Representability of infinite matroids and the structure of linkages in digraphs

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Contents

Introduction

This is a PhD thesis in infinite combinatorics consisting of the results in four papers [1], [2], [3] and [4]. In order to read it, familiarity with the basics of finite graph theory, finite matroid theory and linear algebra can be helpful.

Finite matroids were introduced in 1935 by Hassler Whitney [35] (also independently by Takeo Nakasawa [29]). To abstract the notion of independence that is common to both graph theory and linear algebra, Whitney suggested a set of axioms, and defined any system of sets satisfying these axioms to be matroids. So, many of the terms used in matroid theory are similar to their counterparts in linear algebra or graph theory.

Central to the basic theory of finite matroids is the fact that they can be axiomatised in many different ways. This includes axiomatisations in terms of independent sets, bases, circuits, closure operators and rank functions. However, when applied to infinite sets, the axiomatisations involving cardinality of sets do not make much sense, a fact leading to the question of whether there is a natural theory of infinite matroids. One attempt to define infinite matroids is to restrict attention to *finitary* matroids: those in which a set is independent if and only if all of its finite subsets are, or equivalently in which all circuits are finite. Some of the basic theory of finite matroids can be extended to this class via compactness arguments. A serious problem with the class of finitary matroids is that it is not closed under duality, namely a finitary matroid need not have a finitary dual.

This led Rado to ask in 1966 whether there is a good theory of non-finitary infinite matroids with duality [34]. A wide range of solutions were proposed, based on different axiomatisations of finite matroids. One of these, called the *B-matroids* which was proposed by Higgs 1969 [25], was shown by Oxley to be the largest class closed under duality and taking minors and whose finite elements are matroids [30]. However, B-matroids remained difficult to handle until a discovery by Bruhn, Diestel, Kriesell, Pendavingh and Wollan [15]. Following the duality in infinite graphs, they arrived at a notion of infinite matroids and a simple way to extend each of the five axiomatisations of finite matroids to the infinite case. The models of their axioms for infinite matroids turned out soon to coincide with that of B-matroids but the simplicity of their axiomatisations rapidly led to many theoretical progresses in infinite matroid theory.

The current work is part of the ongoing project, which tries to extend different aspects of finite matroid theory to the infinite case. The thesis consists of four chapters. After the introduction, we collect a couple of definitions and lemmas about general matroid theory in Chapter 1. In Chapter 2, the problem of representability of infinite matroids over a field is addressed [1], and Chapter 3 is devoted to study a class of matroids, called gammoids [[2],[3] [4]].

In the finite case, representable matroids form perhaps the most important subclass of matroids. A matroid is called *representable* over a field k if the elements of the matroid can be labeled by a family of vectors of a vector space over k such that a set is independent in the matroid if and only if the family consisting of the corresponding labels is linearly independent. An important property of representability in finite matroids is that it is preserved under duality, namely the dual of a representable matroid is always representable.

This notion of representability can easily be extended to finitary matroids and some of the interesting properties of the finite case are preserved under such an extension. However, the representability in finitary matroids violates duality, as the dual of a finitary matroid need not be finitary. So the main thema of Chapter 2 is a question akin to Rado's question: Is there a natural extension of the class of representable matroids which is closed under duality? Note that any such extension must contain the dual of ordinary representable matroids.

A putative answer to this question was suggested by Bruhn and Diestel [14]. By extending the notion of linear combinations to allow for infinite combinations in certain constrained circumstances, they arrived at a new definition of representability, called *thin sums representability*. It is not true that the objects given by this definition are always matroids but if they are, they are called *thin sums matroids*.

In Chapter 2, we study different aspects of thin sums matroids. We give a characterization of the duals of ordinary representable matroids among thin sums matroids. A matroid with no infinite circuit-cocircuit intersection is called *tame*. We show that the class of tame thin sums matroids is closed under duality and so taking minors. As we shall see, most of the matroids associated to graphs turn out to be tame and thin sums representable. So we suggest the class of tame matroids, as a suitably large class of matroids in which one can have a reasonable theory of representability which is preserved under duality.

Next, we turn to another class of matroids, namely the class of gammoids. The concept of a gammoid was introduced and shown to be a matroid by Perfect in 1968 [32], and was studied in more detail and given its name by Mason in 1972 [27].

Gammoids can be viewed as a generalization of a well known class of matroids, termed *transversal matroids*. Recall that a transversal matroid can be defined by taking a fixed vertex class of a bipartite graph as the ground set and the matchable subsets of that class as the independent sets. Replacing bipartite graphs with digraphs, and matchings with linkages, sets of disjoint (directed) paths, we arrive at the definition of a gammoid. A *strict gammoid* is a matroid defined on the vertex set of (the digraph of) a *dimaze*, that is a digraph equipped with a specific set of sinks named the (set of) *exits*, where a set is independent if there is a linkage of that set to the set of exits ([27]). *Gammoids* are defined as matroid restrictions of strict gammoids.

The class of finite gammoids has many pleasant properties. For example, Ingleton and Piff [26] proved constructively that finite strict gammoids and finite transversal matroids are dual to each other, a key fact to the result that the class of finite gammoids is minor-closed.

Contrary to the representable matroids, the definition of finite gammoids verbatim extends to (not necessarily finitary) infinite gammoids. It turns out that the system of independent sets of an infinite dimaze always satisfies the *augmentation* axiom, namely a non-maximal independent set can be extended inside any maximal one to a larger independent set. However, it need not satisfy the matroid axiom (IM), which demands the existence of certain maximal independent sets. The duality of strict gammoids and transversal matroids breaks down as well, so the classical proof of the fact that the class of finite gammoids is minor-closed does not extend to the infinite case. In fact, we do not know if this class is minor-closed.

Regarding the problems discussed in the previous paragraph, an intersting question is that if we can restrict our attention to reasonably large subclasses of gammoids, in which different features of the class of finite gammoids are preserved.

In Chapter 3, we begin to address this question. Our approach is, in a sense, similar to identifying a desired class of graphs via forbidding graphs as topological minors. Roughly speaking, looking closely at a system of linkable sets with an undesired behaviour, we try to find the substructure (subdimaze) in its defining dimaze which causes this undesired behaviour, and then study the class of gammoids definable by the dimazes that do not contain this substructure.

Chapter 1

Basics

In this chapter, we introduce basics of infinite matroid theory that we shall use later. Almost all of the materials here can be found (or follow from) the results in [15] or [31].

For any set E let $\mathcal{P}(E) = 2^E$ be the power set of E and $\mathcal{I} \subseteq \mathcal{P}(E)$. Recall from [15], that the *set system* $M = (E, \mathcal{I})$ is called a matroid if $\mathcal{I}$ satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\mathcal{I}$ is closed under taking subsets.

(I3) For all $I, I' \in \mathcal{I}$ with I' maximal in $\mathcal{I}$ but I not maximal in $\mathcal{I}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} | I \subseteq I' \subseteq X\}$ has a maximal element.

The axiom (I3) is called *augmentation axiom*. E is called the ground set and elements of $\mathcal{I}$ the *independent sets* of M (M-independent sets). We often identify a matroid with its set of independent sets. Subsets of the ground set which are not independent are *dependent*, and minimal dependent sets are called *circuits* of the matroid M (M-circuits). It can be proved that any dependent set contains a minimal one. A Circuit of size one is called a *loop*. Maximal independent sets are called *bases*.

Matroids can also be defined via base or circuit axioms. A collection of subsets $\mathcal{B}$ of E is the set of bases of a matroid if and only if the following three conditions hold:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.

(BM) The set $\mathcal{I}$ of all subsets of elements in $\mathcal{B}$ satisfies (IM).

A set $\mathcal{C} \subseteq \mathcal{P}(E)$ is the set of circuits of a matroid if and only if the following conditions hold:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x | x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \iff x = y$ for all $x, y \in X$, then for every $z \in C \setminus (\bigcup_{x \in X} C_x)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X$.

(CM) The set $\mathcal{I}$ of all the elements of $\mathcal{P}(E)$ that do not contain any element of $\mathcal{C}$ satisfies (IM).

To see the equivalency of these axiomatisations and a couple of other axiomatisations of infinite matroids see [15].

$\mathcal{I}(M)$, $\mathcal{B}(M)$ and $\mathcal{C}(M)$ denote sets of independent sets, bases and circuits of the matroid M, respectively. M^* denotes the dual of M whose bases are complements of the bases of M. Expressions like a *coindependent* set, a *cocircuit* and a *codependent* set of M refer, respectively, to an independent set, a circuit and a dependent set of M^*. *Closure operator* is a map $\mathrm{cl} : 2^E \to 2^E$ defined as follows. For a set $X \subseteq E$,

$$\mathrm{cl}(X) := X \cup \{x \in E | \exists I \subseteq X : I \in \mathcal{I}, I + x \notin \mathcal{I}\}.$$

Closure operator has different properties including $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

For any base B and any $e \in E \setminus B$, there is a *unique* circuit o_e with $e \in o_e \subseteq B + e$, called the *fundamental circuit* of e with respect to B. Dually, since $E \setminus B$ is a base of M^*, for any $f \in B$ there is a unique cocircuit b_f with $f \in b_f \subseteq E \setminus B + f$, called the *fundamental cocircuit* of f. We reprove the following from [15].

Lemma 1.0.1. *There is no matroid M with a circuit o and a cocircuit b such that $|o \cap b| = 1$.*

Proof. Suppose for a contradiction that there were such an M, o and b, with $o \cap b = \{e\}$. Let B be a base of M whose complement includes the coindependent set $b - e$. Let I be a maximal independent set with $o - e \subseteq I \subseteq E \setminus b$ - this can't be a base of M since its complement includes

b, so by (I3), there is some $f \in B \setminus I$ such that $I + f$ is independent. Then by maximality of I, $f \in b$, and so $f = e$, so o is independent. This is the desired contradiction. $\square$

Lemma 1.0.2. *Let M be a matroid and B be a base of M. Let o_e and b_f a fundamental circuit and a fundamental cocircuit with respect to B, then*

1. *$o_e \cap b_f$ is empty or $o_e \cap b_f = \{e, f\}$ and*
2. *$f \in o_e$ if and only if $e \in b_f$.*

Proof. (1) is immediate from Lemma 1.0.1 and the fact that $o_e \cap b_f \subseteq \{e, f\}$. (2) is a straightforward consequence of (1). $\square$

Lemma 1.0.3. *For any circuit o, and any elements e, f of o there is a cocircuit b such that $o \cap b = \{e, f\}$.*

Proof. Let B be a base extending the independent set $o - e$, so that o is the fundamental circuit of e with respect to B. Then the fundamental cocircuit of f has the desired property. $\square$

Given a matroid $M = (E, \mathcal{I})$ and $X \subseteq E$, M restricted to X is the matroid $(X, \mathcal{I} \cap 2^X)$, and is denoted by $M \upharpoonright X$ or $M \setminus X^c$. The contraction of M to X, $M.X$ or equally the matroid obtained by contracting X^c, M/X^c is defined to be $(M^* \upharpoonright X)^*$. Let X and Y be two disjoint subsets of E. Then $M/X \setminus Y = M \setminus Y/X$ is a minor of M obtained by contracting X and deleting Y.

Lemma 1.0.4. *([15]) The following statements are equivalent:*

(i) *I is a base of $M.X$.*

(ii) *There exists a base I' of $M \setminus X$ such that $I \cup I'$ is a base of M.*

(iii) *$I \cup I''$ is a base of M for every base I'' of $M \setminus X$.*

Corollary 1.0.5. *([15]) I is independent in $M.X$ if and only if $I \cup I'$ is independent in M for every independent set I' of $M \setminus X$.*

Lemma 1.0.6. *Let M be a matroid with ground set $E = C \dot\cup X \dot\cup D$ and o' be a circuit of $M' = M/C \setminus D$. Then there is an M-circuit o with $o' \subseteq o \subseteq o' \cup C$.*

Proof. Let B be any base of $M \upharpoonright C$. Then $B \cup o'$ is M-dependent since o' is M'-dependent. On the other hand, $B \cup o' - e$ is M-independent whenever $e \in o'$ since $o' - e$ is M'-independent. Putting this together yields that $B \cup o'$ contains an M-circuit o, and this circuit must not avoid any $e \in o'$, as desired. $\square$

Corollary 1.0.7. *Let M be a matroid with ground set $E = C \dot\cup \{x\} \dot\cup D$. Then either there is a circuit o of M with $x \in o \subseteq C + x$ or there is a cocircuit b of M with $x \in b \subseteq D + x$, but not both.*

Proof. Note that $(M/C\backslash D)^* = M^*/D\backslash C$, and apply Lemmas 1.0.1 and 1.0.6. $\square$

Lemma 1.0.8. *Let M be a matroid, $C, D \subseteq E$ with $C \cap D = \emptyset$ and let $M' := M/C \setminus D$ be a minor. Then there is an independent set S and a coindependent set R such that $M' = M/S \setminus R$.*

Proof. Let S be the union of a base of $M|C$ and a base of $M.D$ and let $R := (C \cup D) \setminus S$. In particular S is independent by Corollary 1.0.5. Since R is disjoint from some base extending S in $E\backslash(C \cup D)$, it is coindependent. In particular, any base of $M/S \setminus R$ spans M/S. For a set $B \subseteq E \setminus (C \cup D)$ we have:

$$
\begin{aligned}
&B \in \mathcal{B}(M \setminus D/C) \\
\Leftrightarrow &B \cup (C \cap S) \in \mathcal{B}(M \setminus D) \\
\Leftrightarrow &B \cup (C \cap S) \cup (D \cap S) \in \mathcal{B}(M) \\
\Leftrightarrow &B \cup S \in \mathcal{B}(M) \\
\Leftrightarrow &B \in \mathcal{B}(M/S) \\
\Leftrightarrow &B \in \mathcal{B}(M/S \setminus R).
\end{aligned}
$$
$\square$

Let $M = (E, \mathcal{I})$ be a set system. The set $\mathcal{I}^{\text{fin}}$ consists of the sets which have all their finite subsets in $\mathcal{I}$. $M^{\text{fin}} = (E, \mathcal{I}^{\text{fin}})$ is called *finitarisation* of M. M is called *finitary* if $M = M^{\text{fin}}$. Applying Zorn's Lemma one see that finitary set systems always satisfy (IM). It is easy to see that a matroid is finitary if and only if each of whose circuits is finite. A class of matroids naturally related to finitary one is that of cofinitary matroids. A matroid is called *cofinitary* if it is the dual of a finitary matroid. M is called *nearly finitary* if for any maximal element $B \in \mathcal{I}^{\text{fin}}$ there is an $I \in \mathcal{I}$ such that $|B \setminus I| < \infty$, or equivalently any base of M can be extended to a base of the finitarisation adding only finitely many elements. Nearly finitary matroids first appeared in [7] as a superclass of finitary matroids in which one can have an infinite matroid union theorem.

Connectivity in finite matroids stems from graph connectivity and is a well established part of the theory. In the infinite setting, Bruhn and Wollan [17] gave the following definition of connectivity that is compatible with the finite one. For an integer $k \geq 0$, a *k-separation* of a matroid is a partition of E into X and Y such that both $|X|, |Y| \geq k$ and for any bases

B_X, B_Y of $M \setminus Y$ and $M \setminus X$ respectively, the number of elements to be deleted from $B_X \cup B_Y$ to get a base of M is less than k. It can be proved that this number does not depend on the choice of B_X and B_Y [17]. A matroid is *k-connected* if there are no l-separations for any $l < k$.

Chapter 2

Thin sums matroids and duality

2.1 Introduction

The question addressed by this chapter is that of how to extend the notion of representability over a field from finitary to non-finitary matroids. The results of the chapter are those of [1].

If we have a (possibly infinite) family of vectors in a vector space over some field k, we get a matroid structure on that family whose independent sets are given by the linearly independent subsets of the family. Matroids arising in this way are called *representable* over k, and are always finitary.

Although many interesting finite matroids (eg. finite cycle matroids whose circuits are finite cycles of a graph) are representable, many interesting examples of infinite matroids cannot be of this type, because they are not finitary. Another problem is that in restricting attention to finitary matroids we would lose the power of duality: if a finite matroid is representable over the field k then so is its dual, but the dual of an infinite matroid representable over k need not be finitary. So it is natural to ask:

Question 2.1.1. Is there a good theory of representability over a field k of infinite matroids which is preserved under duality?

Bruhn and Diestel explored one approach to this question in [14]. They tried extending the notion of linear combinations to allow for infinite combinations in certain constrained circumstances.

The construction relies on taking the vector space to be of the form k^A for some set A. We allow linear combinations of infinitely many vectors. However, we require these linear combinations to be well defined pointwise. This means that for each $a \in A$ there are only finitely many nonzero coefficients at vectors with nonzero component at a (further details are given in Section 2.2). It turns out that there are examples of systems of independent sets definable in this way which are not matroids. Accordingly, we refer to such systems in general as *thin sums systems*, and call them thin sums matroids if they really are matroids. Thin sums matroids need not be finitary.

Bruhn and Diestel proved families of vectors, in which for each $a \in A$ there are only finitely many vectors in the family whose component at a is nonzero, always define matroids. Such families are called *thin*. We prove that a matroid M is the dual of a matroid representable over a field k if and only if M arises as a thin sums system over a thin family for the field k. In particular, a thin family always defines a cofinitary matroid.

It follows that the union of the class of (finitary) representable matroids with the class of thin sums matroids over thin families is closed under duality and under taking minors, suggesting a new definition of representability of (not necessarily finitary) infinite matroids. However, since many of the motivating examples are not finitary and do not have finitary duals, this union is not as comprehensive as one might hope. On the other hand, allowing all thin sums systems is too broad, though, as the class of thin sums matroids over $\mathbb{Q}$ is not closed under duality (this is shown in [12]).

We show that Question 2.1.1 can be resolved by restricting to the class of tame thin sums matroids. Most of the standard examples of infinite matroids are tame, and the class of tame matroids is closed under duality and under taking minors. In fact, the first wild matroids were constructed in [12]. (As we shall see in the next chapter, the class of gammoids also is a source for wild matroids.) In contrast to the bad behaviour of thin sums matroids in general, we prove that the class of tame thin sums matroids over any fixed field is closed under duality and under taking minors.

Any finite graphic matroid is representable over every field. The situation for infinite graphs is more complex, in that there is more than one natural way to build a matroid from an infinite graph. In [14], six matroids associated to a graph are defined in three dual pairs. We show that all six of these matroids are thin sums matroids over any field (this was already known for one of the six, and one of the others was already known to be representable).

In Section 2.2, we will introduce basics of representability and thin sums matroids. We will also introduce the six graphic matroids mentioned above. Section 2.3 will be devoted to thin sums matroids on thin families, and their duality with representable matroids. In Section 2.4, we will prove that the class of tame thin sums matroids is closed under duality and taking minors. Our account of why the various matroids associated to an infinite graph are thin sums matroids will be dispersed over all these sections: we give a summary of this aspect of the theory in Section 2.5.

2.2 Preliminaries

In this section, basics of representability, thin sums matroids and some of the matroids associated to infinite graphs are introduced.

We always use k to denote an arbitrary field. The capital letter V always stands for a vector space over k. For any set A, we write k^A to denote the set of all functions from A to k. For any function $E \xrightarrow{c} k$ the *support* $supp(c)$ of c is the set of all elements $e \in E$ such that $c(e) \neq 0$. A *linear dependence* of $E \xrightarrow{\phi} V$ is a map $E \xrightarrow{c} k$ of finite support such that

$$\sum_{e \in E} c(e)\phi(e) = 0\,.$$

For a subset E' of E, we say such a c is a *linear dependence* of E' if it is zero outside E'.

Let V be a vector space. Then for any function $E \xrightarrow{\phi} V$ we get a matroid $M(\phi)$ on the ground set E, where we take a subset E' of E to be independent if there is no nonzero linear dependence of E'. Such a matroid is traditionally called a *representable* or *vector matroid*. Note that this is essentially the same as taking a family of vectors as the ground set and saying that a subfamily of this family is independent if it is linearly independent.

In [14], there is an extension of these ideas to a slightly different context. Suppose now that we have a function $E \xrightarrow{f} k^A$. A *thin dependence* of f is a map $E \xrightarrow{d} k$, not necessarily of finite support, but such that for each $a \in A$,

$$\sum_{e \in E} d(e)f(e)(a) = 0.$$

(here, as in the rest of this chapter, we take this statement as including the claim that the sum is well-defined, i.e. that only finitely many summands are nonzero). This is subtly different from the concept of a linear dependence (in k^A considered as a vector space over k), since it is possible that the sum above might be well defined for each particular a in A, but the sum

$$\sum_{e \in E} d(e)f(e)$$

might still not be well defined. To put it another way, there might be infinitely many $e \in E$ such that there is *some* $a \in A$ with $d(e)f(e)(a) \neq 0$, even if there are only finitely many such e for each *particular* $a \in A$. We may also say d is a thin dependence of a subset E' of E if it is zero outside of E'.

The word *thin* above originated in the notion of a *thin family* - this is an f as above such that sums of the type given above are always defined; that is, for each a in A, there are only finitely many $e \in E$ so that $f(e)(a) \neq 0$. Notice that, for any $E \xrightarrow{f} k^A$, and any thin dependence c of f, the restriction of f to the support of c is thin.

Now we may define thin sums systems.

Definition 2.2.1. Consider a family $E \xrightarrow{f} k^A$ of functions and declare a subset of E as independent if there is no nonzero thin dependence of that subset. Let $M_{ts}(f)$ be the set system with ground set E and the set of all independent sets given in this way. We call $M_{ts}(f)$ the *thin sums system* corresponding to f. Whenever $M_{ts}(f)$ is a matroid it is called a thin sums matroid.

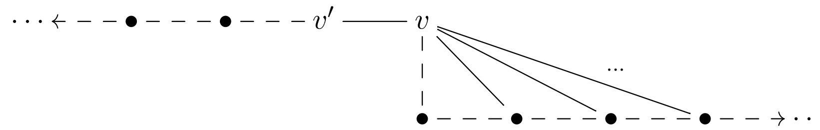
Since a set is dependent in a representable matroid or thin sums system if and only if it has a nonzero linear or thin dependence, we normally talk about such dependences instead of dependent sets.

It follows from Proposition 2.2.2 and the paragraph above it that not every thin sums system is a matroid but it is known that if f is thin then $M_{ts}(f)$ always is a matroid. The existing proof for this is technical and we shall not review it here [14]. However, this fact will follow from the results in Section 2.3. Next we explore the connection between representable and thin sums matroids. Recall that for any infinite matroid M, the finite circuits of M give the circuits of a new matroid on the same ground set, called the finitarisation of M [7].

