

Contractible edges in locally finite graphs

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Chapter 1

Introduction

The study of contractible edges in graphs was initiated by Tutte in 1961 who proved that every 3-connected finite graph of order at least five contains a contractible edge. From that point forward, research on contractible edges have blossomed into various areas: investigating the distribution of contractible edges, studying properties of contraction-critical graphs, discovering sufficient conditions for their existence, and determining which subgraphs contain contractible edges. Contractible edges are extremely useful in inductive arguments for proving a number of graph theory results such as Kuratowski's characterization of planar graphs and Lovász' conjecture that every $(k + 3)$ -connected graph contains a cycle whose deletion results in a k -connected graph. Almost all results about contractible edges were derived solely for finite graphs due to the proof techniques available: induction, *reductio ad absurdum*, and the theory of atoms and ends. Also, 2-connected graphs were often omitted because of their apparent simplicity although some results were obtained for 2-connected matroids.

The purpose of this dissertation is to rectify these situations and provide a deeper understanding of contractible edges in general. First, it fills in the remaining gap for results concerning contractible edges in 2-connected finite graphs. In Chapter 2, the distribution of contractible edges in spanning trees, longest cycles, longest paths and maximum matchings in 2-connected finite graphs is studied. Chapter 3 focuses on vertex covers of contractible edges in 2-connected graphs. Second, results concerning contractible edges in k -connected graphs ($k \geq 3$) are extended to locally finite infinite graphs. In Chapter 4, several results in Chapter 2 are generalized to locally finite infinite 2-connected graphs while contraction-critical 2-connected infinite graphs are studied as well. Chapter 5 deals with the more traditional topic of 3-connected graphs where the structure and distribution of contractible edges around a vertex of finite degree are investigated. Chapter 6 extends the well known characterization of contraction-critical 4-connected graphs by Fontet and Martinov to locally finite infinite graphs. Finally, sufficient conditions for the existence of contractible edges in k -connected locally finite infinite graphs were obtained in Chapter 7.

Chapter 2

Contractible edges in subgraphs of 2-connected finite graphs

2.1 Introduction

The study of contractible edges started with the work of Tutte [15] who proved that every 3-connected finite graph nonisomorphic to K_4 contains a contractible edge. Further results on the number of contractible edges and non-contractible edges in terms of the order and size of a graph were obtained. Ando et al. [1] proved that every 3-connected finite graph G nonisomorphic to K_4 has at least $\frac{|V(G)|}{2}$ contractible edges and characterized all the extremal graphs. Ota [17] proved that every 3-connected finite graph G of order at least 19 has at least $\frac{2|E(G)|+12}{7}$ contractible edges and determined all the extremal graphs. Egawa et al. [13] showed that the number of non-contractible edges in a 3-connected finite graph G nonisomorphic to K_4 is at most $3|V(G)| - \lfloor \frac{3}{2}(\sqrt{24|V(G)| + 25} - 5) \rfloor$.

The existence of contractible edges in certain types of subgraphs in 3-connected finite graphs was also investigated. For any 3-connected finite graphs of order at least seven, Dean et al. [7] proved that for any two distinct vertices x, y , every longest x - y path contains at least two contractible edges and that every longest cycle contains at least three contractible edges. Later, Aldred et al. [1, 2] characterized all 3-connected graphs with a longest path containing exactly two contractible edges and Aldred et al. [3] characterized all 3-connected graphs having a longest cycle containing exactly three contractible edges. Ellingham et al. [10] proved that every non-Hamiltonian 3-connected finite graph has at least six contractible edges in any longest cycle. For any 3-connected graph of order at least five, Fujita [12] proved that there exists a longest cycle C such that C contains at least $\frac{|V(C)|+9}{8}$ contractible edges, and later [13] improved the lower bound to $\frac{|V(C)|+7}{7}$. Maximum matchings were shown to contain a con-

tractible edge by Aldred et al. [4]. They [5] also characterized all 3-connected finite graphs with a maximum matching containing precisely one contractible edge. Recently, Elmasry et al. [11] proved that every depth-first search tree in a 3-connected finite graph nonisomorphic to K_4 contains a contractible edge.

For 2-connected finite graphs, several analogous results on contractible and non-contractible edges were known in the more general context of matroids. Let M be a simple 2-connected finite matroid with rank $r(M)$. Oxley [18] showed that M has at least $r(M) + 1$ contractible elements. Wu [16] characterized the extremal matroids to be precisely the matroids arising from 2-connected finite outerplanar graphs. Kahn and Seymour [15] proved that if M has rank at least two, then M has at least $|E(M)| - r(M) + 2$ contractible elements, and characterized all the matroids where equality holds. When restricted to graphs, these correspond to maximally outerplanar graphs. In Section 3, we will provide graph-theoretical proofs of the above and related results.

Section 4 deals with contractible edges in spanning trees in 2-connected finite graphs. From the above result of Kahn and Seymour, every spanning tree must contain at least two contractible edges. Those graphs having a spanning tree containing exactly two contractible edges are characterized. In Section 5, we study contractible edges in longest cycles and longest paths. It is easy to see that every edge in a longest cycle is contractible, and the first and last edges in any longest path between two given vertices are contractible. Furthermore, we characterize all the graphs with a longest path containing exactly two contractible edges to be the square of a path. For 2-connected non-hamiltonian finite graphs, every longest path is shown to contain at least four contractible edges which is best possible. We also prove that for any 2-connected finite graph nonisomorphic to K_3 , there exists a longest path P containing more than $|E(P)|/2$ contractible edges and this bound is asymptotically optimal. Lastly, in Section 6, every maximum matching is shown to contain a contractible edge. All 2-connected finite graphs with a maximum matching containing precisely one contractible edge are characterized. We also proved that for any 2-connected finite graph nonisomorphic to K_3 , there exists a maximum matching M that contains at least $2(|M| + 1)/3$ contractible edges and the bound is optimal.

2.2 Definitions

All basic graph-theoretical terminology can be found in Diestel [9]. Unless otherwise stated, all graphs G considered in this paper are simple and finite. For any vertex x in G , denote the set of neighbors of x by $N_G(x)$. Let A and B be two disjoint subsets of $V(G)$, define $E_G(A, B)$ to be the set of edges between A and B . The *square* of G , denoted by G^2 , is the graph on $V(G)$ where two vertices are adjacent if and only if they have distance at most two in G . A *matching* is a set of independent edges and a *maximum matching* is a matching of largest cardinality. Let M be a matching in G . An *M -alternating* path is a path whose edges alternate between M and $E(G) \setminus M$. An M -alternating path is called *M -augmenting* if the first and last vertices of the path are not incident

to any edges in M . Let H be a path or a cycle. Two chords x_1x_2 and y_1y_2 of H are *overlapping* if x_1, y_1, x_2, y_2 appear in this order in H .

Let G be a k -connected graph. An edge e of G is said to be *k-contractible* if the graph obtained by its contraction, G/e , is also k -connected. Otherwise, it is called *k-non-contractible*. Since this paper concerns only 2-connected graphs, we write 2-contractible as *contractible*. Let $G = (V, E)$ be a 2-connected graph. Denote the set of contractible edges and non-contractible edges in G by E_C and E_{NC} respectively. Define $G_C := (V, E_C)$ to be the subgraph induced by all the contractible edges and $G_{NC} := (V, E_{NC})$.

A graph is *outerplanar* if it can be embedded in the plane such that all the vertices lie on the boundary of one face. A graph is *maximally outerplanar* if it is outerplanar and the addition of any extra edge results in a non-outerplanar graph.

2.3 Contractible and non-contractible edges in 2-connected finite graphs

Here we group together all the major results concerning contractible and non-contractible edges in 2-connected finite graphs. Most results are well-known but a few are new. We start with three fundamental lemmas applicable to all 2-connected graphs nonisomorphic to K_3 .

Lemma 2.1. *Let G be a 2-connected graph nonisomorphic to K_3 . For every edge e of G , $G - e$ or G/e is 2-connected.*

Lemma 2.2. *Let G be a 2-connected graph nonisomorphic to K_3 . Let e and f be two distinct non-contractible edges of G . Then f is non-contractible in $G - e$.*

Lemma 2.3. *Let G be a 2-connected graph nonisomorphic to K_3 . Let e be a non-contractible edge of G and f be a contractible edge of G . Then f is contractible in $G - e$.*

Proof. Suppose f is non-contractible in $G - e$. Since f is contractible in G , $G - e - V(f)$ has exactly two components, say C and D , and e joins C and D in G . Note that $V(e) \cap V(f) = \emptyset$. Denote the endvertex of e in C by c and the endvertex of e in D by d . Every c - d path except e intersects the endvertices of f . Consider a component B of $G - V(e)$ not containing f . Then $G[B \cup e] - e$ contains a c - d path not intersecting $V(f)$, a contradiction. \square

Using Lemma 2.1 and 2.2, we can prove the following lemma.

Lemma 2.4. *Let G be a 2-connected graph nonisomorphic to K_3 and F be a finite subset of $E(G)$.*

- (a) *If $G - F$ is disconnected, then F contains at least two contractible edges.*
- (b) *If $G - F$ is connected but not 2-connected, then F contains at least one contractible edge.*

As a corollary, we have:

Lemma 2.5. *Let G be a 2-connected graph nonisomorphic to K_3 . Let $\{x, y\}$ be a 2-separator of G and C be a component of $G - x - y$. If $|E_G(x, C)|$ is finite, then $E_G(x, C)$ contains a contractible edge.*

Lemma 2.4 implies that for any 2-connected finite graph nonisomorphic to K_3 , every vertex is incident to at least two contractible edges. Hence, the number of contractible edges is at least the number of vertices. The 2-connected graphs satisfying the lower bound were characterized by Wu [16] to be outerplanar graphs. Since Wu's work concerns simple 2-connected finite matroids, we give a graph-theoretical proof below. This requires the following theorem which can be proved easily by Lemma 2.1 and 2.2.

Theorem 2.6. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then the subgraph G_C induced by all the contractible edges is 2-connected.*

Proof. By Lemma 2.1 and 2.2, we can repeatedly delete all the non-contractible edges so that the resulting graph G_C is 2-connected. \square

Theorem 2.7 (Wu [16]). *Every 2-connected finite graph G nonisomorphic to K_3 has at least $|V(G)|$ contractible edges. The equality holds if and only if G is outerplanar.*

Proof. If G is outerplanar, then G consists of a Hamilton cycle with non-overlapping chords. The edges in the Hamilton cycle are the only contractible edges and the equality holds. Suppose the equality holds. Then by Lemma 2.4, $|V(G)| = |E_C| = \frac{1}{2} \sum_{x \in V(G)} |E_C(x)| \geq \frac{1}{2} \sum_{x \in V(G)} 2 = |V(G)|$. Therefore, every vertex of G is incident to exactly two contractible edges. By Theorem 2.6, G_C is a Hamilton cycle of G . All edges of G outside G_C are chords of G_C and are non-contractible. This implies that no chords of G_C are overlapping. Hence, G is outerplanar. \square

There is also a similar result concerning non-contractible edges in 2-connected finite graphs. As noted in the Introduction, this was already proved by Kahn and Seymour [15]. Here we will adopt Kriesell [7]'s arguments.

