute. A result making actual computations feasible was given by Benoit Fresse in [22] and [23]: He shows that E_n-homology of any E_n-algebra with trivial coefficients can be computed via an iterated bar construction. The bar construction of a differential graded k-algebra was originally defined by Samuel Eilenberg and Saunders Mac Lane in [17] in their study of $K(\pi, n)$-spaces. If the bar construction is applied to a differential graded commutative algebra, it again yields a differential graded commutative algebra, hence the construction can be iterated. In [23] Fresse shows that the structure of an E_n-algebra is sufficient to define an n-fold bar construction. Moreover, up to suspension the homology of the n-fold bar construction of an E_n-algebra computes E_n-homology with trivial coefficients. Unpublished work of Benoit Fresse shows that, at least for a commutative algebra A and coefficients in a symmetric A-bimodule M, one can twist the bar construction to compute E_n-homology and E_n-cohomology with coefficients. We give the details of a proof of this result sketched by Fresse and use this to show that E_n-homology as well as E_n-cohomology of commutative algebras coincides with higher order Hochschild homology and cohomology not only for trivial coefficients, but for coefficients in any symmetric bimodule.
In [41], Muriel Livernet and Birgit Richter use that E_n-homology can be computed via the iterated bar construction to give an interpretation of E_n-homology of commutative algebras with trivial coefficients as functor homology. To be more precise, they show that E_n-homology in these cases can be calculated as Tor groups of certain functors with respect to a category Epi_n of trees encoding the structure of the n-fold bar complex. There is an extension of this result to arbitrary E_n-algebras by Fresse, see [21]. Other interpretations as functor homology have for example been given for Gamma homology in [53], for Hochschild and cyclic homology in [54] and for Leibniz homology of Lie algebras in [35]. Depending on the category, functor homology allows a more combinatorial description of the objects in question. General constructions for Ext and Tor, like for example the construction of the Yoneda pairing, can be carried through. In this thesis, we enlarge the category Epi_n to incorporate the twist needed to compute E_n-homology with coefficients. We show that the functor homology interpretation also holds for E_n-homology as well as for E_n-cohomology of commutative algebras with coefficients in a symmetric bimodule.
Only a few concrete calculations of E_n-homology and -cohomology have been possible until now. Examples include E_n-homology of certain free commutative and certain trivial commutative algebras with trivial coefficients in [23] (based on calculations in [12]) and a comparison of higher order Hochschild homology with the cohomology of iterated loop spaces (see [52], [22]). For commutative algebras and over the rationals Pirashvili proved that E_n-homology admits a decomposition, called the Hodge decomposition, which generalizes the well known λ-decomposition of Hochschild homology (see [52]). In [57] Birgit Richter and the author construct a spectral sequence converging to E_n-homology with trivial coefficients. In characteristic zero for all n and for $n = 2$ in characteristic two the E^2-term

of this spectral sequence can be identified with derived indecomposables with respect to the n-Gerstenhaber structure on the homology of the E_n-algebra. In characteristic zero this spectral sequence allows an identification of the summands of the Hodge decomposition of E_n-homology with these derived indecomposables and an interpretation of these summands as the homology of higher powers of the cotangent complex. The spectral sequence is also used to calculate the E_n-homology of free graded commutative algebras and the E_2-homology of certain Hochschild cochain complexes. In [10] E_n-homology of certain free graded commutative algebras and of truncated polynomial algebras over the finite field $\mathbb{F}_p$ with coefficients in the algebra itself is calculated.
To make more calculations possible one would like to understand the structure of E_n-homology and -cohomology better. A generalized version of the Deligne conjecture says that for a suitable choice $D^*_{E_n}(A;A)$ of a chain complex calculating E_n-cohomology of an E_n-algebra A with coefficients in itself, $D^*_{E_n}(A;A)$ will be an E_{n+1}-algebra. The original Deligne conjecture concentrated on the case $n=1$, with the E_2-structure on the Hochschild cochains giving rise to the well known Gerstenhaber algebra structure on Hochschild cohomology. Proofs have been given, among others, by James McClure and Jeffrey Smith in [51] for $n=1$. The generalized version has for instance been discussed by Po Hu, Igor Kriz and Alexander Voronov in [37] and by Jacob Lurie in [45, 6.1.4]. Grégory Ginot uses a geometric approach in [26] to construct corresponding operations on higher order Hochschild cohomology. There is also a homological variant of the Deligne conjecture which has been proven by Morten Brun, Zbigniew Fiedorowicz and Rainer Vogt in [11] and by Maria Basterra and Michael Mandell [5] for higher topological Hochschild homology in the context of spectra.
In this thesis we use the interpretation of E_n-cohomology as functor cohomology to investigate whether the Yoneda pairing gives rise to cohomology operations on E_n-cohomology which would be part of the induced structure of an E_{n+1}-action, like for example a squaring operation in positive characteristic as discussed by Stefan Schwede in [62] for $n=1$. Unfortunately, the representing object has trivial E_n-cohomology, hence no operations arise this way. Also part of an E_{n+1}-structure are so-called higher cup products $\cup_0, ..., \cup_n$, with the zeroth cup product giving rise to the multiplication every E_{n+1}-algebra is endowed with, while $\cup_{i+1}$ is a homotopy for the commutativity of $\cup_i$. In characteristic two we give an explicit construction of $\cup_1$ on the cochain complex arising via the n-fold bar construction.

Outline Chapter 2 is an introductory chapter in which we recall the basic concepts we will use throughout this thesis and fix notation. We discuss operads and algebras over operads as well as related model structures. We proceed to define representations, the universal enveloping algebra, derivations and Kähler differentials in the operadic context and then give the definition of operadic homology and cohomology. Finally, we recollect some basic material about functor homology.
In chapter 3, we give a proof of an unpublished result by Benoit Fresse which we will need

in the later chapters. We start with a closer look at E_n-operads and E_n-algebras. Then we recall how one can extend the iterated bar complex to E_n-algebras to compute E_n-homology with trivial coefficients. After that we proof that one can extend this result to E_n-homology and E_n-cohomology of commutative algebras with coefficients in a symmetric bimodule. To do this, we define a twist on a certain extension of the iterated bar construction associated to commutative algebras. We mimic methods from [23] to show that this twist lifts to the iterated bar complex for E_∞-algebras and restricts to E_n-structures. Then we show that this twisted complex as well as a standard complex computing E_n-homology both define cofibrant replacements of the module of Kähler differentials and deduce the results.
Chapter 4 is concerned with functor homology. We recall the definition of the category Epi_n encoding the iterated bar construction, the definition of E_n-homology for functors and the Tor-interpretation of E_n-homology with trivial coefficients from [41]. We then construct a category Epi_n^+ that encodes the twisted variant of the iterated bar construction which computes E_n-homology with coefficients in a bimodule. We extend the definition of E_n-homology and -cohomology to functors defined on this category. Like in [41] we use the axiomatic description of Tor and Ext to prove that E_n-homology and -cohomology coincide with functor homology and cohomology. We calculate E_n-homology and -cohomology of a polynomial algebra as an example.
In chapter 5 we recall the definition of the Yoneda pairing. We investigate whether the results of chapter 4 allow us to construct cohomology operations via the Yoneda product: We prove that the E_n-cohomology of the representing object vanishes, hence no cohomology operations arise this way.
In chapter 6, we compare E_n-homology and -cohomology of commutative algebras with coefficients in symmetric bimodules with higher order Hochschild homology and cohomology. After recalling some facts about simplicial structures as well as the relevant definitions, we compare a simplicial variant of the iterated bar construction with the iterated bar construction for differential graded algebras. We deduce that E_n-homology and -cohomology coincides with higher order Hochschild homology and cohomology by using that the relevant complexes coincide for $n = 1$ and by exploiting the fact that simplicial commutative augmented algebras form a pointed simplicial model category.
The n-fold bar complex is equipped with a comultiplication giving rise to the cup product on E_n-cohomology. The last chapter is dedicated to an explicit combinatorial construction of a homotopy for this cup product in characteristic two as part of a possible E_{n+1}-structure on the chain complex calculating E_n-cohomology.

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2 Preliminaries

The goal of this chapter is to recollect the basic concepts that we will need. We start by introducing operads and algebras over them. After that we discuss how these can be endowed with model structures, and how one can then define homology and cohomology of the algebras in question. Finally we recall some facts about homology and cohomology of small categories.

Conventions: We fix a commutative unital ground ring k. We will mainly work in the category dg-mod of $\mathbb{Z}$-graded differential graded k-modules. For a graded k-module C we denote the degree of $c \in C$ by $|c|$. We observe the Koszul sign rule, meaning that for maps $f \colon A \to B$ and $g \colon C \to D$ between graded k-modules and $a \in A, c \in C$

$$(f \otimes g)(a \otimes c) = (-1)^{|g||a|} f(a) \otimes g(c).$$

For an object X we will denote the identity on X by $1, 1_X$ or X. We set $\underline{r} = \{1, ..., r\}$ and $[r] = \{0, ..., r\}$ for $r \geq 0$.

2.1 Operads and algebras over operads

Operads were defined by Boardman-Vogt [9] and May [50] in their study of iterated loop spaces. We recall the definition of an operad as a monad in symmetric sequences as well as an explicit definition. We also recall what an algebra over a given operad is and give some examples. Then we introduce free operads and cofree cooperads and define left and right modules over an operad. Unless stated otherwise the material in this section can be found in [20, ch. 2, ch. 3]. Another important reference is [44]. Information on the history of operads as well as a comprehensive overview of related results and applications can be found in [49].
Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a cocomplete symmetric monoidal category with symmetry isomorphisms $\tau_{c,d} \colon c \otimes d \to d \otimes c$, such that $\otimes$ distributes over colimits, e.g. the category Top of topological spaces or the category dg-mod.

Operads

Definition 2.1. *A Σ_*-module $\mathcal{M}$ in $\mathcal{C}$ is a family $(\mathcal{M}(r))_{r \geq 0}$ of objects $\mathcal{M}(r)$ in $\mathcal{C}$ endowed with a right Σ_r-action. The object $\mathcal{M}(r)$ is said to be the object in arity r. A morphism $f \colon \mathcal{M} \to \mathcal{N}$ of Σ_*-modules is a family $(f_r)_{r \geq 0}$ of Σ_r-equivariant morphisms $f_r \colon \mathcal{M}(r) \to \mathcal{N}(r)$. We denote the category of Σ_*-modules in $\mathcal{C}$ by* $\mathcal{C}\Sigma_*$-mod.

Proposition 2.2. *The category of Σ_*-modules is a symmetric monoidal category with tensor product*

$$(\mathcal{M} \otimes \mathcal{N})(r) = \coprod_{a+b=r} (\mathcal{M}(a) \otimes \mathcal{N}(b)) \otimes_{\Sigma_a \times \Sigma_b} \Sigma_r$$

and unit the Σ_*-module which is zero except in arity 0, where it is k. Here for c an object in $\mathcal{C}$

$$c \otimes \Sigma_r = \coprod_{\Sigma_r} c$$

with the obvious Σ_r-action and

$$(\mathcal{M}(a) \otimes \mathcal{N}(b)) \otimes_{\Sigma_a \times \Sigma_b} \Sigma_r$$

is the coequalizer

$$(\mathcal{M}(a) \otimes \mathcal{N}(b)) \otimes (\Sigma_a \times \Sigma_b) \otimes \Sigma_r \rightrightarrows (\mathcal{M}(a) \otimes \mathcal{N}(b)) \otimes \Sigma_r,$$

where one map is defined via the inclusion $\Sigma_a \times \Sigma_b \to \Sigma_r$ and the product in Σ_r and the other map is given by the Σ_*-structure of $\mathcal{M}$ and $\mathcal{N}$. The symmetry isomorphism $\mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \otimes \mathcal{M}$ is on the component $(\mathcal{M}(a) \otimes \mathcal{N}(b)) \otimes_{\Sigma_a \times \Sigma_b} \Sigma_r$ given by

$$(m \otimes n) \otimes_{\Sigma_a \times \Sigma_b} \sigma \mapsto \tau_{\mathcal{M}(a),\mathcal{N}(b)}(m \otimes n) \otimes_{\Sigma_b \times \Sigma_a} \omega_{a,b}\sigma,$$

where $\omega_{a,b} \in \Sigma_{a+b}$ is the permutation that switches the blocks $\{1, ..., a\}$ and $\{a+1, ..., a+b\}$, i.e.

$$\omega_{a,b}(x) = \begin{cases} x + b & x \leq a, \\ x - a & x > a. \end{cases}$$

Proposition 2.3. *The category of Σ_*-modules in $\mathcal{C}$ is equipped with another structure making it a monoidal category: This product, called plethysm, is given by*

$$\mathcal{M} \circ \mathcal{N} = \coprod_{j \geq 0} \mathcal{M}(j) \otimes_{\Sigma_j} \mathcal{N}^{\otimes j},$$

where Σ_j acts on $\mathcal{N}^{\otimes j}$ by permuting the tensor factors. Hence in each arity r

$$(\mathcal{M} \circ \mathcal{N})(r) = \coprod_{j \geq 0} \mathcal{M}(j) \otimes_{\Sigma_j} \Big(\coprod_{i_1 + ... + i_j = r} (N(i_1) \otimes ... \otimes N(i_j)) \otimes_{\Sigma_{i_1} \times ... \times \Sigma_{i_j}} \Sigma_r \Big).$$

The unit object I for this structure is defined by

$$I(r) = \begin{cases} 1_{\mathcal{C}}, & r = 1, \\ 0, & r \neq 1. \end{cases}$$

We also often will drop $\circ$ from the notation and write $\mathcal{M}\mathcal{N}$ for $\mathcal{M} \circ \mathcal{N}$.

Definition 2.4. *An operad in* $\mathcal{C}$ *is a monoid in the category of* Σ_**-modules in* $\mathcal{C}$ *with respect to the monoidal structure defined in 2.3, i.e. a* Σ_**-module* $\mathcal{P}$ *together with morphisms*

$$\gamma\colon \mathcal{P}\mathcal{P} \to \mathcal{P} \quad \textit{and} \quad \eta\colon I \to \mathcal{P}$$

such that the diagrams

$$\begin{tikzcd} \mathcal{P}\mathcal{P}\mathcal{P} \arrow[r, "\gamma\mathcal{P}"] \arrow[d, "\mathcal{P}\gamma"] & \mathcal{P}\mathcal{P} \arrow[d, "\gamma"] \\ \mathcal{P}\mathcal{P} \arrow[r, "\gamma"] & \mathcal{P} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{P} \arrow[r, "\eta\mathcal{P}"] \arrow[d, "\mathcal{P}\eta"'] & \mathcal{P}\mathcal{P} \arrow[dl, equal] \\ \mathcal{P}\mathcal{P} & \end{tikzcd}$$

commute, where we keep the associativity isomorphism relating $(\mathcal{P}\mathcal{P})\mathcal{P}$ *and* $\mathcal{P}(\mathcal{P}\mathcal{P})$ *as well as the unit isomorphism implicit. A morphism of operads is a morphism of monoids in* Σ_**-modules. We denote the category of operads in* $\mathcal{C}$ *by* $\mathcal{O}_{\mathcal{C}}$*.*

Note that there is also a nonsymmetric version of the notion of operads obtained by considering sequences in $\mathcal{C}$ instead of Σ_*-modules, defining suitable notions of tensor product and plethysm for such sequences and dropping all equivariance requirements. Nonsymmetric operads are for example discussed in [44, 5.9]. We will only consider symmetric operads and hence do not give details about nonsymmetric operads.
Spelling out the above definition yields May's original definition:

Proposition 2.5. *An operad* $\mathcal{P}$ *is a* Σ_**-module together with* $\Sigma_{i_1} \times ... \times \Sigma_{i_r}$*-equivariant morphisms*

$$\gamma_{i_1,...,i_r}\colon \mathcal{P}(r) \otimes_{\Sigma_r} (\mathcal{P}(i_1) \otimes ... \otimes \mathcal{P}(i_r)) \to \mathcal{P}(i_1 + ... + i_r)$$

for all $r \geq 1$ *and all* $i_1, ..., i_r \geq 0$ *as well as a morphism*

$$1_{\mathcal{C}} \to \mathcal{P}(1)$$

satisfying certain relations regarding associativity, unitality and equivariance. We write $\gamma(p; q_1, ..., q_r)$ *for* $\gamma(p \otimes q_1 \otimes ... \otimes q_r)$ *with* $p \in \mathcal{P}(r), q_j \in \mathcal{P}(j)$*.*

Example 2.6. *Standard examples of operads in* Top *include the operad encoding associative topological monoids: In arity* r *the operad* $\mathcal{A}s^{\mathrm{Top}}$ *is given by the discrete set*

$$\mathcal{A}s_+^{\mathrm{Top}}(r) = \Sigma_r$$

The composition morphism γ *is determined by* $\gamma_{i_1,...,i_r}(\mathrm{id}_r, \mathrm{id}_{i_1}, ..., \mathrm{id}_{i_r}) = \mathrm{id}_{i_1+...+i_r}$ *together with the equivariance requirements. Another prominent example is given by the commutative operad. Its topological version is given by*

$$\mathcal{C}om_+^{\mathrm{Top}}(r) = \mathrm{pt}.$$

Similarly we can consider algebraic versions of these examples in the category dg-mod *by setting*

$$\mathcal{A}s_+(r) = k[\Sigma_r]$$

concentrated in degree zero with composition determined as in the topological case, and

$$\mathcal{C}om_+(r) = k$$

where all composition maps are identities. We will later also need the nonunital variants of $\mathcal{A}s_+$ *and* $\mathcal{C}om_+$ *given by*

$$\mathcal{A}s(r) = \begin{cases} \mathcal{A}s_+(r), & r > 0, \\ 0, & r = 0, \end{cases} \quad \text{and} \quad \mathcal{C}om(r) = \begin{cases} \mathcal{C}om_+(r), & r > 0, \\ 0, & r = 0. \end{cases}$$

Relaxing the notions of associativity and commutativity gives rise to the notion of A_∞*- and* E_∞*-operads, see e.g. [49, 1.6,1.7,1.8]. The prototypical example of an* A_∞*-operad is given by the operad formed by Stasheff's associahedra. Examples of* E_∞*-operads include the colimit over the little* n*-cubes operads as well as the Barratt-Eccles operad. We will discuss these operads in subsection 3.1.*

Remark 2.7. *There is a standard method to construct operads in dg*-mod *from topological operads (cf. [49, 1.17]): Given an operad* $\mathcal{P}$ *in* Top*, we obtain an operad* $C_*(\mathcal{P})$ *by setting*

$$C_*(\mathcal{P})(r) = C_*(\mathcal{P}(r))$$

with C_* *denoting the singular chains functor. The composition in* $C_*(\mathcal{P})$ *is defined by applying the Eilenberg-Zilber map*

$$C_*(\mathcal{P}(r)) \otimes C_*(\mathcal{P}(i_1)) \otimes ... \otimes C_*(\mathcal{P}(i_r)) \to C_*(\mathcal{P}(r) \times \mathcal{P}(i_1) \times ... \times \mathcal{P}(i_r))$$

and composing this with $C_*(\gamma_{i_1,...,i_r})$*.*

Example 2.8. *A particular important class of examples arises whenever the category* $\mathcal{C}$ *admits internal hom objects* $\mathrm{hom}_{\mathcal{C}}(-,-)$*. Then we can define an operad* End_c*, the endomorphism operad associated to* c*, for every element* c *of* $\mathcal{C}$ *by setting*

$$\mathrm{End}_c(r) = \mathrm{hom}_{\mathcal{C}}(c^{\otimes r}, c)$$

with right Σ_r*-action given by permuting the factors of* $c^{\otimes r}$ *and composition defined by*

$$\gamma_{i_1,...,i_r}(f; f_1, ..., f_r) = f(f_1 \otimes ... \otimes f_r)$$

for $f \in \mathrm{hom}_{\mathcal{C}}(c^{\otimes r}, c)$ *and* $f_j \in \mathrm{hom}_{\mathcal{C}}(c^{\otimes i_j}, c)$ *for* $1 \leq j \leq r$*.*

Remark 2.9. *Equivalently one can define an operad structure on a Σ_*-module $\mathcal{P}$ via specifying partial composition products*

$$\circ_i \colon \mathcal{P}(a) \otimes \mathcal{P}(b) \to \mathcal{P}(a+b-1)$$

for $a, b \geq 0$ and $1 \leq i \leq a$. If $\mathcal{P}$ is equipped with an operad structure γ, these are given by

$$\circ_i = \gamma_{1,\dots,1,b,1,\dots,1} \colon \mathcal{P}(a) \otimes I(1)^{\otimes i-1} \otimes \mathcal{P}(b) \otimes I(1)^{\otimes a-i} \to \mathcal{P}(a+b-1).$$

Conversely any set of morphisms $\circ_i$ as above satisfying suitable associativity, unitality and equivariance conditions defines an operad. To construct the full composition morphisms

$$\gamma_{i_1,\dots,i_r} \colon \mathcal{P}(r) \otimes \mathcal{P}(i_1) \otimes \dots \otimes \mathcal{P}(i_r) \to \mathcal{P}(i_1 + \dots + i_r)$$

from the partial composition products, set

$$\gamma_{i_1,\dots,i_r} = \circ_{i_1+\dots+i_{r-1}+1} \dots (\circ_{i_1+1} \otimes \mathcal{P}(i_3) \otimes \dots \otimes \mathcal{P}(i_r))(\circ_1 \otimes \mathcal{P}(i_2) \otimes \dots \mathcal{P}(i_r)).$$

For later use we record the following fact, which is proved in [20, 3.1.6]. Recall that a colimit in a category $\mathcal{C}$ is called filtered if it is the colimit over a filtered diagram $G \colon I \to \mathcal{C}$, i.e. a diagram with I a small category such that for all $i, j \in I$ there is $l \in I$ with morphisms $i \to l$ and $j \to l$.

Proposition 2.10. *The category of operads is complete and cocomplete. Limits and filtered colimits are created by the forgetful functor V from operads to Σ_*-modules, i.e.*

$$V(\lim{}_J F) = \lim{}_J VF \quad \text{and} \quad V(\mathrm{colim}_I G) = \mathrm{colim}_I VG$$

for a diagram $F \colon J \to \mathcal{O}_{\mathcal{C}}$ and a filtered diagram $G \colon I \to \mathcal{O}_{\mathcal{C}}$.

Algebras over operads Operads abstractly encode algebraic structures. Concrete instances of the encoded structure are called the algebras over a given operad.

Definition 2.11. *Let $\mathcal{P}$ be an operad in $\mathcal{C}$. A $\mathcal{P}$-algebra A is an object A in $\mathcal{C}$ together with morphisms*

$$\gamma_A \colon \mathcal{P}A \to A \quad \text{and} \quad \eta_A \colon A \to \mathcal{P}A,$$

where we consider A as a Σ_*-module concentrated in arity 0, such that the diagrams

$$\begin{tikzcd} \mathcal{P}\mathcal{P}A \arrow[r, "\gamma A"] \arrow[d, "\mathcal{P}\gamma_A"'] & \mathcal{P}A \arrow[d, "\gamma_A"] \\ \mathcal{P}A \arrow[r, "\gamma_A"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & \mathcal{P}A \arrow[d, "\gamma_A"] \\ & A \end{tikzcd}$$

commute.

Remark 2.12. *Just as for operads from the above description one can derive a hands-on definition: For A an object of $\mathcal{C}$ the structure of a $\mathcal{P}$-algebra is equivalent to giving maps*

$$\gamma_{A,r}\colon \mathcal{P}(r) \otimes_{\Sigma_r} A^{\otimes r} \to A$$

for $r \geq 0$ satisfying certain associativity and unitality conditions.

Example 2.13. *Spelling out what consititutes an $\mathcal{A}s_+^{\mathrm{Top}}$-algebra yields that these are exactly associative unital monoids in topological spaces. Similarly, $\mathcal{C}om_+^{\mathrm{Top}}$-algebras are commutative associative topological monoids with unit. In the differential graded setting we find that $\mathcal{A}s_+$-algebras are differential graded algebras with unit and that $\mathcal{C}om_+$-algebras are graded commutative differential graded algebras with unit, while $\mathcal{A}s$- and $\mathcal{C}om$-algebras are differential graded algebras and commutative differential graded algebras without unit.*
As in 2.7, for an operad $\mathcal{P}$ in Top *applying the singular chains functor to a $\mathcal{P}$-algebra X yields a $C_*(\mathcal{P})$-algebra $C_*(X)$.*

The importance of the endomorphism operad introduced in example 2.8 lies in the following proposition.

Proposition 2.14. *Let $\mathcal{C}$ be a category that admits internal hom objects and let $\mathcal{P}$ be an operad in $\mathcal{C}$. Then there is a bijection between $\mathcal{P}$-algebra structures on an object c in $\mathcal{C}$ and operad morphisms*

$$\mathcal{P} \to \mathrm{End}_c.$$

Indexing by finite sets In the above definitions we consider arity graded Σ_*-modules, where the object in arity r can be thought of as corresponding to operations with r inputs. Let the category Bij be the category with finite sets as objects and morphisms the bijections between them. For a finite set $\underline{e}$ let $|\underline{e}|$ denote its cardinality. It is also possible to carry out the constructions above for functors $\mathcal{M}\colon \mathrm{Bij} \to \mathcal{C}$, see [23, 0.2, 0.3, 0.5, 0.8], [20, 5.1.6]. More precisely, the tensor product of two such functors $\mathcal{M}$ and $\mathcal{N}$ is given by

$$(\mathcal{M} \otimes \mathcal{N})(\underline{e}) = \bigsqcup_{\underline{e}=\underline{e}' \sqcup \underline{e}''} \mathcal{M}(\underline{e}') \otimes \mathcal{N}(\underline{e}'')$$

for a finite set $\underline{e}$. The plethysm is then defined by

$$(\mathcal{M}\mathcal{N}) = \bigsqcup_{r \geq 0} \mathcal{M}(\{1, ..., r\}) \otimes_{\Sigma_r} \mathcal{N}^{\otimes r}.$$

We have the following relation between Σ_*-modules and functors from Bij to $\mathcal{C}$, which allows us to freely switch between these categories.

Proposition 2.15. *There is an equivalence between the category of Σ_*-modules and the category* $\mathrm{Fun}(\mathrm{Bij}, \mathcal{C})$ *defined as follows: For a given Σ_*-module $\mathcal{M}$ set*

$$\mathcal{M}(\underline{e}) = \mathcal{M}(|\underline{e}|) \otimes_{\Sigma_{|\underline{e}|}} \mathrm{Bij}(\underline{e}, \{1, ..., |\underline{e}|\})$$

for any finite set $\underline{e}$, with

$$(m \otimes \sigma).\omega = m \otimes \sigma\omega$$

for $\omega \in \mathrm{Bij}(\underline{e}, \underline{e}')$. Conversely, every functor $\mathcal{N} \colon \mathrm{Bij} \to \mathcal{C}$ defines a Σ_*-module by restricting $\mathcal{N}$ to the sets $\{1, ..., r\}$ and their endomorphisms Σ_r. Moreover this equivalence respects the respective tensor products and plethysms.

Cooperads There is a notion dual to the notion of operad: Cooperads abstractly encode classes of coalgebras with a certain structure.

Definition 2.16. *A cooperad $\mathcal{D}$ in $\mathcal{C}$ is a comonoid in* $(\mathcal{C}\Sigma_*\text{-mod}, \circ)$*, i.e. $\mathcal{D}$ is equipped with morphisms*

$$\Delta \colon \mathcal{D} \to \mathcal{D} \circ \mathcal{D} \quad \textit{and} \quad \epsilon \colon \mathcal{D} \to I$$

such that Δ is coassociative and ϵ is a counit for Δ. A morphism of cooperads is a morphism of comonoids in $\mathcal{C}\Sigma_*\text{-mod}$*.*

Again it is possible to derive a more hands-on definition from this one. One can define coalgebras over a cooperad $\mathcal{D}$ and for example retrieve the category of coassociative coalgebras and the category of cocommutative coassociative coalgebras this way. Since we will not be concerned with coalgebras over a cooperad we refer the reader to [18].

Free operads and cofree cooperads

Definition 2.17. *Let $\mathcal{M}$ be a Σ_*-module in $\mathcal{C}$. We call an operad $\mathcal{P}$ together with a morphism $\iota \colon \mathcal{M} \to \mathcal{P}$ the free operad generated by $\mathcal{M}$ if for every operad $\mathcal{O}$ and every morphism $f \colon \mathcal{M} \to \mathcal{O}$ of Σ_*-modules there is exactly one morphism $\bar{f} \colon \mathcal{P} \to \mathcal{O}$ of operads such that the diagram*

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\iota"'] \arrow[dr, "f"] & \\ \mathcal{P} \arrow[r, "\bar{f}"] & \mathcal{O} \end{tikzcd}$$

commutes.

Proposition 2.18. *For every* $\mathcal{M} \in \mathcal{C}\Sigma_*\text{-mod}$ *there exists an operad $\mathcal{P}$ and a morphism $\iota \colon \mathcal{M} \to \mathcal{P}$ such that $(\mathcal{P}, \iota)$ is the free operad generated by $\mathcal{M}$. Up to a unique isomorphism the pair $(\mathcal{P}, \iota)$ is unique. We denote $\mathcal{P}$ by $\mathcal{F}(\mathcal{M})$.*

We will not give a formal construction here, but intuitively elements of $\mathcal{F}(\mathcal{M})$ are given by formal composites

$$(...((m_1 \circ_{i_1} m_2) \circ_{i_2} ...) \circ_{i_{r-1}} m_r$$

of elements in $\mathcal{M}$. In particular, $\mathcal{F}(\mathcal{M})$ admits a weight grading

$$\mathcal{F}(\mathcal{M}) = \bigsqcup_{l \geq 0} \mathcal{F}_{(l)}(\mathcal{M})$$

with $\mathcal{F}_{(l)}(\mathcal{M})$ the Σ_*-submodule of $\mathcal{F}(\mathcal{M})$ generated by expressions as above with $r = l$. For a formal construction we refer the reader to [18, 3.4].

Similarly one can define the cofree cooperad cogenerated by a Σ_*-module $\mathcal{M}$ as a cooperad satisfying an appropriate dual universal property. We denote the cofree cooperad cogenerated by $\mathcal{M}$ by $\mathcal{F}^c(\mathcal{M})$. Again we refer to [18] for an explicit construction and only note that $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}^c(\mathcal{M})$ are isomorphic as Σ_*-modules, in particular $\mathcal{F}^c(\mathcal{M})$ admits a weight grading as well.

For Σ_*-modules $\mathcal{M}$ and $\mathcal{N}$ let $(\mathcal{M};\mathcal{N})$ be the Σ_*-module given by

$$(\mathcal{M};\mathcal{N}) = \bigsqcup_{i \geq 1} \bigsqcup_{a+b=i-1} \mathcal{M}^{\otimes a} \otimes \mathcal{N} \otimes \mathcal{M}^{\otimes b}.$$

Observe that for a third Σ_*-module $\mathcal{L}$

$$(\mathcal{L}\mathcal{L})(\mathcal{M};\mathcal{N}) \cong \mathcal{L}(\mathcal{L}(\mathcal{M});\mathcal{L}(\mathcal{M};\mathcal{N})).$$

Morphisms $f\colon \mathcal{L} \to \mathcal{L}'$, $g\colon \mathcal{M} \to \mathcal{N}$ of Σ_*-modules give rise to a morphism

$$f \circ' g\colon \mathcal{L}\mathcal{M} \to \mathcal{L}'(\mathcal{M};\mathcal{N})$$

of Σ_*-modules defined by

$$f \circ' g|_{\mathcal{L}(j)\otimes_{\Sigma_j}\mathcal{M}^{\otimes j}} = \sum_{i=1}^{j} f \otimes (\mathcal{M} \otimes ... \otimes \mathcal{M} \otimes g \otimes \mathcal{M} \otimes ... \otimes \mathcal{M}).$$

Definition 2.19.

- *Let $\mathcal{P}$ be an operad in graded modules. A morphism $f\colon \mathcal{P} \to \mathcal{P}$ of Σ_*-modules is called a derivation of $\mathcal{P}$ if the diagram*

$$\begin{tikzcd} \mathcal{P}\mathcal{P} \arrow[r, "f\mathcal{P}+\pi(\mathcal{P}\circ' f)"] \arrow[d, "\gamma"] & \mathcal{P}\mathcal{P} \arrow[d, "\gamma"] \\ \mathcal{P} \arrow[r, "f"] & \mathcal{P} \end{tikzcd}$$

commutes. Here $\pi\colon \mathcal{P}(\mathcal{P};\mathcal{P}) \to \mathcal{P}\mathcal{P}$ is the projection. Denote the derivations on $\mathcal{P}$ by $\mathrm{Der}(\mathcal{P})$.

- *Let $\mathcal{D}$ be a cooperad in graded modules. A coderivation of $\mathcal{D}$ is a morphism $g\colon \mathcal{D} \to \mathcal{D}$ of Σ_*-modules such that the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "g"] \arrow[d, "\Delta"] & \mathcal{D} \arrow[d, "\Delta"] \\ \mathcal{D}\mathcal{D} \arrow[r, "g\mathcal{D}+\pi(\mathcal{D}\circ' g)"] & \mathcal{D}\mathcal{D} \end{tikzcd}$$

commutes. We denote the coderivations on $\mathcal{D}$ by $\mathrm{Coder}(\mathcal{D})$*.*

Example 2.20. *If $\mathcal{P}$ is an operad in dg-*mod*, the differential of $\mathcal{P}$ is by definition a derivation of the operad in graded modules underlying $\mathcal{P}$. Similarly, the differential of a cooperad in dg-*mod *is a coderivation.*

Proposition 2.21. *For a free operad $\mathcal{F}(\mathcal{M})$ in dg-*mod

$$\mathrm{Der}(\mathcal{F}(\mathcal{M})) \cong \mathrm{Hom}_{dg\text{-mod}\Sigma_*\text{-mod}}(\mathcal{M}, \mathcal{F}(\mathcal{M})),$$

i.e. a derivation of $\mathcal{F}(\mathcal{M})$ is determined by its restriction to the generators $\mathcal{M}$. This isomorphism is natural in $\mathcal{M}$.
Similarly, for a cofree cooperad $\mathcal{F}^c(\mathcal{M})$ there is a natural isomorphism

$$\mathrm{Coder}(\mathcal{F}^c(\mathcal{M})) \cong \mathrm{Hom}_{dg\text{-mod}\Sigma_*\text{-mod}}(\mathcal{F}^c(\mathcal{M}), \mathcal{M}).$$

We denote the derivation of $\mathcal{F}(\mathcal{M})$ induced by a map $\alpha\colon \mathcal{M} \to \mathcal{F}(\mathcal{M})$ by ∂_α, and adopt a similar convention for coderivation of cofree cooperads.

Definition 2.22. *An operad $\mathcal{P}$ such that $\mathcal{P} = \mathcal{F}(\mathcal{M})$ as a graded module is called a quasifree operad. The differential of $\mathcal{P}$ is then a sum $d_\mathcal{P} = d_{\mathcal{F}(\mathcal{M})} + \delta$ with $d_{\mathcal{F}(\mathcal{M})}$ induced by the differential of M. We write $\mathcal{P} = (\mathcal{F}(\mathcal{M}), \delta)$ in this situation and call δ a twisting morphism or twist. A quasifree cooperad is defined similarly.*

Modules over operads

Definition 2.23. *Let $(\mathcal{P}, \gamma, \eta)$ be an operad. A left $\mathcal{P}$-module in $\mathcal{C}$ is a Σ_*-module L in $\mathcal{C}$ equipped with a morphism*

$$\gamma_L\colon \mathcal{P}L \to L$$

such that the diagrams

$$\begin{tikzcd} \mathcal{P}\mathcal{P}L \arrow[r, "\mathcal{P}\gamma_L"] \arrow[d, "\gamma L"] & \mathcal{P}L \arrow[d, "\gamma_L"] \\ \mathcal{P}L \arrow[r, "\gamma_L"] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \cong IL \arrow[r, "\eta L"] \arrow[dr, equal] & \mathcal{P}L \arrow[d, "\gamma_L"] \\ & L \end{tikzcd}$$

commute. Similarly, a right $\mathcal{P}$-module is a Σ_*-module R with a morphism

$$\gamma_R \colon R\mathcal{P} \to R$$

such that the diagrams

$$\begin{tikzcd} R\mathcal{P}\mathcal{P} \arrow[r, "\gamma_R \mathcal{P}"] \arrow[d, "R\gamma"] & R\mathcal{P} \arrow[d, "\gamma_R"] \\ R\mathcal{P} \arrow[r, "\gamma_R"] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \cong RI \arrow[r, "R\eta"] \arrow[dr, equal] & R\mathcal{P} \arrow[d, "\gamma_R"] \\ & R \end{tikzcd}$$

commute.

Observe that although the definitions of left and right $\mathcal{P}$-modules look symmetric, they describe quite different structures since the plethysm is not symmetric: While a left $\mathcal{P}$-module L admits maps

$$\mathcal{P}(i) \otimes_{\Sigma_i} L^{\otimes i} \to L,$$

a right $\mathcal{P}$-module structure on R yields maps

$$R(i) \otimes_{\Sigma_i} \mathcal{P}^{\otimes i} \to R.$$

The following lemma allows us to endow the tensor product of two right $\mathcal{P}$-modules again with the structure of a right $\mathcal{P}$-module.

Lemma 2.24. *Let M, N be right $\mathcal{P}$-modules. Then there is an isomorphism*

$$(M \otimes N)\mathcal{P} \cong (M\mathcal{P}) \otimes (N\mathcal{P})$$

of Σ_*-modules.

Definition 2.25. *For right $\mathcal{P}$-modules M and N the tensor product $M \otimes N$ is again a right $\mathcal{P}$-module, with right $\mathcal{P}$-module structure given by*

$$(M \otimes N)\mathcal{P} \xrightarrow{\cong} (M\mathcal{P}) \otimes (N\mathcal{P}) \longrightarrow M \otimes N.$$

Just as for modules over a ring derivations of free modules are determined by the image of their generators. The following material can be found in [18].

Definition 2.26. *1. Let L be a left $\mathcal{P}$-module over an operad $\mathcal{P}$ in dg-*mod*. Denote the differential on $\mathcal{P}$ by $d_{\mathcal{P}}$. A map $f \colon L \to L$ is called a derivation of L if the diagram*

$$\begin{tikzcd} \mathcal{P}L \arrow[r, "d_{\mathcal{P}}L + \pi(\mathcal{P} \circ' f)"] \arrow[d, "\gamma_L"] & \mathcal{P}L \arrow[d, "\gamma_L"] \\ L \arrow[r, "f"] & L \end{tikzcd}$$

commutes, with $\pi \colon \mathcal{P}(L; L) \to \mathcal{P}L$ the projection. We denote the derivations of L by $\mathrm{Der}(L)$.

2. *Let R be a left $\mathcal{P}$-module over an operad $\mathcal{P}$ in dg-*mod*. Denote the differential on $\mathcal{P}$ by $d_{\mathcal{P}}$. A map $f\colon R \to R$ is called a derivation of R if the diagram*

$$\begin{tikzcd} R\mathcal{P} \arrow[r, "f\mathcal{P}+\pi(R\circ' d_{\mathcal{P}})"] \arrow[d, "\gamma_R"] & R\mathcal{P} \arrow[d, "\gamma_R"] \\ R \arrow[r, "f"] & R \end{tikzcd}$$

commutes, with $\pi\colon R(\mathcal{P};\mathcal{P}) \to R\mathcal{P}$ the projection. Denote the derivations of R by $\mathrm{Der}(R)$.

Given a Σ_*-module M, the left $\mathcal{P}$-module $\mathcal{P}M$ satisfies the universal property of a free left $\mathcal{P}$-module. Similarly, $M\mathcal{P}$ is the free right $\mathcal{P}$-module associated to M. An important example arises for an object M in $\mathcal{C}$, i.e. if M is concentrated in arity zero: A left $\mathcal{P}$-module concentrated in arity zero is exactly a $\mathcal{P}$-algebra, and $\mathcal{P}M$ is the free $\mathcal{P}$-algebra generated by M with structure map

$$\mathcal{P}\mathcal{P}M \xrightarrow{\gamma_{\mathcal{P}}M} \mathcal{P}M.$$

For a map $\theta\colon M \to L$ to a left $\mathcal{P}$-module L we denote by $\partial_\theta\colon \mathcal{P}M \to L$ the induced morphism of left $\mathcal{P}$-modules. We adopt a similar notation for right modules.

Proposition 2.27. 1. *For a free left $\mathcal{P}$-module $\mathcal{P}M$ in dg-*mod *there is a natural isomorphism*

$$\mathrm{Der}(\mathcal{P}M) \cong \mathrm{Hom}_{dg\text{-mod}\Sigma_*\text{-mod}}(M, \mathcal{P}M).$$

2. *Similarly, for a free right $\mathcal{P}$-module $M\mathcal{P}$ in dg-*mod *there is a natural isomorphism*

$$\mathrm{Der}(M\mathcal{P}) \cong \mathrm{Hom}_{dg\text{-mod}\Sigma_*\text{-mod}}(M, M\mathcal{P}).$$

In the following we will often abuse notation and, for a map $\alpha\colon M \to \mathcal{P}M$, denote by ∂_α the induced derivation on the left $\mathcal{P}$-module $\mathcal{P}M$. It will be clear from the context whether ∂_β denotes the morphism of left $\mathcal{P}$-modules induced by $\beta\colon M \to \mathcal{P}M$ or the associated derivation. A similar notation will be used for right $\mathcal{P}$-modules.

Definition 2.28. *A quasifree left $\mathcal{P}$-module L is a left $\mathcal{P}$-module such that $L = \mathcal{P}M$ as a graded left $\mathcal{P}$-module. In this case the differential of L is of the form $d_{\mathcal{P}M} + \delta$ with $d_{\mathcal{P}M}$ induced by the differentials of $\mathcal{P}$ and M. We write $L = (\mathcal{P}M, \delta)$ and call δ a twisting morphism or twist.*
A quasifree right $\mathcal{P}$-module $R = (M\mathcal{P}, \delta)$ is defined and denoted similarly. Note that in this case $d_{M\mathcal{P}}$ is a derivation of right $\mathcal{P}$-modules, while δ is a morphism of right $\mathcal{P}$-modules.

Remark 2.29. *Since a $\mathcal{P}$-algebra is precisely a left $\mathcal{P}$-module concentrated in arity zero, the terminology and results above can in particular be applied to algebras over an operad.*

2.2 Cofibrantly generated model categories

There is a standard way of transporting cofibrantly generated model structures along adjunctions, which is frequently used to define model structures in the context of operads. In the following we recall the relevant definitions which can be found in [20, ch. 11]. As a general reference for the language of model categories we refer the reader to [36].

Definition 2.30. *An ordinal κ is a set such that every element of κ is also a subset of κ and that is strictly wellordered with respect to the order defined by*

$$I \leq I' \Leftrightarrow I \subset I'.$$

Definition 2.31. *Let I be a given set of morphisms in a cocomplete category $\mathcal{C}$.*

- *A morphism $A \to B$ is called an I-cell attachment if it is obtained by a pushout*

$$\begin{tikzcd} \coprod_\alpha C_\alpha \arrow[r] \arrow[d, "\coprod_\alpha i_\alpha"] & A \arrow[d] \\ \coprod_\alpha D_\alpha \arrow[r] & B \end{tikzcd}$$

  with $i_\alpha \in I$ for all α.

- *A map $f\colon A \to B$ is called a relative I-cell complex if f is a (possibly transfinite) composite of I-cell attachments over an ordinal κ, i.e. if f is the map $A \to \mathrm{colim}_{\lambda<\kappa} B_\lambda$ associated to*

$$A = B_0 \to B_1 \to \ldots \to B_{\lambda-1} \to B_\lambda \to \ldots \to \mathrm{colim}_{\lambda<\kappa} B_\lambda = B.$$

Definition 2.32. *We say that the small object argument holds for a set of morphisms I of $\mathcal{C}$ if the following conditions are satisfied: There exists an ordinal ω such that, for all $\kappa \geq \omega$ and for all $f\colon A \to B$ in $\mathcal{C}$ with $0 \to B = \mathrm{colim}_{\lambda<\kappa} B_\lambda$ a relative I-cell complex and A the domain of a map in I, the morphism f admits a factorization*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \\ & B_\lambda \arrow[u, "i"'] \end{tikzcd}$$

for some $\lambda < \kappa$, where $i\colon B_\lambda \to B$ is the canonical map into the colimit B.

Definition 2.33. *A model category $\mathcal{C}$ is called cofibrantly generated if there are two sets I and J, called the set I of generating cofibrations and the set J of generating acyclic cofibrations such that*

- *the small object argument holds for I and for J,*
- *a map is a fibration if and only if it has the right lifting property with respect to the maps in J,*
- *a map is an acyclic fibration if and only if it has the right lifting property with respect to the maps in I.*

Theorem 2.34. *[20, 11.1.13] Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J, $\mathcal{D}$ a complete and cocomplete category and let*

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$$

be an adjunction. Assume that the following conditions hold:

- *The small object argument holds for the sets FI and FJ.*
- *For every relative FJ-cell f the map $G(f)$ is a weak equivalence.*

Then $\mathcal{D}$ is a cofibrantly generated model category with generating cofibrations FI and acyclic generating cofibrations FJ. The weak equivalences and fibrations are created by G, i.e. a map f in $\mathcal{D}$ is a weak equivalence (fibration) if and only if $G(f)$ is a weak equivalence (fibration).

