Connected Tree-width and Infinite Gammoids

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Contents

Summary

This thesis in combinatorics consists of two parts: the first part is about connected tree-width [23] and the second part investigates infinite gammoids [2, 3]. In both parts a known result is transferred to a new area and the differences are analyzed.

A tree-decomposition of a graph G is called *connected* if all its parts induce connected subgraphs in G and the minimum width that a connected tree-decomposition can have is the *connected tree-width* of G. Tree-decompositions are a well established concept in graph theory and the standard minimum width tree-decomposition of many simple example graphs, including the grid, complete graphs or trees, are connected. On the other hand the connected tree-width of a cycle is about half of the length of that cycle. Obviously, the connected tree-width is an upper bound for the tree-width, and the cycles, having a tree-width of 2, show that the tree-width and the connected tree-width of a graph can be arbitrary far away from each other.

It was conjectured in [24] and is proved in [23] that for any graph, a large geodesic cycle is the only reason for the connected tree-width to be much larger than the tree-width. This is used to show that a qualitative version of a "connected tree-width duality theorem" holds.

The second part concerns *gammoids*, a class of matroids investigated in the late 1960's [27]. Ingleton and Piff [20] gave a construction that transforms a presentation of a finite strict gammoid to a transversal matroid presentation of its dual, a bipartite graph. This is used in the proof that the class of finite gammoids is closed under minors and under duality. In 2010 Bruhn et al [10] found a notion of infinite matroids that allows for duality. This suggests the question of extending gammoids to infinite ground sets by a verbatim transfer of linkability.

Contrary to the finite case, not every infinite dimaze, digraph together with a specific set of sinks, defines a matroid. One obstruction is a dimaze termed an alternating comb [2]. For such a strict gammoid the construction of Ingleton and Piff (transferred to the infinite case) provides a presentation of the dual and, if the dimaze does not contain an incoming ray, that dual is transversal. The class of gammoids definable by a dimaze without any

outgoing comb is minor closed and the class of gammoids definable by a dimaze without any ray is, like that of finite gammoids, closed under minors and under duality.

Chapter 1

Connected tree-width

The results in this chapter come from [23, 24].

1.1 Introduction

Let us call a tree-decomposition $(T, (V_t)_{t \in T})$ of a graph G *connected* if its parts V_t are connected in G. For example, the standard minimum width tree-decomposition of a tree or a grid has connected parts. The *connected tree-width* $ctw(G)$ of G is the minimum width that a connected tree-decomposition of G can have.

Obviously $tw(G) \leq ctw(G)$, because every connected tree-decomposition is a tree-decomposition. So having large tree-width is a reason for a graph to have large connected tree-width. But it is not the only possible reason.

It is not hard to show that a cycle of length n has connected tree-width $\lceil \frac{n}{2} \rceil$ (see Example 1.5.2). Indeed, any graph containing such a cycle geodesically[1] has connected tree-width $\lceil \frac{n}{2} \rceil$; this will follow from Lemma 1.5.3 below.

The following main theorem of this chapter shows that large tree-width and large geodesic cycles are the only two reasons for a graph to have large connected tree-width.

Theorem 1.1.1. *The connected tree-width of a graph G is bounded above by a function of its tree-width and the maximum length k of its geodesic cycles. Specifically*

$$ctw(G) \leq tw(G) + \binom{tw(G)+1}{2} \cdot (k \cdot tw(G) - 1).$$

(If G is a forest, we define k to be 1)

[1] A subgraph H of a given graph G is called *geodesic* if $d_H(x,y) = d_G(x,y)\ \forall\ x, y \in V(H)$, i.e. there is no shortcut in G between two vertices of H.

Theorem 1.1.1 is qualitatively best possible in that the two reasons are independent: a large cycle (as a graph) contains a large geodesic cycle but has small tree-width, while a large grid has large tree-width but all its geodesic cycles are small.

Among the many obstructions to small tree-width there is only one that gives a tight duality theorem: the existence of a large-order bramble. A *bramble* is a set of pairwise touching connected subsets of $V(G)$, where two such subsets *touch* if they have a vertex in common or G contains an edge between them. A subset of $V(G)$ *covers* (or is a *cover* of) a bramble $\mathcal{B}$ if it meets every element of $\mathcal{B}$. The *order* of a bramble is the least number of vertices needed to cover it.

Tree-width duality theorem (Seymour and Thomas [30]). *Let $k \geq 0$ be an integer. A graph has tree-width $\geq k$ if an only if it contains a bramble of order $> k$.*

Let the *connected order* of a bramble $\mathcal{B}$ be the least order of a *connected cover*, a cover of $\mathcal{B}$ spanning a connected subgraph. Since every bramble is covered by a part in any given tree-decomposition, graphs of connected tree-width $< k$ cannot have brambles of connected order $> k$. I conjecture that the converse of this holds too:

Conjecture 1.1.2 (connected tree-width duality conjecture). *Let $k \geq 0$ be an integer. A graph has connected tree-width $\geq k$ if and only if it contains a bramble of connected order $> k$.*

The second main result of this chapter is a qualitative version of the above conjecture:

Theorem 1.1.3. *Let $k \geq 0$ be an integer. There is a function $g : \mathbb{N} \to \mathbb{N}$ such that any graph with no bramble of connected order $> k$ has connected tree-width $< g(k)$.*

The proof of Theorem 1.1.1 goes roughly as follows. We start with a tree-decomposition of minimum width and enlarge its parts by replacing them with connected supersets. In order to retain a tree-decomposition, we shall have to make sure that vertices which are used to make one part connected also appear in certain other parts of the tree-decomposition (compare axiom (T3) in the definition of a tree-decomposition, e.g. in [13]). Our task will be to find extensions whose sizes are bounded by a function in the maximum length of a geodesic cycle in the graph and its tree-width, regardless of its number of vertices.

All the graphs we consider in this chapter will be finite and nonempty. The notation and terminology we use are explained in [13], in particular we shall assume familiarity with the basic theory of tree-decompositions as described in [13, Ch.12.3.].

The layout of this chapter is as follows. In Section 1.2 we introduce our main technical tool for finding paths in a graph that can be used to make disconnected parts of its tree-decompositions connected: a *navigational path system*, or *nav* for short. In Section 1.3 we introduce tree-decompositions whose parts cannot be split, we call such tree-decompositions *atomic*. For such atomic tree-decompositions we then find cycles in the graph that are separated by its adhesion sets. In Section 1.4 we use those cycles to get an upper bound for the part sizes of our connected tree-decomposition which completes the proof of Theorem 1.1.1. In Section 1.5, this result will be used to prove Theorem 1.1.3.

1.2 Navs

How do we get an upper bound for the connected tree-width of a graph G? The easiest algorithmic way is to start with a tree-decomposition of minimum width and enlarge a part (which does not yet induce a connected subgraph) by adding a path of G (reducing the number of components). This might result in a violation of (T3), which can be repaired by adding the corresponding vertices also to other parts. Now we can go on and make the next part (a little bit more) connected until we have a connected tree-decomposition. If we don't choose the connecting paths carefully, we might add an unbounded number of vertices to one part while repairing (T3). Take the graph and tree-decomposition indicated in Figure 1.1, for instance. If we choose the path containing x_i for making V_{t_i} connected (for every i), we will have to add all the x_i to V_{t_0} while repairing (T3), because V_{t_0} lies between the part containing x_i and V_{t_i} which contains x_i as well (after we added the connecting path).

Obviously we made a bad choice here. If we use the path containing x_1 for every V_{t_i} we don't need to enlarge V_{t_0} arbitrarily often. This is the idea of the following definition: If we already know a path connecting two vertices a and b, then we can reuse it whenever we have a path going through a and b.

Definition 1.2.1 (navigational path-system (short: nav))**.** Let G be a connected graph and $K \subseteq [V(G)]^{\leq 2}$ a subset of the set of all at most 2-element subsets of $V(G)$. A system $\mathcal{N} := (P_{xy})_{\{x,y\}\in K}$ of x–y paths is called *sub-nav*, if for every path P_{xy} in $\mathcal{N}$ and for any two vertices a, b on that path $\{a, b\}$ is in K and $P_{ab} = aP_{xy}b$.

A *nav* is a sub-nav satisfying $K = [V(G)]^{\leq 2}$.

If $\mathcal{D} := (T, (V_t)_{t\in T})$ is a tree-decomposition of G, then a sub-nav satisfying $[V_t]^{\leq 2} \subseteq K \;\forall\; t \in T$ is called a *$\mathcal{D}$-nav*.

A sub-nav is called *geodesic* if for all $x, y \in K$ the length of P_{xy} is $d_G(x, y)$.

The length of a longest path used in a sub-nav is called the *length* of the sub-nav $l(\mathcal{N}) := max_{\{x,y\}\in K} \|P_{xy}\|$.

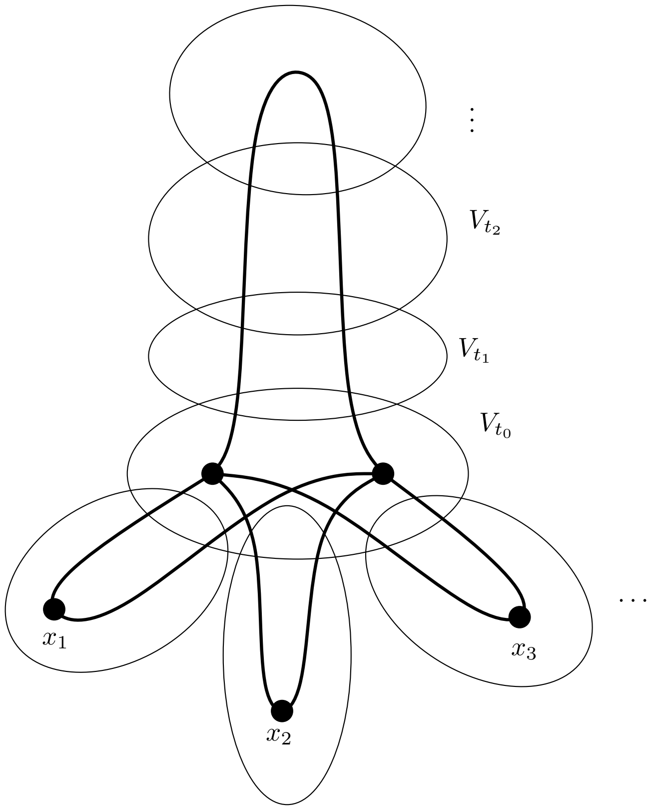

Figure 1.1: V_{t_0} might grow arbitrarily.

A nav knows some connection between every two vertices. A sub-nav might not know all connections, but the known ones are stored in K. If a sub-nav knows a path connecting x and y, then it knows the connections of all vertex-pairs on that path (they are induced by the original x–y path). A $\mathcal{D}$-nav knows the connection of two vertices if they are in a common part of the tree-decomposition $\mathcal{D}$. A geodesic nav does not only know some path between the vertices but a shortest possible. Note that P_{xy} stands for $P_{\{x,y\}}$, so P_{xy} is P_{yx} and in the case of $x = y$ the path P_{xy} is trivial. Let us now see how a nav helps making a tree-decomposition connected:

Theorem 1.2.2. *Let G be a connected graph, $\mathcal{D} = (T, (V_t)_{t\in T})$ a tree-decomposition of G of width w and $\mathcal{N} = (P_{xy})_{\{x,y\}\in K}$ a $\mathcal{D}$-nav of G. Define $W_t := \bigcup_{\{x,y\}\in[V_t]^{\leq 2}} V(P_{xy})$ for all $t \in T$. Then $(T, (W_t)_{t\in T})$ is a connected tree-decomposition of G of width $\leq w + \binom{w+1}{2} \cdot (l(\mathcal{N}) - 1)$.*

Proof. Since $\mathcal{N}$ is a $\mathcal{D}$-nav, all W_t are defined. V_t is a subset of W_t for all $t \in T$ because P_{xy} contains x and y. So (T1) and (T2) are easy to see. For (T3) let t_1, t_2 and t_3 be distinct vertices of T with $t_2 \in t_1Tt_3$ and let s be in $W_{t_1} \cap W_{t_3}$. We need to show $s \in W_{t_2}$: According to the definition of W_t there must be some x_1 and $y_1 \in V_{t_1}$ and some x_3 and $y_3 \in V_{t_3}$ such that $s \in P_{x_1y_1}$ and $s \in P_{x_3y_3}$. The set V_{t_2} separates V_{t_1} from V_{t_3}, in particular, V_{t_2} is an $\{x_1, y_1\}$–$\{x_3, y_3\}$ separator. If $s \in V_{t_2}$, then s is in W_{t_2} too, as required. So V_{t_2} now has to be a separator without using s. This is only possible if s is separated by V_{t_2} from at least one of the sets $\{x_1, y_1\}$ or $\{x_3, y_3\}$ (say $\{x_1, y_1\}$), since otherwise there would be an $\{x_1, y_1\}$–$\{x_3, y_3\}$ path in the union of the $\{x_1, y_1\}$–s path and the s–$\{x_3, y_3\}$ path avoiding V_{t_2}. Hence there have to be two vertices x_2 and y_2 in V_{t_2} such that $x_2 \in x_1P_{x_1y_1}s$ and $y_2 \in sP_{x_1y_1}y_1$. By definition of sub-nav $P_{x_2y_2} = x_2P_{x_1y_1}y_2$ and therefore $s \in V(P_{x_2y_2}) \subseteq W_{t_2}$.

All W_t are connected and their size is bounded by "size of V_t + all vertices added". Every P_{xy} has at most $l(\mathcal{N}) - 1$ vertices besides x and y and at most $\binom{w+1}{2}$ of those paths P_{xy} have been added. □

In order to construct a connected tree-decomposition of small width we need to search a $\mathcal{D}$-nav of small length, which is achieved by a geodesic nav. The existence of an arbitrary nav is easy to show, because a spanning tree gives rise to a nav. A bit more surprising is that it is always possible to find a geodesic nav.

Theorem 1.2.3. *Every connected graph has a geodesic nav.*

Proof. Let $G = (V, E)$ be the connected graph with a fixed linear order of the vertex set. The set of characteristic vectors of geodesic paths in G is by lexicographical order again linearly ordered. Since there are no two different geodesic paths on the same vertex set, there is a 1-1-correspondence between the characteristic vectors of geodesic paths and the paths themselves. So the set of geodesic paths is ordered lexicographically too. Note that there is a geodesic path between any two vertices as G is connected. Hence for every two vertices x and y in G there is exactly one minimal geodesic x–y path. Declare this path to be P_{xy}. Then $\mathcal{N} := (P_{xy})_{\{x,y\}\in[V(G)]^{\leq 2}}$ is a path-system consisting of geodesic paths.

Assume that $\mathcal{N}$ is not a nav. Then there are two vertices x and y in V and $a, b \in P_{xy}$ such that $Q_{ab} := aP_{xy}b \neq P_{ab}$.

Observe that Q_{ab} is a geodesic a–b path and $Q_{xy} := xP_{xy}aP_{ab}bP_{xy}y$ is a geodesic x–y path. So they were considered when declaring P_{xy} and P_{ab}, but were not chosen because $P_{ab} < Q_{ab}$ and $P_{xy} < Q_{xy}$. Since $P_{xy} - Q_{ab} = Q_{xy} - P_{ab}$, we can extend Q_{ab} to P_{xy} and P_{ab} to Q_{xy} using the same paths (i.e. without changing the lexicographical ordering). This is a contradiction, so $\mathcal{N}$ is in fact a geodesic nav of G. □

Given a tree-decomposition $\mathcal{D} = (T, (V_t)_{t\in T})$ and a geodesic nav $\mathcal{N} = (P_{xy})_{\{x,y\}\in[V(G)]^{\leq 2}}$ we can define a geodesic $\mathcal{D}$-nav by collecting only the needed paths: $\mathcal{N}_{\mathcal{D}} := (P_{xy})_{\{x,y\}\in K_{\mathcal{D}}}$ with $K_{\mathcal{D}} := \bigcup_{t\in T} \bigcup_{\{x,y\}\in V_t} [P_{xy}]^{\leq 2}$. The length of this nav $\mathcal{N}_{\mathcal{D}}$ is bounded by the maximal distance of two vertices, which live inside a common part of $\mathcal{D}$. The task has now changed into finding a tree-decomposition of width $tw(G)$ such that two vertices living inside a common part have a distance bounded by the tree-width of G and the length of a longest geodesic cycle.

1.3 Atomic tree-decompositions

In a contradiction proof it might be useful to not be able to refine a tree-decomposition. Technically this can be achieved by considering the descending ordered sequences of part-sizes of the possible tree-decompositions of the graph. A lexicographically minimal such sequence shall be called *atomic*. An equivalent version of the same idea that shortens the argument can be found in [14] (in the proof of Theorem 3 on page 3):

Definition 1.3.1 (atomic tree-decomposition as in [14])**.** Let G be a graph and $n := |G|$. Let the *fatness* of a tree-decomposition of G be the n-tuple $(a_0, \dots, a_n)$, where a_h denotes the number of parts that have exactly $n-h$ vertices. A tree-decomposition of lexicographically minimal fatness is called an *atomic* tree-decomposition.

Since there always exists a tree-decomposition that has no part of size $> tw(G)+1$ it is clear that an atomic tree-decomposition has width $tw(G)$.

1.3.1 Rearranging tree-decompositions

Let us introduce some constructions that will reveal useful properties of atomic tree-decompositions. One possible way of rearranging a tree-decomposition is contracting an edge in its tree:

Lemma 1.3.2. *Let G be a graph, $\mathcal{D} = (T, (V_t)_{t\in T})$ a tree-decomposition of G and $e = rs$ an edge of T. Define $T' := T/e$, $W_t := V_t\ \forall t \in T - \{r, s\}$ and $W_{t_e} := V_r \cup V_s$. Then $\mathcal{D}' := (T', (W_t)_{t\in T'})$ is a tree-decomposition of G.*

Proof. (T1) and (T2): Every vertex and every edge of G was inside one V_t, which now lives inside a W_t. (T3): Let t_1, t_2 and t_3 be distinct vertices of T' with $t_2 \in t_1T't_3$. Consider the contracted vertex t_e: If $t_e \notin \{t_1, t_2, t_3\}$, then $W_{t_1} \cap W_{t_3} = V_{t_1} \cap V_{t_3} \subseteq V_{t_2} = W_{t_2}$. If $t_e = t_2$, then either r or s has to be on the path t_1Tt_3, say r. Since $\mathcal{D}$ is a tree-decomposition $W_{t_1} \cap W_{t_3} = V_{t_1} \cap V_{t_3} \subseteq V_r \subseteq W_{t_2}$ follows. In the case $t_e = t_1$ (and analog $t_e = t_3$) we know $t_2 \in rTt_3$ and $t_2 \in sTt_3$, which implies $V_{t_2} \subseteq V_r \cap V_{t_3}$ and $V_{t_2} \subseteq V_s \cap V_{t_3}$. By taking the union on both sides we get $(W_{t_2} =)V_{t_2} \subseteq (V_r \cap V_{t_3}) \cup (V_s \cap V_{t_3}) = (V_r \cup V_s) \cap V_{t_3} = W_{t_1} \cap W_{t_3}$, completing the proof of (T3). □

For atomic tree-decompositions this means, that parts are not contained in each other:

Corollary 1.3.3. *Let G be a graph and $\mathcal{D} := (T, (V_t)_{t\in T})$ an atomic tree-decomposition of G, then $V_r \nsubseteq V_s$ for all distinct $r, s \in T$.*

Proof. Assume there are two distinct vertices r and s in T with $V_r \subseteq V_s$. By (T3) every vertex from $V_r \cap V_s$ (which is V_r by assumption) is contained in every part on the path rTs. Especially the neighbor t_0 of r in rTs satisfies $V_r \subseteq V_{t_0}$. Contract the edge $e = rt_0$ in the tree-decomposition $\mathcal{D}$ using Lemma 1.3.2 and note that the contracted part W_{t_e} equals $V_r \cup V_{t_0} = V_{t_0}$. This means that $\mathcal{D}'$ has exactly one part of size $|V_r|$ less than $\mathcal{D}$ (the other sizes of parts are the same). So $\mathcal{D}'$ has a smaller fatness than the atomic tree-decomposition $\mathcal{D}$, which cannot be. $\square$

Another tool is "separating the components of a subtree-decomposition". In order to formalize this we need some notation:

Definition 1.3.4. Let G be a connected graph, $\mathcal{D} = (T, (V_t)_{t\in T})$ a tree-decomposition of G and $e = st_0 \in E(T)$. Let T_0 be the component of $T - e$ containing t_0 and T_s the other one (containing s). Define $G_0 := G[\bigcup_{t\in T_0} V_t]$, $G_s := G[\bigcup_{t\in T_s} V_t]$ and $X := V_s \cap V_{t_0}$. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be the set of components of $G_0 - X$ (equivalently, of $G - G_s$) and $N_1, \ldots, N_n$ their neighborhoods (in X) i.e. $N(C_j) = N_j$, $j = 1, \ldots, n$. Let $T_1, \ldots, T_n$ be disjoint copies of T_0 and $\varphi_i : T_0 \longrightarrow T_i$ be the canonical map, mapping every vertex $t \in T_0$ to its copy in T_i.

Define $G_i := V(C_i) \cup N_i$ and $W_{\varphi_i(t)} := V_t \cap G_i$ for $t \in T_0$ and $1 \leq i \leq n$. Set $W_t := V_t$ for $t \in T_s$ and furthermore, $T' := T - T_0 + T_1 + \ldots + T_n + s\varphi_1(t_0) + \ldots + s\varphi_n(t_0)$.

Lemma 1.3.5. *Let the situation of Definition 1.3.4 be given. Then $\mathcal{D}' := (T', (W_t)_{t\in T'})$ is a tree-decomposition of G.*

Proof. (T2): Let $e = xy \in E(G)$ be an edge of G, then one part V_t of $\mathcal{D}$ contains both ends of e. If x and y are in G_s, then they are in one unchanged $V_t = W_t$ (for some $t \in T_s$). If they are not both in G_s, then one of them, say x, is in one component C_i of $G - G_s$. Since all the neighbors of x, in particular y, lie in C_i or in N_i, the ends of the edge e are contained in $V_t \cap G_i = W_{\varphi_i(t)}$. This shows (T1) as well.

(T3): Let t_1', t_2' and $t_3' \in T'$ be given with $t_2' \in t_1'T't_3'$ and let t_1, t_2 and t_3 be their counterparts in T. If there is an index $k \in \{1, \ldots, n\}$ with $\{t_1', t_2', t_3'\} \subseteq T_s \cup T_k \subseteq T'$, then we can find the path $t_1'T't_3'$ in a canonical way in T:

- If t_2' is in T_s, then so is at least one of t_1' and t_3', say t_1'. (T3) for $\mathcal{D}$ implies $W_{t_1'} \cap W_{t_3'} = V_{t_1} \cap W_{t_3'} \subseteq V_{t_1} \cap V_{t_3} \subseteq V_{t_2} = W_{t_2'}$ as desired.
- If, on the other side, $t_2' \in T_k$, then so is at least one of t_1' and t_3', say t_3'. This implies $W_{t_1'} \cap W_{t_3'} \subseteq V_{t_1} \cap (V_{t_3} \cap G_k) = (V_{t_1} \cap V_{t_3}) \cap G_k \subseteq V_{t_2} \cap G_k = W_{t_2'}$ as desired.

In the other case there are distinct indices $k, l \in \{1, \dots, n\}$ such that $t'_1 \in T_k$ and $t'_3 \in T_l$. Because the N_i are disjoint from the C_i we get the inclusion:

$$\begin{aligned}
&W_{t'_1} \cap W_{t'_3} \\
&= (V_{\varphi_k^{-1}(t_1)} \cap G_k) \cap (V_{\varphi_l^{-1}(t_3)} \cap G_l) \\
&\subseteq G_k \cap G_l \\
&= (V(C_k) \cup N_k) \cap (V(C_l) \cup N_l) \\
&= (\underbrace{V(C_k) \cap V(C_l)}_{=\emptyset}) \cup (\underbrace{V(C_k) \cap N_l}_{=\emptyset}) \cup (\underbrace{N_k \cap V(C_l)}_{=\emptyset}) \cup (\underbrace{N_k \cap N_l}_{\subseteq X}) \\
&\subseteq X
\end{aligned}$$

- If $t'_2 \in T_s$ (which means $t'_2 = s$), then $W_{t'_1} \cap W_{t'_3} \subseteq X \subseteq V_s = W_{t'_2}$.
- If $t'_2 \notin T_s$, then it is without loss of generality in $sT't'_1$ (the case $t'_2 \in sT't'_3$ is analog). Consider the vertices s, t'_2 and t'_1 and use the fact, that they are all in $T_s \cup T_k$. We therefore already know that $W_s \cap W_{t'_1} \subseteq W_{t'_2}$ implying $W_{t'_1} \cap W_{t'_3} \subseteq X(\cap W_{t'_1}) \subseteq W_s \cap W_{t'_1} \subseteq W_{t'_2}$

This completes the proof of (T3). So $\mathcal{D}'$ is a tree-decomposition of G. □

1.3.2 Properties of atomic tree-decompositions

Given the situation of Definition 1.3.4, we say that a part V_t with $t \in T_0$ is *split*, if $|V_t \cap G_i| < |V_t| \forall i \in \{1, \dots, n\}$. Note that there is a G_i containing V_t if and only if V_t is not split: If there is a G_i containing V_t, then $V_t \cap G_i$ is V_t, which means that $|V_t \cap G_i|$ is not smaller than $|V_t|$ for this special i, so V_t is not split. If V_t is not split, then there is an $i \in \{1, \dots, n\}$ such that $|V_t \cap G_i| = |V_t|$. Since $V_t \cap G_i$ is a subset of V_t, they can only have the same size, if G_i contains V_t.