Theorem 2.8. *Every 2-connected finite graph G nonisomorphic to K_3 has at most $|V(G)| - 3$ non-contractible edges. The equality holds if and only if G is maximally outerplanar.*

Proof. The first part was proved by Kriesell [7]. Here we prove the second part using the same inductive arguments. (\Leftarrow) Obvious. (\Rightarrow) For $|V(G)| = 4$, the result is true. Consider a non-contractible edge xy in G . Let C_1 be a component of $G - \{x, y\}$ and $C_2 := G - \{x, y\} - C_1$. Suppose $|C_2| = 1$ and let $V(C_2) := \{a\}$. Then $\deg_G(a) = 2$, and ax and ay are contractible in G by Lemma 2.1. Also, $G - a$ is 2-connected and xy is a contractible edge in $G - a$. Therefore, $G - a$ has $|V(G)| - 4$ non-contractible edges. By induction hypothesis, $G - a$ is maximally outerplanar and so is G .

Now, assume $|C_1| > 1$ and $|C_2| > 1$. Let G_i be the graph obtained from G by contracting C_{3-i} to a vertex a_i for $i = 1, 2$. Then G_1 and G_2 are 2-connected and xy is a non-contractible edge in both G_1 and G_2 . By the first part of the theorem, G_i has at most $|V(G_i)| - 3$ non-contractible edges. Since $|V(G)| - 3 = |E_{NC}(G)| = |E_{NC}(G_1)| + |E_{NC}(G_2)| - 1 \leq |V(G_1)| - 3 + |V(G_2)| - 3 - 1 = |V(C_1)| + |V(C_2)| - 1 = |V(G)| - 3$, each G_i has exactly $|V(G_i)| - 3$ non-contractible edges. By induction hypothesis, each G_i is maximally outerplanar and so is G . \square

Combining Theorem 2.7 and 2.8, we obtain a lower bound for the number of contractible edges and an upper bound for the number of non-contractible edges in terms of the size of a graph.

Theorem 2.9. *Every 2-connected finite graph G nonisomorphic to K_3 has at least $\frac{|E(G)|+3}{2}$ contractible edges and at most $\frac{|E(G)|-3}{2}$ non-contractible edges. In both cases, the equality holds if and only if G is maximally outerplanar.*

Proof. By Theorem 2.7 and 2.8, $|V(G)| \leq |E_C|$ and $|E_{NC}| \leq |V(G)| - 3 \leq |E_C| - 3$. Therefore, $2|E_{NC}| + 3 \leq |E(G)| = |E_C| + |E_{NC}| \leq 2|E_C| - 3$. We have $|E_C| \geq \frac{|E(G)|+3}{2}$ and $|E_{NC}| \leq \frac{|E(G)|-3}{2}$. In both cases, the equality holds if and only if $|E_C| = |V(G)|$ and $|E_{NC}| = |V(G)| - 3$ which is equivalent to G being maximally outerplanar by Theorem 2.7 and 2.8. \square

Theorem 2.7 gives a characterization of 2-connected finite graphs of order at least four having exactly $|V(G)|$ contractible edges. Here we characterize all 2-connected finite graphs G having exactly $|V(G)| + 1$ and $|V(G)| + 2$ contractible edges.

Theorem 2.10. *Let G be a 2-connected finite graph. Then G contains exactly $|V(G)| + 1$ contractible edges if and only if G consists of two vertices joined by three internally disjoint paths each of length at least two together with non-overlapping chords on each path.*

Proof. Suppose G contains exactly $|V(G)| + 1$ contractible edges. Then $\sum_{x \in V(G)} |E_C(x)| = 2|E_C| = 2|V(G)| + 2$. By Lemma 2.4, $|E_C(x)| \geq 2$ for all $x \in V(G)$. Therefore, G_C contains exactly two vertices of degree three with the remaining vertices of degree two. Hence, G_C consists of two vertices x, y joined by three internally disjoint paths, say P_1, P_2, P_3 . Since all edges not in G_C are non-contractible, there are no edges joining \dot{P}_i and \dot{P}_j for $i \neq j$, the length of each P_i is at least two, and all chords on P_i are non-overlapping.

Suppose G consists of two vertices x, y joined by three internally disjoint paths P_1, P_2, P_3 each of length at least two together with non-overlapping chords on each path. Then G_C equals to $(V(G), E(P_1 \cup P_2 \cup P_3))$. Hence, $|E_C| = \frac{1}{2} \sum_{x \in V(G)} |E_C(x)| = \frac{1}{2}[2(|V(G)| - 2) + 3(2)] = |V(G)| + 1$. \square

Theorem 2.11. *Let G be a 2-connected finite graph. Then G contains exactly $|V(G)| + 2$ contractible edges if and only if G is one of the following graphs:*

1. G consists of two degree-4 vertices joined by four internally disjoint paths each of length at least two together with non-overlapping chords on each path.
2. G has one degree-4 vertex and two degree-3 vertices. The degree-4 vertex joins each degree-3 vertex by two internally disjoint paths each of length at least two. The two degree-3 vertices are joined by a path. All the paths are internally disjoint and each path has no overlapping chords.
3. G consists of a K_4 subdivision together with non-overlapping chords on each path between any two branch vertices of the K_4 subdivision.

Proof. Suppose G contains exactly $|V(G)|+2$ contractible edges. Then $\sum_{x \in V(G)} |E_C(x)| = 2|E_C| = 2|V(G)| + 4$. By Lemma 2.4, $|E_C(x)| \geq 2$ for all $x \in V(G)$. There are three possibilities: (1) G_C has two vertices of degree four and the rest of degree two; (2) G_C has one vertex of degree four, two vertices of degree three and the rest of degree two; (3) G_C has four vertices of degree three and the rest of degree two. In the first case, G_C consists of two vertices joined by four internally disjoint paths. The remaining edges of G , being non-contractible, can only be chords of the four paths and no two of which are overlapping. This also implies that each of the four paths are of length at least two. In the second case, since G_C is 2-connected, the degree-4 vertex in G_C joins each degree-3 vertex in G_C by two internally disjoint paths each of length at least two, and the two degree-3 vertices are joined by a path where all five paths are internally disjoint. The remaining edges of G , being non-contractible, can only be chords of the five paths, and no two chords on the same path are overlapping. In the third case, G_C is a K_4 subdivision. The remaining edges, being non-contractible, can only be chords on one of the six paths between two branch vertices in G_C . Again no two chords on the same path are overlapping.

Suppose G is one of the three graphs in the above list. It follows easily that $\sum_{x \in V(G)} |E_C(x)| = 2|V(G)| + 4$ and $|E_C| = |V(G)| + 2$. \square

2.4 Contractible edges in spanning trees

Another question that can be asked about contractible edges in a 2-connected finite graph is: How many contractible edges are there in certain types of subgraphs? By Theorem 2.6, every 2-connected finite graph nonisomorphic to K_3 contains a spanning tree consisting of contractible edges only. Theorem 2.8 implies that every spanning tree of a 2-connected finite graph nonisomorphic to K_3 contains at least two contractible edges. Below we characterize all 2-connected graphs having a spanning tree containing exactly two contractible edges.

Theorem 2.12. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then every spanning tree of G contains at least two contractible edges. Moreover, G has a spanning tree containing exactly two contractible edges if and only if G is maximally outerplanar and G_{NC} is acyclic.*

Proof. As noted above, the first part follows from Theorem 2.8. Suppose G has a spanning tree T containing exactly two contractible edges. Since $|E(T)| = |V(G)| - 1$ and $|E_{NC}| \geq |E(T)| - 2$, $|E_{NC}| \geq |V(G)| - 3$. By Theorem 2.8, $|E_{NC}| = |V(G)| - 3 = |E(T)| - 2$ and G is maximally outerplanar. Also, G_{NC} is acyclic as $E_{NC} \subseteq E(T)$.

Suppose G is maximally outerplanar and G_{NC} is acyclic. Then G being maximally outerplanar implies that $G[E_{NC}]$ is connected. Since G_{NC} is acyclic, $G[E_{NC}]$ is acyclic and hence is a tree of order $|E_{NC}| + 1 = |V(G)| - 2$ by Theorem 2.8. Now, $G[E_{NC}]$ can be extended to a spanning tree of G containing exactly two contractible edges. \square

Suppose l is the minimum number of contractible edges a spanning tree of G can contain. It is easy to show that there exists a spanning tree containing exactly k contractible edges for $l \leq k \leq |V(G)| - 1$.

Theorem 2.13. *Let G be a 2-connected finite graph nonisomorphic to K_3 and l be the minimum number of contractible edges a spanning tree of G contains. Then, for $l \leq k \leq |V(G)| - 1$, G has a spanning tree containing exactly k contractible edges.*

Proof. Suppose we have proved that G has a spanning tree T containing exactly k contractible edges. Let xy be a non-contractible edge in T . Denote the subtree of $T - xy$ containing x by T_x and that containing y by T_y . By Lemma 2.4, $E_G(T_x, T_y)$ contains a contractible edge, say uv . Then $T - xy + uv$ is a spanning tree containing exactly $k + 1$ contractible edges. By induction, the theorem follows. \square

2.5 Contractible edges in longest cycles and paths

Inspired by the papers of Dean et al. [7] and Aldred et al. [3], we also study contractible edges in longest paths and longest cycles in 2-connected finite graphs.

Lemma 2.14. *Let G be a 2-connected finite graph nonisomorphic to K_3 , and x, y be two vertices in G . Suppose $P := x_1x_2 \dots x_n$ is a longest x - y path in G ($x = x_1$ and $y = x_n$). If x_ix_{i+1} is non-contractible, then $G - x_i - x_{i+1}$ has exactly two components, each of which intersects P , and there is no $x_1Px_{i-1} - x_{i+2}Px_n$ path in $G - x_i - x_{i+1}$. In particular, x_1x_2 and $x_{n-1}x_n$ are contractible.*

Proof. Suppose $G - x_i - x_{i+1}$ contains a component C disjoint from P . Let y_i be a neighbor of x_i in C , y_{i+1} be a neighbor of x_{i+1} in C , and Q be a $y_i - y_{i+1}$ path in C . Then $P - x_ix_{i+1} + x_iy_i + y_iQy_{i+1} + y_{i+1}x_{i+1}$ is a x - y path longer than P which is impossible. \square

Lemma 2.14 immediately leads to the following theorem.

Theorem 2.15. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then the first and the last edges in a longest path in G are contractible, and all edges in a longest cycle in G are contractible.*

Proof. The first part follows from Lemma 2.14. Let C be a longest cycle in G . Suppose C contains a non-contractible edge xy . Let z be a neighbor of y in C other than x . Then $C - yz$ is a longest y - z path in G . By Lemma 2.14, yx is contractible, a contradiction. \square

As a natural step, we characterize all 2-connected finite graphs having a longest path containing exactly two contractible edges.

Theorem 2.16. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then G has a longest path containing exactly two contractible edges if and only if G is the square of a path.*

Proof. (\Leftarrow) Obvious. (\Rightarrow) Suppose $P := x_1x_2 \dots x_n$ is a longest path in G containing exactly two contractible edges. By Lemma 2.14, x_1x_2 and $x_{n-1}x_n$ are the only contractible edges in P . Note that $n \geq 4$. For $k = 1, 2, \dots, n-3$, define P_k to be the subpath $x_1x_2 \dots x_k$ of P and C_k to be the component of $G - x_{k+1} - x_{k+2}$ containing x_1 .

If $C_1 \neq x_1$, then there exists a vertex in C_1 , say y_1 , adjacent to x_1 . By applying Lemma 2.14 to x_2x_3 , $y_1 \notin P$ and $y_1x_1x_2 \dots x_n$ is a path longer than P , a contradiction. Therefore, $C_1 = x_1$ and $N_G(x_1) = \{x_2, x_3\}$.