If we endow $\mathcal{D}$ with this model structure, the functors F and G form a Quillen adjunction.

2.3 Model structures on operads and algebras over operads

In the following we recall the model structure on the category of operads and on the category of algebras over operads in differential graded modules as defined by Hinich in [32] and [31], see also [20, ch. 11, 12, 14].
We will focus our attention on the case $\mathcal{C} = dg$-mod. Recall that dg-mod is a model category with weak equivalences the quasiisomorphisms and fibrations given by degreewise surjections. This model structure is cofibrantly generated: Let D^l be the chain complex with one generator x_l in degree l, one generator x_{l-1} in degree $l-1$ and differential mapping x_l to x_{l-1}. Let S^l be the chain complex with one generator in degree l. Then the generating cofibrations in dg-mod can be chosen to be the inclusions $S^{l-1} \to D^l$, while the generating acyclic cofibrations are the maps $0 \to D^l$.
There is an adjunction between the category of Σ_*-modules in a given model category $\mathcal{C}$ and the category $\mathcal{C}^{\mathbb{N}}$ of collections of objects in $\mathcal{C}$ indexed by $\mathbb{N}$, given by forgetting the action of the symmetric groups in one direction and by associating to c in $\mathcal{C}$ the free Σ_n-object $\sqcup_{\Sigma_n} c$ in the other direction. Applying Theorem 2.34 to this adjunction yields the following model structure on $\mathcal{C}\Sigma_*$-mod.

Proposition 2.35. *Let $\mathcal{C}$ be a cofibrantly generated model category. Then the category of Σ_*-objects in $\mathcal{C}$ inherits a model structure with weak equivalences and fibrations given by degreewise weak equivalences and degreewise fibrations.*

We call Σ_*-modules which are cofibrant Σ_*-cofibrant. A Σ_*-module M in $\mathcal{C}$ is called $\mathcal{C}$-cofibrant if it is a cofibrant object in $\mathcal{C}$ in each arity. We use a similar terminology for morphisms of Σ_*-modules. We apply theorem 2.34 to the free-forgetful adjunction between the category $\mathcal{C}\Sigma_*$-mod and the category of right $\mathcal{P}$-modules. The condition that $\mathcal{P}$ is $\mathcal{C}$-cofibrant ensures that the free functor maps (acyclic) $\mathcal{C}$-cofibrations to (acyclic) $\mathcal{C}$-cofibrations, which facilitates checking the requirements of Theorem 2.34.

Theorem 2.36. *Let $\mathcal{P}$ be $\mathcal{C}$-cofibrant. The category $\mathcal{M}_{\mathcal{P}}$ of right $\mathcal{P}$-modules is a cofibrantly generated model category with*

- *weak equivalences the maps which are levelwise quasiisomorphisms in $\mathcal{C}$,*
- *fibrations the maps which are levelwise fibrations in $\mathcal{C}$.*

The generating (acyclic) cofibrations are given by the maps

$$i \otimes F_r \circ \mathcal{P} \colon C \otimes F_r \circ \mathcal{P} \to D \otimes F_r \circ \mathcal{P}$$

where $i \colon C \to D$ is a generating (acyclic) cofibration in $\mathcal{C}$ and

$$F_r(l) = \begin{cases} \coprod_{\Sigma_r} 1_{\mathcal{C}}, & l = r, \\ 0, & l \neq r. \end{cases}$$

Unfortunately we need to proceed with more care if we want to define a model structure on the category of operads in $\mathcal{C}$ or the category of algebras over a given operad. Not only do we need to assume that the model structure on $\mathcal{C}$ respects the symmetric monoidal structure, but the category of algebras over operads indeed only forms a so called semi-model category: In general, the right lifting properties for (acyclic) fibrations as well as the factorization axioms only hold if the domain of the map in question is cofibrant, see [20, ch. 12].

Definition 2.37. *Let $\mathcal{C}$ be a complete and cocomplete category with inital object $\emptyset$. We say that $\mathcal{C}$ is a semi-model category if there are three distinguished classes of morphisms, called weak equivalences, fibrations and cofibrations, satisfying the following properties. As for model categories, we call an object c of $\mathcal{C}$ cofibrant if the morphism $\emptyset \to c$ is a cofibration. We also call a fibration (cofibration) acyclic if it is a weak equivalence.*

- *The initial object $\emptyset$ is cofibrant.*

- *If f and g can be composed and if two out of the three morphisms f, g and fg are weak equivalences, then so is the third.*

- *The distinguished classes of morphisms are closed under retracts.*

- *Every cofibration has the left lifting property with respect to acyclic fibrations. Every acyclic cofibration has the left lifting property with respect to fibrations.*

- *Every fibration has the right lifting property with respect to acyclic cofibrations with cofibrant domain. Every acyclic fibration has the right lifting property with respect to cofibrations with cofibrant domain.*

- *Every morphism f with a cofibrant domain can be factored as $f = pi$ with p an acyclic fibration and i a cofibration. The morphism f can also be factored as $f = qj$ with q a fibration and j an acyclic cofibration.*

Since we will later define operadic (co)homology as derived funtors, for our purposes it is important to note that since the initital object is cofibrant, we can cofibrantly replace any object c of $\mathcal{C}$ in a semi-model category, and that there always is a weak equivalence from one cofibrant replacements of c to another.
Again the semi-model structure on the category of algebras over a given operad $\mathcal{P}$ is constructed using the free-forgetful adjunction between the category of $\mathcal{P}$-algebras in $\mathcal{C}$ and $\mathcal{C}$ itself, see [20, Theorem 12.1.4]. As before, the adjunction we use to construct the semi-model structure becomes a Quillen adjunction of semi-model categories, see [20, 12.1.8].

Proposition 2.38. *Let $\mathcal{P}$ be a Σ_*-cofibrant operad in the cofibrantly generated symmetric monoidal model category $\mathcal{C}$. Then the category of $\mathcal{P}$-algebras forms a semi-model category. A morphism of $\mathcal{P}$-algebras is a fibration (respectively a weak equivalence) if it is a fibration (respectively a weak equivalence) in $\mathcal{C}$.*

Examples include $\mathcal{P} = \mathcal{A}s$, which is clearly Σ_*-cofibrant. In this case, the semi-model structure given by proposition 2.38 is indeed a model structure, it coincides with the model structure on differential graded algebras exhibited by Jardine in [38]. On the other hand, if we consider $\mathcal{P} = \mathcal{C}om$ over $k = \mathbb{Z}$, one easily sees that there can be no semi-model structure on $\mathcal{C}om$-algebras such that the free-forgetful adjunction between $\mathcal{C}om$-algebras and $\mathbb{Z}$-modules is a Quillen adjunction: Consider the acyclic complex D^2 of free abelian groups with one generator x in degree 1 and one generator y in degree 2, with differential sending y to x. Then $\mathcal{C}om(H_*(D^2)) = 0$, while $H_*(\mathcal{C}om(D^2))$ does not vanish. But a Quillen adjunction between semi-model categories would preserve acyclic cofibrations between cofibrant objects. One way to deal with this problem is to replace $\mathcal{C}om$ by a Σ_*-cofibrant weakly equivalent operad, see for example [47, §2].

2.4 Homology and cohomology for algebras over operads

From now on we work with the category $\mathcal{C} = dg$-mod as a ground category. Since for a Σ_*-cofibrant operad $\mathcal{P}$ we know what cofibrant replacements in the category of $\mathcal{P}$-algebras are, we can define their homology and cohomology as appropriate derived functors

$$H_*^{\mathcal{P}}(A; M) = H_*(\mathrm{Der}_{\mathcal{P}}(Q_A, M)) \quad \text{and} \quad H_{\mathcal{P}}^*(A; M) = H_*(M \otimes_{U_{\mathcal{P}}(Q_A)} \Omega^1_{\mathcal{P}}(Q_A)).$$

Hence every Σ_*-cofibrant operad comes with a suitable notion of homology and cohomology of algebras over this operad. Note the similarity of these definitions with the classical definitions of André-Quillen homology and cohomology. In particular, if A is concentrated in degree zero $H^0_{\mathcal{P}}(A; M) = \mathrm{Der}_{\mathcal{P}}(A; M)$, while for algebras augmented over k the $\mathcal{P}$-homology with coefficients in k computes the $\mathcal{P}$-indecomposables of A in degree zero. The $\mathcal{P}$-cohomology group $H^1_{\mathcal{P}}(A; M)$ is connected to abelian extension of A by M (see [44, 12.4.3]). The $\mathcal{P}$-cohomology of A is also related to deformation theory (see [44, 12.2]) and obstruction theory ([34], [13]).
In this section we will give a brief overview of the objects involved in the above constructions and state the definition of homology and cohomology of an algebra over an operad. The material in this section stems from [20, ch. 4, ch. 10, ch. 13]. We fix a Σ_*-cofibrant operad $\mathcal{P}$.

Definition 2.39. *Let (A, γ_A) be a $\mathcal{P}$-algebra, $M \in dg$-mod. Then M is called a representation of A over $\mathcal{P}$ if there is a map*

$$\gamma_M \colon \mathcal{P}(A; M) \to M$$

such that the diagrams

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(A); \mathcal{P}(A; M)) \cong (\mathcal{P}\mathcal{P})(A; M) \arrow[r, "\gamma_{\mathcal{P}}(1_A; 1_M)"] \arrow[d, "\mathcal{P}(\gamma_A; \gamma_M)"] & \mathcal{P}(A; M) \arrow[d, "\gamma_M"] \\ \mathcal{P}(A; M) \arrow[r, "\gamma_M"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\eta_{\mathcal{P}}"] \arrow[dr, equal] & \mathcal{P}(A; M) \arrow[d, "\gamma_M"] \\ & M \end{tikzcd}$$

commute. A chain map $f \colon M \to N$ between representations of A is called a morphism of representations if

$$f\gamma_M = \gamma_N(\mathrm{id}_A; f).$$

We denote the category of representations of A by $\mathcal{R}_{\mathcal{P}}(A)$.

Example 2.40. *For $\mathcal{P} = \mathcal{A}s_+$ a representation of the associative algebra A is the same as an A-bimodule. For $\mathcal{P} = \mathcal{C}om_+$, we can identify representations with left modules over a commutative algebra or equivalently with symmetric bimodules. Representations of algebras over $\mathcal{A}s$ and $\mathcal{C}om$ correspond to their nonunital versions. The operadic and the traditional notion of representations of Lie algebras agree as well.*

Like for $\mathcal{P} = Lie$ there is an associative algebra such that representations of A correspond to left modules over that algebra.

Definition 2.41. *Let L be a left and R be a right $\mathcal{P}$-module. The Σ_*-module $R \circ_{\mathcal{P}} L$ is the coequalizer of*

$$R \circ \mathcal{P} \circ L \underset{R\gamma_L}{\overset{\gamma_R L}{\rightrightarrows}} R \circ L,$$

i.e. the quotient of $R \circ L$ by the relation generated by

$$r(l_1, ..., l_{i-1}, \gamma_L(p; l_i, ..., l_{i+j}), l_{i+j+1}, ..., l_s) - \gamma_R(r; 1, ..., 1, p, 1, ..., 1)(l_1, ..., l_s).$$

Note that for right $\mathcal{P}$-modules R, R' and a left $\mathcal{P}$-module L

$$(R \otimes R') \circ_{\mathcal{P}} L \cong (R \circ_{\mathcal{P}} L) \otimes (R' \circ_{\mathcal{P}} L).$$

Definition 2.42. *Let $\mathcal{P}[l]$ be the right $\mathcal{P}$-module with $\mathcal{P}[l](i) = \mathcal{P}(l + i)$, with the action of Σ_i induced by $\{1, ..., i\} \cong \{l + 1, ..., i + l\} \subset \{1, ..., i + l\}$. The right $\mathcal{P}$-module structure is given by*

$$\gamma(p; p_1, ..., p_i) = \gamma_{\mathcal{P}}(p; 1, ..., 1, p_1, ..., p_i).$$

Definition 2.43. *The universal enveloping algebra $U_{\mathcal{P}}(A)$ associated to the $\mathcal{P}$-algebra A is the associative algebra*

$$U_{\mathcal{P}}(A) = \mathcal{P}[1] \circ_{\mathcal{P}} A.$$

The multiplication is induced by

$$\bigoplus_{r,s\geq 0}(\mathcal{P}(r + 1) \otimes \mathcal{P}(s + 1)) \otimes_{k[\Sigma_r]\otimes k[\Sigma_s]} A^{\otimes r+s}$$

$$\xrightarrow{\circ_1 \otimes_{k[\Sigma_r]\otimes k[\Sigma_s]} A^{\otimes r+s}} \bigoplus_{r,s\geq 0} \mathcal{P}(r + s + 1) \otimes_{k[\Sigma_r]\otimes k[\Sigma_s]} A^{\otimes r+s}$$

$$\longrightarrow \bigoplus_{t\geq 0} \mathcal{P}(t + 1) \otimes_{k[\Sigma_t]} A^{\otimes t}.$$

Proposition 2.44. *The category $\mathcal{R}_{\mathcal{P}}(A)$ of A-representations over $\mathcal{P}$ is isomorphic to the category of left $U_{\mathcal{P}}(A)$-modules.*

Lemma 2.45. *Set $U_{\mathcal{P}} = \mathcal{P}[1]$. This is an associative algebra in right $\mathcal{P}$-modules with multiplication*

$$\mathcal{P}[1] \otimes \mathcal{P}[1] \to \mathcal{P}[1]$$

restricted to

$$(\mathcal{P}[1] \otimes \mathcal{P}[1])(l) = \bigoplus_{a+b=l} (\mathcal{P}[1](a) \otimes \mathcal{P}[1](b)) \otimes_{k[\Sigma_a]\otimes k[\Sigma_b]} k[\Sigma_l]$$

given by equivariantly extending

$$\circ_1 \colon \bigoplus_{a+b=l} \mathcal{P}(a+1) \otimes \mathcal{P}(b+1) \to \mathcal{P}(a+b+1).$$

Then for any $\mathcal{P}$-algebra A we have that $U_{\mathcal{P}}(A) = U_{\mathcal{P}} \circ_{\mathcal{P}} A$ with the induced algebra structure. Denoting the right $\mathcal{P}$-module structure map of $\mathcal{P}[1]$ by $\gamma_{\mathcal{P}[1]}$, we will often write

$$p(p_1, ..., p_{i-1}, x, p_i, ..., p_{l-1})$$

for the element $\gamma_{\mathcal{P}[1]}(p.(1....i); p_1, ..., p_{l-1}) \in \mathcal{P}[1]$.

There also is an operadic generalization of the module of Kähler differentials, i.e. a representation that is a representing object for derivations.

Definition 2.46. *For a $\mathcal{P}$-algebra A and a representation M a map $f \colon A \to M$ is called a $\mathcal{P}$-derivation if the diagram*

$$\begin{tikzcd}
\mathcal{P}(A) \arrow[r, "\mathcal{P}\circ' A"] \arrow[d, "\gamma_A"] & \mathcal{P}(A;A) \arrow[r, "1_{\mathcal{P}}(1_A;f)"] & \mathcal{P}(A;M) \arrow[d, "\gamma_M"] \\
A \arrow[rr, "f"] & & M
\end{tikzcd}$$

commutes. We denote the k-module of derivations $A \to M$ by $\mathrm{Der}_{\mathcal{P}}(A, M)$.

Definition 2.47. *Let A be a $\mathcal{P}$-algebra. The module of Kähler differentials $\Omega^1_{\mathcal{P}}(A)$ is the differential graded module generated as a k-module by expressions*

$$p(a_1, ..., da_i, ..., a_n), \quad p \in \mathcal{P}(n), a_1, ..., a_n \in A$$

with equivariance relations

$$(p.\sigma)(a_1, ..., da_i, ..., a_n) - p(\sigma.(a_1, ..., da_i, ..., a_n)),$$

for all $\sigma \in \Sigma_n$, where

$$\sigma.(b_1, ..., b_n) = \left(\prod_{1 \le i < j \le n : \sigma(i) > \sigma(j)} (-1)^{|b_i||b_j|} \right) (b_{\sigma^{-1}(1)}, ..., b_{\sigma^{-1}(n)}),$$

and further relations

$$\begin{aligned} & p(a_1,...,a_{j-1},\gamma_A(q;a_j,...,a_{j+m-1}),a_{j+m},...,da_i,...,a_{m+n-1}) \\ - & (-1)^{|q|(|a_1|+...+|a_{j-1}|)}(p\circ_j q)(a_1,...,a_{m+n-1}) \quad \text{for } j\neq i, \\ & p(a_1,...,a_{i-1},d\gamma_A(q;a_i,...,a_{i+m-1}),a_{i+m},...,a_{m+n-1}) \\ - & \sum_{l=0}^{m-1}(-1)^{|q|(|a_1|+...+|a_{i-1}|)}(p\circ_i q)(a_1,...,da_{i+l},...,a_{n+m-1}) \end{aligned}$$

with $q\in\mathcal{P}(m)$. The differential on $\Omega^1_{\mathcal{P}}(A)$ is induced by the differentials of $\mathcal{P}$ and A. The k-module $\Omega^1_{\mathcal{P}}(A)$ is a representation of A with left $U_{\mathcal{P}}(A)$-action given by

$$q(x,a_{n+1},...,a_{n+m})\cdot p(a_1,...,da_i,...,a_n)=(q\circ_1 p)(a_1,...,da_i,...,a_{n+m})$$

for $p\in\mathcal{P}(n), q\in\mathcal{P}[1](m), a_j\in A$.

Proposition 2.48. *Up to isomorphism the module of Kähler differentials $\Omega_{\mathcal{P}}(A)$ is determined by being a representation of A such that there is a natural isomorphism*

$$\mathrm{Der}_{\mathcal{P}}(A,-)\cong\mathrm{Hom}_{U_{\mathcal{P}}(A)}(\Omega_{\mathcal{P}}(A),-).$$

In the following proposition we assume that $\mathcal{P}(i)$ is a k-module to avoid additional signs.

Proposition 2.49. *Let $\mathcal{P}(i)$ be concentrated in degree zero for each $i\geq 0$. There is a right $\mathcal{P}$-module $\Omega^1_{\mathcal{P}}$ such that for all $\mathcal{P}$-algebras A*

$$\Omega^1_{\mathcal{P}}(A)=\Omega^1_{\mathcal{P}}\circ_{\mathcal{P}}A.$$

More precisely $\Omega^1_{\mathcal{P}}$ is as a k-module generated by expressions

$$p(x_{i_1},...,dx_{i_j},...,x_{i_n})$$

with $p\in\mathcal{P}(n)$, $\{i_1,...,i_n\}=\underline{n}$ and indeterminates $x_1,...,x_n$. The equivariance relations are generated by

$$(p.\sigma)(x_1,...,dx_i,...,x_n)-p(\sigma.(x_1,...,dx_i,...,x_n))$$

for all $\sigma\in\Sigma_n$. The right Σ-action is defined by

$$(p(x_1,...,dx_i,...,x_n)).\sigma=p(x_{\sigma(1)},...,dx_{\sigma(i)},...,x_{\sigma(n)})$$

and the right $\mathcal{P}$-module structure is determined by

$$p(x_1,...,dx_i,...,x_n)\circ_l q=\begin{cases}(p\circ_l q)(x_1,...,dx_{i+m-1},...,x_{n+m-1}), & l<i,\\ (p\circ_l q)(x_1,...,dx_i,...,x_{n+m-1}), & l>i,\\ \sum_{j=0}^{m-1}(p\circ_l q)(x_1,...,dx_{i+j},...,x_{n+m-1}), & l=i.\end{cases}$$

for $p \in \mathcal{P}(n), q \in \mathcal{P}(m)$. The left $U_{\mathcal{P}}$-module structure is given by

$$q \cdot p(x_1, ..., dx_i, ..., x_n) = (p \circ_1 q)(x_1, ..., dx_i, ..., x_{n+i})$$

for $q \in \mathcal{P}(i+1) = \mathcal{P}[1](i)$.

Lemma 2.50. *Let $A = (\mathcal{P}(Y), \partial_\alpha)$ be a quasifree $\mathcal{P}$-algebra with differential induced by $\alpha\colon Y \to \mathcal{P}(Y)$. Then*

$$\Omega_{\mathcal{P}}(A) = (U_{\mathcal{P}}(A) \otimes Y, \partial'_\alpha)$$

as a $U_{\mathcal{P}}(A)$-algebra with differential

$$\partial'_\alpha(u \otimes y) = \sum_j \sum_{i=1}^{n} (-1)^{|y_i^{(j)}|(|y_{i+1}^{(j)}|+...+|y_n^{(j)}|)} u \cdot p^{(j)}(y_1^{(j)}, ..., y_{i-1}^{(j)}, x, y_{i+1}^{(j)}, ..., y_n^{(j)}) \otimes y_i^{(j)},$$

where $\alpha(y) = \sum_j p^{(j)}(y_1^{(j)}, ..., y_n^{(j)})$.

Proof. The isomorphism is induced by

$$U_{\mathcal{P}}(\mathcal{P}(Y)) \otimes Y \to \Omega_{\mathcal{P}}(\mathcal{P}(Y)), \quad p(x, y_2, ..., y_n) \otimes y \mapsto (-1)^{|y|(|y_2|+...+|y_n|)} p(dy, y_2, ..., y_n).$$

A calculation shows that this is a morphism of $U_{\mathcal{P}}(A)$-modules and that the map respects the differentials, see [33, 2.1.1]. $\square$

After defining representations, derivations, the enveloping algebra and the module of Kähler differentials we are finally in the position to talk about operadic homology and cohomology.

Definition 2.51. *Let A be a $\mathcal{P}$-algebra, M a representation of A over $\mathcal{P}$. Then the $\mathcal{P}$-homology of A with coefficients in M is given by*

$$H_*^{\mathcal{P}}(A; M) = H_*(M \otimes_{U_{\mathcal{P}}(Q_A)} \Omega^1_{\mathcal{P}}(Q_A))$$

for a cofibrant replacement Q_A of A as a $\mathcal{P}$-algebra.
Similarly, the $\mathcal{P}$-cohomology of A with coefficients in M is given by

$$H^*_{\mathcal{P}}(A; M) = H_*(\mathrm{Der}_{\mathcal{P}}(Q_A, M))$$

where Q_A is again a cofibrant replacement of A as a $\mathcal{P}$-algebra.

Remark 2.52. *That these definitions are indeed independent of the choice of a specific cofibrant replacement is proved in [20, 13.1.2]. Furthermore, one easily sees that $\mathcal{P}$-homology and $\mathcal{P}$-cohomology are functorial in A and M and that a weak equivalence $A \to B$ of $\mathcal{P}$-algebras induces isomorphisms*

$$H_*^{\mathcal{P}}(A; M) \to H_*^{\mathcal{P}}(B; M) \quad \text{and} \quad H^*_{\mathcal{P}}(B; M) \to H^*_{\mathcal{P}}(A; M).$$

In [33, Theorem 2.5] it is shown that a quasiisomorphism $\mathcal{P} \to \mathcal{Q}$ of Σ_*-cofibrant operads induces a natural isomorphism between $\mathcal{P}$- and $\mathcal{Q}$-homology.

Example 2.53. *For the classical case $\mathcal{P} = \mathcal{A}s_+$ we retrieve Hochschild (co)homology up to suspension. If* $\mathrm{char}(k) = 0$ *the operad $\mathcal{C}om_+$ is Σ_*-cofibrant and the (co)homology of commutative algebras defined this way is, again up to suspension, Harrison (co)homology.*

2.5 Homological algebra for small categories

Let $\mathcal{C}$ be a small category and consider the category $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$ of covariant functors from $\mathcal{C}$ to k-mod. We recall the construction of Ext and Tor in this context, which can be for example found in [66, ch. 2] or [52]. The category of functors $\mathcal{C} \to k$-mod is an abelian category: All limits and colimits exist and are formed objectwise, a natural transformation $\eta\colon F \to G$ is a monomorphism (respectively an epimorphism) if η_c is injective (respectively surjective) for each object $c \in \mathcal{C}$.
The category $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$ has enough injectives and projectives since k-mod has enough injective and projective objects and the abelian structure is defined objectwise. More precisely, by the Yoneda lemma, natural trandsformations $\eta\colon k[\mathcal{C}(c, -)] \to F$ correspond to a choice of $\eta(\mathrm{id}_c) \in F(c)$ for any $c \in \mathcal{C}$ and any $F \in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$. Similarly a natural transformation $F \to \mathrm{Hom}_k(k[\mathcal{C}(-, c)], k)$ corresponds to choosing a value for $F(c)(\mathrm{id}_c)$. Hence $k[\mathcal{C}(c, -)]$ is projective, and every functor from $\mathcal{C}$ to k-mod receives a surjection from a sum of these representables. Seeing that $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$ has enough injectives is more involved, cf. [66, 2.3.13]. For this reason we will restrict ourselves to the case that $\mathcal{C}(c, d)$ is a finite set for all c, d in $\mathcal{C}$. Under these assumptions, the observation about $\mathrm{Nat}(F, \mathrm{Hom}_k(k[\mathcal{C}(-, c)], k))$ we made above yields that the functor $\mathrm{Hom}_k(k[\mathcal{C}(-, c)], k)$ is injective (cf [52]) and that hence $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$ has enough injectives. The same holds for the category of contravariant functors from $\mathcal{C}$ to k-mod if we consider the contravariant representables $k[\mathcal{C}(-, c)]$.
In particular we can talk about projective and injective resolutions of such functors: Given a covariant functor $F\colon \mathcal{C} \to k$-mod a projective resolution of F is an exact sequence

$$... \to P_i \to P_{i-1} \to ... \to P_1 \to P_0 \to F \to 0$$

such that all the P_i are projective. Analogously, an injective resolution of F is an exact sequence of functors

$$0 \to F \to I_0 \to I_1 \to ... \to I_i \to I_{i+1} \to ...$$

with all I_i injective. Projective and injective resolutions of contravariant functors are defined similarly.
Hence we can define derived functors in this setting. We will stick to the cases we need to define the derived functors we will use later.

Definition 2.54. *Let* $\mathcal{F}\colon \mathrm{Fun}(\mathcal{C}, k\text{-mod}) \to k\text{-mod}$ *be a covariant right exact functor and* $F \in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$. *Then*

$$\mathcal{F}(P) = (... \to \mathcal{F}(P_i) \to \mathcal{F}(P_{i-1}) \to ... \to \mathcal{F}(P_1) \to \mathcal{F}(P_0) \to 0)$$

with $\mathcal{F}(P_i)$ in degree i is a chain complex in the abelian category $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$. *The left derived functors of $\mathcal{F}$ are defined as*

$$(\mathbb{L}\mathcal{F})_s(F) = H_s(\mathcal{F}(P)).$$

Similarly, for a left exact functor $\mathcal{G}\colon \mathrm{Fun}(\mathcal{C}, k\text{-mod}) \to k\text{-mod}$ *we define the right derived functors of $\mathcal{G}$ to be*

$$(\mathbb{R}\mathcal{G})_s(F) = H^s(\mathcal{G}(I))$$

for an injective resolution I of F. For a left exact contravariant functor $\mathcal{H}$ from the category $\mathrm{Fun}(\mathcal{C}, k\text{-mod})$ *to* k-mod *we define its right derived functor applied to* $F \in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$ *as*

$$(\mathbb{R}\mathcal{H})_s(F) = H^s(\mathcal{H}(P)),$$

where P again is a projective resolution of F.

Remark 2.55. *As usual one can show that, up to isomorphism, derived functors are independent of the choice of projective or injective resolution.*

We now define Tor and Ext for functors from $\mathcal{C}$ to k-mod by deriving the tensor product functor and the functor of natural transformations. That these functors satisfy the appropriate exactness requirements follows for example from [61, ch. 16].

Definition 2.56. *For* $F \in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$ *and* $G \in \mathrm{Fun}(\mathcal{C}^{op}, k\text{-mod})$ *we define $G \otimes_{\mathcal{C}} F$ as the k-module given by*

$$G \otimes_{\mathcal{C}} F = \bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} G(c) \otimes F(c)/\sim$$

with the relation $\sim$ defined by $(x, F(f)(y)) \sim (G(f)(x), y)$ for $f \in \mathrm{Mor}_{\mathcal{C}}(c, d), y \in F(c), x \in G(d)$. We set

$$\mathrm{Tor}_*^{\mathcal{C}}(G, F) = (\mathbb{L}(G \otimes -))_*(F).$$

Similarly, for $F, H \in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$ *we set*

$$\mathrm{Ext}_{\mathcal{C}}^*(F, H) = (\mathbb{R}\mathrm{Nat}(F, -))_*(H).$$

Remark 2.57. *As in the classical setting, one can compute* Tor *and* Ext *as*

$$\mathrm{Tor}_*^{\mathcal{C}}(G, F) \cong (\mathbb{L}(- \otimes F))_*(G)$$

and

$$\mathrm{Ext}_{\mathcal{C}}^*(F, H) \cong (\mathbb{R}\mathrm{Nat}(-, H))_*(F)$$

as well.

3 E_n-homology and cohomology

In [23] Benoit Fresse proves that the n-fold bar construction for commutative algebras can be extended to E_n-algebras, and that one can calculate E_n-homology with trivial coefficients up to a shift as the homology of this iterated bar construction. Muriel Livernet and Birgit Richter use this in [41] to prove that one can interpret E_n-homology as functor homology. There is also an extension of Fresse's result to E_n-homology and cohomology of a commutative algebra A with coefficients in a symmetric A-bimodule M, which we will use in section 4 to extend the results of Livernet and Richter. The goal of this chapter is to give the details of the proof of Fresse's unpublished result, based on a sketch of the proof provided by Benoit Fresse to the author. The ideas of this section mostly stem from [20] and [23], but the arguments for this extension to the non-trivial coefficient case have not been written down yet.
We first define E_n-operads and E_n-algebras, then recall the constructions of [23]. Finally we give a proof of the unpublished result by Fresse for coefficients in M.

3.1 E_n-algebras

The interest in E_n-structures originated from the study of n-fold loop spaces. Around 1970 Boardman-Vogt [9] and May [50] showed that under some conditions n-fold loop spaces correspond to algebras over a certain operad. We will define this operad and then describe the algebraic analogue of n-fold loop spaces, E_n-algebras.

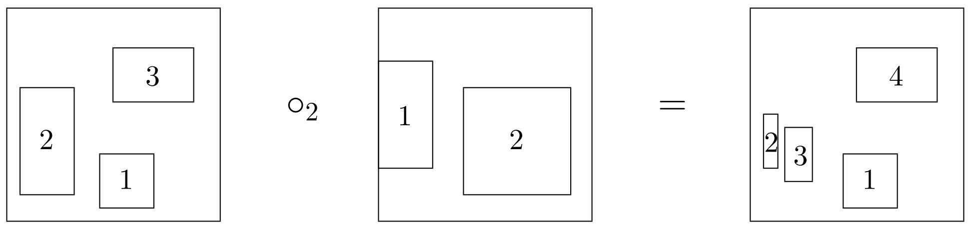
E_n-operads

Definition 3.1. *([9, Example (2.49)],[50, Definition 4.1]) Let $1 \leq n < \infty$ and let $I = [0,1]$ denote the unit interval. As a set the little n-cubes operad $\mathcal{C}_n$ is given in arity r by linear embeddings with parallel axes of r n-dimensional cubes I^n into an n-cube such that the images of the interiors of the r embedded cubes are disjoint. This set is endowed with the topology inherited from being a subspace of* $\mathrm{Top}((I^n)^{\sqcup r}, I^n)$*. The operad structure is defined as follows: Let $f = f_1 + ... + f_r \colon (I^n)^{\sqcup r} \to I^n$ and $g = g_1 + ... + g_s \colon (I^n)^{\sqcup s} \to I^n$ be embeddings as above. Then for $1 \leq i \leq r$ we define the partial composition $\circ_i$ by*

$$f \circ_i g = f_1 + ... + f_{i-1} + f_i \circ (g_1 + ... + g_s) + f_{i+1} + ... + f_r.$$

The symmetric group Σ_r acts on $\mathcal{C}_n(r)$ by permuting the embedded n-cubes.

The following picture illustrates the composition for $n = 2$.



Define an operad morphism $\mathcal{C}_n \to \mathcal{C}_{n+1}$ as the morphism induced by interpreting I^n as a fixed face of I^{n+1}. More precisely, map a little n-cube $f_1 + ... + f_r \colon (I^n)^{\sqcup r} \to I^n$ to the little $n+1$-cube

$$f_1 \times \mathrm{id}_I + ... + f_r \times \mathrm{id}_I \colon (I^{n+1})^{\sqcup r} \to I^{n+1}.$$

We set $\mathcal{C}_\infty = \mathcal{C} = \mathrm{colim}_i \mathcal{C}_i$.
The reason for the interest in $\mathcal{C}_n$ lies in the following example and theorem.

Example 3.2. *Let X be a topological space. Then $\Omega^n X$ is a $\mathcal{C}_n$-algebra: For*

$$\gamma_1, ..., \gamma_r \colon (I^n, \partial I^n) \to X$$

and a little n-cube $f_1 + ... + f_r \colon (I^n)^{\sqcup r} \to I^n$ let

$$\gamma_{\Omega_n X}(f_1 + ... + f_r; \gamma_1, ..., \gamma_r)(x) = \begin{cases} \gamma_i(y), & x = f_i(y) \text{ for some } i, \\ *, & \text{otherwise.} \end{cases}$$

Theorem 3.3 ([50, Theorem 1.3], see also [9, Theorem 6.31, Theorem 6.24])**.** *For $1 \leq n \leq \infty$, a connected space has the weak homotopy type of an n-fold loop space if and only if it is a $\mathcal{C}_n$-algebra.*

By remark 2.7 we pass from the topological to the algebraic world via the singular chains functor C_*.

Definition 3.4. *We call an operad $\mathcal{P}$ in dg*-mod *an E_n-operad if there is a zig-zag of quasiisomorphisms connecting $\mathcal{P}$ and the operad $C_*(\mathcal{C}_n)$, i.e. if there are operads $\mathcal{P}_1, ..., \mathcal{P}_k$ and quasiisomorphisms*

$$\mathcal{P} \xleftarrow{\sim} \mathcal{P}_1 \xrightarrow{\sim} ... \xleftarrow{\sim} \mathcal{P}_k \xrightarrow{\sim} C_*(\mathcal{C}_n).$$

Example 3.5. *Since $\mathcal{C}_1$ is homotopy equivalent to $\mathcal{A}s_+^{\mathrm{Top}}$, an operad $\mathcal{P}$ is an E_1-operad if it is connected to $\mathcal{A}s_+$ by quasiisomorphisms. In the other extreme case, $n = \infty$, one can show (see e.g. [50, 4.8]) that each $\mathcal{C}(r)$ is contractible. Hence in the terminology above an operad $\mathcal{P}$ is an E_∞-operad if it is connected to $\mathcal{C}om_+$ by quasiisomorphisms. Note however that E_∞-operads are often assumed to be Σ_*-cofibrant or even Σ_r-free in each arity r.*

The Barratt-Eccles operad The Barratt-Eccles operad was originally introduced as a simplicial operad $E\Sigma$ by Barratt and Eccles in [3, §3]. In arity r

$$E\Sigma(r) = E\Sigma_r$$

is the nerve of the translation category associated to Σ_r, i.e. the nerve of the category with objects the elements in Σ_r and with exactly one morphism from one object to another. The operation of Σ_r on $E\Sigma_r$ is the diagonal one. Composition in the simplicial Barratt-Eccles operad is defined as follows: For $\sigma \in \Sigma_r$ and $\tau \in \Sigma_s$ the ith composite $\sigma \circ_i \tau \in \Sigma_{r+s-1}$ is given by

$$\sigma_{(1,...,1,s,1,...,1)}(\mathrm{id}_{\{1,...,i-1\}} \oplus \tau \oplus \mathrm{id}_{\{i+s,...,r+s-1\}}),$$

where $\sigma_{(1,...,1,s,1,...,1)}$ permutes the r blocks $\{1\}, ..., \{i-1\}, \{i, ..., i+s-1\}, \{i+s\}, ..., \{r+s-1\}$ like σ permutes $\{1, ..., r\}$, and

$$(\mathrm{id}_{\{1,...,i-1\}} \oplus \tau \oplus \mathrm{id}_{\{i+s,...,r+s-1\}})(l) = \begin{cases} l, & l < i, \\ \tau(l-i+1)+i-1, & i \leq l \leq i+s-1, \\ l, & l \geq i+s. \end{cases}$$

The partial composition

$$\circ_i \colon (E\Sigma(r) \times E\Sigma(s))_l \to (E\Sigma(r+s-1))_l$$

of $E\Sigma$ is then given by

$$(\omega_0, ..., \omega_l) \circ_i (\tau_0, ..., \tau_l) = (\omega_0 \circ_i \tau_0, ..., \omega_l \circ_i \tau_l)$$

for $\omega_0, ..., \omega_l \in \Sigma_r$ and $\tau_0, ..., \tau_l \in \Sigma_s$.
Recall that for a simplicial k-module $K \colon \Delta^{\mathrm{op}} \to k$-mod with face maps $d_i \colon K_l \to K_{l-1}$ and degeneracies $s_i \colon K_l \to K_{l+1}$ the normalized Moore complex NK of K is the chain complex with

$$(NK)_l = K_l / \sum_{i=0}^{l-1} s_i(K_{l-1})$$

and differential

$$\sum_{i=0}^{l} (-1)^i d_i \colon (NK)_l \to (NK)_{l-1}.$$

This gives rise to the following differential graded version.

Definition 3.6. *The Barratt-Eccles operad $\mathcal{E}$ is in arity r given by*

$$\mathcal{E}(r) = N_*(k[E\Sigma_r]).$$

Hence $\mathcal{E}(r)_l$ is k-free with a basis given by $l+1$-tuples $(\sigma_0, ..., \sigma_l) \in \Sigma_r^{l+1}$ such that $\sigma_i \neq \sigma_{i+1}$, and the differential is given by

$$d(\sigma_0, ..., \sigma_l) = \sum_{i=0}^{l} (-1)^i (\sigma_0, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_l).$$

The action of $\sigma \in \Sigma_r$ on $\mathcal{E}(r)$ is the diagonal one, i.e.

$$(\sigma_0, ..., \sigma_l).\sigma = (\sigma_0\sigma, ..., \sigma_l\sigma).$$

The composition induced by the composition in $E\Sigma$ is then

$$\circ_i \colon \mathcal{E}(r)_l \otimes \mathcal{E}(s)_m \to \mathcal{E}(r+s-1)_{l+m},$$

$$(\sigma_0, ..., \sigma_l) \circ_i (\tau_0, ..., \tau_m) = \sum_{\substack{(x_0,...,x_{l+m}),\\ (y_0,...,y_{l+m})}} \pm(\sigma_{x_0} \circ_i \tau_{y_0}, ..., \sigma_{x_{l+m}} \circ_i \tau_{y_{l+m}}),$$

where the sum is taken over all paths (x, y) from $(0, 0)$ to (l, m), i.e. sequences $x = (x_0, ..., x_{l+m})$ and $y = (y_0, ..., y_{l+m})$ in $[l+m] = \{0, ..., l+m\}$ with $x_j \leq x_{j+1}, y_j \leq y_{j+1}$ and $x_j - x_{j+1} + y_j - y_{j+1} = -1$. The concrete signs are described in [8, 1.1.3].

In particular, $\mathcal{E}(r)$ is obviously a Σ_r-free simplicial set. Since $E\Sigma_r$ is the nerve of a category with a terminal object, the maps $E\Sigma_r \to *$ induce isomorphisms on homotopy groups. Hence it is clear that the map

$$N_*(k[E\Sigma_r]) \to \mathcal{C}om$$

sending $N_l(k[E\Sigma_r])$ to 0 for $l > 0$ and given on $N_0(k[E\Sigma_r]) = k[\Sigma_r]$ by $\sigma \mapsto 1$ is a Σ_r-equivariant quasiisomorphism. We see that $\mathcal{E}$ is a Σ_*-cofibrant E_∞-operad.
In [63] Smith defines a filtration

$$E\Sigma^{(1)} \subset E\Sigma^{(2)} \subset ... \subset E\Sigma^{(n)} \subset E\Sigma^{(n+1)} \subset ... \subset E\Sigma$$

of the simplicial Barratt-Eccles operad by suboperads $E\Sigma^{(n)}$. The simplicial operad $E\Sigma^{(n)}$ consists in arity r of simplices $(\omega_0, ..., \omega_l) \in (E\Sigma_r)_l$ such that for all $1 \leq i, j \leq r$ the sequence

$$((\omega_0)_{ij}, ..., (\omega_l)_{ij})$$

has at most $n-1$ variations, where for $\omega \in \Sigma_r$ we define ω_{ij} to be $\mathrm{id}_{(2)} \in \Sigma_2$ if $\omega(i) > \omega(j)$ and $(12) \in \Sigma_2$ otherwise.
This filtration gives rise to the following filtration of $\mathcal{E}$ by E_n-operads, see [7, 1.13].

Proposition 3.7. *Let $\mathcal{E}_n$ be the suboperad of $\mathcal{E}$ which in arity r and degree l is as a k-module generated by sequences $(\omega_0, ..., \omega_l) \in \Sigma_r^{l+1}$ such that*

$$((\omega_0)_{ij}, ..., (\omega_l)_{ij})$$

has at most $n-1$ variations for all $i, j \in \underline{r}$. This defines a filtration

$$\mathcal{A}s = \mathcal{E}_1 \subset \mathcal{E}_2 \subset ... \subset \mathcal{E}_n \subset \mathcal{E}_{n+1} \subset ... \subset \mathcal{E}$$

of $\mathcal{E}$ by suboperads such that $\mathcal{E}_n$ is an E_n-operad.

E_n-algebras We give some examples of E_n-algebras. The obvious class of examples arises from the original motivation for studying E_n-algebras in the topological world.

Example 3.8. *Let X be an arbitrary topological space. Then by example 3.2 the space $\Omega^n X$ is a $\mathcal{C}_n$-algebra, and consequently $C_*(\Omega^n X)$ is an algebra over $C_*(\mathcal{C}_n)$.*

Example 3.9. *Let A be an algebra over an E_∞-operad E admitting a filtration*

$$E_1 \subset ... \subset E_n \subset E_{n+1} \subset ... \subset E$$

by E_n-operads, e.g. $E = C_*(\mathcal{C})$ or $E = \mathcal{E}$. Then A is an E_n-algebra for any $1 \leq n \leq \infty$ by restricting the given E-algebra structure.

Example 3.10. *Often E_∞-operads E are not only connected to $\mathcal{C}om$ via a zig-zag of quasi-isomorphisms, but indeed admit a morphism*

$$E \to \mathcal{C}om.$$

Examples include $C_*(\mathcal{C})$ and $\mathcal{E}$. Then every commutative algebra is an E-algebra as well.

Example 3.11. *Another important example arises for $n = 1$: Algebras over an E_1-operad are A_∞-algebras. A cofibrant (nonsymmetric) E_1-operad $\mathcal{A}_\infty$ is given by the operad formed by the cellular chains on the topological operad of Stasheff's associahedra ([49, 1.8]).*

Example 3.12. *The famous Deligne conjecture (see [49, 1.19] for an overview) states that the standard cochain complex $C^*_{HH}(A; A)$ computing Hochschild cohomology of an associative algebra (or more generally an A_∞-algebra) A is an algebra over a suitable E_2-operad. In particular, the Gerstenhaber algebra structure on Hochschild cohomology stems from an action of the operad $H_*(E_2)$ governing Gerstenhaber algebras. A proof has been given, amongst others, by McClure and Smith in [51]. The E_2-operad acting on $C^*_{HH}(A; A)$ exhibited by them is in fact up to signs a quotient of the suboperad $\mathcal{E}_2$ of the Barratt-Eccles operad (see [8]).*

Example 3.13. *There is also a generalized version of the Deligne conjecture, stating that for a suitable cochain complex $D^*(A; A)$ computing E_n-cohomology $H^*_{E_n}(A; A)$ for an E_n-algebra A, the complex $D^*(A; A)$ is an E_{n+1}-algebra. A construction of such a complex $D^*(A; A)$ is discussed in [37] and [45, 6.1.4].*

3.2 E_n-homology with trivial coefficients via the iterated bar construction

We will discuss the iterated bar complex constructed in [23] and its relation to E_n-homology with trivial coefficients in this subsection. To simplify matters, we restrict our attention to the Barratt-Eccles operad and set $E = \mathcal{E}$ and $E_n = \mathcal{E}_n$ for the rest of this chapter, although the results in [23] are proved for a more general class of E_n-operads. Furthermore, we will work with nonunital E_n-algebras and hence only consider operads $\mathcal{P}$ with $\mathcal{P}(0) = 0$. The trivial action of an E_n-algebra A on k is given by setting

$$E_n(r) \otimes_{\Sigma_r} (A; M)^{\otimes r} \to M$$

to be zero for all $r \geq 2$ and to be the identity on $E_n(1) \otimes (A; M) \cong M$. We will still denote this reduced version of E_n by E_n. We will also frequently switch between considering Σ_*-modules and functors defined on the category Bij of finite sets and bijections as explained in proposition 2.15.

