# Low Mach number equations with a heat source on networks

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Auf Grund der Gutachten von Prof. Dr. Ingenuin Gasser
und Prof. Dr. Rinaldo M. Colombo

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Prof. Dr. Armin Iske
Leiter des Fachbereichs Mathematik

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# Introduction

During the last decades, there have been some devastating fire incidents in pedestrian, vehicular and train tunnels, e.g. in 1999 in the Mont Blanc road tunnel, in 2000 in the tunnel of the Kaprun Gletscherbahn 2 or in 2003 in a subway station in Daegu in South Korea [18], to mention just a few.

These fire disasters show the necessity to find a way to minimize security risks if not to prevent these events altogether. Further research on these situations is highly demanded.

One possibility to find out more about security risks is to use a mathematical model in order to predict the propagation of fires in a tunnel. Based on this model, one could build up better safety arrangements like an additional ventilation, a sprinkler system or an improved evacuation plan. This would permit an important improvement for new tunnels as well as for existent ones [62].

A lot of different mathematical models are available to tackle this task. Essentially, they can be classified in three categories, as stated in more detail in [53].

The first class contains empirical models based on algebraic relations to estimate the critical velocity preventing smoke to move upstream against the hot air flow. But these models are not able to determine the downstream movement of the smoke.

In contrast, the class of phenomenological models can describe up- and downstream movements of the smoke. The idea of that class of models is to split the tunnel into different zones and to describe the exchange of mass, momentum and energy at the zone boundaries. This approach leads to a system of ordinary differential equations.

A third class of models uses time and space dependent fluid dynamical models, as e.g. the Navier-Stokes equations, to describe the flow inside the tunnel. In recent years, this approach has turned out to be the most sophisticated. In these models, the fire itself is usually described as a prescribed heat source. The main drawback of the models of this class is the high computational effort that is needed to compute a solution. Since the time step in most explicit numerical schemes depends on the ratio of the velocity of the flow and the speed of sound, called Mach number, it has to be chosen very small in this particular application. A numerical method for these flows in the context of tunnel fires was studied by Birken in his PhD thesis [6] in 2008 (see also [7]). Especially, the high computational costs for low Mach number flows are discussed in his work.

In order to reduce the computational costs, Gasser and Struckmeier [49] proposed an asymptotic approach based on the low Mach number of the involved flow in 2002. Their ansatz simplifies the compressible Navier-Stokes equations by choosing an adequate time and space scaling and performing a limit. This leads to a set of equations differing from the incompressible Navier-Stokes equations due to the heat source. Another simplification introduced by Gasser and Struckmeier concerns the geometry of the tunnels. A one-dimensional model which reproduces the main features of the flow is obtained by an integration over the cross-sectional area of the tunnel. Obviously, the shear flow orthogonal to the tunnel direction is neglected by this procedure. In [47], it was observed that this model can be rewritten as an ODE coupled with the continuity equation for the mass conservation. Based on this observation, the question of existence and uniqueness of solutions of this model was investigated in 2006 by Gasser and Steinrück [48]. A numerical study for this model was performed in [50].

An advantage of the one-dimensional model is that it can be expanded rather simply to networks of tunnels as they occur very often in real applications. For example, most subway networks consist of more than one tunnel and therefore, they have to be modelled by several one-dimensional objects. The main goal of this work is to study this model for the air flow in tunnels systems. Such a network can be described by an oriented graph and the air flow on each edge of the graph can then be modelled by a system of partial differential equations. These equations are coupled at the nodes of the graph using physical principles in order to complete the model. The choice of the orientation of the graph does not determine the direction of the flow. It merely represents the positive direction of the flow. This expansion of the model to networks and a numerical study can be found in the article [44] by Gasser and Kraft. Our work extends the article by a derivation of the model from the three-dimensional Euler equations and by a well-posedness analysis.

Moreover, our study is not restricted to the particular application of tunnel fires. Indeed, we analyse the whole set of low Mach number models on networks which can be written as

$$\begin{aligned} \rho_t + (u\rho)_x &= 0 \\ (\rho u)_t + \left(\rho u^2 + \pi\right)_x &= -\zeta \rho \alpha(u) u + f\rho \\ u_x &= q \end{aligned} \tag{1}$$

on each edge of the graph. Here, $\rho$ is the density, $u$ is the velocity, $\pi$ is the pressure, $f$ is an external force and $\zeta$ and $\alpha$ are friction parameters. A very important feature of this model is that the space derivative of the velocity is given by the known energy source $q$ so that the unknown velocity reduces to a space-independent quantity. The coupling at the nodes is formulated as algebraic constraints stating the conservation of the mass and of the (internal) energy and as a heuristic principle describing the mixture of the flow.

There are several other applications where models like (1) are used. For example, similar models are proposed in [43] in the context of solar updraft towers and in [41] to describe the flow in a chimney. Recently, an application of model (1) in the field of nuclear reactors was presented by Penel et al. in [73]. For certain initial and boundary conditions, they have proven the existence of solutions on a single edge for the case without friction, i.e. for $\zeta = 0$. Whereas the previous examples concern single gases, there are also extensions of the low Mach number model (1) treating gas mixtures or gas water mixtures. For example, this has been done in [45] by Gasser and Rybicki to model an exhaust pipe. The same model was used in [46] to optimize the fuel combustion in order to heat up catalytic converters in exhaust systems of vehicles. Another example modelling an energy tower was given by Bauer and Gasser [5]. Moreover, an overview of the applications of the low Mach number model for vertical structures is given by Bauer et al. [4].

As mentioned before, the extension of the equations to networks is a crucial point in modelling low Mach number flows in tunnel systems. Since this approach provides a rather simple and efficient method to study phenomena on complex geometries, such extensions of one-dimensional partial differential equations to graphs have been addressed by many researchers during the last three decades. For example, this idea has been used in the context of traffic flow [55, 42], gas flow in pipe lines [2] or river flow [29, 74].

In many cases these models consist of hyperbolic $2 \times 2$ systems. For these systems, the question of the existence of solutions is partially answered since there are compatibility conditions ensuring the local in time existence of solutions for initial conditions in a neighbourhood of a subcritical steady state [21]. In that existence result, the restriction to the subcritical case is essential since it prohibits the change of the direction of the characteristics. A general review of the state of the art in theory and applications of hyperbolic balance laws on networks is provided by Bressan et al. [17].

Another crucial point of the model (1) is the low Mach number itself. For non-network models, the study of low Mach number flows started with the works of Ebin [38, 39] and Klainerman and Majda [57, 58] and was continued by many others. The central point of these works is the question whether one can well describe low Mach number flows with incompressible equations. Those are the formal limit of the compressible equations. For the isentropic equations without an energy source, this question is answered. In 1998, Lions and Nader [66] proved for periodic boundary conditions that solutions of the compressible Navier-Stokes equations converge to solutions of the incompressible Navier-Stokes equations as the density becomes constant and the Mach number converges to zero. This result was extended to Dirichlet boundary conditions in 1999 by Desjardins et al. [32]. A detailed introduction to low Mach number flows can be found in the mini course [1] by Alazard.

While the isentropic case is well-understood, the non-isothermal and non-isentropic case, that is needed in our application, is much more complex to handle. Due to the presence of the energy source $q$ in the low Mach number equations (1), the velocity $u$ is not divergence free and thus, the density $\rho$ needs not to be constant. Therefore, even the analysis of the existence of solutions of these formal limit equations is more complicated since one has to take into account both, the continuity and the momentum equation.

In this thesis, we address that problem on a network. In particular, we will show the local in time existence of solutions. Furthermore, for certain networks, including all paths and all networks with at most one inner node, the global in time existence will be proven.

An important step in the existence proofs consists of the analysis of the transport equation

$$\rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho = f \tag{2}$$

on $(0, T) \times (0, 1)$ for a network with $n$ edges. The coupling conditions of this equation can be expressed as a linear boundary operator mapping from the outflow part of the boundary to the inflow part. We will prove the existence, uniqueness and stability of solutions $\rho \in C([0,T], L^p((0,1))^n)$ for non-smooth velocities $\mathbf{U} = \mathrm{diag}(u)$ with $u \in L^1((0,T), W^{1,1}((0,1))^n)$ and $u_x \in L^1((0,T), L^\infty((0,1))^n)$ without any restrictions on the sign of the velocity. Every change of the sign of the velocity implies a change of the direction of a characteristic. This can lead to additional difficulties. Our results on the transport equation are not only a step in the analysis of the model, in fact, they are interesting themselves since such transport equations on networks occur in other contexts as well. To the best of our knowledge, results like this have not been published before.

Our proofs of the statements on the transport equation rely on the concept of the renormalization property which was introduced in 1989 in a famous paper of Lions and DiPerna [34] and which has gained a lot of attention over the last years. For an overview on the property and some applications see e.g. [28]. In the original work, the velocity field $u$ was restricted to be tangential to the boundary of the domain. For non-tangential velocity fields, the situation is more complex. One has to take into account boundary conditions on the inflow part of the boundary. Boyer [13] has shown how to extend the theory to this case in 2005. We will show that this approach is also well suited to study the equations on a network.

Beside the analysis, the derivation of the low Mach number model is a main goal of this thesis. To this aim, we present a non-standard but elegant motivation of the one-dimensional Euler equations for non-straight tunnels with varying cross-sectional areas based on a coordinate transform and on an asymptotic approach. A transform of the three-dimensional Euler equations to the Frenet-Serret-frame, a local coordinate frame attaching a tangential vector and two normal vectors of the tunnel to each point, gives us the possibility to scale the tangential and normal components of the flow in different ways. Neglecting lower order terms and averaging over the cross-sectional area will complete the derivation of the section-averaged Euler equations.

This work is divided into five chapters. In the first chapter, we introduce some basic concepts and notations from the graph theory, needed for the definition of the equations on a network, and from the theory of functional analysis.

In Chapter 2, we present the low Mach number model on a network. First, we recall the three-dimensional Navier-Stokes and Euler equations in order to use the above mentioned asymptotic approach to derive the one-dimensional Euler equations. Then, the Euler equations are extended to networks, where the required coupling conditions are again obtained by an asymptotic approach from the three-dimensional Euler equations. Subsequently, the low Mach number limit is formally performed and the model is reformulated as an ODE for the velocity, which is coupled with the continuity equation for the density.

Chapter 3 deals with the study of the transport equation with coupled boundary conditions. After a precise definition of the boundary operator, we recall the renormalization property and prove the boundedness and uniqueness of the solutions before the continuous dependence of the solution on the data is shown. At the end, the existence of a solution for general networks is constructively proven using the method of characteristics and the continuous dependence on the data.

In the fourth chapter, we analyse the existence of solutions of general ODEs on a network, which are coupled with a transport equation. To this aim, we first study the local existence and uniqueness of solutions of ODEs on networks as a special class of differential algebraic equations. Subsequently, the local in time existence of solutions of the coupled equations is proven using the Schauder fixed point theorem. Furthermore, the local solution is shown to be extendible to a global solution in case of the existence of an energy functional. At the end of this chapter, such functionals are explicitly constructed for certain networks including all networks with exactly one inner node and all paths.

In the last chapter, the developed theory is applied to the low Mach number model, introduced in Chapter 2. In particular, we show that the reformulation of the model as a transport equation coupled with the ODE for the velocity is indeed equivalent to the original asymptotic model and that there exists at least one solution of the model locally in time. For the same networks as in the previous chapter, the solutions are proven to exist globally in time.

A list of used symbols and a list of occurring linear spaces can be found at the end of this work.


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# 1 Basics

## 1.1 Graph theory

We will start with a brief introduction to some basic graph theoretical concepts. More detailed information can be found in fundamental books as e.g. [33] or [52]. We use the same notation as in [33] to a large extend.

**Definition 1.1.** A *graph* is a pair $G = (\mathfrak{V}, E)$ of sets such that $E \subset \mathfrak{V} \times \mathfrak{V}$. The elements of $\mathfrak{V}$ are called vertices or nodes and the elements of $E$ edges.

To avoid some notational problems, we will assume $\mathfrak{V} \cap E = \emptyset$ during the whole thesis. A vertex $\mathfrak{v} \in \mathfrak{V}$ is called *incident* with an edge $e \in E$ if $\mathfrak{v} \in e$. To simplify the notation, we will shortly write $\mathfrak{v}\mathfrak{w} = \mathfrak{w}\mathfrak{v}$ for an edge instead of $\{\mathfrak{v}, \mathfrak{w}\}$. Two vertices $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}$ are *adjacent* (or neighbours) if $\mathfrak{v}\mathfrak{w}$ is an edge, i.e. $\mathfrak{v}\mathfrak{w} \in E$.

The vertex set of the graph $G$ is denoted by $\mathfrak{V}(G)$ and the edge set by $E(G)$. Furthermore, we write $E(\mathfrak{v})$ for the set of neighbours of a vertex $\mathfrak{v} \in \mathfrak{V}$. The number of neighbours is called degree $d(\mathfrak{v}) = |E(\mathfrak{v})|$ of a vertex $\mathfrak{v} \in \mathfrak{V}$.

**Definition 1.2.** The pair $G' = (\mathfrak{V}', E')$ is a *subgraph* of the graph $G = (\mathfrak{V}, E)$ if $\mathfrak{V}' \subset \mathfrak{V}$ and $E' \subset E$. We write $G' \subset G$.

For a given graph $G = (\mathfrak{V}, E)$ and any set of vertices $\mathfrak{U} \subset \mathfrak{V}$, we write $G - \mathfrak{U}$ for the subgraph with vertex set $\mathfrak{V} \setminus \mathfrak{U}$ and all edges of $G$ such that both ends are in $\mathfrak{V} \setminus \mathfrak{U}$.

For the study of differential equations on networks, we consider graphs with special properties as e.g. connectivity which we will introduce on the following pages.

**Definition 1.3.** A non-empty graph $P = (\mathfrak{V}, E)$ of the form

$$\mathfrak{V} = \{x_0, x_1, \ldots, x_k\}, \quad E = \{x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}, x_{k-1}x_k\}$$

with distinct vertices $x_i$ is called a *path*. We say the vertices $x_0$ and $x_k$ are *linked* by $P$.

**Definition 1.4.** A non-empty graph $G$ is *connected* if any two distinct vertices are linked by a path in $G$.

**Definition 1.5.** A connected subgraph $G'$ of $G$ is called maximal connected if there exists no connected subgraph $G'' \neq G'$ with $G' \subset G'' \subset G$.

**Definition 1.6.** A maximal connected subgraph of a graph $G$ is called a *component*.

During the next chapters we assume graphs always to be connected. Additionally, we need to fix the directions of the edges for our study. To this end, we introduce the concepts of directed and oriented graphs.

**Definition 1.7.** A *directed graph* is a pair $(\mathfrak{V}, E)$ of disjoint sets together with two maps $\mathrm{init}\colon E \to \mathfrak{V}$ and $\mathrm{ter}\colon E \to \mathfrak{V}$ which assigns to each edge $e \in E$ an *initial vertex* (or tail) $\mathrm{init}(e)$ and a *terminal vertex* (or head) $\mathrm{ter}(e)$.

This definition does not exclude loops or parallel edges since $E$ is not necessarily a subset of $\mathfrak{V} \times \mathfrak{V}$. That is why we also need the notion of an orientation.

**Definition 1.8.** A directed graph $D$ is called an *orientation* of an undirected graph $G$ if the vertex and the edge sets coincide, i.e. $\mathfrak{V}(G) = \mathfrak{V}(D)$ and $E(G) = E(D)$ and if $\{\mathrm{init}(e), \mathrm{ter}(e)\} = \{x, y\}$ for all edges $e = xy \in E(D)$. Such graphs are called *oriented graphs.*

Intuitively, each edge of an undirected graph is provided with a direction. In oriented graphs, the set of neighbouring edges of a vertex $\mathfrak{v} \in \mathfrak{V}$ can be divided into two parts, the set of *incoming edges* $E^-(\mathfrak{v}) = \{e|\,\mathrm{ter}(e) = \mathfrak{v}\}$ and the set of *outgoing edges* $E^+(\mathfrak{v}) = \{e|\,\mathrm{init}(e) = \mathfrak{v}\}$.

**Definition 1.9.** An (oriented or unoriented) graph $G = (\mathfrak{V}, E)$ together with a weight function $w : E \to (0, \infty)$ is called *weighted (oriented or unoriented) graph.*

In this thesis, we assume $\mathfrak{V} = \{\mathfrak{v}_1, \dots, \mathfrak{v}_m\}$ to be a finite set and denote the elements of $E$ by $e_1, \dots, e_n$. The structure of a graph can be represented by some matrices:

**Definition 1.10.**

- The *incidence matrix* $\mathbf{B} = (b_{ij}) \in \{-1, 0, 1\}^{m \times n}$ of an oriented graph is defined by

$$b_{ij} = \begin{cases} 1 & \text{if } \mathrm{init}(e_j) = \mathfrak{v}_i \\ -1 & \text{if } \mathrm{ter}(e_j) = \mathfrak{v}_i \\ 0 & \text{otherwise.} \end{cases}$$

- The *adjacency matrix* $\mathbf{A} = (a_{ij}) \in \{0, 1\}^{m \times m}$ of a graph is defined by

$$a_{ij} = \begin{cases} 1 & \text{if there exists an edge } e = \mathfrak{v}_i\mathfrak{v}_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

- The *weighted adjacency matrix* $\mathbf{A}_w = (a_{ij}) \in [0, \infty)^{m \times m}$ of a weighted oriented graph is defined by

$$a_{ij} = \begin{cases} w(e) & \text{if there exists an edge } e = \mathfrak{v}_i\mathfrak{v}_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

- The *degree matrix* $\mathbf{D} = (d_{ij}) \in \mathbb{N}^{m \times m}$ of a graph is defined by

$$d_{ij} = \begin{cases} d(\mathfrak{v}_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

- The *weighted degree matrix* $\mathbf{D}_w = (d_{ij}) \in [0,\infty)^{m\times m}$ of a graph is defined by

$$d_{ij} = \begin{cases} \sum_{e\in E(\mathfrak{v}_i)} w(e) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

- The *weight matrix* $\mathbf{W} = (w_{ij}) \in [0,\infty)^{n\times n}$ of a weighted graph is defined by

$$w_{ij} = \begin{cases} w(e_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The (weighted) adjacency matrix is symmetric and the (weighted) degree matrix and the weight matrix are in diagonal form.

*Remark* 1.11. The degree of a vertex $\mathfrak{v}_i$ can be determined from the rows of the incidence matrix by $d(\mathfrak{v}_i) = \sum_{j=1}^{n} |b_{ij}|$.

In our application, it is important to distinguish between nodes with degree one and nodes with degree larger than one. For this reason, we introduce two submatrices of the incidence matrix. The first one contains all rows of $\mathbf{B}$ corresponding to nodes with degree greater than one. Therefore, let $\mathfrak{v}_{i_1},\ldots,\mathfrak{v}_{i_k}$ be the nodes with degree $d(\mathfrak{v}_{i_l}) > 1$ and define the matrix $\mathbf{B}_{>1} = (b^{>1}_{jl}) \in \{-1,0,1\}^{k\times n}$ by

$$b^{>1}_{jl} = \begin{cases} 1 & \text{if } \operatorname{init}(e_l) = \mathfrak{v}_{i_j} \\ -1 & \text{if } \operatorname{ter}(e_l) = \mathfrak{v}_{i_j} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In the same way, the second matrix corresponds to the nodes with degree one, i.e. for the nodes $\mathfrak{v}_{i_{k+1}},\ldots,\mathfrak{v}_{i_m}$ with degree $d(\mathfrak{v}_{i_l}) = 1$ the matrix $\mathbf{B}_{=1} = (b^{=1}_{jl}) \in \{-1,0,1\}^{(m-k)\times n}$ is specified as

$$b^{=1}_{jl} = \begin{cases} 1 & \text{if } \operatorname{init}(e_l) = \mathfrak{v}_{i_{j+k}} \\ -1 & \text{if } \operatorname{ter}(e_l) = \mathfrak{v}_{i_{j+k}} \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

In the further course of this work, we use a special matrix, called Laplacian matrix, which we will introduce in the next part where we also prove some of its properties. Therefore, we need a fundamental characterization of the extremal eigenvalues of a symmetric matrix using the Rayleigh quotient. The proof of the following theorem and more details can be found e.g. in [56].

**Theorem 1.12 (Rayleigh quotient).** *Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Define the Rayleigh quotient for $x \neq 0$ as*

$$R(x) = \frac{x^T \mathbf{A} x}{x^T x}$$

*and denote the minimal eigenvalue of* $\mathbf{A}$ *by* $\lambda_{\min}(\mathbf{A})$ *and the maximal eigenvalue by* $\lambda_{\max}(\mathbf{A})$*. Then, it holds*

$$\lambda_{\min}(\mathbf{A}) = \min_{x \in \mathbb{R}^n \setminus \{0\}} R(x)$$

*and*

$$\lambda_{\max}(\mathbf{A}) = \max_{x \in \mathbb{R}^n \setminus \{0\}} R(x).$$

*Proof.* See [56]. □

We will use this characterization to estimate the minimal and maximal eigenvalues of some products of matrices:

**Lemma 1.13**. *Le* $\mathbf{A} \in \mathbb{R}^{k \times n}$ *and* $\mathbf{B} \in \mathbb{R}^{k \times k}$ *be matrices where* $\mathbf{B}$ *is symmetric. Then, for the minimal and maximal eigenvalues the inequalities*

$$\lambda_{\max}(\mathbf{A}^T\mathbf{B}\mathbf{A}) \leq \lambda_{\max}(\mathbf{A}^T\mathbf{A})\lambda_{\max}(\mathbf{B})$$

*and*

$$\lambda_{\min}(\mathbf{A}^T\mathbf{B}\mathbf{A}) \geq \lambda_{\min}(\mathbf{A}^T\mathbf{A})\lambda_{\min}(\mathbf{B}).$$

*are valid.*

*Proof.* Using twice the previous theorem, it holds for all $x \in \mathbb{R}^n \setminus \{0\}$

$$x^T\mathbf{A}^T\mathbf{B}\mathbf{A}x \leq \lambda_{\max}(\mathbf{B})x^T\mathbf{A}^T\mathbf{A}x \leq \lambda_{\max}(\mathbf{B})\lambda_{\max}(\mathbf{A}^T\mathbf{A})x^Tx$$

and hence

$$\lambda_{\max}(\mathbf{A}^T\mathbf{B}\mathbf{A}) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T\mathbf{A}^T\mathbf{B}\mathbf{A}x}{x^Tx} \leq \lambda_{\max}(\mathbf{B})\lambda_{\max}(\mathbf{A}^T\mathbf{A}).$$

The inequality for the minimal eigenvalue follows by the same argument. □

**Definition 1.14.** The Laplacian matrix $\mathbf{L} = (l_{ij}) \in \mathbb{Z}^{m \times m}$ is defined by $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and the weighted Laplacian matrix $\mathbf{L}_w \in \mathbb{R}^{m \times m}$ by $\mathbf{L}_w = \mathbf{D}_w - \mathbf{A}_w$.

*Remark* 1.15. To each unweighted graph $G = (\mathfrak{V}, E)$, we can assign the trivial weight function $w : E \to (0, \infty)$ with $w(e) = 1$ such that the weighted and unweighted matrices coincide. Thus, all proven statements for weighted graphs are also true for unweighted graphs.

**Lemma 1.16 (Laplacian matrix)**. *Let* $G_w = (\mathfrak{V}, E, w)$ *be a weighted graph and denote by* $G = (\mathfrak{V}, E)$ *the corresponding unweighted graph. Then, it holds:*

1. *The weighted Laplacian matrix can be decomposed into*

$$\mathbf{L}_w = \mathbf{B}\mathbf{W}\mathbf{B}^T$$

   *for any orientation of the graph.*

2. *The weighted Laplacian matrix is weakly diagonally dominant and positive semidefinite.*

3. *The dimension of the kernel of $\mathbf{L}_w$ is equal to the number of components.*

4. *If the graph is additionally connected, all principal submatrices of $\mathbf{L}_w$ are positive definite, i.e. for each non-empty set of indices $J \subsetneq \{1, \dots, m\}$, the submatrix $\mathbf{L}_w^J$, resulting from removing all columns and rows with indices in $\{1, \dots, m\} \setminus J$ of the weighted Laplacian matrix $\mathbf{L}_w$, is positive definite.*

5. *Let the graph again be connected and let $w_{\min} = \min_{e \in E} w(e)$ be the minimal weight of an edge and $J \subsetneq \{1, \dots, m\}$ be a non-empty index set. The smallest eigenvalue of the principal submatrix of the weighted Laplacian matrix can be bounded from below by*
$$\lambda_{\min}(\mathbf{L}_w^J) \geq w_{\min} \lambda_{\min}(\mathbf{L}^J)$$
*where $L^J$ denotes the principal submatrix of the unweighted Laplacian matrix. Additionally, the spectral norm of the inverse can be bounded from above by*
$$\left\| \left(\mathbf{L}_w^J\right)^{-1} \right\|_2 \leq \frac{\left\| \left(\mathbf{L}^J\right)^{-1} \right\|_2}{w_{\min}}.$$

*Proof.* 1. Let the incidence matrix $\mathbf{B}$ describe any orientation of the graph $G_w$. Since in each column of the incidence matrix $B$ there is exactly one $+1$ and one $-1$, we conclude for $1 \leq i, j \leq m$

$$\begin{aligned}
\left(\mathbf{B}\mathbf{W}\mathbf{B}^T\right)_{ij} &= \sum_{k=1}^{n} b_{ik} w_{kk} b_{jk} \\
&= \begin{cases} \sum_{e \in E(\mathfrak{v}_i)} w(e) & \text{if } i = j \\ -w(e) & \text{if } i \neq j \text{ and if there exists an edge } e = \mathfrak{v}_i \mathfrak{v}_j \end{cases} \\
&= d_{ij} - a_{ij} \\
&= (\mathbf{L}_w)_{ij}.
\end{aligned}$$

2. The weak diagonal dominance is clear since $(\mathbf{L}^w)_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{m} (\mathbf{L}^w)_{ij}$, by definition. The positive semidefiniteness follows directly from the decomposition $\mathbf{L}_w = \mathbf{B}\mathbf{W}\mathbf{B}^T$ because $\mathbf{W}$ is positive definite.

3. For each component $C \subset \mathfrak{V}$ define a vector $x^C \in \mathbb{R}^m$ with

$$x_i^C = \begin{cases} 1 & \text{if } \mathfrak{v}_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

By construction, it holds $\mathbf{B}^T x^C = 0$ and thus, it is also $\mathbf{L}_w x^C = \mathbf{B}\mathbf{W}\mathbf{B}^T x^C = 0$. Hence, $x^C$ is an element of the kernel of the Laplacian and the dimension of the kernel is at least as large as the number of components. Let now be $x \in \ker \mathbf{L}_w$. Then, it follows

$$\begin{aligned} 0 &= x^T \mathbf{L}_w x \\ &= (\mathbf{B}^T x)^T \mathbf{W} (\mathbf{B}^T x) \end{aligned}$$

and because of the positive definiteness of $\mathbf{W}$ it is $\mathbf{B}^T x = 0$. The only possibility for this to be true is given by a vector $x$ which is constant on the components of the graph. This proves the statement.

4. Let $\mathbf{P}^J \in \mathbb{R}^{|J| \times m}$ be the matrix resulting from the $m \times m$ identity matrix removing all rows with indices not in $J$. Then, the principal submatrix can be written as

$$\mathbf{L}_w^J = \mathbf{P}^J \mathbf{L}_w (\mathbf{P}^J)^T.$$

Left be $x \in \mathbb{R}^{|J|}$. Using the positive definiteness of $\mathbf{W}$ and the decomposition of statement 1, we conclude

$$\begin{aligned} x^T \mathbf{L}_w^J x &= x^T \mathbf{P}^J \mathbf{L}_w (\mathbf{P}^J)^T x \\ &= x^T \mathbf{P}^J \mathbf{B} \mathbf{W} \mathbf{B}^T (\mathbf{P}^J)^T x \\ &= \left( \mathbf{B}^T (\mathbf{P}^J)^T x \right)^T \mathbf{W} \left( \mathbf{B}^T (\mathbf{P}^J)^T x \right) \\ &\geq 0. \end{aligned}$$

Now, let be $x^T \mathbf{L}_w^J x = 0$. We conclude $\mathbf{B}^T (\mathbf{P}^J)^T x = 0$, i.e. $(\mathbf{P}^J)^T x \in \ker \mathbf{B}^T$ from the same computation as before

Since the graph is connected, the null space of the transposed incidence matrix is only one-dimensional because it holds $\ker \mathbf{B}^T = \operatorname{span}(\mathbf{1})$. Thus, there exists a constant $c \in \mathbb{R}$ with

$$(\mathbf{P}^J)^T x = c\mathbf{1}.$$

By construction of the matrix $\mathbf{P}^J$, there exists at least one row of $(\mathbf{P}^J)^T$ which is zero. Thus, it follows $\operatorname{im}((\mathbf{P}^J)^T) \cap \operatorname{span}(\mathbf{1}) = \{0\}$, i.e. $c = 0$.

This yields $x \in \ker(\mathbf{P}^J)^T$ and, since the rank of $(\mathbf{P}^J)^T$ is equal to $|J|$, it holds

$$\ker(\mathbf{P}^J)^T = \{0\}.$$

Thus, we have proven the positive definiteness of the principal submatrix $\mathbf{L}_w^J$.

5. To prove the last property, we use again the decomposition

$$\mathbf{L}_w^J = \mathbf{P}^J \mathbf{L}_w \left( \mathbf{P}^J \right)^T = \mathbf{P}^J \mathbf{B} \mathbf{W} \mathbf{B}^T \left( \mathbf{P}^J \right)^T.$$

By Lemma 1.13 we conclude

$$\begin{aligned} \lambda_{\min}(\mathbf{L}_w^J) &= \lambda_{\min} \left( \mathbf{P}^J \mathbf{B} \mathbf{W} \mathbf{B}^T \left( \mathbf{P}^J \right)^T \right) \\ &\geq \lambda_{\min}(\mathbf{W}) \lambda_{\min} \left( \mathbf{P}^J \mathbf{B} \mathbf{B}^T \left( \mathbf{P}^J \right)^T \right) \\ &= w_{\min} \lambda_{\min}(\mathbf{L}^J). \end{aligned}$$

The statement for the spectral norm of the inverse follows directly since the norm of the inverse of a symmetric positive definite matrix is given by the reciprocal of the smallest eigenvalue, i.e.

$$\left\|(\mathbf{L}_w^J)^{-1}\right\|_2 = \frac{1}{\lambda_{\min}(\mathbf{L}_w^J)}.$$

□

## 1.2 Functional analysis

In this second introductory section, we will provide some functional analytical concepts and results, which will turn out to be useful in the course of this thesis. Most of the results are stated without a proof but at least a reference is always named.

This section briefly deals with six different topics: We begin with the existence theory of ordinary differential equations for discontinuous right-hand sides. Then, we recall Grönwall's inequality and the Schauder fixed point theorem. Afterwards, we shortly discuss the continuity of matrix valued functions and their inverses. Thereafter, we treat the concept of equi-integrability. The end of this section concerns a generalization of the Radon-Riesz property to continuous functions with values in a uniformly convex Banach space with a uniformly convex dual space. Since we did not find this useful generalization in the literature, we will provide a proof of it.

As mentioned before, we want to consider ordinary differential equations

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{1.3}$$

with a right-hand side $f$, which may be discontinuous in $t$. Therefore, we introduce the concept of Carathéodory differential equations. Due to the discontinuity, the equations (1.3) have to be formulated in an extended sense. We will recall a general and well-known theorem on the existence and uniqueness of solutions of such equations. More details can be found e.g. in [63] or [20]. In the following, we use the notation $\mathrm{pr}_i$ for the projection on the $i$-th component.

**Definition 1.17.** Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open set. A function $f\colon D \to \mathbb{R}^n$ is called *Carathéodory-Lipschitz vector field* if the following properties hold:

1. $f$ is a *Carathéodory mapping*, i.e. $f(t,x)$ is measurable in $t$ for fixed $x \in \mathrm{pr}_2 D$, $f(t,x)$ is continuous in $x$ for almost all fixed $t \in \mathrm{pr}_1 D$ and $f$ is locally integrable bounded. This means that for any compact subset $D_0 \subset D$ there exist a function $m \in L^1(\mathrm{pr}_1 D_0, [0,\infty))$ and a null subset $I_0 \subset \mathrm{pr}_1 D_0$ such that

   $$\|f(t,x)\| \leq m(t)$$

   holds for all $(t,x) \in D_0$ with $t \notin I_0$.

2. $f$ is *locally integrable Lipschitz* with respect to the second variable. More precisely, for each compact subset $D_0 \subset D$ there exist a function $L \in L^1(\mathrm{pr}_1 D_0, [0,\infty))$ and a null subset $I_0 \subset \mathrm{pr}_1 D_0$ with

   $$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|$$

for all $(t,x),(t,y) \in D_0$ with $t \notin I_0$.

It is possible to prove the local existence and uniqueness of solutions of (1.3) for this class of right-hand sides $f$:

**Theorem 1.18 (Local existence for ODEs).** *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open set and $f\colon D \to \mathbb{R}^n$ a Carathéodory-Lipschitz vector field. Then, it holds:*

1. *For each $(t_0, x_0) \in D$, there exists a unique maximal (i.e. non-continuable) locally absolutely continuous (i.e. absolutely continuous on each compact interval) solution $x(\cdot; t_0, x_0)\colon I(t_0, x_0) \subset \mathbb{R} \to \mathbb{R}^n$ of the equation (1.3) in the sense*

$$\dot{x}(t; t_0, x_0) = f(t, x(t; t_0, x_0))$$

*for almost all $t \in I(t_0, x_0)$ with*

$$x(t_0; t_0, x_0) = x_0.$$

2. *$I(t_0, x_0)$ is an open interval containing $t_0$, the domain $D_f = \{(t, t_0, x_0) | (t_0, x_0) \in D, t \in I(t_0, x_0)\} \subset \mathbb{R} \times D$ is an open subset and the mapping $x\colon D_f \to \mathbb{R}^n$ is continuous.*

3. *The mapping $x(t; t_0, \cdot)$ is locally Lipschitz continuous and the mappings $x(\cdot; t_0, x_0)$ and $x(t; \cdot, x_0)$ are locally absolutely continuous. Furthermore, they satisfy the integral equation*

$$x(t; t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x(s; t_0, x_0)) \mathrm{d}s$$

*for all $(t, t_0, x_0) \in D_f$.*

*Proof.* See e.g. [63]. □

The second topic in this section is Grönwall's inequality. In the literature, there are known various versions of this integral inequality. During this thesis, we will need a quite general form allowing $L^1$-functions on the right-hand side. For this reason, we will provide a proof of it, although the inequality can be found in many textbooks (e.g. [15] or [31]).

**Lemma 1.19 (Gronwall's inequality).** *Let $y \in L^\infty((0,T))$, $g \in L^1((0,T))$ be non-negative and $h \in L^1((0,T))$ such that it is*

$$y(t) \leq h(t) + \int_0^t g(s) y(s) \mathrm{d}s \tag{1.4}$$

*for almost all $t \in (0,T)$. Then, it holds*

$$y(t) \leq h(t) + \int_0^t h(s) g(s) \exp\left(\int_s^t g(r) \mathrm{d}r\right) \mathrm{d}s$$

*for almost all $t \in (0,T)$.*

*If, in addition, the function $h$ is non-decreasing, then it holds*

$$y(t) \leq h(t) \exp\left(\int_0^t g(s)\mathrm{d}s\right).$$

*Proof.* This proof relies on the proof of a similar version of Grönwall's inequality in [15]. We define the function

$$v(t) = \exp\left(-\int_0^t g(s)\mathrm{d}s\right)\int_0^t g(s)y(s)\mathrm{d}s,$$

which belongs to $W^{1,1}((0,T))$. For almost all $t \in (0,T)$, it is

$$v'(t) = \exp\left(-\int_0^t g(s)\mathrm{d}s\right) g(t)\left(y(t) - \int_0^t g(s)y(s)\mathrm{d}s\right)$$

and thus, using the assumption $g \geq 0$ and inequality (1.4) leads to

$$v'(t) \leq \exp\left(-\int_0^t g(s)\mathrm{d}s\right) g(t)h(t).$$

With $v(0) = 0$ it follows

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(s)\mathrm{d}s \\ &\leq \int_0^t h(s)g(s)\exp\left(-\int_0^s g(r)\mathrm{d}r\right)\mathrm{d}s \end{aligned}$$

and hence, again with inequality (1.4), we end up with

$$\begin{aligned} y(t) &\leq h(t) + \int_0^t g(s)y(s) \\ &= h(t) + v(t)\exp\left(\int_0^t g(r)\mathrm{d}r\right) \\ &\leq h(t) + \int_0^t h(s)g(s)\exp\left(-\int_0^s g(r)\mathrm{d}r\right)\mathrm{d}s \exp\left(\int_0^t g(r)\mathrm{d}r\right) \\ &= h(t) + \int_0^t h(s)g(s)\exp\left(\int_s^t g(r)\mathrm{d}r\right)\mathrm{d}s \end{aligned}$$

for almost all $t \in (0,T)$.

If we do now additionally assume that $h$ is non-decreasing, i.e. for almost all $s < t$ it holds $h(s) \leq h(t)$, then we conclude

$$\begin{aligned} y(t) &\leq h(t) + \int_0^t h(s)g(s)\exp\left(\int_s^t g(r)\mathrm{d}r\right)\mathrm{d}s \\ &\leq h(t) + h(t)\int_0^t g(s)\exp\left(\int_s^t g(r)\mathrm{d}r\right)\mathrm{d}s \\ &= h(t) + h(t)\left[-\exp\left(\int_s^t g(r)\mathrm{d}r\right)\right]_{s=0}^{s=t} \\ &= h(t)\exp\left(\int_0^t g(r)\mathrm{d}r\right) \end{aligned}$$

for almost all $t \in (0,T)$. $\square$

As a next topic, we introduce a very important tool for the proof of existence of solutions of nonlinear partial differential equations: The Schauder fixed point theorem ensures the existence of a fixed point of compact self-mappings on a Banach space. The theorem was originally proven in 1930 by Schauder [76]. We need the notion of compact mappings to formulate the result:

**Definition 1.20.** Let $X$ and $Y$ be normed vector spaces, let $M$ be a subset of $X$ and let $F\colon M \to Y$ be a mapping. $F$ is called a *compact mapping* if it is continuous and if bounded subsets of $M$ are mapped on relatively compact subsets of $Y$.

One can show the following result for these mappings.

**Theorem 1.21 (Schauder fixed point theorem** [76]**).** *Let $C \neq \emptyset$ be a closed, convex and bounded subset of a Banach space $X$ and let $F\colon C \to C$ be a compact mapping. Then, there exists at least one fixed point $x \in C$ of $F$, i.e. there exists $x \in C$ with $F(x) = x$.*

*Proof.* See e.g. [75]. $\square$

The fourth part of this section deals with the regularity of matrix valued functions. The precise statement, we will need later in this work, is that an invertible matrix $\mathbf{A}(v_0)$ is also invertible in a neighbourhood of $v_0$ if the matrix-valued function $\mathbf{A}$ is continuous. We will present a proof of this classical result.

**Lemma 1.22 (Regularity and Continuity).** *Let $U \subset \mathbb{R}^n$ be an open subset and let $\mathbf{A} \in C(U, \mathbb{R}^{n\times n})$ be a matrix-valued function. Furthermore, let be $v_0 \in U$ such that $\mathbf{A}(v_0)$ is regular.*

*Then, the matrix $\mathbf{A}(v_1)$ is also regular for all $v_1 \in U$ with $\|\mathbf{A}^{-1}(v_0)\|\|\mathbf{A}(v_0) - \mathbf{A}(v_1)\| < 1$ and $\mathbf{A}^{-1}$ is continuous in $v_1$.*

*Proof.* We will use a Neumann series to construct the inverse. For details about this kind of series see e.g. [82].

Let $v, v_0 \in U$ be given such that $\mathbf{A}(v_0)$ is invertible and

$$\left\|\mathbf{A}^{-1}(v_0)\right\| \|\mathbf{A}(v_0) - \mathbf{A}(v)\| < 1.$$

We write

$$\mathbf{A}(v_1) = \mathbf{A}(v_0)\left(\mathbf{Id} - \left(\mathbf{Id} - \mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)\right)\right).$$

The Neumann series $\sum_{k=0}^{\infty} \left(\mathbf{Id} - \mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)\right)^k$ converges since

$$\left\|\mathbf{Id} - \mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)\right\| \leq \left\|\mathbf{A}(v_0)^{-1}\right\| \|\mathbf{A}(v_0) - \mathbf{A}(v_1)\| < 1$$

holds due to the assumption on $v_1$. The limit of this series is equal to the inverse of the matrix $\mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)$ (see e.g. [82]), i.e. it holds

$$\sum_{k=0}^{\infty} \left(\mathbf{Id} - \mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)\right)^k = \left(\mathbf{A}(v_0)^{-1}\mathbf{A}(v_1)\right)^{-1},$$

which proves the invertibility of $\mathbf{A}(v_1)$.

Furthermore, we observe that we find a neighbourhood $V$ of $v_1$, where the matrix $\mathbf{A}(v)$ is regular. For $v \in V$, it holds

$$\begin{aligned}
&\left\|\mathbf{A}^{-1}(v_1) - \mathbf{A}^{-1}(v)\right\| \\
&= \left\|\mathbf{A}^{-1}(v_1)\left(\mathbf{A}(v) - \mathbf{A}(v_1)\right)\mathbf{A}^{-1}(v)\right\| \\
&\leq \left\|\mathbf{A}^{-1}(v_1)\right\| \|\mathbf{A}(v) - \mathbf{A}(v_1)\| \left(\left\|\mathbf{A}^{-1}(v_1)\right\| + \left\|\mathbf{A}^{-1}(v) - \mathbf{A}^{-1}(v_1)\right\|\right)
\end{aligned}$$

and hence, it follows

$$\left\|\mathbf{A}^{-1}(v_1) - \mathbf{A}^{-1}(v)\right\| \leq \frac{\|\mathbf{A}^{-1}(v_1)\|^2 \|\mathbf{A}(v) - \mathbf{A}(v_1)\|}{1 - \|\mathbf{A}^{-1}(v_1)\|\|\mathbf{A}(v_1) - \mathbf{A}(v)\|} \tag{1.5}$$

for $v \in U$ with $\|\mathbf{A}^{-1}(v_1)\|\|\mathbf{A}(v_1) - \mathbf{A}(v)\| < 1$. This estimate yields the continuity of the inverse since $\mathbf{A}$ itself is continuous. □

If the matrix-valued function is even Lipschitz continuous we can strengthen the result to obtain also the Lipschitz continuity of the inverse matrix.

**Corollary 1.23 (Lipschitz continuity).** *Additionally to the requirements of the previous lemma, let the matrix $\mathbf{A}(v)$ be locally Lipschitz continuous in $v_0 \in U$. Then, the inverse matrix is also locally Lipschitz continuous in $v_0$.*

*Proof.* The Lipschitz continuity can also be concluded from estimate (1.5). □

As next step, we recall the concept of equi-integrability from the integration theory. We refer to the textbook [3] of Bauer for a general introduction to the topic.

**Definition 1.24.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A set $M$ of $\mathcal{A}$-measurable functions on $\Omega$ is called *($\mu$-)equi-integrable* if for every $\varepsilon > 0$ there exists a $\mu$-integrable function $g = g_\varepsilon \geq 0$ on $\Omega$ such that every $f \in M$ satisfies

$$\int_{\{|f| \geq g\}} |f| \mathrm{d}\mu \leq \varepsilon.$$

A consequence of this definition is the following characterization of equi-integrable sets, which will be important in our analysis during the following chapters. We will often use the subsequent remark, in particular.

**Theorem 1.25 (Equi-integrability [3]).** *Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and let $h \in L^1(\Omega)$ be a strictly positive function. Then, for any set $M$ of $\mathcal{A}$-measurable functions on $\omega$ the following two statements are equivalent:*

1. *The set $M$ is equi-integrable.*

2. *The set $M$ satisfies*

$$\sup_{f \in M} \int_\Omega |f| \mathrm{d}\mu < \infty$$

*as well as the following: Given $\varepsilon > 0$ there exists $\delta > 0$ such that it holds*

$$\int_A h \mathrm{d}\mu \leq \delta \Rightarrow \int_A |f| \mathrm{d}\mu \leq \varepsilon \tag{1.6}$$

*for all $A \in \mathcal{A}$ and all $f \in M$.*

*Proof.* See e.g. [3]. □

*Remark* 1.26. For a finite measure, i.e. a measure with $\mu(\Omega) < \infty$, one can choose $h = 1$ in the previous theorem. Then, the implication (1.6) simplifies to

$$\mu(A) \leq \delta \Rightarrow \int_A |f| \mathrm{d}\mu \leq \varepsilon.$$

Important examples for equi-integrable sets are the convergent sequences in $L^1$ as stated in the next lemma.

**Lemma 1.27**. *Every sequence $(f_k)_k \subset L^1(\Omega)$ converging in the $L^1(\Omega)$-norm is equi-integrable.*

*Proof.* See e.g. [3]. □

To end this section, we present a result generalizing the Radon-Riesz property, which is a useful tool in order to prove the strong convergence of a sequence. In certain Banach spaces this property ensures the strong convergence of a weakly convergent sequence $x_k \rightharpoonup x$, if the sequence of the norm $\|x_k\|$ is converging to $\|x\|$. Details on this property can be found e.g. in [67]. We generalize this concept to continuous functions with values in a Banach space. To this end, we define the notion of uniform convexity first.

**Definition 1.28.** A normed vector space $V$ is called *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in V$ with $\|x\| = \|y\| \leq 1$ the following implication is true:

$$\|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

The proof of the desired generalization follows the same lines as the proof of the original property. We begin with the following auxiliary lemma ensuring the existence of a certain continuous function with values in the dual space.

**Lemma 1.29**. *Let $V$ be a Banach space with a uniformly convex dual space $V'$ and let be $f \in C([0,T], V)$. Then, there exists for each $\varepsilon > 0$ a function $\varphi \in C([0,T], V')$ with $\langle \varphi(t), f(t) \rangle = \|f(t)\|_V$ and $\|\varphi(t)\|_{V'} = 1$ for all $t$ with $\|f(t)\|_V \geq \varepsilon$.*

*Proof.* Let be $M = \{t \in [0,T] | \|f(t)\|_V \geq \varepsilon\}$. For each $t \in M$, we use the Hahn-Banach theorem (see e.g. [82]) to find a vector $\varphi(t) \in V'$ with $\langle \varphi(t), f(t)\rangle = \|f(t)\|_V$ and $\|\varphi(t)\|_{V'} = 1$. We will show the continuity of $\varphi$ on the closed set $M$. Afterwards, Dugundji's extension theorem [37] yields the existence of a continuous extension of $\varphi$ on $[0,T]$.

To show the continuity of $\varphi$ on $M$ with values in $V'$, let be $\tilde{\varepsilon} > 0$. We choose $\delta > 0$ like in the Definition 1.28 of the uniform convexity of $V'$. Because of the continuity of $\frac{f(t)}{\|f(t)\|_V}$ with values in $V$, we find $\gamma > 0$ such that

$$\left\| \frac{f(s)}{\|f(s)\|_V} - \frac{f(t)}{\|f(t)\|_V} \right\|_V < 2\delta$$

holds for all $s, t \in M$ with $|s-t| < \gamma$. Then, we compute

$$\begin{aligned}
\left\| \frac{\varphi(t)+\varphi(s)}{2} \right\|_{V'} &= \frac{1}{2} \sup_{\substack{v \in V \\ \|v\| \leq 1}} |\langle \varphi(t), v\rangle + \langle \varphi(s), v\rangle| \\
&\geq \frac{1}{2} \left| \left\langle \varphi(t), \frac{f(t)}{2\|f(t)\|_V} + \frac{f(s)}{2\|f(s)\|_V} \right\rangle + \left\langle \varphi(s), \frac{f(t)}{2\|f(t)\|_V} + \frac{f(s)}{2\|f(s)\|_V} \right\rangle \right| \\
&= \frac{1}{2} \left| \frac{\langle \varphi(t), f(t)\rangle}{\|f(t)\|_V} + \frac{\langle \varphi(s), f(s)\rangle}{\|f(s)\|_V} + \frac{1}{2} \left\langle \varphi(t), \frac{f(s)}{\|f(s)\|_V} - \frac{f(t)}{\|f(t)\|_V} \right\rangle \right. \\
&\qquad \left. + \frac{1}{2} \left\langle \varphi(s), \frac{f(t)}{\|f(t)\|_V} - \frac{f(s)}{\|f(s)\|_V} \right\rangle \right| \\
&\geq \frac{1}{2} \left( 2 - \frac{1}{2} \left| \left\langle \varphi(t), \frac{f(s)}{\|f(s)\|_V} - \frac{f(t)}{\|f(t)\|_V} \right\rangle \right| \right. \\
&\qquad \left. - \frac{1}{2} \left| \left\langle \varphi(s), \frac{f(t)}{\|f(t)\|_V} - \frac{f(s)}{\|f(s)\|_V} \right\rangle \right| \right) \\
&\geq 1 - \frac{1}{2} \left\| \frac{f(s)}{\|f(s)\|_V} - \frac{f(t)}{\|f(t)\|_V} \right\|_V \\
&> 1 - \delta.
\end{aligned}$$

Using the uniform convexity of the dual space $V'$ we conclude

$$\|\varphi(t) - \varphi(s)\|_{V'} < \tilde{\varepsilon},$$

which proves the desired continuity of $\varphi$. □

With this preliminary result, we are able to prove the following generalization of the Radon-Riesz property for uniformly convex Banach spaces with uniformly convex dual spaces.

**Lemma 1.30**. *Let $V$ be a uniformly convex Banach space with a uniformly convex dual space $V'$. Let $(f_n)_n \subset C([0,T], V)$ be a sequence and $f \in C([0,T], V)$ a function. Let the pair $((f_n)_n, f)$ fulfil the following two properties:*

1. $\|f_n(t)\|_V$ *converges uniformly to* $\|f(t)\|_V$ *and*

2. $\langle\varphi(t), f_n(t)\rangle$ *converges uniformly to* $\langle\varphi(t), f(t)\rangle$ *for all* $\varphi \in C([0,T], V')$.

*Then,* $f_n$ *converges to* $f$ *in the norm of* $C([0,T], V)$*, i.e.*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|f_n(t) - f(t)\|_V = 0.$$

*Proof.* If $f(t) = 0$ for all $t$, the statement is clearly true due to the uniform convergence of the norm. Thus, let be $f \neq 0$. We suppose that $f_n$ does not converge uniformly to $f$ in order to prove the lemma by contradiction. Then, there exists a subsequence, also denoted by $(f_n)_n$, such that there is an $\varepsilon > 0$ with

$$\|f_n - f\|_{C([0,T],V)} \geq \varepsilon \tag{1.7}$$

for all $n$. We choose $\delta \in (0,1]$ as in the Definition 1.28 of the uniform convex Banach space $V$ for $\tilde{\varepsilon} = \frac{\varepsilon}{2\|f\|_{C([0,T],V)}}$ and $N_1$ such that it holds

$$\Big| \|f_n(t)\|_V - \|f(t)\|_V \Big| < \frac{\varepsilon\delta}{3}$$

for all $t$ and all $n \geq N_1$. Using Lemma 1.29, we find a function $\varphi \in C([0,T], V')$ with $\|\varphi(t)\|_{V'} = 1$ and $\langle\varphi(t), f(t)\rangle = \|f(t)\|_V$ for all $t \in [0,T]$ with $\|f(t)\|_V \geq \frac{\varepsilon}{3}$.

Because of the uniform convergence of $\langle\varphi(t), f_n(t)\rangle$, there is also an integer $N_2$ such that it holds

$$\sup_{t\in[0,T]} \left| \frac{\langle\varphi(t), f_n(t) - f(t)\rangle}{2} \right| < \frac{\delta^2\varepsilon}{3}$$

for all $n \geq N_2$.

Due to (1.7), we can choose $\bar{t}$ with

$$\|f_N(\bar{t}) - f(\bar{t})\|_V \geq \varepsilon$$

for $N = \max(N_1, N_2)$. Now, it follows $\|f(\bar{t})\|_V > \frac{\varepsilon}{3}$, since if we suppose $\|f(\bar{t})\|_V \leq \frac{\varepsilon}{3}$, then we get the contradiction

$$\begin{aligned}
\|f_N(\bar{t}) - f(\bar{t})\|_V &\leq \|f_N(\bar{t})\|_V + \|f(\bar{t})\|_V \\
&\leq \Big| \|f_N(\bar{t})\|_V - \|f(\bar{t})\|_V \Big| + 2\|f(\bar{t})\|_V \\
&< \frac{\delta\varepsilon}{3} + \frac{2\varepsilon}{3} \\
&\leq \varepsilon.
\end{aligned}$$

From $\|f(\bar{t})\|_V > \frac{\varepsilon}{3}$ we conclude

$$\begin{aligned}
\|f_N(\bar{t})\|_V &\leq \Big| \|f_N(\bar{t})\|_V - \|f(\bar{t})\|_V \Big| + \|f(\bar{t})\|_V \\
&\leq \frac{\delta\varepsilon}{3} + \|f(\bar{t})\|_V \\
&< (1+\delta)\|f(\bar{t})\|_V
\end{aligned}$$

and furthermore, we compute

$$\left\| \frac{f_N(\bar{t})}{(1+\delta)\|f(\bar{t})\|} - \frac{f(\bar{t})}{(1+\delta)\|f(\bar{t})\|} \right\|_V \geq \frac{\varepsilon}{(1+\delta)\|f(\bar{t})\|_V} \geq \frac{\varepsilon}{2\|f\|_{C([0,T]),V)}}.$$

Therefore, we can use the uniform convexity of $V$ to conclude

$$\begin{aligned}
\left| \frac{\langle \varphi(\bar{t}), f_N(\bar{t}) + f(\bar{t}) \rangle}{2} \right| &\leq \frac{\|f_N(\bar{t}) + f(\bar{t})\|_V}{2} \\
&= (1+\delta)\|f(\bar{t})\|_V \left\| \frac{\frac{f_N(\bar{t})}{(1+\delta)\|f(\bar{t})\|} + \frac{f(\bar{t})}{(1+\delta)\|f(\bar{t})\|}}{2} \right\|_V \\
&\leq \|f(\bar{t})\|_V (1+\delta)(1-\delta) \\
&= \|f(\bar{t})\|_V (1-\delta^2).
\end{aligned}$$

Altogether, this yields the contradiction

$$\begin{aligned}
\|f(\bar{t})\|_V &= \langle \varphi(\bar{t}), f(\bar{t}) \rangle \\
&\leq \left| \frac{\langle \varphi(\bar{t}), f_N(\bar{t}) + f(\bar{t}) \rangle}{2} - \langle \varphi(\bar{t}), f(\bar{t}) \rangle \right| + \left| \frac{\langle \varphi(\bar{t}), f_N(\bar{t}) + f(\bar{t}) \rangle}{2} \right| \\
&< \frac{\delta^2 \varepsilon}{3} + \|f(\bar{t})\|_V (1-\delta^2) \\
&< \|f(\bar{t})\|_V \delta^2 + \|f(\bar{t})\|_V (1-\delta^2) \\
&= \|f(\bar{t})\|_V.
\end{aligned}$$

□

Later, we want to apply this result to functions with values in some Lebesgue spaces. Therefore, we recall that all $L^p$-spaces for $p \in (1, \infty)$ are uniformly convex.

**Lemma 1.31**. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let be $p \in (1, \infty)$. Then, the Banach space $L^p(\Omega)$ is uniformly convex.*

*Proof.* See e.g. [67]. □

# 2 Model

In this chapter we will introduce the low Mach number model we want to study in this thesis. Although there are a lot of textbooks dealing with the Navier-Stokes or Euler equations, we will give an introduction to this topic. In the first section, we briefly recall the derivation of the three-dimensional equations starting from the physical conservation principles using Reynold's transport theorem. The second section deals with the one-dimensional case. Our procedure in this section differs from most textbooks since we derive the one-dimensional equations from the three-dimensional equations using an asymptotic approach. In Section 2.3 we introduce the concept of partial differential equations on networks and we derive coupling conditions by another scaling of the three-dimensional Euler equations. The fourth section describes the above-mentioned low Mach number model and in the last section we reformulate the model formally such that it becomes more convenient for the further analysis.

## 2.1 Fundamental fluid dynamics

In the beginning of this section, we want to shortly recapitulate the derivation of the Euler and Navier-Stokes equations. More details on the derivation can be found e.g. in the books of Lions [65], Chorin and Marsden [19] or Boyer and Fabrie [15]. In this section we follow the latter to a large extend. We first introduce the notation of fluid elements together with Reynolds' transport theorem to formulate, in a second step, the physical conservation laws for the mass, the momentum and the energy. For simplicity we postpone the study of boundary conditions by initially considering the whole space $\mathbb{R}^N$ as domain.

We assume the fluid to be a continuous medium, i.e. we can describe all flow quantities such as the density, momentum, energy, temperature and pressure by functions depending on space and time without considering the single molecules. Let $\omega \subset \mathbb{R}^N$ be the volume filled by the fluid at the initial time $t_0 = 0$ and let the movement of the fluid be described by a family of injective maps $(\varphi_t)_t : \omega \to \mathbb{R}^N$ such that for any set $\Omega_0 \subset \omega$ the set $\varphi_t(\Omega_0)$ contains at time $t$ the same molecules as the set $\Omega_0$ at time $t = 0$. The family $\varphi = (\varphi_t)_t$ is called *flow*.