If $C_2 \neq x_1x_2$, then since $N_G(x_1) = \{x_2, x_3\}$, there exists a vertex other than x_1 in C_2 , say y_2 , adjacent to x_2 . By applying Lemma 2.14 to x_3x_4 , $y_2 \notin P$ and $y_2x_2x_1x_3 \dots x_n$ is a path longer than P , a contradiction. Therefore, $C_2 = x_1x_2$ and $N_G(x_2) = \{x_1, x_3, x_4\}$ as $G - x_3$ is connected.

If $C_3 \neq P_3^2$, then since $N_G(x_1) = \{x_2, x_3\}$ and $N_G(x_2) = \{x_1, x_3, x_4\}$, there exists a vertex other than x_1, x_2 in C_3 , say y_3 , adjacent to x_3 . By applying Lemma 2.14 to x_4x_5 , $y_3 \notin P$ and $y_3x_3x_1x_2x_4 \dots x_n$ is a path longer than P , a contradiction. Therefore, $C_3 = P_3^2$ and $N_G(x_3) = \{x_1, x_2, x_4, x_5\}$ as $G - x_4$ is connected.

Suppose we have proved that for $k = 3, 4, \dots, m-1$, $C_k = P_k^2$ and $N_G(x_k) = \{x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}\}$. If $C_m \neq P_m^2$, then there exists a vertex other than x_1, \dots, x_{m-1} in C_m , say y_m , adjacent to x_m . If m is odd, by applying Lemma 2.14 to $x_{m+1}x_{m+2}$, $y_m \notin P$ and $y_mx_mx_{m-2} \dots x_1x_2x_4 \dots x_{m-1}x_{m+1} \dots x_n$ is a path longer than P , a contradiction. If m is even, by applying Lemma 2.14 to $x_{m+1}x_{m+2}$, $y_m \notin P$ and $y_mx_mx_{m-2} \dots x_2x_1x_3 \dots x_{m-1}x_{m+1} \dots x_n$ is a path longer than P , a contradiction. Therefore, $C_m = P_m^2$ and $N_G(x_m) = \{x_{m-2}, x_{m-1}, x_{m+1}, x_{m+2}\}$ as $G - x_{m+1}$ is connected.

By induction, for $k = 3, 4, \dots, n-3$, $C_k = P_k^2$ and $N_G(x_k) = \{x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}\}$. Since the component of $G - x_{n-2} - x_{n-1}$ other than C_{n-3} is x_n , for otherwise we can find a longer path than P , $G = P^2$. \square

Since the square of a path is Hamiltonian, the above theorem implies that every longest path in a 2-connected non-Hamiltonian finite graph contains at least three contractible edges. In fact, the correct lower bound is four. This is best possible as demonstrated by $K_{2,n}$ where $n \geq 3$.

Theorem 2.17. *Let G be a 2-connected non-Hamiltonian finite graph. Then every longest path contains at least four contractible edges.*

Proof. Suppose $P := x_1x_2 \dots x_n$ is a longest path in G containing exactly three contractible edges. By Lemma 2.14, x_1x_2 and $x_{n-1}x_n$ are contractible. Let x_kx_{k+1} be the third contractible edge in P . By arguing as in the proof of Theorem 2.16, we have $N_G(x_1) = \{x_2, x_3\}$, $N_G(x_2) = \{x_1, x_3, x_4\}$, $N_G(x_3) = \{x_1, x_2, x_4, x_5\}, \dots, N_G(x_{k-2}) = \{x_{k-4}, x_{k-3}, x_{k-1}, x_k\}$, $N_G(x_{k+3}) = \{x_{k+1}, x_{k+2}, x_{k+4}, x_{k+5}\}, \dots, N_G(x_{n-2}) = \{x_{n-4}, x_{n-3}, x_{n-1}, x_n\}$, $N_G(x_{n-1}) = \{x_{n-3}, x_{n-2}, x_n\}$, $N_G(x_n) = \{x_{n-2}, x_{n-1}\}$. By the maximality of P , $N_G(x_{k-1}) \subseteq P$ and $N_G(x_{k+2}) \subseteq P$. Since x_kx_{k+1} is contractible, $G - x_k - x_{k+1}$ is connected, and x_{k-1} and x_{k+2} are adjacent. Again by the maximality of P , $N_G(x_k) \subseteq P$ and $N_G(x_{k+1}) \subseteq P$. Now, $V(G) = V(P)$ and G is Hamiltonian, a contradiction. \square

Theorem 2.15 tells us that every longest path has at least two contractible edges but is it possible to find a longest path that contains many contractible edges? The following theorem provides an affirmative answer.

Theorem 2.18. *Let G be a 2-connected finite graph nonisomorphic to K_3 and P be a longest path in G containing as many contractible edges as possible. Then P has more than $|E(P)|/2$ contractible edges.*

Proof. By Theorem 2.15, the result is true if $|E(P)| \leq 4$. Therefore, we can assume $|E(P)| \geq 5$. Let $P := x_1x_2 \dots x_n$.

Claim 2.19. *The first four and last four edges of P are contractible.*

Proof. Suppose x_1x_2 is contractible and x_2x_3 is non-contractible. By the maximality of P and by applying Lemma 2.14 to x_2x_3 , $N_G(x_1) = \{x_2, x_3\}$. Then x_1x_3 is a contractible edge and $x_2x_1x_3Px_n$ has more contractible edges than P , a contradiction.

Suppose x_1x_2, x_2x_3 are contractible and x_3x_4 is non-contractible. Then by the maximality of P and by applying Lemma 2.14 to x_3x_4 , $N_G(x_1) \subseteq \{x_2, x_3, x_4\}$. Since x_2x_3 is contractible, $G - x_2 - x_3$ is connected and x_1 is adjacent to x_4 . Suppose x_1x_4 is non-contractible. By Lemma 4.6, there exists a contractible edge incident to x_1 , say x_1y , such that $y \notin \{x_2, x_3, x_4\}$ which is impossible. Therefore, x_1x_4 is contractible and $x_3x_2x_1x_4Px_n$ has more contractible edges than P , a contradiction.

Suppose x_1x_2, x_2x_3, x_3x_4 are contractible and x_4x_5 is non-contractible. Let C be the component of $G - x_4 - x_5$ containing x_1 . Then by the maximality of P and by applying Lemma 2.14 to x_4x_5 , $N_G(x_1) \subseteq \{x_2, x_3, x_4, x_5\}$. Suppose $x_5 \in N_G(x_1)$. By arguing as above, x_1x_5 is contractible and $x_4x_3x_2x_1x_5Px_n$ has more contractible edges than P , a contradiction. Therefore, $x_5 \notin N_G(x_1)$. Since x_2x_3 is contractible, $G - x_2 - x_3$ is connected and $x_1x_4 \in E(G)$. If x_1x_4 is non-contractible, then by Lemma 4.6, there exists a contractible edge x_1y such that $y \notin \{x_2, x_3, x_5\}$, a contradiction. Hence, x_1x_4 is contractible. Since x_3x_4 is contractible, there exists a x_2 - x_5 path Q . By applying Lemma 2.14 to x_4x_5 , Q lies in $G[(C \cup x_5) - \{x_1, x_3, x_4\}]$. Then $x_3x_4x_1x_2Qx_5Px_n$ is a longer path than P unless $Q = x_2x_5$. Suppose x_2x_5 is non-contractible. Let D be a component of $G - x_2 - x_5$ not containing x_1 . Then there exists a x_2 - x_5 path Q'

in $G[D \cup \{x_2, x_5\}]$ such that $|E(Q')| \geq 2$ which is impossible. Therefore, x_2x_5 is contractible. Now, $x_3x_4x_1x_2x_5Px_n$ has more contractible edges than P , a contradiction. \square

Claim 2.20. *Let x_ix_{i+1} and $x_{i+1}x_{i+2}$ be two consecutive non-contractible edges in P . Then x_ix_{i+2} is a contractible edge.*

Proof. By Lemma 2.14, $i > 1$ and $i + 2 < n$. Let C be the component of $G - x_i - x_{i+1}$ containing x_{i+2} . By Lemma 4.6, there exists a contractible edge x_iy_i such that $y_i \in C$. Let Q be a $y_i - x_{i+2}$ path in C . By applying Lemma 2.14 to $x_{i+1}x_{i+2}$, $Q \cap P = x_{i+2}$. Define $R := x_1Px_iy_iQx_{i+2}Px_n$. If $|E(Q)| \geq 2$, then R is a longer path than P , a contradiction. If $|E(Q)| = 1$, then R and P have the same length, but R has more contractible edges than P , a contradiction. Therefore, $|E(Q)| = 0$ and x_ix_{i+2} is a contractible edge. \square

Claim 2.21. *There are no three consecutive non-contractible edges in P .*

Proof. Suppose there are three consecutive non-contractible edges x_ix_{i+1} , $x_{i+1}x_{i+2}$ and $x_{i+2}x_{i+3}$ in P . By Claim 4.22, x_ix_{i+2} and $x_{i+1}x_{i+3}$ are contractible edges. But then $x_1Px_ix_{i+2}x_{i+1}x_{i+3}Px_n$ has more contractible edges than P , a contradiction. \square

Below we will represent contractible and non-contractible edges in P using the following notation. For example, $x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5} := CNCNN$ denotes that x_ix_{i+1} and $x_{i+2}x_{i+3}$ are contractible, and $x_{i+1}x_{i+2}$, $x_{i+3}x_{i+4}$ and $x_{i+4}x_{i+5}$ are non-contractible. Note that NNN is impossible in P by Claim 4.23.

Claim 2.22. *There is no $NN(CN)_kN$ in P .*

Proof. The case $k = 0$ is Claim 4.23. Suppose $x_ix_{i+1} \dots x_{i+2k+2}x_{i+2k+3} := NN(CN)_kN$ appears in P where $k \geq 1$. Since $x_{i+2j}x_{i+2j+1}$ is contractible ($1 \leq j \leq k$), $G - x_{i+2j} - x_{i+2j+1}$ is connected and contains a $x_1Px_{i+2j-1}x_{i+2j+2}Px_n$ path internally disjoint from P , denoted by Q_j . By applying Lemma 2.14 to $x_{i+1}x_{i+2}$, $x_{i+3}x_{i+4}$, \dots , $x_{i+2k+1}x_{i+2k+2}$, for $1 \leq j \leq k$, $Q_j \cap P = \{x_{i+2j-1}, x_{i+2j+2}\}$ and all Q_j 's are pairwise disjoint. Consider $P' := x_1Px_ix_{i+2}x_{i+1}Q_1x_{i+4}x_{i+3}Q_2x_{i+6} \dots x_{i+2k-3}Q_{k-1}x_{i+2k}x_{i+2k-1}Q_kx_{i+2k+2}x_{i+2k+1}x_{i+2k+3}Px_n$. By the maximality of P , all Q_j 's are contractible edges. But then P' has more contractible edges than P , a contradiction. \square

Claim 2.23. *Every $2k+1$ consecutive edges in P contain at least k contractible edges.*

Proof. For $k = 0$, it is trivial. By Claim 4.23, $k = 1$ is true. Suppose we have proved that for all $0 \leq l \leq k$, every $2l+1$ consecutive edges in P contain at least l contractible edges. Consider any $2k+3$ consecutive edges in P , say $Q := x_ix_{i+1} \dots x_{i+2k+3}$. Assume Q contains only k contractible edges. Since $x_{i+1}Qx_{i+2k+2}$ contains at least k contractible edges, x_ix_{i+1} and

$x_{i+2k+2}x_{i+2k+3}$ are non-contractible. Since x_iQx_{i+2k+1} contains at least k contractible edges, $x_{i+2k+1}x_{i+2k+2}$ is non-contractible. Similarly, $x_{i+1}x_{i+2}$ is non-contractible. By applying Claim 4.24 to x_iQx_{i+3} , $x_{i+2}x_{i+3}$ is contractible. Similarly, $x_{i+2k}x_{i+2k+1}$ is contractible.

