# Solutions of the pentagon equation from quantum (super)groups

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# 1 Introduction

Associative algebras are one of the most basic notions in modern mathematics. They are characterised by the presence of an operation called multiplication which satisfies an associativity property. Those abstract objects can be studied using representation theoretical methods on various modules, of which special mention deserve vector spaces and Hilbert spaces. When considering representations one may ask whether there is a natural action of an algebra on a tensor product of modules and without endowing them with additional structures it is impossible to answer that question.

That is why one can define an operation called co-multiplication which is in some sense "dual" to the ordinary algebraic product: while for an algebra $\mathcal{A}$ the multiplication is a map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, the co-multiplication $\Delta$ is a map in the reversed direction $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. The co-multiplication provides a canonical way of acting on tensor products of representations and on the intertwiners mapping between them. It is important to note however that those two operations ought to be supplemented with additional axioms because in general they can be not compatible with each other.

As the algebraic product is by definition associative, i.e. $m(m \otimes 1) = m(1 \otimes m)$, the co-product is co-associative, which means that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta. \tag{1}$$

Co-associativity has an important consequences for the representation theory: it ensures that the action of Hopf algebra on the modules respects different ways in which one can take tensor products of representations. Tensor products is indeed a binary operation, so when one consideres a product of more than two representations $\pi_i$, it is indeed important whether one means $(\pi_1 \otimes \pi_2) \otimes \pi_3$ or $\pi_1 \otimes (\pi_2 \otimes \pi_3)$. Equation (1) ensures that the result of taking tensor product does not depend on the position of brackets.

The isomorphism between $(\pi_1 \otimes \pi_2) \otimes \pi_3$ and $\pi_1 \otimes (\pi_2 \otimes \pi_3)$, called a Racah-Wigner map or an associator $\alpha$, is not an arbitrary one - it ought to satisfy a number of consistency conditions, among which one important is the pentagon equation stemming from the consideration of a quadruple tensor product. One can present this equation in the form of a commutative diagram as follows:

Since the associator can be equivalently expressed in terms of a summation/integration kernels, called in that case Racah-Wigner or 6j coefficients, one can translate this equation into an integral equation for functions.

The introduction of compatibility axioms and additional canonical structures (like an antipodal map, which can be regarded as an "algebraic inverse" and is in fact unique, if it exists) on top of multiplication and co-multiplication leads naturally to the notion of Hopf algebra (an its special case known by the name of a quantum

group) [1, 2, 3, 4] — which are an interesting, and fruitful in applications, generalisation of standard associative algebras. Moreover, a substantial amount of attention worthy examples has been identified. Semi-simple Lie algebras admit a one-parameter deformation (so called q-deformation) which makes them into a non-trivial Hopf algebras — therefore one can look at Hopf algebras as a "quantisation" of sorts of more familiar algebraic structures.

Yet q-deformation of classical algebras is not the only way to construct interesting Hopf algebras — a double construction dating back to Drinfeld allows one to obtain Hopf algebras belonging to a special class distinguished by a property of quasi-triangularity. One calls a Hopf algebra $\mathcal{A}$ quasi-triangular if there exists an element $R \in \mathcal{A} \otimes \mathcal{A}$ called an universal R-matrix which satisfies specific axioms which lead to Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2}$$

The Yang-Baxter equation manifests itself in various different contexts in theoretical physics, especially in two dimensional integrable systems [5] as well as in four dimensional superconformal field theories. Therefore just by evaluating the relevant quantum groups on representation spaces one can find physically relevant solutions which otherwise could be difficult to compute!

The Drinfeld double construction is not the only one providing solutions to equations of physical interest. Indeed, starting from a Hopf algebra one can use the so-called Heisenberg double construction [6] to obtain an algebra (in general not a Hopf algebra in this case) for which there exists a canonical element $S$ similar to the universal R-matrix, however it verifies not the Yang-Baxter equation but the pentagon one

$$S_{12}S_{13}S_{23} = S_{23}S_{12}. \tag{3}$$

This fact leads to an interesting question: since the associators and canonical elements of Heisenberg doubles verify the same relation, is it possible that for some representation categories of the Hopf algebras associators can be indeed realised as the same operators as the canonical elements $S$? The answer is, suprisingly, yes: it has been shown that at least in the case of representation category of the so called quantum plane, i.e. the Borel half of $\mathcal{U}_q(sl(2))$, the Racah-Wigner map is indeed related by a simple unitary transformation to the canonical element of Heisenberg double constructed from a quantum plane [7]. Since those two object a priori have no obvious relation between them, it is intriguing issue whether this observed equality is an accidental one or is valid for more general classes of Hopf algebras.

The pentagon equation is not however present only in the context of the theory of Hopf algebras — it appears profusely in modern mathematical physics, in particular two-dimensional conformal field theory. Conformal field theories (CFTs) [8, 9, 10] are a special class of quantum field theories in which correlators and fields are not transforming covariantly under usual Poincaré algebra, but under a conformal algebra of which the former is a subalgebra. Conformal algebra is composed of translations, boosts, rotations, dialations and the so called special conformal transformations, all of which can be shown to preserve angles in space (from which fact the algebra derives its name). Although in a generic dimension the conformal algebra is only slightly larger than the Poincaré one, in two dimensions something magnificent happens — it acquires an infinite number of generators. Such an extensive number of symmetries has an important consequences: it provides a possibility to

study theories analitycally in a robust and rigorous way, which is impossible for the higher dimensional quantum field theories, and yet is not restrictive enough as to make CFTs trivial.

The structure of conformal field theories is governed by the representation theory of Virasoro algebra defined by the generators $L_n$ satisfying the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0},\tag{4}$$

(which is a central extension of the conformal algebra with the central element $c$, called the central charge): the space of states decomposes into a direct sum (or a direct integral, if the spectrum is continuous) of irreducible representations of the Virasoro algebra of weight $\alpha$

$$\mathcal{H} = \int^{\oplus} \mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha.$$

Of special importance are the highest weight states of the modules $\mathcal{V}_\alpha$, which are can be obtained by an action of the so-called primary fields on the conformal vacuum $|0\rangle$

$$\lim_{z \to 0} V_\alpha(z, \bar{z})|0\rangle = |\alpha\rangle.$$

It can be indeed shown that the behaviour of the primary states determines uniquely all other fields, so their study is of the utmost importance in the context of CFT. The knowledge of the correlation function of the primary fields allows one to find an arbitrary correlator present in the theory.

Moreover, one often studies CFTs on a Riemann surfaces instead of the two-dimensional Minkowski space in order to be able to use the toolkit of complex analysis. It is possible to independently study chiral parts of the full theory, where the fields and correlators are holomorphic (or anti-holomorphic) functions on the Riemann surface, and only consistently bring them together at the end to obtain the physical results. Therefore, the chiral primary fields and their correlators, called conformal blocks, are essential building blocks for any CFT.

Because of the conformal invariance one can use the so-called operator product expansion (OPE). The product of two fields in the theory can be expanded in terms of other fields, provided that the insertion points are sufficiently close

$$V_\alpha(z)V_\beta(w) = \sum_\gamma a_{\alpha\beta}^\gamma(z - w)V_\gamma(w) + \dots$$

By using OPE one can reduce the problem of computing the correlation functions of $n$ fields to the three point correlators. However, one can perform the expansion of multipoint correlation functions in many different ways, and the value of the physical correlators should not depend on a particular decomposition. In particular, the four point correlation function $\langle V_\alpha(0, 0)V_\beta(1, 1)V_\gamma(z, \bar{z})V_\delta(\infty, \infty)\rangle$ can be reduced to three point correlators by e.g. taking the OPE of fields $V_\alpha(0, 0)V_\beta(1, 1)$ or $V_\gamma(z, \bar{z})V_\delta(\infty, \infty)$. On the level of chiral fields this is encoded in an isomorphism between different four points conformal blocks $\mathcal{F}$, known by the name of the fusion matrix $F$

$$\mathcal{F}_{\Delta_s}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix}(z) = \int d\alpha_t F_{\alpha_s\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1\end{bmatrix}\mathcal{F}_{\Delta_t}\begin{bmatrix}\Delta_1 & \Delta_2 \\ \Delta_4 & \Delta_3\end{bmatrix}(1 - z),\tag{5}$$

where the four point conformal blocks are related to the four point CFT correlators

$$\langle V_{\alpha_1}(0,0)V_{\alpha_2}(1,1)V_{\alpha_3}(z,\bar{z})V_{\alpha_4}(\infty,\infty)\rangle =$$
$$= \int \mathrm{d}\alpha_s C(\alpha_4,\alpha_3,\alpha_s)C(\bar{\alpha}_s,\alpha_2,\alpha_1)\mathcal{F}_{\Delta_s}\begin{bmatrix}\Delta_3\ \Delta_2\\\Delta_4\ \Delta_1\end{bmatrix}(z)\bar{\mathcal{F}}_{\Delta_s}\begin{bmatrix}\Delta_3\ \Delta_2\\\Delta_4\ \Delta_1\end{bmatrix}(\bar{z}),$$
$$\langle V_{\alpha_1}(0,0)V_{\alpha_2}(1,1)V_{\alpha_3}(z,\bar{z})V_{\alpha_4}(\infty,\infty)\rangle =$$
$$= \int \mathrm{d}\alpha_t C(\alpha_4,\alpha_t,\alpha_1)C(\bar{\alpha}_t,\alpha_3,\alpha_2)\mathcal{F}_{\Delta_t}\begin{bmatrix}\Delta_3\ \Delta_2\\\Delta_4\ \Delta_1\end{bmatrix}(1-z)\bar{\mathcal{F}}_{\Delta_t}\begin{bmatrix}\Delta_3\ \Delta_2\\\Delta_4\ \Delta_1\end{bmatrix}(1-\bar{z}),$$

(6)

and constants $C$ are structure constants encoding the three point correlators. The fusion matrix essentially provides one with the associativity of operator product expansion, so it too verifies the pentagon equation as a self-consistency relation.

$$\int_{\mathbb{S}} \mathrm{d}\delta_1 F_{\beta_1\delta_1}\begin{bmatrix}\alpha_3\ \alpha_2\\\beta_2\ \alpha_1\end{bmatrix} F_{\beta_2\gamma_2}\begin{bmatrix}\alpha_4\ \delta_1\\\alpha_5\ \alpha_1\end{bmatrix} F_{\delta_1\gamma_1}\begin{bmatrix}\alpha_4\ \alpha_3\\\gamma_2\ \alpha_2\end{bmatrix} = F_{\beta_2\gamma_1}\begin{bmatrix}\alpha_4\ \alpha_3\\\alpha_5\ \beta_1\end{bmatrix} F_{\beta_1\gamma_2}\begin{bmatrix}\gamma_1\ \alpha_2\\\alpha_5\ \alpha_1\end{bmatrix}.$$

(7)

Therefore one sees that constructing the fusion matrix provides an important stepping stone in the proof of the crossing-symmetry, i.e. the self-consistency of a particular CFT under study.

One can ask a question: are associators in tensor categories of Hopf algebras and fusion matrices in some way related to each other? Both of them do indeed stem from representation theory: former from Hopf algebras while latter from Virasoro algebras. They both verify the same equations. The inquiry seems justified. Indeed, for so-called rational CFTs, i.e. CFT which have only discrete, finite number of the primary fields and correspond to the value of the central charge $c < 1$, it has been shown [11, 5] that the fusion matrices can be identified with 6j symbols of the finite-dimensional representation of $\mathcal{U}_q(\mathfrak{g})$. However, the study of CFTs with $c > 1$ proved to be more elusive — the presence of infinite number of primary fields makes evaluating of general claims rather difficult. There is an evidence that the connection between 6j symbols and fusion matrices should hold as well in the non-rational case. The first compelling insight into that was showing that Liouville theory, which can be regarded as the simplest non-trivial non-rational CFT, has a fusion matrix which can be obtained from the representation theory of $\mathcal{U}_q(sl(2))$ [12, 13].

The Liouville theory [14] is a conformal field theory with classical action of the form

$$S = \frac{1}{4\pi} \int \left[(\partial_\alpha\phi)^2 + 4\pi\mu e^{2b\phi}\right] \mathrm{d}^2x,$$

(8)

where $Q = b + \frac{1}{b}$, $\mu$ is a cosmological constant and $b \in \mathbb{R}$ is the Liouville coupling constant. The central charge of the theory is equal to $c = 1 + 6Q^2$ and, since $b$ is real, it is greater than 25. The theory admits a particular symmetry with respect to a change of the coupling constant $b \to \frac{1}{b}$, which is not at all apparent form the classical action and is purely quantum phenomenon — indeed, the weak and strong coupling limits of the quantum theory lead to the same classical one, which is a behaviour uncommon for more physically relevant, four-dimensional quantum field theories, like quantum chromodynamics.

The Hilbert space of states $\mathcal{H}$ decomposes into direct sum of the tensor products $\mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha$ of highest weight representations of holomorphic and antiholomorphic part of Virasoro algebra:

$$\mathcal{H} = \int_{\frac{Q}{2}+i\mathbb{R}^+}^{\oplus} \mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha,$$

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where $\mathcal{V}_\alpha$ contains primary states $v_\alpha$ with spins $\alpha \in \frac{Q}{2} + i\mathbb{R}^+$. The structure constants $C$ [15, 16], which encode the three point correlation functions, can be expressed by

$$C(\alpha_3, \alpha_2, \alpha_1) = \left[\pi\mu\gamma(b^2)b^{2(1-b^2)}\right]^{(Q-\alpha_3-\alpha_2-\alpha_1)/b} \times$$

$$\times \frac{\Upsilon_0 \Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_3 + \alpha_2 + \alpha_1 - Q)\Upsilon_b(\alpha_2 + \alpha_1 - \alpha_3)\Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)\Upsilon_b(\alpha_3 + \alpha_2 - \alpha_1)},$$

where $\Upsilon_b$ is defined in terms of the Barnes double gamma functions defined in the appendix:

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}.$$

It has been shown that the fusion matrix of Liouville theory can be identified with a 6j symbol for a category of one-parameter self-dual infinite dimensional representations of $\mathcal{U}_q(sl(2))$ [12, 13]. The move to extend this result would be to consider either higher rank cases, corresponding to Toda field theories and $\mathcal{U}_q(sl(n))$ Hopf algebras, or the $\mathbb{Z}_2$-graded case, for which one had to study supersymmetric Liouville theory and representation theory of $\mathcal{U}_q(osp(1|2))$.

Conformal field theories are not the only ones where objects satisfying pentagon equation play an important role. One can consider the Teichmüller theory [17, 34, 19], i.e. the theory of complex structures on Riemann surfaces, or equivalently the theory of $SL(2, \mathbb{R})$-valued connections. The set of local coordinates on Riemann surfaces relevant for this theory is assigned to triangulations of surfaces instead of the surfaces themselves — therefore, one ought to make sure that the constructions using them are independent of the choice of a particular triangulation. As a consequence the Ptolemy groupoid, which relates different triangulations, has a natural representation on the Hilbert spaces assigned to particular triangulations.

For us, one of the generators of the Ptolemy groupoid is of special interest: the operator $T$ relating two possible triangulations of a quadrilateral (i.e. two triangle sharing one edge). One can show that if one consideres how operator $T$ could act on a collection of 3 triangles sharing edges one necessarily arrives at a consistency condition that $T$ ought to verify the pentagon equation!
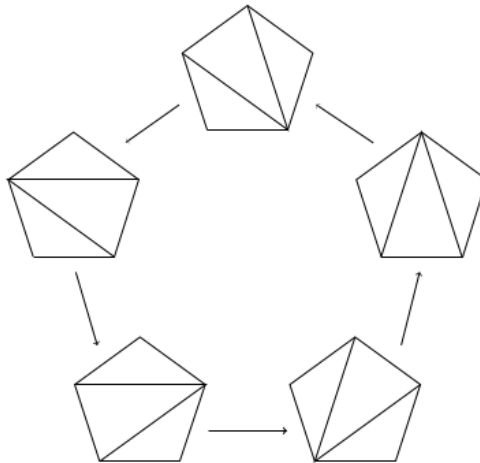


Figure 1: Pentagon equation relating different triangulations of a surface.

It has been proven that the defining operators of the Teichmüller theory, among them $T$, can be obtained using the representation theory of Hopf algebras and the representation theory of Heisenberg algebras [20]. Indeed, the canonical element $S$ of the Heisenberg double of the Borel half of $\mathcal{U}_q(sl(2))$ is nothing but the triangulations changing operator $T$. Moreover, since the canonical element $S$ and the associators for the representation category of a quantum plane were shown to be the same, the defining data of the theory of complex structures on Riemann surfaces can be found also there.

Finally, the pentagon equation has its natural place in the world of topological field theories and knot invariants [21, 22]. There are many different ways to look at topological field theories — it is possible to regard them as mappings which assign Hilbert spaces to 2-cobordism, or as a state-sum (or state-integral) models which assign operators to fundamental tetrahedra into which one decomposes the three-dimensional space. Clearly, the issue of providing at the end a theory which is independent of a decomposition into tetrahedra imposes constraints on the operators assigned to one tetrahedron. On the other side, if one looks at the TFTs from a more mathematical point of view, is has been already shown that the topological invariants associated to the 3-manifolds can be obtained also from the representation theory of Hopf algebra [23, 24, 25, 26], with a seminal result that employing finite dimensional representations of $\mathcal{U}_q(sl(2))$ one reproduced arguably the most famous knot invariant, i.e. Jones polynomial.

It is clear that there exists a multitude of deep bonds between quantum field theories and representation theory where the pentagon equation lies in a central place in a way that has been sketched above. However, what is an explicit realisation of all this? Abstract operators have to have to be able to be written in terms of some functions after all.

The central player here is a Faddeev's quantum dilogarithm [27, 28],

$$\Phi_{\mathrm{b}}(z) = \exp\left(\int_C \frac{e^{-2izw}}{\sinh(w\mathrm{b})\sinh(w/\mathrm{b})} \frac{dw}{4w}\right) . \tag{9}$$

This special function plays an important role in mathematical physics. Firstly, we can regard it as a quantisation of the Roger's dilogarithm, which is a frequent guest in the computations in four-dimensional field theories. The Roger's dilogarithm satisfies moreover a Roger's five-term identity, which for Faddeev's quantum dilogarithm becomes a pentagon equation

$$\Phi_{\mathrm{b}}(X)\Phi_{\mathrm{b}}(P) = \Phi_{\mathrm{b}}(P)\Phi_{\mathrm{b}}(X+P)\Phi_{\mathrm{b}}(X), \tag{10}$$

where $X, P$ are non-commutative variables with $[P.X] = \frac{1}{2\pi i}$. Moreover, this equation can be reformulated into a form of an integral identity, known as the integral analogoue of Ramanujan summation formula.

Moreover, Faddeev's quantum dilogarithm is a non-compact extension of a compact quantum dilogarith, which has found prolific use in the context of representation theory of finite-dimensional representations of Hopf algebras, link invariants etc. Indeed, it is then not surprising that $\Phi_{\mathrm{b}}$ would show up in the related contexts, but with a stress put on the non-compact or infinite-dimensional nature of the problems. Indeed, many elegant integral identities for quantum dilogarithm, among them the so-called star-triangle relation, are the reasons why the representation theoretical or field theoretical constraints put upon objects like 6j symbols or fusion

matrices are verified — they can be in fact reduced to the integral identities for Faddeev's quantum dilogarithm.

The goal of this thesis is to focus on the study of Hopf algebraic structures relevant in the context of two dimensional non-rational conformal field theory and Teichmüller theory, with the special stress put on the supersymmetric (or equivalently $\mathbb{Z}_2$-graded) case. In chapter 1, we will introduce the basic notions behind Hopf algebras and quasi-triangular Hopf algebras, as well as present the quantum deformation of the usual Lie algebras. We will also define the notion of Drinfeld and Heisenberg doubles, which provide the solutions for Yang-Baxter and pentagon equations. In chapter 2, we recall the self-dual representations of $\mathcal{U}_q(sl(2))$, which were shown to be relevant for one of the non-rational conformal field theories called Liouville theory, where the associators were identified with the fusion matrix of the theory. In chapter 3 we introduce the Heisenberg double related to $\mathcal{U}_q(sl(2))$ which was shown to be important in the construction of a quantised Teichmüller theory of Riemann surfaces.

Chapter 5 examines the representations of quantum plane algebra, which independently was shown to provide the construction of the defining objects of Teichmüller theory.

With chapter 6 the part of the thesis discussing the supersymmetric, or $\mathbb{Z}_2$-graded, generalisations of the results presented in the previous chapters opens. Those results are in fact our original work [29, 30] obtained during the doctoral project of which this thesis is a fruit, and were not before hand shown to be true. The basics of graded Hopf algebras and the quantum deformations of the Lie superalgebras are presented, and the generalisation of Drinfeld and Heisenberg constructions are shown - this time with the relevant equations replaced by their graded equivalents. In chapter 7, we study the self-dual representations of $\mathcal{U}_q(osp(1|2))$ and we show that the fusion matrix of $N = 1$ supersymmetric Liouville theory corresponds to the 6j symbols relating different tensor product decompositions of representations. In the following two chapters we study the graded Heisenberg double and present the quantum superplane, which are thought to be relevant for quantisation of the theory of super Riemann surfaces, i.e. super Teichmüller theory. Lastly, the appendices summarise the basics of Lie algebras and discuss the hyperbolic special functions and their integral identities which have been used in the calculations from the body of the text.

# 2 Non-graded Hopf algebras

The algebraic methods have found their place in the toolkit of modern theoretical physics as its essential part, especially in the context of the quantum theory. Quantum mechanics and quantum field theory use profusely the representation theory of Lie algebras — it is used to define the crucial quantities appearing in the theories, as well as a method to tackle the symmetries which are present. Especially in quantum field theory one knows that the spaces of states form modules of particular algebras, like Virasoro algebra and affine Lie algebras in the case of conformal field theory.

In this section we intend to present the basic notions concerning Hopf algebras, including the presentation theorem and several well known examples. The Drinfeld double construction of quasi-triangular Hopf algebras will also be given, as well as the related Heisenberg double construction — both of which will be illustrated by relevant examples.

## 2.1 Quantum groups

In this section we present brief introduction to the nongraded quantum groups. For more detailed treatment one can consult [1, 2, 3].

We will start by introducing the abstract notions of algebra and coalgebra and then follow up with the definition of Hopf algebra, which marries those two objects in a self-consistent way. Among Hopf algebras one finds the special class which distinguishes themselves by the existence of a special objects called universal R-matrices.

### 2.1.1 Algebras and bialgebras

Let $k$ be a field.

**Definition 1** *The unital associative algebra is a triple $(\mathcal{A}, m, \eta)$, where $A$ is a vector space, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a multiplication map and $\eta : k \to \mathcal{A}$ is an unital map, such that the following axioms are satisfied:*

$$m(m \otimes id) = m(id \otimes m), \tag{11}$$

$$m(\eta \otimes id) = id = m(id \otimes \eta). \tag{12}$$

**Definition 2** *The counital coassociative coalgebra is a triple $(\mathcal{A}, \Delta, \epsilon)$, where $A$ is a vector space, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a comultiplication map and $\epsilon : \mathcal{A} \to k$ is a counital map, such that the following axioms are satisfied:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta, \tag{13}$$

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta. \tag{14}$$

### 2.1.2 Hopf algebras

**Definition 3** *A Hopf algebra is a collection $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$, where:*

- *$(\mathcal{A}, m, \eta)$ is an unital associative algebra, $(\mathcal{A}, \Delta, \epsilon)$ — a counital coassociative coalgebra.*

- $\Delta, \epsilon$ *are unital algebra homomorphism (i.e. function $f$ such that $m(f \otimes f) = fm$ and $f\eta = \eta$).*

- *The linear antipodal map $S : \mathcal{A} \to \mathcal{A}$ satisfies $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\epsilon$.*

One can show (c.f. [2]) that the condition of $\Delta, \epsilon$ being unital algebra homomorphisms is equivalent to the condition that $m, \eta$ are counital coalgebra homomorphisms (i.e. functions $f$ such that $(f \otimes f)\Delta = \Delta f$ and $\epsilon f = \epsilon$).

**Proposition 1** *Let $\mathcal{A}$ be a Hopf algebra, and $S$ its antipode. $S$ is a unital anti-algebra morphism and a counital anti-coalgebra morphism (i.e. $Sm = m(S \otimes S)\Sigma$ and $\Delta S = \Sigma(S \otimes S)\Delta$, where $\Sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Sigma(a \otimes b) = b \otimes a$).* PROOF: *C.f. for example [2].* $\square$

One can introduce the notion of a dual to a Hopf algebra.

**Proposition 2** *Let $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$ be a finite-dimensional Hopf algebra. Then $(\mathcal{A}^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is also a Hopf algebra.*
PROOF: *One can use the standard algebraic bracket $\langle , \rangle : \mathcal{A} \otimes \mathcal{A}^* \to k$. Then, multiplication axioms of $m$ correspond to comultiplication axioms of $m^*$, and comultiplication axioms of $\Delta$ to multiplication ones of $\Delta^*$:*

$$\langle m(m \otimes id)(a \otimes b \otimes c), d \rangle = \langle m(1 \otimes m)(a \otimes b \otimes c), d \rangle,$$

$$\langle (m \otimes id)(a \otimes b \otimes c), m^*(d) \rangle = \langle (1 \otimes m)(a \otimes b \otimes c), m^*(d) \rangle,$$

$$\langle a \otimes b \otimes c, (m^* \otimes id)m^*(d) \rangle = \langle a \otimes b \otimes c, (1 \otimes m^*)m^*(d) \rangle.$$

*for $a, b, c \in \mathcal{A}, d \in \mathcal{A}^*$, and analogous for $\Delta$. Unitality of $\eta$ corresponds to counitality of $\eta^*$ and counitality of $\epsilon$ corresponds to unitality of $\epsilon^*$.* $\square$

In the case of infinite-dimensional Hopf algebras, since in general $\mathcal{A}^* \otimes \mathcal{A}^*$ and $(\mathcal{A} \otimes \mathcal{A})^*$ are not isomorphic, $m^*$ is not properly defined as a coproduct. However by more subtle consideration one can properly define the duality in case of infinite-dimensional Hopf algebras (c.f. [4]).

### 2.1.3 Classical examples of Hopf algebras

One can consider the classical examples of Hopf algebras derived from a finite group $G$. Starting from a finite group $G$, one can construct two unital algebras:

- The function algebra $\mathcal{F}(G) = \{f : G \to k\}$ with algebra structure:

$$(f_1 + \lambda f_2)(g) = f_1(g) + \lambda f_2(g),$$

$$m(f_1 \otimes f_2)(g) = f_1(g)f_2(g),$$

$$\eta(\lambda) = \lambda 1,$$

  where $f, g \in \mathcal{F}(G), \lambda \in k$. One can endow this with coalgebra structures: a coproduct $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \times \mathcal{F}(G)$:

$$\Delta(f)(g, h) = f(gh),$$

and a counit $\epsilon : \mathcal{F}(G) \to k$:

$$\epsilon(f) = f(e),$$

where $e \in G$ is the neutral element of $G$. Finally, the antipode is given by

$$S(f)(x) = f(x^{-1}).$$

PROOF: One can easily show the associativity of $m$:

$$[m(m \otimes 1)](f, g, h)(x) = [f(x)g(x)]h(x) = f(x)[g(x)h(x)] = [m(1 \otimes m)](f, g, h)(x),$$

and the coassociativity of $\Delta$:

$$(\Delta \otimes 1)\Delta(f)(x, y, z) = \Delta(f)(xy, z) = f((xy)z) =$$
$$= f(x(yz)) = \Delta(f)(x, yz) = (1 \otimes \Delta)\Delta(f)(x, y, z),$$

which follows from the associativity of multiplication in $G$. The axiom for the unit is satisfied as follows

$$m(f \otimes \eta(\lambda))(x) = m(f \otimes \lambda 1)(x) = f(x)\lambda =$$
$$= \lambda f(x) = m(\lambda 1 \otimes f)(x) = m(\eta(\lambda) \otimes f)(x),$$

and for the counit:

$$(1 \otimes \epsilon)\Delta(f)(x) = f(xe) = f(x) = f(ex) = (\epsilon \otimes 1)\Delta(f)(x).$$

Also one has to consider the consistency conditions for the bialgebra, among others:

$$\Delta(fg)(x, y) = f(xy)g(xy) = \Delta(f)(x, y)\Delta(g)(x, y) =$$
$$= (\Delta(f)\Delta(g))(x, y),$$
$$\eta(\epsilon(f))(x) = f(e)1(x) = f(e)1 = \epsilon(f)1.$$

The axioms for $S$ are verified:

$$\begin{aligned} S(fg)(x) &= f(x^{-1})g(x^{-1}) = g(x^{-1})f(x^{-1}) = S(g)(x)S(f)(x), \\ m(S \otimes 1)\Delta(f)(x) &= f(x^{-1}x) = f(e) = f(xx^{-1}) = \\ &= m(1 \otimes S)\Delta(f)(x) = \epsilon(f)1 = f(e)1. \end{aligned}$$

$\square$

- The group algebra $k[G]$, where $k[G]$ is a vector space freely generated by $G$ with a product induced from the product of $G$:

$$(\sum_g \lambda_g g)(\sum_h \mu_h h) = \sum_{g,h} \lambda_g \mu_h (gh),$$

where $g, h \in G, \lambda \in k$ (and the sums are properly defined because of the finiteness of $G$). One has a coproduct $\Delta : k[G] \to k[G] \otimes k[G]$:

$$\Delta(\lambda_g g) = \lambda_g g \otimes g,$$

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and a counit $\epsilon : k[G] \to k$:

$$\epsilon(g) = 1.$$

The antipode is given by

$$S(g) = g^{-1}.$$

One can define a duality between $\mathcal{F}(G)$ and $k[G]$ by a non-degenerate bracket $\langle,\rangle : \mathcal{F}(G) \otimes k[G] \to k$ such that $\langle f, g \rangle = f(g)$ is just an evaluation of the function, where $f \in \mathcal{F}(G), g \in G$ (with extention to $k[G]$ by linearity).

The bracket $\langle,\rangle$ indeed induces linear isomorphisms $k[G] \to \mathcal{F}(G)^*, g \to \langle,g \rangle$ and $\mathcal{F}(G) \to k[G]^*, f \to \langle f, \rangle$. Then two structures are dual to each other, i.e. $\mathcal{F}(G)^* = k[G]$ and $k[G]^* = \mathcal{F}(G)$. Using that one can transport structures between $\mathcal{F}(G)$ and $k[G]$ — one can show that algebra structures on one side correspond to coalgebra structures on the other side of the bracket.

### 2.1.4 Quasi-triangular Hopf algebras

**Definition 4** *Let $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ be a bialgebra. An invertible element $R = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}$ is called a universal R-matrix if it satisfies*

$$\Delta^{op}(a) = R\Delta(a)R^{-1}, \tag{15}$$

$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{16}$$

$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{17}$$

*where $\Delta^{op} = \Sigma\Delta, a \in \mathcal{A}, R_{12} = R \otimes 1, R_{23} = 1 \otimes R, R_{13} = \sum_i a_i \otimes 1 \otimes b_i$ and $\Sigma$ is a flip map defined as in proposition 1.*

**Proposition 3** *For the universal R-matrix the quantum Yang-Baxter equation is satisfied*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{18}$$

PROOF: *One has*

$$((\Sigma\Delta) \otimes id)R = \sum_i (\Sigma\Delta(a_i)) \otimes b_i = \sum_i R_{12}\Delta(a_i)R_{12}^{-1} \otimes b_i =$$

$$= R_{12} \left( \sum_i \Delta(a_i) \otimes b_i \right) R_{12}^{-1} = R_{12}((\Delta \otimes id)R)R_{12}^{-1} =$$

$$= R_{12}R_{13}R_{23}R_{12}^{-1},$$

*and on the other hand*

$$((\Sigma\Delta) \otimes id)R = \Sigma_{12}(\Delta \otimes id)R = \Sigma_{12}(R_{13}R_{23}),$$

*where $\Sigma_{12} = \Sigma \otimes id$. Comparison gives the claim.* $\square$

**Definition 5** *Let $\mathcal{A}$ be a Hopf algebra. If there exists the universal R-matrix $R \in \mathcal{A} \otimes \mathcal{A}$ then $\mathcal{A}$ is called a quasi-triangular Hopf algebra.*

**Definition 6** *Let $\mathcal{A}$ be a quasi-triangular Hopf algebra. If $\mathcal{A}$ is noncocommutative, then $\mathcal{A}$ is called a quantum group.*

### 2.1.5  Lie algebras $\mathfrak{g}$ and the enveloping algebras $U(\mathfrak{g})$

The basic notions about Lie algebras, like Cartan matrices and root systems, are presented in appendix A.

Let $\mathfrak{g}$ be a Lie algebra. It has been shown by Serre that one can present any simple Lie algebra by the generators and relations between them, which depend only on the choice of a Cartan matrix of $\mathfrak{g}$.

**Theorem 1** *(Serre presentation theorem) Let $A = [A_{ij}]$ be a Cartan matrix of the root system $\mathfrak{R}$ of rank $n$. Let $\mathfrak{g}$ be the Lie algebra defined by $3n$ generators $X_i, Y_i, H_i$ and by the relations*

$$[H_i, H_j] = 0, \tag{19}$$

$$[X_i, Y_j] = \delta_{i,j} H_i, \tag{20}$$

$$[H_i, X_j] = A_{ij} X_j, \quad [H_i, Y_j] = -A_{ij} Y_j, \tag{21}$$

$$ad(X_i)^{1-A_{ij}}(X_j) = 0, \qquad i \neq j, \tag{22}$$

$$ad(Y_i)^{1-A_{ij}}(Y_j) = 0, \qquad i \neq j, \tag{23}$$

*where the two last conditions are called the Serre relations. $\mathfrak{g}$ is a simple Lie algebra, with subalgebra $\mathfrak{h}$ generated by the elements $H_i$ as a Cartan subalgebra; its Cartan matrix is $A$.*

**Example:** $sl(2)$. One can choose $A_{ij} = 2$, $i, j = 1$. Then one has three generators $H, X, Y$, which satisfy commutation relations:

$$[X_i, Y_j] = H,$$

$$[H_i, X_j] = 2X_j,$$

$$[H_i, Y_j] = -2Y_j.$$

These operators form the Chevalley basis and generate the Lie algebra $sl(2)$.

**Definition 7** *Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an associative algebra for which:*

- *there exists a linear map $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that $\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x), x, y \in \mathfrak{g}$;*

- *for every associative algebra $\mathcal{A}$ with a homomorphism $j : g \to \mathcal{A}$ as above, there exists a unique homomorphism of algebras $\phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ such that $j = \phi \circ \iota$.*

**Theorem 2** *(Poincaré-Birkhoff-Witt theorem) Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ be a universal enveloping algebra. Let $\{x_i\}_{i=1}^{n}$ be the basis of $\mathfrak{g}$. Then $\mathcal{U}(\mathfrak{g})$ is infinite dimensional and a set $\{\prod_{i=1}^{k} \iota(x_i)^{a_i}\}_{k=1}^{n}, a_i \in \mathbb{N}$ is the basis of $\mathcal{U}(\mathfrak{g})$.*

**Example:** $\mathcal{U}(sl(2))$ is generated by the elements $i(X), i(Y), i(H)$. As a vector space, it has the basis vectors of the form $i(X)^a i(H)^b i(Y)^c$, where $a, b, c \in \mathbb{Z}_{\geq 0}$. One has relations:

$$i(Y)i(X) = i(X)i(Y) - i(H),$$

$$i(H)i(X) = i(X)i(H) + 2i(X),$$

$$i(Y)i(H) = i(H)i(Y) + 2i(Y).$$

There exists an equivalent version of Serre presentation theorem for the universal enveloping algebras, which is a direct generalisation of the presentation theorem for Lie algebras.

**Theorem 3** *(Presentation theorem) Let $A = [A_{ij}]$ be a Cartan matrix of the root system $\mathfrak{R}$ of rank $n$. Let $\mathfrak{g}$ be the Lie algebra defined by $3n$ generators $X_i, Y_i, H_i$ and the generators $x_i, y_i, h_i$ be the images of the former in $\mathcal{U}(\mathfrak{g})$. One obtains the associative algebra generated by $x_i, y_i, h_i$ with relations*

$$h_i h_j - h_j h_i = 0,$$

$$x_i y_j - y_j x_i = \delta_{i,j} h_i,$$

$$h_i x_j - x_j h_i = A_{ij} x_j,$$

$$h_i y_j - y_j h_i = -A_{ij} y_j,$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1 - A_{ij}}{k} x_i^{1-A_{ij}-k} x_j x_i^k = 0, \qquad i \neq j,$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1 - A_{ij}}{k} y_i^{1-A_{ij}-k} y_j y_i^k = 0, \qquad i \neq j.$$

One can add that every representation $\rho : \mathfrak{g} \to EndV$ (where $V$ is a vector space) extends uniquely to a homomorphism $\tilde{\rho} : \mathcal{U}(\mathfrak{g}) \to EndV$. Conversely, every representation of $\mathcal{U}(\mathfrak{g})$ restricted to $\mathfrak{g}$ is the representation of $\mathfrak{g}$.

### 2.1.6 q-deformations $U_q(\mathfrak{g})$ of the enveloping algebras

**Definition 8** *Let $k \in \mathbb{N}$. The q-integer is*

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

*More generally, for $d \in \mathbb{N}/\{0\}$ one has*

$$[k]_{q^d} = \frac{q^{dk} - q^{-dk}}{q^d - q^{-d}}.$$

Let $[A_{ij}]$ be a Cartan matrix of $\mathfrak{g}$. When $A$ is not symmetric, there exist coprime positive integers $\{d_i\}_{i=1}^n$ (where $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$, where $\alpha_i$ are roots) such that $d_i A_{ij} = d_j A_{ji}$. One sets $q_i = q^{d_i}$.