Lemma 1.3.6. *Let the situation of Definition 1.3.4 be given. If a part V_t with $|V_t| > |X|$ is split, then the resulting tree-decomposition $\mathcal{D}'$ has a smaller fatness than $\mathcal{D}$.*

Proof. At first let V_r be a part, which is not split (note: $r \in T_0$). As we will see there is at most one $k \in \{1, \dots, n\}$ such that $|W_{\varphi_k(r)}| > |X|$: In the case $|V_r| \leq |X|$ we even know $|W_{\varphi_i(r)}| \leq |X|$ for all $i \in \{1, \dots, n\}$, since V_r contains every $W_{\varphi_i(r)}$. In the other case there has to be at least one vertex a of G which is in V_r but not in X. This vertex is contained in one of the components of $G_0 - X$ and hence in one G_k. Since V_r is not split we know that there is a k such that G_k contains V_r, hence the intersection of V_r and a G_i with $i \neq k$ is a subset of X and therefore $|W_{\varphi_i(r)}| \leq |X|$ for all $i \neq k$.

Let V_r now be a part of maximal size that is split (i.e. all the $W_{\varphi_i(r)}$ are smaller than V_r). By prerequisites $|V_r| > |X|$, therefore every part of

$\mathcal{D}$, which has at least size $|V_r|$ and is not split, induces only one part of its original size and the other induced parts are smaller than X. For the comparison of the fatnesses $(a_0, \ldots, a_n)$ of $\mathcal{D}$ and $(a'_0, \ldots, a'_n)$ of $\mathcal{D}'$ this means, that the entries before[2] $a_{n-|V_r|}$ are equal and that $a'_{n-|V_r|}$ is by at least one smaller than $a_{n-|V_r|}$. So $\mathcal{D}'$ has a (lexicographically) smaller fatness than $\mathcal{D}$. $\square$

If a "big"[3] part V_t is split, then the resulting tree-decomposition is smaller than the original one, which therefore was not atomic. So in an atomic tree-decomposition no "big" part is split. In particular the "first part in G_0" V_{t_0} is such a big part, because it contains $V_{t_0} \cap V_s = X$ and at least one more vertex in $G - G_s$ (since otherwise V_{t_0} would be a subset of V_s, contradicting Corollary 1.3.3). For justification of the term "atomic" we will show that even "small" parts are not split in an atomic tree-decomposition:

Lemma 1.3.7. *Let the situation of Definition 1.3.4 be given, where $\mathcal{D}$ is an atomic tree-decomposition. Then V_t is not split for all $t \in T_0$.*

Proof. Suppose $\tilde{t}_0$ is a vertex in T_0 corresponding to a split part. Let $\tilde{s}$ be the neighbor of $\tilde{t}_0$ on the path $\tilde{t}_0 T s$ and $\tilde{e} := \tilde{s}\tilde{t}_0$. Let $\tilde{T}_0$ be the component of $T - \tilde{e}$ containing $\tilde{t}_0$ and $\tilde{T}_s$ the other one (containing $\tilde{s}$). Define $\tilde{G}_0 := G[\bigcup_{t \in \tilde{T}_0} V_t]$ and $\tilde{G}_s := G[\bigcup_{t \in \tilde{T}_s} V_t]$ furthermore $\tilde{X} := V_{\tilde{s}} \cap V_{\tilde{t}_0}$. Let us first check that every component of $G - \tilde{G}_s$ is contained in a component of $G - G_s$:

By choice of $\tilde{s}$ there is an $\tilde{s}$–s path in $T - \tilde{e}$. Combining this path with another path connecting s with a vertex of T_s we get a path from every vertex of T_s to $\tilde{s}$ in $T - \tilde{e}$, since T_s does not contain $\tilde{t}_0$ and therefore cannot contain $\tilde{e}$. This means that every vertex of T_s lives in the component of $T - \tilde{e}$ which contains $\tilde{s}$. So we have $T_s \subseteq \tilde{T}_s$ which implies $G_s \subseteq \tilde{G}_s$. Every component of $G - \tilde{G}_s$ is disjoint from $\tilde{G}_s \supseteq G_s$. These components are connected and are therefore contained in a maximal connected subset of $G - G_s$. So every component of $G - \tilde{G}_s$ is contained in a component of $G - G_s$. Now we construct another situation as in Definition 1.3.4 at the edge $\tilde{e}$.

By Lemma 1.3.6 we now know that $V_{\tilde{t}_0}$, being the "big" part next to $V_{\tilde{s}}$, is not split in this new situation. So there is a component $\tilde{C}$ of $G - \tilde{G}_s$ such that $V(\tilde{C}) \cup \tilde{N}$ contains $V_{\tilde{t}_0}$, where $\tilde{N}$ is the neighborhood of $\tilde{C}$. The component $\tilde{C}$ is contained in a component C of $G - G_s$ and therefore $C \cup N$ contains $\tilde{N}$, where N is the neighborhood of C. There is an $i \in \{1, \ldots, m\}$, such that $V(C) \cup N = G_i$. Now we know $V_{\tilde{t}_0} \subseteq V(\tilde{C}) \cup \tilde{N} \subseteq V(C) \cup N = G_i$. So $V_{\tilde{t}_0}$ is not split even in the original situation. $\square$

After we have seen that there are no split parts in atomic tree-decomposition, we shall now see why this is useful.

[2]where the parts of size larger than $|V_r|$ are counted

[3]big means $|V_t| > |X|$

Lemma 1.3.8. *Let G be a connected graph, $\mathcal{D} = (T, (V_t)_{t\in T})$ an atomic tree-decomposition of G and $e = st_0 \in E(T)$. Use the notation of Definition 1.3.4. Then the neighborhood of C_0, the component of $G_0 - X$ meeting V_{t_0}, is all of X.*

Proof. Since $\mathcal{D}$ is an atomic tree-decomposition, by Lemma 1.3.7, V_{t_0} is not split. This means that there is a component C_i such that the corresponding G_i (which is $V(C_i) \cup N_i$) contains V_{t_0} and since X does not contain all of V_{t_0} we get an element a in $V_{t_0} \cap V(C_i)$. If there would be another component meeting V_{t_0} (in b), then V_{t_0} would be split, because then every G_i misses at least one of the vertices a or b and therefore every $|V_{t_0} \cap G_i|$ is smaller than $|V_{t_0}|$. Now we are allowed to speak of "*the* component C_0 meeting V_{t_0}".

As we have seen V_{t_0} is a subset of $V(C_0) \cup N_0$, where N_0 is the neighborhood of C_0. Since V_{t_0} contains X we know $X \subseteq V_{t_0} \subseteq V(C_0) \cup N_0$. This implies $X \subseteq N_0$ because X is disjoint from the component C_0. So every vertex of X is a neighbor of a vertex in C_0. □

Given two vertices u and v living inside one common part V_s. If there is an edge st_0 in T such that both vertices live in V_{t_0} too, then there is (by Lemma 1.3.8) a u–v path P (going through the component C_0), whose inner vertices are all in $G_0 - X$. Changing the roles of t_0 and s we get another u–v path Q, whose inner vertices are all in $G_s - X$. Combining those paths we get a cycle $C := P \cup Q$ containing u and v, which lies "nice" in G (with respect to the tree-decomposition). An even nicer fact is, that the used intersection X always exists, if needed.

Lemma 1.3.9. *Let $\mathcal{D} = (T, (V_t)_{t\in T})$ be an atomic tree-decomposition of a connected graph G. If u and v are two vertices living inside a common part V_s, then at least one of the following holds:*

- *uv is an edge of G.*
- *There is a neighbor t_0 of s in T, such that $\{u, v\} \subseteq V_s \cap V_{t_0}$*

Proof. Assume both statements are false, then there is a part V_s containing two non-adjacent vertices u and v, such that for every neighbor t of s either u or v (or both) is missing in $V_s \cap V_t$.

Define a new tree-decomposition $(T', \mathcal{W} = (W_t)_{t\in T'})$ by "de-contracting V_s" as follows:

The new tree lives on $V(T') := V(T) - s + t_u + t_v$ where t_u and t_v are two new vertices. Let N be the neighborhood of s in T and $U := \{t \in N : v \notin V_t\}$ the set of neighbors lacking v. Let e be the edge $t_u t_v$ then the edge set of T' is $E(T') := E(T - s) + \{tt_u : t \in U\} + \{tt_v : t \in N - U\} + e$. Since the old neighbors of s are distributed among t_u and t_v, we know that T' is a tree. Let $W_t := V_t \ \forall\ t \in T - s$, $W_{t_u} := V_s - v$, $W_{t_v} := V_s - u$ and $\mathcal{D}' := (T', (W_t)_{t\in T'})$.

Let T_u be the component of $T' - e$ containing t_u and analog T_v the other one (containing t_v), then every part corresponding to a vertex in T_u does not contain v (and vice versa): The parts W_t with $t \in U \cup \{t_u\}$ do not contain v by definition. For the other parts $W_{t'}$ we consider the path $P := sTt'$ in $\mathcal{D}$ and note that it contains a vertex u' of U by construction. If v would be in $W_{t'} = V_{t'}$, then it would be in $V_s \cap V_{t'}$ but not in $V_{u'}$, which is a contradiction. The other statement $u \notin W_t \ \forall t \in T_v$ can be shown in an analog way. Now we will see that $\mathcal{D}'$ is a tree-decomposition of G.

(T1) holds, because $V_s = W_{t_u} \cup W_{t_v}$. (T2) holds, because u and v are not adjacent. For (T3) let t_1, t_2 and t_3 be vertices of T' with $t_2 \in t_1T't_3 =: P'$. By contraction of e we get a t_1–t_3 path P in T containing t_2 (we identify t_u and t_v in T' with s in T and everything else is unchanged): If t_2 is none of t_u and t_v, then we know $W_{t_1} \cap W_{t_3} \subseteq V_{t_1} \cap V_{t_3} \subseteq V_{t_2} = W_{t_2}$. If t_2 is t_u, then $W_{t_1} \cap W_{t_3} \subseteq V_{t_1} \cap V_{t_3} \subseteq V_{t_2} = W_{t_u} \cup \{v\}$. This would only be a problem if $v \in W_{t_1} \cap W_{t_3}$, but in this case both vertices t_1 and t_3 cannot be in T_u. So they are in T_v, which means that t_2 is not on P'. This contradiction shows $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$. The last case $t_2 = t_v$ is analog.

Hence $\mathcal{D}'$ is a tree-decomposition which has exactly one part of size $|V_s|$ less than $\mathcal{D}$ and two smaller parts are added. So $\mathcal{D}'$ has a (lexicographically) smaller fatness than the atomic $\mathcal{D}$. This contradiction shows that at least one of the statements has to be true. □

1.4 $\mathcal{C}$-Closure

The results in this section come from [23].

Now that we have a suitable (atomic) tree-decomposition and know how to turn it into a connected tree-decomposition (using a nav), we just have to show that its width is bounded (by a number only depending on the tree-width and the length of a longest geodesic cycle). The following definition will be a useful tool, because it exhibits the subgraph that will contain the desired path of bounded length:

Definition 1.4.1. Let G be a graph and $\mathcal{C}$ a set of cycles in G. Define the *$\mathcal{C}$-Closure* of a vertex-set X, to be the union of the cycles in $\mathcal{C}$ meeting X. In signs: $\mathcal{C}l(X) := \bigcup_{C \in \mathcal{C}_X} C$ with $\mathcal{C}_X := \{C \in \mathcal{C} : C \cap X \neq \emptyset\}$.

If every $x \in X$ is on a cycle in $\mathcal{C}$ then obviously $X \subseteq \mathcal{C}l(X)$. If, on the other hand, there is a vertex x in X which misses every cycle in $\mathcal{C}$ then $x \notin \mathcal{C}l(X)$, consequently $X \nsubseteq \mathcal{C}l(X)$. If $X \subseteq Y$ then $C_X \subseteq C_Y$ and therefore $\mathcal{C}l(X) \subseteq \mathcal{C}l(Y)$. Since the inclusion $\mathcal{C}l(\mathcal{C}l(X)) \subseteq \mathcal{C}l(X)$ is false in general, the $\mathcal{C}$-Closure is not a closure-operator. The $\mathcal{C}$-closure of a set helps us finding an upper bound for the distance of the vertices in that set.

Lemma 1.4.2. *Let G be a graph and $\mathcal{C}$ a set of cycles in G whose length is bounded by k. Let $X \subseteq V(G)$ be a vertex-set with $X \subseteq \mathcal{C}l(X)$. If $\mathcal{C}l(X)$ is connected then every two vertices in X have a distance $\leq k \cdot (|X| - 1)$ in G.*

Proof. Let us first show that for every bipartition $\{A, B\}$ of X their $\mathcal{C}$-closures meet, i.e. $\mathcal{C}l(A) \cap \mathcal{C}l(B) \neq \emptyset$. Since $X \subseteq \mathcal{C}l(X)$ is equivalent to every vertex in X (which is $A \cup B$) being on a cycle in $\mathcal{C}$, we know $A \subseteq \mathcal{C}l(A)$ and $B \subseteq \mathcal{C}l(B)$. Every edge xy in $\mathcal{C}l(X)$ lies on a cycle C of $\mathcal{C}$ meeting X (in A or B (or both) since $\{A, B\}$ is a partition of X). This shows that x and y are in $\mathcal{C}l(A)$ or $\mathcal{C}l(B)$ (or both), so $\mathcal{C}l(X) = \mathcal{C}l(A) \cup \mathcal{C}l(B)$.

Choose two vertices $a \in A$ and $b \in B$ and an a–b path $P \subseteq \mathcal{C}l(X)$ (there is one, since $\{a, b\} \subseteq A \cup B = X \subseteq \mathcal{C}l(X)$ and $\mathcal{C}l(X)$ is connected). Consider the the first (i.e. closest to a) vertex y in P which is in $\mathcal{C}l(B)$ (there is one, since b is a candidate). In the case that y equals a we have found a vertex in the intersection of $\mathcal{C}l(A)$ and $\mathcal{C}l(B)$. In the other case the predecessor x of y on P has to be in $\mathcal{C}l(A)$. If $\mathcal{C}l(A)$ contains the edge xy, then y lies in $\mathcal{C}l(A) \cap \mathcal{C}l(B)$, in the other case the intersection contains x.

Now construct an auxiliary tree T on X, such that the distance (in G) of every pair of vertices that are adjacent in T is bounded by k. The construction begins with an arbitrary vertex of X as a single vertex tree T_0. If the tree T_i is constructed, we can consider the partition $\{V(T_i), X - V(T_i)\}$. Now we know that their $\mathcal{C}$-closures meet, i.e. there are vertices $x \in V(T_i)$ and $y \in X - V(T_i)$ and intersecting cycles C_x and C_y in $\mathcal{C}$ with $x \in C_x$ and $y \in C_y$. Applying the triangle inequality to x, y and a vertex z in the intersection of the cycles C_x and C_y, we get an upper bound for the distance between x and y:

$$d_G(x, y) \leq d_{C_x}(x, z) + d_{C_y}(z, y) \leq \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor = 2 \left\lfloor \frac{k}{2} \right\rfloor \leq k$$

In order to get the tree T_{i+1} we add y and the edge xy to T_i.

At the end of this iteration we get a tree T (living on all of X). For every pair of vertices in X there is a path connecting them in T. This path has at most $|V(T)| - 1$ edges and every pair of adjacent vertices has a distance of at most k in G. Combining this we get $d_G(x, y) \leq k \cdot (|X| - 1)$ for every two vertices x and y in X. $\square$

Now we want to combine the $\mathcal{C}$-closure with the "nice" cycle that we found in Section 1.3 (before Lemma 1.3.9). We will use the cycle space, so notations like "generate" or "+" have to be read in the sense of the edge space here (whereas "−" remains set-deletion).

Lemma 1.4.3. *Let G be a graph and $\{G_s, G_0\}$ a separation of G and $X := G_s \cap G_0$ the separator. Let $C = P \cup Q$ be a cycle consisting of two x-y paths P and Q such that $P - \{x, y\} \subseteq G_0 - X$ and $Q \subseteq G_s$. Let $\mathcal{C}$ be a set of cycles such that C lies in the subspace they generate.*

Then there exists an x–y path in the $\mathcal{C}$-closure $\mathcal{C}l(X)$ of X.

Proof. Let us proof the following statement first:

Claim. *Let G be a graph, P an x–y path in G and Z an element of the cycle space of G. Then there is an x–y path in $P + Z$.*

Let $e := xy$ be the (theoretical) edge connecting x and y. In the case $e \notin (P + e) + Z$, we know $e \in P + Z$ and hence e is the desired x–y path in $P + Z$. In the other case $e \in (P + e) + Z$ there are two possibilities: On the one hand the path P might be just the edge e, then $P + e$ is empty. On the other hand the path P might be not that edge e, then P (being an x–y path) does not even contain e so $P + e = P \cup e$ is a cycle. In both cases $(P + e) + Z$ is an element of the cycle space of $G \cup e$ and therefore a disjoint union of cycles in $G \cup e$. One of these cycles C' has to contain e. So $P' := C' - e \subseteq ((P + e) + Z) - e \subseteq (G \cup e) - e \subseteq G$ is the desired x–y path in $P + Z$ completing the proof of the claim.

Coming back to the proof of the lemma we write C as a sum of cycles in $\mathcal{C}$, i.e. $C = \sum_{i \in I} C_i$. Divide I into the cycles on the "left", "right" and "middle" of X by $J_0 := \{j \in I : C_j \subseteq G_0 - X\}$, $J_s := \{j \in I : C_j \subseteq G_s - X\}$ and $J := \{j \in I : C_j \cap X \neq \emptyset\}$. Since every cycle is connected and $\{G_s, G_0\}$ is a separation of G, every cycle avoiding X lies either in $G_0 - X$ or in $G_s - X$ (not in both). Therefore $\{J_s, J, J_0\}$ is a partition of I.

By the claim there is an x–y path P' in $P + \sum_{j \in J_0} C_j$ (whose inner vertices have to lie in $G_0 - X$). Since Q and the J_s-cycles are separated by X from P and the J_0-cycles, adding them is taking the disjoint union. So we have the following inclusion:

$$\begin{array}{rcl} P' \subseteq & P + Q & + \sum_{j \in J_0 \cup J_s} C_j \\ = & C & + \sum_{j \in J_0 \cup J_s} C_j \\ = & \sum_{j \in J_0 \cup J \cup J_s} C_j & + \sum_{j \in J_0 \cup J_s} C_j \\ = & \sum_{j \in J} C_j & + \emptyset \\ \subseteq & \mathcal{C}l(X) & \end{array}$$

The last inclusion holds, because all cycles from J hit X and are therefore contained in $\mathcal{C}l(X)$. So P' is the desired path in the $\mathcal{C}$-closure of X. $\square$

Now we have all the tools needed to prove the main theorem:

Theorem 1.1.1. *The connected tree-width of a graph G is bounded above by a function of its tree-width and the maximum length k of its geodesic cycles. Specifically*

$$ctw(G) \leq tw(G) + \binom{tw(G) + 1}{2} \cdot (k \cdot tw(G) - 1).$$

Proof. It is easy to see that the upper bound for the connected tree-width holds if the graph is a forest and k is defined to be > 0 , so without loss of generality $k > 2$ and $tw(G) > 1$.

It suffices to prove the theorem for 2-connected graphs:

Let G be a (possibly not 2-connected) graph and $\mathcal{B}$ be a (2-connected) block of G. Then the tree-width and the maximum length l of geodesic cycles of $\mathcal{B}$ are bounded above by $tw(G)$ and k, respectively.

So the "2-connected version" of the theorem yields a connected tree-decomposition of $\mathcal{B}$ (for bridges and isolated vertices take a one vertex tree-decomposition) of width $\leq tw(\mathcal{B}) + \binom{tw(\mathcal{B})+1}{2} \cdot (l \cdot tw(\mathcal{B}) - 1) \leq tw(G) + \binom{tw(G)+1}{2} \cdot (k \cdot tw(G) - 1)$. We can construct a connected tree-decomposition of the whole graph, by adding edges (according to the block structure[4]) to the disjoint union of the trees (of the connected tree-decompositions of the blocks of the graph) until we get a tree.

So let G be a 2-connected graph (In particular every vertex and every edge of G lies on a (geodesic) cycle). We know how to construct a connected tree-decomposition of width $\leq tw(G) + \binom{tw(G)+1}{2} \cdot (l(\mathcal{N}) - 1)$ using a nav and an atomic tree-decomposition $\mathcal{D}$ (Theorem 1.2.2). Because of the existence of a geodesic nav (Theorem 1.2.3), the length of the used $\mathcal{D}$-nav is bounded by the maximum distance of two vertices living in a common part of the used tree-decomposition.

Let $\mathcal{C}$ be the set of all geodesic cycles of G and V_s be a part of $\mathcal{D}$. If we show that $\mathcal{C}l(V_s)$ is connected, then we know (by Lemma 1.4.2), that every two vertices in V_s have a distance of at most $k \cdot (|V_s| - 1) \leq k \cdot tw(G)$ in G. So let u and v be two vertices in V_s. By Lemma 1.3.9 there is either the edge uv (which is then contained in $\mathcal{C}l(V_s)$) or there is a neighbor t_0 of s in T, such that u and v are contained in the intersection $X := V_s \cap V_{t_0}$. In this case a corollary of Lemma 1.3.8 is the existence of two u–v paths P and Q, that form a cycle $C = P \cup Q$ such that $P - \{x, y\} \subseteq G_0 - X$ and $Q \subseteq G_s$ (G_0 and G_s are defined as in Definition 1.3.4 and form a separation of G). Since C, being a cycle, lies in the cycle space which is generated by the geodesic cycles of G (see exercise 32 of chapter 1 in [13]), we can apply Lemma 1.4.3 and get a u–v path in $\mathcal{C}l(X) \subseteq \mathcal{C}l(V_s)$. So for every two vertices of V_s there is a path in $\mathcal{C}l(V_s)$ connecting them. Since the other vertices of $\mathcal{C}l(V_s)$ lie on cycles which hit V_s, the $\mathcal{C}$-closure of V_s is connected, as required.

Combining all these pieces, we have shown that G has a connected tree-width of at most $tw(G) + \binom{tw(G)+1}{2} \cdot (k \cdot tw(G) - 1)$. □

[4]For each cutvertex x of the graph we choose for every block $\mathcal{B}_i$, that contains x, one vertex t_i in the tree (of the tree-decomposition of $\mathcal{B}_i$), such that V_{t_i} contains x. Then we add the edges of a (arbitrary) tree in order to connect all the chosen t_i vertices. When this is done, we do the same procedure for the empty cutset (i.e. we connect the tree-decompositions of the components of the graph).

1.5 Duality

1.5.1 Brambles

A useful tool for determining the tree-width of an unknown graph is a bramble:

If we know a tree-decomposition of width k, then we know that the tree-width of G is $\leq k$, but we don't know how much smaller the tree-width is. If we additionally know a bramble of order $k+1$, then (by tree-width duality theorem) the tree-width has to be $\geq k$, hence it equals k.

Definition 1.5.1. Two vertex sets are touching if they either intersect or if there is an edge from one to the other. A *bramble* is a set of pairwise touching connected vertex sets. A *cover* of the bramble is a vertex set which intersects every set of the bramble. The *order* of the bramble is the smallest size that a cover of the bramble may have.

The *connected order* of the bramble is the smallest possible size of a connected vertex set covering it.

The tree-width duality theorem says that the only reason for large tree-width is a bramble of large order: It is a reason, because if the graph contains a bramble of large order ($> k$), then it has large tree-width ($\geq k$). And it is the only one, since if it is gone (no bramble of order $> k$), then the tree-width is small ($< k$), so there can be no other reason which rises the tree-width.

1.5.2 Making it connected

The obvious thing to try is finding a "connected tree-width duality theorem", i.e. write a "connected" in front of "tree-width" and see what fits on the bramble side. The natural guess is the connected order:

Conjecture 1.1.2. *Let $k \geq 0$ be an integer. A graph has connected tree-width $\geq k$ if and only if it contains a bramble of connected order $> k$.*

The backward-direction is an easy corollary of the (easy part of the) proof of the tree-width duality theorem, because this direction is shown by the following claim:

Claim (from the proof of theorem 12.3.9. in [13])**.** *Given a bramble $\mathcal{B}$ and a tree-decomposition $\mathcal{D}$, then there is a part of $\mathcal{D}$ which covers $\mathcal{B}$.*

This direction can be used to determine the connected tree-width of a cycle (for example):

Example 1.5.2. For a cycle of length n let $\mathcal{B} := [V(C^n)]_c^{\lfloor \frac{n}{2} \rfloor}$ be the set of all connected subsets of size $\lfloor \frac{n}{2} \rfloor$. Let us show that $\mathcal{B}$ is a bramble of connected order $\lceil \frac{n}{2} \rceil + 1$. After deletion of $X \in \mathcal{B}$ and its neighborhood, there are at most (exactly) $\lceil \frac{n}{2} \rceil - 2$ vertices left. Because those are less than $\lfloor \frac{n}{2} \rfloor$ vertices, there is no element of $\mathcal{B}$ inside this rest (i.e. X touches every other element of $\mathcal{B}$, which is therefore a bramble). After deletion of a connected set of size $\leq \lceil \frac{n}{2} \rceil$, there are at least $\lfloor \frac{n}{2} \rfloor$ connected vertices left (containing a non-covered set of $\mathcal{B}$). One more vertex is sufficient to cover $\mathcal{B}$. So $\mathcal{B}$ is a bramble of connected order $\lceil \frac{n}{2} \rceil + 1$ (i.e. the connected tree-width of a cycle of length n is at least $\lceil \frac{n}{2} \rceil$). On the other hand there is a connected tree-decomposition of that cycle consisting of two connected parts of size $\leq \lceil \frac{n}{2} \rceil + 1$ which cover it (see Figure 1.2 for an example). So the connected tree-width of a cycle of length n is $\lceil \frac{n}{2} \rceil$.