Suppose we have proved that the first $2l + 1$ edges of Q are of the form: $N(NC)_l$ and the last $2l + 1$ edges of Q are of the form: $(CN)_lN$. By considering contractible edges in $x_iQx_{i+2(k-l)+1}$, $x_{i+2(k-l)+1}x_{i+2(k-l)+2}$ is non-contractible. Similarly, $x_{i+2l+1}x_{i+2l+2}$ is non-contractible. By applying Claim 4.24 to x_iQx_{i+2l+3} , $x_{i+2l+2}x_{i+2l+3}$ is contractible. Similarly, $x_{i+2(k-l)}x_{i+2(k-l)+1}$ is contractible. Therefore, by induction, $Q = NN(CN)_kN$, which is impossible by Claim 4.24. \square

Claim 2.24. P has more than $|E(P)|/2$ contractible edges.

Proof. This immediately follows from Claim 4.21 and Claim 4.25. \square

Finally, the bound in Theorem 2.18 is asymptotically best possible as demonstrated by the family of graphs, H_k ($k \geq 0$), constructed below. Define $V(H_k) := \{x_1, x_2, \dots, x_{2k+10}\}$ and $E(H_k) := \bigcup_{i=1}^{2k+9} x_i x_{i+1} \cup \{x_1 x_4, x_2 x_6, x_{2k+5} x_{2k+9}, x_{2k+7} x_{2k+10}\} \cup \bigcup_{i=1}^k x_{2i+3} x_{2i+6}$. It is not difficult to see that the longest path of H_k is either $x_1 x_2 \dots x_{2k+10}$ or $(x_1 x_4 x_3 x_2 / x_3 x_4 x_1 x_2) x_6 x_5 x_8 x_7 \dots x_{6+4i} x_{5+4i} x_{8+4i} x_{7+4i} \dots x_{2k+4} x_{2k+3} x_{2k+6} x_{2k+5} (x_{2k+9} x_{2k+8} x_{2k+7} x_{2k+10} / x_{2k+9} x_{2k+10} x_{2k+7} x_{2k+8})$, and has the contractible/non-contractible edge pattern: $CCCCN(CN)_kCCCC$.

2.6 Contractible edges in maximum matchings

Here, we prove several results concerning contractible edges in maximum matching. First, it is shown that every maximum matching in a 2-connected finite graph nonisomorphic to K_3 contains a contractible edge.

Lemma 2.25. *Let G be a 2-connected finite graph nonisomorphic to K_3 and M be a matching in G such that all of its edges are non-contractible. Then for every edge e in M , there exists an M -augmenting path containing e .*

Proof. Denote e by $x_0 y_0$. Let X and Y be two components of $G - x_0 - y_0$. Let x_1 be a neighbor of x_0 in X and y_1 be a neighbor of y_0 in Y . Note that $x_0 x_1 \notin M$, $y_0 y_1 \notin M$ and $x_1 x_0 y_0 y_1$ is an M -alternating path. Let $P := x_{2k+1} x_{2k} \dots x_1 x_0 y_0 y_1 \dots y_{2l} y_{2l+1}$ be a longest M -alternating path such that $x_{2k+1} x_{2k} \notin M$ and $y_{2l} y_{2l+1} \notin M$. If $x_{2k+1} \in V(M)$, then there exists an edge in M incident to x_{2k+1} , say $x_{2k+1} x_{2k+2}$. Since $x_{2k+1} x_{2k+2}$ is non-contractible, x_{2k+2} is adjacent to a vertex not in P , say x_{2k+3} . But then $x_{2k+3} x_{2k+2} x_{2k+1} x_{2k} \dots x_1 x_0 y_0 y_1 \dots y_{2l} y_{2l+1}$ is an M -alternating path longer than P such that $x_{2k+3} x_{2k+2} \notin M$ and $y_{2l} x_{2l+1} \notin M$, a contradiction. Hence, $x_{2k+1} \notin V(M)$. Similarly, $y_{2l+1} \notin V(M)$. Therefore, P is an M -augmenting path containing e . \square

Since an M -augmenting path enables one to construct a larger matching than M , Lemma 2.25 immediately implies the following.

Theorem 2.26. *Every maximum matching in a 2-connected finite graph nonisomorphic to K_3 contains a contractible edge.*

Next, we characterize all 2-connected finite graphs with a maximum matching containing precisely one contractible edge. For such purpose, we define the following type of graphs $R_n (n \geq 1)$ with $V(R_n) := \{x_0, y_0, x_1, y_1, \dots, x_n, y_n, z\}$ and $E(R_n) := \{x_i y_i, x_i x_{i+1}, y_i y_{i+1} : 0 \leq i \leq n-1\} \cup \{x_n y_n, x_n z, y_n z\} \cup F$ where $F \subseteq \{x_i y_{i+1}, y_i x_{i+1} : 0 \leq i \leq n-1\}$.

Theorem 2.27. *Let G be a 2-connected finite graph. Then G has a maximum matching containing precisely one contractible edge if and only if $G \cong R_n$.*

Proof. (\Leftarrow) $\{x_i y_i : 0 \leq i \leq n\}$ is the desired matching. (\Rightarrow) Let M be a maximum matching containing precisely one contractible edge $x_0 y_0$. Since M contains a contractible edge, G has at least four vertices. There exists two distinct vertices x_1 and y_1 such that x_1 is adjacent to x_0 and y_1 is adjacent to y_0 . Note that $x_0 x_1 \notin M$ and $y_0 y_1 \notin M$.

We claim that $x_1 y_1 \in M$ and $N_G(x_0, y_0) = \{x_1, y_1\}$. There are three cases to consider. (1) If $x_1 \notin V(M)$ and $y_1 \notin V(M)$, then $x_1 x_0 y_0 y_1$ is an M -augmenting path, contradicting M being maximum. (2) If either $x_1 \notin V(M)$ and $y_1 \in V(M)$ or $x_1 \in V(M)$ and $y_1 \notin V(M)$, then as in the proof of Lemma 2.25, we can construct an M -augmenting path, a contradiction. (3) Suppose $x_1 \in V(M)$ and $y_1 \in V(M)$. Let $x_1 x'_1$ and $y_1 y'_1$ be the edges in M incident to x_1 and y_1 respectively. Again as in the proof of Lemma 2.25, we can construct an M -augmenting path unless $x_1 x'_1 = x_1 y_1 = y_1 y'_1$. Hence, $x_1 y_1$ is an edge in M . Suppose there exists a vertex u in $N_G(x_0, y_0)$ other than x_1 and y_1 , and without loss of generality, assume $u \in N_G(x_0)$. Consider the M -alternating path $u x_0 y_0 y_1 x_1$. Then we can construct an M -augmenting path as in the proof of Lemma 2.25, which is impossible. Therefore, $N_G(x_0, y_0) = \{x_1, y_1\}$.

Consider $x_1 y_1$. Then $G - x_1 - y_1$ has exactly two components for otherwise, there exists an M -augmenting path by the construction in Lemma 2.25. Denote the two components by C_1 and D_1 , and without loss of generality, assume $C_1 := x_0 y_0$. If $|V(D_1)| = 1$, then $G \cong R_1$. If $|V(D_1)| > 1$, then there exists two distinct vertices x_2 and y_2 such that x_2 is adjacent to x_1 and y_2 is adjacent to y_1 . By arguments as above, $x_2 y_2 \in M$ and $N_G(x_1, y_1) = \{x_0, y_0, x_2, y_2\}$. We can continue this process with $x_2 y_2, x_3 y_3, \dots$ and prove that $G \cong R_n$. \square

Note that R_n contains not only a maximum matching with exactly one contractible edge but also a maximum matching all of whose edges are contractible. It is natural to ask whether every 2-connected finite graph nonisomorphic to K_3 contains a maximum matching with many contractible edges. The answer is given by Theorem 2.29 below, and we need a result by Grossman and Häggkvist [14] concerning properly colored cycles in edge-colored graphs. A cycle is *properly colored* if adjacent edges have different colors.

Theorem 2.28 (Grossman and Häggkvist [14]). *Let G be a 2-connected finite graph with its edges colored by two colors. If every vertex is incident to at least one edge of each color, then G has a properly colored cycle.*

Theorem 2.29. *Let G be a 2-connected finite graph nonisomorphic to K_3 and M be a maximum matching that contains as many contractible edges as possible. Then M contains at least $2(|M| + 1)/3$ contractible edges.*

Proof. First, define $M_{NC} := M \cap E_{NC}$ and $M_C := M \cap E_C$. We say that a subgraph H in G has property $(*)$ if for each edge $e \in M_{NC} \cap H$, $H - V(e)$ is connected. Define \mathcal{H} to be the set of all maximal induced 2-connected subgraphs in G having property $(*)$.

Claim 2.30. *Every vertex and edge in G belongs to at least one element of \mathcal{H} .*

Proof. Consider a shortest cycle C containing the vertex or the edge. Then C has property $(*)$. \square

Claim 2.31. *Every edge $e \in M_{NC}$ belongs to at least two elements of \mathcal{H} .*

Proof. Let C_1 and C_2 be two components of $G - V(e)$. Consider a shortest cycle D_i in $G[C_i \cup e]$ containing e . Then D_i has property $(*)$. But no element of \mathcal{H} contains both D_1 and D_2 since e is non-contractible. \square

Claim 2.32. *Let H_1 and H_2 be two distinct elements of \mathcal{H} such that $H_1 \cap H_2 \neq \emptyset$. Then $H_1 \cap H_2$ is an edge in M_{NC} .*

Proof. Suppose $|H_1 \cap H_2| = 1$ and let $x := H_1 \cap H_2$. Then there exists a shortest H_1 - H_2 path in $G - x$, say P . Let $x_1 := P \cap H_1$ and $x_2 := P \cap H_2$. Since $H' := G[H_1 \cup H_2 \cup P]$ is 2-connected and does not belong to \mathcal{H} , there exists an edge $e \in H' \cap M_{NC}$ such that $H' - V(e)$ is not connected. This implies that $x \in V(e)$ and $V(e) \cap P \neq \emptyset$. Let $y := V(e) \cap P$. But, then both $G[H_1 \cup x_1Py]$ and $G[H_2 \cup x_2Py]$ are 2-connected with property $(*)$, a contradiction.

Suppose $|H_1 \cap H_2| \geq 2$. Then $G[H_1 \cup H_2]$ has property $(*)$ unless $H_1 \cap H_2$ is an edge in M_{NC} . \square

Now, define the auxillary bipartite graph A with the bipartite vertex sets \mathcal{H} and M_{NC} respectively such that there exists an edge between $H \in \mathcal{H}$ and $e \in M_{NC}$ in A if and only if $e \in H$.