**Definition 2.1.** We call a family of sets $(\Omega_t)_t$ with $\Omega_t = \varphi(\Omega_0)$ *fluid element*. If $\Omega_0$ consists only of a single element $x_0$ it is called *fluid particle*.

**Definition 2.2.** The mapping $t \mapsto X(t, t_0, x_0) = \varphi_t\left(\varphi_{t_0}^{-1}(\{x_0\})\right)$, describing the movement of a particle, which was at position $x_0$ at time $t_0$, is called *trajectory*.

**Theorem 2.3 (Reynolds' transport theorem).** *Let $\varphi$ be a flow such that for each $t$ the map $\varphi_t$ is a smooth diffeomorphism and such that $t \mapsto \varphi_t$ is smooth. Then, for any function $f \in C^1(\mathbb{R} \times \mathbb{R}^N)$ and any $t$ the change of $f$ integrated over a fluid element is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} f(t,x)\mathrm{d}x = \int_{\Omega_t} \left( \frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u}) \right) \mathrm{d}x$$

*where $\mathbf{u}$ is the velocity field defined by*

$$\mathbf{u}(t,x) = \left. \frac{\partial X(s,t,x)}{\partial s} \right|_{s=t}.$$

*Proof.* See e.g. [15]. □

With this theorem we can easily state the physical conservation laws for mass, momentum and energy:

**Mass conservation:** Because of the continuous medium assumption, there exists a density function $\rho(t,x)$ such that for all sets $\Omega \subset \mathbb{R}^N$ the integral

$$\int_{\Omega} \rho(t,x)\mathrm{d}x$$

is equal to the mass contained in $\Omega$ at time $t$. Let $(\Omega_t)_t$ be an arbitrary fluid element and assume there is no mass source or sink in the domain, then the mass contained in the fluid element does not change in time, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho(t,x)\mathrm{d}x = 0.$$

Thus, with Reynolds' transport theorem it holds

$$\int_{\Omega_t} \left( \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{u}) \right) \mathrm{d}x = 0$$

and, since the fluid element is arbitrary, we also have

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{2.1}$$

where we used the short notation $\rho_t = \frac{\partial \rho}{\partial t}$ for the partial derivative. This equation is called *continuity equation*.

**Momentum balance:** To derive the momentum equation we use Newton's second law of motion: The change of linear momentum is equal to the sum of all acting forces. The total linear momentum in a fluid element $(\Omega_t)_t$ at time $t$ is given by

$$\int_{\Omega_t} \rho\mathbf{u}\mathrm{d}x.$$

The acting forces on this fluid element consist of two parts: The *volume forces*

$$\int_{\Omega_t} \rho \mathbf{f} \mathrm{d}x,$$

where $\mathbf{f}$ denotes the mass density of forces, and the *surface forces*

$$\int_{\partial\Omega_t} \boldsymbol{\tau}.\boldsymbol{\nu} \mathrm{d}\omega,$$

acting on the boundary $\partial\Omega_t$ of the fluid element. Here, $\boldsymbol{\nu}$ denotes the outer unit normal vector. The stress tensor $\boldsymbol{\tau}$ also consists of two parts, one corresponding to compression and the other to viscous effects. The first part is given by the hydrostatic pressure $-p\mathbf{Id}$ and in the case of Newtonian fluids the second part can be expressed by

$$\mathcal{T} = 2\mu\mathbf{D}(\mathbf{u}) + \lambda \operatorname{div}(\mathbf{u})\mathbf{Id},$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla u + \nabla \mathbf{u}^T)$ denotes the *strain rate tensor*. The coefficient $\mu > 0$ is called *dynamic viscosity* and defines together with $\lambda$ the *bulk viscosity* $\frac{2}{3}\mu + \lambda \geq 0$. Now, Newton's second law of motion implies together with Stokes' formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho \mathbf{u} \mathrm{d}x &= \int_{\Omega_t} \rho \mathbf{f} \mathrm{d}x + \int_{\partial\Omega_t} \left(-p\mathbf{Id} + \mathcal{T}\right).\boldsymbol{\nu} \mathrm{d}\omega \\ &= \int_{\Omega_t} \rho \mathbf{f} \mathrm{d}x + \int_{\Omega_t} \nabla\left(-p + (\lambda+\mu)\operatorname{div}(\mathbf{u})\right) + \mu\Delta\mathbf{u} \mathrm{d}x.\end{aligned}$$

Thus, again using Reynold's transport theorem, we end up with the vector-valued *momentum equation*

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla\left(p - (\lambda+\mu)\operatorname{div}(\mathbf{u})\right) = \rho\mathbf{f}. \tag{2.2}$$

**Energy balance:** The first law of thermodynamics states that the total energy of an isolated closed system is constant, i.e. the change of total energy in a fluid element $(\Omega_t)_t$ at time $t$ is equal to the sum of the work $W$ done by the forces and the heat exchange rate $Q$ with the exterior. The total energy in $\Omega_t$ can be expressed as the sum of internal and kinetic energy

$$\int_{\Omega_t} \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) \mathrm{d}x.$$

Here, $e$ denotes the specific internal energy. The work $W$ consists of two parts

$$W = \int_{\Omega_t} \rho\mathbf{f}^T\mathbf{u}\mathrm{d}x + \int_{\partial\Omega_t} \mathbf{u}^T(\boldsymbol{\tau}.\boldsymbol{\nu})\mathrm{d}\omega,$$

where the first part corresponds to the work done by the volume forces and the second one to the work done by the surface forces. The heat exchange rate $Q$ is also built from two summands

$$Q = \int_{\Omega_t} q\mathrm{d}x - \int_{\partial\Omega_t} \Phi(t,\omega,\boldsymbol{\nu})^T\boldsymbol{\nu}\mathrm{d}\omega.$$

Here, the first summand describes the heat sources and sinks in the fluid element and the second one describes the heat transfer through the boundary. The last term can be modelled with Fourier's law by

$$\Phi = -k\nabla T,$$

which states that the heat transfer is proportional to the change of the temperature with *thermal conductivity* $k$. Thus, the first law of thermodynamics and Reynolds' transport theorem yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)\mathrm{d}x &= \int_{\Omega_t}\left[\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)\right]_t + \operatorname{div}\left(\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}\right)\mathrm{d}x\\ &= \int_{\Omega_t}\rho\mathbf{f}^T\mathbf{u}+q\mathrm{d}x+\int_{\partial\Omega_t}\mathbf{u}\cdot(-p\mathbf{Id}+\mathcal{T})\boldsymbol{\nu}-\Phi^T\boldsymbol{\nu}\mathrm{d}\omega\\ &= \int_{\Omega_t}\rho\mathbf{f}^T\mathbf{u}+q-\operatorname{div}(p\mathbf{u})+\operatorname{div}(\mathcal{T}^T\mathbf{u})+\operatorname{div}(k\nabla T)\mathrm{d}x.\end{aligned}$$

Since the fluid element was arbitrary we receive the *energy equation*

$$\begin{aligned}&\left[\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)\right]_t+\operatorname{div}\left(\left[\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)+p\right]\mathbf{u}\right)-\operatorname{div}(\mathcal{T}\mathbf{u})-\operatorname{div}(k\nabla T)\\ &=\rho\mathbf{f}^T\mathbf{u}+q.\end{aligned}\tag{2.3}$$

The three equations (2.1), (2.2) and (2.3) contain five unknown variables. Therefore, we need a closure relation. From the theory of thermodynamics it is known that a thermodynamical state is fully determined by the *equation of state* if two out of the four state variables density $\rho$, pressure $p$, internal energy $e$ and temperature $T$ are known. As equation of state we will use the *ideal gas law*

$$p = R\rho T \quad \text{and} \quad e = c_v T,$$

where $R > 0$ is the specific gas constant and $c_v > 0$ is the specific heat capacity at constant volume. The law can be deduced from the kinetic theory of gases and it is a good approximation for the behaviour of many real gases. Moreover, we denote by $c_p = R + c_v$ the specific heat capacity at constant pressure and by $\gamma = \frac{c_p}{c_v}$ the adiabatic exponent of the gas. With the kinetic theory of gases one can develop a relation between the degrees of freedom $f$ of a gas and its adiabatic exponent

$$\gamma = 1 + \frac{2}{f}.$$

Thus, for a diatomic gas like oxygen it holds

$$\gamma = \frac{7}{5},$$

since it has three translational and two rotational degrees of freedom.

To end this section we summarize the derived *Navier-Stokes equations* with heat source $q$ and volume force $\mathbf{f}$:

$$\begin{aligned}\rho_t + \operatorname{div}(\rho\mathbf{u}) &= 0\\(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla\left(p - (\lambda+\mu)\operatorname{div}(\mathbf{u})\right) &= \rho\mathbf{f}\\\left[\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)\right]_t + \operatorname{div}\left(\left[\rho\left(e+\frac{1}{2}|\mathbf{u}|^2\right)+p\right]\mathbf{u}\right) - \operatorname{div}\left(\mathcal{T}\mathbf{u}\right) - \operatorname{div}\left(k\nabla T\right) &= \rho\mathbf{f}^T\mathbf{u} + q\\p &= R\rho T\\e &= c_v T.\end{aligned}$$

If we plug in the equation of state in the third equation we find

$$\left[\frac{p}{\gamma-1} + \frac{1}{2}\rho|\mathbf{u}|^2\right]_t + \operatorname{div}\left(\left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\gamma}{\gamma-1}p\right]\mathbf{u}\right) - \operatorname{div}(\mathcal{T}\mathbf{u}) - \operatorname{div}(k\nabla T) = \rho\mathbf{f}^T\mathbf{u} + q.$$

Setting $\lambda = \mu = k = 0$ leads to the interesting special case of the *Euler equations*:

$$\begin{aligned}\rho_t + \operatorname{div}(\rho\mathbf{u}) &= 0\\(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p &= \rho\mathbf{f}\\\left[\frac{p}{\gamma-1} + \frac{1}{2}\rho|\mathbf{u}|^2\right]_t + \operatorname{div}\left(\left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\gamma}{\gamma-1}p\right]\mathbf{u}\right) &= \rho\mathbf{f}^T\mathbf{u} + q.\end{aligned} \tag{2.4}$$

Both models have to be completed by suitable initial and boundary conditions.

## 2.2 Section-averaged Euler equations

As in our application we study the flow along an object which is almost one-dimensional, i.e. it is very long compared to its diameter. For example, we think of a tunnel. Therefore, in this section we average the three-dimensional Euler equations (2.4) over the cross-sectional area to receive a one-dimensional model which keeps the main characteristics of the flow. This one-dimensional object can be described by a smooth curve, which is defined as the geometric centers of the areas orthogonal to the curve. Of course, such a curve needs not to be straight. Therefore, we will use a local coordinate system where one coordinate is always in tangential direction to the curve. Namely, we will consider the Frenet-Serret-frame and transform the Euler equations in this new coordinate system. During this transform we scale the tangential and normal components differently since the main velocity is in tangential direction. By neglecting small terms and averaging over the cross-sectional area we end up with a set of one-dimensional equations. This procedure is very common, e.g. in the context of the shallow water equations it is used frequently [12, 30, 51]. The use of the Frenet-Serret-frame originates from the works of Bourdarias et al. [12] about water pipes and of Bouchut et al. [10, 11] about avalanches.

The above described procedure is different from the usual motivation of the one-dimensional Euler equations. Often the flow is assumed to be one-dimensional and the equations are derived from the physical conservation principles. One advantage of our approach is that we do not have to assume the normal velocity to be zero. Instead we assume the normal velocity to be of another order of magnitude than the tangential velocity as it can be observed in many situations. Furthermore, we do not need to restrict to straight objects. As we will see, the procedure works as long the curvature is not too big.

### 2.2.1 The geometry of a tunnel

To begin with, we introduce the local coordinate frame and discuss some of its properties. More details can be found in Spivak's textbook [78]. Let $\mathbf{c}\colon [0, L] \to \mathbb{R}^3$ be a smooth regular curve parametrized by its arc length. As usual, the unit tangent vector of the curve at the point $\mathbf{c}(x)$ is given by

$$\mathbf{T}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{c}(x) = \mathbf{c}'(x).$$

The *curvature* is defined as $\kappa(x) = \|\mathbf{c}''(x)\| = \|\mathbf{T}'(x)\|$ and describes the acceleration which is needed to move a particle along the curve with unit speed. If the curvature does not vanish, the direction of the acceleration is the normal vector

$$\mathbf{N}(x) = \frac{1}{\kappa(x)}\mathbf{T}'(x) = \frac{\mathbf{T}'(x)}{\|\mathbf{T}'(x)\|}.$$

Naturally, the binormal vector can be defined as

$$\mathbf{B}(x) = \mathbf{T}(x) \times \mathbf{N}(x).$$

The speed of the rotation of the binormal vector is measured by the *torsion* $\tau(x) = -\mathbf{N}(x)^T\mathbf{B}'(x)$. The vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ provide an orthonormal coordinate frame at each point.

Another way to obtain the same coordinate frame without the restriction of a non-vanishing curvature is to use the Frenet-Serret formulas: Assume the starting point $\mathbf{c}_0 \in \mathbb{R}^3$, the continuous curvature $\kappa\colon [0, L] \to [0, \infty)$ and the continuous torsion $\tau\colon [0, L] \to \mathbb{R}$ are known. Choose an orthonormal, right-handed initial coordinate frame $\mathbf{T}_0$, $\mathbf{N}_0$ and $\mathbf{B}_0$. Then, the Frenet-Serret frame is given as the solution of the ODE system

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T} &= \kappa\mathbf{N} \\
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{B} &= -\tau\mathbf{N}
\end{aligned}$$

with initial conditions

$$\mathbf{T}(0) = \mathbf{T}_0, \quad \mathbf{N}(0) = \mathbf{N}_0 \quad \text{and } \mathbf{B}(0) = \mathbf{B}_0.$$

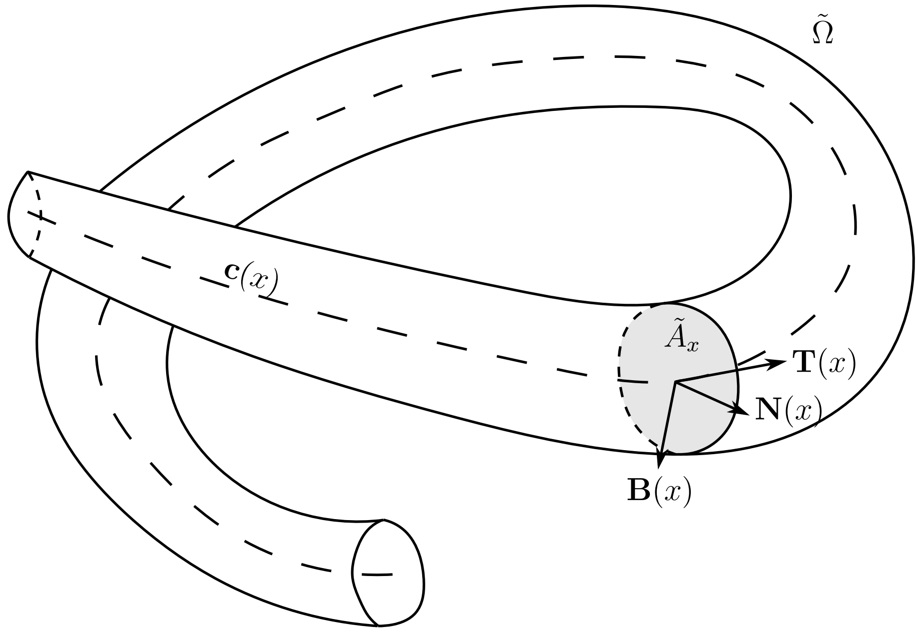



Figure 2.1: The curve $\mathbf{c}$ specifies the domain $\tilde{\Omega}$ and the local coordinate frame ($\mathbf{TNB}$).

Furthermore, the curve $\mathbf{c}$ is uniquely determined by

$$\mathbf{c}(x) = \mathbf{c}_0 + \int_0^x \mathbf{T}(s)\mathrm{d}s.$$

In a more compact way the system can be written in a matrix formulation as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}\begin{pmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix} = \begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}\mathbf{K}.$$

Both approaches can be shown to be equivalent for non-degenerated curves (see [78]).

As a next step, we need to specify the problem setting. With the help of the local coordinate frame, we can define the computational domain (see Figure 2.1). Let $L$ be the length of the tunnel and $\mathbf{c}\colon [0, L] \to \mathbb{R}^3$ a smooth regular curve with attached Frenet-Serret coordinate frame. For each $x \in [0, L]$ let $\tilde{A}_x \subset \mathbb{R}^2$ be a bounded convex domain with geometric center $(0, 0)^T$ and with uniform bounded volume $\mu(\tilde{A}_x) \geq c > 0$. Furthermore, we assume the area to depend smoothly on $x$ and

$$\kappa(x)y < L \tag{2.5}$$

for all $x \in [0, L]$ and $(y, z)^T \in \tilde{A}_x$ and the slices

$$S_x = \left\{\mathbf{c}(x) + y\mathbf{N}(x) + z\mathbf{B}(x) \middle| (y, z)^T \in \overline{\tilde{A}_x}\right\}$$

to be pairwise disjoint, i.e. $S_{x_1} \cap S_{x_2} = \emptyset$ for $x_1, x_2 \in [0, L]$ with $x_1 \neq x_2$. Then, we can define the domain

$$\tilde{\Omega} = \left\{ \mathbf{c}(x) + y\mathbf{N}(x) + z\mathbf{B}(x) \middle| x \in (0, L) \text{ and } (y, z)^T \in \tilde{A}_x \right\}.$$

By construction $\tilde{A}_x$ specifies the cross-sectional area perpendicular to the curve $\mathbf{c}$ and the geometric center of this area is precisely $\mathbf{c}(x)$. The boundary of $\tilde{\Omega}$ is divided into three parts, the tunnel surface

$$\tilde{\Gamma}_s = \left\{ \mathbf{c}(x) + y\mathbf{N}(x) + z\mathbf{B}(x) \middle| x \in (0, L) \text{ and } (y, z)^T \in \partial\tilde{A}_x \right\},$$

the tunnel entrance

$$\tilde{\Gamma}_{en} = \{\mathbf{c}(0) + y\mathbf{N}(0) + z\mathbf{B}(0) | (y, z)^T \in \overline{\tilde{A}_0}\}$$

and the tunnel exit

$$\tilde{\Gamma}_{ex} = \{\mathbf{c}(L) + y\mathbf{N}(L) + z\mathbf{B}(L) | (y, z)^T \in \overline{\tilde{A}_L}\}.$$

Let $(\tilde{\rho}, \tilde{u}, \tilde{p})$ be a strong solution of the non-scaled Euler equations (2.4), i.e.

$$\begin{aligned} \tilde{\rho}_{\tilde{t}} + \widetilde{\operatorname{div}}(\tilde{\rho}\tilde{\mathbf{u}}) &= 0 \\ (\tilde{\rho}\tilde{\mathbf{u}})_{\tilde{t}} + \widetilde{\operatorname{div}}(\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \tilde{\nabla}\tilde{p} &= \tilde{\rho}\tilde{\mathbf{f}} \\ \left[\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}|\tilde{\mathbf{u}}|^2\right]_{\tilde{t}} + \widetilde{\operatorname{div}}\left(\left[\frac{1}{2}\tilde{\rho}|\tilde{\mathbf{u}}|^2 + \frac{\gamma}{\gamma - 1}\tilde{p}\right]\tilde{\mathbf{u}}\right) &= \tilde{\rho}\tilde{\mathbf{f}}^T\tilde{\mathbf{u}} + \tilde{q} \end{aligned}$$

on $(0, \tilde{T}) \times \tilde{\Omega}$ with the three following boundary conditions:

**Impermeable walls:** There is no flow through the tunnel walls, i.e.

$$\tilde{\mathbf{u}}^T\tilde{\boldsymbol{\nu}} = 0 \tag{2.6}$$

on $\tilde{\Gamma}_s$ with the outer normal vector $\tilde{\boldsymbol{\nu}}$.

**Pressure:** At the tunnel ends the pressure is set to

$$\tilde{p} = \tilde{p}_{en} \quad \text{on } \tilde{\Gamma}_{en} \tag{2.7}$$

and

$$\tilde{p} = \tilde{p}_{ex} \quad \text{on } \tilde{\Gamma}_{ex}. \tag{2.8}$$

**Inflow:** For the density we prescribe the inflow at the tunnel ends, i.e.

$$\tilde{\rho}(t, \mathbf{x}) = \tilde{\rho}_{en}(t, \mathbf{x}) \quad \text{for } \mathbf{x} \in \tilde{\Gamma}_{en} \text{ with } \tilde{\mathbf{u}}(t, \mathbf{x})^T\tilde{\boldsymbol{\nu}} < 0 \tag{2.9}$$

and

$$\tilde{\rho}(t, \mathbf{x}) = \tilde{\rho}_{ex}(t, \mathbf{x}) \quad \text{for } \mathbf{x} \in \tilde{\Gamma}_{ex} \text{ with } \tilde{\mathbf{u}}(t, \mathbf{x})^T\tilde{\boldsymbol{\nu}} < 0. \tag{2.10}$$

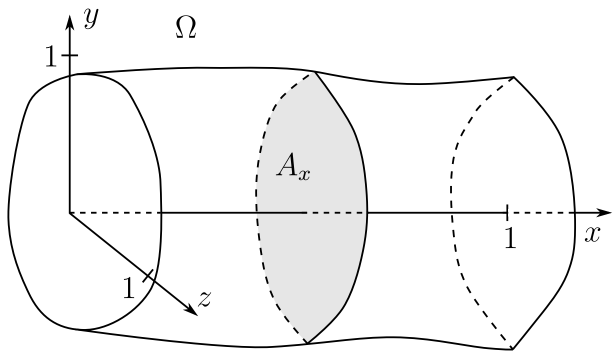



Figure 2.2: The scaled domain Ω.

### 2.2.2 Scaling of the three-dimensional Euler equations

In the next step, we will introduce a coordinate transform to transform the equations on a simpler domain. Therefore, let $d$ be a reference value for the diameter of the cross-sectional areas $\tilde{A}$ and define the smallness parameter $\varepsilon = \frac{d}{L} \ll 1$. We introduce for $x \in [0,1]$ the scaled cross-sectional area

$$A_x = \left\{ (y,z)^T \in \mathbb{R}^2 \middle| (yd, zd)^T \in \tilde{A}_{Lx} \right\}$$

and we set

$$\Omega = \left\{ (x,y,z)^T \in \mathbb{R}^3 \middle| x \in (0,1) \text{ and } (y,z)^T \in A_x \right\}$$

(see Figure 2.2). The diameter of both sets is by assumptions of order $\mathcal{O}(1)$. With the domain $\Omega$ we define the transform

$$\begin{aligned} \mathcal{T} \colon \Omega &\to \tilde{\Omega} \\ (x,y,z)^T &\mapsto \mathbf{c}(Lx) + yd\mathbf{N}(Lx) + zd\mathbf{B}(Lx), \end{aligned}$$

which carries out the different treatment of the normal and tangential component and which defines the non-scaled domain $\tilde{\Omega} = \mathcal{T}(\Omega)$. The Jacobian of $\mathcal{T}$ is given by

$$\begin{aligned} D\mathcal{T}(\mathbf{x}) &= L \begin{pmatrix} \mathbf{T} + \varepsilon y \left(-\kappa \mathbf{T} + \tau \mathbf{B}\right) - \varepsilon z \tau \mathbf{N} & \varepsilon \mathbf{N} & \varepsilon \mathbf{B} \end{pmatrix} \\ &= L \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix} \begin{pmatrix} 1 - \varepsilon \kappa y & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\tau z & 1 & 0 \\ \tau y & 0 & 1 \end{pmatrix} \\ &= L \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix} \mathbf{E}\mathbf{D} \end{aligned} \tag{2.11}$$

with

$$\mathbf{E} = \mathbf{E}_0 + \varepsilon \mathbf{E}_1, \quad \mathbf{E}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_1 = \begin{pmatrix} -\kappa y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ -\tau z & 1 & 0 \\ \tau y & 0 & 1 \end{pmatrix}.$$

The Jacobian is invertible because of

$$\det(D\mathcal{T}) = L\varepsilon^2(1 - \varepsilon\kappa y) = \varepsilon^2(L - \kappa y d),$$

which is always positive by assumption (2.5). Explicitly, the inverse is

$$\begin{aligned} D\mathcal{T}(\mathbf{x})^{-1} &= \frac{1}{L}\mathbf{D}^{-1}\mathbf{E}^{-1}\begin{pmatrix} \mathbf{T}^T \\ \mathbf{N}^T \\ \mathbf{B}^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \tau z & 1 & 0 \\ -\tau y & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\varepsilon\kappa y} & 0 & 0 \\ 0 & \frac{1}{\varepsilon} & 0 \\ 0 & 0 & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \mathbf{T}^T \\ \mathbf{N}^T \\ \mathbf{B}^T \end{pmatrix}. \end{aligned}$$

Now, we are able to specify the scaled variables. Let $t_r$ be a reference time and $U$ a reference velocity in tangential direction and assume the normal velocities to be of order $\varepsilon U$. We set

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= \frac{L}{U} D\mathcal{T}^{-1}(\mathbf{x})\tilde{\mathbf{u}}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \\ &= \frac{1}{U}\mathbf{D}^{-1}\mathbf{E}^{-1}\begin{pmatrix} \mathbf{T}^T \\ \mathbf{N}^T \\ \mathbf{B}^T \end{pmatrix} \tilde{\mathbf{u}}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \end{aligned}$$

and denote the velocity vector $\mathbf{u} = (u, v, w)^T$. From the second equality, we see that this realizes exactly the desired scaling of the tangential and normal components. The density, the pressure, the external force and the heat source are scaled such that

$$\begin{aligned} \rho(t, \mathbf{x}) &= \frac{1}{\rho_r}(1 - \varepsilon\kappa y)\tilde{\rho}\left(t_r t, \mathcal{T}(\mathbf{x})\right), \\ p(t, \mathbf{x}) &= \frac{1}{p_r}\tilde{p}\left(t_r t, \mathcal{T}(\mathbf{x})\right), \\ \mathbf{f}(t, \mathbf{x}) &= \frac{1}{f_r}\tilde{\mathbf{f}}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \end{aligned}$$

and

$$q(t, \mathbf{x}) = \frac{1}{q_r}\tilde{q}\left(t_r t, \mathcal{T}(\mathbf{x})\right).$$

Here, $\rho_r$, $p_r$, $f_r$ and $q_r$ denote reference values for the density, the pressure, the forces and the heat sources. For the following studies we require the scaled variables and all needed derivatives to be of order 1 and introduce the dimensionless *Mach number*

$$M = \frac{\sqrt{\rho_r}U}{\sqrt{\gamma p_r}}$$

and the parameters

$$\hat{f} = \frac{f_r t_r}{U}, \quad \hat{q} = \frac{q_r t_r(\gamma - 1)}{p_r} \quad \text{and} \quad h = \frac{U t_r}{L}.$$

*Remark* 2.4. For the time scaling $t_r = \frac{L}{U}$ is frequently used. However, this might cause different reference values for the time scale depending on the length of the edges of the network. In order to scale the whole network with the same reference time, we introduce the parameter $h$. This leads to a system of PDEs which is defined on the domain $(0,1) \times (0,T)$. The only assumption we need is $h$ to be of order $\mathcal{O}(\varepsilon^0)$.

To transform the Euler equations to the domain $\Omega$, we need the well-known chain rule for the divergence and the gradient that we recall in the following lemma.

**Lemma 2.5 (Chain rule).** *Let $\phi\colon \tilde{\Omega} \to \mathbb{R}^3$ and $\psi\colon \tilde{\Omega} \to \mathbb{R}$ be continuously differentiable. Then, it holds*

$$\operatorname{div}\left[\det D\mathcal{T}(\mathbf{x}) D\mathcal{T}(\mathbf{x})^{-1}\phi(\mathcal{T}(\mathbf{x}))\right] = \det D\mathcal{T}(\mathbf{x})\widetilde{\operatorname{div}}(\phi(\tilde{\mathbf{x}}))\Big|_{\mathcal{T}(\mathbf{x})}$$

*and*

$$\nabla(\psi(\mathcal{T}(\mathbf{x}))) = D\mathcal{T}(\mathbf{x})^T \tilde{\nabla}(\psi(\tilde{\mathbf{x}}))\Big|_{\mathcal{T}(\mathbf{x})}.$$

*Proof.* See e.g. [10]. $\square$

Using this lemma, we transform the equations starting with the continuity equation

$$\begin{aligned}
\rho_t(t,\mathbf{x}) &= \frac{t_r}{\rho_r}(1 - \varepsilon\kappa y)\tilde{\rho}_{\tilde{t}}\,(t_r t, \mathcal{T}(\mathbf{x})) \\
&= -\frac{t_r}{\rho_r}(1 - \varepsilon\kappa y)\widetilde{\operatorname{div}}(\tilde{\rho}\tilde{\mathbf{u}})\Big|_{\mathcal{T}(\mathbf{x})} \\
&= -\frac{t_r}{\rho_r}\operatorname{div}\left[(1 - \varepsilon\kappa y) D\mathcal{T}(\mathbf{x})^{-1}\tilde{\rho}\,(t_r t, \mathcal{T}(\mathbf{x}))\,\tilde{\mathbf{u}}\,(t_r t, \mathcal{T}(\mathbf{x}))\right] \\
&= -h\operatorname{div}(\rho\mathbf{u})\Big|_{(t,\mathbf{x})}.
\end{aligned}$$

In a similar way, we compute the scaled version of the momentum equation, omitting the arguments,

$$\begin{aligned}
\mathbf{D}^T\mathbf{E}^2\mathbf{D}(\rho\mathbf{u})_t &= \frac{t_r}{U\rho_r}(1 - \varepsilon\kappa y)\mathbf{D}^T\mathbf{E}\begin{pmatrix}\mathbf{T}^T \\ \mathbf{N}^T \\ \mathbf{B}^T\end{pmatrix}(\tilde{\rho}\tilde{\mathbf{u}})_{\tilde{t}} \\
&= -\frac{t_r}{UL\rho_r}(1 - \varepsilon\kappa y) D\mathcal{T}^T\left[\widetilde{\operatorname{div}}(\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^T)\Big|_{\mathcal{T}(\mathbf{x})} + \tilde{\nabla}\tilde{p}\Big|_{\mathcal{T}(\mathbf{x})} - \tilde{\rho}\tilde{\mathbf{f}}\right] \\
&= -\frac{t_r}{UL\rho_r}\Big[D\mathcal{T}^T\operatorname{div}\Big((1 - \varepsilon\kappa y)\tilde{\rho}(\mathcal{T}(\mathbf{x}))\tilde{\mathbf{u}}(\mathcal{T}(\mathbf{x}))\tilde{\mathbf{u}}(\mathcal{T}(\mathbf{x}))^T D\mathcal{T}^{-T}\Big) \\
&\qquad + (1 - \varepsilon\kappa y)\nabla\tilde{p}(\mathcal{T}(\mathbf{x})) - (1 - \varepsilon\kappa y)\tilde{\rho} D\mathcal{T}^T\tilde{\mathbf{f}}\Big]
\end{aligned} \tag{2.12}$$

$$= -h\mathbf{D}^T\mathbf{E}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\operatorname{div}\left(\rho\begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}\mathbf{EDuu}^T\right) - (1-\varepsilon\kappa y)\frac{h}{\gamma M^2}\nabla p$$
$$+ \hat{f}\rho\mathbf{D}^T\mathbf{E}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{f}.$$

Since $\begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}$ depends only on $x$ we compute for continuously differentiable matrix valued functions $\mathbf{W}$

$$\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\operatorname{div}(\mathbf{W}) = \operatorname{div}\left(\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{W}\right) - \left(\frac{\partial}{\partial x}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\right)\mathbf{W}\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$= \operatorname{div}\left(\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{W}\right) - \mathbf{K}^T\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{W}\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$= \operatorname{div}\left(\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{W}\right) + \mathbf{K}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{W}\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Hence, the momentum equation (2.12) simplifies to

$$\mathbf{D}^T\mathbf{E}^2\mathbf{D}(\rho\mathbf{u})_t + h\mathbf{D}^T\mathbf{E}\operatorname{div}\left(\rho\mathbf{EDuu}^T\right) + h\mathbf{D}^T\mathbf{EKEDu}u + (1-\varepsilon\kappa y)\frac{h}{\gamma M^2}\nabla p$$
$$= \hat{f}\rho\mathbf{D}^T\mathbf{E}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}\mathbf{f}.$$

Lastly, we also have to transform the energy equation. Therefore, we compute in the same manner

$$\left[\frac{1-\varepsilon\kappa y}{\gamma M^2(\gamma-1)}p + \frac{1}{2}\rho\mathbf{u}^T\mathbf{D}^T\mathbf{E}^2\mathbf{Du}\right]_t$$
$$= \frac{t_r}{U^2\rho_r}\left[(1-\varepsilon\kappa y)\left(\frac{\tilde{p}}{\gamma-1} + \frac{1}{2}\tilde{\rho}\tilde{\mathbf{u}}^T\tilde{\mathbf{u}}\right)\right]_{\tilde{t}}$$
$$= -\frac{t_r}{U^2\rho_r}(1-\varepsilon\kappa y)\left[\widetilde{\operatorname{div}}\left(\left[\frac{1}{2}\tilde{\rho}\tilde{\mathbf{u}}^T\tilde{\mathbf{u}} + \frac{\gamma}{\gamma-1}\tilde{p}\right]\tilde{\mathbf{u}}\right) - \tilde{\rho}\tilde{\mathbf{u}}^T\tilde{\mathbf{f}} - \tilde{q}\right]$$
$$= -h\operatorname{div}\left(\left[\frac{1}{2}\rho\mathbf{u}^T\mathbf{D}^T\mathbf{E}^2\mathbf{Du} + \frac{1-\varepsilon\kappa y)}{(\gamma-1)M^2}p\right]\mathbf{u}\right)$$
$$+ \hat{f}\rho\mathbf{u}^T\mathbf{D}^T\mathbf{E}\begin{pmatrix}\mathbf{T}^T\\ \mathbf{N}^T\\ \mathbf{B}^T\end{pmatrix}f + \frac{\hat{q}(1-\varepsilon\kappa y)}{(\gamma-1)\gamma M^2}q.$$

To transform the boundary conditions we need to determine the form of the three boundary parts of $\Omega$. They are now given by the tunnel surface

$$\Gamma_s = \mathcal{T}^{-1}(\tilde{\Gamma}_s) = \left\{(x, y, z) \in \mathbb{R}^3 \middle| x \in (0, 1) \text{ and } (y, z) \in \partial A_x\right\},$$

the tunnel entrance

$$\Gamma_{en} = \mathcal{T}^{-1}(\tilde{\Gamma}_{en}) = \left\{(0, y, z) \in \mathbb{R}^3 \middle| (y, z) \in \overline{A_0}\right\}$$

and the tunnel exit

$$\Gamma_{ex} = \mathcal{T}^{-1}(\tilde{\Gamma}_{ex}) = \left\{(1, y, z) \in \mathbb{R}^3 \middle| (y, z) \in \overline{A_L}\right\}.$$

If $\tilde{\boldsymbol{\nu}}$ is an outer normal vector on $\tilde{\Gamma}_s$ at the point $\tilde{\mathbf{x}} = \mathcal{T}(\mathbf{x})$, then $\boldsymbol{\nu} = D\mathcal{T}(\mathbf{x})^{-T}\tilde{\boldsymbol{\nu}}$ is an outer normal vector on $\Gamma_s$ at the point $\mathbf{x}$. Thus, the boundary conditions transform into:

**Impermeable walls**

$$0 = \frac{U}{L}\tilde{\mathbf{u}}^T\tilde{\boldsymbol{\nu}} = \mathbf{u}^T D\mathcal{T}^T D\mathcal{T}^{-T}\boldsymbol{\nu} = \mathbf{u}^T\boldsymbol{\nu} \qquad \text{on } \Gamma_s \tag{2.13}$$

**Pressure**

$$p(t, \mathbf{x}) = p_{en}(t, \mathbf{x}) = \frac{1}{p_r}\tilde{p}_{en}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \qquad \text{for } \mathbf{x} \in \Gamma_{en}$$

and

$$p(t, \mathbf{x}) = p_{ex}(t, \mathbf{x}) = \frac{1}{p_r}\tilde{p}_{ex}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \qquad \text{for } \mathbf{x} \in \Gamma_{ex}$$

**Inflow**

$$\rho(t, \mathbf{x}) = \rho_{en}(t, \mathbf{x}) = \frac{1}{\rho_r}\tilde{\rho}_{en}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \qquad \text{for } x \in \Gamma_{en} \text{ with } \mathbf{u}(t, \mathbf{x})^T\nu < 0$$

and

$$\rho(t, \mathbf{x}) = \rho_{ex}(t, \mathbf{x}) = \frac{1}{\rho_r}\tilde{\rho}_{ex}\left(t_r t, \mathcal{T}(\mathbf{x})\right) \qquad \text{for } x \in \Gamma_{ex} \text{ with } \mathbf{u}(t, \mathbf{x})^T\nu < 0.$$

Due to the fact that the scaled variables are assumed to be of order one, considering an approximation of order zero is reasonable, i.e we can neglect all terms of order $\mathcal{O}(\varepsilon)$. To this aim, we will have a closer look on the matrix product $\mathbf{ED}$ arising in the Jacobian (2.11). It holds

$$\begin{aligned}
\mathbf{ED} &= (\mathbf{E}_0 + \varepsilon\mathbf{E}_1)\,\mathbf{D} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\tau z & 1 & 0 \\ \tau y & 0 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon) \\
&= \mathbf{E}_0 + \mathcal{O}(\varepsilon)
\end{aligned}$$

and thus due to $\mathbf{E}_0\mathbf{K}\mathbf{E}_0 = \mathbf{0}$ it follows

$$\mathbf{D}^T\mathbf{E}\mathbf{K}\mathbf{E}\mathbf{D} = \mathbf{E}_0\mathbf{K}\mathbf{E}_0 + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon).$$

Thus, the Euler equations simplify in the zeroth order to

$$\begin{aligned} \rho_t + h\,\mathrm{div}(\rho\mathbf{u}) &= 0 \\ \mathbf{E}_0(\rho\mathbf{u})_t + h\mathbf{E}_0\,\mathrm{div}\left(\rho\mathbf{E}_0\mathbf{u}\mathbf{u}^T\right) + \frac{h}{\gamma M^2}\nabla p &= \hat{f}\rho\begin{pmatrix}\mathbf{T}^T\mathbf{f}\\ 0\\ 0\end{pmatrix} \\ \left[\frac{1}{\gamma M^2(\gamma-1)}p + \frac{1}{2}\rho u^2\right]_t + h\,\mathrm{div}\left(\left[\frac{1}{2}\rho u^2 + \frac{1}{(\gamma-1)M^2}p\right]\mathbf{u}\right)& \\ = \hat{f}\rho u\mathbf{T}^T\mathbf{f} &+ \frac{\hat{q}}{(\gamma-1)\gamma M^2}q \end{aligned} \tag{2.14}$$

on $(0,T)\times\Omega$. Explicitly, the three components of the momentum equation are

$$\begin{aligned} (\rho u)_t + h\,\mathrm{div}(\rho u\mathbf{u}) + \frac{h}{\gamma M^2}p_x &= \hat{f}\rho\mathbf{T}^T\mathbf{f} \\ \frac{h}{\gamma M^2}p_y &= 0 \\ \frac{h}{\gamma M^2}p_z &= 0. \end{aligned}$$

### 2.2.3 Averaging of the three-dimensional Euler equations

Since the set of equations (2.14) still depends on the normal components $v$ and $w$ of the velocity, we will average the equations over the cross-sectional area in the next step. But first of all, we need to specify the cross-sectional area $A_x$ in more detail. Similar as for the notion of fluid elements, we assume the existence of a family $(\varphi_x)_x$ of smooth injective mappings

$$\varphi_x\colon \overline{A_0} \to \mathbb{R}^2$$

with

$$A_x = \varphi_x(A_0)$$

and

$$\partial A_x = \varphi_x(\partial A_0). \tag{2.15}$$

We also demand the mapping $x \mapsto \varphi_x$ to be smooth. In this setting we can apply Reynold's transport theorem, which states for any function $f \in C^1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{A_x} f(x,y,z)\mathrm{d}y\mathrm{d}z = \int_{A_x}\left(\frac{\partial f}{\partial x} + \mathrm{div}_{(y,z)}(f\mathbf{a})\right)\mathrm{d}y\mathrm{d}z$$

with the vector field

$$\mathbf{a}(x,y,z) = \left.\frac{\partial\varphi_p\left(\varphi_x^{-1}(y,z)\right)}{\partial p}\right|_{p=x}.$$

By means of this vector field, we can characterize the outer normal vector of the tunnel surface $\Gamma_s$.

**Lemma 2.6**. *Let* $\mathbf{x} = (x, y, z)^T \in \Gamma_s$ *on the surface be given. Consider* $\boldsymbol{\nu}_{A_x} \in \mathbb{R}^2$*, an outer normal vector of the cross sectional area* $\partial A_x$ *at the point* $(y, z)$*. Then,*

$$\boldsymbol{\nu} = \begin{pmatrix} -\mathbf{a}(\mathbf{x})^T \boldsymbol{\nu}_{A_x}(y,z) \\ \boldsymbol{\nu}_{A_x}(y,z) \end{pmatrix} \in \mathbb{R}^3 \tag{2.16}$$

*is an outer normal vector to the surface* $\Gamma_s$ *at the point* $\mathbf{x}$*.*

*Proof.* Let $\mathbf{x} = (x, y, z) \in \Gamma_s$ be a point and let $\mathbf{c}(t)$ be a curve in $\partial A_x \subset \mathbb{R}^2$ with $\mathbf{c}(0) = (y, z)$. Define the curve

$$t \mapsto \mathbf{c}_1(t) = \begin{pmatrix} x \\ \mathbf{c}(t) \end{pmatrix}$$

in $\Gamma_s \subset \mathbb{R}^3$. Hence, the vector

$$\dot{\mathbf{c}}_1(0) = \begin{pmatrix} 0 \\ \dot{\mathbf{c}}(0) \end{pmatrix}$$

is tangential to the surface at the point $\mathbf{x}$ and any normal vector $\boldsymbol{\nu} = (n_1, n_2, n_3)^T$ has to be orthogonal to it, i.e.

$$0 = \dot{\mathbf{c}}_1(0)^T \boldsymbol{\nu} = \dot{\mathbf{c}}(0)^T \begin{pmatrix} n_2 \\ n_3 \end{pmatrix}.$$

Since $\dot{\mathbf{c}}(0)$ is a tangential vector on $\partial A_x$, this implies

$$\boldsymbol{\nu} = \begin{pmatrix} n_1 \\ d\boldsymbol{\nu}_{A_x} \end{pmatrix}$$

for some constant $d$. Now, let us have a look at the curve

$$t \mapsto \mathbf{c}_2(t) = \begin{pmatrix} t + x \\ \varphi_{t+x}(\varphi_x^{-1}(y, z)) \end{pmatrix}.$$

Because of the condition (2.15) it is $\varphi_{t+x}(\varphi_x^{-1}(y, z)) \in \partial A(t + x)$ and consequently $\mathbf{c}_2(t) \in \Gamma_s$ with $\mathbf{c}_2(0) = \mathbf{x}$. Using the definition of the vector field $\mathbf{a}$ we find the tangential vector

$$\dot{\mathbf{c}}_2(0) = \begin{pmatrix} 1 \\ \mathbf{a}(\mathbf{x}) \end{pmatrix},$$

which has to be orthogonal to $\boldsymbol{\nu}$, i.e.

$$0 = \dot{\mathbf{c}}_2(0)^T \boldsymbol{\nu} = n_1 + d\mathbf{a}(\mathbf{x})^T \boldsymbol{\nu}_{A_x}.$$

Concluding $n_1 = -d\mathbf{a}(\mathbf{x})^T \boldsymbol{\nu}_{A_x}$ finishes the proof. □

We define the *area*

$$A(x) = \int_{A_x} \mathrm{d}y\mathrm{d}z,$$

the *(average) density*

$$\bar{\rho}(t,x) = \frac{1}{A(x)} \int_{A_x} \rho \mathrm{d}y\mathrm{d}z,$$

the *(average) volume flow*

$$\overline{\rho u}(t,x) = \frac{1}{A(x)} \int_{A_x} \rho u \mathrm{d}y\mathrm{d}z,$$

the *(average) kinetic energy*

$$\frac{1}{2}\overline{\rho u^2} = \frac{1}{2A(x)} \int_{A_x} \rho u^2 \mathrm{d}y\mathrm{d}z,$$

the *(average) kinetic energy flux*

$$\overline{\rho u^3} = \frac{1}{A(x)} \int_{A_x} \rho u^3 \mathrm{d}y\mathrm{d}z,$$

the *(average) velocity*

$$\bar{u}(t,x) = \frac{1}{A(x)} \int_{A_x} u \mathrm{d}y\mathrm{d}z$$

and the *(average) heat source*

$$\bar{q}(t,x) = \frac{1}{A(x)} \int_{A_x} q \mathrm{d}y\mathrm{d}z$$

With the previous lemma, Reynold's transport theorem, the Gauss theorem (see e.g. [60]) and the boundary condition (2.13) we observe for any scalar quantity $f(t,\mathbf{x})$

$$\begin{aligned}
\int_{A_x} \mathrm{div}(f\mathbf{u})\mathrm{d}y\mathrm{d}z &= \int_{A_x} (fu)_x + \mathrm{div}_{(y,z)}\left(f\begin{pmatrix} v \\ w \end{pmatrix}\right)\mathrm{d}y\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{A_x} fu\mathrm{d}y\mathrm{d}z\right) - \int_{A_x} \mathrm{div}_{(y,z)}(fu\mathbf{a}))\mathrm{d}y\mathrm{d}z \\
&\quad + \int_{A_x} \mathrm{div}_{(y,z)}\left(f\begin{pmatrix} v \\ w \end{pmatrix}\right)\mathrm{d}y\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{A_x} fu\mathrm{d}y\mathrm{d}z\right) - \int_{\partial A_x}\left(fu\mathbf{a}^T\boldsymbol{\nu}_{A_x} - f\begin{pmatrix} v \\ w \end{pmatrix}^T \boldsymbol{\nu}_{A_x}\right)\mathrm{d}\omega \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{A_x} fu\mathrm{d}y\mathrm{d}z\right) + \int_{\partial A_x} f\mathbf{u}^T\boldsymbol{\nu}\mathrm{d}\omega \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{A_x} fu\mathrm{d}y\mathrm{d}z\right).
\end{aligned} \tag{2.17}$$

Equipped with the preceding considerations we are able to average the equations (2.14) over the cross-sectional area. Integrating the continuity equation leads to

$$(A\bar{\rho})_t + h(A\overline{\rho u})_x = 0. \tag{2.18}$$

For the momentum equation, we observe

$$\int_{A_x} p_x \mathrm{d}y\mathrm{d}z = A\bar{p}_x,$$

using the second and third component of the momentum equation. Assuming the force $\mathbf{f}$ to be independent of $y$ and $z$, the integration of the first component of the momentum equation yields

$$(A\overline{\rho u})_t + h\left(A\overline{\rho u^2} + \frac{A\bar{p}}{\gamma M^2}\right)_x = \frac{h}{\gamma M^2} A_x \bar{p} + \hat{f} A\bar{\rho}\mathbf{T}^T\mathbf{f}. \tag{2.19}$$

The average of the energy equation is given by

$$\left[\frac{A\bar{p}}{\gamma M^2(\gamma-1)} + \frac{1}{2}A\overline{\rho u^2}\right]_t + h\left(\frac{1}{2}A\overline{\rho u^3} + \frac{A\bar{p}\bar{u}}{(\gamma-1)M^2}\right)_x = \hat{f}A\overline{\rho u}\mathbf{T}^T\mathbf{f} + \frac{\hat{q}A\bar{q}}{(\gamma-1)\gamma M^2}. \tag{2.20}$$

In order to get a closed system of differential equations, we need to approximate the average kinetic energy $\overline{\rho u^2}$, the average kinetic energy flux $\overline{\rho u^3}$ and the average velocity $\bar{u}$ in terms of the density $\bar{\rho}$ and of the momentum $\overline{\rho u}$. Therefore, we define the ratios

$$\alpha_1 = \frac{\overline{\rho u}}{\bar{\rho}\bar{u}}, \quad \alpha_2 = \frac{\overline{\rho u^2}}{\bar{\rho}\bar{u}^2} \quad \text{and} \quad \alpha_3 = \frac{\overline{\rho u^3}}{\bar{\rho}\bar{u}^3}.$$

The equations (2.18) - (2.20) then read

$$\begin{aligned}
(A\bar{\rho})_t + h(\alpha_1 A\bar{\rho}\bar{u})_x &= 0\\
(\alpha_1 A\bar{\rho}\bar{u})_t + h\left(\alpha_2 A\bar{\rho}\bar{u}^2 + \frac{A\bar{p}}{\gamma M^2}\right)_x &= \frac{hA_x\bar{p}}{\gamma M^2} + \hat{f}A\bar{\rho}\mathbf{T}^T\mathbf{f}\\
\left[\frac{hA\bar{p}}{\gamma M^2(\gamma-1)} + \frac{1}{2}\alpha_2 A\bar{\rho}\bar{u}^2\right]_t + h\left(\frac{1}{2}\alpha_3 A\bar{\rho}\bar{u}^3 + \frac{A\bar{p}\bar{u}}{(\gamma-1)M^2}\right)_x &\\
&= \alpha_1\hat{f}A\bar{\rho}\bar{u}\mathbf{T}^T\mathbf{f} + \frac{\hat{q}A\bar{q}}{(\gamma-1)\gamma M^2}.
\end{aligned}$$

For practical purposes, we set the parameters $\alpha_i$ equal to one. This can be justified similar to the considerations in [30], where the shallow water equations are derived from the Reynolds-averaged Navier-Stokes equations. There it is shown that the variation in $z$ of the tangential velocity is of order $\mathcal{O}(\varepsilon)$. Thus, one can approximate $u = \bar{u} + \mathcal{O}(\varepsilon)$. In our case, including the viscous effects as well, leads to an additional term of order $\mathcal{O}(\frac{1}{\varepsilon^2})$ in equation (2.12) and thus to the condition

$$\operatorname{div}_{(y,z)}\Big((1-\varepsilon\kappa y)\nabla_{(y,z)}\left[(1-\varepsilon\kappa y)u\right]\Big) = 0$$

with boundary condition $\left(\nabla_{(y,z)}u\right)\boldsymbol{\nu}_{A_x} = 0$. The unique solution of this equation satisfying

$$\bar{u} = \frac{1}{A(x)}\int_{A_x} u\mathrm{d}y\mathrm{d}z$$

is given by

$$u = \frac{A(x)}{\int_{A_x} \frac{1}{1-\varepsilon\kappa y} \mathrm{d}y\mathrm{d}z} \frac{\bar{u}}{1-\varepsilon\kappa y}.$$

Therefore, we get $u = \bar{u} + \mathcal{O}(\varepsilon)$ as in [30] and it is reasonable to set $\alpha_i = 1$ for an approximation of order $\mathcal{O}(\varepsilon)$.

After averaging the equations, we also need to take into account the boundary conditions on the tunnel entrance and exit. For the pressure, this leads to the conditions

$$\bar{p}(t,0) = \frac{1}{A(0)} \int_{A_0} p_{en}(t,0,y,z)\mathrm{d}y\mathrm{d}z$$

and

$$\bar{p}(t,1) = \frac{1}{A(1)} \int_{A_1} p_{ex}(t,1,y,z)\mathrm{d}y\mathrm{d}z.$$

The inflow boundary conditions for the density are transformed into

$$\bar{\rho}(t,0) = \frac{1}{A(0)} \int_{A_0} \rho_{en}(t,0,y,z)\mathrm{d}y\mathrm{d}z \qquad \text{if } \bar{u}(t,0) > 0$$

and

$$\bar{\rho}(t,1) = \frac{1}{A(1)} \int_{A_1} \rho_{ex}(t,1,y,z)\mathrm{d}y\mathrm{d}z \qquad \text{if } \bar{u}(t,1) < 0.$$

At this point, the formal derivation of the section-averaged Euler equations and the corresponding boundary conditions is completed. Nevertheless, we should additionally take into account the physical phenomena of friction and heat loss at the walls. Neglecting the viscous effects does not permit to directly include the wall friction in the model. This is the main drawback of using the Euler equations in the derivation. In [30], Decoene et al. have shown for the shallow water equations how to asymptotically derive the friction using the Reynold-averaged Navier-Stokes equations with the boundary condition

$$\mu \mathbf{t}^T \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \boldsymbol{\nu} = -\alpha \xi(\mathbf{u}) \mathbf{t}^T \mathbf{u}$$

relating the rate of strain at the wall to the tangential velocity. Here, $\mathbf{t}$ denotes a tangential vector. This kind of condition was originally proposed in a similar way by Navier in 1823 [72].

Since we decided to start with the Euler equations, we can only include the effect of wall friction heuristically a posteriori. Therefore, we will use an external force

$$\tilde{\mathbf{f}}_f = -\frac{\xi}{d} \alpha(\tilde{u}) \tilde{u} \mathbf{T}$$

acting in the opposite direction of the flow. Here, $\xi$ is a positive friction parameter and typically it is either $\alpha(\tilde{u}) = |\tilde{u}|$ or $\tilde{\alpha}(\tilde{u}) = 1$. For more details see e.g. the textbook [68]. There are also discussed different approximative formulas for the parameter $\xi$. As a reference value for the force we choose $f_r = \frac{U^2}{d}\xi$. Thus, we have with $\zeta = \frac{f_r t_r}{U} = \xi \frac{t_r U}{d}$ the scaled force

$$\hat{f} \mathbf{T}^T \mathbf{f}_f = -\zeta \alpha(u) u.$$

For some application, it is also interesting to take the heat loss at the walls into account. Again, this cannot be derived from the Euler equations and has to be included afterwards. Often, this effect is modelled by a heat source

$$\tilde{q}_w = \alpha(\tilde{T} - \tilde{T}_w),$$

where $\alpha$ denotes the heat loss coefficient and $\tilde{T}_w$ is the surface temperature (see e.g. [81]). Using the ideal gas law and the reference value $q_r = \frac{\alpha p_r}{R \rho_r}$ we find with $\eta = \frac{q_r t_r(\gamma-1)}{p_r}$ and $T_w = \frac{R\rho_r}{p_r}\tilde{T}_w$ the scaled source term

$$\hat{q}\bar{q} = \eta q_w = \eta\left(\frac{p}{\rho} - T_w\right).$$

To conclude the section, we summarize once more the equations omitting the bars

$$\begin{aligned}
(A\rho)_t + h(A\rho u)_x &= 0\\
(A\rho u)_t + h\left(A\rho u^2 + \frac{Ap}{\gamma M^2}\right)_x &=\\
&\frac{hA_x p}{\gamma M^2} - \zeta A\rho\alpha(u)u + \hat{f}A\rho\mathbf{T}^T\mathbf{f}\\
\left[\frac{Ap}{\gamma M^2(\gamma-1)} + \frac{1}{2}A\rho u^2\right]_t + h\left(\frac{1}{2}A\rho u^3 + \frac{Apu}{(\gamma-1)M^2}\right)_x &=\\
&\hat{f}A\rho u\mathbf{T}^T\mathbf{f} - \zeta A\rho\alpha(u)u^2 + \frac{\eta A\left(\frac{p}{\rho} - T_w\right) + \hat{q}Aq}{(\gamma-1)\gamma M^2}
\end{aligned} \tag{2.21}$$

on $(0,1)\times(0,T)$. These are for $h = 1$ the usual one-dimensional section-averaged Euler equations with source terms.

## 2.3 Section-averaged Euler equations on networks

In this section, we want to describe the flow in a tunnel network, i.e. a system of tunnels that are connected at some nodes (see Figure 2.3). Since the edges of such a network are almost one-dimensional, the idea is to use the section-averaged equations derived in the previous section. At the nodes, the equations need to be coupled. Often, the coupling conditions are derived heuristically from some physical principles. We will take a different approach using a scaling of the three-dimensional Euler equations.

In recent years, the research on differential equations on networks has gained a lot of interest. A general review for hyperbolic balance laws on networks has been written by Bressan et al. [17]. For general $2\times 2$ hyperbolic systems a well-posedness result is presented by Colombo et al. in [21]. Colombo and Mauri [24] and Colombo and Marcellini [23] treated the full $3\times 3$ Euler equations. A numerical study of this problem is provided by Herty in [54].

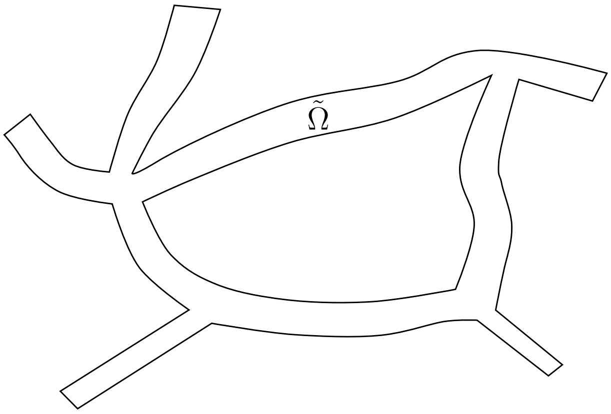



Figure 2.3: A typical domain $\tilde{\Omega}$ which can be simplified by the network approach.

### 2.3.1 Geometry of a network

The first step in the derivation of the Euler equations on a network is the precise specification of the geometry. This is quite technical, but the idea is rather simple. We will define sets of inner and outer nodes and afterwards connect the nodes by edges, which are constructed as in the previous section. Here, *inner node* refers to a node with more than one adjacent edge and *outer node* to one with exact one adjacent edge.