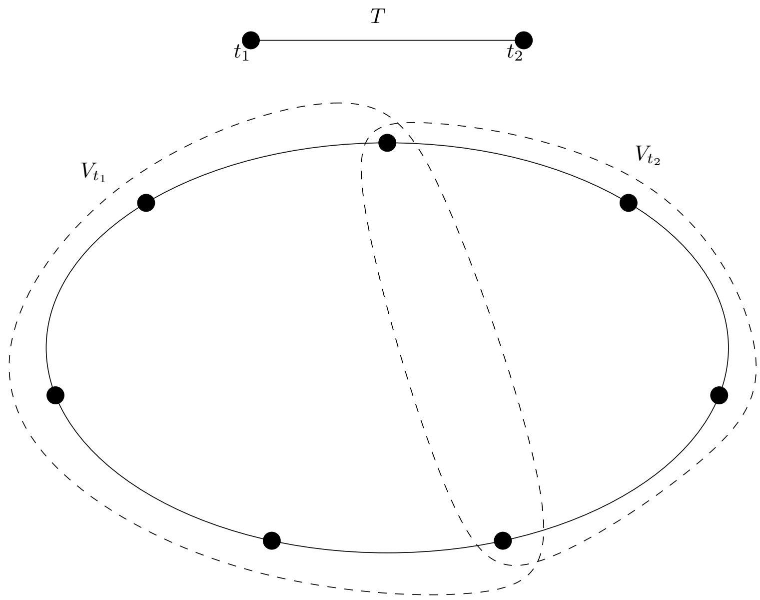

Figure 1.2: A connected minimum width tree-decomposition of a cycle.

The difficult direction is not that easy to change into the connected version.

By Theorem 1.1.1 the only two reasons for large connected tree-width are large tree-width and a long geodesic cycle. If we can show that the absence of a bramble of large connected order prevents both these reasons, then we know that the connected tree-width is small (which is the difficult direction of Conjecture 1.1.2, at least qualitatively). So the next thing to proof is, that a graph with a long geodesic cycle contains a bramble of large connected order (i.e. that a long geodesic cycle really is a reason for large connected tree-width):

Lemma 1.5.3. *If a graph G contains a geodesic cycle C of length n, then G has a bramble of connected order $\geq \lceil \frac{n}{2} \rceil + 1$, namely: $\mathcal{B} := [V(C)]_c^{\lfloor \frac{n}{2} \rfloor}$, the set of all connected subsets of C which have size exactly $\lfloor \frac{n}{2} \rfloor$.*

Proof. Let X be a connected vertex set in G covering $\mathcal{B}$. We want to show $|X| > \lceil \frac{n}{2} \rceil$:

1. Case: $|X \cap C| = 2$. Then n has to be even and the two vertices x_0 and x_1 in this intersection $|X \cap C|$ have a distance of $\frac{n}{2}$ in C, because otherwise there would be a bramble set not covered by X. Since C is geodesic, the distance of x_0 and x_1 in G is (at least) $\frac{n}{2}$. Because X is connected, there is an x_0–x_1 path inside X, which has at least $\frac{n}{2} + 1$ vertices, so $|X| > \lceil \frac{n}{2} \rceil$.

2. Case: $|X \cap C| > 2$. Then there are three vertices x_0, x_1 and x_2 in $X \cap C$. Let P_i be the x_{i-1}–x_{i+1} path in C not containing the vertex x_i, $i \in \{0, 1, 2\}$ (indices modulo 3). By minimization of the maximal length of these paths we can achieve that P_i is the shorter x_{i-1}–x_{i+1} path in C: Choose the three vertices in $X \cap C$ such that the maximal size m of the three corresponding paths P_i is minimal. Suppose $|P_1| = m \geq \lfloor \frac{n}{2} \rfloor + 2$, then there is enough space for a bramble set on P_1 between x_0 and x_2. This set is not covered by x_0, x_1 and x_2, so there has to be another vertex in X which takes care of it. Replacing x_1 by this vertex we get three new paths Q_0, Q_1 and Q_2 which have all less than m vertices. Q_0 and Q_2 are proper subpaths of P_1 and therefore have less than m vertices. Q_1 flipped from P_1 to the other side of the cycle, so there are only $n - (\lfloor \frac{n}{2} \rfloor + 2) + 2 = \lceil \frac{n}{2} \rceil < \lfloor \frac{n}{2} \rfloor + 2 \leq m$ vertices left for it. This contradiction to the minimality of m shows that $|P_1| \leq \lfloor \frac{n}{2} \rfloor + 1$ which means that P_i is the shorter of the two x_{i-1}–x_{i+1} paths in C.

Since X is connected there is an x_1–x_2 path $P \subseteq X$ and an x_0–P path $X_0 \subseteq X$. Let $z := P \cap X_0$, $X_1 := x_1 P z$ and $X_2 := x_2 P z$. So X_i is a path inside X starting at x_i and ending in z (for every $i \in \{0, 1, 2\}$). Since P_i is geodesic and $X_{i-1} \cup X_{i+1}$ is another x_{i-1}–x_{i+1} path, we know $|X_{i-1}| + |X_{i+1}| - 1 \geq |P_i|$. Since all the P_i together form the cycle C, we know $|P_0| + |P_1| + |P_2| - 3 = n$. Combining this, we get:

$$\begin{aligned} & 2(|X_0| + |X_1| + |X_2|) - 3 \\ =& (|X_1| + |X_2| - 1) + (|X_0| + |X_2| - 1) + (|X_0| + |X_1| - 1) \\ \geq & |P_0| + |P_1| + |P_2| \\ =& n + 3 \end{aligned}$$

Rearranging this, we get:

$$(|X_0| + |X_1| + |X_2|) \geq \frac{n}{2} + 3$$

We can use this to estimate the size of X, because all X_i are contained in X and have only z in common:

$$|X| \geq (|X_0| + |X_1| + |X_2|) - 2 \geq \frac{n}{2} + 1 > \left\lceil \frac{n}{2} \right\rceil$$

This shows, that whenever X is a connected set in G which covers $\mathcal{B}$, its size has to be larger than $\lceil \frac{n}{2} \rceil$. So $\mathcal{B}$ is indeed a bramble of connected order $\geq \lceil \frac{n}{2} \rceil + 1$. $\square$

Note that Lemma 1.5.3 does not naively extend to arbitrary geodesic subgraphs: Let G be the graph indicated in Figure 1.3 and $H = G - x$ the considered geodesic subgraph. Then there is a bramble of maximal connected order 5 in H, namely all 9-element connected subsets of the outer 18-cycle C, which has connected order 4 in G.

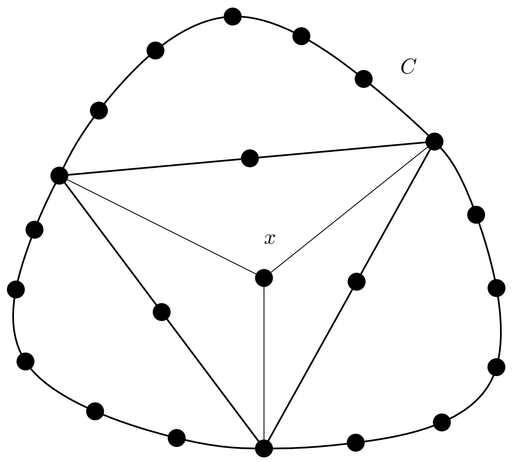

Figure 1.3: A drawing of the example graph G.

It is unknown if there is a graph and a geodesic subgraph such that every maximal connected order bramble of the subgraph has a smaller connected order in the whole graph. In the above example, a bramble of maximal connected order in H whose connected order does not go down in G is the set of all connected 4-element subsets of an 8-cycle in H.

Now we can show the qualitative version of the difficult direction of Conjecture 1.1.2:

Theorem 1.1.3. *Let $k \geq 0$ be an integer. There is a function $g : \mathbb{N} \to \mathbb{N}$, such that any graph with no bramble of connected order $> k$ has connected tree-width $< g(k)$.*

Proof. Let G be a graph which has no bramble of connected order $> k$. Since $k \leq 2$ implies that G is a forest, we can assume $k > 2$.

If the graph has a geodesic cycle of length $\geq 2k$, then, by Lemma 1.5.3, it has a bramble of connected order $\geq \lceil \frac{2k}{2} \rceil + 1 = k + 1$ (which is a contradiction). So there is no geodesic cycle of length $> 2k - 1$ in G. The tree-width of G is bounded too, because:

$$
\begin{aligned}
& G \text{ has no bramble of connected order} > k \\
\Rightarrow\ & G \text{ has no bramble of order} > k \\
\Rightarrow\ & tw(G) < k
\end{aligned}
$$

The last implication follows from the tree-width duality theorem.

By Theorem 1.1.1 the connected tree-width of G is bounded by $tw(G) + \binom{tw(G)+1}{2} \cdot ((2k-1) \cdot tw(G) - 1)$ which is smaller than $k + \binom{k+1}{2} \cdot ((2k-1) \cdot k - 1) =: g(k)$, a function only depending on k (note that this function even works for the cases $k \leq 2$). $\square$

Chapter 2

Infinite gammoids

The results in this chapter come from [2, 3].

2.1 Introduction

Infinite matroid theory has seen vigorous development since Bruhn et al [10] in 2010 gave five equivalent sets of axioms for infinite matroids in response to a problem proposed by Rado [29] (see also Higgs [18] and Oxley [25]). In this chapter, we continue this ongoing project by focusing on the class of gammoids, which originated from the transversal matroids introduced by Edmonds and Fulkerson [16]. A *transversal matroid* can be defined by taking as its independent sets the subsets of a fixed vertex class of a bipartite graph matchable to the other vertex class. Perfect [27] generalized transversal matroids to gammoids by replacing matchings in bipartite graphs with disjoint directed paths in digraphs. Later, Mason [21] started the investigation of a subclass of gammoids known as strict gammoids.

To be precise, let a *dimaze* (short for *di*rected *maze*) be a digraph with a fixed subset of the vertices of out-degree 0, called the *exits*. A dimaze *contains* another dimaze, if, in addition to digraph containment, the exits of the former include those of the latter. In the context of digraphs, any path or ray (i.e. infinite path) is forward oriented. A set of vertices of (the digraph of) the dimaze is called *independent* if it is *linkable* to the exits by a *linkage*, i.e. a collection of disjoint paths. The set of all linkable sets is the *linkability system* of the dimaze. A *strict gammoid* is a matroid isomorphic to one defined on the vertex set of a dimaze, whose set of independent sets is the linkability system. Any dimaze defining a given strict gammoid is a *presentation* of that strict gammoid. A *gammoid* is a matroid restriction of a strict gammoid.

Mason proved that every finite dimaze defines a matroid. When a dimaze is infinite, Perfect gave sufficient conditions for when some subset of the linkability system gives rise to a matroid. Any such matroid is *fini-*

tary, in the sense that a set is independent as soon as all its finite subsets are. Since finitary matroids were the only ones known at that time, infinite dimazes whose linkability systems are non-finitary were not considered to define matroids.

With infinite matroids canonically axiomatized in a way that allows for non-finitary matroids (see the definition in Section 2.2), a natural question is whether every infinite dimaze now defines a matroid. In general, the answer to this question is still negative, as the linkability system may fail to satisfy one of the infinite matroid axioms (IM), which asks for the existence of certain maximal independent sets. However, in Section 2.3 we show that the other matroid axioms hold in any linkability system (where (I1) and (I2) are trivial). Furthermore, investigating a proof of Pym's linkage theorem [28], we prove that for a dimaze, containing an *alternating comb* (see Section 2.2 for the definition) is the unique obstruction to a characterization of maximally linkable (vertex) sets as being linkable onto the exits. This is used to show that any dimaze that does not contain an alternating comb defines a matroid.

In other words, a dimaze whose linkability system fails to define a matroid contains an alternating comb. Conversely, a dimaze containing an alternating comb may still define a matroid. An alternating comb itself does, in fact that matroid has another presentation which does not contain an alternating comb. This does not hold in general, since in Section 2.4.2 we construct a strict gammoid such that any dimaze defining this matroid contains an alternating comb.

Recall that by definition, the class of gammoids is closed under matroid deletion. A pleasant property of the class of finite gammoids is that it is also closed under matroid contractions, and hence, under taking minors. In contrast, whether the class of all gammoids, possibly infinite, is minor-closed is an open question investigated in Section 2.5.

A standard proof of the fact that finite gammoids are minor-closed as a class of matroids proceeds via duality [20]. The proof of this fact can be extended to infinite dimazes whose underlying (undirected) graph does not contain any ray, but it breaks down when rays are allowed. However, by developing the concept of $\mathcal{Q}$-shifting we are able to prove that the class of gammoids that admit a presentation not containing any outgoing comb is minor-closed. If we do allow outgoing combs, combining $\mathcal{Q}$-shifting with a proof of Pym's linkage theorem [28], we can still show that any finite-rank minor of an infinite gammoid is a gammoid.

In [12], Carmesin used a topological approach to extending finite gammoids, in order to allow infinite paths in a linkage. It turned out that these *(strict) topological gammoids* are finitary, which leads to a characterization of finitary strict gammoids in terms of the defining dimaze. In Section 2.5.2, we use $\mathcal{Q}$-shifting to show that every topological gammoid is a gammoid.

This implies that the class of topological gammoids coincides with that of finitary gammoids and is used to show that the class of topological gammoids is minor-closed.

In Section 2.6, we turn to duality. Recall that a transversal matroid is a matroid isomorphic to one defined by taking a fixed vertex class of a bipartite graph as the ground set and its matchable subsets as the independent sets. Ingleton and Piff [20] proved constructively that finite strict gammoids and finite transversal matroids are dual to each other, a key fact to the result that the class of finite gammoids is closed under duality. In contrast, an infinite strict gammoid need not be dual to a transversal matroid, and vice versa (Examples 2.6.12 and 2.6.19). Despite these examples, it might still be possible that the class of infinite gammoids is closed under duality. However, we will see in Section 2.6.3 that there is a gammoid, which is not dual to any gammoid.

In Section 2.6.1, we aim to describe the duals of the strict gammoids that admit a presentation not containing any alternating comb. It turns out that there exists a strict gammoid in this class that is not dual to any transversal matroid. For this reason, we first extend transversal matroids to a larger class termed path-transversal matroids. Then we prove that a strict gammoid that admits a presentation not containing any alternating comb is dual to a path-transversal matroid. We remark that the theorem is used in [1] to characterize cofinitary transversal matroids and cofinitary strict gammoids.

2.2 Preliminaries

In this section, we present relevant definitions. For notions not found here, we refer to [10] and [26] for matroid theory, and [13] for graph theory.

2.2.1 Infinite matroids

Given a set E and a family of subsets $\mathcal{I} \subseteq 2^E$, let $\mathcal{I}^{\max}$ denote the maximal elements of $\mathcal{I}$ with respect to set inclusion. For a set $I \subseteq E$ and $x \in E$, we also write $I + x, I - x$ for $I \cup \{x\}$ and $I \setminus \{x\}$ respectively.

Definition 2.2.1. [10] A *matroid* M is a pair $(E,\mathcal{I})$ where E is a set and $\mathcal{I} \subseteq 2^E$ which satisfies the following:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \subseteq I'$ and $I' \in \mathcal{I}$, then $I \in \mathcal{I}$.

(I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

For $M = (E,\mathcal{I})$ a matroid, E is the ground set, a subset of which is *independent* if it is in $\mathcal{I}$; otherwise *dependent*. A *base* of M is a maximal independent subset of E, while a *circuit* is a minimal dependent subset. Let $\mathcal{C}(M)$ be the set of circuits of M. A circuit of size one is called a *loop*. We usually identify a matroid with its set of independent sets, and so write an independent set I is in M.

Equivalently, matroids can be defined with base axioms. A collection $\mathcal{B}$ of subsets of E is the set of bases of a matroid if and only if the following three axioms hold:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.

(BM) The set $\mathcal{I}$ of all subsets of elements in $\mathcal{B}$ satisfies (IM).

The *dual* matroid M^* of M has as bases precisely the complements of bases of M. Given $X \subseteq E$, M *restricted* to X is the matroid $(X, \mathcal{I} \cap 2^X)$, and is denoted by $M \restriction X$ or $M \setminus X^c$. The *contraction* of M to X, $M.X$ or equally the matroid obtained by contracting X^c, M/X^c is defined to be $(M^* \restriction X)^*$. Let X and Y be two disjoint subsets of E. Then $M/X \setminus Y = M \setminus Y/X$ is a *minor* of M obtained by contracting X and deleting Y. The following standard fact simplifies investigations of minors.

Lemma 2.2.2. *Let M be a matroid, $C, D \subseteq E$ with $C \cap D = \emptyset$ and let $M' := M/C \setminus D$ be a minor. Then there is an independent set S and a coindependent set R such that $M' = M/S \setminus R$.*

Proof. Let S be the union of a base of $M \restriction C$ and a base of $M.D$ and let $R := (C \cup D) \setminus S$. In particular S is independent (by [10, Corollary 3.6]). Since R is disjoint from some base extending S in $E \setminus (C \cup D)$, it is coindependent.

In particular, any base of $M/S \setminus R$ spans M/S. For a set $B \subseteq E \setminus (C \cup D)$ we have:

$$\begin{aligned}
&B \in \mathcal{B}(M \setminus D/C) \\
\Leftrightarrow &B \cup (C \cap S) \in \mathcal{B}(M \setminus D) \\
\Leftrightarrow &B \cup (C \cap S) \cup (D \cap S) \in \mathcal{B}(M) \\
\Leftrightarrow &B \cup S \in \mathcal{B}(M) \\
\Leftrightarrow &B \in \mathcal{B}(M/S) \\
\Leftrightarrow &B \in \mathcal{B}(M/S \setminus R).
\end{aligned}$$

□

Let $M = (E, \mathcal{I})$ be a set system. The set $\mathcal{I}^{\text{fin}}$ consists of the sets which have all their finite subsets in $\mathcal{I}$. $M^{\text{fin}} = (E, \mathcal{I}^{\text{fin}})$ is called *finitarisation* of M. M is called *finitary* if $M = M^{\text{fin}}$; or equivalently if all circuits of M are finite. Applying Zorn's Lemma one see that finitary set systems always satisfy (IM). M is called *nearly finitary* if for any maximal element $B \in \mathcal{I}^{\text{fin}}$ there is an $I \in \mathcal{I}$ such that $|B \setminus I| < \infty$, or equivalently any base of M can be extended to a base of the finitarisation adding only finitely many elements. Nearly finitary matroids first appeared in [5] as a superclass of finitary matroids in which one can have an infinite matroid union theorem.

2.2.2 Linkability system

All the digraphs considered in this chapter do not have any loops or parallel edges. Given a digraph D, let $V := V(D)$ and $B_0 \subseteq V$ be a set of sinks. Call the pair (D, B_0) a *dimaze*[1] and B_0 the (set of) *exits*. Given a (directed) path or ray P, $\text{Ini}(P)$ and $\text{Ter}(P)$ denote the initial and the terminal vertex (if exists) of P, respectively. Let $\mathcal{P}$ be a set of paths and rays, then $\text{Ini}(\mathcal{P}) = \{\text{Ini}(P) : P \in \mathcal{P}\}$ and $\text{Ter}(\mathcal{P}) = \{\text{Ter}(P) : P \in \mathcal{P}\}$. A *linkage* $\mathcal{P}$ is a set of (vertex disjoint) paths ending in B_0. A set $A \subseteq V$ is *linkable* if there is a linkage $\mathcal{P}$ from A to B, i.e. $\text{Ini}(\mathcal{P}) = A$ and $\text{Ter}(\mathcal{P}) \subseteq B$; $\mathcal{P}$ is *onto* B if $\text{Ter}(\mathcal{P}) = B$.

Note that, by adding trivial paths if needed:

$$\text{Any linkable set in } (D, B_0) \text{ can be extended to one linkable onto } B_0. \tag{2.1}$$

Definition 2.2.3. Let (D, B_0) be a dimaze. The pair of $V(D)$ and the set of linkable subsets is denoted by $M_L(D, B_0)$. A *strict gammoid* is a matroid isomorphic to $M_L(D, B_0)$ for some (D, B_0). A *gammoid* is a restriction of a strict gammoid. Given a gammoid M, (D, B_0) is called a *presentation* of M if $M = M_L(D, B_0)\restriction X$ for some $X \subseteq V(D)$.

[1]Dimaze is short for directed maze.

If D' is a subdigraph of D and $B'_0 \subseteq B_0$, then (D, B_0) contains (D', B'_0) as a *subdimaze*. A dimaze (D', B'_0) is a *subdivision* of (D, B_0) if it can be obtained from (D, B_0) as follows. We first add an extra vertex b_0 and the edges $\{(b, b_0) : b \in B_0\}$ to D. Then the edges of this resulting digraph are subdivided to define a digraph D''. Set B'_0 as the in-neighbourhood of b_0 in D'' and D' as $D'' - b_0$. Note that this defaults to the usual notion of subdivision if $B_0 = \emptyset$.

The following dimazes play an important role in our investigation. An undirected ray is a graph with an infinite vertex set $\{x_i : i \geq 1\}$ and the edge set $\{x_i x_{i+1} : i \geq 1\}$. We orient the edges of an undirected ray in different ways to construct three dimazes:

1. R^A by orienting (x_{i+1}, x_i) and (x_{i+1}, x_{i+2}) for each odd $i \geq 1$ and the set of exits is empty;
2. R^I by orienting (x_{i+1}, x_i) for each $i \geq 1$ and x_1 is the only exit;
3. R^O by orienting (x_i, x_{i+1}) for each $i \geq 1$ and the set of exits is empty.

A subdivision of R^A, R^I and R^O is called *alternating ray*, *incoming ray* and *(outgoing) ray*, respectively.

Let $Y = \{y_i : i \geq 1\}$ be a set disjoint from X. We extend the above types of rays to combs by adding edges (and their terminal vertices) and declaring the resulting sinks to be the exits:

1. C^A by adding no edges to R^A;
2. C^I by adding the edges (x_i, y_i) to R^I for each $i \geq 2$;
3. C^O by adding the edges (x_i, y_i) to R^O for each $i \geq 2$.

Furthermore we define the dimaze F^∞ by declaring the sinks of the digraph $(\{v, v_i : i \in \mathbb{N}\}, \{(v, v_i) : i \in \mathbb{N}\})$ to be the exits.

Any subdivision of C^A, C^I, C^O and F^∞ is called *alternating comb*, *incoming comb*, *outgoing comb* and *linking fan*, respectively. The subdivided ray in any comb is called the *spine* and the paths to the exits are the *spikes*.

A dimaze (D, B_0) is called *$\mathcal{H}$-free* for a set $\mathcal{H}$ of dimazes if it does not have a subdimaze isomorphic to a subdivision of an element in $\mathcal{H}$. A (strict) gammoid is called *$\mathcal{H}$-free* if it admits an $\mathcal{H}$-free presentation. In general, an $\mathcal{H}$-free gammoid may admit a presentation that is not $\mathcal{H}$-free (see Figure 2.5 for $\mathcal{H} = \{C^A\}$).

Given a path P and a vertex w on P, Pw denotes the segment from the initial vertex up to w and $P\mathring{w}$ the same segment with w excluded. We use PwQ for the concatenation of Pw and wQ where Q is a path containing w; and other similar notations. We also identify P with its vertex set.

Let (D, B_0) be a dimaze and $\mathcal{Q}$ a set of disjoint paths or rays (usually a linkage). A *$\mathcal{Q}$-alternating walk* is a sequence $W = w_0 e_0 w_1 e_1 \ldots$ of vertices

w_i and distinct edges e_i of D not ending with an edge, such that every $e_i \in W$ is incident with w_i and w_{i+1}, and the following properties hold for each $i \geq 0$ (and $i < n$ in case W is finite, where w_n is the last vertex):

(W1) $e_i = (w_{i+1}, w_i)$ if and only if $e_i \in E(\mathcal{Q})$;

(W2) if $w_i = w_j$ for any $j \neq i$, then $w_i \in V(\mathcal{Q})$;

(W3) if $w_i \in V(\mathcal{Q})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{Q}) \neq \emptyset$ (with $e_{-1} := e_0$).

Let $\mathcal{P}$ be another set of disjoint paths or rays. A *$\mathcal{P}$-$\mathcal{Q}$-alternating walk* is a $\mathcal{Q}$-alternating walk whose edges are in $E(\mathcal{P})\Delta E(\mathcal{Q})$, and such that any interior vertex w_i satisfies

(W4) if $w_i \in V(\mathcal{P})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{P}) \neq \emptyset$.