Observe that working with the reduced variant of E_n does not affect E_n-homology and -cohomology. Indeed, there is a Quillen equivalence between the category of nonunital E_n-algebras and augmented E_{n+}-algebras, where $E_{n+}(0) = k$: Every augmented E_{n+}-algebra can be interpreted as a E_n-algebra, while on the other hand to an E_n-algebra A we can associate the augmented E_{n+}-algebra $A_+ = A \oplus k$. This is an E_{n+}-algebra if we set $p(a_1, ...a_{i-1}, 1_k, a_{i+1}, ..., a_l) = (p \circ_i \eta)(a_1, ..., a_{i-1}, a_{i+1}, ..., a_l)$ for $p \in E_n$, with 1_k denoting the unit of A_+ and η denoting the generator of $E_{n+}(0)$. Also each representation of A is a representation of A_+ as well. Using that $(Q_A)_+$ is a cofibrant replacement of A_+, if Q_A is a cofibrant replacement of the E_n-algebra A, and that

$$\mathrm{Der}_{E_n}(Q_A, M) \cong \mathrm{Der}_{E_{n+}}((Q_A)_+, M),$$

we see that for example $H^*_{E_n}(A; M) \cong H^*_{E_{n+}}(A_+; M)$.

The iterated bar complex associated to E-algebras The (unreduced) bar construction was defined bei Eilenberg-MacLane in [16, II.7]. We recall the definition of the nonunital reduced bar construction, which we will be working with. For $a \in A$ we denote the corresponding element in ΣA by sa.

Definition 3.14. *Let A be a nonunital differential graded k-algebra. The reduced bar construction BA is the differential graded k-module*

$$(\overline{T}^c(\Sigma A) = \bigoplus_{i \geq 1} (\Sigma A)^{\otimes i}, \partial_s).$$

The twist ∂_s is given by

$$\partial_s(sa_1 \otimes ... \otimes sa_l) = \sum_{i=1}^{l-1} (-1)^{i+|a_1|+...+|a_i|} sa_1 \otimes ... \otimes sa_i a_{i+1} \otimes ... \otimes sa_l.$$

If A is graded commutative, BA is a differential graded commutative algebra as well, with product given by the shuffle product

$$\mathrm{sh}(sa_1 \otimes ... \otimes sa_p, sa_{p+1} \otimes ... \otimes sa_{p+q}) = \sum_{\sigma \in \mathrm{sh}(p,q)} (-1)^{\epsilon} sa_{\sigma^{-1}(1)} \otimes ... \otimes sa_{\sigma^{-1}(p+q)},$$

where $\mathrm{sh}(p,q) \subset \Sigma_{p+q}$ *denotes the set of permutations σ such that $\sigma(1) < ... < \sigma(p)$ and $\sigma(p+1) < ... < \sigma(p+q)$. The sign ϵ is determined by picking up a factor $(-1)^{(|a_i|+1)(|a_j|+1)}$ whenever $i < j$ and $\sigma(i) > \sigma(j)$. In particular we can then iterate the construction and define an n-fold bar complex $B^n(A)$.*

Remark 3.15. *The bar construction BA is the augmentation ideal of the normalized Moore complex associated to a simplicial differential graded algebra whose l-simplices are given by $(A_+)^{\otimes l}$, face maps defined similar to the summands of the differential ∂_s and degeneracies given by inserting the unit of A_+. The construction originally given in [16, II.7] corresponds to the unnormalized Moore complex associated to this simplicial differential graded algebra. There also is a reduced bar construction with coefficients in a nonunital right A-module M and a nonunital left A-module N, as well as an unreduced version. The case discussed above corresponds to the reduced bar construction with coefficients in $M = N = k$.*

Definition 3.16. *Let $\mathcal{P}$ be an operad in differential graded modules and let R be an $\mathcal{A}s$-algebra in right $\mathcal{P}$-modules. Just as for usual algebras the bar construction associated to R is the right $\mathcal{P}$-module given by $BR = (\overline{T}^c(\Sigma R), \partial_s)$ with*

$$\overline{T}^c(\Sigma R) = \bigoplus_{i \geq 1} (\Sigma R)^{\otimes i}.$$

and twist defined for

$$sr_1 \otimes ... \otimes sr_l \in \Sigma R(\underline{e_1}) \otimes ... \otimes \Sigma R(\underline{e_l}) \subset (\Sigma R)^{\otimes l}(\underline{e_1 \sqcup ... \sqcup e_l})$$

by

$$\partial_s(sr_1 \otimes ... \otimes sr_l) = \sum_{i=1}^{k-1} (-1)^{i+|r_1|+...|r_i|} sr_1 \otimes ... \otimes s\gamma(\mathrm{id}_{\underline{2}}; r_i, r_{i+1}) \otimes ... \otimes sr_l.$$

Here the tensor product is the tensor product of right $\mathcal{P}$-modules and $\mathrm{id}_{\underline{2}} \in \mathcal{A}s(2)$.
If R is commutative, i.e. if the action of $\mathcal{A}s$ on R factors through $\mathcal{C}om$, then BR is a $\mathcal{C}om$-algebra in right $\mathcal{P}$-modules with multiplication again given by the shuffle product.

Applying this to $\mathcal{C}om$ itself we define the commutative algebra $B^n_{\mathcal{C}om}$ in right $\mathcal{C}om$-modules by

$$B^n_{\mathcal{C}om} := B^n(\mathcal{C}om).$$

According to [23, 2.7, 2.8] the iterated bar module $B^n_{\mathcal{C}om}$ is a quasifree right $\mathcal{C}om$-module

$$B^n_{\mathcal{C}om} = (T^n \mathcal{C}om, \partial_\gamma).$$

with $T^n = (\overline{T}^c \Sigma)^n(I)$ a free Σ_*-module. Recall that, for a free right $\mathcal{P}$-module $K\mathcal{P}$ and a morphism $f\colon K \to R$ with target a right $\mathcal{P}$-module, $\partial_f\colon K\mathcal{P} \to R$ denotes the induced morphism of right $\mathcal{P}$-modules.
By [23, 2.5] it is possible to lift ∂_γ to a twisting differential

$$\partial_\epsilon\colon T^n E \to T^n E$$

and set $B^n_E = (T^n E, \partial_\epsilon)$. For an E-algebra A we call

$$B^n_E(A) = B^n_E \circ_E A$$

the n-fold bar complex of A.

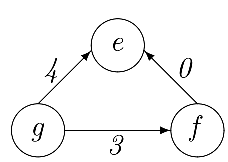
The complete graph operad To prove that it is possible to extend the definition of the n-fold bar complex from E-algebras to E_n-algebras, Fresse shows that $\partial_\epsilon\colon T^n E \to T^n E$ restricts to

$$\partial_\epsilon\colon T^n E_n \to T^n E_n.$$

To do this one uses that E is equipped with a cell structure indexed by complete graphs. Since we will need the complete graph operad $\mathcal{K}$ in subsection 3.3 to prove a similar result for E_n-homology and -cohomology with coefficients, we revisit the relevant definitions and results. The complete graph operad and its relation to E_n-operads has been discussed by Berger in [6].

Definition 3.17. *Let $\underline{e}$ be a finite set with r elements. A complete graph $\kappa = (\sigma, \mu)$ on $\underline{e}$ consists of an ordering $\sigma\colon \{1, ..., r\} \to \underline{e}$ together with a symmetric matrix $\mu = (\mu_{ef})_{e,f \in \underline{r}}$ of elements $\mu_{ef} \in \mathbb{N}_0$ with all diagonal entries 0. We think of σ as a globally coherent orientation of the edges and of the matrix μ as the weights of the edges.*

Example 3.18. *The complete graph*



on the set $\{e, f, g\}$ corresponds to

$$\sigma\colon \{1,2,3\} \to \{e,f,g\}, \quad \sigma(1) = g, \sigma(2) = f, \sigma(3) = e$$

and $\mu_{ef} = 0, \mu_{eg} = 4, \mu_{fg} = 3$.

Definition 3.19. *The set of complete graphs on $\underline{e}$ is partially ordered if we set*

$$(\sigma,\mu) \leq (\sigma',\mu')$$

whenever for all $e,f \in \underline{e}$ either $\mu_{ef} < \mu'_{ef}$ or $(\sigma_{ef},\mu_{ef}) = (\sigma'_{ef},\mu'_{ef})$. The poset of complete graphs on $\underline{e}$ is denoted by $\mathcal{K}(\underline{e})$.

Proposition 3.20. *[6] The collection $\mathcal{K} = (\mathcal{K}(\underline{e}))_{\underline{e}}$ of complete graphs forms an operad in posets: A bijection $\omega\colon \underline{e} \to \underline{e}'$ acts by relabeling the vertices. The partial composition*

$$\circ_e\colon \mathcal{K}(\underline{e}) \times \mathcal{K}(\underline{f}) \to \mathcal{K}(\underline{e} \sqcup \underline{f} \setminus \{e\})$$

is given by substituting the vertex e in a complete graph $(\sigma,\mu) \in \mathcal{K}(\underline{e})$ by the complete graph $(\tau,\nu) \in \mathcal{K}(\underline{f})$, i.e. by inserting (τ,ν) at the position of e, orienting the edges between $g \in \underline{e} \setminus \{e\}$ and $g' \in \underline{f}$ like the edge between g and e and giving them the weight μ_{ge}.

Definition 3.21. *[23, 3.4,3.8] Let $(\mathcal{P}(\underline{e}))_{\underline{e}}$ be a collection of functors $\mathcal{K}(\underline{e}) \to dg\text{-mod}$. Such a collection is called a $\mathcal{K}$-operad if for all finite sets $\underline{e}$ and all bijections $\omega\colon \underline{e} \to \underline{e}'$ there is a natural transformation $\mathcal{P}(\underline{e}) \to \mathcal{P}(\underline{e}')$ with components*

$$\mathcal{P}_\kappa \to \mathcal{P}_{\kappa.\omega}$$

for $\kappa \in \mathcal{K}(\underline{e})$, as well as partial composition products given by natural transformations

$$\circ_e\colon \mathcal{P}(\underline{e}) \otimes \mathcal{P}(\underline{f}) \to \mathcal{P}(\underline{e} \setminus \{e\} \sqcup \underline{f})$$

for $e \in \underline{e}$ with components

$$\circ_e\colon \mathcal{P}_\kappa \otimes \mathcal{P}_{\kappa'} \to \mathcal{P}_{\kappa \circ_e \kappa'},$$

satisfying suitable associativity, unitality and equivariance conditions. A morphism $\mathcal{P} \to \mathcal{P}'$ consists of natural transformations with components $\mathcal{P}_\kappa \to \mathcal{P}'_\kappa$ commuting with composition and the actions of bijections.
Similarly, a right $\mathcal{K}$-module R over a $\mathcal{K}$-operad consists of collections $(R(\underline{e}))_{\underline{e}}$ together with natural transformations $R_{\underline{e}} \to R_{\underline{e}'}$ for all bijections $\omega\colon \underline{e} \to \underline{e}'$ with components

$$R_\kappa \to R_{\kappa.\omega},$$

as well as natural transformations

$$\circ_e\colon \mathcal{P}(\underline{e}) \otimes R(\underline{f}) \to R(\underline{e} \setminus \{e\} \sqcup \underline{f})$$

with components

$$\circ_e\colon R_\kappa \otimes \mathcal{P}_{\kappa'} \to R_{\kappa \circ_e \kappa'}$$

for $e \in \underline{e}$ satisfying again suitable associativity, unitality and equivariance conditions.

Remark 3.22. *[23, 3.4,3.8] There is an adjunction*

$$\operatorname{colim}\colon \mathcal{O}_{dg\text{-mod}} \rightleftarrows {}_{\mathcal{K}}\mathcal{O}\colon \operatorname{const}$$

between the category $\mathcal{O}_{dg\text{-mod}}$ of operads in differential graded k-modules and the category ${}_{\mathcal{K}}\mathcal{O}$ of $\mathcal{K}$-operads defined as follows: For a given $\mathcal{K}$-operad $\mathcal{P}$ let

$$(\operatorname{colim}\mathcal{P})(\underline{e}) = \operatorname{colim}_{\kappa\in\mathcal{K}(\underline{e})}\mathcal{P}_\kappa,$$

while for an ordinary operad $\mathcal{Q}$ we set

$$\operatorname{const}(\mathcal{Q})_\kappa = \mathcal{Q}(\underline{e})$$

for $\kappa \in \mathcal{K}(\underline{e})$. We say that an operad $\mathcal{Q}$ has a $\mathcal{K}$-structure if $\mathcal{Q} = \operatorname{colim}\mathcal{P}$. A similar adjunction exists between the category of right $\mathcal{K}$-modules over a $\mathcal{K}$-operad $\mathcal{P}$ and the category of right modules over $\operatorname{colim}\mathcal{P}$*, and we call right modules of the form* $\operatorname{colim}R$ *right* $\operatorname{colim}\mathcal{P}$*-modules with $\mathcal{K}$-structure.*

Example 3.23. *[23, 3.5] Besides considering constant $\mathcal{K}$-operads the main example we are interested in is the Barratt-Eccles operad E. For $\kappa = (\sigma,\mu) \in \mathcal{K}(r)$ the k-module E_κ is generated by l-tuples*

$$(\omega_0, ..., \omega_l) \in \Sigma_r^{l+1}$$

such that for all $i,j \in \underline{r}$ the sequence

$$((\omega_0)_{ij}, ..., (\omega_l)_{ij})$$

has either less than μ_{ij} variations or has exactly μ_{ij} variations and $(\omega_l)_{ij} = \sigma_{ij}$. It is obvious that the differential of E respects the $\mathcal{K}$-structure, that $\kappa \le \kappa'$ induces an inclusion $E_\kappa \to E_{\kappa'}$ and that the action of $\tau \in \Sigma_r$ restricts to $E_\kappa \to E_{\kappa.\tau}$. Furthermore

$$\operatorname{colim}_{\kappa\in\mathcal{K}(\underline{r})}E_\kappa = E(\underline{r}).$$

To check that the composition in the Barratt-Eccles operad restricts to morphisms

$$\circ_a\colon E_\kappa \otimes E_{\kappa'} \to E_{\kappa\circ_a\kappa'}$$

for $1 \le a \le r$, observe that for $(\omega_0, ..., \omega_l) \in E_\kappa$ and $(\tau_0, ..., \tau_m) \in E_{\kappa'}$ and a path (x,y) from $(0,0)$ to (l,m) the sequence

$$(\omega_{x_0} \circ_a \tau_{y_0}, ..., \omega_{x_{l+m}} \circ_a \tau_{y_{l+m}})$$

has the same number of variations restricted to vertices i,j of κ as $(\omega_0, ..., \omega_l)$. A similar statement holds for vertices i,j of κ'. Also $\omega_{x_{l+m}} \circ_a \tau_{y_{l+m}} = \omega_l \circ_a \tau_m$ and

$$(\omega_l \circ_a \tau_m)_{ij} = \begin{cases} (\omega_l)_{ij}, & i,j \text{ vertices of } \kappa, \\ (\tau_m)_{ij}, & i,j \text{ vertices of } \kappa'. \end{cases}$$

For i a vertex of κ and j a vertex of κ' the corresponding weight and orientation in the graph $(\kappa \circ_a \kappa') = (\tilde{\sigma}, \tilde{\mu})$ is

$$\tilde{\mu}_{ij} = \mu_{ia} \quad \text{and} \quad \tilde{\sigma}_{ia} = \sigma_{ia}.$$

Also the variations of i and j in $(\omega_{x_0} \circ_a \tau_{y_0}, ..., \omega_{x_{l+m}} \circ_a \tau_{y_{l+m}})$ are in bijection with the variations of i and a in $(\omega_0, ..., \omega_l)$, while $(\omega_l \circ_a \tau_m)_{ij} = (\omega_l)_{ia}$. Hence E is equipped with a $\mathcal{K}$-structure.

Definition 3.24. *Let $\mathcal{K}_n(r)$ be the poset of complete graphs $\kappa = (\sigma, \mu)$ such that $\mu_{ij} \leq n-1$ for all pairs $i, j \in \underline{r}$. This defines a filtration*

$$\mathcal{K}_1 \subset ... \subset \mathcal{K}_n \subset \mathcal{K}_{n+1} \subset ... \subset \mathcal{K} = \mathrm{colim}_n \mathcal{K}_n$$

of $\mathcal{K}$ by suboperads.

Remark 3.25. *Considering $\mathcal{K}$-structures allows more control over the operads in question. A closer look at the $\mathcal{K}$-structure of E yields that $\mathrm{colim}_{\kappa \in \mathcal{K}_n} E_\kappa = E_n$: If $x = (\omega_0, ..., \omega_l) \in \Sigma_r^{l+1}$ is in E_n, then $((\omega_0)_{ij}, ..., (\omega_l)_{ij})$ has at most $n-1$ variations for all $i, j \in \underline{r}$. Hence $x \in E_\kappa$ for $\kappa = (\omega_l, \mu)$ with $\mu_{ef} = n-1$ for all $i, j \in \underline{r}$. This will allow us to show that the differentials we are interested in restrict to E_n.*

The n-fold bar complex for E_n-algebras and E_n-homology Recall that $T^n = (\overline{T}^c \Sigma)^n(I)$ and hence has an expansion

$$T^n(\underline{e}) = \bigoplus_{\underline{e} = \underline{e_1} \sqcup ... \sqcup \underline{e_l}} \Sigma T^{n-1}(\underline{e_1}) \otimes ... \otimes \Sigma T^{n-1}(\underline{e_l})$$

with

$$T^1(\underline{e}) = (I^{\otimes s})(\underline{e}) = \bigoplus_{\sigma\colon \underline{s} \to \underline{e}} k \cdot \sigma$$

for $|\underline{e}| = s$. Elements in $T^n(\underline{e})$ correspond to n-level trees with leaves decorated by $\underline{e}$:

Definition 3.26. *A planar fully grown n-level tree is a sequence of order-preserving surjections*

$$t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1] .$$

We call the elements in $[r_i]$ the vertices in level i. The elements of $[r_n]$ are also called the leaves of t. A decoration of t by a finite set $\underline{e}$ is a bijection $\underline{e} \to [r_n]$.

For $n = 1$ it is obvious that $T^n(\underline{e})$ has generators corresponding to decorated 1-level trees. For $n > 1$, let $sx_1 \otimes ... \otimes sx_l \in \Sigma T^{n-1}(\underline{e_1}) \otimes ... \otimes \Sigma T^{n-1}(\underline{e_l})$. Then the corresponding n-level tree is the n-level tree with l vertices in level 1 such that the ith vertex is the root of the $n-1$-level tree defined by x_i.

To show that ∂_ϵ restricts to $\Sigma^{-n}T^nE_n$ Fresse shows that T^n can be interpreted as a $\mathcal{K}$-diagram and that hence $\Sigma^{-n}T^nE$ is a right E-module with $\mathcal{K}$-structure. Since we will use the same decomposition in 3.3 we recall the relevant definition. For a complete graph $\kappa = (\sigma, \mu) \in \mathcal{K}(\underline{e})$ and $\underline{f} \subset \underline{e}$ let

$$\kappa_{|\underline{f}} = (\sigma', \mu_{\underline{f}\times\underline{f}})$$

be the complete subgraph of κ with vertices $\underline{f}$, with $\sigma'\colon \{1, ..., |\underline{f}|\} \to \underline{f}$ defined as the composite

$$\{1, ..., |\underline{f}|\} \xrightarrow{\cong} \sigma^{-1}(\underline{f}) \xrightarrow{\sigma} \underline{f}.$$

Proposition 3.27. *[23, 4.2] There is a $\mathcal{K}$-diagram associated to T^n defined as follows: For $n = 1$ and $\kappa = (\sigma, \mu) \in \mathcal{K}(\underline{e})$ set $T^n_\kappa = k \cdot \sigma \subset T^1(\underline{e})$. For general n and $\kappa = (\sigma, \mu) \in \mathcal{K}(\underline{e})$ an element*

$$sx_1 \otimes ... \otimes sx_l \in \Sigma T^{n-1}(\underline{e}_1) \otimes ... \otimes \Sigma T^{n-1}(\underline{e}_l) \subset \overline{T}^c(\Sigma T^{n-1})(\underline{e})$$

is in T^n_κ if the following conditions hold:

1. *For $1 \leq i \leq l$ we have that x_i is an element of $T^{n-1}_{\kappa|_{\underline{e_i}}}$.*
2. *If $e, f \in \underline{e}$ with $\mu_{ef} < n-1$ then there exists i such that $e, f \in \underline{e_i}$.*
3. *If $e, f \in \underline{e}$ with $\mu_{ef} = n-1$ and with $e \in \underline{e_i}, j \in \underline{e_j}$ with $i < j$ then $\sigma_{ef} = \mathrm{id}$.*

Remark 3.28. *The idea behind this definition is the following [19]: Interpreting $t \in T^n(\underline{e})$ as a tree, the smallest complete graph κ with $t \in T^n_\kappa$ has vertices ordered like the inputs of t and weights μ_{ef} such that $n - 1 - \mu_{ef}$ equals the level on which the paths from e and f to the root first join.*

For a $\mathcal{K}$-operad $\mathcal{P}$ let $(T^n\mathrm{colim}\mathcal{P})_\kappa$ for κ a complete graph be generated as a k-module by elements $t(p_1, ..., p_l) \in T^n\mathrm{colim}\mathcal{P}$ such that $t \in T^n_{\kappa'}, p_i \in \mathcal{P}_{\kappa_i}$ with

$$\kappa'(\kappa_1, ..., \kappa_l) \leq \kappa.$$

This endows $T^n\mathcal{P}$ with the structure of a right $\mathcal{P}$-module with $\mathcal{K}$-structure. Having decomposed T^nE in this way, Fresse proves in [23, 5.3] that

$$\partial_\epsilon((T^nE)_\kappa) \subset (T^nE)_\kappa.$$

From this one deduces that the twist $\partial_\epsilon\colon T^nE \to T^nE$ restricts to E_n, i.e. satisfies

$$\partial_\epsilon(T^nE_n) \subset T^nE_n.$$

Hence we can set

$$B^n_{E_n} = (T^n E_n, \partial_\epsilon)$$

and define the n-fold bar construction

$$B^n(A) = B^n_{E_n} \circ_{E_n} A$$

for any E_n-algebra A [23, 5.4, 5.5].

As a quasifree right E_n-module in nonnegatively graded chain complexes $B^n_{E_n}$ is a cofibrant right E_n-module. The augmentation of the desuspended n-fold bar complex is defined as the composite

$$\epsilon\colon \Sigma^{-n} B^n_{E_n} \to \Sigma^{-n} T^n I \to \Sigma^{-n}\Sigma^n(I) = I$$

with the first map induced by the operad morphism $E_n \to \mathcal{C}om \to I$ and the second map an iteration of the projection $\overline{T}^c(\Sigma I) \to \Sigma I$. In [23, 8.21] Fresse shows that ϵ is a quasiisomorphism, hence $\Sigma^{-n} B^n_{E_n}$ is a cofibrant replacement of I. This yields the desired result:

Theorem 3.29. *[23, 8.22] Let A be an E_n-algebra which is degreewise k-projective. Then*

$$H^{E_n}_*(A; k) = H_*(\Sigma^{-n} B^n_{E_n}(A)).$$

3.3 E_n-homology and E_n-cohomology of commutative algebras via the iterated bar construction

After sketching the construction of the iterated bar complex for E_n-algebras and the proof that one can calculate E_n-homology with trivial coefficients via the iterated bar complex, we are now in the position to generalize this result to coefficients in a symmetric A-bimodule for a nonunital commutative algebra A. This is done by twisting the differential on $U_{\mathcal{C}om} \otimes B^n_{\mathcal{C}om}$, lifting this twist to $U_{\mathcal{C}om} \otimes B^n_{E_n}$ and comparing $U_{\mathcal{C}om} \otimes B^n_{E_n}$ endowed with this twist to a complex which can be used to compute $H^{E_n}_*(A; M)$. This comparison will be a comparison of left $U_{\mathcal{C}om}$-modules in right E_n-modules, hence we start with discussing this category and its model structure. After that we define the needed twist on $U_{\mathcal{C}om} \otimes B^n_{\mathcal{C}om}$ and show that it lifts to $U_{\mathcal{C}om} \otimes B^n_{E_n}$. Then we define the complex we will compare to $U_{\mathcal{C}om} \otimes B^n_{E_n}$ and finally deduce the result. The result and the outline of this proof is due to Benoit Fresse.

The model category of left U-modules in right $\mathcal{P}$-modules Let $(U, \mu_U, 1_U)$ be an associative unital algebra in right $\mathcal{P}$-modules with $\mathcal{P}$-action $\gamma_U\colon U\mathcal{P} \to U$. In particular, the diagram

$$\begin{tikzcd}
(U \otimes U)\mathcal{P} \arrow[r, "\mu_U \mathcal{P}"] \arrow[d, "\cong"] & U\mathcal{P} \arrow[dd, "\gamma_U"] \\
U\mathcal{P} \otimes U\mathcal{P} \arrow[d, "\gamma_U \otimes \gamma_U"] & \\
U \otimes U \arrow[r, "\mu_U"] & U
\end{tikzcd}$$

commutes.

Definition 3.30. *The category ${}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})$ of left U-modules in right $\mathcal{P}$-modules consists of*

- *objects right $\mathcal{P}$-modules (M, γ_M) with a left U-action $\mu_M : U \otimes M \to M$ which is a morphism of right $\mathcal{P}$-modules, i.e. the diagrams*

$$\begin{tikzcd}
(U \otimes M)\mathcal{P} \arrow[r, "\mu_M \mathcal{P}"] \arrow[d, "\cong"] & M\mathcal{P} \arrow[dd, "\gamma_M"] \\
U\mathcal{P} \otimes M\mathcal{P} \arrow[d, "\gamma_U \otimes \gamma_M"] & \\
U \otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
U \otimes U \otimes M \arrow[r, "\mu_U \otimes M"] \arrow[d, "U \otimes \mu_M"] & U \otimes M \arrow[d, "\mu_M"] \\
U \otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd}$$

  commute,

- *morphisms which are morphism of right $\mathcal{P}$-modules and respect the U-action.*

Example 3.31. *Let $\mathcal{P}$ be an operad. Then the right $\mathcal{P}$-module $U_{\mathcal{P}}$ modeling universal enveloping algebras is an algebra in right $\mathcal{P}$-modules and the Kähler differentials $\Omega^1_{\mathcal{P}}$ form a left $U_{\mathcal{P}}$-module in right $\mathcal{P}$-modules.*

Proposition 3.32. *Let M be a Σ_*-module. The free object in ${}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})$ generated by M is*

$$U \otimes M\mathcal{P}$$

with U-action $\mu_{U \otimes M\mathcal{P}}$ given by

$$U \otimes U \otimes M\mathcal{P} \xrightarrow{\mu_U \otimes M\mathcal{P}} U \otimes M\mathcal{P}$$

and right $\mathcal{P}$-module structure $\gamma_{U \otimes M\mathcal{P}}$ defined by

$$(U \otimes M\mathcal{P})\mathcal{P} \xrightarrow{\cong} U\mathcal{P} \otimes M\mathcal{P}\mathcal{P} \xrightarrow{\gamma_U \otimes M\gamma_{\mathcal{P}}} U \otimes M\mathcal{P} .$$

Proof. A calculation shows that $U \otimes M\mathcal{P}$ is an object in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$. For a map $f\colon M \to N$ of Σ_*-modules to $(N, \gamma_N, \mu_N) \in {}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ define

$$\hat{f}\colon U \otimes M\mathcal{P} \to N$$

by

$$\hat{f}(u \otimes (m; p_1, ..., p_r)) = u \cdot \gamma_N(f(m); p_1, ..., p_r),$$

where we abbreviate $\mu_N(u \otimes n)$ by $u \cdot n$. This is obviously the only choice possible if we want to define a morphism in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$. For $v \in U$

$$\begin{aligned} \hat{f}(v \cdot (u \otimes (m; p_1, ..., p_r))) &= \hat{f}(vu \otimes (m; p_1, ..., p_r)) \\ &= (vu) \cdot \gamma_N(f(m); p_1, ..., p_r) \\ &= v \cdot (u \cdot \gamma_N(f(m); p_1, ..., p_r)) \\ &= v \cdot \hat{f}(u \otimes (m; p_1, ..., p_r)). \end{aligned}$$

For suitable $p_1, ..., p_r, q_1, ..., q_s \in \mathcal{P}$, $u \in U(a)$ we have

$$\begin{aligned} & \hat{f}(\gamma_{U \otimes M\mathcal{P}}(u \otimes (m; p_1, ..., p_r); q_1, ..., q_s)) \\ = \quad & (-1)^\epsilon \hat{f}(\gamma_U(u; q_1, ..., q_a) \otimes (m; \gamma_\mathcal{P}(p_1; q_{a+1}, .., q_{a+b_1}), ..., \gamma_\mathcal{P}(p_r; q_{a+b_1+...+b_{r-1}+1}, ..., q_s))) \\ = \quad & (-1)^\epsilon \gamma_U(u; q_1, ..., q_a) \cdot \gamma_N(f(m); \gamma_\mathcal{P}(p_1; q_{a+1}, .., q_{a+b_1}), ..., \gamma_\mathcal{P}(p_r; q_{a+b_1+...+b_{r-1}+1}, ..., q_s))) \\ = \quad & (-1)^\epsilon \gamma_U(u; q_1, ..., q_a) \cdot \gamma_N(\gamma_N(f(m); p_1, ..., p_r); q_{a+1}, ..., q_s) \\ = \quad & \gamma_N(u \cdot \gamma_N(f(m); p_1, ..., p_r); q_1, ..., q_s) \\ = \quad & \gamma_N(\hat{f}(u \otimes (m; p_1, ..., p_r)); q_1, ..., q_s) \end{aligned}$$

with $(-1)^\epsilon$ the sign acquired by applying the isomorphism $(U \otimes M\mathcal{P})\mathcal{P} \cong U\mathcal{P} \otimes M(\mathcal{P}\mathcal{P})$. $\square$

Remark 3.33. *One could also guess that the free object in* ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ *generated by a* Σ_**-module* M *is given by*

$$(U \otimes M)\mathcal{P}$$

with U*-action*

$$U \otimes (U \otimes M)\mathcal{P} \hookrightarrow (U \otimes U \otimes M)\mathcal{P} \xrightarrow{(\mu_U \otimes M)\mathcal{P}} (U \otimes M)\mathcal{P}$$

and right $\mathcal{P}$*-module structure*

$$(U \otimes M)\mathcal{P}\mathcal{P} \xrightarrow{(U \otimes M)\gamma_\mathcal{P}} (U \otimes M)\mathcal{P}.$$

But this is not an object in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ *because the* U*-action is not a right* $\mathcal{P}$*-module morphism. As an example consider* $M = I$, $\mathcal{P} = \mathcal{C}om$, $U = U_{\mathcal{C}om}$. *We denote the generator of* $\mathcal{C}om(j) = k$ *by* μ_j *and the corresponding element in* $U_{\mathcal{C}om}(j) = \mathcal{C}om[1](j)$ *by* μ_j^U. *Then for*

$$x = (\mu_2^U \otimes (\mu_2^U; \mu_1) \otimes \mu_1; \mu_2, \mu_1, \mu_1) \in (U_{\mathcal{C}om} \otimes U_{\mathcal{C}om}\mathcal{C}om \otimes \mathcal{C}om)\mathcal{C}om.$$

we have

$$\gamma_{U_{\mathcal{C}om}\otimes\mathcal{C}om}(\mu_{U_{\mathcal{C}om}\otimes\mathcal{C}om}\mathcal{C}om)(x) = (\mu_3^U;\mu_2,\mu_1)\otimes\mu_1,$$

while

$$\mu_{U_{\mathcal{C}om}\otimes\mathcal{C}om}(\gamma_{U_{\mathcal{C}om}}\otimes\gamma_{U_{\mathcal{C}om}\otimes\mathcal{C}om})((\mu_2^U;\mu_2)\otimes((\mu_2^U;\mu_1);\mu_1)\otimes(\mu_1;\mu_1)) = (\mu_4;\mu_1,\mu_1,\mu_1)\otimes\mu_1.$$

In subsection 2.3 we discussed the standard method to transport cofibrantly generated model categories along adjunctions and recalled the definition of the model structure on right $\mathcal{P}$-modules. We will define a model structure on ${}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})$ by applying theorem 2.34 to the adjunction

$$F = U\otimes - \colon \mathcal{M}_{\mathcal{P}} \rightleftarrows {}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})\colon V$$

with V the corresponding forgetful functor. As we explained in subsection 2.2 the generating cofibrations in dg-mod are of the form $S^{n-1}\to D^n$ and the generating acyclic cofibrations are of the form $0\to D^n$. We start by determining the FI- and FJ-cell complexes, where FI (respectively FJ) is the set of maps

$$U\otimes(i\otimes F_r\mathcal{P})\colon U\otimes(C\otimes F_r\mathcal{P})\to U\otimes(D\otimes F_r\mathcal{P})$$

with $i\colon C\to D$ a generating cofibration (respectively a generating acyclic cofibration) in dg-mod and F_r is as in theorem 2.36.
Note that the underlying differential graded module of the direct sum of left U-modules in right $\mathcal{P}$-modules is the direct sum of their underlying differential graded modules.

Proposition 3.34. *A FI-cell attachment in ${}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})$ is an inclusion $K\to(K\oplus G,\partial)$ with $G = U\otimes M\mathcal{P}$ for a free Σ_*-module M with trivial differential and $\partial\colon G\to K$. A FJ-cell attachment in ${}_U\mathrm{Mod}(\mathcal{M}_{\mathcal{P}})$ is an inclusion $K\to K\oplus G'$ with $G' = U\otimes(\bigoplus_\alpha D^{n_\alpha}\otimes F_r\mathcal{P})$.*

Proof. The second claim is obvious. To see that the first claim holds, consider a FI-cell attachment

$$\begin{tikzcd} \bigoplus_\alpha U\otimes(S^{n_\alpha-1}\otimes F_{r_\alpha}\mathcal{P}) \arrow[r, "\sum f_\alpha"] \arrow[d, "\bigoplus_\alpha U\otimes(i_\alpha\otimes F_{r_\alpha}\mathcal{P})"] & K \\ \bigoplus_\alpha U\otimes(D^{n_\alpha}\otimes F_{r_\alpha}\mathcal{P}). & \end{tikzcd}$$

This diagram is isomorphic to

$$\begin{tikzcd} U\otimes(\bigoplus_{r\geq 0}\bigoplus_{\alpha\in I_r} S^{n_\alpha-1}\otimes F_r\mathcal{P}) \arrow[r, "\sum f_\alpha"] \arrow[d, "U\otimes(\bigoplus i_\alpha\otimes F_r\mathcal{P})"] & K \\ U\otimes(\bigoplus_{r\geq 0}\bigoplus_{\alpha\in I_r} D^{n_\alpha}\otimes F_r\mathcal{P}). & \end{tikzcd}$$

Let e_n be the generator of degree n of D^n and s_n the generator of degree n of S^n. Then with $M(r) = \bigoplus_{\alpha \in I_r} S^{n_\alpha}$ and $\partial(e_{n_\alpha}) = f_\alpha(s_{n_\alpha - 1})$ the FI-cell attachment is exactly of the form above. Conversely one easily shows that each inclusion like above can be constructed as an FI-cell attachment. $\square$

Corollary 3.35. *A relative FI-cell complex in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is an inclusion*

$$K \to (K \oplus (U \otimes M\mathcal{P}), \partial)$$

with M a Σ_*-free Σ_*-module with trivial differential, such that $K \oplus (U \otimes M\mathcal{P})$ is filtered by $G_\lambda, \lambda < \kappa$ for a given ordinal κ, with $\partial(G_\lambda) \subset G_{\lambda-1}$ and $G_0 = K$. A relative FJ-cell complex in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is the same as a FJ-cell attachment.

Now we are in the position to prove that the adjunction between ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ and the category of right $\mathcal{P}$-modules gives rise to a model structure on ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$.

Theorem 3.36. *Let $\mathcal{P}$ be cofibrant in dg-mod. Let ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ be the category of left U-modules in right $\mathcal{P}$-modules. Then ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is a cofibrantly generated model category with weak equivalences and fibrations created by $V\colon {}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P}) \to \mathcal{M}_\mathcal{P}$. The generating (acyclic) cofibrations FI and FJ are of the form*

$$U \otimes (C \otimes F_r\mathcal{P}) \to U \otimes (D \otimes F_r\mathcal{P})$$

with $C \to D$ a generating (acyclic) cofibration in dg-mod.

Proof. The category ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is complete and cocomplete with limits and colimits created by ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P}) \to \mathcal{M}_\mathcal{P}$. Let f be a relative FJ-cell complex. Since V creates colimits, we have that $f = F(g)$ for a relative J-cell complex g, which is an acyclic cofibration by [20, 11.1.8]. But the functor F sends acyclic cofibrations to weak equivalences by [4, Lemma 5.6]. The small object argument holds trivially for FJ since the domains of FJ are all 0. The domains of FI are of the form $U \otimes (D^l \otimes F_r\mathcal{P})$, hence a morphism to $K \in {}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is equivalent to picking an element $x \in K(r)$ of degree l. If $K = \mathrm{colim}_{\lambda<\kappa} L_\lambda$ is a relative FI-cell-complex, it is clear that $x \in L_\lambda$ for some $\lambda < \kappa$. But since $L_\lambda \in {}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ we see that $U \otimes (D^l \otimes F_r\mathcal{P})$ as a whole gets mapped to L_λ. Hence by theorem 2.34 the category ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ is a model category with the properties stated above. $\square$

By [20, 11.1.8] corollary 3.35 implies:

Corollary 3.37. *Let $(U \otimes M\mathcal{P}, \partial)$ be a quasifree object in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ such that M is Σ_*-free and such that there is an ordinal κ and a filtration $(G_\lambda)_{\lambda<\kappa}$ of $U \otimes M\mathcal{P}$ with $\partial(G_\lambda) \subset G_{\lambda-1}$. Then $(U \otimes M\mathcal{P}, \partial)$ is cofibrant. In particular such quasifree objects in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ which are bounded below as chain complexes are cofibrant.*

Finally we examine how an operad morphism $\mathcal{Q} \to \mathcal{P}$ allows us to compare left modules in right $\mathcal{Q}$-modules and in right $\mathcal{P}$-modules. First we record the following fact, which can be directly calculated.

Proposition 3.38. *Given a morphism $\mathcal{Q} \to \mathcal{P}$ of operads, let (V, μ_V, η_V) be an algebra in right $\mathcal{Q}$-modules and $(N, \mu_N, \gamma_N) \in {}_V\mathrm{Mod}(\mathcal{M}_\mathcal{Q})$. Then $V \circ_\mathcal{Q} \mathcal{P}$ is an algebra in right $\mathcal{P}$-modules with multiplication*

$$V \circ_\mathcal{Q} \mathcal{P} \otimes V \circ_\mathcal{Q} \mathcal{P} \xrightarrow{\cong} (V \otimes V) \circ_\mathcal{Q} \mathcal{P} \xrightarrow{\mu_V \circ_\mathcal{Q} \mathcal{P}} V \circ_\mathcal{Q} \mathcal{P}$$

and unit defined via the inclusion $V \to V \circ_\mathcal{Q} \mathcal{P}$. Furthermore, $N \circ_\mathcal{Q} \mathcal{P}$ is a left $V \circ_\mathcal{Q} \mathcal{P}$-module in right $\mathcal{P}$-modules with structure maps

$$V \circ_\mathcal{Q} \mathcal{P} \otimes N \circ_\mathcal{Q} \mathcal{P} \xrightarrow{\cong} (V \otimes N) \circ_\mathcal{Q} \mathcal{P} \xrightarrow{\mu_N \circ_\mathcal{Q} \mathcal{P}} N \circ_\mathcal{Q} \mathcal{P}$$

and

$$(N \circ_\mathcal{Q} \mathcal{P})\mathcal{P} \cong N \circ_\mathcal{Q} (\mathcal{P}\mathcal{P}) \xrightarrow{N \circ_\mathcal{Q} \gamma_\mathcal{P}} N \circ_\mathcal{Q} \mathcal{P}.$$

For categories of right modules a morphism of operads gives rise to a Quillen adjunction, see [20, Theorem 16.B]. In our setting we find a similar result:

Proposition 3.39. *Let V be an algebra in right $\mathcal{Q}$-modules. A morphism $\mathcal{Q} \to \mathcal{P}$ of operads gives rise to an adjunction*

$$- \circ_\mathcal{Q} \mathcal{P} \colon {}_V\mathrm{Mod}(\mathcal{M}_\mathcal{Q}) \rightleftarrows {}_{V \circ_\mathcal{Q} \mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P}) \colon \mathrm{res},$$

where for $M \in {}_{V \circ_\mathcal{Q} \mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ the structure maps of $\mathrm{res}(M)$ are defined by restricting the right $\mathcal{P}$-module structure to $\mathcal{Q}$ and via the inclusion $V \to V \circ_\mathcal{Q} \mathcal{P}$.

Proof. For a morphism $f \colon N \to \mathrm{res}(M)$ of left V-modules in right $\mathcal{Q}$-modules define the corresponding map from $N \circ_\mathcal{Q} \mathcal{P}$ to M as

$$N \circ_\mathcal{Q} \mathcal{P} \xrightarrow{f\mathcal{P}} M \circ_\mathcal{Q} \mathcal{P} \xrightarrow{\gamma_M} M.$$

This is a welldefined map since f is a morphism of right $\mathcal{Q}$-modules and due to the associativity of the right $\mathcal{P}$-action on M, and defines a morphism in ${}_{V \circ_\mathcal{Q} \mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P})$. Conversely, for $g \colon N \circ_\mathcal{Q} \mathcal{P} \to M$ a given morphism in ${}_{V \circ_\mathcal{Q} \mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P})$, define a morphism in ${}_V\mathrm{Mod}(\mathcal{M}_\mathcal{Q})$ as

$$N \longrightarrow N \circ_\mathcal{Q} \mathcal{P} \xrightarrow{g} M.$$

This correspondence defines a natural bijection between $\mathrm{Mor}_{{}_{V \circ_\mathcal{Q} \mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P})}(N \circ_\mathcal{Q} \mathcal{P}, M)$ and $\mathrm{Mor}_{{}_V\mathrm{Mod}(\mathcal{M}_\mathcal{Q})}(N, \mathrm{res}(M))$. □

Proposition 3.40. *Let $\mathcal{P}, \mathcal{Q}$ be cofibrant as differential graded modules in each arity. Let $\mathcal{Q} \to \mathcal{P}$ be a morphism of operads. Then the adjunction described in proposition 3.39 is a Quillen adjunction.*

Proof. Since both in ${}_V\mathrm{Mod}(\mathcal{M}_\mathcal{Q})$ and ${}_{V\circ_\mathcal{Q}\mathcal{P}}\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ fibrations and quasifibrations are created by the forgetful functor to arity-graded differential graded modules, it is clear that the adjunction is a Quillen adjunction. $\square$

Twisting the n-fold bar complex: We now define the twist ∂_θ on $M \otimes \Sigma^{-n} B^n_{\mathcal{C}om}(A)$ which will incorporate the action of the nonunital commutative algebra A on the symmetric A-bimodule M. As we will see later the general case follows from the case of universal coefficients $M = U_{\mathcal{C}om}(A) = A_+$.
Recall from lemma 2.45 that for an operad $\mathcal{P}$ there is an associative algebra $U_\mathcal{P}$ in right $\mathcal{P}$-modules such that

$$U_\mathcal{P} \circ_\mathcal{P} A = U_\mathcal{P}(A)$$

for any $\mathcal{P}$-algebra A. For $\mathcal{P} = \mathcal{C}om$ we have

$$U_{\mathcal{C}om} = \mathcal{C}om_+,$$

see e.g. [20, 10.2.1]. Set $\underline{e}_+ = \underline{e} \sqcup \{+\}$. Denote the generator $\mu_{\underline{e}_+} \in U_{\mathcal{C}om}(\underline{e}) = \mathcal{C}om(\underline{e} \sqcup +)$ by $\mu^U_{\underline{e}}$. Then

$$\mu^U_{\underline{e}} \cdot \mu^U_{\underline{f}} = \mu^U_{\underline{e} \sqcup \underline{f}},$$

with right $\mathcal{C}om$-module structure given by

$$\mu^U_{\underline{e}} \circ_e \mu_{\underline{f}} = \mu^U_{(\underline{e} \sqcup \underline{f}) \setminus \{e\}}$$

for $e \in \underline{e}$.
Since

$$U_{\mathcal{C}om}(A) \otimes \Sigma^{-n} B^n(A) = (U_{\mathcal{C}om} \otimes \Sigma^{-n} B^n_{\mathcal{C}om}) \circ_{\mathcal{C}om} A$$

we only need to define

$$\partial_\theta \colon U_{\mathcal{C}om} \otimes \Sigma^{-n} B^n_{\mathcal{C}om} \to U_{\mathcal{C}om} \otimes \Sigma^{-n} B^n_{\mathcal{C}om}.$$

Recall that as a k-module $B^n_{\mathcal{C}om} = (T^n \mathcal{C}om, \partial_\alpha)$ is generated by planar fully grown n-level trees in T^n with leaves labeled by elements in $\mathcal{C}om$. In the following, for a map $f \colon M \to N$ from a Σ_*-module M to $N \in {}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$ we will denote by

$$\partial_f \colon U \otimes M\mathcal{P} \to N$$

the associated morphism in ${}_U\mathrm{Mod}(\mathcal{M}_\mathcal{P})$. The same notation has already been used (and we will continue to use it) for induced morphisms and derivations defined on free right $\mathcal{P}$-modules. Which version applies will be clear from the context.