**Definition 9** *The $\mathcal{U}_q(\mathfrak{g})$ is an associative algebra generated by $x_i, y_i, K_i, K_i^{-1}$ with relations*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{24}$$

$$K_i K_j - K_j K_i = 0, \tag{25}$$

$$x_i y_j - y_j x_i = \delta_{i,j} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}, \tag{26}$$

$$K_i x_j = q^{\frac{A_{ij}}{2}} x_j K_i, \tag{27}$$

$$K_i y_j = q^{-\frac{A_{ij}}{2}} y_j K_i. \tag{28}$$

**Theorem 4** *Let $\mathcal{U}_q(\mathfrak{g})$ be a algebra generated by $x_i, y_i, K_i, K_i^{-1}$ with appropriate relations. Then $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ with*

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(x_i) = x_i \otimes K_i + K_i^{-1} \otimes x_i,$$

$$\Delta(y_i) = y_i \otimes K_i + K_i^{-1} \otimes y_i,$$

$$\epsilon(K_i) = 1, \quad \epsilon(x_i) = \epsilon(y_i) = 0,$$

$$S(K_i) = K_i^{-1}, \quad S(x_i) = -q_i x_i, \quad S(y_i) = -q_i^{-1} y_i,$$

*is a noncocommutative Hopf algebra.*

PROOF: *For $\Delta$ it is enough to show that $\Delta x_i, \Delta K_i, \Delta y_i$ satisfy the defining relations of $\mathcal{U}_q(\mathfrak{g})$ and the axioms for the generators are verified. For example:*

$$\Delta(K_i)\Delta(x_j) = (K_i \otimes K_i)(x_j \otimes K_j + K_j^{-1} \otimes x_j) =$$
$$= q^{\frac{A_{ij}}{2}} x_j K_i \otimes K_j K_i + q^{\frac{A_{ij}}{2}} K_j^{-1} K_i \otimes x_j K_i =$$
$$= q^{\frac{A_{ij}}{2}} \Delta(x_j)\Delta(K_i),$$

$$\Delta(K_i)\Delta(y_j) = (K_i \otimes K_i)(y_j \otimes K_j + K_j^{-1} \otimes y_j) =$$
$$= q^{-\frac{A_{ij}}{2}} y_j K_i \otimes K_j K_i + q^{-\frac{A_{ij}}{2}} K_j^{-1} K_i \otimes y_j K_i =$$
$$= q^{-\frac{A_{ij}}{2}} \Delta(y_j)\Delta(K_i),$$

$$[\Delta(x_i), \Delta(y_j)] = (x_i \otimes K_i + K_i^{-1} \otimes x_i)(y_j \otimes K_j + K_j^{-1} \otimes y_j) -$$
$$- (y_j \otimes K_j + K_j^{-1} \otimes y_j)(x_i \otimes K_i + K_i^{-1} \otimes x_i) =$$
$$- y_j x_i \otimes K_j K_i + y_j K_i^{-1} \otimes K_j x_i + K_j^{-1} x_i \otimes y_j K_i + K_j^{-1} K_i^{-1} \otimes y_j x_i =$$
$$= [x_i, y_j] \otimes K_i K_j + K_i^{-1} K_j^{-1} \otimes [x_i, y_i] =$$
$$= \frac{\delta_{ij}}{q - q^{-1}} \left( K_i^2 \otimes K_i^2 - K_i^{-2} \otimes K_i^{-2} \right) =$$
$$= \delta_{ij} \frac{\Delta(K_i)^2 - \Delta(K_i)^{-2}}{q - q^{-1}}.$$

*The proof of other axioms is straightforward.* $\square$

**Theorem 5** *The $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ as above is a quantum group, i.e. there exists the universal R-matrix R.*

For reference consult e.g. [2].

### 2.1.7 Examples of quantum groups

One can consider an example of quantum group relevant for future consideration, i.e. $\mathcal{U}_q(sl(2))$.

**Example:** $\mathcal{U}_q(sl(2))$

$\mathcal{U}_q(sl(2))$ is generated by $K, K^{-1}, x, y$ satisfying relations:

$$KK^{-1} = K^{-1}K = 1,$$
$$xy - yx = \frac{K^2 - K^{-2}}{q - q^{-1}}, \tag{29}$$
$$Kx = qxK, \quad Ky = q^{-1}yK,$$

and with $\Delta, \epsilon, S$ such that

$$\Delta(K) = K \otimes K,$$
$$\Delta(x) = x \otimes K + K^{-1} \otimes x,$$
$$\Delta(y) = y \otimes K + K^{-1} \otimes y, \tag{30}$$
$$\epsilon(K) = 1, \quad \epsilon(x) = \epsilon(y) = 0,$$
$$S(K) = K^{-1}, \quad S(x) = -qx, \quad S(y) = -qy.$$

In the future, we will be using a more convenient set of generators $K, K^{-1}, E^{\pm}$ such that $E^+ = ix, E^- = iy$, better suited to the study of non-compact versions of $\mathcal{U}_q(sl(2))$. In that case the commutation relations have the form:

$$KK^{-1} = K^{-1}K = 1,$$

$$\left[E^+, E^-\right] = -\frac{K^2 - K^{-2}}{q - q^{-1}}, \tag{31}$$

$$KE^{\pm} = q^{\pm 1}E^{\pm}K.$$

One can consider $\mathcal{U}(sl(2))$ as a classical limit of $\mathcal{U}_q(sl(2))$. One can formally define $K = e^{\frac{1}{2}hH}, q = e^h$ and take the limit $h \to 1$, that is $q \to 1$. Then one has the following commutation relations

$$(1 + \frac{1}{2}hH + O(h^2))E^+ = (1 + h + O(h^2))E^+(1 + \frac{1}{2}hH + O(h^2)),$$

$$\left[H, E^+\right] = 2E^+ + O(h),$$

$$(1 + \frac{1}{2}hH + O(h^2))E^- = (1 - h + O(h^2))E^-(1 + \frac{1}{2}hH + O(h^2)),$$

$$\left[H, E^-\right] = -2E^- + O(h),$$

and

$$\left[E^+, E^-\right] = -\frac{1 + hH + O(h^2) - 1 + hH + O(h^2)}{1 + h + O(h^2) - 1 + h + O(h^2)} =$$

$$= -H + O(h),$$

what indeed is nothing else than the commutation relations for $sl(2)$ (or $U(sl(2))$) in the $q \to 1$ limit. The other structures have the limits as follows

$$\Delta(u) = u \otimes 1 + 1 \otimes u,$$

$$\epsilon(u) = 0,$$

$$S(u) = -u,$$

where $u = H, E^{\pm}$.

**Proposition 4** *One can show that operator $C$ which has the form*

$$C = E^-E^+ - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2}, \tag{32}$$

*is the Casimir operator for $\mathcal{U}_q(sl(2))$.*

PROOF:

$$
\begin{aligned}
[C, K] &= \left[E^-E^+, K\right] - \frac{1}{(q - q^{-1})^2}(q\left[K^2, K\right] + q^{-1}\left[K^{-2}, K\right] - 2\left[1, K\right] = \\
&= E^-KE^+(q^{-1} - 1) + E^-K(1 - q^{-1})E^+ = 0, \\
[C, E^+] &= \left[E^-E^+, E^+\right] - \frac{1}{(q - q^{-1})^2}(q\left[K^2, E^+\right] + q^{-1}\left[K^{-2}, E^+\right] - 2\left[1, E^+\right]) = \\
&= \frac{1}{(q - q^{-1})}(K^2E^+ - K^{-2}E^+) - \frac{1}{(q - q^{-1})^2}((q - q^{-1})K^2E^+ - (q - q^{-1})K^{-2}E^+) = 0, \\
[C, E^-] &= \left[E^-E^+, E^-\right] - \frac{1}{(q - q^{-1})^2}(q\left[K^2, E^-\right] + q^{-1}\left[K^{-2}, E^-\right] - 2\left[1, E^-\right]) = \\
&= -\frac{1}{(q - q^{-1})}(E^-K^2 - E^-K^{-2}) - \frac{1}{(q - q^{-1})^2}((q - q^{-1}) - E^-K^2 + (q - q^{-1})E^-K^{-2}) = 0.
\end{aligned}
$$

□

**Proposition 5** *The classical limit $h \to 0$ of the Casimir operator is as follows:*

$$C = E^- E^+ - \frac{1}{4} H^2 - \frac{1}{2} H - \frac{1}{4}.$$

PROOF:

$$
\begin{aligned}
C &= E^- E^+ - \left( 1 + hH + \frac{1}{2}(hH)^2 + O(h^3))(1 + h + \frac{1}{2}h^2 + O(h^3)) + \right. \\
&\quad + (1 - hH + \frac{1}{2}(hH)^2 + O(h^3))(1 - h + \frac{1}{2}h^2 + O(h^3)) - 2 \bigg) \times \\
&\quad \times \frac{1}{(1 + h + O(h^2) - 1 + h + O(h^2))^2} = \\
&= E^- E^+ - \frac{1}{4} H^2 - \frac{1}{2} H - \frac{1}{4} + O(h).
\end{aligned}
$$

□

## 2.2 Drinfeld double

There exist various methods of obtaining examples of quasi-triangular Hopf algebras. In particular, Drinfeld presented a construction which allows to acquire a quasi-triangular Hopf algebra starting from an arbitrary Hopf algebra.

Starting from a pair of Hopf algebras (which are dual to each other with respect to the usual bracket) one can construct a larger Hopf algebra, which contains the starting pair as Hopf subalgebras. The specific extension of multiplication and comultiplication of subalgebras to the entire Hopf algebra ensures the existence of a universal R-matrix.

**Definition 10** *Let $\mathcal{A}$ and $\mathcal{A}^*$ be a pair of dual Hopf algebras generated by basis elements $E_\alpha$, $E^\alpha$, $\alpha \in I$ respectively with multiplication and co-multiplication*

$$E_\alpha E_\beta = m_{\alpha\beta}^\gamma E_\gamma, \tag{33}$$

$$\Delta(E_\alpha) = \mu_\alpha^{\beta\gamma} E_\beta \otimes E_\gamma, \tag{34}$$

$$S(E_\alpha) = S_\alpha^\beta E_\beta, \tag{35}$$

*and*

$$E^\alpha E^\beta = \mu_\gamma^{\alpha\beta} E^\gamma, \tag{36}$$

$$\Delta(E^\alpha) = m_{\gamma\beta}^\alpha E^\beta \otimes E^\gamma, \tag{37}$$

$$S(E^\alpha) = (S^{-1})_\beta^\alpha E^\beta. \tag{38}$$

*One can define the Drinfeld double $D(\mathcal{A})$ as a vector space $D(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^*$ with basis elements $E_\alpha \otimes E^\beta$ which satisfy the double's defining relations*

$$(E_\alpha \otimes 1)(E_\beta \otimes 1) = m_{\alpha\beta}^\gamma (E_\gamma \otimes 1), \tag{39}$$

$$(1 \otimes E^\alpha)(1 \otimes E^\beta) = \mu_\gamma^{\alpha\beta}(1 \otimes E^\gamma), \tag{40}$$

$$(1 \otimes E^\alpha)(E_\beta \otimes 1) = (S^{-1})_\phi^\epsilon \mu_\beta^{\mu\nu} \mu_\mu^{\rho\gamma} m_{\nu\sigma}^\alpha m_{\delta\epsilon}^\sigma (E_\gamma \otimes 1)(1 \otimes E^\delta), \tag{41}$$

*and coproducts and antipodes inherited from $\mathcal{A}$ and $\mathcal{A}^*$ in usual way. Alternatively, instead of the last equation one can use*

$$\mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^\beta (E_\sigma \otimes 1)(1 \otimes E^\rho) = m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma} (1 \otimes E^\rho)(E_\sigma \otimes 1),$$

*as a defining formula.*

It is clear that the Drinfeld double defined as above is a Hopf algebra, however, we want to show something more — that it is a quasi-triangular Hopf algebra.

**Theorem 6** *Consider the canonical element $R = (E_\alpha \otimes 1)(1 \otimes E^\alpha$. R satisfies Yang-Baxter relation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{42}$$

PROOF: *With a slight abuse of notation, allow us to denote element $E_\alpha \otimes 1$ just as $E_\alpha$ and $1 \otimes E^\alpha$ just as $E^\alpha$.*

*Then, using the definition of $R$ one has*

$$R_{12}R_{13}R_{23} = (E_\alpha \otimes E^\alpha \otimes 1)(E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\gamma \otimes E^\gamma) =$$

$$= E_\alpha E_\beta \otimes E^\alpha E_\gamma \otimes E^\beta E^\gamma = E_\sigma \otimes m^\sigma_{\alpha\beta}\mu^{\beta\gamma}_\rho E^\alpha E_\gamma \otimes E^\rho =$$

$$= E_\sigma \otimes m^\sigma_{\beta\alpha}\mu^{\gamma\beta}_\rho E_\gamma E^\alpha \otimes E^\rho = m^\sigma_{\beta\alpha}E_\sigma \otimes E_\gamma E^\alpha \otimes \mu^{\gamma\beta}_\rho E^\rho =$$

$$= E_\beta E_\alpha \otimes E_\gamma E^\alpha \otimes E^\gamma E^\beta = (1 \otimes E_\gamma \otimes E^\gamma)(E_\beta \otimes 1 \otimes E^\beta)(E_\alpha \otimes E^\alpha \otimes 1) =$$

$$= R_{23}R_{13}R_{12}.$$

$\square$

Let us illustrate this construction with an example. Previously we cosidered the q-deformed universal enveloping algebras of Lie algebras, in particular $\mathcal{U}_q(sl(2))$. It is possible to construct it as a Drinfeld double of the the Borel half $U_q(\mathcal{B})$ of $U_q(sl(2))$, which we will take as the algebra $\mathcal{A}$. Let us begin from the elements $H, E$ satisfying the following relations

$$[H, E] = E,$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(E) = E \otimes e^{hH} + 1 \otimes E.$$

Additionally, let us set $q = e^{-h}$. Then, the algebra $\mathcal{A}$ will have basis elements of the form

$$E_{m,n} = \frac{1}{m!(q)_n}H^m E^n,$$

where the q-factorial in the normalisation is defined as

$$(q)_0 = 1,$$

$$(q)_n = (1 - q)...(1 - q^n), \quad n > 0.$$

The multiplication and comultiplication for those elements can be found from those for the elements $H$ and $E$ and has the form

$$E_{m,n}E_{l,k} = \sum_{j=0}^{l} \binom{m+j}{j}\binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!}E_{m+j,n+k},$$

$$\Delta(E_{n,m}) = \sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{p=0}^{\infty} \binom{k+p}{k}(m-l)^p h^p E_{n-k,m-l} \otimes E_{k+p,l},$$

where the quantum Newton symbol is defined as $\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$. Now, let us consider the dual algebra $\mathcal{A}^*$. It is generated by the elements $\bar{H}, F$ satisfying

$$[\bar{H}, F] = -hF,$$

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H},$$

$$\Delta(F) = F \otimes e^{-\bar{H}} + 1 \otimes F,$$

and the basis elements have the form

$$E^{n,m} = \bar{H}^n F^m.$$

The multiplication and comultiplication for the above have the form

$$E^{m,n}E^{l,k} = \sum_{j=0}^{l} \binom{l}{j}(n)^{l-j}h^{l-j}E^{m+j,n+k},$$

$$\Delta(E^{n,m}) = \sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{p=0}^{\infty} \binom{n}{k}\binom{m}{l}_q \frac{(-m+l)^p}{p!}E^{n-k,m-l} \otimes E^{k+p,l}.$$

It is clear that the bases $\{E_{n,m}\}$ and $\{E^{n,m}\}$ are dual to each other in a sense that the map $E_{n,m} \to (E^{n,m})^*$ is an isomorphism of Hopf algebras and the multiplication and comultiplication coefficients are given explicitly by

$$m_{m,n;l,k}^{r,s} = \sum_{j=0}^{l} \binom{m+j}{j}\binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!}\delta_{r,m+j}\delta_{s,n+k} =$$

$$= \binom{r}{r-m}\binom{n+k}{k}_q \frac{(-n)^{l-r+m}}{(l-r+m)!}\Theta(r-m)\Theta(l-r+m)\delta_{s,n+k},$$

$$\mu_{r,s}^{m,n;l,k} = \sum_{j=0}^{l} \binom{l}{j}(n)^{l-j}h^{l-j}\delta_{r,m+j}\delta_{s,n+k} =$$

$$= \binom{l}{r-m}(n)^{l-r+m}h^{l-r+m}\Theta(r-m)\Theta(l-r+m)\delta_{s,n+k}.$$

Then one can show that Drinfeld double is isomorphic to the q-deformation of $U(sl(2))$

$$U_q(sl(2)) \cong D(U_q(\mathcal{B}))/(\bar{H} - hH).$$

Now one can consider the universal R-matrix. Using the formula for a canonical element of the Drinfeld double we obtain

$$R = \exp(H \otimes \bar{H})(E \otimes F; q)_{\infty}^{-1},$$

where one uses the fact that

$$(x; q)_{\infty}^{-1} = \prod_{k=0}^{\infty} \frac{1}{1 - xq^k}.$$

## 2.3 Heisenberg double

The Drinfeld double construction allows one to construct the solution to the Yang-Baxter equation, which in this case is an universal R-matrix. However, there are more interesting and physically relevant equations for which algebraic methods of constructing solutions would be extremely useful. In particular, one of them is the pentagon equation.

In the same way as the universal R-matrix in the case of Drinfeld double, the Heisenberg double is defined so that the existence of a canonical element satisfying the pentagon equation is ensured [6]. However, even though the starting point in both constructions are the same, the Heisenberg double is only an algebra, and not a Hopf algebra. Nonetheless, those two notions are indeed related, which will be specified more precisely below.

Let $\mathcal{A}$ be a bialgebra spanned by the basis vectors $\{e_\alpha\}$ with the following multiplication and comultiplication:

$$e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \tag{43}$$

$$\Delta(e_\alpha) = \mu_\alpha^{\beta\gamma} e_\beta \otimes e_\gamma. \tag{44}$$

Moreover, the bialgebra $\mathcal{A}^*$ spanned by the basis vectors $\{e^\alpha\}$ with:

$$e^\alpha e^\beta = \mu_\gamma^{\alpha\beta} e^\gamma, \tag{45}$$

$$\Delta(e^\alpha) = m_{\beta\gamma}^\alpha e^\beta \otimes e^\gamma, \tag{46}$$

and is dual to $\mathcal{A}$ with respect to a duality bracket $(,): \mathcal{A} \times \mathcal{A}^* \to \mathbb{C}$ such that

$$(e_\alpha, e^\beta) = \delta_\alpha^\beta,$$

and it preserve the algebraic structures

$$(e_\alpha, e^\rho e^\sigma) = (\Delta(e_\alpha), e^\rho \otimes e^\sigma),$$

$$(e_\alpha e_\beta, e^\rho) = (e_\alpha \otimes e_\beta, \Delta(e^\rho)).$$

**Definition 11** *The Heisenberg double $H(\mathcal{A})$ is an algebra s.t. as a vector space $H(\mathcal{A}) \cong \mathcal{A} \otimes \mathcal{A}^*$ generated by the elements $\{e_\alpha \otimes e^\beta\}$, $\alpha, \beta \in I$, with multiplication*

$$(e_\alpha \otimes 1)(e_\beta \otimes 1) = m_{\alpha\beta}^\gamma(e_\gamma \otimes 1), \tag{47}$$

$$(1 \otimes e^\alpha)(1 \otimes e^\beta) = \mu_\gamma^{\alpha\beta}(1 \otimes e^\gamma), \tag{48}$$

$$(e_\alpha \otimes 1)(1 \otimes e^\beta) = m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma}(1 \otimes e^\rho)(e_\sigma \otimes 1). \tag{49}$$

**Theorem 7** *Then the canonical element $S = e_\alpha \otimes 1 \otimes 1 \otimes e^\alpha \in H(\mathcal{A}) \otimes H(\mathcal{A})$ satisfies the pentagon equation*

$$S_{12}S_{13}S_{23} = S_{23}S_{12}.$$

PROOF: *Let us denote $e_\beta \otimes 1$ as $e_\beta$ and $1 \otimes e^\gamma$ as $e^\gamma$. Using the definition of the canonical element*

$$S_{12}S_{13}S_{23} = (e_\alpha \otimes e^\alpha \otimes 1)(e_\beta \otimes 1 \otimes e^\beta)(1 \otimes e_\gamma \otimes e^\gamma) = e_\alpha e_\beta \otimes e^\alpha e_\gamma \otimes e^\beta e^\gamma =$$

$$= m_{\alpha\beta}^\rho e_\rho \otimes e^\alpha e_\gamma \otimes \mu_\sigma^{\beta\gamma} e^\sigma = e_\rho \otimes m_{\alpha\beta}^\rho \mu_\sigma^{\beta\gamma} e^\alpha e_\gamma \otimes e^\sigma = e_\rho \otimes e_\sigma e^\rho \otimes e^\sigma =$$

$$= (1 \otimes e_\rho \otimes e^\rho)(e_\sigma \otimes e^\sigma \otimes 1) = S_{23}S_{12}.$$

$\square$

Let us consider some Heisenberg algebras as examples.

- Consider the Hopf algebra of monomials $\mathcal{A}$ with a basis $\{e_n\}$, $n \in \mathbb{N}$ s.t.

$$e_m = \frac{x^m}{m!},$$

$$e_n e_m = \binom{n+m}{n} e_{n+m},$$

$$\Delta(e_n) = \sum_{k=0}^n e_{n-k} \otimes e_k,$$

where $\binom{n+m}{n} = \frac{(n+m)!}{n!m!}$, and the Hopf algebra $\mathcal{A}^*$ with a basis $\{e^n\}$, $n \in \mathbb{N}$ s.t.

$$e^m = \bar{x}^m,$$

$$e^n e^m = e^{n+m},$$

$$\Delta(e^n) = \sum_{k=0}^{n} \binom{n}{k} e^{n-k} \otimes e^k.$$

Then, the Heisenberg double $H(\mathcal{A})$ is generated by the basis elements $\{e_n \otimes e^m\}$, $n, m \in \mathbb{N}$ s.t.

$$(e_n \otimes 1)(e_m \otimes 1) = \binom{n+m}{n}(e_{n+m} \otimes 1),$$

$$(1 \otimes e^n)(1 \otimes e^m) = (1 \otimes e^{n+m}),$$

$$(e_n \otimes 1)(1 \otimes e^m) = \sum_{s=0}^{n} \binom{m}{n-s}(1 \otimes e^{m-n+s})(e_s \otimes 1).$$

In particular,

$$x\bar{x} - \bar{x}x = 1.$$

The canonical element has the form

$$S = \exp(x \otimes \bar{x}).$$

- Now, consider the Borel half $U_q(\mathcal{B})$ of $U_q(sl(2))$ as the algebra $\mathcal{A}$. It is generated by the elements $H, E$ satisfying the following relations

$$[H, E] = E,$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(E) = E \otimes e^{hH} + 1 \otimes E.$$

As usual $q = e^{-h}$. The algebra $\mathcal{A}$ will have basis elements of the form

$$e_{m,n} = \frac{1}{m!(q)_n} H^m E^n.$$

The multiplication and comultiplication for those elements has the form

$$e_{m,n} e_{l,k} = \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!} e_{m+j,n+k},$$

$$\Delta(e_{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{k+p}{k}(m-l)^p h^p e_{n-k,m-l} \otimes e_{k+p,l}.$$

The dual algebra $\mathcal{A}^*$ is generated by the elements $\bar{H}, F$ satisfying

$$[\bar{H}, F] = -hF,$$

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H},$$

$$\Delta(F) = F \otimes e^{-\bar{H}} + 1 \otimes F,$$

and the basis elements have the form

$$e^{n,m} = \bar{H}^n F^m.$$

The multiplication and comultiplication for the above have the form

$$e^{m,n}e^{l,k} = \sum_{j=0}^{l} \binom{l}{j}(n)^{l-j}h^{l-j}e^{m+j,n+k},$$

$$\Delta(e^{n,m}) = \sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{p=0}^{\infty} \binom{n}{k}\binom{m}{l}_q \frac{(-m+l)^p}{p!}e^{n-k,m-l} \otimes e^{k+p,l}.$$

By inspection is it clear that the bases $\{e_{n,m}\}$ and $\{e^{n,m}\}$ are dual to each other. From the relation (49) one finds the rest of the commutation relations

$$H\bar{H} = 1 + \bar{H}H,$$
$$E\bar{H} = \bar{H}E,$$
$$HF - FH = -F,$$
$$EF - FE = (1-q)q^{-H}.$$

The canonical element in this case has the form:

$$S = \exp(H \otimes \bar{H})(E \otimes F; q)_{\infty}^{-1}.$$

It was previously stated that the Heisenberg double $H(\mathcal{A})$ is an algebra, but not a Hopf algebra — it follows from the fact that the coproducts inherited from the Hopf algebraic structure of $\mathcal{A}$ and $\mathcal{A}^*$ are not algebra homomorphisms of the multiplication on $H(\mathcal{A})$. This is a substantial difference between the Heisenberg double and the Drinfeld double. However, one can realise the Drinfeld double as a subalgebra of the tensor square of Heisenberg algebras.

Let $H(\mathcal{A})$ be a Heisenberg double from definition 11. Moreover, let us define another Heisenberg double $\tilde{H}(\mathcal{A})$ generated by basis vectors $\{\tilde{e}_\alpha \otimes \tilde{e}_\beta\}$, $\alpha, \beta \in I$ with

$$(\tilde{e}_\alpha \otimes 1)(\tilde{e}_\beta \otimes 1) = m_{\alpha\beta}^{\gamma}(\tilde{e}_\gamma \otimes 1),$$
$$(1 \otimes \tilde{e}^\alpha)(1 \otimes \tilde{e}^\beta) = \mu_\gamma^{\alpha\beta}(1 \otimes \tilde{e}^\gamma),$$
$$(1 \otimes \tilde{e}^\beta)(\tilde{e}_\alpha \otimes 1) = \mu_\alpha^{\sigma\gamma}m_{\gamma\rho}^{\beta}(\tilde{e}_\sigma \otimes 1)(1 \otimes \tilde{e}^\rho),$$

**Theorem 8** *The canonical element $\tilde{S} = \tilde{e}_\alpha \otimes 1 \otimes 1 \otimes \tilde{e}^\alpha$ for the $\tilde{H}(\mathcal{A})$ satisfies "reversed" pentagon equation:*

$$\tilde{S}_{12}\tilde{S}_{23} = \tilde{S}_{23}\tilde{S}_{13}\tilde{S}_{12}.$$

The Heisenberg algebra defined above together with the one from the definition 11 will allow one to make a connection to the Drinfeld double. Let us denote $e_\beta \otimes 1$ as $e_\beta$ and $1 \otimes e^\gamma$ as $e^\gamma$ in the subsequent.

**Theorem 9** *The Drinfeld double $D(\mathcal{A})$ is realised as a subalgebra of $H(\mathcal{A}) \otimes \tilde{H}(\mathcal{A})$ generated by the elements $\{E_\alpha \otimes E^\beta\}$ s.t.*

$$E_\alpha = \mu_\alpha^{\beta\gamma}e_\beta \otimes \tilde{e}_\gamma, \tag{50}$$
$$E^\alpha = m_{\gamma\beta}^{\alpha}e^\beta \otimes \tilde{e}^\gamma. \tag{51}$$

PROOF: *One ought to show that the double's defining relations (39)-(41) are satisfied. Using the compatibility condition:*

$$\Delta \circ m = (m \otimes m)(id \otimes \tau \otimes id)(\Delta \otimes \Delta),$$

*where $\tau(a \otimes b) = b \otimes a$, which on coordinates reads*

$$m_{\alpha\beta}^{\gamma} \mu_{\gamma}^{\sigma\rho} = \mu_{\alpha}^{\delta\epsilon} \mu_{\beta}^{\eta\xi} m_{\delta\eta}^{\sigma} m_{\epsilon\xi}^{\rho},$$

*one shows:*

$$E_\alpha E_\beta = \mu_\alpha^{\pi\rho} \mu_\beta^{\sigma\tau} (e_\pi \otimes \tilde{e}_\rho)(e_\sigma \otimes \tilde{e}_\tau) =$$
$$= \mu_\alpha^{\pi\rho} \mu_\beta^{\sigma\tau} e_\pi e_\sigma \otimes \tilde{e}_\rho \tilde{e}_\tau = \mu_\alpha^{\pi\rho} \mu_\beta^{\sigma\tau} m_{\pi\sigma}^{\mu} m_{\rho\tau}^{\nu} e_\mu \otimes \tilde{e}_\nu = m_{\alpha\beta}^{\gamma} \mu_\gamma^{\mu\nu} e_\mu \otimes \tilde{e} =$$
$$= m_{\alpha\beta}^{\gamma} E_\gamma,$$

$$E^\alpha E^\beta = m_{\rho\pi}^{\alpha} m_{\tau\sigma}^{\beta} (e^\pi \otimes \tilde{e}^\rho)(e^\sigma \otimes \tilde{e}^\tau) =$$
$$= m_{\rho\pi}^{\alpha} m_{\tau\sigma}^{\beta} e^\pi e^\sigma \otimes \tilde{e}^\rho \tilde{e}^\tau = m_{\rho\pi}^{\alpha} m_{\tau\sigma}^{\beta} \mu_\mu^{\pi\sigma} \mu_\nu^{\rho\tau} e^\mu \otimes \tilde{e}^\nu = m_{\nu\mu}^{\gamma} \mu_\gamma^{\alpha\beta} e^\mu \otimes \tilde{e}^\nu =$$
$$= \mu_\gamma^{\alpha\beta} E^\gamma,$$

*and in addition using associativity and coassociativity:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta,$$
$$\mu_\alpha^{\gamma\beta} \mu_\gamma^{\rho\sigma} = \mu_\alpha^{\rho\gamma} \mu_\gamma^{\sigma\beta}$$
$$m(m \otimes id) = m(id \otimes m),$$
$$m_{\alpha\beta}^{\delta} m_{\delta\gamma}^{\sigma} = m_{\alpha\delta}^{\sigma} m_{\beta\gamma}^{\delta},$$

*it can be shown that*

$$\mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} E_\sigma E^\rho = \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} \mu_\sigma^{ij} (e_i \otimes \tilde{e}_j) m_{lk}^{\rho} (e^k \otimes \tilde{e}^l) =$$
$$= \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} \mu_\sigma^{ij} m_{lk}^{\rho} m_{rg}^{k} \mu_i^{gs} e^r e_s \otimes \tilde{e}_j \tilde{e}^l = (\mu_\alpha^{\sigma\gamma} \mu_\sigma^{ij})(m_{\gamma\rho}^{\beta} m_{lk}^{\rho}) m_{rg}^{k} \mu_i^{gs} e^r e_s \otimes \tilde{e}_j \tilde{e}^l =$$
$$= (\mu_\alpha^{i\sigma} \mu_\sigma^{j\gamma})(m_{\rho k}^{\beta} m_{\gamma l}^{\rho}) m_{rg}^{k} \mu_i^{gs} e^r e_s \otimes \tilde{e}_j \tilde{e}^l = \mu_\alpha^{i\sigma} m_{\rho k}^{\beta} m_{rg}^{k} \mu_i^{gs} e^r e_s \otimes (\mu_\sigma^{j\gamma} m_{\gamma l}^{\rho} \tilde{e}_j \tilde{e}^l) =$$
$$= (\mu_\alpha^{i\sigma} \mu_i^{gs})(m_{\rho k}^{\beta} m_{rg}^{k}) e^r e_s \otimes \tilde{e}^\rho \tilde{e}_\sigma = (\mu_\alpha^{gi} \mu_i^{s\sigma})(m_{kg}^{\beta} m_{\rho r}^{k})(e^r \otimes \tilde{e}^\rho)(e_s \otimes \tilde{e}_\sigma) =$$
$$= \mu_\alpha^{gi} m_{kg}^{\beta} (m_{\rho r}^{k} e^r \otimes \tilde{e}^\rho)(\mu_i^{s\sigma} e_s \otimes \tilde{e}_\sigma) = \mu_\alpha^{gi} m_{kg}^{\beta} E^k E_i.$$

*which is exactly what we set out to prove.* □

Moreover, the universal R-matrix can be expressed in this case by elements $S, \tilde{S}, S' = \tilde{e}_\alpha \otimes e^\alpha, S'' = e_\alpha \otimes \tilde{e}^\alpha$:

$$R_{12,34} = S''_{14} S_{13} \tilde{S}_{24} S'_{23}.$$

PROOF: Using the definitions of canonical elements and multiplication and comultiplication on $H(\mathcal{A})$ and $\tilde{H}(\mathcal{A})$

$$S''_{14} S_{13} \tilde{S}_{24} S'_{23} = (e_\alpha \otimes 1 \otimes 1 \otimes \tilde{e}^\alpha)(e_\beta \otimes 1 \otimes e^\beta \otimes 1)(1 \otimes \tilde{e}_\gamma \otimes 1 \otimes \tilde{e}^\gamma)(1 \otimes \tilde{e}_\delta \otimes e^\delta \otimes 1) =$$
$$= e_\alpha e_\beta \otimes \tilde{e}_\gamma \tilde{e}_\delta \otimes e^\beta e^\delta \otimes \tilde{e}^\alpha \tilde{e}^\gamma = m_{\alpha\beta}^{a} e_a \otimes m_{\gamma\delta}^{b} \tilde{e}_b \otimes \mu_c^{\beta\delta} e^c \otimes \mu_d^{\alpha\gamma} \tilde{e}^d =$$
$$= (\mu_\gamma^{ab} e_a \otimes \tilde{e}_b) \otimes (m_{dc}^{\gamma} e^c \otimes \tilde{e}^d) = E_\alpha \otimes E^\alpha = R_{12,34}.$$

which gives the claim. □

# 3 Representation theory of $\mathcal{U}_q(sl(2))$

## 3.1 Self-dual continuous series for $\mathcal{U}_q(sl(2))$

The goal of this section is to introduce a continuous series of representations of $\mathcal{U}_q(sl(2))$ that has first appeared in a paper by Schmuedgen [31]. A self-dual class among these representations has been discovered by Faddeev [32] and later was analysed by Ponsot and Teschner [12, 13] in the context of Liouville theory.

Let us remind the defining relations for the q-deformed universal enveloping algebra $\mathcal{U}_q(sl(2))$ of the Lie algebra $sl(2)$ introduced in previous section. It is generated by the elements $K, K^{-1}, E^{\pm}$, with relations

$$KE^{\pm} = q^{\pm 1}E^{\pm}K, \tag{52}$$

$$[E^+, E^-] = -\frac{K^2 - K^{-2}}{q - q^{-1}}, \tag{53}$$

where $q = e^{i\pi b^2}$ is the deformation parameter. The deformation will be parametrised by a real number b so that $q$ takes values on the unit circle. Given such a choice, the Hopf algebra comes equipped with the following *-structure

$$\begin{aligned} K^* &= K \quad, \\ (E^{\pm})^* &= E^{\pm} . \end{aligned} \tag{54}$$

The tensor product of any two representations can be built with the help of the following co-product

$$\begin{aligned} \Delta(K) &= K \otimes K \quad, \\ \Delta(E^{\pm}) &= E^{\pm} \otimes K + K^{-1} \otimes E^{\pm} . \end{aligned} \tag{55}$$

Let us as well remind the form of a quadratic Casimir element $C$ of $\mathcal{U}_q(sl(2))$ which reads

$$C = E^- E^+ - \frac{qK^2 + q^{-1}K^{-2} + 2}{(q - q^{-1})^2} .$$

Now we will proceed to study a class of self-dual representations of $\mathcal{U}_q(sl(2))$. It is parametrized by a label $\alpha$ taking values in $\frac{Q}{2} + i\mathbb{R}$, where $Q$ is related to the deformation parameter through $Q = b + \frac{1}{b}$. The carrier spaces $\mathcal{P}_\alpha$ of the associated representations consist of entire analytic functions $f(x)$ in one variable $x$ whose Fourier transform $\hat{f}(\omega)$ is meromorphic in the complex plane with possible poles in

$$\mathcal{S}_\alpha := \{ \omega = \pm i(\alpha - Q - nb - mb^{-1}); n, m \in \mathbb{Z}^{\leq 0} \} . \tag{56}$$

On this space, we represent the element $K$ through a shift operator in the imaginary direction,

$$\pi_\alpha(K) = e^{\frac{ib}{2}\partial_x} =: T_x^{\frac{ib}{2}} . \tag{57}$$

By construction, the operator $T_x^{ia}$ defined in the previous equation acts on functions $f \in \mathcal{P}_\alpha$ as

$$T_x^a f(x) := f(x + a) . \tag{58}$$

The expressions for the remaining two generators $E^{\pm}$ are linear combinations of two shift operators in opposite directions

$$\pi_\alpha(E^{\pm}) = e^{\pm 2\pi bx} \frac{e^{\pm i\pi b\alpha} T^{\frac{ib}{2}} - e^{\mp i\pi b\alpha} T^{-\frac{ib}{2}}}{q - q^{-1}} =: e^{\pm 2\pi bx}[(2\pi)^{-1}\partial_x \pm \bar{\alpha}]_b . \tag{59}$$

Here and in the following we shall use the symbol $\bar{\alpha}$ to denote $\bar{\alpha} = Q - \alpha$ and we introduced the following notation

$$[x]_{\mathrm{b}} \;=\; \frac{\sin(\pi \mathrm{b} x)}{\sin(\pi \mathrm{b}^2)}. \tag{60}$$

The representations $\pi_\alpha$ are self-dual in a following sense: let us define a second action $\tilde{\pi}_\alpha$ of $\mathcal{U}_{\tilde{q}}(sl(2))$ with $\tilde{q} = \exp(i\pi/\mathrm{b}^2)$ on the space $\mathcal{P}_\alpha$ through the formulae (57) and (59) with b replaced by $\mathrm{b}^{-1}$. Then, those two actions $\pi_\alpha$ and $\tilde{\pi}_\alpha$ commute with each other.