Proposition 2.2.2. *For any thin sums matroid $M_{ts}(f)$, the finitarisation of $M_{ts}(f)$ is a representable matroid.*

Proof. For any family $E \xrightarrow{f} k^A$ of functions, a thin dependence of f with finite support is also a linear dependence of f as a family of vectors, and conversely any linear dependence of f as a family of vectors is a thin dependence of f. $\square$

Now let's try to answer to the question: Which matroids arising from graphs are representable or thin sums matroids? It is easy to see that any algebraic cycle matroid is a thin sums matroid (in fact, this was one motivation for the definition of thin sums matroids). Recall that for any graph G which does not contain a subdivision of the Bean graph,



the edge sets of cycles and double rays of G are circuits of a matroid [24] $M_A(G)$ on the edge set of G, called the *algebraic cycle matroid* of G (we call finite cycles and double rays *algebraic cycles*). In fact, even when G does contain a subdivision of the Bean graph we shall still denote this set system by $M_A(G)$, and call it the *algebraic cycle system* of G.

The algebraic cycle system of this graph doesn't satisfy (I3) - the dashed edges above form a maximal independent set, but there is no way to extend the nonmaximal independent set consisting of the edges meeting v (except vv') and those to the left of v' by an edge from this set.

Proposition 2.2.3. *For any graph G the algebraic cycle system of G is a thin sums system over every field.*

Proof. First we give an arbitrary orientation to every edge of G, making G a digraph. For any edge e of G define a function $V(G) \xrightarrow{f(e)} k$ where for any $v \in V(G)$ $f(e)(v)$ is 1 if e originates from v, -1 if it terminates in v, and 0 if e and v are not incident. We show that D is dependent in $M_A(G)$ if and only if it is dependent in $M_{ts}(f)$. If D is dependent in $M_A(G)$, then it contains a cycle or a double ray. Let $D' \subseteq D$ be the edge set of this cycle or double ray. Give a direction to D'. For any edge $e \in D$, define $c(e)$ to be 1 if e is an edge of D' and they have the same directions, -1 if e is in D' and they have different directions, and 0 if $e \notin D'$. Now clearly we have $\sum_{e \in D'} c(e)f(e)(v) = 0$ for any vertex v of G, so c is a thin dependence of D. Conversely if D is dependent in $M_{ts}(f)$, then whenever a vertex v is an end of an edge in D, it has to be the end of at least two edges in D. Now it is not difficult to see that D has to contain a cycle or a double ray. $\square$

Recall that the edge sets of finite cycles give the circuits of a matroid $M_{FC}(G)$ on the edge set of G, the *finite cycle matroid* of G. An argument almost identical to the one above shows that this matroid is always representable. Dually, the edge sets of finite bonds give the circuits of a matroid $M_{FB}(G)$, the *finite bond matroid* of G. Similar ideas allow us to show that for any graph G, $M_{FB}(G)$ is also representable.

Proposition 2.2.4. *For any graph G $M_{FB}(G)$ is representable over every field k.*

Proof. We start by giving fixed directions to every edge, cycle and finite bond. Let O be the set of all cycles of G and for any edge $e \in E(G)$ define a function $O \xrightarrow{\phi(e)} k$ such that for any $o \in O$, $\phi(e)(o)$ is 1 if $e \in o$ and they have the same directions, -1 if $e \in o$ and they have different directions, and 0 if e isn't an edge of o. This defines a map $E(G) \xrightarrow{\phi} k^O$. We will show $M(\phi) = M_{FB}(G)$.

We need to show that $D \subseteq E(G)$ is dependent in $M_{FB}(G)$ if and only if it is dependent in $M(\phi)$. If D is dependent in $M_{FB}(G)$ then it contains a finite bond D'. For any edge $e \in D'$ define $c(e)$ to be 1 if D' and e have the same directions, and -1 if they have different directions, and 0 if $e \notin D'$. Now consider a fixed cycle o which meets D'. Clearly D' has two sides and this cycle has to traverse D' from the first side to the second side as many times as it traverses D' from the second side to the first. As a result, for any $o \in O$ we have $\sum_{e \in E} c(e)\phi(e)(o) = 0$ and so c is a linear dependence of D.

Conversely, suppose that D is dependent in $M(\phi)$, and let D' be the support of any thin dependence of D. Whenever the edge set of a cycle meets D', they have to meet in at least two edges, which means D' (and so also D) meets every spanning tree. Thus D includes a bond and so it is a dependent set in $M_{FB}(G)$. □

Recall that for any graph G the (possibly infinite) bonds of G are the circuits of a matroid $M_B(G)$ on the edge set of G. In the above proof, we could exchange the role of finite bonds and arbitrary bonds and see that $M_B(G)$ is a thin sums matroid. We could also exchange the role of finite cycles and arbitrary bonds, and finite bonds and finite cycles, to get another proof of the fact that $M_{FC}(G)$ is representable. Recall that the finite cycle matroid and the bond matroid of a graph G are dual to each other [14].

As has been shown in [14], for any graph G the circuits of the dual of the finite bond matroid of G are given by the topological circles in a topological space associated to G. For this reason, $M^*_{FB}(G)$ is called the topological cycle matroid of G, and denoted $M_C(G)$. In the next section, we shall show that $M_C(G)$ is also a thin sums matroid.

We will only give a brief summary of the construction of the topological space behind the topological cycle matroid. A ray is a one-way infinite path. Two rays are edge-equivalent if for any finite set F of edges there is a connected component of $G \setminus F$ that contains subrays of both rays. The equivalence classes of this relation are the edge-ends of G; we denote the set of these edge-ends by $\mathcal{E}(G)$. Let us view the edges of G as disjoint topological copies of [0,1], and let X_G be the quotient space obtained by identifying these copies at their common vertices. The set of inner points of an edge e will be denoted by $\overset{o}{e}$. We now define a topological space $||G||$ on the point set of $X_G \cup \mathcal{E}(G)$ by taking as our open sets the union of sets $\tilde{C}$, where C is a connected component of $X_G \setminus Z$ for some finite set $Z \subset X_G$ of inner points of edges, and $\tilde{C}$ is obtained from C by adding all the edge-ends represented by a ray in C. For any $X \subseteq ||G||$ we call $\{e \in E(G) | \overset{o}{e} \subseteq X\}$ the edge set of X. A subspace C of $||G||$ that is homeomorphic to S^1 is a *topological circle* in $||G||$. In [14], it is shown that the edge sets of these circles in $||G||$ are the circuits of $M^*_{FB}(G)$.

2.3 Representable matroids and thin sums

In this section, we first show that any representable matroid is a thin sums matroid. After that we will characterise the dual of an arbitrary representable matroid and show that not only every representable matroid is a thin sums matroid but every matroid whose dual is representable is also a thin sums matroid. In fact, our last result is stronger; we show that the duals of representable matroids are precisely the thin sums matroids for thin families. Since the finite bond matroid of any graph is representable, this implies in particular that its dual, the topological cycle matroid, is a thin sums matroid.

As usual, let V^* be the dual of the vector space V (that is, the vector space consisting of all linear maps from V to k).

Theorem 2.3.1. *Consider a map $E \xrightarrow{\phi} V$ and the representable matroid $M(\phi)$. For any $e \in E$ and $\alpha \in V^*$ define $E \xrightarrow{f} k^{V^*}$ by $f(e)(\alpha) := \alpha(\phi(e))$. Then,*

$$M(\phi) = M_{ts}(f).$$

In particular, $M(\phi)$ is a thin sums matroid.

Proof. We show that I is independent in $M_{ts}(f)$ if and only if I is independent in $M(\phi)$. Suppose that I is independent in $M_{ts}(f)$. Suppose that $E \xrightarrow{c} k$ is any linear dependence of ϕ that is 0 outside I. For any $\alpha \in V^*$ we have,

$$\sum_{e \in E} c(e) f(e)(\alpha) = \sum_{e \in E} c(e) \alpha(\phi(e)) = \alpha \left(\sum_{e \in E} c(e) \phi(e) \right) = 0.$$

Thus c is a thin dependence of f, and since I is independent in $M_{ts}(f)$ we get that c must be the 0 map. So I is also independent in $M(\phi)$.

Conversely, suppose that I is independent in $M(\phi)$. Suppose $E \xrightarrow{c} k$ is any thin dependence of f that is 0 outside I. Let $I' = supp(c)$. Since $I' \subseteq I$, I' is also independent in $M(\phi)$, so (by extending the image of I' by ϕ to a basis of V) we can define a linear map $V \xrightarrow{\alpha_{I'}} k$ such that for any $i \in I'$, $\alpha_{I'}(\phi(i)) = 1$. As the restriction of f to $I' = supp(c)$ is thin and for any $i \in I'$ $f(i)(\alpha_{I'}) = \alpha_{I'}(\phi(i)) = 1$, I' has to be finite. So for every $\alpha \in V^*$,

$$\alpha\left(\sum_{e\in E} c(e)\phi(e)\right) = \sum_{e\in E} c(e)\alpha(\phi(e)) = \sum_{e\in E} c(e)f(e)(\alpha) = 0.$$

Since this is true for every $\alpha \in V^*$, we get that $\sum_{e\in I'} c(e)\phi(e) = 0$ which means c must be a linear dependence and so must be 0. Therefore I is also independent in $M_{ts}(f)$. $\square$

Now let's see how we can move from a representable matroid to its dual. Let's start with a family $E \xrightarrow{\phi} V$. Let C_ϕ be the set of all linear dependences of ϕ. We now define a map $E \xrightarrow{\widehat{\phi}} k^{C_\phi}$ by setting $\widehat{\phi}(e)(c) := c(e)$ for any $e \in E$ and $c \in C_\phi$. Clearly $\widehat{\phi}$ is a thin family of functions. On the other hand, if we let D_f be the set of thin dependences of a thin family $E \xrightarrow{f} k^A$, we get a map $E \xrightarrow{\overline{f}} k^{D_f}$ by setting $\overline{f}(e)(d) := d(e)$ for $e \in E$ and $d \in D_f$. These processes are, in a sense, inverse to each other.

Lemma 2.3.2. *For any thin family $E \xrightarrow{f} k^A$, a map $E \xrightarrow{d} k$ is a thin dependence of f if and only if it is a thin dependence of $\widehat{\overline{f}}$.*

Proof. First, suppose that d is a thin dependence of f. Then for any $c \in C_{\overline{f}}$ we have

$$\sum_{e\in E} d(e)\widehat{\overline{f}}(e)(c) = \sum_{e\in E} d(e)c(e) = \sum_{e\in E} c(e)\overline{f}(e)(d) = 0,$$

so d is also a thin dependence of $\widehat{\overline{f}}$.

Now suppose that d is a thin dependence of $\widehat{\overline{f}}$. For any $a \in A$, let $E \xrightarrow{c_a} k$ be defined by the equation $c_a(e) := f(e)(a)$. Since f is thin, $c_a(e)$ is nonzero for only finitely many values of e. Now for any thin dependence d' of f we have

$$\sum_{e\in E} c_a(e)\overline{f}(e)(d') = \sum_{e\in E} c_a(e)d'(e) = \sum_{e\in E} d'(e)f(e)(a) = 0,$$

and so $c_a \in C_{\overline{f}}$. Now, since d is a thin dependence of $\widehat{\overline{f}}$, we have

$$\sum_{e \in E} d(e) f(e)(a) = \sum_{e \in E} d(e) c_a(e) = \sum_{e \in E} d(e) \widehat{\overline{f}}(e)(c_a) = 0.$$

Since a was arbitrary, this says exactly that d is a thin dependence of f. $\square$

An analogous argument shows that for any map $E \xrightarrow{\phi} V$, the linear dependences of $\widehat{\widehat{\phi}}$ are exactly those of ϕ. We can also show that these inverse processes correspond to duality of matroids.

Theorem 2.3.3. *For any map $E \xrightarrow{\phi} V$ we have,*

$$M^*(\phi) = M_{ts}(\widehat{\phi}).$$

Proof. Suppose we have a set E_1 which is dependent in the dual of $M(\phi)$: that is, it meets every base of $M(\phi)$. Let $E_2 = E \backslash E_1$, so E_2 doesn't include any base of E - that is, E_2 doesn't span this matroid. Thus we can pick $e_1 \in E_1$ such that $\phi(e_1)$ isn't in the linear span of the family $(\phi(e)|e \in E_2)$. Consider a basis B_2 for this linear span, and extend $B_2 + \phi(e_1)$ to a basis B for V, and define a map $B \xrightarrow{h_0} k$ such that $h_0(\phi(e_1)) := 1$, and otherwise 0. Finally, extend h_0 to a linear map $V \xrightarrow{h} k$. Now, for any linear dependence c of ϕ we have

$$\sum_{e \in E} (h \cdot \phi)(e) \widehat{\phi}(e)(c) = h \left(\sum_{e \in E} c(e) \phi(e) \right) = 0$$

So $h \cdot \phi$ is a thin dependence of $\widehat{\phi}$, and since it is 0 outside E_1, E_1 is dependent with respect to $\widehat{\phi}$.

Conversely, suppose that E_1 is dependent in $M_{ts}(\widehat{\phi})$, so that there is a nonzero thin dependence d of $\widehat{\phi}$ which is 0 outside E_1. We want to show that E_1 meets every base of $M(\phi)$, so suppose for a contradiction that there is such a base B which it doesn't meet. Pick $e_1 \in E_1$ so that d is nonzero at e_1. We can express $\phi(e_1)$ as a linear combination of vectors from the family $(\phi(e)|e \in B)$ - that is, there is a linear dependence c of ϕ which is nonzero only on B and at e_1, with $c(e_1) = 1$. But then

$$d(e_1) = \sum_{e \in E} d(e) c(e) = \sum_{e \in E} d(e) \widehat{\phi}(e)(c) = 0,$$

which is the desired contradiction. Thus E_1 does meet every basis of $M(\phi)$, so it is dependent in the dual of $M(\phi)$. $\square$

Corollary 2.3.4. *For any thin family $E \xrightarrow{f} k^A$ we have,*

$$M_{ts}(f) = M^*(\overline{f}).$$

In particular $M_{ts}(f)$ is a cofinitary matroid.

Proof. This is immediate from Theorem 2.3.3, since by Lemma 2.3.2 we have $M_{ts}(f) = M_{ts}(\widehat{\overline{f}})$. $\square$

2.4 Finite circuit-cocircuit intersection and duality

A natural question about the class of thin sums matroids is whether or not it is closed under matroid duality: the fact that the class of representable matroids was not closed under duality was a key motivation for introducing extensions of this class, such as the class of thin sums matroids. Sadly, the class of thin sums matroids is not closed under duality: a counterexample is given in [12]. However, that counterexample involves a matroid with a very unusual property: it has a circuit and a cocircuit whose intersection is infinite. Matroids with this property are called *wild* matroids, and those in which every circuit-cocircuit intersection is finite are called *tame*.

Proposition 2.4.1. *The class of tame matroids is closed under duality and taking minors.*

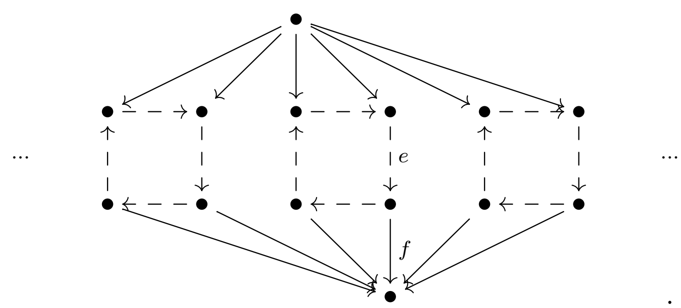
Proof. Closure under duality follows from the symmetry of the definition. For closure under taking minors, let M be a tame matroid with ground set $E = C \dot\cup X \dot\cup D$. We must show that $M' = M/C\backslash D$ is also tame. Let o be any circuit and b any cocircuit of M'. By Lemma 1.0.6 and its dual we can find a circuit o' of M with $o' \subseteq o \subseteq o' \cup C$ and a cocircuit b' of M with $b' \subseteq b \subseteq b' \cup D$. Thus $o \cap b = o' \cap b'$ is finite. $\square$

The main result of this section will be that the class of tame thin sums matroids is closed under duality. It will then quickly follow that it is also closed under taking minors.

The class of tame thin sums matroids includes most of the interesting examples arising from graphs: any finitary or cofinitary matroid must be tame, and this includes the finite and topological cycle matroids as well as the bond and finite bond matroids of a given graph. Recall that a non-empty cut C is called *skew* if (at least) one of its sides does not contain any ray (a one-way infinite path) and C is minimal with this property. In [14], it was proved that given a graph G with no subdivision of the bean graph, the skew cuts of G form the circuits of a matroid called *skew cut matroid* which is the dual of the algebraic cycle matroid of G. A straightforward application of Star-Comb Lemma [19] shows that algebraic cycle matroids and skew cut matroids also are tame,

A first attempt for showing that the dual of a thin sums matroid is again a thin sums matroid is suggested by the results of Section 2.3. These results suggest that in attempting to construct the representation $E \xrightarrow{\overline{f}} k^{\overline{A}}$ of $M^*_{ts}(f)$ we should take $\overline{A}$ to be the set of all thin dependencies of f, and define $\overline{f}(e)(c)$ to be $c(e)$. However, this natural attack fails to work, even if $M_{ts}(f)$ is tame, as our next example shows.

Example 2.4.2. Let G be the graph



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We may represent the algebraic cycle matroid of G as $M_{ts}(f)$ as in the proof of Proposition 2.2.3. Recall that for any edge e of G the function $V(G) \xrightarrow{f(e)} k$ is given by taking $f(e)(v)$ to be 1 if e originates from v, -1 if it terminates in v, and 0 if e and v are not incident. Thus the function which takes the value 1 on the dotted edges and 0 elsewhere is a thin dependence of f. So no function with support given by the skew cut consisting of the vertical dotted edges can be a thin dependence of $\overline{f}$ as given above. That is, for this matroid and this definition of $\overline{f}$, we have $M^* \neq M_{ts}(\overline{f})$.

Despite the existence of this example, as we shall see, the restriction of the $\overline{f}$ defined above to the set of thin dependences whose supports are circuits *does* give a representation of the dual of $M_{ts}(f)$.

Definition 2.4.3. An *affine equation* over a set I with coefficients in k consists of a family $(\lambda_i \in k | i \in I)$ such that only finitely many of the λ_i are nonzero and an element κ of k.

A family $x = (x_i | i \in I)$ is a *solution* of the equation (λ, κ) if $\sum_{i \in I} \lambda_i x_i = \kappa$. Accordingly, we shall use the expression $\ulcorner \sum_{i \in I} \lambda_i x_i = \kappa \urcorner$ to denote the equation (λ, κ). x is a solution of a set Q of equations if it is a solution of every equation in Q.

The following lemma is based on ideas of Bruhn and Georgakopoulos [16], though the proof we give is a little simpler.

Lemma 2.4.4. *If every finite subset of a set Q of affine equations over I with coefficients in k has a solution then so does Q.*

Proof. The set V of affine equations over I can be given the structure of a vector space over k, with $\mu((\lambda_i|i \in I), \kappa) = ((\mu\lambda_i|i \in I), \mu\kappa)$ and $((\lambda_i|i \in I), \kappa) + ((\lambda'_i|i \in I), \kappa') = ((\lambda_i + \lambda'_i|i \in I), \kappa + \kappa')$. Let W be the subspace generated by the equations in Q, and let q_0 be the affine equation $0 = 1$ (that is, $((0|i \in I), 1)$), which has no solutions. It is clear that if $(a_i|i \in I)$ is a solution of all the equations in some finite set Q' then it is also a solution of everything in their linear span in V. So q_0 can't be in W. By choosing a basis of W and extending it to a basis of V that contains q_0, we can construct a linear map $V \xrightarrow{\alpha} k$ which is 0 on W but with $\alpha(q_0) = 1$. For each $i \in I$ let $a_i = -\alpha(\ulcorner x_i = 0 \urcorner)$. Then for each equation $q \in Q$, given as $\ulcorner \sum_{i \in I} \lambda_i x_i = \kappa \urcorner$, we have $\sum_{i \in I} \lambda_i a_i = \alpha(\kappa q_0 - q) = \kappa$, so a is a solution of every equation in Q.

$\square$

Lemma 2.4.5. *Let d be a thin dependence of f. Then $supp(d)$ is a union of minimal dependent sets of $M_{ts}(f)$.*

Proof. Let $I = supp(d)$. It suffices to show that for any $e_0 \in I$ there is a minimal dependent set which contains e_0 and is a subset of I. We begin by fixing such an e_0.

For any $a \in A$ there are only finitely many $e \in I$ with $f(e)(a) \neq 0$, so for any $a \in A$ we get an affine equation $\ulcorner \sum_{e \in I} f(e)(a) x_e = 0 \urcorner$ over I. Let $\mathcal{Q}$ be the set of all affine equations arising in this way. Let $\mathcal{E}$ be the set of all subsets I' of I such that every finite subset of $\mathcal{Q} \cup \{\ulcorner x_e = 0 \urcorner | e \in I'\} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$ has a solution. Since $d \restriction I$ is a solution of all equations in $\mathcal{Q}$, $(d \restriction I)/d(e_0)$ is a solution of all equations in $\mathcal{Q} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$, so $\emptyset \in \mathcal{Q}$. $\mathcal{E}$ is also closed under unions of chains, so by Zorn's lemma it has a maximal element E_m. Now by Lemma 2.4.4 there is some solution d' of all the equations in $\mathcal{Q} \cup \{\ulcorner x_e = 0 \urcorner | e \in E_m\} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$. Since d' solves all the equations in $\mathcal{Q}$, its extension to E taking the value 0 outside I is a thin dependence of f.

We shall show that $D := supp(d') = E \setminus E_m$ is the desired minimal dependent set. If it were not, there would have to be a nonzero thin dependence d'' with $supp(d'') \subseteq supp(d') - e_0$. But then for any $e_1 \in$

$supp(d'')$, we have that $d' - \frac{d'(e_1)}{d''(e_1)} d'' \restriction I$ is a solution of $\ulcorner x_{e_1} = 0 \urcorner$ in addition to the equations solved by d', which contradicts the maximality of E_m. $\square$

Corollary 2.4.6. *If $M_{ts}(f)$ is a matroid, and $E' \subseteq E$, then $e \notin E'$ is in the closure of E' if and only if there is a thin dependence d with $supp(d) \subseteq E' \cup \{e\}$ and $d(e) = 1$.*

Proof. If there is such a d, by Lemma 2.4.5 we can find a minimal dependent set D with $e \in D \subseteq supp(d)$. As $D \setminus \{e\} \subseteq E'$ is independent, $e \in cl(E')$. If $e \in cl(E')$ then there is a dependent set D with $e \in D \subseteq E' \cup \{e\}$ and so there is thin dependence d with $supp(d) \subseteq E' \cup \{e\}$ and $d(e) = 1$. $\square$

Corollary 2.4.7. *Let $M_{ts}(f)$ be a matroid. Then a subset I is independent in $M^*_{ts}(f)$ if and only if for every $i \in I$ there is a thin dependence d_i of f such that $d_i(i) = 1$ and d_i is 0 on the rest of I.*

Proof. We recall that I is independent in $M^*_{ts}(f)$ if and only if $cl(I^c) = E$. Now apply Corollary 2.4.6. $\square$

Lemma 2.4.8. *Let $M_{ts}(f)$ be a matroid, I coindependent and $i_0 \notin I$. If there is a thin dependence d which is nonzero at i_0 and 0 on I, then $I \cup \{i_0\}$ is coindependent.*

Proof. By Corollary 2.4.7, suppose $(d_i | i \in I)$ witnesses the coindependence of I. Let $d'_{i_0} = d/d(i_0)$, and for $i \in I$, $d'_i = d_i - d_i(i_0) d'_{i_0}$. Then $(d'_i | i \in I \cup \{i_0\})$ witnesses the coindependence of $I \cup \{i_0\}$. $\square$

We can now achieve the goal of this section.