Definition 3.41. *The morphism ∂_θ in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{\mathcal{C}om})$ is defined on the generators $\Sigma^{-n}T^n$ of $\Sigma^{-n}B^n_{\mathcal{C}om}$ by*

$$\theta\colon \Sigma^{-n}T^n \to U_{\mathcal{C}om} \otimes \Sigma^{-n}B^n_{\mathcal{C}om}.$$

The map θ sends $s^{-n}(t,\sigma) \in \Sigma^{-n}T^n(\underline{e})$ with $t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$ labeled by $\sigma\colon [r_n] \to \underline{e}$ to

$$\sum_{\substack{0\le l\le r_{n-1},\\ |f_n^{-1}(l)|>1,\\ x=\min f_n^{-1}(l)}} (-1)^{s_{n,x-1}} \mu^U_{\{\sigma^{-1}(x)\}} \otimes s^{-n}(t\setminus x, \sigma'_{\sigma^{-1}(x)})$$
$$+ \sum_{\substack{0\le l\le r_{n-1}\\ |f_n^{-1}(l)|>1,\\ y=\max f_n^{-1}(l)}} (-1)^{s_{n,y}} \mu^U_{\{\sigma^{-1}(y)\}} \otimes s^{-n}(t\setminus y, \sigma'_{\sigma^{-1}(y)}).$$

Here for $s \in [r_n]$ such that s is not the only element in the 1-fiber of t containing s we let $(t \setminus s, \sigma'_{\sigma^{-1}(s)})$ be the tree obtained by deleting the leaf s. To be more precise,

$$t \setminus s = [r_n - 1] \xrightarrow{f'_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1]$$

with

$$f'_n(x) = \begin{cases} f_n(x), & x < s,\\ f_n(x+1), & x \ge s, \end{cases}$$

and $\sigma'_{\sigma^{-1}(s)}\colon [r_n - 1] \to \underline{e} \setminus \{\sigma^{-1}(s)\}$ defined similarly by

$$\sigma'_{\sigma^{-1}(s)}(x) = \begin{cases} \sigma(x), & x < s,\\ \sigma(x+1), & x \ge s. \end{cases}$$

The sign $(-1)^{s_{n,i}}$ is determined by counting the edges in the tree t from bottom to top and from left to right. Then $s_{n,i}$ is the number assigned to the edge connected to the ith leave. We will discuss the reason for introducing this sign in definition 4.4.

Remark 3.42. *On*

$$M \otimes \Sigma^{-n}B^n(A) = M \otimes_{U_{\mathcal{C}om}(A)} ((U_{\mathcal{C}om} \otimes \Sigma^{-n}B^n_{\mathcal{C}om}) \circ_{\mathcal{C}om} (A))$$

the map ∂_θ induces

$$\begin{aligned} m \otimes s^{-n} t(a_0, ..., a_{r_n}) \quad \mapsto \quad & \sum_{\substack{0 \leq l \leq r_{n-1}, \\ |f_n^{-1}(l)|>1, \\ x = \min f_n^{-1}(l)}} (-1)^{s_{n,x-1}} m a_x \otimes s^{-n} (t \setminus x)(a_0, ..., \hat{a_x}, ..., a_{r_n}) \\ + \quad & \sum_{\substack{0 \leq l \leq r_{n-1} \\ |f_n^{-1}(l)|>1, \\ y = \max f_n^{-1}(l)}} (-1)^{s_{n,y}} a_y m \otimes s^{-n} (t \setminus y)(a_0, ..., \hat{a_y}, ..., a_{r_n}) \end{aligned}$$

for a tree $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ *labeled by* $a_0, ..., a_{r_n} \in A$.

Let $\eta_U \colon k \to U_{\mathcal{C}om}(0)$ denote the unit map of $U_{\mathcal{C}om}$. We want to lift

$$\partial_{\theta + \eta_U \otimes \gamma} = \partial_\theta + U_{\mathcal{C}om} \otimes \partial_\gamma$$

to $U_{\mathcal{C}om} \otimes B^n_E$. To achieve this we mimic the construction that is used in [23, 2.4] to lift $\partial_\gamma \colon B^n_{\mathcal{C}om} \to B^n_{\mathcal{C}om}$ to B^n_E.
Recall from proposition 2.15 that there is an adjunction between Σ_*-modules and functors Bij $\to$ *dg*-mod given by extending a Σ_*-modules $\mathcal{M}$ to a functor $\mathcal{M} \colon \mathrm{Bij} \to dg$-mod by

$$\mathcal{M}(\underline{e}) = M(r) \otimes_{\Sigma_r} \mathrm{Bij}(\underline{e}, \underline{r})$$

for a finite set $\underline{e}$ with r elements.
In particular, the Barratt-Eccles operad has an extension to finite sets and bijections given by

$$\mathcal{E}(\underline{e})_l = k < \mathrm{Bij}(\underline{e}, \underline{r})^{l+1} > .$$

In our constructions we often need to choose a distinguished element in $\mathrm{Bij}(\underline{e}, \underline{r})$ for all $\underline{e}$, corresponding to an ordering of $\underline{e}$. We fix a choice of a family $(\tau_{\underline{e}})_{\underline{e} \in \mathrm{Fin}}$ such that for $\underline{e} \subset \mathbb{N}_0$ the element $\tau_{\underline{e}}$ corresponds to the canonical order.

Proposition 3.43. *[23, 1.4] The quasiisomorphism between the commutative operad $\mathcal{C}om$ and the Barratt-Eccles operad E takes the following form: Not only is there a quasifibration*

$$\psi \colon E \to \mathcal{C}om$$

of operads given by

$$E(\underline{e})_l \ni (\sigma_0, ..., \sigma_l) \mapsto \begin{cases} 1, & l = 0, \\ 0, & l \neq 0, \end{cases}$$

but also a section

$$\iota \colon \mathcal{C}om \to E, \mathcal{C}om(\underline{e}) \ni 1 \mapsto \tau_{\underline{e}}$$

which is a morphism of arity-graded differential graded modules, and a k-linear homotopy $\nu\colon E \to E$ between $\iota\psi$ and id_E given by

$$E(\underline{e})_l \ni (\sigma_0, ..., \sigma_l) \mapsto (\sigma_0, ..., \sigma_l, \tau_{\underline{e}})$$

such that in addition $\psi\nu = 0$. Note that this is not a homotopy retract of operads since ι is not a morphism of operads.

For a map $f\colon C \to D$ of degree p between chain complexes (C, d_C) and (D, d_D) let $\delta(f)\colon C \to D$ be

$$\delta(f) = d_D f - (-1)^p f d_C.$$

The following proposition extends [23, 2.5].

Proposition 3.44. *Let U be an algebra in right R-modules and let R, S be operads equipped with differentials d_R and d_S. Suppose there are maps*

$$\begin{tikzcd} R \arrow[loop left, "\nu"] \arrow[r, shift left, "\psi"] & S \arrow[l, shift left, "\iota"] \end{tikzcd}$$

such that ψ is a morphism of operads, ι is a chain map and

$$\psi\iota = \mathrm{id}, \quad d_R\nu - \nu d_R = \mathrm{id} - \iota\psi \quad \textit{and} \quad \psi\nu = 0.$$

Let $K = G \otimes \Sigma_*$ be a free Σ_*-module and

$$\beta\colon G \to U \otimes KS$$

a twisting morphism which additionally satisfies

$$\delta(\beta) = d_{U\otimes KS}\beta + \beta d_G = 0 \quad \textit{and} \quad \partial_\beta\beta = 0$$

with $d_{U\otimes KS}$ and d_G denoting the differentials on $U \otimes KS$ and G. If K, U and S are nonnegatively graded there always exists a twisting morphism $\alpha\colon K \to U \otimes KR$ such that

$$\begin{tikzcd} U\otimes KR \arrow[r, "\partial_\alpha"] \arrow[d, "U\otimes K\psi"] & U\otimes KR \arrow[d, "U\otimes K\psi"] \\ U\otimes KS \arrow[r, "\partial_\beta"] & U\otimes KS. \end{tikzcd}$$

Proof. Extend the homotopy $\nu\colon R \to R$ to $R^{\otimes l}$ by setting

$$\hat{\nu}^{(l)} = \sum_{i=1}^{l} (-1)^{i-1} (\iota\psi)^{\otimes i-1} \otimes \nu \otimes \mathrm{id}_R^{\otimes l-i}$$

and extend ι to $\tilde{\iota}\colon S^{\otimes l} \to R^{\otimes l}$ by

$$\hat{\iota}^{(l)} = \iota^{\otimes l}.$$

Since K is Σ_*-free and hence $KR = \bigoplus_{i\geq 0} G(i) \otimes R^{\otimes i}$ and similarly $KS = \bigoplus_{i\geq 0} G(i) \otimes S^{\otimes i}$ we can then define

$$\tilde{\nu}\colon U \otimes KR \to U \otimes KR \quad \text{and} \quad \tilde{\iota}\colon U \otimes KS \to U \otimes KR$$

as $\tilde{\nu} = \mathrm{id}_U \otimes \bigoplus_{l\geq 0} G(l) \otimes \hat{\nu}^{(l)}$ and $\tilde{\iota} = \mathrm{id}_U \otimes \bigoplus_{l\geq 0} G(l) \otimes \hat{\iota}^{(l)}$. Note that with this definition

$$(U \otimes K\psi)\tilde{\iota} = \mathrm{id}, \quad \delta(\tilde{\nu}) = \mathrm{id}_{U\otimes KR} - \tilde{\iota}(U \otimes K\psi) \quad \text{and} \quad (U \otimes K\psi)\tilde{\nu} = 0.$$

We define $\alpha\colon G \to U \otimes KR$ by setting $\alpha_0 = \tilde{\iota}\beta$,

$$\alpha_m = \sum_{a+b=m-1} \tilde{\nu}\partial_{\alpha_a}\alpha_b$$

and

$$\alpha = \sum_{m\geq 0} \alpha_m.$$

Observe that $\beta\colon K \to U \otimes KS$ lowers the degree in K by at least 1, hence α_m lowers the degree in K by $m+1$. Since K is bounded below, α is well defined. Then for $m \geq 1$

$$(U \otimes K\psi)\alpha_m = (U \otimes K\psi) \sum_{a+b=m-1} \tilde{\nu}\partial_{\alpha_a}\alpha_b = 0,$$

while $(U \otimes K\psi)\alpha_0 = \beta$, hence

$$(U \otimes K\psi)\alpha = \beta$$

and the diagram above commutes.
To show that ∂_α is indeed a twisting differential we proceed by induction to show that

$$\delta(\alpha_m) = \sum_{a+b=m-1} \partial_{\alpha_a}\alpha_b,$$

which then yields the claim for α. For $m = 0$ observe that

$$\delta(\alpha_0) = \tilde{\iota}\delta(\beta) = 0.$$

For $m > 0$ we calculate

$$\begin{aligned} \delta(\alpha_m) &= \sum_{a+b=m-1} d_{U\otimes KR}\tilde{\nu}\partial_{\alpha_a}\alpha_b + \sum_{a+b=m-1} \tilde{\nu}\partial_{\alpha_a}\alpha_b d_G \\ &= \sum_{a+b=m-1} (-\tilde{\nu}d_{U\otimes KR} + \mathrm{id}_{U\otimes KR} - \tilde{\iota}(U \otimes K\psi))\partial_{\alpha_a}\alpha_b + \sum_{a+b=m-1} \tilde{\nu}\partial_{\alpha_a}\alpha_b d_G. \end{aligned}$$

But ψ is a morphism of operads, hence

$$\tilde{\iota}(U \otimes K\psi)\partial_{\alpha_a}\alpha_b = \tilde{\iota}\partial_{(U\otimes K\psi)\alpha_a}(U \otimes K\psi)\alpha_b.$$

Since

$$(U \otimes K\psi)\alpha_j = \begin{cases} \beta, & j = 0, \\ 0, & j > 0, \end{cases}$$

we find that $\tilde{\iota}(U \otimes K\psi)\partial_{\alpha_a}\alpha_b = 0$ for all $a, b \geq 0$. It is well known that δ is a derivation with respect to the operation defined by $f \star g = \partial_f g$, hence using the induction hypothesis we see that

$$\begin{aligned} \sum_{a+b=m-1} (d_{U\otimes KR}\partial_{\alpha_a}\alpha_b - \partial_{\alpha_a}\alpha_b d_G) &= \sum_{a+b=m-1} \delta(\partial_{\alpha_a}\alpha_b) \\ &= \sum_{a+b=m-1} (\delta(\partial_{\alpha_a})\alpha_b - \partial_{\alpha_a}\delta(\alpha_b)) \\ &= \sum_{a+b=m-1} (\sum_{r+s=a-1} \partial_{\alpha_r}\partial_{\alpha_s})\alpha_b - \partial_{\alpha_a}(\sum_{r+s=b-1} \partial_{\alpha_r}\alpha_s)) \\ &= 0 \end{aligned}$$

which concludes the proof. □

Proposition 3.45. *There is a map $\lambda \colon \Sigma^{-n}T^n \to U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n E$ such that*

$$\begin{tikzcd} U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE \arrow[r, "\partial_\lambda"] \arrow[d, "U_{\mathcal{C}om}\otimes\Sigma^{-n}T^n(\psi)"'] & U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE \arrow[d, "U_{\mathcal{C}om}\otimes\Sigma^{-n}T^n(\psi)"] \\ U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n\mathcal{C}om \arrow[r, "\partial_\theta + U_{\mathcal{C}om}\otimes\partial_\gamma"] & U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n\mathcal{C}om \end{tikzcd}$$

commutes and such that $\partial_\lambda^2 = 0$.

Proof. We know that T^n is Σ_*-free. We need to show that the maps

$$\begin{tikzcd} E \arrow[loop left, "\nu"] \arrow[r, shift left, "\psi"] & \mathcal{C}om \arrow[l, shift left, "\iota"] \end{tikzcd}$$

discussed in proposition 3.43 and the map $\theta + \eta_U \otimes \gamma$ fulfill the requirement of proposition 3.44. Since the differential of $U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n\mathcal{C}om$ is zero, it is trivially true that

$$d_{U_{\mathcal{C}om}\otimes\Sigma^{-n}T^n\mathcal{C}om}(\theta + \eta_U \otimes \gamma) + (\theta + \eta_U \otimes \gamma)d_{\Sigma^{-n}T^n} = 0.$$

Hence we only need to show that

$$(\partial_\theta + U_{\mathcal{C}om} \otimes \partial_\gamma)(\theta + \eta_U \otimes \gamma) = 0.$$

Since we already know that γ defines a differential, this amounts to proving that

$$\partial_\theta \theta + \partial_\theta(U_{\mathcal{C}om} \otimes \gamma) + (U_{\mathcal{C}om} \otimes \partial_\gamma)\theta = 0.$$

We omit the proof since it will be carried out in detail in lemma 4.14, with $U_{\mathcal{C}om} \otimes \partial_\gamma$ corresponding to $\tilde{\partial}^{(n)} + \sum_{i=1}^{n-1} \partial^{(i)}$ and ∂_θ corresponding to $\delta_{\min} + \delta_{\max}$. □

Lemma 3.46. *The twist ∂_θ satifies*

$$\partial_\theta((\Sigma^{-n}T^n\mathcal{C}om)_\kappa) \subset \bigoplus_{e \in \underline{e}} U_{\mathcal{C}om}(\{e\}) \otimes (\Sigma^{-n}T^n\mathcal{C}om)_{\kappa|(\underline{e}\setminus\{e\})}$$

for $\kappa = (\sigma, \mu) \in \mathcal{K}(\underline{e})$.

Proof. We proceed by induction. For $n = 1$ we see that $x \in T^1_\kappa$ if and only if x corresponds to the 1-level tree $t_r(e_1, ..., e_r)$ with r leaves decorated by $e_1, ..., e_r$, where $\sigma^{-1}(i) = e_i$. If $r > 1$ the map θ sends $s^{-n}t_r(e_1, ..., e_r)$ to

$$-\mu^U_{\{e_1\}} \otimes s^{-n}t_{r-1}(e_2, ..., e_r) + (-1)^r \mu^U_{\{e_r\}} \otimes s^{-n}t_{r-1}(e_1, ..., e_{r-1}).$$

Denote by $\partial^{(a)}_\theta$ the morphism ∂_θ defined on $T^a\mathcal{C}om$. For $n > 1$ observe that

$$\partial^{(n)}_\theta(s^{-n}(sx_1 \otimes ... \otimes sx_r)) = \sum_j \pm\tau_j s^{-n}(sx_1 \otimes ... \otimes s\partial^{(n-1)}_\theta(x_j) \otimes ... \otimes sx_r)$$

for $sx_1 \otimes ... \otimes sx_r \in (\Sigma T^{n-1})^{\otimes r}$, where τ_j is the isomorphism

$$(\Sigma^{-n}\Sigma T^{n-1} \otimes ... \otimes \Sigma(U_{\mathcal{C}om} \otimes T^{n-1}) \otimes ... \otimes \Sigma T^{n-1}) \to U_{\mathcal{C}om} \otimes \Sigma^{-n}(\Sigma T^{n-1})^{\otimes r}.$$

Let $x_i \in T^{n-1}_{\underline{f_i}}$. By the induction hypothesis

$$\partial_\theta(x_i) \in \bigoplus_{e \in \underline{f}_i} U_{\mathcal{C}om}(\{e\}) \otimes \Sigma^{n-1}(T^{n-1}\mathcal{C}om)_{\kappa|\underline{f}_i\setminus\{e\}},$$

which yields the claim. □

The $\mathcal{K}$-structure on $\mathcal{E}$ can be extended to finite sets and bijections. Concretely, for a finite set $\underline{e}$ with r elements and $\kappa = (\sigma, \mu) \in \mathcal{K}(\underline{e})$ an element $(\omega_0, ..., \omega_l) \in \mathrm{Bij}(\underline{r}, \underline{e})$ is in $\mathcal{E}_\kappa$ if for all $e, f \in \underline{e}$ the sequence

$$((\omega_0)_{ef}, ..., (\omega_l)_{ef})$$

has either less than μ_{ef} variations or has exactly μ_{ef} variations and $(\omega_l)_{ef} = \sigma_{ef}$. Observe that in

$$\nu \circlearrowright E \underset{\iota}{\overset{\psi}{\rightleftarrows}} \mathcal{C}om,$$

the map ψ respects the $\mathcal{K}$-structures on E and $\mathcal{C}om$ and that for $\kappa = (\tau_e, \mu)$

$$\iota(\mathcal{C}om_\kappa) \subset E_\kappa \quad \text{and} \quad \nu(E_\kappa) \subset E_\kappa.$$

We already noted that T^n is Σ_*-free: There are graded modules $G(\underline{e})$ with

$$T^n(\underline{e}) \cong G(\underline{e}) \otimes \Sigma_{\underline{e}}.$$

The functor G is defined inductively by

$$G^0(\underline{e}) = \begin{cases} k, & |\underline{e}| = 1, \\ 0, & |\underline{e}| \neq 1, \end{cases}$$

and by

$$G^n = \bigoplus_{i \geq 1} (\Sigma G^{n-1})^{\otimes i}.$$

Intuitively, G associates to a finite set $\underline{e}$ the set of trees with $|\underline{e}|$ leaves with a tree $t \in G(r)$ having degree equal to the number of its edges. The inclusion $G^n(\underline{e}) \to T^n(\underline{e})$ is given by mapping a tree t to the tree with leaves labeled by $\underline{e}$ according to the chosen order $\tau_{\underline{e}}$.
We now deduce from proposition 3.46 that ∂_λ restricts to $U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n$. Note that the lemmata 3.47, 3.48 and proposition 3.49 are completely analougous to [18, 5.2], [18, 5.3] and [18, 5.4].
For a complete graph κ denote by G^n_κ the arity-graded k-submodule of G^n generated by elements $g \in G^n$ with $g \in T^n_\kappa$.

Lemma 3.47. *The map λ_0 satisfies*

$$\lambda_0(\Sigma^{-n} G^n_\kappa) \subset \bigoplus_{\underline{e} = \underline{e}' \sqcup \underline{e}''} U_{\mathcal{C}om}(\underline{e}') \otimes \Sigma^{-n} (T^n E)_{\kappa|_{\underline{e}''}}$$

for $\kappa = (\tau_{\underline{e}}, \mu) \in \mathcal{K}(\underline{e})$ with $\underline{e} \subset \mathbb{N}_0$.

Proof. We know that $\lambda_0 = \tilde{\iota}(\theta + \eta_U \otimes \gamma)$ and that

$$\theta(\Sigma^{-n} T^n_\kappa) \subset \bigoplus_{e \in \underline{e}} U_{\mathcal{C}om}(\{e\}) \otimes \Sigma^{-n} (T^n \mathcal{C}om)_{\kappa|_{\underline{e} \setminus \{e\}}},$$

while according to [23, 4.6]

$$(\eta_U \otimes \gamma)(\Sigma^{-n} G^n_\kappa) \subset U_{\mathcal{C}om}(\emptyset) \otimes \Sigma^{-n} (T^n \mathcal{C}om)_\kappa.$$

But by [23, 4.5] $(T^n \mathcal{C}om)_\kappa$ is spanned by elements $t(c_1, ..., c_l)$ with $t \in T^n_{\kappa'}, c_i \in \mathcal{C}om_{\kappa_i}$ such that κ_i is also of the form $(\tau_{\underline{e}^{(i)}}, \mu^{(i)})$ for some $\underline{e}^{(i)} \subset \mathbb{N}_0$. Hence we find that

$$(T^n \iota)((T^n \mathcal{C}om)_\kappa) \subset (T^n E)_\kappa.$$

Observe that $\kappa|_{\underline{e}\setminus\{e\}} = (\tau_{\underline{e}\setminus\{e\}}, \mu')$. Hence

$$\begin{aligned}
&(U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n\iota)(\bigoplus_{e\in\underline{e}} U_{\mathcal{C}om}(\{e\}) \otimes \Sigma^{-n}(T^n\mathcal{C}om)_{\kappa|_{\underline{e}\setminus\{e\}}}) \\
\subset \quad & \bigoplus_{e\in\underline{e}} U_{\mathcal{C}om}(\{e\}) \otimes \Sigma^{-n}(T^nE)_{\kappa|_{\underline{e}\setminus\{e\}}}
\end{aligned}$$

as well. This proves the claim. □

Lemma 3.48. *The twist ∂_λ satisfies*

$$\partial_\lambda(U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_\kappa) \subset \bigoplus_{\underline{e}'\subset\underline{e}} U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_{\kappa|\underline{e}'}$$

for all $\kappa \in \mathcal{K}(\underline{e})$ with $\underline{e} \subset \mathbb{N}_0$.

Proof. We show by induction that

$$\partial_{\lambda_m}(U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_\kappa) \subset \bigoplus_{\underline{e}'\subset\underline{e}} U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_{\kappa|\underline{e}'}$$

for all m. Let $t(e_1, ..., e_l) \in (T^nE)_\kappa$. Then $t \in T^n_{\kappa'}$ and $e_i \in E_{\kappa_i}$ with

$$\kappa'(\kappa_1, ..., \kappa_l) \leq \kappa.$$

Note that $(T^nE)(\underline{e}) = \bigoplus_{i\geq 1} T^n(i) \otimes_{\Sigma_i} (E^{\otimes i}(\underline{e}))$, hence we can assume that $\kappa' = (\mathrm{id}_{\underline{l}}, \mu')$ and $t \in G^n_{\kappa'}$. Writing down the definition of ∂_{λ_0} and using 3.47 yields that ∂_{λ_0} maps $U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_\kappa$ to $U_{\mathcal{C}om} \otimes \bigoplus_{\{i_1<...<i_j\}=\underline{e}'\subset\underline{l}} \Sigma^{-n}(T^nE)_{\kappa'|_{\underline{e}'}(\kappa_{i_1},...,\kappa_{i_j})}$. Since

$$\kappa'|_{\underline{e}'}(\kappa_{i_1}, ..., \kappa_{i_j}) \leq \kappa|_{\underline{e}'}$$

the claim holds for $n = 0$. For $m > 0$ recall that

$$\partial_{\lambda_m} = \sum_{a+b=m-1} \tilde{\nu}\partial_{\lambda_a}\partial_{\lambda_b}.$$

The induction hypothesis yields that

$$\partial_{\lambda_a}\partial_{\lambda_b}(U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_\kappa) \subset \bigoplus_{\underline{e}'\subset\underline{e}} U_{\mathcal{C}om} \otimes (T^nE)_{\kappa|\underline{e}'}.$$

Since

$$\tilde{\nu}(u \otimes s^{-n}t(e_1, ..., e_l)) = \sum_i \pm u \otimes s^{-n}t(\iota\psi(e_1), ..., \iota\psi(e_{i-1}), \nu(e_i), e_{i+1}, ..., e_l)$$

for $\xi \in T^n$ and $y_r \in E$ the same reasoning as in lemma 3.47 together with our assumptions on the interaction between ψ, ι, ν and the $\mathcal{K}$-structure yields the claim. □

Proposition 3.49. *We have*

$$\partial_\lambda(U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n) \subset U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n.$$

Proof. We need to show that $\partial_\lambda(\Sigma^{-n}T^n) \subset U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n$. Observe that for every $r \geq 0$ and $\xi \in T^n(r)$ there is a complete graph $\kappa = (\sigma, \mu)$ with $\xi \in T^n_\kappa$ such that $\mu_{ef} \leq n-1$ for all vertices e, f. Hence

$$\partial_\lambda(\Sigma^{-n}T^n(r)) = \partial_\lambda(\operatorname{colim}_{\kappa \in \mathcal{K}_n(\underline{r})} T^n_\kappa) = \operatorname{colim}_{\kappa \in \mathcal{K}_n(\underline{r})}(\bigoplus_{\underline{e}' \subset \underline{r}} U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_{\kappa|\underline{e}'}).$$

But

$$\begin{aligned} & \operatorname{colim}_{\kappa \in \mathcal{K}_n(\underline{r})}(\bigoplus_{\underline{e}' \subset \underline{r}} U_{\mathcal{C}om} \otimes \Sigma^{-n}(T^nE)_{\kappa|\underline{e}'}) \\ = \quad & U_{\mathcal{C}om} \otimes \bigoplus_{\underline{e}' \subset \underline{r}} \operatorname{colim}_{\kappa \in \mathcal{K}_n(\underline{r})} \Sigma^{-n}(T^nE)_{\kappa|\underline{e}'} \\ \subset \quad & U_{\mathcal{C}om} \otimes \bigoplus_{\underline{e}' \subset \underline{r}} (\Sigma^{-n}T^nE_n)(\underline{e}'), \end{aligned}$$

since $\kappa(\kappa_1, ..., \kappa_l) \in \mathcal{K}_n$ implies $\kappa_1, ..., \kappa_l \in \mathcal{K}_n$. $\square$

The bar construction with coefficients and Kähler differentials We now introduce the operadic bar construction with coefficients and use it to construct an object in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{E_n})$ which can be used to calculate $H^{E_n}_*(A; A_+)$. We then show that this object admits a quasifibration to $\Omega^1_{E_n}$, which will allow us to compare it with $(U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n, \partial_\lambda)$.

Definition 3.50. *([18, 3.1.9]) Let $\mathcal{P}$ be an operad with composition $\gamma_{\mathcal{P}}$. Let $\bar{\mathcal{P}}$ be the augmentation ideal of $\mathcal{P}$. The reduced bar construction $\bar{B}(\mathcal{P})$ is the quasifree cooperad*

$$\bar{B}(\mathcal{P}) = (\mathcal{F}^c(\Sigma\bar{\mathcal{P}}), \partial_B)$$

with

$$\partial_B \colon \mathcal{F}^c(\Sigma\bar{\mathcal{P}}) \to \mathcal{F}^c(\Sigma\bar{\mathcal{P}})$$

the coderivation of cooperads corresponding to the map

$$\mathcal{F}^c(\Sigma\bar{\mathcal{P}}) \to \bar{\mathcal{P}}$$

of degree -1 given by

$$\mathcal{F}^c(\Sigma\bar{\mathcal{P}}) \twoheadrightarrow \mathcal{F}^c_{(2)}(\Sigma\bar{\mathcal{P}}) \xrightarrow{\cong} \Sigma^2\bar{\mathcal{P}}(I; \bar{\mathcal{P}}) \xrightarrow{\Sigma^2\gamma_{\mathcal{P}}} \Sigma^2\bar{\mathcal{P}} \xrightarrow{\cong} \Sigma\bar{\mathcal{P}}.$$

Here $\mathcal{F}^c_{(2)}(\Sigma\bar{\mathcal{P}})$ denotes the summand $\Sigma\bar{\mathcal{P}}(I; \Sigma\bar{\mathcal{P}})$ of weight 2 in the decomposition $\mathcal{F}^c(\Sigma\bar{\mathcal{P}}) = \bigoplus_{i \geq 0} \mathcal{F}^c_{(i)}(\Sigma\bar{\mathcal{P}})$ of the cofree cooperad.

Set $\bar{B}_{(i)}(\mathcal{P}) = \mathcal{F}^c_{(i)}(\Sigma\bar{\mathcal{P}})$. Note that this weight grading is not respected by the differential of $\bar{B}(\mathcal{P})$.

Definition 3.51. *([18, 4.4]) The differential graded $\mathcal{P}$-bimodule*

$$B(\mathcal{P},\mathcal{P},\mathcal{P})$$

is given by

$$B(\mathcal{P},\mathcal{P},\mathcal{P}) = (\mathcal{P}\bar{B}(\mathcal{P})\mathcal{P}, \partial_L + \partial_R)$$

The twisting differentials are defined as follows: The left and right $\mathcal{P}$-module derivation

$$\partial_L \colon \mathcal{P}\bar{B}(\mathcal{P})\mathcal{P} \to \mathcal{P}\bar{B}(\mathcal{P})\mathcal{P}$$

is induced by the map

$$\bar{B}_{(i)}(\mathcal{P}) \twoheadrightarrow \bar{B}_{(i-1)}(\mathcal{P})(I;\Sigma\bar{\mathcal{P}}) \xrightarrow{\cong} \bar{B}_{(i-1)}(\mathcal{P})(I;\bar{\mathcal{P}}) \hookrightarrow \bar{B}(\mathcal{P})\mathcal{P}.$$

Here the first map sends an element $x \in \bar{B}(\mathcal{P})$ of the form $x = (b; p_1, ..., p_r)$ with $p_i \in \Sigma\bar{\mathcal{P}}$ and $b \in \bar{B}(\mathcal{P})$ to

$$\sum_{j=1}^{r} \pm((b; p_1, ..., p_{j-1}, 1, p_{j+1}, ..., p_r); 1, ..., 1, p_j, 1, ..., 1).$$

The left and right $\mathcal{P}$-module derivation

$$\partial_R \colon \mathcal{P}\bar{B}(\mathcal{P})\mathcal{P} \to \mathcal{P}\bar{B}(\mathcal{P})\mathcal{P}$$

is induced by the map

$$\bar{B}(\mathcal{P}) \longrightarrow \mathcal{P}\bar{B}(\mathcal{P})$$

which maps $(p; b_1, ..., b_s) \in \bar{B}(\mathcal{P})$ with $p \in \Sigma\bar{\mathcal{P}}$ and $b_i \in \bar{B}(\mathcal{P})$ to $(s^{-1}p; b_1, ..., b_s)$. For the exact signs see [18, 4.4.3].

Definition 3.52. *([18, 4.4]) Let $\mathcal{P}$ be an operad, L a left $\mathcal{P}$-module and R a right $\mathcal{P}$-module. The differential graded Σ_*-module*

$$B(R,\mathcal{P},L)$$

is given by

$$B(R,\mathcal{P},L) = R \circ_{\mathcal{P}} B(\mathcal{P},\mathcal{P},\mathcal{P}) \circ_{\mathcal{P}} L.$$

Denote by ϵ_B the augmentation

$$\epsilon_B \colon B(R,\mathcal{P},L) \to R \circ_{\mathcal{P}} L.$$

The object $B(R,\mathcal{P},L)$ inherits a grading by weight components $B_{(i)}(R,\mathcal{P},L)$ from $\bar{B}(\mathcal{P})$. The summand $B_{(i)}(R,\mathcal{P},L)$ corresponds to expressions in $R\mathcal{F}^c(\Sigma\bar{\mathcal{P}})L$ with i occurences of elements in $\bar{\mathcal{P}}$.

Lemma 3.53. *Let $\epsilon_{\mathcal{C}om}$ denote the map $E_n \to \mathcal{C}om$. Given $b \in B(I,E_n,I)$ considered as an element of $B(E_n,E_n,E_n)$, assume that $\partial_R(b)$ has an expansion*

$$\partial_R(b) = \sum_i e^{(i)}(b_1^{(i)},...,b_{k_i}^{(i)}) \in B(E_n,E_n,I)$$

with $e^{(i)} \in E_n$, $b_1^{(i)},...,b_{k_i}^{(i)} \in B(I,E_n,I)$. Consider the element

$$\sum_i e^{(i)}(b_1^{(i)},...,b_{j-1}^{(i)},x,b_{j+1}^{(i)},...,b_{k_i}^{(i)})$$

in $U_{E_n}(B(I,E_n,I))$. We define

$$\partial_{\theta_B} : U_{\mathcal{C}om} \otimes B(I,E_n,E_n) \to U_{\mathcal{C}om} \otimes B(I,E_n,E_n)$$

to be the morphism in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{E_n})$ induced by

$$\theta_B(b) = \sum_{i,j} \epsilon_{\mathcal{C}om}(e^{(i)})(\epsilon_{\mathcal{C}om}\epsilon_B b_1^{(i)},...,\epsilon_{\mathcal{C}om}\epsilon_B b_{j-1}^{(i)},x,\epsilon_{\mathcal{C}om}\epsilon_B b_{j+1}^{(i)},...,\epsilon_{\mathcal{C}om}\epsilon_B b_{k_i}^{(i)}) \otimes b_j^{(i)}$$

Then

$$(U_{\mathcal{C}om} \otimes B(I,E_n,E_n),\theta_B) \circ_{E_n} A \cong U_{\mathcal{C}om}(A) \otimes_{U_{E_n}(Q_A)} \Omega^1_{E_n}(Q_A)$$

with $Q_A = B(E_n,E_n,A))$.

Proof. This follows from lemma 2.50 and from $U_{\mathcal{C}om} \circ_{E_n} A = U_{\mathcal{C}om}(A)$. □

Since $(U_{\mathcal{C}om} \otimes B(I,E_n,E_n),\partial_{\theta_B})$ is quasifree in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{E_n})$ we know from corollary 3.37 that:

Proposition 3.54. *The object $(U_{\mathcal{C}om} \otimes B(I,E_n,E_n),\partial_{\theta_B})$ is cofibrant in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_E)$.*

Recall that there is a left $U_{\mathcal{P}}$-module $\Omega^1_{\mathcal{P}}$ in right $\mathcal{P}$-modules introduced in lemma 2.49 such that

$$\Omega^1_{\mathcal{P}} \circ_{\mathcal{P}} A = \Omega^1_{\mathcal{P}}(A)$$

for A a $\mathcal{P}$-algebra. Applied to $\mathcal{P} = \mathcal{C}om$ we have that

$$\Omega^1_{\mathcal{C}om}(\underline{e}) = k < \mu_{\underline{e}}(dx_e,x,...,x) | e \in \underline{e} > .$$

The morphism induced by a bijection $\phi\colon \underline{e} \to \underline{f}$ maps $\mu_{\underline{e}}(dx_e, x, ..., x)$ to $\mu_{\underline{f}}(dx_{\phi(e)}, x, ..., x)$. The right $\mathcal{C}om$-module structure is given by

$$\mu_{\underline{e}}(dx_e, x, ..., x) \circ_g \mu_{\underline{f}} = \begin{cases} \mu_{(\underline{e} \sqcup \underline{f}) \setminus \{g\}}(dx_e, ..., dx, ..., x), & g \neq e, \\ \sum_{f \in \underline{f}} \mu_{(\underline{e} \sqcup \underline{f}) \setminus \{e\}}(dx_f, x, ..., x), & g = e \end{cases}$$

for $e, g \in \underline{e}$. The $U_{\mathcal{C}om}$-module structure is given by

$$\mu^U_{\underline{e}} \cdot \mu_{\underline{f}}(dx_f, x, ..., x) = \mu_{\underline{e} \sqcup \underline{f}}(dx_f, x, ..., x).$$

Proposition 3.55. *We define a morphism*

$$\mathrm{ev}\colon (U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B}) \to \Omega^1_{\mathcal{C}om}$$

of left $U_{\mathcal{C}om}$-modules in right E_n-modules as follows: Restricted to $B(I, E_n, I)$ the map ev *is*

$$B(I, E_n, I) \twoheadrightarrow B_{(0)}(I, E_n, I) = I \longrightarrow \Omega^1_{\mathcal{C}om}$$

where the last map sends $1 \in I(\{e\})$ to $\mu_{\{e\}}(dx_e)$. This yields a well defined morphism in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{E_n})$.

Proof. By definition ev maps $B_{(i)}(I, E_n, I)$ to zero for $i \geq 1$. The internal differential induced by the differential of E_n respects the weight splitting. The differential on $B(I, E_n, I) \subset B(I, E_n, E_n)$ is the sum of the differential ∂_B of $\bar{B}(E_n)$ and the twist ∂_L. They both map $B_{(i)}(I, E_n, E_n)$ to $B_{(i-1)}(I, E_n, E_n)$. In addition the complex $(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B})$ is twisted by θ_B. Since $B(I, E_n, E_n)$ is a quasifree right E_n-module it suffices to show that

$$ev(\partial_B + \partial_L + \theta_B) = 0$$

on $B_{(1)}(I, E_n, I) \subset B(I, E_n, E_n)$. Let $a \in \bar{E}_n(\underline{e}) = (I\bar{E}_n I)(\underline{e}) = B_{(1)}(I, E_n, I)(\underline{e})$. Note that ∂_B vanishes on $B_{(1)}(I, E_n, E_n)$. On the other hand ∂_L maps a to the element $\hat{a}$ represented by e in $IIE_n = B_{(0)}(I, E_n, E_n)$ with

$$\mathrm{ev}(\hat{a}) = \mathrm{ev}(1_{I(1)} \circ_1 \hat{a}) = \mu_{\{1\}}(dx_1) \circ_1 a = \mu_{\{1\}}(dx_1) \circ_1 \epsilon_{\mathcal{C}om}(a).$$

This is

$$\sum_{e \in \underline{e}} \epsilon_{\mathcal{C}om}(a)(dx_e, x, ..., x).$$

Let $\tilde{a}$ be the element in $E_n II$ represented by a. Then the map $\mathrm{ev}\theta_B$ maps a to the element

$$-\sum_{e \in \underline{e}} \epsilon_{\mathcal{C}om}(a) \cdot \mu_{\{e\}}(dx_e) = -\sum_{e \in \underline{e}} \epsilon_{\mathcal{C}om}(a)(dx_e, x, ..., x)$$

(see [18, 4.4.3] for the sign). $\square$

Lemma 3.56. *([20, 10.3]) There is an isomorphism*

$$U_{\mathcal{P}} \otimes I \to \Omega^1_{\mathcal{P}}$$

of left $U_{\mathcal{P}}$-modules given by mapping $1 \in I(\{e\})$ to $\mu_{\{e\}}(dx_e)$.

Proposition 3.57. *The morphism*

$$(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B}) \to \Omega^1_{\mathcal{C}om}$$

is a weak equivalence.

Proof. Filter $(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B})$ by

$$F^p = \bigoplus_{i \geq p} U_{\mathcal{C}om}(i) \otimes B(I, E_n, E_n)$$

and $\Omega^1_{\mathcal{C}om}$ by

$$G^p = \bigoplus_{i \geq p} \mathrm{Im}(U_{\mathcal{C}om}(i) \otimes I)$$

where $\mathrm{Im}(U_{\mathcal{C}om}(i) \otimes I)$ is the image of $U_{\mathcal{C}om}(i) \otimes I$ under the isomorphism $U_{\mathcal{C}om} \otimes I \to \Omega^1_{\mathcal{C}om}$ defined in lemma 3.56. The morphism ev respects this filtration. We consider the associated spectral sequences. Observe that the only part of the differential of $(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B})$ that maps F^p to F^{p+1} is the part induced by θ_B. Hence the E^1-term of the spectral sequence associated to the filtration F is given by

$$E^1_{p,q} = U_{\mathcal{C}om}(p) \otimes H_q(B(I, E_n, E_n)).$$

But the map $E^1(\mathrm{ev})$ coincides with the tensor product of the identity and the augmentation $B(I, E_n, E_n) \to I$ composed with the isomorphism defined in lemma 3.56 . According to [18, 4.1.3] this is a quasiisomorphism. $\square$

The iterated bar module and Kähler differentials We defined a twisting morphism ∂_θ on $B^n_{\mathcal{C}om}$ in definition 3.41 and showed in proposition 3.49 that the sum of ∂_θ and the differential ∂_γ of $B^n_{\mathcal{C}om}$ can be lifted to a differential

$$\partial_\lambda \colon U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n \to U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n.$$

We will construct a quasifibration

$$(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda) \to \Omega^1_{E_n}$$

which will then allow us to compare $(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda)$ and $(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B})$ and deduce that $(M \otimes_{A_+} A_+ \otimes \Sigma^{-n} B^n(A), \partial_\theta)$ computes E_n-homology of A with coefficients in M.

Definition 3.58. *We define a morphism*

$$\Phi\colon (U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda) \to \Omega^1_{\mathcal{C}om}$$

in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_E)$ *as follows: Restricted to the generators* $\Sigma^{-n}T^n$ *let* Φ *be the map*

$$\Sigma^{-n}T^n \longrightarrow I \longrightarrow \Omega^1_{\mathcal{C}om}$$

with the first map given by mapping the trunk tree $[0] \to ... \to [0]$ *labeled by* e *to* $1 \in I(\{e\})$ *and the second map sending* $1 \in I(\{e\})$ *to* $\mu_{\{e\}}(dx_e)$*.*

Lemma 3.59. *The map* Φ *is well defined.*

Proof. Observe that Φ is zero on $\Sigma^{-n}T^n(\underline{e})$ unless $\underline{e}$ is a singleton. Also note that Φ factors as

$$(U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n E_n, \partial_\lambda) \xrightarrow{U \otimes \Sigma^{-n}T^n \epsilon_{\mathcal{C}om}} (U_{\mathcal{C}om} \otimes \Sigma^{-n}T^n \mathcal{C}om, \partial_\theta + U_{\mathcal{C}om} \otimes \partial_\gamma) \xrightarrow{\Phi'} \Omega^1_{\mathcal{C}om}$$

with Φ' the morphism in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{\mathcal{C}om})$ induced by $\Phi_{|\Sigma^{-n}T^n}$. The map θ maps $\Sigma^{-n}T^n(\underline{e})$ to

$$\bigoplus_{e \in \underline{e}} U_{\mathcal{C}om}(\{e\}) \otimes \Sigma^{-n}T^n(\underline{e} \setminus \{e\}),$$

while γ maps $\Sigma^{-n}T^n(\underline{e})$ to

$$\Sigma^{-n}T^n(\underline{e}) \oplus \bigoplus_{e,f \in \underline{e}, e \neq f} \Sigma^{-n}T^n(\underline{e} \setminus \{e,f\} \sqcup \{h\}) \circ_h \mathcal{C}om(\{e,f\})$$

with $\circ_h$ denoting the formal partial composite with respect to h. Also observe that γ is zero restricted to $T^n(\underline{e})$ for $|\underline{e}| = 1$. Hence we only have to check that

$$\Phi(\theta + \eta_Q \otimes \gamma) = 0$$

restricted to $\Sigma^{-n}T^n(\underline{e})$ for $\underline{e} = \{e_1, e_2\}$ containing 2 elements. Consider the decorated tree $t(e_1, e_2) \in T^n(\{e_1, e_2\})$ with $t = [1] \to [0] \to ... \to [0]$ decorated by e_1 and e_2. Denote the trunk tree $[0] \to ... \to [0]$ decorated by e in $T^n(\{e\})$ by $r(e)$. Then

$$\gamma(s^{-n}t(e_1, e_2)) = (-1)^n s^{-n} r(h) \circ_h \mu_{\{e_1,e_2\}} \in T^n(\{h\}) \circ_h \mathcal{C}om(\{e_1, e_2\}) \subset (T^n\mathcal{C}om)(\{e_1, e_2\})$$

with $\circ_h$ again denoting the formal composite. On the other hand,

$$\theta(s^{-n}t(e_1, e_2)) = (-1)^{n-1}(\mu^U_{\{e_1\}} \otimes s^{-n}r(e_2)) + (-1)^{n+1}(\mu^U_{\{e_2\}} \otimes s^{-n}r(e_1)).$$

Hence

$$\begin{aligned}\Phi(\eta_U \otimes \gamma)(s^{-n}t(e_1,e_2)) &= (-1)^n \mu_{\{h\}}(dx_h) \circ_h \mu_{\{e_1,e_2\}} \\ &= (-1)^n \mu_{\{e_1,e_2\}}(dx_{e_1}, x) + (-1)^n \mu_{\{e_1,e_2\}}(dx_{e_2}, x),\end{aligned}$$

while

$$\begin{aligned}\Phi\theta(t) &= (-1)^{n-1}\Phi(\mu^U_{\{e_1\}} \otimes s^{-n}r(e_2)) + (-1)^{n+1}\Phi(\mu^U_{\{e_2\}} \otimes s^{-n}r(e_1)) \\ &= (-1)^{n-1}\mu^U_{\{e_1\}} \cdot \mu_{\{e_2\}}(dx_{e_2}) + (-1)^{n+1}\mu^U_{\{e_2\}} \cdot \mu_{\{e_1\}}(dx_{e_1}) \\ &= (-1)^{n-1}\mu_{\{e_1,e_2\}}(dx_{e_2}, x) + (-1)^{n+1}\mu_{\{e_1,e_2\}}(dx_{e_1}, x).\end{aligned}$$

For all other trees $t \in T^n(\{e_1, e_2\})$ the differential $\partial_\theta + U_{\mathcal{C}om} \otimes \partial_\gamma$ has no summand with nontrivial image in $T^n(\underline{e})$ for $|\underline{e}| = 1$. □

Proposition 3.60. *The morphism*

$$\Phi\colon (U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n, \partial_\lambda) \to \Omega^1_{\mathcal{C}om}$$

is a weak equivalence.