## 3.2 The Clebsch-Gordan coefficients for $\mathcal{U}_q(sl(2))$

The action $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$ of the quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$ on the tensor product of any two representations $\pi_{\alpha_1}$ and $\pi_{\alpha_2}$ is defined in terms of the coproduct. Such a tensor product is reducible and its decomposition into a direct sum of irreducibles is what defines the Clebsh-Gordan coefficients. In this case at hand, one has the following decomposition,

$$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \;\simeq\; \int_{\frac{Q}{2}+i\mathbb{R}^+}^{\oplus} \mathrm{d}\alpha_3 \, \mathcal{P}_{\alpha_3}.$$

We will provide the explicit expression and derivation of the homomorphism

$$f(x_2, x_1) \to F_f(\alpha_3, x_3) \;=\; \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} f(x_2, x_1) \;.$$

Here, $f(x_2, x_1)$ denotes an element in $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ and $F_f(\alpha_3, x_3)$ is its image in $\mathcal{P}_{\alpha_3}$. In order to state a formula for the Clebsch-Gordan map, we build

$$D(z;\alpha) = \frac{S_{\mathrm{b}}(z)}{S_{\mathrm{b}}(z+\alpha)}, \tag{61}$$

from the special function $S_{\mathrm{b}}$, see appendix B.1 for a precise definition, and we introduce

$$z_{21} = ix_{12} - Q + \frac{1}{2}(2\bar{\alpha}_3 + \bar{\alpha}_1 + \bar{\alpha}_2),$$

$$z_{31} = ix_{13} + \frac{1}{2}(\bar{\alpha}_1 - \bar{\alpha}_3),$$

$$z_{32} = ix_{32} + \frac{1}{2}(\bar{\alpha}_2 - \bar{\alpha}_3),$$

where $\bar{\alpha}_i \in Q/2 + i\mathbb{R}$ is defined as before and we used $x_{ij} = x_i - x_j$. The symbols $\alpha_{ij}$ stand for

$$\alpha_{21} = \alpha_1 + \alpha_2 + \alpha_3 - Q,$$

$$\alpha_{31} = Q + \alpha_1 - \alpha_2 - \alpha_3,$$

$$\alpha_{32} = Q - \alpha_1 + \alpha_2 - \alpha_3 \;.$$

With all these notations, we are finally able to spell out the relevant Clebsh-Gordan coefficients [13],

$$\begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} \;=\; \mathcal{N} D(z_{21}; \alpha_{21}) D(z_{23}; \alpha_{23}) D(z_{13}; \alpha_{13}) \;, \tag{62}$$

where

$$\mathcal{N} \;=\; \exp\left[ -\frac{i\pi}{2}(\bar{\alpha}_3\alpha_3 - \bar{\alpha}_2\alpha_2 - \bar{\alpha}_1\alpha_1) \right] \;. \tag{63}$$

Let us note that this product form of the Clebsch-Gordan coefficients is familiar e.g. from the 3-point functions in conformal field theory which may be written as a product. Although the representations we study here are not obtained by deforming discrete series representations of $sl(2)$, i.e. of those representations that fields of a conformal field theory transform in, the familiar product structure of the Clebsch-Gordan coefficients survives.

### 3.2.1 The intertwining property

The fundamental intertwining property of the Clebsch-Gordan coefficients takes the following form

$$\pi_{\alpha_3}(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} (\pi_{\alpha_2} \otimes \pi_{\alpha_1})\Delta(X) \tag{64}$$

for $X = K, E^{\pm}$. The equation should be interpreted as an identity of operators on the representation space $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. While the operators $K$ and $E^{\pm}$ may be expressed through multiplication and shift operators, the Clebsch-Gordan map itself provides the kernel of an integral transform. With the help of partial integration, we can re-write the intertwining relation as an identity for the integral kernel,

$$\pi_{\alpha_3}(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = (\pi_{\alpha_2} \otimes \pi_{\alpha_1})\Delta^t(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}, \tag{65}$$

where the superscript $^t$ means that we should replace all shift operators by shifts in the opposite direction, i.e. $(T_x^{ia})^t = T_x^{-ia}$ and exchange the order between multiplication and shifts, i.e $(f(x)T_x^{ia})^t = T_x^{-ia}f(x)$. In this new form, the intertwining property is simply an identity of functions in the variables $x_i$.

One can check eq. (65) by direct computation. This is particularly easy for the element $K$ for which eq. (65) reads

$$T_{x_3}^{\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = T_{x_2}^{-\frac{ib}{2}} T_{x_1}^{-\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}. \tag{66}$$

Since the Clebsch-Gordan maps depend only in the differences $x_{ij}$ we can replace $T_{x_1} = T_{12}T_{13}$ etc. where $T_{ij}$ denotes a shift operator acting on $x_{ij}$. Consequently, the intertwining property for $K$ becomes

$$T_{13}^{-\frac{ib}{2}} T_{23}^{-\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = T_{12}^{\frac{ib}{2}} T_{23}^{-\frac{ib}{2}} T_{12}^{-\frac{ib}{2}} T_{13}^{-\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}, \tag{67}$$

which is trivially satisfied since all shifts commute. This concludes the proof of the intertwining property (65) for $X = K$.

For $X = E^+$ the check is a bit more elaborate. Using the anti-symmetry $[-x]_b = -[x]_b$ of the function (60) and the property $\partial_x^t = -\partial_x$ of derivatives, we obtain

$$e^{2\pi b x_3}[\delta_{x_3} + \bar{\alpha}_3]_b \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} =$$

$$- [\delta_{x_2} - \bar{\alpha}_2]_b \, e^{2\pi b x_2} T_{x_1}^{\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} - [\delta_{x_1} - \bar{\alpha}_1]_b \, e^{2\pi b x_1} T_{x_2}^{-\frac{ib}{2}} \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}.$$

where $\delta_x = (2\pi)^{-1}\partial_x$. After a bit of rewriting we find

$$\left[ e^{i\pi b(\bar{\alpha}_1 - \bar{\alpha}_2)/2}[-ix_{21} + Q - \frac{1}{2}(\bar{\alpha}_2 + \bar{\alpha}_1)]_b T_{21}^{ib} T_{23}^{ib} \right.$$

$$+ e^{-\pi b x_{23}} e^{-i\pi b(\bar{\alpha}_3 + \bar{\alpha}_1)/2}[-ix_{13} + Q + \frac{1}{2}(\bar{\alpha}_3 - \bar{\alpha}_1)]_b T_{13}^{ib} T_{23}^{ib}$$

$$\left. - e^{-i\pi b Q} e^{-\pi b x_{13}} e^{i\pi b(\bar{\alpha}_2 + \bar{\alpha}_3)/2}[-ix_{23} + \frac{1}{2}(\bar{\alpha}_2 - \bar{\alpha}_3)]_b \right] \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = 0.$$

Now, because of the shift properties of the function $S_b$, see Appendix A.1, we have

$$T_x^{ib} \frac{S_b(-ix + a_1)}{S_b(-ix + a_2)} = \frac{[-ix + a_1]_b}{[-ix + a_2]_b} \frac{S_b(-ix + a_1)}{S_b(-ix + a_2)} T_x^{ib}.$$

With the help of this equation it is easy to check that our Clebsch-Gordan coefficients obey the desired intertwining relation with $E^+$. For the intertwining property involving $X = E^-$ one proceeds in a similar way.

### 3.2.2 Orthogonality and Completeness

The Clebsh-Gordan coefficients for the self-dual series of $\mathcal{U}_q(sl(2))$ satisfy the following orthogonality and completeness relation

$$\int_{\mathbb{R}} dx_2 dx_1 \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}^* \begin{bmatrix} \beta_3 \, \alpha_2 \, \alpha_1 \\ y_3 \, x_2 \, x_1 \end{bmatrix} = |S_{\mathrm{b}}(2\alpha_3)|^{-2} \delta(\alpha_3 - \beta_3) \delta(x_3 - y_3), \tag{68}$$

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} d\alpha_3 \int_{\mathbb{R}} dx_3 |S_{\mathrm{b}}(2\alpha_3)|^2 \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}^* \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, y_2 \, y_1 \end{bmatrix} = \delta(x_2 - y_2) \delta(x_1 - y_1) . \tag{69}$$

In writing the first equation we have assumed that $i(Q/2 - \alpha_3) \geq 0$ and $i(Q/2 - \beta_3) \geq 0$. Without this assumption, there would be a second term on the left hand side involving the delta function $\delta(Q - \alpha_3 - \beta_3)$. Except for the normalizing factor on the right hand side of eq. (68), these relations follow from the intertwining properties of Clebsch-Gordan maps. We shall discuss a derivation of eq. (68) in some detail. This will allow us to skip over some painful details later when we discuss the corresponding issues for the deformed superalgebra. In order to compute the integral on the left hand side of eq. (68) we shall employ a star-triangle relation for the functions $S_{\mathrm{b}}$ along with several of its corollaries. All necessary integral formulae are collected in Appendix B.1.[1]

Before we proceed proving the orthogonality relations, let us point out that the equations (68) involve products of the Clebsch-Gordan kernels. Since these are distributional kernels, one must take some care when multiplying two of them. Following [13], the strategy is to regularize the Clebsch-Gordan maps through some $\epsilon$ prescription, then to multiply the regularized kernels before we send the parameter $\epsilon$ to zero in the very end of the computation. For the problem at hand, one appropriate regularization takes the form

$$\begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}_\epsilon = \mathcal{N} D(z_{21} + \frac{\epsilon}{2}; \alpha_{21} - \frac{\epsilon}{2}) D(z_{32} + \epsilon; \alpha_{32} - \frac{3\epsilon}{2}) D(z_{31} + \frac{\epsilon}{2}; \alpha_{31} + \frac{\epsilon}{2}), \tag{70}$$

with the same normalisation (63) as above. Our prescription is different from the one used in [13].

Inserting the regularized Clebsch-Gordan maps into the orthogonality relation (68), we obtain

$$\int dx_2 dx_1 \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}_\epsilon^* \begin{bmatrix} \beta_3 \, \alpha_2 \, \alpha_1 \\ y_3 \, x_2 \, x_1 \end{bmatrix}_\epsilon = \eta \int dx_2 dx_1 \frac{S_{\mathrm{b}}(-ix_{21} + Q - \frac{1}{2}(\bar{\alpha}_2 + \bar{\alpha}_1))}{S_{\mathrm{b}}(-ix_{21} + Q - \frac{1}{2}(\bar{\alpha}_2 + \bar{\alpha}_1))} \times$$

$$\times S_{\mathrm{b}}(-ix_{21} - Q + \frac{1}{2}(2\bar{\beta}_3 + \bar{\alpha}_1 + \bar{\alpha}_2) + \frac{\epsilon}{2}) \, S_{\mathrm{b}}(ix_{21} + Q - \frac{1}{2}(2\bar{\alpha}_3 + \bar{\alpha}_1 + \bar{\alpha}_2) + \frac{\epsilon}{2}) \times$$

$$\times S_{\mathrm{b}}(-i(x_1 - y_3) + Q + \frac{1}{2}(\bar{\alpha}_1 - \bar{\beta}_3 - 2\bar{\alpha}_2) - \epsilon) S_{\mathrm{b}}(ix_{13} - \frac{1}{2}(\bar{\alpha}_1 - \bar{\alpha}_3 - 2\bar{\alpha}_2) - \epsilon) \times$$

$$\times S_{\mathrm{b}}(-ix_{13} + \frac{1}{2}(\bar{\alpha}_3 - \bar{\alpha}_1) + \frac{\epsilon}{2}) \, S_{\mathrm{b}}(-i(x_1 - y_3) - \frac{1}{2}(\bar{\beta}_3 - \bar{\alpha}_1) + \frac{\epsilon}{2}) \times$$

$$\times D^*(z_{32} + \epsilon, \alpha_{32} - \frac{3\epsilon}{2}) \, D(\tilde{z}_{32} + \epsilon, \tilde{\alpha}_{32} - \frac{3\epsilon}{2}) =: I_1^\epsilon .$$

In writing this expression we have expressed all the D-functions that contain some dependence on the variable $x_1$ in terms of $S_{\mathrm{b}}$, see eq. (61). We brought all but three of the $S_{\mathrm{b}}$ functions to the numerator with the help of the property $S_{\mathrm{b}}^{-1}(x) = S_{\mathrm{b}}(Q - x)$. In taking the complex conjugate, we used that the variables $x_i, y_3$ and our regulator $\epsilon$ are real. The labels $\alpha_i$ and $\beta_3$, on the other hand, satisfy $\alpha_i^* = \bar{\alpha}_i = Q - \alpha_i$ and $\beta_3^* = \bar{\beta}_3 = Q - \beta_3$. Finally, we introduced $\tilde{\mathcal{N}}, \tilde{z}_{32}$ and $\tilde{\alpha}_{32}$. These are obtained from $\mathcal{N}, z_{32}$ and $\alpha_{32}$ by the substitution $x_3 \to y_3$ and $\alpha_3 \to \beta_3$. The constant prefactor $\eta$ is given by $\eta = \mathcal{N}\tilde{\mathcal{N}}$.

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[1]Our derivation resembles a similar calculation for the undeformed group $SL(2, \mathbb{C})$ performed by Lev Lipatov in [43].

Before we continue our evaluation of the integrals we note that the fraction of $S$-functions in the first line of the previous equation cancels out. Hence, we are left with a product of six $S$-functions that contain all the $x_1$ dependence of the integrand. It turns out that we can actually evaluate the $x_1$ integral with the help of the following star-triangle equation, see e.g. [20],

$$\int dx_1 \prod_{i=1}^{3} S_b(ix_1 + \gamma_i) S_b(-ix_1 + \delta_i) = \prod_{i,j=1}^{3} S_b(\gamma_i + \delta_j) \tag{71}$$

which holds as long as the arguments on the left hand side add up to $Q$, i.e. if

$$\sum_{i=1}^{3}(\gamma_i + \delta_i) = Q,$$

It is not difficult to check that the arguments which appear in our formula for $I_1^\epsilon$ above satisfy this condition. Hence, we can perform the integral over $x_1$ to obtain

$$I_1^\epsilon = \eta S_b(\bar\beta_3 - \bar\alpha_3 + \epsilon) \frac{S_b(-i(y_3 - x_3) + \frac{1}{2}(\bar\alpha_3 - \bar\beta_3) + \epsilon)}{S_b(-i(y_3 - x_3) - \frac{1}{2}(\bar\alpha_3 - \bar\beta_3) + 2\epsilon)} \frac{S_b(\bar\alpha_3 + \bar\alpha_2 - \bar\alpha_1 - \frac{\epsilon}{2})}{S_b(\bar\beta_3 + \bar\alpha_2 - \bar\alpha_1 + \frac{\epsilon}{2})} \times$$

$$\times \int dx_2 \frac{S_b(-ix_{23} - Q + \frac{1}{2}(2\bar\beta_3 + \bar\alpha_2 + \bar\alpha_3) + \epsilon)}{S_b(-i(x_2 - y_3) + \frac{1}{2}(2\bar\alpha_3 + \bar\alpha_2 + \bar\beta_3) - \epsilon))} \frac{S_b(-i(x_2 - y_3) + \frac{1}{2}(\bar\alpha_2 - \bar\beta_3) + \epsilon)}{S_b(-ix_{23} + Q + \frac{1}{2}(\bar\alpha_2 - \bar\alpha_3) - \epsilon)} =$$

$$= \eta S_b(\bar\beta_3 - \bar\alpha_3 + \epsilon) \frac{S_b(-i(y_3 - x_3) + \frac{1}{2}(\bar\alpha_3 - \bar\beta_3) + \epsilon)}{S_b(-i(y_3 - x_3) - \frac{1}{2}(\bar\alpha_3 - \bar\beta_3) + 2\epsilon)} \frac{S_b(\bar\alpha_3 + \bar\alpha_2 - \bar\alpha_1 - \frac{\epsilon}{2})}{S_b(\bar\beta_3 + \bar\alpha_2 - \bar\alpha_1 + \frac{\epsilon}{2})} \times$$

$$\times \int \frac{d\tau}{i} \frac{S_b(\tau + \xi_1 + \epsilon)}{S_b(Q + \tau + \xi_2 - \epsilon)} \frac{S_b(\tau - \xi_1 + \epsilon)}{S_b(Q + \tau - \xi_2 - \epsilon)} =: I_2^\epsilon .$$

In the first step we evaluated the right hand side of the star-triangle relation (71) and we expressed the remaining two $D$-functions that appear in $I_1^\epsilon$ through the functions $S_b$. After these two steps, the formula for $I_1^\epsilon$ should contain a total number of $9 + 4 = 13$ functions $S_b$. It turns out that four of them cancel against each other so that we are left with the nine factors in the first two lines of the previous formula. In passing to the lower lines we simply performed the substitutions

$$\tau = -ix_2 + \bar\alpha_2/2 - i(x_3 + y_3)/2 - Q/2 + (\bar\beta_3 + \bar\alpha_3)/4,$$

$$\xi_1 = -\frac{i}{2}(y_3 - x_3) + \frac{1}{4}(3\bar\beta_3 + \bar\alpha_3) - \frac{Q}{2},$$

$$\xi_2 = \frac{i}{2}(y_3 - x_3) + \frac{1}{4}(\bar\beta_3 + 3\bar\alpha_3) - \frac{Q}{2}.$$

In this form we can now also carry out the integral of the variable $\tau$ using a limiting case of the Saalschütz formula, see Appendix B.1, to find

$$I_2^\epsilon = \eta S_b(\gamma + \epsilon) \frac{S_b(-\xi_- - \gamma + \epsilon)}{S_b(2\epsilon - \xi_-)} \frac{S_b(\bar\alpha_3 + \bar\alpha_2 - \bar\alpha_1 - \frac{\epsilon}{2})}{S_b(\gamma + \bar\alpha_3 + \bar\alpha_2 - \bar\alpha_1 + \frac{\epsilon}{2})} \times$$

$$\times e^{-i\pi\xi_-\xi_+} \frac{S_b(2\epsilon - \xi_-)S_b(2\epsilon + \xi_-)S_b(2\epsilon - \xi_+)S_b(2\epsilon + \xi_+)}{S_b(4\epsilon)}$$

where $\gamma = \bar\beta_3 - \bar\alpha_3 \in i\mathbb{R}$ and

$$\xi_- = \xi_2 - \xi_1 = i(y_3 - x_3) - \frac{1}{2}\gamma \in i\mathbb{R},$$

$$\xi_+ = \xi_2 + \xi_1 = \bar\beta_3 + \bar\alpha_3 - Q \in i\mathbb{R} \setminus \{0\}.$$

Having performed both integrations, it remains to remove our regulator $\epsilon$. The most nontrivial part of this computation is to show that

$$\lim_{\epsilon \to 0} \frac{S_{\rm b}(\epsilon + \gamma)S_{\rm b}(\epsilon - \xi_- - \gamma)S_{\rm b}(2\epsilon + \xi_-)}{S_{\rm b}(4\epsilon)} = \delta(i\gamma)\delta(i\xi_-) \ . \tag{72}$$

A full proof is given in Appendix C. The remaining factors in $I_2^\epsilon$ possess a regular limit. In particular we find

$$\lim_{\epsilon \to 0} S_{\rm b}(2\epsilon - \xi_+)S_{\rm b}(2\epsilon + \xi_+) = \frac{1}{S_{\rm b}(Q + \xi_+)S_{\rm b}(Q - \xi_+)} = |S_{\rm b}(\bar{\beta}_3 + \bar{\alpha}_3)|^{-2}. \tag{73}$$

Finally, for the normalisation factor $\eta = \mathcal{N}\tilde{\mathcal{N}}$ we obtain

$$\eta = e^{-\frac{i\pi}{2}(\bar{\alpha}_3\alpha_3 - \bar{\alpha}_2\alpha_2 - \bar{\alpha}_1\alpha_1)} e^{\frac{i\pi}{2}(\bar{\beta}_3\beta_3 - \bar{\alpha}_2\alpha_2 - \bar{\alpha}_1\alpha_1)} = e^{-\frac{i\pi}{2}\gamma(\gamma + 2\bar{\alpha}_3 - Q)},$$

Putting all these results together we have shown that

$$\lim_{\epsilon \to 0} \int {\rm d}x_2 {\rm d}x_1 \begin{bmatrix} \alpha_3 \ \alpha_2 \ \alpha_1 \\ x_3 \ x_2 \ x_1 \end{bmatrix}_\epsilon^* \begin{bmatrix} \beta_3 \ \alpha_2 \ \alpha_1 \\ y_3 \ x_2 \ x_1 \end{bmatrix}_\epsilon =$$

$$= e^{-i\pi\xi_-\xi_+} \frac{e^{-\frac{i\pi}{2}\gamma(\gamma + 2\bar{\alpha}_3 - Q)}}{|S_{\rm b}(\bar{\beta}_3 + \bar{\alpha}_3)|^2} \frac{S_{\rm b}(\bar{\alpha}_3 + \bar{\alpha}_2 - \bar{\alpha}_1)}{S_{\rm b}(\gamma + \bar{\alpha}_3 + \bar{\alpha}_2 - \bar{\alpha}_1)} \delta(i\gamma)\delta(i\xi_-)$$

$$= |S_{\rm b}(\bar{\beta}_3 + \bar{\alpha}_3)|^{-2}\delta(i(\bar{\beta}_3 - \bar{\alpha}_3)\delta(y_3 - x_3) \ .$$

This is the orthonormality relation we set out to prove. The proof of eq. (69) can be also be worked out, following the helpful comments in [13]. An alternative proof of the orthonormality and completeness relations was published recently in [45] and [46]. The latter is particularly elegant.

## 3.3 The Racah-Wigner coefficients for $\mathcal{U}_q(sl(2))$

The Racah-Wigner coefficients describe a change of basis in the 3-fold tensor product of representations. Let us denote these three representations by $\pi_{\alpha_i}, i = 1, 2, 3$. In decomposing their product into irreducibles $\pi_{\alpha_4}$ there exists two possible fusion paths, denoted by $t$ and $s$, which are described by the following combination of Clebsch-Gordan coefficients

$$\Phi^t_{\alpha_t} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4 \ \alpha_1 \end{bmatrix}_\epsilon (x_4; x_i) = \int {\rm d}x_t \begin{bmatrix} \alpha_4 \ \alpha_t \ \alpha_1 \\ x_4 \ x_t \ x_1 \end{bmatrix}_\epsilon \begin{bmatrix} \alpha_t \ \alpha_3 \ \alpha_2 \\ x_t \ x_3 \ x_2 \end{bmatrix}_\epsilon, \tag{74}$$

$$\Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4 \ \alpha_1 \end{bmatrix}_\epsilon (x_4; x_i) = \int {\rm d}x_s \begin{bmatrix} \alpha_4 \ \alpha_3 \ \alpha_s \\ x_4 \ x_3 \ x_s \end{bmatrix}_\epsilon \begin{bmatrix} \alpha_s \ \alpha_2 \ \alpha_1 \\ x_s \ x_2 \ x_1 \end{bmatrix}_\epsilon, \tag{75}$$

which are related by

$$\Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4 \ \alpha_1 \end{bmatrix}_\epsilon (x_4; x_i) = \int {\rm d}\alpha_t |S_{\rm b}(2\alpha_t)|^2 \chi \begin{bmatrix} \alpha_4 \ \alpha_s \ x_4 \\ \alpha_4' \ \alpha_t \ x_4' \end{bmatrix} \Phi^t_{\alpha_t} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4 \ \alpha_1 \end{bmatrix}_\epsilon (x_4; x_i), \tag{76}$$

where the above kernel $\chi$ is related to 6j symbol by

$$\chi \begin{bmatrix} \alpha_4 \ \alpha_s \ x_4 \\ \alpha_4' \ \alpha_t \ x_4' \end{bmatrix} = \left\{ \begin{matrix} \alpha_1 \ \alpha_3 \\ \alpha_2 \ \alpha_4 \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\rm b} |S_{\rm b}(2\alpha_4)|^{-2}\delta(\alpha_4' - \alpha_4)\delta(x_4' - x_4). \tag{77}$$

The regularization we use here is the same as in the previous section. From the two objects $\Phi^s$ and $\Phi^t$ we obtain the Racah-Wigner coefficients as

$$\chi \begin{bmatrix} \alpha_4 \ \alpha_s \ x_4 \\ \alpha_4' \ \alpha_t \ x_4' \end{bmatrix} = \lim_{\epsilon \to 0} \int {\rm d}^3 x_i \Phi^t_{\alpha_t} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4' \ \alpha_1 \end{bmatrix}_\epsilon^* (x_4'; x_i) \Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 \ \alpha_2 \\ \alpha_4 \ \alpha_1 \end{bmatrix}_\epsilon (x_4; x_i). \tag{78}$$

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After inserting the concrete expressions (70) for the regularised Clebsch-Gordan coefficients one may evaluate the integrals to obtain [13]

$$\left\{ \begin{matrix} \alpha_1 \ \alpha_3 \\ \alpha_2 \ \alpha_4 \end{matrix} \Big| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\rm b} = |S_{\rm b}(2\alpha_t)|^2 \frac{S_{\rm b}(a_4)S_{\rm b}(a_1)}{S_{\rm b}(a_2)S_{\rm b}(a_3)} \times$$

$$\times \int_{i\mathbb{R}} {\rm d}t \frac{S_{\rm b}(u_4+t)S_{\rm b}(\tilde{u}_4+t)S_{\rm b}(u_3+t)S_{\rm b}(\tilde{u}_3+t)}{S_{\rm b}(u_{23}+t)S_{\rm b}(\tilde{u}_{23}+t)S_{\rm b}(2\alpha_s+t)S_{\rm b}(Q+t)}, \tag{79}$$

where the four variables $a_i$ are associated with the four Clebsch-Gordan maps that appear in eqs. (74) and (75)

$$a_1 = \alpha_1 - \bar{\alpha}_t + \alpha_4 \quad, \qquad\qquad\qquad a_2 = \alpha_2 - \alpha_3 + \alpha_t,$$

$$a_3 = \alpha_3 - \bar{\alpha}_s + \alpha_4 \quad, \qquad\qquad\qquad a_4 = \alpha_2 - \alpha_1 + \alpha_s,$$

and similarly for the remaining set of variables,

$$u_4 = \alpha_s + \alpha_1 - \alpha_2 \quad, \qquad\qquad\qquad \tilde{u}_4 = \alpha_s + \bar{\alpha}_1 - \alpha_2,$$

$$u_3 = \alpha_s + \alpha_4 - \alpha_3 \quad, \qquad\qquad\qquad \tilde{u}_3 = \alpha_s + \alpha_4 - \bar{\alpha}_3, \tag{80}$$

$$u_{23} = \alpha_s + \alpha_t + \alpha_4 - \alpha_2 \quad, \qquad\qquad \tilde{u}_{23} = \alpha_s + \bar{\alpha}_t + \alpha_4 - \alpha_2 \ .$$

The derivation of eq. (79) from eq. (78) is in principle straightforward, though a bit cumbersome. One simply has to evaluate the integrals. The integrals over the variables $x_i, i = 1, 2, 3$, are performed with the help of Cauchy's integral formula. The resulting integral expression involves delta functions in both the difference $\alpha_4 - \alpha_4'$ and $x_4 - x_4'$. Hence the integrals over $x_4'$ and $\alpha_4'$ are easy to perform at the end of the computation. So, let us get back to the integrals over $x_i, i = 1, 2, 3$. It is convenient so start with $x_1$. In order to perform the integration, one needs to keep track of all the poles in the integrand along with their residues. Since the functions $\Phi^s$ and $\Phi^t$ are ultimately built from $S_{\rm b}$ through equations (61), (70), (74) and (75), this step only requires knowledge of the poles and residues of $S_{\rm b}$. All this information on $S_{\rm b}$ can be found in Appendix A.1. Once the integration over $x_1$ has been performed, one focuses on the variable $x_3$. There are a few poles that have been around before we integrated over $x_1$. In addition, the integration over $x_1$ brought in some new poles through the usual pole collisions (pinching). These must all be accounted for before we can apply Cauchy's formula again to perform the integration over $x_3$. Similar comments apply to the final integral over $x_2$. Many more details on this computations can be found in section 5 of the paper by Ponsot and Teschner [13]. Let us stress again that no fancy identities are needed at any stage of the calculation.

After these comments on the derivation of eq. (79), let us list a few more properties of the Racah-Wigner symbols. To begin with, they can be shown to satisfy the following orthogonality relations

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} {\rm d}\alpha_s |S_{\rm b}(2\alpha_s)|^2 \left( \left\{ \begin{matrix} \alpha_1 \ \alpha_2 \\ \alpha_3 \ \alpha_4 \end{matrix} \Big| \begin{matrix} \alpha_s \\ \beta_t \end{matrix} \right\}_{\rm b} \right)^* \left\{ \begin{matrix} \alpha_1 \ \alpha_2 \\ \alpha_3 \ \alpha_4 \end{matrix} \Big| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\rm b} = |S_{\rm b}(2\alpha_t)|^2 \delta(\alpha_t - \beta_t).$$

As a consequence of their very definition, the Racah-Wigner symbols must also satisfy the pentagon equation

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} {\rm d}\delta_1 \left\{ \begin{matrix} \alpha_1 \ \alpha_2 \\ \alpha_3 \ \alpha_4 \end{matrix} \Big| \begin{matrix} \beta_1 \\ \delta_1 \end{matrix} \right\}_{\rm b} \left\{ \begin{matrix} \alpha_1 \ \delta_1 \\ \alpha_4 \ \alpha_5 \end{matrix} \Big| \begin{matrix} \beta_2 \\ \gamma_2 \end{matrix} \right\}_{\rm b} \left\{ \begin{matrix} \alpha_2 \ \alpha_3 \\ \alpha_4 \ \gamma_2 \end{matrix} \Big| \begin{matrix} \delta_1 \\ \gamma_1 \end{matrix} \right\}_{\rm b} = \left\{ \begin{matrix} \beta_1 \ \alpha_3 \\ \alpha_4 \ \alpha_5 \end{matrix} \Big| \begin{matrix} \beta_2 \\ \gamma_1 \end{matrix} \right\}_{\rm b} \left\{ \begin{matrix} \alpha_1 \ \alpha_2 \\ \gamma_1 \ \alpha_5 \end{matrix} \Big| \begin{matrix} \beta_1 \\ \gamma_2 \end{matrix} \right\}_{\rm b} .$$

More recently, Teschner and Vartanov found an interesting alternative expression for the Racah-Wigner coefficients [47]. We will discuss this representation in the following section.

## 3.4 Teschner-Vartanov form of Racah-Wigner coefficients

In this section we will start from a recent integral formula for the Racah-Wigner symbol of a self-dual series representations of $\mathcal{U}_q(sl(2))$ with $q = e^{i\pi b^2}$ which was presented in [47] and was shown to agree with formula (79) for the Racah-Wigner symbol of $\mathcal{U}_q(sl(2))$ that was established by Teschner and Ponsot [12, 13].

This symbol turns out to simplify when the representation labels $\alpha = Q/2 + i\mathbb{R}, Q = b + b^{-1}$, assume a value $-2\alpha b^{-1} \in \mathbb{N}$. In fact, it can be written as a sum over finitely many pole contributions. We compare the resulting expressions with the formulae for Racah-Wigner coefficients of finite dimensional representations of $\mathcal{U}_q(sl(2))$ and find complete agreement, at least up to some normalisation dependent prefactors.

Let us begin our discussion by reviewing the formulae for the universal Racah-Wigner coefficients of $\mathcal{U}_q(sl(2))$ which were proposed by Teschner and Vartanov [47]

$$\left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{array} \right\} = \Delta(\alpha_1, \alpha_2, \alpha_s)\Delta(\alpha_s, \alpha_3, \alpha_4)\Delta(\alpha_t, \alpha_3, \alpha_2)\Delta(\alpha_4, \alpha_t, \alpha_1) \tag{81}$$

$$\times \int_{\mathcal{C}} du \, S_b(u - \alpha_{12s}) \, S_b(u - \alpha_{s34}) \, S_b(u - \alpha_{23t}) \, S_b(u - \alpha_{1t4})$$
$$S_b(\alpha_{1234} - u) \, S_b(\alpha_{st13} - u) \, S_b(\alpha_{st24} - u) \, S_b(2Q - u),$$

where

$$\Delta(\alpha_3, \alpha_2, \alpha_1) = \left( \frac{S_b(\alpha_{123} - Q)}{S_b(\alpha_{12} - \alpha_3) \, S_b(\alpha_{23} - \alpha_1) \, S_b(\alpha_{31} - \alpha_2)} \right)^{\frac{1}{2}}. \tag{82}$$

The contour $\mathcal{C}$ crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity.

Let us begin our analysis of the Racah-Wigner symbols (81) with the prefactor of the integral in the first line. Insertion of the definition (82) gives

$$\Delta(\alpha_1, \alpha_2, \alpha_s)\Delta(\alpha_s, \alpha_3, \alpha_4)\Delta(\alpha_t, \alpha_3, \alpha_2)\Delta(\alpha_4, \alpha_t, \alpha_1) = \tag{83}$$

$$= \left( \frac{S_b(\alpha_{12s} - Q)S_b(\alpha_{s34} - Q)}{S_b(\alpha_{12} - \alpha_s) \, S_b(\alpha_{2s} - \alpha_1) \, S_b(\alpha_{1s} - \alpha_2) \, S_b(\alpha_{34} - \alpha_s) \, S_b(\alpha_{3s} - \alpha_4) \, S_b(\alpha_{4s} - \alpha_3)} \right)^{\frac{1}{2}} \times$$
$$\times \left( \frac{S_b(\alpha_{23t} - Q)S_b(\alpha_{1t4} - Q)}{S_b(\alpha_{23} - \alpha_t) \, S_b(\alpha_{2t} - \alpha_3) \, S_b(\alpha_{3t} - \alpha_2) \, S_b(\alpha_{14} - \alpha_t) \, S_b(\alpha_{1t} - \alpha_4) \, S_b(\alpha_{4t} - \alpha_1)} \right)^{\frac{1}{2}}.$$

We observe that the prefactor vanishes each time one of the external weights approaches a degenerate value $\alpha_i \to -\frac{nb}{2} - \frac{n'}{2b}$ where $n, n' \in \mathbb{Z}_{\geq 0}$, and one of the intermediate weights may be obtained by fusion of $\alpha_j$ with the degenerate weight, i.e.

$$\alpha_s \to \alpha_j - \frac{sb}{2} - \frac{s'}{2b}, \qquad \text{or} \qquad \alpha_t \to \alpha_k - \frac{tb}{2} - \frac{t'}{2b}, \tag{84}$$

$$s, t \in \{-n, -n+2, \ldots, n\}, \qquad s', t' \in \{-n', -n'+2, \ldots, n'\}.$$

As we shall show below, the full Racah-Wigner symbol does not vanish for these special values because the integral in eq. (81) contributes singular terms such that the limit of the Racah-Wigner symbols is finite and non-zero.

In order to see how this works in detail, let us consider the limit of degenerate weight $\alpha_2 \to -\frac{nb}{2}$ $(n > 0)$

and $\alpha_s \to \alpha_1 - \frac{sb}{2}$. The zero in the prefactor comes from the first two terms in the denominator of eq. (83)

$$\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} (S_b(\alpha_{12} - \alpha_s) S_b(\alpha_{2s} - \alpha_1))^{-\frac{1}{2}} = \left( S_b\left( \frac{s-n}{2}b \right) S_b\left( -\frac{s+n}{2}b \right) \right)^{-\frac{1}{2}} =$$

$$= \left( -2\sin(\pi b^2) \right)^{\frac{n}{2}} \left( \left[ \frac{n-s}{2} \right]! \left[ \frac{n+s}{2} \right]! \right)^{\frac{1}{2}} S_b(0)^{-1},$$

where we used the shift relation (223) for the double sine function. For integer $x$ the factorial $[x]!$ is defined as

$$[x]! = \prod_{a=1}^{x} [a] = \left( \sin \pi b^2 \right)^{-x} \prod_{a=1}^{x} \sin(\pi b^2 a) . \tag{85}$$

In order to obtain a finite non-zero limit for the full Racah-Wigner symbol, the integral must contribute a divergent factor $S_b(0)$ to cancel the corresponding term from the prefactor. Let us therefore have a closer look at the integral

$$\int_{\mathcal{C}} du \quad S_b(u - \alpha_{12s}) S_b(u - \alpha_{s34}) S_b(u - \alpha_{23t}) S_b(u - \alpha_{1t4}) \times \tag{86}$$

$$\times S_b(\alpha_{1234} - u) S_b(\alpha_{st13} - u) S_b(\alpha_{st24} - u) S_b(2Q - u) .$$

The first contribution to a singular result comes from two terms of the integrant $S_b(u - \alpha_{s34}) S_b(\alpha_{1234} - u)$. The points $u = \alpha_{s34}$ and $u = \alpha_{1234}$ are situated on the left and right sides of the contour, respectively, see figure 1. Taking the limit $\alpha_2 \to -\frac{nb}{2}$ and $\alpha_s \to \alpha_1 - \frac{sb}{2}$ requires a certain deformation of the contour. Let us first consider the case of $s \geq 0$. Then $\alpha_{s34}$ can reach the point $\alpha_{134} - \frac{sb}{2}$ without crossing through the contour. On the other hand the point $\alpha_{1234}$ meets the contour on the way to $\alpha_{134} - \frac{nb}{2}$. We can deform the contour as long as it does not pass through one of the double poles of $S_b(u - \alpha_{134} + \frac{sb}{2}) S_b(\alpha_{134} - \frac{nb}{2} - u)$ in $u = \alpha_{134} - \frac{sb}{2} - pb$ ( $0 \leq p \leq \frac{n-s}{2}$). From each pole we get a singular term due to the so called "pinching mechanism", see e.g. [13], Lemma 3 and [52, 40] for similar calculations. This is illustrated on the right hand side of figure 1. In the end we obtain the following sum

$$\sum_{p=0}^{\frac{n-s}{2}} \left( \frac{\left( -2\sin(\pi b^2) \right)^{\frac{s-n}{2}} S_b(0)}{[p]! \left[ \frac{n-s}{2} - p \right]!} S_b(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb) S_b(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) \times \right. \tag{87}$$

$$\left. \times S_b(\alpha_3 - \alpha_t - \frac{sb}{2} - pb) S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) S_b(\alpha_{1t} - \alpha_4 + pb) S_b(2Q - \alpha_{134} + \frac{sb}{2} + pb) \right).$$
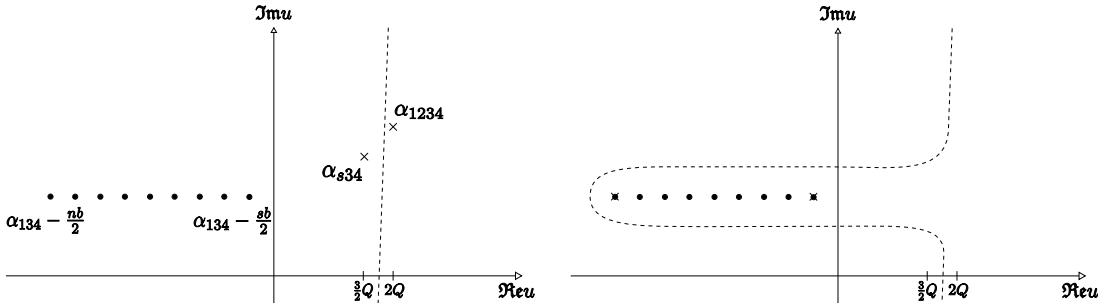


Figure 2: The original integration contour passes between the points $u = \alpha_{s34}$ and $u = \alpha_{1234}$. As we move the point $\alpha_{1234}$ to its limiting value, the shown poles contribute to the integral due to the pinching mechanism.