Theorem 2.4.9. *Suppose that $M_{ts}(f)$ is a tame matroid, and let D be the set of thin dependences of f whose supports are circuits of $M_{ts}(f)$. We get a map $E \xrightarrow{g} k^D$ where for $e \in E$ and $d \in D$, $g(e)(d) := d(e)$. Then,*

$$M^*_{ts}(f) = M_{ts}(g).$$

Proof. Suppose that I is independent in $M^*_{ts}(f)$ with $(d_i | i \in I)$ given as in Corollary 2.4.7. Without loss of generality, by Lemma 2.4.5, we can suppose that d_i are in D. Suppose that d is a thin dependence of g which is 0 outside I. For any $i_0 \in I$ we have,

$$d(i_0) = \sum_{e \in E} d(e) d_{i_0}(e) = \sum_{e \in E} d(e) g(e)(d_{i_0}) = 0,$$

and so d is 0 which means I is independent in $M_{ts}(g)$.

Conversely, suppose that I is dependent in $M^*_{ts}(f)$. Let $C \subseteq I$ be a circuit of $M^*_{ts}(f)$, and $i_0 \in C$. $C - i_0$ is independent in $M^*_{ts}(f)$ and so ,there is a family $(d'_i | i \in C - i_0)$ as defined in Corollary 2.4.7. Now, define the map $E \xrightarrow{d'} k$ to be 1 at i_0, $-d'_i(i_0)$ for $i \in C - i_0$ and 0 on the rest of E. We claim that d' is a thin dependence of g, and as it is 0 outside I, I is dependent in $M_{ts}(g)$.

First, note that the map $E \xrightarrow{d''} k$ given by $d''(e) := d(e) - \sum_{i \in C - i_0} d'_i(e)d(i)$ is a linear combination of thin dependences of f and so is a thin dependence. Clearly for any $i \in C - i_0$ we have $d''(i) = 0$, and as $C - i_o$ is coindependent, if d'' takes a nonzero value at i_0, applying Lemma 2.4.8, C must be independent in $M^*_{ts}(f)$ which is a contradiction. Therefore, $d''(i_0)$ has to be 0, and so

$$\sum_{i \in E} d'(i)g(i)(d) = \sum_{i \in C} d'(i)d(i) = d(i_0) - \sum_{i \in C - i_0} d'_i(i_0)d(i) = d''(i_0) = 0$$

which means d' is a thin dependence of g, and so I is dependent in $M_{ts}(g)$. □

Corollary 2.4.10. *The class of tame thin sums matroids is closed under duality, restriction, and so taking minors.*

Proof. Clear by Theorem 2.4.9 and the fact that thin sums matroids are closed under matroid restrictions. □

2.5 Overview of the connections to graphic matroids

Our results on graphic matroids have been scattered through the paper. We can now make use of Proposition 2.2.2 and Theorem 2.4.10, and go on a short tour of the standard matroids arising from an infinite graph G. We shall recall why all of them are tame thin sums matroids over any field, and a couple of them are representable over any field. Since we want our results to apply to any field, we continue to work over an arbitrary fixed field k.

Our starting point is the most algebraic of examples, the algebraic cycle matroid, for which we gave a thin sums representation in Proposition 2.2.3. Applying Theorem 2.4.10, we deduce that the dual $M^*_A(G)$, the skew cuts matroid of G, is also a thin sums matroid. We can also apply Proposition

2.2.2 to deduce that the finitarisation of $M_A(G)$, that is, the finite cycle matroid $M_{FC}(G)$, whose circuits are the cycles of G, is representable.

Applying Theorem 2.4.10, its dual, the bond matroid $M_B(G)$, whose circuits are (possibly infinite) bonds, is a thin sums matroid. So by Proposition 2.2.2, the finite bond matroid $M_{FB}(G)$ is representable. Applying Theorem 2.4.10 one more time, we recover the fact that the topological cycle matroid $M_C(G)$ is a thin sums matroid.

We could, of course, continue this process further, but it quickly becomes periodic, as sketched out in the following diagram:

$$
\begin{tikzcd}
\text{Algebraic cycles} \arrow[r, "dual"] \arrow[d, "fin"'] & \text{Skew cuts} \\
\begin{array}{c}\text{Finite cycles}\\ \text{(representable)}\end{array} \arrow[r, "dual"] & \text{Bonds} \arrow[d, "fin"] \\
\text{Topological cycles} \arrow[d, "fin"'] & \begin{array}{c}\text{Finite bonds}\\ \text{(representable)}\end{array} \arrow[l, "dual"] \\
\begin{array}{c}\text{Finite cycles}\\ \text{of FSep}(G)\\ \text{(representable)}\end{array} \arrow[r, "dual"'] & \begin{array}{c}\text{Bonds}\\ \text{of FSep}(G)\end{array} \arrow[u, "fin"']
\end{tikzcd}
$$

Here $\mathrm{FSep}(G)$ is the finitely separable quotient of G, obtained from G by identifying any two vertices which cannot be separated by removing only finitely many edges from G.

Chapter 3

Infinite gammoids

3.1 Introduction

This chapter mainly studies the system of linkable sets in a dimaze and consists of the results in [2], [3] and [4].

A *dimaze* is a digraph equipped with a fixed set of sinks which are called the *exits* (of the dimaze). A dimaze *contains* another dimaze, if, in addition to digraph containment, the exits of the former include those of the latter. In the context of digraphs, any path or ray (a one-way infinite path) is forward oriented.

A *strict gammoid* is a matroid isomorphic to the one defined on the vertex set of (the digraph of) a dimaze, whose independent sets are those subsets that are linkable to the exits by a set of disjoint paths ([27]). A *gammoid* is a matroid restriction of a strict gammoid. Any dimaze defining a given gammoid is a *presentation* of that gammoid.

Mason [27] proved that every finite dimaze defines a strict gammoid. When a dimaze is infinite, Perfect [32] gave sufficient conditions for when some subset of system of linkable sets gives rise to a finitary matroid. Since finitary matroids were the only ones known at that time, infinite dimazes which define non-finitary gammoids were not considered to define matroids.

With infinite matroids canonically axiomatized in a way that allows for non-finitary ones, a natural question is whether the set of linkable sets in a dimaze satisfies matroid axioms (Section 3.3). Clearly, the system of linkable sets of any dimaze satisfies (I1) and (I2). We use a result of Grünwald [22] to prove that it also satisfies (I3). However, the matroid axiom (IM) may fail. An *alternating comb* is a dimaze obtained by taking the union of a ray (with directed edges) whose first vertex has indegree one and which has infinitely many vertices of indegree two, and a set of disjoint paths from these vertices meeting the ray precisely at their initial vertices and declaring the sinks of the resulting digraph as the set of exits. We invoke a result of Pym [33] to prove that the system of linkable sets of any dimaze not containing alternating comb satisfies (IM). Section 3.4 is devoted to construct examples of gammoids with certain properties including a strict gammoid with every presentation containing an alternating comb.

Recall that, by definition, the class of gammoids is closed under matroid deletion. A pleasant property of the class of finite gammoids is that it is also closed under matroid contraction, and hence, under taking minor. In contrast, whether the class of possibly infinite gammoids is minor-closed is an open question. In Section 3.5, we begin to address this and related questions.

A standard proof of the fact that the finite gammoids are minor-closed proceeds via duality. The proof of this fact can be extended to infinite dimazes whose underlying (undirected) graph does not contain any ray (Section 3.6.3), but it breaks down when rays are allowed. An *outgoing comb* is a dimaze obtained from a ray, called the *spine*, by adding infinitely many non-trivial disjoint paths, that meet the ray precisely at their initial vertices, and declaring the sinks of the resulting digraph to be the exits. By developing an infinite version of a construction in [9], we prove that the class of gammoids that admit a presentation not containing any outgoing comb is minor-closed.

Topological gammoids were introduced by Carmesin in [18], where, developing a new notion of linkability, he proved that any dimaze gives rise to a finitary matroid on the vertex set of the dimaze, called a *topological gammoid* (see Section 3.2.1 for the definition). Applying the tools developed in Section 3.5.1, it is proved in Section 3.5.2 that the class of topological gammoids coincides with the class of finitary gammoids, which yields the fact that the class of topological gammoids is minor-closed.

In Section 3.6, we turn to duality. Recall that a *transversal matroid* is a matroid isomorphic to a matroid defined by taking a fixed vertex class of a bipartite graph as the ground set and its matchable subsets as the independent sets. A result due to Ingleton and Piff [26] states that finite strict gammoids and finite transversal matroids are dual to each other, a key fact to the result that the class of finite gammoids is closed under duality. In contrast, an infinite strict gammoid need not be dual to any transversal matroid, and vice versa (Examples 3.6.11 and 3.6.18). Despite these examples, it might still be possible that the class of infinite gammoids is closed under duality. However, we will see in Section 3.6.3 that there is a gammoid, which is not dual to any gammoid.

In Section 3.6.1, we aim to describe the dual of the strict gammoids introduced in Section 3.3; the strict gammoids which admit a presentation not containing any alternating comb. It turns out that there exists a strict gammoid in this class that is not dual to any transversal matroid. For this reason, we first extend the class of transversal matroids to that of *path-transversal matroids*. Then we prove that a strict gammoid that admits a presentation not containing any alternating comb is dual to a path-transversal matroid (Theorem 3.6.9).

In Section 3.7, we restrict our attention to several subclasses of the class of gammoids and transversal matroids. We first use the tools mainly extended in Section 3.6.1 to give characterizations of cofinitary strict gammoids and cofinitary transversal matroids. Then we investigate the class of nearly finitary gammoids and nearly finitary transversal matroids which are two superclasses of the finitary ones.

3.2 Preliminaries

We collect definitions, basic results and examples. Many of the definitions and the notations come from [19].

3.2.1 Linkability system

From now on, digraphs do not have any loop or parallel adges. Given a digraph D, let $V := V(D)$ and $B_0 \subseteq V$ be a set of sinks. Call the pair (D, B_0) a *dimaze*[1] and B_0 the (set of) *exits*. Given a (directed) path or ray P, $Ini(P)$ and $Ter(P)$ denote the initial and the terminal vertex (if exists) of P, respectively. For a set $\mathcal{P}$ of paths and rays, then $Ini(\mathcal{P}) = \{Ini(P) : P \in \mathcal{P}\}$ and $Ter(\mathcal{P}) = \{Ter(P) : P \in \mathcal{P}\}$. A *linkage* $\mathcal{P}$ is a set of (vertex disjoint) paths ending in B_0. A set $A \subseteq V$ is *linkable* if there is a linkage $\mathcal{P}$ from A to B, i.e. $Ini(\mathcal{P}) = A$ and $Ter(\mathcal{P}) \subseteq B$; $\mathcal{P}$ is *onto* B if $Ter(\mathcal{P}) = B$.

Note that, by adding trivial paths if required:

Any linkable set in (D, B_0) can be extended to one linkable onto B_0. (3.1)

Definition 3.2.1. Let (D, B_0) be a dimaze. The pair of $V(D)$ and the set of linkable subsets is called the *linkability system* of (D, B_0) and denoted by $M_L(D, B_0)$. When $M_L(D, B_0)$ is a matroid, it is called a *strict gammoid.* A

[1] Dimaze is short for directed maze.

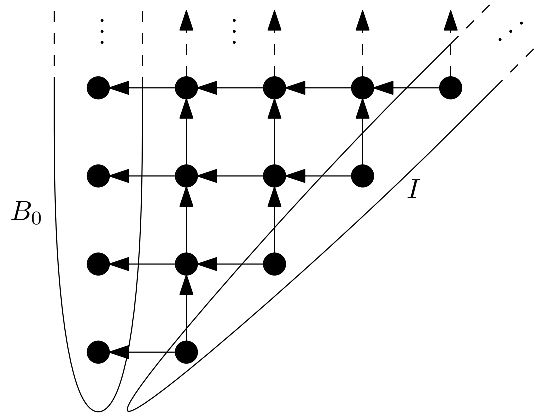

Figure 3.1: A locally finite dimaze which does not define a matroid

gammoid is a (matroid) restriction of a strict gammoid. Given a gammoid M, (D, B_0) is a presentation of M if $M = M_L(D, B_0) \restriction X$ for some $X \subseteq V(D)$.

If D' is a subdigraph of D and $B_0' \subseteq B_0$, then (D, B_0) contains (D', B_0') as a *subdimaze*. A dimaze (D', B_0') is a *subdivision* of (D, B_0) if it can be obtained from (D, B_0) as follows. We first add an extra vertex b_0 and the edges $\{(b, b_0) : b \in B_0\}$ to D. Then the edges of this resulting digraph are subdivided to define a digraph D''. Set B_0' as the in-neighbourhood of b_0 in D'' and D' as $D'' - b_0$. Note that this defaults to the usual notion of subdivision if $B_0 = \emptyset$.

Mason [27] (see also [32]) showed that $M_L(D, B_0)$ is a matroid for any finite dimaze (D, B_0). However, this is not the case for infinite dimazes. For example, let D be a complete bipartite graph between an uncountable set X and a countably infinite set B_0 with all the edges directed towards B_0. Then $I \subseteq X$ is independent if and only if I is countable. So there is no maximal independent set in X, hence $M_L(D, B_0)$ does not satisfy the axiom (IM).

Example 3.2.2. Here is a counterexample whose digraph is locally finite. Let D be the digraph obtained by directing upwards or leftwards the edges of the subgraph of the grid $\mathbb{Z} \times \mathbb{Z}$ induced by $\{(x, y) : y > 0 \text{ and } y \geq x \geq 0\}$ and let $B_0 := \{(0, y) : y > 0\}$, see Figure 3.1. Then $I := \{(x, x) : x > 0\}$ is linkable onto a set $J \subseteq B_0$ if and only if J is infinite. Therefore, $I \cup (B_0 J)$ is independent if and only if J is infinite. Hence, I does not extend to a maximal independent set in $X := I \cup B_0$.

The following dimazes play an important role in our investigation (see

Figure 3.6). An undirected ray is a graph with an infinite vertex set $\{x_i : i \geq 1\}$ and the edge set $\{x_i x_{i+1} : i \geq 1\}$. We orient the edges of a ray in different ways to construct three dimazes:

1. R^A by orienting (x_{i+1}, x_i) and (x_{i+1}, x_{i+2}) for each odd $i \geq 1$ and the set of exits is empty;

2. R^I by orienting (x_{i+1}, x_i) for each $i \geq 1$ and x_1 is the only exit;

3. R^O by orienting (x_i, x_{i+1}) for each $i \geq 1$ and the set of exits is empty.

Any subdivision of R^A, R^I and R^O is called an *alternating ray*, an *incoming ray*, and a *ray*, respectively.

Let $Y = \{y_i : i \geq 1\}$ be a set disjoint from X. We extend the above rays to combs by adding edges (and their terminal vertices) and declaring the resulting sinks to be the exits:

1. C^A by adding no edges to R^A;

2. C^I by adding the edges (x_i, y_i) to R^I for each $i \geq 2$;

3. C^O by adding the edges (x_i, y_i) to R^O for each $i \geq 2$.

Furthermore, we define the dimaze F^∞ by declaring the sinks of the digraph $(\{v, v_i : i \in \mathbb{N}\}, \{(v, v_i) : i \in \mathbb{N}\})$ to be the exits.

Any subdivision of C^A, C^I, C^O and F^∞ is called an *alternating comb*, an *incoming comb*, a *comb* and a *linking fan*, respectively. The subdivided ray in any comb is called the *spine* and the paths to the exits are the *spikes*.

A dimaze (D, B_0) is called *$\mathcal{H}$-free* for a set $\mathcal{H}$ of dimazes if it does not have a subdimaze isomorphic to a subdivision of an element in $\mathcal{H}$. A (strict) gammoid is called *$\mathcal{H}$-free* if it admits an $\mathcal{H}$-free presentation.

In general, a $\mathcal{H}$-free gammoid may admit a presentation which contains a subdivision of an element of $\mathcal{H}$ (see Figure 3.6 for $\mathcal{H} = C^A$). This class is also a fruitful source for *wild* matroids ([12]).

Let (D, B_0) be a dimaze and $\mathcal{Q}$ a set of disjoint paths or rays (usually a linkage). A *$\mathcal{Q}$-alternating walk* is a sequence $W = w_0 e_0 w_1 e_1 \ldots$ of vertices w_i and distinct edges e_i of D not ending with an edge, such that every $e_i \in W$ is incident with w_i and w_{i+1}, and the following properties hold for each $i \geq 0$ (and $i < n$ in case W is finite, where w_n is the last vertex):

(W1) $e_i = (w_{i+1}, w_i)$ if and only if $e_i \in E(\mathcal{Q})$;

(W2) if $w_i = w_j$ for any $j \neq i$, then $w_i \in V(\mathcal{Q})$;

(W3) if $w_i \in V(\mathcal{Q})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{Q}) \neq \emptyset$ (with $e_{-1} := e_0$).

Let $\mathcal{P}$ be another set of disjoint paths or outgoing rays. A $\mathcal{P}$-$\mathcal{Q}$-*alternating walk* is a $\mathcal{Q}$-alternating walk whose edges are in $E(\mathcal{P})\Delta E(\mathcal{Q})$, and such that any interior vertex w_i satisfies

(W4) if $w_i \in V(\mathcal{P})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{P}) \neq \emptyset$.

Two $\mathcal{Q}$-alternating walks W_1 and W_2 are *disjoint* if they are edge disjoint, $V(W_1) \cap V(W_2) \subseteq V(\mathcal{Q})$ and $Ter(W_1) \neq Ter(W_2)$.

Suppose that a dimaze (D, B_0), a set $X \subseteq V$ and a linkage $\mathcal{P}$ from a subset of X to B_0 are given. An X–B_0 (vertex) separator S is a set of vertices such that every path from X to B_0 intersects S, and S is *on* $\mathcal{P}$ if it consists of exactly one vertex from each path in $\mathcal{P}$.

We recall two classical results. The first one is due to Grünwald [22], and can be formulated as follows (see also [19, Lemmas 3.3.2 and 3.3.3])[2].

Lemma 3.2.3. *Let (D, B_0) be a dimaze, $\mathcal{Q}$ a linkage, and $Ini(\mathcal{Q}) \subseteq X \subseteq V$.*

(i) If there is a $\mathcal{Q}$-alternating walk from $X \setminus Ini(\mathcal{Q})$ to $B_0 \setminus Ter(\mathcal{Q})$, then there is a linkage $\mathcal{Q}'$ with $Ini(\mathcal{Q}) \subsetneq Ini(\mathcal{Q}') \subseteq X$ onto $Ter(\mathcal{Q}) \subsetneq Ter(\mathcal{Q}') \subseteq B_0$.

(ii) If there is not any $\mathcal{Q}$-alternating walk from $X \setminus Ini(\mathcal{Q})$ to $B_0 \setminus Ter(\mathcal{Q})$, then there is a X–B_0 separator on $\mathcal{Q}$.

The second one is the linkage theorem of Pym [33], which a sketch of whose proof is given.

Linkage Theorem. *Let D be a digraph and two linkages be given: the "red" one, $\mathcal{P} = \{P_x : x \in X_\mathcal{P}\}$, from $X_\mathcal{P}$ onto $Y_\mathcal{P}$ and the "blue" one, $\mathcal{Q} = \{Q_y : y \in Y_\mathcal{Q}\}$, from $X_\mathcal{Q}$ onto $Y_\mathcal{Q}$. Then there is a set X^∞ satisfying $X_\mathcal{P} \subseteq X^\infty \subseteq X_\mathcal{P} \cup X_\mathcal{Q}$ which is linkable onto a set Y^∞ satisfying $Y_\mathcal{Q} \subseteq Y^\infty \subseteq Y_\mathcal{Q} \cup Y_\mathcal{P}$.*

Proof outline. We construct a sequence of linkages converging to a linkage with the desired properties. For each integer $i \geq 0$, we will specify a vertex on each path in $\mathcal{P}$. For each $x \in X_\mathcal{P}$, let $f_x^0 := x$. Let $\mathcal{Q}^0 := \mathcal{Q}$. For each $i > 0$ and each $x \in X_\mathcal{P}$, let f_x^i be the last vertex v on $f_x^{i-1} P_x$ such that

[2] With alternating walks defined as here, the proofs given in [19] work almost verbatim for general dimazes.

$(f_x^{i-1} P_x \mathring{v}) \cap V(\mathcal{Q}^{i-1}) = \emptyset$. For $y \in Y_{\mathcal{Q}}$, let t_y^i be the first vertex $v \in Q_y$ such that the terminal segment $\mathring{v} Q_y$ does not contain any f_x^i. Let

$$\begin{aligned}
\mathcal{A}^i &:= \{Q_y \in \mathcal{Q} : t_y^i \neq f_x^i \ \forall x \in X_{\mathcal{P}}\}, \\
\mathcal{B}^i &:= \{P_x f_x^i Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^i = t_y^i\}, \\
\mathcal{C}^i &:= \{P_x \in \mathcal{P} : f_x^i \in Y_{\mathcal{P}} \text{ and } f_x^i \neq t_y^i \ \forall y \in Y_{\mathcal{Q}}\},
\end{aligned}$$

and $\mathcal{Q}^i := \mathcal{A}^i \cup \mathcal{B}^i \cup \mathcal{C}^i$. It can be shown that $\mathcal{Q}^i$ is a linkage. Moreover, for any $x \in X_{\mathcal{P}}$, $\{f_x^i\}_{i \geq 0}$ eventually settles at a vertex f_x^∞ as $i \to \infty$; similarly for any $y \in Y_{\mathcal{Q}}$, $\{t_y^i\}_{i \geq 1}$ settles at some t_y^∞. Then $\mathcal{Q}^\infty$, defined as the union of the following three sets,

$$\begin{aligned}
\mathcal{A}^\infty &:= \{Q_y \in \mathcal{Q} : t_y^\infty \neq f_x^\infty \ \forall x \in X_{\mathcal{P}}\}, \\
\mathcal{B}^\infty &:= \{P_x f_x^\infty Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^\infty = t_y^\infty\}, \\
\mathcal{C}^\infty &:= \{P_x \in \mathcal{P} : f_x^\infty \in Y_{\mathcal{P}} \text{ and } f_x^\infty \neq t_y^\infty \ \forall y \in Y_{\mathcal{Q}}\},
\end{aligned}$$

is a linkage satisfying the requirements. □

A set $X \subseteq V$ in (D, B_0) is *topologically linkable* if X admits a *topological linkage*, which means that from each vertex $x \in X$, there is a *topological path* P_x, i.e. P_x is the spine of a comb, a path ending in the centre of a linking fan, or a path ending in B_0, such that P_x is disjoint from P_y for any $y \neq x$. Clearly, a finite topologically linkable set is linkable. Denote by $M_{TL}(D, B_0)$ the pair of V and the set of the topologically linkable subsets. Carmesin gave the following connection between dimazes (not necessarily defining a matroid) and topological linkages.