Let $\widetilde{\mathfrak{V}}_{\text{in}} = \{\widetilde{\mathfrak{V}}_1, \ldots, \widetilde{\mathfrak{V}}_k\}$ be a set of convex open subsets $\widetilde{\mathfrak{V}}_j$ of $\mathbb{R}^3$ with pairwise disjoint closure, i.e. $\overline{\widetilde{\mathfrak{V}}_{j_1}} \cap \overline{\widetilde{\mathfrak{V}}_{j_2}} = \emptyset$ for $j_1 \neq j_2$. We call these subsets inner nodes and assume their diameter to be of order $\mathcal{O}(d)$. Furthermore, let $\widetilde{\mathfrak{V}}_{\text{out}} = \{\tilde{\mathfrak{v}}_{k+1}, \ldots, \tilde{\mathfrak{v}}_m\} \subset \mathbb{R}^3$ be the set of outer nodes such that $\widetilde{\mathfrak{V}}_{\text{out}} \cap \widetilde{\mathfrak{V}}_j = \emptyset$. Note that the outer nodes are points whereas the inner nodes are sets.

The edges are defined by $n$ smooth regular curves $\mathbf{c}_i \colon [0, L_i] \to \mathbb{R}^3$ and their cross-sectional-areas $\tilde{A}^i_x \subset \mathbb{R}^2$. For the curve $\mathbf{c}_i$ we denote the curvature by $\kappa_i$, the torsion by $\tau_i$ and the corresponding Frenet-Serret frame by $\begin{pmatrix}\mathbf{T}_i & \mathbf{N}_i & \mathbf{B}_i\end{pmatrix}$. For technical reasons, the edges are supposed to be straight at the ends, i.e. it is $\operatorname{supp}(\kappa_i) \subset (0, L_i)$ and $\operatorname{supp}(\tau_i) \subset (0, L_i)$. As in Section 2.2 we make five assumptions on the area $\tilde{A}^i_x$: 1. $\tilde{A}^i_x$ depends smoothly on $x$. 2. The geometrical center is $(0,0)^T$. 3. The volume is uniformly bounded, i.e. $\mu(\tilde{A}^i_x) \geq c > 0$. 4. The curvature is bounded in order to avoid intersections of neighbouring cross-sectional areas, i.e. $\kappa_i(x)y < L_i$ for all $x \in [0, L_i]$ and $(y,z)^T \in \tilde{A}^i_x$. 5. The slices

$$S^i_x = \left\{\mathbf{c}_i(x) + y\mathbf{N}_i(x) + z\mathbf{B}_i(x) \middle| (y,z)^T \in \overline{\tilde{A}^i_x}\right\}$$

are pairwise disjoint, i.e. $S^i_{x_1} \cap S^i_{x_2} = \emptyset$ for $x_1, x_2 \in [0, L_i]$ with $x_1 \neq x_2$. The fifth assumption guarantees that the tunnel is not intersecting itself.

Then, the $i$-th edge or tunnel is given by the set

$$\tilde{\Omega}_i = \left\{\mathbf{c}_i(x) + y\mathbf{N}_i(x) + z\mathbf{B}_i(x) \middle| x \in (0, L_i) \text{ and } (y,z)^T \in \tilde{A}^i_x\right\}$$

with its tunnel entrance

$$\tilde{\Gamma}^i_{en} = \left\{ \mathbf{c}_i(0) + y\mathbf{N}_i(0) + z\mathbf{B}_i(0) \middle| (y,z)^T \in \overline{\tilde{A}^i_0} \right\}$$

and its tunnel exit

$$\tilde{\Gamma}^i_{ex} = \left\{ \mathbf{c}_i(L_i) + y\mathbf{N}_i(L_i) + z\mathbf{B}_i(L_i) \middle| (y,z)^T \in \overline{\tilde{A}^i_{L_i}} \right\}.$$

Now, that we have defined sets of nodes and edges, we need to couple them. Thus, we assume for each edge $\tilde{\Omega}_i$ that the ends are either equal to an outer node, i.e. $\mathbf{c}_i(0) \in \tilde{\mathfrak{V}}_{\text{out}}$ or $\mathbf{c}_i(L_i) \in \tilde{\mathfrak{V}}_{\text{out}}$, respectively, or the tunnel ends lie on the boundary $\partial\tilde{\mathfrak{V}}_j$ of an inner node, i.e. $\Gamma^i_{en} \subset \partial\tilde{\mathfrak{V}}_j$ or $\Gamma^i_{ex} \subset \partial\tilde{\mathfrak{V}}_j$, respectively. Additionally, we require that each outer node $\tilde{\mathfrak{v}}_j$ is the end of exact one edge and for each inner node $\tilde{\mathfrak{V}}_j$ there are at least two adjacent edges.

Of course, the only intersections of the edges should be at the nodes. Hence, it is necessary to assume the edges to be pairwise disjoint, i.e. $\tilde{\Omega}_{i_1} \cap \tilde{\Omega}_{i_2} = \emptyset$ for $i_1 \neq i_2$, and to assume the edges and nodes to be also disjoint, i.e. $\tilde{\Omega}_i \cap \tilde{\mathfrak{V}}_j = \emptyset$. Last but not least, we exclude that a slice corresponding to an outer node intersects the closure of an inner node, i.e. for all $\tilde{\mathfrak{v}}_j$ and the corresponding curve $\mathbf{c}_i$ with $\mathbf{c}_i(0) = \tilde{\mathfrak{v}}_j$ or $\mathbf{c}_i(L_i) = \tilde{\mathfrak{v}}_j$ holds $\Gamma^i_{en} \cap \overline{\tilde{\mathfrak{V}}_l} = \emptyset$ or $\Gamma^i_{ex} \cap \overline{\tilde{\mathfrak{V}}_l} = \emptyset$, respectively for all $\tilde{\mathfrak{V}}_l \in \tilde{\mathfrak{V}}_{\text{in}}$.

Now, the whole network can be defined as

$$\tilde{\Omega} = \operatorname{Int}\left( \bigcup_{i=1}^{n} \tilde{\Omega}_i \cup \bigcup_{j=1}^{k} \overline{\tilde{\mathfrak{V}}_j} \right).$$

This construction using the closure and the interior is necessary in order to include the connection between the edges and the inner nodes in the domain $\tilde{\Omega}$. The boundary of $\tilde{\Omega}$ is naturally divided into two parts. The first one consists of the outer nodes and the corresponding tunnel ends

$$\tilde{\Gamma}_{\text{out}} = \left( \bigcup_{\substack{j=k+1 \\ \exists i: \mathbf{c}_i(0) = \tilde{\mathfrak{v}}_j}}^{m} \tilde{\Gamma}^i_{en} \right) \cup \left( \bigcup_{\substack{j=k+1 \\ \exists i: \mathbf{c}_i(L_i) = \tilde{\mathfrak{v}}_j}}^{m} \tilde{\Gamma}^i_{ex} \right)$$

and the second one consists of the tunnel surface

$$\tilde{\Gamma}_s = \partial\tilde{\Omega} \setminus \tilde{\Gamma}_{\text{out}}.$$

### 2.3.2 Scaling and coupling conditions

After setting up the geometry we can start studying the three-dimensional Euler equations (2.4) on the whole domain $(0,\tilde{T}) \times \tilde{\Omega}$ with the boundary conditions (compare (2.6) - (2.10))

$$\tilde{\mathbf{u}}^T \tilde{\boldsymbol{\nu}} = 0 \qquad \text{on } \tilde{\Gamma}_s \tag{2.22}$$

$$\tilde{p} = \tilde{p}_{\text{out}} \qquad \text{on } \tilde{\Gamma}_{\text{out}} \tag{2.23}$$

and

$$\tilde{\rho}(t,\mathbf{x}) = \tilde{\rho}_{\text{out}}(t,\mathbf{x}) \qquad\qquad \text{for } \mathbf{x} \in \tilde{\Gamma}_{\text{out}} \text{ with } \tilde{\mathbf{u}}(t,\mathbf{x})^T \tilde{\boldsymbol{\nu}} < 0. \tag{2.24}$$

For each edge $\tilde{\Omega}_i$ we can use the scaling and construction of the transform $\mathcal{T}_i$ of Section 2.2 to conclude the one-dimensional equations on the edges

$$\begin{aligned}
(A_i\rho_i)_t + h_i(A_i\rho_i u_i)_x &= 0\\
(A_i\rho_i u_i)_t + h_i\left(A_i\rho_i u_i^2 + \frac{A_i p_i}{\gamma M^2}\right)_x &= \\
\frac{h_i(A_i)_x p_i}{\gamma M^2} &- \zeta_i A_i\rho_i\alpha(u_i)u_i + \hat{f}A_i\rho_i f_i\\
\left(\frac{A_i p_i}{(\gamma-1)\gamma M^2} + \frac{1}{2}A_i\rho_i u_i^2\right)_t + h_i\left(\frac{1}{2}A_i\rho_i u_i^3 + \frac{A_i p_i u_i}{(\gamma-1)M^2}\right)_x &= \\
\hat{f}A_i\rho_i u_i f_i - \zeta_i A_i\rho_i\alpha(u_i)u_i^2 &+ \frac{\eta_i A_i\left(\frac{p_i}{\rho_i} - T_w^i\right) + \hat{q}A_i q_i}{(\gamma-1)\gamma M^2}
\end{aligned} \tag{2.25}$$

on $(0,T)\times(0,1)$. Here, $A_i$, $\rho_i$, $u_i$, $p_i$, $q_i$ and $f_i = \mathbf{T}_i^T\mathbf{f}$ denote the section averaged quantities as defined in the previous section. At this stage, the advantage of the introduction of the parameter $h_i = \frac{Ut_r}{L_i}$ in the last section becomes evident. Using $h_i$, we are able to scale all edges to the domain $(0,T)\times(0,1)$ using the same reference time and reference velocity although the lengths of the edges are not the same.

The main part of this section is devoted to the study of the boundary conditions for the one-dimensional equations. An elegant way to formulate these conditions uses notations of the graph theory (see Section 1.1) and a graph, which reflects the geometry of $\tilde{\Omega}$. Therefore, we build a weighted oriented graph $G = (\mathfrak{V}, E, w, \text{init}, \text{ter})$ in the following way (compare Figure 2.4): Let $\mathfrak{V} = \{\mathfrak{v}_1,\ldots,\mathfrak{v}_m\}$ be the vertex set with $m$ elements. The first $k$ elements represent the inner nodes and the other the outer nodes. The edge set $E = \{e_1,\ldots,e_n\}$ and the orientation reflect the adjacency relations of the edges $\tilde{\Omega}_i$ and the nodes. To be precise, it is $e_i = \mathfrak{v}_{j_1}\mathfrak{v}_{j_2}$, $\text{init}(e_i) = \mathfrak{v}_{j_1}$ and $\text{ter}(e_i) = \mathfrak{v}_{j_2}$ with

$$j_1 = \begin{cases} j & \text{if } \mathbf{c}_i(0) = \tilde{\mathfrak{v}}_j\\ j & \text{if } \mathbf{c}_i(0) \in \bar{\tilde{\mathfrak{V}}}_j\end{cases}$$

and

$$j_2 = \begin{cases} j & \text{if } \mathbf{c}_i(L_i) = \tilde{\mathfrak{v}}_j\\ j & \text{if } \mathbf{c}_i(L_i) \in \bar{\tilde{\mathfrak{V}}}_j.\end{cases}$$

Clearly, either $\mathbf{c}_i(0) = \tilde{\mathfrak{v}}_j$ or $\mathbf{c}_i(0) \in \bar{\tilde{\mathfrak{V}}}_j$ is true for some $j$ and the values of $j_1$ and $j_2$ are well-defined. The weights of the graph describe the lengths of the edges, this means $w(e_i) = L_i$.

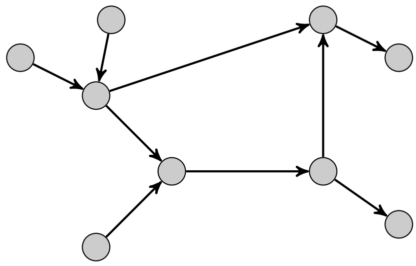


Figure 2.4: Graph describing the same geometry as $\tilde{\Omega}$ in Figure 2.3.

Now, it is rather simple to transform the boundary conditions at the outer nodes. Therefore, let $\mathfrak{v}_j$ be an outer node with adjacent edge $e_i \in E(\mathfrak{v}_j)$. If $\mathfrak{v}_j$ is the initial vertex of the edge $e_i$, i.e. $e_i \in E^+(\mathfrak{v}_j)$, we find by averaging the following boundary conditions

$$p_i(t,0) = \bar{p}^j_{\text{out}}(t) = \frac{1}{p_r A_i(0)} \int_{A^i_0} \tilde{p}_{\text{out}}(t_r t, \mathcal{T}_i(0,y,z))\mathrm{d}y\mathrm{d}z \tag{2.26}$$

and

$$\rho_i(t,0) = \rho^j_{\text{out}}(t) = \frac{1}{\rho_r A_i(0)} \int_{A^i_0} \tilde{\rho}_{\text{out}}(t_r t, \mathcal{T}_i(0,y,z))\mathrm{d}y\mathrm{d}z \tag{2.27}$$

if $u_i(t,0) > 0$. In the other case, $\mathfrak{v}_j$ is the terminal vertex and the boundary conditions are given by

$$p_i(t,1) = \bar{p}^j_{\text{out}}(t) = \frac{1}{p_r A_i(1)} \int_{A^i_1} \tilde{p}_{\text{out}}(t_r t, \mathcal{T}_i(1,y,z))\mathrm{d}y\mathrm{d}z \tag{2.28}$$

and

$$\rho_i(t,1) = \rho^j_{\text{out}}(t) = \frac{1}{\rho_r A_i(1)} \int_{A^i_1} \tilde{\rho}_{\text{out}}(t_r t, \mathcal{T}_i(1,y,z))\mathrm{d}y\mathrm{d}z \tag{2.29}$$

for $t$ with $u(t,1) < 0$.

For the boundary conditions at the inner nodes we will have a closer look at a node $\widetilde{\mathfrak{V}}_j$ with center $\tilde{\mathbf{x}}_j$ (see Figure 2.5). The typical diameter of the node is very small compared to the size of the network. Thus, it seems reasonable to introduce the scaling

$$\begin{aligned}
\mathbf{u}(t,\mathbf{x}) &= \frac{1}{U}\tilde{\mathbf{u}}(t_r t, \tilde{\mathbf{x}}_j + d\mathbf{x}),\\
\rho(t,\mathbf{x}) &= \frac{1}{\rho_r}\tilde{\rho}(t_r t, \tilde{\mathbf{x}}_j + d\mathbf{x}),\\
p(t,\mathbf{x}) &= \frac{1}{p_r}\tilde{p}(t_r t, \tilde{\mathbf{x}}_j + d\mathbf{x}),\\
\mathbf{f}(t,\mathbf{x}) &= \frac{1}{f_r}\tilde{\mathbf{f}}(t_r t, \tilde{\mathbf{x}}_j + d\mathbf{x})
\end{aligned}$$

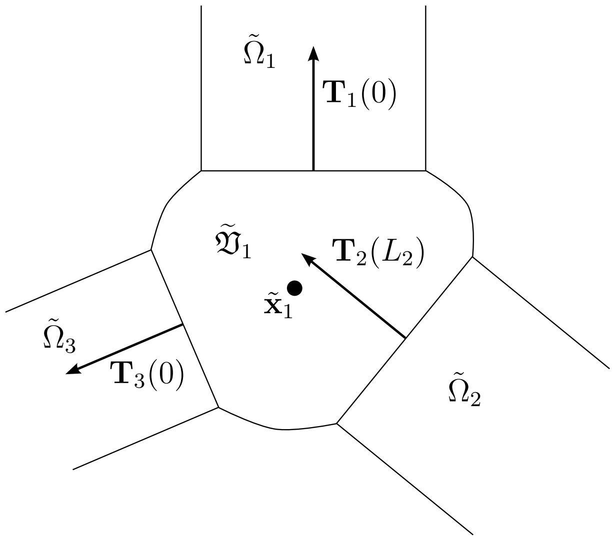



Figure 2.5: The inner node $\tilde{\mathfrak{V}}_1$ with center $\tilde{\mathbf{x}}_1$ and three adjacent edges $\tilde{\Omega}_i$ and their tangential vectors $\mathbf{T}_i$.

and

$$q(t, \mathbf{x}) = \frac{1}{q_r} \tilde{q}(t_r t, \tilde{\mathbf{x}}_j + d\mathbf{x})$$

with the same reference values as before. In contrast to the scaling for the edges, this scaling treats all coordinate directions in the same manner.

Using theses scaled quantities the equations (2.4) are transformed into

$$\begin{aligned}
\varepsilon \rho_t + h \operatorname{div}(\rho \mathbf{u}) &= 0 \\
\varepsilon (\rho \mathbf{u})_t + h \operatorname{div}(\rho \mathbf{u}\mathbf{u}^T) + h \frac{\nabla p}{\gamma M^2} &= \varepsilon \hat{f} \rho \mathbf{f} \\
\varepsilon \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{p}{(\gamma - 1)\gamma M^2} \right)_t + h \operatorname{div} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} + \frac{p\mathbf{u}}{(\gamma - 1) M^2} \right) &= \\
\varepsilon \left( \hat{f} \rho \mathbf{f}^T \mathbf{u} + \frac{\hat{q}}{(\gamma - 1)\gamma M^2} q \right)
\end{aligned}$$

in the domain $(0, T) \times \mathfrak{V}_j$ with scaled inner nodes $\mathfrak{V}_j = \left\{ \mathbf{x} \in \mathbb{R}^3 \middle| \tilde{\mathbf{x}}_j + d\mathbf{x} \in \tilde{\mathfrak{V}}_j \right\}$. Hence, neglecting all terms of order $\mathcal{O}(\varepsilon)$, we see that a solution fulfils the steady Euler equations without source terms at each inner node $\mathfrak{V}_j$, i.e.

$$\operatorname{div}(\rho \mathbf{u}) = 0 \tag{2.30}$$

$$\operatorname{div}(\rho \mathbf{u}\mathbf{u}^T) + \frac{\nabla p}{\gamma M^2} = 0 \tag{2.31}$$

$$\operatorname{div} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} + \frac{p\mathbf{u}}{(\gamma - 1) M^2} \right) = 0. \tag{2.32}$$

In the next step, we want to use these relations to obtain boundary conditions at the inner nodes, so-called coupling conditions. We observe that the outer normal vectors on the parts of the boundary belonging to the $i$-th edge is given by $\mathbf{T}_i(0)$ or $-\mathbf{T}_i(L_i)$, respectively. To simplify the notation, we will often omit the argument of $\mathbf{T}_i$ if it is clear from the context. Since the edges are straight in a neighbourhood of the nodes, the scaling on the edges for $\rho_i$, $p_i$ and $u_i$ coincides at the transition boundary $\Gamma^i_{en}$ or $\Gamma^i_{ex}$ with the scaling on the node for $\rho$, $p$ and the normal velocity $\mathbf{u}^T\mathbf{T}_i$.

Thus, an integration of the steady continuity equation (2.30) over the node $\mathfrak{V}_j$ leads together with the divergence theorem and the boundary condition (2.22) of the three-dimensional equations to

$$
\begin{aligned}
0 &= \int_{\mathfrak{V}_j} \operatorname{div}(\rho\mathbf{u})\mathrm{d}\mathbf{x} \\
&= \int_{\partial\mathfrak{V}_j} \rho\mathbf{u}^T\boldsymbol{\nu}\mathrm{d}\omega \\
&= \sum_{i:e_i\in E^+(\mathfrak{v}_j)} \int_{A^i_0} \rho_i u_i \mathrm{d}y\mathrm{d}z - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} \int_{A^i_1} \rho_i u_i \mathrm{d}y\mathrm{d}z \\
&= \sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\overline{\rho_i u_i}(0) - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\overline{\rho_i u_i}(1).
\end{aligned}
$$

This condition states the conservation of mass at the node $\mathfrak{v}_j$, this means the mass which flows into the node at time $t$ has to flow out at the same time. Assuming $\overline{\rho_i u_i} = \bar{\rho}_i\bar{u}_i$ as in the previous section we end up with

$$
\sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\bar{\rho}_i(0)\bar{u}_i(0) = \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\bar{\rho}_i(1)\bar{u}_i(1). \tag{2.33}
$$

Similarly, we get a kind of energy conservation law. We integrate the steady energy equation (2.32) to obtain

$$
\begin{aligned}
0 &= \int_{\mathfrak{V}_j} \operatorname{div}\left(\left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{p}{(\gamma-1)M^2}\right]\mathbf{u}\right)\mathrm{d}\mathbf{x} \\
&= \int_{\partial\mathfrak{V}_j} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{p}{(\gamma-1)M^2}\right]\mathbf{u}^T\boldsymbol{\nu}\mathrm{d}\omega \\
&= \sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\left[\frac{1}{2}\bar{\rho}_i(0)\bar{u}_i(0)^3 + \frac{p_i(0)\bar{u}_i(0)}{(\gamma-1)M^2}\right] \\
&\quad - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\left[\frac{1}{2}\bar{\rho}_i(1)\bar{u}_i(1)^3 + \frac{p_i(1)\bar{u}_i(1)}{(\gamma-1)M^2}\right] + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{2.34}
$$

In the last step, we neglected the components $\mathbf{u}^T\mathbf{N}_i$ and $\mathbf{u}^T\mathbf{B}_i$ of the velocity which are transversal to the tunnel since they were assumed to be of order $\mathcal{O}(\varepsilon)$.

For the momentum equation, the situation is more complicated. Intuitively, it is clear that the momentum will not be conserved at a node. Let $\mathfrak{d} = \dim\left(\operatorname{span}_{i:e_i\in E(\mathfrak{v}_j)} \mathbf{T}_i\right) \in$

$\{1,2,3\}$ be the dimension of the subspace spanned by the tangential vectors $\mathbf{T}_i$ and let $\mathbf{T}_{k_l}$, $1 \leq l \leq \mathfrak{d}$ be a base of this subspace. Multiplication of the momentum equation (2.31) by the constant vectors $\mathbf{T}_{k_l}$ leads to

$$\operatorname{div}\left(\rho\mathbf{u}\mathbf{u}^T\mathbf{T}_{k_l} + \frac{p\mathbf{T}_{k_l}}{\gamma M^2}\right) = 0$$

and after integration it holds

$$\begin{aligned} 0 &= \int_{\mathfrak{V}_j} \operatorname{div}\left(\rho\mathbf{u}\mathbf{u}^T\mathbf{T}_{k_l} + \frac{p\mathbf{T}_{k_l}}{\gamma M^2}\right)\mathrm{d}\mathbf{x} \\ &= \int_{\partial\mathfrak{V}_j} \rho\mathbf{u}^T\mathbf{T}_{k_l}\mathbf{u}^T\boldsymbol{\nu} + \frac{p}{\gamma M^2}\mathbf{T}_{k_l}^T\boldsymbol{\nu}\mathrm{d}\omega. \end{aligned} \tag{2.35}$$

This integral can not be evaluated by using just the density, the pressure and the tangential velocity at the transition boundary, as we did for the continuity and energy equation. For this reason, we need to use two approximations: The first one concerns the velocity. We have to compute $\mathbf{u}^T\mathbf{T}_{k_l}$ on all parts of the boundary, even on those not belonging to the edge $\Omega_{k_l}$. Let $e_p \in E^+(\mathfrak{v}_j)$ be an arbitrary outgoing edge. Since $\begin{pmatrix}\mathbf{T}_p & \mathbf{N}_p & \mathbf{B}_p\end{pmatrix}$ is an orthonormal base, the assumption, that the normal velocity is of order $\mathcal{O}(\varepsilon)$, yields

$$\mathbf{T}_{k_l} = \mathbf{T}_{k_l}^T\mathbf{T}_p\mathbf{T}_p + \mathbf{T}_{k_l}^T\mathbf{N}_p\mathbf{N}_p + \mathbf{T}_{k_l}^T\mathbf{B}_p\mathbf{B}_p$$

and thus

$$\mathbf{u}^T\mathbf{T}_{k_l} = \mathbf{u}^T\left(\mathbf{T}_{k_l}^T\mathbf{T}_p\mathbf{T}_p + \mathbf{T}_{k_l}^T\mathbf{N}_p\mathbf{N}_p + \mathbf{T}_{k_l}^T\mathbf{B}_p\mathbf{B}_p\right) = \mathbf{T}_{k_l}^T\mathbf{T}_p(0)u_p(0) + \mathcal{O}(\varepsilon)$$

on the part of the boundary belonging to $e_p$. Similarly, for an incoming edge $e_p \in E^-(\mathfrak{v}_j)$ we find

$$\mathbf{u}^T\mathbf{T}_{k_l} = \mathbf{T}_{k_l}^T\mathbf{T}_p(L_p)u_p(1) + \mathcal{O}(\varepsilon)$$

and thus, it holds in order $\mathcal{O}(\varepsilon^0)$

$$\begin{aligned} &\int_{\partial\mathfrak{V}_j} \rho\mathbf{u}^T\mathbf{T}_{k_l}\mathbf{u}^T\boldsymbol{\nu}\mathrm{d}\omega \\ &\quad= \sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\mathbf{T}_{k_l}^T\mathbf{T}_i\bar{\rho}_i(0)\bar{u}_i(0)^2 - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\mathbf{T}_{k_l}^T\mathbf{T}_i\bar{\rho}_i(1)\bar{u}_i(1)^2 \end{aligned}$$

The second approximation concerns the pressure and is necessary to compute the integral $\int_{\partial\mathfrak{V}_j} p\mathbf{T}_{k_l}^T\boldsymbol{\nu}\mathrm{d}\omega$. Denote by $\partial\mathfrak{V}_j^s$ the part of the boundary of the node which belongs to no edge. Then, it holds

$$\begin{aligned} &\int_{\partial\mathfrak{V}_j} \frac{p}{\gamma M^2}\mathbf{T}_{k_l}^T\boldsymbol{\nu}\mathrm{d}\omega = \\ &\sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\frac{p_i(0)}{\gamma M^2}\mathbf{T}_{k_l}^T\mathbf{T}_i - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\frac{p_i(1)}{\gamma M^2}\mathbf{T}_{k_l}^T\mathbf{T}_i + \int_{\partial\mathfrak{V}_j^s} \frac{p}{\gamma M^2}\mathbf{T}_{k_l}^T\boldsymbol{\nu}\mathrm{d}\omega. \end{aligned}$$

In general, it is not possible to compute the last integral $\hat{P}_j^{k_l} = \int_{\partial\mathfrak{V}_j^s} p\mathbf{T}_{k_l}^T\boldsymbol{\nu}\mathrm{d}\omega$ without solving the steady Euler equations (2.30) - (2.32), since there is no boundary condition for the pressure $p$ on $\partial\mathfrak{V}_j^s$ prescribed. For this reason, we will present two different heuristic approaches to handle the $\mathfrak{d}$ linear independent equations (2.35) for the *dynamic pressure* $P_i = \bar{\rho}_i\bar{u}_i^2 + \frac{p_i}{\gamma M^2}$

$$\sum_{i:e_i\in E^+(\mathfrak{v}_j)} A_i(0)\mathbf{T}_{k_l}^T\mathbf{T}_iP_i(0) - \sum_{i:e_i\in E^-(\mathfrak{v}_j)} A_i(1)\mathbf{T}_{k_l}^T\mathbf{T}_iP_i(1) = -\frac{\hat{p}_j^{k_l}}{\gamma M^2}. \tag{2.36}$$

**Case I:** $d(\mathfrak{v}_j) = \mathfrak{d}+1$**.** Let the degree of the vertex $d(\mathfrak{v}_j)$ be equal to $\mathfrak{d}+1$ and let the pressure loss $\hat{p}_j^{k_l}$ be a known quantity, which is modelled by some heuristics. In this case, we have $\mathfrak{d}$ linear independent equations (2.36) for the $\mathfrak{d}+1$ unknowns $P_i$. Hence, the solution lies in a one-dimensional affine subspace of $\mathbb{R}^{\mathfrak{d}+1}$. This subspace is defined by a particular solution $P_j^p$ and a homogeneous solution $P_j^h$, where $P_j^p$ depends on $\hat{p}_j$ and $P_j^h$ only on the geometry. Then, the solution of the linear equations (2.36) is given by

$$P_i(0) = \frac{\left(P_j^p\right)_i}{\gamma M^2} + P_j^*(P_j^h)_i \qquad \text{for } e_i \in E^+(\mathfrak{v}_j)$$

and

$$P_i(1) = \frac{\left(P_j^p\right)_i}{\gamma M^2} + P_j^*(P_j^h)_i \qquad \text{for } e_i \in E^-(\mathfrak{v}_j)$$

for some $P_j^* \in \mathbb{R}$.

As a special case, we want to mention a sudden contraction or expansion of the cross-sectional area (see Figure 2.6). In the engineering literature, minor loss factors are often used to describe the pressure drop at nodes with two adjacent edges pointing in opposite directions, i.e. $\mathbf{T}_1 = -\mathbf{T}_2$, (see e.g. [25] or [70]). More precisely, let $e_1$ be the edge with the smaller cross-sectional $A_1 < A_2$ and define the ratio of the diameters $\beta = \sqrt{\frac{A_1}{A_2}}$. The pressure drop is estimated empirical as

$$P_2 = P_1 + K\frac{\bar{\rho}_1\bar{u}_1^2}{2}, \tag{2.37}$$

where $K$ denotes the so-called $K$-factor. Typically, this factor is determined by

$$K = \left(1-\beta^2\right)^2 \qquad \text{for a sudden expansion, i.e. for } \bar{u}_1 > 0$$

and

$$K = -\frac{1}{2}\left(1-\beta^2\right) \qquad \text{for a sudden contraction, i.e. for } \bar{u}_1 < 0.$$

We see that the pressure relation (2.37) is exactly of the same form as the solution of (2.36) in the case where $d(\mathfrak{v}_j) = \mathfrak{d}+1$. Explicitly, if we set

$$\hat{p}_1^1 = A_2\left(1-\beta^2\right)P_1 - A_2K\frac{\bar{\rho}_1\bar{u}_1^2}{2},$$

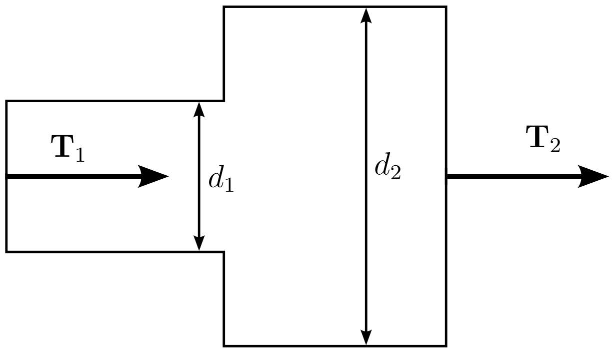



Figure 2.6: A sudden expansion or contraction of the cross-sectional areas with diameter $d_1$ and $d_2$.

all solutions of (2.36) are given by (2.37) for some $P_1 = P_1^*$.

**Case II:** $d(\mathfrak{v}_j) > \mathfrak{d} + 1$. Let the number of adjacent edges be larger than $\mathfrak{d}+1$ and let the node $\mathfrak{V}_j$ be symmetric in the sense

$$\sum_{i:e_i \in E^+(\mathfrak{v}_j)} A_i(1)\mathbf{T}_i(L_i) = \sum_{i:e_i \in E^-(\mathfrak{v}_j)} A_i(0)\mathbf{T}_i(0),$$

i.e. for incoming and outgoing edges the sums of the products of area and tunnel direction are balanced. This definition extends the usual concept of symmetry to nodes with different cross-sectional areas. In this case, the linear equations (2.36) for the dynamic pressures $P_i$ are in general under-determined, but the symmetry allows finding one specific solution for the case $\hat{p}_j = 0$, namely the solution of constant dynamical pressure

$$P_i(0) = P_j^* \qquad \text{if } e_i \in E^+(\mathfrak{v}_j)$$

and

$$P_i(1) = P_j^* \qquad \text{if } e_i \in E^-(\mathfrak{v}_j)$$

for an arbitrary $P_j^* \in \mathbb{R}$. Of course, the assumption $\hat{p}_j = 0$ is a real drawback in this approach.

*Remark* 2.7. Instead of choosing the dynamic pressure $P_i$ to be the same for all $i$ with $e_i \in E(\mathfrak{v}_j)$, the equality of the pressure $p_i$ at a node $\mathfrak{v}_j$ is often postulated. See e.g. [22] for a comparison of the different approaches.

All approaches have in common that we can express the pressures $p_i$ as

$$p_i(0) = \left(P_j^p\right)_i + \gamma M^2 P_j^* (P_j^h)_i - b\gamma M^2 \bar{\rho}_i(0)\bar{u}_i^2(0) \qquad \text{for } e_i \in E^+(\mathfrak{v}_j) \tag{2.38}$$

and

$$p_i(1) = \left(P_j^p\right)_i + \gamma M^2 P_j^* (P_j^h)_i - b\gamma M^2 \bar{\rho}_i(1)\bar{u}_i^2(1) \qquad \text{for } e_i \in E^-(\mathfrak{v}_j)$$

depending on a single pressure variable $P_j^* \in \mathbb{R}$ and the product $\rho_i u_i^2$. Here, $P_j^p$ and $P_j^h$ are some known vectors. The introduction of the parameter $b \in \{0,1\}$ yields the possibility to include the equality of the pressure in our framework (see Remark 2.7).

In the case of a node with only one outflow edge, i.e. there is exactly one $i$ with $u_i(0) > 0$ if $e_i \in E^+(\mathfrak{v}_j)$ or $u_i(1) < 0$ if $e_i \in E^-(\mathfrak{v}_j)$, the coupling conditions (2.33), (2.34) and (2.38) are sufficient for the well-posedness study of the Euler equations as shown in [24]. But since we do not want to restrict ourselves to this kind of nodes, we need an additional constraint controlling the distribution of the density to the different outflow edges. We will assume a good mixing of the gas flowing in the edges, which means the density should be the same in all outflow edges at the transition boundary. More precisely, there exists $\rho^j(t)$ with

$$\rho_i(t,0) = \rho^j(t) \qquad \text{if } e_i \in E^+(\mathfrak{v}_j) \text{ and } u_i(t,0) > 0 \tag{2.39}$$

and

$$\rho_i(t,1) = \rho^j(t) \qquad \text{if } e_i \in E^-(\mathfrak{v}_j) \text{ and } u_i(t,1) < 0.$$

*Remark* 2.8. Restricting to the subsonic case, the number of required boundary conditions is well-known from the theory of hyperbolic systems. For a node $\mathfrak{v}$ with $d(\mathfrak{v})$ adjacent edges one needs $d(\mathfrak{v}) + r$ independent conditions where $r$ is the number of outflow edges, this means the number of edges with velocity $u_i$ pointing from the node to the edge. Thus, it is easy to check that the number of defined boundary conditions fits to the Euler equations.

### 2.3.3 Matrix formulation

Up to now, we have formulated equations for each edge $e_i$ and coupling conditions for each node $\mathfrak{v}_j$ of the system. To simplify these notations, we want to introduce a matrix formulation of the equations and the coupling conditions. At this point, we change the notation, vectors are not any longer denoted by bold letters, whereas for a vector valued quantity $w = (w_1, \ldots, w_n)$, we denote the diagonal matrix with the same entries by the capital bold letter $\mathbf{W}$, i.e.

$$\mathbf{W} = \operatorname{diag}(w) = \begin{pmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & w_n \end{pmatrix}.$$

For Greek letters, we simply use the small bold letter to avoid ambiguities.

Then, the section-averaged Euler equations (2.25) read for the vector-valued unknowns

$\rho$, $u$ and $p$

$$
\begin{aligned}
(\mathbf{A}\rho)_t + \mathbf{H}(\mathbf{A}\mathbf{U}\rho)_x &= 0\\
(\mathbf{A}\mathbf{U}\rho)_t + \mathbf{H}\left(\mathbf{A}\mathbf{U}^2\rho + \frac{1}{\gamma M^2}\mathbf{A}p\right)_x &=\\
&\frac{1}{\gamma M^2}\mathbf{H}\mathbf{A}_x p - \boldsymbol{\zeta}\mathbf{A}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{A}\mathbf{F}\rho\\
\left(\frac{\mathbf{A}p}{(\gamma-1)\gamma M^2} + \frac{1}{2}\mathbf{A}\mathbf{U}^2\rho\right)_t + \mathbf{H}\left(\frac{1}{2}\mathbf{A}\mathbf{U}^3\rho + \frac{\mathbf{A}\mathbf{U}p}{(\gamma-1)M^2}\right)_x &=\\
&\hat{f}\mathbf{A}\mathbf{U}\mathbf{F}\rho - \boldsymbol{\zeta}\mathbf{A}\alpha(\mathbf{U})\mathbf{U}^2\rho + \frac{\boldsymbol{\eta}\mathbf{A}\left(\boldsymbol{\rho}^{-1}p - T_w\right) + \hat{q}\mathbf{A}q}{(\gamma-1)\gamma M^2}
\end{aligned}
\tag{2.40}
$$

on $(0,T)\times(0,1)$. Here, $\mathbf{A}$ is the matrix of the areas, $\boldsymbol{\zeta}$ the matrix of the friction parameters and $\mathbf{F}$ of the forces, to mention just a few.

To specify the boundary conditions, we will use the incidence matrix $\mathbf{B}$ and its submatrices $\mathbf{B}_{>1}$ and $\mathbf{B}_{=1}$, which were defined in Section 1.1. Denoting by $x^+ = \frac{1}{2}(x+|x|)$ the positive part of $x$ and by $x^- = (-x)^+$ the negative part, we can write the inflow boundary conditions (2.27) and (2.29) at the outer nodes and the good mixing (2.39) at the inner nodes in a simple way: There exists a time-dependent vector $\rho_V(t) \in \mathbb{R}^k$ describing the densities at the inner nodes such that

$$
\mathbf{U}(t,0)^+\rho(t,0) = \mathbf{U}(t,0)^+\left(\mathbf{B}^+_{>1}\right)^T\rho_V(t) + \mathbf{U}(t,0)^+\left(\mathbf{B}^+_{=1}\right)^T\rho_{\text{out}} \tag{2.41}
$$

and

$$
\mathbf{U}(t,1)^-\rho(t,1) = \mathbf{U}(t,1)^-\left(\mathbf{B}^-_{>1}\right)^T\rho_V(t) + \mathbf{U}(t,1)^-\left(\mathbf{B}^-_{=1}\right)^T\rho_{\text{out}}. \tag{2.42}
$$

If the velocity points outwards of the domain, this relation is obviously fulfilled because we multiply by $\mathbf{U}(t,0)^+$ or $\mathbf{U}(t,1)^-$, respectively. If the velocity is directed into the domain, this formulation is equivalent to the original good mixing of the density and the original inflow boundary conditions. The mass conservation (2.33) and the energy conservation (2.34) can be transformed directly into

$$
\mathbf{B}^+_{>1}\mathbf{A}(0)\mathbf{U}(t,0)\rho(t,0) = \mathbf{B}^-_{>1}\mathbf{A}(1)\mathbf{U}(t,1)\rho(t,1) \tag{2.43}
$$

and

$$
\begin{aligned}
&\mathbf{B}^+_{>1}\mathbf{A}(0)\mathbf{U}(t,0)\left(\frac{1}{2}\mathbf{U}(t,0)^2\rho(t,0) + \frac{p(t,0)}{(\gamma-1)M^2}\right)\\
&\quad = \mathbf{B}^-_{>1}\mathbf{A}(1)\mathbf{U}(t,1)\left(\frac{1}{2}\mathbf{U}(t,1)^2\rho(t,1) + \frac{p(t,1)}{(\gamma-1)M^2}\right).
\end{aligned}
\tag{2.44}
$$

For the pressure condition, the situation is slightly more complicated due to the difference between the pressure and the dynamic pressure. We prescribed the pressure at the outer nodes by the equations (2.26) and (2.28) and we have a coupling condition (2.38) for the dynamic pressure at the inner nodes. To fit these two conditions into one closed form, we assume the vector $P_j^h > 0$ to be positive and denote by $\mathbf{B}_{>1,h} = (b_{il}^{>1,h}) \in \mathbb{R}^{k\times n}$ the weighted submatrix of the incidence matrix belonging to the inner nodes. The $j$-row of $\mathbf{B}_{>1}$ is weighted by the homogeneous solution $P_j^h$ of (2.36), i.e.

$$b_{jl}^{>1,h} = b_{jl}^{>1}(P_j^h)_l. \tag{2.45}$$

We combine the vector $\bar{p}_{\text{out}}$ from the pressure boundary condition with the vector $P_j^p$ coming from the coupling condition (2.38) to the new vectors $p^0$ and $p^1 \in \mathbb{R}^n$ with

$$\left(p^0\right)_i = \begin{cases} (\bar{p}_{\text{out}})_j & \text{if } \operatorname{init}(e_i) = \mathfrak{v}_j \text{ and } d(\mathfrak{v}_j) = 1 \\ \left(P_j^p\right)_i & \text{if } \operatorname{init}(e_i) = \mathfrak{v}_j \text{ and } d(\mathfrak{v}_j) > 1 \end{cases}$$

and

$$\left(p^1\right)_i = \begin{cases} (\bar{p}_{\text{out}})_j & \text{if } \operatorname{ter}(e_i) = \mathfrak{v}_j \text{ and } d(\mathfrak{v}_j) = 1 \\ \left(P_j^p\right)_i & \text{if } \operatorname{ter}(e_i) = \mathfrak{v}_j \text{ and } d(\mathfrak{v}_j) > 1. \end{cases}$$

This is a reordering of the vectors $\bar{p}_{\text{out}} \in \mathbb{R}^{m-k}$ and $P_j^p \in R^k$ to construct an $n$-dimensional vector. Then, the conditions (2.26), (2.28) and (2.38) are equivalent to the existence of a time-dependent vector $P_V(t) \in \mathbb{R}^k$ with

$$p(t,0) = p^0(t) + \gamma M^2 \left(\mathbf{B}_{>1,h}^{+}\right)^T P_V(t) - b\gamma M^2 \mathbf{K}_0 \mathbf{U}(t,0)^2 \rho(t,0) \tag{2.46}$$

and

$$p(t,1) = p^1(t) + \gamma M^2 \left(\mathbf{B}_{>1,h}^{-}\right)^T P_V(t) - b\gamma M^2 \mathbf{K}_1 \mathbf{U}(t,1)^2 \rho(t,1). \tag{2.47}$$

Here, the two matrices $\mathbf{K}_q = (k_{ij}^q) \in \{0,1\}^{n\times n}$ denote the diagonal matrices, which are zero if the end $q \in \{0,1\}$ of the edge is at a node with degree 1 and which are 1 otherwise. In formula, it is

$$k_{ij}^0 = \delta_{ij} \min\left(d(\operatorname{init}(e_i)) - 1, 1\right)$$

and

$$k_{ij}^1 = \delta_{ij} \min\left(d(\operatorname{ter}(e_i)) - 1, 1\right).$$

The main message of these lines is that we are able to write the pressure conditions in the form

$$p(t,0) = g\left(t, 0, -\gamma M^2 \mathbf{U}(t,0)^2 \rho(t,0)\right) + \gamma M^2 \left(\mathbf{B}_{>1,h}^{+}\right)^T P_V(t) \tag{2.48}$$

and

$$p(t,1) = g\left(t, 1, \gamma M^2 \mathbf{U}(t,1)^2 \rho(t,1)\right) + \gamma M^2 \left(\mathbf{B}_{>1,h}^{-}\right)^T P_V(t), \tag{2.49}$$

such that the pressure $p$ depends linearly on the node pressure $P_V$. Here, $g$ denotes a function.

## 2.4 Low Mach number asymptotic

In this section we want to do a further step to simplify the in the previous section derived section-averaged Euler equations (2.40). Therefore, we consider an asymptotic expansion of the equations and its limit, called the low Mach number limit. In several physical relevant applications, the Mach number $M$, describing the ratio between velocity and speed of sound, is very small. As an example you can think of the air flow in a vehicle tunnel in the emergency case of a fire [49, 44], the flow in an exhaust pipe [45] or in a nuclear reactor [73], to mention just a few applications. The advantage of considering the low Mach number limit is that one does not have to use small time steps in numerical computations of the solution. In contrast, these small time steps are necessary solving the Euler equations with small Mach number by explicit numerical schemes.

We introduce the small parameter $\varepsilon = \gamma M^2 \ll 1$ and we consider the following expansion of the physical quantities and source terms

$$\begin{aligned}
p(t,x) &= p_0(t,x) + \varepsilon\pi(t,x) + \mathcal{O}(\varepsilon^2),\\
\rho(t,x) &= \rho_0(t,x) + \mathcal{O}(\varepsilon),\\
u(t,x) &= u_0(t,x) + \mathcal{O}(\varepsilon),\\
f(t,x) &= f_0(t,x) + \mathcal{O}(\varepsilon),\\
\alpha(u) &= \alpha_0(u_0) + \mathcal{O}(\varepsilon),\\
q(t,x) &= q_0(t,x) + \mathcal{O}(\varepsilon),\\
T_w(t,x) &= T_{w,0}(t,x) + \mathcal{O}(\varepsilon)
\end{aligned}$$

and

$$g(t,x,\varepsilon\mathbf{U}^2\rho) = g_0(t,x) + \varepsilon g_1(t,x,\mathbf{U}_0^2\rho_0) + \mathcal{O}(\varepsilon).$$

If we plug this ansatz into the system (2.40) and if we neglect all terms of order $\mathcal{O}(\varepsilon)$, we formally find

$$\begin{aligned}
(\mathbf{A}\rho_0)_t + \mathbf{H}\left(\mathbf{A}\mathbf{U}_0\rho_0\right)_x &= \mathcal{O}(\varepsilon)\\
(\mathbf{A}\mathbf{U}_0\rho_0)_t + \mathbf{H}\left(\mathbf{A}\left[\mathbf{U}_0^2\rho_0 + \frac{1}{\varepsilon}p_0 + \pi\right]\right)_x &=\\
\mathbf{H}\mathbf{A}_x\left[\frac{1}{\varepsilon}p_0 + \pi\right] &- \boldsymbol{\zeta}\mathbf{A}\alpha_0(\mathbf{U}_0)\mathbf{U}_0\rho_0 + \hat{f}\mathbf{A}\mathbf{F}_0\rho_0 + \mathcal{O}(\varepsilon)\\
(\mathbf{A}p_0)_t + \mathbf{H}\gamma\left(\mathbf{A}\mathbf{U}_0p_0\right)_x &= \boldsymbol{\eta}\mathbf{A}\left({\boldsymbol{\rho_0}}^{-1}p_0 - T_{w,0}\right) + \hat{q}\mathbf{A}q_0 + \mathcal{O}(\varepsilon).
\end{aligned} \tag{2.50}$$

The analysis of the low Mach number expansion is done in four steps. First, we derive a differential equation for the zeroth order pressure $p_0$. Then, we expand the coupling condition for the pressure and solve the equation for $p_0$. In a third step, we extract equations for the first order pressure and zeroth order density and velocity. The derivation is completed with the expansion of the remaining coupling and boundary conditions.

Comparing the terms of orders $\mathcal{O}\left(\varepsilon^{-1}\right)$, we extract

$$\left(\mathbf{A}p_0\right)_x = \left(\mathbf{A}\right)_x p_0$$

from the second equation of (2.50). This equation implies that the zeroth order pressure $p_0(t,x) = p_0(t)$ is constant in space. The coupling conditions for the pressure (2.48) and (2.49) read

$$p_0(t) + \varepsilon\pi(t,0) = g_0\,(t,0) + \varepsilon g_1\left(t,0,-\mathbf{U}_0^2(t,0)\rho(t,0)\right) + \varepsilon\left(\mathbf{B}_{>1,h}^{+}\right)^T P_V(t)$$

and

$$p_0(t) + \varepsilon\pi(t,1) = g_0\,(t,1) + \varepsilon g_1\left(t,1,\mathbf{U}_0^2(t,1)\rho(t,1)\right) + \varepsilon\left(\mathbf{B}_{>1,h}^{-}\right)^T P_V(t)$$

up to terms of order $\mathcal{O}(\varepsilon)$. To fulfil this condition in the zeroth order we require the compatibility condition

$$g_0(t,0) = g_0(t,1).$$

Then, it is

$$p_0(t) = g_0(t,0) = g_0(t,1)$$

and the remaining parts of the coupling conditions are

$$\pi(t,0) = g_1\left(t,0,-\mathbf{U}_0^2(t,0)\rho(t,0)\right) + \left(\mathbf{B}_{>1,h}^{+}\right)^T P_V(t) \tag{2.51}$$

and

$$\pi(t,1) = g_1\left(t,1,\mathbf{U}_0^2(t,1)\rho(t,1)\right) + \left(\mathbf{B}_{>1,h}^{-}\right)^T P_V(t). \tag{2.52}$$

*Remark* 2.9. For the coupling condition stating the equality of the (dynamic) pressure at the nodes (compare Remark 2.7), the compatibility condition simplifies to the equality of the pressure $\bar{p}_{\text{out}}$ at the outer nodes.

The comparison of the terms of order $\mathcal{O}(\varepsilon^0)$ in (2.50) leads to the *low Mach number equations*

$$\begin{aligned}
(\mathbf{A}\rho_0)_t + \mathbf{H}\left(\mathbf{A}\mathbf{U}_0\rho_0\right)_x &= 0\\
(\mathbf{A}\mathbf{U}_0\rho_0)_t + \mathbf{H}\left(\mathbf{A}\left[\mathbf{U}_0^2\rho_0 + \pi\right]\right)_x &= \mathbf{H}\mathbf{A}_x\pi - \boldsymbol{\zeta}\mathbf{A}\alpha_0(\mathbf{U}_0)\mathbf{U}_0\rho_0 + \hat{f}\mathbf{A}\mathbf{F}_0\rho_0\\
\mathbf{H}\gamma\left(\mathbf{A}\mathbf{U}_0\right)_x p_0 &= \boldsymbol{\eta}\mathbf{A}\left(\boldsymbol{\rho_0}^{-1}p_0 - T_{w,0}\right) + \hat{q}\mathbf{A}q_0 - \mathbf{A}\left(p_0\right)_t.
\end{aligned} \tag{2.53}$$

The energy conservation (2.44) at the nodes leads to the relation

$$\mathbf{B}_{>1}^{+}\mathbf{A}(0)\mathbf{U}_0(t,0)p_0(t) = \mathbf{B}_{>1}^{-}\mathbf{A}(1)\mathbf{U}(t,1)p_0(t)$$

up to order $\mathcal{O}(\varepsilon)$. Assuming that the ground pressure never vanishes, i.e. $p_0(t) \neq 0$ in all components, the previous equation implies the simpler condition

$$\mathbf{B}_{>1}^{+}\mathbf{A}(0)u_0(t,0) = \mathbf{B}_{>1}^{-}\mathbf{A}(1)u_0(t,1), \tag{2.54}$$

which states that the sum of the products of velocity and area should vanish at each node.

*Remark* 2.10. Using the conservation of internal energy at a node as it was done by Gasser and Kraft [44] instead of using the conservation of the total energy (2.44) yields exactly the same coupling condition.

The mass conservation (2.43) and the boundary conditions (2.41) and (2.42) just remain the same. In the following chapters, we will omit the subindex 0 and write again $\rho$, $u$, $p$ , $f$, $\alpha$, $q$ and $T_w$ and $g_1 = g$ instead.

## 2.5 Formal computations

In the previous section, we have derived the low Mach number equations (2.53). While there are boundary conditions for the pressure $\pi$, there is no differential equation for it. Therefore, we want to perform some formal computations on the low Mach number system (2.53) and the boundary conditions in order to eliminate the quantity $\pi$. This will be done without taking care of the regularity of the involved functions, but later in Chapter 5, we will justify why we are allowed to do this.

We start from the momentum equation to derive a velocity equation. Using the product rule and the fact that all matrices have diagonal form, the second equation of (2.53) yields

$$\begin{aligned}&\mathbf{U}\left(\mathbf{A}\rho\right)_t + \mathbf{H}\mathbf{U}\left(\mathbf{A}\mathbf{U}\rho\right)_x + \mathbf{A}\mathbf{U}_t\rho + \mathbf{H}\mathbf{A}\mathbf{U}_x\mathbf{U}\rho + \mathbf{H}\left(\mathbf{A}_x\pi + \mathbf{A}\pi_x\right)\\ &= \mathbf{H}\mathbf{A}_x\pi - \boldsymbol{\zeta}\mathbf{A}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{A}\mathbf{F}\rho.\end{aligned}$$

Plugging in the first equation of (2.53) and multiplying by $\mathbf{A}^{-1}$ we find the *velocity equation*

$$\mathbf{U}_t\rho + \mathbf{H}\mathbf{U}\mathbf{U}_x\rho + \mathbf{H}\pi_x = -\boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{F}\rho. \tag{2.55}$$

As a second step, we want to use the third equation of (2.53) to obtain a new unknown $v(t) \in \mathbb{R}^n$, which depends only on the time. Therefore, we multiply the energy equation by $\frac{1}{\gamma}\mathbf{P}^{-1}\mathbf{H}^{-1}$ to conclude

$$\left(\mathbf{A}u\right)_x = \frac{1}{\gamma}\mathbf{P}^{-1}\mathbf{H}^{-1}\mathbf{A}\left(-\boldsymbol{\eta}T_w + \hat{q}q - p_t\right) + \frac{1}{\gamma}\mathbf{H}^{-1}\boldsymbol{\rho}^{-1}\eta.$$

Then, an integration from $x_0$ to $x$ leads to

$$\begin{aligned}&\mathbf{A}(x)u(t,x)\\ &= \mathbf{A}(x_0)v(t) + \frac{1}{\gamma}\mathbf{P}^{-1}(t)\mathbf{H}^{-1}\int_{x_0}^{x}\mathbf{A}\left(-\boldsymbol{\eta}T_w + \hat{q}q - p_t\right)\mathrm{d}y + \frac{1}{\gamma}\mathbf{H}^{-1}\int_{x_0}^{x}\boldsymbol{\rho}^{-1}\eta\mathrm{d}y\\ &= \mathbf{A}(x_0)v(t) + \mathbf{A}(x)Q(t,x) + \frac{1}{\gamma}\mathbf{H}^{-1}\int_{x_0}^{x}\boldsymbol{\rho}^{-1}\eta\mathrm{d}y.\end{aligned} \tag{2.56}$$

Here, $\mathbf{P}^{-1}\mathbf{A}\mathbf{U}p = \mathbf{A}u$ holds due to the structure of the involved matrices. In the expression (2.56) the value $x_0 \in [0,1]$ is arbitrary and the vector-valued function $Q(t,x) \in \mathbb{R}^n$

is a known quantity, which can be computed directly from the heat source, the cross-sectional area, the ground pressure and the wall temperature by

$$Q(t,x) = \frac{1}{\gamma}\mathbf{A}^{-1}(x)\mathbf{P}^{-1}(t)\mathbf{H}^{-1}\int_{x_0}^{x}\mathbf{A}\left(-\boldsymbol{\eta}T_w + \hat{q}q - p_t\right)\mathrm{d}y.$$

The function $v$ describes the velocity at point $x_0$, i.e.

$$v(t) = u(t,x_0).$$

In principle, it is also possible to choose a different position $x_0$ for each edge.

We concentrate on the special case without heat loss at the wall, i.e. $\eta = 0$, and we plug (2.56) into equation (2.55). To simplify the notation, we define the vector $\bar{A}(x) = \mathbf{A}^{-1}(x)A(x_0)$. Then, an integration from 0 to 1 yields

$$\begin{aligned}
&\int_0^1 \hat{f}\mathbf{F}\rho - \boldsymbol{\zeta}\alpha\left(\bar{\mathbf{A}}\mathbf{V}+\mathbf{Q}\right)\left(\bar{\mathbf{A}}\mathbf{V}+\mathbf{Q}\right)\rho\mathrm{d}x\\
&= \int_0^1 \left(\bar{\mathbf{A}}\mathbf{V}_t+\mathbf{Q}_t\right)\rho + \mathbf{H}\left(\bar{\mathbf{A}}\mathbf{V}+\mathbf{Q}\right)\left(\bar{\mathbf{A}}\mathbf{V}+\mathbf{Q}\right)_x\rho + \mathbf{H}\pi_x\mathrm{d}x\\
&= \mathbf{V}_t\int_0^1 \bar{\mathbf{A}}\rho\mathrm{d}x + \int_0^1 \mathbf{Q}_t\rho\mathrm{d}x - \mathbf{V}^2\mathbf{H}\int_0^1 \bar{\mathbf{A}}\bar{\mathbf{A}}_x\rho\mathrm{d}x\\
&\quad + \mathbf{V}\mathbf{H}\int_0^1\left(\mathbf{Q}\bar{\mathbf{A}}\right)_x\rho\mathrm{d}x + \mathbf{H}\int_0^1 \mathbf{Q}\mathbf{Q}_x\rho\mathrm{d}x + \mathbf{H}\Delta p.
\end{aligned} \tag{2.57}$$

In the last term $\Delta p$ denotes the pressure difference

$$\Delta p(t) = \pi(t,1) - \pi(t,0).$$

With the coupling condition (2.51) and (2.52) this difference can be expressed as

$$\begin{aligned}
\Delta p(t) &= -\mathbf{B}^T_{>1,h}P_V(t) + g\left(t,1,\mathbf{U}^2(t,1)\rho(t,1)\right) - g\left(t,0,-\mathbf{U}^2(t,0)\rho(t,0)\right)\\
&= -\mathbf{B}^T_{>1,h}P_V(t) + g\left(t,1,\left(\bar{\mathbf{A}}(1)\mathbf{V}(t)+\mathbf{Q}(t,1)\right)^2\rho(t,1)\right)\\
&\quad - g\left(t,0,-\left(\bar{\mathbf{A}}(0)\mathbf{V}(t)+\mathbf{Q}(t,0)\right)^2\rho(t,0)\right).
\end{aligned} \tag{2.58}$$

With this expression we are able to determine the pressure difference without the pressure $\pi$. Now, we introduce the abbreviation for the mass

$$R(t) = \int_0^1 \rho(t,y)\mathrm{d}y$$

and for the weighted mass

$$R_f(t) = \int_0^1 \mathbf{F}(t,y)\rho(t,y)\mathrm{d}y$$

for any vector-valued function $f$. Then, the equation (2.57) can be written as

$$\mathbf{R}_{\bar{\mathbf{A}}} v_t - \mathbf{R}_{\mathbf{H}\bar{\mathbf{A}}\bar{\mathbf{A}}_x} \mathbf{V} v + \mathbf{R}_{\mathbf{H}\left(\mathbf{Q}\bar{\mathbf{A}}\right)_x} v + R_{Q_t + \mathbf{H}\mathbf{Q}Q_x} - \mathbf{H}\mathbf{B}_{>1,h}^T P_V + \mathbf{H}\Delta g \\ = R_{\hat{f}f - \zeta\alpha(\bar{\mathbf{A}}\mathbf{V}+\mathbf{Q})(\bar{\mathbf{A}}v+Q)}. \tag{2.59}$$

We already see that this is for a given density $\rho$ an ordinary differential equation for the velocity $v$. The advantage of this formulation is that only the pressure difference $\Delta p$ but not the pressure $\pi$ itself is involved.