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When $s < 0$ the poles $u = \alpha_{134} - \frac{sb}{2} - pb$ for $0 \leq p < -\frac{s}{2}$ and $-\frac{s}{2} \leq p \leq \frac{n-s}{2}$ are located on the right and left side of the contour, respectively, see figure 2. Taking the limit $\alpha_s \to \alpha_1 - \frac{sb}{2}$ we have to deform contour such that it passes the poles with $0 \leq p < -\frac{s}{2}$. By taking $\alpha_2 \to -\frac{nb}{2}$ we get contributions from the rest of the poles $(-\frac{s}{2} \leq p \leq \frac{n-s}{2}$ ), see figure 2. The final result will be the same as in the case of $s \geq 0$ (87).

The second contribution to the singular result of the integral (86) comes from the functions $S_b(u - \alpha_{1t4}) \, S_b(\alpha_{st24} - u)$ with common poles in $u = \alpha_{1t4} - p'b$ for $0 \leq p' \leq \frac{n+s}{2}$. Since $s > -n$, all the poles lie on the left side of the contour independently of the sign of the parameter $s$. The point $\alpha_{st24}$ lies on the right side of the contour and before reaching $\alpha_{1t4} - \frac{(s+n)b}{2}$ one needs to pass with the contour thought all the double poles obtaining the sum of singular terms,

$$\sum_{p'=0}^{\frac{n+s}{2}} \left( \frac{\left(-2\sin(\pi b^2)\right)^{-\frac{n+s}{2}} S_b(0)}{[p']! \left[\frac{n+s}{2} - p'\right]!} S_b(\alpha_{t4} - \alpha_1 + \frac{(s+n)b}{2} - p'b) \, S_b(\alpha_{14} - \alpha_3 + \frac{nb}{2} - p'b) \times \right. \tag{88}$$
$$\left. \times S_b(\alpha_t - \alpha_3 + \frac{sb}{2} - p'b) \, S_b(\alpha_3 - \alpha_t - \frac{nb}{2} + p'b) \, S_b(\alpha_{13} - \alpha_4 - \frac{sb}{2} + p'b) \, S_b(2Q - \alpha_{1t4} + p'b) \right).$$

Combining the two above sums (87, 88) with the prefactor (83) we get a finite result for the limit:

$$\left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_1 - \frac{sb}{2} \\ -\frac{nb}{2} & \alpha_4 & \alpha_t \end{array} \right\} \equiv \lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{array} \right\} =$$

$$= \left( \frac{S_b(\alpha_{14} + \alpha_t - Q) S_b(\alpha_3 + \alpha_t - \frac{nb}{2} - Q)}{S_b(\alpha_3 - \alpha_t - \frac{nb}{2}) S_b(\alpha_t - \alpha_3 - \frac{nb}{2}) S_b(\alpha_{14} - \alpha_t) S_b(\alpha_{1t} - \alpha_4) S_b(\alpha_{4t} - \alpha_1) S_b(\alpha_{3t} + \frac{nb}{2})} \right)^{\frac{1}{2}} \times$$

$$\times \left( \frac{\left[\frac{n-s}{2}\right]! \left[\frac{n+s}{2}\right]! S_b(2\alpha_1 - \frac{(s+n)b}{2} - Q) S_b(2\alpha_{134} - \frac{sb}{2} - Q)}{S_b(2\alpha_1 + \frac{(n-s)b}{2}) S_b(\alpha_{34} - \alpha_1 + \frac{sb}{2}) S_b(\alpha_{13} - \alpha_4 - \frac{sb}{2}) S_b(\alpha_{14} - \alpha_3 - \frac{sb}{2})} \right)^{\frac{1}{2}} \times$$

$$\times \left\{ \sum_{q=0}^{\frac{n-s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{\frac{s}{2}}}{[q]! \left[\frac{n-s}{2} - q\right]!} S_b(\alpha_{34} - \alpha_1 + \frac{nb}{2} - qb) \, S_b(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - qb) \times \right.$$
$$\times S_b(\alpha_3 - \alpha_t - \frac{sb}{2} - qb) \, S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + qb) \, S_b(\alpha_{1t} - \alpha_4 + qb) \, S_b(2Q - \alpha_{134} + \frac{sb}{2} + qb) +$$

$$+ \sum_{p'=0}^{\frac{n+s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{-\frac{s}{2}}}{[p']! \left[\frac{n+s}{2} - p'\right]!} S_b(\alpha_{t4} - \alpha_1 + \frac{(s+n)b}{2} - p'b) \, S_b(\alpha_{14} - \alpha_3 + \frac{nb}{2} - p'b) \times$$

$$\left. \times S_b(\alpha_t - \alpha_3 + \frac{sb}{2} - p'b) \, S_b(\alpha_3 - \alpha_t - \frac{nb}{2} + p'b) \, S_b(\alpha_{13} - \alpha_4 - \frac{sb}{2} + p'b) \, S_b(2Q - \alpha_{1t4} + p'b) \right\}.$$

Let us now consider the case when the other intermediate weight $\alpha_t$ also satisfies fusion rules (84) i.e. $\alpha_t \to \alpha_3 - \frac{tb}{2}$. Then prefactor in the formula above gives zero. On the other hand in each term of the sums there are double poles for $t \in \{-n + 2p, -n + 2p + 2, \ldots, s + 2p\}$ and $t \in \{s - 2p', s - 2p' + 2, \ldots, n - 2p'\}$ coming from $S_b(\alpha_3 - \alpha_t - \frac{sb}{2} - pb) \, S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + pb)$ and $S_b(\alpha_t - \alpha_3 - p'b + \frac{sb}{2}) \, S_b(\alpha_3 - \alpha_t + \alpha_2 - p'b)$, respectively. The
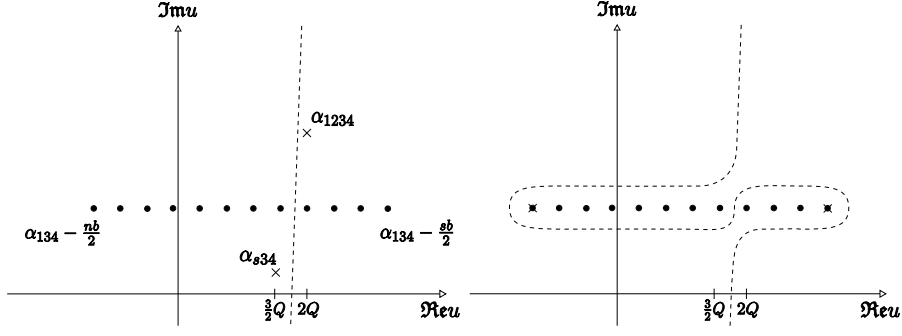
Figure 3: When $s < 0$ poles appear to both sides of the integration contour. While moving $\alpha_s$ and $\alpha_2$ to their final values, we need to deform the contour such that it picks up contributions from all these poles.

residue for a given $\alpha_t \to \alpha_3 - \frac{tb}{2}$ takes the form

$$
\operatorname*{Res}_{\alpha_t \to \alpha_3 - \frac{tb}{2}} \left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_1 - \frac{sb}{2} \\ -\frac{nb}{2} & \alpha_4 & \alpha_t \end{array} \right\} = 2 \left( \frac{S_b(2\alpha_1 - \frac{(s+n)b}{2} - Q)S_b(2\alpha_3 - \frac{(t+n)b}{2} - Q)}{S_b(2\alpha_1 + \frac{(n-s)b}{2}) S_b(2\alpha_3 + \frac{(n-t)b}{2})} \right)^{\frac{1}{2}} \times \tag{89}
$$

$$
\times \sum_{p=max\{0,\frac{t-s}{2}\}}^{min\{\frac{n-s}{2},\frac{n+t}{2}\}} \frac{\left( \left[\frac{n-s}{2}\right]! \left[\frac{n+s}{2}\right]! \left[\frac{n-t}{2}\right]! \left[\frac{n+t}{2}\right]! \right)^{\frac{1}{2}}}{[p]! \left[\frac{n-s}{2} - p\right]! \left[\frac{s-t}{2} + p\right]! \left[\frac{n+t}{2} - p\right]!} \frac{S_b(\alpha_{13} - \alpha_4 + pb - \frac{tb}{2})}{\left(S_b(\alpha_{13} - \alpha_4 - \frac{sb}{2})S_b(\alpha_{13} - \alpha_4 - \frac{tb}{2})\right)^{\frac{1}{2}}} \times
$$

$$
\times \frac{S_b(\alpha_{34} - \alpha_1 - pb + \frac{nb}{2})}{\left(S_b(\alpha_{34} - \alpha_1 + \frac{sb}{2}) S_b(\alpha_{34} - \alpha_1 - \frac{tb}{2})\right)^{\frac{1}{2}}} \frac{S_b(\alpha_{14} - \alpha_3 - pb + \frac{(n+t-s)b}{2})}{\left(S_b(\alpha_{14} - \alpha_3 - \frac{sb}{2}) S_b(\alpha_{14} - \alpha_3 + \frac{tb}{2})\right)^{\frac{1}{2}}} \times
$$

$$
\times \frac{\left(S_b(\alpha_{134} - \frac{sb}{2} - Q)S_b(\alpha_{134} - \frac{tb}{2} - Q)\right)^{\frac{1}{2}}}{S_b(\alpha_{134} - \frac{sb}{2} - pb - Q)},
$$

where we redefined the second summation parameter $p' = p - \frac{t-s}{2}$ in order to obtain two identical sums. Now one can take a limit where all external weights have degenerate values $\alpha_i \to -j_i b$, $2j_i \in \mathbb{Z}_{\geq 0}$. We will denote this limit as

$$
\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}, \tag{90}
$$

remembering that it is a residue of the Racah-Wigner symbol with one degenerate external weight and both intermediate weights satisfying fusion rules.

Assuming that $\frac{n}{2} - \frac{\alpha_{134}}{b} = j_{134} + \frac{n}{2}$ in eq. (89) takes integer values one can write the $S_b$ functions in terms of the [.]-factorials (85)

$$
\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_1 b - \frac{sb}{2} \\ -\frac{nb}{2} & -j_4 b & -j_3 b - \frac{t}{2} b \end{array} \right\} = 2 \left( \frac{[2j_1 + \frac{s-n}{2}]!}{[2j_1 + \frac{n+s}{2} + 1]!} \frac{[2j_3 + \frac{t-n}{2}]!}{[2j_3 + \frac{n+t}{2} + 1]!} \right)^{\frac{1}{2}} \times
$$

$$
\times \sum_{p=max\{0,\frac{t-s}{2}\}}^{min\{\frac{n-s}{2},\frac{t+n}{2}\}} (-1)^{j_1+j_3-p+\frac{n+t}{2}} \frac{\left( \left[\frac{n-s}{2}\right]! \left[\frac{n+s}{2}\right]! \left[\frac{n-t}{2}\right]! \left[\frac{n+t}{2}\right]! \right)^{\frac{1}{2}}}{[p]! \left[p + \frac{s-t}{2}\right]! \left[\frac{n-s}{2} - p\right]! \left[\frac{n+t}{2} - p\right]!} \times
$$

$$
\times \frac{[j_{134} + p + \frac{s}{2} + 1]!}{\left([j_{134} + \frac{s}{2} + 1]! \left[j_{134} + \frac{t}{2} + 1\right]!\right)^{\frac{1}{2}}} \frac{\left([j_{13} - j_4 + \frac{s}{2}]! \left[j_{13} - j_4 + \frac{t}{2}\right]!\right)^{\frac{1}{2}}}{[j_{13} - j_4 - p + \frac{t}{2}]!} \times
$$

$$
\times \frac{\left([j_{34} - j_1 - \frac{s}{2}]! \left[j_{34} - j_1 + \frac{t}{2}\right]!\right)^{\frac{1}{2}}}{[j_{34} - j_1 + p - \frac{n}{2}]!} \frac{\left([j_{14} - j_3 + \frac{s}{2}]! \left[j_{14} - j_3 - \frac{t}{2}\right]!\right)^{\frac{1}{2}}}{[j_{14} - j_3 + p - \frac{t+n-s}{2}]!},
$$

38

where the minus sign comes from the difference in the shift relations (223) concerning $S_b(-xb)$ and $S_b(-xb+Q)$. Denoting $j_2 = \frac{n}{2}, j_s = j_1 + \frac{s}{2}, j_t = j_3 + \frac{t}{2}$ and shifting the summation parameter to $z = p + j_{s34}$, one obtains the 6j symbol of the finite dimensional representations of the quantum deformed algebra $\mathcal{U}_q(sl(2))$,

$$\left\{ \begin{array}{cc|c} -j_1b & -j_3b & -j_sb \\ -j_2b & -j_4b & -j_tb \end{array} \right\} = \frac{(-1)^{j_s+j_t}([2j_s+1]_q[2j_t+1]_q)^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \left( \begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right)_q, \tag{91}$$

where the deformation parameter $q$ is given in terms of $b$ as $q = e^{i\pi b^2}$ and the quantum numbers $[.]_q$ of $\mathcal{U}_q(sl(2))$ are equal those defined in eq. (60), i.e.

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = [x] . \tag{92}$$

The 6j symbol of finite dimensional representations of $U_q(sl(2))$ is given by the following sum [21, 62, 11]

$$\left( \begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right)_q = \sqrt{[2j_s+1]_t[2j_t+1]_q}\,(-1)^{j_{12}-j_{34}-2j_s} \times \tag{93}$$

$$\times \sum_{z \geq 0} (-1)^z \frac{\Delta_q(j_s,j_2,j_1)\Delta_q(j_s,j_3,j_4)\Delta_q(j_t,j_3,j_2)\Delta_q(j_4,j_t,j_1)\,[z+1]_q!}{[z-j_{12s}]_q!\,[z-j_{34s}]_q!\,[z-j_{14t}]_q!\,[z-j_{23t}]_q![j_{1234}-z]_q!\,[j_{13st}-z]_q!\,[j_{24st}-z]_q!} .$$

Here, the summation extend over those values of $z$ for which all arguments of the quantum number $[.]_q$ are non-negative. In addition we used the shorthand

$$\Delta_q(a,b,c) = \sqrt{[-a+b+c]_q!\,[a-b+c]_q!\,[a+b-c]_q!/[a+b+c+1]_q!} .$$

It is worth pointing out the similarities between the expression (93) and the original formula (81). In passing to equation (93), the four factors $\Delta$ got replaced by $\Delta_q$ while the eight functions $S_b$ have contributed the same number of quantum factorials. In addition, the integration over $u$ became a summation over $z$.

Let as finally note that it is also possible to consider a limit of all weights approaching general degenerate values $\alpha_i \to -j_ib - j_i'b^{-1}$. In that case the limit is proportional to product of two 6j symbols of finite dimensional representations of the quantum deformed algebra $\mathcal{U}_q(sl(2))$

$$\left\{ \begin{array}{cc|c} -j_1b - j_1'b^{-1} & -j_3b - j_3'b^{-1} & -j_sb - j_s'b^{-1} \\ -j_2b - j_2'b^{-1} & -j_4b - j_4'b^{-1} & -j_tb - j_t'b^{-1} \end{array} \right\} = (-1)^{j_{st}+j_{st}'+3j_{1234st}j_{1234st}'-j_{13}j_{13}'-j_{24}j_{24}'-j_{st}j_{st}'} \times$$

$$\times \frac{([2j_s+1]_q[2j_t+1]_q[2j_s'+1]_{q'}[2j_t'+1]_{q'})^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \left( \begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right)_q \left( \begin{array}{ccc} j_1' & j_2' & j_s' \\ j_3' & j_4' & j_t' \end{array} \right)_{q'}, \tag{94}$$

where deformation parameters are $q = e^{i\pi b^2}$ and $q' = e^{i\pi b^{-2}}$.

# 4   Heisenberg double of $U_q(sl(2, \mathbb{R}))$

In this section we will consider the Heisenberg double of the Borel half of $U_q(sl(2))$ and a class of its self-dual representations first considered in [20] in the context of Teichmüller theory of Riemann surfaces. Kashaev has shown that the Heisenberg double canonical element evaluated on this representations can be identified with a flip operator which is a quantised transformation relating two different triangulations of a fixed quadrilateral. Moreover, he showed that the algebra isomorphism of Heisenberg double is an operator changing the marked corner of a triangle belonging to a triangulation of a Riemann surface.

Since we are considering the non-compact version of Heisenberg double, our previous, general considerations are not directly applicable. However, one can make the following mathematically precise if one understands the algebraic formulae always as the realisations on some representations. We will recall a self-dual series of representations of Heisenberg double and evaluate on it the canonical element satisfying pentagon equation. We will show that $\mathcal{U}_q(sl(2))$ can be realised as a subalgebra of the tensor square of Heisenberg doubles in a manner slightly different than in section 2.3, repeating [20].

The Heisenberg double $H(U_q(\mathcal{B}))$ with a deformation parameter $q = e^{i\pi b^2}$ will be generated by elements $O, P$ from the Borel half $\mathcal{A}$ of $\mathcal{U}_q(sl(2))$ and $\hat{O}, \hat{P}$ of the dual Borel half $\mathcal{A}^*$ which have the following commutation relations:

$$
\begin{aligned}
[O, \hat{O}] &= \frac{1}{2\pi i}, \\
[O, P] &= -ibP, \\
[O, \hat{P}] &= ib\hat{P}, \\
[\hat{O}, P] &= 0, \\
[\hat{O}, \hat{P}] &= +ib\hat{P}, \\
[P, \hat{P}] &= q(1 - q^{-2})e^{2\pi bO},
\end{aligned}
\tag{95}
$$

and coproducts

$$
\begin{aligned}
\Delta(O) &= 1 \otimes O + O \otimes 1, \\
\Delta(\hat{O}) &= 1 \otimes \hat{O} + \hat{O} \otimes 1, \\
\Delta(P) &= P \otimes e^{2\pi bO} + 1 \otimes P, \\
\Delta(\hat{P}) &= \hat{P} \otimes e^{-2\pi b\hat{O}} + 1 \otimes \hat{P}.
\end{aligned}
\tag{96}
$$

Then the Heisenberg double is spanned by the basis $\{e(s, t) \otimes \hat{e}(s', t')\}$, where $s, s', t, t' \in \mathbb{R}$ such that

$$
e(s, t) = \frac{1}{2\pi}\Gamma(-is)G_{\mathrm{b}}(-it)q^{-(ib^{-1}t)^2}(2\pi iO)^{is}P^{ib^{-1}t},
$$
$$
\hat{e}(s, t) = \hat{O}^{is}\hat{P}^{ib^{-1}t},
$$

and the canonical element can be expressed by those generators in the following way

$$
S = \exp(2\pi iO \otimes \hat{O})g_{\mathrm{b}}^{-1}(P \otimes \hat{P}),
\tag{97}
$$

where the function $g_{\mathrm{b}}$ is related to the double sine and defined in the appendix B.1.

Now we will consider the self-dual representation $\pi : H(U_q(\mathcal{B})) \to L^2(\mathbb{R})$ of the Heisenberg double. Its generators can be expressed as an operators on $L^2(\mathbb{R})$ in the following way

$$
\begin{aligned}
O &= p, \\
P &= e^{2\pi bq}, \\
\hat{O} &= q, \\
\hat{P} &= e^{2\pi b(p-q)},
\end{aligned}
\tag{98}
$$

where $[p,q] = \frac{1}{2\pi i}$ are usual operators on $L^2(\mathbb{R})$. The self-duality $b \leftrightarrow \frac{1}{b}$ should be understood in the same way as in the case of $\mathcal{U}_q(sl(2))$, i.e. the second action of the Heisenberg double with $\tilde{q} = e^{i\pi b^{-2}}$ representented on $L^2(\mathbb{R})$ by (98) with $b$ replaced by $b^{-1}$ commutes with the one above. One can easily evaluate our canonical element in this representation:

$$
S = e^{2\pi i p_1 q_2} g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1+p_2-q_2)}).
\tag{99}
$$

By construction, the canonical element satisfies the pentagon equation. It is almost immediate to see that from the pentagon equation follows the usual pentagon equation for Fadeev's quantum dilogarithm (and vice-versa). Additionally, $S$ encodes the coproduct

$$
\Delta(e(s,t)) = Ad(S^{-1})(1 \otimes e(s,t)),
\tag{100}
$$

$$
\Delta(\hat{e}(s,t)) = Ad(S)(\hat{e}(s,t) \otimes 1),
\tag{101}
$$

which follows from the relations for the generators $O, P, \hat{O}, \hat{P}$ and it is easy to show, using a shift relation for the quantum dilogarithm

$$
\begin{aligned}
\Delta(O) &= S^{-1}(1 \otimes O)S = g_{\mathrm{b}}(P \otimes \hat{P})e^{-2\pi iO\otimes\hat{O}}(1 \otimes O)e^{2\pi iO\otimes\hat{O}}g_{\mathrm{b}}^{-1}(P \otimes \hat{P}) = \\
&= g_{\mathrm{b}}(P \otimes \hat{P})(1 \otimes O + O \otimes 1)g_{\mathrm{b}}^{-1}(P \otimes \hat{P}) = 1 \otimes O + O \otimes 1, \\
\Delta(P) &= S^{-1}(1 \otimes P)S = g_{\mathrm{b}}(P \otimes \hat{P})(1 \otimes P)g_{\mathrm{b}}^{-1}(P \otimes \hat{P}) = \\
&= g_{\mathrm{b}}(e^{2\pi b(q_1+p_2-q_2)})e^{2\pi bq_2}g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1+p_2-q_2)}) = \\
&= e^{\pi bq_2}g_{\mathrm{b}}(e^{-i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})g_{\mathrm{b}}^{-1}(e^{+i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})e^{\pi bq_2} = e^{\pi bq_2}(1 + e^{2\pi b(q_1+p_2-q_2)})e^{\pi bq_2} = \\
&= e^{2\pi bq_2} + e^{2\pi b(q_1+p_2)} = 1 \otimes P + P \otimes e^{2\pi bO},
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta(\hat{O}) &= S(\hat{O} \otimes 1)S^{-1} = e^{2\pi iO\otimes\hat{O}}g_{\mathrm{b}}^{-1}(P \otimes \hat{P})(\hat{O} \otimes 1)g_{\mathrm{b}}(P \otimes \hat{P})e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}(\hat{O} \otimes 1)e^{-2\pi iO\otimes\hat{O}} = 1 \otimes \hat{O} + \hat{O} \otimes 1, \\
\Delta(\hat{P}) &= S(\hat{P} \otimes 1)S^{-1} = e^{2\pi iO\otimes\hat{O}}g_{\mathrm{b}}^{-1}(P \otimes \hat{P})(\hat{P} \otimes 1)g_{\mathrm{b}}(P \otimes \hat{P})e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1+p_2-q_2)})e^{2\pi b(p_1-q_1)}g_{\mathrm{b}}(e^{2\pi b(q_1+p_2-q_2)})e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}e^{\pi b(p_1-q_1)}g_{\mathrm{b}}^{-1}(e^{+i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})g_{\mathrm{b}}(e^{-i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})e^{\pi b(p_1-q_1)}e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}e^{\pi b(p_1-q_1)}(1 + e^{2\pi b(q_1+p_2-q_2)})e^{\pi b(p_1-q_1)}e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}(e^{2\pi b(p_1-q_1)} + e^{2\pi b(p_1+p_2-q_2)})e^{-2\pi iO\otimes\hat{O}} = \\
&= e^{2\pi iO\otimes\hat{O}}(\hat{P} \otimes 1 + e^{2\pi bO} \otimes \hat{P})e^{-2\pi iO\otimes\hat{O}} = 1 \otimes \hat{P} + \hat{P} \otimes e^{-2\pi b\hat{O}}.
\end{aligned}
$$

Moreover, there exists an algebra automorphism $A = e^{-i\pi/3} e^{3\pi i q^2} e^{i\pi(p+q)^2}$, which has a following action on the operators $p$ and $q$

$$AqA^{-1} = p - q,$$
$$ApA^{-1} = -q,$$
$$A^{-1}qA = -p,$$
$$A^{-1}pA = q - p.$$

One can show that the elements $\tilde{e}(s,t), \hat{\tilde{e}}(s,t)$ defined by the action of $A$ as follows

$$\tilde{e}(s,t) = Ae(s,t)A^{-1},$$
$$\hat{\tilde{e}}(s,t) = A\hat{e}(s,t)A^{-1},$$

satisfy the same Heisenberg double relations.

The algebra automorphism can be used to establish the morphism between a tensor product of two Heisenberg doubles and $\mathcal{U}_q(sl(2))$. In particular, one can define the elements $E(s,t), \hat{E}(s,t)$ which are representented on $L^2(\mathbb{R}^2)$ as follows

$$E(a) = \mu(a;b,c)e(b) \otimes A_2 e(c) A_2^{-1},$$
$$\hat{E}(a) = m(a;c,b)\hat{e}(b) \otimes A_2^{-1} \hat{e}(c) A_2,$$

where $m$ and $\mu$ are the multiplication and comultiplication coefficients of the Borel half $\mathcal{A}$ of $\mathcal{U}_q(sl(2))$ respectively. In particular, the lowest lying elements of this type are as follows

$$E(1,0) = 2\pi i(p_1 - q_2),$$
$$E(0,1) = (q - q^{-1})^{-1}(e^{2\pi b(p_2 - q_2)} + e^{2\pi b(q_1 - q_2)}),$$
$$\hat{E}(1,0) = q_1 - p_2,$$
$$\hat{E}(0,1) = e^{2\pi b(q_2 - q_1)} + e^{2\pi b(p_1 - q_1)}.$$

It is clear that those elements do have a particular normalisation factors. It would be useful to define another set of elements $u(i,j)$, for which those normalisation factors has been removed, and which generate an algebra that we will denote by $\mathcal{G}$

$$u(1,0) = p_1 - q_2,$$
$$u(0,1) = e^{2\pi b(p_2 - q_2)} + e^{2\pi b(q_1 - q_2)},$$
$$\hat{u}(1,0) = q_1 - p_2,$$
$$\hat{u}(0,1) = e^{2\pi b(q_2 - q_1)} + e^{2\pi b(p_1 - q_1)}.$$

Those generators satisfy commutation relations

$$[u(1,0), \hat{u}(1,0)] = 0,$$
$$[u(1,0), \hat{u}(0,1)] = +ib\hat{u}(0,1),$$
$$[u(1,0), u(0,1)] = -ibu(0,1),$$
$$[\hat{u}(1,0), \hat{u}(0,1)] = +ib\hat{u}(0,1),$$
$$[\hat{u}(1,0), u(0,1)] = -ibu(0,1),$$
$$[u(1,0), \hat{u}(1,0)] = (q - q^{-1})(e^{2\pi b u(1,0)} - e^{-2\pi b \hat{u}(1,0)}).$$

Finally, there exists an algebra homomorphism $\mathcal{U}_q(sl(2)) \to \mathcal{G}^2$ with

$$K = e^{\pi b(u(1,0)+\hat{u}(1,0))/2},$$
$$E = ie^{-\pi b(c_b - \hat{u}(1,0))}\frac{\hat{u}(0,1)}{q - q^{-1}},$$
$$F = i\frac{u(0,1)}{q - q^{-1}}e^{\pi b(c_b - \hat{u}(1,0))},$$

which satisfy commutation relations of $\mathcal{U}_q(sl(2))$ with deformation parameter $q = e^{i\pi b^2}$. Therefore, one can manufacture representation of $\mathcal{U}_q(sl(2))$ (or, more generally, of arbitrary Drinfeld double) from representations of the Heisenberg double of the Borel half of $\mathcal{U}_q(sl(2))$ (or, respectively, of arbitrary Heisenberg double). Moreover, this map allows one to construct $R$-matrix of the Drinfeld double from the canonical element $S$.

# 5 Nonsupersymmetric quantum plane

The infinite-dimensional self-dual representations of a quantum plane were considered by Frenkel and Kim [7] and were shown to be relevant to the construction of Teichmüller theory. In particular, they showed that they furnish the solution to the pentagon equation corresponding to the flip operator of the quantum Teichmüller theory as 6j symbol ensuring the associativity of triple tensor product of the representations, and to the operator changing the marked corner of a triangle as a particular automorphism of the representations.

Frenkel and Kim has shown that the results obtained from the consideration of the category of representations of quantum plane is related to the one obtained by Kashaev in [20]. In fact the operators defining the Teichmüller theory found by Kashaev using the Heisenberg double construction are related by a similarity transformation to the ones from the representation theory of a quantum plane. In this section we will focus on the one of them, specifically 6j symbol.

## 5.1 Self-dual continuous series for a quantum plane

The quantum plane is essentialy the Borel half of a q-deformed universal enveloping algebra $\mathcal{U}_q(sl(2))$ of the Lie algebra $sl(2)$. It is generated by the elements $X, X^{-1}, Y$, with relations

$$XY = q^2 YX, \tag{102}$$

$$\Delta(X) = X \otimes X, \tag{103}$$

$$\Delta(Y) = Y \otimes X + 1 \otimes Y, \tag{104}$$

where the deformation parameter $q = e^{i\pi b^2}$. Again we parametrise the deformation through a real number b so that $q$ takes values on the unit circle. We also equip this algebra with the following *-structure

$$X^* \;=\; X \quad , \quad Y^* \;=\; Y \;.$$

Now we want to introduce the series of representations relevant for the quantum plane. The carrier spaces $\mathcal{H}$ of the associated representations are $L^2(\mathbb{R})$. Then, the generators $X, Y$ are expressed as

$$\pi(X) = e^{-2\pi bp} = T_x^{ib}, \tag{105}$$

$$\pi(Y) = e^{2\pi bx}, \tag{106}$$

where $[p, x] = \frac{1}{2\pi i}$ are usual operators on $L^2(\mathbb{R})$. $\pi(X), \pi(Y)$ have self-adjoint extensions in $L^2(\mathbb{R})$ [7]. This representation is self-dual in the usual sense which has been discussed in the previous sections.

## 5.2 The Clebsch-Gordan coefficients for a quantum plane

The tensor product of two representations $\pi$ is defined in terms of the coproduct, is reducible and its decomposition into a direct sum of irreducibles is what defines the Clebsh-Gordan coefficients. In this case one has the following decomposition,

$$\mathcal{H} \otimes \mathcal{H} \;\simeq\; \int_{\mathbb{R}}^{\oplus} \mathcal{H},$$

however we will establish that this tensor decomposition can be understood as

$$\mathcal{H} \otimes \mathcal{H} \;\simeq\; M \otimes \mathcal{H},$$

with the help of a multiplicity space such that $M = L^2(\mathbb{R})$. We are going to spell out and prove an explicit formula for the maps $\mathcal{H} \otimes \mathcal{H} \to M \otimes \mathcal{H}$ and $M \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$f(x_1, x_2) \to F_f(\alpha, x) \; = \; \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha & x \\ x_1 & x_2 \end{bmatrix} f(x_1, x_2) \, ,$$

$$F(\alpha, x) \to f_F(x_1, x_2) \; = \; \int_{\mathbb{R}} \mathrm{d}\alpha \mathrm{d}x \begin{bmatrix} \alpha & x \\ x_1 & x_2 \end{bmatrix}^{-1} F(\alpha, x) \, .$$

The kernels of the Clebsh-Gordan map are expressed in terms of $G_{\mathrm{b}}$ functions

$$\begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = e^{2\pi i \alpha(x_3 - x_1)} e^{2\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mathrm{b}}^{-1}(i(x_3 - x_2) + Q), \tag{107}$$

and

$$\begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}^{-1} = e^{-2\pi i \alpha(x_3 - x_1)} e^{-2\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mathrm{b}}(i(x_3 - x_2)). \tag{108}$$

The above expressions can be shown to be equivalent to those found by Frenkel and Kim, however we decided to rewrite them in terms of special functions used more profusely in other sections of this thesis.

## 5.3 The intertwining property

The fundamental intertwining property of the Clebsch-Gordan coefficients takes the following form

$$\pi(u) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} (\pi \otimes \pi) \Delta(u), \tag{109}$$

for $u = X, Y$. The equation should be interpreted in the same sense as the intertwining property (64). Rewriting this as an identity of functions in the variables $x_i$ one obtains

$$\pi(u) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = (\pi \otimes \pi) \Delta^t(u) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}. \tag{110}$$

One can check eq. (110) by direct computation. This is almost immediate for the element $X$

$$\Delta_{12}^t(X) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = T_{x_3}^{ib} \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = X_3 \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix},$$

$$\Delta_{12}^t(Y) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = (e^{2\pi b x_1} T_{x_2}^{ib} + e^{2\pi b x_2})^t \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} =$$

$$= e^{2\pi i \alpha(x_3 - x_1)} e^{2\pi i (x_3 - x_1)(x_2 - x_1)} \left\{ e^{2\pi b x_1} e^{2\pi b (x_3 - x_1)} G_{\mathrm{b}}^{-1}(Q + i(x_3 - x_2) - b) + e^{2\pi b x_2} G_{\mathrm{b}}^{-1}(Q + i(x_3 - x_2)) \right\} =$$

$$= e^{2\pi i \alpha(x_3 - x_1)} e^{2\pi i (x_3 - x_1)(x_2 - x_1)} \left\{ e^{2\pi b x_3}(1 - e^{2\pi i b(i(x_3 - x_2) + b^{-1})}) + e^{2\pi b x_2} \right\} G_{\mathrm{b}}^{-1}(Q + i(x_3 - x_2)) =$$

$$= e^{2\pi i \alpha(x_3 - x_1)} e^{2\pi i (x_3 - x_1)(x_2 - x_1)} e^{2\pi b x_3} G_{\mathrm{b}}^{-1}(i(x_3 - x_2) + Q) = e^{2\pi b x_3} \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} =$$

$$= Y_3 \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix},$$

while the computation for $Y$ is slightly more involved and uses the shift property of $G_{\mathrm{b}}$ function. One can also perform analogous computation for the inverse kernel.

## 5.4 Orthogonality and Completeness

The Clebsch-Gordan coefficients for the self-dual series of $\mathcal{U}_q(sl(2))$ satisfy the following orthogonality and completeness relation

$$\int_{\mathbb{R}^2} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \beta & y_3 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}^{-1} = \delta(\alpha_3 - \beta_3) \delta(x_3 - y_3), \tag{111}$$

$$\int_{\mathbb{R}^2} \mathrm{d}\alpha \mathrm{d}x \begin{bmatrix} \alpha & x \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha & x \\ y_1 & y_2 \end{bmatrix} = \delta(x_2 - y_2)\delta(x_1 - y_1) . \tag{112}$$

Let us focus on the proof of the orthogonality relation. Starting from the left hand side of (111) we can gather the exponentials depending on the $x_1$. Hence those exponentials are the only terms depending on the first variable, we can perform the integration which results in a Dirac delta function

$$\int \mathrm{d}x_1 e^{-2\pi i x_1(\beta - \alpha) - 2\pi i x_1(y_3 - x_3)} = \delta(\beta - \alpha + y_3 - x_3).$$

The resulting expression involves only an integration over $x_2$. In order to perform it, we use the representation of the Dirac delta function as a regularised integral involving two $G_\mathrm{b}$ functions, which we have stated and proven in the appendix B.1. Using it one can simplify the remaining terms of our computation

$$e^{2\pi i(\beta y_3 - \alpha x_3)} \int \mathrm{d}x_2 e^{2\pi i x_2(y_3 - x_3)} \frac{G_\mathrm{b}(i(x_3 - x_2))}{G_\mathrm{b}(Q + i(y_3 - x_2))} =$$

$$= e^{2\pi i(\beta y_3 - \alpha x_3) + 2\pi i(y_3 - x_3)y_3} \int \frac{\mathrm{d}\tau}{i} e^{2\pi i(-i(x_3 - y_3))\tau} \frac{G_\mathrm{b}(\tau + i(x_3 - y_3))}{G_\mathrm{b}(Q + \tau)} =$$

$$= \delta(y_3 - x_3),$$

which leads directly to

$$\int \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \beta & y_3 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}^{-1} = \delta(\beta - \alpha + y_3 - x_3)\delta(y_3 - x_3) =$$

$$= \delta(\beta - \alpha)\delta(y_3 - x_3),$$

which is in fact nothing else than our claim. The proof of completeness relation can be performed in a similar way.