Corollary 3.2.4. *[18, Corollary 5.7] Given a dimaze (D, B_0), $M_{TL}(D, B_0) = M_L(D, B_0)^{\text{fin}}$. In particular, $M_{TL}(D, B_0)$ is always a finitary matroid.*

A *strict topological gammoid* is a matroid of the form $M_{TL}(D, B_0)$, and a restriction of which is called a *topological gammoid*.

3.2.2 Transversal system

Let $G = (V, W)$ be a bipartite graph and call V and W, respectively, the *left* and the *right* vertex class of G. A subset I of V is *matchable onto* $W' \subseteq W$ if there is a matching m of I such that $m \cap V = I$ and $m \cap W = W'$; where we are identifying a set of edges (and sometimes more generally a subgraph) with its vertex set.

Definition 3.2.5. Let $G = (V, W)$ be a bipartite graph. The pair of V and the set of all its matchable subsets is called the *transversal system* of G and denoted by $M_T(G)$. When $M_T(G)$ is a matroid, it is called a *transversal matroid*. Given a transversal matroid M, G is a *presentation* of M if $M = M_T(G)$.

In general, a transversal system may admit different presentations. The following is a well-known fact (see [13]).

Lemma 3.2.6. *Let $G = (V, W)$ be a bipartite graph. Suppose there is a maximal element in $M_T(G)$, witnessed by a matching m_0. Then $M_T(G) = M_T(G \setminus (W - m_0))$, and $N(W - m_0)$ is a subset of every maximal element in $M_T(G)$.*

In case $M_T(G)$ is a matroid, the second part states that $N(W - m_0)$ is a set of coloops. From now on, wherever there is a maximal element in $M_T(G)$, we assume that W is covered by a matching. The following is easy.

Lemma 3.2.7. *Given a bipartite graph $G = (V, W)$ and $X \subseteq V$, $M_T(G) \upharpoonright X = M_T(G')$, where G' is the induced bipartite graph on $X \cup N(X)$.*

If G is finite, Edmonds and Fulkerson [21] showed that $M_T(G)$ satisfies (I3), and so is a matroid. When G is infinite, $M_T(G)$ still satisfies (I3), but need not be a matroid. Given a matching m, an edge in m is called an *m-edge*, and an *m-alternating walk* is a walk such that the consecutive edges alternate in and out of m in G. Given another matching m', an *m-m'-alternating walk* is a walk such that consecutive edges alternate between the two matchings.

Lemma 3.2.8. *For any bipartite graph G, $M_T(G)$ satisfies (I3).*

Proof. Let $I, B \in M_T(G)$ such that B is maximal but I is not. As I is not maximal, there is a matching m of $I + x$ for some $x \in V \setminus I$. Let m' be a matching of B to W. Consider a miximal m–m'-alternating walk P starting from x. By maximality of B, this walk is not infinite and cannot end in $W \setminus m'$. So it has to end at some $y \in B \setminus I$. Then $m \Delta E(P)$ is a matching of $I + y$, which completes the proof. □

A standard compactness proof shows that a left locally finite bipartite graph $G = (V, W)$, i.e. every vertex in V has finite degree, defines a finitary transversal matroid.

Lemma 3.2.9 ([28])**.** *Every left locally finite bipartite graph defines a finitary transversal matroid.*

The following corollary is a tool to show that a matroid is not transversal.

Lemma 3.2.10. *Any infinite circuit of a transversal matroid contains an element which does not lie in any finite circuit.*

Proof. Let C be an infinite circuit of some $M_T(G)$. Applying Lemma 3.2.9 on the restriction of $M_T(G)$ to C, we see that there is a vertex in C having infinite degree. However, such a vertex cannot not lie in any finite circuit. $\square$

3.3 Dimazes and matroid axioms

The next two sections investigate basics of infinite gammoids and consist of the results in [2].

This section aims to give a sufficient condition for a dimaze (D, B_0) to define a matroid. As (I1) and (I2) hold for $M_L(D, B_0)$, we need only consider (I3) and (IM).

We prove that (I3) holds in any $M_L(D, B_0)$ using a result due to Grünwald [22].

Proposition 3.3.1. *Let (D, B_0) be a dimaze. Then $M_L(D, B_0)$ satisfies (I3).*

Proof. Let $I, B \in M_L(D, B_0)$ such that B is maximal but I is not. Then we have a linkage $\mathcal{Q}$ from B and another $\mathcal{P}$ from I. We may assume $\mathcal{P}$ misses some $v_0 \in B_0$.

If there is an alternating walk with respect to $\mathcal{P}$ from $(B \cup I) \setminus V(\mathcal{P})$ to $B_0 \setminus V(\mathcal{P})$, then by Lemma 3.2.3, we can extend I in $B \setminus I$.

On the other hand, if no such walk exists, we draw a contradiction to the maximality of B. In this case, by Lemma 3.2.3, there is a $(B \cup I)$–B_0 separator S on $\mathcal{P}$. For every $v \in B$, let Q_v be the path in $\mathcal{Q}$ starting from v. Let s_v be the first vertex of S that Q_v meets and P_v the path in $\mathcal{P}$ containing s_v. Let us prove that $\mathcal{Q}' := \{Q_v s_v P_v : v \in B\}$ is a linkage.

Suppose v and v' are distinct vertices in B such that $Q_v s_v P_v$ and $Q_{v'} s_{v'} P_{v'}$ meet each other. As $\mathcal{P}$ and $\mathcal{Q}$ are linkages, without loss of generality, we may assume $Q_v s_v$ meets $s_{v'} P_{v'}$ at some $s \notin S$. Then $Q_v s P_{v'}$ is a path from B to B_0 avoiding the separator. This contradiction shows that $\mathcal{Q}'$ is indeed a linkage from B to B_0. As $\mathcal{Q}'$ does not cover v_0, $B + v_0$ is independent which contradicts to the maximality of B. $\square$

If D is finite, then the following holds:

A set is maximally independent if and only if it is linkable onto the exits. (†)

When D is infinite, (†) need not hold; for instance, the dimaze in Example 3.2.2, which does not even define a matroid. Before giving a sufficient condition on a dimaze so that the linkability system satisfies (IM), using the Aharoni-Berger-Menger's theorem [6] and the linkage theorem [33] (see also [20]), we prove that when (†) holds, $M_L(D, B_0)$ is a matroid.

Proposition 3.3.2. *Given a dimaze (D, B_0), suppose that every independent set linkable onto the exits is maximal, then the dimaze defines a matroid.*

Proof. Proposition 3.3.2 Since (I1) and (I2) are obviously true for $M_L(D, B_0)$, and that (I3) holds by Proposition 3.3.1, to prove the theorem, it remains to check that (IM) holds.

Let I be independent and a set $X \subseteq V$ such that $I \subseteq X$ be given. Suppose there is a "red" linkage from I to B_0. Apply the Aharoni-Berger-Menger's theorem on X and B_0 to get a "blue" linkage $\mathcal{Q}$ from $B \subseteq X$ to B_0 and an X–B_0 separator S on the blue linkage. Let H be the digraph obtained from the subgraph of D induced by those vertices separated from B_0 by S via deleting the edges with initial vertex in S. Since every linkage from H to B_0 goes through S, a subset of $V(H)$ is linkable in (D, B_0) if and only if it is linkable in (H, S). Use the linkage theorem to find a linkage $\mathcal{Q}^\infty$ from X^∞ with $I \subseteq X^\infty \subseteq I \cup B \subseteq X$ onto S.

Let $Y \supseteq X^\infty$ be any independent set in $M_L(H, S)$. By applying the linkage theorem on a linkage from Y to S and $\mathcal{Q}^\infty$ in (H, S), we may assume that Y is linkable onto S by a linkage $\mathcal{Q}'$. Concatenating $\mathcal{Q}'$ with segments of paths in $\mathcal{Q}$ starting from S and adding trivial paths from $B_0 \setminus V(\mathcal{Q})$ gives us a linkage from $Y \cup (B_0 \setminus V(\mathcal{Q}))$ onto B_0. By the hypothesis, $Y \cup (B_0 \setminus V(\mathcal{Q}))$ is a maximal independent set in $M_L(D, B_0)$.

Applying the above statement on X^∞ shows that $X^\infty \cup (B_0 \setminus V(\mathcal{Q}))$ is also maximal in $M_L(D, B_0)$. It follows that Y cannot be a proper superset of X^∞. Hence, X^∞ is maximal in $M_L(H, S)$, and so also in $M_L(D, B_0) \cap 2^X$. This completes the proof that $M_L(D, B_0)$ is a matroid. □

Now the natural question is: in which dimazes is every set, that is linkable onto the exits, a maximal independent set? Consider the alternating comb given in Figure 3.6a. Using the notation there, the set $X := \{x_i : i \geq 1\}$ can be linked onto B_0 by the linkage $\{(x_i, y_{i-1}) : i \geq 1\}$ or to $B_0 - x_0$ by the linkage $\{(x_i, y_i) : i \geq 1\}$. Hence, X is a non-maximal independent set that is linkable onto B_0. More generally, if a dimaze (D, B_0) contains an alternating comb C, then the vertices of out-degree 2 on C together with $B_0 - C$ is a non-maximal set linkable onto B_0. So an answer to the above question must exclude dimazes containing an alternating comb. Invoking a *proof* of the linkage theorem of Pym [33], we will prove that dimazes not containing any alternating comb are precisely the answer. We use the notations in the proof of the linkage theorem sketched in Section 3.2.

Lemma 3.3.3. *Let (D, B_0) be a C^A-free dimaze. Then a set $B \subseteq V$ is maximal in $M_L(D, B_0)$ if and only if it is linkable onto B_0.*

Proof. The forward direction follows trivially from (3.1).

For the backward direction, let I be a non-maximal subset that is linkable onto B_0, by a "blue" linkage $\mathcal{Q}$. Since I is not maximal, there is $x_0 \notin I$ such that $I + x_0$ is linkable to B_0 as well, by a "red" linkage $\mathcal{P}$. Construct an alternating comb inductively as follows:

Use (the proof of) the linkage theorem to get a linkage $\mathcal{Q}^\infty$ from $I + x_0$ onto B_0. Since $Y_\mathcal{P} \subseteq Y_\mathcal{Q}$ and $X_\mathcal{Q} \subseteq X_\mathcal{P}$, $\mathcal{A}^\infty = \mathcal{C}^\infty = \emptyset$. So each path in $\mathcal{Q}^\infty$ consists of a red initial and a blue terminal segment.

Start the construction with x_0. For $k \geq 1$, if x_{k-1} is defined, let Q_k be the blue path containing $p_{k-1} := f^\infty_{x_{k-1}}$. We will prove that $p_{k-1} \notin I$ so that we can define q_k to be the last vertex on $Q_k \mathring{p}_{k-1}$ that is on a path in $\mathcal{Q}^\infty$. Since the blue segments of $\mathcal{Q}^\infty$ are disjoint, q_k lies on a red path P_{x_k}. We continue the construction with x_k.

Claim 3.3.4. *$p_{k-1} \notin I$ and hence, the blue segment $q_k Q_k p_{k-1}$ is non-trivial. The red segment $q_k P_{x_k} p_k$ is also non-trivial.*

Proof. We prove by induction. Clearly, $p_0 \notin I$, so the claim holds for $k = 1$. For $k \geq 2$, assume that $p_{k-2} \notin I$. We argue that $q_{k-1} \neq p_{k-1}$. Suppose they are equal for a contradiction. Then the path $P_{x_{k-1}} q_{k-1} Q_{k-1}$ is in $\mathcal{B}^\infty$. Since $q_{k-1} Q_{k-1} p_{k-2}$ is non-trivial, p_{k-2} and p_{k-1} are distinct vertices of the form f^∞_x on $P_{x_{k-1}} q_{k-1} Q_{k-1}$. This contradicts that $P_{x_{k-1}} q_{k-1} Q_{k-1}$ is in $\mathcal{B}^\infty$. Hence, we have $p_{k-1} \neq q_{k-1}$. This shows that the red segment $q_{k-1} P_{x_{k-1}} p_{k-1}$ is non-trivial, and so $p_{k-1} \notin I$. $\square$

We now show that $p_1 Q_1 \cup \bigcup_{k=2}^\infty q_j Q_j p_{j-1} \cup q_k P_{x_k} p_k Q_{k+1}$ is an AC.

Claim 3.3.5. *$x_j \neq x_k$ for any distinct j and k.*

Proof. For $l \geq 0$, let i_l be the least integer such that $f^{i_l}_{x_l} = f^\infty_{x_l}$. We show that $i_{k-1} < i_k$. By the definition of q_k and i_{k-1}, $q_k Q_k$ is a segment of a path in $\mathcal{Q}^i$ for any $i < i_{k-1}$, so $f^i_{x_k}$ is on the segment $P_{x_k} q_k$, and

$P_{x_k} f_{x_k}^{i_{k-1}} \subseteq P_{x_k} q_k$. Since $q_k P_{x_k} p_k$ is non-trivial, $P_{x_k} q_k \subsetneq P_{x_k} f_{x_k}^{\infty}$. We conclude that $f_{x_k}^{i_{k-1}} \neq f_{x_k}^{\infty}$. By the definition of i_k, we have $i_k > i_{k-1}$. Hence, $x_j \neq x_k$ for any $j \neq k$. □

Since $q_k P_{x_k} p_k Q_{k+1}$ is a segment of the path on x_k in the linkage $\mathcal{Q}^{\infty}$, it is disjoint from $\mathring{q}_j Q_j \mathring{p}_{j-1}$ by the definition of q_j. Moreover, by Claim 3.3.5, all the segments of the form $q_k P_{x_k} p_k Q_{k+1}$ are disjoint, and so are those of the form $q_j Q_j p_{j-1}$. Hence, we have an alternating comb. This contradiction shows that I is maximal. □

We have all the ingredients to prove the following.

Theorem 3.3.6. *Given a C^A-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid.*

Proof. The proof follows from Proposition 3.3.2 and Lemma 3.3.3. □

3.4 Examples

We have seen in Section 3.3 that forbidding alternating comb in a dimaze guarantees that it defines a strict gammoid. However, the alternating comb in Figure 3.6 defines a matroid. On the other hand, this strict gammoid is isomorphic to the one defined by the incoming comb via the isomorphism given in the figure. So one might hope that *every* strict gammoid is isomorphic to a C^A-free strict gammoid. However, as we shall see in Corollary 3.4.7, this is not true in general. Examples 3.4.9 and 3.4.12 show that after forbidding even undirected rays in the underlying graph of (the digraph of) a dimaze, the dimaze might define an interesting strict gammoid (e.g. a highly connected, non-nearly finitary and non-nearly cofinitary matroid or a wild matroid).

3.4.1 Finite circuits, cocircuits and alternating combs

We first give a necessary condition of any strict gammoid which is not C^A-free.

Lemma 3.4.1. *If a dimaze (D, B_0) is C^A-free, then $M_L(D, B_0)$ contains a finite circuit or a finite cocircuit.*

Proof. Suppose the lemma does not hold. Then every finite subset of V is independent and coindependent, and B_0 is infinite. We construct a sequence $(R_k : k \geq 1)$ of subdigraphs of D that gives rise to a subdivision of C^A for a contradiction.

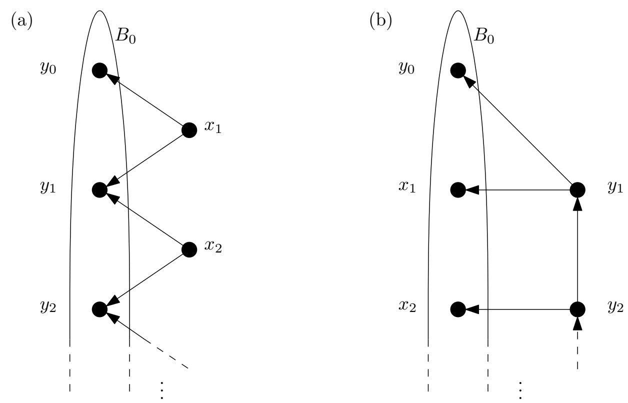

Figure 3.2: An alternating comb and an incoming comb defining isomorphic strict gammoids

Let $v_1 \notin B_0$ and $R_1 := v_1$. For $k \geq 1$, we claim that there is a path P_k from v_k to B_0 such that $P_k \cap V(R_k) = \{v_k\}$, a vertex w_k on $\mathring{v}_k P_k$, and a vertex $v_{k+1} \notin V(R_k) \cup P_k$ with $(v_{k+1}, w_k) \in E(D)$. Let $R_{k+1} := R_k \cup P_k \cup (v_{k+1}, w_k)$.

Indeed, since any finite set containing v_k is independent, there is a path from v_k avoiding any given finite set disjoint from v_k. Hence, there is a set $\mathcal{F}$ of $|V(R_k)| + 1$ disjoint paths (except at v_k) from v_k to B_0 avoiding the finite set $V(R_k) - v_k$. Since $V(\mathcal{F}) \cup R_k$ is coindependent, its complement contains a base B, witnessed by a linkage $\mathcal{P}$. Since $|V(\mathcal{F}) \cap B_0| > |V(R_k)|$ and $Ter(\mathcal{P}) = B_0$, there is a path $P \in \mathcal{P}$ that is disjoint from R_k and ends in $V(\mathcal{F}) \cap B_0$. Then the last vertex v_{k+1} of P before hitting $V(\mathcal{F})$, the next vertex w_k, and the segment $P_k := w_k P$ satisfy the requirements of the claim. By induction, the claim holds for all $k \geq 1$.

Let $R := \bigcup_{k \geq 1} R_k$. Then $(R, V(R) \cap B_0)$ is an alternating comb in (D, B_0). This contradiction completes the proof. □

A matroid is *infinitely connected* if it does not have any k-separation for any integer k. The only infinitely connected finite matroids are uniform matroids of rank about half of the size of the ground set (see [31, Chapter 8]) and they are strict gammoids. It seems natural to look for an infinitely connected infinite matroid among strict gammoids, but the previous lemma gives us a partial negative result because the bipartition of any finite circuit of size k against the rest is a k-separation. It remains open whether there is an infinitely connected infinite gammoids.

Corollary 3.4.2. *If an infinite dimaze (D, B_0) is C^A-free, then $M_L(D, B_0)$ is not infinitely connected.*

3.4.2 Trees and transversal matroids

To give a strict gammoid that is not C^A-free, we need only construct a strict gammoid without any finite circuit or cocircuit. An example is furnished by turning a transversal matroid defined on a tree to a strict gammoid. We prove a more general result that any tree gives rise to a transversal matroid.

Suppose now G is a tree rooted at a vertex in W. By upwards (downwards), we mean towards (away from) the root. For any vertex set Y, let $N^\uparrow(Y)$ be the upward neighbourhood of Y, and $N^\downarrow(Y)$ the set of downward neighbours. An edge is called *upward* if it has the form $\{v, N^\uparrow(v)\}$ where $v \in V$, otherwise it is *downward*.

We will prove that $M_T(G)$ is a matroid. For a witness of (IM), we inductively construct a sequence of matchings $(m^\alpha : \alpha \geq 0)$, indexed by ordinals.

Given $m^{\beta-1}$, to define m^β, we consider the vertices not matched by $m^{\beta-1}$ that do not have unmatched children for the first time at step $\beta - 1$. We ensure that any such vertex v that is also in I is matched in step β, by exchanging v with a currently matched vertex r_v that is not in I.

When every vertex that has not been considered has an unmatched child, we stop the algorithm. We then prove that the union of the matched vertices and those unconsidered vertices is a maximal independent superset of I.

Theorem 3.4.3. *For any tree G with an ordered bipartition (V, W), $M_T(G)$ is a transversal matroid.*

Proof. To prove that $M_T(G)$ is a matroid, by Lemma 3.2.8, it suffices to prove that (IM) holds. Let an independent set $I \subseteq X \subseteq V$ be given. Without loss of generality, by Lemma 3.2.7, we may assume that $X = V$.

We start by introducing some notations. Root G at some vertex in W. Given an ordinal α and a matching m^α, let $I^\alpha := V(m^\alpha) \cap V$ and $W^\alpha := V(m^\alpha) \cap W$. Given a sequence of matchings $(m^{\alpha'} : \alpha' \leq \alpha)$, let

$$C^\alpha := \{v \in V \setminus I^\alpha : N^\downarrow(v) \subseteq W^\alpha \text{ but } N^\downarrow(v) \not\subseteq W^{\alpha'} \ \forall \alpha' < \alpha\}.$$

Note that $C^\alpha \cap C^{\alpha'} = \emptyset$ for $\alpha' \neq \alpha$. For each $w \in W \setminus W^\alpha$, choose one vertex v_w in $N^\downarrow(w) \cap C^\alpha$ if it is not empty. Let

$$S^\alpha := \{v_w : w \in W \setminus W^\alpha \text{ and } N^\downarrow(w) \cap C^\alpha \neq \emptyset\}.$$

Denote the following statement by $A(\alpha)$:

There is a pairwise disjoint collection $\mathcal{P}^\alpha := \{P_v : v \in I \cap C^\alpha \setminus S^\alpha\}$ of m^α–alternating paths such that each P_v starts from $v \in I \cap C^\alpha \setminus S^\alpha$ with a downward edge and ends at the first vertex r_v in $I^\alpha \setminus I$.

Start the inductive construction with m^0, which is the set of upward edges that is contained in every matching of I. It is not hard to see that $C^0 \cap I = \emptyset$, so that $A(0)$ holds trivially.

Let $\beta > 0$. Given the constructed sequence of matchings $(m^\alpha : \alpha < \beta)$, suppose that $A(\alpha)$ holds for each $\alpha < \beta$. Construct a matching m^β as follows.

If β is a successor ordinal, let

$$m^\beta := E(S^{\beta-1}, N^\uparrow(S^{\beta-1})) \cup (m^{\beta-1} \Delta E(\mathcal{P}^{\beta-1})).$$

By $A(\beta-1)$, the paths in $\mathcal{P}^{\beta-1}$ are disjoint. So $m^{\beta-1}\Delta E(\mathcal{P}^{\beta-1})$ is a matching. Using the definition of $S^{\beta-1}$, we see that m^β is indeed a matching. Observe also that

$$\begin{aligned} I^{\beta-1} \cap I &\subseteq I^\beta \cap I; &(3.2)\\ W^{\beta-1} &\subseteq W^{\beta-1} \cup N^\uparrow(S^{\beta-1}) = W^\beta. &(3.3)\end{aligned}$$

If β is a limit ordinal, define m^β by

$$e \in m^\beta \iff \exists \beta' < \beta \text{ such that } e \in m^\alpha \ \forall \alpha \text{ with } \beta' \leq \alpha < \beta. \qquad (3.4)$$

As m^α is a matching for every ordinal $\alpha < \beta$, we see that m^β is a matching in this case, too.