Proof. Recall from the proof of proposition 3.44 that $\lambda = \sum_{m\geq 0}\lambda_m$ with

$$\lambda_0 = \tilde{\imath}(\theta + \eta_{U_{\mathcal{C}om}} \otimes \lambda)$$

and

$$\lambda_m = \sum_{a+b=m-1} \tilde{\nu}\partial_{\alpha_a}\alpha_b.$$

Hence $\partial_\lambda = U_{\mathcal{C}om} \otimes \partial_\epsilon + \partial'$ with ∂_ϵ the differential of B^n_E and such that ∂' lowers the arity of $U_{\mathcal{C}om}$. Filter $(U_{\mathcal{C}om} \otimes \Sigma^{-n}T^nE_n, \partial_\lambda)$ by the subcomplexes

$$F^p = \bigoplus_{i\geq p} U_{\mathcal{C}om}(i) \otimes \Sigma^{-n}T^nE_n$$

and filter $\Omega^1_{\mathcal{C}om}$ by

$$G^p = \bigoplus_{i\geq p} \mathrm{Im}(U_{\mathcal{C}om}(i) \otimes I).$$

Here $\mathrm{Im}(U_{\mathcal{C}om}(i) \otimes I)$ again is the image of $U_{\mathcal{C}om}(i) \otimes I$ under the isomorphism given in lemma 3.56. The morphism Φ respects these filtrations. The spectral sequence associated to the filtration F has E_1-term

$$E^1_{p,q} = U_{\mathcal{C}om}(p) \otimes H_q(\Sigma^{-n}B^n_{E_n}).$$

Let ϵ denote the quasiisomorphism

$$\Sigma^{-n} B^n_{E_n} \to I$$

exhibited in [23, 8.1]. The map Φ factors as the map $U_{\mathcal{C}om} \otimes \epsilon$ followed by the isomorphism from lemma 3.56, hence induces an isomorphism at the E^1-stage of the spectral sequences. □

E_n-homology and E_n-cohomology with coefficients

Theorem 3.61. *For a k-projective nonunital commutative algebra A*

$$H_*^{E_n}(A; U_{\mathcal{C}om}(A)) = H_*(A_+ \otimes \Sigma^{-n} B^n(A), \partial_\theta).$$

Proof. By definition $H_*^{E_n}(A; U_{\mathcal{C}om}(A))$ is the homology of

$$U_{\mathcal{C}om}(A) \otimes_{U_{E_n}(Q_A)} \Omega^1_{E_n}(Q_A)$$

for a cofibrant replacement Q_A of A as an E_n-algebra. Pick $Q_A = B(E_n, E_n, A)$ as a cofibrant replacement. There exists a lift f such that

$$\begin{tikzcd}
 & (U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B}) \arrow[d, two heads, "\mathrm{ev}", "\sim"'] \\
(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda) \arrow[ur, "f"] \arrow[r, two heads, "\Phi", "\sim"'] & \Omega^1_{\mathcal{C}om}
\end{tikzcd}$$

commutes since $(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda)$ is cofibrant according to proposition 3.37. The map f is a quasiisomorphism of left $U_{\mathcal{C}om}$-modules in right E_n-modules. But note that while these are cofibrant objects in ${}_{U_{\mathcal{C}om}}\mathrm{Mod}(\mathcal{M}_{E_n})$ they are not cofibrant as right E_n-modules, hence we can not deduce from [20, 15.1.A] that $f \circ_{E_n} B$ is a quasiisomorphism for any E_n-algebra B.

However, consider $f \circ_{E_n} \mathcal{C}om$ and note that $- \circ_{E_n} \mathcal{C}om$ is the left adjoint in the Quillen adjunction discussed in 3.40. Hence $- \circ_{E_n} \mathcal{C}om$ preserves cofibrant objects and weak equivalences between between them, and therefore $f \circ_{E_n} \mathcal{C}om$ is a weak equivalence. Now

$$(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B}) \circ_{E_n} \mathcal{C}om \cong (U_{\mathcal{C}om} \otimes B(I, E_n, \mathcal{C}om), \partial_{\theta_B})$$

and

$$(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda) \circ_{E_n} \mathcal{C}om \cong (U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n \mathcal{C}om, U_{\mathcal{C}om} \otimes \partial_\gamma + \partial_\theta)$$

are quasi-free right $\mathcal{C}om$-modules because $U_{\mathcal{C}om}$ is a free right $\mathcal{C}om$-module generated by μ_1^U in arity zero and μ_2^U in arity one. Therefore, according to Theorem [20, 15.1.A.(a)], for a commutative algebra A the map $f \circ_{E_n} \mathcal{C}om \circ_{\mathcal{C}om} A$ is a quasiisomorphism as well. But for commutative A we have $f \circ_{E_n} \mathcal{C}om \circ_{\mathcal{C}om} A = f \circ_{E_n} A$ and $U_{\mathcal{C}om} \circ_{E_n} A = U_{\mathcal{C}om}(A) = A_+$. Since

$$(U_{\mathcal{C}om} \otimes \Sigma^{-n} T^n E_n, \partial_\lambda) \circ_{E_n} A = (A_+ \otimes \Sigma^{-n} T^n A, \partial_\theta + \mathrm{id}_{A_+} \otimes \partial_\gamma),$$

while we know from lemma 3.53 that

$$(U_{\mathcal{C}om} \otimes B(I, E_n, E_n), \partial_{\theta_B}) \circ_{E_n} A = U_{\mathcal{C}om}(A) \otimes_{U_{E_n}(Q_A)} \Omega^1_{E_n}(Q_A),$$

this yields an isomorphism

$$H_*^{E_n}(A; U_{\mathcal{C}om}(A)) \cong H_*(A_+ \otimes \Sigma^{-n} T^n(A), \partial_\theta + \mathrm{id}_{A_+} \otimes \partial_\gamma).$$

□

Theorem 3.62. *Let A be a k-projective nonunital commutative algebra and M a symmetric A-bimodule. Then*

$$H_*^{E_n}(A; M) = H_*(M \otimes \Sigma^{-n} B^n(A), \partial_\theta).$$

Proof. For Q_A a cofibrant replacement of A as an E_n-algebra $H_*^{E_n}(A; M)$ is the homology of the complex

$$M \otimes_{A_+} A_+ \otimes_{U_{\mathcal{C}om}(Q_A)} \Omega^1_{E_n}(Q_A).$$

Again, we set $Q_A = B(E_n, E_n, A)$ and see that this equals

$$M \otimes_{A_+} (A_+ \otimes B(I, E_n, A), \partial_{\theta_B}).$$

Since both $A_+ \otimes B(I, E_n, A)$ as well as $(A_+ \otimes \Sigma^{-n} B^n(A), \partial_\theta)$ are A_+-free in each degree and all our objects are concentrated in nonnegative degrees, the result follows directly from the quasiisomorphism exhibited in the proof of theorem 3.61 via the Künneth spectral sequence, see e.g. [59, 10.90]. □

Theorem 3.63. *Let A be a k-projective nonunital commutative algebra and M a symmetric A-bimodule. Then*

$$H^*_{E_n}(A; M) = H_*(\mathrm{Hom}_{A_+}((A_+ \otimes \Sigma^{-n} B^n(A), \partial_\theta), M).$$

Proof. By definition $H^*_{E_n}(A; M) = H_*(\mathrm{Der}_{\mathcal{P}}(Q_A, M))$ for a cofibrant replacement Q_A of A as an E_n-algebra. Choose $Q_A = B(E_n, E_n, A)$. Since $B(E_n, E_n, A)$ is quasifree,

$$\mathrm{Der}_{E_n}(Q_A, M) = (\mathrm{Hom}_k(B(I, E_n, A), M), \partial)$$

with $\partial(f)$ the composite

$$B(I, E_n, A) \xrightarrow{\partial_R} E_n B(I, E_n, A) \xrightarrow{E_n \circ' B(I, E_n, A)} E_n(B(I, E_n, A); B(I, E_n, A))$$

$$\xrightarrow{E_n(\epsilon, f)} E_n(A; M) \xrightarrow{\gamma_M} M$$

for $f\colon B(I,E_n,A) \to M$, where the first map is defined by using that $B(I,E_n,A) \subset B(E_n,E_n,A)$, the map $\epsilon\colon B(I,E_n,A) \subset B(E_n,E_n,A) \to A$ is the standard augmentation and γ_M is the structure map of the E_n-representation M of A. There is a commuting diagram of differential graded modules

$$\begin{tikzcd}
E_n(B(I,E_n,A);B(I,E_n,A)) \arrow[r, "E_n(\epsilon,f)"] \arrow[d, "\cong"] & E_n(A;M) \arrow[rr, "\gamma_M"] \arrow[d, "\cong"] & & M \arrow[d, equal] \\
U_{E_n}(B(I,E_n,A)) \otimes B(I,E_n,A) \arrow[r, "U_{E_n}(\epsilon)\otimes f"] & U_{E_n}(A) \otimes M \arrow[r] & U_{\mathcal{C}om}(A) \otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd}$$

with μ_M defined by M being an E_n-representation of A. The vertical isomorphisms are given by identifying

$$p(x_1,...,x_{i-1},y,x_{i+1},...,x_l) \in \mathcal{P}(X;Y)$$

with

$$p(x_1,...,x_{i-1},x,x_{i+1},...,x_l) \otimes y \in U_{\mathcal{P}}(X) \otimes Y$$

for $p \in \mathcal{P}, x_1,...,x_l \in X$ and $y \in Y$. Note that

$$(\mathrm{Hom}_k(B(I,E_n,A),M),\partial) \cong (\mathrm{Hom}_{U_{\mathcal{C}om}(A)}(U_{\mathcal{C}om}(A) \otimes B(I,E_n,A),M),\tilde{\partial})$$

with $\tilde{\partial}(f)$ the left vertical map in the commutative diagram

$$\begin{tikzcd}
U_{\mathcal{C}om}(A) \otimes B(I, E_n, A) \arrow[d, "U_{\mathcal{C}om}(A)\otimes\partial_R"] & \\
U_{\mathcal{C}om}(A) \otimes E_n B(I, E_n, A) \arrow[d, "U_{\mathcal{C}om}(A)\otimes E_n \circ' B(I,E_n,A)"] & \\
U_{\mathcal{C}om}(A) \otimes E_n(B(I, E_n, A); B(I, E_n, A)) \arrow[d, "\cong"] & \\
U_{\mathcal{C}om}(A) \otimes U_{E_n}(B(I, E_n, A)) \otimes B(I, E_n, A) \arrow[r, equal] \arrow[dd, "U_{\mathcal{C}om}(A)\otimes U_{E_n}(\epsilon)\otimes f"] & U_{\mathcal{C}om}(A) \otimes U_{E_n}(B(I, E_n, A)) \otimes B(I, E_n, A) \arrow[d, "U_{\mathcal{C}om}(A)\otimes U_{E_n}(\epsilon)\otimes B(I,E_n,A)"'] \\
 & U_{\mathcal{C}om}(A) \otimes U_{E_n}(A) \otimes B(I, E_n, A) \arrow[d] \\
U_{\mathcal{C}om}(A) \otimes U_{E_n}(A) \otimes M \arrow[d] & U_{\mathcal{C}om}(A) \otimes U_{\mathcal{C}om}(A) \otimes B(I, E_n, A) \arrow[d, "\mu_{U_{\mathcal{C}om}(A)\otimes B(I,E_n,A)}"] \\
U_{\mathcal{C}om}(A) \otimes U_{\mathcal{C}om}(A) \otimes M \arrow[d, "U_{\mathcal{C}om}(A)\otimes\mu_M"] & U_{\mathcal{C}om}(A) \otimes B(I, E_n, A) \arrow[d, "U_{\mathcal{C}om}(A)\otimes f"] \\
U_{\mathcal{C}om}(A) \otimes M \arrow[d, "\mu_M"] & U_{\mathcal{C}om}(A) \otimes M \arrow[d, "\mu_M"] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

Calculating the composite

$$U_{\mathcal{C}om}(A) \otimes B(I, E_n, A) \to U_{\mathcal{C}om}(A) \otimes B(I, E_n, A)$$

of the maps in the diagram above shows that this coincides with the map ∂_{θ_B}. Since both $(A_+ \otimes B(I, E_n, A), \partial_{\theta_B})$ and $(A_+ \otimes B^n(A), \partial_\theta)$ are quasifree A_+-modules, the universal coefficients spectral sequence (see e.g. [40, 2.3]) yields that the quasiisomorphism exhibited in the proof of theorem 3.61 induces a quasiisomorphism from

$$\mathrm{Hom}_{A_+}((A_+ \otimes B(I, E_n, A), \partial_{\theta_B}), M)$$

to

$$\mathrm{Hom}_{A_+}((A_+ \otimes B^n(A), \partial_\theta), M).$$

$\square$

4 E_n-homology and cohomology as functor homology and cohomology

4.1 The category Epi_n and E_n-homology with trivial coefficients as functor homology

In [41] Livernet and Richter use that E_n-homology with trivial coefficients can be calculated via the iterated bar complex to give an interpretation of E_n-homology as functor homology. They encode the information necessary to define an iterated bar complex in a category Epi_n of trees. This enables them to define E_n-homology for arbitrary functors from this category to k-modules, and they proceed to prove that E_n-homology of these functors can be calculated as certain Tor-groups. We recall the relevant definitions and results in order to fix notation and to give the reader the necessary background to understand in what sense the results in the rest of this chapter are analogous to the results of Livernet-Richter.
In the following we assume that $1 \leq n < \infty$, that A is a commutative nonunital k-algebra over the commutative unital ring k and that M is a symmetric A-module. We start by fixing some terminology regarding trees and defining the category Epi_n^+.

Definition 4.1. *A planar fully grown n-level tree t is a sequence*

$$t = \; [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$$

of order preserving surjections. The element $i \in [r_j]$ is called the ith vertex of the jth level, the elements in $[r_n]$ are also called leaves. The degree of a tree t is given by the number of its edges, i.e. by

$$d(t) = \sum_{j=1}^{n} r_j + 1.$$

For a given vertex $i \in [r_j]$ the subtree $t_{j,i}$ is the $(n-j)$-level subtree of t with root i, i.e.

$$t_{j,i} = \; [|f_n^{-1} \dots f_{j+1}^{-1}(i)| - 1] \xrightarrow{g_n} [|f_{n-1}^{-1} \dots f_{j+1}^{-1}(i)| - 1] \xrightarrow{g_{n-1}} \dots \xrightarrow{g_{j+2}} [|f_{j+1}^{-1}(i)| - 1]$$

with g_l the map making the diagram

$$\begin{tikzcd}
{[|f_l^{-1} \dots f_{j+1}^{-1}(i)| - 1]} \arrow[r, "g_l"] \arrow[d, "\cong"] & {[|f_{l-1}^{-1} \dots f_{j+1}^{-1}(i)| - 1]} \arrow[d, "\cong"] \\
f_l^{-1} \dots f_{j+1}^{-1}(i) \arrow[r, "f_l"] & f_{l-1}^{-1} \dots f_{j+1}^{-1}(i)
\end{tikzcd}$$

commute.

Definition 4.2. *The category* Epi_n *is given by the following data:*

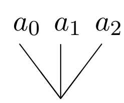
- *The objects are planar fully grown trees with* n *levels.*
- *A morphism from* $[r_n] \xrightarrow{f_n^r} \dots \xrightarrow{f_2^r} [r_1]$ *to* $[s_n] \xrightarrow{f_n^s} \dots \xrightarrow{f_2^s} [s_1]$ *consists of surjections* $h_i \colon [r_i] \to [s_i], 1 \leq i \leq n$ *such that the diagram*

$$\begin{tikzcd} {[r_n]} \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}^r"] \arrow[d, "h_{n-1}"] & \dots \arrow[r, "f_2^r"] & {[r_1]} \arrow[d, "h_1"] \\ {[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r, "f_{n-1}^s"] & \dots \arrow[r, "f_2^s"] & {[s_1]} \end{tikzcd}$$

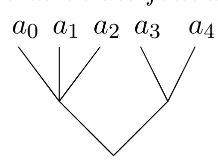
commutes and such that h_i *is order-preserving on the fibers* $(f_i^r)^{-1}(l)$ *of* f_i^r *for all* $l \in [r_i]$*. For* $i = 1$ *the map* h_1 *has to be order-preserving on* $[r_1]$*.*

The composite of two morphisms $(g_n, ..., g_1) \colon t^q \to t^r$ *and* $(h_n, ..., h_1) \colon t^r \to t^s$ *is given by* $(h_n g_n, ..., h_1 g_1)$*.*

Remark 4.3. *The connection between planar fully grown* n*-level trees and the* n*-fold bar construction that we already sketched in subsection 3.2 is easy to see: A typical element in the bar complex* $B(A)$ *defined in 3.14 is a tensor power of elements in* ΣA*, hence can be thought of as a* 1*-level tree whose leaves are labeled by elements in* A*:*



With the same reasoning a typical element in $B^2(A)$ *is a* 1*-level tree labeled with elements in* $\Sigma B(A)$*. But these are* 1*-level trees with leaves labeled by elements in* A*, hence a typical element in* $B^2(A)$ *is a labeled* 2*-level tree as in the following example:*



Iterating this description we see that a typical element in $B^n(A)$ *is a tensor power of elements in* $\Sigma B^{n-1}(A)$*, hence can be described as a* 1*-level tree whose leaves are labeled by elements in* $\Sigma B^{n-1}(A)$*, i.e. by* $n-1$*-level trees with leaves labeled by elements in* A*.*
Observe that since A *is concentrated in degree zero, the degree of a labeled tree viewed as an element in* $B^n(A)$ *is given by the number of edges of the tree. Lemma 3.5 in [41] says that the maps in* Epi_n *decreasing the number of edges by one are exactly the summands of the differential of* $B^n(A)$*.*

Hence the category Epi_n encodes precisely what is needed to make sense of an n-fold bar construction: the trees correspond to elements in $B^n(A)$ while a closer inspection reveals that the morphisms are generated by the summands of the differential of $B^n(A)$. Since the n-fold bar construction computes E_n-homology with trivial coefficients up to suspension we can hence define E_n-homology for functors from Epi_n to k-modules.

Definition 4.4. *Let $F\colon \mathrm{Epi}_n \to k\text{-mod}$ be a covariant functor. Let $\tilde{C}^{E_n}(F)$ be the $\mathbb{N}_0^n$-graded k-module with*

$$\tilde{C}^{E_n}_{(r_n,\dots,r_1)}(F) = \bigoplus_{t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]} F(t).$$

For $1 \le j \le n$ let $\tilde{\partial}_j\colon \tilde{C}^{E_n} \to \tilde{C}^{E_n}$ be the following map lowering the jth degree by one:

- *For $j = n$ define $\tilde{\partial}_j$ restricted to $F(t)$ as*

$$\sum_{\substack{0 \le i < r_n,\\ f_n(i) = f_n(i+1)}} (-1)^{s_{n,i}} F(d_i, \mathrm{id}_{[r_{n-1}]}, \dots, \mathrm{id}_{[r_1]})$$

  with $d_i\colon [r_n] \to [r_n - 1]$ the order-preserving surjection which maps i and $i+1$ to i.

- *For $1 \le j < n$ let $\tilde{\partial}_j$ be the map which restricted to $F(t)$ equals*

$$\sum_{\substack{0 \le i < r_j,\\ f_j(i) = f_j(i+1)}} \sum_{\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} F(h_{i,\sigma}),$$

  with $h = h_{i,\sigma}$ the unique morphism of trees exhibited in [41, Lemma 3.5] with $h_j = d_i\colon [r_j] \to [r_j - 1]$ and h_{j+1} restricted to $f_{j+1}^{-1}(\{i, i+1\})$ acting like σ.

The signs are determined by the number of suspensions the degree -1 map d_i has to be switched with before we actually apply it as well as by the graded signature of the permutation σ in the cases $j < n$.

To be more precise, for any j we number the edges in the tree t from bottom to top and from left to right. Then for $j < n$ we aquire a sign $(-1)^{s_{j,i}}$ where $s_{j,i}$ is the number of the rightmost top edge of the $n-j$-level subtree $t_{j,i}$. For $j = n$ we set $s_{n,i}$ to be the label of the edge whose leaf is the ith leaf for $0 \le i \le n$.

For $j < n$ the map $F(h_{i,\sigma})$ is not only decorated by $(-1)^{s_{j,i}}$ but also by a sign associated to $\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))$: Let $t_1, \dots, t_a$ be the $n-j-1$-level subtrees of t above the j-level vertex i, i.e. the $n-j-1$-level subtrees forming $t_{j,i}$, similarly let $t_{a+1}, \dots, t_{a+b}$ denote the $n-j-1$-level subtrees above $i+1$. Then σ determines a shuffle of $\{t_1, \dots, t_a\}$ and $\{t_{a+1}, \dots, t_{a+b}\}$. The sign $\epsilon(\sigma; t_{j,i}, t_{j,i+1})$ picks up a factor $(-1)^{(d(t_x)+1)(d(t_y)+1)}$ whenever $x < y$ and $\sigma(x) > \sigma(y)$.

Lemma 4.5. *For any functor* $F\colon \mathrm{Epi}_n \to k\text{-mod}$ *the* $\mathbb{N}_0^n$*-graded module* $\tilde{C}^{E_n}(F)$ *together with* $\tilde{\partial}_1, ..., \tilde{\partial}_n$ *forms a multicomplex which we again denote by* $\tilde{C}^{E_n}(F)$*, i.e. the identities*

$$\tilde{\partial}_i^{\,2} = 0 \quad \text{and} \quad \tilde{\partial}_i\tilde{\partial}_j = -\tilde{\partial}_j\tilde{\partial}_i$$

hold for all $1 \leq i, j \leq n$.

Definition 4.6. *We call the homology*

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}\tilde{C}^{E_n}(F))$$

of the total complex associated to $\tilde{C}^{E_n}(F)$ *the* E_n*-homology of* $F\colon \mathrm{Epi}_n \to k\text{-mod}$.

After establishing this generalization of E_n-homology and showing that there is a Loday functor

$$\mathcal{L}(A;k)\colon \mathrm{Epi}_n \to k\text{-mod}$$

associated to every k-projective nonunital commutative algebra A such that

$$H_*^{E_n}(\mathcal{L}(A;k)) = H_*^{E_n}(A;k),$$

Livernet and Richter prove that E_n-homology of functors is indeed functor homology.

Theorem 4.7. *Let* $\tilde{b}\colon \mathrm{Epi}_n^{\mathrm{op}} \to k\text{-mod}$ *be the functor given by*

$$\tilde{b}(t) = \begin{cases} k, & t = [0] \to ... \to [0], \\ 0 & \textit{else.} \end{cases}$$

Then for $F\colon \mathrm{Epi}_n \to k\text{-mod}$

$$H_*^{E_n}(F) = \mathrm{Tor}_*^{\mathrm{Epi}_n}(\tilde{b}, F).$$

Remark 4.8. *The proof relies on the axiomatic description of* $\mathrm{Tor}^{\mathrm{Epi}_n}$ *which we will recall in 4.27: Livernet-Richter show that short exact sequences of functors give rise to long exact sequences of homology groups, that* $H_*^{E_n}$ *vanishes on projectives and construct* $\tilde{b}$ *so that* $H_0^{E_n} = \mathrm{Tor}_0^{\mathrm{Epi}_n}(\tilde{b}, -)$.

4.2 The category Epi_n^+ and E_n-homology and cohomology of functors

We would like to establish a functor homology interpretation not only for E_n-homology of a commutative algebra A with trivial coefficients but for arbitrary coefficients in a symmetric A-bimodule M as well as for E_n-cohomology. We know from 3.62 that to compute E_n-homology with coefficients we need to twist the chain complex $M \otimes B^n(A)$ by a twisting cochain

$$\delta\colon M \otimes B^n(A) \to M \otimes B^n(A).$$

To model E_n-homology with coefficients as functor homology we hence have to enlarge the category Epi_n to incorporate the summands of this twisting cochain.

Definition 4.9. *The objects of the category* Epi_n^+ *are given by planar fully grown trees with n levels. A morphism from* $t^r = [r_n] \xrightarrow{f_n^r} \dots \xrightarrow{f_2^r} [r_1]$ *to* $t^s = [s_n] \xrightarrow{f_n^s} \dots \xrightarrow{f_2^s} [s_1]$ *is represented by a sequence of maps* $(h_n, ..., h_1)$*, where*

- *for* $i = 2, ..., n-1$*, the map* $h_i\colon [r_i] \to [s_i]$ *is a surjection which is order-preserving on the fibers* $f_i^{-1}(l)$ *for all* $l \in [r_{i-1}]$*. For* $i = 1$ *we require* $h_1\colon [r_1] \to [s_1]$ *to be order-preserving.*
- *The map* h_n *is a map*
$$h_n\colon [r_n] \to [s_n]_+ := [s_n] \sqcup \{+\}$$
such that $[s_n]$ *lies in the image of* h_n *and such that the restriction of* h_n *to* $h_n^{-1}([s_n])$ *is order-preserving on the fibers of* f_n*. Furthermore the intersection of* $h_n^{-1}([s_n])$ *with a fiber* $f_n^{-1}(l)$ *is a (potentially empty) interval for all* $l \in [r_{n-1}]$*, i.e. is of the form* $\{a, a+1, ..., a+l\}$ *with* $l \geq -1$*.*
- *The diagram*
$$\begin{tikzcd} h_n^{-1}([s_n]) \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r] \arrow[d, "h_{n-1}"] & \dots \arrow[r] & {[r_2]} \arrow[r, "f_1^r"] \arrow[d, "h_2"] & {[r_1]} \arrow[d, "h_1"] \\ {[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r] & \dots \arrow[r] & {[s_2]} \arrow[r, "f_1^s"] & {[s_1]} \end{tikzcd}$$
commutes.

Finally we identify certain morphisms by imposing the following equivalence relation on the set of morphisms from t^r *to* t^s*: We identify morphisms* h *and* h' *if*

- ${h_n}^{-1}(+) = {h'_n}^{-1}(+)$ *and*
- h_i *and* h'_i *coincide if restricted to* $f_{i+1}^r ... f_n^r([r_n] \setminus h_n^{-1}(+))$*.*

The composition of two morphism $(g_n, ..., g_1)\colon t^q \to t^r$ *and* $(h_n, ..., h_1)\colon t^r \to t^s$ *is defined by composing componentwise and sending* $+$ *to* $+$*, i.e.*

$$(h_n, ..., h_1) \circ (g_n, ..., g_1) := ((hg)_n, h_{n-1}g_{n-1}, ..., h_1 g_1)$$

with $(hg)_n(x) = \begin{cases} +, & g_n(x) = +, \\ h_n g_n(x), & \textit{otherwise.} \end{cases}$

Lemma 4.10. *Composition in* Epi_n^+ *is well defined and associative.*

Proof. It is easy to check that the relation defined above is indeed an equivalence relation and that the composition is associative. To show that composition is well defined, consider morphisms $g\colon t^q \to t^r$ and $h\colon t^r \to t^s$ as above and another morphism g' equivalent to g. Then $g_n^{-1}(+) = {g'}_n^{-1}(+)$ and hence $(hg)_i$ and $(hg')_i$ agree on $f_{i+1}^q ... f_n^q([q_n] \setminus (hg)_n^{-1}(+)) \subset f_{i+1}^q ... f_n^q([q_n] \setminus g_n^{-1}(+))$ because g_i and g'_i do. Since g_n and g'_n coincide on $f_{i+1}^q ... f_n^q([q_n] \setminus g_n^{-1}(+))$ we see that $(hg)_n^{-1}(+) = (hg')_n^{-1}(+)$ as well, hence hg and hg' are equivalent.
If h is equivalent to h', then

$$(hg)_n^{-1}(+) = g_n^{-1}(+) \cup g_n^{-1}(h_n^{-1}(+)) = g_n^{-1}(+) \cup g_n^{-1}({h'}_n^{-1}(+)) = (h'g)_n^{-1}(+).$$

If $x \in [q_i]$ with $x \in f_{i+1}^q ... f_n^q([q_n] \setminus (hg)_n^{-1}(+))$ we find that $g_i(x) \in f_{i+1}^r ... f_n^r([r_n] \setminus h_n^{-1}(+))$, hence $h_i g_i(x) = h'_i g_i(x)$. $\square$

Remark 4.11. 1. *It is clear that* Epi_n *is a subcategory of* Epi_n^+ *and that both categories share the same objects. Intuitively the category* Epi_n^+ *is built from* Epi_n *by adding morphisms of the form*

$$\begin{tikzcd}
[r_n] \arrow[r, "f_n"] \arrow[d, "\delta_i"'] & {[r_{n-1}]} \arrow[r, "f_{n-1}"] \arrow[d, "\mathrm{id}"'] & ... \arrow[r, "f_2"] & {[r_1]} \arrow[d, "\mathrm{id}"'] \\
{[r_n - 1]} \arrow[r, "\hat{f}_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}"] & ... \arrow[r, "f_2"] & {[r_1]}
\end{tikzcd}$$

and adding all newly arising compositions with such morphisms. Here δ_i *is the map*

$$\delta_i(x) = \begin{cases} x, & x < i, \\ +, & x = i \\ x-1, & x > i \end{cases}$$

with i *the minimal or maximal element of a fiber* $f_n^{-1}(l)$ *containing at least two elements, and*

$$\hat{f}_n(x) = \begin{cases} f_n(x), & x < i, \\ f_n(x+1), & x \geq i. \end{cases}$$

The requirement that the elements of a fiber of f_n *that get not mapped to* $+$ *form an interval reflects the fact that we have only added morphisms of the aforementioned kind.*

2. *Our motivation for defining* Epi_n^+ *is to model the complex calculating* E_n*-homology of* A *with coefficients in* M*. Hence imposing the equivalence relation is necessary because it should not matter what precisely happens to a subtree of a tree* t *if all its leaves get mapped to* $+$*, i.e. in which order and on what side a family of elements of*

A acts on an element of M. Otherwise we would encounter pathologies, for example the composition of

$$\begin{tikzcd} {[2]} \arrow[r] \arrow[d] & {[2]} \arrow[d, "0\mapsto 0;1,2\mapsto 1"] \\ {[2]} \arrow[r, "0\mapsto 0;1,2\mapsto 1"'] & {[1]} \end{tikzcd} \qquad \textit{with} \qquad \begin{tikzcd} {[2]} \arrow[r, "0\mapsto 0;1,2\mapsto 1"] \arrow[d, "0\mapsto 0;1\mapsto +;2\mapsto 1"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & {[1]} \end{tikzcd}$$

and the composition of

$$\begin{tikzcd} {[2]} \arrow[r] \arrow[d] & {[2]} \arrow[d, "0,1\mapsto 0;2\mapsto 1"] \\ {[2]} \arrow[r, "0\mapsto 0;1,2\mapsto 1"'] & {[1]} \end{tikzcd} \qquad \textit{with} \qquad \begin{tikzcd} {[2]} \arrow[r, "0,1\mapsto 0;2\mapsto 1"] \arrow[d, "0\mapsto 0;1\mapsto +;2\mapsto 1"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & {[1]} \end{tikzcd}$$

would not coincide, which is not in accordance with what we try to model: In $M \otimes B^2(A)$ the first composite corresponds to the map sending $m \otimes [[a_0]|[a_1]|[a_2]]$ to $m \otimes [[a_0]|[a_1|a_2]]$ and then to $a_1 m \otimes [[a_0]|[a_2]]$, while the second composite corresponds to mapping $m \otimes [[a_0]|[a_1]|[a_2]]$ to $m \otimes [[a_0|a_1]|[a_2]]$ and then to $ma_1 \otimes [[a_0]|[a_2]]$. Hence the two compositions should coincide.

After defining a category which also models the summands of the twisting cochain δ we can proceed to define E_n-homology of a functor.

Definition 4.12. *Let $F\colon \mathrm{Epi}_n^+ \to k\text{-mod}$ be a functor. As in 4.4 set*

$$C_{r_n,\ldots,r_1}(F) := \bigoplus_{t=[r_n]\to\ldots\to[r_1]} F(t).$$

Define maps ∂_j lowering the jth degree by one by

$$\partial_j = \tilde{\partial}_j \quad \text{for } i < n, \quad \partial_n = \tilde{\partial}_n + \delta_{\min} + \delta_{\max},$$

with

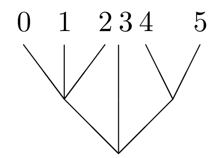
$$\delta_{\min} = \sum_{\substack{0 \le l \le r_{n-1}, \\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\min f_n^{-1}(l)}-1} F(\delta_{\min f_n^{-1}(l)}, \mathrm{id}, \ldots, \mathrm{id})$$

and

$$\delta_{\max} = \sum_{\substack{0 \le l \le r_{n-1} \\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\max f_n^{-1}(l)}} F(\delta_{\max f_n^{-1}(l)}, \mathrm{id}, \ldots, \mathrm{id}).$$

Here δ_i is as in remark 4.11.

Example 4.13. *Let t be the 2-level tree*



Then $\delta_{\min}$ is the sum of the morphism induced by mapping the leaf labeled 0 to $+$, equipped with the sign $(-1)^1$, and the morphism induced by mapping 4 to $+$, decorated by $(-1)^7$. The map δ_{max} is induced by sending 2 to $+$ with sign $(-1)^4$ and by mapping 5 to $+$ which yields the sign$(-1)^9$.

Now we face the task of proving that $(C^{E_n}, \partial_1, ..., \partial_n)$ is in fact a multicomplex. Since we already know from [41, Lemma 3.8] that $(C^{E_n}, \tilde{\partial}_1, ..., \tilde{\partial}_n)$ is a multicomplex it suffices to prove the following lemma.

Lemma 4.14. *The differentials defined above satisfy the identities*

$$\begin{aligned}
\delta_{\min}\partial_j + \partial_j\delta_{\min} &= 0 \quad \textit{for } j < n, \\
\delta_{\max}\partial_j + \partial_j\delta_{\max} &= 0 \quad \textit{for } j < n, \\
\delta_{\min}\delta_{\max} + \delta_{\max}\delta_{\min} &= 0, \\
\delta_{\min}^2 + \tilde{\partial}_n\delta_{\min} + \tilde{\partial}_n\delta_{\min} &= 0, \\
\delta_{\max}^2 + \tilde{\partial}_n\delta_{\max} + \tilde{\partial}_n\delta_{\max} &= 0.
\end{aligned}$$

We will frequently encounter notational difficulties during this proof, since for example in the first identity above the two maps $\delta_{\min}$ are not equal. In particular the trees defining the occuring signs do not coincide. We will try to circumvent defining a new galaxy of notation by sticking to the following rule: We examine each summand in the identity above one by one and then compare them, and while investigating the composition $f \circ g$ we will always use the undecorated notation introduced above to denote anything having to do with g, while signs, trees, etc. ocurring in the evaluation of f will be decorated with a "ˆ". Note that the trees on which for example ∂_j in $\delta_{\min}\partial_j$ and $\delta_{\min}$ in $\partial_j\delta_{\min}$ are defined coincide, while this does not need to be the case for the other two maps, although according to our convention we will denote their signs by the same symbol. We will solve this notational problem by always comparing the signs equipped with a "ˆ" with signs in the original source tree.
We also will frequently refer to morphisms in Epi_n^+ instead of the parts of the differentials they induce, including the signs they carry.

Proof. Let $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ be a given tree. Let us start with proving the first identity. Fix $l \in [r_{n-1}]$ with $|f_n^{-1}(l)| > 1$ and set $i := \min f_n^{-1}(l)$. Consider $a \in [r_j]$

with $f_j(a) = f_j(a+1)$ and fix a shuffle $\tau^{a,j}_{j+1} \in \mathrm{sh}(f^{-1}_{j+1}(a), f^{-1}_{j+1}(a+1))$, defining a morphism $(\tau^{a,j}_n, ..., \tau^{a,j}_{j+1}, d_a, \mathrm{id}, ..., \mathrm{id})$ as explained in [41, Lemma 3.5]. We distinguish a couple of cases:

- If the ith leaf is to the right of the leaves of the subtree $t_{j,a+1}$, first applying

$$(\tau^{a,j}_n, ..., \tau^{a,j}_{j+1}, d_a, \mathrm{id}, ..., \mathrm{id})$$

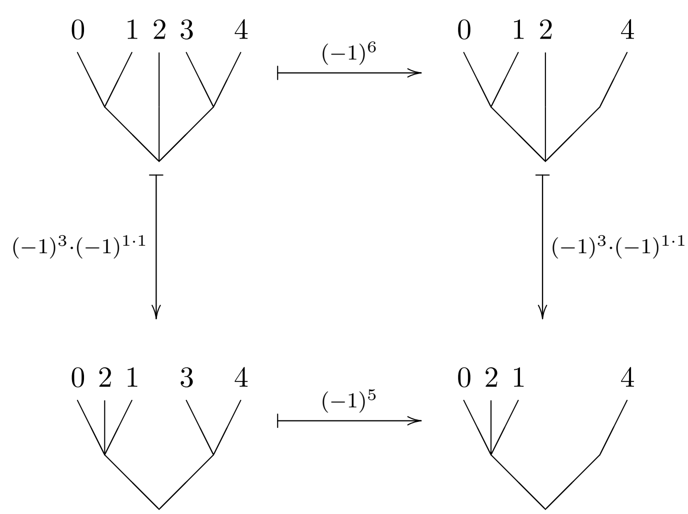
  and then deleting the ith leaf results in a sign

$$(-1)^{\tilde{s}_{n,i}-1}(-1)^{s_{j,a}}\epsilon(\tau^{a,j}_{j+1}; t_{j,a}, t_{j,a+1})$$

  with $(-1)^{\tilde{s}_{n,i}-1} = (-1)^{s_{n,j}-2}$ since two edges left and below of i got merged. If we first delete the ith leaf and then apply the corresponding shuffle and merge maps we get a sign

$$(-1)^{\tilde{s}_{j,a}}\epsilon(\tilde{\tau}^{a,j}_{j+1}; \tilde{t}_{j,a}, \tilde{t}_{j,a+1})(-1)^{s_{n,i}-1}.$$

  But since deleting a leaf right of $t_{j,a+1}$ does nothing to the signs associated with the shuffle and merge operation this is $(-1)^{s_{j,a}}\epsilon(\tau^{a,j}_{j+1}; t_{j,a}, t_{j,a+1})(-1)^{s_{n,i}-1}$ and the two operations anticommute. The following picture illustrates this for $n = 2, j = 1, a = 0$ and $i = 3$ together with the signs the operations produce:

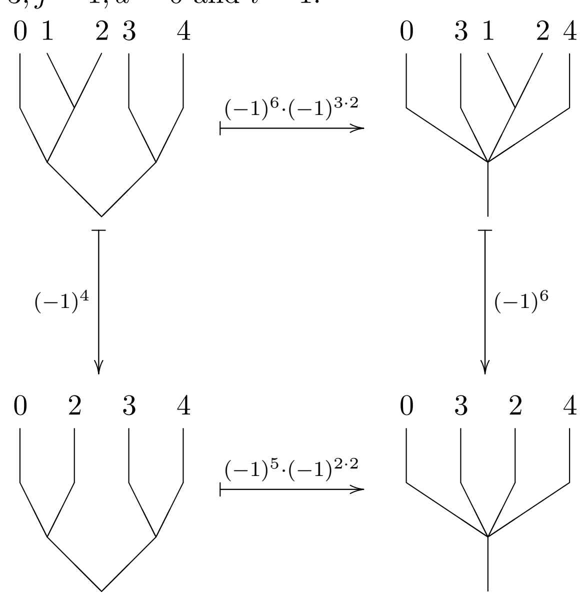


- The same holds if the ith leaf is left to $t_{j,a}$, with the difference that here the sign associated to deleting the leaf stays the same and the signs $(-1)^{s_{j,a}}$ and $(-1)^{\tilde{s}_{j,a}}$ differ due to one edge getting deleted left of the rightmost leaf of $t_{j,a}$.

- Now let us consider the case where the ith leaf is actually a leaf of the subtrees that get shuffled. We have to distinguish the cases $j = n-1$ and $j < n-1$. Start with the latter and assume that i is a leaf of $t_{j,a}$. If we first want to apply the merge and shuffle operation and then delete what was the ith leaf in the original tree we have to delete the $\tau_n^{a,j}(i)$th leaf, which is a suitable minimum again since $j < n-1$. The difference in sign between deleting this leaf and deleting the ith leaf is determined by how many subtrees $t_{p+1}, ..., t_{p+q}$ of $t_{j,a+1}$ get moved past the subtree t_c of $t_{j,a}$ containing i, hence changing the labeling. For each of these subtrees $\tilde{s}_{n,\tau_n^{j,a}(i)}$ gains $d(t_d)+1$ compared to $s_{n,i}$. On the other hand there clearly exists a $\tilde{\tau}_n^{a,j}$ such that

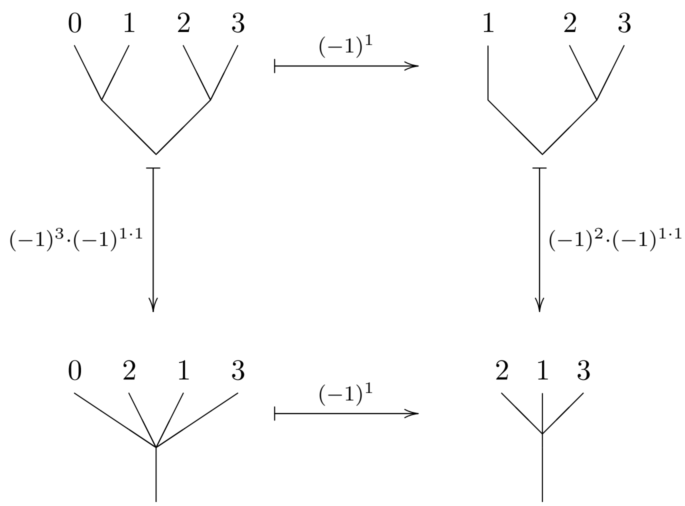
$$\begin{aligned} & (\tilde{\tau}_n^{a,j}, \tau_{n-1}^{a,j}, ..., \tau_{j+1}^{a,j}, d_a, \mathrm{id}, ..., \mathrm{id})(\delta_i, \mathrm{id}, ..., \mathrm{id}) \\ = \; & (\delta_{\tau_n^{j,a}(i)}, \mathrm{id}, ..., \mathrm{id})(\tau_n^{a,j}, ..., \tau_{j+1}^{a,j}, d_a, \mathrm{id}, ..., \mathrm{id}). \end{aligned}$$

  The sign associated to $(\tilde{\tau}_n^{a,j}, ..., \tau_{j+q}^{a,j}, d_a, \mathrm{id}, ..., \mathrm{id})$ differs from the one associated to $(\tau_n^{a,j}, ..., \tau_{j+1}^{a,j}, d_a, \mathrm{id}, ..., \mathrm{id})$ in two aspects: First, we apply the former after the ith leaf has been deleted, and this leaf sits left to the rightmost leaf of $t_{j,a}$. Hence $\tilde{s}_{j,a} = s_{j,a} - 1$. On the other hand all subtrees of $\tilde{t}_{j,a}$ and $t_{j,a}$ are equal except for the subtree t_c containing the ith leaf: $\tilde{t}_c$ has one edge less. Hence $d(\tilde{t}_c) = d(t_c) - 1$ and so $\epsilon(\tau_{j+1}^{a,j}, \tilde{t}_{j,a}, \tilde{t}_{j,a+1})$ picks up a sign $(-1)^{(d(t_c)-1+1)(d(t_d)+1)}$ for $d \in \{p+1, ..., p+q\}$ with $\tau_{j+1}^{a,j}(c) > \tau_{j+1}^{a,j}(d)$, whereas $\epsilon(\tau_{j+1}^{a,j}, t_{j,a}, t_{j,a+1})$ gains a sign $(-1)^{(d(t_c)+1)(d(t_d)+1)}$ in the same case. Hence the two shuffle signs compare to each other just like $(-1)^{\tilde{s}_{n,\tau_n^{j,a}(i)}}$ to $(-1)^{s_{n,i}}$, and since $\tilde{s}_{j,a} = s_{j,a} - 1$ the desired anticommutativity is proven. We give an example for $n = 3, j = 1, a = 0$ and $i = 1$.