## 5.5 The Racah-Wigner coefficients for a quantum plane

The Racah-Wigner coefficients describe an isomorphism between different 3-fold tensor product of representations. In decomposing their product into irreducibles $\pi$ there exists two possible fusion paths, denoted by $t$ and $s$, which are described by the following combination of Clebsch-Gordan coefficients

$$f(\alpha_s, \alpha_4, x_4) = \int \mathrm{d}x_s \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_s & x_s \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha_4 & x_4 \\ x_s & x_3 \end{bmatrix} f(x_1, x_2, x_3),$$

$$f(x_1, x_2, x_3) = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_s \mathrm{d}x_4 \mathrm{d}x_s \begin{bmatrix} \alpha_s & x_s \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_4 & x_4 \\ x_s & x_3 \end{bmatrix}^{-1} f(\alpha_s, \alpha_4, x_4),$$

for isomorphisms $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \cong M_{12}^s \otimes M_{s3}^4 \otimes \mathcal{H}_4$ and

$$f(\alpha_4, \alpha_t, x_4) = \int \mathrm{d}x_t \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_t & x_t \\ x_2 & x_3 \end{bmatrix} \begin{bmatrix} \alpha_4 & x_4 \\ x_1 & x_t \end{bmatrix} f(x_1, x_2, x_3),$$

$$f(x_1, x_2, x_3) = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_t \mathrm{d}x_4 \mathrm{d}x_t \begin{bmatrix} \alpha_t & x_t \\ x_2 & x_3 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_4 & x_4 \\ x_1 & x_t \end{bmatrix}^{-1} f(\alpha_4, \alpha_t, x_4),$$

for $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \cong M_{23}^t \otimes M_{1t}^4 \otimes \mathcal{H}_4$. Then one can relate the elements corresponding to those two decompositions using the map

$$f(\alpha_4', \alpha_t, x_4') = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_s \mathrm{d}x_4 \left\{ \begin{matrix} \alpha_4 & \alpha_s \\ \alpha_4' & \alpha_t \end{matrix} \right\}_\mathrm{b} f(\alpha_s, \alpha_4, x_4), \tag{113}$$

where the kernel of this map is 6j symbol and it is expressed using the Clebsch-Gordan coefficients in the following way

$$\left\{ \begin{matrix} \alpha_4 \\ \alpha_4' \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} = \int \mathrm{d}x_s \mathrm{d}x_t \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_t \ x_t \\ x_2 \ x_3 \end{bmatrix} \begin{bmatrix} \alpha_4' \ x_4' \\ x_1 \ x_t \end{bmatrix} \begin{bmatrix} \alpha_s \ x_s \\ x_1 \ x_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_4 \ x_4 \\ x_s \ x_3 \end{bmatrix}^{-1}. \tag{114}$$

The 6j symbol is expressed in terms of $G_{\mathrm{b}}$ functions by the formula below

$$\left\{ \begin{matrix} \alpha_4 \\ \alpha_4' \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} = \int \mathrm{d}x_s e^{2\pi i[(\alpha_4' - \alpha_s - x_s)^2 + x_s(\alpha_4 - \alpha_4') - \alpha_t(\alpha_4' - \alpha_s - x_s)]} G_{\mathrm{b}}(i(2x_s - \alpha_4' + \alpha_s))\delta(x_4' - x_4). \tag{115}$$

The derivation of eq. (115) from equation (114) can be performed directly using various integral identities summarised in the appendix. Integration over $x_3$ can be performed using the integral analogue of the Ramanujan summation formula as

$$\int \mathrm{d}x_3 e^{2\pi i x_3[x_t - x_2 - x_4 + x_s]} \frac{G_{\mathrm{b}}(i(x_4 - x_3))}{G_{\mathrm{b}}(i(x_t - x_3) + Q)} =$$
$$= e^{2\pi i x_t(x_t + x_s - x_4 - x_2)} \frac{G_{\mathrm{b}}(i(x_4 - x_t))G_{\mathrm{b}}(i(x_t + x_s - x_4 - x_2))}{G_{\mathrm{b}}(i(x_s - x_2))},$$

which results in the following form of the left hand site of equation (115)

$$\left\{ \begin{matrix} \alpha_4 \\ \alpha_4' \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} = \int \mathrm{d}x_s \mathrm{d}x_t \mathrm{d}x_2 \mathrm{d}x_1 e^{2\pi i[\alpha_t(x_t - x_2) + \alpha_4'(x_4' - x_1) - \alpha_s(x_s - x_1) - \alpha_4(x_4 - x_s)] + 2\pi i x_t(x_t + x_s - x_4 - x_2)} \times$$
$$\times e^{2\pi i[-x_2(x_t - x_2) + (x_4' - x_1)(x_t - x_1) - (x_s - x_1)(x_2 - x_1) + (x_4 - x_s)x_s]} \frac{G_{\mathrm{b}}(i(x_4 - x_t))G_{\mathrm{b}}(i(x_t + x_s - x_4 - x_2))}{G_{\mathrm{b}}(i(x_4' - x_t) + Q)}.$$

We see that the only dependence on $x_1$ lies in the exponentials, so we can easily perform the integration over this variable

$$\int \mathrm{d}x_1 e^{2\pi i x_1(-\alpha_4' + \alpha_s - x_t - x_4' + x_s + x_2)} = \delta(-\alpha_4' + \alpha_s - x_t - x_4' + x_s + x_2),$$

what results in the delta function which we use to perform the integration over $x_2$ as well. The expression for 6j symbol involves only two integrations, i.e. over $x_t$ and $x_s$

$$\left\{ \begin{matrix} \alpha_4 \\ \alpha_4' \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} = \int \mathrm{d}x_s \mathrm{d}x_t e^{2\pi i[\alpha_t(x_s - x_4' - \alpha_4' + \alpha_s) + \alpha_4' x_4' - \alpha_s x_s - \alpha_4(x_4 - x_s)] + 2\pi i x_t(x_t + x_s - x_4 - x_2)} \times$$
$$\times e^{2\pi i[-(\alpha_4' - \alpha_s + x_t + x_4' - x_s)(x_s - x_4' - \alpha_4' + \alpha_s) + x_4' x_t - x_s(\alpha_4' - \alpha_s + x_t + x_4' - x_s) + (x_4 - x_s)x_s]} \times$$
$$\times \frac{G_{\mathrm{b}}(i(x_4 - x_t))G_{\mathrm{b}}(i(2x_s - x_4 - x_4' - \alpha_4' + \alpha_s))}{G_{\mathrm{b}}(i(x_4' - x_t) + Q)}.$$

The final integration over $x_t$ can be performed using the relation which we have already used in the proof of orthogonality an completeness of Clebsch-Gordan coefficients

$$\int \mathrm{d}x_t e^{2\pi i x_t(2x_s - x_4 - \alpha_4' + \alpha_s) - 2\pi i x_t(x_s - x_4' - \alpha_4' + \alpha_s + x_s)} \frac{G_{\mathrm{b}}(i(x_4 - x_t))}{G_{\mathrm{b}}(i(x_4' - x_t) + Q)} = \delta(x_4' - x_4).$$

Above expression provides us immediately with the formula (115).

The Racah-Wigner coefficients are integration kernels between the $M_{23}^t \otimes M_{1t}^4 \otimes \mathcal{H}_4$ and $M_{12}^s \otimes M_{s3}^4 \otimes \mathcal{H}_4$ and we see that the action on the spaces $\mathcal{H}_4$ is trivial because of the presence of Dirac delta function. Therefore, we can define a map between the multiplicity spaces $M_{12}^s \otimes M_{s3}^4 \to M_{23}^t \otimes M_{1t}^4$ which is encoded in the integration kernel $T$

$$T \begin{bmatrix} \alpha_4 \ \alpha_s \\ \alpha_4' \ \alpha_t \end{bmatrix} = \int \mathrm{d}x_s e^{2\pi i[(\alpha_4' - \alpha_s - x_s)^2 + x_s(\alpha_4 - \alpha_4') - \alpha_t(\alpha_4' - \alpha_s - x_s)]} G_{\mathrm{b}}(i(2x_s - \alpha_4' + \alpha_s)).$$

Frenkel and Kim have showed that this operator is related by a linear transformation to the canonical element $S$ (99) of Heisenberg double evaluated on the self-dual representations by Kashaev in [20].

# 6 $\mathbb{Z}_2$-graded Hopf algebras

## 6.1 Graded quantum groups

In this section we present a brief introduction to the $\mathbb{Z}_2$-graded quantum groups. For more detailed treatment one can consult [55, 57, 56] on the topic of graded Lie algebras, and [58, 59, 22] for quantum supergroups.

### 6.1.1 Graded algebras and co-algebras

Let $k$ be a field. We want to generalise the notions of algebra and co-algebra by introducing a $\mathbb{Z}_2$ grading. To do that, we divide the (co-)algebra $\mathcal{A}$ into direct sum of two parts $\mathcal{A}_0 \oplus \mathcal{A}_1$ in such a way that the (co-)multiplication preserves this grading. These notions can be put into more strict definition as follows.

**Definition 12** *The unital associative $\mathbb{Z}_2$-graded algebra (also called superalgebra) is a triple $(\mathcal{A}, m, \eta)$, where $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a vector space, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is multiplication map and $\eta : k \to \mathcal{A}$ is unital map, such that the following axioms are satisfied:*

$$m(m \otimes id) = m(id \otimes m) \tag{116}$$

$$m(\eta \otimes id) = id = m(id \otimes \eta) \tag{117}$$

*and if $a \in \mathcal{A}_i, b \in \mathcal{A}_j$ then $m(a, b) \in \mathcal{A}_{i+j}$, where $i, j \in \mathbb{Z}_2$.*

If the element $a$ belongs to the subalgebra $\mathcal{A}_i$ we say that it is homogenous of degree $|a| = i$. We call the elements belonging to $\mathcal{A}_0$ even, while those belonging to $\mathcal{A}_1$ — odd.

**Definition 13** *The counital coassociative $\mathbb{Z}_2$-graded coalgebra (also called supercoalgebra) is a triple $(\mathcal{A}, \Delta, \epsilon)$, where $\mathcal{A}$ is a vector space such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is comultiplication map and $\epsilon : \mathcal{A} \to k$ is counital map, such that the following axioms are satisfied:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \tag{118}$$

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta \tag{119}$$

*and if $a \in \mathcal{A}_i$ then $\Delta(a) \in \mathcal{A}_i$, where $i \in \mathbb{Z}_2$.*

The definition of the tensor product is slightly different than in the case of algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. Then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is the superalgebra which as a vector space is the tensor product of $\mathcal{A}$ and $\mathcal{B}$ as vector spaces, with the induced $\mathbb{Z}_2$-grading and the operation defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2, \tag{120}$$

where $a_i \in \mathcal{A}, b_i \in \mathcal{B}$.

### 6.1.2 Graded Hopf algebras

**Definition 14** *A $\mathbb{Z}_2$-graded Hopf algebra is a collection $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$, where:*

- $(\mathcal{A}, m, \eta)$ *is an unital associative superalgebra, $(\mathcal{A}, \Delta, \epsilon)$ — a counital coassociative supercoalgebra.*

- $\Delta, \epsilon$ *are unital superalgebra homomorphism (i.e. function $f$ such that $m(f \otimes f) = fm$ and $f\eta = \eta$).*

- *There exists a linear antipodal map $S : \mathcal{A} \to \mathcal{A}$ satisfying $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\epsilon$.*

**Proposition 6** *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded Hopf algebra, and $S$ its antipode. $S$ is a unital and counital morphism such that $Sm = m(S \otimes S)\Sigma$ and $\Delta S = \Sigma(S \otimes S)\Delta$, where $\Sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Sigma(a \otimes b) = (-1)^{|a||b|}b \otimes a$.*

### 6.1.3 q-deformations

Let $[A_{ij}]$, $i, j = 1, \ldots, n$ be a Cartan matrix of superalgebra $\mathfrak{g}$ (where $n$ is a rank of $\mathfrak{g}$). The elements of the Cartan matrix can be expressed by the scalar products of the roots $\alpha_i$, which span the dual space $\mathfrak{h}^*$ to a Cartan subalgebra $\mathfrak{h}$.

$$a_{ij} = \begin{cases} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, & \text{if } (\alpha_i, \alpha_i) \neq 0, \\ (\alpha_i, \alpha_j), & \text{if } (\alpha_i, \alpha_i) = 0, \end{cases}$$

Moreover, if one defines a space

$$\mathfrak{g}_\alpha = \{\alpha \in \mathfrak{g} | [h, a] = \alpha(h)a, h \in \mathfrak{h}\},$$

for $\alpha$ being a nonzero root, then we call the root $\alpha$ even if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0$ and odd if $\mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq 0$.

Lets define

$$d_i = \begin{cases} \frac{1}{2}(\alpha_i, \alpha_i), & \text{if } (\alpha_i, \alpha_i) \neq 0, \\ 0, & \text{if } (\alpha_i, \alpha_i) = 0, \end{cases}$$

One sets $q_i = q^{d_i}$.

**Definition 15** *Let $\mathfrak{g}$ be a Lie superalgebra. The $\mathcal{U}_q(\mathfrak{g})$ is an associative superalgebra generated by $x_i, y_i, K_i, K_i^{-1}$ with relations*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{121}$$

$$K_i K_j - K_j K_i = 0, \tag{122}$$

$$x_i y_j - (-1)^{|x_i||y_j|} y_j x_i = \delta_{i,j} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}, \tag{123}$$

$$K_i x_j = q^{\frac{A_{ij}}{2}} x_j K_i, \quad K_i y_j = q^{-\frac{A_{ij}}{2}} y_j K_i, \tag{124}$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{Bmatrix} 1 - A_{ij} \\ k \end{Bmatrix}_{q_i} x_i^{1-A_{ij}-k} x_j x_i^k = 0, \qquad i \neq j, \quad (\alpha_i, \alpha_i) \neq 0, \tag{125}$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{Bmatrix} 1 - A_{ij} \\ k \end{Bmatrix}_{q_i} y_i^{1-A_{ij}-k} y_j y_i^k = 0, \qquad i \neq j, \quad (\alpha_i, \alpha_i) \neq 0, \tag{126}$$

$$(e_i)^2 = (f_i)^2 = 0, \qquad (\alpha_i, \alpha_i) = 0, \tag{127}$$

*where $|K_i| = 0$, $|x_i| = |y_i| = 0$ if $\alpha_i$ is an even root and $|x_i| = |y_i| = 1$ if $\alpha_i$ is an odd root and*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{q_i} = \begin{cases} \frac{[n]_{q_i}!}{[k]_{q_i}![n-k]_{q_i}!}, & \text{if } \alpha_i \text{ is even,} \\ (-1)^{\frac{k}{2}(k-(-1)^{|e_j|})} \frac{[n]_{q_i}^{(+)}!}{[k]_{q_i}^{(+)}![n-k]_{q_i}^{(+)}!}, & \text{if } \alpha_i \text{ is odd,} \end{cases}$$

*where $[n]_q^{(+)} = \frac{q^n + q^{-n}}{q + q^{-1}}$ for $n \in \mathbb{N}$.*

**Theorem 10** *Let $\mathcal{U}_q(\mathfrak{g})$ be a superalgebra generated by $x_i, y_i, K_i, K_i^{-1}$ with appropriate relations. Then $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ with*

$$\Delta(K_i) = K_i \otimes K_i, \tag{128}$$

$$\Delta(x_i) = x_i \otimes K_i + K_i^{-1} \otimes x_i, \tag{129}$$

$$\Delta(y_i) = y_i \otimes K_i + K_i^{-1} \otimes y_i, \tag{130}$$

$$\epsilon(K_i) = 1, \quad \epsilon(x_i) = \epsilon(y_i) = 0, \tag{131}$$

$$S(K_i) = K_i^{-1}, \quad S(x_i) = -q_i x_i, \quad S(y_i) = -q_i^{-1} y_i, \tag{132}$$

*is a noncocommutative $\mathbb{Z}_2$-graded Hopf algebra.*

**Theorem 11** *The $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ as above is a quasi-triangular $\mathbb{Z}_2$-graded Hopf algebra (also called a quantum supergroup), i.e. there exists the universal R-matrix $R \in \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ such that*

$$\Delta^{op}(a) = R\Delta(a)R^{-1}, \tag{133}$$

$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{134}$$

$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{135}$$

*where $\Delta^{op} = \Sigma\Delta, a \in \mathcal{U}_q(\mathfrak{g})$ and $\Sigma$ is a flip map defined as in proposition 6.*

### 6.1.4   Examples of graded quantum groups

**Example: $\mathcal{U}_q(osp(1|2))$.**

Lets consider one of the simplest quantum supergroups. It is a quasi-triangular $\mathbb{Z}_2$-graded super Hopf algebra $\mathcal{U}_q(osp(1|2))$ comming from the super Lie algebra $osp(1|2)$, which was first considered by Kulish and Reshetikhin [60]. It is generated by $K, K^{-1}, v^{(+)}, v^{(-)}$ satisfying relations:

$$Kv^{(\pm)} = q^{\pm\frac{1}{2}}v^{(\pm)}K,$$
$$\{v^{(+)}, v^{(-)}\} = -\frac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{136}$$

with the comultiplication

$$\Delta(K) = K \otimes K,$$
$$\Delta(v^{(\pm)}) = K \otimes v^{(\pm)} + v^{(\pm)} \otimes K^{-1}, \tag{137}$$

The generators are graded such that

$$|K| = 0,$$
$$|v^{(\pm)}| = 1.$$

PROOF: One can check if the comultiplication preserves the algebra structure:

$$\begin{aligned}
\Delta(K)\Delta(v^{(\pm)}) &= (K \otimes K)(v^{(\pm)} \otimes K^{-1} + K \otimes v^{(\pm)}) = Kv^{(\pm)} \otimes 1 + K^2 \otimes Kv^{(\pm)} = \\
&= q^{\pm\frac{1}{2}}(K^2 \otimes v^{(\pm)}K + v^{(\pm)}K \otimes 1) = q^{\pm\frac{1}{2}}\Delta(v^{(\pm)})\Delta(K),
\end{aligned}$$

$$\Delta(v^{(+)})\Delta(v^{(-)}) \;=\; (K \otimes v^{(+)} + v^{(+)} \otimes K^{-1})(K \otimes v^{(-)} + v^{(-)} \otimes K^{-1}) =$$
$$=\; K^2 \otimes v^{(+)}v^{(-)} - v^{(-)}K \otimes K^{-1}v^{(+)} + Kv^{(+)} \otimes v^{(-)}K^{-1} + v^{(+)}v^{(-)} \otimes K^{-2},$$
$$\Delta(v^{(-)})\Delta(v^{(+)}) \;=\; (K \otimes v^{(-)} + v^{(-)} \otimes K^{-1})(K \otimes v^{(+)} + v^{(+)} \otimes K^{-1}) =$$
$$=\; K^2 \otimes v^{(-)}v^{(+)} - v^{(+)}K \otimes v^{(-)}K^{-1} + v^{(-)}K \otimes K^{-1}v^{(+)} + v^{(-)}v^{(+)} \otimes K^{-2},$$

therefore

$$\{\Delta(v^{(+)}), \Delta(v^{(-)})\} \;=\; K^2 \otimes \{v^{(+)}, v^{(-)}\} + \{v^{(+)}, v^{(-)}\} \otimes K^{-2} =$$
$$=\; -\frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} (\Delta(K)^2 - \Delta(K)^{-2}).$$

Showing that the other axioms hold is left for the reader. $\qquad\square$

**Proposition 7** *The Casimir operator has the form*

$$C = -\frac{qK^4 + q^{-1}K^{-4} + 2}{q - q^{-1}} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(qK^2 + q^{-1}K^{-2})v^{(-)}v^{(+)} + (q - q^{-1})v^{(-)2}v^{(+)2}. \qquad (138)$$

PROOF: *For $v^{(+)}$ one has:*

$$-(q - q^{-1})[v^{(+)}, C] = -(q - q^{-1})(v^{(+)}C - Cv^{(+)}) =$$
$$= (q^{-1} - q)K^4 v^{(+)} + (q - q^{-1})K^{-4}v^{(+)} + (q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(qK^2 + q^{-1}K^{-2})v^{(-)}v^{(+)2} +$$
$$+ (q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(K^2 + K^{-2})v^{(-)}v^{(+)2} + (q - q^{-1})(K^+ K^{-2})(K^2 - K^{-2})v^{(+)} +$$
$$- \frac{(q - q^{-1})^2}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(K^2 - K^{-2})v^{(+)2} + \frac{(q - q^{-1})^2}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(K^2 - K^{-2})v^{(-)}v^{(+)2} =$$
$$= 0,$$

*and for $v^{(-)}$:*

$$-(q - q^{-1})[C, v^{(-)}] =$$
$$= (qK^4 + q^{-1}K^{-4})v^{(-)} + (q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(qK^2 + q^{-1}K^{-2})v^{(-)2}v^{(+)} +$$
$$+ \frac{(q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(qK^2 + q^{-1}K^{-2})(qK^2 - q^{-1}K^{-2})v^{(-)} - (q - q^{-1})^2 v^{(-)3}v^{(+)2} +$$
$$- \frac{(q - q^{-1})^2}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}v^{(-)2}(K^2 - K^{-2})v^{(+)} + \frac{(q - q^{-1})^2}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(qK^2 - q^{-1}K^{-2})v^{(-)2}v^{(+)} +$$
$$- (q^3 K^4 + q^{-3}K^{-4})v^{(-)} + (q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q^2 K^2 + q^{-2}K^{-2})v^{(-)2}v^{(+)} +$$
$$+ (q - q^{-1})^2 v^{(-)3}v^{(+)2} =$$
$$= 0.$$

*A proof for the commutator $[C, K]$ is straightforward.* $\qquad\square$

**Proposition 8** *Let $K = e^{\frac{1}{2}hH}$ i $q = e^{hb^2}$. Taking the classical limit $h \to 0$ one obtains*

$$[H, v^{(\pm)}] = \pm v^{(\pm)},$$
$$\{v^{(+)}, v^{(-)}\} = -2H,$$

*with coproduct*

$$\Delta(H) = 1 \otimes H + H \otimes 1,$$
$$\Delta(v^{(\pm)}) = 1 \otimes v^{(\pm)} + v^{(\pm)} \otimes 1.$$

The Casimir operator $C$ in the form as above does not have the classical limit, but there exists other Casimir operator:

$$\tilde{C} = \frac{C + \frac{4}{q-q^{-1}}}{q - q^{-1}} \rightarrow -H^2 - H - \frac{1}{4} + v^{(-)}v^{(+)} + v^{(-)2}v^{(+)2},$$

which commutes with $H, v^{(\pm)}$.

## 6.2   Graded Drinfeld double

As in the case of non-graded Hopf algebras, there is a method of obtaining quantum supergroups from an arbitrary $\mathbb{Z}_2$-graded Hopf algebras. This method is a generalisation of Drinfeld double construction, presented in [61]. In this section we will present this construction.

**Definition 16** *Let $\mathcal{A}$ and $\mathcal{A}^*$ be a pair of dual $\mathbb{Z}_2$-graded Hopf algebras generated by basis elements $E_\alpha$, $E^\alpha$, $\alpha \in I$ respectively with multiplication and co-multiplication*

$$E_\alpha E_\beta = m_{\alpha\beta}^{\gamma} E_\gamma, \tag{139}$$

$$\Delta(E_\alpha) = \mu_\alpha^{\beta\gamma} E_\beta \otimes E_\gamma, \tag{140}$$

$$S(E_\alpha) = S_\alpha^{\beta} E_\beta, \tag{141}$$

*and*

$$E^\alpha E^\beta = (-1)^{|\alpha||\beta|} \mu_\gamma^{\alpha\beta} E^\gamma, \tag{142}$$

$$\Delta(E^\alpha) = m_{\gamma\beta}^{\alpha} E^\beta \otimes E^\gamma, \tag{143}$$

$$S(E^\alpha) = (S^{-1})_\beta^{\alpha} E^\beta. \tag{144}$$

*Alternatively, with respect to the bracket $(,) : \mathcal{A} \times \mathcal{A}^* \to \mathbb{C}$ satisfying*

$$(m(a,b), c) = (a \otimes b, T\Delta(c)),$$

$$(\Delta(a), c \otimes d) = (a, m(c,d)),$$

$$(a \otimes b, c \otimes d) = (-1)^{|b||c|}(a,c)(b,d),$$

*where $a, b \in \mathcal{A}$, $c, d \in \mathcal{A}^*$ and $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$ the basis elements $\{E_\alpha\}$ and $\{E^\alpha\}$ ought to satisfy*

$$(E_\alpha, E^\beta) = \delta_\alpha^{\beta}.$$

*One can define Drinfeld double $D(\mathcal{A})$ as a vector space $D(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^*$ with basis elements $E_\alpha \otimes E^\beta$ which satisfy the double's defining relations*

$$(E_\alpha \otimes 1)(E_\beta \otimes 1) = m_{\alpha\beta}^{\gamma}(E_\gamma \otimes 1), \tag{145}$$

$$(1 \otimes E^\alpha)(1 \otimes E^\beta) = \mu_\gamma^{\alpha\beta}(1 \otimes E^\gamma), \tag{146}$$

$$(-1)^{|\gamma||\sigma|+|\rho||\sigma|} \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta}(E_\sigma \otimes 1)(1 \otimes E^\rho) = (-1)^{|\rho||\gamma|} m_{\rho\gamma}^{\beta} \mu_\alpha^{\gamma\sigma}(1 \otimes E^\rho)(E_\sigma \otimes 1)., \tag{147}$$

It is clear that the Drinfeld double defined as above is a Hopf superalgebra, however, we want to show something more — that it is a quasi-triangular Hopf superalgebra. With a slight abuse of notation, allow us to denote element $E_\alpha \otimes 1$ just as $E_\alpha$ and $1 \otimes E^\alpha$ just as $E^\alpha$.

**Theorem 12** *Lets consider the canonical element $R = E_\alpha \otimes E^\alpha$. $R$ is an universal R-matrix.*

PROOF: *Using the definition of $R$ one has to show that the following equations are satisfied*

$$T\Delta(a)R = R\Delta(a),$$

$$(\Delta \otimes id)R = R_{13}R_{23},$$

$$(id \otimes \Delta)R = R_{13}R_{12},$$

$$(T\Delta \otimes id)R = R_{23}R_{13},$$

$$(id \otimes T\Delta)R = R_{12}R_{13},$$

*which can be easily proven*

$$(\Delta \otimes id)R = (\Delta \otimes id)(E_\alpha \otimes E^\alpha) = \mu_\alpha^{\beta\gamma} E_\beta \otimes E_\gamma \otimes E^\alpha =$$

$$= E_\beta \otimes E_\gamma \otimes \mu_\alpha^{\beta\gamma} E^\alpha = (-1)^{|\beta||\gamma|} E_\beta \otimes E_\gamma \otimes E^\beta E^\gamma =$$

$$= (E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\gamma \otimes E^\gamma) = R_{13}R_{23},$$

$$(id \otimes \Delta)R = (id \otimes \Delta)(E_\alpha \otimes E^\alpha) = E_\gamma E_\beta \otimes E^\beta \otimes E^\gamma =$$

$$= (E_\gamma \otimes 1 \otimes E^\gamma)(E_\beta \otimes E^\beta \otimes 1) = R_{13}R_{12},$$

$$(T\Delta \otimes id)R = (T\Delta \otimes id)(E_\alpha \otimes E^\alpha) = (T \otimes id)(\mu_\alpha^{\beta\gamma} E_\beta \otimes E_\gamma \otimes E^\alpha) =$$

$$= E_\gamma \otimes E_\beta \otimes (-1)^{|\beta||\gamma|} \mu_\alpha^{\beta\gamma} E^\alpha = E_\gamma \otimes E_\beta \otimes E^\beta E^\gamma =$$

$$= (1 \otimes E_\beta \otimes E^\beta)(E_\gamma \otimes 1 \otimes E^\gamma) = R_{23}R_{13},$$

$$(id \otimes T\Delta)R = (id \otimes T\Delta)(E_\alpha \otimes E^\alpha) = (-1)^{|\beta||\gamma|} m_{\beta\gamma}^\alpha E_\alpha \otimes E^\beta \otimes E^\gamma =$$

$$= (-1)^{|\beta||\gamma|} E_\beta E_\gamma \otimes E^\beta \otimes E^\gamma = (E_\beta \otimes E^\beta \otimes 1)(E_\gamma \otimes 1 \otimes E^\gamma) = R_{12}R_{13}.$$

$$R\Delta(E_i) = (E_\alpha \otimes E^\alpha)\mu_i^{\beta\gamma} E_\beta \otimes E_\gamma = (-1)^{|\beta||\alpha|} \mu_i^{\beta\gamma} E_\alpha E_\beta \otimes E^\alpha E_\gamma =$$

$$= E_\delta \otimes ((-1)^{|\beta||\alpha|} \mu_i^{\beta\gamma} m_{\alpha\beta}^\delta E^\alpha E_\gamma) = E_\delta \otimes ((-1)^{|\beta||\gamma|+|\alpha||\gamma|} \mu_i^{\gamma\beta} m_{\beta\alpha}^\delta E_\gamma E^\alpha) =$$

$$= (-1)^{|\beta||\gamma|+|\alpha||\gamma|} \mu_i^{\gamma\beta} E_\beta E_\alpha \otimes E_\gamma E^\alpha = (-1)^{|\beta||\gamma|} \mu_i^{\gamma\beta}(E_\beta \otimes E_\gamma)(E_\alpha \otimes E^\alpha) =$$

$$= T(\mu_i^{\gamma\beta}(E_\gamma \otimes E_\beta))R = T\Delta(E_i)R,$$

*and analogously for $a = E^i$. This provides us with the claim.* □

**Theorem 13** *From the above follows that the canonical element $R$ satisfies Yang-Baxter relation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

PROOF:*Alternatively, one can show that YB equation is satisfied directly using the definition of $R$*

$$R_{12}R_{13}R_{23} = (E_\alpha \otimes E^\alpha \otimes 1)(E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\gamma \otimes E^\gamma) =$$

$$= (-1)^{|\beta||\gamma|+|\alpha||\beta|} E_\alpha E_\beta \otimes E^\alpha E_\gamma \otimes E^\beta E^\gamma = (-1)^{|\alpha||\beta|} E_\sigma \otimes m_{\alpha\beta}^\sigma \mu_\rho^{\beta\gamma} E^\alpha E_\gamma \otimes E^\rho =$$

$$= (-1)^{|\beta||\gamma|+|\gamma||\alpha|} E_\sigma \otimes m_{\beta\alpha}^\sigma \mu_\rho^{\gamma\beta} E_\gamma E^\alpha \otimes E^\rho = (-1)^{|\beta||\gamma|+|\gamma||\alpha|} m_{\beta\alpha}^\sigma E_\sigma \otimes E_\gamma E^\alpha \otimes \mu_\rho^{\gamma\beta} E^\rho =$$

$$= (-1)^{|\gamma||\alpha|} E_\beta E_\alpha \otimes E_\gamma E^\alpha \otimes E^\gamma E^\beta = (1 \otimes E_\gamma \otimes E^\gamma)(E_\beta \otimes 1 \otimes E^\beta)(E_\alpha \otimes E^\alpha \otimes 1) =$$

$$= R_{23}R_{13}R_{12}.$$

□

Let us illustrate this construction with an example. In the previous section we considered $\mathcal{U}_q(osp(1|2))$. It is possible to construct it as a Drinfeld double of the the Borel half $U_q(\mathcal{B})$ of $U_q(osp(1|2))$, which we will take as the algebra $\mathcal{A}$. Lets begin from the elements $H, v^{(+)}$ satisfying the following relations

$$[H, v^{(+)}] = v^{(+)},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(v^{(+)}) = v^{(+)} \otimes e^{hH} + 1 \otimes v^{(+)},$$

with $|H| = 0, |v^{(+)}| = 1$. Additionally, lets set $q = e^{-h}$. Then, the algebra $\mathcal{A}$ will have basis elements of the form

$$E_{m,n} = (-1)^{-n^2/2} \frac{1}{m!(-q)_n} H^m v^{(+)^n}.$$

The multiplication and comultiplication for those elements can be found from those for the elements $H$ and $v^{(+)}$ and has the form

$$E_{m,n} E_{l,k} = (-1)^{nk} \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_{-q} \frac{(-n)^{l-j}}{(l-j)!} E_{m+j,n+k},$$

$$\Delta(E_{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^{l(l-m)} \sum_{p=0}^{\infty} \binom{k+p}{k} (m-l)^p h^p E_{n-k,m-l} \otimes E_{k+p,l}.$$

Now, let us move to the consideration of the dual algebra $\mathcal{A}^*$. It is generated by the elements $\bar{H}, v^{(-)}$ satisfying

$$[\bar{H}, v^{(-)}] = -h v^{(-)},$$

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H},$$

$$\Delta(v^{(-)}) = v^{(-)} \otimes e^{-\bar{H}} + 1 \otimes v^{(-)},$$

with $|\bar{H}| = 0, |v^{(-)}| = 1$, and the basis elements have the form

$$E^{n,m} = \bar{H}^n v^{(-)^m}.$$

The multiplication and comultiplication for above has the form

$$E^{m,n} E^{l,k} = \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} E^{m+j,n+k},$$

$$\Delta(E^{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{n}{k} \binom{m}{l}_{-q} \frac{(-m+l)^p}{p!} E^{n-k,m-l} \otimes E^{k+p,l}.$$

It is clear that the bases $\{E_{n,m}\}$ and $\{E^{n,m}\}$ are dual to each other and the multiplication and comultiplication coefficients are given explicitly by

$$m_{m,n;l,k}^{r,s} = (-1)^{nk} \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!} \delta_{r,m+j} \delta_{s,n+k} =$$

$$= (-1)^{nk} \binom{r}{r-m} \binom{n+k}{k}_q \frac{(-n)^{l-r+m}}{(l-r+m)!} \Theta(r-m) \Theta(l-r+m) \delta_{s,n+k},$$

$$\mu_{r,s}^{m,n;l,k} = (-1)^{l(l-m)} \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} \delta_{r,m+j} \delta_{s,n+k} =$$

$$= (-1)^{l(l-m)} \binom{l}{r-m} (n)^{l-r+m} h^{l-r+m} \Theta(r-m) \Theta(l-r+m) \delta_{s,n+k}.$$

Then one can show that Drinfeld double is isomorphic to the q-deformation of $U(osp(1|2))$

$$U_q(osp(1|2)) \cong D(U_q(\mathcal{B}))/(\bar{H} - hH).$$

Now one can consider the universal R-matrix. Using the formula for a canonical element of the Drinfeld double we obtain

$$R = \exp(H \otimes \bar{H})(-iv^{(+)} \otimes v^{(-)}; -q)_\infty^{-1}.$$

## 6.3 Graded Heisenberg double

Not only the Drinfeld double construction generalises to the $\mathbb{Z}_2$-graded setting — Heisenberg double definition can be extended for the $\mathbb{Z}_2$-graded Hopf algebras as well. Then, instead of a canonical element satisfying the pentagon equation one gets a canonical element which satisfies graded analogue of it.

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded Hopf algebra spanned by the basis vectors $\{e_\alpha\}$ with the following multiplication and comultiplication:

$$e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \tag{148}$$

$$\Delta(e_\alpha) = \mu_\alpha^{\beta\gamma} e_\beta \otimes e_\gamma. \tag{149}$$

Moreover, a $\mathbb{Z}_2$-graded Hopf algebra $\mathcal{A}^*$ spanned by the basis vectors $\{e^\alpha\}$ with:

$$e^\alpha e^\beta = (-1)^{|\alpha||\beta|} \mu_\gamma^{\alpha\beta} e^\gamma, \tag{150}$$

$$\Delta(e^\alpha) = (-1)^{|\beta||\gamma|} m_{\beta\gamma}^\alpha e^\beta \otimes e^\gamma. \tag{151}$$

and is dual to $\mathcal{A}$ with respect to a duality bracket $(,) : \mathcal{A} \times \mathcal{A}^* \to \mathbb{C}$ such that

$$(e_\alpha, e^\beta) = \delta_\alpha^\beta.$$

and it preserve the algebraic structures

$$(m(a,b), c) = (a \otimes b, \Delta^*(c)),$$

$$(\Delta(a), c \otimes d) = (a, m^*(c,d)),$$

$$(a \otimes b, c \otimes d) = (-1)^{|b||c|}(a,c)(b,d),$$

where $a, b \in \mathcal{A}$, $c, d \in \mathcal{A}^*$.

**Definition 17** *The Heisenberg double $H(\mathcal{A})$ is a superalgebra s.t. as a vector space $H(\mathcal{A}) \cong \mathcal{A} \oplus \mathcal{A}^*$ generated by the elements $\{e_\alpha \otimes e^\beta\}$, $\alpha, \beta \in I$, with multiplication*

$$(e_\alpha \otimes 1)(e_\beta \otimes 1) = m_{\alpha\beta}^\gamma(e_\gamma \otimes 1), \tag{152}$$

$$(1 \otimes e^\alpha)(1 \otimes e^\beta) = (-1)^{|\alpha||\beta|} \mu_\gamma^{\alpha\beta}(1 \otimes e^\gamma), \tag{153}$$

$$(-1)^{|\alpha||\beta|}(e_\alpha \otimes 1)(1 \otimes e^\beta) = (-1)^{|\rho||\gamma|} m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma}(1 \otimes e^\rho)(e_\sigma \otimes 1). \tag{154}$$

**Theorem 14** *Then canonical element $S = e_\alpha \otimes 1 \otimes 1 \otimes e^\alpha \in H(\mathcal{A}) \otimes H(\mathcal{A})$ satisfies the pentagon equation*

$$S_{12} S_{13} S_{23} = S_{23} S_{12}. \tag{155}$$

Let us consider an example of the graded Heisenberg algebra. Lets consider the Borel half $U_q(\mathcal{B})$ of $U_q(osp(1|2))$ as the algebra $\mathcal{A}$. It is generated by the elements $H, v^{(+)}$ satisfying the following relations

$$[H, v^{(+)}] = v^{(+)},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(v^{(+)}) = v^{(+)} \otimes e^{hH} + 1 \otimes v^{(+)},$$

and $|H| = 0, |v^{(+)}| = 1$. As usual $q = e^{-h}$. The algebra $\mathcal{A}$ will have basis elements of the form

$$e_{m,n} = (-1)^{-n^2/2} \frac{1}{m!(-q)_n} H^m v^{(+)n}.$$

The multiplication and comultiplication for those elements have the form

$$e_{m,n} e_{l,k} = (-1)^{nk} \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!} e_{m+j,n+k},$$

$$\Delta(e_{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} (-1)^{l(l-m)} \binom{k+p}{k} (m-l)^p h^p e_{n-k,m-l} \otimes e_{k+p,l},$$

The dual algebra $\mathcal{A}^*$ is generated by the elements $\bar{H}, v^{(-)}$ satisfying

$$[\bar{H}, v^{(-)}] = -h v^{(-)},$$

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H},$$

$$\Delta(v^{(-)}) = v^{(-)} \otimes e^{-\bar{H}} + 1 \otimes v^{(-)},$$

with $|\bar{H}| = 0, |v^{(-)}| = 1$, and the basis elements have the form

$$e^{n,m} = \bar{H}^n v^{(-)m}.$$

The multiplication and comultiplication for above has the form

$$e^{m,n} e^{l,k} = \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} e^{m+j,n+k},$$

$$\Delta(e^{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{n}{k} \binom{m}{l}_{-q} \frac{(-m+l)^p}{p!} e^{n-k,m-l} \otimes e^{k+p,l}.$$

By inspection is it clear that the bases $\{e_{n,m}\}$ and $\{e^{n,m}\}$ are dual to each other. From the relation (154) one finds the rest of commutation relations

$$H\bar{H} = 1 + \bar{H}H,$$

$$v^{(+)}\bar{H} = \bar{H}v^{(+)},$$

$$Hv^{(-)} - v^{(-)}H = -v^{(-)},$$

$$v^{(+)}v^{(-)} + v^{(-)}v^{(+)} = -i(1+q)q^{-H}.$$

The canonical element in this case has the form:

$$S = \exp(H \otimes \bar{H})(-iv^{(+)} \otimes v^{(-)}; -q)_\infty^{-1}.$$

The result is rather compelling given the known connection [54] between quantities for finite dimensional representations of $U_q(sl(2))$ and $U_{-q}(osp(1,2))$.