Suppose that a vertex $u \in (V \cap I) \cup W$ is matched to different vertices by m^α and $m^{\alpha'}$ for some $\alpha, \alpha' \leq \beta$. Then there exists some ordinal $\alpha'' + 1$ between α and α' such that u is matched by an upward $m^{\alpha''}$–edge and by a downward $m^{\alpha''+1}$–edge. Hence, the change of the matching edges is unique. This implies that for any α, α' with $\alpha \leq \alpha' \leq \beta$, by (3.2) and (3.3), we have

$$\begin{aligned} I^\alpha \cap I &\subseteq I^{\alpha'} \cap I; &(3.5)\\ W^\alpha &\subseteq W^{\alpha'}. &(3.6)\end{aligned}$$

Moreover, for an upward m^β–edge vw with $v \in V$, we have

$$v \in I^0 \text{ or } \exists \alpha < \beta \text{ such that } v \in C^\alpha \text{ and } w \notin W^\alpha. \qquad (3.7)$$

We now prove that $A(\beta)$ holds. Given $v_0 = v \in I \cap C^\beta \setminus S^\beta$, we construct a decreasing sequence of ordinals starting from $\beta_0 := \beta$. For an integer $k \geq 0$, suppose that $v_k \in I \cap C^{\beta_k}$ with $\beta_k \leq \beta$ is given. By (3.5), $I^0 \subseteq I^{\beta_k}$, so $v_k \notin I^0$ and hence there exists $w_k \in N^\downarrow(v_k) \setminus W^0$.[3] Since $N^\downarrow(v_k) \subseteq W^{\beta_k} \subseteq W^\beta$, w_k is matched by m^β to some vertex v_{k+1}. In fact, as $w_k \notin W^0$, $v_{k+1} \notin I^0$. Let β_{k+1} be the ordinal with $v_{k+1} \in C^{\beta_{k+1}}$. Since $v_{k+1}w_k$ is an upward edge and $N^\downarrow(v_k) \subseteq W^{\beta_k}$, we have by (3.7) that $w_k \in W^{\beta_k} \setminus W^{\beta_{k+1}}$. By (3.6), $\beta_k > \beta_{k+1}$.

As there is no infinite decreasing sequence of ordinals, we have an m^β–alternating path $P_v = v_0w_0v_1w_1 \cdots$ that stops at the first vertex $r_v \in V \setminus I$.

The disjointness of the P_v's follows from that every vertex has a unique upward neighbour and, as we just saw, that $\mathring{v}P_v$ cannot contain any vertex $v' \in C^\beta$. So $A(\beta)$ holds.

We can now go on with the construction.

Let $\gamma \leq |V|$[4] be the least ordinal such that $C^\gamma = \emptyset$. Let $C := \bigcup_{\alpha<\gamma} C^\beta$ and $U := V \setminus (I^0 \cup C)$; so V is partitioned into I^0, C and U. As $C^\gamma = \emptyset$, every vertex in U can be matched downwards to a vertex that is not in W^γ. These edges together with m^γ form a matching m^B of $B := U \cup I^\gamma$, which we claim to be a witness for (IM). By (3.5), $I^0 \cup (C \cap I) \subseteq I^\gamma$, hence, $I \subseteq B$.

Suppose B is not maximally independent for a contradiction. Then there is an m^B–alternating path $P = v_0w_0v_1w_1 \cdots$ such that $v_0 \in V \setminus B$ that is either infinite or ends with some $w_n \in W \setminus V(m^B)$. We show that neither occurs.

Claim 3.4.4. *P is finite.*

Proof. Suppose P is infinite. Since $v_0 \notin B$, P has a subray $R = w_iP$ such that w_iv_{i+1} is an upward m^B–edge. So $w_jv_{j+1} \in m^B$ for any $j \geq i$. As vertices in U are matched downwards, $R \cap U = \emptyset$. As $m^B \Delta E(R)$ is a matching of $B \supseteq I$ in which every vertex in $R \cap V$ is matched downwards, $R \cap I^0 = \emptyset$ too. So for any $j \geq i$, there exists a unique β_j such that $v_j \in C^{\beta_j}$.

Choose $k \geq i$ such that β_k is minimal. But with a similar argument used to prove $A(\beta)$, we have $\beta_k > \beta_{k+1}$. Hence P cannot be infinite. □

Claim 3.4.5. *P does not end in $W \setminus V(m^B)$.*

[3] For a vertex $v \notin I$, $N^\downarrow(v) \setminus W^0$ may be empty.

[4] For example, fix a well ordering of V and map each β to the least element in C^β.

Proof. Suppose that P ends with $w_n \in W \setminus V(m^B)$. Certainly, v_n can be matched downwards (either to w_{n-1} or w_n) in a matching of $B \supseteq I$. Hence, $v_n \notin I^0$. It is easy to check that for $v \in C^\alpha$, $N(v) \subseteq W^{\alpha+1}$. Hence, as $w_n \in W \setminus W^\gamma$, $v_n \notin C$. Hence, $v_n \in U$. It follows that for each $0 < i \leq n$, v_i is matched downwards and so does not lie in I^0. As $v_0 \notin B$, $v_0 \in C$. It follows that $w_0 \in W^\gamma$ and $v_1 \in C$. Repeating the argument, we see that $v_n \in C$, which is a contradiction. □

We conclude that B is maximal. So (IM) holds and $M_T(G)$ is a matroid. □

Corollary 3.4.6. *Let (D, B_0) be a dimaze such that the underlying graph of D is a tree and B_0 is a vertex class of a bipartition of D with edges directed towards B_0. Then $M_L(D, B_0)$ is a matroid.*

Proof. By the theorem, we need only present $M_L(D, B_0)$ as a transversal matroid defined on a tree. Define a tree G with bipartition $((V \setminus B_0) \cup B_0', B_0)$, where B_0' is a copy of B_0, from D by ignoring the directions and joining each vertex in B_0 to its copy with an edge. Then $M_L(D, B_0) \cong M_T(G)$. □

Consider the infinitely branching rooted tree, i.e. a rooted tree such that each vertex has infinitely many children. Let B_0 consist of the vertices on alternate levels, starting from the root. Define $\mathcal{T}$ by directing all edges towards B_0. Corollary 3.4.6 shows that $M_L(\mathcal{T}, B_0)$ is a matroid. Clearly, this matroid does not contain any finite circuit. Moreover, as any finite set C^* misses a base obtained by adding finitely many vertices to $B_0 \setminus C^*$, any cocircuit must be infinite. With Lemma 3.4.1, we conclude the following.

Corollary 3.4.7. *Every dimaze that defines a strict gammoid isomorphic to $M_L(\mathcal{T}, B_0)$ contains an alternating comb.*

We remark that although forbidding alternating comb in a dimaze ensures that we get a strict gammoid, as we just saw, not every strict gammoid arises this way. So it is natural to look for other conditions so that the linkability system is a matroid. In [2] it was proved that when a dimaze gives rise to a nearly finitary linkability system, the dimaze defines a matroid. On the other hand, it follows from Carmesin's result (Corollary 3.2.4) that if a dimaze contains only finitely many linking fans with distinct centers and finitely many spine-disjoint combs, then the linkability system is nearly finitary and hence is a matroid. The following two examples show that this result and Theorem 3.3.6 do not imply each other.

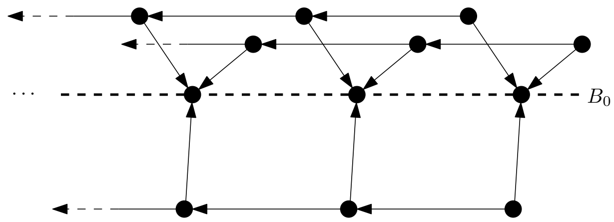

Figure 3.3: A dimaze that defines a nearly finitary linkability system

The first example is the dimaze depicted in Figure 3.3 which contains only finitely many linking fans and combs but it contains an alternating comb. To give the second example, we need an easy lemma.

Lemma 3.4.8. *Given a matroid M and a base B, if $E \setminus B$ contains infinitely many elements which are not in any finite circuit, then M is not nearly finitary.*

Proof. Extend B to a base B^{fin} of M^{fin}. As $|B^{\text{fin}} \setminus B| = \infty$, so is $|B^{\text{fin}} \setminus B'|$ for any other base B' of M inside B^{fin}. So M is not nearly finitary. $\square$

Example 3.4.9. For any integer $k \geq 2$, there is a k-connected strict gammoid $M = M_L(D, B_0)$ such that the underlying graph of D is rayless, and neither M nor its dual is nearly finitary.

Consider a rooted tree T of depth 3 where each internal vertex has infinitely many children, and each edge is directed towards $L_0 \cup L_2$ where L_i is the set of vertices at distance i from the root for $0 \leq i \leq 3$. Let D be a digraph with $V = V(T) \cup X \cup Y$, where each of X and Y is an extra set of k vertices; and $E(D) = E(T) \cup \{(x, b), (v, y) : x \in X, b \in B_0, v \in V \setminus B_0, y \in Y\}$, where $B_0 = L_0 \cup L_2 \cup Y$. Since (D, B_0) does not contain any alternating comb, by Theorem 3.3.6, $M = M_L(D, B_0)$ is a matroid.

As no vertex in L_1 lies in a finite circuit, applying Lemma 3.4.8 with the base B_0 shows that M is not nearly finitary. Similarly, as no vertex in L_2 lies in a finite cocircuit, the same lemma with $V \setminus B_0$ shows that M^* is not nearly finitary.

For any $l < k$, it is not difficult to see that in any bipartition of V into sets P, Q of size at least l, there is a linkage from $P_1 \subseteq P \setminus B_0$ to $Q \cap B_0$ and from $Q_1 \subseteq Q \setminus B_0$ to $P \cap B_0$ of size at least l. It follows that $P_1 \cup (P \cap B_0) \cup Q_1 \cup (Q \cap B_0)$ contains at least l vertices more than B_0. Hence, (P, Q) is not an l-separation. So M is k-connected.

So far we know that if a dimaze (D, B_0) does not contain any alternating comb or that it contains only finitely many linking fans with distinct centers and and finitely many spine-disjoint combs, then $M_L(D, B_0)$ is a matroid. However, there are examples of strict gammoids that lie in neither of the two classes. All our examples of dimazes that do not define a matroid share a common feature, namely there is an independent set I that cannot be extended to a maximal in $I \cup B_0$. In view of this, we propose the following.

Conjecture 3.4.10. Suppose that for all $I \in M_L(D, B_0)$ and $B \subseteq B_0$, there is a maximal independent set in $I \cup B$ extending I. Then (IM) holds for $M_L(D, B_0)$.

We remark that even after forbidding alternating comb (or any ray at all), there are dimazes defining interesting strict gammoids. The existence of wild matroids, was first demonstrated in [12]. It turns out that strict gammoids are a rich source of wild matroids.

Lemma 3.4.11. *Suppose that $M_L(D, B_0)$ is a strict gammoid such that there is a circuit containing infinitely many vertices linkable to a fixed exit b in B_0. Then $M_L(D, B_0)$ is a wild matroid.*

Proof. The fundamental cocircuit of b with respect to B_0, consisting of all the vertices linkable to b, intersects the given circuit at infinitely many vertices. □

Example 3.4.12. A concrete example is that $V(D) = \{v_i, b_i : i \geq 1\}$ with $B_0 = \{b_i : i \geq 1\}$ and $E(D) = \{(v_i, b_i), (v_1, b_i), (v_i, b_1) : i \geq 1\}$. Then (D, B_0) is a C^A-free dimaze, and $\{v_i, b_1 : i \geq 1\}$ is an infinite circuit satisfying the lemma.

3.5 Minor

The next two sections investigate minor and duality in the class of gammoids and consist of the materials in [3].

The class of gammoids is closed under deletion by definition. In fact, finite gammoids are minor-closed. To see this, note that matroid deletion and contraction commute, so it suffices to show that a contraction minor M/X of a strict gammoid M is also a gammoid. Indeed, in [26] it was shown that finite strict gammoids are precisely the dual of finite transversal matroid. Moreover, they provided a construction to turn a dimaze to a bimaze presentation of the dual, and vice versa (essentially Definitions 3.6.1 and 3.6.2). Thus, we apply the construction to a presentation of M and get one of M^*. By deleting X, we get a presentation of the transversal matroid $M^* \setminus X$. Reversing the construction with any base of $M^* \setminus X$ gives us a dimaze presentation of $(M^* \setminus X)^* = M/X$.

In case of general gammoids, we can no longer appeal to duality, since, as we shall see, strict gammoids need not be cotransversal (Example 3.6.11) and the dual of transversal matroids need not be strict gammoids (Example 3.6.18). We will instead investigate the effect of the construction sketched above on a dimaze directly. We are then able to show that the class of C^O-free gammoids, i.e. gammoids that admit a C^O-free presentation, is minor-closed. In combination with the linkage theorem, we can also prove that finite rank minors of gammoids are gammoids.

It remains open whether the class of gammoids is minor-closed.

Topological gammoids are introduced in [18] and are always finitary. The independent set systems are always finitary and define matroids. It turns out that such matroids are precisely the finitary gammoids. By investigating the structure of dimaze presentations of such gammoids, we then show that finitary strict gammoids, or equivalently, topological gammoids, are closed under taking minors.

3.5.1 Matroid contraction and shifting along a linkage

Our aim is to show that a contraction minor M/S of a strict gammoid M is a strict gammoid. By Lemma 1.0.8, we may assume that S is independent. The first case is that S is a subset of the exits.

Lemma 3.5.1. *Let $M = M_L(D, B_0)$ be a strict gammoid and $S \subseteq B_0$. Then a dimaze presentation of M/S is given by $M_L(D - S, B_0 \setminus S)$.*

Proof. Since $S \subseteq B_0$ is independent, $I \in \mathcal{I}(M/S) \iff I \cup S \in \mathcal{I}(M)$. Moreover,

$$\begin{aligned} I \in \mathcal{I}(M/S) &\iff I \cup S \text{ admits a linkage in } (D, B_0) \\ &\iff I \text{ admits a linkage } \mathcal{Q} \text{ with } Ter(\mathcal{Q}) \cap S = \emptyset \text{ in } (D, B_0) \\ &\iff I \in \mathcal{I}(M_L(D - S, B_0 \setminus S). \end{aligned}$$ □

Thus, it suffices to give a dimaze presentation of M such that S is a subset of the exits. For this purpose we consider the process of "shifting along a linkage", which replaces the previously discussed detour via duality.

Throughout the section, (D, B_0) denotes a dimaze, $\mathcal{Q}$ a set of disjoint paths or rays, where by rays we mean outgoing rays, $S := Ini(\mathcal{Q})$ and $T := Ter(\mathcal{Q})$. Next, we define various maps which are dependent on $\mathcal{Q}$.

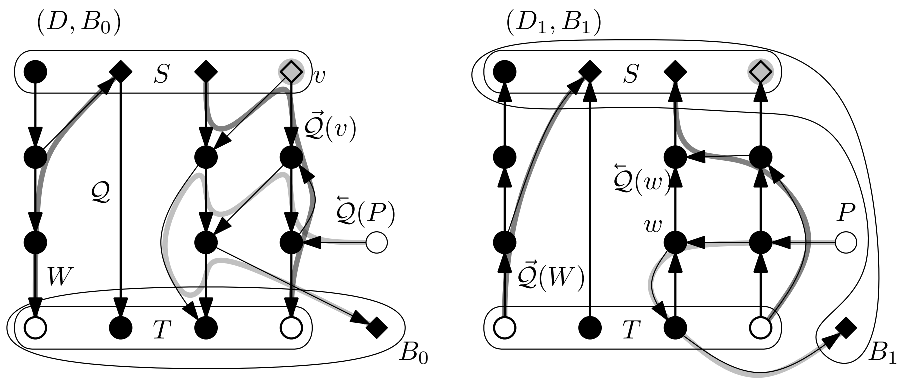

Figure 3.4: A $\mathcal{Q}$-shifted dimaze: $D_1 = \vec{\mathcal{Q}}(D)$, $B_1 = (B_0 \setminus T) \cup S$, where $\mathcal{Q}$ consists of the vertical downward paths. Outlined circles and diamonds are respectively initial and terminal vertices of $\mathcal{Q}$-alternating walks (left) and their $\vec{\mathcal{Q}}$-images (right).

Define a bijection between $V \setminus T$ and $V \setminus S$ as follows: $\vec{\mathcal{Q}}(v) := v$ if $v \notin V(\mathcal{Q})$; otherwise $\vec{\mathcal{Q}}(v) := u$ where u is the unique vertex such that $(v, u) \in E(\mathcal{Q})$. The inverse is denoted by $\overleftarrow{\mathcal{Q}}$.

Construct the digraph $\vec{\mathcal{Q}}(D)$ from D by replacing each edge $(v, u) \in E(D) \setminus E(\mathcal{Q})$ with $(\vec{\mathcal{Q}}(v), u)$ and each edge $(v, u) \in \mathcal{Q}$ with (u, v). Set for the rest of this section

$$D_1 := \vec{\mathcal{Q}}(D) \text{ and } B_1 := (B_0 \setminus T) \cup S$$

and call (D_1, B_1) the $\mathcal{Q}$-*shifted* dimaze.

Given a $\mathcal{Q}$-alternating walk $W = w_0 e_0 w_1 e_1 w_2 \dots$ in D, let $\vec{\mathcal{Q}}(W)$ be obtained from W by deleting all e_i and each $w_i \in W$ such that $w_i \in V(\mathcal{Q})$ but $e_i \notin E(\mathcal{Q})$.

For a path or ray $P = v_0 v_1 v_2 \dots$ in D_1, let $\overleftarrow{\mathcal{Q}}(P)$ be obtained from P by inserting after each $v_i \in P \setminus Ter(P)$ the following:

(v_i, v_{i+1}) if $v_i \notin V(\mathcal{Q})$;

(v_{i+1}, v_i) if $v_i \in V(\mathcal{Q})$ and $(v_{i+1}, v_i) \in E(\mathcal{Q})$;

$(w, v_i) w (w, v_{i+1})$ with $w := \overleftarrow{\mathcal{Q}}(v_i)$ if $v_i \in V(\mathcal{Q})$ but $(v_{i+1}, v_i) \notin E(\mathcal{Q})$.

We examine the relation between alternating walks in D and paths/rays in $\vec{\mathcal{Q}}(D)$.

Lemma 3.5.2. *(i) A $\mathcal{Q}$-alternating walk in D that is infinite or ends in $t \in B_1$ is respectively mapped by $\vec{\mathcal{Q}}$ to a ray or a path ending in t in D_1. Disjoint such walks are mapped to disjoint paths/rays.*

(ii) A ray or a path ending in $t \in B_1$ in D_1 is respectively mapped by $\overleftarrow{\mathcal{Q}}$ to an infinite $\mathcal{Q}$-alternating walk or a finite $\mathcal{Q}$-alternating walk ending in t in D. Disjoint such paths/rays are mapped to disjoint $\mathcal{Q}$-alternating walks.

Proof. We prove (i) since a proof of (ii) can be obtained by reversing the construction.

Let $W = w_0e_0w_1e_1w_2\dots$ be a $\mathcal{Q}$-alternating walk in D. If a vertex v in W is repeated, then v occurs twice and there is i such that $v = w_i$ with $e_{i-1} = (w_i, w_{i-1}) \in E(\mathcal{Q})$ and $e_i \notin E(\mathcal{Q})$. Hence, w_i is deleted in $P := \vec{\mathcal{Q}}(W)$ and so v does not occur more than once in P, that is, P consists of distinct vertices.

By construction, the last vertex of a finite W is not deleted, hence P ends in t. In case W is infinite, by (W3), no tail of W is deleted so that P remains infinite.

Next, we show that (v_i, v_{i+1}) is an edge in D_1. Let $w_j = v_i$ be the non-deleted instance of v_i. If w_{j+1} has been deleted, then the edge (w_{j+1}, w_{j+2}) (which exists since the last vertex cannot be deleted) in D has been replaced by the edge $(\vec{\mathcal{Q}}(w_{j+1}), w_{j+2}) = (v_i, v_{i+1})$ in D_1. If both w_j and $v_{i+1} = w_{j+1}$ are in $V(\mathcal{Q})$ then the edge $(w_{j+1}, w_j) \in E(\mathcal{Q})$ has been replaced by (v_i, v_{i+1}) in D_1. In the other cases $(w_j, w_{j+1}) = (v_i, v_{i+1})$ is an edge of D and remains one in D_1.

Let W_1, W_2 be disjoint $\mathcal{Q}$-alternating walks. By construction, $\vec{\mathcal{Q}}(W_1) \cap \vec{\mathcal{Q}}(W_2) \subseteq W_1 \cap W_2 \subseteq V(\mathcal{Q})$. By disjointness, at any intersecting vertex, one of W_1 and W_2 leaves with an edge not in $E(\mathcal{Q})$. Thus, such a vertex is deleted upon application of $\vec{\mathcal{Q}}$. Hence, $\vec{\mathcal{Q}}(W_1)$ and $\vec{\mathcal{Q}}(W_2)$ are disjoint paths/rays. $\square$

Note that for a path P in D_1 and a $\mathcal{Q}$-alternating walk W in D, we have

$$\vec{\mathcal{Q}}(\overleftarrow{\mathcal{Q}}(P)) = P; \qquad \overleftarrow{\mathcal{Q}}(\vec{\mathcal{Q}}(W)) = W.$$

This correspondence of sets of disjoint $\mathcal{Q}$-alternating walks in (D, B_0) and sets of disjoint paths or rays in the $\mathcal{Q}$-shifted dimaze will be used in various situations in order to show that the independent sets associated with (D, B_0) and the $\mathcal{Q}$-shifted dimaze are the same.

Given a set $\mathcal{W}$ of $\mathcal{Q}$-alternating walks, define the graph $\mathcal{Q}\Delta\mathcal{W} := (V(\mathcal{Q}) \cup V(\mathcal{W}), E(\mathcal{Q})\Delta E(\mathcal{W}))$.

Lemma 3.5.3. *Let $J \subseteq V \setminus S$ and $\mathcal{W}$ a set of disjoint $\mathcal{Q}$-alternating walks, each of which starts from J and does not end outside of B_1. Then* *there is a set of disjoint rays or paths from* $X := J \cup (S \setminus Ter(\mathcal{W}))$ *to* $Y := T \cup (Ter(\mathcal{W}) \cap B_0)$ *in* $\mathcal{Q}\Delta\mathcal{W}$.

Proof. Every vertex in $\mathcal{Q}\Delta\mathcal{W} \setminus (X \cup Y)$ has in-degree and out-degree both 1 or both 0. Moreover, every vertex in X has in-degree 0 and out-degree 1 (or 0, if it is also in Y) and every vertex in Y has out-degree 0 and in-degree 1 (or 0, if it is also in X). Therefore every (weakly) connected component of $\mathcal{Q}\Delta\mathcal{W}$ meeting X is either a path ending in Y or a ray. □

The following will be used to complete a ray to an outgoing comb in various situations.