- If i is a leaf of $t_{j,a+1}$, the same argument holds, just that in this case $\tilde{s}_{j,a}$ and $s_{j,a}$ coincide while the difference between $\tilde{s}_{n,\tau_n^{j,a}(i)}$ and $s_{n,i}$ increases by one since an edge got merged left to $\tau_n^{j,a}(i)$.

- Now let $j = n-1$ and let the ith leaf be a leaf of $t_{n-1,a}$. Since $t_{n-1,a} = f_n^{-1}(a)$ this is the leftmost leaf of $t_{n-1,a}$. Consider deleting the ith leaf first and then applying a merge and shuffle operation $(\tilde{\sigma}, d_a, \mathrm{id}, ..., \mathrm{id})$. The shuffle $\tilde{\sigma}$ is a shuffle of $f_n^{-1}(a) \setminus \{i\}$ and $f_n^{-1}(a+1)$. There is exactly one corresponding shuffle $\sigma \in \mathrm{sh}(f_n^{-1}(a), f_n^{-1}(a+1))$ fixing i, and if we apply $(\sigma, d_a, \mathrm{id}, ..., \mathrm{id})$ first and then delete the ith leaf the result coincides with the operation considered first, up to sign. Since the position of i did not change and no edge to the left of i got merged or deleted, the signs associated to the deletion operations coincide. The same is valid for the shuffle signs, since the ith leaf is fixed by σ and hence does not contribute any inversions. On the other hand $s_{n-1,a}$ and $\tilde{s}_{n-1,a}$ differ by one due to the deleted leaf. Hence the operations anticommute. An example for $n = 2, j = 1$ and $i = 0$ is shown in the following picture:



- If i is a leaf of $t_{n-1,a+1}$, we can identify exactly one shuffle $\sigma \in \mathrm{sh}(f_n^{-1}(a), f_n^{-1}(a+1))$ which sends i to the minimum of $t_{n-1,a}$, and the same reasoning as above holds, with the following modifications: $\tilde{s}_{n,\sigma(i)}$ and $s_{n,i}$ now differ by $(-1)^{d(t_{n-1,a})+1}$, whereas $\epsilon(\sigma; t_{n-1,a}, t_{n-1,a+1})$ and $\epsilon(\tilde{\sigma}; t_{n-1,a}, \tilde{t}_{n-1,a+1})$ differ by $(-1)^{d(t_{n-1,a})}$, while $\tilde{s}_{n-1,a}$ and $s_{n-1,a}$ coincide.

Similar arguments hold when we consider deleting a rightmost leaf, hence the first two identities hold.

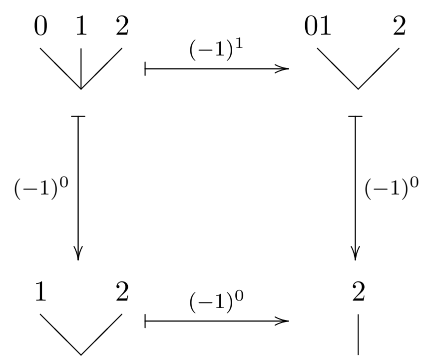
Considering the last three identities note that all the operations merge or delete one edge. So whenever the operations in question are operating on different fibers of f_n it is obvious that the operations commute up to sign, and that applying the operation acting further to the left after the operation acting further to the right yields exactly the sign opposite to that of applying the right one after the left one, picking up the merging or deletion of edges. Hence we only need to prove the identities for operations acting on the same 1-level subtree and may assume without loss of generality that we are considering a tree of the form $t = [r] \longrightarrow [0] \longrightarrow ... \longrightarrow [0]$ for $r > 1$.
The third identity is easy to see: deleting the leftmost leaf and then the rightmost leaf yields the sign $(-1)^{s_{n,r}-1}(-1)^{s_{n,0}-1}$, whereas performing these operations in the opposite order yields $(-1)^{s_{n,0}-1}(-1)^{s_{n,r}}$.
To prove the fourth identity fix $a \in \{0, ..., r-1\}$. Assume first that $a \neq 0$. Then

$$(\delta_0, \mathrm{id}, ..., \mathrm{id})(d_a, \mathrm{id}, ..., \mathrm{id}) = (d_{a-1}, \mathrm{id}, ..., \mathrm{id})(\delta_0, \mathrm{id}, ..., \mathrm{id}).$$

The first composition induces maps decorated with the sign $(-1)^{s_{n,0}-1}(-1)^{s_{n,a}}$, while the other yields $(-1)^{s_{n,a-1}}(-1)^{s_{n,0}-1}$. Since $s_{n,a-1} = s_{n,a} - 1$ the induced maps anticommute. Hence all terms except those originating from $\delta^2_{\min}$ and those which are of the form $(\delta_0, \mathrm{id}, ..., \mathrm{id})(d_0, \mathrm{id}, ..., \mathrm{id})$ cancel out. But it is clear that

$$(\delta_0, \mathrm{id}, ..., \mathrm{id})(d_0, \mathrm{id}, ..., \mathrm{id}) = (\delta_0, \mathrm{id}, ..., \mathrm{id})(\delta_0, \mathrm{id}, ..., \mathrm{id}),$$

the first yielding the sign $(-1)^{s_{n,0}-1}(-1)^{s_{n,0}}$, the latter giving $(-1)^{s_{n,0}-1}(-1)^{s_{n,0}-1}$. We give an example for $n = 1$ and $a = 0$:



Similar arguments prove the last identity. □

After establishing that $C^{E_n}(F)$ is in fact a multicomplex we can define E_n-homology:

Definition 4.15. *Let $F\colon \mathrm{Epi}_n^+ \to k\text{-mod}$ be a functor. The E_n-homology of F is*

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}C^{E_n}(F)).$$

Remark 4.16. *Given a functor* $\tilde{F}\colon \mathrm{Epi}_n \to k\text{-mod}$*, we can extend* $\tilde{F}$ *to* $F\colon \mathrm{Epi}_n^+ \to k\text{-mod}$ *by setting* $F(h) = 0$ *for every morphism* $h\colon t^r \to t^s$ *in* Epi_n^+ *such that* $h([r_n]) \cap \{+\} \neq \emptyset$*. With these definitions* $H^{E_n}(F)$ *coincides with the* E_n*-homology of* $\tilde{F}$ *as defined in 4.6. In this sense the definition of* E_n*-homology we just gave extends the definition given in [41, Definition 3.7].*

We are specifically interested in calculating E_n-homology of commutative algebras, which is the E_n-homology of the following functors.

Definition 4.17. *The Loday functor*

$$\mathcal{L}(A, M)\colon \mathrm{Epi}_n^+ \to k\text{-mod}$$

is the following functor: For a given tree $t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$ *set*

$$\mathcal{L}(A, M)(t) = M \otimes A^{\otimes r_n + 1}.$$

If $(h_n, ..., h_1)\colon t^r \to t^s$ *is a morphism let*

$$\mathcal{L}(A, M)(h_n, ..., h_1)\colon M \otimes A^{\otimes r_n+1} \to M \otimes A^{\otimes s_n+1}$$

be given by

$$m \otimes a_0 \otimes ... \otimes a_{r_n} \mapsto \left(m \cdot \prod_{i:h_n(i)=+} a_i\right) \otimes \left(\prod_{i:h_n(i)=0} a_i\right) \otimes ... \otimes \left(\prod_{i:h_n(i)=s_n} a_i\right).$$

Observe that $\mathcal{L}(A, M)$ *could also be considered as a functor from finite pointed sets and surjections to* k*-modules: The values and induced morphisms really only depend on the leaves of the trees we consider. We will encounter this point of view later in our discussion of higher order Hochschild homology in 6.3.*

Remark 4.18. *It is easily seen that* $\mathrm{Tot}(C^{E_n}(\mathcal{L}(A, M))) = \Sigma^{-n}(M \otimes B^n(A), \partial_\theta)$ *as defined in 3.42: That*

$$\Sigma^{-n}B^n(A) = \mathrm{Tot}(C^{E_n}(\mathcal{L}(A, k)))$$

has already been noted in [41, 3.1]. Hence $\Sigma^{-n}(M \otimes B^n(A)) = \mathrm{Tot}(M \otimes C^{E_n}(\mathcal{L}(A, k)))$*. But* $M \otimes C^{E_n}(\mathcal{L}(A, k))$ *and* $C^{E_n}(\mathcal{L}(A, M))$ *coincide as graded modules and their differentials only differ by* $\delta_{\min} + \delta_{\max}$*, the part of the differentials actually incorporating the action of* A *on the coefficient module. The twist* $\delta_{\min} + \delta_{\max}$ *on* $C^{E_n}(\mathcal{L}(A, M))$ *corresponds to* ∂_θ *on* $\Sigma^{-n}(M \otimes B^n(A))$*. In particular,*

$$H_*^{E_n}(\mathcal{L}(A, M)) = H_*^{E_n}(A; M)$$

if A *is* k*-projective.*

We now consider E_n-cohomology.

Definition 4.19. *Let* $G\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$ *be a functor. The* E_n*-cohomology of* G *is defined as*

$$H^*_{E_n}(G) = H_*(\mathrm{Tot}(C_{E_n}(G))),$$

where $C_{E_n}(G)$ *is the multicomplex with*

$$C^{r_n,\dots,r_1}_{E_n}(G) = \bigoplus_{t=[r_n]\xrightarrow{f_n}\dots\xrightarrow{f_2}[r_1]} G(t)$$

and differentials $\partial_j\colon C^{r_n,\dots,r_1}_{E_n}(G) \to C^{r_n,\dots r_j+1,\dots,r_1}_{E_n}(G)$ *raising the* j*th degree by one defined as follows:*

- *For* $j = n$ *define* ∂_n *restricted to* $G(t)$ *as*

$$\begin{aligned}
&\sum_{\substack{0\le i<r_n,\\ f_n(i)=f_n(i+1)}} (-1)^{s_{n,i}} G(d_i, \mathrm{id}_{[r_{n-1}]}, \dots, \mathrm{id}_{[r_1]})\\
+&\sum_{\substack{0\le l\le r_{n-1},\\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\min f_n^{-1}(l)}-1} G(\delta_{\min f_n^{-1}(l)}, \mathrm{id}, \dots, \mathrm{id})\\
+&\sum_{\substack{0\le l\le r_{n-1}\\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\max f_n^{-1}(l)}} G(\delta_{\max f_n^{-1}(l)}, \mathrm{id}, \dots, \mathrm{id}).
\end{aligned}$$

- *For* $1 \le j < n$ *the map* ∂_j *restricted to* $F(t)$ *is given by*

$$\sum_{\substack{0\le i<r_j,\\ f_j(i)=f_j(i+1)}} \sum_{\sigma\in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} G(h_{i,\sigma}),$$

  with $h = h_{i,\sigma}$ *again denoting the unique morphism of trees exhibited in [41, Lemma 3.5] with* $h_j = d_i\colon [r_j] \to [r_{j-1}]$ *and* h_{j+1} *restricted to* $f_{j+1}^{-1}(\{i, i+1\})$ *acting like* σ.

Remark 4.20. *With the same reasoning as in the homological case one sees that* $C_{E_n}(G)$ *is in fact a multicomplex.*

As was the case for E_n-homology this definition generalizes E_n-cohomology of commutative algebras with coefficients in a bimodule:

Definition 4.21. *Let*

$$\mathcal{L}^c(A, M)\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$$

be defined on a tree $t = \ [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ *as*

$$\mathcal{L}^c(A, M)(t) = \mathrm{Hom}_k(A^{\otimes r_n+1}, M).$$

If $(h_n, ..., h_1)$ *is a morphism from* t^r *to* t^s *define*

$$\mathcal{L}^c(A, M)(h_n, ..., h_1)\colon \mathrm{Hom}_k(A^{\otimes s_n+1}, M) \to \mathrm{Hom}_k(A^{\otimes r_n+1}, M)$$

by

$$\mathcal{L}^c(A, M)(h_n, ..., h_1)(f)(a_0 \otimes ... \otimes a_{r_n}) = \left(\prod_{i:h_n(i)=+} a_i\right) \cdot f(\left(\prod_{i:h_n(i)=0} a_i\right) \otimes ... \otimes \left(\prod_{i:h_n(i)=s_n} a_i\right)).$$

Then $\mathrm{Tot}(C_{E_n}(\mathcal{L}^c(A, M)))$ *coincides with the complex computing* E_n*-cohomology of* A *with coefficients in* M *introduced in 3.63, in particular*

$$H^*_{E_n}(\mathcal{L}^c(A, M)) = H^*_{E_n}(A, M)$$

if A *is* k*-projective.*

Remark 4.22. *For* $n = 1$ *the category* Epi_1^+ *can be identified with the image of the semisimplicial part of the simplicial circle* $C : \Delta^{\mathrm{op}} \to \mathrm{Fin}_*$ *in the category* Fin_* *of finite pointed sets up to a shift: In the terminology of [54] the tree* $[r]$ *corresponds to the finite pointed set* $[r + 1]$ *with* 0 *as basepoint, the morphisms* δ_0 *and* δ_r *correspond to* d_0 *and* d_{r+1}*, while the merging operation* d_i *on the tree* $[r]$ *is the counterpart to the map* d_{i+1} *in the simplicial circle. Hence* E_1*-homology agrees with Hochschild homology for functors from* Fin_* *to* k-mod *as defined by Pirashvili and Richter up to a shift. As we will see later there is a Loday functor* $\mathcal{L}_+(A, M)\colon \mathrm{Fin}_* \to k\text{-mod}$ *which is an unreduced version of* $\mathcal{L}(A, M)$*. This yields in particular that*

$$H_*^{E_1}(\mathcal{L}(A, M)) = HH_{*+1}(A_+, M)$$

if A *is* k*-projective, where* $HH_{*+1}(A_+, M)$ *denotes classical Hochschild homology of the commutative unital algebra* A_+ *with coefficients in the symmetric* A*-bimodule* M*. Similar considerations hold for* E_1*-cohomology.*

Every functor $F\colon \mathrm{Epi}_n^+ \to k\text{-mod}$ gives rise to a functor $F^*\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$, its dual, by setting $F^*(t) = \mathrm{Hom}_k(F(t), k)$. The following universal coefficient spectral sequence relates E_n-homology with E_n-cohomology.

Proposition 4.23 ([40, Theorem 2.3])**.** *If $F(t)$ is k-free for every $t \in \mathrm{Epi}_n^+$, there is a first quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Ext}^q_k(H^{E_n}_p(F), k) \Rightarrow H^{p+q}_{E_n}(F^*).$$

In particular whenever k is injective as a k-module, E_n-homology of F and E_n-cohomology of its dual are dual to each other.

Examples of commutative self-injective rings include fields, group algebras of finite commutative groups over a self-injective ring, quotients R/I of a principal ideal domain R with $I \neq 0$ and commutative Frobenius rings [1, ch.5, §18]. The product of self-injective rings is again self-injective.

4.3 E_n-cohomology as functor cohomology

In [41, Theorem 4.1] Livernet and Richter show that E_n-homology with trivial coefficients can be interpreted as functor homology. We want to extend this result to E_n-homology with arbitrary coefficients.
To prove that E_n-homology coincides with functor homology we first show that certain projective functors are acyclic. For $t \in \mathrm{Epi}_n^+$ set

$$P_t = k\langle \mathrm{Epi}_n^+(t, -)\rangle \colon \mathrm{Epi}_n^+ \to k\text{-mod} \quad \text{and} \quad P^t = k\langle \mathrm{Epi}_n^+(-, t)\rangle \colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}.$$

In the proof of the following lemma, we will consider trees obtained by restricting a given tree to certain leaves.

Definition 4.24. *Let $t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$ be a tree. For fixed $I \subset [r_n]$ set $r_i^I = |f_n \dots f_{i+1}(I)| - 1$. Define a tree t^I as the upper row in*

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & \dots \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] & f_n(I) \arrow[r, "f_{n-1}"] & \dots \arrow[r, "f_2"] & f_2 \dots f_n(I)
\end{tikzcd}$$

Here the vertical morphisms are determined by requiring that they are bijective and order-preserving, while the maps f_n^I are given by requiring that all squares commute. Intuitively t^I is the subtree of t given by restricting t to edges connecting leaves labeled by I with the root.

Lemma 4.25. *Let t and I be as above. Then we can define a morphism $h^I : t \to t^I$ in Epi_n^+ as the vertical maps in*

$$\begin{tikzcd}
[r_n] \arrow[r, "f_n"] \arrow[d, "h_n^I"] & {[r_{n-1}]} \arrow[r, "f_{n-1}"] \arrow[d, "h_{n-1}^I"] & \dots \arrow[r, "f_2"] & {[r_1]} \arrow[d, "h_1^I"] \\
{[r_n^I]} \arrow[r, "f_n^I"] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] & \dots \arrow[r, "f_2^I"] & {[r_1^I],}
\end{tikzcd}$$

where the h_i^I are defined as follows: The map h_n^I maps all $x \in [r_n] \setminus I$ to $+$ and is an order-preserving bijection restricted to I. For $i < n$ we require that h_i^I restricted to $f_{i+1} ... f_n(I)$ is the order-preserving bijection to $[r_i^I]$ and that h_i^I be order-preserving on the whole set $[r_i]$.

Proof. Since $I = [r_n] \setminus (h_n^I)^{-1}(+)$ this determines h^I up to equivalence. The maps h_i^I assemble to a morphism in Epi_n^+ since they are chosen to be order-preserving and the squares

$$\begin{tikzcd}
f_{i+1} ... f_n(I) \arrow[r, "f_i"] \arrow[d, "h_i^I"] & f_i ... f_n(I) \arrow[d, "h_{i-1}^I"] \\
{[r_i^I]} \arrow[r, "f_i^I"] & {[r_{i-1}^I]}
\end{tikzcd}$$

commute by definition of f_i^I. Furthermore $(h_n^I)^{-1}(+) \cap f_n^{-1}(i) = I \cap f_n^{-1}(i)$ is an interval if $D^I \neq \emptyset$. $\square$

Now we are in the position to compute the E_n-homology of the representable projectives.

Lemma 4.26. *Fix a tree $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$. Then*

$$H_*^{E_n}(P_t) = \begin{cases} 0, & * > 0, \\ \bigoplus_{i \in [r_n]} k, & * = 0. \end{cases}$$

Proof. Set $C := C^{E_n}(P_t)$. We filter C by the submulticomplexes

$$F^p C_{s_n,...,s_1} := \bigoplus_{t^s = [s_n] \xrightarrow{f_n^s} ... \xrightarrow{f_2^s} [s_1]} k[\{(h_n, ..., h_1) \in P_t(t^s) || h_n^{-1}([s_n])| \leq p + 1\}].$$

so that $F^p C$ is generated by morphisms that map at least $r_n - p$ leaves to $+$. This defines an ascending filtration of C by subcomplexes and therefore yields a first quadrant spectral sequence

$$E_{p,q}^1 = H_{p+q}(\mathrm{Tot}(F^p C / F^{p-1} C)) \Rightarrow H_{p+q}(\mathrm{Tot} C).$$

The quotient $F^pC/F^{p-1}C$ can be identified with the free k-module generated by morphisms $(h_n, ..., h_1) \in P_t(t^s)$ with $|h_n^{-1}([s_n])| = p+1$. The differentials $\delta_{\min}$ and $\delta_{\max}$ vanish on the quotient. The remaining summands of ∂_n and the differentials $\partial_{n-1}, ..., \partial_1$ do not change the number of leaves that get mapped to $+$. We conclude that $F^pC/F^{p-1}C$ is isomorphic to D as a multicomplex, where

$$D_{s_n,...,s_1} = \bigoplus_{t^s = [s_n] \xrightarrow{f_n^s} ... \xrightarrow{f_2^s} [s_1]} k[\{(h_n, ..., h_1) \in P_t(t^s) || h_n^{-1}([s_n])| = p+1\}]$$

with differentials $\partial_1, ..., \partial_{n-1}, \hat{\partial}_n$, where $\hat{\partial}_n = \partial_n - \delta_{\min} - \delta_{\max}$. The multicomplex D can be decomposed further: The remaining differentials do not only respect the number of deleted leaves but the set of deleted leaves itself. Hence D is the direct sum of submulticomplexes D^I with

$$D^I_{s_n,...,s_1} = \bigoplus_{t^s = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]^{s}} k[\{(h_n, ..., h_1) \in P_t(t^s) | h_n^{-1}([s_n]) = I\}]$$

such that I is a subset of $[r_n]$ of cardinality $p+1$.
Notice that the differentials of D and D^I look like the differentials used in 4.4 to define E_n-homology of functors from Epi_n to k-mod. We will show that D^I in fact can be identified with the multicomplex associated to such a functor. More precisely, D^I is the multicomplex computing E_n-homology of the representable functor P^{t^I}: Denote by $h^I : t \to t^I$ the morphism defined in lemma 4.26. We define

$$\Psi : \tilde{C}^{E_n}(\mathrm{Epi}_n(t^I, -)) \to D^I$$

by mapping $j \in \mathrm{Epi}_n(t^I, t^s)$ to $\Psi(j) = j \circ h^I$. Since j does not delete any leaves this yields an element of D^I. We define an inverse Φ to Ψ by mapping $h \in D^I$ to the composite of the columns in

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & ... \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] \arrow[d] & f_n(I) \arrow[r, "f_{n-1}"] \arrow[d] & ... \arrow[r, "f_2"] & f_2...f_n(I) \arrow[d] \\
{[r_n]} \arrow[r, "f_n"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}"] \arrow[d, "h_{n-1}"] & ... \arrow[r, "f_2"] & {[r_1]} \arrow[d] \\
{[s_n]} \arrow[r, "g_n"] & {[s_{n-1}]} \arrow[r, "g_{n-1}"] & ... \arrow[r, "g_2"] & {[s_1]}
\end{tikzcd}$$

Here the upper vertical maps are order-preserving bijections while the vertical maps in the middle are inclusions. We see that $\Phi(h)_i$ only depends on $h_i|_{f_{i+1}...f_n(I)}$, i.e. Φ is well defined on equivalence classes. It is obvious that each $\Phi(h)_i$ is surjective and that the usual requirements on commutativity are satisfied. Consider a fiber $(f_i^I)^{-1}(l)$: The map $\Phi(h)_i$ first sends it order-preservingly and surjectively to $f_{i+1}...f_n(I) \cap f_i^{-1}(j) \subset [r_i]$, where j denotes the image of l under the map $[r_{i-1}^I] \to f_i...f_n(I)$. Since h_i preserves the order on fibers of f_i we see that $\Phi(h)_i$ is order-preserving on the fibers of f_i^I. Finally we note that obviously $\Phi \circ \Psi$ is the identity. To show that Ψ is a left inverse for Φ one writes down $\Psi \circ \Phi(h)$ for a given h and uses that $\Psi \circ \Phi(h)_i$ only needs to coincide with h_i on $f_{i+1}...f_n(I)$. The maps Φ and Ψ commute with composition, hence also with applying the differentials. Since the signs in the differentials applied to a morphism h are determined by the target tree t^s of h there is no trouble with signs either. Hence we have constructed an isomorphism

$$D^I \cong C^{E_n}(\mathrm{Epi}_n(t^I, -))$$

of multicomplexes. We know from [41] that $H_*(\mathrm{Tot}\tilde{C}^{E_n}(\mathrm{Epi}_n(t^I, -))) = 0$ for $* > 0$ and that

$$H_0(\mathrm{Tot}\tilde{C}^{E_n}(\mathrm{Epi}_n(t^I, -))) = \begin{cases} k, & t^I = [0] \to [0] \to ... \to [0], \\ 0, & \text{else.} \end{cases}$$

Since $t^I = [0] \to [0] \to ... \to [0]$ implies $p + 1 = |I| = 1$ we see that the E^1-term of our spectral sequence is

$$E^1_{p,q} = H_{p+q}(\mathrm{Tot}(F^pC/F^{p-1}C)) = \begin{cases} \bigoplus_{i \in [r_n]} k, & p = q = 0, \\ 0, & \text{else.} \end{cases}$$

The spectral sequence collapses at E^1 and the claim follows. □

Having proved that $H_*^{E_n}(P_t)$ is acyclic we can use the axiomatic decription of Tor (see e.g. [29, ch. 2], [46, III.10]).

Proposition 4.27. *Let $E_i \colon \mathrm{Fun}(\mathrm{Epi}_n^+, k\text{-mod}) \to k\text{-mod}, i \geq 0$ be a sequence of functors with the following properties:*

1. *For every short exact sequence*

$$0 \longrightarrow F_1 \xrightarrow{f} F_2 \xrightarrow{g} F_3 \longrightarrow 0$$

of functors from Epi_n^+ to k-mod there are natural k-linear maps

$$\delta \colon E_{i+1}(F_3) \to E_i(F_1)$$

for all $i \geq 0$ such that the sequence

$$\ldots \xrightarrow{g} E_{i+1}(F_3) \xrightarrow{\delta} E_i(F_1) \xrightarrow{f} E_i(F_2) \xrightarrow{g} E_i(F_3) \xrightarrow{\delta} \ldots \xrightarrow{g} E_0(F_3) \longrightarrow 0$$

is exact.

2. *The k-modules $E_i(P)$ are zero for all $i \geq 1$ whenever $P \colon \mathrm{Epi}_n^+ \to k\text{-mod}$ is projective.*

3. *There exists $G \colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$ such that there is a natural isomorphism*

$$E_0 \cong G \otimes_{\mathrm{Epi}_n^+} -.$$

Then E is naturally isomorphic to $\mathrm{Tor}_*^{\mathrm{Epi}_n^+}(G, -)$.

Theorem 4.28. *Denote by $b \colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$ the functor given by the cokernel of*

$$(\delta_0, \mathrm{id}, ..., \mathrm{id})_* - (d_0, \mathrm{id}, ..., \mathrm{id})_* + (\delta_1, \mathrm{id}, ..., \mathrm{id})_* \colon P^{[1] \to [0] \to ... \to [0]} \to P^{[0] \to ... \to [0]}.$$

Then for any $F \colon \mathrm{Epi}_n^+ \to k\text{-mod}$

$$H_*^{E_n}(F) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n^+}(b, F),$$

and this isomorphism is natural in F.

Proof. We apply the axiomatic description of Tor. A short exact sequence of functors

$$0 \to F \to G \to H \to 0$$

yields a short exact sequence of chain complexes

$$0 \to C^{E_n}(F) \to C^{E_n}(G) \to C^{E_n}(H) \to 0$$

and therefore gives rise to the desired long exact sequence. We already showed that $H_*^{E_n}(P_t)$ is zero in positive degrees. Every projective functor from Epi_n^+ to k-mod receives a surjection from a sum of functors of the form of P_t and hence is a direct summand of this sum. Hence $H_*^{E_n}(P)$ vanishes in positive degrees for all projective functors P. Finally the zeroth E_n-homology of a functor F is given by the cokernel of

$$(-1)^{n-1} F(\delta_0, \mathrm{id}, ..., \mathrm{id}) + (-1)^n F(d_0, \mathrm{id}, ..., \mathrm{id}) + (-1)^{n+1} F(\delta_1, \mathrm{id}, ..., \mathrm{id}).$$

Using the natural isomorphism $P^t \otimes_{\mathrm{Epi}_n^+} F \cong F(t)$ of k-modules this is the cokernel of

$$\big((-1)^{n-1}(\delta_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^n (d_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^{n+1}(\delta_1, \mathrm{id}, ..., \mathrm{id})_*\big) \otimes_{\mathrm{Epi}_n^+} \mathrm{id}_F \colon$$

$$P^{[1]\to[0]\to\ldots\to[0]} \otimes_{\mathrm{Epi}_n^+} F \to P^{[0]\to\ldots\to[0]} \otimes_{\mathrm{Epi}_n^+} F.$$

But tensor products are right exact, and hence this cokernel coincides with the tensor product of F with the cokernel of

$$(-1)^{n-1}(\delta_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^n(d_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^{n+1}(\delta_1, \mathrm{id}, ..., \mathrm{id})_*,$$

hence with $b \otimes_{\mathrm{Epi}_n^+} F$. $\square$

Theorem 4.29. *Suppose that k is injective as a k-module and let $G\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$ be a functor. Then there is an isomorphism*

$$H^*_{E_n}(G) \cong \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(b, G).$$

This isomorphism is natural in G.

Proof. There is an axiomatic description of $\mathrm{Ext}_{\mathrm{Epi}_n^+}(b, -)$ which is analogous to the description of Tor in proposition 4.27, see [29, ch. 2], [46, III.10]. That $H^*_{E_n}$ maps short exact sequences to long exact sequences follows as in the homological case. Since the projective functor P_t is finitely generated and k-free, the functor P_t^* is injective. The universal coefficient spectral sequence 4.23 yields that these modules are acyclic. But then all other injective modules are acyclic, too, since they are direct summands of products of these. Finally let $G\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{-mod}$ be an arbitrary functor. Then the zeroth E_n-cohomology of G is by definition the kernel of

$$(-1)^{n-1}G(\delta_0, \mathrm{id}, ..., \mathrm{id}) + (-1)^n G(d_0, \mathrm{id}, ..., \mathrm{id}) + (-1)^{n+1}G(\delta_1, \mathrm{id}, ..., \mathrm{id}).$$

The Yoneda lemma identifies this with the kernel of

$$(-1)^{n-1}((\delta_0, \mathrm{id}, ..., \mathrm{id})_*)^* + (-1)^n((d_0, \mathrm{id}, ..., \mathrm{id})_*)^* + (-1)^{n+1}((\delta_1, \mathrm{id}, ..., \mathrm{id})_*)^*\colon$$

$$\mathrm{Nat}_{\mathrm{Epi}_n^+}(P^{[0]\to\ldots\to[0]}, G) \to \mathrm{Nat}_{\mathrm{Epi}_n^+}(P^{[1]\to[0]\to\ldots\to[0]}, G).$$

Since the functor $\mathrm{Nat}_{\mathrm{Epi}_n^+}(-, G)$ is left exact it takes finite colimits to limits, and in particular maps cokernels to kernels. Hence the above kernel is the result of applying $\mathrm{Nat}_{\mathrm{Epi}_n^+}(-, G)$ to the cokernel of

$$(-1)^{n-1}(\delta_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^n(d_0, \mathrm{id}, ..., \mathrm{id})_* + (-1)^{n+1}(\delta_1, \mathrm{id}, ..., \mathrm{id})_*,$$

i.e. to b. $\square$

4.4 Spectral sequences and examples

We need methods to calculate E_n-homology and -cohomology. First we record the following universal coefficient spectral sequence from [20, Proposition 13.2.2]

Proposition 4.30. *Let A be a commutative nonunital k-algebra and M a symmetric A-bimodule. There are spectral sequences*

$$E^2_{p,q} = \mathrm{Tor}_p^{A_+}(H_q^{E_n}(A;A_+), M) \Rightarrow H^{A_+}_{p+q}(A;M)$$

and

$$E_2^{p,q} = \mathrm{Ext}^p_{A_+}(H_q^{E_n}(A;A_+), M) \Rightarrow H^{p+q}_{A_+}(A;M).$$

There also is the following generalization of the spectral sequence in [41, Proposition 3.13] which allows to compute E_n-homology iteratively for even n.

Proposition 4.31. *For even n let $1 \leq j < n$. Let A be k-projective. If $H_*^{E_j}(A;A_+)$ is A_+-flat then there is a spectral sequence of A_+-modules with*

$$E^1_{p,q} = \bigoplus_{\substack{T \text{ an } n-j\text{-level tree with } s+1 \text{ leaves,}\\ d(T)=q+n-j}} (\Sigma^{sj}(H^{E_j}(A;A_+)^{\otimes_{A_+} s+1}))_p \Rightarrow H^{E_n}_{p+q}(A;A_+).$$

The differential d^1 is induced by $\partial_1 + ... + \partial_{n-j}$.

Proof. We can write $\mathrm{Tot}(C^{E_n}(A;A_+))$ as the total complex associated to the bicomplex

$$D_{a,b}(A;A_+) = \bigoplus_{a=r_n+..+r_{n-j+1}} \bigoplus_{b=r_{n-j}+...+r_1} C_{r_n,r_{n-1},...,r_1}(A;A_+)$$

with horizontal differential $\partial_{n-j+1} + ... + \partial_n$ and vertical differential $\partial_1 + ... + \partial_{n-j}$. Thus we get a spectral sequence with

$$E^1_{p,q} = \bigoplus_{q=r_{n-j}+...+r_1} H_p(C_{*,...,*,r_{n-j},...,r_1}(A;A_+), \partial_{n-j+1} + ... + \partial_n) \Rightarrow H^{E_n}_{p+q}(A;A_+).$$

Now observe that a n-level tree $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ is built from the $(n-j)$-level tree $T = [r_{n-j}] \xrightarrow{f_{n-j}} [r_{n-2}] \longrightarrow ... \longrightarrow [r_1]$ and the j-level trees $t_0, ..., t_{r_{n-j}}$ over the leaves of T. We will write $t = ((t_0, ..., t_{r_{n-j}}), T)$ to indicate this. Observe that applying $\partial_{n-j+1} + ... + \partial_n$ does not change the associated $(n-j)$-tree, i.e. we can decompose D into the subcomplexes

$$D_a^T = \bigoplus_{\substack{t=((t_0,...,t_{r_{n-j}}),T),\\ d(t_0)+...+d(t_{r_{n-j}})-j=a}} \mathcal{L}(A;A_+)(t)$$

as

$$D_{a,b} = \bigoplus_{d(T)-n+j=b} D_a^T.$$

To understand the grading, recall that elements in $\mathrm{Tot}(C^{E_n}(\mathcal{L}(A;A_+)))$ corresponding to a tree t have degree $d(t)-n$. Hence

$$E^1_{p,q} = \bigoplus_{\deg(T)=q} H_p(D^T, \partial_{n-j+1}+...+\partial_n).$$

To identify this last term consider

$$\mathrm{Tot}(C^{E_j}_*(A;A_+)) \otimes_{A_+} (\mathrm{Tot}(C^{E_j}_*(A;A_+)))^{\otimes_{A_+} r_{n-j}} \to (\Sigma^{-r_{n-j}j} D^T, \partial_{n-j+1}+...+\partial_n)$$

given by identifying the summand

$$\mathcal{L}(A;A_+)(t_0) \otimes_{A_+} \mathcal{L}(A;A_+)(t_1) \otimes_{A_+} ... \otimes_{A_+} \mathcal{L}(A;A_+)(t_{r_{n-j}})$$

of $((\mathrm{Tot}(C^{E_j}(A;A)))^{\otimes r_{n-j}+1})_{d(t_0)+...+d(t_{r_{n-j}})-r_{n-j}(j+1)}$ with

$$\mathcal{L}(A;A_+)(t) \subset D^T_{d(t_0)+...+d(t_{r_{n-j}})-j}.$$

To see that this is coherent with signs, we make the following comparison for $0 \le i \le r_{n-j}$ and $1 \le l \le j$: In $\mathcal{L}(A;A_+)(t) \subset C^{E_n}_*(\mathcal{L}(A;A_+))$ the signs associated to the summands of ∂_{n-j+l} applied to the j-level subtree t_i differ from the signs of the corresponding differential ∂_l on $\mathcal{L}(A;A_+)(t_i) \subset C^{E_j}_*(\mathcal{L}(A;A_+))$ by $(-1)^{n-j+\sum_{x=0}^{i-1}(d(t_x)+n-j)}$. On the other hand, we pick up a factor $(-1)^{\sum_{x=0}^{i-1}(d(t_x)-j)}$ if we want to apply ∂_l to the $(i+1)$th tensor factor of $\mathcal{L}(A;A_+)(t_0) \otimes_{A_+} \mathcal{L}(A;A_+)(t_1) \otimes_{A_+} ... \otimes_{A_+} \mathcal{L}(A;A_+)(t_{r_{n-j}})$. Since n is even, the difference in signs does not depend on t. Under the conditions stated in the theorem this yields the result. □

We use this result to compute E_2-homology and cohomology of a polynomial algebra. Hochschild homology of polynomial algebras is well known, see [43, 3.2].

Proposition 4.32. *The E_1-homology of $\overline{k[x]}$ with coefficients in itself is concentrated in degree zero, where it is*

$$H_0^{E_1}(\overline{k[x]}, k[x]) = k[x].$$

The following result agrees with the calculations in characteristic 0 and 2 in [57] and with the results for $k = \mathbb{F}_p$ in [10].

Proposition 4.33. *The E_2-homology of $\overline{k[x]}$ with coefficients in $k[x]$ is given by*

$$H_l^{E_2}(\overline{k[x]}, k[x]) = \begin{cases} k[x], & l \text{ even},\\ 0, & l \text{ odd}.\end{cases}$$

Proof. We find that for the spectral sequence 4.31 applied to $n = 2$ and $j = 1$

$$E^1_{p,q} = \bigoplus_{\substack{T \text{ a 1-level tree with } s+1 \text{ leaves,} \\ \deg(T)=q}} (\Sigma^s k[x])_p.$$

But if $\deg(T) = q$ then T has $q + 1$ leaves. Hence we see that

$$E^1_{p,q} = \begin{cases} k[x], & p = q \geq 0, \\ 0, p \neq q \end{cases}$$

and the spectral sequence collapses. □

Since $H^{E_2}_*(\overline{k[x]}, k[x])$ is $k[x]$-free, the universal coefficient spectral sequences allows more general calculations.

Corollary 4.34. *Let M be a symmetric $\overline{k[x]}$-module. Then*

$$H^{E_2}_l(\overline{k[x]}; M) = \begin{cases} M, & l \text{ even}, \\ 0, & l \text{ odd} \end{cases}$$

and similarly

$$H^l_{E_2}(\overline{k[x]}; M) = \begin{cases} M, & l \text{ even}, \\ 0, & l \text{ odd}. \end{cases}$$

Remark 4.35. *Let us exhibit explicit generating cycles in $C^{E_2}_*(\overline{k[x]}, k[x])$: Consider the fork tree $F_l = [l] \xrightarrow{\mathrm{id}} [l]$ and*

$$1 \otimes x^{\otimes l+1} \in \mathcal{L}(\overline{k[x]}, k[x])(F_l).$$

This is a cycle. An element in $\mathcal{L}(\overline{k[x]}, k[x])(F_l)$ can only be hit by the differential of an element in $\mathcal{L}(\overline{k[x]}, k[x])(\ [l+1] \xrightarrow{f_i} [l]\ )$ with

$$f_i(j) = \begin{cases} j, & j \leq i, \\ j - 1, & j > i. \end{cases}$$

A calculation shows that no such element gets mapped to $1 \otimes x^{\otimes l+1}$. Since we know that $H^{E_2}_{2l}(\overline{k[x]}, k[x])$ is $k[x]$-free, we see that the above cycle is a $k[x]$-generator. It follows that if we endow $H^{E_2}_*(\overline{k[x]}, k[x])$ with the shuffle product (see lemma 6.10) arising from the bar construction, it is the augmentation ideal of the shifted $k[x]$-algebra $k[x] \otimes \Sigma^{-2}\Gamma(y)$ with $|y| = 2$.

5 Functor cohomology and cohomology operations

We recall the definition of the Yoneda pairing on Ext. The Yoneda pairing is usually defined in the context of modules over a ring (see e.g. [46, III.5, III.6]), but is well known to be easily generalized to suitable abelian categories with enough projectives and injectives. Since we are interested in E_n-cohomology we assume that k is k-injective in this section.

Definition 5.1. *Let F, G and H be functors from $\mathrm{Epi}_n^{+\mathrm{op}}$ to k-mod. Let P_F denote a projective resolution of F and I_H an injective resolution of H. There is a pairing*

$$\mu\colon \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(G,H) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(F,G) \to \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(F,H),$$

defined as the composite

$$\begin{tikzcd}
\mathrm{Ext}^m_{\mathrm{Epi}_n^+}(G,H) \otimes \mathrm{Ext}^n_{\mathrm{Epi}_n^+}(F,G) \arrow[d, equal] \\
H_m(\mathrm{Nat}_{\mathrm{Epi}_n^+}(G,I_H)) \otimes H_n(\mathrm{Nat}_{\mathrm{Epi}_n^+}(P_F,G)) \arrow[d] \\
H_{n+m}(\mathrm{Nat}_{\mathrm{Epi}_n^+}(G,I_H) \otimes \mathrm{Nat}_{\mathrm{Epi}_n^+}(P_F,G)) \arrow[d] \\
H_{n+m}(\mathrm{Nat}_{\mathrm{Epi}_n^+}(P_F,I_H)) = \mathrm{Ext}^{n+m}_{\mathrm{Epi}_n^+}(F,H).
\end{tikzcd}$$

Here the second map is induced by composing natural transformations. This pairing is associative, i.e. the diagram

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathrm{Epi}_n^+}(G,H) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(F,G) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(E,F) \arrow[r, "\mu\otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(E,F)"] \arrow[d, "\mathrm{Ext}^*_{\mathrm{Epi}_n^+}(G,H)\otimes\mu"] & \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(F,H) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(E,F) \arrow[d, "\mu"] \\
\mathrm{Ext}^*_{\mathrm{Epi}_n^+}(G,H) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(E,G) \arrow[r, "\mu"] & \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(E,H)
\end{tikzcd}$$

commutes. The pairing is called the Yoneda pairing.

Example 5.2. *Similarly one can define a Yoneda pairing for the* Ext*-groups associated to modules over a unital ring R. In particular, if R is a projective k-algebra, we can define the cup product on Hochschild cohomology via the Yoneda product. Recall that since R is k-projective we can calculate its Hochschild cohomology as*

$$HH^*(R;R) = \mathrm{Ext}^*_{R\otimes R^{\mathrm{op}}}(R;R)$$

with R^{op} the opposite of R. The usual cup product on Hochschild cohomology and the Yoneda product coincide ([62]).

In particular there is a natural action of $\mathrm{Ext}^*_{\mathrm{Epi}_n^+}(b,b) = H^*_{E_n}(b)$ on E_n-cohomology. To identify $H^*_{E_n}(b)$ we first examine b and its dual b^*. For the remainder of this section we will denote $b\colon \mathrm{Epi}_n^+ \to k\text{-mod}$ by b_n since we will have to consider trees of varying levels.

Proposition 5.3. *The representing functor b_n can be identified with the functor mapping a tree $t = [r_n] \xrightarrow{f_n} \ldots \xrightarrow{f_2} [r_1]$ to the free k-module $k\langle [r_n]\rangle$ generated by the set $[r_n]$. Denote the generators of $k\langle[r_n]\rangle$ by $\overline{0}, \ldots, \overline{r_n}$. Then with respect to this identification the functor b_n induces the following morphisms:*

$$b_n(\tau_n, \ldots, \tau_{j+1}, d_i, \mathrm{id}, \ldots, \mathrm{id})\colon k\langle[r_n]\rangle \to k\langle[r_n]\rangle, \qquad \overline{m} \mapsto \overline{\tau_n^{-1}(m)}$$
$$\text{for permutations } \tau_{j+1}, \ldots, \tau_n,$$

$$b_n(d_i, \mathrm{id}, \ldots, \mathrm{id})\colon k\langle[r_n]\rangle \to k\langle[r_n+1]\rangle, \qquad \overline{m} \mapsto \begin{cases} \overline{m}, & m < i \\ \overline{m} + \overline{m+1}, & m = i, , \\ \overline{m+1}, & m > i, \end{cases}$$

$$b_n(\delta_i, \mathrm{id}, \ldots, \mathrm{id})\colon k\langle[r_n]\rangle \to k\langle[r_n+1]\rangle, \qquad \overline{m} \mapsto \begin{cases} \overline{m}, & m < i, \\ \overline{m+1}, & m \geq i. \end{cases}$$

Proof. The k-module $b_n(t)$ is generated by elements that can be represented by a map from t to $[0] \to \ldots. \to [0]$. Such morphisms are completely determined by which interval in $[r_n]$ is mapped to $0 \in [0]$. In the same manner a morphism from t to $[1] \to [0] \to \ldots \to [0]$ can be identified with two disjoint intervals in $[r_n]$, and such a pair of subsets (A, B) is sent to $A - A \cup B + B$ by the map

$$(\delta_0, \mathrm{id}, \ldots, \mathrm{id})_* - (d_0, \mathrm{id}, \ldots, \mathrm{id})_* + (\delta_1, \mathrm{id}, \ldots, \mathrm{id})_*$$

of which b_n is the cokernel. In particular, if $M = \{m_1 < \ldots < m_l\} \subset [r_n]$ represents a morphism to the tree with a single leaf and if $l > 1$, there is a morphism to the palm tree $[1] \longrightarrow [0] \longrightarrow \ldots \longrightarrow [0]$ with two leaves represented by $(\{m_1\}, \{m_2, \ldots, m_l\})$. Hence M and the formal sum $\{m_1\} + \{m_2, \ldots, m_l\}$ coincide in $b_n(t)$. Iterating this we see that M is equivalent to $\{m_1\} + \ldots + \{m_l\}$. On the other hand every singleton $\{m\}$ for $0 \leq m \leq r_n$ represents the following map from t to the tree with one leaf:

$$\begin{tikzcd}
{[r_n]} \arrow[r] \arrow[d, "m \neq x \mapsto +, m \mapsto 0"'] & {[r_{n-1}]} \arrow[r] \arrow[d, "x \mapsto 0"'] & \ldots \arrow[r] & {[r_1]} \arrow[d, "x \mapsto 0"'] \\
{[0]} \arrow[r] & {[0]} \arrow[r] & \ldots \arrow[r] & {[0]}
\end{tikzcd}$$

Hence all sets $\{m\}$ do indeed represent a morphism. Since all of the imposed relations in the cokernel consist of splitting up a set into two subsets, $b_n(t)$ can be identified with $k\langle[r_n]\rangle$, with $\overline{m}$ representing the morphism above for $m \in [r_n]$. Using this the induced maps are easily calculated to be the ones above. For example, precomposing the morphism represented by $\overline{m}$ with $(\delta_i, \mathrm{id}, ..., \mathrm{id})$ and evaluating this on $x \in [r_n]$ yields

$$\begin{aligned}
\overline{m} \circ (\delta_i, \mathrm{id}, ..., \mathrm{id})(x) &= \begin{cases} \overline{m}(x), & x < i \\ \overline{m}(+), & x = i, \\ \overline{m}(x-1), & x > i. \end{cases} \\
&= \begin{cases} +, & x < i \text{ and } m \geq i, \\ +, & x < i \text{ and } m < i \text{ and } x \neq m, \\ 0, & x < i \text{ and } m < i \text{ and } x = m, \\ +, & x = i, \\ +, & x > i \text{ and } m \geq i \text{ and } x - 1 \neq m, \\ 0, & x > i \text{ and } m \geq i \text{ and } x - 1 = m, \\ +, & x > i \text{ and } m < i. \end{cases} \\
&= \begin{cases} \overline{m}(x), & m < i, \\ \overline{m+1}(x), & m \geq i. \end{cases}
\end{aligned}$$

$\square$

Since we are going to work homologically we determine b_n^* as well.