Claim 2.33. *A is a tree.*

Proof. First, we show that A is connected. By Claim 2.31, without loss of generality, it suffices to prove that for any $H_1, H_2 \in \mathcal{H}$, there is a path between H_1 and H_2 in A . For $H_1 = H_2$, it is trivial. For $H_1 \cap H_2 \neq \emptyset$, it is true by Claim 2.32. For $H_1 \cap H_2 = \emptyset$, let $P := x_1x_2 \dots x_k$ be a H_1 - H_2 path in G . By Claim 2.30, every edge $x_i x_{i+1}$ belongs to an element of \mathcal{H} , say G_i . Note that $G_1 := H_1$ and $G_{k-1} := H_2$. By Claim 2.32, if $G_i \neq G_{i+1}$, then $G_i \cap G_{i+1} \in M_{NC}$. Therefore, there exists a path between H_1 and H_2 in A .

Next, we show that A is acyclic. Suppose there is a cycle in A , say $H_1 e_1 H_2 e_2 \dots H_k e_k H_1$. But then $G[H_1 \cup H_2 \cup \dots \cup H_k]$ has property $(*)$, a contradiction. \square

Claim 2.34. *For any $H \in \mathcal{H}$, H is not K_3 .*

Proof. Suppose H is K_3 with vertices x, y, z . Since G is not isomorphic to K_3 , there exists a $H' \in \mathcal{H}$ other than H such that $H \cap H' \neq \emptyset$. By Claim 2.32, one edge of H , say xy , belongs to M_{NC} . This means that H is a leaf in A , and xz and yz are contractible in G . But then $M - xy + xz$ contains more contractible edges than M , a contradiction. \square

Claim 2.35. *Let $H \in \mathcal{H}$ and e be an edge in H . If e is non-contractible in H , then e is non-contractible in G . If e is contractible in H , then either e is contractible in G or $e \in M_{NC}$.*

Proof. Suppose e is contractible in G and $H - V(e)$ is non-connected. Let C_1 and C_2 be two components of $H - V(e)$. Since $G - V(e)$ is connected, there exists a shortest C_1 - C_2 path in $G - V(e)$, say P . But then $G[H \cap P]$ has property (*), a contradiction.

Suppose e is contractible in H and non-contractible in G . Let C be a component of $G - V(e)$ not containing $H - V(e)$. Let D be a shortest cycle in $G[C \cup e]$ containing e , and H' be an element of \mathcal{H} containing D . Obviously, $H \neq H'$. By Claim 2.32, $e \in M_{NC}$. \square

Claim 2.36. $|\mathcal{H}| \geq |M_{NC}| + 1$.

Proof. By Claim 2.33 and 2.31, we have $2(|\mathcal{H}| + |M_{NC}| - 1) = 2(|V(A)| - 1) = \sum_{H \in \mathcal{H}} \deg_A(H) + \sum_{e \in M_{NC}} \deg_A(e) = 2 \sum_{e \in M_{NC}} \deg_A(e) \geq 4|M_{NC}|$. Therefore, $|\mathcal{H}| \geq |M_{NC}| + 1$. \square

Claim 2.37. *For each $H \in \mathcal{H}$, H contains at least two edges in M_C .*

Proof. Suppose H contains at most one edge in M_C . Since H is not K_3 by Claim 2.34, by applying Lemma 2.2 and 2.3 to H , we can delete all non-contractible edges in H so that the resulting graph H' is 2-connected and all edges in H' are contractible in H' . Note that by Claim 2.35, every edge in H' is contractible in G or belongs to M_{NC} . By the definition of (*) and Claim 2.35, none of the edges in $M \cap H$ are deleted. Consider any vertex x in H' . Suppose x is incident to an edge in M , say xy . If $y \notin H'$, then xy belongs to an element of \mathcal{H} other than H , say I . By Claim 2.32, $I \cap H$ is an edge in M_{NC} incident to x , which is impossible. Therefore, any edge in M incident to a vertex in H' lies in H' .

We claim that every vertex in H' is incident to an edge in $M \cap H'$. Suppose x is a vertex in H' not incident to any edges in $M \cap H'$. Let y be any neighbor of x in H' . Therefore, xy is contractible in G . By the maximality of M , y is incident to an edge in $M \cap H'$, say yz . If $yz \in M_{NC}$, then $M - yz + xy$ contains more contractible edges than M , a contradiction. Hence, $yz \in M_C$. Since y is an arbitrary neighbor of x and H' contains at most one edge in M_C , this implies that y and z are the only neighbors of x in H' . But then yz is non-contractible in H' , a contradiction.

Summing up, every edge in H' belongs to either M or $E_C \setminus M$, and every vertex in H' is incident to an edge in $M \cap H'$ and an edge in $(E_C \setminus M) \cap H'$.

By Theorem 2.28, there exists a cycle $x_1x_2 \dots x_{2k}x_1$ in H' such that $F := \{x_1x_2, x_3x_4, \dots, x_{2k-1}x_{2k}\} \subseteq M$ and $F' := \{x_2x_3, x_4x_5, \dots, x_{2k}x_1\} \subseteq E_C \setminus M$. Since H contains at most one edge in M_C , $M - F + F'$ has more contractible edges than M , a contradiction. \square

Claim 2.38. *M contains at least $2(|M| + 1)/3$ contractible edges.*

Proof. By Claim 2.37 and 2.36, $|M_C| \geq 2|\mathcal{H}| \geq 2(|M_{NC}| + 1)$. Therefore, $3|M_C| \geq 2(|M_C| + |M_{NC}| + 1) = 2(|M| + 1)$. \square

Lastly, the bound in Theorem 2.29 is best possible as demonstrated by the family of graphs below. The building blocks are cycle of length four, C_4 , and K_2 . Define $V(C_4^i) := \{x_1^i, y_1^i, x_2^i, y_2^i\}$ and $E(C_4^i) := \{x_1^iy_1^i, y_1^ix_2^i, x_2^iy_2^i, y_2^ix_1^i\}$, and $V(K_2^i) := \{z_1^i, z_2^i\}$ and $E(K_2^i) := \{z_1^iz_2^i\}$. Now, we construct the family of graphs G_n inductively. Define $V(G_1) := V(C_4^1) \cup V(K_2^1) \cup V(C_4^2)$ and $E(G_1) := E(C_4^1) \cup E(K_2^1) \cup E(C_4^2) \cup \{x_1^1z_1^1, x_2^1z_2^1, z_1^1x_1^2, z_2^1x_2^2\}$. Suppose we have constructed G_n . Define $V(G_{n+1}) := V(G_n) \cup V(K_2^{n+1}) \cup V(C_4^{n+2})$ and $E(G_{n+1}) := E(G_n) \cup E(K_2^{n+1}) \cup E(C_4^{n+2}) \cup \{x_1^{n+1}z_1^{n+1}, x_2^{n+1}z_2^{n+1}, z_1^{n+1}x_1^{n+2}, z_2^{n+1}x_2^{n+2}\}$. Notice that any maximum matching of G_n is in fact a perfect matching, and must contain two independent edges of every C_4 and all the K_2 's.

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Chapter 3

Covering contractible edges in 2-connected graphs

Covers for contractible edges in 3-connected graphs were first studied by Ota and Saito [3] who proved that the set of contractible edges $E_C(G)$ in a 3-connected graph G of order at least six cannot be covered by two vertices (see also Saito [4]). Later, Hemminger and Yu [2] characterized all 3-connected graphs of order at least ten whose contractible edges can be covered by three vertices. Yu [5] showed that for any 3-connected graph G nonisomorphic to K_4 , if S covers $E_C(G)$ such that $|V(G)| \geq 3|S| - 1$, then $G - S$ is not connected. Hemminger and Yu [1] provided upper bounds for the order, size and number of non-c-components of $G - S$ (refer to the paper for the definition) in terms of $|S|$. Inspired by the above work, we prove the corresponding results for 2-connected graphs.

All graphs considered in this note are finite and simple. Consider any 2-connected graph G . An edge is *contractible* if its contraction results in a 2-connected graph. Denote the set of contractible edges of G by $E_C(G)$. Let S be a subset of $V(G)$. A component of $G - S$ is *trivial* if its order is one. A *fragment* of S is a union of at least one but not all components of $G - S$. Denote the vertex set, edge set and component set of all non-trivial components of $G - S$ by $VN(G, S)$, $EN(G, S)$ and $CN(G, S)$ respectively. We say S is a *cover* of $E_C(G)$ if every contractible edge in G is incident to a vertex in S . For any two disjoint subsets A and B of $V(G)$, denote $E_G(A, B)$ to be the set of all edges between A and B in G . Consider the complete bipartite graph $K_{2,k}$ and let $\{x, y\}$ be the partition class of the two vertices. Define $K_{2,k}^+ := K_{2,k} + xy$. Also, we define the following construction of a new 2-connected graph based on G which will be useful later. For each edge e in a subset F of $E(G)$, add a vertex x_e together with two edges from x_e to $V(e)$. Denote the resulting graph by $G \# F$.

We begin with two basic results concerning contractible edges in 2-connected graphs.

Lemma 3.1. *Let G be any 2-connected graph nonisomorphic to K_3 and e be*

an edge of G . Then $G - e$ or G/e is 2-connected.

Lemma 3.2. *Let G be any 2-connected graph nonisomorphic to K_3 , and e and f be two non-contractible edges of G . Then f is a non-contractible edge of $G - e$.*

By the above two fundamental lemmas, every vertex of G is incident to at least two contractible edges and hence $|V(G)| \leq |E_C(G)|$. Also, the subgraph induced by all the contractible edges $(V(G), E_C(G))$ is 2-connected.

Lemma 3.3. *Consider any 2-connected graph G nonisomorphic to K_3 . Let x, y be any two vertices of G and C be a component of $G - x - y$. Then $E_G(x, C)$ contains a contractible edge and so does $E_G(y, C)$. Moreover, if $|C| > 1$, then there exist two independent contractible edges in $E_G(\{x, y\}, C)$.*

Proof. Suppose all edges in $E_G(x, C)$ are non-contractible. By Lemma 4.3 and 4.4, we can delete all edges in $E_G(x, C)$ and the resulting graph $H := G - E_G(x, C)$ is 2-connected. However, either x is an isolated vertex of H or y is a cutvertex of H , a contradiction.

Now, assume $|C| > 1$. Suppose $E_G(\{x, y\}, C) \cap E_C(G)$ can be covered by a vertex z in C . From above, xz and yz are contractible edges. By the 2-connectedness of G , there exists an edge joining $\{x, y\}$ to a vertex w of C other than z . Without loss of generality, assume w is adjacent to y . Then wy is non-contractible. Let D be a component of $G - w - y$ not containing x . Then $D \subsetneq C$ and from above, $E_G(y, D)$ contains a contractible edge not covered by z , a contradiction. Therefore, there exist two independent contractible edges in $E_G(\{x, y\}, C)$. \square

Lemma 3.4. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Suppose $G - S$ contains two vertices x and y . Let C be any component of $G - x - y$. Then the following statements hold.*

- (a) $C \cap S \neq \emptyset$.
- (b) If $|C \cap S| = 1$, then $|C| = 1$.
- (c) If $|C \cap S| > 1$, then there exist two independent contractible edges in $E_G(\{x, y\}, C)$.