Lemma 3.5.4. *Suppose $\mathcal{Q}$ is a topological linkage. Any ray R that hits infinitely many vertices of $V(\mathcal{Q})$ is the spine of an outgoing comb.*

Proof. The first step is to inductively construct an infinite linkable subset of $V(R)$. Let $\mathcal{Q}_0 := \mathcal{Q}$ and $A_0 := \emptyset$. For $i \geq 0$, assume that $\mathcal{Q}_i$ is a topological linkage that intersects $V(R)$ infinitely but avoids the finite set of vertices A_i. Since it is not possible to separate a vertex on a topological path from B_0 by a finite set of vertices disjoint from that topological path, there exists a path P_i from $V(R) \cap V(\mathcal{Q}_i)$ to B_0 avoiding A_i. Let $A_{i+1} := A_i \cup V(P_i)$ and $\mathcal{Q}_{i+1}$ obtained from $\mathcal{Q}_i$ by deleting from each of its elements the minimal initial segment that intersects A_{i+1}. As $\mathcal{Q}_{i+1}$ remains a topological linkage that intersects $V(R)$ infinitely, we can continue the procedure. By construction $\{P_i : i \in \mathbb{N}\}$ is an infinite set of disjoint finite paths from a subset of $V(R)$ to B_0. Let $p_i \in P_i$ be the last vertex of R on P_i, then R is the spine of the outgoing comb: $R \cup \bigcup_{i \in \mathbb{N}} p_i P_i$. □

Corollary 3.5.5. *Any ray provided by Lemma 3.5.3 is in fact the spine of an outgoing comb if $\mathcal{Q}$ is a topological linkage, and the infinite forward segments of the walks in $\mathcal{W}$ are the spines of outgoing combs.*

Proof. Observe that a ray R constructed in Lemma 3.5.3 is obtained by alternately following the forward segments of the walks in $\mathcal{W}$ and the forward segments of elements in $\mathcal{Q}$.

Either a tail of R coincides with a tail of a walk in $\mathcal{W}$, and we are done by assumption; or R hits infinitely many vertices of $V(\mathcal{Q})$, and Lemma 3.5.4 applies. □

With Lemma 3.5.3 we can transform disjoint alternating walks into disjoint paths or rays. A reverse transform is described as follows.

Lemma 3.5.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of disjoint paths or rays. and $\mathcal{W}$ be a set of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in distinct vertices of $Ini(\mathcal{P})$. Then the walks in $\mathcal{W}$ are disjoint and can only end in $(Ter(\mathcal{P}) \setminus T) \cup S$.*

Proof. Let $W = w_0 e_0 w_1 \dots$ be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk. Then W is a trivial walk if and only if $w_0 \in (Ter(\mathcal{P}) \setminus T) \cup S$. If W is nontrivial then $e_0 \in E(\mathcal{Q})$ if and only if $w_0 \in V(\mathcal{Q})$.

Let W_1 and $W_2 \in \mathcal{W}$. Note that for any interior vertex w_i of a $\mathcal{P}$-$\mathcal{Q}$-alternating walk, it follows from the definition that either edge in $\{e_{i-1}, e_i\}$ determines uniquely the other. So if W_1 and W_2 share an edge, then a reduction to their common initial vertex shows that they are equal by their maximality. Moreover if the two walks share a vertex $v \notin V(\mathcal{Q})$, then they are equal since they share the edge of $\mathcal{P}$ whose terminal vertex is v.

Therefore, if $W_1 \neq W_2$ and they end at the same vertex v, then $v \in V(\mathcal{P}) \cap V(\mathcal{Q})$. More precisely, we may assume that v is the initial vertex of an edge in $E(\mathcal{Q}) \cap E(W_1)$ and the terminal vertex of an edge $e \in E(\mathcal{P}) \cap E(W_2)$ (both the last edges of their alternating walk). Since v is the initial vertex of some edge, it cannot be in B_0, so the path (or ray) in $\mathcal{P}$ containing e does not end at v. Hence we can extend W_1 contradicting its maximality.

Similarly we can extend a $\mathcal{P}$-$\mathcal{Q}$-alternating walk that ends in some vertex $v \in Ter(\mathcal{P}) \cap Ter(\mathcal{Q})$ by the edge in $E(\mathcal{Q})$ that has v as its terminal vertex, unless $v \in Ini(\mathcal{Q})$. So $\mathcal{W}$ is a set of disjoint $\mathcal{P}$-$\mathcal{Q}$-alternating walks that can only end in $(Ter(\mathcal{P}) \setminus T) \cup S$. □

Now we investigate when a dimaze and its $\mathcal{Q}$-shifted dimaze present the same strict gammoid.

Lemma 3.5.7. *Suppose that $\mathcal{Q}$ is a linkage from S onto T and I a set linkable in (D_1, B_1). Then I is linkable in (D, B_0) if (i) $I \setminus S$ is finite or (ii) (D, B_0) is C^O-free.*

Proof. There is a set of disjoint finite paths from I to B_1 in (D_1, B_1), which, by Lemma Lemma 3.5.2, gives rise to a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1 in (D, B_0). Let $\mathcal{W}$ be the subset of those walks starting in $J := I \setminus S$. Then Lemma 3.5.3 provides a set $\mathcal{P}$ of disjoint paths or rays from $J \cup (S \setminus Ter(\mathcal{W})) \supseteq I$ to $Y \subseteq B_0$. It remains to argue that $\mathcal{P}$ does not contain any ray. Indeed, any such ray meets infinitely many paths in $\mathcal{Q}$. But by Lemma 3.5.4, the ray is the spine of an outgoing comb, which is a contradiction. □

In fact the converse of (ii) holds.

Lemma 3.5.8. *Suppose that (D, B_0) is C^O-free, and $\mathcal{Q}$ is a linkage from S onto T such that there exists no linkage from S to a proper subset of T. Then a linkable set I in (D, B_0) is also linkable in (D_1, B_1), and (D_1, B_1) is C^O-free.*

Proof. For the linkability of I it suffices by Lemma 3.5.2 to construct a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1. Let $\mathcal{P}$ be a linkage of I in (D, B_0).

For each vertex $v \in I$ let W_v be the maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting in v. By Lemma 3.5.6, $\mathcal{W} := \{W_v : v \in I\}$ is a set of disjoint $\mathcal{Q}$-alternating walks that can only end in $(Ter(\mathcal{P}) \setminus T) \cup S \subseteq B_1$.

If there is an infinite alternating walk $W = W_{v_0}$ in $\mathcal{W}$, then Lemma 3.5.3 applied on just this walk gives us a set $\mathcal{R}$ of disjoint paths or rays from $S + v_0$ to T. Since the forward segments of W are subsegments of paths in $\mathcal{P}$, by Corollary Corollary 3.5.5 any ray in $\mathcal{R}$ would extend to a forbidden outgoing comb. Thus, $\mathcal{R}$ is a linkage of $S + v_0$ to T. In particular, S is linked to a proper subset of T contradicting the minimality of T. Hence $\mathcal{W}$ consists of finite disjoint $\mathcal{Q}$-alternating walks, as desired.

For the second statement suppose that (D_1, B_1) contains an outgoing comb whose spine R starts at $v_0 \notin S$. Then $W := \overleftarrow{\mathcal{Q}}(R)$ is a $\mathcal{Q}$-alternating walk in (D, B_0) by Lemma 3.5.2. Any infinite forward segment R' of W contains an infinite subset linkable to B_1 in (D_1, B_1). By Lemma 3.5.7(ii) this subset is also linkable in (D, B_0), so R' is the spine of an outgoing comb by Lemma 3.5.4, which is a contradiction.

On the other hand, suppose that W does not have an infinite forward tail. By investigating W as we did with W_{v_0} above, we arrive at a contradiction. Hence, there does not exist any outgoing comb in (D_1, B_1). $\square$

For later applications, we note the following refinement.

Corollary 3.5.9. *If (D, B_0) is F^∞-free as well, then so is (D_1, B_1).*

Proof. Suppose that (D_1, B_1) contains a subdivision of F^∞ with centre v_0. Then an infinite subset X of the out-neighbourhood of v_0 in (D_1, B_1) is linkable. By Lemma 3.5.7(ii), X is also linkable in (D, B_0). As X is a subset of the out-neighbourhood of $\overleftarrow{\mathcal{Q}}(v_0)$, a forbidden linking fan in (D, B_0) results. $\square$

Proposition 3.5.10. *Suppose (D, B_0) is C^O-free and $\mathcal{Q}$ is a linkage from S onto T such that S cannot be linked to a proper subset of T. Then $M_L(D_1, B_1) = M_L(D, B_0)$.*

Proof. By Lemma 3.5.7(ii) and Lemma 3.5.8, a set $I \subseteq V$ is linkable in (D, B_0) if and only if it is linkable in (D_1, B_1). $\square$

We remark that in order to show that $M_L(D, B_0) = M_L(D_1, B_1)$, the assumption in Proposition 3.5.10 that (D, B_0) is C^O-free can be slightly relaxed. Only outgoing combs constructed in the proofs of Lemma 3.5.7(ii) and Lemma 3.5.8 which have the form that all the spikes are terminal segments of paths in the linkage $\mathcal{Q}$ need to be forbidden.

Theorem 3.5.11. *The class of C^O-free gammoids is minor-closed.*

Proof. Let $N := M_L(D, B_0)$ be a strict gammoid. It suffices to show that any minor of N is a gammoid. By Lemma 1.0.8, such has the form $M := N/S \setminus R$ for some independent set S and coindependent set R. First extend S in B_0 to a base B_1. This gives us a linkage $\mathcal{Q}$ from S onto $T := B_0 \setminus B_1$ such that there exists no linkage from S to a proper subset of T.

Assume that (D, B_0) is C^O-free. Then by Lemma 3.5.8, (D_1, B_1) is C^O-free, and by Proposition 3.5.10, $M_L(D, B_0) = M_L(D_1, B_1)$. Since $S \subseteq B_1$, $M = M_L(D_1, B_1)/S \setminus R = M_L(D_1 - S, B_1 \setminus S) \setminus R$ is a C^O-free gammoid. □

3.5.2 Topological gammoids

A topological notion of linkability is introduced in [18]. Roughly speaking, a topological path from a vertex v does not need to reach the exits as long as no finite vertex set avoiding that path can prevent an actual connection of v to B_0.

Here we show that in fact, topological gammoids (see Section 3.2) coincide with the finitary gammoids. As a corollary, we see that topological gammoids are minor-closed.

The difference between a topological linkage and a linkage is that paths ending in the centre of a linking fan and spines of outgoing combs are allowed. Thus, to prove the following, it suffices to give a $\{C^O, F^\infty\}$-free dimaze presentation for the strict topological gammoid.

Lemma 3.5.12. *Every strict topological gammoid is a strict gammoid.*

Proof. Let (D', B_0') be a dimaze and F be the set of all vertices that are the centre of a subdivision of F^∞. Let (D, B_0) be obtained from (D', B_0') by deleting all edges whose initial vertex is in F from D' and $B_0 := B_0' \cup F$.

We claim that $M_{TL}(D, B_0) = M_{TL}(D', B_0')$. Let $\mathcal{P}$ be a topological linkage of I in (D', B_0'). Then the collection of the initial segments of each element of $\mathcal{P}$ up to the first appearance of a vertex in F forms a topological linkage of I in (D, B_0). Conversely, let $\mathcal{P}$ be a topological linkage of I in (D, B_0). Note that any linkage in (D, B_0) is a topological linkage in (D', B'_0). In particular the spikes of an outgoing comb whose spine R is in $\mathcal{P}$ form a topological linkage. Hence, R is also the spine of an outgoing comb in (D', B'_0) by Lemma 3.5.4. So I is topologically linkable in (D, B_0).

Let $S \cup B_0$ be a base of $M_{TL}(D, B_0)$ and $\mathcal{Q}$ a set of disjoint spines of outgoing combs starting from S. We show that a set I is topologically linkable in (D, B_0) if and only if it is linkable in the $\mathcal{Q}$-shifted dimaze (D_1, B_1).

Let $\mathcal{P}$ be a topological linkage of I in (D, B_0). By Lemma 3.5.6, the set $\mathcal{W}$ of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in I is a set of disjoint $\mathcal{Q}$-alternating walks possibly ending in $Ter(\mathcal{P}) \cup S \subseteq B_1$. If there were an infinite walk, then it would have to start outside S and give rise to a topologically linkable superset of $S \cup B_0$, by Lemma 3.5.3 and Lemma 3.5.4. So each walk in $\mathcal{W}$ is finite. By Lemma 3.5.2, I is linkable in (D_1, B_1).

Conversely let I be linkable in (D_1, B_1) and $\mathcal{W}$ a set of disjoint finite $\mathcal{Q}$-alternating walks in (D, B_0) from I to B_1 provided by Lemma 3.5.2. By Lemma 3.5.3, $\mathcal{Q} \Delta \mathcal{W}$ contains a set $\mathcal{R}$ of disjoint paths or rays in (D, B_0) from I to B_0. By Corollary 3.5.5, any ray in $\mathcal{R}$ is in fact the spine of an outgoing comb, so I is topologically linkable in (D, B_0). $\square$

Now we can characterize strict topological gammoids among strict gammoids.

Theorem 3.5.13. *The following are equivalent:*

1. *M is a strict topological gammoid;*
2. *M is a finitary strict gammoid;*
3. *M is a strict gammoid such that any presentation is $\{C^O, F^\infty\}$-free;*
4. *M is a $\{C^O, F^\infty\}$-free strict gammoid.*

Proof. 1. $\Rightarrow$ 2. : By Corollary 3.2.4, M is a finitary matroid and by Lemma 3.5.12 it is a strict gammoid.

2. $\Rightarrow$ 3. : Let $M_L(D, B_0)$ be any presentation of M. Note that the union of any vertex $v \in V \setminus B_0$ and all the vertices in B_0 to which v is linkable forms a circuit in M (the fundamental circuit of v and B_0). Suppose (D, B_0) is not $\{C^O, F^\infty\}$-free, then there is a vertex linkable to infinitely many vertices in B_0. But then M contains an infinite circuit and is not finitary.

3. $\Rightarrow$ 4. : Trivial.

4. $\Rightarrow$ 1. : Take a $\{C^O, F^\infty\}$-free presentation of M. Then topological linkages coincide with linkages. Hence M is a topological gammoid. $\square$

Next we also characterize topological gammoids among gammoids.

Corollary 3.5.14. *The following are equivalent:*

1. *M is a topological gammoid;*
2. *M is a finitary gammoid;*
3. *M is a $\{C^O, F^\infty\}$-free gammoid.*

Proof. 1. ⇒ 3. : There exist a dimaze (D, B_0) and $X \subseteq V$ such that $M = M_{TL}(D, B_0) \setminus X$. By Theorem 3.5.13, there is a $\{C^O, F^\infty\}$-free dimaze (D_1, B_1) such that $M_L(D_1, B_1) = M_{TL}(D, B_0)$. Hence, M is a $\{C^O, F^\infty\}$-free gammoid.

3. ⇒ 2. : There exists a $\{C^O, F^\infty\}$-free presentation of a strict gammoid N of which M is a restriction. By Theorem 3.5.13, N is finitary, thus, so is M.

2. ⇒ 1. : There exist (D, B_0) and $X \subseteq V$ such that $M = M_L(D, B_0) \setminus X$. Since $M \setminus X$ is finitary, $\mathcal{C}(M \setminus X) = \mathcal{C}(M^{\text{fin}} \setminus X)$. By Corollary 3.2.4, the latter is equal to $\mathcal{C}(M_{TL}(D, B_0) \setminus X)$. Hence, M is a topological gammoid. □

Theorem 3.5.15. *The class of finitary gammoids (or equivalently topological gammoids) is closed under taking minors.*

Proof. Let M be a finitary gammoid. By Corollary 3.5.14, M is a $\{C^O, F^\infty\}$-free gammoid. Any minor of M is a C^O-free gammoid by Theorem 3.5.11, and also F^∞-free by Corollary 3.5.9. So any minor of M is a finitary gammoid by Corollary 3.5.14. □

3.6 Duality

Finite strict gammoids and finite transversal matroids are dual to each other [26]. However, this is not the case in the infinite setting (Examples 3.6.11 and 3.6.18). So to describe the dual of C^A-free strict gammoids, we first introduce a natural extension of the class of transversal matroids, and then show that the dual of every C^A-free strict gammoid is of this type. Nevertheless this extension does not contain the dual of all strict gammoids (Example 3.6.13).

A standard result [26] states that the dual of a finite gammoid is a gammoid. This can be proved using duality between finite strict gammoids and transversal matroids and the fact that the class of finite gammoids is closed under contraction minors. In Section 3.6.3, we see that this proof remains valid for the class of those infinite gammoids that admit a presentation (D, B_0) with the underlying graph of D rayless. However, we finally see that there is a strict gammoid which is not dual to any gammoid (Example 3.6.24).

3.6.1 Strict gammoids and path-transversal matroid

The class of path-transversal matroids is introduced as a superclass of transversal matroids, and proved to contain the dual matroids of any C^A-free strict gammoid. We shall see that an extra condition forces C^A-free strict gammoids to be dual to transversal matroids. On the other hand, even though path-transversal matroids extend transversal matroids, they do not capture the dual of all strict gammoids, as we shall see in Example 3.6.13.

Let us introduce a dual object of a dimaze. Given a bipartite graph $G = (V, W)$, we call a matching m_0 onto W an *identity matching*, and the pair (G, m_0) a *bimaze*[5]. We adjust two constructions of [26] for our purposes.

Definition 3.6.1. Given a dimaze (D, B_0), define a bipartite graph $D^\star_{B_0}$, with bipartition $(V, (V \setminus B_0)^\star)$, where $(V \setminus B_0)^\star := \{v^\star : v \in V \setminus B_0\}$ is disjoint from V; and $E(D^\star_{B_0}) := m_0 \cup \{vu^\star : (u, v) \in E(D)\}$, where $m_0 := \{vv^\star : v \in V \setminus B_0\}$. Call $(D, B_0)^\star := (D^\star_{B_0}, m_0)$ the *converted bimaze* of (D, B_0).

Starting from a dimaze (D, B_0), we write $(V \setminus B_0)^\star$, m_0 and $v^\star$ for the corresponding objects in Definition 3.6.1.

Definition 3.6.2. Given a bimaze (G, m_0), where $G = (V, W)$, define a digraph $G^\star_{m_0}$ such that $V(G^\star_{m_0}) := V$ and $E(G^\star_{m_0}) := \{(v, w) : wv^\star \in E(G) \setminus m_0\}$, where $v^\star$ is the vertex in W that is matched by m_0 to $v \in V$. Let $B_0 := V \setminus V(m_0)$. Call $(G, m_0)^\star := (G^\star_{m_0}, B_0)$ the *converted dimaze* of (G, m_0).

Starting from a bimaze (G, m_0), we write B_0 and $v^\star$ for the corresponding objects in Definition 3.6.2 and $(V \setminus B_0)^\star$ for the right vertex class of G.

Note that these constructions are inverse to each other (see Figure 3.5). In particular, let (G, m_0) be a bimaze, then

$$(G, m_0)^{\star\star} = (G, m_0). \tag{3.8}$$

[5]Short for *bi*partite *maze*.

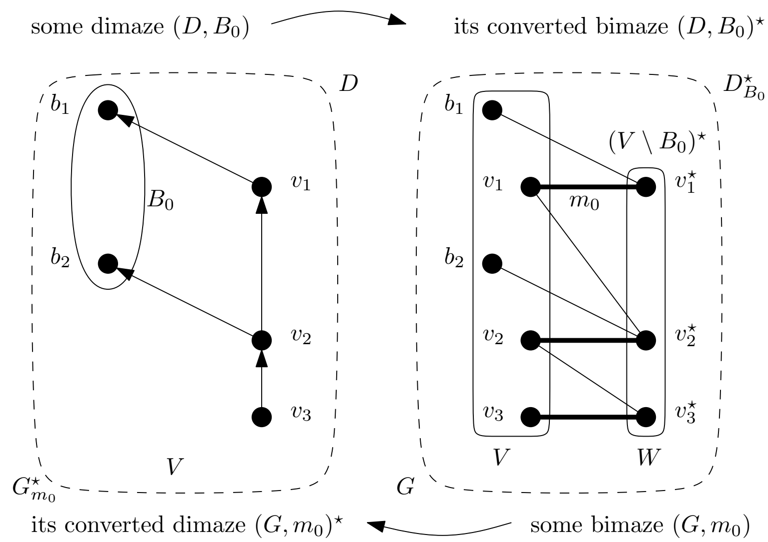

Figure 3.5: Converting a dimaze to a bimaze and vice versa

Given a bimaze (G, m_0), note that for any matching m, each infinite component of $G[m_0 \cup m]$ is either a ray or a double ray. We say m is an m_0-*matching*, if $G[m_0 \cup m]$ has no infinite component. A set $I \subseteq V$ is m_0-*matchable*, if there is an m_0-matching of I.

Definition 3.6.3. Given a bimaze (G, m_0), the pair of V and the set of all m_0-matchable subsets of V is denoted by $M_{PT}(G, m_0)$. If $M_{PT}(G, m_0)$ is a matroid, it is called a *path-transversal matroid*.

The correspondence between finite paths and m_0-matchings is depicted in the following lemma.

Lemma 3.6.4. *Let (D, B_0) be a dimaze. Then B is linkable onto B_0 in (D, B_0) iff $V \setminus B$ is m_0-matchable onto $(V \setminus B_0)^\star$ in $(D, B_0)^\star$.*

Proof. Suppose a linkage $\mathcal{P}$ from B onto B_0 is given. Let

$$m := \{vu^\star : (u, v) \in E(\mathcal{P})\} \cup \{ww^\star : w \notin V(\mathcal{P})\}.$$

Note that m is a matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ in $D^\star_{B_0}$. Any component induced by $m_0 \cup m$ is finite, since any component which contains more than one edge corresponds to a path in $\mathcal{P}$. So m is a required m_0-matching in $(D, B_0)^\star$.

Conversely let m be an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. Define a linkage from B onto B_0 as follows. From every vertex $v \in B$, start an m_0-m

alternating walk, which is finite because m is an m_0-matching. Moreover, the walk cannot end with an m_0-edge because m covers $(V \setminus B_0)^\star$. So the walk is either trivial or ends with an m-edge in B_0. As the m-edges on each walk correspond to a path from B to B_0, together they give us a required linkage in (D, B_0). $\square$

Proposition 3.6.5. *Let $M_T(G)$ be a transversal matroid and m_0 a matching of a base B. Then $M_T(G) = M_{PT}(G, m_0)$.*

Proof. Suppose $I \subseteq V$ admits a matching m. By the maximality of B, any infinite component of $m \cup m_0$ does intersect $V \setminus B$. Replacing the m-edges of all the infinite components by the m_0-edges gives an m_0-matching of I. $\square$

In fact, the class of path-transversal matroids contains the class of transversal matroids as a proper subclass; see Example 3.6.11 and Figure 3.6. Just as we can extend a linkage to cover the exits by trivial paths, any m_0-matching can be extended to cover W.