To end this section, we want to analyse the coupling conditions for the density (2.41), (2.42) and (2.43). We plug the first two equations into the third. This yields

$$\begin{aligned} 0 &= \mathbf{B}_{>1}^{-}\mathbf{A}(1)\mathbf{U}(t,1)\rho(t,1) - \mathbf{B}_{>1}^{+}\mathbf{A}(0)\mathbf{U}(t,0)\rho(t,0) \\ &= \mathbf{B}_{>1}^{-}\mathbf{A}(1)\left(\mathbf{U}(t,1)^{+}\rho(t,1) - \mathbf{U}(t,1)^{-}\left(\left(\mathbf{B}_{>1}^{-}\right)^T \rho_V(t) + \left(\mathbf{B}_{=1}^{-}\right)^T \rho_{\text{in}}(t)\right)\right) \\ &\quad - \mathbf{B}_{>1}^{+}\mathbf{A}(0)\left(-\mathbf{U}(t,0)^{-}\rho(t,0) + \mathbf{U}(t,0)^{+}\left(\left(\mathbf{B}_{>1}^{+}\right)^T \rho_V(t) + \left(\mathbf{B}_{=1}^{+}\right)^T \rho_{\text{in}}(t)\right)\right) \\ &= -\left(\mathbf{B}_{>1}^{-}\mathbf{A}(1)\mathbf{U}(t,1)^{-}\left(\mathbf{B}_{>1}^{-}\right)^T + \mathbf{B}_{>1}^{+}\mathbf{A}(0)\mathbf{U}(t,0)^{+}\left(\mathbf{B}_{>1}^{+}\right)^T\right)\rho_V \\ &\quad + \mathbf{B}_{>1}^{-}\mathbf{A}(1)\mathbf{U}(t,1)^{+}\rho(t,1) + \mathbf{B}_{>1}^{+}\mathbf{A}(0)\mathbf{U}(t,0)^{-}\rho(t,0). \end{aligned} \tag{2.60}$$

Here, we used the fact that it holds $\mathbf{B}_{>1}^{\pm}\mathbf{W}\left(\mathbf{B}_{=1}^{\pm}\right)^T = 0$ for arbitrary diagonal weight matrices $\mathbf{W} = (w_{ij})_{ij}$, which can be easily derived from the definition of $\mathbf{B}_{>1}$ and $\mathbf{B}_{=1}$:

$$\left(\mathbf{B}_{>1}^{\pm}\mathbf{W}\left(\mathbf{B}_{=1}^{\pm}\right)^T\right)_{ij} = \sum_{l=1}^{n}\left(b_{il}^{>1}\right)^{\pm} w_{ll}\left(b_{jl}^{=1}\right)^{\pm} = 0. \tag{2.61}$$

In a similar way, we see that the matrix

$$\mathbf{M}(t) = (m_{ij}(t))_{ij} = \mathbf{B}_{>1}^{-}\mathbf{A}(1)\mathbf{U}(t,1)^{-}\left(\mathbf{B}_{>1}^{-}\right)^T + \mathbf{B}_{>1}^{+}\mathbf{A}(0)\mathbf{U}(t,0)^{+}\left(\mathbf{B}_{>1}^{+}\right)^T$$

has diagonal form:

$$\begin{aligned} m_{ij}(t) &= \sum_{l=1}^{n}\left(b_{il}^{>1}\right)^{-} A_l(1)u_l(t,1)^{-}\left(b_{jl}^{>1}\right)^{-} + \left(b_{il}^{>1}\right)^{+} A_l(0)u_l(t,0)^{+}\left(b_{jl}^{>1}\right)^{+} \\ &= \delta_{ij}\left(\sum_{l=1}^{n}\left(b_{il}^{>1}\right)^{-} A_l(1)u_l(t,1)^{-} + \left(b_{il}^{>1}\right)^{+} A_l(0)u_l(t,0)^{+}\right) \\ &= \delta_{ij}\left(\mathbf{B}_{>1}^{-}\mathbf{A}(1)u(t,1)^{-} + \mathbf{B}_{>1}^{+}\mathbf{A}(0)u(t,0)^{+}\right)_i. \end{aligned}$$

Using the coupling condition for the energy conservation (2.54) we conclude

$$m_{ij}(t) = \delta_{ij}\left(\mathbf{B}_{>1}^{-}\mathbf{A}(1)u(t,1)^{+} + \mathbf{B}_{>1}^{+}\mathbf{A}(0)u(t,0)^{-}\right)_i.$$

From these two representations of $\mathbf{M}(t)$ we see that $m_{jj}(t)$ is zero if and only if in all adjacent edges $E(\mathfrak{v}_j)$ the velocity is zero, more precisely if

$$u_i(0,t) = 0 \qquad\qquad \text{for all } e_i \in E^+(\mathfrak{v}_j)$$

and

$$u_i(1,t) = 0 \qquad\qquad \text{for all } e_i \in E^-(\mathfrak{v}_j)$$

holds. For this reason, we can use the Moore-Penrose pseudoinverse $\mathbf{M}(t)^{-1}$ (for details see e.g. [79]) in (2.60) to write the inflow boundary conditions (2.41) and (2.42) as

$$\begin{aligned}\mathbf{U}(t,0)^+\rho(t,0) = \mathbf{U}(t,0)^+ \left(\mathbf{B}^+_{>1}\right)^T \mathbf{M}(t)^{-1} \Big(&\mathbf{B}^-_{>1}\mathbf{A}(1)\mathbf{U}(t,1)^+\rho(t,1) \\ &+\mathbf{B}^+_{>1}\mathbf{A}(0)\mathbf{U}(t,0)^-\rho(t,0)\Big) + \mathbf{U}(t,0)^+ \left(\mathbf{B}^+_{=1}\right)^T \rho_{\text{out}}(t)\end{aligned} \tag{2.62}$$

and

$$\begin{aligned}\mathbf{U}(t,1)^-\rho(t,1) = \mathbf{U}(t,1)^- \left(\mathbf{B}^-_{>1}\right)^T \mathbf{M}(t)^{-1} \Big(&\mathbf{B}^-_{>1}\mathbf{A}(1)\mathbf{U}(t,1)^+\rho(t,1) \\ &+\mathbf{B}^+_{>1}\mathbf{A}(0)\mathbf{U}(t,0)^-\rho(t,0)\Big) + \mathbf{U}(t,1)^- \left(\mathbf{B}^-_{=1}\right)^T \rho_{\text{out}}(t).\end{aligned} \tag{2.63}$$

The advantage of this formula is that the inflow density of an edge is expressed as a linear combination of the outflow densities of the adjacent edges, i.e. no term on the right hand side involves the density of parts of the network where the velocity points from an edge to a node. This can be seen more clearly if we write the formula componentwise. Therefore, let $\mathfrak{v}_l$ be an inner node with degree $d(\mathfrak{v}_l) > 1$. Then, it holds for all adjacent edges $e_i \in E^-(\mathfrak{v}_l)$ with $u_i(t,1) < 0$

$$\rho_i(t,1) = \frac{\sum_{j:e_j\in E^-(\mathfrak{v}_l)} A_j(1)u_j(t,1)^+\rho_j(t,1) + \sum_{j:e_j\in E^+(\mathfrak{v}_l)} A_j(0)u_j(0,t)^-\rho_j(t,0)}{\sum_{j:e_j\in E^-(\mathfrak{v}_l)} A_j(1)u_j(t,1)^+ + \sum_{j:e_j\in E^+(\mathfrak{v}_l)} A_j(0)u_j(t,0)^-}.$$

Obviously the same is also true for $e_i \in E^+(\mathfrak{v}_l)$ with $u_i(t,0) > 0$

$$\rho_i(t,0) = \frac{\sum_{j:e_j\in E^-(\mathfrak{v}_l)} A_j(1)u_j(t,1)^+\rho_j(t,1) + \sum_{j:e_j\in E^+(\mathfrak{v}_l)} A_j(0)u_j(0,t)^-\rho_j(t,0)}{\sum_{j:e_j\in E^-(\mathfrak{v}_l)} A_j(1)u_j(t,1)^+ + \sum_{j:e_j\in E^+(\mathfrak{v}_l)} A_j(0)u_j(t,0)^-}.$$

This will be a crucial point in the next chapters.

# 3 Transport equation

In this chapter, we consider transport equations on a network for a given velocity, since the continuity equation fits in this class of equations. The theory of this chapter will be used in Chapter 4 and 5 to solve the low Mach number equations of Chapter 2. One of the main difficulties concerning transport equations on networks is a possible change of the sign of the velocity in the evolution of time which has two effects:

First, even if the data is smooth, there does not have to be a classical smooth solution as the domain is bounded. For example, by the method of characteristics the solution of

$$\rho_t + ((2t-1)\rho)_x = 0$$

on $(0,T) \times (0,1)$ is given by

$$\rho(t,x) = \begin{cases} \rho_0\left(x - t(t-1)\right) & \text{if } t < \frac{1}{2} \text{ or } x \geq \left(t - \frac{1}{2}\right)^2 \\ \rho_{\text{in}}\left(\frac{1}{2} + \sqrt{\left(t-\frac{1}{2}\right)^2 - x}\right) & \text{otherwise,} \end{cases}$$

which has possibly a discontinuity if no further conditions are imposed. The only way to get smooth solutions is to require additional conditions on the initial and boundary data, which depend on the characteristics and thus on the velocity. In our application however, the velocity is not known a priori.

The second issue concerns the network. In this case, we can compute the characteristics, but due to the coupling conditions, we are not able to use them directly to find a solution if the velocity has an infinite number of changes of the sign in a finite time. This can lead to a characteristic of the form of an infinite tree. Each time the characteristic intersects an inner node it divides itself into multiple parts.

For these reasons we will use a different and very general approach looking for weak solutions. This approach is based on the concept of the renormalization property, which was introduced in 1989 by Lions and DiPerna [34] for tangential velocity fields. This concept was extended to bounded domains in $\mathbb{R}^N$ with inflow boundary conditions and velocity fields with a kind of Sobolev regularity by Boyer [13] in 2005 and Boyer and Fabrie [15]. Recently, these results were generalized to velocity fields with $BV$ regularity by Crippa et al. [26, 27]. For constant velocity fields, the transport equation on a network was studied by Sikolya in her dissertation [77] in 2004. In 2008, this was expanded to infinite networks by Dorn [35, 36]. Especially, the long time behaviour was analysed by a semigroup approach.

In the following, we generalize the results of Boyer and Fabrie to the network case. In particular, we will study a vector valued transport equation with affine linear coupled boundary conditions and a time and space dependent velocity field. This means we consider even more general coupling conditions than described in the previous chapter.

## 3.1 Assumptions and requirements

In the beginning, we have to introduce the precise setting. For simplicity, we will only consider the one-dimensional domain $\Omega = (0,1)$ and denote the boundary $\Gamma = \partial\Omega = \{0,1\}$ and $\Gamma_T = [0,T] \times \Gamma$. Since we consider one-dimensional edges of a graph this is no restriction for our application. The outer normal vector on $\Gamma$ is called $\nu$ and is given by

$$\nu = -(-1)^{\omega}$$

for $\omega \in \Gamma$. The aim of this chapter is to find and characterize a solution $\rho(t,x) \in \mathbb{R}^n$ of the problem

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= f && \text{in } (0,T)\times\Omega \\ \rho(0,\cdot) &= \rho_0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^-\mathcal{H}(\rho|_{\Gamma_T^+}) && \text{on } \Gamma_T \end{aligned} \tag{3.1}$$

with initial conditions $\rho_0 \in L^\infty(\Omega)^n$ with $\rho_0 \geq 0$ almost everywhere and with an affine linear boundary operator $\mathcal{H}$. For this chapter we assume

$$\begin{aligned} u &\in L^1((0,T), W^{1,1}(\Omega)^n), \\ c &\in L^1((0,T)\times\Omega)^n, \\ (u_x + c)^- &\in L^1((0,T), L^\infty(\Omega)^n), \\ (u_x)^+ &\in L^1((0,T), L^\infty(\Omega)^n) \end{aligned} \tag{3.2}$$

and

$$f \in L^1((0,T), L^\infty(\Omega)^n)$$

and denote by $\mathbf{U}$ and $\mathbf{C}$ again diagonal matrix with the diagonal entries $u^j$ and $c^j$, respectively. On $\Gamma_T$ we define component-by-component the measure

$$\mathrm{d}\mu_u = (\nu u)\mathrm{d}\omega\mathrm{d}t \tag{3.3}$$

and introduce its positive part $\mathrm{d}\mu_u^+ = (\nu u)^+\mathrm{d}\omega\mathrm{d}t$, its negative part $\mathrm{d}\mu_u^- = (\nu u)^-\mathrm{d}\omega\mathrm{d}t$ and its absolute value $|\mathrm{d}\mu_u| = \mathrm{d}\mu_u^+ + \mathrm{d}\mu_u^-$. These measures divide the boundary $\Gamma_T$ into two parts, the inflow part $\Gamma_T^-$ with $\nu u < 0$ and the outflow part $\Gamma_T^+$ with $\nu u > 0$. For $p \in [1,\infty)$ the space $L^p(\Gamma_T, \mathrm{d}\mu_u^\pm) = L^p(\Gamma_T, \mathbb{R}^n, \mathrm{d}\mu_u^\pm)$ is provided with the norm

$$\|g\|_{p,w,\pm} = \left(\int_0^T \int_\Gamma \beta(|g|)^T \overline{\mathbf{W}}(\nu u)^\pm \mathrm{d}\omega\mathrm{d}t\right)^{\frac{1}{p}}$$

with the function $\beta(x) = \begin{pmatrix} x_1^p & \cdots & x_n^p \end{pmatrix}^T$ and with a positive definite diagonal weight matrix $\overline{\mathbf{W}} \in \mathbb{R}^{n\times n}$. Here, the absolute value has to be understood component-by-component. We do also consider the space $L^\infty(\Gamma_T, \mathrm{d}\mu_u^\pm) = L^\infty(\Gamma_T, \mathbb{R}^n, \mathrm{d}\mu_u^\pm)$ equipped with the maximum norm of the componentwise $L^\infty$-norm, i.e.

$$\|g\|_{\infty,\pm} = \max_j \left( \operatorname*{ess\,sup}_{\substack{(\omega,t)\in\Gamma_T \\ u^j(t,\omega)^\pm \neq 0}} |g^j| \right).$$

The boundary operator $\mathcal{H}$ has to assign a boundary value to the inflow part of the boundary $\Gamma_T^-$, i.e. $\mathcal{H}$ is a mapping

$$\mathcal{H}\colon L^\infty(\Gamma_T, \mathrm{d}\mu_u^+) \to L^\infty(\Gamma_T, \mathrm{d}\mu_u^-).$$

We assume the mapping to be affine linear, i.e. it is given as $\mathcal{H}(\rho) = \rho_{\mathrm{in}} + \mathcal{G}(\rho)$ where it is $\rho_{\mathrm{in}} \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ with $\rho_{\mathrm{in}}(t,x) \geq 0$ for $\mathrm{d}\mu_u^-$-almost all $(t,\omega)$. Here, $\mathcal{G}$ is a linear operator fulfilling the following conditions:

- The operator $\mathcal{G}\colon L^\infty(\Gamma_T, \mathrm{d}\mu_u^+) \to L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ is weakly-$\star$ continuous, i.e. for all $\rho_n \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ with $\rho_n \overset{\star}{\rightharpoonup} \rho$ in $L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ it holds
$$\mathcal{G}(\rho_n) \overset{\star}{\rightharpoonup} \mathcal{G}(\rho)$$
in $L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$.
- There is a positive definite diagonal weight matrix $\overline{\mathbf{W}} \in \mathbb{R}^{n\times n}$ such that the $L^1$-operator norm of $\mathcal{G}$ is less or equal one, i.e. for all $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ it holds
$$\|\mathcal{G}(\rho)\|_{1,w,-} \leq \|\rho\|_{1,w,+}.$$
- The image of $\mathcal{G}$ does not depend on the future time, i.e. for all $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ and almost all $t \in [0,T]$ it holds
$$\mathcal{G}(\chi_{[0,t]}\rho) = \chi_{[0,t]}\mathcal{G}(\rho). \tag{3.4}$$
- The operator $\mathcal{G}$ is $\mathrm{d}\mu_u^+$-almost everywhere positive, i.e. $\mathrm{d}\mu_u^-$-almost everywhere it holds $\mathcal{G}(\rho) \geq 0$ for all $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ with $\rho \geq 0$ $\mathrm{d}\mu_u^+$-almost everywhere.
- There exists a constant vector $\rho_{\max} \in \mathbb{R}^n$ such that component-by-component the inequalities
$$\rho_{\mathrm{in}} \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right) - F_+ + \mathcal{G}(F_+) + \mathcal{G}(\rho_{\max}) \leq \rho_{\max} \tag{3.5}$$
for $\mathrm{d}\mu_u^-$-almost all $(t,\omega) \in \Gamma_T$ and
$$\rho_0 \leq \rho_{\max}$$
for almost all $x \in \Omega$ are valid. Here, it is $\alpha(s) = \|(u_x(s,\cdot) + c(s,\cdot))^-\|_{L^\infty(\Omega)^n}$ and
$$F_+(t) = \int_0^t \exp\left(-\int_0^s \alpha(r)\mathrm{d}r\right) \begin{pmatrix} \|(f_1(s,\cdot))^+\|_{L^\infty} \\ \vdots \\ \|(f_n(s,\cdot))^+\|_{L^\infty} \end{pmatrix} \mathrm{d}s.$$

This construction of $\mathcal{H}$ is motivated by the expressions (2.62) and (2.63) for the coupling conditions for the density. These formulas fit perfectly into the affine linear framework.

To conclude this section, we observe three properties of the operator $\mathcal{G}$. First, for all $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ and all $t \in [0,T]$ the inequality

$$\begin{aligned}
\int_0^t \int_\Gamma |\mathcal{G}(\rho)|^T \overline{\mathbf{W}}(\nu u)^- \mathrm{d}\omega \mathrm{d}t &= \int_0^T \int_\Gamma \chi_{[0,t]} |\mathcal{G}(\rho)|^T \overline{\mathbf{W}}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \int_\Gamma |\mathcal{G}(\chi_{[0,t]}\rho)|^T \overline{\mathbf{W}}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&= \|\mathcal{G}(\chi_{[0,t]}\rho)\|_{1,w,-} \\
&\leq \|\chi_{[0,t]}\rho\|_{1,w,+} \\
&= \int_0^T |\chi_{[0,t]}\rho|^T \overline{\mathbf{W}}(\nu u)^+ \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^t \int_\Gamma |\rho|^T \overline{\mathbf{W}}(\nu u)^+ \mathrm{d}\omega \mathrm{d}t
\end{aligned} \tag{3.6}$$

is true. Second, the equation (3.4) holds in fact for all essential bounded functions $g \in L^\infty([0,T])$: Let $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$, $\rho \neq 0$ and $\varepsilon > 0$ be given. Since the step functions are dense in $L^\infty([0,T])$ we can choose a step function $g_n = \sum_{k=1}^n a_k \chi_{[t_k, t_{k+1}]}$ with

$$\|g - g_n\|_{L^\infty([0,T])} < \frac{\varepsilon}{2\|\rho\|_{1,w,+}}.$$

Then, because of $\mathcal{G}(f_n\rho) = f_n\mathcal{G}(\rho)$ it follows

$$\begin{aligned}
\|\mathcal{G}(g\rho) - g\mathcal{G}(\rho)\|_{1,w,-} &\leq \|\mathcal{G}((g-g_n)\rho)\|_{1,w,-} + \|(g-g_n)\mathcal{G}(\rho)\|_{1,w,-} \\
&\leq \|(g-g_n)\rho\|_{1,w,+} + \|g-g_n\|_{L^\infty([0,T])}\|\mathcal{G}(\rho)\|_{1,w,-} \\
&\leq \|g-g_n\|_{L^\infty([0,T])}\|\rho\|_{1,w,+} + \|g-g_n\|_{L^\infty([0,T])}\|\rho\|_{1,w,+} \\
&< \varepsilon
\end{aligned}$$

and thus

$$\mathcal{G}(g\rho) = g\mathcal{G}(\rho) \tag{3.7}$$

holds $\mathrm{d}\mu_u^-$-almost everywhere.

The third observation concerns the positivity of the operator. Using the positivity of $\mathcal{G}$ and the triangle inequality, we estimate for $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$

$$\begin{aligned}
(\mathcal{G}(\rho))^+ &= \frac{1}{2}\left(\mathcal{G}(\rho) + |\mathcal{G}(\rho)|\right) \\
&= \frac{1}{2}\Big(\mathcal{G}(\rho^+) - \mathcal{G}(\rho^-) + \Big|\mathcal{G}(\rho^+) - \mathcal{G}(\rho^-)\Big|\Big) \\
&\leq \frac{1}{2}\Big(\mathcal{G}(\rho^+) - \mathcal{G}(\rho^-) + \Big|\mathcal{G}(\rho^+)\Big| + |\mathcal{G}(\rho^-)|\Big) \\
&= \mathcal{G}(\rho^+)
\end{aligned} \tag{3.8}$$

almost everywhere in $\Gamma_T$. In the same way, we can prove that

$$(\mathcal{G}(\rho))^- \leq \mathcal{G}(\rho^-) \tag{3.9}$$

holds. In the following, we often extend functions in $L^p(\Gamma_T, \mathrm{d}\mu_u^\pm)$ by zero to construct functions in $L^p(\Gamma_T, |\mathrm{d}\mu_u|)$ or $L^p(\Gamma_T)^n$. Especially, we extend, if necessary, the operator $\mathcal{G}$ to an operator mapping from $L^\infty(\Gamma_T, |\mathrm{d}\mu_u|) \to L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$.

## 3.2 Existence, uniqueness and stability

In this section, we will prove the uniqueness, stability and existence of solutions of the initial-boundary-value problem (3.1). This is an extension of the current state of research since it generalizes the existence theory to networks. To this end, we first introduce the concept of weak solutions before we discuss regularity and the renormalization property of the solutions. Equipped with this, we are able to prove uniqueness and boundedness. For the latter, we will give explicit lower and upper bounds. We continue with one of the main results of this thesis concerning the stability of the solution operator. Finally, to end the chapter, we prove the existence of solutions.

To find a definition for a weak solution, let $\rho \in C^1([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ be a strong solution of (3.1) and let $\varphi \in C_c^{0,1}([0,T) \times \overline{\Omega}, \mathbb{R}^n)$ be a Lipschitz continuous compactly supported test function. By partial integration we conclude

$$\begin{aligned}
-\int_0^T \int_\Omega \varphi^T f \mathrm{d}x\mathrm{d}t &= -\int_0^T \int_\Omega \varphi^T \left(\rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho\right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\varphi_t^T + \varphi_x^T \mathbf{U} - \varphi^T \mathbf{C}\right) \rho \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \varphi(0,\cdot)^T \rho_0 \mathrm{d}x - \int_0^T \int_\Gamma \varphi^T (\nu \mathbf{U}) \rho \mathrm{d}\omega\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\varphi_t^T + \varphi_x^T \mathbf{U} - \varphi^T \mathbf{C}\right) \rho \mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(0,\cdot)^T \rho_0 \mathrm{d}x \\
&\quad - \int_0^T \int_\Gamma \varphi^T (\nu \mathbf{U})^+ \rho \mathrm{d}\omega\mathrm{d}t + \int_0^T \int_\Gamma \varphi^T (\nu \mathbf{U})^- \mathcal{H}(\rho|_{\Gamma_T}) \mathrm{d}\omega\mathrm{d}t.
\end{aligned}$$

*Remark* 3.1. We use the integral notation $\int_\Gamma$ although the boundary consists only of two single points, i.e. $\mathrm{d}\omega$ is the counting measure. An advantage of this notation is the easy generalization to higher dimensions.

**Definition 3.2.** We call a function

$$\rho \in L^\infty([0,T] \times \Omega)^n$$

a *weak solution of the transport equation* $\rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho = 0$ if it holds

$$0 = \int_0^T \int_\Omega \rho^T \left(\varphi_t + \mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + f^T \varphi \mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ with $\varphi(0,\cdot) = \varphi(T,\cdot) = 0$ and $\varphi = 0$ on $\Gamma_T$.

A priori, it is not clear how to define boundary conditions for this class of weak solutions since $\rho$ is not a Sobolev function and thus, the usual trace theory (see e.g. [40]) is not applicable.

In [15], Boyer has proven some properties of such weak solutions for scalar equations, especially the existence of a trace and the renormalization property of the solution (see the following theorem).

**Theorem 3.3 (Trace theorem**, Boyer [15]). *Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional bounded Lipschitz domain and let be $u \in L^1((0,T), W^{1,1}(\Omega)^d)$ and $c, f \in L^1((0,T) \times \Omega)$ with $(c+u_x)^- \in L^1((0,T), L^\infty(\Omega))$ and $(u_x)^+ \in L^1((0,T), L^\infty(\Omega))$. Then, for each weak solution $\rho \in L^\infty((0,T)\times\Omega)$ of the scalar transport equation*

$$\rho_t + \operatorname{div}(u\rho) + c\rho = f$$

*the following properties hold:*

***Time continuity:*** *The function $\rho$ lies in $C([0,T], L^p(\Omega))$ for all $p \in [1,\infty)$.*

***Existence and uniqueness of a trace:*** *There exists a unique essentially bounded function $\gamma\rho \in L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$, called* trace*, such that for any $[t_0,t_1] \subset [0,T]$ and for all test functions $\varphi \in C^{0,1}([0,T]\times\overline{\Omega})$*

$$\begin{aligned} 0 = &\int_{t_0}^{t_1}\int_\Omega \rho(\varphi_t + u^T\nabla\varphi - c\varphi) + f\varphi \mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1}\int_\Gamma \gamma\rho\varphi(u^T\nu)\mathrm{d}\omega\mathrm{d}t \\ &+ \int_\Omega \rho(t_0)\varphi(t_0)\mathrm{d}x - \int_\Omega \rho(t_1)\varphi(t_1)\mathrm{d}x \end{aligned} \tag{3.10}$$

*holds.*

***Renormalization property:*** *For any continuous and piecewise $C^1$ function $\beta$ the renormalization property holds, i.e. $\rho$ satisfies for any $[t_0,t_1] \subset [0,T]$ and for all test functions $\varphi \in C^{0,1}([0,T]\times\overline{\Omega})$*

$$\begin{aligned} 0 = &\int_{t_0}^{t_1}\int_\Omega \beta(\rho)(\varphi_t + u^T\nabla\varphi)\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1}\int_\Omega \beta'(\rho)\left(\rho c - f\right)\varphi\mathrm{d}x\mathrm{d}t \\ &- \int_{t_0}^{t_1}\int_\Omega \varphi u_x(\beta'(\rho)\rho - \beta(\rho))\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1}\int_\Gamma \beta(\gamma\rho)(u^T\nu)\varphi\mathrm{d}\omega\mathrm{d}t \\ &+ \int_\Omega \beta(\rho(t_0))\varphi(t_0)\mathrm{d}x - \int_\Omega \beta(\rho(t_1))\varphi(t_1)\mathrm{d}x. \end{aligned} \tag{3.11}$$

*Remark* 3.4. The renormalization property implies equation (3.10) for $\beta(s) = s$.

*Sketch of the proof.* The proof is based on a space-dependent mollifying procedure and carefully done technical estimates. Here, we only present the main ideas. For a detailed version of the proof we refer to [15].

The first step is to construct a space-dependent mollifying operator $\mathcal{S}_\varepsilon$ for $\varepsilon > 0$ such that for any $p \in [1, \infty]$ and $f \in L^p(\Omega)$ it holds

$$f_\varepsilon = \mathcal{S}_\varepsilon f \in C_c^\infty(\overline{\Omega}), \quad \|\mathcal{S}_\varepsilon f\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} \quad \text{and } \|\nabla(\mathcal{S}_\varepsilon f)\|_{L^p(\Omega)} \leq \frac{C}{\varepsilon}\|f\|_{L^p(\Omega)}$$

and for $\varepsilon \to 0$ we have the convergence $\mathcal{S}_\varepsilon f \to f$ in $L^p(\Omega)$. Applying this operator to the transport equation yields

$$\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div}(u\rho_\varepsilon) + c\rho_\varepsilon - f_\varepsilon = R_\varepsilon$$

in the distributional sense with $\|R_\varepsilon\|_{L^1((0,T)\times\Omega)} \to 0$ for $\varepsilon \to 0$. From this equation, one can deduce $\frac{\partial \rho_\varepsilon}{\partial t} \in L^1((0,T) \times \Omega)$ and thus we observe the higher regularity $\rho_\varepsilon \in W^{1,1}((0,T) \times \Omega) \cap C^0([0,T], L^p(\Omega))$.

The next aim is to prove that $\rho_\varepsilon$ is a Cauchy sequence in $C^0([0,T], L^p(\Omega))$. Therefore, let be $\varepsilon_1, \varepsilon_2 > 0$. Then, it holds

$$\frac{\partial}{\partial t}\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right) + \operatorname{div}\left(u\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\right) + c\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right) - \left(f_{\varepsilon_1} - f_{\varepsilon_2}\right) = R_{\varepsilon_1} - R_{\varepsilon_2}.$$

Let $\beta \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with bounded derivative. Since we gained the regularity $\rho_\varepsilon \in W^{1,1}((0,T) \times \Omega)$, we can multiply the equation by $\beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})$ to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\beta\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\right) + \operatorname{div}\left(u\beta\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\right) + c\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\beta'\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right) \\
&\quad - \left(f_{\varepsilon_1} - f_{\varepsilon_2}\right)\beta'\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right) + \operatorname{div}(u)\left(\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\beta'\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right) - \beta\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\right) \\
&= \beta'\left(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\right)\left(R_{\varepsilon_1} - R_{\varepsilon_2}\right).
\end{aligned}$$

We choose a time-independent test function $\varphi_h \in C^{0,1}(\overline{\Omega})$ with $0 \leq \varphi \leq 1$ and $\varphi = 0$ on $\Gamma$ such that $\varphi \to 1$ for $h \to 0$ holds. Furthermore, we consider $\beta = \beta_\delta$ such that $\beta_\delta$ approximates the absolute value. After some computations, this allows with the help of Gronwall's inequality to find the estimate

$$\begin{aligned}
\sup_{[0,T]} \|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^1(\Omega)} \leq C\Bigg( & h + \|\rho_{\varepsilon_1}(0) - \rho_{\varepsilon_2}(0)\|_{L^1(\Omega)} + \|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{L^1((0,T)\times\Omega)} \\
& + \|R_{\varepsilon_1} - R_{\varepsilon_2}\|_{L^1((0,T)\times\Omega)} + \frac{1}{h}\|v - w\|_{L^1((0,T)\times\Omega)} \\
& + \|w\|_{L^\infty((0,T)\times\Omega)}\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^1((0,T)\times\Omega)}\Bigg)
\end{aligned}$$

for any smooth function $w$. Choosing $h$ and $w$ carefully, one can conclude that $\rho_\varepsilon$ is indeed a Cauchy sequence and thus $\rho \in C([0,T], L^1(\Omega))$ holds. This proves the regularity of the solution.

The existence of the trace is proven similarly. Using $\beta(s) = s^2$ and another test function, one shows that the restriction of $\rho_\varepsilon$ to $\Gamma_T$ is a Cauchy sequence in $L^2(\Gamma_T, |\mathrm{d}\mu_u|)$.

The limit of this sequence naturally fulfils equation (3.11) and hence, the renormalization property is also proven for smooth functions $\beta$. The generalization to piecewise $C^1$ functions is done by an approximation argument.

To complete the proof, one assumes the existence of two trace functions $\gamma_1\rho, \gamma_2\rho \in L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$ satisfying (3.10). Then, for the difference, it holds

$$\int_0^T \int_\Gamma (\gamma_1\rho - \gamma_2\rho)\, \varphi(u^T\nu)\mathrm{d}\omega\mathrm{d}t = 0$$

for any $\varphi \in C^{0,1}([0,T] \times \overline{\Omega})$. By a density argument one finds a sequence of Lipschitz continuous functions $\varphi_n$ with

$$\sup_n \|\varphi_n\|_{L^\infty(\Gamma_T)} \leq \infty$$

and

$$\varphi_n \to (\gamma_1\rho - \gamma_2\rho)\,\mathrm{sgn}(u^T\nu)$$

almost everywhere in $\Gamma_T$. Using these test functions $\varphi_n$, it follows

$$\int_0^T \int_\Gamma |\gamma_1\rho - \gamma_2\rho|^2 |u^T\nu|\mathrm{d}\omega\mathrm{d}t = 0$$

with the dominated convergence theorem, which proves the uniqueness of the trace. □

*Remark* 3.5. A component-by-component consideration shows the validity of Theorem 3.3 for vector-valued equations with $n > 1$. The renormalization property then reads:

There exists a trace $\gamma\rho \in L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$ such that for any continuous and piecewise $C^1$ function $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ with $D\beta$ diagonal, for any $[t_0, t_1] \subset [0,T]$ and for any test function $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ it holds

$$\begin{aligned}
0 = &\int_{t_0}^{t_1} \int_\Omega \beta(\rho)^T(\varphi_t + \mathbf{U}\varphi_x)\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1} \varphi^T\left(\mathbf{C}D\beta(\rho)\rho - D\beta(\rho)f\right)\mathrm{d}x\mathrm{d}t \\
&- \int_{t_0}^{t_1} \int_\Omega \varphi^T \mathbf{U}_x(D\beta(\rho)\rho - \beta(\rho))\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1} \int_\Gamma \beta(\gamma\rho)^T(\nu\mathbf{U})\varphi d\omega\mathrm{d}t \\
&+ \int_\Omega \beta(\rho(t_0))^T\varphi(t_0)\mathrm{d}x - \int_\Omega \beta(\rho(t_1))^T\varphi(t_1)\mathrm{d}x.
\end{aligned}$$

This theorem is no existence result, it only classifies solutions. But due to the continuity of $\rho$ with values in $L^p(\Omega)$ and the existence of the trace we can define a solution of the initial-boundary-value problem as following:

**Definition 3.6.** The function

$$\rho \in L^\infty((0,T) \times \Omega)^n$$

is called a solution of the initial-boundary-value problem (3.1) if and only if

- $\rho$ is a weak solution of the transport equation,
- the initial conditions are fulfilled, i.e. $\rho(0) = \rho_0$ and
- the trace $\gamma\rho$ satisfies the boundary conditions, i.e. $\mathcal{H}(\gamma\rho|_{\Gamma_T^+}) = \gamma\rho|_{\Gamma_T^-}$.

As a consequence of the renormalization property we can prove the uniqueness of the above defined solution.

**Theorem 3.7 (Uniqueness).** *Under the assumptions from Section 3.1, there is at most one solution of the initial-boundary-value problem* (3.1)*.*

*Proof.* Let $\rho^1$ and $\rho^2$ be two solutions of (3.1) with its traces $\gamma\rho_1$ and $\gamma\rho_2$ and define $\rho = \rho^1 - \rho^2$. Then, $\rho$ is a weak solution of the homogeneous transport equation. Because of the uniqueness of the trace $\gamma\rho = \gamma\rho_1 - \gamma\rho_2$ is the trace of $\rho$. We see that $\rho$ is a solution of the following homogeneous initial-boundary-value problem

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= 0 && \text{in } (0,T)\times\Omega \\ \rho(0,\cdot) &= 0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^-\mathcal{G}(\rho|_{\Gamma_T}) && \text{on } \Gamma_T. \end{aligned}$$

Using the renormalization property for $\beta(s) = |s|$ and $\varphi = \overline{\mathbf{W}}\mathbf{1}$ with $\mathbf{1} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$ we conclude for all $t \in [0,T]$

$$\begin{aligned} 0 &= -\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\rho(t))\mathrm{d}x - \int_0^t\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\mathbf{C}\beta(\rho)\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^t\int_\Gamma \mathbf{1}^T\overline{\mathbf{W}}(\nu\mathbf{U})\beta(\gamma\rho)\mathrm{d}\omega\mathrm{d}t \\ &= -\|\overline{\mathbf{W}}\rho(t)\|_{L^1(\Omega)^n} - \int_0^t\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\mathbf{C}|\rho|\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^t\int_\Gamma |\gamma\rho|^T\overline{\mathbf{W}}(\nu u)^+ - |\mathcal{G}(\gamma\rho|_{\Gamma_T})|^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t. \end{aligned}$$

With inequality (3.6) we get

$$\|\overline{\mathbf{W}}\rho(t)\|_{L^1(\Omega)^n} \le -\int_0^t\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\mathbf{C}|\rho|\mathrm{d}x\mathrm{d}t$$

and thus

$$\begin{aligned} \|\overline{\mathbf{W}}\rho(t)\|_{L^1(\Omega)^n} &\le \int_0^t \|c^-\|_{L^\infty(\Omega)^n}\|\overline{\mathbf{W}}\rho\|_{L^1(\Omega)^n}\mathrm{d}t \\ &\le \int_0^t \Big(\|(u_x+c)^-\|_{L^\infty(\Omega)^n} + \|(u_x)^+\|_{L^\infty(\Omega)^n}\Big)\|\overline{\mathbf{W}}\rho\|_{L^1(\Omega)^n}\mathrm{d}t. \end{aligned}$$

By Gronwall's inequality (Lemma 1.19) we conclude $\|\overline{\mathbf{W}}\rho(t)\|_{L^1(\Omega)^n} = 0$ for all $t$ and thus $\rho = 0$ almost everywhere since $\overline{\mathbf{W}}$ is positive definite. □

*Remark* 3.8. We can slightly weaken the assumptions on the source term $f$. In fact, for the existence of the trace, the renormalization property and the uniqueness it is sufficient to require $f \in L^1((0,T) \times \Omega)^n$ (see [15]).

Now, we are able to prove an upper and lower bound of the defined solution.

**Lemma 3.9 (Upper bound).** *Let the assumptions from Section 3.1 be valid. Then, a solution $\rho$ of the initial-boundary-value problem* (3.1) *and its trace $\gamma\rho$ are componentwise bounded, i.e. it holds for all $t \in [0,T]$*

$$-\left(F_-(t) + F_-(T)\right) \exp\left(\int_0^t \alpha(s)\mathrm{d}s\right) \le \rho(t,\cdot) \le (\rho_{\max} + F_+(t)) \exp\left(\int_0^t \alpha(s)\mathrm{d}s\right) \tag{3.12}$$

*almost everywhere in $\Omega$ and*

$$-\left(F_-(t) + F_-(T)\right) \exp\left(\int_0^t \alpha(s)\mathrm{d}s\right) \le \gamma\rho(t,\omega) \le (\rho_{\max} + F_+(t)) \exp\left(\int_0^t \alpha(s)\mathrm{d}s\right) \tag{3.13}$$

*for $|\mathrm{d}\mu_u|$-almost all $(t,\omega) \in \Gamma_T$. Here, it is $\alpha(t) = \left\|(u_x(t,\cdot) + c(t,\cdot))^-\right\|_{L^\infty(\Omega)^n}$ and*

$$F_\pm(t) = \int_0^t \exp\left(-\int_0^s \alpha(r)\mathrm{d}r\right) \begin{pmatrix} \|(f_1(s,\cdot)^\pm\|_{L^\infty} \\ \vdots \\ \|(f_n(s,\cdot))^\pm\|_{L^\infty} \end{pmatrix} \mathrm{d}s.$$

*Proof.* Let $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ be an arbitrary test function with $\varphi(T) = 0$. In order to prove that $r\rho$ with $r(t) = \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right)$ solves also a transport equation, we would like to use $\varphi r$ as a test function. Since $r$ is only absolutely continuous, but not necessarily Lipschitz continuous, we need to approximate $r$ by smoother functions. Therefore, let $\alpha_k \in C^1([0,T])$ be a sequence converging to $\alpha$ in $L^1((0,T))$. We define

$$r_k(t) = \exp\left(-\int_0^t \alpha_k(s)\mathrm{d}s\right).$$

The mean value theorem (see e.g. [59]) states for all $a \le b \in \mathbb{R}$ the existence of $\zeta \in [a,b]$ with

$$\exp(b) - \exp(a) = \exp(\zeta)(b-a).$$

Thus, we find for each $t \in [0,T]$ a constant $\zeta(t)$ with

$$|\zeta(t)| \le C_k := \max\left(\|\alpha_k\|_{L^1((0,T))}, \|\alpha\|_{L^1((0,T))}\right)$$

such that it holds

$$\begin{aligned} |r_k(t) - r(t)| &= \exp(\zeta(t)) \left|\int_0^t \alpha_k(s) - \alpha(s)\mathrm{d}s\right| \\ &\le \exp(C_k)\|\alpha_k - \alpha\|_{L^1((0,T))}. \end{aligned}$$

The sequence $C_k$ is bounded as $\|\alpha_k\|_{L^1((0,T))}$ is convergent, thus $r_k \to r$ in $L^\infty((0,T))$. Especially, it also holds $\alpha_k r_k \to \alpha r$ in $L^1((0,T))$. Now, we will use $\varphi r_k$ as test functions in the weak formulation of the transport equation and take the limit for $k \to \infty$. Because of $(r_k)_t = -\alpha_k r_k$ this yields

$$\begin{aligned}
0 &= \int_0^T \int_\Omega r_k \rho^T \left(\varphi_t + \mathbf{U}\varphi_x - (\mathbf{C} - \alpha_k \mathbf{Id})\,\varphi\right) + r_k f^T \varphi \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Gamma r_k \gamma \rho^T (\nu \mathbf{U}) \varphi \mathrm{d}\omega \mathrm{d}t + \int_\Omega \rho_0^T \varphi(0)\mathrm{d}x \\
&\to \int_0^T \int_\Omega r \rho^T \left(\varphi_t + \mathbf{U}\varphi_x - (\mathbf{C} - \alpha \mathbf{Id})\,\varphi\right) + r f^T \varphi \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Gamma r \gamma \rho^T (\nu \mathbf{U}) \varphi \mathrm{d}\omega \mathrm{d}t + \int_\Omega \rho_0^T \varphi(0)\mathrm{d}x \\
&= \int_0^T \int_\Omega (r\rho - F_+)^T \left(\varphi_t + \mathbf{U}\varphi_x - (\mathbf{C} - \alpha \mathbf{Id})\,\varphi\right) \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (rf - (F_+)_t)^T \varphi - F_+^T \left(\mathbf{U}_x + \mathbf{C} - \alpha \mathbf{Id}\right) \varphi \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Gamma (r\gamma\rho - F_+)^T (\nu \mathbf{U}) \varphi \mathrm{d}\omega \mathrm{d}t + \int_\Omega \rho_0^T \varphi(0)\mathrm{d}x.
\end{aligned}$$

Keeping in mind Remark 3.8, this shows that $\bar{\rho}(t,x) = r(t)\rho(t,x) - F_+(t)$ with its trace $\gamma\bar{\rho}(t,\omega) = r(t)\gamma\rho(t,\omega) - F_+(t)$ is the unique solution of the following initial-boundary-value problem:

$$\begin{aligned}
\bar{\rho}_t + (\mathbf{U}\bar{\rho})_x + \bar{\mathbf{C}}\bar{\rho} &= \bar{f} && \text{in } (0,T) \times \Omega \\
\bar{\rho}(0,x) &= \rho_0(x) && \text{in } \Omega \\
(\nu \mathbf{U})^- \bar{\rho} &= (\nu \mathbf{U})^- \bar{\mathcal{H}}(\bar{\rho}|_{\Gamma_T^+}) && \mathrm{d}\mu_u^- \text{on } \Gamma_T
\end{aligned}$$

with reaction term $\bar{c} = c + \alpha \mathbf{1}$, boundary operator $\bar{\mathcal{H}}(\rho) = r\rho_{\text{in}} - F_+ + \mathcal{G}(F_+) + \mathcal{G}(\rho)$ and source term $\bar{f}_i = rf_i - \|(rf_i)^+\|_{L^\infty(\Omega)} - ((u_i)_x + \bar{c}_i)\,(F_+)_i$. For the boundary operator $\bar{\mathcal{H}}$ this conclusion needs further explanations. Since $\rho$ is a solution of (3.1) we find

$$\begin{aligned}
\gamma\bar{\rho}|_{\Gamma_T^-} &= r\gamma\rho|_{\Gamma_T^-} - F_+ \\
&= r\mathcal{H}(\gamma\rho|_{\Gamma_T^+}) - F_+ \\
&= r\rho_{\text{in}} + \mathcal{G}(r\gamma\rho|_{\Gamma_T^+}) - F_+ \\
&= r\rho_{\text{in}} - F_+ + \mathcal{G}(F_+) + \mathcal{G}(\gamma\bar{\rho}|_{\Gamma_T^+}) \\
&= \bar{\mathcal{H}}(\gamma\bar{\rho}|_{\Gamma_T^+}).
\end{aligned}$$

The advantage of introducing $\bar{\rho}$ is that proving the upper bound (3.12) reduces to proving $\bar{\rho} \leq \rho_{\max}$, where $\rho_{\max}$ is known from the assumptions in Section 3.1. To this end, we define $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$\beta(s) = (s - \rho_{\max})^+$$

and we will show $\beta(\bar\rho) = 0$ almost everywhere.

For the test function $\varphi = \overline{\mathbf{W}}\mathbf{1}$ and all $t_0 \in [0, T]$, the renormalization property yields

$$
\begin{aligned}
&-\int_0^{t_0}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\left(\bar{\mathbf{C}}D\beta(\bar\rho)\bar\rho - D\beta(\bar\rho)\bar f\right)\mathrm{d}x\mathrm{d}t - \int_0^{t_0}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\mathbf{U}_x(D\beta(\bar\rho)\bar\rho - \beta(\bar\rho))\mathrm{d}x\mathrm{d}t \\
&-\int_0^{t_0}\int_\Gamma \beta(\gamma\bar\rho)^T\overline{\mathbf{W}}(\nu u)\mathrm{d}\omega\mathrm{d}t - \int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\bar\rho(t_0))\mathrm{d}x = 0
\end{aligned}
\tag{3.14}
$$

because of $\beta(\rho_0) = 0$. The boundary term of this equation is non-negative. This can be shown using the componentwise monotonicity of $\beta$, the assumption (3.5) and the inequalities (3.6) and (3.8):

$$
\begin{aligned}
&\int_0^{t_0}\int_\Gamma \beta(\bar{\mathcal{H}}(\gamma\bar\rho))^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^{t_0}\int_\Gamma \beta\left(r\rho_{\mathrm{in}} - F_+ + \mathcal{G}(F_+) + \mathcal{G}(\gamma\bar\rho)\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^{t_0}\int_\Gamma \beta\left(r\rho_{\mathrm{in}} + \mathcal{G}(\rho_{\max}) + \mathcal{G}(\gamma\bar\rho - \rho_{\max})\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&\le \int_0^{t_0}\int_\Gamma \beta\left(\rho_{\max} + \mathcal{G}(\gamma\bar\rho - \rho_{\max})\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^{t_0}\int_\Gamma \left((\mathcal{G}(\gamma\bar\rho - \rho_{\max}))^+\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&\le \int_0^{t_0}\int_\Gamma \mathcal{G}\left((\gamma\bar\rho - \rho_{\max})^+\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&\le \int_0^{t_0}\int_\Gamma \left((\gamma\bar\rho - \rho_{\max})^+\right)^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^{t_0}\int_\Gamma \beta(\gamma\bar\rho)^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t.
\end{aligned}
$$

Because of $\beta(s) \ge 0$, $D\beta(s)s \ge 0$, $\bar c + u_x = c + u_x + \|(c+u_x)^-\|_{L^\infty(\Omega)^n}\mathbf{1} \ge 0$ and $(D(\beta)\bar f)_i = D(\beta)_{ii}\left(rf_i - \|(rf_i)^+\|_{L^\infty(\Omega)} - ((u_i)_x + \bar c_i)F_i\right) \le 0$ we conclude from equation (3.14) that it holds

$$
\begin{aligned}
\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\bar\rho(t_0))\mathrm{d}x = &-\int_0^{t_0}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\left(\bar{\mathbf{C}} + \mathbf{U}_x\right)D\beta(\bar\rho)\bar\rho\mathrm{d}x\mathrm{d}t \\
&+ \int_0^{t_0}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}D\beta(\bar\rho)\bar f\mathrm{d}x\mathrm{d}t + \int_0^{t_0}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\mathbf{U}_x\beta(\bar\rho)\mathrm{d}x\mathrm{d}t \\
&- \int_0^{t_0}\int_\Gamma \beta(\gamma\bar\rho)^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t + \int_0^{t_0}\int_\Gamma \beta(\bar{\mathcal{H}}(\gamma\bar\rho))^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
\le &\int_0^{t_0}\|(u_x)^+\|_{L^\infty(\Omega)^n}\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\bar\rho)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.15}
$$

With Gronwall's inequality (Lemma 1.19) this leads to $\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\bar\rho(t_0))\mathrm{d}x = 0$ and thus $\beta(\bar\rho(t_0)) = 0$ and $\bar\rho(t_0) \le \rho_{\max}$ almost everywhere. Using the renormalization property and the uniqueness of the trace of $\beta(\bar{\rho}) = 0$, we also conclude

$$0 = \gamma\beta(\bar{\rho}) = \beta(\gamma\bar{\rho})$$

and thus $\gamma\bar{\rho}(t,\omega) \leq \rho_{\max}$ holds for $|\mathrm{d}\mu_u|$-almost all $(t,\omega)$. Addition of $F$ and multiplication by $\exp\left(\int_0^t \alpha(s)\mathrm{d}s\right)$ shows the desired upper bound for $\rho$ and $\gamma\rho$.

In order to prove the lower bound, we use the same procedure for $\bar{\rho} = r\rho + F_-$ and $\beta(s) = (s + F_-(T))^-$. In this case, it is $\bar{f}_i = rf_i + \|(rf_i)^-\|_{L^\infty(\Omega)} + ((u_i)_x + c)(F_-)_i$ and $\bar{\mathcal{H}}(\rho) = r\rho_{\text{in}} + F_- - \mathcal{G}(F_-) + \mathcal{G}(\rho)$. The only step we have to take again into account is the transition from equation (3.14) to inequality (3.15). With the monotonicity of $\beta$ and $F$, the positivity of $\rho_{\text{in}}$ and $F_-$ and the inequalities (3.6) and (3.9), it follows

$$\begin{aligned}
&\int_0^{t_0}\int_\Gamma \beta(\bar{\mathcal{H}}(\gamma\bar{\rho}))^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t\\
&= \int_0^{t_0}\int_\Gamma \beta\left(r\rho_{\text{in}} + F_- + \mathcal{G}(\gamma\bar{\rho}) - \mathcal{G}(F_-)\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t\\
&= \int_0^{t_0}\left((r\rho_{\text{in}} + F_- + F_-(T) + \mathcal{G}(\gamma\bar{\rho} - F_-))^-\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t\\
&\leq \int_0^{t_0}\int_\Gamma \left((\mathcal{G}(\gamma\bar{\rho} - F_-))^-\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t\\
&\leq \int_0^{t_0}\int_\Gamma \mathcal{G}\left((\gamma\bar{\rho} - F_-)^-\right)^T\overline{\mathbf{W}}(\nu u)^-\mathrm{d}\omega\mathrm{d}t\\
&\leq \int_0^{t_0}\int_\Gamma \left((\gamma\bar{\rho} - F_-)^-\right)^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t\\
&\leq \int_0^{t_0}\int_\Gamma \left((\gamma\bar{\rho} - F_-(T))^-\right)^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t\\
&= \int_0^{t_0}\int_\Gamma \beta(\gamma\bar{\rho})^T\overline{\mathbf{W}}(\nu u)^+\mathrm{d}\omega\mathrm{d}t.
\end{aligned}$$

Therefore, we can conclude as before $\bar{\rho}(t_0) \geq -F_-(T)$ almost everywhere in $\Omega$ and $\gamma\bar{\rho}(t,\omega) \geq -F_-(T)$ for $|\mathrm{d}\mu_u|$-almost all $(t,\omega) \in \Gamma_T$. This completes the proof. □

Beside the upper bound we just proved a lower bound. This lower bound can be sharpened if we ask for an additional assumption on $\mathcal{G}$. For simplicity, we will formulate this result only for a vanishing source term $f = 0$, since in this case the left-hand side of the inequalities (3.12) and (3.13) is zero and thus $\rho \geq 0$ almost everywhere.

**Lemma 3.10 (Lower bound).** *In addition to the requirements from Section 3.1, we assume $(c + u_x) \in L^1((0,T), L^\infty(\Omega)^n)$ and $f = 0$. Furthermore, let $\rho_{\min} \in \mathbb{R}^n_{>0}$ be a vector such that component-by component it holds*

$$\rho_{\text{in}}\exp\left(\int_0^t \zeta(s)\mathrm{d}s\right) + \mathcal{G}(\rho_{\min}) \geq \rho_{\min} \tag{3.16}$$

*for $\mathrm{d}\mu_u^-$-almost all $(t,\omega) \in \Gamma_T$ and*

$$\rho_0(x) \geq \rho_{\min}$$

*for almost all $x \in \Omega$ with $\zeta(s) = \|(u_x(s,\cdot) + c(s,\cdot))^+\|_{L^\infty(\Omega)^n}$. Then, a solution of the initial-boundary-value problem* (3.1) *is bounded from below. More precisely, it is for all $t \in [0,T]$*

$$\rho(t,\cdot) \geq \rho_{\min} \exp\left(-\int_0^t \zeta(s)\mathrm{d}s\right) > 0$$

*almost everywhere in $\Omega$ and*

$$\gamma\rho(t,\omega) \geq \rho_{\min} \exp\left(-\int_0^t \zeta(s)\mathrm{d}s\right) > 0$$

*for $|\mathrm{d}\mu_u|$-almost all $(t,\omega) \in \Gamma_T$.*

*Proof.* Keeping in mind the non-negativity of $\rho$, this lemma can be proven in exactly the same manner as Lemma 3.9. □

The previous statements, especially the renormalization property of Theorem 3.3, provide us with the necessary tools to prove a central result of this chapter: a kind of sequential continuity of the solution operator. We formulate this statement as a theorem with two parts. In the first part the weak-$\star$ convergence of a sequence of solutions to a solution of the limit problem is obtained for very weak assumptions. This result will play a crucial role in the proof of existence of solutions of the transport equation, whereas the second part, proving the convergence in $C([0,T], L^p(\Omega)^n)$ under more restrictive assumptions, will be used in Chapter 5 for the existence result of solutions of the low Mach number equations.

**Theorem 3.11 (Stability).** *1. For all $k \in \mathbb{N}$ let $u_k$, $c_k$, $f_k$, $\rho_{0,k}$, $\rho_{\mathrm{in},k}$ and $\mathcal{G}_k$ be defined as in Section 3.1. Assume there exists for each $k$ a solution*

$$\rho_k \in L^\infty((0,T) \times \Omega)^n$$

*of the initial-boundary-value problem*

$$\begin{aligned} (\rho_k)_t + (\mathbf{U}_k\rho_k)_x + \mathbf{C}_k\rho_k &= f_k && \text{in } (0,T)\times\Omega \\ \rho_k(0) &= \rho_{0,k} && \text{in } \Omega \\ (\nu\mathbf{U}_k)^-\rho_k &= (\nu\mathbf{U}_k)^-\mathcal{H}_k(\rho_k|_{\Gamma_T^+}) && \text{on } \Gamma_T \end{aligned}$$

*with trace $\gamma\rho_k \in L^\infty(\Gamma_T, |\mathrm{d}\mu_{u_k}|)$.*

*Moreover, we assume:*

***uniform boundedness:*** *The sequence $(\rho_{\max,k})_k \subset \mathbb{R}^n_{\geq 0}$ from assumption* (3.5) *is bounded, i.e. there is $\rho_{\max} \in \mathbb{R}^n_{\geq 0}$ with $\rho_{\max,k} \leq \rho_{\max}$.*

***convergence of the reaction term:*** *$(c_k)_k$ converges strongly to $c \in L^1((0,T)\times\Omega)^n$ in $L^1((0,T)\times\Omega)^n$.*

***convergence of the velocity:*** *The sequence $(u_k)_k$ converges strongly in $L^1((0,T)\times\Omega)^n$ to $u \in L^1((0,T), W^{1,1}(\Omega)^n)$ with $(u_x + c)^- \in L^1((0,T), L^\infty(\Omega)^n)$ and $(u_x)^+ \in L^1((0,T), L^\infty(\Omega)^n)$.*

***convergence of the velocity at the boundary:*** *The sequence* $(\nu u_k)_k$ *converges strongly to* $\nu u$ *in* $L^1(\Gamma_T)^n$.