Proof. Suppose $C \cap S = \emptyset$. By Lemma 4.6, $E_G(x, C)$ contains a contractible edge not covered by S , a contradiction. Now, (b) and (c) follow directly from the second part of Lemma 4.6. \square

Theorem 3.5. *For any 2-connected graph G nonisomorphic to K_3 , $E_C(G)$ cannot be covered by one vertex.*

Proof. Suppose x is a vertex in G that covers $E_C(G)$. Obviously, there exists an edge yz that is not incident to x . Therefore, yz is non-contractible. But this contradicts Lemma 3.4(a) by considering a component of $G - y - z$ not containing x . \square

Theorem 3.6. *Let G be any 2-connected graph nonisomorphic to K_3 . Then $E_C(G)$ can be covered by two vertices if and only if G is isomorphic to $K_{2,k}$ or to $K_{2,k}^+$ where $k \geq 2$.*

Proof. (\Leftarrow) Easy.

(\Rightarrow) Let $S := \{x, y\}$ be a cover of $E_C(G)$. Consider any component C of $G - S$. If $|C| > 1$, then C contains a non-contractible edge, say uv . By Lemma 3.4, $G - u - v$ has exactly two components both of order one, namely x and y . We have $G = K_{2,2}^+$.

Now, assume that every component of $G - S$ consists of exactly one vertex. Then G is isomorphic to $K_{2,k}$ or $K_{2,k}^+$ where $k \geq 2$. \square

Theorem 3.7. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. If $|V(G)| \geq 2|S| + 1$, then $G - S$ is not connected.*

Proof. The proof is by induction on $|V(G)|$. The result is true for $|V(G)| = 4$ by Theorem 3.5. Suppose the theorem is true for all 2-connected graphs with less than k vertices. Consider any 2-connected graph G with k vertices. Let S be a cover of $E_C(G)$ such that $|S| \leq \frac{k-1}{2}$. Suppose $G - S$ is connected. Note that all edges in $G - S$ are non-contractible. Let xy be any edge in $G - S$ and C_1, C_2, \dots, C_m be the components of $G - x - y$. For each C_i , define $G_i := (V(C_i) \cup \{x, y, x_i\}, E(G[C_i \cup xy]) \cup \{x_i x, x_i y\})$.

Suppose $m \geq 3$, or $m = 2$ and both C_1 and C_2 contain at least two vertices. Then $|V(G_i)| < |V(G)|$. Now, $S_i := (S \cap C_i) \cup x_i$ is a vertex cover of all contractible edges of G_i . Since $G - S$ is connected, $G_i - S_i$ is also connected. By induction, $|V(G_i)| \leq 2|S_i| = 2|S \cap C_i| + 2$. Now, $|V(G)| = 2 + \sum_i |V(C_i)| = 2 + \sum_i (|V(G_i)| - 3) \leq 2 + \sum_i (2|S \cap C_i| - 1) = 2 - m + 2|S| \leq 2|S|$, a contradiction. Therefore, $m = 2$, and one of C_1 and C_2 contains exactly one vertex.

For each edge e in $G - S$, define x_e to be the single vertex component of $G - V(e)$. Note that $x_e \in S$, $N_G(x_e) = V(e)$, and for any two distinct edges e, f in $G - S$, $x_e \neq x_f$. Therefore, $|S| \geq |E(G - S)| \geq |V(G - S)| - 1 = |V(G)| - |S| - 1$ implying $|V(G)| \leq 2|S| + 1$. Consequently, $|V(G)| = 2|S| + 1$, $|S| = |E(G - S)|$ and $G - S$ is a tree. But then G is not 2-connected, a contradiction. \square

The bound $2|S| + 1$ is best possible as demonstrated by K_4^- (K_4 minus an edge) for $|S| = 2$ and $K_3 \# E(K_3)$ for $|S| = 3$. For $|S| = k \geq 4$, let H be any 2-connected outerplanar graph of order k . Consider $H \# E_C(H)$. Take S to be the set of vertices not in H .

Theorem 3.8. *Let G be any 2-connected graph nonisomorphic to K_3 . Suppose S is a cover of $E_C(G)$ of order three. Then either $G - S$ is independent or $G - S$ contains exactly one non-trivial component such that $|VN(G, S)| \leq 3$ and $|EN(G, S)| \leq 3$.*

Proof. Let $S := \{x, y, z\}$. Suppose $G - S$ contains an edge uv . Obviously, uv is non-contractible. By Lemma 3.4(a), $G - u - v$ contains exactly two or three components. Suppose $G - u - v$ consists of three components. By Lemma 3.4(b), the components are precisely x, y and z , and $G[u, v]$ is the non-trivial component

of $G - S$. Otherwise, let C and D be the two components of $G - u - v$. Without loss of generality, by Lemma 3.4(a) and (b), assume $C = z$ and $x, y \in D$. Then uz and vz are contractible edges. By Lemma 3.4(c), we can assume ux and vy are contractible edges. Denote $T := S \cup \{u, v\}$. Note that $G[T]$ is connected. Suppose $G - T$ contains an edge e . Obviously, e is non-contractible. By Lemma 4.6, there exists a contractible edge not covered by S , a contradiction. Therefore, $G - T$ is independent.

Suppose $G - T = \emptyset$. Then xy is an edge and $G[u, v]$ is the non-trivial component of $G - S$. Now, let $G - T = \{a_1, a_2, \dots, a_k\}$. Then the neighbors of a_i belong to $\{u, v, x, y\}$. Obviously, a_iu and a_iv , if exist, are non-contractible edges. Therefore, a_ix and a_iy are contractible edges. If the edge xy exists, then both a_iu and a_iv must be absent, and $G[u, v]$ is the non-trivial component of $G - S$. Therefore, we can assume xy is absent. Suppose $k = 1$. Then $G[u, v]$ is the non-trivial component of $G - S$ if both a_1u and a_1v are absent. Otherwise, $G[u, v, a_1]$ is the non-trivial component of $G - S$ and $|VN(G, S)| = 3$. Now, $|EN(G, S)| = 3$ if and only if both a_1u and a_1v are present. Suppose $k \geq 2$. Then none of a_iu and a_iv exist, and $G[u, v]$ is the non-trivial component of $G - S$. \square

Theorem 3.9. *Let G be any 2-connected graph nonisomorphic to K_3 . Suppose S is a cover of $E_C(G)$ of order four. Then $|VN(G, S)| \leq 4$, $|EN(G, S)| \leq 5$ and $|CN(G, S)| \leq 2$.*

Proof. Let $S := \{w, x, y, z\}$. If $G - S$ is independent, then $|VN(G, S)| = |EN(G, S)| = |CN(G, S)| = 0$. Suppose $G - S$ contains an edge uv . Obviously, uv is non-contractible. By Lemma 3.4(a), $G - u - v$ contains exactly two, three or four components.

Suppose $G - u - v$ consists of four components. By Lemma 3.4(b), each component is precisely one vertex of S . We have $|VN(G, S)| = 2$, $|EN(G, S)| = 1$ and $|CN(G, S)| = 1$.

Suppose $G - u - v$ consists of three components. Then by Lemma 3.4(b), two components consist of one vertex of S while the third contains two vertices of S . By arguing as in the proof of Theorem 3.8, we have $|VN(G, S)| \leq 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$.

Suppose $G - u - v$ consists of two components, namely C and D . If $|C \cap S| = 2$ and $|D \cap S| = 2$, by arguing as in the proof of Theorem 3.8, we have $|VN(G, S)| \leq 4$, $|EN(G, S)| \leq 5$ and $|CN(G, S)| = 1$. Without loss of generality, suppose uw, vx, uy and vz are contractible edges where $w, x \in C$ and $y, z \in D$. If $|VN(G, S)| = 4$ and $|EN(G, S)| = 5$, then both C and D have order three. Let c be the vertex of C other than w and x , and d be the vertex of D other than y and z . Now, $|VN(G, S)| = 4$ if and only if wx and yz are absent, c is adjacent to u or v , and d is adjacent to u or v . Whereas $|EN(G, S)| = 5$ if and only if wx and yz are absent, c is adjacent to both u and v , and d is adjacent to both u and v .

Suppose $|C \cap S| = 1$ and $|D \cap S| = 3$. By Lemma 3.4(b), $|C| = 1$ and let $C = w$. By Lemma 3.4(c), there exist two independent contractible edges in $E_G(\{u, v\}, D)$, say ux and vy . Let $T := \{u, v, w, x, y\}$. Note that $G[T]$

is connected and $z \in G - T$. Let $V(G) - T := \{a_1, a_2, \dots, a_m\}$ where $a_1 = z$. Suppose $G - T$ is independent. Every vertex a_i other than z , if exists, is adjacent to both x and y such that $a_i x$ and $a_i y$ are contractible. If $m = 1$, then $|VN(G, S)| = 2$, $|EN(G, S)| = 1$ and $|CN(G, S)| = 1$. If $m = 2$, then $|VN(G, S)| \leq 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$. If $m > 2$, then for every $i > 1$, both $a_i u$ and $a_i v$ are absent. We have $|VN(G, S)| = 2$, $|EN(G, S)| = 1$ and $|CN(G, S)| = 1$. Now, assume that $G - T$ is not independent.

Suppose $G - T$ contains a non-contractible edge ab . Then by Lemma 3.4(a) and (b), $G - a - b$ consists of exactly two components, one of which is z . Without loss of generality, by Lemma 3.4(c), assume ax and by are contractible edges. By Lemma 4.6, every non-contractible edge of G lies in $G[u, v, x, y, a, b]$. Every vertex in $H := G - S - u - v - a - b$, if exists, is adjacent to x and y only. Therefore, $|VN(G, S)| = 4$. We also have $|CN(G, S)| \leq 2$ with equality holds if and only if ua , ub , va and vb are all absent. Lastly, $|EN(G, S)| \leq 5$ with equality holds if and only if $H = \emptyset$, xy is absent, ua and vb are present, and exactly one of ub and va is present.

Therefore, we can assume that all edges in $G - T$ are contractible, and are thus incident to z . Let a_2, \dots, a_l be all the neighbors of z in $V(G) - T$. Note that $l \geq 2$. Suppose there exists a vertex a in $G - T - z$ that is not adjacent to z . Then ax and ay are contractible edges. Now, suppose there exists a vertex b in $G - T - z - a$ that is adjacent to u . Obviously, ub is non-contractible. By Lemma 3.4(b), one of the components of $G - u - b$ is z . Hence, $l = 2$ and $a_2 = b$. By considering the contractible edge ux , by exists and we have $|VN(G, S)| = 3$, $|EN(G, S)| = 2$ and $|CN(G, S)| = 1$. Therefore, assume no vertex in $G - T - z - a$ is adjacent to $\{u, v\}$. Then $|VN(G, S)| \leq 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$.

Suppose every vertex in $G - T - z$ is adjacent to z . Since every vertex is incident to at least two contractible edges, every vertex in $G - T - z$ is adjacent to x or y . For $m = 2$, $|VN(G, S)| \leq 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$. For $m = 3$, $|VN(G, S)| \leq 4$, $|EN(G, S)| \leq 4$ and $|CN(G, S)| = 1$ where $|VN(G, S)| = 4$ if and only if a_2 is adjacent to x and u , a_3 is adjacent to y and v , and zx and zy are absent. For $m \geq 4$, without loss of generality, assume $a_2 x$ and $a_3 x$ exist and are both contractible. Suppose $a_2 u$ exists. Then wvy and $xa_3 z$ belong to two different components of $G - a_2 - u$, and by Lemma 4.6, $a_2 y$ is a contractible edge. For $i \geq 3$, $a_i u$ and $a_i v$ are absent. We have $|VN(G, S)| = 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$. Suppose $a_2 v$ exists. Then $wuxa_3 z$ and y belongs to two different components of $G - a_2 - v$, and by Lemma 4.6, $a_2 y$ is a contractible edge. For $i \geq 3$, $a_i u$ and $a_i v$ are absent. Again, $|VN(G, S)| = 3$, $|EN(G, S)| \leq 3$ and $|CN(G, S)| = 1$. \square

Finally, we derive tight upper bounds for the order, size and number of non-trivial components of $G - S$ in terms of $|S|$, and characterize all the extremal graphs.