# 7 Representation theory of graded algebras

## 7.1 Self-dual continuous series for $\mathcal{U}_q(osp(1|2))$

After the extended review of the quantum deformed enveloping algebra $\mathcal{U}_q(sl(2))$ and its self-conjugate series of representations in the section 3 and of quantum superalgebras in the previous section we are now well prepared to turn to the algebra $\mathcal{U}_q(osp(1|2))$. The content of the following sections is based on the original research presented in [29, 30].

Following [38], the quantum deformed superalgebra $\mathcal{U}_q(osp(1|2))$ is generated by the bosonic generators $K, K^{-1}$ along with two fermionic (odd) ones $v^{(\pm)}$. These satisfy the relations

$$Kv^{(\pm)} = q^{\pm 1}v^{(\pm)}K, \tag{156}$$

$$\{v^{(+)}, v^{(-)}\} = -\frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}, \tag{157}$$

where $q = e^{i\pi b^2}$. We demand also the following star structure

$$K^* = K \quad , \quad v^{(\pm)*} = \pm v^{(\pm)}, \tag{158}$$

and with the coproduct

$$\Delta(K) = K \otimes K \quad , \quad \Delta(v^{(\pm)}) = v^{(\pm)} \otimes K + K^{-1} \otimes v^{(\pm)}, \tag{159}$$

that can be used to define tensor products of representations. It is easy to verify that the following even element of $\mathcal{U}_q(osp(1|2))$ commutes with all generators,

$$C = -\frac{qK^4 + q^{-1}K^{-4} + 2}{(q - q^{-1})^2} + \frac{(qK^2 + q^{-1}K^{-2})v^{(-)}v^{(+)}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} + v^{(-)2}v^{(+)2},$$

i.e. $C$ is a Casimir element. In addition, the algebra $\mathcal{U}_q(osp(1|2))$ also contains an element $Q$ which is defined as

$$Q = \frac{1}{2}(v^{(-)}v^{(+)} - v^{(+)}v^{(-)}) + \frac{1}{2}\frac{K^2 + K^{-2}}{q^{1/2} + q^{-1/2}}.$$

Up to some shift, the element $Q$ may be considered as the square root of the quadratic Casimir element $C$,

$$C = -\left(Q + \frac{2i}{q - q^{-1}}\right)\left(Q - \frac{2i}{q - q^{-1}}\right).$$

This concludes our short description of the algebraic setup so that we can begin to discuss the representations we are about to analyse. Following the intuition developed from the non-supersymmetric case, we shall introduce representation on carrier spaces $\mathcal{Q}_\alpha$ which are parametrized by a single parameter $\alpha$ of the form $\alpha \in \frac{Q}{2} + i\mathbb{R}$ where $Q = b + \frac{1}{b}$ and the relation between b and $q$ is the same as before. The spaces $\mathcal{Q}_\alpha$ are graded vector spaces. By definition, they consist of pairs $(f^0(x), f^1(x))$ of functions $f^j$ which are entire analytic and whose Fourier transform $\hat{f}^j(\omega)$ is allowed to possess poles in the set $\mathcal{S}_\alpha$ that was defined in eq. (56). The upper index $j$ indicates the parity of the element, i.e. vectors of the form $(f^0, 0)$ are considered even while we declare elements of the form $(0, f^1)$ to be odd. On these carrier spaces, we represent the elements $K$ and $v^\pm$ through

$$\pi_\alpha(K) = T_x^{\frac{ib}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \pi_\alpha\left(v^{(\pm)}\right) = ie^{\pm\pi bx}\begin{pmatrix} 0 & [\delta_x \pm \bar{\alpha}]_- \\ [\delta_x \pm \bar{\alpha}]_+ & 0 \end{pmatrix}, \tag{160}$$

where $T_x^{ia}$ denotes the shift operator that was defined in eq. (58) and we introduced

$$[x]_- = \frac{\sin(\frac{\pi b x}{2})}{\sin(\frac{\pi b^2}{2})} \quad , \quad [x]_+ = \frac{\cos(\frac{\pi b x}{2})}{\cos(\frac{\pi b^2}{2})} . \tag{161}$$

Consequently, the matrix elements in our expression for $\pi_\alpha(v^\pm)$ are given by

$$[\delta_x + \bar\alpha]_- = \frac{e^{\frac{i\pi b \alpha}{2}} T^{\frac{ib}{2}} - e^{\frac{-i\pi b \alpha}{2}} T^{-\frac{ib}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$[\delta_x + \bar\alpha]_+ = \frac{e^{\frac{i\pi b \alpha}{2}} T^{\frac{ib}{2}} + e^{\frac{-i\pi b \alpha}{2}} T^{-\frac{ib}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}.$$

It is not difficult to check that our prescription respects the algebraic relations in the universal enveloping algebra $\mathcal{U}_q(osp(1|2))$ and hence provides a family of representations. In these representations, we can evaluate the Casimir element $C$ and its square root $Q$,

$$\pi_\alpha(C) = [\frac{Q}{2} - \alpha]_-^2 [\frac{Q}{2} - \alpha]_+^2 \sigma_0 \quad , \quad \pi_\alpha(Q) = [\bar\alpha - \frac{b}{2}]_- [\bar\alpha - \frac{b}{2}]_+ \sigma_3,$$

where $\sigma_0$ denotes the 2-dimensional identity matrix and $\sigma_3$ is the Pauli matrix that is diagonal in our basis. Note that the value of the Casimir element $C$ is the same in the representations $\pi_\alpha$ and $\pi_{\bar\alpha}$. This is because the representations are actually equivalent. In fact, one may easily check that the following unitary operator

$$\mathcal{I}_\alpha = \begin{pmatrix} 0 & \frac{S_1(\alpha - i\omega)}{S_1(\bar\alpha - i\omega)} \\ \frac{S_0(\alpha - i\omega)}{S_0(\bar\alpha - i\omega)} & 0 \end{pmatrix},$$

involving the special functions $S_\nu$ defined in Appendix B.1, satisfies

$$\pi_{\bar\alpha}(X)\mathcal{I}_\alpha = \mathcal{I}_\alpha \pi_\alpha(X) \quad , \quad \text{for} \quad X = K, v^{(\pm)}.$$

In order to discuss the reality properties of our representation, we need to introduce the following matrix

$$\lambda^2 = \begin{pmatrix} \varrho & 0 \\ 0 & \varrho^{-1} \end{pmatrix} \quad \text{where} \quad \varrho = i\frac{q^{1/2} - q^{-1/2}}{q^{1/2} + q^{-1/2}} . \tag{162}$$

A square root $\lambda$ of the matrix $\lambda^2$ appears in the definition of the scalar product

$$\langle g, f \rangle = \sum_{i,j} \int \mathrm{d}x\, g^i(x)^* \lambda_{ij} f^j(x) . \tag{163}$$

A short calculation shows that the adjoint with respect to this scalar product implements the $*$ operation we defined above. Once again one can check that the representations $\pi_\alpha$ admit duality $b \to b^{-1}$ in the same sense as above. More concretely, our formulae can be used to define a representation of $\mathcal{U}_{\tilde q}(osp(1|2))$ with $\tilde q = e^{i\pi b^{-2}}$ on $\mathcal{Q}_\alpha$ such that the corresponding operators (anti-)commute with the representation operators for the original action of $\mathcal{U}_q(osp(1|2))$ on $\mathcal{Q}_\alpha$. Let us mention that a similar series of representations was recently discussed in [48], though the precise relation to the ones we consider here is not clear to us.

## 7.2 The Clebsch-Gordan coefficients for $\mathcal{U}_q(osp(1|2))$

As in the case of $\mathcal{U}_q(sl(2))$, we are interested in the Clebsch-Gordan decomposition of the representation $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$,

$$\pi_{\alpha_2} \otimes \pi_{\alpha_1} \simeq \int_{\frac{Q}{2} + i\mathbb{R}^+}^{\otimes} \mathrm{d}\alpha_3\, \pi_{\alpha_3} .$$

We shall show below that there exist two independent intertwiners for any given choice of $\alpha_1, \alpha_2$ and $\alpha_3$. We shall label these by an index $\nu$. The corresponding Clebsch-Gordan coefficients are defined as

$$F_f^{(\nu)\,j_3}(\alpha_3, x_3) =: \sum_{j_2, j_1} \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{j_2 j_1}^{(\nu)\,j_3} f^{j_2 j_1}(x_2, x_1) \ .$$

In order to construct these coefficients, we introduce the following products

$$D_{\tau\sigma}^{(\nu);\epsilon}(x_i; \alpha_i) = (-1)^{\nu(\tau+1)(\sigma+1)} \frac{S_{\tau+\nu+1}(z_{21} + \frac{\epsilon}{2})}{S_\tau(z_{21} + \alpha_{21})} \frac{S_{\nu+\sigma+1}(z_{23} + \epsilon)}{S_\sigma(z_{23} - \frac{\epsilon}{2} + \alpha_{23})} \frac{S_{\tau+\sigma+\nu}(z_{13} + \frac{\epsilon}{2})}{S_{\tau+\sigma+1}(z_{13} + \epsilon + \alpha_{13})}, \tag{164}$$

where $z_{ij}$ and $\alpha_{ij}$ in the same way as in the $\mathcal{U}_q(sl(2))$ case, and $\nu, \tau, \sigma = 0, 1 \mod 2$. In addition, we have introduced a parameter $\epsilon$ that will serve as a regulator in products of Clebsch-Gordan coefficients later on, just as in the case of $\mathcal{U}_q(sl(2))$. The Clebsch-Gordan maps are now obtained as

$$\begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{j_2 j_1}^{(\nu)\,j_3} = \sum_{\tau, \sigma} (-1)^{(j_1+\sigma)(j_2+\tau+\sigma)} (-|\rho|)^{\nu(1-(j_1-j_2)^2) - j_1 j_2} \delta_{j_1+j_2+\nu, j_3} \mathcal{N}^{1/2} D_{\tau\sigma}^{(\nu)}(x_i; \alpha_i) \ . \tag{165}$$

The normalizing factor $\mathcal{N}^{1/2}$ is the square root of the $\mathcal{N}$ we defined in eq. (63). Regularization is understood whenever it is necessary. If we remove the regulator $\epsilon$, we obtain the Clebsch-Gordan coefficients

$$\begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{j_2 j_1}^{(\nu)\,j_3} = \lim_{\epsilon \to 0} \left( \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{\epsilon} \right)_{j_2 j_1}^{(\nu)\,j_3} \ .$$

The intertwiner properties and orthogonality relations for these Clebsch-Gordan coefficients are established following the same steps as in the case of $\mathcal{U}_q(sl(2))$. Our discussion in the subsequent section will therefore focus on equations containing additional signs and on the final results.

### 7.2.1   Intertwiner property

The Clebsh-Gordan coefficients satisfy the intertwining properties for $X = K, v^{(\pm)}$

$$\pi_{\alpha_3}(X)^i{}_{j_3} \delta_j^{j_2} \delta_k^{j_1} \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{j_2 j_1}^{j_3} = \delta_{j_3}^i (\pi_{\alpha_2} \otimes \pi_{\alpha_1}) \Delta^t(X)^{j_2\ j_1}{}_{j\ k} \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{j_2 j_1}^{j_3} \ .$$

The transpose on the right hand side is defined with respect to the scalar product (163). All these equations may be checked by direct computations. For $X = K$ the analysis is identical to the one outlined in section 3.1. So, let us proceed to $X = v^{(+)}$ right away. When written out in components, our basic intertwining relation reads

$$(v_{\alpha_3}^{(+)})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{00}^{(0)\,0} = \Delta_{12}^t(v^{(+)})^0{}_0{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{01}^{(0)\,1} + \Delta_{12}^t(v^{(+)})^1{}_0{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{12}^{(0)\,1} \ ,$$

$$(v_{\alpha_3}^{(+)})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{01}^{(0)\,1} = \Delta_{12}^t(v^{(+)})^0{}_0{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{00}^{(0)\,0} + \Delta_{12}^t(v^{(+)})^1{}_0{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{11}^{(0)\,0} \ ,$$

$$(v_{\alpha_3}^{(+)})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{12}^{(0)\,1} = -\Delta_{12}^t(v^{(+)})^1{}_1{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{11}^{(0)\,0} + \Delta_{12}^t(v^{(+)})^0{}_1{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{00}^{(0)\,0} \ ,$$

$$(v_{\alpha_3}^{(+)})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{11}^{(0)\,0} = \Delta_{12}^t(v^{(+)})^0{}_1{}^1{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{01}^{(0)\,1} - \Delta_{12}^t(v^{(+)})^1{}_1{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_{12}^{(0)\,1} \ .$$

For the second set of Clebsch-Gordan coefficients we find,

$$(v_{\alpha_3}^{(+)})^0{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{11}^{(1)\,1} = \Delta_{12}^t (v^{(+)})^0{}_1{}^1{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{01}^{(1)\,0} - \Delta_{12}^t (v^{(+)})^1{}_1{}^0{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{12}^{(1)\,0},$$

$$(v_{\alpha_3}^{(+)})^0{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{00}^{(1)\,1} = \Delta_{12}^t (v^{(+)})^0{}_0{}^1{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{01}^{(1)\,0} + \Delta_{12}^t (v^{(+)})^1{}_0{}^0{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{12}^{(1)\,0},$$

$$(v_{\alpha_3}^{(+)})^1{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{01}^{(1)\,0} = \Delta_{12}^t (v^{(+)})^0{}_0{}^0{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{00}^{(1)\,1} + \Delta_{12}^t (v^{(+)})^1{}_0{}^1{}_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{11}^{(1)\,1},$$

$$(v_{\alpha_3}^{(+)})^1{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{12}^{(1)\,0} = -\Delta_{12}^t (v^{(+)})^1{}_1{}^1{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{11}^{(1)\,1} + \Delta_{12}^t (v^{(+)})^0{}_1{}^0{}_0 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}_{00}^{(1)\,1},$$

As in the nonsupersymetric case one may employ the identities

$$T_x^{ib} \frac{S_1(-ix+a_1)}{S_1(-ix+a_2)} = \frac{[-ix+a_1]_1}{[-ix+a_2]_1} \frac{S_0(-ix+a_1)}{S_0(-ix+a_2)} T_x^{ib},$$

$$T_x^{ib} \frac{S_0(-ix+a_1)}{S_0(-ix+a_2)} = \frac{[-ix+a_1]_0}{[-ix+a_2]_0} \frac{S_1(-ix+a_1)}{S_1(-ix+a_2)} T_x^{ib},$$

$$T_x^{ib} \frac{S_0(-ix+a_1)}{S_1(-ix+a_2)} = -i \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \frac{[-ix+a_1]_0}{[-ix+a_2]_1} \frac{S_1(-ix+a_1)}{S_0(-ix+a_2)} T_x^{ib}.$$

to check that all the intertwining relation for $X = v^{(+)}$ is satisfied. The same steps are carried out to discuss $X = v^{(-)}$. Details are left to the reader.

### 7.2.2 Orthogonality and Completeness

The most difficult part in the analysis of the Clebsch-Gordan coefficients is once again concerning their orthonormality relations. The intertwining relations we have established in the previous subsection guarantee that

$$\sum_{j_2,j_2} \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \overline{\left[\begin{smallmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\,x_2\,x_1 \end{smallmatrix}\right]}_{j_2 j_1}^{(\nu),j_3} \left[\begin{smallmatrix} \beta_3\,\alpha_2\,\alpha_1 \\ y_3\,x_2\,x_1 \end{smallmatrix}\right]_{k_2 k_1}^{(\mu),i_3} \lambda^{j_2 k_2} \lambda^{j_1 k_1} =$$

$$= 32\sqrt{\rho}^{(-1)^{(\nu+1)}} \sum_\sigma (-1)^{(\nu+1)(\sigma+1)} |S_\sigma(2\alpha_3)|^{-2} \delta_{\nu,\mu} \lambda^{j_3,i_3} \delta(\bar{\beta}_3 - \bar{\alpha}_3)\delta(y_3 - x_3), \tag{166}$$

up to an overall factor. This normalisation is established with the help of a set of integral identities which follow from a supersymmetric version of the star-triangle identity, see Appendix B.2. In particular one uses

$$\sum_{\tau,\sigma} \int dx_2 dx_1 (-1)^{(\rho+\nu+\mu)\tau} \left(\tilde{D}_{\tau\sigma}^{(\mu)\,\epsilon}\right)^* D_{\tau(\sigma+\rho)}^{(\nu)\,\epsilon} = \frac{16(-1)^{\rho\nu}}{|S_{\rho+1}(2\bar{\alpha}_3)|^2} \delta_{\mu,\nu} \delta(\bar{\beta}_3 - \bar{\alpha}_3)\delta(y_3 - x_3),$$

where we introduced

$$D_{\tau\sigma}^{(\nu)\,\epsilon} = D_{\tau\sigma}^{(\nu)\,\epsilon}(x_3, x_2, x_1; \alpha_3, \alpha_2, \alpha_1) \quad , \quad \tilde{D}_{\tau\sigma}^{(\nu)\,\epsilon} = D_{\tau\sigma}^{(\nu)\,\epsilon}(y_3, x_2, x_1; \beta_3, \alpha_2, \alpha_1) .$$

Since the analogous computation for $\mathcal{U}_q(sl(2))$ was described in great detail in section 3.2 we can leave the derivation of eq. (166) as an exercise.

## 7.3 The Racah-Wigner coefficients for $\mathcal{U}_q(osp(1|2))$

The definition and computation of the Racah-Wigner coefficients for $\mathcal{U}_q(osp(1|2))$ proceeds very much along the same lines as for $\mathcal{U}_q(sl(2))$, see section 3.3. After giving a precise definition in the Racah-Wigner coefficients, we will state an explicit formula. It resembles the one for $\mathcal{U}_q(sl(2))$, see eq. (79), except that all special functions carry an additional label $\nu \in \{0,1\}$.

As in the case of $\mathcal{U}_q(sl(2))$ be begin by defining the following two maps for the decomposition of triple tensor products,

$$
\left( \Phi_{\alpha_t}^t \begin{bmatrix} \alpha_3\,\alpha_2 \\ \alpha_4\,\alpha_1 \end{bmatrix}_\epsilon^{\nu_1\nu_2} \right)_{jkl}^i (x_4;x_i) \;\; = \;\; \sum_n \int dx_t \begin{bmatrix} \alpha_4\,\alpha_t\,\alpha_1 \\ x_4\,x_t\,x_1 \end{bmatrix}_{\epsilon\;nl}^{(\nu_1)\,i} \begin{bmatrix} \alpha_t\,\alpha_3\,\alpha_2 \\ x_t\,x_3\,x_2 \end{bmatrix}_{\epsilon\;jk}^{(\nu_2)\,n} \tag{167}
$$

$$
\left( \Phi_{\alpha_s}^s \begin{bmatrix} \alpha_3\,\alpha_2 \\ \alpha_4\,\alpha_1 \end{bmatrix}_\epsilon^{\nu_3\nu_4} \right)_{jkl}^i (x_4;x_i) \;\; = \;\; \sum_m \int dx_s \begin{bmatrix} \alpha_4\,\alpha_3\,\alpha_s \\ x_4\,x_3\,x_s \end{bmatrix}_{\epsilon\;jm}^{(\nu_3)\,i} \begin{bmatrix} \alpha_s\,\alpha_2\,\alpha_1 \\ x_s\,x_2\,x_1 \end{bmatrix}_{\epsilon\;kl}^{(\nu_4)\,m} \;. \tag{168}
$$

From these two maps we can compute the Racah-Wigner coefficients through the usual prescription

$$
\left( \left\{ \begin{matrix} \alpha_1\,\alpha_3 \\ \alpha_2\,\alpha_4 \end{matrix} \Big| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} \right)_{\nu_1\nu_2}^{\nu_3\nu_4} |S_1(2\alpha_4)|^{-2} \delta(\alpha_4' - \alpha_4)\delta(x_4' - x_4) \;\; = \tag{169}
$$

$$
= \lim_{\epsilon\to 0} \sum_{jklm} \int d^3x \left( \left( \Phi_{\alpha_t}^t \begin{bmatrix} \alpha_3\,\alpha_2 \\ \alpha_4'\,\alpha_1 \end{bmatrix}_\epsilon^{\nu_1\nu_2} \right)_{jkl}^m (x_4';x_i) \right)^* \left( \Phi_{\alpha_s}^s \begin{bmatrix} \alpha_3\,\alpha_2 \\ \alpha_4\,\alpha_1 \end{bmatrix}_\epsilon^{\nu_3\nu_4} \right)_{jkl}^n (x_4;x_i)
$$

where the integration measure is $d^4x = \prod_{i=1}^4 dx_i$. After integration and summation, the right hand side turns out to be independent of $\alpha_4', x_4'$ and $n$. Using the explicit formulae for the regularized Clebsch-Gordan maps along with our knowledge of poles and residues of the special functions $S_\nu$, see Appendix B.2, we can perform the integrations with the help of Cauchy's integral formula to obtain,

$$
\left( \left\{ \begin{matrix} \alpha_1\,\alpha_3 \\ \alpha_2\,\alpha_4 \end{matrix} \Big| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{\mathrm{b}} \right)_{\nu_1\nu_2}^{\nu_3\nu_4} \;\; = \;\; \delta_{\sum_i \nu_i = 0 \; mod \; 2} \, (-1)^{\nu_2\nu_3 + \nu_4} \, \frac{S_{\nu_4}(a_4)S_{\nu_1}(a_1)}{S_{\nu_2}(a_2)S_{\nu_3}(a_3)} |S_1(2\alpha_t)|^2 \times \tag{170}
$$

$$
\times \int_{i\mathbb{R}} dt \sum_{\nu=0}^1 (-1)^{\nu(\nu_2+\nu_4)} \frac{S_{1+\nu+\nu_4}(u_4 + t)S_{1+\nu+\nu_4}(\tilde u_4 + t)S_{1+\nu+\nu_3}(u_3 + t)S_{1+\nu+\nu_3}(\tilde u_3 + t)}{S_{\nu+\nu_2+\nu_3}(u_{23} + t)S_{\nu+\nu_2+\nu_3}(\tilde u_{23} + t)S_\nu(2\alpha_s + t)S_\nu(Q + t)} \;.
$$

All the variables $a_i$ and $u_i, \tilde u_i$ etc. where defined in section 4. Note that they are associated to the four vertices which in turn correspond to the indices $\nu_i$. In this form, our result appears as a natural extension of the expression (79) for the Racah-Wigner coefficients of $\mathcal{U}_q(sl(2))$. The sum over $\nu$ accompanies the integral over t. The shift $\nu \to \nu + 1$ in the index of $S_\nu$ appears for those $S_\nu$ that we decided to write into the numerator. The parameters $\nu_i$ are placed such that they mimic the arguments of the $S_\nu$.

## 7.4 Teschner-Vartanov-like form of Racah-Wigner coefficients

We are now prepared to study the extension of the Teschner-Vartanov form of Racah-Wigner coefficients to the supersymmetric case. We shall define the supersymmetric Racah-Wigner symbol in the next few paragraphs and comment a bit on its relation with N=1 Liouville field theory and the Racah-Wigner symbol for self-dual representations of $\mathcal{U}_q(osp(1|2))$. Then we perform an analysis along the lines of section 3.4, i.e. we compute the limit of the Racah-Wigner symbol for a discrete set of representation labels. The interpretation of the results is a bit more subtle than in the example of $\mathcal{U}_q(sl(2))$.

As a supersymmetric extension of the Racah-Wigner symbol (81) we propose the following integral formula

$$
\left\{
\begin{array}{cc|c}
\alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\
\alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t}
\end{array}
\right\}^{\nu_3\nu_4}_{\nu_1\nu_2}
= \delta_{\sum_i \nu_i = a_s + a_t \, mod \, 2} \Delta_{\nu_4}(\alpha_s, \alpha_2, \alpha_1) \Delta_{\nu_3}(\alpha_s, \alpha_3, \alpha_4) \Delta_{\nu_2}(\alpha_t, \alpha_3, \alpha_2)
$$

(171)

$$
\times \Delta_{\nu_1}(\alpha_4, \alpha_t, \alpha_1) \int_{\mathcal{C}} du \sum_{\nu=0}^{1} \Big( (-1)^X S_{1+\nu+\nu_4+a_s}(u - \alpha_{12s}) S_{1+\nu+\nu_3+a_s}(u - \alpha_{s34})
$$

$$
S_{1+\nu+\nu_2+a_t}(u - \alpha_{23t}) S_{1+\nu+\nu_1+a_t}(u - \alpha_{1t4}) S_{\nu+\nu_1+\nu_2+a_t}(\alpha_{1234} - u)
$$

$$
S_{\nu+\nu_1+\nu_3+a_2}(\alpha_{st13} - u) S_{\nu+\nu_1+\nu_4+a_3}(\alpha_{st24} - u) S_{\nu}(2Q - u) \Big)
$$

where

$$
\Delta_{\nu}(\alpha_3, \alpha_2, \alpha_1) = \left( \frac{S_{\nu+\frac{1}{2}a_{123}}(\alpha_{123} - Q)}{S_{\nu+\frac{1}{2}(a_{12}-a_3)}(\alpha_{12} - \alpha_3) S_{\nu+\frac{1}{2}(a_{23}-a_1)}(\alpha_{23} - \alpha_1) S_{\nu+\frac{1}{2}(a_{31}-a_2)}(\alpha_{31} - \alpha_2)} \right)^{\frac{1}{2}}
$$

and the contour $\mathcal{C}$, as in the bosonic case, crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity. Note that the arguments $\alpha^a$ of the Racah-Wigner symbol contain a continuous quantum number $\alpha \in Q/2 + i\mathbb{R}$ along with a superscript $a$ that can take the values $a = 0$ and $a = 1$. The discrete label $a$ keeps track on whether the corresponding representation is taken from the NS or R sector, respectively. We will comment a bit more on this below. Let us agree that the Racah-Wigner symbol is zero unless the discrete labels $a_i$ satisfy the following conditions

$$
a_s = a_1 + a_2 = a_3 + a_4 \, mod \, 2, \qquad a_t = a_1 + a_4 = a_2 + a_3 \, mod \, 2, \qquad \sum_{i=1}^{4} a_i = 0 \, mod \, 2.
$$

(172)

The sign factor

$$
(-1)^X = (-1)^{\nu(a_s\nu_1 + a_1\nu_3 + a_4\nu_4 + a_1 a_s + a_2 a_4 + a_s + a_t)}
$$

(173)

kicks in as soon as some of the discrete labels $a_i$ are nonzero.

Our symbol (171) with $a_i = 0$ was defined to extend the Teschner-Vartanov version of the non-supersymmetric symbol to $\mathcal{U}_q(osp(1|2))$. At the moment we cannot prove that the expression (171), $a_i = 0$, agrees with the formula (171) simply because we are missing certain supersymmetric analogues of the integral identities employed in [47]. On the other hand our results below make it seem highly plausible that both formulae agree. In [29] no attempt was made to extend the constructions to the R sector of $N = 1$ Liouville field theory. It is likely that $\mathcal{U}_q(osp(1|2))$ indeed possesses another self-dual series of representations which can mimic the R sector and that the fusing matrix involving R sector fields may be obtained from the Racah-Wigner symbol in an extended class of self-dual representations, but the details have not been worked out. Here we just make a proposal for the extension of the Racah-Wigner symbol cases with some $a_i \neq 0$. Our results below strongly support a relation with the R sector if N=1 Liouville field theory.

After these comments on the Racah-Wigner symbol (171), we would like to repeat the analysis we have performed in section 3.4. By analogy with the bosonic case we expect that the prefactor vanishes each time one of the external weights approaches a degenerate value $\alpha_{n,n'} = -\frac{nb}{2} - \frac{n'}{2b}$. Weights $\alpha_{n,n'}$ with even $n + n'$ are degenerate in the NS sector while those with $n + n'$ odd degenerate in the R sector of the theory. Suppose now that the external weight $\alpha_i$ degenerates. Fusion with a generic weight $\alpha_j$ gives the following finite set of

intermediate weights

$$\alpha_s \to \alpha_j - \frac{sb}{2} - \frac{s'}{2b}, \qquad \text{or} \qquad \alpha_t \to \alpha_j - \frac{tb}{2} - \frac{t'}{2b}, \tag{174}$$

$$s, t \in \{-n, -n+2, \dots, n\}, \qquad s', t' \in \{-n', -n'+2, \dots, n'\}.$$

Let us now take a closer look at the prefactor of our Racah-Wigner symbols. When written in terms of the double sine function, it takes the from

$$\Delta_{\nu_4}(\alpha_s, \alpha_2, \alpha_1) \Delta_{\nu_3}(\alpha_s, \alpha_3, \alpha_4) \Delta_{\nu_2}(\alpha_t, \alpha_3, \alpha_2) \Delta_{\nu_1}(\alpha_4, \alpha_t, \alpha_1) \tag{175}$$

$$= \left( S_{\nu_4 + a_s}(\alpha_{12s} - Q) S_{\nu_3 + a_s}(\alpha_{s34} - Q) S_{\nu_2 + a_t}(\alpha_{23t} - Q) S_{\nu_1 + a_t}(\alpha_{14t} - Q) \right)^{\frac{1}{2}}$$

$$\left( S_{\nu_4}(\alpha_{12} - \alpha_s) \, S_{\nu_4 + a_1}(\alpha_{1s} - \alpha_2) \, S_{\nu_4 + a_2}(\alpha_{2s} - \alpha_1) \right.$$

$$S_{\nu_3}(\alpha_{34} - \alpha_s) \, S_{\nu_3 + a_4}(\alpha_{s4} - \alpha_3) \, S_{\nu_3 + a_3}(\alpha_{3s} - \alpha_4)$$

$$S_{\nu_2}(\alpha_{23} - \alpha_t) \, S_{\nu_2 + a_2}(\alpha_{t2} - \alpha_3) \, S_{\nu_2 + a_3}(\alpha_{3t} - \alpha_2)$$

$$\left. S_{\nu_1}(\alpha_{14} - \alpha_t) \, S_{\nu_1 + a_1}(\alpha_{1t} - \alpha_4) \, S_{\nu_1 + a_4}(\alpha_{4t} - \alpha_1) \right)^{-\frac{1}{2}}$$

This factor vanishes every time we hit a pole of the denominator. As one can easily see, there are four different cases in which this can happen, each corresponding to the consecutive products of three double sine functions,

$$\frac{n-s}{2} + \frac{n'-s'}{2} \in 2\mathbb{N} + 1 + \begin{cases} \nu_4, & \text{degenerate} \quad \alpha_i, \quad i = 1, 2 \\ \nu_3, & \text{degenerate} \quad \alpha_i, \quad i = 3, 4 \end{cases}$$

$$\frac{n+s}{2} + \frac{n'+s'}{2} \in 2\mathbb{N} + 1 + \begin{cases} \nu_4 + a_i, & \text{degenerate} \quad \alpha_i, \quad i = 1, 2 \\ \nu_3 + a_i, & \text{degenerate} \quad \alpha_i, \quad i = 3, 4 \end{cases} \tag{176}$$

As one example, let us discuss the first line and suppose that $\alpha_i = \alpha_1$ for definiteness. It follows that $\alpha_j = \alpha_2$ because $\alpha_1$ and $\alpha_s$ appear only in combination with $\alpha_2$ in the arguments of the double sine functions. According to the properties listed in Appendix A the first double sine function $S_{\nu_4}(\alpha_{12} - \alpha_s)$ runs into a pole provided that its argument $\alpha_{12} - \alpha_s = \frac{s-n}{2}b + \frac{s'-n'}{2}b^{-1}$ satisfies $\frac{n-s}{2} + \frac{n'-s'}{2} \in 2\mathbb{N} + 1 + \nu_4$. This gives the first line above. The analysis for the other cases is similar.

The analysis for the t-channel, i.e. for terms involving $\alpha_t$, is performed in exactly the same way and it leads to

$$\frac{n-t}{2} + \frac{n'-t'}{2} \in 2\mathbb{N} + 1 + \begin{cases} \nu_1, & \text{degenerate} \quad \alpha_i, \quad i = 1, 4 \\ \nu_2, & \text{degenerate} \quad \alpha_i, \quad i = 2, 3 \end{cases}$$

$$\frac{n+t}{2} + \frac{n'+t'}{2} \in 2\mathbb{N} + 1 + \begin{cases} \nu_1 + a_i, & \text{degenerate} \quad \alpha_i, \quad i = 1, 4 \\ \nu_2 + a_i, & \text{degenerate} \quad \alpha_i, \quad i = 2, 3 \end{cases} \tag{177}$$

This implies a relation between the sets of parameters $\{\nu_i\}$, $\{a_i\}$ ($i = 1, \dots, 4$) and the type of fusion rules satisfied by $\alpha_s, \alpha_t$ in the limit of a degenerate weight. We see that the prefactor has the fusion rules of $N = 1$ Liouville field theory built in. This provides a first non-trivial test for our proposal.

Let us anticipate that the numerator of the normalisation factor furnishes residues that can be related to $S_\nu(0)$ by shift equations, see Appendix A. All four double sine functions satisfy this condition simultaneously, provided that the external weights obey

$$j_{1234} + j'_{1234} \in 2\mathbb{N} + \nu_3 + \nu_4 + a_s, \quad \text{and} \quad j_{1234} + j'_{1234} \in 2\mathbb{N} + \nu_1 + \nu_2 + a_t \tag{178}$$

64

at the same time. This is guaranteed by the Kronecker $\delta$ we have put into our definition of the Racah-Wigner symbol (171).