Corollary 5.4. *The dual b_n^* of b_n assigns $k\langle[r_n]\rangle$ to the tree $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$. Denoting the generators of $k\langle[r_n]\rangle$ by $\alpha_0, ..., \alpha_{r_n}$, it induces the maps*

$$b_n^*(\tau_n, ..., \tau_{j+1}, d_i, \mathrm{id}, ..., \mathrm{id})\colon k\langle[r_n]\rangle \to k\langle[r_n]\rangle, \qquad \alpha_m \mapsto \alpha_{\tau_n^{-1}(m)}$$
for permutations $\tau_{j+1}, ..., \tau_n$,

$$b_n^*(d_i, \mathrm{id}, ..., \mathrm{id})_*\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \qquad \alpha_m \mapsto \begin{cases} \alpha_m, & m \leq i, \\ \alpha_{m-1}, & m > i, \end{cases}$$

$$b_n^*(\delta_i, \mathrm{id}, ..., \mathrm{id})_*\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \qquad \alpha_m \mapsto \begin{cases} \alpha_m, & m < i, \\ 0, & m = i, \\ \alpha_{m-1}, & m > i. \end{cases}$$

We start with calculating the E_1-cohomology of b_1 and then deduce the general result from this calculation.

Proposition 5.5. *For $n = 1$ we have*

$$H^r_{E_1}(b_1) \cong H^{E_1}_r(b_1^*) = 0$$

for $r > 0$ and

$$H^0_{E_1}(b_1) \cong H^{E_1}_0(b_1^*) = k.$$

Proof. Since the complex calculating $H^{E_1}_*(b_1^*)$ is k-free, the result for cohomology follows from the homological result via the universal coefficient spectral sequence 4.23. Let us determine the differentials of the homological chain complex $C^{E_1}_*(b_1^*)$: The rth differential is

$$d^{(r)} := \delta_0 - d_0 + ... + (-1)^{r+1} d_r + (-1)^{r+2}\delta_{r+1} \colon r\langle \alpha_0, ..., \alpha_{r+1}\rangle \to r\langle \alpha_0, ..., \alpha_r\rangle.$$

For $m \in [r+1]$ this yields

$$\begin{aligned}
& d^{(r)}(\alpha_m) \\
= \;& \delta_0(\alpha_m) - \sum_{i=0}^{m-1}(-1)^i \alpha_{m-1} - \sum_{i=m}^{r}(-1)^i\alpha_m + (-1)^r\delta_{r+1}(\alpha_m) \\
= \;& \begin{cases} -\sum_{i=0}^r(-1)^i\alpha_0 + (-1)^r\alpha_0, & m = 0, \\ \alpha_{m-1} - \sum_{i=0}^{m-1}(-1)^i\alpha_{m-1} - \sum_{i=m}^r(-1)^i\alpha_m + (-1)^r\alpha_m, & 0 < m < r+1, \\ \alpha_r - \sum_{i=0}^r(-1)^i\alpha_k, & m = r+1, \end{cases} \\
= \;& \begin{cases} -\delta_{r,\mathrm{even}}\alpha_0 + (-1)^r\alpha_0, & m = 0, \\ \alpha_{m-1} - \delta_{m-1,\mathrm{even}}\alpha_{m-1} - \delta_{r-m,\mathrm{even}}(-1)^r\alpha_m + (-1)^r\alpha_m, & 0 < m < r+1, \\ \alpha_r - \delta_{r,\mathrm{even}}\alpha_r, & m = r+1. \end{cases}
\end{aligned}$$

Hence for r even we get

$$d^{(r)}(\alpha_m) = \begin{cases} 0, & m = 0, \\ \alpha_{m-1}, & 0 < m < r+1, m \text{ even}, \\ \alpha_m, & 0 < m < r+1, m \text{ odd}, \\ 0, & m = r+1, \end{cases}$$

whereas for r odd we get

$$d^{(r)}(\alpha_m) = \begin{cases} -\alpha_0, & m = 0, \\ \alpha_{m-1} - \alpha_m, & 0 < m < r+1, m \text{ even}, \\ 0, & 0 < m < r+1, m \text{ odd}, \\ \alpha_k, & m = r+1. \end{cases}$$

Accordingly the kernel of $d^{(2l)}$ is generated by α_0, α_{2l+1} and elements of the form $\alpha_{2j-1} - \alpha_{2j}, j = 1, ..., l$, which is exactly the image of $d^{(2l+1)}$. On the other hand, the image of $d^{(2l+2)}$ is generated by those α_m with $m \in [2l+2]$ odd, while $\sum_{i=0}^{2l+1} \lambda_i \alpha_i$ is an element of the kernel of $d^{(2l+1)}$ if and only if $\lambda_{2m} = 0$ for all m. Hence the complex in question is acyclic, with $\mathrm{Im}(d^{(0)}) = 0$ and hence $H_0(C_*^{E_1}) = b_1^*([0]) = k$ as claimed. $\square$

To prove the result for $n > 1$ we first show that $H_*(C_{(*,r_{n-1},...,r_1)}(b_n^*), \partial_n)$ vanishes whenever $r_{n-1} \geq 1$. For this we need the following lemma.

Lemma 5.6. *Let $F\colon \mathrm{Epi}_n^+ \to k$-mod be a functor and $r_1, ..., r_{n-1} \geq 0$. Then*

$$\Sigma^{-r_1-...-r_{n-1}}(C^{E_n}_{(*,r_{n-1},...,r_1)}(F), \partial_n)$$

is isomorphic to the total complex associated to the r_{n-1}-fold multicomplex

$$D_{a_0,...,a_{r_{n-1}}}(F) = \bigoplus_{\substack{t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1], \\ |f_n^{-1}(0)| = a_0+1, |f_n^{-1}(i)| = a_i}} F(t)$$

with ith differential d^i the part of ∂_n induced by morphisms operating on the fiber $f_n^{-1}(i)$. Furthermore we can split D into submulticomplexes corresponding to the underlying $(n-1)$-level tree, i.e.

$$D_{a_1,...,a_{r_{n-1}}}(F) = \bigoplus_{T = [r_{n-1}] \xrightarrow{f_{n-1}} ... \xrightarrow{f_2} [r_1]} (D^T_{a_1,...,a_{r_{n-1}}}, d^0, ..., d^{r_{n-1}})$$

with

$$D^T_{a_1,...,a_{r_{n-1}}} = \bigoplus_{\substack{t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1], \\ |f_n^{-1}(0)| = a_0+1, |f_n^{-1}(i)| = a_i}} F(t).$$

Proof. The differential ∂_n is the sum of the maps d_i acting on one of the fibers $f_n^{-1}(i)$. Two such differentials d_i and d_j commute except for their signs: Since d_i deletes an edge left of $f_n^{-1}(j)$ for $i < j$, we find that $d_i d_j = -d_j d_i$. Hence it is clear that up to a shift we can interpret $C^{E_n}_{(*,r_{n-1},...,r_1)}(F)$ as a total complex as above. Let us check that we chose the right shift of degrees: If $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ such that $|f_n^{-1}(0)| = a_0 + 1$ and $|f_n^{-1}(i)| = a_i$ for $i > 0$ then $r_n = a_0 + ... + a_{r_{n-1}}$. The summand $F(t)$ has total degree $r_1 + ... + r_n - (r_1 - ... - r_{n-1}) = r_n$ in $\Sigma^{-r_1-...-r_{n-1}} C^{E_n}_{(*,r_{n-1},...,r_1)}(F)$ and hence this degree and the total degree in D of elements in $F(t)$ coincide. Since all the differentials d^i leave the lower levels of a tree t as they were it is clear that the splitting above holds, allowing us to consider one $(n-1)$-tree shape at a time. $\square$

Theorem 5.7. *For all $n \geq 0$*

$$H_s^{E_n}(b_n^*) = \begin{cases} k, & s = 0, \\ 0, & s > 0. \end{cases}$$

Proof. Fix $r_{n-1} \geq 1, r_{n-2}, ..., r_1 \geq 0$. We will prove that $H_*(C^{E_n}_{(*,r_{n-1},...,r_1)}(F), \partial_n)$ vanishes. Let T be a $(n-1)$-level tree $T = [r_{n-1}] \xrightarrow{f_{n-1}} ... \xrightarrow{f_2} [r_1]$. Consider the corresponding summand D^T of the multicomplex $D(b_n^*)$. According to lemma 5.6 it suffices to show that the homology of the total complex associated to D^T is trivial for all trees T as above.

Let us start by calculating the homology of D^T in the zeroth direction, i.e. for each given $a_1, ..., a_{r_{n-1}} \geq 1$ we consider the complex

$$(D^T_{*,a_1,...,a_{r_{n-1}}}, d^0) = \Big(\bigoplus_{\substack{t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1], \\ |f_n^{-1}(0)| = *+1, |f_n^{-1}(i)| = a_i}} F(t), d^0 \Big).$$

Since we fixed T there is exactly one tree $t = [r_n] \xrightarrow{f_n} ... \xrightarrow{f_2} [r_1]$ with $|f_n^{-1}(0)| = p+1$ and $|f_n^{-1}(i)| = a_i$ for each p. Let $p + q = r_n$. The differential d^0 maps $\alpha_j \in b_n^*(t) = k\langle \alpha_0, ..., \alpha_{p+q} \rangle$ to

$$\begin{aligned} & (-1)^{n-1} b_n^*(\delta_0, \mathrm{id}, ..., \mathrm{id})(\alpha_j) \\ + \ & \sum_{i=0}^{p-1} (-1)^{n+i} b_n^*(d_i, \mathrm{id}, ..., \mathrm{id})(\alpha_j) \\ + \ & (-1)^{n+p} b_n^*(\delta_p, \mathrm{id}, ..., \mathrm{id})(\alpha_j). \end{aligned}$$

Applied to elements with $j \leq p$ this coincides up to a sign $(-1)^{n-1}$ with the image of $\alpha_j \in b_1^*([p])$ under the differential d^{HH} of $C_*^{E_1}(b_1^*)$ calculated in proposition 5.5. If $j > p$ all the induced morphisms are the identity. One easily checks that those add up to 0 if p is even and that α_j is sent to $(-1)^{n-1}\alpha_{j-1}$ if p is odd. Hence $(D^T_{*,a_1,...,a_{r_{n-1}}}, d^0)$ is isomorphic to

$$... \xrightarrow{d^{HH} \oplus 0} b_1^*([3]) \oplus k^q \xrightarrow{d^{HH} \oplus \mathrm{id}} b_1^*([2]) \oplus k^q \xrightarrow{d^{HH} \oplus 0} b_1^*([1]) \oplus k^q \xrightarrow{d^{HH} \oplus \mathrm{id}} b_1^*([0]) \oplus k^q$$

and $H_p(D^T_{*,a_1,...,a_{r_{n-1}}}, d^0)$ is concentrated in degree $p = 0$ where it is k. We showed in proposition 5.5 that $H_0^{E_1}(b_1^*) = b_1^*([0])$, hence a cycle in $H_0(D^T_{*,a_1,...,a_{r_{n-1}}}, d^0)$ is given by $\alpha_0 \in b_n^*(t^{0,a_1,...,a_{r_{n-1}}})$ where $t^{0,a_1,...,a_{r_{n-1}}}$ is the tree extending T with top fibers of arity $1, a_1, ..., a_{r_{n-1}}$.

We now determine how d^1 acts on these cycles. The differential d^1 is induced by morphisms acting on leaves in the second to left top fiber. All of these leave the leftmost leaf invariant and hence each of the induced maps sends α_0 to α_0. Hence for fixed $a_2, ..., a_{r_{n-1}} \geq 1$ the chain complex $(H_0(D^T_{*,*,a_2...,a_{r_{n-1}}}, d^0), d^1)$ is one-dimensional on the generator α_0 in each degree r with differential

$$\begin{aligned} & d^1(\alpha_0) \\ = \quad & (-1)^n(-1)^{n-1} b_n^*(\delta_1, \mathrm{id}, ..., \mathrm{id})(\alpha_0) + (-1)^n(-1)^n \sum_{i=1}^{r-1} (-1)^{i-1} b_n^*(d_i, \mathrm{id}, ..., \mathrm{id})(\alpha_0) \\ & + (-1)^n(-1)^{n+r-1} b_n^*(\delta_r, \mathrm{id}, ..., \mathrm{id})(\alpha_0) \\ = \quad & (-1)^{2n-1} \sum_{i=0}^{r} (-1)^i \alpha_0. \end{aligned}$$

Hence the homology of $(H_0(D^T_{*,*,a_2...,a_{r_{n-1}}}, d^0), d^1)$ vanishes completely and the homology of the total complex of D^T is zero. This holds for all trees $T = [r_{n-1}] \xrightarrow{f_{n-1}} ... \xrightarrow{f_2} [r_1]$ with $r_{n-1} \geq 1$. Hence $(C^{E_n}_{(*,r_{n-1},...,r_1)}(b_n^*), \partial_n)$ has trivial homology as well, whenever $r_{n-1} \geq 1$. If $r_{n-1} = 0$ this forces $r_{n-2}, ..., r_1 = 0$. But one easily sees that $(C^{E_n}_{(*,0,...,0)}(b_n^*), \partial_n)$ is isomorphic to $C^{E_1}_*(b_1^*)$, hence has homology concentrated in degree zero where it is k.
Since E_n-homology is the homology of the total complex associated to a multicomplex with $(C^{E_n}_{(*,r_{n-1},...,r_1)}(b_n^*), \partial_n)$ the complex in nth direction, a standard spectral sequence argument yields the result. $\square$

Corollary 5.8. *E_n-cohomology of b_n is trivial in positive degrees and equals k in degree zero. In particular no nontrivial cohomology operations arise on E_n-cohomology via the Yoneda pairing defined in 5.1.*

Remark 5.9. *One possible explanation for the vanishing of $H^*_{E_n}(b_n) = \mathrm{Ext}^*_{\mathrm{Epi}_n^+}(b_n, b_n)$ is that $b_n \colon \mathrm{Epi}_n^+ \to k\text{-mod}$ might be injective, but we were not able to verify this.*

6 Higher order Hochschild homology

Higher order Hochschild homology is a generalization of Hochschild homology based on the observation that Hochschild homology can be computed as the homotopy groups of a simplicial k-module obtained by evaluating the Loday functor on a sphere. In [41, 3.1] it is shown that E_n-homology of a commutative algebra coincides with higher order Hochschild homology of that algebra up to a shift in degree. In this section we establish that this comparison also holds for arbitrary coefficients and for cohomology.

6.1 Definition of higher order Hochschild homology

Higher order Hochschild homology has been introduced by Teimuraz Pirashvili in [52, 5]. Pirashvili defines higher order Hochschild homology $HH^{[n]}$ and proves that over the rationals $HH^{[n]}$ has a decomposition, called the Hodge decomposition, which generalizes the well known λ-decomposition for ordinary Hochschild homology (see e.g. [43, 4.5]). In [26] Grégory Ginot defines higher order Hochschild cohomology $HH_{[n]}$ and uses a geometric approach to study additional structures on $HH_{[n]}$ like Adams operations and a Lie bracket. Higher order Hochschild homology is also related to factorization homology, see [27].
Let A be a nonunital algebra and M a symmetric A-bimodule. It is well known (see e.g. [54, 1.5]) that Hochschild homology of $A_+ = A \oplus k$ with coefficients in M can be computed as

$$HH(A_+; M) = \pi_*(\mathcal{L}_+(A, M)(S^1))$$

where we need to consider an unreduced version $\mathcal{L}_+(A, M)$ of $\mathcal{L}(A, M)$ in order to get a functor from the category Δ_+ of finite pointed sets and order preserving pointed maps to k-modules. Indeed, if A is commutative, $\mathcal{L}_+(A, M)$ is a functor from finite pointed sets Fin_* to k-mod. This observation led to defining higher order Hochschild homology as the homotopy groups of $\mathcal{L}_+(A, M)$ evaluated on a higher dimensional sphere.
For the rest of this chapter we fix a commutative nonunital algebra A and a symmetric A-bimodule M. We first recall some basic facts about simplicial and cosimplicial k-modules and then give the definitions of higher order Hochschild homology and cohomology. Background on the following simplicial constructions can be found in [28, III.2].

Definition 6.1. *For a cosimplicial k-module $C\colon \Delta \to k\text{-mod}$ with coface maps δ_i and codegeneracies σ_i the associated normalized Moore cochain complex $N(C)$ is given by*

$$N(C)^l = \bigcap_{i=0}^{l-1} \ker(\sigma_i) \subset C^l$$

and differential

$$\sum_{i=0}^{l+1} (-1)^i \delta_i \colon N(C)^l \to N(C)^{l+1}.$$

Lemma 6.2 ([28, p.153, p.393])**.** *For any simplicial k-module K there is a natural isomorphism*

$$\pi_*(K) \cong H_*(N(K))$$

between the homotopy groups of K and the homology of the normalized Moore chain complex $N(K)$ associated to K. Similarly, the cohomotopy groups of a cosimplicial k-module C can be computed as

$$\pi^*(C) \cong H^*(N(C))$$

and this isomorphism is natural.

We need versions of $\mathcal{L}(A, M)$ and $\mathcal{L}^c(A, M)$ which are defined on Fin_*, hence we need to work with A_+ rather than with A as in definition 4.17.

Definition 6.3. *Consider $\underline{r}_+ = \{1, ..., r\} \sqcup \{+\}$ as a finite pointed set with basepoint $+$ for $r \geq 0$. The functors*

$$\mathcal{L}_+(A, M)\colon \mathrm{Fin}_* \to k\text{-mod} \quad \textit{and} \quad \mathcal{L}^c_+(A, M)\colon \mathrm{Fin}_*^{\mathrm{op}} \to k\text{-mod}$$

are defined on the skeleton r_+, $r \geq 0$ by setting

$$\mathcal{L}_+(A, M)(\underline{r}_+) = M \otimes A_+^{\otimes r}$$

and

$$\mathcal{L}^c(A, M)(\underline{r}_+) = \mathrm{Hom}_k(A_+^{\otimes r}, M).$$

A basepoint-preserving map $f\colon \underline{r}_+ \to \underline{s}_+$ induces maps given by

$$(\mathcal{L}_+(A, M)(f))(m \otimes a_1 \otimes ... \otimes a_r) = \left(m \cdot \prod_{i:f(i)=+} a_i\right) \otimes \left(\prod_{i:f(i)=1} a_i\right) \otimes ... \otimes \left(\prod_{i:f(i)=s} a_i\right)$$

and

$$(\mathcal{L}^c_+(A, M)(f))(g) = \left(a_1 \otimes ... \otimes a_r \mapsto \left(\prod_{i:f(i)=+} a_i\right) \cdot g(\prod_{i:f(i)=1} a_i, ..., \prod_{i:f(i)=s} a_i)\right)$$

where $g \in \mathrm{Hom}_k(A_+^{\otimes s}, M)$ and $\prod_\emptyset a_i = 1 \in A_+$.

Definition 6.4 ([52, 5.1])**.** *Denote by $\mathcal{L}_+(A, M)(S^n)\colon \Delta^{\mathrm{op}} \to k\text{-mod}$ the simplicial k-module obtained by composing a simplicial n-sphere S^n with $\mathcal{L}_+(A, M)$. The Hochschild homology of order n of A_+ with coefficients in M is*

$$HH_*^{[n]}(A_+; M) = \pi_*(\mathcal{L}_+(A, M)(S^n)).$$

Similarly, let $\mathcal{L}_+^c(A,M)(S^n)\colon \Delta \to k\text{-mod}$ be the cosimplicial k-module obtained by composing a simplicial n-sphere with the contravariant functor $\mathcal{L}_+^c(A,M)$. The Hochschild cohomology of order n is given by

$$HH^*_{[n]}(A_+;M) = \pi_*(\mathcal{L}_+^c(A,M)(S^n)).$$

Remark 6.5. *The higher order Hochschild homology and cohomology groups are independent of the choice of a concrete model for S^n, see [52], [26].*

In the rest of this section we will prove a generalization of the following result.

Proposition 6.6 ([41, 3.1])**.** *For a k-projective nonunital k-algebra A and trivial coefficients E_n-homology coincides with higher order Hochschild homology up to a degree shift, i.e.*

$$H_*^{E_n}(A;k) \cong HH_{*+n}^{[n]}(A_+;k)$$

for $* \geq 0$.

6.2 Simplicial commutative algebras

Denote by sS the category of simplicial sets and by sS_* the category of pointed simplicial sets. The category sS is a simplicial model category by [55, II.3]. Hence sS_* is what one might call a pointed simplicial model category, see e.g. [36, Proposition 4.2.9]. In particular there is an operation

$$\wedge\colon \mathrm{sS}_* \times \mathrm{sS}_* \to \mathrm{sS}_*, \quad (K,L) \mapsto K \wedge L$$

and an internal hom object

$$\mathrm{Hom}_{\underline{sS_*}}(K,L)_i = \mathrm{Hom}_{\mathrm{sS}_*}(K \rtimes \Delta_i, L),$$

as well as an adjunction

$$\mathrm{Hom}_{\underline{sS_*}}(K \wedge L, M) \cong \mathrm{Hom}_{\underline{sS_*}}(K, \mathrm{Hom}_{\underline{sS_*}}(L,M)).$$

Here Δ_i is the simplicial i-simplex with $\Delta_i(a) = \Delta([a],[i])$ and for pointed simplicial sets K and L we set

$$K \rtimes L = K \times L / * \times L.$$

A corresponding structure is present for the category sAlg_k of simplicial commutative augmented k-algebras by [55, II.4]: We define an action of sS_* on sAlg_k by

$$\overline{\otimes}\colon \mathrm{sAlg}_k \times \mathrm{sS}_* \to \mathrm{sAlg}_k, \quad (X,K) \mapsto \bigotimes_{K_n \setminus *} X_n,$$

using that the tensor product is the coproduct in the category of commutative augmented k-algebras. We will also write $\overline{\otimes}_k$ for $\overline{\otimes}$ to indicate the ground ring we are working with. The simplicial mapping space $\mathrm{Hom}_{\underline{\mathrm{sAlg}_k}}$ is given by

$$\mathrm{Hom}_{\underline{\mathrm{sAlg}_k}}(X,Y)_i = \mathrm{Hom}_{\mathrm{sAlg}_k}(X \otimes \Delta_i, Y),$$

where $(X \otimes \Delta_i)_m = \bigotimes_{\alpha \in \Delta_i(m)} X_m$. The simplicial set $\mathrm{Hom}_{\underline{\mathrm{sAlg}_k}}(X,Y)$ is pointed with basepoints given by the composite $\eta\epsilon$ of the augmentation ϵ of $X \otimes \Delta_i$ and the unit η of Y. According to [36, 4.2.19] these definitions make sAlg_k a sS_*-model category, which in particular implies the following adjunction:

Proposition 6.7. *There is an adjunction*

$$\mathrm{Hom}_{\underline{\mathrm{sAlg}_k}}(X\overline{\otimes}K, Y) \cong \mathrm{Hom}_{\underline{sS_*}}(K, \mathrm{Hom}_{\underline{\mathrm{sAlg}_k}}(X,Y)).$$

We will also need the following result stating that sS_* indeed acts on sAlg_k, which also follows from [36, 4.2.19].

Proposition 6.8. *For any simplicial commutative augmented k-algebra X*

$$(X\overline{\otimes}_k K)\overline{\otimes}_k L \cong X\overline{\otimes}_k (K \wedge L)$$

as simplicial commutative augmented k-algebras.

6.3 Higher order Hochschild homology and cohomology coincides with E_n-homology and cohomology

We want to compare higher order Hochschild homology and cohomology with E_n-homology and cohomology. Our approach, especially the proof of proposition 6.13, is based on ideas contained in recent work by Bobkova-Lindenstrauss-Poirier-Richter-Zakharevich. In [10, Corollary 8.4] they establish a comparison of higher order Hochschild homology and an iterated bar construction in a simplicial setting.
For $X, Y \in \mathrm{sAlg}_k$ we denote by $X \otimes Y$ the bisimplicial augmented commutative algebra with

$$(X \otimes Y)_{a,b} = X_a \otimes Y_b.$$

For a bisimplical object Z let $d(Z)$ be its diagonal. Recall that the shuffle map

$$\nabla \colon N(X) \otimes N(Y) \to N(d(X \otimes Y))$$

is defined by

$$\nabla(x \otimes y) = \sum_{\sigma \in \mathrm{sh}(\{0,\dots,p-1\},\{p,\dots,p+q-1\})} \mathrm{sgn}(\sigma) s_{\sigma(p+q-1)} \dots s_{\sigma(p)} x \otimes s_{\sigma(p-1)} \dots s_{\sigma(0)} y$$

for $x \in N_p(X)$ and $y \in N_q(Y)$. The shuffle map makes N a lax symmetric monoidal functor from the category of simplicial k-modules to nonnegatively graded chain complexes. Hence N maps a simplicial augmented commutative k-algebra to a differential graded commutative augmented k-algebra. More precisely for $X \in \mathrm{sAlg}_k$ with product $\mu\colon d(X \otimes X) \to X$ we can make $N(X)$ into a differential graded commutative augmented k-algebra if we define the product as

$$N(X) \otimes N(X) \xrightarrow{\nabla} N(d(X \otimes X)) \xrightarrow{N(\mu)} N(X) ,$$

see [17, II.7], see also [56].
We will be using several variants of the bar construction in this subsection. Recall that the bar construction $B(M, A, N)$ can be defined in any monoidal category for a monoid A, a right module M over A and a left module N over A. The cases we will be interested in are the following:

1. We will use the bar construction $B(k, X, k)$ of a simplicial commutative augmented k-algebra X with coefficients in k. This is the bisimplicial commutative augmented k-algebra with
$$B_{i,j}(k, X, k) = X_j^{\otimes i}$$
with multiplication
$$X_j^{\otimes i} \otimes X_j^{\otimes i} \cong (X_j \otimes X_j)^{\otimes i} \to X_j^{\otimes i}$$
induced by the multiplication of X.

2. We will also need the bar construction $B(k, C_*, k)$ associated to a differential graded commutative augmented algebra C_*. This is the simplicial differential graded commutative augmented algebra with
$$B_i(k, C_*, k)_j = (C_*^{\otimes i})_j$$
with multiplication again induced by the multiplication of C_*. By $\mathrm{Tot}(N(B(k, C_*, k)))$ we denote the total complex of its normalization with respect to the simplicial structure, this is then again a differential graded commutative augmented k-algebra via the shuffle map.

3. We denote by $B(D_*)$ the reduced bar construction as defined in definition 3.14 associated to a nonunital differential graded commutative algebra D_*. The shuffle product makes $B(D_*)$ a (nonunital) differential graded commutative k-algebra. Note that $B(D_*)$ is the augmentation ideal of $\mathrm{Tot}(N(B(k, (D_*)_+, k)))$.

Hence (see also [41, 3.1]) for a commutative nonunital k-algebra A the n-fold bar construction $B^n(A)$ can be expressed via the simplicial bar construction as the augmentation ideal of

$$\mathrm{Tot}(N(B(k, N(B(k, ..., N(B(k, A_+, k)), ..., k)), k))),$$

which implies that

$$H_*^{E_n}(A;k) = H_*(\Sigma^{-n}\mathrm{Tot}(N(B(k, N(B(k, ..., N(B(k, A_+, k)), ..., k)), k))))$$

if A is k-projective. On the other hand,

$$HH_*^{[n]}(A_+;k) = H_*(N(\mathcal{L}_+(A,k)(S^n))).$$

We will compare both $N(B(k, N(B(k, ..., N(B(k, A_+, k)), ..., k)), k))$ and $N(\mathcal{L}_+(A,k)(S^n))$ with the normalized Moore complex

$$N(d(B(k, B(k, ...B(k, A_+, k)..., k), k)))$$

of the diagonal of the n-fold simplicial k-algebra $B(k, B(k, ...B(k, A_+, k), k), k))$. The key to this comparison is the following lemma.

Lemma 6.9. *Let X be an augmented simplicial commutative k-algebra. Then there is a quasiisomorphism*

$$\mathrm{Tot}(N(B(k, N(X), k))) \to N(d(B(k, X, k)))$$

of commutative differential graded augmented k-algebras.

Before we prove the lemma, let us recall the following correspondence between compositions and shuffles. A (p,q)-composition consists of two disjoint subsets P and Q of $\{0, ..., p+q-1\}$ with $|P| = p$ and $|Q| = q$. There is a bijection between the set of shuffles of $\{0, ..., p-1\}$ with $\{p, ..., p+q-1\}$ and the set $\mathrm{comp}(p,q)$ of (p,q)-compositions given by identifying $\sigma \in \mathrm{sh}(\{0, ..., p-1\}, \{p, ..., p+q-1\})$ with the composition (P,Q) with $P = \{\sigma(0), ..., \sigma(p-1)\}$ and $Q = \{\sigma(p), ..., \sigma(p+q-1)\}$. The signum of σ can be calculated from (P,Q) by noting that it picks up a factor -1 for every pair $x, y \in \{0, ..., p+q-1\}$ with $x < y$, but $x \in Q$ and $y \in P$. We set $\mathrm{sgn}(P,Q) = \mathrm{sgn}(\sigma)$.
Generalizing this we call pairwise disjoint subsets $A^{(1)}, ..., A^{(s)}$ of $\{0, ..., j_1 + ... + j_s - 1\}$ a $(j_1, ..., j_s)$-composition if $|A^{(i)}| = j_i$ for all $1 \le i \le s$. The signum of $(A^{(1)}, ..., A^{(s)})$ picks up a factor -1 for every pair $x, y \in \{0, ..., j_1 + ... + j_s - 1\}$ such that $x < y$ but $x \in A^{(a)}, y \in A^{(b)}$ with $b < a$. The set of $(j_1, ..., j_s)$-compositions can be identified with the set of shuffles of $\{0, ..., j_1 - 1\}$,...,$\{j_1 + ... + j_{s-1}, ..., j_1 + ... + j_s - 1\}$.
Using this we can write the shuffle map $\nabla \colon N(X) \otimes N(Y) \to N(d(X \otimes Y))$ as

$$\nabla(x \otimes y) = \sum_{(P,Q) \in \mathrm{comp}(p,q)} \mathrm{sgn}(\sigma) s_Q x \otimes s_P y$$

with $s_I = s_{i_t} ... s_{i_1}$ for a set $I = \{i_1 < ... < i_t\}$ of nonnegative integers.

Proof. Let Y be a bisimplicial commutative augmented algebra with horizontal and vertical degeneracies s'_i and s''_i and horizontal and vertical face maps d'_i and d''_i. Then the associated normalized total complex $\mathrm{Tot}(Y)$ is the total complex associated to the bicomplex $B_{*,*}$ given by

$$B_{p,q} = Y_{p,q}/Z$$

with $Z = \sum_{i=0}^{p-1} \mathrm{Im} s'_i + \sum_{i=0}^{q-1} \mathrm{Im} s''_i$ and differentials

$$d^h = \sum_{i=0}^{p} (-1)^p d'_i \quad \text{and} \quad d^v = \sum_{i=0}^{q} (-1)^q d''_i.$$

Consider the bisimplicial shuffle map $\nabla^{\mathrm{bi}} \colon \mathrm{Tot}(Y) \to N(d(Y))$ given by mapping $y \in Y_{p,q}$ to

$$\nabla^{bi}(y) = \sum_{(P,Q)\in \mathrm{comp}(p,q)} \mathrm{sgn}(P,Q) s'_Q s''_P(y).$$

Then ∇^{bi} is a quasiisomorphism, see e.g. [15, Satz 2.9]. On the other hand, the complex $\mathrm{Tot}(N(B(k, N(X), k)))$ is the total complex associated to the double complex

$$C_{p,*} = N(X)^{\otimes p} / \sum_{i=0}^{p-1} \mathrm{Im} s_i^B$$

with s_i^B the degeneracies of the bar construction $B(k, N(X), k)$ and differentials induced by the differential of $N(X)$ and by the face maps of $B(k, N(X), k)$. Now let $Y = B(k, X, k)$. For every $l \geq 0$ there is a quasiisomorphism

$$\nabla^{(l)} \colon N(X)^{\otimes l} \to N(d(X^{\otimes l}))$$

given by iterated application of $\nabla^X \colon N(X) \otimes N(X) \to N(d(X \otimes X))$. Since ∇^X is strictly associative (see [14, Exercise 12.2]) the concrete choices made for $\nabla^{(l)}$ do not matter and we see that

$$\nabla^{(l)}(x_1, ..., x_l) = \sum_{(A^{(1)},...,A^{(l)})\in \mathrm{comp}(j_1,...,j_l)} (s^X_{A^{(2)}\sqcup ...\sqcup A^{(l)}}(x_1), ..., s^X_{A^{(1)}\sqcup ...\sqcup A^{(l-1)}}(x_l))$$

for $(x_1, ..., x_l) \in X_{l_1} \otimes ... \otimes X_{j_l}$. Applying $\nabla^{(p)}$ to the pth column in the double complex $C_{*,*}$ yields a map from $C_{*,*}$ to $B_{*,*}$. Since this map induces a quasiisomorphism on each column it induces a quasiisomorphism $\mathrm{Tot}(C_{*,*}) \to \mathrm{Tot}(B_{*,*})$, see [59, Exercise 10.12]. Hence

$$\mathrm{Tot}(N(B(k, N(X), k))) \xrightarrow{\nabla^{(*)}} \mathrm{Tot}(Y) \xrightarrow{\nabla^{\mathrm{bi}}} N(d(B(k, X, k)))$$

is a quasiisomorphism.

To see that this quasiisomorphism respects products, observe that the shuffle map with respect to the simplicial structure of the bar construction

$$\nabla^B\colon N(B(k,N(X),k)) \otimes N(B(k,N(X),k)) \to N(d(B(k,N(X),k)^{\otimes 2}))$$

is a morphism of bicomplexes. We will denote the induced map $\mathrm{Tot}(N(B(k,N(X),k))) \otimes \mathrm{Tot}(N(B(k,N(X),k))) \to \mathrm{Tot}(N(d(B(k,N(X),k)^{\otimes 2})))$ by ∇^B as well. The product sh on $\mathrm{Tot}(N(B(k,N(X),k)))$ is given by

$$\begin{aligned}
\mathrm{Tot}(N(B(k,N(X),k))) \otimes \mathrm{Tot}(N(B(k,N(X),k))) &\xrightarrow{\nabla^B} \mathrm{Tot}(N(d(B(k,N(X),k)^{\otimes 2}))) \\
\xrightarrow{\quad\cong\quad} \mathrm{Tot}(N(B(k,N(X)^{\otimes 2},k))) &\xrightarrow{\mathrm{Tot}(N(B(k,\nabla^X,k)))} \mathrm{Tot}(N(B(k,N(d(X^{\otimes 2})),k))) \\
\xrightarrow{\mathrm{Tot}(N(k,N(\mu),k))} \mathrm{Tot}(N(B(k,N(X),k)))
\end{aligned}$$

with μ denoting the multiplication of X. The product on $N(d(B(k,X,k)))$ is defined via the shuffle map ∇^{diag} with respect to the diagonal simplicial structure of $d(B(k,X,k))$ and the bisimplicial commutative algebra structure

$$(X_q)^{\otimes p} \otimes (X_q)^{\otimes p} \cong (X_q \otimes X_q)^{\otimes p} \to (X_q)^{\otimes p}$$

on $B(k,X,k)$ induced by μ. We will denote the multiplication on $d(B(k,X,k))$ also by μ. We need to show that the diagram

$$\begin{tikzcd}
\begin{matrix}\mathrm{Tot}(N(B(k,N(X),k))) \\ \otimes\mathrm{Tot}(N(B(k,N(X),k)))\end{matrix} \arrow[r, "\nabla^{(*)}\otimes\nabla^{(*)}"] \arrow[dd, "\mathrm{sh}"] & \mathrm{Tot}(Y) \otimes \mathrm{Tot}(Y) \arrow[r, "\nabla^{bi}\otimes\nabla^{bi}"] & N(d(Y)) \otimes N(d(Y)) \arrow[d, "\nabla^{\mathrm{diag}}"] \\
 & & N(d(d(Y) \otimes d(Y))) \arrow[d, "N(\mu)"] \\
\mathrm{Tot}(N(B(k,N(X),k))) \arrow[r, "\nabla^{(*)}"] & \mathrm{Tot}(Y) \arrow[r, "\nabla^{bi}"] & N(d(Y))
\end{tikzcd}$$

commutes. Let $d_\otimes(Y \otimes Y)$ be the diagonal bisimplicial set obtained by taking the diagonal with respect to the tensor product, i.e.

$$d_\otimes(Y \otimes Y)_{pq} = Y_{pq} \otimes Y_{pq}.$$

Since $\mu\colon d_\otimes(Y\otimes Y)\to Y$ is a morphism of bisimplicial sets the diagram

$$\begin{tikzcd}
\mathrm{Tot}(N(B(k,N(d(X^{\otimes 2})),k))) \arrow[r, "\nabla^{(*)}"] \arrow[d, "\mathrm{Tot}(N(B(k,N(\mu),k)))"] & \mathrm{Tot}(d_\otimes(Y\otimes Y)) \arrow[r, "\nabla^{bi}"] \arrow[d, "\mathrm{Tot}(\mu)"] & N(d(d_\otimes(Y\otimes Y))) \arrow[d, "N(\mu)"] \\
\mathrm{Tot}(N(B(k,N(X),k))) \arrow[r, "\nabla^{(*)}"] & \mathrm{Tot}(Y) \arrow[r, "\nabla^{bi}"] & N(d(Y))
\end{tikzcd}$$

commutes due to naturality of the occuring shuffle maps. Hence it suffices to show that the diagram

$$\begin{tikzcd}
\begin{matrix}\mathrm{Tot}(N(B(k,N(X),k)))\\ \otimes\mathrm{Tot}(N(B(k,N(X),k)))\end{matrix} \arrow[r, "\nabla^{(*)}\otimes\nabla^{(*)}"] \arrow[d, "\nabla^{B}"] & \mathrm{Tot}(Y)\otimes\mathrm{Tot}(Y) \arrow[r, "\nabla^{bi}\otimes\nabla^{bi}"] & N(d(Y))\otimes N(d(Y)) \arrow[d, "\nabla^{\mathrm{diag}}"] \\
\mathrm{Tot}(N(d(B(k,N(X),k)^{\otimes 2}))) \arrow[d, "\cong"] & & N(d(d(Y)\otimes d(Y))) \arrow[ddd, equal] \\
\mathrm{Tot}(N(B(k,N(X)^{\otimes 2},k))) \arrow[d, "\mathrm{Tot}(N(B(k,\nabla^{X},k)))"] & & \\
\mathrm{Tot}(N(B(k,N(d(X^{\otimes 2})),k))) \arrow[r, "\nabla^{(*)}"] & \mathrm{Tot}(d_\otimes(Y\otimes Y)) \arrow[r, "\nabla^{bi}"] & N(d(d_\otimes(Y\otimes Y)))
\end{tikzcd}$$

commutes. We first show that

$$\begin{tikzcd}
(X_r)^{\otimes a}\otimes(X_s)^{\otimes b} \arrow[r, "\nabla^{bi}\otimes\nabla^{bi}"] \arrow[d, "\nabla^{B,\mathrm{Tot}}"] & (X_{a+r})^{\otimes a+r}\otimes(X_{b+s})^{\otimes b+s} \arrow[dd, "\nabla^{\mathrm{diag}}"] \\
(X_r\otimes X_s)^{\otimes a+b} \arrow[d, "\nabla^{X}"] & \\
(X_{r+s}^{\otimes a+b})^{\otimes 2} \arrow[r, "\nabla^{bi}"] & (X_{a+b+r+s}^{\otimes a+b+r+s})^{\otimes 2}
\end{tikzcd}$$

commutes for all $a,b,r,s\geq 0$, where for $(x_1,...,x_{a+b})\in(X_r)^{\otimes a}\otimes(X_s)^{\otimes b}$ the map $\nabla^{B,\mathrm{Tot}}$ is defined by

$$\nabla^{B,\mathrm{Tot}}(x_1,...,x_a,x_{a+1},...,x_{a+b})=\sum_{\sigma\in\mathrm{sh}(a,b)}\mathrm{sgn}(\sigma)\sigma.(x_1\otimes 1_s,...,x_a\otimes 1_s,1_r\otimes x_{a+1},...,1_r\otimes x_{a+b})$$

with 1_r the unit element of X_r, 1_s the unit of X_s and the usual left action by the symmetric group given by $\sigma.(y_1,...,y_{a+b})=(y_{\sigma^{-1}(1)},...,y_{\sigma^{-1}(a+b)})$. A quick calculation shows that $\nabla^X\nabla^{B,\mathrm{Tot}}$ defines a map

$$\mathrm{Tot}(Y)\otimes\mathrm{Tot}(Y)\to\mathrm{Tot}(d_\otimes(Y\otimes Y)).$$

We denote the degeneracies originating from the bar construction $B(k, X, k)$ by s^B and the degeneracies of X by s^X. The maps in the diagram are

$$\begin{aligned}
& \nabla^{\mathrm{diag}}(\nabla^{bi} \otimes \nabla^{bi})(x_1, ..., x_a, x_{a+1}, ..., x_{a+b}) \\
= \quad & \nabla^{\mathrm{diag}} \left(\sum_{\substack{(A,R) \in \mathrm{comp}(a,r) \\ (B,S) \in \mathrm{comp}(b,s)}} \pm s_A^X s_R^B(x_1, ..., x_a) \otimes s_B^X s_S^B(x_{a+1}, ..., x_{a+b}) \right) \\
= \quad & \sum_{\substack{(U,V) \in \mathrm{comp}(a+r,b+s) \\ (A,R) \in \mathrm{comp}(a,r) \\ (B,S) \in \mathrm{comp}(b,s)}} \pm s_V^X s_V^B s_A^X s_R^B(x_1, ..., x_a) \otimes s_U^X s_U^B s_B^X s_S^B(x_{a+1}, ..., x_{a+b})
\end{aligned}$$

and

$$\begin{aligned}
& \nabla^{bi} \nabla^X \nabla^{B,\mathrm{Tot}}(x_1, ..., x_a, x_{a+1}, ..., x_{a+b}) \\
= \quad & \nabla^{bi} \left(\sum_{\substack{(A,B) \in \mathrm{comp}(a,b) \\ (R,S) \in \mathrm{comp}(r,s)}} \pm s_S^X s_B^B(x_1, ..., x_a) \otimes s_R^X s_A^B(x_{a+1}, ..., x_{a+b}) \right) \\
= \quad & \sum_{\substack{(I,J) \in \mathrm{comp}(a+b,r+s) \\ (A,B) \in \mathrm{comp}(a,b) \\ (R,S) \in \mathrm{comp}(r,s)}} \pm s_I^X s_J^B (s_S^X s_B^B(x_1, ..., x_a) \otimes s_R^X s_A^B(x_{a+1}, ..., x_{a+b})).
\end{aligned}$$

Here the summands of the first map are decorated by

$$\mathrm{sgn}(U, V)\mathrm{sgn}(A, R)\mathrm{sgn}(B, S),$$

while those of the second map carry the sign

$$\mathrm{sgn}(I, J)\mathrm{sgn}(A, B)\mathrm{sgn}(R, S).$$

The degeneracies originating from the bar construction and from X commute, hence the two maps above are given by

$$\sum_{\substack{(U,V) \in \mathrm{comp}(a+r,b+s) \\ (A,R) \in \mathrm{comp}(a,r) \\ (B,S) \in \mathrm{comp}(b,s)}} \pm s_V^X s_A^X s_V^B s_R^B(x_1, ..., x_a) \otimes s_U^X s_B^X s_U^B s_S^B(x_{a+1}, ..., x_{a+b})$$

and

$$\sum_{\substack{(I,J) \in \mathrm{comp}(a+b,r+s) \\ (A,B) \in \mathrm{comp}(a,b) \\ (R,S) \in \mathrm{comp}(r,s)}} \pm s_I^X s_S^X s_J^B s_B^B(x_1, ..., x_a) \otimes s_I^X s_R^X s_J^B s_A^B(x_{a+1}, ..., x_{a+b}).$$