Two $\mathcal{Q}$-alternating walks W_1 and W_2 are *disjoint* if they are edge disjoint, $V(W_1) \cap V(W_2) \subseteq V(\mathcal{Q})$ and $\mathrm{Ter}(W_1) \neq \mathrm{Ter}(W_2)$.

Suppose that a digraph D, a set $A \subseteq V(D)$ and a linkage $\mathcal{P}$ from a subset of A to some $B \subseteq V$ are given. An A–B (vertex) separator S is a set of vertices such that every path from A to B intersects S, and S is *on* $\mathcal{P}$ if it consists of exactly one vertex from each path in $\mathcal{P}$. Given $A, B \subseteq V$, the Aharoni-Berger-Menger's theorem [4] states that there exists a linkage from a subset of A to B and an A–B separator on this linkage.

We recall a classical result due to Grünwald [17], which can be formulated as follows (see also [13, Lemmas 3.3.2 and 3.3.3]).

Lemma 2.2.4. *Let (D, B_0) be a dimaze, $\mathcal{Q}$ a linkage, and $\mathrm{Ini}(\mathcal{Q}) \subseteq X \subseteq V$.*

(i) If there is a $\mathcal{Q}$-alternating walk from $X \setminus \mathrm{Ini}(\mathcal{Q})$ to $B_0 \setminus \mathrm{Ter}(\mathcal{Q})$, then there is a linkage $\mathcal{Q}'$ with $\mathrm{Ini}(\mathcal{Q}) \subsetneq \mathrm{Ini}(\mathcal{Q}') \subseteq X$ onto $\mathrm{Ter}(\mathcal{Q}) \subsetneq \mathrm{Ter}(\mathcal{Q}') \subseteq B_0$.

(ii) If there is not any $\mathcal{Q}$-alternating walk from $X \setminus \mathrm{Ini}(\mathcal{Q})$ to $B_0 \setminus \mathrm{Ter}(\mathcal{Q})$, then there is a X–B_0 separator on $\mathcal{Q}$.

A set $X \subseteq V$ in (D, B_0) is *topologically linkable* if X admits a *topological linkage*, which means that from each vertex $x \in X$, there is a *topological path* P_x, i.e. P_x is the spine of an outgoing comb, a path ending in the centre of a linking fan, or a path ending in B_0, such that P_x is disjoint from P_y for any $y \neq x$. Clearly, a finite topologically linkable set is linkable. Denote by $M_{TL}(D, B_0)$ the pair of V and the set of the topologically linkable subsets. Carmesin gave the following connection between dimazes (not necessarily defining a matroid) and topological linkages.

Corollary 2.2.5. *[12, Corollary 5.7] Given a dimaze (D, B_0), $M_{TL}(D, B_0) = M_L(D, B_0)^{\mathrm{fin}}$. In particular, $M_{TL}(D, B_0)$ is always a finitary matroid.*

A *strict topological gammoid* is a matroid of the form $M_{TL}(D, B_0)$, and a restriction of which is called a *topological gammoid*.

2.2.3 Transversal system

Let $G = (V, W)$ be a bipartite graph and call V and W, respectively, the *left* and the *right* vertex class of G. A subset I of V is *matchable onto* $W' \subseteq W$ if there is a matching m of I such that $m \cap V = I$ and $m \cap W = W'$; where we are identifying a set of edges (and sometimes more generally a subgraph) with its vertex set. Given a set $X \subseteq V$ or $X \subseteq W$, write $m(X)$ for the set of vertices matched to X by m and $m{\restriction}X$ for the subset of m incident with vertices in X.

Definition 2.2.6. Given a bipartite graph $G = (V, W)$, the pair of V and all its matchable subsets is denoted by $M_T(G)$. A *transversal matroid* is a matroid isomorphic to $M_T(G)$ for some G. Given a transversal matroid M, G is a *presentation* of M if $M = M_T(G)$.

In general, a transversal matroid may have different presentations. The following is a well-known fact (see [9]).

Lemma 2.2.7. *Let $G = (V, W)$ be a bipartite graph. Suppose there is a maximal element in $M_T(G)$, witnessed by a matching m_0. Then $M_T(G) = M_T(G \setminus (W - m_0))$, and $N(W - m_0)$ is a subset of every maximal element in $M_T(G)$.*

In case $M_T(G)$ is a matroid, the second part states that $N(W - m_0)$ is a set of coloops. From now on, wherever there is a maximal element in $M_T(G)$, we assume that W is covered by a matching.

Given a matching m, an *m-alternating walk* is a walk such that the consecutive edges alternate in and out of m in G. Given another matching m', an *m-m'-alternating walk* is a walk such that consecutive edges alternate between the two matchings.

A standard compactness proof shows that a left locally finite bipartite graph $G = (V, W)$, i.e. every vertex in V has finite degree, defines a finitary transversal matroid.

Lemma 2.2.8 ([22])**.** *Every left locally finite bipartite graph defines a finitary transversal matroid.*

The following corollary is a tool to show that a matroid is not transversal.

Lemma 2.2.9. *Any infinite circuit of a transversal matroid contains an element which does not lie in any finite circuit.*

Proof. Let C be an infinite circuit of some $M_T(G)$. Applying Lemma 2.2.8 on the restriction of $M_T(G)$ to C, we see that there is a vertex in C having infinite degree. However, such a vertex does not lie in any finite circuit. $\square$

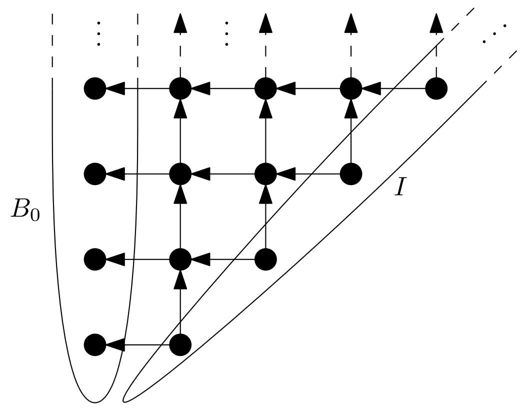

Figure 2.1: A locally finite dimaze which does not define a matroid

2.3 Dimazes and matroid axioms

The results in the following two sections come from [2].

Mason [21] (see also [27]) showed that given a finite digraph D, for any $B_0 \subseteq V$, $M_L(D, B_0)$ is a matroid. However, this is not the case for infinite digraphs. For example, let D be a complete bipartite graph between an uncountable set X and a countably infinite set B_0 with all the edges directed towards B_0. Then $I \subseteq X$ is independent if and only if I is countable, so there is not any maximal independent set in X. Hence, $M_L(D, B_0)$ does not satisfy the axiom (IM).

Example 2.3.1. A counterexample with a locally finite digraph is the half-grid. Define a digraph D by directing upwards or leftwards the edges of the subgraph of the grid $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ induced by $\{(x, y) : y > 0 \text{ and } y \geq x \geq 0\}$. The *half-grid* is the dimaze (D, B_0) where $B_0 := \{(0, y) : y > 0\}$; see Figure 2.1. Then $I := \{(x, x) : x > 0\}$ is linkable onto a set $J \subseteq B_0$ if and only if J is infinite. Therefore, $I \cup (B_0 \setminus J)$ is independent if and only if J is infinite. Hence, I does not extend to a maximal independent set in $X := I \cup B_0$.

The aim of this section is to give a sufficient condition for a dimaze (D, B_0) to define a matroid. As (I1) and (I2) hold for $M_L(D, B_0)$, we need only consider (I3) and (IM).

2.3.1 Linkability system and proof of (I3)

We prove that (I3) holds in any $M_L(D, B_0)$ using a result due to Grünwald [17].

Lemma 2.3.2. *Let (D, B_0) be a dimaze. Then $M_L(D, B_0)$ satisfies (I3).*

Proof. Let $I, B \in M_L(D, B_0)$ such that B is maximal but I is not. Then we have a linkage $\mathcal{Q}$ from B and another $\mathcal{P}$ from I. We may assume $\mathcal{P}$ misses some $v_0 \in B_0$.

If there is an alternating walk with respect to $\mathcal{P}$ from $(B \cup I) \setminus V(\mathcal{P})$ to $B_0 \setminus V(\mathcal{P})$, then by Lemma 2.2.4(i), we can extend I in $B \setminus I$.

On the other hand, if no such walk exists, we draw a contradiction to the maximality of B. In this case, by Lemma 2.2.4(ii), there is a $(B \cup I)$–B_0 separator S on $\mathcal{P}$. For every $v \in B$, let Q_v be the path in $\mathcal{Q}$ starting from v. Let s_v be the first vertex of S that Q_v meets and P_v the path in $\mathcal{P}$ containing s_v. Let us prove that the set $\mathcal{Q}' = \{Q_v s_v P_v : v \in B\}$ is a linkage.

Suppose v and v' are distinct vertices in B such that $Q_v s_v P_v$ and $Q_{v'} s_{v'} P_{v'}$ meet each other. As $\mathcal{P}$ and $\mathcal{Q}$ are linkages, without loss of generality, we may assume $Q_v s_v$ meets $s_{v'} P_{v'}$ at some $s \notin S$. Then $Q_v s P_{v'}$ is a path from B to B_0 avoiding the separator. This contradiction shows that $\mathcal{Q}'$ is indeed a linkage from B to B_0. As $\mathcal{Q}'$ does not cover v_0, $B + v_0$ is independent which contradicts to the maximality of B. $\square$

2.3.2 Linkage theorem and (IM)

Since (I3) holds for any $M_L(D, B_0)$, it remains to investigate (IM). Recall that for any finite digraph D and $B_0 \subseteq V$, the following holds for (D, B_0):

A set is maximally independent if and only if it is linkable onto the exits. (†)

When D is infinite, (†) need not hold; for instance, the half grid in Example 2.3.1, which does not even define a matroid. It turns out that there are always non-maximal independent sets linkable onto the exits in a dimaze that does not define a matroid. To prove this, we will use the linkage theorem [28] (see also [15]) and the infinite Menger's theorem [4].

Now the natural question is: in which dimazes is every set, that is linkable onto the exits, a maximal independent set? Consider the alternating comb given in Figure 2.5a. Using the notation there, the set $A = \{a_i : i \geq 1\}$ can be linked onto B_0 by the linkage $\{(a_i, b_{i-1}) : i \geq 1\}$ or to $B_0 - b_0$ by the linkage $\{(a_i, b_i) : i \geq 1\}$. Hence, A is a non-maximal independent set that is linkable onto B_0. More generally, if a dimaze (D, B_0) contains an alternating comb C, then the vertices of out-degree 2 on C together with $B_0 - C$ is a non-maximal set linkable onto B_0. So an answer to the above question must exclude dimazes containing an alternating comb. We will prove that dimazes without any alternating comb are precisely the answer.

One might think that the following proof strategy should work: If the characterization of maximal independent sets does not hold, then there are two linkages, a blue one from a set and a red one from a proper superset, both covering the exits. To construct an alternating comb, one starts with finding an alternating ray. For that, a first attempt is to "alternate" between the red and blue linkages, i.e. to repeat the following: go forward along the red linkage, change to the blue one at some common vertex, and then go backwards on the blue linkage, and change again to the red one. It is not the case that this construction always gives rise to an alternating ray (because vertices might be visited twice). But supposing that we do get an alternating ray, a natural way to extend it to an alternating comb is to use the terminal segments of one fixed linkage. However, this alternating ray can have two distinct vertices of in-degree 2 which lie on the same path of the fixed linkage.

Appropriate choices to alternate between the linkages will be provided by the *proof* of the linkage theorem of Pym [28]. So we give a sketch of the proof, rephrased for our purpose.

Linkage Theorem. *Let D be a digraph and two linkages be given: the "red" one, $\mathcal{P} = \{P_x : x \in X_{\mathcal{P}}\}$, from $X_{\mathcal{P}}$ onto $Y_{\mathcal{P}}$ and the "blue" one, $\mathcal{Q} = \{Q_y : y \in Y_{\mathcal{Q}}\}$, from $X_{\mathcal{Q}}$ onto $Y_{\mathcal{Q}}$. Then there is a set X^∞ satisfying $X_{\mathcal{P}} \subseteq X^\infty \subseteq X_{\mathcal{P}} \cup X_{\mathcal{Q}}$ which is linkable onto a set Y^∞ satisfying $Y_{\mathcal{Q}} \subseteq Y^\infty \subseteq Y_{\mathcal{Q}} \cup Y_{\mathcal{P}}$.*

Sketch of proof. [2] We construct a sequence of linkages converging to a linkage with the desired properties. For each integer $i \geq 0$, we will specify a vertex on each path in $\mathcal{P}$. For each $x \in X_{\mathcal{P}}$, let $f_x^0 := x$. Let $\mathcal{Q}^0 := \mathcal{Q}$. For each $i > 0$ and each $x \in X_{\mathcal{P}}$, let f_x^i be the last vertex v on $f_x^{i-1} P_x$ such that

[2] We find that the following story makes the proof more intuitive. Imagine that a directed path in a linkage corresponds to a pipeline which transports water backwards from a pumping station located at the terminal vertex of this path to its initial vertex. At the initial vertex of every red pipeline, there is a farm whose farmer is *happy* if and only if his farm is supplied with water. The water from every blue pipeline flows into the desert at the initial vertex, even if there is a farm.

The story starts on day 0 when suddenly all the red pumping stations are broken. At the beginning of each day, every farmer follows the rule:

> "If you are *unhappy*, then move onward along your red pipeline until you can potentially get some water."

So the unhappy farmers take their toolboxes and move along their pipelines. Every farmer stops as soon as he comes across a pipeline/pumping station which still transports water and manipulates it in such a way that all the water flows into his red pipeline and then to his farm. If a pipeline has been manipulated by more than one farmer, then the one who is the closest to the pumping station gets the water, "stealing" it from the others. When a farmer arrives at his pumping station, he repairs it, but only if he cannot steal. We assume that every farmer needs a full day for the whole attempt to get water; so he realizes that someone else has stolen his water only at the end of the day. The story ends when every farmer is happy.

A proof of the linkage theorem can be derived by examining the flow of water and the movement of the farmers. In particular, each pumping station supplies its final farm and each farmer reaches his final position (where he stays happily) after only finitely many days. It turns out that the water flow at the end is a linkage covering the red initial and the blue terminal vertices, as required by the theorem.

$(f_x^{i-1} P_x \mathring{v}) \cap V(\mathcal{Q}^{i-1}) = \emptyset$.[3] For $y \in Y_{\mathcal{Q}}$, let t_y^i be the first vertex $v \in Q_y$ such that the terminal segment $\mathring{v} Q_y$ does not contain any f_x^i.[4] Let

$$\begin{aligned}
\mathcal{A}^i &:= \{Q_y \in \mathcal{Q} : t_y^i \neq f_x^i \ \forall x \in X_{\mathcal{P}}\}, \\
\mathcal{B}^i &:= \{P_x f_x^i Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^i = t_y^i\}, \\
\mathcal{C}^i &:= \{P_x \in \mathcal{P} : f_x^i \in Y_{\mathcal{P}} \text{ and } f_x^i \neq t_y^i \ \forall y \in Y_{\mathcal{Q}}\},
\end{aligned}$$

and $\mathcal{Q}^i := \mathcal{A}^i \cup \mathcal{B}^i \cup \mathcal{C}^i$.

Inductively, one can show that $\mathcal{Q}^i$ is a linkage which covers $Y_{\mathcal{Q}}$. Since for any x and $j \geq i$, $P_x f_x^i \supseteq P_x f_x^j$, it can be shown that $t_y^i Q_y \subseteq t_y^j Q_y$ for any y. As all the paths P_x and Q_y are finite, there exist integers $i_x, i_y \geq 0$ such that $f_x^{i_x} = f_x^k$, $t_y^{i_y} = t_y^l$ for all integers $k \geq i_x$ and $l \geq i_y$. Thus, we define $f_x^\infty := f_x^{i_x}, t_y^\infty := t_y^{i_y}$ and

$$\begin{aligned}
\mathcal{A}^\infty &:= \{Q_y \in \mathcal{Q} : t_y^\infty \neq f_x^\infty \ \forall x \in X_{\mathcal{P}}\}, \\
\mathcal{B}^\infty &:= \{P_x f_x^\infty Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^\infty = t_y^\infty\}, \\
\mathcal{C}^\infty &:= \{P_x \in \mathcal{P} : f_x^\infty \in Y_{\mathcal{P}} \text{ and } f_x^\infty \neq t_y^\infty \ \forall y \in Y_{\mathcal{Q}}\}.
\end{aligned}$$

Then $\mathcal{Q}^\infty := \mathcal{A}^\infty \cup \mathcal{B}^\infty \cup \mathcal{C}^\infty$ covers $Y_{\mathcal{Q}}$. Moreover, $\mathcal{Q}^\infty$ is a linkage. Indeed, as $t_y^\infty Q_y \subseteq t_y^i Q_y$ for any i, the intersection of $P_x f_x^\infty$ and $t_y^\infty Q_y$ is either empty or the singleton of $f_x^\infty = t_y^\infty$. It remains to argue that $X_{\mathcal{P}} \subseteq \mathrm{Ini}(\mathcal{Q}^\infty)$. Let $x \in X_{\mathcal{P}}$. If $f_x^\infty = t_y^\infty$ for some y, then $x \in \mathrm{Ini}(\mathcal{B}^\infty)$. Otherwise, there exists an integer j such that $f_x^\infty = f_x^j$ and $f_x^\infty \neq t_y^j$ for any y. Since $f_x^{j+1} = f_x^j$, it follows that f_x^j is on a path in $\mathcal{C}^j$, so $f_x^\infty \in Y_{\mathcal{P}}$. Hence, $x \in \mathrm{Ini}(\mathcal{C}^\infty)$. $\square$

We can now prove the following.

Theorem 2.3.3. *Given a dimaze (D, B_0), suppose that every independent set linkable onto the exits is maximal, then the dimaze defines a matroid.*

Proof. Since (I1) and (I2) are obviously true for $M_L(D, B_0)$, and that (I3) holds by Lemma 2.3.2, to prove the theorem, it remains to check that (IM) holds.

Let I be independent and a set $X \subseteq V$ such that $I \subseteq X$ be given. Suppose there is a "red" linkage from I to B_0. Apply the Aharoni-Berger-Menger's theorem on X and B_0 to get a "blue" linkage $\mathcal{Q}$ from $B \subseteq X$ to B_0 and an X–B_0 separator S on the blue linkage. Let H be the subgraph induced by those vertices separated from B_0 by S with the edges going out of S deleted. Since every linkage from H to B_0 goes through S, a subset of $V(H)$ is linkable in (D, B_0) if and only if it is linkable in (H, S). Use the

[3] The farmer at f_x^{i-1} moves onward to the closest vertex where there was still water at the beginning of day $i-1$ or, if no such vertex exists, to the pumping station of P_x.

[4] If $f_x^i = t_y^i$, the farmer at f_x^i is the closest to the pumping station at y and steals the water from any other farmer on Q_y.

linkage theorem to find a linkage $\mathcal{Q}^\infty$ from X^∞ with $I \subseteq X^\infty \subseteq I \cup B \subseteq X$ onto S.

Let $Y \supseteq X^\infty$ be any independent set in $M_L(H,S)$. By applying the linkage theorem on a linkage from Y to S and $\mathcal{Q}^\infty$ in (H,S), we may assume that Y is linkable onto S by a linkage $\mathcal{Q}'$. Concatenating $\mathcal{Q}'$ with segments of paths in $\mathcal{Q}$ starting from S and adding trivial paths from $B_0 \setminus V(\mathcal{Q})$ gives us a linkage from $Y \cup (B_0 \setminus V(\mathcal{Q}))$ onto B_0. By the hypothesis, $Y \cup (B_0 \setminus V(\mathcal{Q}))$ is a maximal independent set in $M_L(D, B_0)$.

Applying the above statement on X^∞ shows that $X^\infty \cup (B_0 \setminus V(\mathcal{Q}))$ is also maximal in $M_L(D, B_0)$. It follows that $Y = X^\infty$. Hence, X^∞ is maximal in $M_L(H,S)$, and so also in $M_L(D,B_0) \cap 2^X$. This completes the proof that $M_L(D,B_0)$ is a matroid. □

Next we show that containing an alternating comb is the only reason that the criterion (†) fails.

Lemma 2.3.4. *Let (D, B_0) be a dimaze without any alternating comb. Then a set $B \subseteq V$ is maximal in $M_L(D,B_0)$ if and only if it is linkable onto B_0.*

Proof. The forward direction follows trivially from (2.1).

For the backward direction, let I be a non-maximal subset that is linkable onto B_0, by a "blue" linkage $\mathcal{Q}$. Since I is not maximal, there is $x_0 \notin I$ such that $I + x_0$ is linkable to B_0 as well, by a "red" linkage $\mathcal{P}$. Construct an alternating comb inductively as follows:

Use (the proof of) the linkage theorem to get a linkage $\mathcal{Q}^\infty$ from $I + x_0$ onto B_0. Since $Y_\mathcal{P} \subseteq Y_\mathcal{Q}$ and $X_\mathcal{Q} \subseteq X_\mathcal{P}$, $\mathcal{A}^\infty = \mathcal{C}^\infty = \emptyset$. So each path in $\mathcal{Q}^\infty$ consists of a red initial and a blue terminal segment.

Start the construction with x_0. For $k \geq 1$, if x_{k-1} is defined, let Q_k be the blue path containing $p_{k-1} := f^\infty_{x_{k-1}}$. We will prove that $p_{k-1} \notin I$ so that we can choose a last vertex q_k on $Q_k \mathring{p}_{k-1}$ that is on a path in $\mathcal{Q}^\infty$. Since the blue segments of $\mathcal{Q}^\infty$ are disjoint, q_k lies on a red path P_{x_k}. We continue the construction with x_k.

Claim 2.3.5. *For each $k \geq 1$, $p_{k-1} \notin I$ and hence, the blue segment $q_k Q_k p_{k-1}$ is non-trivial. The red segment $q_k P_{x_k} p_k$ is also non-trivial.*

Proof. We prove by induction that $p_{k-1} \notin I$. This will guarantee that $q_k Q_k p_{k-1}$ is non-trivial since $Q_k \cap I \in V(\mathcal{Q}^\infty)$. Clearly, $p_0 \notin I$. Given $k \geq 1$, assume that $p_{k-1} \notin I$. We argue that $q_k \neq p_k$. Suppose not for a contradiction. Then the path $P_{x_k} q_k Q_k$ is in $\mathcal{B}^\infty$. Since $q_k Q_k p_{k-1}$ is non-trivial, p_{k-1} and p_k are distinct vertices of the form f^∞_x on $P_{x_k} q_k Q_k$. This contradicts that $P_{x_k} q_k Q_k$ is in $\mathcal{B}^\infty$. Hence, we have $p_k \neq q_k$,[5] and so $p_k \notin I$. This also shows that the red segment $q_k P_{x_k} p_k$ is non-trivial. □

[5]On day i_{k-1}, the farmer at p_{k-1} prevents the water supplied by Q_k from flowing to q_k. Hence, the farmer on P_{x_k} must go further to get water, so $p_k \neq q_k$.

Claim 2.3.6. *For any $j < k$, $x_j \neq x_k$. Therefore, $q_k Q_k p_{k-1}$ is disjoint from $q_j Q_j p_{j-1}$, and so is $q_k P_{x_k} p_k$ from $q_j P_{x_j} p_j$.*

Proof. For $k \geq 0$, let i_k is the least integer such that $f_{x_k}^{i_k} = f_{x_k}^{\infty}$. We now show that $i_{k-1} < i_k$ for any given $k \geq 1$. By the choice of q_k, for any $i \leq i_{k-1}$, $f_{x_k}^{i}$ is on the segment $P_{x_k} q_k$. Since $P_{x_k} q_k$ is a red segment of $\mathcal{Q}^{\infty}$, p_k is in the segment $q_k P_{x_k}$. Since $p_k \neq q_k$, it follows that $P_{x_k} f_{x_k}^{i_{k-1}} \subseteq P_{x_k} q_k \subsetneq P_{x_k} f_{x_k}^{\infty}$. This implies that $f_{x_k}^{i_{k-1}} \neq f_{x_k}^{\infty}$, so that by definition of i_k, we have $i_k > i_{k-1}$.[6] Hence, $x_k \neq x_j$ for any $j \neq k$. □

We now show that $\bigcup_{k=1}^{\infty} q_k Q_k \cup q_k P_{x_k} p_k$ is an alternating comb. Indeed, by the claims, it remains to check that $q_k P_{x_k} p_k$ does not meet any $q_j Q_j$ for any $j < k$. But if there is such an intersection, it can neither lie in $\mathring{q}_j Q_j \mathring{p}_{j-1}$ by the choice of q_j; nor in $p_{j-1} Q_j$ since $x_k \neq x_{j-1}$ (Claim 2.3.6) and $\mathcal{Q}^{\infty}$ is a linkage. Hence, we have constructed an alternating comb. This contradiction shows that I is maximal. □

We have all the ingredients to prove the main result.