Theorem 3.10. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|VN(G, S)| \leq 2|S| - 4$ for $|S| \geq 4$.*

Proof. The statement is true for $|S| = 4$ by Theorem 3.9. Suppose the theorem holds for all $|S| < k$ where $k \geq 5$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S| = k$. If $G - S$ is independent, then the theorem is trivially true. Let xy be any edge in $G - S$. Suppose $G - x - y$ consists of two fragments F_1 and F_2 , each of which contains at least two vertices in S . For each F_i , define $G_i := (V(F_i) \cup \{x, y, x_i\}, E(G[F_i \cup xy]) \cup \{x_ix, x_iy\})$ and $S_i := x_i \cup (S \cap F_i)$. Note that S_i covers $E_C(G_i)$. Suppose $|F_1 \cap S| \geq 3$ and $|F_2 \cap S| \geq 3$. Then $|VN(G_1, S_1)| \leq 2|S_1| - 4$ and $|VN(G_2, S_2)| \leq 2|S_2| - 4$. We have $|VN(G, S)| = |VN(G_1, S_1)| + |VN(G_2, S_2)| - 2 \leq 2|S_1| - 4 + 2|S_2| - 4 - 2 = 2(|S_1| + |S_2| - 2) - 6 = 2|S| - 6 < 2|S| - 4$. Suppose $|F_1 \cap S| = 2$ and $|F_2 \cap S| \geq 3$. Then $|VN(G_1, S_1)| \leq 3$ by Theorem 3.8 and $|VN(G_2, S_2)| \leq 2|S_2| - 4$. We have $|VN(G, S)| = |VN(G_1, S_1)| + |VN(G_2, S_2)| - 2 \leq 3 + 2|S_2| - 4 - 2 = 2(3 + |S_2| - 2) - 5 = 2|S| - 5 < 2|S| - 4$.

Suppose for every edge e in $G - S$, $G - V(e)$ consists of two components, one of which consists of exactly one vertex denoted by x_e and the other is denoted by C_e . Note that $x_e \in S$, $x_e \neq x_f$ for any two distinct edges in $G - S$, and C_e contains at least two vertices in S . Therefore, $|S| \geq |E(G - S)| + 2$ and $|VN(G, S)| \leq 2|E(G - S)| \leq 2|S| - 4$. Equality holds if and only if each edge e in $G - S$ corresponds to a non-trivial component of $G - S$ and $|\bigcup_{e \in G-S} C_e \cap S| = 2$. Equivalently, for $k \geq 5$, $V(G) := \{x, y\} \cup \bigcup_{i=1}^{k-2} \{x_i, y_i, z_i\} \cup \bigcup_{j=1}^l \{a_j\}$, $E(G) := \bigcup_{i=1}^{k-2} \{z_ix_i, z_iy_i, x_iy_i, x_ix, y_iy\} \cup \bigcup_{j=1}^l \{a_jx, a_jy\} \cup F$ where $F \subseteq xy \cup \bigcup_{i=1}^{k-2} \{x_iy, y_ix\}$, and $S := \{x, y\} \cup \bigcup_{i=1}^{k-2} \{z_i\}$. \square

Theorem 3.11. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|EN(G, S)| \leq 2|S| - 3$ for $|S| \geq 2$. Equality holds if and only if $G = K_4^-$ for $|S| = 2$, $G = K_3 \# E(K_3)$ for $|S| = 3$, and $G = H \# E_C(H)$ for $|S| \geq 4$ where H is any 2-connected maximally outerplanar graph of order $|S|$.*

Proof. The statement is true for $|S| = 2$ and $|S| = 3$ by Theorem 3.6 and Theorem 3.8. Suppose the theorem holds for all $|S| < k$ where $k \geq 4$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S| = k$. If $G - S$ is independent, then the theorem is trivially true. Let xy be any edge in $G - S$. Suppose $G - x - y$ consists of two fragments F_1 and F_2 , each of which contains at least two vertices in S . For each F_i , define $G_i := (V(F_i) \cup \{x, y, x_i\}, E(G[F_i \cup xy]) \cup \{x_ix, x_iy\})$ and $S_i := x_i \cup (S \cap F_i)$. Note that S_i covers $E_C(G_i)$ and $|EN(G_i, S_i)| \leq 2|S_i| - 3$. Now, $|EN(G, S)| = |EN(G_1, S_1)| + |EN(G_2, S_2)| - 1 \leq 2|S_1| - 3 + 2|S_2| - 3 - 1 = 2(|S_1| + |S_2| - 2) - 3 = 2|S| - 3$.

Suppose for every edge e in $G - S$, $G - V(e)$ consists of two components, one of which consists of exactly one vertex denoted by x_e . Note that $x_e \in S$ and $x_e \neq x_f$ for any two distinct edges in $G - S$. Therefore, $|EN(G, S)| \leq |S| < 2|S| - 3$.

It follows easily by induction that the equality holds if and only if G is one of the graphs stated above. \square

Theorem 3.12. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|CN(G, S)| \leq |S| - 2$ for $|S| \geq 3$.*

Proof. The statement is true for $|S| = 3$ by Theorem 3.8. Suppose the theorem holds for all $|S| < k$ where $k \geq 4$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S| = k$. If $G - S$ is independent, then the theorem is trivially true. Let xy be any edge in $G - S$. Suppose $G - x - y$ consists of two fragments F_1 and F_2 , each of which contains at least two vertices in S . For each F_i , define $G_i := (V(F_i) \cup \{x, y, x_i\}, E(G[F_i \cup xy]) \cup \{x_i x, x_i y\})$ and $S_i := x_i \cup (S \cap F_i)$. Note that S_i covers $E_C(G_i)$ and $|CN(G_i, S_i)| \leq |S_i| - 2$. Now, $|CN(G, S)| = |CN(G_1, S_1)| + |CN(G_2, S_2)| - 1 \leq |S_1| - 2 + |S_2| - 2 - 1 = (|S_1| + |S_2| - 2) - 3 = |S| - 3 < |S| - 2$.

Suppose for every edge e in $G - S$, $G - V(e)$ consists of two components, one of which consists of exactly one vertex denoted by x_e and the other is denoted by C_e . Note that $x_e \in S$, $x_e \neq x_f$ for any two distinct edges in $G - S$, and C_e contains at least two vertices in S . Therefore, $|S| \geq |E(G - S)| + 2$ and $|CN(G, S)| \leq |E(G - S)| \leq |S| - 2$. Equality holds if and only if each edge e in $G - S$ corresponds to a non-trivial component of $G - S$ and $|\bigcup_{e \in G - S} C_e \cap S| = 2$. Equivalently, for $k \geq 4$, $V(G) := \{x, y\} \cup \bigcup_{i=1}^{k-2} \{x_i, y_i, z_i\} \cup \bigcup_{j=1}^l \{a_j\}$, $E(G) := \bigcup_{i=1}^{k-2} \{z_i x_i, z_i y_i, x_i y_i, x_i x, y_i y\} \cup \bigcup_{j=1}^l \{a_j x, a_j y\} \cup F$ where $F \subseteq xy \cup \bigcup_{i=1}^{k-2} \{x_i y, y_i x\}$, and $S := \{x, y\} \cup \bigcup_{i=1}^{k-2} \{z_i\}$. \square

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Chapter 4

Contractible edges in 2-connected locally finite graphs

4.1 Introduction

Since the pioneering work of Tutte [15] who proved that every 3-connected finite graph nonisomorphic to K_4 contains a contractible edge, a lot of research has been done on contractible edges in finite graphs. One may consult the survey paper by Kriesell [11] for details.

For any 2-connected graph nonisomorphic to K_3 , we have the well-known fact that every edge can either be deleted or contracted so that the resulting graph remains 2-connected. This immediately leads to the following result.

Theorem 4.1. *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then the subgraph induced by all the contractible edges in G is 2-connected.*

Wu [16] investigated the distribution of contractible elements in matroids and extended Theorem 4.1 to simple 2-connected matroids. He also characterized all simple 2-connected matroids M having exactly $r(M) + 1$ contractible elements (where $r(M)$ is the rank of M) as those matroids isomorphic to a graphic matroid of an outerplanar Hamiltonian graph.

Theorem 4.2 (Wu [16]). *Let G be a 2-connected finite graph nonisomorphic to K_3 . Then every vertex of G is incident to exactly two contractible edges if and only if G is outerplanar.*

On the other hand, only a few results were known for contractible edges in infinite graphs. For example, Mader [18] showed that every contraction-critical locally finite infinite graph has infinitely many triangles. Kriesell [9] provided a method of constructing contraction-critical k -connected infinite graphs ($k \geq 2$).

In Section 3, we will prove that every contraction-critical 2-connected infinite graph contains vertices of infinite degree only and has uncountably many ends.

A natural way to extend Theorems 4.1 and 4.2 is to consider locally finite infinite graphs. Notice that Theorem 4.1 is no longer true as demonstrated by the infinite double ladder (the cartesian product of a double ray and K_2). The subgraph G_C induced by all the contractible edges is the disjoint union of two double rays and is not even connected. Interestingly, the situation changes dramatically by looking at the graph from a topological viewpoint as introduced by Diestel and Kühn [10, 11, 12]. By adding the two ends of the double ladder to G_C , the resulting closure $\overline{G_C}$ is a circle and is 2-arc-connected. In Section 4, we will prove that for every 2-connected locally finite infinite graph G , $\overline{G_C}$ is 2-arc-connected.

Returning to Theorem 4.2, the backward direction is straightforward. For the forward direction, by Theorem 4.1, G_C is spanning and 2-connected. Since every vertex is incident to exactly two contractible edges, G_C is a Hamilton cycle. Then it is easy to see that G is outerplanar. When extending to locally finite infinite graphs, we now need the non-trivial statement that if G is a 2-connected locally finite infinite graph such that every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This will be proved in Section 5. We will use it to prove an infinite analog of Theorem 4.2 for any 2-connected locally finite graph G nonisomorphic to K_3 . Also we will show that G is outerplanar if and only if every finite bond of G contains exactly two contractible edges.

4.2 Definitions

All basic graph-theoretical terminology can be found in Diestel [9]. Unless otherwise stated, all graphs considered in this paper can be finite or infinite. An edge of a k -connected graph is said to be *k -contractible* if its contraction results in a k -connected graph. Otherwise, it is called *k -non-contractible*. A k -connected graph in which every edge is k -non-contractible is called *contraction-critical k -connected*. We simply write 2-contractible as *contractible*. Let $G = (V, E)$ be a 2-connected graph. Denote the set of all contractible edges in G by E_C and the subgraph induced by all the contractible edges by $G_C := (V, E_C)$. Let X and Y be two disjoint subsets of V . An X - Y *path* P is a path such that only the starting vertex of P lies in X and only the ending vertex of P lies in Y . Denote the set of all edges between X and Y by $E_G(X, Y)$. If X and Y form a partition of V , then $E_G(X, Y)$ is called a *cut*. A minimal non-empty cut is a *bond*. Denote the set of all edges incident to a vertex x by $E_G(x)$ and the set of all neighbors of x by $N_G(x)$. Define $N_G(X) := (\bigcup_{x \in X} N_G(x)) \setminus X$. A set S of k vertices is called an *k -separator* if $G - S$ is not connected.