We claim that both of these terms equal

$$\sum_{(A,R,B,S)\in\mathrm{comp}(a,r,b,s)} \pm s^X_{A\sqcup B\sqcup S} s^B_{B\sqcup R\sqcup S}(x_1,...,x_a) \otimes s^X_{A\sqcup B\sqcup R} s^B_{A\sqcup R\sqcup S}(x_{a+1},...,x_{a+b}).$$

Since the arguments are analogous we will only consider the first map in question. Note that

$$\mathrm{comp}(a+r,b+s)\times\mathrm{comp}(a,r)\times\mathrm{comp}(b,s)\cong\mathrm{comp}(a,r,b,s).$$

The bijection is given concretely by assigning to compositions

$$(U = u_1 < ... < u_{a+r}, V = v_1 < ... < v_{b+s}),$$

$$(A = a_1 < ... < a_a, R = r_1 < ... < r_r),$$

$$(B = b_1 < ... < b_b, S = s_1 < ... < s_s)$$

the composition

$$(\tilde{A} = u_{a_1} < ... < u_{a_a}, \tilde{R} = u_{r_1} < ... < u_{r_r}, \tilde{B} = v_{b_1} < ... < v_{b_b}, \tilde{S} = v_{s_1} < ... < v_{s_s}).$$

Observe that $\mathrm{sgn}(U,V)\mathrm{sgn}(A,R)\mathrm{sgn}(B,S) = \mathrm{sgn}(\tilde{A},\tilde{R},\tilde{B},\tilde{S})$. We need to show that

$$s^X_V s^X_A = s^X_{\tilde{A}\sqcup\tilde{B}\sqcup\tilde{S}}$$

and that similar identities hold for the other degeneracies in question. Let

$$\alpha_V(a_j) = |\{i | v_i \leq a_j + i - 1\}|,$$

i.e. $\alpha_V(a_j)$ equals the number of degeneracies the degeneracy s_{a_j} has to be switched with in order to write $s^X_V s^X_A$ as a product $s_{t_{a+b+s}}...s_{t_1}$ with $t_1 < ... < t_{a+b+s}$. Observe that $a_j + \alpha_V(a_j) \notin V$. Hence

$$\begin{aligned} \{t_1 < ... < t_{a+b+s}\} &= \{v_1,...,v_r, a_1+\alpha_V(a_1),...,a_a+\alpha_V(a_a)\} \\ &= \tilde{B}\sqcup\tilde{S}\sqcup\{a_1+\alpha_V(a_1),...,a_a+\alpha_V(a_a)\} \end{aligned}$$

Since $a_i + \alpha_V(a_i) \neq r_j + \alpha_V(r_j)$ for all i,j the same argument also yields that

$$U = \{a_1+\alpha_V(a_1),...,a_a+\alpha_V(a_a), r_1+\alpha_V(r_1),...,r_r+\alpha(r_r)\}.$$

Since the map $\underline{a+r} \to U, x \mapsto x + \alpha_V(x)$ is clearly order-preserving, it coincides with $\underline{a+r} \to U, y \mapsto u_y$. Therefore $a_j + \alpha_V(a_j) = u_{a_j}$ and hence

$$s^X_V s^X_A = s^X_{\tilde{A}\sqcup\tilde{B}\sqcup\tilde{S}}.$$

Repeated application of this argument yields the desired commutativity of the diagram above.
Finally we consider the diagram

$$\begin{tikzcd}
(X_{r_1} \otimes ... \otimes X_{r_l}) \otimes (X_{r_{l+1}} \otimes ... \otimes X_{r_{l+k}}) \arrow[r, "\nabla^{(l)} \otimes \nabla^{(k)}"] \arrow[d, "\nabla^B"] & (X_{r_1+...+r_l})^{\otimes l} \otimes (X_{r_{l+1}+...+r_{l+k}})^{\otimes k} \arrow[d, "\nabla^{B,\mathrm{Tot}}"] \\
\begin{array}{c} \bigoplus_{\sigma \in \mathrm{sh}(l,k)} \sigma.(X_{r_1} \otimes X_0, ..., X_{r_l} \otimes X_0, \\ X_0 \otimes X_{r_{l+1}}, ..., X_0 \otimes X_{r_{l+k}}) \end{array} \arrow[d, "N(B(k,\nabla^X,k))"] & (X_{r_1+...+r_l} \otimes X_{r_{l+1}+...+r_{l+k}})^{\otimes l+k} \arrow[d, "\nabla^X"] \\
\bigoplus_{\sigma \in \mathrm{sh}(l,k)} \sigma.(X_{r_1}^{\otimes 2}, ..., X_{r_l}^{\otimes 2}, X_{r_{l+1}}^{\otimes 2}, ..., X_{r_{l+k}}^{\otimes 2}) \arrow[r, "\nabla^{(l+k)}"] & (X_{r_1+...+r_{l+k}}^{\otimes l+k})^{\otimes 2}
\end{tikzcd}$$

Fix an (l,k)-shuffle σ corresponding to $(L,K) \in \mathrm{comp}(l,k)$. We can restrict our attention to showing that the diagram

$$\begin{tikzcd}
(X_{r_1} \otimes ... \otimes X_{r_l}) \otimes (X_{r_{l+1}} \otimes ....X_{r_{l+k}}) \arrow[r, "\nabla^{(l)} \otimes \nabla^{(k)}"] \arrow[d, "\tilde{s}_K^B \otimes \tilde{s}_L^B"] & (X_{r_1+...+r_l})^{\otimes l} \otimes (X_{r_{l+1}+...+r_{l+k}})^{\otimes k} \arrow[d, "s_K^B \otimes s_L^B"] \\
\begin{array}{c} \sigma.(X_{r_1} \otimes X_0, ..., X_{r_l} \otimes X_0, \\ X_0 \otimes X_{r_{l+1}}, ..., X_0 \otimes X_{r_{l+k}}) \end{array} \arrow[d, "N(B(k,\nabla^X,k))"] & (X_{r_1+...+r_l} \otimes X_{r_{l+1}+...+r_{l+k}})^{\otimes l+k} \arrow[d, "\nabla^X"] \\
\sigma.(X_{r_1}^{\otimes 2}, ..., X_{r_l}^{\otimes 2}, X_{r_{l+1}}^{\otimes 2}, ..., X_{r_{l+k}}^{\otimes 2}) \arrow[r, "\nabla^{(l+k)}"] & (X_{r_1+...+r_{l+k}}^{\otimes l+k})^{\otimes 2}
\end{tikzcd}$$

commutes. The maps $\tilde{s}_i^B$ denote the degeneracies of the bar construction $B(k, N(X), k)$.

Computing the two maps yields that

$$\begin{aligned}
& \nabla^X(s_K^B \otimes s_L^B)(\nabla^{(l)} \otimes \nabla^{(k)})(x_1, ..., x_{l+k}) \\
= \; & \nabla^X(\sum_{\substack{(R_1,...,R_l)\in\mathrm{comp}(r_1,...,r_l) \\ (R_{l+1},...,R_{l+k})\in\mathrm{comp}(r_{l+1},...,r_{l+k})}} \pm s_K^B(s^X_{R_2\sqcup...\sqcup R_l}(x_1), ..., s^X_{R_1\sqcup...\sqcup R_{l-1}}(x_l)) \\
& \otimes s_L^B(s^X_{R_{l+2}\sqcup...\sqcup R_{l+k}}(x_{l+1}), ..., s^X_{R_{l+1}\sqcup...\sqcup R_{l+k-1}}(x_{l+k}))) \\
= \; & \sum_{\substack{(R_1,...,R_l)\in\mathrm{comp}(r_1,...,r_l) \\ (R_{l+1},...,R_{l+k})\in\mathrm{comp}(r_{l+1},...,r_{l+k}) \\ (U,V)\in\mathrm{comp}(r_1+...+r_l, r_{l+1}+...+r_{l+k})}} \pm s_K^B(s_V^X s^X_{R_2\sqcup...\sqcup R_l}(x_1), ..., s_V^X s^X_{R_1\sqcup...\sqcup R_{l-1}}(x_l)) \\
& \otimes s_L^B(s_U^X s^X_{R_{l+2}\sqcup...\sqcup R_{l+k}}(x_{l+1}), ..., s_U^X s^X_{R_{l+1}\sqcup...\sqcup R_{l+k-1}}(x_{l+k}))
\end{aligned}$$

and

$$\begin{aligned}
& \nabla^{(l+k)} N(B(k, \nabla^X, k))(\tilde{s}_K^B \otimes \tilde{s}_L^B)(x_1, ..., x_{l+k}) \\
= & \sum_{\substack{(R_1,...,R_{l+k}) \\ \in\mathrm{comp}(r_1,...,r_{l+k})}} \pm(s_K^B \otimes s_L^B)(s^X_{R_2\sqcup...\sqcup R_{l+k}}(x_1), ..., s^X_{R_1\sqcup...\sqcup R_{l+k-1}}(x_{l+k})).
\end{aligned}$$

As before a longish calculation shows that these maps are equal. Since we also saw that

$$\nabla^{\mathrm{bi}}\nabla^X\nabla^{B,\mathrm{Tot}} = \nabla^{\mathrm{diag}}(\nabla^{\mathrm{bi}} \otimes \nabla^{\mathrm{bi}})$$

the two maps from $\mathrm{Tot}(N(B(k, N(X), k))) \otimes \mathrm{Tot}(N(B(k, N(X), k)))$ to $N(d(d_\otimes(Y \otimes Y)))$ exhibited in the main diagram above coincide. This yields the claim. □

For a simplicial commutative augmented A_+-algebra X let $B^{A_+}(A_+, X, A_+)$ denote the bar construction of X with coefficients in A_+ with respect to A_+ as a ground ring, i.e.

$$B^{A_+}(A_+, X, A_+)_i = A_+ \otimes_{A_+} X^{\otimes i} \otimes_{A_+} A_+.$$

Denote by $\bar{N}(X)$ the augmentation ideal of $N(X)$.

Lemma 6.10. *Let $\partial_\theta^{(n)}$ denote the twist on $A_+ \otimes B^n(A)$ defined in definition 3.41. The shuffle product* sh *on $B^n(A)$ induces a differential graded A_+-algebra structure on $(A_+ \otimes B^n(A), \partial_\theta)$ via*

$$(A_+ \otimes B^n(A)) \otimes_{A_+} (A_+ \otimes B^n(A)) \cong A_+ \otimes B^n(A) \otimes B^n(A) \xrightarrow{\mathrm{sh}} A_+ \otimes B^n(A).$$

We again denote this product by sh.

Proof. We only need to check that the shuffle product respects $\partial_\theta^{(n)}$. We define a map

$$i\colon B^n(A)\otimes B^n(A)\to B^{n+1}(A)$$

by identifying an n-level tree $t(a_1,...,a_l)$ and an n-level tree $s(a_{l+1},...,a_{l+m})$ labeled by elements $a_i\in A$ with the $(n+1)$-level tree $(t,s)(a_1,...,a_{l+m})$ with two vertices in level one and whose subtrees above these vertices are t and s. Similarly we define a map $j\colon B^n(A)\to B^{n+1}(A)$ as the composite

$$B^n(A)\xrightarrow{\cong}\Sigma B^n(A)\hookrightarrow B^{n+1}(A)\ .$$

Then $\mathrm{id}_{A_+}\otimes j$ maps $\mathrm{sh}((a'\otimes t(a_1,...,a_l))\otimes_{A_+}(a''\otimes s(a_{l+1},...,a_{l+m})))$ to

$$(-1)^{d(t)+1}\partial_1(\mathrm{id}_{A_+}\otimes i)(a'a''\otimes t(a_1,...,a_l)\otimes s(a_{l+1},...,a_{l+m}))$$

with ∂_1 as defined in 4.12. We already know from 4.14 that ∂_1 and $\partial_\theta^{(n+1)}$ anticommute. Furthermore we have

$$\partial_\theta^{(n+1)}(\mathrm{id}_{A_+}\otimes j)=-(\mathrm{id}_{A_+}\otimes j)\partial_\theta^{(n)}$$

and

$$(\mathrm{id}_{A_+}\otimes i)(\partial_\theta^{(n)}\otimes_{A_+}\mathrm{id}_{A_+\otimes B^n(A)}+\mathrm{id}_{A_+\otimes B^n(A)}\otimes_{A_+}\partial_\theta^{(n)})=\partial_\theta^{(n+1)}(\mathrm{id}_{A_+}\otimes i).$$

This yields that $\partial_\theta^{(n)}$ is a derivation with respect to sh. □

Lemma 6.11. *We have*

$$(A_+\otimes B^n(A),\partial_\theta^{(n)})\cong\mathrm{Tot}(\bar N(B^{A_+}(A_+,(A_+\otimes B^{n-1}(A),\partial_\theta^{(n-1)}),A_+)))$$

as differential graded A_+-algebras for $n\geq 2$.

Proof. By definition

$$A_+\otimes B^n(A)=\mathrm{Tot}(\bar N(A_+\otimes B^k(k,B^{n-1}(A)_+,k)))$$

with $N_j(A_+\otimes(B^k(k,B^{n-1}(A)_+,k))=A_+\otimes B^{n-1}(A)^{\otimes j}$. On the other hand,

$$N_j(B^{A_+}(A_+,A_+\otimes B^{n-1}(A),A_+))=(A_+\otimes B^{n-1}(A))^{\otimes_{A_+}j}\cong A_+\otimes B^{n-1}(A)^{\otimes j},$$

and under this isomorphism $\partial_\theta^{(n-1)}$ induces $\partial_\theta^{(n)}$. This identification also respects the corresponding signs. □

Since $d(B(k,X,k))=X\overline{\otimes}S^1$ for a simplicial commutative augmented algebra X, we get the following result.

Proposition 6.12. *There is a quasiisomorphism*

$$(A_+ \otimes B^n(A), \partial_\theta) \to \bar{N}(\mathcal{L}_+(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-1})$$

of differential graded A_+-algebras.

Proof. For $n = 1$, that $(A_+ \otimes B(A), \partial_\theta) = \bar{N}(\mathcal{L}_+(A; A_+)(S^1))$ can be checked directly. For $n > 1$ we know from proposition 6.8 that

$$N(\mathcal{L}_+(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-1}) \cong N((\mathcal{L}_+(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-2})\overline{\otimes}_{A_+} S^1).$$

We know from lemma 6.9 that there is a quasiisomorphism of algebras from

$$\mathrm{Tot}(N(B^{A_+}(A_+, N((\mathcal{L}_+(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-2}), A_+)))$$

to

$$N((\mathcal{L}_+(A; A)(S^1)\overline{\otimes}_{A_+} S^{n-2})\overline{\otimes}_{A_+} S^1).$$

A quasiisomorphism $C \to D$ of augmented differential graded degreewise A_+-projective A_+-algebras yields a quasiisomorphism $\mathrm{Tot}(N(B^{A_+}(A_+, C, A_+))) \to \mathrm{Tot}(N(B^{A_+}(A_+, D, A_+)))$ of augmented differential graded A_+-algebras (see [30, 1.8, A.8]) which are again A_+-projective. Hence by induction we get a quasiisomorphism from

$$\mathrm{Tot}(N(B^{A_+}(A_+, (A_+ \otimes B^{n-1}(A)_+, \partial_\theta), A_+)))$$

to

$$N(\mathcal{L}_+(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-1}).$$

By lemma 6.11

$$\mathrm{Tot}(\bar{N}(B^{A_+}(A_+, (A_+ \otimes B^{n-1}(A)_+, \partial_\theta), A_+))) \cong (A_+ \otimes B^n(A), \partial_\theta)$$

as differential graded algebras. □

Proposition 6.13. *For a nonunital k-projective commutative k-algebra A*

$$H_*^{E_n}(A; A_+) \cong HH_{*+n}^{[n]}(A_+; A_+)$$

in nonnegative degrees.

Proof. By 6.12 we know that $H_*^{E_n}(A; A_+) \cong H_*(\Sigma^{-n}\bar{N}(\mathcal{L}_+^k(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-1}))$. Note that

$$\mathcal{L}_+^k(A; A_+) = \mathcal{L}_+^{A_+}(\overline{A_+ \otimes A_+}; A_+)$$

as functors from the category of finite sets to the category of A_+-modules. Here $\overline{A_+ \otimes A_+}$ denotes the augmentation ideal of $A_+ \otimes A_+$. The A_+-module structure is in both cases given by the action on the coefficient copy of A_+ and the A_+-module structure of $A_+ \otimes A_+$ is given by multiplication on the right factor. We know that

$$\mathcal{L}_+^{A_+}(\overline{A_+ \otimes A_+}; A_+)(S^1) = (A_+ \otimes A_+)\overline{\otimes}_{A_+} S^1$$

and hence

$$\mathcal{L}_+^k(A; A_+)(S^1)\overline{\otimes}_{A_+} S^{n-1} = (A_+ \otimes A_+)\overline{\otimes}_{A_+} S^n,$$

which again by the comparison above is $\mathcal{L}_+^k(A; A_+)(S^n)$. Since

$$H_*(N_{*+n}(\mathcal{L}_+^k(A; A_+)(S^n))) = \pi_*(\mathcal{L}_+^k(A; A_+)(S^n)),$$

which is by definition $HH_{*+n}^{[n]}(A_+; A_+)$, the result follows. □

Theorem 6.14. *For a nonunital k-projective commutative k-algebra A and a symmetric A-bimodule M*

$$H_*^{E_n}(A; M) \cong HH_{*+n}^{[n]}(A_+; M)$$

in nonnegative degrees.

Proof. Proposition 6.12 and the arguments used in the proof of 6.13 yield a quasiisomorphism

$$(A_+ \otimes B^n(A), \partial_\theta) \to \bar{N}(\mathcal{L}_+(A; A_+)(S^n))$$

of differential graded A_+-algebras. Note that $(M \otimes B^n(A), \partial_\theta) \cong M \otimes_{A_+} (A_+ \otimes B^n(A), \partial_\theta)$. There is a quasiisomorphism given by the Alexander-Whitney map

$$N(\mathcal{L}_+(A; M)(S^n)) = N(M \otimes_{A_+} \mathcal{L}_+(A; A_+)(S^n)) \to M \otimes_{A_+} N(\mathcal{L}_+(A; A_+)(S^n)).$$

Since $(A_+ \otimes B^n(A), \partial_\theta)$ as well as $N(\mathcal{L}_+(A; A_+)(S^n))$ are A_+-free as A_+-modules the claim follows from the Künneth spectral sequence [59, 10.90]. □

Theorem 6.15. *For a nonunital k-projective commutative k-algebra A and a symmetric A-bimodule M*

$$H_{E_n}^*(A; M) \cong HH_{[n]}^{*+n}(A_+; M)$$

in nonnegative degrees.

Proof. Observe that

$$C_{E_n}^*(A; M) = \mathrm{Hom}_{A_+}((A_+ \otimes \Sigma^{-n} B^n(A), \partial_\theta), M).$$

Consider the quasiisomorphism of A_+-modules

$$(A_+ \otimes B^n(A), \partial_\theta) \to \bar{N}(\mathcal{L}_+^k(A; A_+)(S^n))$$

exhibited in 6.12 and 6.13. Since this is a quasiisomorphism between chain complexes of A_+-free A_+-modules the universal coefficient spectral sequence (see e.g. [40, Theorem 2.3]) yields that we get a quasiisomorphism

$$\mathrm{Hom}_{A_+}(N_{*+n}(\mathcal{L}_+^k(A; A_+)(S^n)), M) \to C^*_{E_n}(A; M).$$

One calculates that

$$\mathrm{Hom}_{A_+}(N_{*+n}(\mathcal{L}_+^k(A; A_+)(S^n)), M) \cong N^{*+n}(\mathrm{Hom}_{A_+}(\mathcal{L}_+^k(A; A_+)(S^n), M)).$$

But

$$\mathrm{Hom}_{A_+}(\mathcal{L}_+^k(A; A_+)(S^n), M) = \mathrm{Hom}_{A_+}(\mathcal{L}_+^k(A; A_+), M)(S^n),$$

with $\mathrm{Hom}_{A_+}(\mathcal{L}_+^k(A; A_+), M) = \mathcal{L}_+^c(A; M)$ being precisely the contravariant Loday functor associated to A and M as defined in 6.3. □

Remark 6.16. *The complex $N_*(\mathcal{L}_+(A; A_+)(S^n))$ is trivial in degrees $0 < * < n$ and is k in degree zero. In particular Hochschild homology $HH_s^{[n]}(A_+; A_+)$ of order n is trivial in degrees s with $0 < s < n$ and is k for $s = 0$, see [52, 5.1], and hence higher order Hochschild homology can be reconstructed from E_n-homology.*

7 An explicit homotopy for the commutativity of the cup product

Let A be a nonunital commutative k-algebra. It is classical that Hochschild cohomology $HH^*(A_+; A_+)$ is a Gerstenhaber algebra (see [24]), i.e. is a graded commutative k-algebra equipped with a Lie bracket of degree 1 satisfying a Poisson relation. Gerstenhaber algebras are governed by $H_*(E_2)$, the operad formed by taking the homology of an E_2-operad. The Deligne conjecture, proven amongst others by McClure and Smith in [51], states that the Gerstenhaber structure on $HH^*(A_+; A_+)$ indeed stems from an action of an E_2-operad on the Hochschild cochain complex $C^*_{HH}(A_+, A_+)$. There is a generalized version of the Deligne conjecture, for example discussed in [37] and [45, 6.1.4], which states that for a suitable choice of a complex $D^*_{E_n}(A_+; A_+)$ calculating E_n-cohomology of A_+ with coefficients in A_+ this complex is an E_{n+1}-algebra.
In this section we show that $C^*_{E_n}(A; A_+)$ exhibits at least a small part of the structure of an E_{n+1}-algebra. We start by recalling the definition of the cup product on $C^*_{E_n}(A; A_+)$ in terms of trees. The cup product is induced by the comultiplication of the tensor coalgebra.

Definition 7.1. *For a graded nonunital noncounital A_+-bialgebra C write $[c_0|...|c_l]$ for $c_0 \otimes_{A_+} .... \otimes_{A_+} c_l \in (\Sigma C)^{\otimes_{A_+} l} \subset B^{A_+}(C)$. We define*

$$\Delta_0 \colon B^{A_+}(C) \to B^{A_+}(C) \otimes_{A_+} \otimes B^{A_+}(C)$$

by

$$\Delta_0([c_0|...|c_l]) = \sum_{j=0}^{l-1} [c_0|...|c_j] \otimes_{A_+} [c_{j+1}|...|c_l].$$

It is classical that the map Δ_0 is a chain map. We show that for $C = A_+ \otimes B^{n-1}(A)$ the map Δ_0 also respects the twist on $B^n(A)$.

Lemma 7.2. *The map Δ_0 yields a morphism*

$$\Delta_0 \colon (A_+ \otimes B^n(A), \partial_\theta) \to (A_+ \otimes B^n(A), \partial_\theta) \otimes_{A_+} (A_+ \otimes B^n(A), \partial_\theta)$$

of chain complexes.

Proof. We will not distinguish morphisms of trees and the maps they induce on $C^{E_n}_*(A; A_+)$ in our notation used in this proof. For $n = 1$, that Δ_0 respects the differential is a standard computation, see [24, §7]. For $n > 1$, we proved in lemma 6.11 that

$$(A_+ \otimes B^n(A), \partial^{(n)}_\theta) \cong B^{A_+}(A_+ \otimes B^{n-1}(A), \partial^{(n-1)}_\theta),$$

with B^{A_+} the bar construction as defined in 3.14 with respect to A_+ as a ground ring. Since Δ_0 is induced by the comultiplication of the tensor coalgebra, it is a chain map. □

Corollary 7.3. *The map Δ_0 induces an associative product $\cup = \cup_0$ of degree n on the cochain complex $C^*_{E_n}(A, A_+)$ given by*

$$\begin{array}{c} (C^*_{E_n}(A;A_+))^{\otimes_{A_+} 2} = \mathrm{Hom}_{A_+}(C^{E_n}_*(A;A_+), A_+)^{\otimes_{A_+} 2} \\ \big\downarrow \\ \mathrm{Hom}_{A_+ \otimes_{A_+} A_+}((C^{E_n}_*(A;A_+))^{\otimes_{A_+} 2}, A_+ \otimes_{A_+} A_+) \\ \big\downarrow \\ \mathrm{Hom}_{A_+}((C^{E_n}_*(A;A_+))^{\otimes_{A_+} 2}, A_+) \\ \big\downarrow \Delta_0^* \\ \mathrm{Hom}_{A_+}((C^{E_n}_*(A;A_+), A_+) = C^*_{E_n}(A;A_+). \end{array}$$

For differential graded k-modules C and D let

$$T\colon C \otimes D \to D \otimes C$$

be the twist $T(c \otimes d) = (-1)^{|c||d|} d \otimes c$ making the category of differential graded k-modules a symmetric monoidal category. It is well-known that the product $\cup_0$ is commutative up to homotopy. Part of the structure of an E_{n+1}-algebra is a sequence $\cup_1, ..., \cup_n$ of higher cup products such that

$$\cup_i \circ (\mathrm{id} + T) = \delta(\cup_{i+1})$$

for $0 \leq i \leq n-1$. Hence the generalized Deligne conjecture implies that for a suitable choice of a complex $D^*_{E_n}(A_+; A_+)$ calculating E_n-cohomology, these higher cup products exist. We will now construct an explicit possible choice for $\cup_1$ for the complex $C_{E_n}(A; A_+)$. We assume that we are working in characteristic two to avoid dealing with signs.

Definition 7.4. *Let the characteristic of k be two and let C be a graded nonunital noncounital A_+-bialgebra with comultiplication Δ. We will use an abbreviated Sweedler notation and denote $\Delta(c)$ by $c^{(1)} \otimes_{A_+} c^{(2)}$. Let C_+ denote the unital augmented counital coaugmented A_+-bialgebra obtained by setting $C_+ = C \oplus A_+$. Extend Δ to a map*

$$\Delta_+\colon C \to C_+ \otimes_{A_+} C$$

by setting $\Delta_+(c) = 1 \otimes_{A_+} c + \Delta(c)$ and write $\Delta_+(c) = c' \otimes_{A_+} c''$. We define a map

$$\Delta_1\colon B^{A_+}(C) \to B^{A_+}(C) \otimes_{A_+} B^{A_+}(C)$$

of degree 1 by setting

$$\Delta_1([c_0|...|c_l]) = \sum_{0 \leq i \leq i+j \leq l} [c_0|...|c'_i...c'_{i+j}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{i+j}],$$

where we set $[c_0|...|1|...|c_l] = 0$, i.e. project from the unnormalized to the normalized bar construction.

Lemma 7.5. *In characteristic two, the map Δ_1 is a homotopy for the cocommutativity of the map Δ_0, i.e.*

$$\delta(\Delta_1) = (T + id)\Delta_0.$$

Proof. Decompose Δ_1 by setting

$$\rho_{i,j}([c_0|...|c_l]) = [c_0|...|c'_i...c'_{i+j}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{i+j}]$$

for $0 \leq i \leq i+j \leq l$. We first determine how the simplicial differential ∂ of the bar construction and Δ_1 interact. Recall that the simplicial part of the differential of the bar construction is given by

$$\partial([c_0|...|c_l]) = \sum_{r=0}^{l-1} [c_0|...|c_r c_{r+1}|...|c_l].$$

Denote the summand $[c_0|...|c_r c_{r+1}|...|c_l]$ by $d_r([c_0|...|c_l])$. Let $0 \leq r \leq l-1$ and $0 \leq i \leq i+j \leq l-1$. Then

$$\rho_{i,j} d_r([c_0|...|c_l]) = \begin{cases} (d_r \otimes_{A_+} \mathrm{id})\rho_{i+1,j}([c_0|...|c_l]), & r < i, \\ (d_{r-j} \otimes_{A_+} \mathrm{id})\rho_{i,j}([c_0|...|c_l]), & r > i+j. \end{cases}$$

Now let $0 \leq i \leq r < i+j \leq l$. Then

$$\begin{aligned} & \rho_{i,j-1} d_r([c_0|...|c_l]) \\ = \;& \rho_{i,j-1}[c_0|...|c_r c_{r+1}|...|c_l] \\ = \;& [c_0|...|c_{i-1}|c'_i...(c_r c_{r+1})'...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|(c_r c_{r+1})''|...|c''_{i+j}]. \end{aligned}$$

From our definition of Δ_+ we calculate that

$$\begin{aligned} & (c_r c_{r+1})' \otimes_{A_+} (c_r c_{r+1})'' + c'_r c'_{r+1} \otimes_{A_+} c''_r c''_{r+1} \\ = \;& c_r c_{r+1}^{(1)} \otimes_{A_+} c_{r+1}^{(2)} + c_{r+1} \otimes_{A_+} c_r + c_r \otimes_{A_+} c_{r+1} + c_r^{(1)} c_{r+1} \otimes_{A_+} c_r^{(2)}. \end{aligned}$$

Hence

$$\begin{aligned} & \rho_{i,j-1} d_r([c_0|...|c_l]) \\ = \;& [c_0|...|c_{i-1}|c'_i...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{r-1}|c''_r c''_{r+1}|c''_{r+2}|...|c''_{i+j}] \\ + \;& [c_0|...|c_{i-1}|c'_i...c'_{r-1} c_r c_{r+1}^{(1)} c'_{r+2}...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{r-1}|c_{r+1}^{(2)}|c''_{r+2}|...|c''_{i+j}] \\ + \;& [c_0|...|c_{i-1}|c'_i...c'_{r-1} c_{r+1} c'_{r+2}...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{r-1}|c_r|c''_{r+2}|...|c''_{i+j}] \\ + \;& [c_0|...|c_{i-1}|c'_i...c'_{r-1} c_r c'_{r+2}...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{r-1}|c_{r+1}|c''_{r+2}|...|c''_{i+j}] \\ + \;& [c_0|...|c_{i-1}|c'_i...c'_{r-1} c_r^{(1)} c_{r+1} c'_{r+2}...c'_{i+j}|c_{i+j+1}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{r-1}|c_r^{(2)}|c''_{r+2}|...|c''_{i+j}]. \end{aligned}$$

The first summand equals $(\mathrm{id} \otimes_{A_+} d_{r-i})\rho_{i,j}$. Summing over all r with $i \le r \le i+j$ and using that

$$c' \otimes_{A_+} c'' = 1 \otimes_{A_+} c + c^{(1)} \otimes_{A_+} c^{(2)}$$

we see that the other summands cancel each other except for $r = i$ and $r = i+j$, leaving

$$\begin{aligned}
& \sum_{r=i}^{i+j} \rho_{i,j-1} d_r([c_0|...|c_l]) \\
= \ & \sum_{r=i}^{i+j} (\mathrm{id} \otimes_{A_+} d_{r-i})\rho_{i,j} + \sum_{r=i}^{i+j} [c_0|...|c_i c'_{i+1} ... c'_{i+j}|...|c_l] \otimes_{A_+} [c''_{i+1}|...|c''_{i+j}] \\
+ \ & \sum_{r=i}^{i+j} [c_0|...|c'_i ... c'_{i+j-1} c_{i+j}|...|c_l] \otimes_{A_+} [c''_i|...|c''_{i+j-1}].
\end{aligned}$$

But the last two summands equal

$$\begin{aligned}
& \sum_{r=i}^{\min(l-1,i+j)} (d_i \otimes_{A_+} \mathrm{id})\rho_{i+1,j-1} + \sum_{r=i}^{\min(l-1,i+j)} (d_{i+1} \otimes_{A_+} \mathrm{id})\rho_{i,j-1} \\
+ & \sum_{r=i}^{\min(l-1,i+j)} [c_0|...|c_i|c_{i+j+1}|...|c_l] \otimes_{A_+} [c_{i+1}|...|c_{i+j}] \\
+ & \sum_{r=i}^{\min(l-1,i+j)} [c_0|...|c_{i-1}|c_{i+j}|...|c_l] \otimes_{A_+} [c_i|...|c_{i+j-1}].
\end{aligned}$$

If we sum over all $0 \le r \le l-1$ and all $0 \le i \le i+j \le l-1$ we hence see that

$$\partial \rho_{i,j} + \rho_{i,j}(\partial \otimes_{A_+} \mathrm{id} + \mathrm{id} \otimes_{A_+} \partial) = \sum_{r=0}^{l-1} [c_0|...|c_i] \otimes_{A_+} [c_{i+1}|...|c_{i+j}].$$

Finally, consider the differential δ of C. For $r < i$ it is clear that

$$\rho_{i,j}(\mathrm{id}^{\otimes_{A_+} r} \otimes_{A_+} \delta \otimes_{A_+} \mathrm{id}^{\otimes_{A_+} l-r}) = (\mathrm{id}^{\otimes_{A_+} r} \otimes_{A_+} \delta \otimes_{A_+} \mathrm{id}^{\otimes_{A_+} l-r-j})\rho_{i,j}.$$

Similarly for $r > i+j$

$$\rho_{i,j}(\mathrm{id}^{\otimes_{A_+} r} \otimes_{A_+} \delta \otimes_{A_+} \mathrm{id}^{\otimes_{A_+} l-r}) = (\mathrm{id}^{\otimes_{A_+} r-j} \otimes_{A_+} \delta \otimes_{A_+} \mathrm{id}^{\otimes_{A_+} l-r})\rho_{i,j}.$$

Finally, let $i \le r \le i+j$. Since δ is a coderivation,

$$(\delta(c_r))' \otimes_{A_+} (\delta(c_r))'' = \delta(c'_r) \otimes_{A_+} c''_r + c'_r \otimes_{A_+} \delta(c''_r).$$

Hence we see that

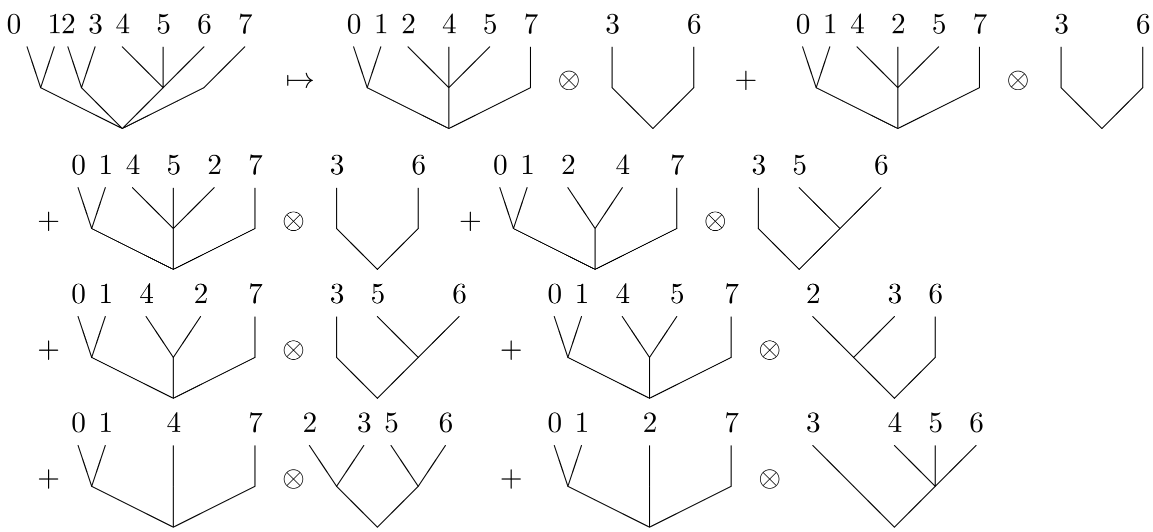
$$\begin{aligned} & \rho_{i,j}(\mathrm{id}^{\otimes_{A_+} r} \otimes_{A_+} \delta \otimes_{A_+} \mathrm{id}^{\otimes_{A_+} l-r})([c_0|...|c_l]) \\ = \quad & (\delta(c_r))' \otimes_{A_+} (\delta(c_r))'' = \delta(c_r') \otimes_{A_+} c_r'' + c_r' \otimes_{A_+} \delta(c_r''). \end{aligned}$$

□

Recall from lemma 6.10 that with the shuffle product $(A_+ \otimes B^{n-1}(A), \partial_\theta)$ is a k-algebra. It is well known, see for example [65, 0.6], that the shuffle product and the coproduct derived from the tensor coalgebra structure turn $B^{n-1}(A)$ into a differential graded k-bialgebra, hence $(A_+ \otimes B^{n-1}(A), \partial_\theta)$ is an A_+-bialgebra. Setting $C = (A_+ \otimes B^{n-1}(A), \partial_\theta)$ in definition 7.4 yields

$$\Delta_1 \colon (A_+ \otimes B^n(A), \partial_\theta) \to (A_+ \otimes B^n(A), \partial_\theta) \otimes_{A_+} (A_+ \otimes B^n(A), \partial_\theta)$$

for $n \geq 2$. An example of the summand $\rho_{i,j}$ of Δ_1 defined in the proof of lemma 7.5 is shown in the following picture, for $n = 2, l = 3$ and $i = j = 1$.



We can construct $\cup_1$ in a fashion similar to the construction of $\cup_0$ from Δ_0 in corollary 7.3.

Corollary 7.6. *For $n \geq 2$ the map Δ_1 induces a map*

$$\cup_1 \colon C^*_{E_n}(A; A_+) \otimes_{A_+} C^*_{E_n}(A; A_+) \to C^*_{E_n}(A; A_+)$$

of degree $n-1$ such that

$$\delta(\cup_1) = \cup_0 \circ (T + id).$$

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Summary

This thesis studies E_n-homology and E_n-cohomology. These are invariants associated to algebraic analogues of n-fold loop spaces for $1 \leq n \leq \infty$: Iterated loop spaces can be described via topological operads, from which one can construct corresponding operads in differential graded modules. Algebras over such an algebraic operad are called E_n-algebras. More concretely, an E_n-algebra is a differential graded module equipped with a product which is associative up to a coherent system of higher homotopies for associativity, but commutative only up to homotopies of a certain level, depending on n. In particular, every commutative k-algebra over a commutative unital ring k is an E_n-algebra.

Using the operadic description, one can construct suitable homological invariants for E_n-algebras, called E_n-homology and -cohomology. For $n = 1$ and $n = \infty$ this gives rise to familiar invariants: E_1-homology and -cohomology coincide with Hochschild homology and cohomology, while for $n = \infty$ one retrieves Γ-homology and -cohomology. Note that in characteristic zero Γ-homology and -cohomology equal André-Quillen-homology and -cohomology.

Although Hochschild homology and André-Quillen-homology are classical invariants and have been extensively studied, very little is known in the intermediate cases $1 < n < \infty$. In this thesis we extend results known for special cases of E_n-homology and -cohomology to a broader context. We use these extensions to examine E_n-cohomology for additional structures.

In [23] Benoit Fresse proved that E_n-homology with trivial coefficients can be computed via a generalized iterated bar construction. By unpublished work of Fresse, if one assumes that the E_n-algebra in question is strictly commutative, this is also possible for cohomology and for coefficients in a symmetric bimodule. We give the details of a proof of this result based on a sketch of a proof by Benoit Fresse.

Hochschild homology and cohomology can be interpreted as functor homology and cohomology. In [41] Muriel Livernet and Birgit Richter prove that this is always possible for E_n-homology of commutative algebras with trivial coefficients. We extend the category defined by Livernet and Richter in their work to a category which also incorporates the action of a commutative algebra A on a symmetric A-bimodule M. We then show that E_n-homology as well as E_n-cohomology of A with coefficients in M can be calculated as functor homology and cohomology, i.e. as derived functors Tor and Ext.

Hence E_n-cohomology of such objects is representable in a derived sense. In this case the Yoneda pairing yields a natural action of the E_n-cohomology of the representing object on E_n-cohomology. We prove that E_n-cohomology of the representing object is trivial, therefore no operations arise this way.

Livernet and Richter showed in [41] that E_n-homology of commutative algebras with trivial coefficients coincides with higher order Hochschild cohomology. We extend this result to cohomology and to coefficients in a symmetric bimodule.

It is well known that for a suitable choice of a chain complex calculating E_n-cohomology of an algebra with coefficients in the algebra itself, this chain complex is an E_{n+1}-algebra. For $n = 1$ this is the classical Deligne conjecture. For $n > 1$, the constructions of the E_{n+1}-action given so far have not been very explicit. We show that in characteristic two the chain complex defined via the n-fold bar construction admits at least a part of an E_{n+1}-structure, namely a homotopy for the cup product, and give an explicit formula for this homotopy.

Zusammenfassung

Diese Dissertation beschäftigt sich mit E_n-Homologie und E_n-Kohomologie. Dies sind Invarianten, die einem algebraischen Analogon n-facher Schleifenräume zugeordnet sind, wobei $1 \leq n \leq \infty$ ist. Es gibt topologische Operaden, die n-fache Schleifenräume modellieren. Diese Operaden lassen sich in den algebraischen Kontext differentiell graduierter Moduln übertragen, und Algebren über einer solchen Operade heißen E_n-Algebren. Eine E_n-Algebra ist ein differentiell graduierter Module mit einem Produkt, welches assoziativ bis auf ein kohärentes System aller höherer Homotopien für Assoziativität ist, aber kommutativ nur bis auf Homotopien eines bestimmten Levels, abhängig von n. Insbesondere ist jede kommutative k-Algebra über einem kommutativen unitären Grundring k eine E_n-Algebra. Die auf E_n-Algebren zugeschnittenen homologischen Invarianten sind E_n-Homologie und -Kohomologie. Bekannte Spezialfälle treten für $n = 1$ und für $n = \infty$ auf, in diesen Fällen stimmen E_n-Homologie und -Kohomologie mit Hochschildhomologie und -kohomologie beziehungsweise mit Γ-Homologie und -Kohomologie überein. In Charakteristik null ist Γ-Homologie gleich André-Quillen-Homologie.
Obwohl Hochschildhomologie und André-Quillen-Homologie klassische, viel studierte Invarianten sind, ist für $1 < n < \infty$ nur wenig über E_n-Homologie und -Kohomologie bekannt. Inhalt dieser Dissertation ist, für Spezialfälle bekannte Resultate zu erweitern und diese zu nutzen, um E_n-Kohomologie auf zusätzliche Struktur zu untersuchen.
Benoit Fresse hat in [23] gezeigt, dass E_n-Homologie mit trivialen Koeffizienten sich über eine verallgemeinerte n-fache Bar-Konstruktion berechnen lässt. Ein unveröffentlichtes Resultat von Fresse besagt, dass dies auch für E_n-Homologie sowie für E_n-Kohomologie gewöhnlicher kommutativer Algebren mit Koeffizienten in einem symmetrischen Bimodul möglich ist. Wir geben einen Beweis dieses Resultats basierend auf einer Beweisskizze von Benoit Fresse.
Hochschildhomologie und -kohomologie lassen sich als Funktorhomologie und -kohomologie berechnen. In [41] beweisen Muriel Livernet und Birgit Richter, dass dies auch für E_n-Homologie kommutativer Algebren mit trivialen Koeffizienten gilt. Wir erweitern die hierzu von Livernet und Richter definierte Kategorie zu einer Kategorie, die die Wirkung einer kommutativen Algebra A auf einem symmetrischen Bimodul M mit einbezieht und zeigen, dass sich sowohl E_n-Homologie als auch E_n-Kohomologie von A und M als abgeleitete Funktoren Tor und Ext bezüglich dieser Kategorie interpretieren lassen.
Als Funktorkohomologie ist E_n-Kohomologie darstellbar in einem abgeleiteten Sinne. Daher gibt es eine natürliche Wirkung der E_n-Kohomologie des darstellenden Objektes auf E_n-Kohomologie, definiert über die Yoneda-Paarung. Wir zeigen, dass E_n-Kohomologie des darstellenden Objektes trivial ist und folglich keine Kohomologieoperationen auf diese Weise entstehen.
Für die E_n-Homologie kommutativer Algebren mit trivialen Koeffizienten haben Muriel Livernet und Birgit Richter in [41] gezeigt, dass E_n-Homologie mit höherer Hochschildho-

mologie übereinstimmt. Wir zeigen, dass auch dieses Resultat für E_n-Kohomologie und für Koeffizienten in einem symmetrischen Bimodul gilt.
Es ist bekannt, dass füer Koeffizienten in der Algebra selbst und für eine geeignete Wahl eines E_n-Kohomologie berechnenden Kettenkomplexes dieser Kettenkomplex eine E_{n+1}-Algebra ist. Für $n = 1$ ist dies der Inhalt der klassischen Deligne-Vermutung, für $n > 1$ sind die bisher gegebenen Konstruktionen dieser Struktur nicht sehr explizit. Wir zeigen, dass zumindest ein Teil einer solchen E_{n+1}-Struktur, nämlich eine Homotopie für das cup-Produkt, in Charakteristik zwei auch auf dem über den iterierten Barkomplex definierten Kettenkomplex vorhanden ist und geben eine explizite Formel für diese Homotopie an.

Lebenslauf

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