Theorem 2.3.7. *Given a dimaze, the vertex sets linkable onto the exits form the bases of a matroid if and only if the dimaze contains no alternating comb. The independent sets of this matroid are precisely the linkable sets of vertices.*

Proof. The backward direction of the first statement follows from Theorem 2.3.3 and Lemma 2.3.4. To see the forward direction, suppose there is an alternating comb C. Let B_1 be the union of the vertices of out-degree 2 on C with $B_0 - C$. Then B_1 is linkable onto B_0, and so is $B_1 + v$ for any $v \in B_0 \cap C$. But B_1 and $B_1 + v$ violate the base axiom (B2). The second statement follows from the first and (2.1). □

Corollary 2.3.8. *Any dimaze which does not define a matroid contains an alternating comb.*

2.3.3 Nearly finitary linkability system

Although forbidding alternating combs ensures that we get a strict gammoid, not every strict gammoid arises this way. It turns out that when a dimaze gives rise to a nearly finitary ([5]) linkability system, the dimaze defines a matroid regardless of whether it contains an alternating comb or not. In order to show this we prove the following.

Lemma 2.3.9. *Let (D, B_0) be a dimaze. Then $M_L(D, B_0)$ satisfies the following:*

[6] Before day i_{k-1}, the water supplied by Q_k ensures that the farmer on P_{x_k} need not go beyond q_k. But eventually he moves past q_k and arrives at p_k on day i_k; so $i_k > i_{k-1}$.

$(*)$ *For all independent sets I and J with $J \setminus I \neq \emptyset$, for every $v \in I \setminus J$ there exists $u \in J \setminus I$ such that $J + v - u$ is independent.*

Proof. We may assume that $I \setminus J = \{v\}$. Let $\mathcal{Q} = (Q_y)_{y \in Y_{\mathcal{Q}}}$ be a "blue" linkage from J onto some $Y_{\mathcal{Q}} \subseteq B_0$ and $\mathcal{P}$ a "red" one from I. The linkage theorem yields a linkage $\mathcal{Q}^\infty$, which we will show to witness the independence of a desired set. We use the notations introduced in its proof. For each $y \in Y_{\mathcal{Q}}$, let t_y^0 be the initial vertex of Q_y.

For $i > 0$ it is not hard to derive the following facts from the definitions of $\mathcal{Q}^i$, f_x^i and t_y^i:

$$x \in I \cap \mathrm{Ini}(\mathcal{Q}^{i-1}) \implies f_x^i = f_x^{i-1}; \tag{2.2}$$
$$t_y^0 \in \mathrm{Ini}(\mathcal{A}^i) \iff \forall x \in I, f_x^i \notin Q_y; \tag{2.3}$$
$$x \in I \setminus \mathrm{Ini}(\mathcal{Q}^i) \iff \exists y \in Y_{\mathcal{Q}}, x' \in I \text{ s.t. } f_x^i \in Q_y \mathring{f}_{x'}^i. \tag{2.4}$$

Claim. *For $i \geq 0$, either $\mathcal{Q}^i = \mathcal{Q}^\infty$ or there is some x^i such that:*[7]

U_i*:* $(J + v) \setminus \mathrm{Ini}(\mathcal{Q}^i) = \{x^i\}$ *and* $I - x^i \subseteq \mathrm{Ini}(\mathcal{B}^i)$;

D_i*:* $\forall y \in Y_{\mathcal{Q}}$*, if* $t_y^0 \in I$ *then* $\exists! x \in I - x^i$ *s.t.* $f_x^i \in Q_y$*; no such x otherwise.*

Proof. With $x^0 := v$, the claim clearly holds for $i = 0$. Given $i > 0$, to prove the claim, we may assume that $\mathcal{Q}^{i-1} \neq \mathcal{Q}^\infty$ and U_{i-1} and D_{i-1} hold.

By definition of $f_{x^{i-1}}^i$, either

$$f_{x^{i-1}}^i \in t_{y^i}^{i-1} Q_{y^i} \text{ for some unique } y^i \in Y_{\mathcal{Q}} \quad \text{or} \quad f_{x^{i-1}}^i \in Y_{\mathcal{P}} \setminus Y_{\mathcal{Q}}.$$

Note that by (2.2) only x^{i-1} can be a vertex such that $f_{x^{i-1}}^i \neq f_{x^{i-1}}^{i-1}$. Hence, $t_y^i = t_y^{i-1}$ for all $y \in Y_{\mathcal{Q}}$ except possibly y^i which satisfies $t_{y^i}^i = f_{x^{i-1}}^i$. So by (2.4), we have $x^{i-1} \in \mathrm{Ini}(\mathcal{Q}^i)$.[8]

Case (i): Suppose that there exists $x \in I - x^{i-1}$ such that $f_{x^{i-1}}^i$ and f_x^i are on the same path Q_{y^i}. By D_{i-1}, $t_{y^i}^0 \in I$ and x is unique. Then D_i holds for $x^i := x$. In particular, by (2.3), $J \setminus I \subseteq \mathrm{Ini}(\mathcal{A}^i)$.[9]

We now prove U_i. As $x^i \in \mathrm{Ini}(\mathcal{B}^{i-1})$, $t_{y^i}^{i-1} = f_{x^i}^{i-1}$, so $f_{x^{i-1}}^i \in \mathring{f}_{x^i}^i Q_{y^i}$, which implies that $x^i \notin \mathrm{Ini}(\mathcal{Q}^i)$ by (2.4).[10] Given $x \in I - \{x^{i-1}, x^i\}$, then $x \in \mathrm{Ini}(\mathcal{B}^{i-1})$ by U_{i-1}. So there exists $y \neq y^i$, such that $f_x^i = f_x^{i-1} = t_y^{i-1} = t_y^i$. It follows that $x \in \mathrm{Ini}(\mathcal{B}^i)$, and $I - x^i \subseteq \mathrm{Ini}(\mathcal{B}^i)$. Therefore, $(J + v) \setminus \mathrm{Ini}(\mathcal{Q}^i) = \{x^i\}$.

[7] There is a unique unhappy farmer and the others are distributed such that there is exactly one happy farmer on each blue pipeline leading to a farm.

[8] The unhappy farmer becomes happy, either by repairing or because he is the only one who moves to steal water.

[9] The whole scene only changes at Q_{y^i}, so there is still exactly one unhappy farmer and the pipelines not leading to a farm are untouched.

[10] The farmer at $f_{x^i}^i$ becomes unhappy as his water has been stolen by the farmer at $f_{x^{i-1}}^i$.

Case (ii): Suppose that there does not exist any $x \in I - x^{i-1}$ such that f^i_x is on the path Q_{y^i} containing $f^i_{x^{i-1}}$, if such a path exists. In this case, $t^0_{y^i} \in J \setminus I$. By D_{i-1}, (2.2) and (2.4), we have $I - x^{i-1} \subseteq \mathrm{Ini}(\mathcal{B}^i)$. Hence, $I \subseteq \mathrm{Ini}(\mathcal{Q}^i)$, and $\mathcal{Q}^\infty = \mathcal{Q}^i$. □

If for some integer $i > 0$, case (ii) holds, then by (2.3), only $u := t^0_{y^i} \in J \setminus I$ can fail to be in $\mathrm{Ini}(\mathcal{A}^\infty)$. Otherwise, case (i) holds for each integer $i \geq 0$, so that $J \setminus I$ is a subset of $\mathrm{Ini}(\mathcal{A}^i)$ and hence a subset of $\mathrm{Ini}(\mathcal{Q}^\infty)$. In either situation, since $I = X_{\mathcal{P}} \subseteq \mathrm{Ini}(\mathcal{Q}^\infty)$, we conclude that there is some $u \in J \setminus I$ such that $J + v - u$ is independent. □

Now we can use a result in [5] to find a sufficient condition for a linkability system to define a matroid.

Theorem 2.3.10. *Let (D, B_0) be a dimaze. If $M_L(D, B_0)$ is nearly finitary, then it is a matroid.*

Proof. Since $M_L(D, B_0)$ satisfies (I1), (I2) and $(*)$, by [5, Lemma 4.15], it also satisfies (IM). Hence, by Lemma 2.3.2, it is a matroid. □

The theorem shows that dimazes which contain an alternating comb may also define matroids.

Example 2.3.11. We construct a dimaze (D, B_0) which defines a nearly finitary linkability system, by identifying the corresponding exits of $n > 1$ copies of C^O (see Figure 2.2). Note that (D, B_0) contains an alternating comb; and $M_L(D, B_0)$ is not finitary (a vertex not in B_0 together with all reachable vertices in B_0 form an infinite circuit).

We check that $M_L(D, B_0)$ is nearly finitary. Let B be a maximal element in $M_L(D, B_0)^{\mathrm{fin}}$. Let I be the set obtained from B by deleting the last vertex, if exists, of B on each ray in $D - B_0$. Fix an enumeration $i_1, i_2, \ldots$ for I such that D contains a ray starting in i_{k+1} that avoids $I_k := \{i_1, \ldots, i_k\}$ for each $k \geq 0$. For any integer $k \geq 1$, let T_k consist of exactly one vertex on each ray in $D - B_0$ (that hits I_k): the first one in B after the last vertex of I_k. Note that there are only finitely many linkages from I_k to B_0 disjoint from T_k. In fact, there is at least one: the restriction to I_k of a linkage of the finite subset $I_k \cup T_k$ of B. Applying the infinity lemma ([13, Proposition 8.2.1]), with the kth set consisting of the finite non-empty collection of linkages from I_k to B_0 disjoint from T_k, we obtain a linkage from I to B_0. Hence, $I \in M_L(D, B_0)$. As B is arbitrary and $|T| \leq n$, we conclude that $M_L(D, B_0)$ is nearly finitary.

On the other hand, Theorem 2.3.10 does not imply Theorem 2.3.7.

Example 2.3.12. Let (for this example) a *comb* be the digraph obtained by identifying the edge (x_1, x_2) of a C^O with the edge (x_2, x_1) of a disjoint C^I

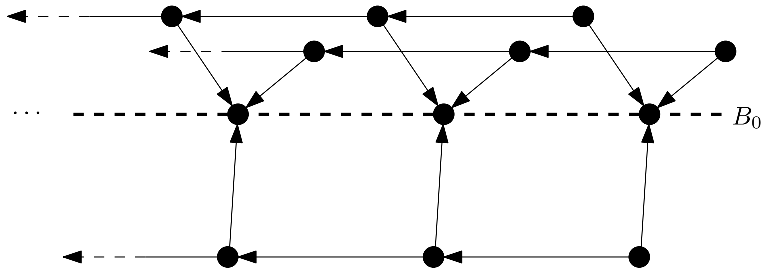

Figure 2.2: A dimaze that defines a nearly finitary linkability system

(a comb on a directed double ray). Take infinitely many disjoint copies of a comb. Add two extra vertices, and an edge from every vertex on the double ray of each comb to those two vertices. Take all the vertices of out-degree 0 as the exits. Then this dimaze defines a matroid that is 3–connected, not nearly finitary and whose dual is not nearly finitary.

So far we have seen that if a dimaze (D, B_0) does not contain any alternating comb or that $M_L(D, B_0)$ is nearly finitary, then $M_L(D, B_0)$ is a matroid. However, there are examples of strict gammoids that lie in neither of the two classes. All our examples of dimazes that do not define a matroid share another feature other than possessing an alternating comb: there is an independent set I that cannot be extended to a maximal in $I \cup B_0$. In view of this, we propose the following.

Conjecture 2.3.13. *Suppose that for all $I \in M_L(D, B_0)$ and $B \subseteq B_0$, there is a maximal independent set in $I \cup B$ extending I. Then (IM) holds for $M_L(D, B_0)$.*

2.4 Dimazes with alternating combs

We have seen in Section 2.3 that forbidding C^A in a dimaze guarantees that it defines a strict gammoid. However, the alternating comb in Figure 2.5 defines a finitary strict gammoid. On the other hand, this strict gammoid is isomorphic to the one defined by an incoming comb via the isomorphism given in the figure. So one might think that every strict gammoid has a defining dimaze which does not contain C^A. We will prove that this is not the case with two intermediate steps. In Section 2.4.1, we derive a property satisfied by any strict gammoid defined by a dimaze without any alternating comb. In Section 2.4.2, we show that any tree defines a transversal matroid. We then construct a strict gammoid which cannot be defined by a dimaze without any alternating comb.

2.4.1 Connectivity

Connectivity in finite matroids stems from graph connectivity and is a well established part of the theory. In the infinite setting, Bruhn and Wollan [11] gave the following rank-free definition of connectivity that is compatible with the finite case. For an integer $k \geq 0$, a *k–separation* of a matroid is a partition of E into X and Y such that both $|X|, |Y| \geq k$ and for any pair of bases B_X, B_Y of $M \upharpoonright X$ and $M \upharpoonright Y$ respectively, the number of elements to be deleted from $B_X \cup B_Y$ to get a base of M is less than k. It was shown there that this number does not depend on the choice of B_X or B_Y or the deleted set. A matroid is *k–connected* if there are no l–separations for any $l < k$. If a matroid does not have any k–separations for any integer k, then it is *infinitely connected*. Recall that the only infinitely connected finite matroids are uniform matroids of rank about half of the size of the ground set (see [26, Chapter 8]) and they are strict gammoids. It seems natural to look for an infinitely connected infinite matroid among strict gammoids, but here we give a partial negative result. It remains open whether there is an infinitely connected infinite gammoids.

Lemma 2.4.1. *If a dimaze (D, B_0) does not contain any alternating comb, then $M_L(D, B_0)$ contains a finite circuit or cocircuit.*

Proof. Suppose the lemma does not hold. Then every finite subset of V is independent and coindependent, and B_0 is infinite. We construct a sequence $(R_k : k \geq 1)$ of subdigraphs of D that gives rise to an alternating comb for a contradiction.

Let $v_1 \notin B_0$ and $R_1 := v_1$. For $k \geq 1$, we claim that there is a path P_k from v_k to B_0 such that $P_k \cap V(R_k) = \{v_k\}$, a vertex w_k on $\mathring{v}_k P_k$, and a vertex $v_{k+1} \notin V(R_k) \cup P_k$ with $(v_{k+1}, w_k) \in E(D)$. Let $R_{k+1} := R_k \cup P_k \cup (v_{k+1}, w_k)$.

Indeed, since any finite set containing v_k is independent, there is a path from v_k avoiding any given finite set disjoint from v_k. Hence, there is a set $\mathcal{F}$ of $|V(R_k)| + 1$ disjoint paths (except at v_k) from v_k to B_0 avoiding the finite set $V(R_k) - v_k$. Since $V(\mathcal{F}) \cup R_k$ is coindependent, its complement contains a base B, witnessed by a linkage $\mathcal{P}$. Since $|V(\mathcal{F}) \cap B_0| > |V(R_k)|$ and $\mathrm{Ter}(\mathcal{P}) = B_0$, there is a path $P \in \mathcal{P}$ that is disjoint from R_k and ends in $V(\mathcal{F}) \cap B_0$. Then the last vertex v_{k+1} of P before hitting $V(\mathcal{F})$, the next vertex w_k, and the segment $P_k := w_k P$ satisfy the requirements of the claim. By induction, the claim holds for all $k \geq 1$.

Let $R := \bigcup_{k \geq 1} R_k$. Then $(R, V(R) \cap B_0)$ is an alternating comb in (D, B_0). This contradiction completes the proof. $\square$

In an infinite matroid that is infinitely connected, the bipartition of the ground set into any finite circuit of size k against the rest is a k–separation. Hence, such a matroid must not have finite circuits or cocircuits.

Corollary 2.4.2. *If an infinite dimaze (D, B_0) does not contain any alternating comb, then $M_L(D, B_0)$ is not infinitely connected.*

2.4.2 Trees and transversal matroids

In this section, the aim is to prove that a tree defines a transversal matroid. We then construct a strict gammoid which cannot be defined by a dimaze without any alternating comb.

Given a bipartite graph G, fix an ordered bipartition (V, W) of $V(G)$; this induces an ordered bipartition of any subgraph of G. A subset of V is *independent* if it is matchable to W. Let $M_T(G)$ be the pair of V and the collection of independent sets. It is clear that (I1), (I2) hold for $M_T(G)$. When G is finite, (I3) also holds [16]. The proof of this fact which uses alternating paths can be extended to show that (I3) also holds when G is infinite.

Let m be a matching. An edge in m is called an m–edge. An m–*alternating path* is a path or a ray that starts from a vertex in V such that the edges alternate between the m–edges and the non-m–edges. An m–m' *alternating path* is defined analogously with m', also a matching, replacing the role of the non-m–edges.

Lemma 2.4.3. *For any bipartite graph G, $M_T(G)$ satisfies (I3).*

Proof. Let $I, B \in M_T(G)$ such that B is maximal but I is not. As I is not maximal, there is a matching m of $I + x$ for some $x \in V \setminus I$. Let m' be a matching of B to W. Start an m–m' alternating path P from x. By maximality of B, the alternating path is not infinite and cannot end in $W \setminus V(m')$. So we can always extend it until it ends at some $y \in B \setminus I$. Then $m \Delta E(P)$ is a matching of $I + y$, which completes the proof. □

If $M_T(G)$ is a matroid, it is called a *transversal matroid*. For $X \subseteq V$, the restriction of $M_T(G)$ to X is also a transversal matroid, and can be defined by the independent sets of the subgraph of G induced by $X \cup N(X)$.

Suppose now G is a tree rooted at a vertex in W. By upwards (downwards), we mean towards (away from) the root. For any vertex set Y, let $N^{\uparrow}(Y)$ be the upward neighbourhood of Y, and $N^{\downarrow}(Y)$ the set of downward neighbours. An edge is called *upward* if it has the form $\{v, N^{\uparrow}(v)\}$ where $v \in V$, otherwise it is *downward*.

We will prove that $M_T(G)$ is a matroid. For a witness of (IM), we inductively construct a sequence of matchings $(m^\alpha : \alpha \geq 0)$, indexed by ordinals, of $I^\alpha := V(m^\alpha) \cap V$.

Given $m^{\beta-1}$, to define a matching for β, we consider the vertices in $V \setminus I^{\beta-1}$ that do not have unmatched children for the first time at step $\beta - 1$. We ensure that any such vertex v that is also in I is matched in step β, by exchanging v with a currently matched vertex r_v that is not in I.

When every vertex that has not been considered has an unmatched child, we stop the algorithm, at some step γ. We then prove that the union of all these unconsidered vertices and I^γ is a maximal independent superset of I.

Theorem 2.4.4. *For any tree G with an ordered bipartition (V, W), $M_T(G)$ is a transversal matroid.*

Proof. To prove that $M_T(G)$ is a matroid, it suffices to prove that (IM) holds. Let an independent set $I \subseteq X \subseteq V$ be given. Without loss of generality, we may assume that $X = V$.

We start by introducing some notations. Root G at some vertex in W. Given an ordinal α and a matching m^α, let $I^\alpha := V(m^\alpha) \cap V$ and $W^\alpha := V(m^\alpha) \cap W$. Given a sequence of matchings $(m^{\alpha'} : \alpha' \leq \alpha)$, let

$$C^\alpha := \{v \in V \setminus I^\alpha : N^\downarrow(v) \subseteq W^\alpha \text{ but } N^\downarrow(v) \not\subseteq W^{\alpha'} \ \forall \alpha' < \alpha\}.$$

Note that $C^\alpha \cap C^{\alpha'} = \emptyset$ for $\alpha' \neq \alpha$. For each $w \in W \setminus W^\alpha$, choose one vertex v_w in $N^\downarrow(w) \cap C^\alpha$ if it is not empty. Let

$$S^\alpha := \{v_w : w \in W \setminus W^\alpha \text{ and } N^\downarrow(w) \cap C^\alpha \neq \emptyset\}.$$

Denote the following statement by $A(\alpha)$:

> There is a pairwise disjoint collection $\mathcal{P}^\alpha := \{P_v : v \in I \cap C^\alpha \setminus S^\alpha\}$ of m^α–alternating paths such that each P_v starts from $v \in I \cap C^\alpha \setminus S^\alpha$ with a downward edge and ends at the first vertex r_v in $I^\alpha \setminus I$.

Start the inductive construction with m^0, which is the set of upward edges that is contained in every matching of I. It is not hard to see that $C^0 \cap I = \emptyset$, so that $A(0)$ holds trivially.

Let $\beta > 0$. Given the constructed sequence of matchings $(m^\alpha : \alpha < \beta)$, suppose that $A(\alpha)$ holds for each $\alpha < \beta$. Construct a matching m^β as follows.

If β is a successor ordinal, let

$$m^\beta := E(S^{\beta-1}, N^\uparrow(S^{\beta-1})) \cup (m^{\beta-1} \Delta E(\mathcal{P}^{\beta-1})).$$

By $A(\beta-1)$, the paths in $\mathcal{P}^{\beta-1}$ are disjoint. So $m^{\beta-1}\Delta E(\mathcal{P}^{\beta-1})$ is a matching. Using the definition of $S^{\beta-1}$, we see that m^β is indeed a matching. Observe also that

$$I^{\beta-1} \cap I \subseteq I^\beta \cap I; \tag{2.5}$$
$$W^{\beta-1} \subseteq W^{\beta-1} \cup N^\uparrow(S^{\beta-1}) = W^\beta. \tag{2.6}$$

If β is a limit ordinal, define m^β by

$$e \in m^\beta \iff \exists \beta' < \beta \text{ such that } e \in m^\alpha \ \forall \alpha \text{ with } \beta' \leq \alpha < \beta. \tag{2.7}$$

As m^α is a matching for every ordinal $\alpha < \beta$, we see that m^β is a matching in this case, too.

Suppose that a vertex $u \in (V \cap I) \cup W$ is matched to different vertices by m^α and $m^{\alpha'}$ for some $\alpha, \alpha' \leq \beta$. Then there exists some ordinal $\alpha'' + 1$ between α and α' such that u is matched by an upward $m^{\alpha''}$–edge and by a downward $m^{\alpha''+1}$–edge. Hence, the change of the matching edges is unique. This implies that for any α, α' with $\alpha \leq \alpha' \leq \beta$, by (2.5) and (2.6), we have

$$\begin{aligned} I^\alpha \cap I &\subseteq I^{\alpha'} \cap I; && (2.8) \\ W^\alpha &\subseteq W^{\alpha'}. && (2.9) \end{aligned}$$

Moreover, for an upward m^β–edge vw with $v \in V$, we have

$$v \in I^0 \text{ or } \exists \alpha < \beta \text{ such that } v \in C^\alpha \text{ and } w \notin W^\alpha. \tag{2.10}$$

We now prove that $A(\beta)$ holds. Given $v_0 = v \in I \cap C^\beta \setminus S^\beta$, we construct a decreasing sequence of ordinals starting from $\beta_0 := \beta$. For an integer $k \geq 0$, suppose that $v_k \in I \cap C^{\beta_k}$ with $\beta_k \leq \beta$ is given. By (2.8), $I^0 \subseteq I^{\beta_k}$, so $v_k \notin I^0$ and hence there exists $w_k \in N^\downarrow(v_k) \setminus W^0$.[11] Since $N^\downarrow(v_k) \subseteq W^{\beta_k} \subseteq W^\beta$, w_k is matched by m^β to some vertex v_{k+1}. In fact, as $w_k \notin W^0$, $v_{k+1} \notin I^0$. Let β_{k+1} be the ordinal with $v_{k+1} \in C^{\beta_{k+1}}$. Since $v_{k+1}w_k$ is an upward edge and $N^\downarrow(v_k) \subseteq W^{\beta_k}$, we have by (2.10) that $w_k \in W^{\beta_k} \setminus W^{\beta_{k+1}}$. By (2.9), $\beta_k > \beta_{k+1}$.

As there is no infinite decreasing sequence of ordinals, we have an m^β–alternating path $P_v = v_0 w_0 v_1 w_1 \cdots$ that stops at the first vertex $r_v \in V \setminus I$.

The disjointness of the P_v's follows from that every vertex has a unique upward neighbour and, as we just saw, that $\mathring{v}P_v$ cannot contain any vertex $v' \in C^\beta$. So $A(\beta)$ holds.

We can now go onwards with the construction.

Let $\gamma \leq |V|$[12] be the least ordinal such that $C^\gamma = \emptyset$. Let $C := \bigcup_{\alpha<\gamma} C^\beta$ and $U := V \setminus (I^0 \cup C)$; so V is partitioned into I^0, C and U. As $C^\gamma = \emptyset$, every vertex in U can be matched downwards to a vertex that is not in W^γ. These edges together with m^γ form a matching m^B of $B := U \cup I^\gamma$, which we claim to be a witness for (IM). By (2.8), $I^0 \cup (C \cap I) \subseteq I^\gamma$, hence, $I \subseteq B$.

Suppose B is not maximally independent for a contradiction. Then there is an m^B–alternating path $P = v_0 w_0 v_1 w_1 \cdots$ such that $v_0 \in V \setminus B$ that is either infinite or ends with some $w_n \in W \setminus V(m^B)$. We show that neither occurs.

Claim 2.4.5. *P is finite.*

[11] For a vertex $v \notin I$, $N^\downarrow(v) \setminus W^0$ may be empty.

[12] For example, fix a well ordering of V and map each β to the least element in C^β.

Proof. Suppose P is infinite. Since $v_0 \notin B$, P has a subray $R = w_iP$ such that w_iv_{i+1} is an upward m^B–edge. So $w_jv_{j+1} \in m^B$ for any $j \geq i$. As vertices in U are matched downwards, $R \cap U = \emptyset$. As $m^B \Delta E(R)$ is a matching of $B \supseteq I$ in which every vertex in $R \cap V$ is matched downwards, $R \cap I^0 = \emptyset$ too. So for any $j \geq i$, there exists a unique β_j such that $v_j \in C^{\beta_j}$.