***boundedness of reaction term and velocity derivative:*** *The sequence* $((c_k + (u_k)_x)^-)_k$ *is bounded in* $L^1((0,T), L^\infty(\Omega))$.

***weak convergence and boundedness of the source term:*** *The sequence* $(f_k)_k$ *converges weakly to* $f \in L^1((0,T), L^\infty(\Omega)^n)$ *in* $L^1((0,T) \times \Omega)^n$ *and it is bounded in* $L^1((0,T), L^\infty(\Omega)^n)$.

***weak-$\star$ convergence of the initial conditions:*** *The sequence* $(\rho_{0,k})_k$ *converges weakly-$\star$ to* $\rho_0 \in L^\infty(\Omega)^n$ *in* $L^\infty(\Omega)^n$.

***weak-$\star$ convergence of the inflow density:*** *The sequence* $(\rho_{\text{in},k})_k$ *converges weakly-$\star$ to* $\rho_{\text{in}} \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ *in* $L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$.

***weak-$\star$ convergence/continuity of the boundary operator:*** *There is an operator* $\mathcal{G}$ *fulfilling the assumptions of Section 3.1 such that for each weak-$\star$ convergent sequence* $q_k \in L^\infty(\Gamma_T)^n$ *with limit* $q$ *it holds*

$$(\nu \mathbf{U}_k)^- \mathcal{G}_k(q_k)(\nu u_k)^- \rightharpoonup (\nu \mathbf{U})^- \mathcal{G}(q)$$

*in* $L^1(\Gamma_T)^n$.

*Then, there exists a solution* $\rho \in C([0,T], L^p(\Omega)^n)$ *with trace* $\gamma\rho \in L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$ *of the transport equation*

$$\begin{aligned} \rho_t + (\mathbf{U})\rho_x + \mathbf{C}\rho &= f && \text{in } (0,T) \times \Omega \\ \rho(0) &= \rho_0 && \text{in } \Omega \\ (\nu \mathbf{U})^- \rho &= (\nu \mathbf{U})^- \mathcal{H}(\rho|_{\Gamma_T^+}) && \text{on } \Gamma_T. \end{aligned} \tag{3.17}$$

*Furthermore, it holds*

$$\begin{aligned} \rho_k &\overset{\star}{\rightharpoonup} \rho && \text{in } L^\infty((0,T) \times \Omega)^n, \\ \gamma\rho_k &\overset{\star}{\rightharpoonup} \gamma\rho && \text{in } L^\infty(\Gamma_T, |\mathrm{d}\mu_u|) \end{aligned}$$

*and*

$$\rho_k(t) \rightharpoonup \rho(t) \qquad \text{in } L^p(\Omega)^n$$

*for all* $t \in [0,T]$ *and* $p \in [1, \infty)$.

*2. We assume additionally:*

***convergence of the velocity derivative:*** *The sequence* $((u_k)_x)_k$ *converges strongly to* $u_x$ *in the* $L^1((0,T) \times \Omega)^n$*-norm.*

***convergence of reaction term and velocity derivative:*** *The sequence* $(\bar{\alpha}_k)_k$ *with* $\bar{\alpha}_k = \|(2c_k + (u_k)_x)^-\|_{L^\infty(\Omega)^n}$ *converges strongly to* $\bar{\alpha} = \|(2c + u_x)^-\|_{L^\infty(\Omega)^n}$ *in* $L^1(0,T)$.

***convergence of the source term:*** *The sequence* $(f_k)_k$ *converges to* $f$ *in* $L^1((0,T)\times\Omega)^n$.

***convergence of the initial conditions:*** *The sequence* $(\rho_{0,k})_k$ *converges strongly to* $\rho_0$ *in the* $L^1(\Omega)^n$*-norm.*

***"weak lower semi-continuity" of the boundary operator:*** *For each weak-*$\star$ *convergent sequence* $\rho_k \overset{\star}{\rightharpoonup} \rho$ *in* $L^\infty(\Gamma_T)^n$ *it holds*

$$\begin{aligned}&\int_0^T \int_\Gamma \beta(\rho)^T \overline{\mathbf{W}}(\nu u)^+ - \beta(\bar{\mathcal{H}}(\rho))^T \overline{\mathbf{W}}(\nu u)^- \mathrm{d}\omega \mathrm{d}t\\ &\quad \le \liminf_k \int_0^T \int_\Gamma \beta(\rho_k)^T \overline{\mathbf{W}}(\nu u_k)^+ - \beta(\bar{\mathcal{H}}_k(\rho_k))^T \overline{\mathbf{W}}(\nu u_k)^- \mathrm{d}\omega \mathrm{d}t\end{aligned} \tag{3.18}$$

*with* $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $\beta_j(s) = s_j^2$, $\bar{\mathcal{H}}_k(\rho) = r_k\rho_{\mathrm{in},k} + \mathcal{G}_k(\rho)$ *and* $\bar{\mathcal{H}}(\rho) = r\rho_{\mathrm{in}} + \mathcal{G}(\rho)$. *Here, it is* $r_k(t) = \exp\left(-\frac{1}{2}\int_0^t \bar{\alpha}_k(s)\mathrm{d}s\right)$ *and* $r(t) = \exp\left(-\frac{1}{2}\int_0^t \bar{\alpha}(s)\mathrm{d}s\right)$ *and* $\overline{\mathbf{W}}$ *is the weight matrix introduced in Section 3.1.*

*Then, the convergence is even stronger:*

$$\rho_k \to \rho \qquad\qquad \text{in } C([0,T], L^p(\Omega)^n)$$

*and*

$$\gamma\rho_k \to \gamma\rho \qquad\qquad \text{in } L^p(\Gamma_T, |\mathrm{d}\mu_u|)$$

*for all* $p \in [1,\infty)$.

*Proof.* For the proof we extend $\gamma\rho_k$, $\rho_{\mathrm{in},k}$, $\mathcal{G}_k(\gamma\rho_k)$ and $\mathcal{H}(\gamma\rho_k)$ by zero to functions in $L^\infty(\Gamma_T)^n$.

1. First, we want to prove the existence of a solution of the limit problem and the weak-$\star$ convergence. Because of Lemma 3.9 and the boundedness of $\rho_{\max_k} \le \rho_{\max}$, $\|(c_k + (u_k)_x)^-\|_{L^1((0,T),L^\infty(\Omega)^n)} \le C_1$ and $\|f_k\|_{L^1((0,T),L^\infty(\Omega)^n)} \le C_2$ the sequences $\rho_k$ and $\gamma\rho_k$ are bounded almost everywhere by

$$\begin{aligned}&\max(\|\rho_k\|_{L^\infty((0,T)\times\Omega)^n}, \|\gamma\rho_k\|_{L^\infty(\Gamma_T)^n})\\ &\le \max\left((\rho_{\max,k} + F_k(t), F_k(T) + F_k(t)\right)\exp\left(\|(c_k + (u_k)_x)^-\|_{L^1((0,T),L^\infty(\Omega)^n)}\right)\\ &\le (\max(\rho_{\max}, C_2) + C_2)\exp(C_1)\\ &= \bar{\rho}_{\max}.\end{aligned} \tag{3.19}$$

Here, it is

$$F_k(t) = \int_0^t \exp\left(-\int_0^s \|(c_k + (u_k)_x)^-\|\mathrm{d}r\right)\begin{pmatrix}\|((f_k)_1(s,\cdot))^+\|_{L^\infty(\Omega)}\\ \vdots \\ \|((f_k)_n(s,\cdot))^+\|_{L^\infty(\Omega)}.\end{pmatrix}\mathrm{d}s$$

Thus, we can extract weak-$\star$ convergent subsequences - also denoted by $\rho_k$ and $\gamma\rho_k$ - with

$$\rho_k \overset{\star}{\rightharpoonup} \rho \qquad \text{in } L^\infty((0,T)\times\Omega)^n$$

and

$$\gamma\rho_k \overset{\star}{\rightharpoonup} q \qquad \text{in } L^\infty(\Gamma_T)^n.$$

As a second step, we will show that $\rho$ is a solution of the transport equation and that $q$ is its trace and fulfils the boundary conditions.

Because of the strong convergence of $u_k$ and $c_k$ in $L^1((0,T)\times\Omega)^n$, the sequences $\mathbf{U}_k\rho_k$ and $\mathbf{C}_k\rho_k$ converge weakly in $L^1((0,T)\times\Omega)^n$, i.e.

$$\mathbf{U}_k\rho_k \rightharpoonup \mathbf{U}\rho$$

and

$$\mathbf{C}_k\rho_k \rightharpoonup \mathbf{C}\rho.$$

Similarly, $(\nu\mathbf{U}_k)\gamma\rho_k$ converges weakly to $(\nu\mathbf{U})\rho$ in $L^1(\Gamma_T)^n$. Thus, for a test function $\varphi \in C^{0,1}([0,T]\times\overline{\Omega})$ with $\varphi(T)=0$ it holds

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \rho_k^T\left(\varphi_t + \mathbf{U}_k\varphi_x - \mathbf{C}_k\varphi\right) + f_k^T\varphi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \rho_{0,k}^T\varphi(0)\mathrm{d}x - \int_0^T\int_\Gamma \gamma\rho_k^T(\nu\mathbf{U}_k)\varphi\mathrm{d}\omega\mathrm{d}t \\
&\to \int_0^T \int_\Omega \rho^T\left(\varphi_t + \mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + f^T\varphi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \rho_0^T\varphi(0)\mathrm{d}x - \int_0^T\int_\Gamma q^T(\nu\mathbf{U})\varphi\mathrm{d}\omega\mathrm{d}t.
\end{aligned}$$

In words, this means that $\rho$ is a solution of the transport equation with its unique trace $\gamma\rho = q$. Furthermore, because of Theorem 3.3 it is $\rho \in C([0,T], L^p(\Omega)^n)$ and $\rho$ fulfils the initial condition $\rho(0) = \rho_0$. For the boundary term we have to consider the trace $q$ in more detail. The assumptions on $\mathcal{G}_k$ and $\rho_{\mathrm{in},k}$ yield the weak convergence of $(\nu\mathbf{U}_k)^-\rho_{\mathrm{in},k}$ and $(\nu\mathbf{U}_k)^-\mathcal{G}_k(\gamma\rho_k)$. Thus,

$$\begin{aligned}
\int_0^T\int_\Gamma q^T(\nu\mathbf{U})^-\varphi\mathrm{d}\omega\mathrm{d}t &\leftarrow \int_0^T\int_\Gamma \gamma\rho_k^T(\nu\mathbf{U}_k)^-\varphi\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^T\int_\Gamma \mathcal{H}(\gamma\rho_k)^T(\nu\mathbf{U}_k)^-\varphi\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^T\int_\Gamma \left(\rho_{\mathrm{in},k} + \mathcal{G}_k(\gamma\rho_k)\right)^T(\nu\mathbf{U}_k)^-\varphi\mathrm{d}\omega\mathrm{d}t \\
&\to \int_0^T\int_\Gamma \left(\rho_{\mathrm{in}} + \mathcal{G}(q)\right)^T(\nu\mathbf{U})^-\varphi\mathrm{d}\omega\mathrm{d}t \\
&= \int_0^T\int_\Gamma \mathcal{H}(q)^T(\nu\mathbf{U})^-\varphi\mathrm{d}\omega\mathrm{d}t
\end{aligned}$$

is true and the trace $q = \gamma\rho$ fulfils the boundary condition

$$(\nu\mathbf{U})^{-}\gamma q = (\nu\mathbf{U})^{-}\mathcal{H}(\gamma q).$$

Hence, $\rho$ is the unique solution of the limit problem (3.17). Since the weak-$\star$ limits $\rho$ and $q$ are unique, in fact the whole sequences and not only subsequences converge.

As last step of the first statement, we have to show the weak convergence of $\rho_k(t)$ for each $t \in [0,T]$. However, this is rather simple, as the sequence $\rho_k(t)$ is bounded in $L^p(\Omega)^n$ because of the continuity of $\rho_k$ with values in $L^p(\Omega)^n$ and Lemma 3.9. Thus, there exists a weak convergent subsequence in $L^p(\Omega)^n$ with

$$\rho_k(t) \rightharpoonup q.$$

For this subsequence and for all time-independent test functions $\varphi \in C^{0,1}(\overline{\Omega}, \mathbb{R}^n)$ it holds with Theorem 3.3

$$\begin{aligned}
\int_\Omega q^T\varphi \mathrm{d}x \leftarrow & \int_\Omega \rho_k(t)^T\varphi \mathrm{d}x \\
&= \int_0^t \int_\Omega \rho_k^T\left(\mathbf{U}_k\varphi_x - \mathbf{C}_k\varphi\right) + f_k^T\varphi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \rho_{0,k}^T\varphi \mathrm{d}x - \int_0^t\int_\Gamma \gamma\rho_k^T(\nu\mathbf{U}_k)\varphi \mathrm{d}\omega\mathrm{d}t \\
&\to \int_0^t \int_\Omega \rho^T\left(\mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + f^T\varphi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \rho_0^T\varphi \mathrm{d}x - \int_0^t\int_\Gamma \gamma\rho^T(\nu\mathbf{U})\varphi \mathrm{d}\omega\mathrm{d}t \\
&= \int_\Omega \rho(t)^T\varphi \mathrm{d}x.
\end{aligned}$$

Thus, we conclude

$$q = \rho(t)$$

due to the density of $C^{0,1}(\overline{\Omega}, \mathbb{R}^n)$ in $L^{p'}(\Omega)^n$. Again, by the uniqueness of the solution $\rho$ we see that the whole sequence converges weakly.

2. Now, we consider the stronger assumptions in order to prove both, the uniform convergence of $\rho_k$ with values in $L^p(\Omega)^n$ and the strong convergence of the trace. As a first step, we prove that the convergence of $\rho_k(t)$ is actually strong in $L^p(\Omega)^n$ for each $t \in [0,T]$. To this end, we use the Radon-Riesz property (see e.g. [67]) of the $L^p$-spaces with $p \in (1,\infty)$, i.e. the fact

$$\left.\begin{array}{l} f_k \rightharpoonup f \text{ in } L^p(\Omega)^n \\ \|f_k\|_{L^p(\Omega)^n} \to \|f\|_{L^p(\Omega)^n} \end{array}\right\} \Rightarrow f_k \to f \text{ in } L^p(\Omega)^n. \tag{3.20}$$

We define the auxiliary variable

$$\bar{\rho}_k(t,x) = \rho_k(t,x) r_k(t)$$

with $r_k(t) = \exp\left(-\frac{1}{2}\int_0^t \bar{\alpha}_k(s)\mathrm{d}s\right)$ and $\bar{\alpha}_k(s) = \left\|(2c_k + (u_k)_x)^-\right\|_{L^\infty(\Omega)^n}$. By assumption, it holds $\bar{\alpha}_k \to \bar{\alpha} = \left\|(2c + u_x)^-\right\|_{L^\infty(\Omega)^n}$ in $L^1((0,T))$. Thus, we conclude

$$r_k(t) \to r(t) = \exp\left(-\frac{1}{2}\int_0^t \alpha(s)\mathrm{d}s\right)$$

in $C([0,T])$, as in the proof of Lemma 3.9. We also see, as in that proof, that $\bar{\rho}_k$ solves the equation

$$\begin{aligned}
(\bar{\rho}_k)_t + (\mathbf{U}_k\bar{\rho}_k)_x + \bar{\mathbf{C}}_k\bar{\rho}_k &= \bar{f}_k && \text{in } (0,T)\times\Omega\\
\bar{\rho}_k(0) &= \rho_{0,k} && \text{in } \Omega\\
(\nu\mathbf{U}_k)^-\bar{\rho}_k &= (\nu\mathbf{U}_k)^-\bar{\mathcal{H}}_k(\bar{\rho}_k|_{\Gamma_T}) && \text{on } \Gamma_T
\end{aligned}$$

with $\bar{c}_k = c_k + \frac{1}{2}\bar{\alpha}_k\mathbf{1}$, $\bar{f}_k = r_k f_k$ and $\bar{\mathcal{H}}_k(\bar{\rho}) = r_k(t)\rho_{\text{in},k} + \mathcal{G}_k(\bar{\rho})$. For $\bar{\rho}_k$ and this problem, all requirements of the first part of this theorem are fulfilled. Thus, we conclude

$$\bar{\rho}_k(t) \rightharpoonup \bar{\rho}(t)$$

in $L^p(\Omega)^n$, where $\bar{\rho}$ is the solution of

$$\begin{aligned}
(\bar{\rho})_t + (\mathbf{U}\bar{\rho})_x + \bar{\mathbf{C}}\bar{\rho} &= \bar{f} && \text{in } (0,T)\times\Omega\\
\bar{\rho}(0) &= \rho_0 && \text{in } \Omega\\
(\nu\mathbf{U})^-\bar{\rho}_k &= (\nu\mathbf{U})^-\bar{\mathcal{H}}(\bar{\rho}|_{\Gamma_T}) && \text{on } \Gamma_T
\end{aligned}$$

with $\bar{c} = c + \bar{\alpha}\mathbf{1}$, $\bar{f} = rf$ and $\bar{\mathcal{H}}(\bar{\rho}) = r(t)\rho_{\text{in}} + \mathcal{G}(\bar{\rho})$.

The renormalization property for $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\beta(s)_j = s_j^2$ and $\varphi = \overline{\mathbf{W}}\mathbf{1}$, the weak lower semi-continuity of the $L^2$-Norm, the strong convergence of $\rho_{0,k}$ and the weak convergence of $\bar{f}_k^T\overline{\mathbf{W}}\bar{\rho}_k$ yield

$$\begin{aligned}
&\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\rho_0)\mathrm{d}x - \int_0^t\int_\Gamma \beta(\gamma\bar{\rho})^T\overline{\mathbf{W}}(\nu u)\mathrm{d}\omega\mathrm{d}t\\
&\quad - \int_0^t\int_\Omega (2\bar{c} + u_x)^T\overline{\mathbf{W}}\beta(\bar{\rho})\mathrm{d}x\mathrm{d}t - 2\bar{f}^T\overline{\mathbf{W}}\bar{\rho}\mathrm{d}x\mathrm{d}t\\
&= \int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\bar{\rho}(t))\mathrm{d}x\\
&= \|\overline{\mathbf{W}}\bar{\rho}(t)\|^2_{L^2(\Omega)^n}\\
&\le \liminf_k \|\overline{\mathbf{W}}\bar{\rho}_k(t)\|^2_{L^2(\Omega)^n} \qquad (3.21)\\
&= \liminf_k\Bigg(\int_\Omega \mathbf{1}^T\overline{\mathbf{W}}\beta(\rho_{0,k})\mathrm{d}x - \int_0^t\int_\Gamma \beta(\gamma\bar{\rho}_k)^T\overline{\mathbf{W}}(u_k\nu)\mathrm{d}\omega\mathrm{d}t\\
&\qquad - \int_0^t\int_\Omega ((2\bar{c}_k + u_k)_x)^T\overline{\mathbf{W}}\beta(\bar{\rho}_k) - 2\bar{f}_k^T\overline{\mathbf{W}}\bar{\rho}_k\mathrm{d}x\mathrm{d}t\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \mathbf{1}^T \overline{\mathbf{W}} \beta(\rho_0) \mathrm{d}x \\
&\quad + \liminf_k \Bigg( \int_0^t \int_\Gamma \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k \nu)^- - \beta(\overline{\gamma\rho_k})^T \overline{\mathbf{W}} (u_k\nu)^+ \mathrm{d}\omega \mathrm{d}t \\
&\qquad - \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t \Bigg) + \int_0^t \int_\Omega 2\bar{f}^T \overline{\mathbf{W}} \bar{\rho} \mathrm{d}x \mathrm{d}t
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\limsup_k \Bigg( \int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (u_k\nu)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k\nu)^- \mathrm{d}\omega \mathrm{d}t \\
&\qquad + \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t \Bigg) \\
&\leq \int_0^t \int_\Gamma \beta(\gamma\bar{\rho})^T \overline{\mathbf{W}} (\nu u)^+ - \beta(\bar{\mathcal{H}}(\gamma\bar{\rho}))^T \overline{\mathbf{W}} (\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&\quad + \int_0^t \int_\Omega (2\bar{c} + u_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}) \mathrm{d}x \mathrm{d}t.
\end{aligned} \tag{3.22}
$$

Using the assumption (3.18) for the weak-$\star$ convergent sequence $\chi_{[0,t]}\gamma\bar{\rho}_k$, we conclude

$$
\begin{aligned}
&\int_0^t \int_\Gamma \beta(\gamma\bar{\rho})^T \overline{\mathbf{W}} (\nu u)^+ - \beta(\bar{\mathcal{H}}(\gamma\bar{\rho}))^T \overline{\mathbf{W}} (\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&\leq \liminf_k \int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (\nu u_k)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (\nu u_k)^- \mathrm{d}\omega \mathrm{d}t
\end{aligned} \tag{3.23}
$$

for all $t \in [0,T]$. Because of the strong convergence of $c_k$ and $(u_k)_x$ in $L^1((0,T)\times\Omega)^n$, we have for $j = 1, \dots, n$ also the convergence

$$
\left|2\bar{c}_k^j + (u_k^j)_x\right|^{\frac{1}{2}} \to \left|2\bar{c}^j + (u^j)_x\right|^{\frac{1}{2}}
$$

in $L^2([0,T]\times\Omega)$ and hence also the weak convergence

$$
\left|2\bar{c}_k^j + (u_k^j)_x\right|^{\frac{1}{2}} \bar{\rho}_k^j \rightharpoonup \left|2\bar{c}^j + (u^j)_x\right|^{\frac{1}{2}} \bar{\rho}^j
$$

in $L^2((0,T)\times\Omega)$. Here, the superscripts denote the components of the vectors. Since it holds $2\bar{c}_k + (u_k)_x = 2c_k + (u_k)_x + \|(2c_k + (u_k)_x)^-\|_{L^\infty(\Omega)^n}\mathbf{1} \geq 0$, we can again use the weak lower semi-continuity of the norm to find

$$
\begin{aligned}
\int_0^t \int_\Omega (2\bar{c} + u_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}) \mathrm{d}x \mathrm{d}t &= \sum_{j=1}^n \|(2\bar{c}^j + (u^j)_x)^{\frac{1}{2}} \mathbf{W}_{jj} \bar{\rho}^j\|^2_{L^2((0,t)\times\Omega)} \\
&\leq \liminf_k \sum_{j=1}^n \|(2\bar{c}_k^j + (u_k^j)_x)^{\frac{1}{2}} \overline{\mathbf{W}}_{jj} \bar{\rho}_k^j\|^2_{L^2((0,t)\times\Omega)} \\
&= \liminf_k \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \, \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t.
\end{aligned}
$$

This leads with equation (3.22) and inequality (3.23) to

$$
\begin{aligned}
&\limsup_k \Bigg( \int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (u_k\nu)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k\nu)^- \mathrm{d}\omega \mathrm{d}t \\
&\qquad + \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t \Bigg) \\
&\le \int_0^t \int_\Gamma \beta(\gamma\bar{\rho})^T \overline{\mathbf{W}} (\nu u)^+ - \beta(\bar{\mathcal{H}}(\gamma\bar{\rho}))^T \overline{\mathbf{W}} (\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&\quad + \int_0^t \int_\Omega (2\bar{c} + u_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}) \mathrm{d}x \mathrm{d}t \\
&\le \liminf_k \Bigg( \int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (u_k\nu)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k\nu)^- \mathrm{d}\omega \Bigg) \mathrm{d}t \\
&\quad + \liminf_k \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t \\
&\le \liminf_k \Bigg( \int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (u_k\nu)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k\nu)^- \mathrm{d}\omega \mathrm{d}t \\
&\qquad + \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) \mathrm{d}x \mathrm{d}t \Bigg).
\end{aligned}
$$

Hence, the limit inferior and the limit superior coincide and consequently they are equal to the limit. Altogether this yields

$$
\begin{aligned}
&\lim_{k\to\infty} \|\overline{\mathbf{W}}\bar{\rho}_k(t)\|^2_{L^2(\Omega)^n} \\
&= \lim_{k\to\infty} \Bigg( -\int_0^t \int_\Gamma \beta(\gamma\bar{\rho}_k)^T \overline{\mathbf{W}} (u_k\nu)^+ - \beta(\bar{\mathcal{H}}_k(\gamma\bar{\rho}_k))^T \overline{\mathbf{W}} (u_k\nu)^- \mathrm{d}\omega \mathrm{d}t \\
&\qquad + \int_\Omega \mathbf{1}^T \overline{\mathbf{W}} \beta(\rho_{0,k}) \mathrm{d}x - \int_0^t \int_\Omega (2\bar{c}_k + (u_k)_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}_k) - 2\bar{f}_k^T \overline{\mathbf{W}} \bar{\rho}_k \mathrm{d}x \mathrm{d}t \Bigg) \\
&= -\int_0^t \int_\Gamma \beta(\gamma\bar{\rho})^T \overline{\mathbf{W}} (\nu u)^+ - \beta(\bar{\mathcal{H}}(\gamma\bar{\rho}))^T \overline{\mathbf{W}} (\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&\quad + \int_\Omega \mathbf{1}^T \overline{\mathbf{W}} \beta(\rho_0) \mathrm{d}x - \int_0^t \int_\Omega (2\bar{c} + u_x)^T \overline{\mathbf{W}} \beta(\bar{\rho}) - 2\bar{f}^T \overline{\mathbf{W}} \bar{\rho} \mathrm{d}x \mathrm{d}t \\
&= \|\overline{\mathbf{W}}\bar{\rho}(t)\|^2_{L^2(\Omega)^n}.
\end{aligned}
$$

Now, the strong convergence of $\bar{\rho}_k(t)$ for all $t \in [0, T]$ follows by the Radon-Riesz property (3.20) of $L^2(\Omega)^n$ and the equivalence of the weighted Euclidean norm $\|\overline{\mathbf{W}} \cdot \|_2$ and the Euclidean norm $\| \cdot \|_2$. Thus, by the convergence of $\bar{r}_k$

$$
\bar{r}_k(t) = \exp\left(\frac{1}{2}\int_0^t \bar{\alpha}_k(s)\mathrm{d}s\right) \to \bar{r}(t) = \exp\left(\frac{1}{2}\int_0^t \bar{\alpha}(s)\mathrm{d}s\right)
$$

in $C([0,T])$ (see proof of Lemma 3.9), the convergence

$$\rho_k(t) = \bar{\rho_k}(t)\bar{r}_k(t) \to \bar{\rho}(t)\bar{r}(t) = \rho(t)$$

in $L^2(\Omega)^n$ is also true. Because of the uniform boundedness of $\rho_k(t)$ in $L^\infty(\Omega)^n$, we see that $\rho_k(t)$ converges in all $L^p(\Omega)^n$-spaces for $p \in [1,\infty)$.

Up to now, we have shown the pointwise in time convergence of $\rho_k$. In the next step, we want to show that the convergence of $\rho_k$ is uniform in $[0,T]$ with values in $L^p(\Omega)^n$. This can be done by applying Lemma 1.30 to $\rho_k$, a Radon-Riesz like property for uniform convergence. Therefore, we first prove that $\int_\Omega \rho_k(t,x)^T\varphi(t,x)\mathrm{d}x$ converges uniformly towards $\int_\Omega \rho(t,x)^T\varphi(t,x)\mathrm{d}x$ for all fixed $\varphi \in C^1([0,T]\times\overline{\Omega},\mathbb{R}^n)$. Recall the upper bound $\bar{\rho}_{\max} \in \mathbb{R}_{>0}$ for $\rho_k$ from the first part of the proof and let be $\varepsilon > 0$. We define $\tilde{\varepsilon} = \frac{\varepsilon}{5\bar{\rho}_{\max}\|\varphi\|_{C^1([0,T]\times\overline{\Omega},\mathbb{R}^n)}}$. Since $u_k$, $c_k$, $f_k$ and $\nu u_k$ are converging in $L^1$, they are equi-integrable (see Lemma 1.27) and we can use the characterization of equi-integrability of Remark 1.26 to obtain the existence of $\delta > 0$ such that $S \subset [0,T]$ with $\mu(S) < \delta$ implies the four inequalities: $\int_S \|u_k\|_{L^1(\Omega)^n}\mathrm{d}t < \tilde{\varepsilon}$, $\int_S \|c_k\|_{L^1(\Omega)^n}\mathrm{d}t < \tilde{\varepsilon}$, $\int_S \|f_k\|_{L^1(\Omega)^n}\mathrm{d}t < \tilde{\varepsilon}$ and $\int_S \|\nu u_k\|_{L^1(\Gamma)^n}\mathrm{d}t < \tilde{\varepsilon}$. Then, we compute for $t_1, t_2 \in [0,T]$ with $|t_2 - t_1| < \min(\delta, \tilde{\varepsilon})$

$$\begin{aligned}
&\left|\int_\Omega \varphi(t_2)^T\rho_k(t_2)\mathrm{d}x - \int_\Omega \varphi(t_1)^T\rho_k(t_1)\mathrm{d}x\right| \\
&= \left|\int_{t_1}^{t_2}\int_\Omega \left(\varphi_t^T + \varphi_x^T\mathbf{U}_k - \varphi^T\mathbf{C}_k\right)\rho_k + f_k^T\varphi\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\int_\Gamma \varphi^T(\nu\mathbf{U}_k)\gamma\rho_k\mathrm{d}\omega\mathrm{d}t\right| \\
&\leq \|\varphi\|_{C^1}\bar{\rho}_{\max}\int_{t_1}^{t_2}\left(1 + \|u_k\|_{L^1(\Omega)^n} + \|c_k\|_{L^1(\Omega)^n} + \|f_k\|_{L^1(\Omega)^n}\right)\mathrm{d}t \\
&\quad + \|\varphi\|_\infty\bar{\rho}_{\max}\int_{t_1}^{t_2}\|\nu u_k\|_{L^1(\Gamma)^n}\mathrm{d}t \\
&\leq \bar{\rho}_{\max}\|\varphi\|_{C^1}\int_{t_1}^{t_2}\left(1 + \|u_k\|_{L^1(\Omega)^n} + \|c_k\|_{L^1(\Omega)^n} + \|f_k\|_{L^1(\Omega)^n} + \|\nu u_k\|_{L^1(\Gamma)^n}\right)\mathrm{d}t \\
&< \varepsilon,
\end{aligned}$$

which proves the weak equicontinuity of $\rho_k$. Together with the weak convergence of $\rho_k(t)$ in $L^p(\Omega)$, we conclude the uniform in time convergence of $\int_\Omega \rho_k(t)^T\varphi(t)\mathrm{d}x$ as each equicontinuous pointwise convergent sequence is uniformly convergent. Due to the density of $C^1([0,T]\times\overline{\Omega})^n$ in $C([0,T],L^2(\Omega)^n)$, this uniform convergence holds in fact for all $\varphi \in C([0,T],L^2(\Omega)^n)$.

To apply Lemma 1.30, we additionally need the uniform convergence of the norm $\|\rho_k(t)\|_{L^2(\Omega)^n}$. Therefore, let again be $\varepsilon > 0$, define $\tilde{\varepsilon} = \frac{\varepsilon}{3\max(\bar{\rho}_{\max},1)\bar{\rho}_{\max}}$ and choose $\delta > 0$ such that for all $S \subset [0,T]$ with $\mu(S) < \delta$ the inequalities $\int_S \|2c_k + (u_k)_x\|_{L^1(\Omega)^n}\mathrm{d}t < \tilde{\varepsilon}$, $\int_S \|f_k\|_{L^1(\Omega)^n}\mathrm{d}t < \tilde{\varepsilon}$ and $\int_S \|\nu u_k\|_{L^1(\Gamma)^n}\mathrm{d}t < \tilde{\varepsilon}$ hold. Then, the renormalization property leads to

$$\begin{aligned}
&\left|\|\rho_k(t_2)\|^2_{L^2(\Omega)^n} - \|\rho_k(t_1)\|^2_{L^2(\Omega)^n}\right| \\
&= \left|\int_{t_1}^{t_2}\int_\Gamma \beta(\gamma\rho_k)^T(\nu u_k)\mathrm{d}\omega\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega (2c_k + (u_k)_x)^T\beta(\rho_k) - 2f_k^T\rho_k\mathrm{d}x\mathrm{d}t\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \max(\bar{\rho}_{\max}, 1)\bar{\rho}_{\max} \int_{t_1}^{t_2} \left( \|\nu u_k\|_{L^1(\Gamma)^n} + \|2c_k + (u_k)_x\|_{L^1(\Omega)^n} + \|f_k\|_{L^1(\Omega)^n} \right) \mathrm{d}t \\
&< \varepsilon
\end{aligned}
$$

for $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| < \delta$. Again, the shown equicontinuity yields together with the pointwise convergence of $\|\rho_k(t)\|_{L^2(\Omega)^n}$ the uniform convergence. As a consequence, we can apply Lemma 1.30 to deduce the convergence of $\rho_k$ in $C([0,T], L^2(\Omega)^n)$. As before, the convergence holds in fact in all the spaces $C([0,T], L^p(\Omega)^n)$ for $p \in [1, \infty)$ because of the uniform boundedness.

To finish the proof, it remains to show the strong convergence of the trace $\gamma\rho_k$. Again, we consider the renormalization property for $\beta$ as before with an arbitrary $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$. Since $\beta(\rho_k)$ converges strongly to $\beta(\rho)$ in $L^p((0,T) \times \Omega)^n$ and $\beta(\rho_k(T))$ to $\beta(\rho(T))$ in $L^p(\Omega)^n$ we conclude

$$
\begin{aligned}
&\int_0^T \int_\Gamma \beta(\gamma\rho_k)^T (\nu \mathbf{U}_k) \varphi \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \int_\Omega \beta(\rho_k)^T \left(\varphi_t + \mathbf{U}_k \varphi_x - (2\mathbf{C}_k + (\mathbf{U}_k)_x)\varphi\right) + 2 f_k^T \rho_k \mathrm{d}x \mathrm{d}t \\
&\quad + \int_\Omega \beta(\rho_{0,k})^T \varphi(0) \mathrm{d}x - \int_\Omega \beta(\rho_k(T))^T \varphi(T) \mathrm{d}x \\
&\to \int_0^T \int_\Omega \beta(\rho)^T \left(\varphi_t + \mathbf{U}\varphi_x - (2\mathbf{C} + \mathbf{U}_x)\varphi\right) + 2 f^T \rho \mathrm{d}x \mathrm{d}t \\
&\quad + \int_\Omega \beta(\rho_0)^T \varphi(0) \mathrm{d}x - \int_\Omega \beta(\rho(T))^T \varphi(T) \mathrm{d}x \\
&= \int_0^T \int_\Gamma \beta(\gamma\rho)^T (\nu \mathbf{U}) \varphi \mathrm{d}\omega \mathrm{d}t.
\end{aligned}
\tag{3.24}
$$

To use the Radon-Riesz-property again, we would like to choose $\operatorname{sgn}(\nu u)$ as a test function in order to receive a norm of $\gamma\rho$. But since this function is not smooth enough, we have to approximate it to ensure the $C^{0,1}$-regularity. Therefore, let be $\varphi_j \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ such that

$$\varphi_j \to \operatorname{sgn}(\nu u)$$

holds almost everywhere in $\Gamma_T$ with $|\varphi(t,x)| \leq 1$. The dominated convergence theorem (see e.g. [60]) immediately yields

$$(\nu \mathbf{U})\varphi_j \to |\nu u|$$

in $L^1(\Gamma_T)^n$. Thus, we find for $\varepsilon > 0$ an index $J \in \mathbb{N}$ with

$$\|(\nu \mathbf{U})\varphi_J - |\nu u|\|_{L^1(\Gamma_T)^n} < \frac{\varepsilon}{4\bar{\rho}_{\max}^2}.$$

With the convergence in (3.24) we find for $\varphi_J$ an index $K_1$ such that

$$\left| \int_0^T \int_\Gamma \left( \beta(\gamma\rho_k)^T (\nu \mathbf{U}_k) - \beta(\gamma\rho)^T (\nu \mathbf{U}) \right) \varphi_J \mathrm{d}\omega \mathrm{d}t \right| < \frac{\varepsilon}{4}$$

for all $k \geq K_1$. Furthermore, because of the convergence of $\nu u_k$ in $L^1(\Gamma_T)^n$ there is also $K_2$ with

$$\|\nu(u_k - u)\|_{L^1(\Gamma_T)^n} < \frac{\varepsilon}{4\bar{\rho}_{\max}^2}$$

for $k \geq K_2$. Finally, for all $k \geq \max(K_1, K_2)$ it holds

$$\begin{aligned}
&\left|\|\gamma\rho_k\|^2_{L^2(\Gamma_T,|\mathrm{d}\mu_u|)} - \|\gamma\rho\|^2_{L^2(\Gamma_T,|\mathrm{d}\mu_u|)}\right| \\
&\leq \left|\int_0^T \int_\Gamma \beta(\gamma\rho_k)^T(\nu\mathbf{U})\left(\operatorname{sgn}(\nu u) - \varphi_J\right)\mathrm{d}\omega\mathrm{d}t\right| \\
&\quad + \left|\int_0^T \int_\Gamma \beta(\gamma\rho_k)^T(\nu\left(\mathbf{U} - \mathbf{U}_k\right))\varphi_J\mathrm{d}\omega\mathrm{d}t\right| \\
&\quad + \left|\int_0^T \int_\Gamma \Big(\beta(\gamma\rho_k)^T(\nu\mathbf{U}_k) - \beta(\gamma\rho)^T(\nu\mathbf{U})\Big)\,\varphi_J\mathrm{d}\omega\mathrm{d}t\right| \\
&\quad + \left|\int_0^T \int_\Gamma \beta(\gamma\rho)^T(\nu\mathbf{U})\left(\varphi_J - \operatorname{sgn}(\nu u)\right)\mathrm{d}\omega\mathrm{d}t\right| \\
&< 2\bar{\rho}_{\max}^2 \||\nu u| - (\nu\mathbf{U})\varphi_J\|_{L^1(\Gamma_T)^n} + \bar{\rho}_{\max}^2\|\nu(u - u_k)\|_{L^1(\Gamma_T)^n} + \frac{\varepsilon}{4} \\
&< \varepsilon
\end{aligned}$$

and we can use the Radon-Riesz-property and the boundedness of $\gamma\rho_k$ to conclude the strong convergence in $L^p(\Gamma_T, |\mathrm{d}\mu_u|)$ for all $p \in [1, \infty)$, which finishes the proof. □

*Remark* 3.12. For the special case of non-coupled boundary conditions Boyer and Fabrie obtained in [15] the same result. Their proof technique is completely different and uses the same mollifier $\mathcal{S}_\varepsilon$ as the proof of the trace theorem (Theorem 3.3) for the uniform convergence. The additional regularity of $\mathcal{S}_\varepsilon\rho$ allows them to find the estimate

$$\begin{aligned}
\|\mathcal{S}_\varepsilon\rho - \rho_k\|^2_{C([0,T],L^2(\Omega)} \leq M\Big(&\|\mathcal{S}_\varepsilon\rho_0 - \rho_0\|^2_{L^2} + \|\rho_0 - \rho_{0,k}\|^2_{L^2} + \|(\nu v_k)^- - (\nu v)^-\|_{L^1(\Gamma_T)} \\
&+ \|\mathcal{R}_{\varepsilon,k}\|_{L^1} + \|\gamma_0(\mathcal{S}_\varepsilon\rho) - \gamma\rho\|^2_{L^2(\Gamma_T,\mathrm{d}\mu_u^-)} + \|\gamma\rho - \rho_{\mathrm{in},k}\|^2_{L^2(\Gamma_T,\mathrm{d}\mu_u^-)}\Big),
\end{aligned}$$

which finally leads after some computations to

$$\begin{aligned}
&\limsup \|\rho - \rho_k\|^2_{C([0,T],L^2(\Omega))} \\
&\leq M\left(\|\mathcal{S}_\varepsilon\rho - \rho\|^2_{C([0,T],L^2(\Omega))} + \|\mathcal{S}_\varepsilon\rho_0 - \rho_0\|^2_{L^2}\right. \\
&\quad\left. + \limsup\|\mathcal{R}_{\varepsilon,k}\|_{L^1} + \|\gamma_0(\mathcal{S}_\varepsilon\rho) - \gamma\rho\|^2_{L^2(\Gamma_T,\mathrm{d}\mu_u^-)}\right).
\end{aligned}$$

The limit $\varepsilon \to 0$ yields the desired result

$$\limsup \|\rho - \rho_k\|^2_{C([0,T],L^2(\Omega))} = 0.$$

The result we presented in Theorem 3.11 is a significant generalization.

Now, we are finally able to prove the existence of solutions of the transport equation (3.1). This will be done in two steps. In a first step, we will consider the equation with non-coupled boundary conditions. In this case, the existence of a solution can be proven in several ways. We will use the classical method of characteristics for smooth data. Then, the first part of the previous theorem yields the existence for general data. In [15], a proof of the existence using a parabolic approximation is presented. In a second step, we will construct a solution for general boundary conditions by an iterative method, where we will again use the first part of Theorem 3.11.

**Lemma 3.13 (Existence).** *For $\mathcal{G} = 0$ there exists a solution of the initial-boundary-value problem* (3.1) *under the assumptions from Section 3.1.*

*Proof.* It is sufficient to consider the scalar case $n = 1$, since the equations are independent from each other. First, we will construct the solution for smooth data by the method of characteristics. Therefore, let be $u, c, f \in C^1([0,T] \times \overline{\Omega})$ and let $\rho_0$ and $\rho_{\text{in}}$ be step functions defined on $\bar{\Omega}$ and $\Gamma_T$, respectively.

Let $\eta = \eta(t; t_0, x_0)$ be the unique solution of the ODE

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\eta &= u(t, \eta)\\ \eta(t_0; t_0, x_0) &= x_0.\end{aligned}$$

For each point $(t, x) \in [0, T] \times \overline{\Omega}$ let $(\bar{t}, \bar{x})$ be the "first" intersection of the characteristic through $(t, x)$ with the boundary $(\{0\} \times \Omega) \cup \Gamma_T$, i.e.

$$(\bar{t}, \bar{x}) = (\bar{t}, \eta(\bar{t}; t, x))$$

with

$$\bar{t} = \inf_{s \in [0,t]} \{s | \eta(r; t, x) \in \Omega \ \forall r \in (s, t)\}.$$

We define

$$F(t, x, r) = -\int_r^t (u_x + c)(s, \eta(s; t, x)) \mathrm{d}s.$$

Then, the solution of the transport equation is given by

$$\rho(t, x) = \begin{cases} \mathrm{e}^{F(t,x,\bar{t})} \left[\rho_{\text{in}}(\bar{t}, \bar{x}) + \int_{\bar{t}}^t f(s, \eta(s; t, x)) e^{-F(s,x,\bar{t})} \mathrm{d}s\right] & \text{for } \bar{t} \neq 0 \\ \mathrm{e}^{F(t,x,0)} \left[\rho_0(\bar{x}) + \int_0^t f(s, \eta(s; t, x)) e^{-F(s,x,0)} \mathrm{d}s\right] & \text{otherwise.} \end{cases} \tag{3.25}$$

To check that this is indeed a solution, we use the substitutions $y = \eta(0; t, x) = \bar{x}$ and $s = \bar{t}(t, x)$, respectively, in the weak formulation of the transport equation. One has to be careful with the second substitution, since $s$ is not smooth on the whole domain. But there are certain subdomains of $(0, T) \times \Omega$, where $\bar{t}$ depends smoothly on $(t, x)$. To be precise, let $\Gamma_T^- = \{(t, \omega) \in \Gamma_T | \nu u(t, \omega) < 0\}$ be the inflow part of the boundary and let $(t_0, \omega_0) \in \Gamma_T^-$ be arbitrary. We denote by $\Gamma_0$ the connected component of $\Gamma_T^-$ containing $(t_0, \omega_0)$. Then, we define

$$\hat{\Omega}_0 = \{(t, x) \in (0, T) \times \Omega | x = \eta(t; t_1, \omega_1) \text{ for some } (t_1, \omega_1) \in \Gamma_0\},$$

which is the domain of influence of $\Gamma_0$. The implicit function theorem yields the smooth dependence of $s = \bar{t}$ on $(t,x)$ in $\hat{\Omega}_0$. Thus, it is easy but technical to verify that (3.25) fulfils for any $\varphi \in C^{0,1}([0,T]\times\overline{\Omega})$ the weak formulation. However, we omit this technical computation since it gives no new insight. Instead, we refer to standard textbooks on the method of characteristics (e.g. [16]).

Now, we will treat the general case for non-smooth data. Therefore, let be

$$\begin{aligned} u &\in L^1((0,T),W^{1,1}(\Omega)),\\ c &\in L^1((0,T)\times\Omega),\\ f &\in L^1((0,T),L^\infty(\Omega)),\\ \rho_0 &\in L^\infty(\Omega) \end{aligned}$$

and

$$\rho_{\mathrm{in}} \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$$

with $(u_x)^+, (u_x+c)^- \in L^1((0,T),L^\infty(\Omega))$. Using a mollifier one can construct sequences $(u_k)_k$ and $(\psi_k)_k$ of smooth functions converging to $u$ in $L^1((0,T),W^{1,1}(\Omega))$ and to $\psi = c+u_x$ in $L^1((0,T)\times\Omega)$, respectively, such that

$$\|\psi_k^-\|_{L^1((0,T),L^\infty(\Omega))} \le C\|(c+u_x)^-\|_{L^1((0,T),L^\infty(\Omega))} \tag{3.26}$$

holds (see Chapter III of [15]). Now, we set $c_k = \psi_k - (u_k)_x$. Furthermore, we extend the boundary value $\rho_{\mathrm{in}}$ by zero to a function in $L^\infty(\Gamma_T)$ and denote by $\rho_{\mathrm{in},k}$ and $\rho_{0,k}$ sequences of step functions converging to $\rho_{\mathrm{in}}$ and $\rho_0$ in $L^\infty(\Gamma_T)$ and $L^\infty(\Omega)$, respectively. The source term $f$ is also approximated with a mollifier by smooth functions $f_k$ converging to $f$ in $L^1((0,T)\times\Omega)$ such that $\|f_k\|_{L^1((0,T),L^\infty(\Omega))}$ is bounded. From the first part of this proof we know that there exists a solution $\rho_k$ with trace $\gamma\rho_k$ of the transport equation with smooth data $u_k$, $f_k$, $c_k$, i.e. of

$$\begin{aligned} (\rho_k)_t + (\mathbf{U}_k\rho_k)_x + \mathbf{C}_k\rho &= f_k && \text{in } (0,T)\times\Omega\\ \rho_k(0) &= \rho_{0,k} && \text{in } \Omega\\ (\nu\mathbf{U}_k)^-\rho_k &= (\nu\mathbf{U}_k)^-\rho_{\mathrm{in},k} && \text{on } \Gamma_T. \end{aligned}$$

Due to (3.26) all requirements of Theorem 3.11 are fulfilled by construction. Thus, there exists a solution $\rho$ of the limit problem

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= f && \text{in } (0,T)\times\Omega\\ \rho(0) &= \rho_0 && \text{in } \Omega\\ (\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^-\rho_{\mathrm{in}} && \text{on } \Gamma_T. \end{aligned}$$

□

Now that we have shown the existence of solutions in the case of non-coupled boundary conditions, we will provide a small lemma concerning the monotonicity of the solution operator in dependence on the inflow values. This lemma will turn out to be useful for the construction of the solution in the general case with coupled boundary conditions.

**Lemma 3.14**. *For $\mathcal{G} = 0$ let the assumptions from Section 3.1 be true and let the functions $\rho_{\mathrm{in},1}, \rho_{\mathrm{in},2} \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ be two different inflow values with*

$$\rho_{\mathrm{in},2} \geq \rho_{\mathrm{in},1} \geq 0$$

$\mathrm{d}\mu_u^-$*-almost everywhere. Then, for the solutions $\rho_i$ of the transport equation* (3.1) *with inflow values $(\nu\mathbf{U})^-\rho = (\nu\mathbf{U})^-\rho_{\mathrm{in},i}$ it holds*

$$\rho_2(t) \geq \rho_1(t)$$

*almost everywhere in $\Omega$. Moreover, for their traces $\gamma\rho_i$ hold*

$$\gamma\rho_2 \geq \gamma\rho_1$$

$|\mathrm{d}\mu_u|$*-almost everywhere in $\Gamma_T$.*

*Proof.* Let $\rho_1$ and $\rho_2$ be the solutions corresponding to the inflow values $\rho_{\mathrm{in},1}$ and $\rho_{\mathrm{in},2}$. Recall that these solutions $\rho_1$ and $\rho_2$ exist due to Lemma 3.13. Then, the difference $\rho = \rho_2 - \rho_1$ with trace $\gamma\rho = \gamma\rho_2 - \gamma\rho_1$ solves the homogeneous transport equation

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= 0 && \text{in } (0,T)\times\Omega \\ \rho(0,\cdot) &= 0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^-\left(\rho_{\mathrm{in},2} - \rho_{\mathrm{in},1}\right) && \text{on } \Gamma_T. \end{aligned}$$

Since the inflow value $\rho_{\mathrm{in},2} - \rho_{\mathrm{in},1}$ is non-negative $\mathrm{d}\mu_u^-$ almost everywhere, we can apply Lemma 3.9 to obtain the lower bound $\rho(t) \geq 0$ almost everywhere in $\Omega$ and $\gamma\rho(t,\omega) \geq 0$ for $|\mathrm{d}\mu_u|$-almost all $(t,\omega) \in \Gamma_T$. □

Equipped with this lemma, we can prove the existence of solutions for general boundary operators $\mathcal{H}$ using an iterative construction in the next theorem.

**Theorem 3.15 (Existence)**. *Under the assumptions* (3.2) *on $u$, $c$ and $f$ and under the assumptions of Section 3.1 on $\mathcal{H}$, there exists a solution of the initial-boundary-value problem* (3.1)*.*

*Proof.* First, due to the linearity of the transport equation, we observe that we can split the problem in the two subproblems

$$\begin{aligned} \rho^1_t + (\mathbf{U}\rho^1)_x + \mathbf{C}\rho^1 &= f^+ && \text{in } (0,T)\times\Omega \\ \rho^1(0,\cdot) &= \rho_0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho^1 &= (\nu\mathbf{U})^-\mathcal{H}(\rho^1|_{\Gamma_T^+}) && \text{on } \Gamma_T \end{aligned}$$

and

$$\begin{aligned} \rho^2_t + (\mathbf{U}\rho^2)_x + \mathbf{C}\rho^2 &= f^- && \text{in } (0,T)\times\Omega \\ \rho^2(0,\cdot) &= 0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho^2 &= (\nu\mathbf{U})^-\mathcal{G}(\rho^2|_{\Gamma_T^+}) && \text{on } \Gamma_T. \end{aligned}$$

Then $\rho = \rho^1 - \rho^2$ is a solution of the original problem. Thus, it is sufficient to consider only non-negative right-hand sides $f \geq 0$. We will iteratively construct a sequence of functions converging weakly-$\star$ to a solution $\rho$ of the transport equation. The idea is to use the outflow value of a solution to compute the inflow value for the problem of the next iteration step. Due to the positivity of the operator $\mathcal{G}$, the obtained sequence will be monotonically increasing.

Therefore, we define the starting value of the sequence $\gamma\rho_0 = 0$. From Lemma 3.13 we know that the problem

$$\begin{aligned} (\rho_k)_t + (\mathbf{U}\rho_k)_x + \mathbf{C}\rho_k &= f && \text{in } (0,T)\times\Omega \\ \rho_k(0) &= \rho_0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho_k &= (\nu\mathbf{U})^-\mathcal{H}((\gamma\rho_{k-1})|_{\Gamma_T^+}) && \text{on } \Gamma_T \end{aligned}$$

has a unique solution $\rho_k$ with trace $\gamma\rho_k$. Once more, like in the proof of Lemma 3.13, we want to use the stability theorem (Theorem 3.11), but this time with $\rho_{\mathrm{in},k} = \mathcal{H}(\gamma\rho_{k-1}|_{\Gamma_T^+})$ and $\mathcal{G}_k = 0$.

As a first step, we have to check the requirements for this theorem. Therefore, we want to analyse the boundedness of the boundary value. From assumption (3.5) we know that there is a constant $\rho_{\max} \in \mathbb{R}^n_{>0}$ with

$$\rho_{\mathrm{in}} \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right) - F + \mathcal{G}(F) + \mathcal{G}(\rho_{\max}) \leq \rho_{\max}$$

and

$$\rho_0 \leq \rho_{\max}$$

almost everywhere. We want to show by induction that an analogue bound for the boundary value $\rho_{\mathrm{in},k} = \mathcal{H}(\gamma\rho_{k-1}|_{\Gamma_T^+})$ and $\mathcal{G}_k = 0$ is valid for all $k$, i.e.

$$\gamma\rho_k \exp\left(\int_0^t \alpha(s)\mathrm{d}s\right) - F \leq \rho_{\max}.$$

The base case is clear since $\mathcal{G}$ and $\rho_{\mathrm{in}}$ are positive and it holds $\gamma\rho_0 = 0$. Using the positivity of $\mathcal{G}$, the induction hypothesis and the assumption (3.5) on $\rho_{\max}$ we calculate for the inductive step

$$\begin{aligned} &\mathcal{H}(\gamma\rho_{k-1}|_{\Gamma_T^+}) \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right) \\ &= \rho_{\mathrm{in}} \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right) + \mathcal{G}(\rho_{\max}) + \mathcal{G}(F) \\ &\quad - \mathcal{G}\left(\rho_{\max} + F - \gamma\rho_{k-1}\exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right)\right) \\ &\leq \rho_{\mathrm{in}} \exp\left(-\int_0^t \alpha(s)\mathrm{d}s\right) + \mathcal{G}(\rho_{\max}) + \mathcal{G}(F) \\ &\leq \rho_{\max} + F. \end{aligned}$$

Thus, Lemma 3.9 yields the desired bound for $\gamma\rho_k$ for all $k$.

The next aim is to show the convergence of the inflow values $\rho_{\text{in},k}$. As we have just seen, this sequence is bounded and it is monotonically increasing, i.e. it holds

$$\rho_{\text{in},k+1}(t,\omega) \geq \rho_{\text{in},k}(t,\omega)$$

for $\mathrm{d}\mu_u^-$-almost all $(t,\omega) \in \Gamma_T$. We will also prove this by induction. Due to the non-negativity of $f$ and Lemma 3.9 it is $\gamma\rho_1(t,\omega) \geq 0$ for $|\mathrm{d}\mu_u|$-almost all $(t,\omega)$ and thus, because of the positivity of $\mathcal{G}$ and $\rho_{\text{in}}$

$$\rho_{\text{in},2} = \mathcal{H}((\gamma\rho_1)|_{\Gamma_T^+}) \geq \rho_{\text{in}} = \mathcal{H}((\gamma\rho_0)|_{\Gamma_T}^+) = \rho_{\text{in},1}$$

holds $\mathrm{d}\mu_u^-$-almost everywhere. For the inductive step, we will use the previous lemma about the monotonicity. Assume, it is $\rho_{\text{in},k} \geq \rho_{\text{in},k-1}$. Then, Lemma 3.14 immediately yields $\gamma\rho_k \geq \gamma\rho_{k-1}$. Thus, the positivity of $\mathcal{G}$ leads to

$$\rho_{\text{in},k+1} - \rho_{\text{in},k} = \mathcal{H}((\gamma\rho_k)|_{\Gamma_T^+}) - \mathcal{H}((\gamma\rho_{k-1})|_{\Gamma_T^+}) = \mathcal{G}((\gamma\rho_k - \gamma\rho_{k-1})|_{\Gamma_T^+}) \geq 0.$$

Since $(\rho_{\text{in},k})_k$ is bounded and monotone, we can apply the monotone convergence theorem (see e.g. [60]) to conclude that there exists a function $q \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ with

$$\rho_{\text{in},k} \to q$$

in $L^p(\Gamma_T, \mathrm{d}\mu_u^-)$. Since the sequence is bounded in $L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$, it is also weak-$\star$ convergent in $L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$. Hence, Theorem 3.11 is applicable and it yields the existence of a solution $\rho$ with trace $\gamma\rho$ of

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= f && \text{in } (0,T)\times\Omega \\ \rho(0) &= \rho_0 && \text{in } \Omega \\ (\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^- q && \text{on } \Gamma_T. \end{aligned}$$

Furthermore, it holds

$$\gamma\rho_k \overset{\star}{\rightharpoonup} \gamma\rho$$

in $L^\infty(\Gamma_T, |\mathrm{d}\mu_u|)$. The weak-$\star$ continuity of the operator $\mathcal{G}$ implies

$$\mathcal{H}(\gamma\rho_k|_{\Gamma_T^+}) \overset{\star}{\rightharpoonup} \mathcal{H}(\gamma\rho|_{\Gamma_T^+})$$

in $L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$. Finally, we conclude

$$\begin{aligned} (\nu\mathbf{U})^-\gamma\rho \leftharpoonup (\nu\mathbf{U})^-\gamma\rho_k \\ &= (\nu\mathbf{U})^-\mathcal{H}(\gamma\rho_{k-1}|_{\Gamma_T^+}) \\ &\rightharpoonup (\nu\mathbf{U})^-\mathcal{H}(\gamma\rho|_{\Gamma_T^+}) \end{aligned}$$

in $L^1(\Gamma_T)$. Because of the uniqueness of the weak limit the boundary condition

$$(\nu\mathbf{U})^-\gamma\rho = (\nu\mathbf{U})^-\mathcal{H}(\gamma\rho|_{\Gamma_T^+})$$

is satisfied. $\square$

To end this chapter, we will provide two small lemmas concerning the momentum $\mathbf{U}\rho$. Especially the second will be useful in Chapter 5.