Lemma 3.6.6. *Let (G, m_0) be a bimaze. For any m_0-matchable I, there is an m_0-matching from some $B \supseteq I$ onto W.*

Proof. Let m be an m_0-matching of I. Take the union of all connected components of $m \cup m_0$ that meet $W - m$. The symmetric difference of m and this union is a desired m_0-matching of a superset of I. $\square$

We find it convenient to abstract two properties of a dimaze and a bimaze. Given a dimaze (D, B_0), let (†) be

$$I \in M_L(D, B_0) \text{ is maximal } \Leftrightarrow \exists \text{ linkage from } I \text{ onto } B_0. \tag{†}$$

Analogously, given a bimaze (G, m_0), let (‡) be

$$I \in M_{PT}(G, m_0) \text{ is maximal } \Leftrightarrow \exists\ m_0\text{-matching from } I \text{ onto } (V \setminus B_0)^\star. \tag{‡}$$

In some sense (†) and (‡) are dual to each other.

Lemma 3.6.7. *A dimaze (D, B_0) satisfies (†) iff $(D, B_0)^\star$ satisfies (‡).*

Proof. Assume (D, B_0) satisfies (†). To prove the backward direction of (‡), suppose there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. By Lemma 3.6.4, there is a linkage from B onto B_0. Therefore, B is maximal in $M_L(D, B_0)$ by (†). By Lemma 3.6.6, any m_0-matchable superset of $V \setminus B$ may be extended to one, say $V \setminus I$, that is m_0-matchable onto $(V \setminus B_0)^\star$.

As before, $I \subseteq B$ is maximal in $M_L(D, B_0)$, so $I = B$ and hence, $V \setminus B$ is a maximal m_0-matchable set. To see the forward direction of (‡), suppose $V \setminus B$ is a maximal m_0-matchable set witnessed by an m_0-matching m, that does not cover $v^\star \in (V \setminus B_0)^\star$. As m is an m_0-matching, a maximal m_0-m alternating walk starting from $v^\star$ ends at some vertex in B. So the symmetric difference of this walk and m is an m_0-matching of a proper superset of $V \setminus B$ which is a contradiction.

Assume $(D, B_0)^\star$ satisfies (‡). The forward direction of (†) is trivial. For the backward direction, suppose there is a linkage from B onto B_0. Then there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ by Lemma 3.6.4. By (‡), $V \setminus B$ is maximal in $M_{PT}(D, B_0)^\star$. With an argument similar to the above, we can conclude that B is maximal in $M_L(D, B_0)$. □

Now let us see how (†) helps to identify the dual of a strict gammoid.

Lemma 3.6.8. *If a dimaze (D, B_0) satisfies (†), then the dual of $M_L(D, B_0)$ is $M_{PT}(D, B_0)^\star$.*

Proof. By Lemma 3.6.7, $(D, B_0)^\star$ satisfies (‡). Let B be an independent set in $M_L(D, B_0)$. Then B is maximal if and only if there is a linkage from B onto B_0. By Lemma 3.6.4, this holds if and only if there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$, which by (‡) is equivalent to $V \setminus B$ being maximal in $M_{PT}(D, B_0)^\star$.

To complete the proof, it remains to see that every m_0-matchable set can be extended to a maximal one, which follows from Lemma 3.6.6 and (‡). □

Note that while we do not need it, the twin of Lemma 3.6.8 is true, namely, if a bimaze (G, m_0) satisfies (‡), then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

To summarize, we have the following.

Theorem 3.6.9. *(i) Given a C^A-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid dual to $M_{PT}(D, B_0)^\star$.*

(ii) Given a bimaze (G, m_0), if $(G, m_0)^\star$ is C^A-free, then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

Proof. (i) This is the direct consequence of Lemmas 3.3.3 and 3.6.8.

(ii) Apply part (i) and (3.8). □

One might hope that in the first part of the theorem the path-transversal matroid $M_{PT}(D, B_0)^\star$ is in fact the transversal matroid $M_T(D, B_0)^\star$. However, the dimaze R^I defines a strict gammoid whose dual is not the transversal matroid defined by the converted bimaze. It turns out that R^I is the only obstruction to this hope.

Theorem 3.6.10. *(i) Given an* $\{R^I, C^A\}$*-free dimaze* (D, B_0)*,* $M_L(D, B_0)$ *is a matroid dual to* $M_T(D^\star_{B_0})$*.*

(ii) Given a bimaze (G, m_0)*, if* $(G, m_0)^\star$ *is* $\{R^I, C^A\}$*-free, then* $M_T(G)$ *is a matroid dual to* $M_L(G, m_0)^\star$*.*

Proof. (i) This follows from Theorem 3.6.9(i) and the fact that for an R^I-free dimaze (D, B_0), we have $M_T(D^\star_{B_0}) = M_{PT}(D, B_0)^\star$. The proof of the latter is similar to the one given to Proposition 3.6.5 and omitted.

(ii) Apply part (i) and (3.8). □

It appears that C^A is a natural constraint in the above theorem.

Example 3.6.11. The strict gammoid defined by the dimaze C^A (Figure 3.6a) is not cotransversal.

Proof. Since $V \setminus B_0 + v$ is a base for every $v \in B_0$, B_0 is an infinite cocircuit. On the other hand, every vertex v of B_0 is contained in a finite cocircuit, namely v and its in-neighbours. So by Lemma 3.2.10, the dual is not transversal. □

Here is a question which is in some sense converse to Theorem 3.6.10(i).

Question 3.6.12. Is every cotransversal strict gammoid $\{C^A, R^I\}$-free?

Although the class of path-transversal matroids contains that of transversal matroids properly, not every strict gammoid has its dual of this type. To show this, we first note that in a path-transversal matroid $M_{PT}(G, m_0)$, if C is the fundamental circuit of u, then $N(C) = m_0(C - u)$. Indeed, $N(u) \subseteq m_0(C - u)$; and for any $v \in C - u$, since there is an m_0-alternating path from u ending in v, v cannot have any neighbour outside $m_0(C - u)$.

Example 3.6.13. Let T be a rooted tree such that each vertex has infinitely many children, with edges directed towards B_0, which consists of the root and vertices on alternating levels. Then $M_L(T, B_0)$ is a strict gammoid that is not dual to any path-transversal matroid.

Proof. In Corollary 3.4.6, it was proved that $M := M_L(T, B_0)$ is a matroid. Suppose that $M^* = M_{PT}(G, m)$. Let $\mathcal{Q}$ a linkage of $B := V - m$ to B_0. Since (T, B_0) is C^O-free, by Proposition 3.5.10, we have $M = M_L(D_1, B_1)$ where (D_1, B_1) is the $\mathcal{Q}$-shifted dimaze. By construction, the underlying graph of D is also a tree.

By Corollary 3.4.7, (D_1, B_1) contains a subdivision R of C^A. Let $\{s_i : i \geq 1\} := R \cap B_1$ and $U := \{u_i : i \geq 1\}$ be the set of vertices of out-degree 2 on R such that u_i is joined to s_i and s_{i+1} in R. Let

$$U_i := \{v \in T : v \text{ is separated from } R \cap B_1 \text{ by } u_i\},$$

$$S_i := \{v \in T : v \text{ can be linked to } s_i \text{ in } D \setminus U\}.$$

Since D is a tree, $\{U_i, S_i : i \geq 1\}$ is a collection of pairwise disjoint sets.

Let $C := \bigcup_{i \geq 1} S_i$. Any linkable set in $V \setminus C$ has a linkage that misses an exit in $R \cap B_1$. Since D is a tree, $(B_1 - R) \cup U + c$ for any $c \in C$ is a base of M. Hence, C is a circuit in M^*. For a contradiction, we construct an m-matching of C in (G, m).

In M^*, the fundamental circuit of s_i with respect to B_1 is $S_i \cup U_{i-1} \cup U_i$ (with $U_0 = \emptyset$). By the remark before the example, $N(S_i \cup U_{i-1} \cup U_i) = m(S_i \cup U_{i-1} \cup U_i - s_i)$ for $i \geq 1$.

We claim that for $i \geq 1$, in any m-matching m' of $\bigcup_{j \leq i} S_j$, the maximal m'-m alternating walk from s_j ends in $m(U_j)$ for $j \leq i$. Note that such a walk cannot end in $m(S_j)$ as those vertices are incident with m'-edges. Since $N(S_1) \subseteq m(S_1 \cup U_1)$, the claim is true for $i = 1$. Assume that it is true for $i - 1$. Consider an m-matching m_1 of $\bigcup_{j \leq i} S_j$. Let P_j be the maximal m_1-m-alternating walk starting from s_j. By assumption, P_j ends in $m(U_j)$ for each $j < i$. As P_i ends in $m(U_{i-1} \cup U_i)$, we are done unless it ends in $m(U_{i-1})$. In that case, the union of an m-matching of $C \setminus \bigcup_{j \leq i} S_j$ with

$$(m \restriction \bigcup_{j \leq i} S_j) \Delta \bigcup_{j \leq i} E(P_j)$$

is an m-matching of C, a contradiction.

Therefore, there is a collection of pairwise disjoint m-alternating walks $\{P'_i : i \geq 1\}$ where P'_i starts from s_i and ends in $m(U_i)$. Then $m \Delta \bigcup_{i \geq 1} E(P'_i)$ is an m-matching of C, a contradiction which completes the proof. $\square$

Here is a question similar to Question 3.6.12 akin to Theorem 3.6.9.

Question 3.6.14. Is every strict gammoid which is dual to a path-transversal matroid C^A-free?

It may be interesting to investigate further path-transversal systems. For example, while they need not satisfy (IM), it may be the case that (I3) always holds.

Conjecture 3.6.15. Given a bimaze (G, m_0), $M_{PT}(G, m_0)$ satisfies (I3).

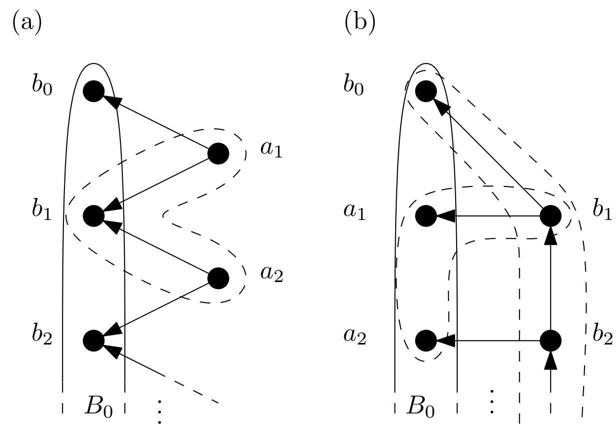

Figure 3.6: An alternating ray and an isomorphic incoming comb

3.6.2 Finitary transversal matroids

Our aim in this section is to give a transversal matroid that is not dual to any strict gammoid. To this end, we extend some results in [10] and [11]. The following identifies edges that may be added to a presentation of a finitary transversal matroid without changing the matroid.

Lemma 3.6.16. *Suppose that $M_T(G)$ is finitary. Let K be a subset of $\{vw \notin E(G) : v \in V, w \in W\}$. Then the following are equivalent:*

1. *$M_T(G) \neq M_T(G+K)$;*
2. *there are $vw \in K$ and a circuit C with $v \in C$ and $w \notin N(C)$;*
3. *there is $vw \in K$ such that v is not a coloop of $M_T(G) \setminus N(w)$.*

Proof. 1. holds if and only if there is a circuit C in $M_T(G)$ which is matchable in $G + K$. This, since C is finite, in turn holds if and only if there is $v \in C$ that can be matched outside $N(C)$ in $G + K$, i.e. 2. holds.

The equivalence between 2. and 3. is clear since a vertex is not a coloop if and only if it lies in a circuit. □

Given a bipartite graph G, recall that a presentation of a transversal matroid M as $M_T(G)$ is maximal if $M_T(G + vw) \neq M_T(G)$ for any $vw \notin E(G)$ with $v \in V, w \in W$. Thus, the previous lemma implies that if $M_T(G)$ is finitary, then G is maximal if and only if $M \setminus N(w)$ is coloop-free for any $w \in W$. Bondy [10] asserted that there is a unique maximal presentation for any finite coloop-free transversal matroid; where two presentations of a transversal matroid by bipartite graphs G and H are *isomorphic* if there is a graph isomorphism from G to H fixing the left vertex class pointwise.

Proposition 3.6.17. *Every finitary transversal matroid M has a unique maximal presentation.*

Proof. Let $M = M_T(G)$. Adding all vw with the property that there is not any circuit C with $v \in C$ and $w \notin N(C)$ gives a maximal presentation of M by Lemma 3.6.16. In particular, any coloop is always adjacent to every vertex in W. So without loss of generality, we assume that M is coloop-free.

Now let G and H be distinct maximal presentations of M.

Claim 1. For any finite subset F of V, the induced subgraphs $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$ are isomorphic.

For every $v \in F$ pick a circuit C_v with $v \in C_v$. By Lemma 3.6.16, for every $vw \in \{xy \notin E(G) : x \in F, y \in N_G(F)\}$, there is a circuit C_{vw} with $v \in C$ and $w \notin N_G(C)$. Let F_G be the union of all C_v's and C_{vw}'s. Analogously define F_H and let $F' = F_G \cup F_H$. Extend the presentations $G[F' \cup N_G(F')]$ and $H[F' \cup N_H(F')]$ of $M|F'$ to maximal ones G' and H' respectively, between which there is a graph isomorphism fixing the left vertex class pointwise by Bondy's result. Restricting the isomorphism to $F \cup N_G(F)$ is an isomorphism of $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$, as by definition of F' and Lemma 3.6.16, no non-edge between F and $N_G(F)$ is an edge in G' (analogously between F and $N_H(F)$ in H').

Without loss of generality, there is an $A \subseteq V$ such that $g := |\{w \in W(G) : N_G(w) = A\}| < |\{w \in W(H) : N_H(w) = A\}| =: h$. Note that as H is a maximal presentation, by Lemma 3.6.16, $M \setminus A$ is coloop-free.

As M is coloop-free, so is $M.A$. Let B_1 be a base of $M.A$ and extend B_1 to a base of M which admits a matching m; thus m contains a matching of a base of $M \setminus A$. Since $M \setminus A$ is coloop-free, by Lemma 3.2.6, the neighbourhood of each vertex matched by m to a vertex in B_1 is a subset of A. Thus, $M.A$ can be presented with the subgraphs induced by $A \cup \{w \in W : N(w) \subseteq A\}$ in both graphs; call these subgraphs G_1 and H_1. For any $w \in W(G_1)$, since $M \setminus N_{G_1}(w)$ is coloop-free, so is $M.A \setminus N_{G_1}(w)$. By Lemma 3.6.16, G_1 (analogously H_1) is a maximal presentation of $M.A$.

Claim 2. Given a family $(N_j)_{j \in J}$ of finite subsets of W, if the intersection of any finite subfamily has size at least k, then the intersection of the family has size at least k.

Let $N = \bigcap_{j \in J} N_j$. Suppose $|N| < k$. Fix some $j_0 \in J$ and for each element $y \in N_{j_0} \setminus N$ pick some N_y such that $y \notin N_y$. Then $|N_{j_0} \cap \bigcap_{y \in N_{j_0} \setminus N} N_y| = |N| < k$, which is a contradiction.

By Claim 2, there is a finite set $F \subseteq A$ such that $|\bigcap_{v \in F} N_{G_1}(v)| = g$. But Claim 1 says that F has at least $h > g$ common neighbours in H_1; this contradiction completes the proof. □

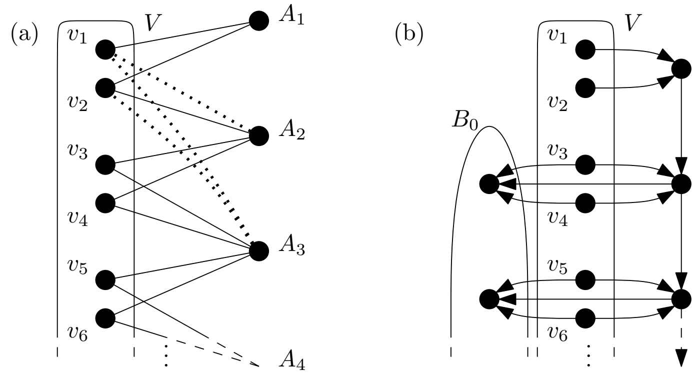

Figure 3.7: A transversal matroid which is not dual to a strict gammoid and a gammoid presentation of its dual

To show that the following finitary transversal matroid is not dual to a strict gammoid, it suffices to show that there is no bimaze presentation whose converted dimaze is C^A-free.

Example 3.6.18. Define a bipartite graph G as $V(G) = \{v_i, A_i : i \geq 1\}$ and $E(G) = \{v_1A_1, v_2A_1, v_1A_3, v_2A_3\} \cup \{v_{2i-3}A_i, v_{2i-2}A_i, v_{2i-1}A_i, v_{2i}A_i : i \geq 2\}$. Then $M = M_T(G)$ is not dual to a strict gammoid.

Proof. As G is left locally finite, M is a finitary matroid. Assume for a contradiction that $M^* = M_L(D, B_0)$. By a characterization of cofinitary strict gammoids in [3], we may assume that (D, B_0) is $\{R^I, C^A\}$-free. Then by Theorem 3.6.10, $M = M_T(D, B_0)^\star$.

Now it can be checked that all $M \backslash N(w_i)$ are coloop-free. By Lemma 3.6.16, G is the maximal presentation of M. The same lemma also implies that any minimal presentation G' is obtained by deleting edges from $\{v_1A_3, v_2A_3\}$ and at most one from $\{v_1A_2, v_2A_2\}$. In particular, all presentations of M differ from G only finitely. It is not difficult to check that with any matching m_0 of a base, $(G, m_0)^\star$ contains a subdivision of C^A. Hence, there is no bimaze presentation of M such that the converted dimaze is C^A-free, contradicting that $(D, B_0)^\star$ is such a presentation. □

We remark that the above transversal matroid is dual to a gammoid, see Figure 3.7. However, in the next section, we give a transversal matroid that is not dual to any gammoid.

3.6.3 Duality in gammoids

Recall that finite strict gammoids and transversal matroids are dual to each other. As a consequence one can see that the class of gammoids is closed under duality. In the infinite case, (as we have seen) there are examples of strict gammoids which are not cotransversal, as well as transversal matroids which are not dual to strict gammoids. However, this is not the case when the (undirected) underlying graphs do not contain any ray, and so, as we shall see, the proof of finite gammoids being closed under duality remains valid if we consider the class of gammoids which admit a presentation whose (undirected) underlying graph does not contain any ray. We finally conclude by giving an example of a gammoid which is not dual to any gammoid.

An undirected graph is called *rayless*, if it does not contain any ray. We call a gammoid *rayless* if it admits a presentation whose (undirected) underlying graph is rayless. A *rayless transversal matroid* is defined analogously.

An infinite version of König duality theorem [5] states that given a bipartite graph G, there is a matching m (König matching) and a cover C such that each edge in m has precisely one vertex of C as an end-point.

Lemma 3.6.19. *If $G = (V, W)$ is a rayless bipartite graph, then $M_T(G)$ is a matroid.*

Sketch of proof. It suffices to check (IM) with $X = V$. We show that any König matching m_0 (with vertex cover C) gives a maximal $I_0 \in M_T(G)$, and that any given $I \in M_T(G)$ can be covered by a König matching.

Suppose there exists $v \notin I_0$ such that there is matching m of $I_0 + v$. The m–m_0 alternating walk from v is a finite path ending in W. As m_0 is a König matching, and $v \notin I_0$, the vertices in C on this path lie in W. But then C does not cover the last edge.

Let m be a matching of I. Consider a component P induced by $m_0 \cup m$ such that P has a starting vertex in $I \setminus I_0$. Let m_1 be the set of all m–edges in such components union with the m_0–edges outside such components. Then m_1 is a matching covering I, indeed, a König matching with the same cover C. □

Given any rayless transversal matroid M with a presentation $G = (V, W)$, give a direction to the edges from V to W and consider W as the set of exits. This will give us a rayless strict gammoid and M is the restriction of this rayless gammoid onto V.

Proposition 3.6.20. *The class of rayless gammoids is closed under taking minor and duality.*

Proof. For the minor part, as restriction and contraction commute, it suffices to prove that any contraction of any rayless strict gammoid M is a rayless gammoid. Let $M = M_L(D, B_0)$ where the underlying graph of D is rayless. As there is no C^A or R^I in (D, B_0), by Theorem 3.6.10, $M_L^*(D, B_0) = M_T(D^\star_{B_0})$ which is a rayless transversal matroid. Any restriction of this transversal matroid is also a rayless transversal matroid, say $M_T(G)$. Pick some identity matching m_0. As the underlying graph of $(G, m_0)^\star$ is rayless, by Theorem 3.6.10, $M_T^*(G) = M_L(G, m_0)^\star$ which is a rayless gammoid.

To show that the class of rayless gammoids is closed under duality, note that $M^* = M_L^*(D, B_0)/X = M_T(D^\star_{B_0})/X$. But the last matroid is a contraction of a rayless gammoid which was just shown to be a rayless gammoid. $\square$

We remark that that Aharoni-Berger-Menger's theorem [6] for digraphs whose underlying graphs are rayless can be easily proved here as follows: Let $A, B \subseteq V(D)$ be given. We prove that $M_L(D, B)$ is a matroid without implicitly using Menger's theorem. First, any $I \in M_L(D, B)$ can be extended to a maximal by adding vertices in B not covered by a linkage from I to B by Lemma 3.3.3 . Next, by Lemmas 3.3.3, 3.6.4 and 3.6.7, a set I is maximal in $M_L(D, B)$ iff $V \setminus I$ is maximal in $M_{PT}(D, B)^\star$. Hence, $M_L^*(D, B) = M_{PT}(D, B)^\star$. As $M_{PT}(D, B)^\star = M_T(D^\star_B)$ and $D^\star_B$ is rayless, by Lemma 3.6.19, $M_T(D^\star_B)$ and so $M_L(D, B)$ is a matroid. Take a maximal element in $M_L(D, B) \restriction A$ witnessed by a linkage $\mathcal{P}$ and apply Lemma 3.2.3 to find a required separator on $\mathcal{P}$.

To show that there is a strict gammoid not dual to a gammoid, we prove the following lemmas, whose common setting is that a given dimaze (D, B_0) defines a matroid $M_L(D, B_0)$. For a linkage $\mathcal{Q}$ and any $X \subseteq Ini(\mathcal{Q})$, $\mathcal{Q} \restriction X := \{Q \in \mathcal{Q} : Ini(Q) \in X\}$; when $X = \{x\}$, we write simply Q_x.

Lemma 3.6.21. *Let b be an element in an infinite circuit C, $\mathcal{Q}$ a linkage from $C - b$. Then b can reach infinitely many vertices in C via $\mathcal{Q}$-alternating walks.*

Proof. Given any $x \in C - b$, let $\mathcal{P}$ be a linkage of $C - x$. Let W be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting from b. If W is infinite, then we are done. Otherwise, W ends in either $Ter(\mathcal{P}) \setminus Ter(\mathcal{Q})$ or $Ini(\mathcal{Q}) \setminus Ini(\mathcal{P}) = \{x\}$. The former case does not occur, since it gives rise to a linkage of C by Lemma 3.2.3(i), contradicting C being a circuit. As x was arbitrary, the proof is complete. $\square$

Lemma 3.6.22. *For $i = 1, 2$, let C_i be a circuit of M, x_i, b_i distinct elements in $C_i \setminus C_{3-i}$. Suppose that $(C_1 \cup C_2) \setminus \{b_1, b_2\}$ admits a linkage*

$\mathcal{Q}$. Then any two $\mathcal{Q}$-alternating walks W_i from b_i to x_i, for $i = 1, 2$, are disjoint.