Choose $k \geq i$ such that β_k is minimal. But with a similar argument used to prove $A(\beta)$, we have $\beta_k > \beta_{k+1}$. Hence P cannot be infinite. □

Claim 2.4.6. *P does not end in $W \setminus V(m^B)$.*

Proof. Suppose that P ends with $w_n \in W \setminus V(m^B)$. Certainly, v_n can be matched downwards (either to w_{n-1} or w_n) in a matching of $B \supseteq I$. Hence, $v_n \notin I^0$. It is easy to check that for $v \in C^\alpha$, $N(v) \subseteq W^{\alpha+1}$. Hence, as $w_n \in W \setminus W^\gamma$, $v_n \notin C$. Hence, $v_n \in U$. It follows that for each $0 < i \leq n$, v_i is matched downwards and so does not lie in I^0. As $v_0 \notin B$, $v_0 \in C$. It follows that $w_0 \in W^\gamma$ and $v_1 \in C$. Repeating the argument, we see that $v_n \in C$, which is a contradiction. □

We conclude that B is maximal. So (IM) holds and $M_T(G)$ is a matroid. □

Corollary 2.4.7. *Let (D, B_0) be a dimaze such that the underlying graph of D is a tree and B_0 is a vertex class of a bipartition of D with edges directed towards B_0. Then $M_L(D, B_0)$ is a matroid.*

Proof. By the theorem, we need only present $M_L(D, B_0)$ as a transversal matroid defined on a tree. Define a tree G with bipartition $((V \setminus B_0) \cup B_0', B_0)$, where B_0' is a copy of B_0, from D by ignoring the directions and joining each vertex in B_0 to its copy with an edge. It can be easily checked that $M_L(D, B_0) \cong M_T(G)$. □

Consider the countably infinite branching rooted tree, i.e. a rooted tree such that each vertex has countably many children. Let B_0 consist of the root and vertices on every other level. Define $\mathcal{T}$ by directing all edges towards B_0. Corollary 2.4.7 shows that $M_L(\mathcal{T}, B_0)$ is a matroid. Clearly, this matroid does not contain any finite circuit. Moreover, as any finite set C^* misses a base obtained by adding finitely many vertices to $B_0 \setminus C^*$, any cocircuit must be infinite. With Lemma 2.4.1, we conclude the following.

Corollary 2.4.8. *Every dimaze that defines a strict gammoid isomorphic to $M_L(\mathcal{T}, B_0)$ contains an alternating comb.*

2.5 Minor

The results in the following two sections come from [3].

The class of gammoids is closed under deletion by definition. In fact, finite gammoids are minor-closed. To see this, note that matroid deletion and contraction commute, so it suffices to show that a contraction minor M/X of a strict gammoid M is also a gammoid. Indeed, in [20] it was shown that finite strict gammoids are precisely the dual of finite transversal matroid. Moreover, they provided a construction to turn a dimaze to a bimaze presentation of the dual, and vice versa (essentially Definitions 2.6.1 and 2.6.2). Thus, we apply the construction to a presentation of M and get one of M^*. By deleting X, we get a presentation of the transversal matroid $M^* \setminus X$. Reversing the construction with any base of $M^* \setminus X$ gives us a dimaze presentation of $(M^* \setminus X)^* = M/X$.

In case of general gammoids, we can no longer appeal to duality, since, as we shall see, strict gammoids need not be cotransversal (Example 2.6.12) and the dual of transversal matroids need not be strict gammoids (Example 2.6.19). We will instead investigate the effect of the construction sketched above on a dimaze directly, similar to what has been done in [6] for the finite case. We are then able to show that the class of C^O-free gammoids, i.e. gammoids that admit a C^O-free presentation, is minor-closed. In combination with the linkage theorem, we can also prove that finite rank minors of gammoids are gammoids.

It remains open whether the class of gammoids is closed under taking minors.

Topological gammoids are introduced in [12] in response to a question raised by Diestel. The independent set systems are always finitary and define matroids. It turns out that such matroids are precisely the finitary gammoids. By investigating the structure of dimaze presentations of such gammoids, we then show that finitary strict gammoids, or equivalently, topological gammoids, are also closed under taking minors.

2.5.1 Matroid contraction and shifting along a linkage

Our aim is to show that a contraction minor M/S of a strict gammoid M is a strict gammoid. By Lemma 2.2.2, we may assume that S is independent. The first case is that S is a subset of the exits.

Lemma 2.5.1. *Let $M = M_L(D, B_0)$ be a strict gammoid and $S \subseteq B_0$. Then a dimaze presentation of M/S is given by $M_L(D - S, B_0 \setminus S)$.*

Proof. Since $S \subseteq B_0$ is independent, $I \in \mathcal{I}(M/S) \iff I \cup S \in \mathcal{I}(M)$.

Moreover,

$$\begin{aligned} I \in \mathcal{I}(M/S) &\iff I \cup S \text{ admits a linkage in } (D, B_0) \\ &\iff I \text{ admits a linkage } \mathcal{Q} \text{ with } \mathrm{Ter}(\mathcal{Q}) \cap S = \emptyset \text{ in } (D, B_0) \\ &\iff I \in \mathcal{I}(M_L(D - S, B_0 \setminus S). \end{aligned} \qquad \square$$

Thus, it suffices to give a dimaze presentation of M such that S is a subset of the exits. For this purpose we consider the process of "shifting along a linkage", which replaces the previously discussed detour via duality.

Throughout the section, (D, B_0) denotes a dimaze, $\mathcal{Q}$ a set of disjoint paths or rays, $S := \mathrm{Ini}(\mathcal{Q})$ and $T := \mathrm{Ter}(\mathcal{Q})$. Next, we define various maps which are dependent on $\mathcal{Q}$.

Define a bijection between $V \setminus T$ and $V \setminus S$ as follows: $\vec{\mathcal{Q}}(v) := v$ if $v \notin V(\mathcal{Q})$; otherwise $\vec{\mathcal{Q}}(v) := u$ where u is the unique vertex such that $(v, u) \in E(\mathcal{Q})$. The inverse is denoted by $\overleftarrow{\mathcal{Q}}$.

Construct the digraph $\vec{\mathcal{Q}}(D)$ from D by replacing each edge $(v, u) \in E(D) \setminus E(\mathcal{Q})$ with $(\vec{\mathcal{Q}}(v), u)$ and each edge $(v, u) \in \mathcal{Q}$ with (u, v). Set for the rest of this section

$$D_1 := \vec{\mathcal{Q}}(D) \text{ and } B_1 := (B_0 \setminus T) \cup S$$

and call (D_1, B_1) the $\mathcal{Q}$-*shifted* dimaze.

Given a $\mathcal{Q}$-alternating walk $W = w_0e_0w_1e_1w_2\ldots$ in D, let $\vec{\mathcal{Q}}(W)$ be obtained from W by deleting all e_i and each $w_i \in W$ such that $w_i \in V(\mathcal{Q})$ but $e_i \notin E(\mathcal{Q})$.

For a path or ray $P = v_0v_1v_2\ldots$ in D_1, let $\overleftarrow{\mathcal{Q}}(P)$ be obtained from P by inserting after each $v_i \in P \setminus \mathrm{Ter}(P)$ the following:

(v_i, v_{i+1}) if $v_i \notin V(\mathcal{Q})$;

(v_{i+1}, v_i) if $v_i \in V(\mathcal{Q})$ and $(v_{i+1}, v_i) \in E(\mathcal{Q})$;

$(w, v_i)w(w, v_{i+1})$ with $w := \overleftarrow{\mathcal{Q}}(v_i)$ if $v_i \in V(\mathcal{Q})$ but $(v_{i+1}, v_i) \notin E(\mathcal{Q})$.

We examine the relation between alternating walks in D and paths/rays in $\vec{\mathcal{Q}}(D)$.

Lemma 2.5.2. *(i) A $\mathcal{Q}$-alternating walk in D that is infinite or ends in $t \in B_1$ is respectively mapped by $\vec{\mathcal{Q}}$ to a ray or a path ending in t in D_1. Disjoint such walks are mapped to disjoint paths/rays.*

(ii) A ray or a path ending in $t \in B_1$ in D_1 is respectively mapped by $\overleftarrow{\mathcal{Q}}$ to an infinite $\mathcal{Q}$-alternating walk or a finite $\mathcal{Q}$-alternating walk ending in t in D. Disjoint such paths/rays are mapped to disjoint $\mathcal{Q}$-alternating walks.

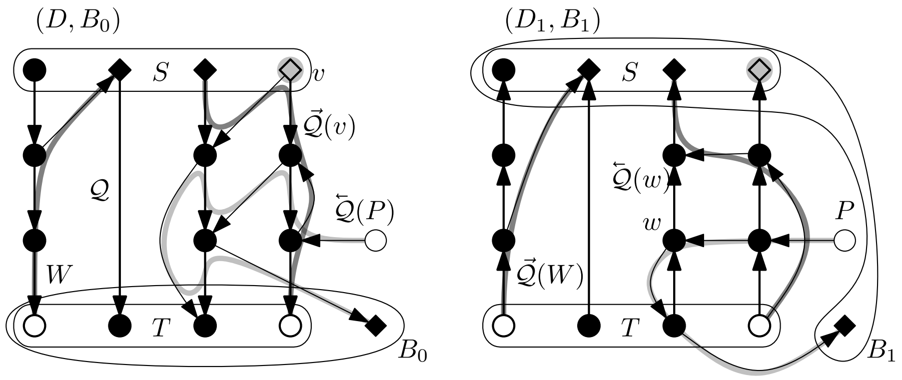

Figure 2.3: A $\mathcal{Q}$-shifted dimaze: $D_1 = \vec{\mathcal{Q}}(D)$, $B_1 = (B_0 \setminus T) \cup S$, where $\mathcal{Q}$ consists of the vertical downward paths. Outlined circles and diamonds are respectively initial and terminal vertices of $\mathcal{Q}$-alternating walks (left) and their $\vec{\mathcal{Q}}$-images (right).

Proof. We prove (i) since a proof of (ii) can be obtained by reversing the construction.

Let $W = w_0 e_0 w_1 e_1 w_2 \ldots$ be a $\mathcal{Q}$-alternating walk in D. If a vertex v in W is repeated, then v occurs twice and there is i such that $v = w_i$ with $e_{i-1} = (w_i, w_{i-1}) \in E(\mathcal{Q})$ and $e_i \notin E(\mathcal{Q})$. Hence, w_i is deleted in $P := \vec{\mathcal{Q}}(W)$ and so v does not occur more than once in P, that is, P consists of distinct vertices.

By construction, the last vertex of a finite W is not deleted, hence P ends in t. In case W is infinite, by (W3), no tail of W is deleted so that P remains infinite.

Next, we show that (v_i, v_{i+1}) is an edge in D_1. Let $w_j = v_i$ be the non-deleted instance of v_i. If w_{j+1} has been deleted, then the edge (w_{j+1}, w_{j+2}) (which exists since the last vertex cannot be deleted) in D has been replaced by the edge $(\vec{\mathcal{Q}}(w_{j+1}), w_{j+2}) = (v_i, v_{i+1})$ in D_1. If both w_j and $v_{i+1} = w_{j+1}$ are in $V(\mathcal{Q})$ then the edge $(w_{j+1}, w_j) \in E(\mathcal{Q})$ has been replaced by (v_i, v_{i+1}) in D_1. In the other cases $(w_j, w_{j+1}) = (v_i, v_{i+1})$ is an edge of D and remains one in D_1.

Let W_1, W_2 be disjoint $\mathcal{Q}$-alternating walks. By construction, $\vec{\mathcal{Q}}(W_1) \cap \vec{\mathcal{Q}}(W_2) \subseteq W_1 \cap W_2 \subseteq V(\mathcal{Q})$. By disjointness, at any intersecting vertex, one of W_1 and W_2 leaves with an edge not in $E(\mathcal{Q})$. Thus, such a vertex is deleted upon application of $\vec{\mathcal{Q}}$. Hence, $\vec{\mathcal{Q}}(W_1)$ and $\vec{\mathcal{Q}}(W_2)$ are disjoint paths/rays. $\square$

Note that for a path P in D_1 and a $\mathcal{Q}$-alternating walk W in D, we have

$$\vec{\mathcal{Q}}(\overleftarrow{\mathcal{Q}}(P)) = P; \qquad \overleftarrow{\mathcal{Q}}(\vec{\mathcal{Q}}(W)) = W.$$

This correspondence of sets of disjoint $\mathcal{Q}$-alternating walks in (D, B_0)

and sets of disjoint paths or rays in the $\mathcal{Q}$-shifted dimaze will be used in various situations in order to show that the independent sets associated with (D, B_0) and the $\mathcal{Q}$-shifted dimaze are the same.

Given a set $\mathcal{W}$ of $\mathcal{Q}$-alternating walks, define the graph $\mathcal{Q}\Delta\mathcal{W} := (V(\mathcal{Q}) \cup V(\mathcal{W}), E(\mathcal{Q})\Delta E(\mathcal{W}))$.

Lemma 2.5.3. *Let $J \subseteq V \setminus S$ and $\mathcal{W}$ a set of disjoint $\mathcal{Q}$-alternating walks, each of which starts from J and does not end outside of B_1. Then there is a set of disjoint rays or paths from $X := J \cup (S \setminus \mathrm{Ter}(\mathcal{W}))$ to $Y := T \cup (\mathrm{Ter}(\mathcal{W}) \cap B_0)$ in $\mathcal{Q}\Delta\mathcal{W}$.*

Proof. Every vertex in $\mathcal{Q}\Delta\mathcal{W} \setminus (X \cup Y)$ has in-degree and out-degree both 1 or both 0. Moreover, every vertex in X has in-degree 0 and out-degree 1 (or 0, if it is also in Y) and every vertex in Y has out-degree 0 and in-degree 1 (or 0, if it is also in X). Therefore every (weakly) connected component of $\mathcal{Q}\Delta\mathcal{W}$ meeting X is either a path ending in Y or a ray. □

The following will be used to complete a ray to an outgoing comb in various situations.

Lemma 2.5.4. *Suppose $\mathcal{Q}$ is a topological linkage. Any ray R that hits infinitely many vertices of $V(\mathcal{Q})$ is the spine of an outgoing comb.*

Proof. The first step is to inductively construct an infinite linkable subset of $V(R)$. Let $\mathcal{Q}_0 := \mathcal{Q}$ and $A_0 := \emptyset$. For $i \geq 0$, assume that $\mathcal{Q}_i$ is a topological linkage that intersects $V(R)$ infinitely but avoids the finite set of vertices A_i. Since it is not possible to separate a vertex on a topological path from B_0 by a finite set of vertices disjoint from that topological path, there exists a path P_i from $V(R) \cap V(\mathcal{Q}_i)$ to B_0 avoiding A_i. Let $A_{i+1} := A_i \cup V(P_i)$ and $\mathcal{Q}_{i+1}$ obtained from $\mathcal{Q}_i$ by deleting from each of its elements the minimal initial segment that intersects A_{i+1}. As $\mathcal{Q}_{i+1}$ remains a topological linkage that intersects $V(R)$ infinitely, we can continue the procedure. By construction $\{P_i : i \in \mathbb{N}\}$ is an infinite set of disjoint finite paths from a subset of $V(R)$ to B_0. Let $p_i \in P_i$ be the last vertex of R on P_i, then R is the spine of the outgoing comb: $R \cup \bigcup_{i \in \mathbb{N}} p_i P_i$. □

Corollary 2.5.5. *Any ray provided by Lemma 2.5.3 is in fact the spine of an outgoing comb if $\mathcal{Q}$ is a topological linkage, and the infinite forward segments of the walks in $\mathcal{W}$ are the spines of outgoing combs.*

Proof. Observe that a ray R constructed in Lemma 2.5.3 is obtained by alternately following the forward segments of the walks in $\mathcal{W}$ and the forward segments of elements in $\mathcal{Q}$.

Either a tail of R coincides with a tail of a walk in $\mathcal{W}$, and we are done by assumption; or R hits infinitely many vertices of $V(\mathcal{Q})$, and Lemma 2.5.4 applies. □

With Lemma 2.5.3 we can transform disjoint alternating walks into disjoint paths or rays. A reverse transform is described as follows.

Lemma 2.5.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of disjoint paths or rays. Let $\mathcal{W}$ be a set of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in distinct vertices of* $\mathrm{Ini}(\mathcal{P})$*. Then the walks in $\mathcal{W}$ are disjoint and can only end in* $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$*.*

Proof. Let $W = w_0 e_0 w_1 \dots$ be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk. Then W is a trivial walk if and only if $w_0 \in (\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$. If W is nontrivial then $e_0 \in E(\mathcal{Q})$ if and only if $w_0 \in V(\mathcal{Q})$.

Let W_1 and $W_2 \in \mathcal{W}$. Note that for any interior vertex w_i of a $\mathcal{P}$-$\mathcal{Q}$-alternating walk, it follows from the definition that either edge in $\{e_{i-1}, e_i\}$ determines uniquely the other. So if W_1 and W_2 share an edge, then a reduction to their common initial vertex shows that they are equal by their maximality. Moreover if the two walks share a vertex $v \notin V(\mathcal{Q})$, then they are equal since they share the edge of $\mathcal{P}$ whose terminal vertex is v.

Therefore, if $W_1 \neq W_2$ and they end at the same vertex v, then $v \in V(\mathcal{P}) \cap V(\mathcal{Q})$. More precisely, we may assume that v is the initial vertex of an edge in $E(\mathcal{Q}) \cap E(W_1)$ and the terminal vertex of an edge $e \in E(\mathcal{P}) \cap E(W_2)$ (both the last edges of their alternating walk). Since v is the initial vertex of some edge, it cannot be in B_0, so the path (or ray) in $\mathcal{P}$ containing e does not end at v. Hence we can extend W_1 contradicting its maximality.

Similarly we can extend a $\mathcal{P}$-$\mathcal{Q}$-alternating walk that ends in some vertex $v \in \mathrm{Ter}(\mathcal{P}) \cap \mathrm{Ter}(\mathcal{Q})$ by the edge in $E(\mathcal{Q})$ that has v as its terminal vertex, unless $v \in \mathrm{Ini}(\mathcal{Q})$. So $\mathcal{W}$ is a set of disjoint $\mathcal{P}$-$\mathcal{Q}$-alternating walks that can only end in $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$. $\square$

Now we investigate when a dimaze and its $\mathcal{Q}$-shifted dimaze present the same strict gammoid.

Lemma 2.5.7. *Suppose that $\mathcal{Q}$ is a linkage from S onto T and I a set linkable in (D_1, B_1). Then I is linkable in (D, B_0) if (i) $I \setminus S$ is finite or (ii) (D, B_0) is C^O-free.*

Proof. There is a set of disjoint finite paths from I to B_1 in (D_1, B_1), which, by Lemma 2.5.2, gives rise to a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1 in (D, B_0). Let $\mathcal{W}$ be the subset of those walks starting in $J := I \setminus S$. Then Lemma 2.5.3 provides a set $\mathcal{P}$ of disjoint paths or rays from $J \cup (S \setminus \mathrm{Ter}(\mathcal{W})) \supseteq I$ to $Y \subseteq B_0$. It remains to argue that $\mathcal{P}$ does not contain any ray. Indeed, any such ray meets infinitely many paths in $\mathcal{Q}$. But by Lemma 2.5.4, the ray is the spine of an outgoing comb, which is a contradiction. $\square$

In fact the converse of (ii) holds.

Lemma 2.5.8. *Suppose that (D, B_0) is C^O-free, and $\mathcal{Q}$ is a linkage from S onto T such that there exists no linkage from S to a proper subset of T. Then a linkable set I in (D, B_0) is also linkable in (D_1, B_1), and (D_1, B_1) is C^O-free.*

Proof. For the linkability of I it suffices by Lemma 2.5.2 to construct a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1. Let $\mathcal{P}$ be a linkage of I in (D, B_0).

For each vertex $v \in I$ let W_v be the maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting in v. By Lemma 2.5.6, $\mathcal{W} := \{W_v : v \in I\}$ is a set of disjoint $\mathcal{Q}$-alternating walks that can only end in $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S \subseteq B_1$.

If there is an infinite alternating walk $W = W_{v_0}$ in $\mathcal{W}$, then Lemma 2.5.3 applied on just this walk gives us a set $\mathcal{R}$ of disjoint paths or rays from $S+v_0$ to T. Since the forward segments of W are subsegments of paths in $\mathcal{P}$, by Corollary 2.5.5 any ray in $\mathcal{R}$ would extend to a forbidden outgoing comb. Thus, $\mathcal{R}$ is a linkage of $S + v_0$ to T. In particular, S is linked to a proper subset of T contradicting the minimality of T. Hence $\mathcal{W}$ consists of finite disjoint $\mathcal{Q}$-alternating walks, as desired.

For the second statement suppose that (D_1, B_1) contains an outgoing comb whose spine R starts at $v_0 \notin S$. Then $W := \overleftarrow{\mathcal{Q}}(R)$ is a $\mathcal{Q}$-alternating walk in (D, B_0) by Lemma 2.5.2. Any infinite forward segment R' of W contains an infinite subset linkable to B_1 in (D_1, B_1). By Lemma 2.5.7(ii) this subset is also linkable in (D, B_0), so R' is the spine of an outgoing comb by Lemma 2.5.4, which is a contradiction.

On the other hand, suppose that W does not have an infinite forward tail. By investigating W as we did with W_{v_0} above, we arrive at a contradiction. Hence, there does not exist any outgoing comb in (D_1, B_1). $\square$

For later applications, we note the following refinement.

Corollary 2.5.9. *If (D, B_0) is F^∞-free as well, then so is (D_1, B_1).*

Proof. Suppose that (D_1, B_1) contains a subdivision of F^∞ with centre v_0. Then an infinite subset X of the out-neighbourhood of v_0 in (D_1, B_1) is linkable. By Lemma 2.5.7(ii), X is also linkable in (D, B_0). As X is a subset of the out-neighbourhood of $\overleftarrow{\mathcal{Q}}(v_0)$, a forbidden linking fan in (D, B_0) results. $\square$

Proposition 2.5.10. *Suppose (D, B_0) is C^O-free and $\mathcal{Q}$ is a linkage from S onto T such that S cannot be linked to a proper subset of T. Then $M_L(D_1, B_1) = M_L(D, B_0)$.*

Proof. By Lemma 2.5.7(ii) and Lemma 2.5.8, a set $I \subseteq V$ is linkable in (D, B_0) if and only if it is linkable in (D_1, B_1). $\square$

We remark that in order to show that $M_L(D, B_0) = M_L(D_1, B_1)$, the assumption in Proposition 2.5.10 that (D, B_0) is C^O-free can be slightly relaxed. Only outgoing combs constructed in the proofs of Lemma 2.5.7(ii) and Lemma 2.5.8 which have the form that all the spikes are terminal segments of paths in the linkage $\mathcal{Q}$ need to be forbidden.

Theorem 2.5.11. *The class of C^O-free gammoids is minor-closed.*

Proof. Let $N := M_L(D, B_0)$ be a strict gammoid. It suffices to show that any minor of N is a gammoid. By Lemma 2.2.2, such has the form $M := N/S \setminus R$ for some independent set S and coindependent set R. First extend S in B_0 to a base B_1. This gives us a linkage $\mathcal{Q}$ from S onto $T := B_0 \setminus B_1$ such that there exists no linkage from S to a proper subset of T.

Assume that (D, B_0) is C^O-free. Then by Lemma 2.5.8, (D_1, B_1) is C^O-free, and by Proposition 2.5.10, $M_L(D, B_0) = M_L(D_1, B_1)$. Since $S \subseteq B_1$, $M = M_L(D_1, B_1)/S \setminus R = M_L(D_1 - S, B_1 \setminus S) \setminus R$ is a C^O-free gammoid. $\square$

A partial converse of Lemma 2.5.7(i) can be proved by analyzing the proof of the linkage theorem.

Lemma 2.5.12. *Let $M = M_L(D, B_0)$ be a strict gammoid, $\mathcal{Q}$ a linkage from S onto T such that B_1 is a base and $I \subseteq V \setminus S$ such that $S \cup I$ is linkable in (D, B_0). If I is finite, then it is linkable in $(D_1 - S, B_1 \setminus S)$.*

Proof. By Lemma 2.5.2 it suffices to construct a set of disjoint finite $\mathcal{Q}$-alternating walks from I to $B_0 \setminus T$.

Let $\mathcal{P}$ be a linkage of $S \cup I$ in (D, B_0). We apply the linkage theorem of Pym [28] to get a linkage $\mathcal{Q}^\infty$ from $S \cup I$ onto some set $Y^\infty \supseteq T$ in the following way:

For each $x \in S \cup I$, let P_x be the path in $\mathcal{P}$ containing x and $f_x^0 := x$. Let $\mathcal{Q}^0 := \mathcal{Q}$. For each $i > 0$ and each $x \in S \cup I$, let f_x^i be the last vertex v on $f_x^{i-1} P_x$ such that $(f_x^{i-1} P_x \mathring{v}) \cap V(\mathcal{Q}^{i-1}) = \emptyset$. For $y \in T$, let Q_y be the path in $\mathcal{Q}$ containing y and t_y^i be the first vertex $v \in Q_y$ such that the terminal segment $\mathring{v} Q_y$ does not contain any f_x^i. Define the linkage $\mathcal{Q}^i := \mathcal{B}^i \cup \mathcal{C}^i$ with

$$\begin{aligned}
\mathcal{B}^i &:= \{P_x f_x^i Q_y : x \in S \cup I, y \in T \text{ and } f_x^i = t_y^i\}, \\
\mathcal{C}^i &:= \{P_x \in \mathcal{P} : f_x^i \in B_0 \setminus T\}.
\end{aligned}$$

There exist integers $i_x, i_y \geq 0$ such that $f_x^{i_x} = f_x^k$, $t_y^{i_y} = t_y^l$ for all integers $k \geq i_x$ and $l \geq i_y$. Define $f_x^\infty := f_x^{i_x}, t_y^\infty := t_y^{i_y}$ and

$$\begin{aligned}
\mathcal{B}^\infty &:= \{P_x f_x^\infty Q_y : x \in S \cup I, y \in T \text{ and } f_x^\infty = t_y^\infty\}, \\
\mathcal{C}^\infty &:= \{P_x \in \mathcal{P} : f_x^\infty \in B_0 \setminus T\}.
\end{aligned}$$

Then $\mathcal{Q}^\infty := \mathcal{B}^\infty \cup \mathcal{C}^\infty$ is the linkage given by the linkage theorem.