**Lemma 3.16**. *Let the assumptions of Section 3.1 be satisfied. Furthermore, let the velocity be continuously differentiable, i.e. $v \in C^1((0,T) \times \Omega, \mathbb{R}^n)$. Denote the solution of the initial-boundary-value problem* (3.1) *by $\rho$. Then, $m = \mathbf{V}\rho$ is a weak solution of the transport equation*

$$m_t + (\mathbf{U}m)_x + \mathbf{C}m = (\mathbf{V}_t + \mathbf{U}\mathbf{V}_x)\,\rho \tag{3.27}$$

*with initial value $m(0) = \mathbf{V}(0)\rho_0$ and trace $\gamma m = \mathbf{V}\gamma\rho$.*

*Proof.* Let $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ be a test function. Then, we compute for any $[t_0, t_1] \subset [0,T]$

$$\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_\Omega \rho^T\left((\mathbf{V}\varphi)_t + \mathbf{U}\,(\mathbf{V}\varphi)_x - \mathbf{C}\,(\mathbf{V}\varphi)\right)\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1}\int_\Gamma \gamma\rho^T(\nu\mathbf{U})\mathbf{V}\varphi\mathrm{d}\omega\mathrm{d}t \\
&\quad + \int_\Omega \rho(t_0)^T\mathbf{V}(t_0)\varphi(t_0)\mathrm{d}x - \int_\Omega \rho(t_1)^T\mathbf{V}(t_1)\varphi(t_1)\mathrm{d}x \\
&= \int_{t_0}^{t_1}\int_\Omega (\mathbf{V}\rho)^T\left(\varphi_t + \mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + \rho^T\left(\mathbf{V}_t + \mathbf{U}\mathbf{V}_x\right)\varphi\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{t_0}^{t_1}\int_\Gamma (\mathbf{V}\gamma\rho)^T(\nu\mathbf{U})\varphi\mathrm{d}\omega\mathrm{d}t + \int_\Omega \left(\mathbf{V}(t_0)\rho(t_0)\right)^T\varphi(t_0)\mathrm{d}x \\
&\quad - \int_\Omega \left(\mathbf{V}(t_1)\rho(t_1)\right)^T\varphi(t_1)\mathrm{d}x.
\end{aligned}$$

Thus, $m = \mathbf{V}\rho$ solves the equation (3.27) with initial value $m(0) = \mathbf{V}(0)\rho_0$ and because of the uniqueness of the trace it holds $\gamma m = \mathbf{V}\gamma\rho$. ☐

**Lemma 3.17**. *Additionally to the assumptions from Section 3.1, let us assume $u \in W^{1,1}((0,T) \times \Omega)^n \cap C([0,T] \times \overline{\Omega})^n$. Denote the solution of the initial-boundary-value problem* (3.1) *by $\rho$. Then, the momentum $m = \mathbf{U}\rho$ is a solution of the transport equation*

$$m_t + (\mathbf{U}m)_x + \mathbf{C}m = (\mathbf{U}_t + \mathbf{U}\mathbf{U}_x)\,\rho \tag{3.28}$$

*with initial value $m(0) = \mathbf{U}(0)\rho_0$ and trace $\gamma m = \mathbf{U}\gamma\rho$.*

*Proof.* Since $u \in W^{1,1}((0,T) \times \Omega)^n \cap C((0,T) \times \Omega)^n$, we can find a sequence of smooth functions $v_k \in C^1((0,T) \times \Omega)^n$ such that $v_k \to u$ in $W^{1,1}((0,T) \times \Omega)^n$ and $v_k \overset{\star}{\rightharpoonup} u$ in $L^\infty((0,T) \times \Omega)^n$ (see Chapter III of [15]). Applying the trace operator for Sobolev functions, it also holds $v_k|_{\Gamma_T} \to u|_{\Gamma_T}$ in $L^1(\Gamma_T)^n$ and $v_k(0) \to u(0)$ in $L^1(\Omega)^n$. According to the previous lemma, $m_k = \mathbf{V}_k\rho$ solves the transport equation

$$(m_k)_t + (\mathbf{U}m_k)_x + \mathbf{C}m_k = ((\mathbf{V}_k)_t + \mathbf{U}(\mathbf{V}_k)_x)\,\rho$$

with initial value $m_k(0) = \mathbf{V}_k(0)\rho_0$ and trace $\gamma m_k = \mathbf{V}_k\gamma\rho$. Therefore, for any test function $\varphi \in C^{0,1}([0,T] \times \overline{\Omega}, \mathbb{R}^n)$ with $\varphi(T) = 0$ it holds

$$\begin{aligned}
0 = &\int_0^T \int_\Omega \rho \mathbf{V}_k^T \left(\varphi_t + \mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + \left((\mathbf{V}_k)_t + \mathbf{U}(\mathbf{V}_k)_x\right) \rho \mathrm{d}x\mathrm{d}t \\
&- \int_0^T \int_\Gamma \gamma\rho \mathbf{V}_k^T (\nu \mathbf{U})\varphi \mathrm{d}\omega \mathrm{d}t + \int_\Omega \rho_0^T \mathbf{V}_k(0)\varphi(0)\mathrm{d}x \\
\to &\int_0^T \int_\Omega \rho \mathbf{U}^T \left(\varphi_t + \mathbf{U}\varphi_x - \mathbf{C}\varphi\right) + \left(\mathbf{U}_t + \mathbf{U}\mathbf{U}_x\right) \rho \mathrm{d}x\mathrm{d}t \\
&- \int_0^T \int_\Gamma \gamma\rho \mathbf{U}^T (\nu \mathbf{U})\varphi \mathrm{d}\omega \mathrm{d}t + \int_\Omega \rho_0^T \mathbf{U}(0)\varphi(0)\mathrm{d}x.
\end{aligned}$$

Thus, $m = \mathbf{U}\rho$ solves equation (3.28) with initial value $m(0) = \mathbf{U}(0)\rho_0$ and trace $\gamma m = \mathbf{U}\gamma\rho$. □

*Remark* 3.18. All considerations in this chapter could also be done for the case of multidimensional domains $\Omega$ with Lipschitz boundary.

# 4 Differential equations on a network

Up to now, we have derived a model for the low Mach number flow on a network (Chapter 2), which consists of a transport equation, discussed in Chapter 3, and an ordinary differential equation, coupled by algebraic relations. In this chapter, we want to study a certain class of differential equations on a network, namely the class of transport equations coupled with an ordinary differential equation. One particular example of this class are the above-mentioned low Mach number equations, as we discussed in Section 2.5 heuristically and as we will analyse rigorously in Chapter 5.

The analysis of the coupled equations is divided into two steps. First, we consider only ordinary differential equations on a network and in a second step we treat the coupling with the transport equation. A general formulation of ordinary differential equations on a network is given by:

$$\begin{aligned} v_t &= \bar{f}(t,v) - \mathbf{R}(t)\Delta p(t,v) \\ \Delta p(t,v) &= -\mathbf{B}_{>1}^T P_V(t) + g(t,v) \\ F(t,v(t)) &= 0 \end{aligned} \tag{4.1}$$

and

$$v(0) = v_0.$$

This class of differential equations is characterized by the additional algebraic constraints for the network structure. Prominent examples for this class of equations are electrical networks. The electrical elements as e.g. capacitors or inductors are considered as the edges and can be modelled with the help of ODEs. Kirchhoff's laws couple these ODEs at the intersections. Another example is the velocity equation (2.59) of the low Mach number model.

The idea of the analysis of (4.1) is to reformulate the differential algebraic equation as an ODE without algebraic constraints. In this way, we can guarantee the existence and uniqueness of a solution local in time. A crucial point in this theory is the global existence on $[0,T]$. For this reason, we will introduce the concept of energy functionals and we can conclude the global existence for certain classes of networks.

In the second step, we discuss the coupling with the transport equation. The right-hand side of the ODE depends on the solution $\rho$ of a transport equation, whereas $\rho$ itself depends on the solution of the ODE, i.e. we consider the equations

$$\begin{aligned} \rho_t + ((\mathbf{V}+\mathbf{Q})\,\rho)_x + \mathbf{C}\rho &= 0 \\ v_t &= \bar{\mathfrak{f}}(t,v,\rho) - \mathbf{R}(t)\Delta p(t,v) \\ \Delta p(t,v) &= -\mathbf{B}_{>1}^T P_V(t) + \mathfrak{g}(t,v,\nu(\mathbf{V}+\mathbf{Q})|_\Gamma \gamma\rho) \\ F(t,v(t)) &= 0 \end{aligned} \tag{4.2}$$

together with suitable initial and boundary conditions. In this part, we will provide the main result of this thesis: the existence of solutions of (4.2). The proof is based on the Schauder fixed point theorem together with the theory of the previous chapter on the transport equation and on the reformulation of the ODE. To ensure the global existence, we will again use an energy functional

The first section of this chapter deals with ODEs on a network and in the second section we treat the coupled equations. In the third section, we will construct energy functionals for different graphs in order to show for which graphs the theory can be applied.

## 4.1 Ordinary differential equations on a network

In this section, we will combine the concept of oriented graphs with the theory of ODEs. The regarded equation is a special semi-explicit differential algebraic equation (DAE), where the algebraic coupling conditions depend on the structure of the graph.

In recent years, there has been done a lot of research focussing on DAEs, e.g. in the context of optimal control or model order reduction. The aim of this section is merely to provide us with the necessary tools for the analysis of coupled equations. We do not want to give a full review of this topic. For more information, we refer to the textbooks [61] by Kunkel and Mehrmann and [64] by Lamour et al. as well as the references therein.

We have in mind to study the velocity equation (2.59). Therefore, we consider an oriented weighted connected graph $G = (\mathfrak{V}, E, w, \mathrm{init}, \mathrm{ter})$ with its incidence matrix $\mathbf{B} \in \{-1, 0, 1\}^{m\times n}$, which has $k < m$ inner nodes $\mathfrak{v} \in \mathfrak{V}$ with $d(\mathfrak{v}) > 1$. We recall the notation $\mathbf{B}_{>1}$ for the submatrix of the incidence matrix corresponding to the inner nodes, which was introduced in equation (1.1).

The problem is to find $v \in W^{1,1}((0,T))^n$ and $P_V \in L^1((0,T))^k$ solving the equation (4.1). Since the right-hand side is not necessarily continuous, the ordinary differential equation has to be understood in the extended sense (see Theorem 1.18), i.e. a solution $v$ has to be locally absolute continuous and has to solve the ODE almost everywhere in $(0,T)$. In this section we assume:

- The matrix $\mathbf{R}\colon [0,T] \to \mathbb{R}^{n\times n}$ is continuous and $\mathbf{R}(t)$ is a diagonal matrix with

  $$R_{\min} = \min_{t\in[0,T],1\leq i\leq n} R_{ii}(t) > 0$$

  and

  $$R_{\max} = \max_{t\in[0,T],1\leq i\leq n} R_{ii}(t).$$

- The function $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(t,v) = \bar{f}(t,v) - \mathbf{R}(t)g(t,v)$ is a Carathéodory-Lipschitz vector field (see Definition 1.17).

- The coupling condition $F\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable. The derivative $D_vF$ is locally Lipschitz continuous and there exist two positive definite diagonal matrices $\mathbf{D}_1 \in C^{0,1}([0,T] \times \mathbb{R}^n, \mathbb{R}^{k\times k})$ and $\mathbf{D}_2 \in C^{0,1}([0,T] \times \mathbb{R}^n, \mathbb{R}^{n\times n})$ with

$$D_v F(t,v) = \mathbf{D}_1(t,v)\mathbf{B}_{>1}\mathbf{D}_2(t,v) \tag{4.3}$$

for all $(t,v) \in [0,T] \times \mathbb{R}^n$. The eigenvalues of $\mathbf{D}_2$ are bounded from below, i.e. there exists a constant $c_1 > 0$ with

$$\inf_{\substack{(t,v)\in[0,T]\times\mathbb{R}^n \\ 1\le i\le n}} (\mathbf{D}_2)_{ii} \ge c_1.$$

- The initial condition $v_0 \in \mathbb{R}^n$ is a solution of

$$F(0, v_0) = 0.$$

*Remark* 4.1. If the underlying unoriented graph has no circles, the existence of $\mathbf{D}_1$ and $\mathbf{D}_2$ with

$$D_v F(t,v) = \mathbf{D}_1(t,v)\mathbf{B}_{>1}\mathbf{D}_2(t,v)$$

is equivalent to the equality

$$\operatorname{sgn}(D_v F(t,v)) = \mathbf{B}_{>1}.$$

Thus, the Jacobian has the same sign pattern as in the linear case

$$F(t,v) = \mathbf{B}^+_{>1}\mathbf{A}(0)\,(v + Q(t,0)) - \mathbf{B}^-_{>1}\mathbf{A}(1)\,(v + Q(t,1)) = 0,$$

which was derived in Section 2.4 from the energy conservation at the nodes.

The following theorem states the local in time existence and uniqueness, which is a crucial step in the direction of the global existence.

**Theorem 4.2 (Local existence).** *Under the above assumptions, there exists a unique solution*

$$(v,p) \in W^{1,1}((0,T_0))^n \times L^1((0,T_0))^k$$

*of* (4.1) *for* $T_0$ *small enough.*

*Proof.* The idea of the proof is to rewrite the DAE as an ODE for $v$. As often done in the context of DAEs, we will differentiate the coupling condition, since this allows a reformulation of the differential equation.

First, we will derive necessary conditions for the existence of a solution. Therefore, let us assume the pair $(v, P_V)$ with $v \in W^{1,1}((0,T_0))^n$ and $P_V \in L^1((0,T_0))^k$ to be a solution. The total derivative of the coupling condition then yields

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} F(t, v(t)) \\ &= F_t(t,v) + D_v F(t,v) v_t \\ &= F_t(t,v) + D_v F(t,v) \left( f(t,v) + \mathbf{R}\mathbf{B}^T_{>1,h} P_V \right) \end{aligned}$$

for almost all $t \in (0, T_0)$. The matrix

$$D_v F \mathbf{R}\mathbf{B}_{>1}^T = \mathbf{D}_1 \mathbf{B}_{>1} \mathbf{D}_2 \mathbf{R}\mathbf{B}_{>1}^T = \mathbf{D}_1 \mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}$$

is regular since we can write it as a product of $\mathbf{D}_1$ and a principal submatrix of the weighted Laplacian matrix with weight matrix $\mathbf{D}_2\mathbf{R}$ (see Lemma 1.16). Thus, we can express the pressure independently of the derivative of $v$ as

$$\begin{aligned} P_V &= -\left(\mathbf{D}_1 \mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \left(\mathbf{D}_1 \mathbf{B}_{>1} \mathbf{D}_2 f(t,v) + F_t(t,v)\right) \\ &= -\left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \left(\mathbf{B}_{>1} \mathbf{D}_2 f(t,v) + \mathbf{D}_1^{-1} F_t(t,v)\right). \end{aligned}$$

Plugging this expression into the ODE (4.1) leads to the necessary condition

$$v_t = \left(\mathbf{Id} - \mathbf{R}\mathbf{B}_{>1}^T \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2\right) f - \mathbf{R}\mathbf{B}_{>1}^T \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{D}_1^{-1} F_t. \tag{4.4}$$

Instead of proving the existence of a solution of the original problem it is sufficient to prove the existence of a solution of (4.4). Therefore, let $D_0 \subset [0,T] \times \mathbb{R}^n$ be a compact subset. Since $f$ is a Carathéodory-Lipschitz vector field, it is locally integrable bounded, i.e. there is $m_f \in L^1(\mathrm{pr}_1 D_0, [0,\infty))$ and a null subset $I_0 \subset \mathrm{pr}_1 D_0$ with

$$\|f(t,v)\|_2 \leq m_f(t)$$

for all $(t,v) \in D_0$ with $t \notin I_0$. Because of the continuity of the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ we find two constants $c_2, c_3 > 0$ with

$$(\mathbf{D}_2)_{ii} \leq c_2$$

and

$$(\mathbf{D}_1)_{jj} \geq c_3$$

for all $(t,v) \in D_0$, $1 \leq i \leq n$ and $1 \leq j \leq k$.

The matrix $\mathbf{T} = \mathbf{Id} - \mathbf{R}\mathbf{B}_{>1}^T \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2$ is continuous in $t$ and Lipschitz continuous in $v$ (see Lemma 1.22 and Corollary 1.23). Furthermore, for each $(t,v)$ this matrix is a projector, since it holds $\mathbf{T}^2 = \mathbf{T}$. Thus, for the spectral norm it holds $\|\mathbf{T}\|_2 = \|\mathbf{Id} - \mathbf{T}\|_2$ (see e.g. [80]) and we can estimate with the Lemmas 1.13 and 1.16 for $(t,v) \in D_0$

$$\begin{aligned} \|\mathbf{T}(t,v)\|_2^2 &= \lambda_{\max}\left(\left(\mathbf{R}\mathbf{B}_{>1}^T \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2\right)^T \mathbf{R}\mathbf{B}_{>1}^T \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2\right) \\ &\leq \lambda_{\max}\left(\mathbf{D}_2^2\right) \lambda_{\max}\left(\mathbf{B}_{>1}^T\mathbf{B}_{>1}\right) \lambda_{\max}\left(\left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-2}\right) \lambda_{\max}\left(\mathbf{B}_{>1}\mathbf{B}_{>1}^T\right) \lambda_{\max}\left(\mathbf{R}^2\right) \\ &\leq c_2^2 \lambda_{\max}\left(\mathbf{B}_{>1}\mathbf{B}_{>1}^T\right)^2 \left\|\left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1}\right\|_2^2 R_{\max}^2 \\ &\leq \left(\frac{c_2 R_{\max}}{c_1 R_{\min}} \left\|\mathbf{L}^{>1}\right\|_2 \left\|\left(\mathbf{L}^{>1}\right)^{-1}\right\|_2\right)^2 \\ &= c_4^2. \end{aligned}$$

Similarly, we find

$$\left\| \mathbf{R}(t)\mathbf{B}_{>1}^{T} \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{D}_1^{-1} \right\|_2 \le \frac{R_{\max}}{R_{\min} c_1 c_3} \sqrt{\|\mathbf{L}^{>1}\|_2} \left\| \left(\mathbf{L}^{>1}\right)^{-1} \right\|_2 = c_5.$$

Thus, the right-hand side of the ODE (4.4) is itself locally integrably bounded with

$$\begin{aligned} &\left\| \left(\mathbf{Id} - \mathbf{R}\mathbf{B}_{>1}^{T} \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2\right) f - \mathbf{R}\mathbf{B}_{>1}^{T} \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{D}_1^{-1} F_t \right\|_2 \\ &\le c_4 m_f(t) + c_5 \sup_{(t,v)\in D_0} \|F_t(t,v)\|_2 \end{aligned}$$

and hence it is a Carathéodory-Lipschitz vector field. Consequently, the ODE locally has a unique solution $\bar{v} \in W^{1,1}((0,T_0))^n$ (see Theorem 1.18).

Due to the construction, the pair

$$\left(\bar{v}, -\left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \left(\mathbf{B}_{>1}\mathbf{D}_2 f(t,\bar{v}) + \mathbf{D}_1^{-1} F_t(t,\bar{v})\right)\right)$$

is the unique solution of the original problem. This completes the proof. □

*Remark* 4.3. Using the implicit function theorem, it is also possible to prove a similar result for more general coupling conditions for the pressure. For example, one could require $\mathfrak{G}(t,v,p) = 0$ together with some regularity constraints instead of $\Delta p = -\mathbf{B}_{>1}^{T} P_V + g$. Beside the more complex notation, the main drawback of using the implicit function theorem is that one does not know the explicit reformulated ODE (4.4) a priori. Thus, we could not define an energy functional as simple as for the coupling condition of equation (4.1), as we will see in the course of this section.

In the next section, we will need a uniform lower bound of the length of the existence interval to analyse the coupled equations. To this end, we will formulate a quite general method to ensure the global existence of the solution on the time interval $[0,T]$ using an energy principle. Later on in Section 4.3, we will illustrate this principle for three special cases, namely arbitrary networks with linearly bounded right-hand sides, networks with at most one inner node and networks with maximum node degree 2. For the analysis we need the two matrices

$$\mathbf{T}(t,v) = \mathbf{Id} - \mathbf{R}\mathbf{B}_{>1}^{T} \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{B}_{>1}\mathbf{D}_2 \tag{4.5}$$

and

$$\mathbf{S}(t,v) = \mathbf{R}\mathbf{B}_{>1}^{T} \left(\mathbf{L}_{\mathbf{D}_2\mathbf{R}}^{>1}\right)^{-1} \mathbf{D}_1^{-1}, \tag{4.6}$$

introduced in the previous proof.

Furthermore, we need the notation of radially unboundedness in order to introduce the energy functionals.

**Definition 4.4.** We call a function $V\colon \mathbb{R}^n \to \mathbb{R}$ *radially unbounded* if for all $M > 0$ there exists $R \ge 0$ such that the implication

$$\|v\| > R \Rightarrow V(v) > M$$

is true for all $v \in \mathbb{R}^n$.

To prove the global existence in time we are going to use the chain rule even for Lipschitz continuous functions which are only piecewise $C^1$. This is motivated by the choice of the energy functionals, where we will see that the 1-norm is appropriate in some cases. Therefore, we recall a definition and a result from Murat and Trombetti [71].

**Definition 4.5** ([71]). A Lipschitz continuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ is called *piecewise $C^1$-function* if the space $\mathbb{R}^n$ is decomposed into a finite union of disjoint Borel sets $P^\alpha$, i.e.

$$\mathbb{R}^n = \cup_{\alpha\in I} P^\alpha, \quad I \text{ finite}, \quad P^\alpha \text{ Borel subset of } \mathbb{R}^n, \quad P^\alpha \cap P^\beta \text{ for all } \alpha \neq \beta,$$

and if there exists for all $\alpha \in I$ a globally Lipschitz continuous function $V^\alpha \in C^1(\mathbb{R}^m, \mathbb{R})$ with

$$V(x) = V^\alpha(x)$$

for all $x \in P^\alpha$.

Clearly, such a function can be written as

$$V(x) = \sum_{\alpha\in I} \chi_{P^\alpha}(x) V^\alpha(x)$$

and from this form the following chain rule can be obtained:

**Theorem 4.6 (Murat and Trombetti [71]).** *Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous, piecewise $C^1$-function and $r \in [1,\infty)$. Then, for any $v \in W^{1,r}((0,T),\mathbb{R}^n)$ the function $V(v)$ belongs to $W^{1,r}((0,T))$ and one has for almost all $t \in [0,T]$*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(v(t)) = \sum_{\alpha\in I} \chi_{P^\alpha}(v(t)) (\nabla V^\alpha)(v(t)) v'(t).$$

*Proof.* See [71]. □

Now we are able to proceed the study of the DAE presenting the main result of this section.

**Theorem 4.7 (Global existence).** *Let $V\colon \mathbb{R}^n \to \mathbb{R}^+$ be a Lipschitz continuous, piecewise $C^1$ and radially unbounded function. Denote $V(x) = \sum_{\alpha\in I} \chi_{P^\alpha}(x) V^\alpha(x)$ with $I$, $P^\alpha$ and $V^\alpha$ as before. Furthermore, there exist two non-negative functions $k_1, k_2 \in L^1((0,T))$ and a null subset $I_0 \subset [0,T]$ with*

$$\nabla V^\alpha(v) \left(\mathbf{T}(t,v) f(t,v) - \mathbf{S}(t,v) F_t(t,v)\right) \leq k_1(t) + k_2(t) V(v) \tag{4.7}$$

*for all $(t,v) \in \{[0,T] \times P^\alpha | F(t,v) = 0$ and $t \notin I_0\}$ and all $\alpha \in I$. Then, the solution $(v,p)$ of the DAE* (4.1) *exists globally on $[0,T]$ and is bounded by*

$$\|v(t)\| \leq R(t)$$

*and*

$$V(v(t)) \leq M(t),$$

*respectively, for all* $t \in [0, T]$. *These bounds are explicitly given by*

$$M(t) = \left(V(v_0) + \int_0^t k_1(s)\mathrm{d}s\right) \exp\left(\int_0^t k_2(s)\mathrm{d}s\right)$$

*and*

$$R(t) = \inf\left\{c \in \mathbb{R}^+ | V(v) > M(t) \textit{ for all } v \textit{ with } \|v\| > c\right\}.$$

*Proof.* We have already proven the local existence of a solution of the DAE (4.1) in Theorem 4.2. Let $v \colon J \to \mathbb{R}^n$ be such a maximal solution. We will show that $v$ is bounded. To this aim, we use Theorem 4.6 to compute the derivative of $V(v(t))$ for almost all $t \in J$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(v(t)) &= \sum_{\alpha \in P^\alpha} \chi_{P^\alpha}(v(t))(\nabla V^\alpha)(v(t)) v_t(t) \\
&= \sum_{\alpha \in P^\alpha} \chi_{P^\alpha}(v(t))(\nabla V^\alpha)(v(t)) \left(\mathbf{T}(t, v) f(t, v) - \mathbf{S}(t, v) F_t(t, v)\right) \\
&\leq k_1(t) + k_2(t) V(v),
\end{aligned}$$

where the inequality holds since it is $F(t, v(t)) = 0$ by construction. Thus, we estimate for all $t \in J$

$$\begin{aligned}
V(v(t)) &= V(v_0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} V(v(s))\mathrm{d}s \\
&\leq V(v_0) + \int_0^t k_1(s)\mathrm{d}s + \int_0^t k_2(s) V(v(s))\mathrm{d}s
\end{aligned}$$

and use Gronwall's inequality (Lemma 1.19) to conclude

$$V(v(t)) \leq \left(V(v_0) + \int_0^t k_1(s)\right) \exp\left(\int_0^t k_2(s)\mathrm{d}s\right) = M(t).$$

Because of the radial unboundedness of $V$ we find for $M(t)$ a value $R(t) > 0$ such that

$$V(v(t)) \leq M(t)$$

implies

$$\|v(t)\| \leq R(t).$$

Then, the formula for the value of $R(t)$ is obvious. Since $v$ is a maximal solution $J = [0, T]$ must be true. □

*Remark* 4.8. A closer view at the proof reveals that instead of the radially unboundedness the following modified condition is also sufficient: For all $M \in (0, M(T)]$ there is a constant $R \geq 0$ such that it holds

$$V(v) \leq M \text{ and } v \in \{v \in \mathbb{R}^n | \exists t \text{ with } F(t, v) = 0\} \Rightarrow \|v\| \leq R.$$

The theorem inspires the following definition:

**Definition 4.9.** A function $V\colon \mathbb{R}^n \to \mathbb{R}^+$ satisfying the same assumptions as in Theorem 4.7 is called an *energy functional* for the graph $G$, the function $f$ and the matrix-valued function $\mathbf{R}$.

In general, it is a difficult task to find such energy functionals $V$. As mentioned earlier, we will construct such functionals for certain types of networks in Section 4.3.

## 4.2 Coupled differential equations on a network

Now, we will use the theory about the transport equation (Chapter 3) together with the knowledge about the ODEs (Section 4.1) to study the coupled system (4.2). For the proof of existence of solutions we will consider a map which is defined by solving the continuity equation and the ordinary differential equation successively. Combining the results of Chapter 3 and Section 4.1, the so constructed map is shown to be a compact self-mapping in order to apply the Schauder fixed point theorem (Theorem 1.21). This proof strategy is often used to show the existence of solutions of non-linear differential equations, for example in the context of the Navier-Stokes equations (see e.g. [65]). It is applied for the non-homogeneous incompressible Navier-Stokes equations with boundary conditions in [14] and [15].

Gasser and Steinrück considered the existence of solutions of the low Mach number equations on a single edge in [48]. Their approach is based on the idea of solving the equations alternately. For the conclusion of the proof, they used the Banach fixed-point theorem.

In [8] and [9], Borsche et al. studied the well-posedness of hyperbolic balance laws coupled with a system of ODEs. One main difference to our setting is that the ODE only depends on the solution of the balance law at the boundary and not on the whole solution.

As a first step, we precisely want to define the settings including all initial and boundary values. We want to study the equations

$$
\begin{aligned}
\rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho &= 0 && \text{in } (0,T)\times\Omega \\
v_t(t) &= \mathfrak{f}\Big(t, v(t), \rho(t), (\nu\mathbf{U}(t)|_\Gamma)\gamma\rho(t)\Big) + \mathbf{R}^{-1}(t)\mathbf{B}_{>1}^T P_V(t) && \text{in } (0,T) \\
u &= v + Q \\
F(t, v(t)) &= 0 \\
(\nu\mathbf{U})^-\rho &= (\nu\mathbf{U})^-\mathcal{H}_u(\rho|_{\Gamma_T}) && \text{on } \Gamma_T \\
v(0) &= v_0 \\
\rho(0) &= \rho_0 && \text{in } \Omega
\end{aligned}
\tag{4.8}
$$

for a given oriented, connected graph $G = (\mathfrak{V}, E, \mathrm{init}, \mathrm{ter})$ with incidence matrix $\mathbf{B}$. Recall the notation $R$ for the integral $R(t) = \int_\Omega \rho(t)\mathrm{d}x$ and the usage of capital bold letters for diagonal matrices. In contrast to the previous section, the right-hand side $\mathfrak{f}$ depends on the density $\rho$ and its trace $\gamma\rho$. During this section we assume:

$$\begin{aligned} Q &\in L^1((0,T), W^{1,1}(\Omega)^n), \\ c &\in L^1((0,T), L^\infty(\Omega)^n, \\ Q_x &\in L^1((0,T), L^\infty(\Omega)^n, \\ \rho_0 &\in L^\infty(\Omega)^n \end{aligned}$$

and

$$v_0 \in \mathbb{R}^n.$$

To give a meaning to the boundary condition for the transport equation, we need for each $u = w+Q$ for $w \in C([0,T], \mathbb{R}^n)$ a boundary operator $\mathcal{H}_u : L^\infty(\Gamma_T, \mathrm{d}\mu_u^+) \to L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ with $\mathcal{H}_u(\rho) = \rho_{\text{in}} + \mathcal{G}_u(\rho)$ as defined in Section 3.1. Additionally, we need a kind of continuous dependence of these operators on $u$. More precisely, for each convergent sequence $(w_k)_k \subset C([0,T], \mathbb{R}^n)$ with $w_k \to w$ uniformly and each weak-$\star$ convergent sequence $(q_k)_k \subset L^\infty(\Gamma_T)^n$ with $q_k \overset{\star}{\rightharpoonup} q$, we require that the operators $\mathcal{G}$ and $\mathcal{H}$ fulfil

$$(\nu(\mathbf{W}_k + \mathbf{Q}))^- \mathcal{G}_{w_k+Q}(q_k) \rightharpoonup (\nu(\mathbf{W} + \mathbf{Q}))^- \mathcal{G}_{w+Q}(q) \tag{4.9}$$

in $L^1(\Gamma_T)^n$ and

$$\begin{aligned} &\int_0^T \int_\Gamma \beta(q)^T \overline{\mathbf{W}}(\nu u)^+ - \beta(\bar{\mathcal{H}}_{w+Q}(q))^T \overline{\mathbf{W}}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\ &\quad \le \liminf_k \int_0^T \int_\Gamma \beta(q_k)^T \overline{\mathbf{W}}(\nu u_k)^+ - \beta(\bar{\mathcal{H}}_{w_k+Q}(q_k))^T \overline{\mathbf{W}}(\nu u_k)^- \mathrm{d}\omega \mathrm{d}t \end{aligned} \tag{4.10}$$

for $\beta : \mathbb{R}^n \to \mathbb{R}^n$ with $\beta_j(s) = s_j^2$ and $\bar{\mathcal{H}}(\rho) = \exp\left(\int_0^t \|2c(s) + Q_x(s))^-\| \mathrm{d}s\right) \rho_{\text{in}} + \mathcal{G}(\rho)$. These are exactly the conditions which are needed to apply both parts of the stability theorem (Theorem 3.11) to an arbitrary convergent sequence $w_k$.

Furthermore, the solution of the transport equation should be bounded from above and below. Thus, we assume that there exist $\rho_{\min} \in \mathbb{R}^n_{>0}$ and $\rho_{\max} \in \mathbb{R}^n_{>0}$ such that for all $w \in C([0,T], \mathbb{R}^n)$ it holds

$$\rho_{\text{in}} \exp\left(-\int_0^t \|(Q_x(s,\cdot) + c(s,\cdot))^-\|_{L^\infty(\Omega)^n} \mathrm{d}s\right) + \mathcal{G}_{w+Q}(\rho_{\max}) \le \rho_{\max}$$

and

$$\rho_{\text{in}} \exp\left(\int_0^t \|(Q_x(s,\cdot) + c(s,\cdot))^+\|_{L^\infty(\Omega)^n} \mathrm{d}s\right) + \mathcal{G}_{w+Q}(\rho_{\min}) \ge \rho_{\min}$$

for $\mathrm{d}\mu^-_{w+Q}$-almost all $(t,\omega) \in \Gamma_T$ and

$$\rho_{\min} \le \rho_0 \le \rho_{\max}$$

almost everywhere in $\Omega$. As we will see this holds for our application and thus this assumption is justified.

Inspired by the upper and lower bounds of the transport equation, we define the sets

$$\mathcal{V}_t = \Big\{\rho \in L^1(\Omega)^n | \bar{\rho}_{\min}(t) \leq \rho \leq \bar{\rho}_{\max}(t) \text{ almost everywhere in } \Omega\Big\}$$

and

$$\mathcal{W}_t = \{\gamma\rho \in L^\infty(\Gamma)^n | \bar{\rho}_{\min}(t) \leq \gamma\rho \leq \bar{\rho}_{\max}(t)\}$$

for

$$\bar{\rho}_{\max}(t) = \rho_{\max} \exp\left(\int_0^t \|(Q_x(s,\cdot) + c(s,\cdot))^-\|_{L^\infty(\Omega)^n} \mathrm{d}s\right)$$

and

$$\bar{\rho}_{\min}(t) = \rho_{\min} \exp\left(-\int_0^t \|(Q_x(s,\cdot) + c(s,\cdot))^+\|_{L^\infty(\Omega)^n} \mathrm{d}s\right).$$

It is sufficient to define the right-hand side $\mathfrak{f}$ of the ODE in (4.8) for densities $\rho$ in the set $\mathcal{V}_T$. Moreover, the function $\mathfrak{f}$ should be a kind of generalized Carathéodory-Lipschitz vector field. To be precise, we assume $\mathfrak{f}\colon [0,T] \times \mathbb{R}^n \times \mathcal{V}_T \times L^1(\Gamma)^n \to \mathbb{R}^n$ to be a function such that the mapping

$$(t,v) \mapsto \mathfrak{f}\Big(t, v, \rho, \nu(\mathbf{V} + \mathbf{Q}(t)|_\Gamma)\gamma\rho\Big)$$

is a Carathéodory-Lipschitz vector field (see Definition 1.17) for each fixed pair $(\rho, \gamma\rho) \in \mathcal{V}_T \times \mathcal{W}_T$. Additionally, we assume $\mathfrak{f}(t,\cdot,\cdot,\cdot)$ to be continuous in $(v,\rho,q) \in \mathbb{R}^n \times \mathcal{V}_T \times L^1(\Gamma)^n$ and to be locally integrable uniformly bounded for almost all $t \in [0,T]$. By the latter, we mean that for each compact set $D_0 \subset [0,T] \times \mathbb{R}^n$, there is an integrable function $m \in L^1(\mathrm{pr}_1 D_0, [0,\infty))$ and a null set $I_0 \in [0,T]$ such that it holds

$$\|\mathfrak{f}(t, v, \rho(t), \nu(\mathbf{V} + \mathbf{Q}(t))\gamma\rho(t))\|_2 \leq m(t) \tag{4.11}$$

for all $(t, v, \rho, \gamma\rho) \in D_0 \times \mathcal{V} \times \mathcal{W}$ with $t \notin I_0$. Here, the sets $\mathcal{V}$ and $\mathcal{W}$ are given as

$$\mathcal{V} = \Big\{\rho \in C([0,T], L^1(\Omega)^n) | \rho(t) \in \mathcal{V}_t\Big\}$$

and

$$\mathcal{W} = \{\gamma\rho \in L^\infty(\Gamma_T)^n | \gamma\rho(t) \in \mathcal{W}_t \text{ almost everywhere}\}.$$

Last but not least, we assume the same conditions as in Section 4.1 for the coupling conditions of the ODE $F\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^k$ and $F(0, v_0) = 0$.

Combining the concepts of solutions for the transport equation of Chapter 3 and of solutions for ODEs on networks of Section 4.1 we define a solution of the whole model.

**Definition 4.10.** A tuple $(\rho, v, P_V, \gamma\rho)$ with

$$\begin{aligned} \rho &\in L^\infty((0,T) \times \Omega)^n \\ v &\in W^{1,1}((0,T))^n \\ P_V &\in L^1((0,T))^k \\ \gamma\rho &\in L^\infty(\Gamma_T, |\mathrm{d}\mu_{v+Q}|) \end{aligned}$$

is called a *solution of the coupled system* (4.8) *on the network* $G$ on $(0,T) \times \Omega$ if the following conditions are satisfied:

1. The density $\rho$ solves the transport equation

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x + \mathbf{C}\rho = 0 \qquad & \text{in } (0,T)\times\Omega \\ \rho(0,\cdot) = \rho_0 \qquad & \text{in } \Omega \\ (\nu\mathbf{U})^-\rho = (\nu\mathbf{U})^-\mathcal{H}_u(\rho|_{\Gamma_T^+}) \qquad & \text{on } \Gamma_T \end{aligned} \tag{4.12}$$

   and $\gamma\rho$ is its trace. Here, $\mathcal{H}_u$ is the boundary operator belonging to $u = v + Q$.

2. The pair $(v, P_V)$ solves the ODE on the network (in the extended sense)

$$\begin{aligned} v_t(t) &= \mathfrak{f}\Big(t, v(t), \rho(t), \nu(\mathbf{V}(t) + \mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) + \mathbf{R}^{-1}(t)\mathbf{B}_{>1}^T P_V(t) \\ F(t, v(t)) &= 0 \\ v(0) &= v_0 \end{aligned}$$

   on $(0,T)$.

An immediate consequence of this definition concerns the regularity and the boundedness of the density, therefore we can directly apply the results of Chapter 3.

**Lemma 4.11 (Boundedness).** *Let $(\rho, v, P_V, \gamma\rho)$ be a solution of the coupled system (4.8). Then, the density $\rho$ is continuous with values in $L^p(\Omega)^n$ and for all $t \in [0,T]$ it is bounded from above and below by*

$$\bar{\rho}_{\min}(t) \leq \rho(t,\cdot) \leq \bar{\rho}_{\max}(t)$$

*almost everywhere in $\Omega$, i.e. $\rho \in \mathcal{V}$. Furthermore, its trace $\gamma\rho$ is also bounded by*

$$\bar{\rho}_{\min}(t) \leq \gamma\rho \leq \bar{\rho}_{\max}(t)$$

$|\mathrm{d}\mu_u|$*-almost everywhere in $\Gamma_T$, i.e. $\gamma\rho \in \mathcal{W}$.*

*Proof.* This is a direct consequence of Theorem 3.3 and of the Lemmas 3.9 and 3.10. □

Before we come to the next step, we will recall some notations from the previous section. According to the equations (4.5) and (4.6) the matrices occurring in the reformulation of the DAE as an ODE are

$$\mathbf{T}(t,v,\rho) = \mathbf{Id} - \mathbf{R}^{-1}\mathbf{B}_{>1}^T\left(\mathbf{B}_{>1}\mathbf{D}_2(t,v)\mathbf{R}^{-1}\mathbf{B}_{>1}^T\right)^{-1}\mathbf{B}_{>1}\mathbf{D}_2 \tag{4.13}$$

and

$$\mathbf{S}(t,v,\rho) = \mathbf{R}^{-1}\mathbf{B}_{>1}^T\left(\mathbf{B}_{>1}\mathbf{D}_2(t,v)\mathbf{R}^{-1}\mathbf{B}_{>1}^T\right)^{-1}\mathbf{D}_1^{-1}(t,v). \tag{4.14}$$

Now, we are finally able to prove the main result of this thesis, the local in time existence of solutions of the coupled system. The proof is mainly based on the Schauder fixed point theorem, on the sequential continuity of the solution operator of the transport equation and on the reformulation of the DAE as an ODE. We will construct a compact self-mapping on a subset of $C([0,T_0],\mathbb{R}^n)$. This result especially shows the local existence of solutions for the low Mach number model (2.53) without heat loss at the walls, as we will see in the next chapter.

**Theorem 4.12 (Local existence)**. *Let all the assumptions mentioned in this section be valid. Then, there exists at least one solution of the system* (4.8) *on the network* $G$ *on* $(0,T_0)\times\Omega$ *for* $T_0$ *small enough.*

*Proof.* The outline of the proof is as follows: We construct a compact-self mapping on a certain Banach space in such a way that the desired solution is a fixed point of this mapping.

First, we will choose an appropriate value for the existence time $T_0$. Therefore, let be $r>0$ and denote by $K=\overline{B_r(v_0)}$ the closed ball around the initial velocity $v_0$ with radius $r$. Since $\mathfrak{f}$ is locally integrable uniformly bounded, there exists a function $m\in L^1([0,T],[0,\infty))$ and a null subset $I_0\subset[0,T]$ such that

$$\left\|\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t))\gamma\rho(t)\Big)\right\|_2\le m(t)$$

is true for all $(t,v,\rho,\gamma\rho)\in([0,T]\setminus I_0)\times K\times\mathcal{V}\times\mathcal{W}$. As in the proof of Theorem 4.2 we find two constants $c_4$ and $c_5$ such that

$$\|\mathbf{T}(t,v,\rho(t))\|_2\le c_4$$

and

$$\|\mathbf{S}(t,v,\rho(t))\|_2\le c_5$$

for all $(t,v,\rho)\in[0,T]\times K\times\mathcal{V}$. Due to the integrability of $m$, we can choose $T_0\in(0,T]$ small enough in order to guarantee

$$\int_0^{T_0} c_4 m(s)\mathrm{d}s+T_0c_5\sup_{(t,v)\in[0,T]\times K}\|F_t(t,v)\|_2\le r$$

(see Remark 1.26).

Now, we will proceed with the construction of the above-mentioned mapping

$$\mathcal{F}\colon C([0,T_0],K))\to C([0,T_0],K))$$

such that each fixed point of this mapping solves the coupled system (4.8). Therefore, let $w\in C([0,T_0],K)$ be arbitrary. We denote the unique solution of the transport equation

$$\begin{aligned}
\rho_t+\left((\mathbf{W}+\mathbf{Q})\,\rho\right)_x+\mathbf{C}\rho&=0 && \text{in } (0,T_0)\times\Omega\\
\rho(0,\cdot)&=\rho_0 && \text{in } \Omega\\
(\nu(\mathbf{W}+\mathbf{Q}))^-\rho&=(\nu(\mathbf{W}+\mathbf{Q}))^-\mathcal{H}_{w+Q}(\rho|_{\Gamma_{T_0}}) && \text{on } \Gamma_{T_0}
\end{aligned}$$

by $\rho\in L^\infty((0,T_0)\times\Omega)^n$ and its trace by $\gamma\rho\in L^\infty(\Gamma_{T_0},|\mathrm{d}\mu_{w+Q}|)$. This solution exists due to Theorem 3.15 and it is continuous with values in $L^1(\Omega)^n$ due to Theorem 3.3. Furthermore, using the Lemmas 3.9 and 3.10 $\rho$ and $\gamma\rho$ are bounded from above and below by

$$\bar{\rho}_{\min}(t)\le\rho(t)\le\bar{\rho}_{\max}(t)$$

almost everywhere in $\Omega$ and

$$\bar{\rho}_{\min}(t) \le \gamma\rho(t,\omega) \le \bar{\rho}_{\max}(t)$$

almost everywhere in $\Gamma_{T_0}$. At this stage, we cannot directly apply the results of Section 4.1 for the ODE since we have no estimate for the length of the existence interval from the previous section. In particular, we do not know if the solution of the ODE exists on the whole time interval $(0,T_0)$. For this reason, we define

$$\begin{aligned} v(t) = v_0 + \int_0^t &\mathbf{T}(s,w(s),\rho(s))\mathfrak{f}\Big(s,w(s),\rho(s),\nu(\mathbf{W}(s)+\mathbf{Q}(s)|_\Gamma)\gamma\rho(s)\Big) \\ &- \mathbf{S}(s,w(s),\rho(s))F_t(s,w(s))\mathrm{d}s, \end{aligned}$$

where we used the reformulation of the DAE with the matrices $\mathbf{T}$ and $\mathbf{S}$. The advantage of this integral formulation is that it is independent on the length of the existence interval of the solution of the ODE. The mapping $\mathcal{F}$ is then defined by $\mathcal{F}(w) = v$. Clearly, $v$ is continuous and due to the choice of $T_0$ it is

$$\|v(t) - v_0\|_2 \le \int_0^{T_0} c_4 m(s) + c_5 \sup_{(t,w)\in[0,T_0]\times K} \|F_t(t,w)\|_2 \mathrm{d}s \le r.$$

Thus, it is $v(t) \in K$ and the mapping is well-defined.

To apply the Schauder fixed point theorem, we will prove that $\mathcal{F}$ is a compact self-mapping on a certain convex, non-empty, closed and bounded subset of $C([0,T_0],K)$. We compute

$$\begin{aligned} &\|v(t) - v(s)\| \\ &= \Big\|\int_s^t \mathbf{T}(r,w(r),\rho(r))\mathfrak{f}\Big(r,w(r),\rho(r),\nu(\mathbf{W}(r)+Q(r)|_\Gamma)\gamma\rho(r)\Big) \\ &\quad - \mathbf{S}(r,w(r),\rho(r))F_t(r,w(r))\mathrm{d}r\Big\|_2 \\ &\le c_4 \int_s^t m(r)\mathrm{d}r + c_5 \sup_{(t,v)\in[0,T_0]\times K} \|F_t(t,v)\|_2 |t-s| \end{aligned}$$

for any $s,t \in [0,T]$. Since $m$ is integrable, we can find a value $\delta > 0$ for each $\varepsilon > 0$ such that for any subset $S \subset [0,T]$ with $\mu(S) < \delta$ it holds

$$\int_S m(t)\mathrm{d}t < \frac{\varepsilon}{2c_4}$$

(see Remark 1.26). Thus, for all $t,s \in [0,T]$ with $|t-s| < \min\left(\delta, \frac{\varepsilon}{2c_5 \sup_{(t,v)\in[0,T]\times K}\|F_t\|_2}\right)$ it follows

$$\|v(t) - v(s)\| < \varepsilon.$$

This shows the equicontinuity of the image of $\mathcal{F}$ and allows the application of the Arzelà-Ascoli theorem (see e.g. [82]). Thus, the image $\mathcal{F}(C([0,T_0],K)) \subset C([0,T],\mathbb{R}^n)$ is relatively compact. Furthermore, the derivative of $v$ is bounded by

$$\begin{aligned}\|v_t(t)\|_2 &= \Big\|\mathbf{T}(t,w(t),\rho(t))\mathfrak{f}\Big(t,w(t),\rho(t),\nu(\mathbf{W}(t)+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)\\ &\quad - \mathbf{S}(t,w(t),\rho(t))F_t(t,w(t))\Big\|_2\\ &\le c_4 m(t) + c_5 \sup_{(t,v)\in[0,T_0]\times K} \|F_t(t,v)\|_2\\ &= k(t)\end{aligned}$$

almost everywhere. Thus, we define the non-empty convex subset

$$C = \Big\{v \in W^{1,1}([0,T],\mathbb{R}^n)\Big|\|v(t)-v_0\|_2 \le r \text{ and } \|v_t(t)\|_2 \le k(t) \text{ a.e.}\Big\} \subset C([0,T],K),$$

such that for the image of $\mathcal{F}$ it holds $\mathcal{F}(C([0,T],K)) \subset C$.

Up to now, we have constructed a bounded non-empty convex set and we have shown that the image of $\mathcal{F}$ is relatively compact. The only requirement of the Schauder fixed point theorem (Theorem 1.21) which remains to be proven is the compactness of the mapping $\mathcal{F} : \overline{C} \to \overline{C}$. Due to the relative compactness of $\mathrm{im}(\mathcal{F})$ it is sufficient to show the continuity of $\mathcal{F}$. Therefore, let $w_j \in \overline{C}$ be a convergent sequence with $w_j \to w$ in $C([0,T_0],\mathbb{R}^n)$. We have to show that $v_j = \mathcal{F}(w_j) \to v = \mathcal{F}(w)$ holds uniformly. To this end, we will apply the stability theorem (Theorem 3.11) to the sequence $\rho_j$ occurring in the definition of $\mathcal{F}(w_j)$. The requirements of this theorem are fulfilled by construction: The only non-trivial requirement, we have to check, concerns the boundary operator $\mathcal{H}$. However, for this operator the requirements of the theorem are valid for all possible uniform convergent sequences due to our assumptions (4.9) and (4.10). Thus, the sequences $\rho_j$ and $\gamma\rho_j$ are convergent. To be more precise, it holds

$$\rho_j \to \rho$$

in $C([0,T_0],L^1(\Omega)^n)$ and

$$\gamma\rho_j \to \gamma\rho$$

in $L^1(\Gamma_{T_0},|\mathrm{d}\mu_{w+Q}|)$, where $\rho$ is the solution of the limit problem

$$\begin{aligned}\rho_t + ((\mathbf{W}+\mathbf{Q})\,\rho)_x + \mathbf{C}\rho &= 0 && \text{in } (0,T_0)\times\Omega\\ \rho(0,\cdot) &= \rho_0 && \text{in } \Omega\\ (\nu(\mathbf{W}+\mathbf{Q}))^-\rho &= (\nu(\mathbf{W}+\mathbf{Q}))^-\mathcal{H}_{w+Q}(\rho|_{\Gamma_{T_0}}) && \text{on } \Gamma_T\end{aligned}$$

and where $\gamma\rho$ is its trace. Since the sequence $\gamma\rho_j$ is bounded in $L^\infty(\Gamma_{T_0})^n$ we also find

$$\nu(\mathbf{W}_j + \mathbf{Q}|_\Gamma)\gamma\rho_j \to \nu(\mathbf{W}+\mathbf{Q}|_\Gamma)\gamma\rho$$

in $L^1(\Gamma_{T_0})^n$ and thus there exists a subsequence $\nu(\mathbf{W}_{j_l} + \mathbf{Q}|_\Gamma)\gamma\rho_{j_l}$ that converges to $\nu(\mathbf{W}+\mathbf{Q}|_\Gamma)\gamma\rho$ almost everywhere in $\Gamma_{T_0}$. Due to the continuity of $\mathfrak{f}(t,\cdot,\cdot,\cdot)$ for almost all $t \in [0,T]$, we conclude for this subsequence

$$\mathfrak{f}\Big(t,w_{j_l}(t),\rho_{j_l}(t),\nu(\mathbf{W}_{j_l}(t)+\mathbf{Q}(t)|_\Gamma)\gamma\rho_{j_l}(t)\Big) \to \mathfrak{f}\Big(t,w(t),\rho(t),\nu(\mathbf{W}(t)+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)$$

for almost all $t \in [0, T_0]$. Thus, using the estimate

$$\begin{aligned}&\Big\| \mathbf{T}(t, w_{j_l}(t), \rho_{j_l}(t)) \mathfrak{f}\Big(t, w_{j_l}(t), \rho_{j_l}(t), \nu(\mathbf{W}_{j_l}(t) + \mathbf{Q}(t)|_\Gamma)\gamma\rho_{j_l}(t)\Big) \\ &\quad - \mathbf{S}(t, w_{j_l}(t), \rho_{j_l}(t)) F_t(t, w_j(t)) \Big\|_2 \\ &\leq c_4 m(t) + c_5 \sup_{(t,v)\in[0,T_0]\times K} \|F_t(t,v)\|_2\end{aligned}$$

we can apply the dominated convergence theorem to obtain

$$\begin{aligned}\mathcal{F}(w_{j_l})(t) &= v_0 + \int_0^t \mathbf{T}(s, w_{j_l}(s), \rho_{j_l}(s)) \mathfrak{f}\Big(s, w_{j_l}(s), \rho_{j_l}(s), \nu(\mathbf{W}_{j_l}(s) + \mathbf{Q}(s)|_\Gamma)\gamma\rho_{j_l}(s)\Big) \\ &\qquad - \mathbf{S}(s, w_{j_l}(s), \rho_j(s)) F_t(s, w_{j_l}(s)) \mathrm{d}s \\ &\to v_0 + \int_0^t \mathbf{T}(s, w(s), \rho(s)) \mathfrak{f}\Big(s, w(s), \rho(s), \nu(\mathbf{W}(s) + \mathbf{Q}(s)|_\Gamma)\gamma\rho(s)\Big) \\ &\qquad - \mathbf{S}(s, w(s), \rho(s)) F_t(s, w(s)) \mathrm{d}s \\ &= \mathcal{F}(w)(t)\end{aligned}$$

for all $t \in [0, T]$. In fact, due to the uniqueness of the limit, the whole sequence is converging and not only a subsequence and due to the equicontinuity the convergence is uniform in $t$. Hence, the mapping $\mathcal{F}$ is continuous and an application of the Schauder fixed point theorem yields the existence of a fixed point $v$ of it. The construction of the corresponding density $\rho$, the trace $\gamma\rho$ and the pressure $P_V$ is straightforward: $\rho$ is the corresponding solution of the transport equation with its trace $\gamma\rho$ and $P_V$ is given by

$$\begin{aligned}P_V = &- \Big(\mathbf{B}_{>1}\mathbf{D}_2(t, v(t))\mathbf{R}^{-1}(t)\mathbf{B}_{>1}^T\Big)^{-1} \\ &\Big(\mathbf{B}_{>1}\mathbf{D}_2(t, v(t))\mathfrak{f}\Big(t, v(t), \rho(t), \nu(\mathbf{V}(t) + \mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) + \mathbf{D}_1(t, v)^{-1} F_t(t, v(t))\Big),\end{aligned} \tag{4.15}$$

as we know from the proof of Theorem 4.2 concerning the local existence of solutions for the ODE on the network. □

In Section 4.1, we have discussed the difficulty that solutions of an ODE on a network need not to exist globally in time. To deal with this, we introduced an energy functional $V$ to ensure the global existence. For the coupled system, we proceed in the same manner. In this case, we need an even more restrictive form of such a functional $V \colon \mathbb{R}^n \to \mathbb{R}$, since the density $\rho$, and thus the right-hand side of the ODE is not known a priori. Therefore, we extend the notion of Definition 4.9:

**Definition 4.13.** Let $V \colon \mathbb{R}^n \to \mathbb{R}$ be a radially unbounded, Lipschitz continuous, piecewise $C^1$-function written as $V(x) = \sum_{\alpha\in I} \chi_{P^\alpha}(x) V^\alpha(x)$ with $I$, $P^\alpha$ and $V^\alpha$ as in definition 4.5. Then, $V$ is called a *uniform energy functional* for the function $\mathfrak{f}$ and the graph $G$, if the following is satisfied: There exist two non-negative functions $k_1, k_2 \in L^1((0,T))$

such that for all $(\rho, \gamma\rho) \in \mathcal{V} \times \mathcal{W}$, there exists a null subset $I_0 \subset [0, T]$ with

$$\begin{aligned} \nabla V^\alpha(v)\Big(\mathbf{T}(t,v,\rho(t))\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) - \mathbf{S}(t,v,\rho(t))F_t(t,v)\Big) \\ \le k_1(t) + k_2(t)V(v) \end{aligned}$$

for all $(t, v) \in \{[0, T] \times P^\alpha | F(t, v) = 0 \text{ and } t \notin I_0\}$.

**Corollary 4.14 (Global existence)**. *Additionally to the assumptions of this section, let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a uniform energy functional for $\mathfrak{f}$ and $G$.*

*Then, any maximal solution of system* (4.8) *is defined on the whole domain* $(0, T) \times \Omega$ *and it is bounded by*

$$\|v(t)\| \le R(t)$$

*and*

$$V(v(t)) \le M(t),$$

*respectively, for all $t \in [0, T]$. As before, the bounds are given by*

$$M(t) = \left(V(v_0) + \int_0^t k_1(s)\mathrm{d}s\right)\exp\left(\int_0^t k_2(s)\mathrm{d}s\right)$$

*and*

$$R(t) = \inf\left\{c \in \mathbb{R}^+ | V(v) > M(t) \textit{ for all } v \textit{ with } \|v\| > c\right\}.$$

*Proof.* Let the tuple $(\rho, v, P_V, \gamma\rho)$ be a maximal solution of (4.8). Then, $(v, P_V)$ is a solution of an ODE on the network. Since $V$ is an energy functional for this ODE, we can apply Theorem 4.7 to ensure the stated boundedness. Thus, $(\rho, v, P_V, \gamma\rho)$ is a maximal bounded solution and it must be defined on the whole domain $(0, T) \times \Omega$, since otherwise we could extend it to a larger domain. □

*Remark* 4.15. The approach using the Schauder fixed point theorem does not yield a uniqueness result. A main problem of proving the uniqueness is the missing Lipschitz continuous dependence of the solution of the transport equation on the velocity. If the density is continuously differentiable it is possible to prove the uniqueness with some further assumptions on $\mathfrak{f}$. Usually such a kind of result is called a “weak=strong” uniqueness result, since the existence of a strong solution implies the uniqueness of the weak solutions. For the Navier-Stokes equations such a result can be found e.g. in [65].