We plan to test our proposal (171) by evaluating it for degenerate weights, as in the previous section. To this end, let us consider the limit where $\alpha_2 \to -\frac{nb}{2}$. Before talking the limit of the Racah-Wigner symbol it is useful to pass from the summation over $\nu$ to a new summation index $\nu' = \nu + \nu_3 + a_s$. The Racah-Wigner symbol then reads,

$$
\left\{ \begin{matrix} \alpha_1^{a_1} & \alpha_3^{a_3} \\ \alpha_2^{a_2} & \alpha_4^{a_4} \end{matrix} \middle| \begin{matrix} \alpha_s^{a_s} \\ \alpha_t^{a_t} \end{matrix} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \, mod \, 2} \Delta_{\nu_4}(\alpha_s, \alpha_2, \alpha_1) \Delta_{\nu_3}(\alpha_s, \alpha_3, \alpha_4) \Delta_{\nu_2}(\alpha_t, \alpha_3, \alpha_2)
$$

(179)

$$
\Delta_{\nu_1}(\alpha_4, \alpha_t, \alpha_1) \int_{\mathcal{C}} du \sum_{\nu'=0}^{1} \Big( (-1)^X S_{1+\nu_3+\nu_4+\nu'}(u - \alpha_{12s}) S_{1+\nu'}(u - \alpha_{s34})
$$

$$
S_{1+\nu_1+\nu_4+\nu'}(u - \alpha_{23t}) S_{1+\nu_2+\nu_4+\nu'}(u - \alpha_{1t4}) S_{\nu_4+\nu'}(\alpha_{1234} - u) S_{\nu_1+\nu'+a_1}(\alpha_{st13} - u)
$$

$$
S_{\nu_2+\nu'+a_2}(\alpha_{st24} - u) S_{\nu_3+\nu'+a_s}(2Q - u) \Big) .
$$

As in the previous section, we need to determine the singular contributions from the integral. Note that the product

$$
S_{1+\nu'}(u - \alpha_{s34}) S_{\nu_4+\nu'}(\alpha_{1234} - u)
$$

has poles in the positions $u = \alpha_{134} - \frac{sb}{2} - pb$ for $p \in \{\nu', \nu'+2, \ldots, \le \frac{n-s}{2} - \nu'\}$ ($\nu'$ keeps track of the parity of $p$). Due to the "pinching mechanism" each pole contributes a sum of singular terms. Once we include the summation over $\nu' = 0, 1$, the sum of singular terms runs through all values of $p \in \{0, 1, \ldots, \frac{n-s}{2}\}$,

$$
\sum_{p=0}^{\frac{n-s}{2}} (-1)^X \frac{\left(2\cos(\frac{\pi b^2}{2})\right)^{\frac{s-n}{2}} S_{1+p}(0)}{[p]_+! \left[\frac{n-s}{2} - p\right]_+!} S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb)
$$

(180)

$$
S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) S_{1+\nu_2+\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)
$$

$$
S_{\nu_1+\nu'+a_1}(\alpha_{1t} - \alpha_4 + pb) S_{\nu_2+\nu'+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) S_{\nu_3+\nu'+a_s}(2Q - \alpha_{134} + \frac{sb}{2} + pb),
$$

where we used the shift relations for the supersymmetric double sine function (233) and the notation

$$
[n]_+! = \begin{cases} \prod_{j=1 \, mod \, 2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1 \, mod \, 2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n-1} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} + 1 . \end{cases}
$$

(181)

With the help of conditions (176) one can verify that the product

$$
S_{1+\nu_2+\nu_4+\nu'}(u - \alpha_{1t4}) S_{\nu_2+\nu'+a_2}(\alpha_{st24} - u)
$$

has common poles which are located in $u = \alpha_{1t4} - p'b$, where $p' \in \{\nu', \nu'+2, \ldots, \le \frac{n+s}{2} - \nu'\}$. They lead to a second sum of singular terms. Once the two singular contributions from the integral are multiplied by the

vanishing prefactor, they give a finite result for the limit of the Racah-Wigner symbol,

$$
\left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & (\alpha_1 - \frac{sb}{2})^{a_s} \\ (-\frac{nb}{2})^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} \equiv \lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \bmod 2}
$$

$$
\left( \frac{\left[\frac{n-s}{2}\right]_+! \left[\frac{n+s}{2}\right]_+! S_{\nu_4+a_s}(2\alpha_1 - \frac{(s+n)b}{2} - Q) S_{\nu_3+a_s}(2\alpha_{134} - \frac{sb}{2} - Q)}{S_{\nu_4+a_1}(2\alpha_1 + \frac{(n-s)b}{2}) S_{\nu_3}(\alpha_{34} - \alpha_1 + \frac{sb}{2}) S_{\nu_3+a_3}(\alpha_{13} - \alpha_4 - \frac{sb}{2}) S_{\nu_3+a_4}(\alpha_{14} - \alpha_3 - \frac{sb}{2})} \right)^{\frac{1}{2}}
$$

$$
\left( \frac{S_{\nu_2+a_t}(\alpha_3 + \alpha_t - \frac{nb}{2} - Q) S_{\nu_1+a_t}(\alpha_{14} + \alpha_t - Q) \left( S_{\nu_1+a_4}(\alpha_{4t} - \alpha_1) S_{\nu_2+a_3}(\alpha_{3t} + \frac{nb}{2}) \right)^{-1}}{S_{\nu_2}(\alpha_3 - \alpha_t - \frac{nb}{2}) S_{\nu_2+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2}) S_{\nu_1}(\alpha_{14} - \alpha_t) S_{\nu_1+a_1}(\alpha_{1t} - \alpha_4)} \right)^{\frac{1}{2}}
$$

$$
\left\{ \sum_{p=0}^{\frac{n-s}{2}} (-1)^X \frac{\left( 2\cos(\frac{\pi b^2}{2}) \right)^{\frac{s}{2}}}{[p]_+! \left[\frac{n-s}{2} - p\right]_+!} S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb) \right. \tag{182}
$$

$$
S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) S_{1+\nu_2+\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)
$$

$$
S_{\nu_1+\nu'+a_1}(\alpha_{1t} - \alpha_4 + pb) S_{\nu_2+\nu'+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) S_{\nu_3+\nu'+a_s}(2Q - \alpha_{134} + \frac{sb}{2} + pb)
$$

$$
+ \sum_{p'=0}^{\frac{n+s}{2}} (-1)^X \frac{\left( 2\cos(\frac{\pi b^2}{2}) \right)^{-\frac{s}{2}}}{[p']_+! \left[\frac{n+s}{2} - p'\right]_+!} S_{\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{nb}{2} + p'b) S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_3 + \frac{nb}{2} - p'b)
$$

$$
\left. \frac{S_{1+\nu'}(\alpha_t - \alpha_3 + \frac{sb}{2} - p'b) S_{1+\nu_3+\nu_4+\nu'}(\alpha_{t4} - \alpha_1 + \frac{(n+s)b}{2} - p'b) S_{\nu_1+\nu'+a_1}(\alpha_{13} - \alpha_4 + p'b - \frac{sb}{2})}{S_{\nu_3+\nu'+a_s}(\alpha_{1t4} - p'b - Q)} \right\}.
$$

This expression has simple poles when the second intermediate weight approach a degenerate value $\alpha_t \to \alpha_3 - \frac{tb}{2}$. The residua are given by the following formula,

$$
\operatorname*{Res}_{\alpha_t \to \alpha_3 - \frac{tb}{2}} \left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & (\alpha_1 - \frac{sb}{2})^{a_s} \\ (-\frac{nb}{2})^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \bmod 2} \tag{183}
$$

$$
2 \left( \frac{S_{\nu_4+a_s}(2\alpha_1 - \frac{(s+n)b}{2} - Q) S_{\nu_2+a_t}(2\alpha_3 - \frac{(t+n)b}{2} - Q)}{S_{\nu_4+a_1}(2\alpha_1 + \frac{(n-s)b}{2}) S_{\nu_2+a_3}(2\alpha_3 + \frac{(n-t)b}{2})} \right)^{\frac{1}{2}}
$$

$$
\sum_{p=max\{0, \frac{t-s}{2}\}}^{min\{\frac{n-s}{2}, \frac{n+t}{2}\}} \left\{ (-1)^X \frac{\left( S_{\nu_3+a_s}(\alpha_{134} - \frac{sb}{2} - Q) S_{\nu_1+a_t}(\alpha_{134} - \frac{tb}{2} - Q) \right)^{\frac{1}{2}}}{S_{\nu_3+\nu'+a_s}(\alpha_{134} - \frac{sb}{2} - pb - Q)} \right.
$$

$$
\frac{\left( \left[\frac{n-s}{2}\right]_+! \left[\frac{n+s}{2}\right]_+! \left[\frac{n-t}{2}\right]_+! \left[\frac{n+t}{2}\right]_+! \right)^{\frac{1}{2}}}{[p]_+! \left[\frac{n-s}{2} - p\right]_+! \left[p + \frac{s-t}{2}\right]_+! \left[\frac{t+n}{2} - p\right]_+!} \frac{S_{\nu_1+\nu'+a_1}(\alpha_{13} - \alpha_4 + pb - \frac{tb}{2})}{\left( S_{\nu_3+a_3}(\alpha_{13} - \alpha_4 - \frac{sb}{2}) S_{\nu_1+a_1}(\alpha_{13} - \alpha_4 - \frac{tb}{2}) \right)^{\frac{1}{2}}}
$$

$$
\left. \frac{S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34} - \alpha_1 - pb + \frac{nb}{2})}{\left( S_{\nu_3}(\alpha_{34} - \alpha_1 + \frac{sb}{2}) S_{\nu_1+a_4}(\alpha_{34} - \alpha_1 - \frac{tb}{2}) \right)^{\frac{1}{2}}} \frac{S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_3 - pb + \frac{(n+t-s)b}{2})}{\left( S_{\nu_3+a_4}(\alpha_{14} - \alpha_3 - \frac{sb}{2}) S_{\nu_1}(\alpha_{14} - \alpha_3 + \frac{tb}{2}) \right)^{\frac{1}{2}}} \right\}.
$$

Now we can send all the external weights to degenerate values,

$$
\alpha_i \to -j_i b, \qquad j_i \in \mathbb{Z}_{\geq 0} + \frac{a_i}{2}.
$$

In complete analogy to the bosonic case, see eq. (90), we shall denote the limit by

$$
\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4},
$$

remembering that it denotes a residue of the Racah-Wigner symbol (171) with one degenerate external weight and both intermediate weights satisfying fusion rules. Assuming that

$$\frac{n}{2} = j_2, \qquad \frac{s}{2} = j_s - j_1, \qquad \frac{t}{2} = j_t - j_3$$

satisfy the conditions (176) to (178), one can use the shift relations (233) for $S_\nu(x)$ to obtain

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = 2(j_s + j_t) \bmod 2} \frac{(-1)^{A(j_i)}}{2 \cos\left(\frac{\pi b^2}{2}\right) \cos\left(\frac{\pi}{2b^2}\right)} \tag{184}$$

$$\Delta_+(j_s, j_2, j_1) \Delta_+(j_s, j_3, j_4) \Delta_+(j_t, j_3, j_2) \Delta_+(j_4, j_t, j_1) \sum_{z \geq 0} \Big\{ (-1)^X (-1)^{\frac{1}{2} z(z-1)} [z+1]_+!$$

$$\Big( [z - j_{12s}]_+! \, [z - j_{34s}]_+! \, [z - j_{14t}]_+! [z - j_{23t}]_+! [j_{1234} - z]_+! \, [j_{13st} - z]_+! \, [j_{24st} - z]_+! \Big)^{-1} \Big\}$$

where the sum is over $z = p + j_{s34}$ such that all arguments $[.]_+$ are non-negative, and

$$\Delta_+(a, b, c) = \sqrt{[-a+b+c]_+! \, [a-b+c]_+! \, [a+b-c]_+!/[a+b+c+1]_+!}.$$

The powers of $(-1)$ come from the relation (233) applied to the terms $S\nu(-xb - Q)$, in particular:

$$A(j_i) = \frac{1}{4} j_{12s}(j_{12s} - 1) + \frac{1}{4} j_{s34}(j_{s34} - 1) + \frac{1}{4} j_{23t}(j_{23t} - 1) + \frac{1}{4} j_{14t}(j_{14t} - 1) + 1.$$

This concludes our computation of the Racah-Wigner symbol (171) for degenerate labels. The final formula looks somewhat similar to the corresponding equation in section 3. We are now going to see that it indeed very closely related.

### 7.4.1 Comparison with the finite dimensional 6j symbols

Our formula (184) for the limiting value of the proposed Racah-Wigner symbol could turn into a strong test of eq. (171) provided we were able to show that the expression (184) gives rise to a solution of the pentagon equation. In our discussion of the Racah-Wigner symbol for $\mathcal{U}_q(sl(2))$ this followed from the comparison with the 6j symbol for finite dimensional representations. By construction, the latter are known to satisfy the pentagon equation. By analogy one might now hope that the coefficients (184) coincide with the 6j symbol for finite dimensional representations of the quantum universal enveloping algebra $\mathcal{U}_q(osp(1|2))$. This, however, is not quite the case. To start the comparison, we quote an expression for the 6j symbols of $\mathcal{U}_q(osp(1|2))$ from [63, 64],

$$\left[ \begin{array}{ccc} l_1 & l_2 & l_s \\ l_3 & l_4 & l_t \end{array} \right]_q = (-1)^{\frac{1}{2}(l_{1234} + l_s + l_t)(l_{1234} + l_s + l_t + 1) + \frac{1}{2}\left(\sum_{i=1}^{4} l_i(l_i - 1) + l_s(l_s - 1) + l_t(l_t - 1)\right)}$$

$$\Delta'_q(l_s, l_2, l_1) \Delta'_q(l_s, l_3, l_4) \Delta'_q(l_t, l_3, l_2) \Delta'_q(l_4, l_t, l_1)$$

$$\sum_{z \geq 0} (-1)^{\frac{1}{2} z(z-1)} [z+1]'_q! \Big( [z - l_{12s}]'_q! \, [z - l_{34s}]'_q! \, [z - l_{14t}]'_q!$$

$$[z - l_{23t}]'_q! [l_{1234} - z]'_q! \, [l_{13st} - z]'_q! \, [l_{24st} - z]'_q! \Big)^{-1}$$

where

$$\Delta'_q(a, b, c) = \sqrt{[-a+b+c]'_q! \, [a-b+c]'_q! \, [a+b-c]'_q!/[a+b+c+1]'_q!}.$$

Let us stress that irreducible finite dimensional representations of $\mathcal{U}_q(osp(1|2))$ are labeled by integers $l$. Hence all the arguments $l_i$ in the above 6j symbols satisfy $l_i \in \mathbb{N}$. In the previous definition the q-number $[.]'_q$ is defined as

$$[n]'_q = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} . \tag{185}$$

For $q = e^{i\pi b^2}$ the quantum factorial takes the form

$$[n]'_q! = \begin{cases} \prod_{j=1 \, mod \, 2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n} \left(i \, \sin(-j\frac{\pi b^2}{2})\right) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1 \, mod \, 2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n-1} \left(i \, \sin(-j\frac{\pi b^2}{2})\right) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1 . \end{cases} \tag{186}$$

It is related to the similar symbol $[.]_+!$ which we defined in eq. (181) through

$$[n]_+! = (-1)^{\frac{1}{12}n(n+1)(2n+1)}(-i)^n \, [n]'_q! . \tag{187}$$

In order to compare the limiting values (184) Racah-Wigner symbols (171) with the 6j symbols (185) we rewrite the latter in terms of the new symbol $[n]'_q$,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t) \, mod \, 2} \frac{(-1)^{A'(j_i)} \Delta'_q(j_s, j_2, j_1) \Delta'_q(j_s, j_3, j_4)}{2 \cos\left(\frac{\pi b^2}{2}\right) \cos\left(\frac{\pi}{2b^2}\right)}$$

$$\Delta'_q(j_t, j_3, j_2) \Delta'_q(j_4, j_t, j_1) \sum_{z \geq 0} (-1)^X (-1)^{\frac{1}{2}z(z-1)+2z(j_{1234st}+j_1 j_3+j_2 j_4+j_s j_t)} [z+1]'_q! \tag{188}$$

$$\left([z - j_{12s}]'_q! \, [z - j_{34s}]'_q! [z - j_{14t}]'_q! \, [z - j_{23t}]'_q! [j_{1234} - z]'_q! \, [j_{13st} - z]'_q! \, [j_{24st} - z]'_q! \right)^{-1}$$

where

$$A'(j_i) = -\frac{1}{2} - (j_{1234st}+1)(j_1 j_3 + j_2 j_4 + j_s j_t + 1)$$
$$+ \frac{1}{2}j_{12s}(j_{12s}-1) + \frac{1}{2}j_{s34}(j_{s34}-1) + \frac{1}{2}j_{23t}(j_{23t}-1) + \frac{1}{2}j_{14t}(j_{14t}-1)$$
$$- F(j_1, j_2, j_s) - F(j_3, j_4, j_s) - F(j_2, j_3, j_t) - F(j_1, j_4, j_t)$$

and

$$F(j_1, j_2, j_3) = \frac{3}{4}j_{123}(j_{123}+1) + j_1 j_2 j_3 + j_1 j_2 + j_1 j_3 + j_2 j_3 .$$

For integer $j_i$ the signs $(-1)^{2z(j_{1234st}+j_1 j_3+j_2 j_4+j_s j_t)}$ and $(-1)^X$ which were defined in eq. (173) vanish so that we can relate the limit of the Racah-Wigner symbol to the $\mathcal{U}_q(osp(1|2))$ 6j coefficients,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t) \, mod \, 2} \frac{(-1)^{A''(j_i)}}{2 \cos\left(\frac{\pi b^2}{2}\right) \cos\left(\frac{\pi}{2b^2}\right)} \left[ \begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right]_q \tag{189}$$

where

$$A''(j_i) = \frac{1}{2} - j_{1234st}(j_1 j_3 + j_2 j_4 + j_s j_t)$$
$$- F(j_1, j_2, j_s) - F(j_3, j_4, j_s) - F(j_2, j_3, j_t) - F(j_1, j_4, j_t) .$$

Let us emphasize that in arriving at the expressions (184) for the limiting values of the Racah-Wigner symbol, the parameters $j_i$ were allowed to take either integer (NS weights) and half-integer (R weights) values. We

have now shown that the limit is proportional to the $\mathcal{U}_q(osp(1|2))$ 6j coefficients, provided all arguments $j_i$ are integer. In order to find an interpretation of the limit (184) in the case of half-integer $j_i$, we will have to bring in a different idea. It is related to an intriguing duality between the 6j symbol of $\mathcal{U}_q(osp(1|2))$ and $\mathcal{U}_q(sl(2))$.

As was originally noticed in [39], [54], the $\mathcal{U}_q(sl(2))$ quantum numbers (92) with the deformation parameter $q' = i\sqrt{q}$ are related to the $\mathcal{U}_q(osp(1|2))$ quantum numbers (185) through,

$$[x]_{q'} = (-1)^{\frac{1-x}{2}} [x]'_q \ . \tag{190}$$

This equation implies a relation between the quantum factorials,

$$[x]'_{q'}! = (-1)^{\frac{x(x-1)}{4}} [x]_{q'}! \ . \tag{191}$$

With its help we can rewrite the $\mathcal{U}_q(osp(1|2))$ 6j symbol in terms of the $\mathcal{U}_q(sl(2))$ quantum factorials,

$$
\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q = (-1)^{\sum_{i=1}^{4} \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}}
$$

$$
\Delta_{q'}(j_s,j_2,j_1)\Delta_{q'}(j_s,j_3,j_4)\Delta_{q'}(j_t,j_3,j_2)\Delta_{q'}(j_4,j_t,j_1)
$$

$$
\sum_{z\geq 0}(-1)^{z+2zj_{1234st}}[z+1]_{q'}!\Big([z-j_{12s}]_{q'}! \, [z-j_{34s}]_{q'}! \, [z-j_{14t}]_{q'}!
$$

$$
[z-j_{23t}]_{q'}![j_{1234}-z]_{q'}! \, [j_{13st}-z]_{q'}! \, [j_{24st}-z]_{q'}!\Big)^{-1} \ .
$$

Due to the condition $j_i \in \mathbb{N}$ in the $\mathcal{U}_q(osp(1|2))$ 6j symbol, the sign $(-1)^{2zj_{1234st}}$ vanishes and one arrives at the following relation between the 6j symbols (93) and (185)

$$
\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q = (-1)^{\sum_{i=1}^{4} \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}}
$$

$$
\frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_{q'} \ .
$$

In a similar way we can relate our limit of Racah-Wigner coefficients (188) to the 6j symbol of $U_{q'}(sl(2))$ even if some of the arguments $j_i$ assume (half-)integer values. When written in terms of $[x]_{q'}$, the Racah-Wigner coefficients (188) take the following form,

$$
\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2} \, \frac{(-1)^{A'''(j_i)}\Delta_{q'}(j_s,j_2,j_1)\Delta_{q'}(j_s,j_3,j_4)}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)}
$$

$$
\Delta_{q'}(j_t,j_3,j_2)\Delta_{q'}(j_4,j_t,j_1)\sum_{z\geq 0}(-1)^X(-1)^{z+2(z+1)(j_1j_3+j_2j_4+j_sj_t)}[z+1]_{q'}! \tag{192}
$$

$$
\Big([z-j_{12s}]_{q'}! \, [z-j_{34s}]_{q'}![z-j_{14t}]_{q'}! \, [z-j_{23t}]_{q'}![j_{1234}-z]_{q'}! \, [j_{13st}-z]_{q'}! \, [j_{24st}-z]_{q'}!\Big)^{-1}
$$

where

$$
A'''(j_i) = \frac{1}{2} - (j_{1234st}+2)(j_1j_3+j_2j_4+j_sj_t)
$$

$$
- F'(j_1,j_2,j_s) - F'(j_3,j_4,j_s) - F'(j_2,j_3,j_t) - F'(j_1,j_4,j_t)
$$

$$
F'(j_1,j_2,j_3) = j_1j_2j_3 + \frac{1}{2}(j_1+j_2+j_3) \ .
$$

Using the relations (176, 177) and (178) one may check that

$$(-1)^{2j_1j_3+2j_2j_4+2j_sj_t} = (-1)^{a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t} \quad . \tag{193}$$

Since the parameter $z$ is related to the summation parameter $p$ (180) as $z = p + j_{34s}$ and the parity of $p$ is tracked by $\nu' = \nu + \nu_3 + a_s$, we may relate the sign under the sum in eq. (192) to the sign factor $(-1)^X$ that was defined in eq. (173),

$$
\begin{aligned}
(-1)^{2(z+1)(j_1j_3+j_2j_4+j_sj_t)} &= (-1)^{2(\nu+\nu_3+a_s+j_{34s}+1)(j_1j_3+j_2j_4+j_sj_t)} \\
&= (-1)^{\nu(a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t)} = (-1)^X,
\end{aligned}
\tag{194}
$$

where we used (178) to check that $\nu + \nu_3 + a_s + j_{34s} + 1 \in 2\mathbb{N} + 2(\nu + \nu_3 + \nu_4 + a_s) + \nu$. Thus the limit (192) is proportional to the 6j symbol of finite dimensional representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$,

$$
\left\{
\begin{array}{cc|c}
-j_1b & -j_3b & -j_sb \\
-j_2b & -j_4b & -j_tb
\end{array}
\right\}^{\nu_3\nu_4}_{\nu_1\nu_2}
= \delta_{\sum_i \nu_i = 2j_s + 2j_t \bmod 2} \frac{(-1)^{A'''(j_i)}}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)}
$$

$$
\frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}}
\left(
\begin{array}{ccc}
j_1 & j_2 & j_s \\
j_3 & j_4 & j_t
\end{array}
\right)_{q'} .
\tag{195}
$$

This concludes our discussion of the limiting Racah-Wigner coefficients (184). Our analysis has shown that the expression we obtained from our proposal (171) is dual to the 6j symbol for finite dimensional representations of the quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$. By construction the latter satisfy the pentagon equation. Even though we have not demonstrated that the original symbol (171) solved the pentagon identity for arbitrary values of the weights $\alpha$, our results provide highly non-trivial evidence in favor of the proposal. Note in particular that our sign factors were rather crucial in making things work as soon as some of the weights were taken from the R sector. It is actually possible to carry things a bit further. In fact, the evaluation of the Racah-Wigner symbols (81) and (171) is possible for all degenerate weights, not just the one parameter series we have studied above. In that case, the limiting values of the Racah-Wigner symbol are no longer given by a single 6j symbol. On the other hand the coefficients obtained from the the symbol (93) are guaranteed to satisfy the pentagon relations, simply because the full symbol does [12, 13, 47]. We have checked in a few examples that the limiting values of the proposed Racah-Wigner symbol (171) are still related to those of the $\mathcal{U}_q(sl(2))$ symbol even when $\alpha \sim -\frac{nb}{2} - \frac{n'}{2b}$ for $n' \neq 0$. With all these non-trivial test being performed, we trust that our formula (171) correctly describes the fusing matrix of $N = 1$ Liouville field theory for both NS and R sector fields.

# 8 Heisenberg double of $\mathcal{U}_q(osp(1|2))$

In this section we will consider the Heisenberg double of the Borel half of $\mathcal{U}_q(osp(1|2))$ and a class of its self-dual representations which will analogous to those from [20]. We conjecture that the Heisenberg double canonical element evaluated on these representations can be identified with a flip operator of the quantised Teichmüller theory of super Riemann surfaces. Moreover, the algebra isomorphism of Heisenberg double should correspond to an operator changing the marked corner of a triangle belonging to a triangulation of a super Riemann surface.

Since again we are considering the non-compact version of Heisenberg double for the transparency we will restate some important formulae which differ in comparison to the compact case. We will recall a self-dual series of representations of Heisenberg double and evaluate on it the canonical element satisfying pentagon equation. We will show that $\mathcal{U}_q(osp(1|2))$ can be realised as a subalgebra of the tensor square of Heisenberg doubles.

The Heisenberg double $H(U_q(\mathcal{B}))$ with a deformation parameter $q = e^{i\pi b^2}$ will be generated by elements $O, P$ from the Borel half $\mathcal{A}$ of $\mathcal{U}_q(osp(1|2))$ and $\hat{O}, \hat{P}$ of the dual Borel half $\mathcal{A}^*$ which have the following commutation relations:

$$
\begin{aligned}
[O, \hat{O}] &= \frac{1}{2\pi i}, \\
[O, P] &= -ibP, \\
[O, \hat{P}] &= ib\hat{P}, \\
[\hat{O}, P] &= 0, \\
[\hat{O}, \hat{P}] &= +ib\hat{P}, \\
\{P, \hat{P}\} &= q(1 + q^{-2})e^{2\pi bO},
\end{aligned}
\tag{196}
$$

and coproducts

$$
\begin{aligned}
\Delta(O) &= 1 \otimes O + O \otimes 1, \\
\Delta(\hat{O}) &= 1 \otimes \hat{O} + \hat{O} \otimes 1, \\
\Delta(P) &= P \otimes e^{2\pi bO} + 1 \otimes P, \\
\Delta(\hat{P}) &= \hat{P} \otimes e^{-2\pi b\hat{O}} + 1 \otimes \hat{P},
\end{aligned}
\tag{197}
$$

and with grading such that $|O| = |\hat{O}| = 0$, $|P| = |\hat{P}| = 1$. Then the Heisenberg double is spanned by the basis $\{e(s,t) \otimes \hat{e}(s',t')\}$, where $s, s', t, t' \in \mathbb{R}$ such that

$$
\begin{aligned}
e(s,t) &= \frac{\mu}{2\pi}\Gamma(-is)G_{\mathrm{b}_*}(-i\mu t)(-q)^{-(ib^{-1}t)^2}(2\pi iO)^{is}P^{ib^{-1}t}, \\
\hat{e}(s,t) &= \hat{O}^{is}\hat{P}^{ib^{-1}t},
\end{aligned}
$$

where $q_*^2 = e^{2\pi ib_*^2} = -q^2$, $\mu = \frac{b_*}{b}$, and the canonical element can be expressed by those generators in the following way

$$
S = \exp(2\pi iO \otimes \hat{O})g_{\mathrm{b}_*}^{-1}((-1)^{-1/2}P \otimes \hat{P}).
\tag{198}
$$

Now we will consider the self-dual representation $\pi : H(U_q(\mathcal{B})) \to L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ of the Heisenberg double.

Its generators can be expressed as an operators on $L^2(\mathbb{R})$ in the following way

$$O = p\mathbb{I}_2,$$

$$P = e^{2\pi bq} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{O} = q\mathbb{I}_2,$$

$$\hat{P} = e^{2\pi b(p-q)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(199)

where $[p,q] = \frac{1}{2\pi i}$ are usual operators on $L^2(\mathbb{R})$. The self-duality $b \leftrightarrow \frac{1}{b}$ should be understood in the same way as in the case of $\mathcal{U}_q(sl(2))$, i.e. the second action of the Heisenberg double with $\tilde{q} = e^{i\pi b^{-2}}$ represententended on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ by (199) with $b$ replaced by $b^{-1}$ commutes with the one above. One can easily evaluate our canonical element in this representation:

$$S = e^{2\pi i p_1 q_2} g_{b_*}^{-1}((-1)^{-\frac{1}{2}} e^{2\pi b(q_1+p_2-q_2)} \xi \otimes \xi).$$

(200)

where $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By construction, the canonical element satisfies the pentagon equation. Additionally, $S$ encodes the coproduct

$$\Delta(e(s,t)) = Ad(S^{-1})(1 \otimes e(s,t)),$$

(201)

$$\Delta(\hat{e}(s,t)) = Ad(S)(\hat{e}(s,t) \otimes 1),$$

(202)

which follows from the relations for the generators $O, P, \hat{O}, \hat{P}$ and it is easy to show, using a shift relation for the quantum dilogarithm

$$\Delta(O) = Ad(S^{-1})(1 \otimes O) = S^{-1}(1 \otimes O)S =$$

$$= g_{b_*}((-1)^{-1/2} P \otimes \hat{P}) e^{-2\pi i O \otimes \hat{O}} (1 \otimes O) e^{2\pi i O \otimes \hat{O}} g_{b_*}^{-1}((-1)^{-1/2} P \otimes \hat{P}) =$$

$$= g_{b_*}((-1)^{-1/2} P \otimes \hat{P})(1 \otimes O + O \otimes 1) g_{b_*}^{-1}((-1)^{-1/2} P \otimes \hat{P}) = 1 \otimes O + O \otimes 1,$$

$$\Delta(P) = Ad(S^{-1})(1 \otimes O) = S^{-1}(1 \otimes P)S =$$

$$= g_{b_*}((-1)^{-1/2} P \otimes \hat{P}) e^{-2\pi i O \otimes \hat{O}} (1 \otimes P) e^{2\pi i O \otimes \hat{O}} g_{b_*}^{-1}((-1)^{-1/2} P \otimes \hat{P}) =$$

$$= g_{b_*}((-1)^{-1/2} P \otimes \hat{P})(1 \otimes P) g_{b_*}^{-1}((-1)^{-1/2} P \otimes \hat{P}) =$$

$$= g_{b_*}((-1)^{-1/2} e^{2\pi b(q_1+p_2-q_2)} (\xi \otimes \xi)) e^{2\pi b q_2} (\mathbb{I}_2 \otimes \xi) g_{b_*}^{-1}((-1)^{-1/2} e^{2\pi b(q_1+p_2-q_2)} (\xi \otimes \xi)) =$$

$$= e^{\pi b q_2} g_{b_*}(e^{-i\pi b_*^2} e^{2\pi b(q_1+p_2-q_2)} (\xi \otimes \xi)) g_{b_*}^{-1}(e^{+i\pi b_*^2} e^{2\pi b(q_1+p_2-q_2)} (\xi \otimes \xi)) e^{\pi b q_2} (\mathbb{I}_2 \otimes \xi) =$$

$$= e^{\pi b q_2} (1 + e^{2\pi b(q_1+p_2-q_2)} (\xi \otimes \xi)) e^{\pi b q_2} (\mathbb{I}_2 \otimes \xi) =$$

$$= e^{2\pi b q_2} (\mathbb{I}_2 \otimes \xi) + e^{2\pi b(q_1+p_2)} (\xi \otimes \mathbb{I}_2) = 1 \otimes P + P \otimes e^{2\pi b O},$$

and

$$\Delta(\hat{O}) = Ad(S)(\hat{O} \otimes 1) = S(\hat{O} \otimes 1)S^{-1} =$$

$$= e^{2\pi i O \otimes \hat{O}} g_{\mathrm{b}_*}^{-1}((-1)^{-1/2} P \otimes \hat{P})(\hat{O} \otimes 1) g_{\mathrm{b}_*}((-1)^{-1/2} P \otimes \hat{P}) e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}}(\hat{O} \otimes 1) e^{-2\pi i O \otimes \hat{O}} = 1 \otimes \hat{O} + \hat{O} \otimes 1,$$

$$\Delta(\hat{P}) = Ad(S)(\hat{P} \otimes 1) = S(\hat{P} \otimes 1)S^{-1} =$$

$$= e^{2\pi i O \otimes \hat{O}} g_{\mathrm{b}_*}^{-1}((-1)^{-1/2} P \otimes \hat{P})(\hat{P} \otimes 1) g_{\mathrm{b}_*}((-1)^{-1/2} P \otimes \hat{P}) e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}} g_{\mathrm{b}_*}^{-1}((-1)^{-1/2} e^{2\pi b(q_1+p_2-q_2)}(\xi \otimes \xi)) e^{2\pi b(p_1-q_1)}(\xi \otimes \mathbb{I}_2) g_{\mathrm{b}_*}((-1)^{-1/2} e^{2\pi b(q_1+p_2-q_2)}(\xi \otimes \xi)) e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}} e^{\pi b(p_1-q_1)}(\xi \otimes \mathbb{I}_2) g_{\mathrm{b}_*}^{-1}(e^{+i\pi b_*^2} e^{2\pi b(q_1+p_2-q_2)}(\xi \otimes \xi)) g_{\mathrm{b}_*}(e^{-i\pi b_*^2} e^{2\pi b(q_1+p_2-q_2)}(\xi \otimes \xi)) e^{\pi b(p_1-q_1)} e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}} e^{\pi b(p_1-q_1)}(\xi \otimes \mathbb{I}_2)(1 + e^{2\pi b(q_1+p_2-q_2)})(\xi \otimes \xi)) e^{\pi b(p_1-q_1)} e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}}(e^{2\pi b(p_1-q_1)}(\xi \otimes \mathbb{I}_2) + e^{2\pi b(p_1+p_2-q_2)})(\mathbb{I}_2 \otimes \xi)) e^{-2\pi i O \otimes \hat{O}} =$$

$$= e^{2\pi i O \otimes \hat{O}}(\hat{P} \otimes 1 + e^{2\pi b O} \otimes \hat{P}) e^{-2\pi i O \otimes \hat{O}} = 1 \otimes \hat{P} + \hat{P} \otimes e^{-2\pi b \hat{O}}.$$

Moreover, there exists an algebra automorphism $A = e^{-i\pi/3} e^{3\pi i q^2} e^{i\pi(p+q)^2} \mathbb{I}_2$, which has a following action on the operators $p$ and $q$

$$A(q\mathbb{I}_2)A^{-1} = (p-q)\mathbb{I}_2,$$
$$A(p\mathbb{I}_2)A^{-1} = -q\mathbb{I}_2,$$
$$A^{-1}(q\mathbb{I}_2)A = -p\mathbb{I}_2,$$
$$A^{-1}(p\mathbb{I}_2)A = (q-p)\mathbb{I}_2.$$

One can show that the elements $\tilde{e}(s,t), \tilde{\hat{e}}(s,t)$ defined by the action of $A$ as follows

$$\tilde{e}(s,t) = Ae(s,t)A^{-1},$$
$$\tilde{\hat{e}}(s,t) = A\hat{e}(s,t)A^{-1},$$

satisfy the same Heisenberg double relations.

The algebra automorphism can be used to establish the morphism between a tensor product of two Heisenberg doubles and $\mathcal{U}_q(osp(1|2))$. In particular, one can define the elements $E(s,t), \hat{E}(s,t)$ which are represented on $L^2(\mathbb{R}^2) \otimes (\mathbb{C}^{1|1})^2$ defined

$$E(a) = \mu(a; b, c)e(b) \otimes A_2 e(c) A_2^{-1},$$
$$\hat{E}(a) = m(a; c, b)\hat{e}(b) \otimes A_2^{-1} \hat{e}(c) A_2,$$

where $m$ and $\mu$ are the multiplication and comultiplication coefficients of the Borel half $\mathcal{A}$ of $\mathcal{U}_q(osp(1|2))$ respectively. In particular, the lowest lying elements of this type are as follows

$$E(1,0) = 2\pi i(p_1 - q_2)\mathbb{I}_2 \otimes \mathbb{I}_2,$$

$$E(0,1) = -(q + q^{-1})^{-1}(e^{2\pi b(p_2-q_2)}\mathbb{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{2\pi b(q_1-q_2)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{I}_2),$$

$$\hat{E}(1,0) = (q_1 - p_2)\mathbb{I}_2 \otimes \mathbb{I}_2,$$

$$\hat{E}(0,1) = (e^{2\pi b(q_2-q_1)}\mathbb{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{2\pi b(p_1-q_1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{I}_2).$$

It is visible that those elements do have a particular normalisation factors. It would be useful to define another set of elements $u(i,j)$, for which those normalisation factors has been removed, and which generate an algebra that we will denote by $\mathcal{G}$

$$u(1,0) = (p_1 - q_2)\mathbb{I}_2 \otimes \mathbb{I}_2,$$

$$u(0,1) = e^{2\pi b(p_2 - q_2)}\mathbb{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{2\pi b(q_1 - q_2)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{I}_2,$$

$$\hat{u}(1,0) = (q_1 - p_2)\mathbb{I}_2 \otimes \mathbb{I}_2,$$

$$\hat{u}(0,1) = e^{2\pi b(q_2 - q_1)}\mathbb{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{2\pi b(p_1 - q_1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{I}_2.$$

Those generators satisfy commutation relations

$$[u(1.0), \hat{u}(1,0)] = 0,$$
$$[u(1.0), \hat{u}(0,1)] = +ib\hat{u}(0,1),$$
$$[u(1.0), u(0,1)] = -ibu(0,1),$$
$$[\hat{u}(1.0), \hat{u}(0,1)] = +ib\hat{u}(0,1),$$
$$[\hat{u}(1.0), u(0,1)] = -ibu(0,1),$$
$$\{u(1.0), \hat{u}(1,0)\} = (q + q^{-1})(e^{2\pi bu(1,0)} + e^{-2\pi b\hat{u}(1,0)}).$$

Finally, there exists an algebra homomorphism $\mathcal{U}_q(osp(1|2)) \to \mathcal{G}^2$ expressed by

$$K = e^{\pi b(u(1,0) + \hat{u}(1,0))/2},$$

$$v^{(+)} = e^{\pi b\hat{u}(1,0)}\frac{\hat{u}(0,1)}{q + q^{-1}},$$

$$v^{(-)} = \frac{u(0,1)}{q + q^{-1}}e^{-\pi b\hat{u}(1,0)},$$

where the generators $K, v^{\pm}$ are slightly redefined generators of $\mathcal{U}_q(osp(1|2))$ and satisfy commutation relations

$$Kv^{(+)} = qv^{(+)}K,$$
$$Kv^{(-)} = q^{-1}v^{(-)}K,$$
$$\{v^{(+)}, v^{(-)}\} = \frac{K^2 + K^{-2}}{q + q^{-1}},$$

in way which is exactly analogous to the non-graded case considered in the section 4.

# 9 Quantum superplane

In this section we will try to find the analogoue of the representation theoretical construction done by Frenkel and Kim [7] in the case of a quantum superplane. We conjecture that the 6j symbols for the category of self-dual representations of the quantum superplane can be identified with the flip operators of the Teichmüller theory. Also, our goal is to show that the operators defining the Teichmüller theory from the previous section concerning the Heisenberg double are related by a similarity transformation to the ones from the representation theory of a quantum plane.

## 9.1 Self-dual continuous series for a quantum superplane

The quantum plane is essentialy the Borel half of a q-deformed universal enveloping algebra $\mathcal{U}_q(osp(1|2))$ of the Lie algebra $osp(1|2)$. It is generated by the elements $X, X^{-1}, Y$, with relations

$$XY = q^2 YX, \tag{203}$$

$$\Delta(X) = X \otimes X, \tag{204}$$

$$\Delta(Y) = Y \otimes X + 1 \otimes Y, \tag{205}$$

and $|X| = 0, |Y| = 1$, where the deformation parameter $q = e^{i\pi b^2}$. Again we parametrise the deformation through a real number b so that $q$ takes values on the unit circle. We also equip this algebra with the following *-structure

$$X^* \;=\; X \quad , \quad Y^* \;=\; Y \;.$$

Now we want to introduce the series of representations relevant for the quantum plane. The carrier spaces $\mathcal{H}$ of the associated representations are $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$. Then, the generators $X, Y$ are expressed as

$$X = e^{-2\pi bp}\mathbb{I}_2 = T_x^{ib}\mathbb{I}_2, \tag{206}$$

$$Y = e^{\pi bx}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{207}$$

where $[p, x] = \frac{1}{2\pi i}$ are usual operators on $L^2(\mathbb{R})$. This representation is self-dual in the usual sense which has been discussed in the previous sections.