Let $Y := Y^\infty \setminus T$ and B_2 an extension to a base of the independent set $(B_0 \setminus Y^\infty) \cup (S \cup I)$ inside B_1. Then $B_2 \setminus B_1 = I$ and $B_1 \setminus B_2 \subseteq Y$ and so, by [10, Lemma 3.7], $|I| = |B_2 \setminus B_1| = |B_1 \setminus B_2| \leq |Y|$.

Let $v \in V(D)$ be a vertex with the property that $v = f^{j+1}_{x_{j+1}}$ for some integer j and a vertex $x_{j+1} \in S \cup I$ such that $f^j_{x_{j+1}} \neq f^{j+1}_{x_{j+1}}$. We backward inductively construct a walk $W(v)$ that starts from I and ends in v as follows:

Given x_{i+1} for a positive integer $i \leq j$, let Q_i be the path in $\mathcal{Q}$ containing $f^i_{x_{i+1}}$ (if there is no such path, then $f^i_{x_{i+1}} \in I$ and $i = 0$). Since $f^i_{x_{i+1}} \neq f^{i+1}_{x_{i+1}}$, it follows that $\mathcal{F}^i \cap \mathring{f}^i_{x_{i+1}} Q_i \neq \emptyset$, where $\mathcal{F}^i := \{f^i_x : x \in S \cup I\}$. Let x_i be such that $f^i_{x_i}$ is the first vertex of $\mathcal{F}^i$ on $\mathring{f}^i_{x_{i+1}} Q_i$. Moreover, since $f^i_{x_{i+1}} \in Q_i$, $\mathcal{F}^{i-1} \cap \mathring{f}^i_{x_{i+1}} Q_i = \emptyset$, so $f^{i-1}_{x_i} \neq f^i_{x_i}$. Hence we can complete the construction down to $i = 1$ and define:

$$W(v) := f^0_{x_1} P_1 f^1_{x_1} \cup \bigcup_{0<i<j} f^i_{x_{i+1}} Q_i f^i_{x_i} \cup f^i_{x_{i+1}} P_{i+1} f^{i+1}_{x_{i+1}}. \tag{2.11}$$

Note that $f^0_{x_1} \neq f^1_{x_1}$ and for any $x \in S$, the definition of f^1_x implies $f^0_x = f^1_x$. Hence, $f^0_{x_1}$, the initial vertex of $W(v)$, is in $(S \cup I) \setminus S = I$. Now we examine the interaction between two such walks:

Claim. *Let $x, x' \in S \cup I$ be given such that $f^{j+1}_x \neq f^j_x$ and $f^{j'+1}_{x'} \neq f^{j'}_{x'}$.*

(i) If $j = j'$ and $f^{j+1}_x \neq f^{j'+1}_{x'}$, then $\mathrm{Ini}(W(f^{j+1}_x)) \neq \mathrm{Ini}(W(f^{j'+1}_{x'}))$.

(ii) If $W(f^{j+1}_x)$ and $W(f^{j'+1}_{x'})$ start at the same vertex in I, then one is a subwalk of the other.

Proof. For (i) we first note that f^{j+1}_x and $f^{j'+1}_{x'}$ are on distinct paths in $\mathcal{P}$ and apply induction on j. If $j = j' = 0$, then $\mathrm{Ini}(W(f^{j+1}_x)) = x \neq x' = \mathrm{Ini}(W(f^{j'+1}_{x'}))$. For $j > 0$ the walk $W(f^{j'+1}_{x'})$ has the form $W(f^j_{x'_j}) \cup f^j_{x'_{j+1}} Q'_j f^j_{x'_j} \cup f^j_{x'_{j+1}} P'_{j+1} f^{j+1}_{x'_{j+1}}$ and analogue $W(f^{j+1}_x)$. The vertices $f^j_{x'_{j+1}}$ and $f^j_{x_{j+1}}$ are on distinct paths in $\mathcal{P}$ and therefore distinct. Then it follows from the definition that $f^j_{x_j} \neq f^j_{x'_j}$ and we use the induction hypothesis to see that $\mathrm{Ini}(W(f^j_{x'_j})) \neq \mathrm{Ini}(W(f^j_{x_j}))$ and hence $\mathrm{Ini}(W(f^{j+1}_{x'_{j+1}})) \neq \mathrm{Ini}(W(f^{j+1}_{x_{j+1}}))$, as desired.

For (ii) suppose that $f^{j+1}_x \neq f^{j'+1}_{x'}$, then (i) implies $j \neq j'$, say $j < j'$. If $f^{j+1}_{x'_{j+1}} \neq f^{j+1}_{x_{j+1}}$, then ,by (i), $\mathrm{Ini}(W(f^{j+1}_x)) \neq \mathrm{Ini}(W(f^{j+1}_{x'_{j+1}})) = \mathrm{Ini}(W(f^{j'+1}_{x'}))$. Hence $W(f^{j+1}_x)$ is a subwalk of $W(f^{j'+1}_{x'})$. □

Each vertex $y \in Y \setminus I$ is on a non-trivial path in $\mathcal{Q}^\infty$, so there exists a least integer $i_y > 0$ such that $y = f^{i_y}_{x_{i_y}}$ for some $x_{i_y} \in S \cup I$. For $y \in Y \cap I$ let $W(y)$ be the trivial walk at y, so that we can define $\mathcal{W} := \{W(y) : y \in Y\}$.

Suppose y and y' are distinct vertices in $Y \setminus I$ such that $\mathrm{Ini}(W(y)) = \mathrm{Ini}(W(y'))$. Since there is no edge of $\mathcal{Q}$ ending in either of these vertices, (ii) implies that $W(y) = W(y')$ and therefore $y = y'$. Since the initial vertex of a non-trivial walk in $\mathcal{W}$ is not in B_0, we have $\mathrm{Ini}(W(y)) \neq \mathrm{Ini}(W(y'))$ for any two distinct vertices y, y' in Y. That means $\mathrm{Ini}(\mathcal{W}) = I$, since $|I| \leq |Y|$.

By Lemma 2.5.6, the maximal $\mathcal{Q}^\infty$-$\mathcal{Q}$-alternating walks starting in I are disjoint. Thus, to complete the proof, it remains to check that each $\mathcal{Q}^\infty$-$\mathcal{Q}$-alternating walk starting in I is finite. To that end, let e be an edge of such a walk. As $E(\mathcal{W})$ is finite, it suffices to show that $e \in E(W)$ for some $W \in \mathcal{W}$. By definition, $e \in E(\mathcal{Q}^\infty)\Delta E(\mathcal{Q})$. The following case analysis completes the proof.

1. $e \in E(\mathcal{Q}^\infty) \setminus E(\mathcal{Q})$: e is on some initial segment $P_x f_x^\infty$ of a path P_x in $\mathcal{P}$. More precisely, there is an integer i, such that $e \in f_x^i P_x f_x^{i+1}$. By construction $e \in W(f_x^{i+1})$ and $\mathrm{Ini}(W(f_x^{i+1})) \in I$. Let W be the walk in $\mathcal{W}$ whose initial vertex is $\mathrm{Ini}(W(f_x^{i+1}))$, then (ii) implies that e is on W.

2. $e \in E(\mathcal{Q}) \setminus E(\mathcal{Q}^\infty)$: e is on some initial segment $Q f_x^\infty$ of a path Q in $\mathcal{Q}$. More precisely, there is an integer i and $x, x' \in S \cup I$, such that $e \in f_x^i Q f_{x'}^i$. Since $f_x^i \neq f_x^{i+1}$, similar to the previous case, there is a walk in $\mathcal{W}$ containing e. $\square$

An immediate corollary of the following is that any forbidden minor, of which there are infinitely many ([19]), for the class of finite gammoids is also a forbidden minor for infinite gammoids.

Theorem 2.5.13. *Any finite-rank minor of a gammoid is also a gammoid.*

Proof. The setting follows the first paragraph of the proof of Theorem 2.5.11. Suppose that M has finite rank r. Since R is coindependent, $V \setminus R$ is spanning in N. Therefore, N/S also has rank r. Let $I \in M_L(D_1 - S, B_1 \setminus S)$, then $r = |B_0 \setminus T| = |B_1 \setminus S| \geq |I|$ and, by Lemma 2.5.7(i), I is in $\mathcal{I}(N/S)$. Conversely, if $I \in \mathcal{I}(N/S)$, then I is finite. By Lemma 2.5.12, I is linkable in $(D_1 - S, B_1 \setminus S)$. Hence $M_L(D_1 - S, B_1 \setminus S)$ is a strict gammoid presentation of N/S and $M = M_L(D_1 - S, B_1 \setminus S) \setminus R$ is a gammoid. $\square$

2.5.2 Topological gammoids

A topological notion of linkability is introduced in [12]. Roughly speaking, a topological path from a vertex v does not need to reach the exits as long as no finite vertex set avoiding that path can prevent an actual connection of v to B_0.

Here we show that in fact, topological gammoids coincide with the finitary gammoids. As a corollary, we see that topological gammoids are minor-closed.

The difference between a topological linkage and a linkage is that paths ending in the centre of a linking fan and spines of outgoing combs are allowed. Thus, to prove the following, it suffices to give a $\{C^O, F^\infty\}$-free dimaze presentation for the strict topological gammoid.

Lemma 2.5.14. *Every strict topological gammoid is a strict gammoid.*

Proof. Let (D', B_0') be a dimaze and F be the set of all vertices that are the centre of a subdivision of F^∞. Let (D, B_0) be obtained from (D', B_0') by deleting all edges whose initial vertex is in F from D' and $B_0 := B_0' \cup F$.

We claim that $M_{TL}(D, B_0) = M_{TL}(D', B_0')$. Let $\mathcal{P}$ be a topological linkage of I in (D', B_0'). Then the collection of the initial segments of each element of $\mathcal{P}$ up to the first appearance of a vertex in F forms a topological linkage of I in (D, B_0). Conversely, let $\mathcal{P}$ be a topological linkage of I in (D, B_0). Note that any linkage in (D, B_0) is a topological linkage in (D', B_0'). In particular the spikes of an outgoing comb whose spine R is in $\mathcal{P}$ form a topological linkage. Hence, R is also the spine of an outgoing comb in (D', B_0') by Lemma 2.5.4. So I is topologically linkable in (D, B_0).

Let $S \cup B_0$ be a base of $M_{TL}(D, B_0)$ and $\mathcal{Q}$ a set of disjoint spines of outgoing combs starting from S. We show that a set I is topologically linkable in (D, B_0) if and only if it is linkable in the $\mathcal{Q}$-shifted dimaze (D_1, B_1).

Let $\mathcal{P}$ be a topological linkage of I in (D, B_0). By Lemma 2.5.6, the set $\mathcal{W}$ of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in I is a set of disjoint $\mathcal{Q}$-alternating walks possibly ending in $\mathrm{Ter}(\mathcal{P}) \cup S \subseteq B_1$. If there were an infinite walk, then it would have to start outside S and give rise to a topologically linkable superset of $S \cup B_0$, by Lemma 2.5.3 and Lemma 2.5.4. So each walk in $\mathcal{W}$ is finite. By Lemma 2.5.2, I is linkable in (D_1, B_1).

Conversely let I be linkable in (D_1, B_1) and $\mathcal{W}$ a set of disjoint finite $\mathcal{Q}$-alternating walks in (D, B_0) from I to B_1 provided by Lemma 2.5.2. By Lemma 2.5.3, $\mathcal{Q} \Delta \mathcal{W}$ contains a set $\mathcal{R}$ of disjoint paths or rays in (D, B_0) from I to B_0. By Corollary 2.5.5, any ray in $\mathcal{R}$ is in fact the spine of an outgoing comb, so I is topologically linkable in (D, B_0). $\square$

Now we can characterize strict topological gammoids among strict gammoids.

Theorem 2.5.15. *The following are equivalent:*

1. *M is a strict topological gammoid;*
2. *M is a finitary strict gammoid;*
3. *M is a strict gammoid such that any presentation is $\{C^O, F^\infty\}$-free;*
4. *M is a $\{C^O, F^\infty\}$-free strict gammoid.*

Proof. 1. $\Rightarrow$ 2. : By Corollary 2.2.5, M is a finitary matroid and by Lemma 2.5.14 it is a strict gammoid.

2. $\Rightarrow$ 3. : Let $M_L(D, B_0)$ be any presentation of M. Note that the union of any vertex $v \in V \setminus B_0$ and all the vertices in B_0 to which v is linkable forms a circuit in M (the fundamental circuit of v and B_0). Suppose (D, B_0) is not $\{C^O, F^\infty\}$-free, then there is a vertex linkable to infinitely many vertices in B_0. But then M contains an infinite circuit and is not finitary.

3. $\Rightarrow$ 4. : Trivial.

4. $\Rightarrow$ 1. : Take a $\{C^O, F^\infty\}$-free presentation of M. Then topological linkages coincide with linkages. Hence M is a topological gammoid. □

Next we also characterize topological gammoids among gammoids.

Corollary 2.5.16. *The following are equivalent:*

1. *M is a topological gammoid;*
2. *M is a finitary gammoid;*
3. *M is a $\{C^O, F^\infty\}$-free gammoid.*

Proof. 1. $\Rightarrow$ 3. : There exist a dimaze (D, B_0) and $X \subseteq V$ such that $M = M_{TL}(D, B_0) \setminus X$. By Theorem 2.5.15, there is a $\{C^O, F^\infty\}$-free dimaze (D_1, B_1) such that $M_L(D_1, B_1) = M_{TL}(D, B_0)$. Hence, M is a $\{C^O, F^\infty\}$-free gammoid.

3. $\Rightarrow$ 2. : There exists a $\{C^O, F^\infty\}$-free presentation of a strict gammoid N of which M is a restriction. By Theorem 2.5.15, N is finitary, thus, so is M.

2. $\Rightarrow$ 1. : There exist (D, B_0) and $X \subseteq V$ such that $M = M_L(D, B_0) \setminus X$. Since $M \setminus X$ is finitary, $\mathcal{C}(M \setminus X) = \mathcal{C}(M^{\text{fin}} \setminus X)$. By Corollary 2.2.5, the latter is equal to $\mathcal{C}(M_{TL}(D, B_0) \setminus X)$. Hence, M is a topological gammoid. □

Theorem 2.5.17. *The class of finitary gammoids (or equivalently topological gammoids) is closed under taking minors.*

Proof. Let M be a finitary gammoid. By Corollary 2.5.16, M is a $\{C^O, F^\infty\}$-free gammoid. Any minor of M is a C^O-free gammoid by Theorem 2.5.11, and also F^∞-free by Corollary 2.5.9. So any minor of M is a finitary gammoid by Corollary 2.5.16. □

2.6 Duality

A standard result [20] states that the dual of a finite gammoid is a gammoid. It can be proved by observing that the dual of any finite strict gammoid is a transversal matroid and finite gammoids are closed under contraction minors. The proof remains valid for the those infinite gammoids that admit a presentation (D, B_0) with the underlying graph of D rayless. However, the proof breaks down when rays are allowed. For example, we shall see that C^I defines a strict gammoid whose dual is not a transversal matroid, but the dual is still a gammoid. The last assertion follows from the fact whose proof is omitted that the dual of a R^A-free strict gammoid is a R^A-free gammoid. However, a more badly behaved example exists: there is a strict gammoid which is not dual to any gammoid.

We first describe the dual of C^A-free strict gammoids. Counterexamples to duality, Examples 2.6.14, 2.6.19 and 2.6.23, are proved in detail.

2.6.1 Strict gammoids and path-transversal matroids

The class of path-transversal matroids is introduced as a superclass of transversal matroids, and proved to contain the dual matroids of of any C^A-free strict gammoid. We shall see that an extra condition forces C^A-free strict gammoids to be dual to transversal matroids. On the other hand, even though path-transversal matroids extend transversal matroids, they do not capture the dual of all strict gammoids, as we shall see in Example 2.6.14.

Let us introduce a dual object of a dimaze. Given a bipartite graph $G = (V, W)$, we call a matching m_0 onto W an *identity matching*, and the pair (G, m_0) a *bimaze*[13]. We adjust two constructions of [20] for our purposes.

Definition 2.6.1. Given a dimaze (D, B_0), define a bipartite graph $D^\star_{B_0}$, with bipartition $(V, (V \setminus B_0)^\star)$, where $(V \setminus B_0)^\star := \{v^\star : v \in V \setminus B_0\}$ is disjoint from V; and $E(D^\star_{B_0}) := m_0 \cup \{vu^\star : (u, v) \in E(D)\}$, where $m_0 := \{vv^\star : v \in V \setminus B_0\}$. Call $(D, B_0)^\star := (D^\star_{B_0}, m_0)$ the *converted bimaze* of (D, B_0).

Starting from a dimaze (D, B_0), we write $(V \setminus B_0)^\star$, m_0 and $v^\star$ for the corresponding objects in Definition 2.6.1.

Definition 2.6.2. Given a bimaze (G, m_0), where $G = (V, W)$, define a digraph $G^\star_{m_0}$ such that $V(G^\star_{m_0}) := V$ and $E(G^\star_{m_0}) := \{(v, w) : wv^\star \in E(G) \setminus m_0\}$, where $v^\star$ is the vertex in W that is matched by m_0 to $v \in V$. Let $B_0 := V \setminus V(m_0)$. Call $(G, m_0)^\star := (G^\star_{m_0}, B_0)$ the *converted dimaze* of (G, m_0).

Starting from a bimaze (G, m_0), we write B_0 and $v^\star$ for the corresponding objects in Definition 2.6.2 and $(V \setminus B_0)^\star$ for the right vertex class of G.

Note that these constructions are inverse to each other (see Figure 2.4). In particular, let (G, m_0) be a bimaze, then

$$(G, m_0)^{\star\star} = (G, m_0). \tag{2.12}$$

[13]Short for *bi*partite *maze*.

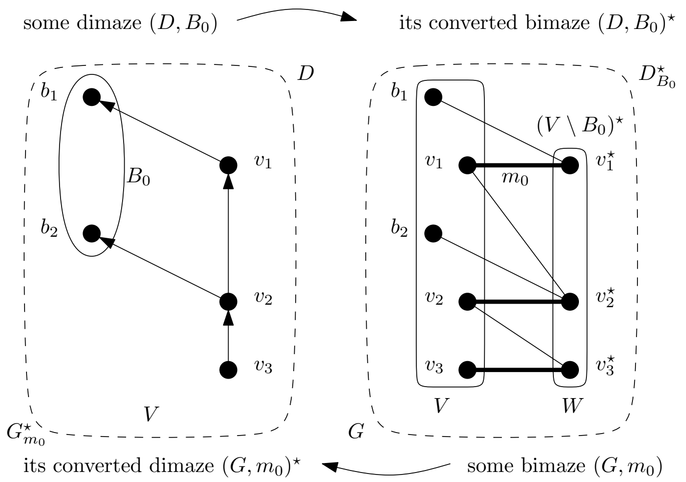

Figure 2.4: Converting a dimaze to a bimaze and vice versa

With the aim of describing the dual of C^A-free strict gammoids, we extend the class of transversal matroids as follows:

Given a bimaze (G, m_0), note that for any matching m, each infinite component of $G[m_0 \cup m]$ is either a ray or a double ray. We say m is an *m_0-matching*, if $G[m_0 \cup m]$ has no infinite component. A set $I \subseteq V$ is *m_0-matchable*, if there is an m_0-matching of I.

Definition 2.6.3. Given a bimaze (G, m_0), the pair of V and the set of all m_0-matchable subsets of V is denoted by $M_{PT}(G, m_0)$. If $M_{PT}(G, m_0)$ is a matroid, it is called a *path-transversal matroid*.

The correspondence between finite paths and m_0-matchings is depicted in the following lemma.

Lemma 2.6.4. *Let (D, B_0) be a dimaze. Then B is linkable onto B_0 in (D, B_0) if and only if $V \setminus B$ is m_0-matchable onto $(V \setminus B_0)^\star$ in $(D, B_0)^\star$.*

Proof. Suppose a linkage $\mathcal{P}$ from B onto B_0 is given. Let

$$m := \{vu^\star : (u, v) \in E(\mathcal{P})\} \cup \{ww^\star : w \notin V(\mathcal{P})\}.$$

Note that m is a matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ in $D^\star_{B_0}$. Any component induced by $m_0 \cup m$ is finite, since any component which contains more than one edge corresponds to a path in $\mathcal{P}$. So m is a required m_0-matching in $(D, B_0)^\star$.

Conversely let m be an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. Define a linkage from B onto B_0 as follows. From every vertex $v \in B$, start an m_0-m alternating walk, which is finite because m is anm_0-matching. Moreover, the walk cannot end with an m_0-edge because m covers $(V \setminus B_0)^\star$. So the walk is either trivial or ends with an m-edge in B_0. As the m-edges on each walk correspond to a path from B to B_0, together they give us a required linkage in (D, B_0). $\square$

Proposition 2.6.5. *Let $M_T(G)$ be a transversal matroid and m_0 a matching of a base B. Then $M_T(G) = M_{PT}(G, m_0)$.*

Proof. Suppose $I \subseteq V$ admits a matching m. By the maximality of B, any infinite component of $m \cup m_0$ does intersect $V \setminus B$. Replacing the m-edges of all the infinite components by the m_0-edges gives an m_0-matching of I. $\square$

In fact, we will see that the class of path-transversal matroids contains the class of transversal matroids as a proper subclass; combine Example 2.6.12 and Theorem 2.6.9. Just as we can extend a linkage to cover the exits by trivial paths, any m_0-matching can be extended to cover W.

Lemma 2.6.6. *Let (G, m_0) be a bimaze. For any m_0-matchable I, there is an m_0-matching from some $B \supseteq I$ onto W.*

Proof. Let m be an m_0-matching of I. Take the union of all connected components of $m \cup m_0$ that meet $W - m$. The symmetric difference of m and this union is a desired m_0-matching of a superset of I. $\square$

We find it convenient to abstract two properties of a dimaze and a bimaze. Recall that in Section 2.3.2 we defined for a dimaze (D, B_0) the property (†) to be the following:

$$I \in M_L(D, B_0) \text{ is maximal } \Leftrightarrow \exists \text{ linkage from } I \text{ onto } B_0. \tag{†}$$

Analogously, given a bimaze (G, m_0), let (‡) be

$$I \in M_{PT}(G, m_0) \text{ is maximal } \Leftrightarrow \exists\ m_0\text{-matching from } I \text{ onto } (V \setminus B_0)^\star. \tag{‡}$$

In some sense (†) and (‡) are dual to each other.

Lemma 2.6.7. *A dimaze (D, B_0) satisfies (†) if and only if $(D, B_0)^\star$ satisfies (‡).*

Proof. Assume (D, B_0) satisfies (†). To prove the backward direction of (‡), suppose there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. By Lemma 2.6.4, there is a linkage from B onto B_0. Therefore, B is maximal in $M_L(D, B_0)$ by (†). By Lemma 2.6.6, any m_0-matchable superset of $V \setminus B$ may be extended to one, say $V \setminus I$, that is m_0-matchable onto $(V \setminus B_0)^\star$. As before, $I \subseteq B$ is maximal in $M_L(D, B_0)$, so $I = B$ and hence, $V \setminus B$ is a maximal m_0-matchable set. To see the forward direction of (‡), suppose $V \setminus B$ is a maximal m_0-matchable set witnessed by an m_0-matching m, that does not cover $v^\star \in (V \setminus B_0)^\star$. As m is an m_0-matching, a maximal m_0-m alternating walk starting from $v^\star$ ends at some vertex in B. So the symmetric difference of this walk and m is an m_0-matching of a proper superset of $V \setminus B$ which is a contradiction.

Assume $(D, B_0)^\star$ satisfies (‡). The forward direction of (†) is trivial. For the backward direction, suppose there is a linkage from B onto B_0. Then there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ by Lemma 2.6.4. By (‡), $V \setminus B$ is maximal in $M_{PT}(D, B_0)^\star$. With an argument similar to the above, we can conclude that B is maximal in $M_L(D, B_0)$. □

Now let us see how (†) helps to identify the dual of a strict gammoid.

Lemma 2.6.8. *If a dimaze (D, B_0) satisfies (†), then the dual of $M_L(D, B_0)$ is $M_{PT}(D, B_0)^\star$.*

Proof. By Lemma 2.6.7, $(D, B_0)^\star$ satisfies (‡). Let B be an independent set in $M_L(D, B_0)$. Then B is maximal if and only if there is a linkage from B onto B_0. By Lemma 2.6.4, this holds if and only if there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$, which by (‡) is equivalent to $V \setminus B$ being maximal in $M_{PT}(D, B_0)^\star$.