## 4.3 Some energy functionals

Until now, we have proven the local existence of solution for both, the ODE and the coupled system on a network. To ensure the global existence in time, we have required the existence of an energy functional $V$ in both sections. These energy functionals highly depend on the form of the network. In this section, we want to construct such functionals for three special cases.

We begin with linearly bounded ODEs on arbitrary networks. In this case, the construction is rather simple, since the reformulated ODE is then linearly bounded itself. The second case deals with networks that have at most one inner node. And in the last case, we study arbitrary networks with maximum node degree 2. Although we will discuss only the case of the coupled system (4.8), all results are also applicable to the solitary ODEs (4.1) with some minor changes.

Before we consider these special cases, we discuss some general properties of the matrices $\mathbf{T}$ and $\mathbf{S}$ used for the reformulation of the ODE. From the definition of the matrix $\mathbf{T}$ in (4.13) we immediately see the idempotence

$$\mathbf{T}\Big(t, v, \rho(t)\Big) = \mathbf{T}\Big(t, v, \rho(t)\Big)\mathbf{T}\Big(t, v, \rho(t)\Big)$$

for each $(t, v, \rho) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}$. Thus, $\mathbf{T}$ is a (in general non-orthogonal) projector with

$$\ker \mathbf{T}(t, v, \rho(t)) = \operatorname{im}\Big(\mathbf{R}(t)\mathbf{B}_{>1}^T\Big)$$

and

$$\operatorname{im} \mathbf{T}(t, v, \rho(t)) = \ker\left(\mathbf{B}_{>1}\mathbf{D}_2(t, v)\right).$$

Let $\bar{\mathbf{B}} = \Big(b_1, \ldots, b_{n-k}\Big)$ be a basis of $\ker \mathbf{B}_{>1}$. Then, it holds

$$\operatorname{im} \mathbf{T}(t, v, \rho(t)) = \operatorname{im}\Big(\mathbf{D}_2(t, v)^{-1}\bar{\mathbf{B}}\Big)$$

and

$$(\ker \mathbf{T}(t, v, \rho(t)))^{\perp} = \operatorname{im}\Big(\mathbf{T}^T(t, v)\Big) = \operatorname{im}\Big(\mathbf{R}(t)^{-1}\bar{\mathbf{B}}\Big)$$

and we can write the projector $\mathbf{T}$ as

$$\mathbf{T}(t, v, \rho(t)) = \mathbf{D}_2(t, v)^{-1}\bar{\mathbf{B}}\left(\bar{\mathbf{B}}^T\mathbf{R}(t)^{-1}\mathbf{D}_2(t, v)\bar{\mathbf{B}}\right)^{-1}\bar{\mathbf{B}}^T\mathbf{R}(t)^{-1} \tag{4.16}$$

(see e.g. [69]). On the next pages, we will switch between both formulations for $\mathbf{T}$, depending on the context.

In the following, we will assume the eigenvalues of $\mathbf{D}_2$ to be bounded from above and the eigenvalues from $\mathbf{D}_1$ from below, i.e. there exist a positive integrable function $c_2 \in L^\infty([0, T], [0, \infty))$ and a constant $c_3 > 0$ with

$$\sup_{\substack{v \in \mathbb{R}^n \\ 1 \leq i \leq n}} (\mathbf{D}_2(t, v))_{ii} \leq c_2(t)$$

and

$$\inf_{\substack{v \in \mathbb{R}^n \\ 1 \leq i \leq k}} (\mathbf{D}_1(t, v))_{ii} \geq c_3$$

for almost all $t \in [0, T]$. Furthermore, we require $F_t$ to be linearly integrable bounded. More precisely, we require the existence of two positive functions $r_1 \in L^2([0, T], [0, \infty))$ and $r_2 \in L^1([0, T], [0, \infty))$ such that it holds

$$\|F_t(t, v)\|_2 \leq r_1(t) + r_2(t)\|v\|_2$$

for almost all $t \in [0,T]$ and all $v \in \mathbb{R}^n$ with $F(t,v) = 0$. For the low Mach number system we want to study in this work, all the assumptions on $\mathbf{D}_1$, $\mathbf{D}_2$ and $F_t$ are fulfilled. We recall the two estimates from the proof of Theorem 4.2

$$\|\mathbf{T}(t,v,\rho(t))\|_2 \leq \frac{c_2(t)\bar{\rho}_{\max}(t)}{c_1\bar{\rho}_{\min}(t)} \left\|\mathbf{L}^{>1}\right\|_2 \left\|\left(\mathbf{L}^{>1}\right)^{-1}\right\|_2 = c_4(t)$$

and

$$\|\mathbf{S}(t,v,\rho(t))\|_2 \leq \frac{\bar{\rho}_{\max}(t)}{c_1 c_3\bar{\rho}_{\min}(t)} \sqrt{\|\mathbf{L}^{>1}\|_2} \left\|\left(\mathbf{L}^{>1}\right)^{-1}\right\|_2 = c_5(t),$$

which are now valid for all $(t,v,\rho) \in [0,T] \times \mathbb{R}^n \times \mathcal{V}$. The defined functions $c_4$ and $c_5$ are essentially bounded, i.e. it holds $c_4, c_5 \in L^\infty([0,T],[0,\infty))$.

### 4.3.1 Arbitrary networks with linearly bounded equations

If the right-hand side $\mathfrak{f}$ of the ODE in (4.8) is linearly bounded for any network, it is simple to construct an energy functional, since the resulting ODE (4.4) is also linearly bounded. As an example for this class, one can think of the velocity equation of the low Mach number model (2.59) with $0 \leq \alpha(u) \leq 1$, i.e. with a linearly bounded friction term.

In other words, we can prove the following corollary:

**Corollary 4.16 (Global existence).** *Let $G$ be a connected, oriented graph and let the right-hand side $\mathfrak{f}$ of the ODE of system* (4.8) *be linearly integrable uniformly bounded, i.e. there exist two functions $d_1 \in L^2([0,T],[0,\infty))$ and $d_2 \in L^1([0,T],[0,\infty))$ such that*

$$\left\|\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}(t)+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)\right\|_2 \leq d_1(t) + d_2(t)\|v\|_2$$

*holds for almost all $t \in [0,T]$ and all $(v,\rho,\gamma\rho) \in \mathbb{R}^n \times \mathcal{V} \times \mathcal{W}$ with $F(t,v) = 0$.*

*Then, any solution $(\rho, v, P_V, \gamma\rho)$ of system* (4.8) *exists on the whole domain $(0,T) \times \Omega$ and its velocity $v$ is bounded by*

$$\|v(t)\|_2^2 \leq \left(\|v_0\|_2^2 + \int_0^t (c_4 d_1 + c_5 r_1)^2 \,\mathrm{d}s\right) \exp\left(\int_0^t (2c_4 d_2 + 2c_5 r_2 + 1)\,\mathrm{d}s\right).$$

*Proof.* We would like to apply Corollary 4.14 in combination with Remark 4.8. The function $V(v) = \frac{1}{2}\|v\|_2^2$ would be a convenient choice for a uniform energy functional since the inequality in Definition 4.13 has been fulfilled by construction, but this function is not globally Lipschitz continuous. In order to obtain a uniform energy functional we will modify this function.

We define the integrable functions $k_1 = \frac{1}{2}(c_4 d_1 + c_5 r_1)^2$ and $k_2 = 2c_4 d_2 + 2c_5 r_2 + 1$ with $k_1, k_2 \in L^1([0,T],[0,\infty))$. Then, we choose an arbitrary value

$$C > \left(\frac{1}{2}\|v_0\|_2 + \|c_1\|_{L^1([0,T])}\right) \exp\left(\|c_2\|_{L^1([0,T])}\right).$$

We introduce the globally Lipschitz $C^1$-function

$$V^1(v) = \begin{cases} \frac{1}{2}\|v\|_2^2 & \text{for } v \in P^1 \\ -\frac{1}{2}C^2 + C\|v\|_2 & \text{otherwise,} \end{cases}$$

where it is $P^1 = \{v \in \mathbb{R}^n | \|v\|_2 \le C\}$. To simplify the following estimation, we cut this function at $\frac{1}{2}C^2$ and define the Lipschitz continuous, piecewise $C^1$-function

$$\tilde{V}(v) = \chi_{P^1}(v)V^1(v) + \chi_{\mathbb{R}^n \setminus P^1}(v)\frac{C^2}{2}.$$

This function satisfies the condition of Remark 4.8 for

$$M(T) = \left(\frac{1}{2}\|v_0\|_2 + \|k_1\|_{L^1([0,T])}\right) \exp\left(\|k_2\|_{L^1([0,T])}\right)$$

and

$$R = \sqrt{2M}.$$

Finally, with the Cauchy-Schwarz inequality and Young's inequality (see e.g. [40]) we estimate for $v \in P^1$

$$\begin{aligned}
&\nabla V^1(v)\Big(\mathbf{T}(t,v,\rho)\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) - \mathbf{S}(t,v,\rho(t))F_t(t,v)\Big) \\
&= v^T\left(\mathbf{T}(t,v,\rho)\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) - \mathbf{S}(t,v,\rho(t))F_t(t,v)\right) \\
&\le \|v\|_2\Big(c_4(t)\|\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}|_\Gamma)\gamma\rho(t)\Big)\|_2 + c_5(t)\|F_t(t,v)\|_2\Big) \\
&\le \|v\|_2\Big(c_4(t)d_1(t) + c_4(t)d_2(t)\|v\|_2 + c_5(t)\Big(r_1(t) + r_2(t)\|v\|_2\Big)\Big) \\
&\le \left(c_4(t)d_2(t) + c_5(t)r_2(t) + \frac{1}{2}\right)\|v\|_2^2 + \frac{1}{2}\Big(c_4(t)d_1(t) + c_5(t)r_1(t)\Big)^2 \\
&= \Big(2c_4(t)d_2(t) + 2c_5(t)r_2(t) + 1\Big)\tilde{V}(v) + \frac{1}{2}\Big(c_4(t)d_1(t) + c_5(t)r_1(t)\Big)^2 \\
&= k_1(t) + k_2(t)\tilde{V}(v)
\end{aligned}$$

and Corollary 4.14 yields the desired result

$$\frac{1}{2}\|v(t)\|_2^2 \le \left(\frac{1}{2}\|v_0\|_2^2 + \frac{1}{2}\int_0^t (c_4d_1 + c_5r_1)^2\,\mathrm{d}s\right)\exp\left(\int_0^t (2c_4d_2 + 2c_5r_2 + 1)\,\mathrm{d}s\right).$$

□

### 4.3.2 Networks with one inner node

The second case we want to consider deals with networks which have exactly one inner node (see Figure 4.1). Since we have in mind to take into account non-linear friction phenomena in the low Mach number model, we will allow a wider class of right-hand sides $\mathfrak{f}$ than in the last example. A typical friction force looks like $-cu|u|$ and acts in the opposite direction of the flow. For this reason, we relax the requirements on $\mathfrak{f}$ by asking for linear boundedness of $\mathfrak{f}$ only where $\mathfrak{f}$ and $v$ have the same sign. Everywhere else $\mathfrak{f}$ is allowed to be arbitrary. In mathematical terms, there exist two non-negative integrable functions $d_1, d_2 \in L^1([0,T],[0,\infty))$ such that

$$\operatorname{sgn}(v_i)\mathfrak{f}_i\Big(t, v, \rho(t), \nu(\mathbf{V} + \mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) \leq d_1(t) + d_2(t)\|v\|_1 \tag{4.17}$$

holds for almost all $t \in [0,T]$, all $(v, \rho, \gamma\rho) \in \mathbb{R}^n \times \mathcal{V} \times \mathcal{W}$ and all $i \in \{1, \ldots, n\}$. For technical reasons, we will enforce the matrix $\mathbf{D}_2$ to be constant in order to use it as part of the energy functional.

Before we formulate the corollary for the global existence of solutions, we will study the matrix $\mathbf{T}$ in the following lemma. In its proof, the restriction to networks with exact one inner node is essential since it guarantees the one-dimensionality of the kernel of the matrix $\mathbf{T}$.

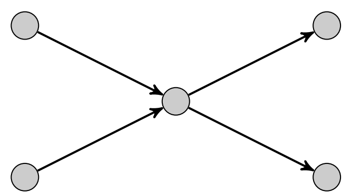


Figure 4.1: A network with one inner node

**Lemma 4.17**. *Let $G$ be an oriented graph with exact one inner node. Then,*

$$\mathbf{T}_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji} \right|$$

*holds for all $(t, v, \rho) \in [0,T] \times \mathbb{R}^n \times \mathcal{V}$. Furthermore, for $v_i \neq 0$ there exists a function $b_i$ with $b_i(t, v, \rho(t)) \in [0,2]$ and*

$$\left(\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)^T \operatorname{sgn} v\right)_i = b_i \operatorname{sgn} v_i.$$

*For $v_i = 0$ it is*

$$\left| \left(\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)^T \operatorname{sgn}(v)\right)_i \right| \leq \mathbf{T}_{ii} \leq 1.$$

*Proof.* Let $\mathfrak{v}_1 \in \mathfrak{V}$ be the inner node of $G = (\mathfrak{V}, E)$. Since $\mathfrak{v}_1$ is the only inner node of $G$, the matrix $\mathbf{B}_{>1}$ consists only of one row and we compute

$$\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1} = \mathbf{Id} - \frac{1}{\mathbf{B}_{>1}\mathbf{D}_2\mathbf{R}\mathbf{B}_{>1}^T}\mathbf{D}_2\mathbf{R}\mathbf{B}_{>1}^T\mathbf{B}_{>1}$$

and

$$\begin{aligned}
\sum_{\substack{j=1\\j\neq i}}^n \left|\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\right| &= \frac{\sum_{\substack{j:e_j\in E(\mathfrak{v}_1)\\j\neq i}}(\mathbf{D}_2)_{jj}\mathbf{R}_{jj}}{\sum_{j:e_j\in E(\mathfrak{v}_1)}(\mathbf{D}_2)_{jj}\mathbf{R}_{jj}}\\
&= 1 - \frac{(\mathbf{D}_2)_{ii}\mathbf{R}_{ii}}{\sum_{j:e_j\in E(\mathfrak{v}_1)}(\mathbf{D}_2)_{jj}\mathbf{R}_{jj}}\\
&= (\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1})_{ii}\\
&= \mathbf{T}_{ii}.
\end{aligned}$$

Using this fact we conclude for $v_i \neq 0$

$$\begin{aligned}
b_i &= \operatorname{sgn}(v_i)\left(\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)^T \operatorname{sgn}(v)\right)_i\\
&= \mathbf{T}_{ii} + \sum_{\substack{j=1\\j\neq i}}^n \left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\operatorname{sgn}(v_j)\operatorname{sgn}(v_i)\\
&\leq \mathbf{T}_{ii} + \sum_{\substack{j=1\\j\neq i}}^n \left|\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\right|\\
&= 2\mathbf{T}_{ii}\\
&\leq 2
\end{aligned}$$

and

$$b_i \geq \mathbf{T}_{ii} - \sum_{\substack{j=1\\j\neq i}}^n \left|\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\right| = 0.$$

Finally, for $v_i = 0$ we have

$$\begin{aligned}
\left|\left(\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)^T \operatorname{sgn}(v)\right)_i\right| &= \left|\sum_{j=1}^n \left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\operatorname{sgn}(v_j)\right|\\
&= \left|\sum_{\substack{j=1\\j\neq i}}^n \left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\operatorname{sgn}(v_j)\right|\\
&\leq \sum_{\substack{j=1\\j\neq i}}^n \left|\left(\mathbf{D}_2\mathbf{T}\mathbf{D}_2^{-1}\right)_{ji}\right|\\
&= \mathbf{T}_{ii}\\
&\leq 1.
\end{aligned}$$

□

With this information concerning the structure of the matrix $\mathbf{T}$, we are able to define an appropriate uniform energy functional $V$ to apply the Corollary 4.14 to prove the global existence of a solution.

**Corollary 4.18 (Global existence).** *Let $G$ be an oriented graph with exact one inner node and let $d_1, d_2 \in L^1([0,T],[0,\infty))$ be two functions such that (4.17) is satisfied. Furthermore, let the matrix $\mathbf{D}_2$ be constant with maximal eigenvalue $c_2 = \lambda_{\max}(\mathbf{D}_2)$.*

*Then, any solution $(\rho, v, P_V, \gamma\rho)$ of system (4.8) exists on the whole domain $(0,T)\times\Omega$ and its velocity $v$ is bounded.*

*Proof.* The function $V(v) = \|\mathbf{D}_2 v\|_1$ is globally Lipschitz continuous and piecewise $C^1$. We define for $\alpha \in I = \{-1,0,1\}^n$

$$P^\alpha = \{v \in \mathbb{R}^n |\, \mathrm{sgn}(v_i) = \alpha_i\}$$

and

$$V^\alpha(v) = \sum_{i=1}^n \alpha_i (\mathbf{D}_2)_{ii} v_i.$$

Then, it holds

$$V(v) = \sum_{\alpha\in I} \chi_{P^\alpha}(v) V^{\alpha(v)}$$

and

$$\nabla V^\alpha(v) = \mathrm{sgn}(\bar{v})^T \mathbf{D}_2$$

for an arbitrary $\bar{v} \in P^\alpha$. Altogether, for almost all $t \in [0,T]$ and all $(v,\rho,\gamma\rho) \in P^\alpha \times \mathcal{V} \times \mathcal{W}$ with $F(t,v) = 0$ we conclude with Lemma 4.17

$$\begin{aligned}
&\nabla V^\alpha(v)\Big(\mathbf{T}(t,v,\rho(t))\mathfrak{f}\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}|_\Gamma)\gamma\rho(t)\Big) - \mathbf{S}(t,v,\rho(t))F_t(t,v)\Big)\\
&= \mathrm{sgn}(v)^T \mathbf{D}_2 \Big(\mathbf{T}\mathfrak{f} - \mathbf{S}F_t\Big)\\
&\le \mathfrak{f}^T \mathbf{D}_2 \mathbf{D}_2^{-1} \mathbf{T}^T \mathbf{D}_2 \,\mathrm{sgn}(v) + \|\,\mathrm{sgn}(v)\|_2 \|\mathbf{D}_2\|_2 \|\mathbf{S}\|_2 \|F_t\|_2\\
&\le \sum_{i=1}^n \mathfrak{f}_i (\mathbf{D}_2)_{ii} \left(\Big(\mathbf{D}_2 \mathbf{T} \mathbf{D}_2^{-1}\Big)^T \mathrm{sgn}(v)\right)_i + \sqrt{n} c_2 c_5 \|F_t\|_2\\
&\le \sum_{\substack{i=1\\ v_i\neq 0}}^n \mathfrak{f}_i\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)(\mathbf{D}_2)_{ii} b_i \,\mathrm{sgn}(v_i)\\
&\quad + \sum_{\substack{i=1\\ v_i=0}}^n (\mathbf{D}_2)_{ii} \left|\mathfrak{f}_i\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)\right| + \sqrt{n} c_2 c_5 \left(r_1 + r_2 \|v\|_1\right)\\
&\le \sum_{\substack{i=1\\ v_i\neq 0}}^n (\mathbf{D}_2)_{ii} b_i (d_1 + d_2\|v\|_1) + c_2 \left\|\mathfrak{f}\Big(t,0,\rho(t),\nu\mathbf{Q}(t)|_\Gamma \gamma\rho(t)\Big)\right\|_1 + \sqrt{n} c_2 c_5 \left(r_1 + r_2\|v\|_1\right)
\end{aligned}$$

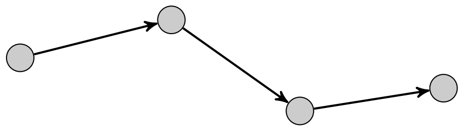


Figure 4.2: A network with maximum node degree 2

$$\leq 2\,\mathrm{tr}(\mathbf{D}_2)d_1 + \sqrt{n}c_2c_5r_1 + c_2\left\|\mathfrak{f}\Big(t,0,\rho(t),\nu\mathbf{Q}(t)|_\Gamma\gamma\rho(t)\Big)\right\|_1$$
$$+ \Big(2\,\mathrm{tr}(\mathbf{D}_2)d_2 + \sqrt{n}c_2c_5r_2\Big)V(v).$$

In the above inequality, arguments of the functions were often omitted for the sake of clarity.

Since $\mathfrak{f}$ is locally integrable uniformly bounded, there exists an integrable function $m \in L^1([0,T],[0,\infty))$ for the compact set $D_0 = [0,T]\times\{0\}$ such that the relation (4.11) is fulfilled. Thus, it is

$$\left\|\mathfrak{f}\Big(t,0,\rho(t),\nu\mathbf{Q}(t)|_\Gamma\gamma\rho(t)\Big)\right\|_1 \leq m(t)$$

for almost all $t \in [0,T]$ and the Corollary 4.14 yields the global existence of the solution on the domain $(0,T)\times\Omega$. Furthermore, the velocity is bounded by

$$\|v(t)\|_1 \leq \left(\|v_0\|_1 + \int_0^t 2\,\mathrm{tr}(\mathbf{D}_2)d_1 + \sqrt{n}c_2c_5r_1 + c_2m\mathrm{d}s\right)$$
$$\exp\left(\int_0^t 2\,\mathrm{tr}(\mathbf{D}_2)d_2 + \sqrt{n}c_2c_5r_2\mathrm{d}s\right).$$

□

### 4.3.3 Networks with maximum node degree 2

As last example, we consider graphs which consist of a single non-closed path (see Figure 4.2) and right-hand sides $\mathfrak{f}$ as described in the previous example. In this case, exactly two nodes have degree one (the ends of the path) and all other nodes have degree two (the inner nodes). Whereas in the previous example, we had to require $\mathbf{D}_2$ from the derivative of $F$ to be constant, now we are able to handle the general case for a nonlinear coupling condition $F$ as introduced in Section 4.1. Due to the special form of the graph, the ODE system reduces essentially to a scalar equation and the projector $\mathbf{T}$ gets a very simple structure, as can be seen from its second formulation (4.16).

**Corollary 4.19 (Global existence).** *Let $G$ be an oriented, connected graph with maximum degree 2 and let $d_1, d_2 \in L^1([0,T],[0,\infty))$ be two non-negative functions such that it holds*

$$\mathrm{sgn}(v_i)\mathfrak{f}_i\Big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) \leq d_1(t) + d_2(t)\|v\|_1$$

*for almost all $t \in [0,T]$, all $(v,\rho,\gamma\rho) \in \mathbb{R}^n\times\mathcal{V}\times\mathcal{W}$ with $F(t,v)=0$ and all $i \in \{1,\dots,n\}$.*

*Then, any solution* $(\rho, v, P_V, \gamma\rho)$ *of system* (4.8) *exists on the whole domain* $(0,T)\times\Omega$ *and its velocity* $v$ *is bounded.*

*Proof.* In this setting, the unoriented graph consists of a single path. To simplify the notation, we choose a natural orientation and enumeration of the graph $G$ in such a way that

$$\mathbf{B}_{>1} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \vdots \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

holds. This means that the vertices and edges are enumerated in the order of the occurrence in the path. Clearly, a basis of the kernel of $\mathbf{B}_{>1}$ is given by

$$\bar{\mathbf{B}} = \mathbf{1}.$$

The first step in the proof will be to express all $v_i$ only in terms of $v_1$: Because of $D_v F = \mathbf{D}_1 \mathbf{B}_{>1} \mathbf{D}_2$ we immediately see

$$\frac{\partial F_j}{\partial v_i} = \begin{cases} -(\mathbf{D}_1)_{jj}(\mathbf{D}_2)_{ii} & \text{if } j = i \\ (\mathbf{D}_1)_{jj}(\mathbf{D}_2)_{ii} & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we conclude for each $j \in \{1, \ldots, n-1\}$ that for fixed $t$ and fixed $v_j$ the function

$$h_j \colon v_{j+1} \mapsto F_j(t, v)$$

is strictly increasing with $h_j'(v_{j+1}) = (\mathbf{D}_1)_{jj}(\mathbf{D}_2)_{j+1j+1} \geq c_1 c_3 > 0$. In particular, there exists a unique zero $\bar{v}_{j+1}$ with $h_j(\bar{v}_{j+1}) = 0$. The implicit function theorem yields the continuously differentiable dependence of $\bar{v}_{j+1}$ on $v_j$ and $t$. In more detail, it is

$$\frac{\partial}{\partial v_j}\bar{v}_{j+1} = -\left(\frac{\partial F_j(t,v)}{\partial v_{j+1}}\right)^{-1} \frac{\partial F_j(t,v)}{\partial v_j} = \frac{(\mathbf{D}_1)_{jj}(\mathbf{D}_2)_{jj}}{(\mathbf{D}_1)_{jj}(\mathbf{D}_2)_{j+1,j+1}} \geq \frac{c_1}{c_2} > 0$$

and thus, $\bar{v}_{j+1}$ is strictly increasing. Since this is true for all $j \in \{1, \ldots, n-1\}$ and for all $t$, these considerations iteratively lead to the existence of a unique componentwise strictly increasing function $w_t \colon \mathbb{R} \to \mathbb{R}^{n-1}$ with

$$F(t, v_1, w_t(v_1)) = 0.$$

Explicitly, this function is given by

$$w_t(v_1) = \begin{pmatrix} \bar{v}_2(v_1) \\ \bar{v}_3(\bar{v}_2(v_1)) \\ \vdots \\ \bar{v}_n(\cdots(\bar{v}_2(v_1))\cdots) \end{pmatrix}.$$

For the derivative of the $j$-th component it holds

$$\frac{\partial}{\partial v_1}(w_t)_j = \frac{(\mathbf{D}_2)_{11}}{(\mathbf{D}_2)_{jj}} \leq \frac{c_2(t)}{c_1}. \tag{4.18}$$

In the second step of the proof, we will construct a uniform energy functional $V$. Therefore, we want to study terms of the form $\operatorname{sgn}(v_1)\mathfrak{f}_j$. If $\operatorname{sgn}(v_1) = \operatorname{sgn}(v_j)$ it is clear by assumption that

$$\operatorname{sgn}(v_1)\mathfrak{f}_j = \operatorname{sgn}(v_j)\mathfrak{f}_j \leq d_1 + d_2\|v\|_1$$

holds. Thus, let us consider the other case: First assume $v_1 \geq 0$ and $v_j \leq 0$. With the monotonicity of $w_t$, we find

$$0 \geq v_j = (w_t(v_1))_{j-1} \geq (w_t(0))_{j-1},$$

since $v$ solves $F(t, v) = 0$. Similarly, for $v_1 \leq 0$ and $v_j \geq 0$ we have

$$0 \leq v_j = (w_t(v_1))_{j-1} \leq (w_t(0))_{j-1}.$$

Hence, in both cases it is

$$|v_j| \leq |(w_t(0))_{j-1}|$$

and we can define the compact set

$$D_0 = \left\{ v \in \mathbb{R}^n \middle| |v_j| \leq \sup_{t\in[0,T]} |(w_t(0))_{j-1}| \right\}.$$

As $\mathfrak{f}$ is locally integrable uniformly bounded we find $m \in L^1([0,T],[0,\infty))$ with

$$\left\| \mathfrak{f}\Big(t, v, \rho(t), \nu(\mathbf{V} + \mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big) \right\|_1 \leq m(t)$$

for almost all $t \in [0,T]$ and all $(v, \rho, \gamma\rho) \in D_0 \times \mathcal{V} \times \mathcal{W}$. With formula (4.16), we observe

$$\mathbf{T}(t, v, \rho(t)) = \frac{1}{\mathbf{1}^T\mathbf{R}(t)^{-1}\mathbf{D}_2(t,v)^{-1}\mathbf{1}}\mathbf{D}_2(t,v)^{-1}\mathbf{1}\mathbf{1}^T\mathbf{R}(t)^{-1}.$$

In particular, it is

$$\mathbf{T}_{ij} = \frac{(\mathbf{D}_2)_{ii}^{-1}(\mathbf{R})_{jj}^{-1}}{\sum_{l=1}^n (\mathbf{D}_2)_{ll}^{-1}(\mathbf{R})_{ll}^{-1}}$$

and thus, it holds

$$0 \leq \mathbf{T}_{ii} \leq 1$$

and

$$0 \leq \mathbf{T}_{ij} \leq \frac{c_2(t)\bar{\rho}_{\max}(t)}{c_1\bar{\rho}_{\min}(t)n}.$$

Now, we are able to define the energy functional $V(v) = |v_1|$. Obviously, this function is globally Lipschitz continuous and piecewise $C^1$. Furthermore, it is radially unbounded on the set $A = \{v \in \mathbb{R}^n | \exists t \text{ with } F(t,v) = 0\}$: Let be $v \in A$ and $t \in [0,T]$ such that $F(t,v) = 0$ holds. As mentioned earlier, it must be $v = (v_1 \; w_t(v_1)^T)^T$ and thus, due to the mean value theorem (see e.g. [59]) there exists a value $\xi \in [0, v_1]$ with

$$w_t(v_1) = w_t(0) + \frac{\partial w_t(\xi)}{\partial v_1} v_1.$$

Thus, the formula (4.18) for the derivative of $w_t$ yields the estimate

$$\begin{aligned} \|v\|_1 &= |v_1| + \|w_t(v_1)\|_1 \\ &\leq |v_1| + \|w_t(0)\|_1 + \left\| \frac{\partial w_t(\xi)}{\partial v_1} \right\|_1 |v_1| \\ &\leq |v_1| \left(1 + \frac{c_2(t)}{c_1}\right) + \|w_t(0)\|_1 \\ &\leq \sup_{t\in[0,T]} \|w_t(0)\|_1 + V(v)\left(1 + \frac{\|c_2\|_{L^\infty([0,T])}}{c_1}\right) \end{aligned} \tag{4.19}$$

for almost all $t \in [0,T]$. We define the two constants $K_1 = \sup_{t\in[0,T]} \|w_t(0)\|_1$ and $K_2 = \left(1 + \frac{\|c_2\|_{L^\infty([0,T])}}{c_1}\right)$, where the first is well-defined since $w_t$ depends continuously on $t$. In particular, this shows the radial unboundedness of $V$ on the set $A$.

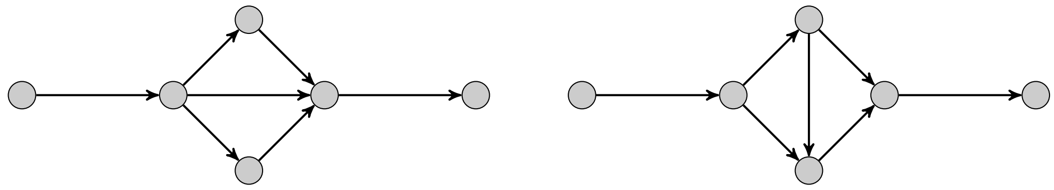
For the function $V$, we estimate

$$\begin{aligned} &\nabla V\Big(\mathbf{T}(t,v,\rho(t))\mathfrak{f}\big(t,v,\rho(t),\nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\big) - \mathbf{S}(t,v,\rho(t))F_t(t,v)\Big) \\ &= \operatorname{sgn}(v_1)\Big(\mathbf{T}\mathfrak{f} - \mathbf{S}F_t\Big)_1 \\ &= \sum_{\substack{j=1 \\ \operatorname{sgn}(v_j)=\operatorname{sgn}(v_1)}}^n \operatorname{sgn}(v_j)\mathbf{T}_{1j}\mathfrak{f}_j + \operatorname{sgn}(v_1)\sum_{\substack{j=1 \\ \operatorname{sgn}(v_j)\neq\operatorname{sgn}(v_1)}}^n \mathbf{T}_{1j}\mathfrak{f}_j - \operatorname{sgn}(v_1)\left(\mathbf{S}F_t\right)_1 \\ &\leq \sum_{\substack{j=1 \\ \operatorname{sgn}(v_j)=\operatorname{sgn}(v_1)}}^n \frac{c_2\bar{\rho}_{\max}}{c_1\bar{\rho}_{\min}n}\left(d_1 + d_2\|v\|_1\right) + \sum_{\substack{j=1 \\ \operatorname{sgn}(v_j)\neq\operatorname{sgn}(v_1)}}^n \frac{c_2\bar{\rho}_{\max}}{c_1\bar{\rho}_{\min}n}|\mathfrak{f}_j| + \|\mathbf{S}F_t\|_2 \\ &\leq \frac{c_2\bar{\rho}_{\max}}{c_1\bar{\rho}_{\min}}\Big(d_1 + d_2\|v\|_1 + m\Big) + c_5\Big(r_1 + r_2\|v\|_1\Big) \\ &= k_1(t) + k_2(t)\|v\|_1 \\ &\leq (k_1(t) + K_1k_2(t)) + K_2k_2(t)V(v) \end{aligned}$$

for almost all $t \in [0,T]$ and all $(v,\rho,\gamma\rho) \in \mathbb{R}^n \times \mathcal{V} \times \mathcal{W}$ with $F(t,v) = 0$. In the last lines we introduced the integrable functions $k_1 = \frac{c_2\bar{\rho}_{\max}}{c_1\bar{\rho}_{\min}}(d_1+m)+c_5r_1$ and $k_2 = \frac{c_2\bar{\rho}_{\max}}{c_1\bar{\rho}_{\min}}d_2+c_5r_2$. Applying Corollary 4.14 provides us with the global existence of the solution on $(0,T)\times\Omega$ and the following bound for the velocity

$$|v_1(t)| \leq \left(|(v_0)_1| + \int_0^t k_1 + K_1k_2 \mathrm{d}s\right) \exp\left(\int_0^t K_2k_2 \mathrm{d}s\right).$$

The estimate (4.19) yields also a bound for $\|v(t)\|_1$. □



(a) Combination is working (b) Combination is not working

Figure 4.3: The combination of the second and third approach to construct a uniform energy functional is working for the network on the left, but not for the network on right side

*Remark* 4.20. In this example, the image of the projector $\mathbf{T}$ is one-dimensional whereas in the previous example, dealing with networks with exactly one inner node, the kernel of the matrix was one-dimensional. Thus, both examples have in common that a one-dimensional structure simplifies the construction of the energy functional.

### 4.3.4 Further networks

Of course, the previous three examples are not the only ones for which it is possible to construct an energy functional. Especially, one can combine the approach for the networks with one inner node with the approach for the paths. With this technique, it is feasible to find a uniform energy functional $V$ e.g. for the graph in Figure 4.3a. We will not perform the construction since it is even more technical and reveals no new insights. In this case, one can express the right number of variables in dependence of the others in order to use the structure of the matrix $\mathbf{T}$ to construct an energy functional.

Unfortunately, until now, we did not succeed in classifying all graphs where this procedure works. For instance, for the network in Figure 4.3b, the approach does not work, since the number of variables which can be expressed in dependence of the others does not fit to the matrix $\mathbf{T}$.

# 5 Low Mach number equations on a network

To apply the developed theory, we return to the low Mach number equations (2.53) on a network as derived in Chapter 2. With the preliminary work of the previous chapters, we are able to analyse the question of existence of solutions of these equations. We will show that the results of Chapter 4 are applicable to this problem. Especially, the existence theorem (Theorem 4.12) will ensure the local in time existence of solutions. For the networks described in Section 4.3, the global existence is proven as well. These results generalize the existence results for the single edge proven by Gasser and Steinrück [48] to the network case.

The chapter is divided into two sections. In the first one, we justify the computations performed in Section 2.5 and answer the question concerning the regularity of the velocity. Furthermore, the boundary operator, which is involved in the continuity equation, is shown to fit into the framework of the previous chapters. In the second section, the existence of solutions is proven. Additionally, we characterize the dependence of the solutions on the initial conditions.

## 5.1 Justification of the formal computations

In Section 2.5, we reformulated the momentum equation of the low Mach number equations (2.53) to obtain the ordinary differential equation for the velocity (2.59). At that stage, we did not analyse if the regularity of the involved functions is sufficient for the computations. Now, we will rigorously prove the equivalence of both formulations. Therein, the simple form of the energy equation $u_x = \bar{q}$ will play an important role. For the computations in this section, we will mainly use the results of Chapter 3 about the transport equation.

During the whole chapter we assume

$$\begin{aligned} \bar{q} = \frac{1}{\gamma}\mathbf{P}^{-1}\mathbf{H}^{-1}\left(\hat{q}q - p_t\right) &\in C^{0,1}([0,T], L^\infty(\Omega)^n), \\ f &\in L^1((0,T)\times\Omega)^n, \\ \zeta &\in L^\infty((0,T)\times\Omega, \mathbb{R}^n_{\geq 0}) \\ \alpha_i &\in C^1(\mathbb{R}_{\geq 0}) \end{aligned} \tag{5.1}$$

and we concentrate on the case without heat loss at the walls, i.e. $\eta = 0$, and with constant cross-sectional areas, i.e. $A = \text{const}$. We recall the notation $R_f$ for the weighted mass

$$R_f = \int_\Omega \mathbf{F}\rho \mathrm{d}x$$

and $\mathbf{R}_f$ for the corresponding diagonal matrix, which was introduced in Section 2.5. In this chapter, the capital bold $\mathbf{Q} = \mathrm{diag}(Q)$ will denote the diagonal matrix corresponding to $Q$ although this is not in accordance to our convention.

For the moment, we do not take into account the initial and boundary values and prove the equivalence of the weak form of both formulations of the low Mach number equations:

**Theorem 5.1 (Equivalence).** *Let be $\rho_{\min} > 0$ and $\Delta p \in L^1((0,T))^n$. Then, the following two statements are true:*

1. *Let the triple $(\rho, u, \pi)$ with*

$$\begin{aligned} &\rho \in L^\infty((0,T)\times\Omega)^n \\ &u \in L^1((0,T), W^{1,1}(\Omega)^n) \cap L^\infty((0,T)\times\Omega)^n \\ &\pi \in L^1((0,T), W^{1,1}(\Omega)^n) \end{aligned}$$

*and $\rho \geq \rho_{\min}$ almost everywhere be a weak solution of*

$$\begin{aligned} \rho_t + (\mathbf{HU}\rho)_x &= 0 \\ (\mathbf{U}\rho)_t + \left(\mathbf{HU}^2\rho + \mathbf{H}\pi\right)_x &= -\boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{F}\rho \\ u_x &= \bar{q} \\ \pi(1) - \pi(0) &= \Delta p. \end{aligned} \tag{5.2}$$

*Then, the pair $(\rho, v) \in L^\infty((0,T)\times\Omega)^n \times W^{1,1}((0,T))^n$ with $v = u - Q$ and $Q = \int_0^x \bar{q}\mathrm{d}x$ solves*

$$\begin{aligned} \rho_t + (\mathbf{H}(\mathbf{V}+\mathbf{Q})\rho)_x &= 0 \\ \mathbf{R}v_t + \mathbf{R}_{\mathbf{H}\bar{q}}v + R_{Q_t + \mathbf{HQ}\bar{q}} + \mathbf{H}\Delta p &= R_{\hat{f}f - \boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(v+Q)} \end{aligned} \tag{5.3}$$

*weakly.*

2. *Let the pair $(\rho, v) \in L^\infty((0,T)\times\Omega)^n \times W^{1,1}((0,T))^n$ with $\rho \geq \rho_{\min}$ almost everywhere be a weak solution of (5.3). Then, the triple $(\rho, u, \pi)$ with*

$$u = v + Q$$

*and*

$$\pi = \bar{\pi} - \mathbf{H}^{-1}\int_0^x \left(\mathbf{U}_t + \mathbf{HUQ}_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U} - \hat{f}\mathbf{F}\right)\rho \mathrm{d}x \tag{5.4}$$

*weakly solves (5.2) for an arbitrary $\bar{\pi} \in L^1((0,T))$. From the definition, it follows*

$$u \in L^1((0,T), W^{1,1}(\Omega)^n) \cap L^\infty((0,T)\times\Omega)^n$$

*and*

$$\pi \in L^1((0,T), W^{1,1}(\Omega)^n).$$

*Remark* 5.2. If $\rho \in L^\infty((0,T)\times\Omega)$ is a solution of the continuity equation, it is continuous with values in $L^p(\Omega)$, i.e. $\rho \in C([0,T], L^p(\Omega))$ (see Theorem 3.3) such that all expressions in the velocity equation (5.3) are well-defined.

*Proof.* 1. We assume $(\rho, u, \pi)$ solves the equations (5.2) and want to show, that $(\rho, v)$ is a solution of (5.3). First, we will show that the regularity of $u$ is indeed higher than required, which allows us to use Lemma 3.17 to conclude that the pair $(\rho, v)$ solves (5.3).

We define $v = u - Q$. For almost all $t$, it is $v_x = u_x - \bar{q} = 0$ almost everywhere because of the energy equation. Thus, $v(t)$ is constant in space and it holds $v \in L^1((0,T))^n$. In fact, $v$ is even weakly differentiable. To show this, we observe that the momentum $m = \mathbf{U}\rho$ solves the transport equation

$$m_t + (\mathbf{H}\mathbf{U}m)_x = -\mathbf{H}\pi_x - \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{F}\rho, \tag{5.5}$$

where the right hand side is a $L^1((0,T)\times\Omega)$-function. From the trace theorem (Theorem 3.3) and Remark 3.8, we know that there exists a trace $\gamma m \in L^\infty(\Gamma_T, |\mathrm{d}\mu_{\mathbf{H}u}|)$ and that we are allowed to use also test functions $\varphi$ with non-compact support in the weak formulation of equation (5.5). In particular, for space-independent test-functions $\varphi \in C_c^1((0,T),\mathbb{R}^n)$ it is

$$0 = \int_0^T \int_\Omega m^T\varphi_t - \left(\mathbf{H}\pi_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho - \hat{f}\mathbf{F}\rho\right)^T \varphi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Gamma \gamma m^T(\nu\mathbf{U})\varphi \mathrm{d}\omega\mathrm{d}t.$$

Thus, we conclude that the mapping

$$t \mapsto R_U(t) = \int_\Omega \rho^T\mathbf{U}\mathrm{d}x$$

is weakly differentiable.

Due to the required regularity of $q$, it is $Q \in C^{0,1}([0,T]\times\overline{\Omega})$ and we can apply the trace theorem 3.3 to the continuity equation with test functions $\varphi = \mathbf{1}$ and $\varphi = Q$ as well. This yields

$$\int_\Omega \mathbf{1}^T\rho(t)\mathrm{d}x = \int_\Omega \mathbf{1}^T\rho(0)\mathrm{d}x - \int_0^t \int_\Gamma \gamma\rho^T(\nu u)\mathrm{d}\omega\mathrm{d}t$$

and

$$\begin{aligned}\int_\Omega Q(t)^T\rho(t)\mathrm{d}x = &\int_\Omega Q(0)^T\rho(0)\mathrm{d}x + \int_0^t \int_\Omega \rho^T\left(Q_t + \mathbf{U}Q_x\right)\mathrm{d}x\mathrm{d}t \\ &- \int_0^t \int_\Gamma \gamma\rho^T(\nu\mathbf{U})Q\mathrm{d}\omega\mathrm{d}t,\end{aligned}$$

which shows that the mappings $t \mapsto R(t) = \int_\Omega \rho(t)\mathrm{d}x$ and $t \mapsto R_Q(t) = \int_\Omega \mathbf{Q}(t)\rho(t)\mathrm{d}x$ are weakly differentiable. Due to the lower bound $\rho \geq \rho_{\min} > 0$, we can invert the matrix $\mathbf{R} = \operatorname{diag}(R_1,\ldots,R_n)$ and thus, we can express the velocity $v$ as

$$v = \mathbf{R}^{-1}\int_\Omega \mathbf{V}\rho\mathrm{d}x = \mathbf{R}^{-1}\left(R_U - R_Q\right).$$

Hence, it is $v \in W^{1,1}((0,T))$ and thus, it holds also $u \in W^{1,1}((0,T)\times\Omega)^n \cap C([0,T]\times\overline{\Omega})$. Therefore, we can apply Lemma 3.17 to the momentum $m = \mathbf{U}\rho$. Thus, it is

$$m_t + (\mathbf{H}\mathbf{U}m)_x = (\mathbf{U}_t + \mathbf{H}\mathbf{U}\mathbf{U}_x)\,\rho \tag{5.6}$$

and it holds $\gamma m = \mathbf{U}\gamma\rho$ for the trace. For any space-independent test function $\varphi \in C_c^1([0,T],\mathbb{R}^n)$, we find

$$\begin{aligned}
0 &= \int_0^T \int_\Omega m^T\varphi_t - \left(\mathbf{H}\pi_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho - \hat{f}\mathbf{F}\rho\right)^T \varphi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Gamma \gamma m^T(\nu\mathbf{U})\varphi \mathrm{d}\omega\mathrm{d}t \\
&= \int_0^T \int_\Omega (-\rho)^T (\mathbf{U}_t + \mathbf{H}\mathbf{U}\mathbf{U}_x)\,\varphi - \left(\mathbf{H}\pi_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho - \hat{f}\mathbf{F}\rho\right)^T \varphi \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \varphi^T \left(\int_\Omega \mathbf{U}_t\rho + \mathbf{H}\mathbf{U}\mathbf{U}_x\rho + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho - \hat{f}\mathbf{F}\rho\mathrm{d}x + \mathbf{H}\Delta p\right)\mathrm{d}t,
\end{aligned}$$

where we used the two equations (5.5) and (5.6) for the momentum. Since $\varphi$ is arbitrary, the pair $(\rho, v)$ solves (5.3).

2. Now, we assume $(\rho, v)$ to be a solution of (5.3) and we have to show that $(\rho, u, \pi)$, as defined in the theorem, solves the equations (5.2).

By construction, the velocity $u$ and the pressure $\pi$ both lie in the claimed spaces, i.e. $u \in L^1((0,T), W^{1,1}(\Omega)^n) \cap L^\infty((0,T)\times\Omega)^n$ and $\pi \in L^1((0,T), W^{1,1}(\Omega)^n)$, respectively. The pressure fulfils

$$\pi_x = -\mathbf{H}^{-1}\left(\mathbf{U}_t + \mathbf{H}\mathbf{U}\mathbf{U}_x - \hat{f}\mathbf{F} + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\right)\rho$$

by definition. To check that the moment equation is fulfilled, let $\varphi \in C^{0,1}([0,T]\times\overline{\Omega},\mathbb{R}^n)$ be an arbitrary test function with $\varphi(0,\cdot) = \varphi(T,\cdot) = 0$ and $\varphi = 0$ on $\Gamma_T$. We would like to use $\bar{\varphi} = \mathbf{U}\varphi = (\mathbf{V}+\mathbf{Q})\varphi$ as a test function for the continuity equation, but since $v$ is not necessarily smooth enough, we need to approximate it by a sequence of smooth functions $v_k \in C^1((0,T),\mathbb{R}^n)$ with $v_k \to v$ in $W^{1,1}((0,T))^n$. Then, we define $u_k = v_k + Q$ and conclude with the definition of $\pi$

$$\begin{aligned}
&\int_0^T \int_\Omega m^T(\varphi_t + \mathbf{H}\mathbf{U}\varphi_x - \boldsymbol{\zeta}\alpha(\mathbf{U})) - \left(\pi_x^T\mathbf{H} - \hat{f}\rho^T\mathbf{F}\right)\varphi\mathrm{d}x\mathrm{d}t \\
&\leftarrow \int_0^T \int_\Omega \rho^T\mathbf{U}_k(\varphi_t + \mathbf{H}\mathbf{U}\varphi_x) - \left(\rho^T\boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U} + \pi_x^T\mathbf{H} - \hat{f}\rho^T\mathbf{F}\right)\varphi\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \rho^T\left((\mathbf{U}_k\varphi)_t + \mathbf{H}\mathbf{U}(\mathbf{U}_k\varphi)_x\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \rho^T\left((\mathbf{U}_k)_t + \mathbf{H}\mathbf{U}(\mathbf{U}_k)_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U} - \hat{f}\mathbf{F}\right)\varphi + \pi_x^T\mathbf{H}\varphi\mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_\Omega \rho^T(\mathbf{U}_k - \mathbf{U})_t\varphi\mathrm{d}x\mathrm{d}t \\
&\to 0
\end{aligned}$$

since it is $u, \rho \in L^\infty((0,T)\times\Omega)^n$ and since $\mathbf{U}_k\varphi$ is a smooth test function for the continuity equation. Thus, $m = \mathbf{U}\rho$ is a weak solution of the momentum equation and it remains to check the pressure boundary conditions. From the definition of $\pi$, using the velocity equation (5.3), we conclude

$$\begin{aligned}\mathbf{H}\left(\pi(1)-\pi(0)\right) &= -\int_0^1 \left(\mathbf{U}_t + \mathbf{H}\mathbf{U}\mathbf{Q}_x + \boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U} - \hat{f}\mathbf{F}\right)\rho \mathrm{d}x \\ &= -\mathbf{R}v_t - \mathbf{R}_{\mathbf{H}q}v + R_{Q_t+\mathbf{H}\mathbf{Q}q} + R_{\hat{f}f-\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(v+Q)} \\ &= \mathbf{H}\Delta p\end{aligned}$$

for almost all $t \in [0, T]$. $\square$

Until now, we have shown the validity of the reformulation of the low Mach number equations as an ODE coupled with the continuity equation. Beside this, in Section 2.5 we analysed the coupling conditions to define inflow boundary values for the density. The therfor necessary computations are now justified since the velocity $u$ is continuous. These computations define naturally a boundary operator $\mathcal{H}_u$, which will be the object of the rest of this section. More precisely, we want to check whether the coupling conditions (2.62) and (2.63) define an affine linear boundary operator $\mathcal{H}_u$ as introduced in Section 3.1 for a given velocity $u$ in order to apply the theory of Chapter 4. Therefore, let $G = (\mathfrak{V}, E, w, \mathrm{init}, \mathrm{ter})$ be an oriented, weighted and connected graph and let $\mathbf{B}$ denote its incidence matrix with corresponding submatrices $\mathbf{B}_{>1}$ and $\mathbf{B}_{=1}$. Furthermore, let $u \in L^1((0,T), W^{1,1}(\Omega))$ with $u_x \in L^1((0,T), L^\infty(\Omega)^n)$ fulfil the coupling condition for the energy conservation $\mathbf{B}^+_{>1}\mathbf{A}u(0) = \mathbf{B}^-_{>1}\mathbf{A}u(1)$. As in Section 2.5, we define the matrix $\mathbf{M} \in L^1((0,T))^{k\times k}$ by

$$\begin{aligned}\mathbf{M}(t) &= \mathbf{B}^-_{>1}\mathbf{A}\mathbf{U}(t,1)^-\left(\mathbf{B}^-_{>1}\right)^T + \mathbf{B}^+_{>1}\mathbf{A}\mathbf{U}(t,0)^+\left(\mathbf{B}^+_{>1}\right)^T \\ &= \int_\Gamma \left(\nu\mathbf{B}_{>1}\right)^- \mathbf{A}\left(\nu\mathbf{U}\right)^-\left(\nu\mathbf{B}^T_{>1}\right)^- \mathrm{d}\omega.\end{aligned}$$

Then, for $\rho_{\mathrm{out}} \in L^\infty((0,T))^{m-k}$ with $\rho_{\mathrm{out}} \geq \rho_{\min} > 0$ almost everywhere, we define the boundary operator

$$\begin{aligned}\mathcal{H}_u \colon L^\infty(\Gamma_T, \mathrm{d}\mu_u^+) &\to L^\infty(\Gamma_T, \mathrm{d}\mu_u^-) \\ \rho &\mapsto \rho_{\mathrm{in}} + \mathcal{G}_u(\rho).\end{aligned}$$

Here, corresponding to the equations (2.62) and (2.63), it is

$$\rho_{\mathrm{in}}(\cdot,\omega) = \begin{cases}\left(\mathbf{B}^+_{=1}\right)^T \rho_{\mathrm{out}} & \text{for } \omega = 0\\ \left(\mathbf{B}^-_{=1}\right)^T \rho_{\mathrm{out}} & \text{for } \omega = 1\end{cases}$$

and

$$\mathcal{G}_u \colon L^\infty(\Gamma_T, \mathrm{d}\mu_u^+) \to L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$$

with

$$\mathcal{G}_u(\rho)(\cdot,\omega) = \begin{cases}\left(\mathbf{B}^+_{>1}\right)^T \mathbf{M}^{-1}\left(\mathbf{B}^-_{>1}\mathbf{A}\mathbf{U}(\cdot,1)^+\rho(\cdot,1) + \mathbf{B}^+_{>1}\mathbf{A}\mathbf{U}(\cdot,0)^-\rho(\cdot,0)\right) & \text{for } \omega = 0\\ \left(\mathbf{B}^-_{>1}\right)^T \mathbf{M}^{-1}\left(\mathbf{B}^-_{>1}\mathbf{A}\mathbf{U}(\cdot,1)^+\rho(\cdot,1) + \mathbf{B}^+_{>1}\mathbf{A}\mathbf{U}(\cdot,0)^-\rho(\cdot,0)\right) & \text{for } \omega = 1.\end{cases}$$

In a closed form, we can also write

$$\mathcal{G}_u(\rho) = \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{M}^{-1} \int_\Gamma (\nu \mathbf{B}_{>1})^- \mathbf{A} (\nu \mathbf{U})^+ \rho \mathrm{d}\omega.$$

Before we check that $\mathcal{H}_u$ fulfils the requirements of Section 3.1, we will show the well-definedness of $\mathcal{G}_u$ and we will introduce another operator $\bar{\mathcal{G}}_u$, to which $\mathcal{G}_u$ is the adjoint operator.

In Section 2.5, it was pointed out that $\mathbf{M} = (m_{ij})$ is a diagonal matrix with

$$m_{ij} = \delta_{ij} \int_\Gamma \left( (\nu \mathbf{B}_{>1})^- \mathbf{A} (\nu u)^+ \right)_i \mathrm{d}\omega.$$

Thus, for all $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ and for almost all $t \in [0, T]$ it is either $m_{jj}(t) = 0$ or

$$m_{jj}(t)^{-1} \int_\Gamma \left( (\nu \mathbf{B}_{>1})^- \mathbf{A} (\nu \mathbf{U}(t))^+ \rho(t,) \right)_j \mathrm{d}\omega \leq \max_{j,\omega}(\rho_j(t,\omega)). \tag{5.7}$$

In the case $m_{jj}(t) = 0$, we also know from Section 2.5 that it holds $u_i(t, 0) = 0$ for all $e_i \in E^+(\mathfrak{v}_j)$ and $u_i(t, 1) = 0$ for all $e_i \in E^-(\mathfrak{v}_j)$.

Altogether, this proves

$$|(\mathcal{G}_u)_i (\rho)| \leq \max_{j,\omega}(\rho_j(t,\omega))$$

for almost all $(t, \omega) \in \Gamma_T$ with $\nu u_i(t, \omega) < 0$. Hence, it is $\mathcal{G}_u(\rho) \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^-)$ and the linear operator $\mathcal{G}_u$ is well-defined. Descriptively, this is not surprising since the coupling condition chooses a weighted mean value of the inflow densities for each $t$. Of course, the weighted mean value is smaller than the maximum.

As a next step, we define the operator $\bar{\mathcal{G}}_u$ as

$$\begin{aligned} \bar{\mathcal{G}}_u \colon L^1(\Gamma_T, \mathrm{d}\mu_u^-) &\to L^1(\Gamma_T, \mathrm{d}\mu_u^+) \\ \bar{\rho} &\mapsto \mathbf{A} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{M}^{-1} \int_\Gamma (\nu \mathbf{B}_{>1})^- (\nu \mathbf{U})^- \bar{\rho} \mathrm{d}\omega. \end{aligned}$$

This operator is well-defined by the same argument since we can also write

$$m_{ij} = \delta_{ij} \int_\Gamma \left( (\nu \mathbf{B}_{>1})^- \mathbf{A} (\nu u)^- \right)_i \mathrm{d}\omega$$

due to the coupling condition for the energy. We observe that $\mathcal{G}_u$ is the adjoint operator of $\bar{\mathcal{G}}_u$. To check this, let be $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$ and $\bar{\rho} \in L^1(\Gamma_T, \mathrm{d}\mu_u^-)$. Then, we compute

$$\begin{aligned} \int_{\Gamma_T} \bar{\rho}^T (\nu \mathbf{U})^- \mathcal{G}_u(\rho) \mathrm{d}\omega \mathrm{d}t &= \int_0^T \int_\Gamma \bar{\rho}^T (\nu \mathbf{U})^- \left(\nu \mathbf{B}_{>1}^T\right)^- \mathrm{d}\omega \mathbf{M}^{-1} \int_\Gamma (\nu \mathbf{B}_{>1})^- \mathbf{A} (\nu \mathbf{U})^+ \rho \mathrm{d}\omega \\ &= \int_{\Gamma_T} \bar{\mathcal{G}}_u(\bar{\rho})^T (\nu \mathbf{U})^+ \rho \mathrm{d}\omega \mathrm{d}t. \end{aligned}$$

Now, we will check all requirements for the boundary operator assumed in Section 3.1.

**Lemma 5.3 (Boundary operator).** *The operator $\mathcal{H}_u$ fulfils all requirements of Section 3.1. In addition, for each $\rho_0 \in L^\infty(\Omega)^n$ with $\rho_0 \geq \rho_{\min} > 0$ almost everywhere, there is a vector $\rho_{\min} \in \mathbb{R}^n_{>0}$ satisfying inequality (3.16), which guarantees the existence of a lower bound of the solution.*

*Proof.* We will verify each of the assumptions of Section 3.1. Therefore, let be $\rho \in L^\infty(\Gamma_T, \mathrm{d}\mu_u^+)$.

**Weak-$\star$ continuity:** Since $\mathcal{G}_u$ is the adjoint of another operator, namely $\bar{\mathcal{G}}_u$, it is weak-$\star$ continuous.