Proof. Suppose that $W_1 = w_0^1 e_0^1 w_1^1 \dots w_n^1$ and $W_2 = w_0^2 e_0^2 w_1^2 \dots w_m^2$ are not disjoint. Then there exists a first vertex $v = w_j^1$ on W_1 such that $v = w_k^2 \in W_2$ and either $v \in V(\mathcal{Q})$ and $e_j^1 = e_k^2 \in E(\mathcal{Q})$ or $v \notin V(\mathcal{Q})$. In both cases $W_3 := W_1 v W_2$ is a $\mathcal{Q}$-alternating walk from b_1 to x_2. Let v' be the first vertex of W_3 in $V(\mathcal{Q} \restriction (C_2 - b_2) \setminus C_1)$ and Q the path in $\mathcal{Q}$ containing v'. Then $W_3 v' Q$ is a $(\mathcal{Q} \restriction (C_1 - b_1))$-alternating walk from b_1 to $B_0 \setminus Ter(\mathcal{Q} \restriction (C_1 - b_1))$, which by Lemma 3.2.3(i) contradicts the dependence of C_1. Hence W_1 and W_2 are disjoint. $\square$

Lemma 3.6.23. *Let $\{C_i : i \in N\}$ be a set of circuits of M; x_i, b_i distinct elements in $C_i \setminus \bigcup_{j \neq i} C_j$. Suppose that $\bigcup_{i \in N} C_i \setminus \{b_i : i \in N\}$ admits a linkage $\mathcal{Q}$. Let W_i be a $\mathcal{Q}$-alternating walk from b_i to x_i. If $X \subseteq V$ is a finite set containing $C_i \cap C_j$ for any distinct i, j, then only finitely many of W_i meet $\mathcal{Q} \restriction X$.*

Proof. By Lemma 3.6.22, the walks W_i are pairwise disjoint. Since $\mathcal{Q} \restriction X$ is finite, it can be met by only finitely many W_i's. $\square$

We are now ready to give a counterexample to classical duality in gammoids.

Example 3.6.24. Let (T, B_0) be the dimaze defined in Example 3.6.13. The dual of the strict gammoid $M = M_L(T, B_0)$ is not a gammoid.

Proof. Suppose that $M^* = M_L(D, B_1) \restriction V$, where $V := V(T)$. Fix a linkage $\mathcal{Q}$ of $V \setminus B_0$ in (D, B_1). For $b \in B_0$, let C_b be the fundamental cocircuit of M with respect to B_0. Then for any (undirected) ray $b_0 x_0 b_1 x_1 \cdots$ in T, $C := \bigcup_{k \in \mathbb{N}} C_{b_k} \setminus \{x_k : k \in \mathbb{N}\}$ is a cocircuit of M. We get a contradiction by building a linkage for C in (D, B_1) inductively using disjoint $\mathcal{Q}$-alternating walks.

Let b_0 be the root of T. By Lemma 3.6.21, there is a $\mathcal{Q}$-alternating walk W_0 from b_0 to one of its children x_0. At step $k > 0$, from each child b of x_{k-1} in T, by Lemma 3.6.21, there is a $\mathcal{Q}$-alternating walk W_b in (D, B_1) to a child x of b. Applying Lemma 3.6.23 on $\{C_i : i \in N^-(x_{k-1}) - b_{k-1}\}$ with $X = \{x_{k-1}\}$, we may choose $b_k := b$, $x_k := x$ such that $W_k := W_b$ avoids $Q_{x_{k-1}}$.

By Lemma 3.6.22, distinct W_k and $W_{k'}$ are disjoint. Moreover, as each W_k avoids $Q_{x_{k-1}}$, Lemma 3.2.3(i) implies that W_k can only meet $\mathcal{Q}$ at Q_x where $x \in C_{b_k} - x_{k-1}$. Then $E(\mathcal{Q}) \triangle \bigcup_{k \in \mathbb{N}} E(W_k)$ contains a linkage of C. $\square$

Note that, by adding a unique in-neighbour to every vertex in B_0, we can also present $M_L(T, B_0)$ as a transversal matroid (the same as the proof of Corollary 3.4.6). Thus, not every transversal matroid is dual to even a restriction of a strict gammoid.

It is possible to show (although we don't prove it) that the dual of any R^A-free strict gammoid is a gammoid. In fact, we expect more.

Conjecture 3.6.25. The class of C^A-free gammoids is closed under duality.

3.7 More on gammoids and transversal matroids

We first give characterizations of cofinitary strict gammoids and cofinitary transversal matroids and then investigate nearly finitary gammoids and nearly finitary transversal matroids. The results here come from [4].

3.7.1 Cofinitary strict gammoids and transversal matroids

Recall that by Lemma 3.2.9, every left locally finite bipartite graph defines a finitary transversal matroid. On the other hand, assume that a finitary transversal matroid M and a presentation of that $G = (V, W)$ is given. For every vertex $v \in V$ of infinite degree, we can delete all the infinitely many edges at v, add a new vertex to W and make it the private neighbour of v. This gives us a left locally finite bipartite graph, say G', and one can check that G and G' define the same transversal matroid. Therefore, finitary transversal matroids are precisely those that admit a left locally finite presentation.

It is also a straightforward consequence of Corollary 3.2.4, that a strict gammoid is finitary if and only if it admits a $\{C^O, F^\infty\}$-free presentation. A characterization of cofinitary strict gammoids is given as follows.

Theorem 3.7.1. *Any strict gammoid is cofinitary if and only if it admits a $\{C^A, R^I\}$-free presentation with in-degree of each vertex finite.*

Proof. Suppose the dimaze (D, B_0) is $\{C^A, R^I\}$-free and the in-degree of every vertex is finite. By Theorem 3.6.10, $M_L^*(D, B_0) = M_T(D^\star_{B_0})$. As the in-degree of D is finite, $D^\star_{B_0}$ is left locally finite. So by Lemma 3.2.9, $M_T(D^\star_{B_0})$ is finitary and hence $M_L(D, B_0)$ is a cofinitary matroid.

Conversely, suppose (D, B_0) is a dimaze defining a cofinitary strict gammoid M. Note that all the vertices which are linkable to a fixed vertex in B_0 form a cocircuit. Therefore, firstly it is R^I-free, and secondly as any vertex of infinite in-degree is a loop, deleting the edges at these vertices gives a new dimaze (D', B_0) defining the same matroid. Finally, for a contradiction assume that R is a subdivision of a C^A in (D', B_0). Let I' be the set of those vertices of R with out-degree two and $I := I' \cup (B_0 - R)$. Note that as I can be extended to a larger independent set, its complement meets every base of M. So the complement of I is codependent and contains some cocircuit C.

If C meets $R \cap B_0$ at only finitely many vertices, then $B_0 \setminus C$ can be extended by some $|C \cap B_0|$ vertices of I' to a base disjoint from C which contradicts codependence of C. So the intersection, and hence C is infinite which is the desired contradiction. □

By Theorem 3.7.1 and Theorem 3.6.10, we immediately have the following.

Corollary 3.7.2. *Cofinitary strict gammoids are dual to transversal matroids.*

We combine Theorem 3.5.13 with duality to determine presentations of cofinitary transversal matroids among all transversal matroids' presentations. Recall that given a matroid M, a base B and an independent set I, if $|B \setminus I| = |I \setminus B| < \infty$, then we can apply base exchange axiom ((B2) in Chapter 1) repeatedly to see that I has to be a base of M as well.

Theorem 3.7.3. *Let $G = (V, W)$ be a bipartite graph defining a transversal matroid and m_0 a matching of a base. Then $M_T(G)$ is cofinitary if and only if $(G, m_0)^\star$ is $\{C^A, R^I, C^O, F^\infty\}$-free.*

Proof. The backward direction follows from Theorem 3.6.10 (ii) and Corollary 3.2.4.

For the forward direction, first note that by Proposition 3.6.5 $M := M_T(G) = M_{LT}(G, m_0)$. Clearly $(G, m_0)^\star$ is R^I-free as that gives rise to an m_0-alternating walk in (G, m_0) which contradicts the maximality of $m_0 \cap V =: V \setminus B_0$ in M.

Let R be a subdivision of an C^A in $(G, m_0)^\star$. As a contradiction we find an infinite cocircuit of M. Let I_1 be the vertices of out-degree two on R, $I_2 = B_0 - R$, $B_1 = I_1 \cup I_2$, and v a vertex in $B_0 \cap R$. As B_1 and $B_1 + v$ are both linkable onto B_0 in $(G, m_0)^\star$, by Lemma 3.6.4 and (3.8), $V \setminus B_1$ and $V \setminus (B_1 + v)$ are both matchable in G which means $B_1 + v$ is codependent and so contains a cocircuit C of M. Assume that $C \cap I_1 =: C_1$ is finite and $\mathcal{P}$ is a linkage from C_1 to $B_0 \setminus (I_2 + v)$ (which exists by definition of I_1). Let $B_2 := (B_0 \setminus Ter(\mathcal{P})) \cup Ini(\mathcal{P})$. As $|(V \setminus B_0) \setminus (V \setminus B_2)| = |B_2 \setminus B_0| = |Ini(\mathcal{P})| = |Ter(\mathcal{P})| = |B_0 \setminus B_2| = |(V \setminus B_2) \setminus (V \setminus B_0)|$, by the sentence before the theorem, $V \setminus B_2$ is a base of M. But $C \cap (V \setminus B_2) = \emptyset$ which contradicts C being a cocircuit of M. Hence C is infinite which contradicts M being cofinitary.

As $(G, m_0)^\star$ is $\{C^A, R^I\}$-free, by Theorem 3.6.10 (ii), $M^\star = M_L(G, m_0)^\star$ and so $(G, m_0)^\star$ does not contain any subdivision of C^O or F^∞ either, as any subdivision of C^O or F^∞ leads to an infinite circuit in M^*. □

Applying Theorem 3.6.10 (ii) and Theorem 3.7.3 the following is immediate.

Corollary 3.7.4. *Let $M_T(G)$ be a cofinitary transversal matroid and m_0 be a matching of a base. Then $M_T(G)$ is dual to $M_L(G, m_0)^\star$.*

We remark that corollaries 3.7.2 and 3.7.4 can be used to show that the classes of cofinitary strict gammoids and cofinitary transversal matroids are representable over sufficiently large fields.

3.7.2 Nearly finitary gammoids

Recall that a nearly finitary matroid [7] is a matroid M such that every base can be extended to a base of the finitarization M^{fin}, by adding finitely many elements. If the number of additions is bounded by an integer $k \geq 0$, then M is called *k-nearly finitary.* The question whether a nearly finitary matroid is k-nearly finitary for some $k \geq 0$ is open [8].

We answer this question for strict gammoids, using Corollary 3.2.4 and the following modified version of a result of Halin [23].

Lemma 3.7.5. *If for any $k \in \mathbb{N}$ there is a set of spine-disjoint outgoing combs of size k, then there is an infinite such set.*

Proof. We mimic a proof given by Andreae (see [19, Theorem 8.2.5]), so we need only highlight the differences. Instead of constructing inductively an infinite set of disjoint rays, we construct an infinite set of spine-disjoint outgoing combs. As an outgoing comb has infinitely many spikes, we ensure that we extend at each step every initial segment of the outgoing combs by at least, not only one vertex, but one spike. This can be done by following the spine of an outgoing comb until we hit a spike disjoint from the already chosen and finite set of spikes and initial segments of spines. Moreover, at each step we choose a new initial segment of an outgoing comb disjoint from the others. □

Proposition 3.7.6. *A nearly finitary strict gammoid is k-nearly finitary for some $k \geq 0$.*

Proof. Let $M = M_L(D, B_0)$ be a nearly finitary strict gammoid. By Corollary 3.2.4, there cannot be infinitely many spine-disjoint outgoing combs, otherwise the union of the infinitely many initial vertices of the combs with B_0 is independent in M^{fin}. It follows then, by Lemma 3.7.5, that the number of spine-disjoint outgoing combs is bounded. Similarly, there cannot be infinitely many vertices which are the centre of an infinite star to B_0, otherwise, B_0 together with the centres form an independent set in M^{fin}. Let k be the sum of the maximum number of spine-disjoint outgoing combs and the number of centres of infinite stars. Let B be an independent set in M^{fin}. By deleting at most k vertices, we get a linkable subset of B. $\square$

Next, we consider the question for transversal matroids.

Lemma 3.7.7. *Let M be a matroid such that no infinite circuit is contained in a union of finite circuits. If M is nearly finitary, then M is k-nearly finitary for some integer $k \geq 0$.*

Proof. Let L be the set of coloops of M^{fin}, it follows that each element in $M \setminus L$ is contained in a finite circuit. Hence, $M \setminus L$ does not contain any infinite circuit, for otherwise, such a circuit is contained in a union of finite circuits.

Let B_1 be a base of $M.L$ and B^f a base of M^{fin}. Then $B^f \setminus L$ is a base of the finitary matroid $M \setminus L$, since it does not contain any finite circuit of M and addition of any element in $M \setminus B^f$ creates one.

Hence, $B = B_1 \cup (B^f \setminus L)$ is a base of M. Since M is nearly finitary, $B^f \setminus B$ is finite. On the other hand, $B^f \setminus B = L \setminus B_1$. It follows that $k := |B^f \setminus B| = |L \setminus B_1|$, which is the corank of $M.L$, is finite. Therefore, any base B^f of M^{fin} contains a base of M which has at most k elements fewer. Hence, M is k-nearly finitary. $\square$

Remark 3.7.8. *In fact, if B is a base of M and B^f is a base of M^{fin} containing B, then $|B^f \setminus B| = k$. Indeed, $B \cap L$ contains a base B_1 of $M.L$, and $B \setminus L$ can be extended in $B^f \setminus L$ to a base B_2 in $M \setminus L$. In fact, $B_2 = B^f \setminus L$, so $|B^f \setminus B| = |B^f \setminus (B_1 \cup B_2)| = k$.*

As a consequence of Lemma 3.2.10 we have the following.

Corollary 3.7.9. *A nearly finitary transversal matroid is k-nearly finitary, for some integer $k \geq 0$.*

Moreover, since the extension of a base of M to one of M^{fin} occurs in L, which is the set of vertices of infinite degree in case M is a transversal matroid, another corollary is the following.

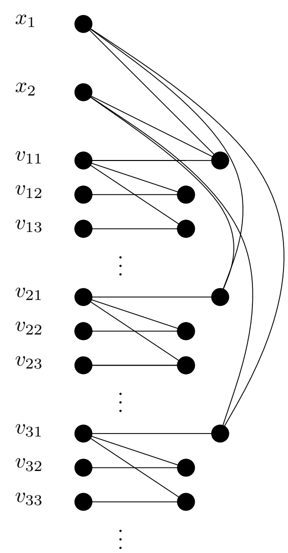

Figure 3.8: A 2-nearly finitary transversal matroid

Corollary 3.7.10. *For any integer $k \geq 0$ and any bipartite graph $G = (V, W)$ such that there are at most k vertices of infinite degree in V, $M_T(G)$ is a k-nearly finitary matroid.*

The converse when $k = 0$ is true by the first paragraph of Section 3.7.2. For $k = 1$, it turns out that the converse of Corollary 3.7.10 is also true. As the proof involves long case analysis, we only sketch it here.

Proposition 3.7.11. *A 1-nearly finitary transversal matroid M admits a presentation $G = (V, W)$ with at most 1 vertex of infinite degree in V.*

Proof sketch. Let $L \subseteq V$ be the set of vertices of infinite degree. Let B_1 be a base of $M.L$. If $L \setminus B_1 = \emptyset$ then M is a finitary matroid and, as already described, we can find a presentation of M without any vertex of infinite degree. On the other hand, as M is 1-nearly finitary, $|L \setminus B_1|$ cannot be greater than one, so $L \setminus B_1 = \{x\}$ for some x in L.

Build the graph G' as follows. Extend B_1 to a base B of M, and let m be a matching of B. For every edge in $vw \in E(G) \setminus m$ with one end in $L \setminus \{v\}$ delete vw from G and add a new edge xw, and if xw is already an edge we do not add any new edge.

It is then possible to inductively prove that $M_T(G) = M_T(G')$. $\square$

On the other hand, a 2-nearly finitary transversal matroid may only have presentations with infinitely many vertices of infinite degree.

Example 3.7.12. The circuits of the transversal matroid defined by the bipartite graph in Figure 3.8 are $\{V - x_1, V - x_2, V \setminus V_i : i \in \mathbb{N}\}$, where $V_i = \{v_{ij} : j \in \mathbb{N}\}$. For any presentation $G = (V, W)$ of M, there are infinitely many vertices of infinite degree in V.

Proof. It is straightforward to compute the set of circuits. Let $X = \{x_1, x_2\}$. Fix a matching m of the base $V \setminus X$. Let $(D, X) = (G, m)^\star$. It suffices to prove that in D, there is a vertex of infinite in-degree in each V_i.

Note that an (infinite) m-alternating walk in G corresponds to an (infinite) directed path in D. As $V - x_1$ and $V - x_2$ are circuits, there is no incoming ray to X. Since $V \setminus V_i$ is a circuit, there are no two disjoint m-alternating walks from X to V_i in G. Therefore, there are no two disjoint paths from V_i to X in D, by Menger's theorem, V_i is separated from X by a vertex v_i. For any $v \in V_i, v' \in V_j$ where $i \neq j$, as $V \setminus \{v, v'\}$ is matchable, there is a pair of disjoint paths from $\{v, v'\}$ to X. It follows that $v_i \in V_i$. For any $v \in V_i - v_i$, if there is an edge (v, v') with $v' \in V_j$, then the edge can be extended to a path from v to X avoiding v_i, using a path in a linkage from $\{v', v_i\}$ to X, which is a contradiction. Hence, within V_i, there is a path from v to v_i. As there is no incoming ray to X, it follows that V_i contains a vertex of infinite in-degree. $\square$

So having at most k vertices of infinite degree is sufficient but not necessary for a transversal matroid to be k-nearly finitary. However, in general, there is a deletion minor of a k-nearly finitary transversal matroid that is finitary.

Proposition 3.7.13. *A transversal matroid M is k-nearly finitary if and only if there is a set of vertices $X \subseteq V$ of size at most k such that $M \setminus X$ is finitary.*

Proof. Let G be a presentation of M. Let L be the set of coloops of M^{fin}. Let B_1 be a base of $M.L$ and B_2 be a base of $M \setminus L$, which is a finitary matroid.

Suppose that M is k-nearly finitary. We claim that $X := L \setminus B_1$ is the required set. As $B_1 \cup B_2 \cup X$ is a base of M^{fin}, $|X| \leq k$. It remains to show that $M \setminus X$ is finitary. Let C be a circuit of $M \setminus X$. As $C \subseteq B_1 \cup C \setminus L$ and B_1 is a base of $M.L$, $C \cap L = \emptyset$. So C is a circuit of the finitary matroid $M \setminus L$.

Suppose X is a minimal set of k vertices such that $M \setminus X$ is finitary. By compactness, every vertex in X has infinite degree. Let B^{fin} be a base of M^{fin}. Then $B^{\text{fin}} \setminus X$ is a base of $M \setminus X$, as $X \subseteq L$. Let $B \subseteq B^{\text{fin}}$ be a base of M. Extend $B^{\text{fin}} \setminus X$ to a base B' of M in B. As $B \setminus B' \subseteq X$, $|B \setminus B'| = |B' \setminus B|$. Hence, $|B^{\text{fin}} \setminus B| = |X \setminus B| + |B' \setminus B| = |X \setminus B| + |B \setminus B'| \leq |X| = k$. $\square$

Problem 3.7.14. Characterize k-nearly finitary transversal matroids in graph theoretic terms.

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Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Seyed Hadi Afzali Borujeni

Summary

This thesis is part of an ongoing project, which tries to extend different aspects of finite matroid theory to the infinite case. The thesis consists of two major parts.

In Chapter 2, the problem of representability of infinite matroids over a field is addressed, where we study different aspects of thin sums matroids. We give a characterization of the duals of ordinary representable matroids among thin sums matroids. We show that the class of tame thin sums matroids is closed under duality and so taking minors. As we shall see, most of the matroids associated to graphs turn out to be tame and thin sums representable. So we suggest the class of tame matroids, as a suitably large class of matroids in which one can have a reasonable theory of representability which is preserved under duality.

In Chapter 3, we look at another class of matroids namely the class of gammoids. These matroids are usually given via their presentations which are digraphs. As graph properties are usually easy to visualise, we are interested in the interaction of properties of gammoids as matroids and their presentations. To give a taste of what we do, our approach is similar to identifying a desired class of graphs via forbidding graphs as topological minors. Roughly speaking, looking closely at a system of linkable sets with an undesired behaviour, we try to find the substructure in its defining digraph which causes this undesired behaviour, and then study the class of gammoids definable by the digraphs that do not contain this substructure.

Zusammenfassung

Diese Arbeit ist teil eines fortlaufenden Projektes, in dem versucht wird unterschiedliche Aspekte endlicher Matroidtheorie auf den unendlichen Fall zu übertragen. Sie besteht aus zwei Teilen.

In Kapitel 2 wird die Darstellbarkeit unendlicher Matroide studiert, insbesondere werden sogenannte thin sums Matroide untersucht. Wir charakterisieren diejenigen Mantroide unter den thin sims Matroiden, die dual zu einem darstellbaren Matroid sind. Wir zeigen, dass die Klasse der zahmen thin sums Matroide unter Dualität abgeschlossen ist, und folglich auch unter Minorbildung. Wie sich herausstellen wird, sind die meisten, mit Hilfe von Graphen definierten, Matroide zahm und thin sums darstellbar. Also schlagen wir die zahmen Matroide vor, als hinreichend große Klasse von Matroiden, die eine, unter Dualität abgeschlossene, Theorie von Darstellbarkeit besitzt.

In Kapitel 3 untersuchen wir eine weitere Klasse von Matroiden, die Gammoide. Diese Matroide werden gewöhnlich durch Digraphen präsentiert. Da Eigenschaften von Graphen üblicherweise leicht zu visualisieren sind, sind wir an deren Interaktion mit den Matroideigenschaften der, durch die Graphen präsentierten, Gammoide interessiert. Unsere Methode ist ähnlich dem Versuch eine Grapheneigenschaft durch verbotene topologische Minoren zu charakterisieren. Hat ein gegebenes System verbindbarer Mengen eine unerwünschte Eigenschaft, so versuchen wir in dem definierenden Digraphen eine Struktur zu finden, die für diese Eigenschaft verantwortlich ist. Anschließend untersuchen wir die Klasse der Gammoide, die durch einen Digraphen definiert werden können, der diese Struktur nicht enthält.