To complete the proof, it remains to see that every m_0-matchable set can be extended to a maximal one, which follows from Lemma 2.6.6 and (‡). □

Note that while we do not need it, the twin of Lemma 2.6.8 is true, namely, if a bimaze (G, m_0) satisfies (‡), then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

To summarize, the dual of a C^A-free strict gammoid is given as follows.

Theorem 2.6.9. *(i) Given a C^A-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid dual to $M_{PT}(D, B_0)^\star$.*

(ii) Given a bimaze (G, m_0), if $(G, m_0)^\star$ is C^A-free, then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

Proof. (i) This is the direct consequence of Lemma 2.3.4 and Lemma 2.6.8.

(ii) Apply part (i) and (2.12). □

One might hope that in the first part of the theorem the path-transversal matroid $M_{PT}(D, B_0)^\star$ is in fact the transversal matroid $M_T(D, B_0)^\star$. However, the dimaze R^I defines a strict gammoid whose dual is not the transversal matroid defined by the converted bimaze. It turns out that R^I is the only obstruction to this hope.

Theorem 2.6.10. *(i) Given an $\{R^I, C^A\}$-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid dual to $M_T(D^\star_{B_0})$.*

(ii) Given a bimaze (G, m_0), if $(G, m_0)^\star$ is $\{R^I, C^A\}$-free, then $M_T(G)$ is a matroid dual to $M_L(G, m_0)^\star$.

Proof. (i) This follows from Theorem 2.6.9(i) and the fact that for an R^I-free dimaze (D, B_0), we have $M_T(D^\star_{B_0}) = M_{PT}(D, B_0)^\star$. The proof of the latter is similar to the one given to Proposition 2.6.5 and omitted.

(ii) Apply part (i) and (2.12). □

As a corollary we can show that the class of rayless gammoids has some nice properties like that of finite gammoids. An undirected graph is called *rayless*, if it does not contain any ray. We call a gammoid *rayless* if it admits a presentation whose (undirected) underlying graph is rayless. A transversal matroid is called *rayless* if there is a rayless bipartite graph defining it.

Proposition 2.6.11. *The class of rayless gammoids is closed under taking minor and duality.*

Proof. For the minor part, as restriction and contraction commute, it suffices to prove that any contraction of any rayless strict gammoid M is a rayless gammoid. Let $M = M_L(D, B_0)$ where the underlying graph of D is rayless. As there is no C^A or R^I in (D, B_0), by Theorem 2.6.10, $M^*_L(D, B_0) = M_T(D^\star_{B_0})$ which is a rayless transversal matroid. Any restriction of this transversal matroid is also a rayless transversal matroid, say $M_T(G)$. Pick some identity matching m_0. As the underlying graph of $(G, m_0)^\star$ is rayless, by Theorem 2.6.10, $M^*_T(G) = M_L(G, m_0)^\star$ which is a rayless gammoid.

To show that the class of rayless gammoids is closed under duality, note that $M^* = M^*_L(D, B_0)/X = M_T(D^\star_{B_0})/X$. But the last matroid is a contraction of a rayless gammoid which was just shown to be a rayless gammoid. □

Given a rayless presentation $G = (V, W)$ of a transversal matroid M, we construct a rayless gammoid presentation of M by directing the edges from V to W and defining W to be the set of exits. So the class of rayless gammoids contains that of rayless transversal matroids.

It appears that C^A is a natural constraint in Theorem 2.6.10.

Example 2.6.12. The strict gammoid defined by the dimaze C^A (Figure 2.5a) is not cotransversal.

Proof. Since $V \setminus B_0 + v$ is a base for every $v \in B_0$, B_0 is an infinite cocircuit. On the other hand, every vertex v of B_0 is contained in a finite cocircuit, namely v and its in-neighbours. So by Lemma 2.2.9, the dual is not transversal. □

Here is a question which is in some sense converse to Theorem 2.6.10(i).

Question 2.6.13. *Is every cotransversal strict gammoid $\{C^A, R^I\}$-free?*

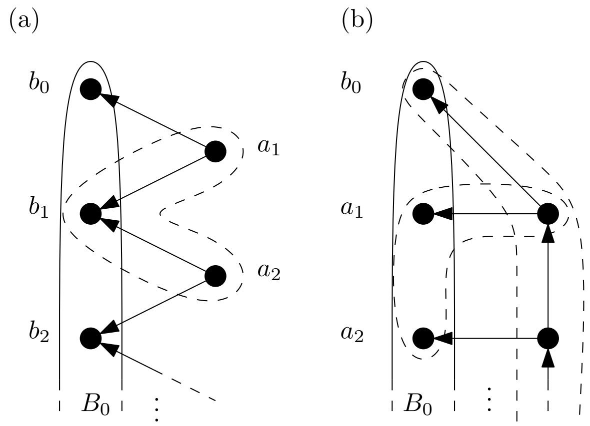

Figure 2.5: An alternating comb and an incoming comb defining isomorphic strict gammoids.

Although the class of path-transversal matroids contains that of transversal matroids properly, not every strict gammoid has its dual of this type. To show this, we first note that in a path-transversal matroid $M_{PT}(G, m_0)$, if C is the fundamental circuit of u, then $N(C) = m_0(C - u)$. Indeed, $N(u) \subseteq m_0(C - u)$; and for any $v \in C - u$, since there is an m_0-alternating path from u ending in v, v cannot have any neighbour outside $m_0(C - u)$.

Example 2.6.14. Let T be a rooted tree such that each vertex has infinitely many children, with edges directed towards B_0, which consists of the root and vertices on alternating levels. Then $M_L(T, B_0)$ is a strict gammoid that is not dual to any path-transversal matroid.

Proof. By Theorem 2.4.4, $M := M_L(T, B_0)$ is a matroid. Suppose that $M^* = M_{PT}(G, m)$. Let $\mathcal{Q}$ be a linkage of $B := V - m$ to B_0. Since (T, B_0) is C^O-free, by Proposition 2.5.10, we have $M = M_L(D_1, B_1)$ where (D_1, B_1) is the $\mathcal{Q}$-shifted dimaze. By construction, the underlying graph of D is also a tree.

By Corollary 2.4.8, (D_1, B_1) contains a subdivision R of C^A. Let $\{s_i : i \geq 1\} = R \cap B_1$ and $U = \{u_i : i \geq 1\}$ be the set of vertices of out-degree 2 on R. Let U_i be the set of vertices such that any path from which to $R \cap B_1$ contains u_i; and S_i be the set of vertices such that there is a path from which to s_i in $D - U$. Since D is a tree, $\{U_i, S_i : i \geq 1\}$ is a collection of pairwise disjoint sets.

Let $C := \bigcup_{i \geq 1} S_i$. Then any linkable set in $V \setminus C$ has a linkage that misses an exit in $R \cap B_1$. Since D is a tree, $(B_1 - R) \cup U + c$ for any $c \in C$ is a base of M. Hence, C is a circuit in M^*. For a contradiction, we construct an m_0-matching of C in (G, m).

In M^*, the fundamental circuit of s_i with respect to B_1 is $S_i \cup U_{i-1} \cup U_i$ (with $U_0 = \emptyset$). By the remark before the example, $N(S_i \cup U_{i-1} \cup U_i) =$

$m(S_i \cup U_{i-1} \cup U_i - s_i)$ for $i \geq 1$.

We claim that for $i \geq 1$, in any m-matching m' of $\bigcup_{j \leq i} S_j$, the maximal m'-m alternating walk from s_j ends in $m(U_j)$ for $j \leq i$. Note that such a walk cannot end in $m(S_j)$ as those vertices are incident with m'-edges. Since $N(S_1) \subseteq m(S_1 \cup U_1)$, the claim is true for $i = 1$. Assume that it is true for $i - 1$. Consider an m-matching m_1 of $\bigcup_{j \leq i} S_j$. Let P_j be the maximal m_1-m-alternating walk starting from s_j. By assumption, P_j ends in $m(U_j)$ for each $j < i$. As P_i ends in $m(U_{i-1} \cup U_i)$, we are done unless it ends in $m(U_{i-1})$. In that case, the union of an m-matching of $C \setminus \bigcup_{j \leq i} S_j$ with

$$(m \restriction \bigcup_{j \leq i} S_j) \Delta \bigcup_{j \leq i} E(P_j)$$

is an m-matching of C, a contradiction.

Therefore, there is a collection of pairwise disjoint m-alternating walks $\{P'_i : i \geq 1\}$ where P'_i starts from s_i and ends in $m(U_i)$. Then $m \Delta \bigcup_{i \geq 1} E(P'_i)$ is an m-matching of C, a contradiction which completes the proof. $\square$

Here is a question similar to Question 2.6.13 akin to Theorem 2.6.9.

Question 2.6.15. *Is every strict gammoid which is dual to a path-transversal matroid C^A-free?*

It may be interesting to investigate further path-transversal systems. For example, while they need not satisfy (IM), it may be the case that (I3) always holds.

Conjecture 2.6.16. *Given a bimaze (G, m_0), $M_{PT}(G, m_0)$ satisfies (I3).*

2.6.2 Finitary transversal matroids

Our aim in this section is to give a transversal matroid that is not dual to any strict gammoid. To this end, we extend some results in [7] and [8]. The following identifies edges that may be added to a presentation of a finitary transversal matroid without changing the matroid.

Lemma 2.6.17. *Suppose that $M_T(G)$ is finitary. Let K be a subset of $\{vw \notin E(G) : v \in V, w \in W\}$. Then the following are equivalent:*

1. *$M_T(G) \neq M_T(G + K)$;*
2. *there are $vw \in K$ and a circuit C with $v \in C$ and $w \notin N(C)$;*
3. *there is $vw \in K$ such that v is not a coloop of $M_T(G) \setminus N(w)$.*

Proof. 1. holds if and only if there is a circuit C in $M_T(G)$ which is matchable in $G + K$. This, since C is finite, in turn holds if and only if there is $v \in C$ that can be matched outside $N(C)$ in $G + K$, i.e. 2. holds.

The equivalence between 2. and 3. is clear since a vertex is not a coloop if and only if it lies in a circuit. $\square$

Given a bipartite graph G, recall that a presentation of a transversal matroid M as $M_T(G)$ is maximal if $M_T(G + vw) \neq M_T(G)$ for any $vw \notin E(G)$ with $v \in V, w \in W$. Thus, the previous lemma implies that if $M_T(G)$ is finitary, then G is maximal if and only if $M \setminus N(w)$ is coloop-free for any $w \in W$. Bondy [7] asserted that there is a unique maximal presentation for any finite coloop-free transversal matroid; where two presentations of a transversal matroid by bipartite graphs G and H are *isomorphic* if there is a graph isomorphism from G to H fixing the left vertex class pointwise.

Proposition 2.6.18. *Every finitary transversal matroid M has a unique maximal presentation.*

Proof. Let $M = M_T(G)$. Adding all vw with the property that there is not any circuit C with $v \in C$ and $w \notin N(C)$ gives a maximal presentation of M by Lemma 2.6.17. In particular, any coloop is always adjacent to every vertex in W. So without loss of generality, we assume that M is coloop-free.

Now let G and H be distinct maximal presentations of M.

Claim 1. For any finite subset F of V, the induced subgraphs $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$ are isomorphic.

For every $v \in F$ pick a circuit C_v with $v \in C_v$. By Lemma 2.6.17, for every $vw \in \{xy \notin E(G) : x \in F, y \in N_G(F)\}$, there is a circuit C_{vw} with $v \in C$ and $w \notin N_G(C)$. Let F_G be the union of all C_v's and C_{vw}'s. Analogously define F_H and let $F' = F_G \cup F_H$. Extend the presentations $G[F' \cup N_G(F')]$ and $H[F' \cup N_H(F')]$ of $M \restriction F'$ to maximal ones G' and H' respectively, between which there is a graph isomorphism fixing the left vertex class pointwise by Bondy's result. Restricting the isomorphism to $F \cup N_G(F)$ is an isomorphism of $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$, as by definition of F' and Lemma 2.6.17, no non-edge between F and $N_G(F)$ is an edge in G' (analogously between F and $N_H(F)$ in H').

Without loss of generality, there is an $A \subseteq V$ such that $g := |\{w \in W(G) : N_G(w) = A\}| < |\{w \in W(H) : N_H(w) = A\}| =: h$. Note that as H is a maximal presentation, by Lemma 2.6.17, $M \setminus A$ is coloop-free.

As M is coloop-free, so is $M.A$. Let B_1 be a base of $M.A$ and extend B_1 to a base of M which admits a matching m; thus m contains a matching of a base of $M \setminus A$. Since $M \setminus A$ is coloop-free, by Lemma 2.2.7, the neighbourhood of each vertex matched by m to a vertex in B_1 is a subset of A. Thus, $M.A$ can be presented with the subgraphs induced by $A \cup \{w \in W : N(w) \subseteq A\}$ in both graphs; call these subgraphs G_1 and H_1. For any $w \in W(G_1)$, since $M \setminus N_{G_1}(w)$ is coloop-free, so is $M.A \setminus N_{G_1}(w)$. By Lemma 2.6.17, G_1 (analogously H_1) is a maximal presentation of $M.A$.

Claim 2. Given a family $(N_j)_{j \in J}$ of finite subsets of W, if the intersection of any finite subfamily has size at least k, then the intersection of the family has size at least k.

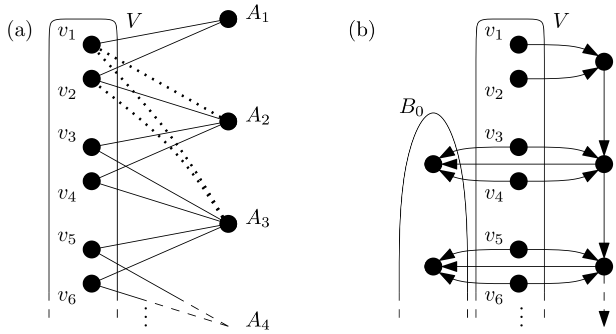

Figure 2.6: A transversal matroid which is not dual to a strict gammoid and a gammoid presentation of its dual

Let $N = \bigcap_{j \in J} N_j$. Suppose $|N| < k$. Fix some $j_0 \in J$ and for each element $y \in N_{j_0} \setminus N$ pick some N_y such that $y \notin N_y$. Then $|N_{j_0} \cap \bigcap_{y \in N_{j_0} \setminus N} N_y| = |N| < k$, which is a contradiction.

By Claim 2, there is a finite set $F \subseteq A$ such that $|\bigcap_{v \in F} N_{G_1}(v)| = g$. But Claim 1 says that F has at least $h > g$ common neighbours in H_1; this contradiction completes the proof. □

To show that the following finitary transversal matroid is not dual to a strict gammoid, it suffices to show that there is no bimaze presentation whose converted dimaze is C^A-free.

Example 2.6.19. Define a bipartite graph G as $V(G) = \{v_i, A_i : i \geq 1\}$ and $E(G) = \{v_1A_1, v_2A_1, v_1A_3, v_2A_3\} \cup \{v_{2i-3}A_i, v_{2i-2}A_i, v_{2i-1}A_i, v_{2i}A_i : i \geq 2\}$ (see Figure 2.6). Then $M = M_T(G)$ is not dual to a strict gammoid.

Proof. As G is left locally finite, M is a finitary matroid. Assume for a contradiction that $M^* = M_L(D, B_0)$. By a characterization of cofinitary strict gammoids in [1], we may assume that (D, B_0) is $\{R^I, C^A\}$-free. Then by Theorem 2.6.10, $M = M_T(D, B_0)^\star$.

Now it can be checked that all $M \backslash N(w_i)$ are coloop-free. By Lemma 2.6.17, G is the maximal presentation of M. The same lemma also implies that any minimal presentation G' is obtained by deleting edges from $\{v_1A_3, v_2A_3\}$ and at most one from $\{v_1A_2, v_2A_2\}$. In particular, all presentations of M differ from G only finitely. It is not difficult to check that with any matching m_0 of a base, $(G, m_0)^\star$ contains a subdivision of C^A. Hence, there is no bimaze presentation of M such that the converted dimaze is C^A-free, contradicting that $(D, B_0)^\star$ is such a presentation. □

We remark that the above transversal matroid is dual to a gammoid, see Figure 2.6(b). However, in the next section, we give a transversal matroid that is not dual to any gammoid.

2.6.3 Infinite tree and gammoid duality

To show that there is a strict gammoid not dual to a gammoid, we prove the following lemmas, whose common setting is that a given dimaze (D, B_0) defines a matroid $M_L(D, B_0)$. For a linkage $\mathcal{Q}$ and any $X \subseteq \mathrm{Ini}(\mathcal{Q})$, $\mathcal{Q} \upharpoonright X := \{Q \in \mathcal{Q} : \mathrm{Ini}(Q) \in X\}$; when $X = \{x\}$, we write simply Q_x.

Lemma 2.6.20. *Let C be an infinite circuit containing b and $\mathcal{Q}$ a linkage from $C - b$. Then b can reach infinitely many vertices in C via $\mathcal{Q}$-alternating walks.*

Proof. Given any $x \in C - b$, let $\mathcal{P}$ be a linkage of $C - x$. Let W be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting from b. If W is infinite, then we are done. Otherwise, W ends in either $\mathrm{Ter}(\mathcal{P}) \setminus \mathrm{Ter}(\mathcal{Q})$ or $\mathrm{Ini}(\mathcal{Q}) \setminus \mathrm{Ini}(\mathcal{P}) = \{x\}$. The former case does not occur, since it gives rise to a linkage of C by Lemma 2.2.4(i), contradicting C being a circuit. As x was arbitrary, the proof is complete. □

Lemma 2.6.21. *For $i = 1, 2$, let C_i be a circuit of M, and $b_i, x_i \in C_i \setminus C_{3-i}$, W_i a $\mathcal{Q}$-alternating walk from b_i to x_i, where $\mathcal{Q}$ is a linkage from $(C_1 \cup C_2) \setminus \{b_1, b_2\}$. Then W_1 and W_2 are disjoint.*

Proof. Suppose that $W_1 = w_0^1 e_0^1 w_1^1 \dots w_n^1$ and $W_2 = w_0^2 e_0^2 w_1^2 \dots w_m^2$ are not disjoint. Then there exists a first vertex $v = w_j^1$ on W_1 such that $v = w_k^2 \in W_2$ and either $v \in V(\mathcal{Q})$ and $e_j^1 = e_k^2 \in E(\mathcal{Q})$ or $v \notin V(\mathcal{Q})$. In both cases $W_3 := W_1 v W_2$ is a $\mathcal{Q}$-alternating walk from b_1 to x_2. Let v' be the first vertex of W_3 in $V(\mathcal{Q} \upharpoonright (C_2 - b_2) \setminus C_1)$ and Q the path in $\mathcal{Q}$ containing v'. Then $W_3 v' Q$ is a $(\mathcal{Q} \upharpoonright (C_1 - b_1))$-alternating walk from b_1 to $B_0 \setminus \mathrm{Ter}(\mathcal{Q} \upharpoonright (C_1 - b_1))$, which by Lemma 2.2.4(i) contradicts the dependence of C_1. Hence W_1 and W_2 are disjoint. □

Lemma 2.6.22. *Let $\{C_i : i \in N\}$ be a collection of circuits of M; x_i, b_i distinct elements in $C_i \setminus \bigcup_{j \neq i} C_j$. Suppose that $\bigcup_{i \in I} C_i \setminus \{b_i : i \in N\}$ admits a linkage $\mathcal{Q}$. Let W_i be a $\mathcal{Q}$-alternating walk from b_i to x_i. If $X \subseteq V$ is a finite set containing $C_i \cap C_j$ for any distinct i, j, then only finitely many of W_i meet $\mathcal{Q} \upharpoonright X$.*

Proof. By Lemma 2.6.21, the walks W_i are pairwise disjoint. Since $\mathcal{Q} \upharpoonright X$ is finite, it can be met by only finitely many W_i's. □

We are now ready to give a counterexample to classical duality in gammoids.

Example 2.6.23. Let (T, B_0) be the dimaze defined in Example 2.6.14. The dual of the strict gammoid $M = M_L(T, B_0)$ is not a gammoid.

Proof. Suppose that $M^* = M_L(D, B_1) \upharpoonright V$, where $V := V(T)$. Fix a linkage $\mathcal{Q}$ of $V \setminus B_0$ in (D, B_1). For $b \in B_0$, let C_b be the fundamental cocircuit of M with respect to B_0. Then for any (undirected) ray $b_0 x_0 b_1 x_1 \cdots$ in T, $C := \bigcup_{k \in \mathbb{N}} C_{b_k} \setminus \{x_k : k \in \mathbb{N}\}$ is a cocircuit of M. We get a contradiction by building a linkage for C in (D, B_1) inductively using disjoint $\mathcal{Q}$-alternating walks.

Let b_0 be the root of T. By Lemma 2.6.20, there is a $\mathcal{Q}$-alternating walk W_0 from b_0 to one of its children x_0. At step $k > 0$, from each child b of x_{k-1} in T, by Lemma 2.6.20, there is a $\mathcal{Q}$-alternating walk W_b in (D, B_1) to a child x of b. Applying Lemma 2.6.22 on $\{C_i : i \in N^-(x_{k-1}) - b_{k-1}\}$ with $X = \{x_{k-1}\}$, we may choose $b_k := b$, $x_k := x$ such that $W_k := W_b$ avoids $Q_{x_{i-1}}$.

By Lemma 2.6.21, distinct W_k and $W_{k'}$ are disjoint. Moreover, as each W_k avoids $Q_{x_{k-1}}$, Lemma 2.2.4(i) implies that W_k can only meet $\mathcal{Q}$ at Q_x where $x \in C_{b_k} - x_{k-1}$. Then $E(\mathcal{Q}) \triangle \bigcup_{k \in \mathbb{N}} E(W_k)$ contains a linkage of C. $\square$

By adding a unique in-neighbour to every vertex in B_0, we can present $M_L(T, B_0)$ as a transversal matroid (as in the proof of Corollary 2.4.7). Thus, not every transversal matroid is dual to a gammoid. It might be the case that the alternating comb is the obstacle to duality.

Conjecture 2.6.24. *The class of C^A-free gammoids is closed under duality.*

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Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den

Malte Müller

Summary

The connected tree-width is an upper bound for the tree-width of a graph, and the cycles (as graphs), having a tree-width of 2, show that the tree-width and the connected tree-width of a graph can be arbitrarily far away from each other.

It is proved that for any graph, a large geodesic cycle is the only reason for the connected tree-width to be much larger than the tree-width. This is used to show that a qualitative version of a "connected tree-width duality theorem" holds.

The second part concerns gammoids, a class of matroids investigated in the late 1960's. Ingleton and Piff gave a construction that transforms a presentation of a finite strict gammoid, a dimaze, to a transversal matroid presentation of its dual, a bipartite graph. This is used in the proof that the class of finite gammoids is closed under minors and under duality. In 2010 Bruhn, Diestel, Kriesell, Pendavingh and Wollan found a notion of infinite matroids that allows for duality. This suggests the question of extending gammoids to infinite ground sets by a verbatim transfer of linkability.

Contrary to the finite case, not every infinite dimaze defines a matroid. One obstruction is a dimaze termed an alternating comb. For such a strict gammoid the construction of Ingleton and Piff (transferred to the infinite case) provides a presentation of the dual and, if the dimaze does not contain an incoming ray, that dual is transversal. The class of gammoids definable by a dimaze without any outgoing comb is minor closed and the class of gammoids definable by a dimaze without any ray is, like that of finite gammoids, closed under minors and under duality.

Zusammenfassung

Die zusammenhängende Baumweite eines Graphen ist eine obere Schranke für seine Baumweite und bei einem Kreis können diese beiden Parameter beliebig weit voneinander entfernt sein. Es wird bewiesen, dass einen langen geodätischen Kreis zu enthalten der einzige Grund für einen großen Abstand dieser Parameter ist. Dies wird benutzt um eine qualitative Version eines "Dualitätssatzes der zusammenhängenden Baumweite" zu beweisen.

Im zweiten Teil beschäftigen wir uns mit den, in den 1960er Jahren entwickelten, Gammoiden. Eine Konstruktion von Ingleton und Piff überführt eine Präsentation eines endlichen strikten Gammoids, ein Dimaze, in eine transversal Matroid-Präsentation des Duals, einen bipartiten Graphen. Sie wurde benutzt um zu beweisen, dass die endlichen Gammoide unter Minorbildung und Dualität abgeschlossen sind. Im Jahr 2010 fanden Bruhn, Diestel, Kriesell, Pendavingh und Wollan Axiome für unendliche Matroide, die Dualität erlauben. Eine natürliche Frage ist, ob sich Gammoide, durch wörtliche Übertragung von Verbindbarkeit, auf unendliche Grundmengen erweitern lassen.

Im Gegensatz zum endlichen Fall definiert nicht jedes unenliche Dimaze ein Matroid. Enthält es allerdings kein Dimaze namens alternating comb, so wird ein Matroid definiert und die Konstuktion von Ingleton und Piff (auf den unendlichen Fall übertragen) liefert eine Präsentation vom Dual des definierten strikten Gammoids. Falls das Dimaze keinen incoming comb enthielt, so ist dieses Dual ein transversal Matroid. Die Klasse der Gammoide, die eine Präsentation ohne outgoing comb haben, ist unter Minorbildung abgeschlossen und die Klasse der Gammoide, die durch ein strahlenloses Dimaze definierbar sind, ist, wie die der endlichen Gammoide, unter Minorbildung und Dualität abgeschlossen.