## 9.2 The Clebsch-Gordan coefficients for a quantum plane

The tensor product of two representations $\pi$ is defined in terms of the coproduct, is reducible and its decomposition into a direct sum of irreducibles is what defines the Clebsh-Gordan coefficients. In this case one has the following decomposition,

$$\mathcal{H} \otimes \mathcal{H} \;\simeq\; \int_{\mathbb{R}}^{\oplus} \mathcal{H},$$

however we will establish that this tensor decomposition can be understood as

$$\mathcal{H} \otimes \mathcal{H} \;\simeq\; M \otimes \mathcal{H},$$

with the help of a multiplicity space such that $M = L^2(\mathbb{R})$. We are going to spell out and prove an explicit formula for the homomorphisms $\mathcal{H} \otimes \mathcal{H} \to M \otimes \mathcal{H}$ and $M \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$
\begin{aligned}
f^{jk}(x_1, x_2) \to F_f^i(\alpha, x) &= \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \left( \begin{bmatrix} \alpha & x \\ x_1 & x_2 \end{bmatrix}_\mu \right)^i_{jk} f^{jk}(x_1, x_2) \,, \\
F^k(\alpha, x) \to f_F^{ij}(x_1, x_2) &= \int_{\mathbb{R}} \mathrm{d}\alpha \mathrm{d}x \left( \begin{bmatrix} \alpha & x \\ x_1 & x_2 \end{bmatrix}_\mu^{-1} \right)^{ij}_k F^k(\alpha, x) \,,
\end{aligned}
\tag{208}
$$

where $\mu = 0, 1$. The kernels of the Clebsh-Gordan map are expressed in terms of $G_\mu$ functions

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^1_{11} = a e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_\mu^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^2_{21} = a e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mu+1}^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^2_{12} = (-1)^{\mu+1} a e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_\mu^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^1_{22} = (-1)^{\mu} a e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mu+1}^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^2_{11} = b e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_\mu^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^1_{21} = b e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mu+1}^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^1_{12} = (-1)^{\mu+1} b e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_\mu^{-1}(i(x_3 - x_2) + Q),
$$

$$
\left( \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix}_\mu \right)^2_{22} = (-1)^{\mu} b e^{\pi i \alpha(x_3 - x_1)} e^{\pi i (x_3 - x_1)(x_2 - x_1)} G_{\mu+1}^{-1}(i(x_3 - x_2) + Q).
$$

where $a, b \in \mathbb{C}$ and $\mu = 0, 1$.

## 9.3   The intertwining property

The fundamental intertwining property of the Clebsch-Gordan coefficients takes the following form

$$
\pi(u) \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \alpha & x_3 \\ x_1 & x_2 \end{bmatrix} (\pi \otimes \pi) \Delta(u),
\tag{209}
$$

for $u = X, Y$. The equation should be interpreted in the same sense as the intertwining property (64). The proof of eq. (209) goes analogously to the one for the case of quantum plane — the only difference is that we use the shift property for the functions $G_\mu$ instead of the one for $G_\mathrm{b}$.

## 9.4   The Racah-Wigner coefficients for a quantum plane

The Racah-Wigner coefficients describe a change of basis in the 3-fold tensor product of representations. In decomposing their product into irreducibles $\pi$ there exists two possible fusion paths, denoted by $t$ and $s$, which

are described by the following combination of Clebsch-Gordan coefficients

$$f^i(\alpha_s, \alpha_4, x_4) = \int \mathrm{d}x_s \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_4 \ x_4 \\ x_s \ x_3 \end{bmatrix}^i_{dc} \begin{bmatrix} \alpha_s \ x_s \\ x_1 \ x_2 \end{bmatrix}^d_{ab} f^{abc}(x_1, x_2, x_3),$$

$$f^{ijk}(x_1, x_2, x_3) = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_s \mathrm{d}x_4 \mathrm{d}x_s \left( \begin{bmatrix} \alpha_s \ x_s \\ x_1 \ x_2 \end{bmatrix}^{-1} \right)^{ij}_b \left( \begin{bmatrix} \alpha_4 \ x_4 \\ x_s \ x_3 \end{bmatrix}^{-1} \right)^{bk}_a f^a(\alpha_s, \alpha_4, x_4),$$

for isomorphisms $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \cong M^s_{12} \otimes M^4_{s3} \otimes \mathcal{H}_4$ and

$$f^i(\alpha_4, \alpha_t, x_4) = \int \mathrm{d}x_t \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_4 \ x_4 \\ x_1 \ x_t \end{bmatrix}^i_{ad} \begin{bmatrix} \alpha_t \ x_t \\ x_2 \ x_3 \end{bmatrix}^d_{bc} f^{abc}(x_1, x_2, x_3),$$

$$f^{ijk}(x_1, x_2, x_3) = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_t \mathrm{d}x_4 \mathrm{d}x_t \left( \begin{bmatrix} \alpha_t \ x_t \\ x_2 \ x_3 \end{bmatrix}^{-1} \right)^{jk}_b \left( \begin{bmatrix} \alpha_4 \ x_4 \\ x_1 \ x_t \end{bmatrix}^{-1} \right)^{ib}_a f^a(\alpha_4, \alpha_t, x_4),$$

for $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \cong M^t_{23} \otimes M^4_{1t} \otimes \mathcal{H}_4$. Then one can relate the elements corresponding to those two decompositions using the map

$$f^i(\alpha'_4, \alpha_t, x'_4) = \int \mathrm{d}\alpha_4 \mathrm{d}\alpha_s \mathrm{d}x_4 \left\{ \begin{matrix} \alpha_4 & \alpha_s \\ \alpha'_4 & \alpha_t \end{matrix} \right\}^i_a f^a(\alpha_s, \alpha_4, x_4),$$

where the kernel of this map is 6j symbol and it is expressed using the Clebsch-Gordan coefficients in the following way

$$\left\{ \begin{matrix} \alpha_4 & \alpha_s \\ \alpha'_4 & \alpha_t \end{matrix} \right\}^i_a = \int \mathrm{d}x_s \mathrm{d}x_t \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha'_4 \ x'_4 \\ x_1 \ x_t \end{bmatrix}^i_{\alpha\delta} \begin{bmatrix} \alpha_t \ x_t \\ x_2 \ x_3 \end{bmatrix}^\delta_{\beta\gamma} \left( \begin{bmatrix} \alpha_s \ x_s \\ x_1 \ x_2 \end{bmatrix}^{-1} \right)^{\alpha\beta}_b \left( \begin{bmatrix} \alpha_4 \ x_4 \\ x_s \ x_3 \end{bmatrix}^{-1} \right)^{b\gamma}_a. \tag{210}$$

# 10   Conclusions

In this text we presented several ways in which liberally understood quantum group methods allowed us to obtain solutions to the pentagon equation relevant for the problems of physical interest and consequently for their deeper study. We reviewed how representation theory of $\mathcal{U}_q(sl(2))$ manifested itself in the context of quantum Liouville field theory and Teichmüller theory, where special class of self-dual representations played a crucial role. The fusion matrix of Liouville theory was shown to be up to normalisation nothing else than the 6j symbols for this family of representations, while important generators of the Ptolemy groupoid of the quantised Teichmüller theory were nothing else than objects (canonical element and algebra automorphism) of Heisenberg double of the Borel half of $\mathcal{U}_q(sl(2))$.

We generalised those result to the supersymmetric case: instead of $\mathcal{U}_q(sl(2))$ we considered the quantum supergroup $\mathcal{U}_q(osp(1|2))$, and analysed the family of self-dual representations of it. Thanks to that we constructed fusion matrix of $N = 1$ supersymmetric Liouville theory. We also considered the representation theory of Heisenberg double of the Borel half of $\mathcal{U}_q(osp(1|2))$ and constructed the canonical element and algebra automorphism, which we expect are related to a quantisation of Teichmüller theory of super Riemann surfaces.

One could ask about some additional avenues of study which would expand on the results presented here. For once it is possible to look at other conformal field theories with non-compact spectra and find their corresponding quantum groups. Starting from the Liouville theory one can generalise it in two different, immidiate ways: either go up with rank or go up with supersymmetry. Since Liouville theory is an example of Toda theory for $sl(n)$ gauge groups for $n = 2$, it is possible to just consider the cases when $n > 2$. Classical Toda field theory Lagrangian has the form

$$L = \frac{1}{8\pi}(\partial\phi)^2 + \mu \sum_{k=1}^{n-1} e^{b(e_k, \phi)},$$

where $e_k$ are the simple roots of $sl(n)$ and $(,)$ is the Killing form on this Lie algebra. It is clear that the case of $n = 2$ would give us the Lagrangian for the Liouville theory. One would expect that the Moore-Seiberg grupoid for this class of theories could be acquired through the means of representation theory of $U_q(sl(n))$ quantum groups. This is especially appealing because, even though the subject is extensively studied, even $sl(3)$ Toda theory is substantially more complicated and lack the explicit form of the fusion, braiding and modular matrices.

One the other hand, one could as well enhance Liouville theory with higher and higher number of supersymmetries: since this thesis was partially devoted to consideration of $N = 1$ supersymmetric Liouville theory, one could try to extend those results to $N = 2$ supersymmetric Liouville — which could be interesting given the known connection of it to black holes. Lastly, one can study Wess-Zumino-Witten models with non-compact spectra.

Using the Teichmüller theory Andersen and Kashaev [37] constructed a topological field theory of 3-dimensional manifolds and consequently link invariants. Extending the framework of Reshetikhin and Turaev, they associated to a tetrahedron of a triangulated 3-manifold an operator

$$T = \Phi_{\mathrm{b}}^{-1}(q_1 - p_1 + p_2)e^{2\pi i p_1 q_2}, \tag{211}$$

which has an interpretation as a flip map between triangulations in the Teichmüller theory and, as we have seen previously, is a canonical element of Heisenberg double of the Borel half of $\mathcal{U}_q(sl(2))$. Because $T$ satisfies the pentagon equation, what follows from the properties of Fadeev's quantum dilogarithm, the partition function obtained by gluing tetrahedra together stays the same even when one chooses different, but equivalent, triangulation of the 3-manifold, which provides one with a properly defined invariant.

One could use the data from supersymmetric generalisation of the Teichmüller theory to define a corresponding invariant for 3-dimensional spin manifolds.

# A  Lie algebras

In this section we will summarise the basic notions concerning the Lie algebras and remind the classification of those objects.

**Definition 18** *Let $\mathfrak{g}$ be a finite dimensional vector space over $\mathbb{C}$. $\mathfrak{g}$ is called a complex finite dimensional Lie algebra when equipped with a bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

$$[x, y] = -[y, x], \tag{212}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{213}$$

*for all $x, y, z \in \mathfrak{g}$. The second condition is called Jacobi identity.*

**Definition 19** *Let $\mathfrak{g}$ be a Lie algebra and $V$ be a vector space. A representation $\rho$ of $\mathfrak{g}$ is a homomorphism $\rho : \mathfrak{g} \to EndV$ that satisfy*

$$\rho([x, y]) = [\rho(x), \rho(y)],$$

*for all $x, y \in \mathfrak{g}$.*

**Definition 20** *An adjoint representation is a representation ad such that $ad : \mathfrak{g} \to End\mathfrak{g}$*

$$ad(x)(y) = [x, y], \tag{214}$$

*for all $x, y \in \mathfrak{g}$.*

**Definition 21** *Let ad be an adjoint representation of a Lie algebra $\mathfrak{g}$. A Killing form is a bilinear symmetric form $\langle, \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$*

$$\langle x, y \rangle = Tr(ad(x)ad(y)). \tag{215}$$

**Definition 22** *Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{g}$ is called semisimple if its Killing form is nondegenerate.*

**Definition 23** *A Lie algebra $\mathfrak{g}$ is simple if there is no subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.*

**Theorem 15** *Every finite dimensional semisimple Lie algebra $\mathfrak{g}$ decomposes into a direct sum of pairwise orthogonal (with respect to the Killing form) simple subalgebras:*

$$\mathfrak{g} = \bigoplus_i \mathfrak{s}_i. \tag{216}$$

**Definition 24** *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. A Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$.*

One can also define a Cartan subalgebra by the means of generalised eigenspaces of regular elements of $\mathfrak{g}$. Then above definition becomes a statement that needs to be proven.

**Theorem 16** *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, for every $h \in \mathfrak{h}$ $ad(h)$ is diagonalisable.*

**Definition 25** *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. A root $\alpha : \mathfrak{h} \to$ is a nonzero linear form such that:*

$$[h, x] = \alpha(h)x.$$

The set of all roots is denoted by $\mathfrak{R} \in \mathfrak{h}^*$. One can decompose $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right),$$

where $g_\alpha = \{ x \in \mathfrak{g} | \forall h \in \mathfrak{h} \quad [h, x] = \alpha(h)x \}$ is a root subspace of $\mathfrak{g}$.

One defines a map $\mathfrak{h} \to \mathfrak{h}^*$ such that $h \mapsto \alpha_h$, $\alpha_h(h') = \langle h, h' \rangle$. From this we can define a bilinear form on $\mathfrak{h}^*$

$$\langle \alpha_h, \alpha_{h'} \rangle = \langle h, h' \rangle.$$

**Properties:** Root system $\mathfrak{R}$ satisfies:

$$\forall \alpha, \beta \in \mathfrak{R} : \langle \alpha, \beta \rangle \in \mathbb{R},$$

$$\forall \alpha, \beta \in \mathfrak{R} : \alpha - \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \beta \in \mathfrak{R},$$

$$\forall \alpha, \beta \in \mathfrak{R} : \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z},$$

$$\forall \alpha \in \mathfrak{R} : \mathbb{R}\alpha \cap \mathfrak{R} = \{ -\alpha, \alpha \}.$$

The root system $\mathfrak{R}$ spans $\mathfrak{h}^*$, but it is not linearly independent. There are many ways in which one can choose a linearly independent subset of $\mathfrak{R}$ which is a basis, but of importance is a basis of simple roots.

**Definition 26** *A basis of simple roots $\mathfrak{S}$ is a linearly independent subset of $\mathfrak{R}$ spanning $\mathfrak{h}^*$ such that for every $\beta \in \mathfrak{R}$ the coefficients of expansion of $\beta$ in $\mathfrak{S}$ are all positive or negative integers. $\beta \in R$ with all positive or zero coefficients is called a positive root, with all negative — a negative root.*

**Definition 27** *The Cartan matrix $A$ of a simple Lie algebra $\mathfrak{g}$ is a matrix such that*

$$A = \left[ \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \right]_{\alpha, \beta \in S} \tag{217}$$

*which depends only on the root system $\mathfrak{R}$ of $\mathfrak{g}$.*

**Properties:** The Cartan matrix satisfies:

$$A_{ii} = 2,$$

$$A_{ij} \leq 0, \qquad i \neq j,$$

$$A \text{ does not have a block diagonal form.}$$

After introducing the notation

$$A_{ij} = A_{ji} = 0 \qquad i \bullet \qquad \bullet j$$

$$A_{ij} = A_{ji} = -1 \qquad i \bullet \!\!-\!\! \bullet j$$

$$A_{ij} = -2, A_{ji} = -1 \qquad i \bullet \Rightarrow \bullet j$$

$$A_{ij} = -3, A_{ji} = -1 \qquad i \bullet \Rrightarrow \bullet j$$

one can represent the Cartan matrix in terms of the Dynkin diagram. This leads us to the classification of the semisimple Lie algebras:

**Theorem 17** *Let $\mathfrak{g}$ be a simple Lie algebra. Its Dynkin diagram is one of the following*

$$A_n \qquad n \geq 1 \qquad \bullet \!-\! \bullet \!-\! \ldots \!-\! \bullet$$

$$B_n \qquad n \geq 2 \qquad \bullet \!-\! \bullet \!-\! \ldots \!-\! \bullet \Rightarrow \bullet$$

$$C_n \qquad n \geq 3 \qquad \bullet \!-\! \bullet \!-\! \ldots \!-\! \bullet \Leftarrow \bullet$$

$$D_n \qquad n \geq 4 \qquad \bullet \!-\! \overset{\displaystyle\bullet}{\overset{|}{\bullet}} \!-\! \ldots \!-\! \bullet \!-\! \bullet$$

$$G_2 \qquad\qquad\quad\; \bullet \Rightarrow \bullet$$

$$F_4 \qquad\qquad\quad\; \bullet \!-\! \bullet \Rightarrow \bullet \!-\! \bullet$$

$$E_6 \qquad\qquad\quad\; \bullet \!-\! \bullet \!-\! \overset{\displaystyle\bullet}{\overset{|}{\bullet}} \!-\! \bullet \!-\! \bullet$$

$$E_7 \qquad\qquad\quad\; \bullet \!-\! \bullet \!-\! \overset{\displaystyle\bullet}{\overset{|}{\bullet}} \!-\! \bullet \!-\! \bullet \!-\! \bullet$$

$$E_8 \qquad\qquad\quad\; \bullet \!-\! \bullet \!-\! \overset{\displaystyle\bullet}{\overset{|}{\bullet}} \!-\! \bullet \!-\! \bullet \!-\! \bullet \!-\! \bullet$$
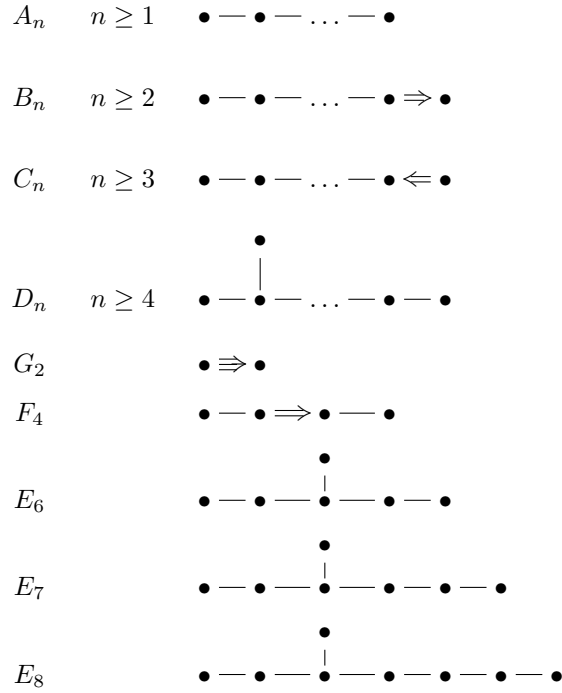
*where all diagrams have n vertices $\bullet$, representing simple roots $\alpha_i$. Conversely, each of these diagrams is the Dynkin diagram of a unique simple Lie algebra, up to isomorphism.*

Moreover, one can present also the Dynkin diagrams of untwisted affine Lie algebras $\hat{\mathcal{G}}$ which are associated with the infinite dimensional loop algebras $\hat{L}(\mathcal{G}_l)$ (for more details c.f. [5]; for the discussion of q-deformation [3]).

# B Special functions

## B.1 Non-graded case

The basic building block for all objects that appear in the context of the quantum algebra $\mathcal{U}_q(sl(2))$ is Barnes' double Gamma function. For $\mathfrak{Re}\,x > 0$ it admits an integral representation

$$\log \Gamma_{\mathrm{b}}(x) = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-xt} - e^{-\frac{Q}{2}t}}{(1 - e^{-t\mathrm{b}})(1 - e^{-\frac{t}{\mathrm{b}}})} - \frac{\left(\frac{Q}{2} - x\right)^2}{2e^t} - \frac{\frac{Q}{2} - x}{t} \right],$$

where $Q = \mathrm{b} + \frac{1}{\mathrm{b}}$. One can analytically continue $\Gamma_{\mathrm{b}}$ to a meromorphic function defined on the entire complex plane $\mathbb{C}$. The most important property of $\Gamma_{\mathrm{b}}$ is its behavior with respect to shifts by $\mathrm{b}^\pm$,

$$\Gamma_{\mathrm{b}}(x + \mathrm{b}) = \frac{\sqrt{2\pi}\,\mathrm{b}^{\mathrm{b}x - \frac{1}{2}}}{\Gamma_{\mathrm{b}}(bx)} \Gamma_{\mathrm{b}}(x) \quad , \quad \Gamma_{\mathrm{b}}(x + \mathrm{b}^{-1}) = \frac{\sqrt{2\pi}\,\mathrm{b}^{-\frac{x}{\mathrm{b}} + \frac{1}{2}}}{\Gamma_{\mathrm{b}}(\frac{x}{\mathrm{b}})} \Gamma_{\mathrm{b}}(x) \; . \tag{218}$$

These shift equation allows us to calculate residues of the poles of $\Gamma_{\mathrm{b}}$. When $x \to 0$, for instance, one finds

$$\Gamma_{\mathrm{b}}(x) = \frac{\Gamma_{\mathrm{b}}(Q)}{2\pi x} + O(1). \tag{219}$$

From Barnes' double Gamma function we can build two other important special functions,

$$S_{\mathrm{b}}(x) = \frac{\Gamma_{\mathrm{b}}(x)}{\Gamma_{\mathrm{b}}(Q - x)}, \tag{220}$$

$$G_{\mathrm{b}}(x) = e^{-\frac{i\pi}{2}x(Q - x)} S_{\mathrm{b}}(x). \tag{221}$$

We shall often refer to the function $S_{\mathrm{b}}$ as double sine function. It is related to Faddeev's quantum dilogarithm through,

$$\Phi_{\mathrm{b}}(x) = A G_{\mathrm{b}}^{-1}(-ix + \frac{Q}{2}),$$

where

$$A = e^{-i\pi(1 - 4c_{\mathrm{b}}^2)/12} \quad , \quad c_{\mathrm{b}} = iQ/2 \; . \tag{222}$$

The $S_{\mathrm{b}}$ function is meromorphic with poles and zeros in

$$S_{\mathrm{b}}(x) = 0 \Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0} \; ,$$

$$S_{\mathrm{b}}(x)^{-1} = 0 \Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0} \; .$$

Using the shift property of quantum dilogarith one can evaluate

$$
\begin{aligned}
S_b(-kb) &= \prod_{j=1}^k \left(2\sin(-\pi j b^2)\right)^{-1} S_b(0) = \left(-2\sin\left(\pi b^2\right)\right)^{-k} \frac{S_b(0)}{[k]!} \\
S_b(-kb - Q) &= \left(2\sin\left(\pi b^2\right)\right)^{-k-1} \left(2\sin\left(-\pi b^{-2}\right)\right)^{-1} \frac{S_b(0)}{[k + 1]!}
\end{aligned}
\tag{223}
$$

for $k \in \mathbb{N}$. From its definition and the shift property of Barnes' double Gamma function it is easy to derive the following shift and reflection properties of $G_{\mathrm{b}}$,

$$G_{\mathrm{b}}(x + \mathrm{b}) = (1 - e^{2\pi i \mathrm{b}x})G_{\mathrm{b}}(x) \; , \tag{224}$$

$$G_{\mathrm{b}}(x)G_{\mathrm{b}}(Q - x) = e^{\pi i x(x - Q)} \; . \tag{225}$$

We also need to the asymptotic behavior of the function $G_{\mathrm{b}}$ along the imaginary axis,

$$G_{\mathrm{b}}(x) \sim 1 \, , \qquad \Im x \to +\infty,$$

$$G_{\mathrm{b}}(x) \sim e^{i\pi x(x-Q)} \, , \quad \Im x \to -\infty. \tag{226}$$

We define as well function

$$g_{\mathrm{b}}(x) = \frac{\bar{\zeta}_{\mathrm{b}}}{G_{\mathrm{b}}(\frac{Q}{2} + \frac{1}{2\pi i b}\log x)},$$

$$g_{\mathrm{b}}(e^{2\pi br}) = \int \mathrm{d}t\, e^{2\pi i tr} \frac{e^{-i\pi t^2}}{G_{\mathrm{b}}(Q+it)},$$

$$g_{\mathrm{b}}^{-1}(e^{2\pi br}) = \int \mathrm{d}t\, e^{2\pi i tr} \frac{e^{-\pi tQ}}{G_{\mathrm{b}}(Q+it)},$$

where $\bar{\zeta}_{\mathrm{b}} = \exp[\frac{i\pi}{4} + \frac{i\pi}{12}(b^2 + b^{-2})]$. The shift and reflection relations that it satisfies are as follows

$$g_{\mathrm{b}}(e^{-i\pi b^2}x) = (1+x)g_{\mathrm{b}}(e^{+i\pi b^2}x),$$

$$g_{\mathrm{b}}(e^{2\pi i bx})g_{\mathrm{b}}(e^{-2\pi i bx}) = e^{\frac{i\pi Q^2}{4}}\bar{\zeta}_{\mathrm{b}}^2 e^{-i\pi x^2}.$$

Also, we know that for noncocommutative variables $U, V$ such that $UV = q^2 VU$ it satisfies the pentagon relation

$$g_{\mathrm{b}}(U)g_{\mathrm{b}}(V) = g_{\mathrm{b}}(V)g_{\mathrm{b}}(q^{-1}UV)g_{\mathrm{b}}(U).$$

### B.1.1  Integral identities for $\mathcal{U}_q(sl(2))$

The most complex identity we need in the main text is the following star triangle relation for double sine function,

$$\int \frac{\mathrm{d}x}{i} \prod_{i=1}^{3} S_{\mathrm{b}}(x + a_i)S_{\mathrm{b}}(-x + b_i) = \prod_{i,j=1}^{3} S_{\mathrm{b}}(a_i + b_j) \, ,$$

which holds provided that the arguments satisfy the balancing condition

$$\sum_{i=1}^{3}(a_i + b_i) = Q.$$

A proof can be found e.g. in [44]. Here, we will only state the necessary results. The star triangle relation can be reduced to the Saalschütz summation formula [28]

$$\frac{1}{i} \int_{-i\infty}^{i\infty} \mathrm{d}\tau e^{2\pi i \tau Q} \frac{G_{\mathrm{b}}(\tau + a)G_{\mathrm{b}}(\tau + b)G_{\mathrm{b}}(\tau + c)}{G_{\mathrm{b}}(\tau + a + b + c - d + Q)G_{\mathrm{b}}(\tau + Q)G_{\mathrm{b}}(\tau + d)} =$$

$$= e^{i\pi d(Q-d)}G_{\mathrm{b}}(a)G_{\mathrm{b}}(b)G_{\mathrm{b}}(c)\frac{G_{\mathrm{b}}(Q+b-d)G_{\mathrm{b}}(Q+c-d)G_{\mathrm{b}}(Q+a-d)}{G_{\mathrm{b}}(Q+b+c-d)G_{\mathrm{b}}(Q+a+c-d)G_{\mathrm{b}}(Q+a+b-d)}.$$

A useful consequence of the Saalschütz summation formula can be obtained by taking the limit $c \to i\infty$

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{2\pi i \tau Q} \frac{G_{\mathrm{b}}(\tau + a)G_{\mathrm{b}}(\tau + b)}{G_{\mathrm{b}}(\tau + d)G_{\mathrm{b}}(\tau + Q)} =$$

$$= e^{i\pi d(Q-d)}G_{\mathrm{b}}(a)G_{\mathrm{b}}(b)G_{\mathrm{b}}(Q+b-d)\frac{G_{\mathrm{b}}(Q+a-d)}{G_{\mathrm{b}}(Q+a+b-d)}.$$

Also, by taking the additional limits $a \to -i\infty$, $d \to -i\infty$ with $a - d + Q$ fixed one may derive the well known Ramanujan summation formula

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{2\pi i \tau \beta} \frac{G_{\mathrm{b}}(\tau + \alpha)}{G_{\mathrm{b}}(\tau + Q)} = \frac{G_{\mathrm{b}}(\alpha) G_{\mathrm{b}}(\beta)}{G_{\mathrm{b}}(\alpha + \beta)}, \tag{227}$$

which holds for arbitrary $\alpha = a - d + Q$ and $\beta = b$. Ramanujan's summation formula is a five-term (pentagon) identity. In may be considered a quantization of the familiar Rogers five-term identity satisfied by dilogarithms.

From (227) follows that we can represent Dirac delta distribution in terms of $G_{\mathrm{b}}$ functions as follows

$$D_\epsilon(a) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{2\pi i \tau (a + \epsilon)} \frac{G_{\mathrm{b}}(\tau + \epsilon - a)}{G_{\mathrm{b}}(\tau + Q)},$$

$$\lim_{\epsilon \to 0} D_\epsilon(a) = \delta(ia).$$

## B.2 Graded case

In discussing the representation theory of the quantum superalgebra $\mathcal{U}_q(osp(1|2))$ we need the following additional special functions

$$\Gamma_1(x) = \Gamma_{\mathrm{NS}}(x) = \Gamma_{\mathrm{b}}\left(\frac{x}{2}\right) \Gamma_{\mathrm{b}}\left(\frac{x + Q}{2}\right),$$

$$\Gamma_0(x) = \Gamma_{\mathrm{R}}(x) = \Gamma_{\mathrm{b}}\left(\frac{x + \mathrm{b}}{2}\right) \Gamma_{\mathrm{b}}\left(\frac{x + \mathrm{b}^{-1}}{2}\right).$$

Furthermore, let us define

$$\begin{aligned}
S_1(x) &= S_{\mathrm{NS}}(x) = \frac{\Gamma_{\mathrm{NS}}(x)}{\Gamma_{\mathrm{NS}}(Q-x)}, & G_1(x) &= G_{\mathrm{NS}}(x) = \zeta_0 e^{-\frac{i\pi}{4} x(Q-x)} S_{\mathrm{NS}}(x), \\
S_0(x) &= S_{\mathrm{R}}(x) = \frac{\Gamma_{\mathrm{R}}(x)}{\Gamma_{\mathrm{R}}(Q-x)}, & G_0(x) &= G_{\mathrm{R}}(x) = e^{-\frac{i\pi}{4}} \zeta_0 e^{-\frac{i\pi}{4} x(Q-x)} S_{\mathrm{R}}(x),
\end{aligned} \tag{228}$$

where $\zeta_0 = \exp(-i\pi Q^2/8)$. The functions $S_\nu$ are related to the supersymmetric analogue of Faddeev's quantum dilogarithm through

$$\Phi_{\mathrm{b}}^\nu(x) = A^2 G_\nu^{-1}\left(-ix + \frac{Q}{2}\right),$$

with a constant $A$ as defined in eq. (222). As for $S_{\mathrm{b}}$, the functions $S_0(x)$ and $S_1(x)$ are meromorphic with poles and zeros in

$$S_0(x) = 0 \Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m + n \in 2\mathbb{Z} + 1,$$

$$S_1(x) = 0 \Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m + n \in 2\mathbb{Z},$$

$$S_0(x)^{-1} = 0 \Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m + n \in 2\mathbb{Z} + 1,$$

$$S_1(x)^{-1} = 0 \Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m + n \in 2\mathbb{Z}.$$

As in the previous subsection, we want to state the shift and reflection properties of the functions $G_1$ and $G_0$,

$$G_\nu(x + \beta^{\pm 1}) = (1 - (-1)^\nu e^{\pi i \mathrm{b}^{\pm 1} x}) G_{\nu+1}(x), \tag{229}$$

$$G_\nu(x) G_\nu(Q - x) = e^{\frac{i\pi}{2}(\nu - 1)} \zeta_0^2 e^{\frac{\pi i}{2} x(x - Q)}. \tag{230}$$

Asymptotically, the functions $G_1$ and $G_0$ behave as

$$G_\nu(x) \sim 1 , \qquad\qquad \mathfrak{Im}x \to +\infty , \qquad\qquad (231)$$

$$G_\nu(x) \sim e^{\frac{i\pi}{2}(\nu-1)}\zeta_0^2 e^{\frac{i\pi}{2}x(x-Q)} , \quad \mathfrak{Im}x \to -\infty . \qquad\qquad (232)$$

For $x$ integer such that $x \in 2\mathbb{N} + (1-\nu)$ the double sine functions can be written as:

$$S_\nu(-xb) = \frac{S_1(0)}{\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^x [x]_+!} \qquad\qquad (233)$$

$$S_\nu(-xb-Q) = \frac{(-1)^{-\frac{x+1}{2}-\frac{1}{2}\delta_{\nu,1}} S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1}[x+1]_+!} = \frac{(-1)^{-\frac{x(x-1)}{2}+1} S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1}[x+1]_+!},$$

where

$$[n]_+! = \begin{cases} \prod_{j=1 \, mod \, 2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1 \, mod \, 2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2 \, mod \, 2}^{n-1} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1 \end{cases}$$

### B.2.1 Integral identities for $\mathcal{U}_q(osp(1|2))$

In the supersymmetric case, the star triangle relations take the following form

$$\sum_{\nu=0,1} (-1)^{\nu(1+\sum_i(\nu_i+\mu_i))/2} \int \frac{\mathrm{d}x}{i} \prod_{i=1}^3 S_{\nu+\nu_i}(x+a_i) S_{1+\nu+\mu_i}(-x+b_i) = 2 \prod_{i,j=1}^3 S_{\nu_i+\mu_j}(a_i+b_j),$$

with

$$\sum_i (\nu_i + \mu_i) = 1 \, mod \, 2, \qquad\qquad (234)$$

and the balancing condition

$$\sum_{i=1}^3 (a_i + b_i) = Q .$$

From these equations one can get 16 "supersymmetric" analogues of the Saalschütz summation formula, some of which are stated with proofs for instance in [40]. As in the non-supersymmetric case, taking the limit $d \to i\infty$ leads to the reduced formulae

$$\sum_{\sigma=0,1} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{i\pi\tau Q} \frac{G_{\sigma+\rho_a}(\tau+a) G_{\sigma+\rho_\beta}(\tau+b)}{G_{\sigma+\rho_c}(\tau+c) G_{1+\sigma}(\tau+Q)} =$$

$$= 2i^{1-\rho_c}\zeta_0^{-3} e^{\frac{i\pi}{2}c(Q-c)} \frac{G_{\rho_a}(a) G_{\rho_b}(b) G_{1+\rho_a-\rho_c}(Q+a-c) G_{1+\rho_b-\rho_c}(Q+b-c)}{G_{\rho_a+\rho_b-\rho_c}(Q+a+b-c)}.$$

where $\zeta_0 = \exp(-i\pi Q^2/8)$ is the same constant factor as before. From these identities one can easily obtain a system of four equations that generalize Ramanujan's formula (227) to the supersymmetric case,

$$\sum_{\sigma=0,1} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} (-1)^{\rho_\beta\sigma} e^{\pi i\tau\beta} \frac{G_{\sigma+\rho_\alpha}(\tau+\alpha)}{G_{\sigma+1}(\tau+Q)} = 2\zeta_0^{-1} \frac{G_{\rho_\alpha}(\alpha) G_{1+\rho_\beta}(\beta)}{G_{\rho_\alpha+\rho_\beta}(\alpha+\beta)} \qquad\qquad (235)$$

The notations are the same as in section B.1. The last identity is is supersymmetric version of the pentagon identity for Faddeev's quantum dilogarithm.

From the supersymmetric analogoue of (227) follows that we can represent Dirac delta distribution in terms of $G_\nu$ functions as follows

$$D_\epsilon^{\nu\bar{\nu}}(a) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{\pi i\tau(a+\epsilon)} \left( \frac{G_{\bar{\nu}}(\tau + \epsilon - a)}{G_\nu(\tau + Q)} + (-1)^{\nu+\bar{\nu}} \frac{G_{\bar{\nu}+1}(\tau + \epsilon - a)}{G_{\nu+1}(\tau + Q)} \right),$$

$$\lim_{\epsilon\to 0} D_\epsilon^{\nu\bar{\nu}}(a) = 4\delta(a)\delta_{\nu,\bar{\nu}}.$$

## B.3   Integral representation of the product of two Dirac delta functions

Lets consider the distribution

$$D(x, \xi_-) = \lim_{\epsilon\to 0} \frac{S_{\mathrm{b}}(\epsilon + x) S_{\mathrm{b}}(\epsilon - \xi_- - x) S_{\mathrm{b}}(2\epsilon + \xi_-)}{S_{\mathrm{b}}(4\epsilon)}.$$

We need to show that the following holds

$$D(x, \xi_-) = \delta(ix)\delta(i\xi_-).$$

In order to do so we integrate this equation along the imaginary axis against an arbitrary test function $f = f(x, y)$ to find

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{i} f(y, x) D(x, y) =$$

$$= \lim_{\epsilon\to 0} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{i} f(y, x) \frac{S_{\mathrm{b}}(\epsilon + x) S_{\mathrm{b}}(\epsilon - y - x) S_{\mathrm{b}}(2\epsilon + y)}{S_{\mathrm{b}}(4\epsilon)} =$$

$$= \lim_{\epsilon\to 0} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{2\pi i} f(y, x) \frac{4\epsilon}{(\epsilon + x)(\epsilon - x - y)(2\epsilon + y)} =$$

$$= \lim_{\epsilon\to 0} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{2\pi i} f(\epsilon y, \epsilon x) \frac{4}{(1 + x)(1 - x - y)(2 + y)} =$$

$$= \left( \int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{2\pi i} \frac{4}{(1 + x)(1 - x - y)(2 + y)} \right) f(0, 0) =$$

$$= f(0, 0) .$$

In the last step of our short computation we have evaluated the double integral using Cauchy's formula.

# C   Summary

In this thesis we present the applications of quantum group representation theoretical methods to two dimensional non-rational conformal field theory and Teichmüller theory. We recall the notion of Hopf algebra and the notions of the Heisenberg and Drinfeld doubles. We use the representation theoretical methods to obtain the pentagon equation solutions from the representation theory of $\mathcal{U}_q(sl(2))$, a quantum plane and the Heisenberg double of a quantum plane, what are known from literature results, however they have high pedagogical value from the point of possible generalisations. We generalise the results to the $\mathbb{Z}_2$-graded case, where we obtain the solutions of the pentagon equation using the representation theory of $\mathcal{U}_q(osp(1|2))$ and the Heisenberg double of quantum superplane.

In diese Dissertation wir präsentieren die Anwendung von die Repräsentationen der Quantengruppen auf der Konforme Feldtheorie aus einem zweidimensionalen Raum und der Teichmüller-Theorie. Wir erinneren uns an die Definitionen des Drinfeld-Doppels und des Heisenberg-Doppels. Wir benutzen die Repräsentationstheorie der Quantengruppen zu den Lösungen der Pentagon Gleichung aus der $\mathcal{U}_q(sl(2))$, der Quantenebene und dem Heisenberg-Doppel aus der Quantenebene erhalten. Das ist aus der Literatur bekannt, aber das has den pädagogisch Wert wann mann die Generalisierung präsentiert. Wir verallgemeinern diese Ergebnisse und wir studieren den $\mathbb{Z}_2$-graduierter Fall. Wir erhalten die Lösungen der Pentagon Gleichung aus der $\mathcal{U}_q(osp(1|2))$, der Quantensuperebene und dem Heisenberg-Doppel aus der Quantensuperebene.

List of publications derived from this dissertation:

1. L. Hadasz, M. Pawelkiewicz and V. Schomerus, *Self-dual Continuous Series of Representations for* $\mathcal{U}_q(sl(2))$ *and* $\mathcal{U}_q(osp(1|2))$, arXiv:1305.4596 [hep-th].

2. M. Pawelkiewicz, V. Schomerus and P. Suchanek, *The universal Racah-Wigner symbol for* $U_q(osp(1\text{---}2))$, JHEP **1404** (2014) 079 [arXiv:1307.6866 [hep-th]].

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