**$L^1$-operator norm:** As weight matrix for the $L^p(\Gamma_T, \mathrm{d}\mu_u^\pm)$ norm we choose the matrix containing the cross-sectional areas $\overline{\mathbf{W}} = \mathbf{A}$. Then, using the definition of $\mathbf{M}$, we estimate

$$
\begin{aligned}
\|\mathcal{G}_u(\rho)\|_{1,w,-} &= \int_{\Gamma_T} |\mathcal{G}_u(\rho)|^T \mathbf{A}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \left| \int_\Gamma \rho^T (\nu \mathbf{U})^+ \mathbf{A} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathrm{d}\omega \right| M^{-1} \int_\Gamma (\nu \mathbf{B}_{>1})^- \mathbf{A}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \left| \int_\Gamma \rho^T (\nu \mathbf{U})^+ \mathbf{A} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathrm{d}\omega \right| \mathbf{1} \mathrm{d}t \\
&\leq \|\rho\|_{1,w,+}.
\end{aligned}
$$

Thus, the $L^1$-operator norm of $\mathcal{G}_u$ is less or equal one.

**Independence on future times:** Equation (3.4) is fulfilled for almost all $t \in [0,T]$ since it holds

$$
\mathcal{G}_u(\chi_{[0,t]}\rho) = \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{M}^{-1} \int_\Gamma (\nu \mathbf{B}_{>1})^- \mathbf{A} \, (\nu \mathbf{U})^+ \chi_{[0,t]} \rho \mathrm{d}\omega = \chi_{[0,t]} \mathcal{G}_u(\rho)
$$

by definition of $\mathcal{G}_u$.

**Positivity:** Let be $\rho \geq 0$ almost everywhere. Then, $\mathcal{G}_u(\rho)$ is positive as sum of products of positive factors.

**Boundedness:** For any $\rho_0 \in L^\infty(\Omega)^n$ define the scalar

$$
\rho_{\max} = \max\left( \|\rho_{\mathrm{out}} r\|_{L^\infty(\Gamma_T)}, \|\rho_0\|_{L^\infty(\Omega)^n} \right)
$$

with $r = \exp(t) \left( -\int_0^t \|(u_x(s,\cdot))^-\|_{L^\infty(\Omega)^n} \mathrm{d}s \right)$. Observe that the reaction term $c$ and the source term $f$ are not present in the continuity equation. Due to the definition of the matrix $\mathbf{M}$, it is

$$
\mathcal{G}_u(\rho_{\max} \mathbf{1}_n) = \rho_{\max} \mathcal{G}_u(\mathbf{1}_n) = \rho_{\max} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{M}^{-1} \mathbf{M} \mathbf{1}_k = \rho_{\max} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{1}_k
$$

and thus, it follows also

$$
\begin{aligned}
\rho_{\mathrm{in}} r + \mathcal{G}_u(\rho_{\max} \mathbf{1}_n) &= \left(\nu \mathbf{B}_{=1}^T\right)^- \rho_{\mathrm{out}} r + \mathcal{G}_u(\rho_{\max} \mathbf{1}_n) \\
&\leq \rho_{\max} \left(\nu \mathbf{B}_{=1}^T\right)^- \mathbf{1}_{m-k} + \rho_{\max} \left(\nu \mathbf{B}_{>1}^T\right)^- \mathbf{1}_k \\
&= \rho_{\max} (\nu \mathbf{B}^T)^- \mathbf{1}_m \\
&= \rho_{\max} \mathbf{1}_n
\end{aligned}
$$

$\mathrm{d}\mu_u^-$-almost everywhere. Here, one has to be careful with the different dimensions of the $\mathbf{1}$-vectors. In the last step, we used that $(\nu\mathbf{B})^-$ has in each column exactly one non-vanishing entry.

Now, we additionally assume $\rho_0 \geq \rho_{\min} > 0$ almost everywhere and denote

$$\bar{r}(t) = \exp\left(\int_0^t \|(u_x(s,\cdot))^+\|\mathrm{d}s\right).$$

Then, we define

$$\rho_{\min} = \min\{\operatorname*{ess\,inf}_{\Omega} \rho_0, \operatorname*{ess\,inf}_{\Gamma_T} (\bar{r}\rho_{\text{out}})\} \geq \rho_{\min} > 0$$

and thus, it holds

$$\begin{aligned}
\rho_{\text{in}}\bar{r} + \mathcal{G}_u(\rho_{\min}\mathbf{1}_n) &= \left(\nu\mathbf{B}_{=1}^T\right)^- \rho_{\text{out}}\bar{r} + \rho_{\min}\mathcal{G}_u(\mathbf{1}_n) \\
&\geq \rho_{\min}\left(\nu\mathbf{B}_{=1}^T\right)^- \mathbf{1}_{m-k} + \rho_{\min}\left(\nu\mathbf{B}_{>1}^T\right)^- \mathbf{1}_k \\
&= \rho_{\min}(\nu\mathbf{B}^T)^-\mathbf{1}_m \\
&= \rho_{\min}\mathbf{1}_n
\end{aligned}$$

$\mathrm{d}\mu_u^-$-almost everywhere. This completes the proof. □

Now, to apply the existence theory of Chapter 4, we additionally have to check the two requirements on the boundary operator, which were demanded in (4.9) and (4.10). These conditions were necessary for the proof of local existence in Section 4.2.

**Lemma 5.4 (Dependence of $\mathcal{G}_u$ on $u$).** *For each uniform convergent sequence $(w_j)_j \subset C([0,T],\mathbb{R}^n)$ with $w_j \to w$ and each weak-⋆ convergent sequence $(q_j)_j \subset L^\infty(\Gamma_T)^n$ with $q_j \overset{\star}{\rightharpoonup} q$, it holds*

$$(\nu(\mathbf{W}_j + \mathbf{Q}))^-\mathcal{G}_{w_j+Q}(q_j) \rightharpoonup (\nu(\mathbf{W}+\mathbf{Q}))^-\mathcal{G}_{w+Q}(q) \tag{5.8}$$

*in $L^1(\Gamma_T)^n$ and*

$$\begin{aligned}
&\int_0^T \int_\Gamma \beta(q)^T\mathbf{A}(\nu u)^+ - \beta(\bar{\mathcal{H}}_{w+Q}(q))^T\mathbf{A}(\nu u)^-\mathrm{d}\omega\mathrm{d}t \\
&\quad\leq \liminf_j \int_0^T \int_\Gamma \beta(q_j)^T\mathbf{A}(\nu u_j)^+ - \beta(\bar{\mathcal{H}}_{w_j+Q}(q_j))^T\mathbf{A}(\nu u_j)^-\mathrm{d}\omega\mathrm{d}t
\end{aligned} \tag{5.9}$$

*for $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\beta_j(s) = s_j^2$ and $\bar{\mathcal{H}}_u(\rho) = \exp\left(\int_0^t \|(Q_x(s))^-\|_{L^\infty(\Omega)}\mathrm{d}s\right)\rho_{\text{in}} + \mathcal{G}_u(\rho)$.*

*Proof.* We introduce the abbreviations $u = w + Q$, $u_j = w_j + Q$, $\mathcal{G}_j = \mathcal{G}_{w_j+Q}$ and $\bar{\mathcal{G}}_j = \bar{\mathcal{G}}_{w_j+Q}$. During this proof, we will use superscripts to denote the components of a vector. We start with proving the weak convergence (5.8). To do so, we will first show the strong convergence

$$(\nu\mathbf{U}_j)^+\bar{\mathcal{G}}_j(\bar{\rho}) \to (\nu\mathbf{U})^+\bar{\mathcal{G}}_u(\bar{\rho}) \tag{5.10}$$

in $L^1(\Gamma_T)^n$ for an arbitrary $\bar{\rho} \in L^\infty(\Gamma_T)^n$ before we use the adjoint operator of $\bar{\mathcal{G}}_u$ to conclude the convergence (5.8). Recalling the definition of $\bar{\mathcal{G}}_j$, the convergence (5.10) is equivalent to the convergence

$$\begin{aligned}&(\nu\mathbf{U}_j)^+\mathbf{A}\left(\nu\mathbf{B}_{>1}^T\right)^-\mathbf{M}_j^{-1}\int_\Gamma(\nu\mathbf{B}_{>1})^-(\nu\mathbf{U}_j)^-\bar{\rho}\mathrm{d}\omega\\ &\to(\nu\mathbf{U})^+\mathbf{A}\left(\nu\mathbf{B}_{>1}^T\right)^-\mathbf{M}^{-1}\int_\Gamma(\nu\mathbf{B}_{>1})^-(\nu\mathbf{U})^-\bar{\rho}\mathrm{d}\omega.\end{aligned} \tag{5.11}$$

To show this convergence, we will use the dominated convergence theorem. Since $w_j$ is uniformly convergent with limit $w$, it is

$$\int_\Gamma(\nu\mathbf{B}_{>1})^-(\nu\mathbf{U}_j)^-\bar{\rho}\mathrm{d}\omega\to\int_\Gamma(\nu\mathbf{B}_{>1})^-(\nu\mathbf{U})^-\bar{\rho}\mathrm{d}\omega \tag{5.12}$$

for almost all $t \in [0, T]$. Furthermore, as mentioned earlier, for almost all $t$ with $m_{ll}(t) = 0$ we have $u^i(t, 0) = 0$ for all $e_i \in E^+(\mathfrak{v}_l)$ and $u^i(t, 1) = 0$ for all $e_i \in E^-(\mathfrak{v}_l)$ and, in particular, it is

$$\int_\Gamma\left((\nu\mathbf{B}_{>1})^-(\nu\mathbf{U})^-\bar{\rho}\right)_l\mathrm{d}\omega=0.$$

Thus, due to the entrywise inequality

$$\left(\mathbf{M}_j^{-1}(\nu\mathbf{B}_{>1})^-\mathbf{A}(\nu\mathbf{U}_j)^+\right)_{i_1i_2}\le 1, \tag{5.13}$$

we conclude the pointwise convergence

$$(\nu\mathbf{U}_j)^+\bar{\mathcal{G}}_j(\bar{\rho})^i(t,\omega)\to 0=(\nu\mathbf{U})^+\bar{\mathcal{G}}_u(\bar{\rho})^i(t,\omega)$$

for almost all $t$ with $m_{ll}(t) = 0$ and all $(i, \omega) \in E^+(\mathfrak{v}_l) \times \{0\} \cup E^-(\mathfrak{v}_l) \times \{1\}$. If it is $m_{ll}(t) \neq 0$, then it holds

$$\left(\mathbf{M}_j^{-1}(\nu\mathbf{B}_{>1})^-\mathbf{A}(\nu\mathbf{U}_j)^+\right)_{li}\to\left(\mathbf{M}^{-1}(\nu\mathbf{B}_{>1})^-\mathbf{A}(\nu\mathbf{U})^+\right)_{li}$$

for all $i$ and almost all $t$ since these are quotients with non-vanishing denominators. Hence, together with the convergence (5.12), we have proven the pointwise convergence (5.11) for almost all $(t, \omega) \in \Gamma_T$. The dominated convergence theorem yields the stated convergence in $L^1(\Gamma_T)^n$ since the sequence $(\nu\mathbf{U}_j)^+\bar{\mathcal{G}}_j(\bar{\rho})$ is bounded. To see this, we use the uniform convergence of $w_j$ to obtain a constant $C$ with $\|w_j\|_{C([0,T])} \le C$. Thus, with the entrywise inequality (5.13), we find the integrable bound

$$\|(\nu\mathbf{U}_j)^+\bar{\mathcal{G}}_j(\bar{\rho})\|_1\le n\left\|\int_\Gamma(\nu\mathbf{B}_{>1})^-(C+(\nu Q)^-)\bar{\rho}\mathrm{d}\omega\right\|_1$$

and the dominated convergence theorem is applicable. In particular, the convergence (5.10) is true.

Now we will turn to the weak convergence (5.8) which is stated in the lemma. Therefore, let be $\bar{\rho} \in L^\infty(\Gamma_T)^n$. Then, with the previous considerations, we compute

$$\begin{aligned}
\int_{\Gamma_T} \bar{\rho}^T (\nu \mathbf{U}_j)^- \mathcal{G}_j(q_j) \mathrm{d}\omega \mathrm{d}t &= \int_{\Gamma_T} \bar{\mathcal{G}}_j(\bar{\rho})^T (\nu \mathbf{U}_j)^+ q_j \mathrm{d}\omega \mathrm{d}t \\
&\to \int_{\Gamma_T} \bar{\mathcal{G}}(\bar{\rho})^T (\nu \mathbf{U})^+ q \mathrm{d}\omega \mathrm{d}t \\
&= \int_{\Gamma_T} \bar{\rho}^T (\nu \mathbf{U})^- \mathcal{G}(q) \mathrm{d}\omega \mathrm{d}t
\end{aligned}$$

since the product of a weak-$\star$ convergent and a strong convergent sequence is weak-$\star$ convergent.

To prove the second statement of the lemma, namely the inequality (5.9), we will use the weak lower semi-continuity of the $L^2$-norm. More precisely, we will rearrange the integrals in (5.9) in such a way that they are $L^2([0,T])$-norms of a sequence of vector-valued functions. Since this sequence will converge weakly, the weak lower semi-continuity will complete the proof.

We begin with the transform of the involved terms in the inequality (5.9) into a more suitable form. We recall from equation (2.61) that it holds

$$(\nu \mathbf{B}_{=1})^- \mathbf{A}(\nu \mathbf{U})^- \left(\nu \mathbf{B}_{>1}^T\right)^- = 0$$

and we define the function $r(t) = \exp\left(-\frac{1}{2}\int_0^t \|Q_x(s)\|_{L^\infty(\Omega)} \mathrm{d}s\right)$. Then, due to the definition of $\mathbf{M}$ and due to the property $\mathbf{M}^{-1}\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}$ of the Moore-Penrose pseudoinverse, it holds

$$\begin{aligned}
&\int_0^T \int_\Gamma \beta(q)^T \mathbf{A}(\nu u)^+ - \beta(\bar{\mathcal{H}}_u)(q)^T \mathbf{A}(\nu u)^- \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \int_\Gamma q^T \mathbf{A}(\nu \mathbf{U})^+ q - \Big(r\rho_{\mathrm{in}} + \mathcal{G}_u(q)\Big)^T \mathbf{A}(\nu \mathbf{U})^- \Big(r\rho_{\mathrm{in}} + \mathcal{G}_u(q)\Big) \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \int_\Gamma q^T \mathbf{A}(\nu \mathbf{U})^+ q - r^2 \rho_{\mathrm{in}}^T \Big(\nu \mathbf{B}_{=1}\Big)^- \mathbf{A}(\nu \mathbf{U})^- \Big(\nu \mathbf{B}_{=1}^T\Big)^- \rho_{\mathrm{in}} \\
&\quad - \int_\Gamma q^T (\nu_1 \mathbf{U})^+ \mathbf{A}\Big(\nu_1 \mathbf{B}_{>1}^T\Big)^- \mathrm{d}\omega_1 \mathbf{M}^{-1} \Big(\nu \mathbf{B}_{>1}\Big)^- \mathbf{A}(\nu \mathbf{U})^- \Big(\nu \mathbf{B}_{>1}^T\Big)^- \mathbf{M}^{-1} \\
&\qquad \int_\Gamma \Big(\nu_2 \mathbf{B}_{>1}\Big)^- \mathbf{A}\left(\nu_2 \mathbf{U}\right)^+ q \mathrm{d}\omega_2 \mathrm{d}\omega \mathrm{d}t \\
&= \int_0^T \underbrace{\int_\Gamma q^T \mathbf{A}(\nu \mathbf{U})^+ q \mathrm{d}\omega}_{S^1} - \underbrace{\int_\Gamma r^2 \rho_{\mathrm{in}}^T \Big(\nu \mathbf{B}_{=1}\Big)^- \mathbf{A}(\nu \mathbf{U})^- \Big(\nu \mathbf{B}_{=1}^T\Big)^- \rho_{\mathrm{in}} \mathrm{d}\omega}_{S^2} \\
&\quad - \underbrace{\int_\Gamma q^T (\nu_1 \mathbf{U})^+ \mathbf{A}\Big(\nu_1 \mathbf{B}_{>1}^T\Big)^- \mathrm{d}\omega_1 \mathbf{M}^{-1} \int_\Gamma \Big(\nu_2 \mathbf{B}_{>1}\Big)^- \mathbf{A}(\nu_2 \mathbf{U})^+ q \mathrm{d}\omega_2}_{S^3} \mathrm{d}t \\
&= \int_0^T S^1 - S^2 - S^3 \mathrm{d}t.
\end{aligned}$$

Thus, the desired inequality (5.9) is equivalent to

$$\int_0^T S^1 - S^2 - S^3 \mathrm{d}t \leq \liminf_j \int_0^T S_j^1 - S_j^2 - S_j^3 \mathrm{d}t,$$

where $S_j^i$ is defined analogously to $S^i$. In the following computations, we will use a componentwise notation in order to avoid confusing matrix products. For a node $\mathfrak{v}_l$ and an edge $e_i \in E(\mathfrak{v}_l)$, we denote

$$\omega_{li} = \begin{cases} 0 & \text{if } \operatorname{init}(e_i) = \mathfrak{v}_l \\ 1 & \text{if } \operatorname{ter}(e_i) = \mathfrak{v}_l \end{cases}$$

and

$$\nu_{li} = (-1)^{\omega_{li}+1}.$$

Taking a closer look at the integrals, we see that $\Gamma$ is a finite set and thus the integrals define a weighted sum over the edges summing up $q$, evaluated at the ends of the edges. Now, we can change the order of summation which results in summing over the nodes and their adjacent ends of the edges instead of summing over the edges. These considerations lead to

$$\begin{aligned} S^1 &= \int_\Gamma q^T \mathbf{A}(\nu \mathbf{U})^+ q \mathrm{d}\omega \\ &= \sum_{e_i \in E} q^i(0)^2 A^i(u^i(0))^- + q^i(1)^2 A^i(u^i(1))^+ \\ &= \sum_{\mathfrak{v}_l \in \mathfrak{V}} \sum_{e_i \in E(\mathfrak{v}_l)} \Big(q^i(\omega_{li})\Big)^2 A^i(\nu_{li} u^i)^+ \\ &= \sum_{\substack{\mathfrak{v}_l \in \mathfrak{V} \\ d(\mathfrak{v}_l)=1}} \sum_{e_i \in E(\mathfrak{v}_l)} \Big(q^i(\omega_{li})\Big)^2 A^i(\nu_{li} u^i)^+ \\ &\quad + \sum_{\substack{\mathfrak{v}_l \in \mathfrak{V} \\ d(\mathfrak{v}_l)>1}} m_{ll}^{-1} \left( \sum_{e_s \in E(\mathfrak{v}_l)} A^s(\nu_{ls} u^s)^+ \right) \left( \sum_{e_i \in E(\mathfrak{v}_l)} \Big(q^i(\omega_{li})\Big)^2 A^i(\nu_{li} u^i)^+ \right). \end{aligned}$$

Here, the inverse of $m_{ll}$ has again to be understood in the sense of the Moore-Penrose pseudoinverse, i.e. it is either 0 or $m_{ll}^{-1}(t)$. Furthermore, using the definition of the incidence matrix, we compute

$$\begin{aligned} S^3 &= \int_\Gamma q^T (\nu_1 \mathbf{U})^+ \mathbf{A} \Big(\nu_1 \mathbf{B}_{>1}^T\Big)^- \mathrm{d}\omega_1 \mathbf{M}^{-1} \int_\Gamma \Big(\nu_2 \mathbf{B}_{>1}\Big)^- \mathbf{A}(\nu_2 \mathbf{U})^+ q \mathrm{d}\omega_2 \\ &= \sum_{\substack{\mathfrak{v}_l \in \mathfrak{V} \\ d(\mathfrak{v}_l)>1}} m_{ll}^{-1} \left( \sum_{e_i \in E(\mathfrak{v}_l)} q^i(\omega_{il}) A^i(\nu_{il} u^i)^+ \right) \left( \sum_{e_s \in E(\mathfrak{v}_l)} q^s(\omega_{sl}) A^s(\nu_{sl} u^i)^+ \right). \end{aligned}$$

Thus, the difference of $S^1$ and $S^3$ is

$$\begin{aligned} S^1 - S^3 = &\frac{1}{2} \sum_{\substack{\mathfrak{v}_l \in \mathfrak{V} \\ d(\mathfrak{v}_l)>1}} \sum_{e_i \in E(\mathfrak{v}_l)} \sum_{e_s \in E(\mathfrak{v}_l)} \underbrace{m_{ll}^{-1} A^i (\nu_{li} u^i)^+ A^s (\nu_{ls} u^s)^+ \left( q^i(\omega_{il}) - q^s(\omega_{sl}) \right)^2}_{\geq 0} \\ &+ \sum_{\substack{\mathfrak{v}_l \in \mathfrak{V} \\ d(\mathfrak{v}_l)=1}} \sum_{e_i \in E(\mathfrak{v}_l)} \underbrace{q^i(\omega_{li})^2 A^i (\nu_{li} u^i)^+}_{\geq 0} \\ &\geq 0. \end{aligned} \tag{5.14}$$

The non-negativity of all summands is an important step in the proof since it allows the usage of the weak lower semi-continuity of the $L^2$-norm. More precisely, the difference $\int_0^T S^1 - S^3 \mathrm{d}t$ is the $L^2([0,T], \mathbb{R}^{2n})$-norm of a vector-valued function $z$, where the different components of the vector $z(t)$ are exactly the square roots of the summands of (5.14). With the uniform convergence of $w_j$ and the weak-$\star$ convergence of $q_j$, we conclude the weak convergence in $L^2([0,T])$, as in the first part of the proof:

$$\begin{aligned} z_j^r &= \left( (m_{ll})_j^{-1} A^i (\nu_{li} u_j^i)^+ A^s (\nu_{ls} u_j^s)^+ \right)^{\frac{1}{2}} \left( q_j^i(\omega_{il}) - q_j^s(\omega_{sl}) \right) \\ &\rightharpoonup \left( m_{ll}^{-1} A^i (\nu_{li} u^i)^+ A^s (\nu_{ls} u^s)^+ \right)^{\frac{1}{2}} \left( q^i(\omega_{il}) - q^s(\omega_{sl}) \right) = z^r. \end{aligned}$$

Thus, due to the non-negativity of the components of $z$ and the weak lower semi-continuity, we obtain

$$\begin{aligned} \int_0^T S^1 - S^3 \mathrm{d}t &= \|z\|^2_{L^2([0,T],\mathbb{R}^{2n})} \\ &\leq \liminf_j \|z_j\|^2_{L^2([0,T],\mathbb{R}^{2n})} \\ &= \liminf_j \int_0^T S_j^1 - S_j^3 \mathrm{d}t. \end{aligned}$$

Now, the desired inequality is proven since it holds

$$\int_0^T S^2 \mathrm{d}t = \lim_j \int_0^T S_j^2 \mathrm{d}t$$

because of the strong convergence of $w_j$. □

## 5.2 Existence and stability

To end this thesis, we want to complete the analysis of the existence of solutions of the low Mach number equations (2.53) on a network. In the previous section, we have shown the validity of the reformulation of the low Mach number equations as an ODE coupled with a transport equation so that we can now apply the existence theorems, which we derived in Section 4.2. This will yield the local existence of solutions for arbitrary networks and the global existence at least for the networks described in Section 4.3.

Due to the complex technical work performed in the previous chapters, the proofs of this section will become rather simple. It only remains to validate all the requirements of the derived theorems.

After proving the existence of solutions, we will show a stability result, describing the dependence of the solutions on the initial values. We are not able to show the continuous dependence because of the missing uniqueness result. Nevertheless, for each convergent sequence of initial conditions, we can prove that there exists at least a subsequence of solutions converging to a solution of the limit problem.

To simplify the notation, we will assume the edges of the unscaled network to have all the same length, i.e. we will restrict ourselves to the case $\mathbf{H} = \mathbf{Id}$. With slight modifications of the results in Chapter 4, it is also possible to handle the general case. Before stating the main result, the local in time existence of solutions of the low Mach number system on a network, we will recall the low Mach number equations (2.53) with all coupling conditions derived in Chapter 2:

$$\begin{aligned} \rho_t + (\mathbf{U}\rho)_x &= 0 \\ (U\rho)_t + \left(\mathbf{U}^2\rho + \pi\right)_x &= -\boldsymbol{\zeta}\alpha(\mathbf{U})\mathbf{U}\rho + \hat{f}\mathbf{F}\rho \\ u_x &= \bar{q} \end{aligned} \tag{5.15}$$

in $(0,T) \times \Omega$ with coupling conditions

$$(\nu\mathbf{U})^- \rho = (\nu\mathbf{U})^- \mathcal{H}_u(\rho|_{\Gamma_T}) \qquad \text{on } \Gamma_T \tag{5.16}$$

$$\pi(t,\omega) = g(t,\omega,\nu\mathbf{U}^2(t,\omega)\rho(t,\omega)) + \left(\nu\mathbf{B}_{>1}^T\right)^- P_V(t) \qquad \text{on } \Gamma_T \tag{5.17}$$

$$\mathbf{B}_{>1}^+\mathbf{A}u(t,0) = \mathbf{B}_{>1}^-\mathbf{A}u(t,1) \qquad \text{in } (0,T) \tag{5.18}$$

and initial conditions

$$\rho(0) = \rho_0 \qquad \text{in } \Omega \tag{5.19}$$

$$u(0,x) = v_0 + \int_0^x \bar{q}(0,y)\mathrm{d}y \qquad \text{in } \Omega. \tag{5.20}$$

Additionally to the assumptions (5.1) introduced at the beginning of Section 5.1, we require $g(\cdot,\omega,\cdot)$ to be a Carathéodory-Lipschitz vector field for each $\omega \in \Gamma$. This is reasonable since the commonly used $g$ in the coupling condition (5.17) is such a vector field. The initial conditions should satisfy

$$\rho_0 \in L^\infty(\Omega)^n$$

and

$$v_0 \in \mathbb{R}^n$$

with

$$\mathbf{B}_{>1}\mathbf{A}v_0 = \mathbf{B}_{>1}\mathbf{A}Q(0,1).$$

As before, it is $Q(t,x) = \int_0^x \bar{q}(t,y)\mathrm{d}y$.

**Theorem 5.5 (Local existence).** *Let $G$ be a connected, oriented graph and let the above-mentioned assumptions be valid.*

*Then, there exists a value $T_0 > 0$ such that there exists at least one solution*

$$(\rho, u, \pi, P_V, \gamma\rho)$$

*of the low Mach number equations* (5.15) *in the domain* $(0, T_0) \times \Omega$ *satisfying the coupling conditions* (5.16) *to* (5.18) *and the initial conditions* (5.19) *and* (5.20).

*Proof.* The outline of the proof is as follows: First, we use Theorem 4.12 to obtain a local solution of the alternative formulation (5.3) of the equations based on the velocity equation. Afterwards, we show the existence of a solution of the equations (5.15) with the help of Theorem 5.1 stating the equivalence of the two formulations.

To apply Theorem 4.12 to the equation (5.3) with the coupling conditions (5.16) to (5.18) and the initial conditions (5.19) and (5.20), we need to check the requirements demanded in Section 4.2. We define the function $\mathfrak{f}\colon [0,T] \times \mathbb{R}^n \times \mathcal{V}_T \times L^1(\Gamma)^n \to \mathbb{R}^n$ by

$$\mathfrak{f}(t, v, \rho, w) = -\mathbf{R}^{-1}\Big(\mathbf{R}_{\bar{q}} v + R_{Q_t + \mathbf{Q}\bar{q}} - R_{\hat{f}f - \zeta\alpha(\mathbf{V}+\mathbf{Q})(V+Q)} + \Delta g(t, v, w)\Big) \tag{5.21}$$

with

$$\Delta g(t, v, w) = g\Big(t, 1, (\mathbf{V} + \mathbf{Q}(t,1))\, w(1)\Big) - g\Big(t, 0, -\mathbf{V}w(0)\Big).$$

For fixed $(\rho, \gamma\rho) \in \mathcal{V}_T \times \mathcal{W}_T$, the mapping

$$(t, v) \mapsto \mathfrak{f}\Big(t, v, \rho, \nu(\mathbf{V} + \mathbf{Q}(t))|_\Gamma \gamma\rho\Big)$$

is a Carathéodory-Lipschitz vector field as a concatenation of Carathéodory-Lipschitz vector fields. Furthermore, the function $\mathfrak{f}(t, \cdot, \cdot, \cdot)$ is continuous in $(v, \rho, w) \in \mathbb{R}^n \times \mathcal{V}_T \times L^1(\Gamma)^n$ for almost all $t \in [0,T]$ since $g(t, \omega, \cdot)$ is continuous. Finally, the mapping $\mathfrak{f}$ is locally integrable uniformly bounded: Let $D_0 \subset [0,T] \times \mathbb{R}^n$ be a compact set and let be $(t, v, \rho, \gamma\rho) \in D_0 \times \mathcal{V} \times \mathcal{W}$. Since $\mathcal{W}$ is bounded and since $\mathbf{Q}$ is continuous, there exists a compact set $\bar{D}_0 \subset [0,T] \times \mathbb{R}^n$ with $(t, \nu(\mathbf{V} + \mathbf{Q}(t))|_\Gamma \gamma\rho(t)) \in \bar{D}_0$. Thus, for the Carathéodory-Lipschitz vector field $g$, there exists a bound $m \in L^1(\mathrm{pr}_1 \bar{D}_0, [0,\infty))$ and a null subset $I_0 \subset [0,T]$ with

$$\|g(t, \omega, z)\| \le m(t)$$

for all $(t, z) \in \bar{D}_0$ with $t \notin I_0$. Therefore, we estimate

$$\begin{aligned}
&\|\mathfrak{f}(t, v, \rho(t), \nu(\mathbf{V} + \mathbf{Q}(t))\gamma\rho(t))\| \\
&\le \frac{1}{\bar{\rho}_{\min}(T)}\left(\bar{\rho}_{\max}(T)\left(\|q\|_{L^1([0,T]\times\overline{\Omega})} \sup_{v \in \mathrm{pr}_2 D_0} \|v\|_\infty + \|Q_t + \mathbf{Q}\bar{q} - \hat{f}f\|_{L^1((0,T)\times\Omega)}\right.\right. \\
&\qquad \left.\left. + \|\zeta\|_{L^\infty((0,T)\times\Omega)} \sup_{(t,v,x)\in D_0\times\overline{\Omega}} \|\alpha(\mathbf{V} + \mathbf{Q}(t,x))(v + Q(t,x))\|_1\right) + 2m(t)\right).
\end{aligned}$$

Hence, the mapping $\mathfrak{f}$ fulfils all requirements of the local existence theorem (Theorem 4.12) and it remains to check the requirements on the coupling condition

$$F(t,v) = \mathbf{B}_{>1}\mathbf{A}v - \mathbf{B}_{>1}^{-}\mathbf{A}Q(t,1) = 0.$$

Since it holds

$$D_vF(t,v) = \mathbf{B}_{>1}\mathbf{A},$$

the assumptions for the matrices $\mathbf{D}_1 = \mathbf{Id}$ and $\mathbf{D}_2 = \mathbf{A}$ are clearly satisfied (see equation (4.3)) as these matrices are constant and positive definite. Thus, applying Theorem 4.12 yields the existence of $T_0 > 0$ and of at least one solution

$$(\rho, v, P_V, \gamma\rho) \in L^\infty((0,T_0)\times\Omega)^n \times W^{1,1}(0,T_0) \times L^1((0,T_0)) \times L^\infty(\Gamma_{T_0}, |\mathrm{d}\mu_{v+Q}|)$$

of

$$\begin{aligned} &\rho_t + ((\mathbf{V}+\mathbf{Q})\rho)_x = 0 \\ &\qquad\qquad\qquad v_t = \mathfrak{f}\Big(t, v, \rho(t), \nu(\mathbf{V}+\mathbf{Q}(t))|_\Gamma \gamma\rho(t)\Big) + \mathbf{R}^{-1}\mathbf{B}_{>1}^T P_V \end{aligned} \tag{5.22}$$

fulfilling the coupling and initial conditions.

Defining $\Delta p = \Delta g(\cdot, v(\cdot), \gamma\rho(\cdot)) \in L^1((0,T_0))^n$ the pair $(\rho, v)$ solves the equation (5.3). Thus, due to Theorem 5.1 the triple $(\rho, v+Q, \pi)$ with

$$\pi = \bar{\pi} - \int_0^x \Big(\mathbf{V}_t + \mathbf{Q}_t + (\mathbf{V}+\mathbf{Q})\mathbf{Q}_x + \boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(\mathbf{V}+\mathbf{Q}) - \hat{f}\mathbf{F}\Big)\,\rho\mathrm{d}x \tag{5.23}$$

is a weak solution of the equations (5.15) satisfying $\pi(1) - \pi(0) = \Delta p$. Choosing

$$\bar{\pi}(t) = g\Big(t, 0, \nu\mathbf{V}(t)^2\gamma\rho(0,t)\Big)$$

the tuple $(\rho, v+Q, \pi, P_V, \gamma\rho)$ satisfies also all initial and coupling conditions (5.16) to (5.20). □

Now that we have shown the local existence, we will validate the requirements for the global existence on the networks described in Section 4.3 in the following. To simplify the notation, we assume $\Delta g(t, v, \gamma\rho(t))$ to be linearly integrable uniformly bounded, i.e. there are two functions $d_{g,1} \in L^2([0,T],[0,\infty))$ and $d_{g,2} \in L^1([0,T],[0,\infty))$ with

$$\|\Delta g(t, v, \gamma\rho(t))\|_2 \le d_1(t) + d_2(t)\|v\|_2$$

for almost all $t \in [0,T]$ and all $\gamma\rho \in \mathcal{W}$. Then, we can prove the following lemma which allows us to conclude the global existence for all graphs presented in Section 4.3.

**Lemma 5.6 (Linearly boundedness).** *Let $\mathfrak{f}$ be the function defined in* (5.21)*.*

*1. Let be $\alpha_i(u_i) \ge 0$. Then, the function $\mathfrak{f}$ is linearly integrable uniformly bounded as long as $\mathfrak{f}$ and $v$ have the same sign, i.e. there exist two functions $d_1, d_2 \in L^1([0,T],[0,\infty))$ with*

$$\operatorname{sgn}(v_i)\mathfrak{f}_i\Big(t, v, \rho(t), \nu(\mathbf{V}(t)+\mathbf{Q}(t))|_\Gamma\Big) \le d_1(t) + d_2(t)\|v\|_1 \tag{5.24}$$

*for almost all $t \in [0,T]$ and all $(\rho, \gamma\rho) \in \mathcal{V} \times \mathcal{W}$.*

*2. If we additionally assume $\alpha_i(u_i) \le 1$, then $\mathfrak{f}$ is linearly integrable uniformly bounded, i.e. there exist two functions $d_1 \in L^2([0,T],[0,\infty))$ and $d_2 \in L^1([0,T],[0,\infty))$ with*

$$\left\|\mathfrak{f}\Big(t, v, \rho(t), \nu(\mathbf{V}(t)+\mathbf{Q}(t))|_\Gamma\Big)\right\|_2 \le d_1(t) + d_2(t)\|v\|_2$$

*for almost all $t \in [0,T]$ and all $(\rho, \gamma\rho) \in \mathcal{V} \times \mathcal{W}$.*

*Proof.* For $\bar{\mathfrak{f}} = \mathfrak{f} + \mathbf{R}^{-1} R_{\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(V+Q)}$, we estimate

$$\begin{aligned}
&\left\|\bar{\mathfrak{f}}\Big(t, v, \rho(t), \nu(\mathbf{V}+\mathbf{Q}(t)|_\Gamma)\gamma\rho(t)\Big)\right\|_2 \\
&= \left\|-\mathbf{R}^{-1}\left(\mathbf{R}_{\bar{q}} v + R_{Q_t + \mathbf{Q}\bar{q}} - R_{\hat{f}f} + \Delta g(t, v, \gamma\rho(t))\right)\right\|_2 \\
&\le \frac{\bar{\rho}_{\max}(t)}{\bar{\rho}_{\min}(t)} \Bigg\| \bar{q}(t)\|_{L^\infty(\Omega)^n}\|v\|_1 + \left(\|Q_t\|_{L^\infty(\Omega)^n} + \|\bar{q}(t)\|^2_{L^\infty(\Omega)^n}\right) \\
&\quad + \hat{f}\|f\|_{L^1(\Omega)^n} + d_{g,1}(t) + d_{g,2}(t)\|v\|_2\Bigg)
\end{aligned}$$

for almost all $t \in [0,T]$ and all $(\rho, \gamma\rho) \in \mathcal{V} \times \mathcal{W}$. Thus, $\bar{\mathfrak{f}}$ is linearly integrable uniformly bounded. The only remaining part to estimate is the term $-\mathbf{R}^{-1} R_{\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(V+Q)}$. Here, we distinguish the two cases 1 and 2.

1. Let be $\alpha_i(u_i) \ge 0$. For $|v_i| \ge \|Q_i(t)\|_{L^\infty(\Omega)^n}$, it is $\operatorname{sgn}(v_i) = \operatorname{sgn}(v_i + Q_i)$ and thus, because of the non-negativity of $\zeta$, $\alpha$ and $\rho$ it holds

$$\begin{aligned}
-\operatorname{sgn}(v_i) R_i^{-1}\left(R_{\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(V+Q)}\right)_i &= -\frac{1}{R_i(t)}\int_0^1 \zeta_i \alpha_i(v_i+Q_i)(v_i+Q_i)\rho_i \operatorname{sgn}(v_i)\mathrm{d}x \\
&= -\frac{1}{R_i(t)}\int_0^1 \zeta_i \alpha_i(v_i+Q_i)\,|v_i+Q_i|\,\rho_i \mathrm{d}x \\
&\le 0.
\end{aligned}$$

Due to the continuity of $\alpha$ and $Q$, the function

$$K(t) = \sup_{|u| \le 2\|Q(t)\|_{L^\infty(\Omega)^n}} \|\alpha(u)\|_\infty$$

is well-defined. Thus, for $|v_i| < \|Q_i(t)\|_{L^\infty(\Omega)^n}$ we can estimate

$$\begin{aligned}
&-\operatorname{sgn}(v_i) R_i^{-1}\left(R_{\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(V+Q)}\right)_i \\
&\le \frac{\bar{\rho}_{\max}(t)}{\bar{\rho}_{\min}(t)}\|\zeta(t)\|_{L^\infty(\Omega)^n} K(t)\left(\|v\|_1 + \|Q(t)\|_{L^1(\Omega)^n}\right),
\end{aligned}$$

which proves the inequality (5.24) for almost all $t \in [0,T]$ and all $(\rho, \gamma\rho) \in \mathcal{V} \times \mathcal{W}$.

2. The case $\alpha_i(u_i) \le 1$ is even simpler since it holds

$$\left\|\mathbf{R}^{-1} R_{\boldsymbol{\zeta}\alpha(\mathbf{V}+\mathbf{Q})(V+Q)}\right\|_2 \le \frac{\bar{\rho}_{\max}(t)}{\bar{\rho}_{\min}(t)}\|\zeta(t)\|_{L^\infty(\Omega)^n}\left(\|Q\|_{L^2(\Omega)^n} + \|v\|_2\right).$$

□

*Remark* 5.7. Clearly, for the second statement, it is sufficient to require $\Delta g$ to be linearly integrable uniformly bounded if $v$ and $\Delta g$ have the same sign.

In most of the common examples, $\Delta g$ depends only on $t$ and thus, it is linearly integrable uniformly bounded. An exception is formed by the minor loss factors, used to model sudden contractions or expansions, which were introduced in (2.37). For networks consisting of a path, these minor loss terms have the form

$$\Delta g_i(t, v, \nu(\mathbf{V} + \mathbf{Q}(t)|_\Gamma)\gamma\rho(t)) = -\tilde{K}_i \frac{\left|v_{i-1} + Q_{i-1}(t,1)\right|\Big(v_{i-1} + Q_{i-1}(t,1)\Big)\gamma\rho_{i-1}(t,1)}{2}$$

for $\tilde{K}_i \geq 0$ (compare equation (2.37)). Using the function $w$ introduced in the construction of the energy functional for paths (see proof of Corollary 4.19), it is easy to verify that $\Delta g$ is linearly bounded if $v$ and $\Delta g$ have the same sign. Therefore, the previous lemma is also valid for paths with sudden contractions or expansions modelled with minor loss terms. Altogether, we have proven the global existence of solutions of (5.15) to (5.20) for the networks described in Section 4.3 since all corollaries of that section can be applied.

To end this section, we characterize the dependence of the solutions on the initial conditions. As mentioned before, we are able to show that for each convergent sequence of initial values, there exists a subsequence of corresponding solutions converging to a solution satisfying the limit initial conditions.

**Theorem 5.8 (Dependence on the initial values)**. *Let $G$ be a graph such that there exists a uniform energy functional $V$ for the low Mach number equations. Furthermore, let $(v_{0,j})_j \subset \mathbb{R}^n$ and $(\rho_{0,j})_j \subset L^\infty(\Omega)^n$ be convergent sequences with*

$$v_{0,j} \to v_0 \qquad \text{in } \mathbb{R}^n$$

*and*

$$\rho_{0,j} \to \rho_0 \qquad \text{in } L^1(\Omega)^n$$

*such that it is $\rho_0 \in L^\infty(\Omega)^n$ and such that $\rho_{0,j}$ is bounded from below and above by*

$$0 < \rho_{\min} \leq \rho_{0,j} \leq \rho_{\max}$$

*almost everywhere. Denote by $(\rho_j, v_j + Q, \pi_j, (P_V)_j, \gamma\rho_j)$ a solution of the equations* (5.15) *satisfying the coupling conditions* (5.16) *to* (5.18) *and the initial values $\rho_j(0) = \rho_{0,j}$ and $v_j(0) = v_{0,j}$.*

*Then, at least a subsequence $(\rho_{j_l}, v_{j_l} + Q_{j_l}, \pi_{j_l}, (P_V)_{j_l}, \gamma\rho_{j_l})$ is convergent to a solution $(\rho, v + Q, \pi, P_V, \gamma\rho)$ of the limit problem, i.e. it holds*

$$\begin{aligned}
\rho_{j_l} &\to \rho && \text{in } C([0,T], L^p(\Omega)^n),\\
v_{j_l} &\to v && \text{in } W^{1,1}([0,T])^n,\\
\pi_{j_l} &\to \pi && \text{in } L^1((0,T), W^{1,1}(\Omega)^n),\\
(P_V)_{j_l} &\to P_V && \text{in } L^1((0,T))^n
\end{aligned}$$

*and*

$$\gamma\rho_{j_l} \to \gamma\rho \qquad\qquad \textit{in } L^p(\Gamma_T, |\mathrm{d}\mu_u|).$$

*Proof.* The proof relies on the theorem of Arzelà-Ascoli to extract a convergent subsequence and on the stability theorem (Theorem 3.11) of the transport equation to prove that the limit is a solution of the equations.

Due to the regularity of $q$ we can switch between both formulations of the equations (see Theorem 5.1). We define $v_j = u_j - Q$ and want to apply the theorem of Arzelà-Ascoli. Since $(\rho_j, v_j)$ is a solution of the velocity equation (5.3) we can apply Lemma 4.11 to conclude the boundedness of the density $\rho_j$ independent on $j$, i.e. to conclude $\bar{\rho}_{\min}(t) \le \rho_j \le \bar{\rho}_{\max}(t)$. Thus, the energy functional $V$ is valid for all $j$ and the velocity $v_j$ is bounded independently on $j$, i.e. there exists $R(t)$ with

$$\|v_j(t)\| \le R(t).$$

As in the proof of Theorem 4.12 of the local existence of solutions, the sequence $v_j$ can be shown to be equicontinuous. Therefore, due to the theorem of Arzelà-Ascoli, there exists a uniform convergent subsequence with $v_{j_l} \to v$.

With the stability theorem 3.11, the sequence of the densities $\rho_{j_l}$ and the sequence of their traces $\gamma\rho_{j_l}$ are convergent, i.e. it holds

$$\rho_{j_l} \to \rho \qquad\qquad \text{in } C([0,T], L^p(\Omega)^n)$$

and

$$\gamma\rho_{j_l} \to \gamma\rho \qquad\qquad \text{in } L^p(\Gamma_T, |\mathrm{d}\mu_{v+Q}|)$$

where the limit $\rho$ is the solution of the transport equation with velocity $v + Q$ and its trace $\gamma\rho$. Using the dominated convergence theorem, the uniform convergence of $v_{j_l}$ and the continuity of $\mathfrak{f}$ yields after passing to a subsequence the following (compare proof of Theorem 4.12):

$$\begin{aligned} v(t) &\leftarrow v_{j_l} \\ &\quad = v_{0,j} \\ &\qquad + \int_0^t \mathbf{T}(s, \rho_{j_l}(s))\mathfrak{f}(s, v_{j_l}(s), \rho_{j_l}(s), \nu(\mathbf{V}_{j_l}(s) + Q(s)|_\Gamma)\gamma\rho_{j_l}(s)) - \mathbf{S}(s, \rho_{j_l}(s))F_t(s)\mathrm{d}s \\ &\quad \to v_0 + \int_0^t \mathbf{T}(s, \rho(s))\mathfrak{f}(s, v(s), \rho(s), \nu(\mathbf{V}(s) + Q(s)|_\Gamma)\gamma\rho(s)) - \mathbf{S}(s, \rho(s))F_t(s)\mathrm{d}s. \end{aligned}$$

In particular, it holds also $(v_{j_l})_t \to v_t$ in $L^1((0,T))$. The stated convergence of $(P_V)_{j_l}$ (see formula (4.15)) and the convergence of $\pi_{j_l}$ (see formula (5.23)) is then clear. □

*Remark* 5.9. The previous result does not state the continuous dependence of the solution on the initial data since a uniqueness result is missing. Assuming the solution of the limit problem to be unique, indeed the whole sequence and not only a subsequence is converging.

# 6 Summary and outlook

In this thesis, we studied a system of partial differential equations on a network. On each edge of the network a transport equation is coupled with an ODE. The velocity field of the transport equation is driven by the ODE whereas the right-hand side of the ODE depends on the solution of the transport equation. To complete the model, the equations are coupled by algebraic relations at the nodes of the network.

Such systems of partial differential equations are well-defined since the solution of the transport equation is continuous with values in a Banach space and thus, the ODE gets a meaning in the classical sense. A problem when studying such systems on networks is that it is not possible to determine the direction of the flow a priori. In this case, the method of characteristics may fail due to the changing flow directions. Nevertheless, we were able to prove the existence of solutions of such equations locally in time. The proof is mainly based on the continuous dependence of the solution of the transport equation on the velocity field.

Since this dependence is not Lipschitz continuous, we were not able to prove a uniqueness result. However, we could establish the global in time existence under additional constraints on the network. The local solution can be expanded to the whole time interval and hence a possible blow-up of the velocity can be excluded if there existed a uniform energy functional for the network. We were able to provide such functionals for different cases. Either, if we restrict the right-hand side of the ODE to be linearly bounded and allow the network to be arbitrary, or if we allow more general right-hand sides and restrict the form of the network, we provided constructions of energy functionals ensuring the global existence. The main problem in order to show the global existence is that the ODE has an infinite speed of propagation, i.e. a change of the ODE on any edge leads to a change of the solution on all edges and all nodes.

Furthermore, we discussed the non-isothermal and non-isentropic low Mach number equations with a heat source as an example, on which the developed theory can be applied. These equations describe the hot air flow in a tunnel system in case of a fire. We formally derived the low Mach number model from the three-dimensional Euler equations. For the passage from the three-dimensional equations to the one-dimensional section-averaged Euler equations, the use of the Frenet-Serret-frame has turned out to be valuable in order to be able to consider curved edges as well. Moreover, the resulting low Mach number model consisting of the continuity, momentum and energy equation was shown to be equivalent to a system consisting of the continuity and the velocity equation. Here, the velocity equation was an ODE such that this model fits into the developed framework and the existence of solutions could be established.

Of course, there are lots of possibilities how to continue the research. To end this thesis, we want to mention a few aspects:

- **Construction of further energy functionals:** From a physical point of view there seems to be no reason why we have to restrict to special classes of networks in order to prevent a blow-up of the velocity. Therefore, the construction of energy functionals for arbitrary networks seems to be feasible. Or, if this is not possible, a classification of networks into those, where such functionals exists and those where not, would be a big progress in understanding the flow on networks.

- **Proof of uniqueness:** The missing Lipschitz continuous dependence of the solution of the transport equation on the velocity makes it difficult to prove the uniqueness of solutions of the coupled partial differential equations. Moreover, it would be an important step in the analysis of the equations to provide a proof of uniqueness or a counterexample disproving it.

- **Justification of the formal asymptotic limits:** In the derivation of the low Mach number model, we formally performed two asymptotic limits. The first concerns the passage from the three-dimensional equations to the one-dimensional section-averaged equations on a network and the second one the low Mach number limit itself. Since both asymptotic limits are only formally taken, it is desirable to identify admissible flow situations for which the use of the model is justified such that the solution of the low Mach number equations really approximates the solution of the three-dimensional Euler or Navier-Stokes equations.

- **Consideration of higher order terms:** In the formal limits in the derivation of the model we have neglected all terms of higher order. Including these terms would possibly lead to a more sophisticated one-dimensional model for the flow on a network including correction terms for non-straight geometries.

# List of symbols

| | |
|---|---|
| $A$ | cross-sectional area, 32 |
| $\alpha$ | function describing the friction model, 34 |
| $\mathbf{B}$ | incidence matrix, 2 |
| $\mathbf{B}_{=1}$ | submatrix of the incidence matrix $\mathbf{B}$ corresponding to the outer nodes, 3 |
| $\mathbf{B}_{>1}$ | submatrix of the incidence matrix $\mathbf{B}$ corresponding to the inner nodes, 3 |
| $c$ | reaction term, 56 |
| $\mathbf{D}_1$, $\mathbf{D}_2$ | matrices to factorize the Jacobian $D_vF$, 89 |
| $E$ | Edge set, 1 |
| $e$ | specific internal energy, 19 |
| $e_i$ | edge, 1 |
| $F$ | coupling condition for the velocity, 88 |
| $\mathfrak{f}$ | right-hand side of the ODE, 94 |
| $G$ | Graph, 1 |
| $\mathcal{G}$ | linear boundary operator, 57 |
| $g$ | coupling condition for the pressure, 47 |
| $\Gamma$ | boundary of $\Omega$, 56 |
| $\gamma$ | adiabatic exponent, 20 |
| $\gamma\rho$ | trace, 60 |
| $\Gamma_T$ | space-time boundary, 56 |
| $\Gamma_T^{\pm}$ | inflow or outflow part of the space-time boundary, 56 |
| $\mathbf{H}$ | diagonal matrix of dimensionless parameters $h$, 46 |
| $h$ | dimensionless parameter, 26 |
| $\mathcal{H}$ | affine linear boundary operator, 57 |
| $\mathbf{L}_w$ | (weighted) Laplacian matrix, 4 |
| $M$ | Mach number, 26 |
| $\Omega$ | computational domain, 56 |
| $p$ | pressure, 19 |
| $\pi$ | first order pressure, 48 |
| $\mathrm{pr}_i$ | projection on the $i$-th component, 7 |
| $P_V$ | pressure at the inner nodes, 47 |
| $Q$ | known space-dependent part of the velocity, 50 |
| $q$ | heat source, 20 |
| $\mathbf{R}_f$ | diagonal matrix of weighted masses $R_f$, 51 |
| $R_f$ | weighted mass, 51 |
| $\rho$ | density, 18 |
| $\rho_0$ | initial density, 56 |
| $\rho_{\mathrm{in}}$ | inflow density, 57 |
| $\rho_{\max}$ | maximal density, 57 |
| $\rho_{\min}$ | minimal density, 67 |
| $\mathbf{S}$ | matrix, 91 |
| $\mathbf{T}$ | projector, 91 |
| $T_w$ | surface temperature, 35 |
| $\mathbf{U}$ | diagonal matrix of velocities $u$, 45 |
| $u$ | velocity, 18 |
| $\mathbf{V}$ | diagonal matrix of velocities $v$, 45 |
| $V$ | (uniform) energy functional, 94 |
| $\mathcal{V}$ | admissible set for the density $\rho$, 96 |
| $\mathfrak{V}$ | Vertex set, 1 |
| $v$ | unknown space-independent part of the velocity, 50 |
| $\mathfrak{v}$ | node, 1 |
| $\overline{\mathbf{W}}$ | weight matrix, 57 |
| $\mathcal{W}$ | admissible set for the trace $\gamma\rho$, 96 |
| $\zeta$ | friction coefficient, 34 |

# List of spaces

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## Summary

In this thesis, we study analytically a low Mach number model on a network describing the flow of fluids or gases driven by a strong heat source. This model can for example be used to describe fires in tunnel systems or to describe exhaust systems of vehicles.

In the first part of the work, the model is formally derived from the three-dimensional Euler equations. Using the Frenet-Serret-frame and asymptotic expansions, the Euler equations are transferred to a non-hyperbolic system of one-dimensional partial differential equations on a graph.

On each edge of the graph, the flow is governed by

$$\begin{aligned} \rho_t + (u\rho)_x &= 0 \\ (\rho u)_t + \left(\rho u^2 + \pi\right)_x &= -\zeta\rho\alpha(u)u + f\rho \\ u_x &= q \end{aligned}$$

where $q$ describes a known heat source, $\zeta\alpha(u)u$ the wall friction and $f$ an external force. The other quantities are as usual the density $\rho$, the velocity $u$ and the pressure $\pi$. The equations are completed by reasonable coupling conditions at the nodes reflecting physical principles like the conservation of mass and the conservation of internal energy.

These equations were shown to be solvable globally in time for single edges in [48]. In the second part of the thesis, we generalize this result to the case of networks. For the analysis, we study the involved equations - a transport equation for the density and a nonlinear ODE for the velocity - separately using a combination of graph theoretical and functional analytical concepts.

# Zusammenfassung

In dieser Arbeit untersuchen wir analytisch ein mathematisches Modell für Gas- oder Fluidströmung mit kleiner Machzahl auf Netzwerken, die durch starke Hitzequellen angetrieben wird. Das Modell kann zum Beispiel verwendet werden, um Brände in Tunnelsystemen oder das Verhalten von Auspuffsystemen zu beschreiben.

Im ersten Teil der Dissertation wird das Modell formal aus den dreidimensionalen Eulergleichungen hergeleitet. Mit Hilfe des Frenet-Serret-Koordinatensystems und asymptotischen Entwicklungen werden die Eulergleichungen in ein nichthyperbolisches System eindimensionaler partieller Differentialgleichungen auf einem Graphen überführt.

Auf jeder Kante des Graphen wird die Strömung durch die Gleichungen

$$\begin{aligned} \rho_t + (u\rho)_x &= 0 \\ (\rho u)_t + \left(\rho u^2 + \pi\right)_x &= -\zeta\rho\alpha(u)u + f\rho \\ u_x &= q \end{aligned}$$

beschrieben, wobei $q$ eine bekannte Hitzequelle, $\zeta\alpha(u)u$ die Reibung und $f$ eine externe Kraft bezeichnet. Die anderen Größen sind, wie üblich, die Dichte $\rho$, die Geschwindigkeit $u$ und der Druck $\pi$. Die Gleichungen werden an den Knoten durch Kopplungsbedingungen, die durch die physikalischen Grundsätze wie z.B. Massen- oder Energieerhaltung gegeben sind, abgeschlossen.

In [48] wurde gezeigt, dass die Gleichungen für einzelne Kanten immer eine Lösung besitzen. Im zweiten Teil der Arbeit verallgemeinern wir dieses Resultat auf Netzwerke. Für die Analysis untersuchen wir die beiden Gleichungen - eine Transportgleichung für die Dichte und eine nichtlineare gewöhnliche Differentialgleichung für die Geschwindigkeit - separat, wobei wir Konzepte der Graphentheorie mit Konzepten der Funktionalanalysis kombinieren.

# Lebenslauf

| | |
|---|---|
| Name | Arne Roggensack |
| Geburtsdatum | 7. April 1985 |
| Nationalität | deutsch |

## Ausbildung

| | |
|---|---|
| 08.2010 – 09.2014 | **Promotionsstudium in Mathematik**, *Universität Hamburg*.<br>Betreuer: Professor Dr. Ingenuin Gasser |
| 10.2005 – 06.2010 | **Studium der Technomathematik**, *Universität Hamburg*. |
| 06.2010 | **Diplom in Technomathematik (Dipl.-Technomath.)**, *Universität Hamburg*. |
| 10.2007 | **Vordiplom in Technomathematik**, *Universität Hamburg*. |
| 06.2004 | **Abitur**, *Bismarckschule Elmshorn*. |

## Berufliche Erfahrung

### Forschung

| | |
|---|---|
| 04.2014 – 09.2014 | **Industrielles Forschungsprojekt**, HITeC Hamburg.<br>Modellierung und numerische Simulation eines chemischen Rohrreaktors |
| 08.2010 – 09.2014 | **Wissenschaftlicher Mitarbeiter**, *Differentialgleichungen und Dynamische Systeme, Fachbereich Mathematik*, Universität Hamburg. |
| 10.2008 – 02.2010 | **studentische Hilfskraft**, *Institut für Numerische Simulation*, Technische Universität Hamburg-Harburg.<br>Implementierung einer numerischen Simulation für Wasserströmung in Rohrsystemen |

### Lehre

| | |
|---|---|
| 10.2010 – 09.2014 | **Übungsgruppenleiter**, *Technische Universität Hamburg-Harburg*.<br>Übungen in Analysis I für Ingenieure, Analysis II für Ingenieure, Differentialgleichungen I für Ingenieure und Differentialgleichungen II für Ingenieure |
| 04.2007 – 07.2008 | **studentischer Übungsgruppenleiter**, *Fachbereich Mathematik, Universität Hamburg*.<br>Übungen in Analyis I und Analysis II |

## Veröffentlichungen

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