Let G be a locally finite graph. A *ray* is a 1-way infinite path, a *double ray* is a 2-way infinite path, and the subrays of a ray or double ray are its *tails*. An *end* is an equivalence class of rays where two rays are *equivalent* if no finite set of vertices separates them. Denote the set of the ends by $\Omega(G)$.

We define a topological space, denoted by $|G|$, on G together with its ends, which is known as the *Freudenthal compactification* of G as follows. View G as a 1-complex. Thus, every edge is homeomorphic to the unit interval. The basic open neighborhoods of a vertex x consists of a choice of half-open half edges $[xz)$, one for each incident edge xy , where z is any interior point of xy . For an end $\omega \in \Omega(G)$, we take as a basic open neighborhood the set of the form: $\hat{C}(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \dot{E}(S, \omega)$, where $S \subseteq V$ is a finite set of vertices, $C(S, \omega)$ is the component of $G - S$ in which every ray from ω has a tail, $\Omega(S, \omega)$ is the set of all ends whose rays have a tail in $C(S, \omega)$, and $\dot{E}(S, \omega)$ is the set of all interior points of edges between S and $C(S, \omega)$. Let H be a subgraph of G . Then the closure of H in $|G|$ is called a *standard subspace* and is denoted by \overline{H} . We say H *contains* a point x of $|G|$ if $x \in \overline{H}$.

Let X and Y be two topological spaces. A continuous map from the unit interval $[0, 1]$ to X is a *path* in X . A homeomorphic image of $[0, 1]$ in X is called an *arc* in X . This induces an ordering $<$ for the points in the arc. The images of 0 and 1 are the *endpoints* of the arc. An arc in X with endpoints x and y is called an *x - y arc*. A homeomorphic image of the unit circle in X is called a *circle* in X . A (*path*)-*component* of X is a maximal (path)-connected set in X . X is *2-connected* (*2-arc-connected*) if for all $x \in X$, $X \setminus x$ is connected (arc-connected). We say X can be *embedded* in Y if there exists an injective continuous function $\phi : X \rightarrow Y$ such that X is homeomorphic to $\phi(X)$ in the subspace topology of Y . Then ϕ is called an *embedding* of X in Y . Take Y to be \mathbb{R}^2 . A component of $\mathbb{R}^2 \setminus \phi(X)$ is called a *face* of $\phi(X)$ in \mathbb{R}^2 . A graph G is *planar* if G can be embedded in \mathbb{R}^2 . A graph G is *outerplanar* if there exists an embedding ϕ of G in \mathbb{R}^2 such that there is a face f of $\phi(G)$ in \mathbb{R}^2 whose boundary ∂f contains all the vertices of G . Chartrand and Harary [2] characterized outerplanar finite graphs as precisely those graphs that do not contain a $K_{2,3}$ - or K_4 - subdivision.

Suppose A is an arc in $|G|$ and x is a vertex in A . Then the vertex immediately before x in A if exists is denoted by x^- and the vertex immediately after x in A if exists is denoted by x^+ . An arc in $|G|$ is an ω -*arc* if the end ω is one of its endpoints and unless otherwise stated, it corresponds to the image of 1. Following Bruhn and Stein [4], we define the *end degree* of an end ω in G as the supremum over the cardinalities of sets of edge-disjoint rays in ω , and denote this number by $\deg_G(\omega)$. In fact, they proved that this is equal to the supremum over the cardinalities of sets of edge-disjoint ω -arcs in $|G|$. For a subgraph H of G , define the degree of ω in H as the supremum over the cardinalities of sets of edge-disjoint ω -arcs in \overline{H} which is denoted by $\deg_H(\omega)$.

4.3 Contraction-critical 2-connected infinite graphs

It is well-known that the only contraction-critical 2-connected finite graph is K_3 . However, there are infinitely many contraction-critical 2-connected infinite graphs as shown by the following construction due to Kriesell [9]. Define $G_0 := \emptyset$ and let G_1 be any 2-connected finite graph. Suppose we have constructed G_n

such that $G_{n-1} \subsetneq G_n$. For each edge xy in $E(G_n) \setminus E(G_{n-1})$, add a new x - y path of length at least 2. The resulting graph is G_{n+1} . Repeat the process inductively. Then the graph $G := \bigcup_{i \geq 1} G_i$ is a contraction-critical 2-connected infinite graph. Note that G has no vertex of finite degree and has uncountably many ends. We will show that this holds in general for any contraction-critical 2-connected infinite graph. First, we state a fundamental fact about contractible edges in 2-connected graphs.

Lemma 4.3. *Let G be a 2-connected graph nonisomorphic to K_3 and e be an edge of G . Then $G - e$ or G/e is 2-connected.*

Now, we can develop some tools that will be used for the rest of the paper.

Lemma 4.4. *Let G be a 2-connected graph nonisomorphic to K_3 , and e and f be two non-contractible edges of G . Then f is a non-contractible edge of $G - e$.*

Proof. By Lemma 4.3, $G - e$ is 2-connected. Since $V(f)$ is a 2-separator of G , $V(f)$ is also a 2-separator of $G - e$ and f is a non-contractible edge of $G - e$. \square

Lemma 4.5. *Let G be a 2-connected graph nonisomorphic to K_3 and F be a finite subset of $E(G)$.*

- (a) *If $G - F$ is disconnected, then F contains at least two contractible edges.*
- (b) *If $G - F$ is connected but not 2-connected, then F contains at least one contractible edge.*

Proof. For (a), suppose F contains at most one contractible edge. Then by Lemma 4.3 and 4.4, we can delete all the non-contractible edges in F and the resulting graph is still 2-connected, a contradiction.

For (b), suppose all edges in F are non-contractible. Then by Lemma 4.3 and 4.4, we can delete all edges in F and $G - F$ is still 2-connected, a contradiction. \square

Lemma 4.6. *Let G be a 2-connected graph nonisomorphic to K_3 . Let $\{x, y\}$ be a 2-separator of G and C be a component of $G - x - y$. If $|E_G(x, C)|$ is finite, then $E_G(x, C)$ contains a contractible edge.*

Proof. Note that y is a cutvertex of $G - E_G(x, C)$. By Lemma 4.5(b), $E_G(x, C)$ contains a contractible edge. \square

Lemma 4.7. *Let G be a 2-connected graph nonisomorphic to K_3 and x be a vertex of G . Suppose all edges incident to x are non-contractible. Then*

- (a) *x has infinite degree.*
- (b) *For any edge xy incident to x , every component of $G - x - y$ contains infinitely many neighbors of x .*

Proof. For (a), Suppose x has finite degree. By applying Lemma 4.5(a) to $E_G(x)$, x is incident to at least two contractible edges, a contradiction.

For (b), let C be a component of $G - x - y$. By Lemma 4.6, $E_G(x, C)$ contains infinitely many edges. \square

Theorem 4.8. *Let G be a contraction-critical 2-connected infinite graph. Then every vertex of G has infinite degree and G has uncountably many ends.*

Proof. By Lemma 4.7(a), every vertex of G has infinite degree.

Next, we will construct a rooted binary infinite tree T in G together with edges incident to each vertex of T with the following properties:

- (1) The root of T is denoted by x .
- (2) The vertices of T are denoted by $x_{n_1 n_2 \dots n_k}$ where $k \in \mathbb{N}$ and $n_i \in \{0, 1\}$ for $1 \leq i \leq k$. For $k = 0$, define $x_{n_1 n_2 \dots n_k} := x$.
- (3) Each vertex $x_{n_1 n_2 \dots n_k}$ of T is adjacent to two vertices $x_{n_1 n_2 \dots n_k 0}$ and $x_{n_1 n_2 \dots n_k 1}$ in T .
- (4) For each vertex $x_{n_1 n_2 \dots n_k}$ of T , there exists an edge $x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k}$ in G such that $y_{n_1 n_2 \dots n_k}$ does not lie in T .
- (5) The subtree of T rooted at $x_{n_1 n_2 \dots n_k}$ is defined as

$$T_{n_1 n_2 \dots n_k} := T[\bigcup_{i=0}^{\infty} \bigcup_{(m_1, m_2, \dots, m_i) \in \{0, 1\}^i} x_{n_1 n_2 \dots n_k m_1 m_2 \dots m_i}].$$

For fixed n_1, n_2, \dots, n_k , $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ separates $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ in G .

Each ray in T starting at x is of the form: $xx_{n_1}x_{n_1 n_2}x_{n_1 n_2 n_3} \dots$. Let $R := xx_{n_1}x_{n_1 n_2}x_{n_1 n_2 n_3} \dots$ and $Q := xx_{m_1}x_{m_1 m_2}x_{m_1 m_2 m_3} \dots$ be two distinct rays in T . Then there exists a smallest k such that $n_i = m_i$ for all $i \leq k$ and $n_{k+1} \neq m_{k+1}$. By property (5) above, $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ separates R and Q in G . Therefore, each ray in T starting at x belongs to a unique end of G , and G has uncountably many ends.

Now, it remains to construct T inductively. Let x be any vertex in G . Define $T_0 := (\{x\}, \emptyset)$. Choose any edge incident to x in G , say xy . Let C_0 and C_1 be any two components of $G - x - y$. Let x_0 be a neighbor of x in C_0 and x_1 be a neighbor of x in C_1 . Define $T_1 := (\{x, x_0, x_1\}, \{xx_0, xx_1\})$. Note that $N_G(C_0) \subseteq \{x, y\}$ and $N_G(C_1) \subseteq \{x, y\}$. Also, $G - C_0$ and $G - C_1$ are both connected.

Suppose we have constructed the rooted binary tree T_k where

$$V(T_k) := \bigcup_{i=0}^k \bigcup_{(n_1, n_2, \dots, n_i) \in \{0, 1\}^i} x_{n_1 n_2 \dots n_i} \text{ and}$$

$E(T_k) := \bigcup_{i=0}^{k-1} \bigcup_{(n_1, n_2, \dots, n_i) \in \{0, 1\}^i} \{x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 0}, x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 1}\}$ such that

- (i) each vertex $x_{n_1 n_2 \dots n_i}$ ($0 \leq i \leq k$) lies in a connected subgraph $C_{n_1 n_2 \dots n_i}$ of G (for $i = 0$, $x_{n_1 n_2 \dots n_i} := x$, $y_{n_1 n_2 \dots n_i} := y$ and $C_{n_1 n_2 \dots n_i} := G$),

- (ii) for each vertex $x_{n_1 n_2 \dots n_i}$ ($0 \leq i < k$), we have found an edge $x_{n_1 n_2 \dots n_i} y_{n_1 n_2 \dots n_i}$ that lies in $C_{n_1 n_2 \dots n_i}$ such that $C_{n_1 n_2 \dots n_i 0}$ and $C_{n_1 n_2 \dots n_i 1}$ are two components of $C_{n_1 n_2 \dots n_i} - x_{n_1 n_2 \dots n_i} - y_{n_1 n_2 \dots n_i}$ that are adjacent to $x_{n_1 n_2 \dots n_i}$,
- (iii) for fixed n_1, n_2, \dots, n_i ($1 \leq i \leq k$), $N_G(C_{n_1 n_2 \dots n_i}) \subseteq \bigcup_{j=0}^{i-1} \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$,
- (iv) for fixed n_1, n_2, \dots, n_i ($1 \leq i \leq k$), $G - C_{n_1 n_2 \dots n_i}$ is connected.

Now, for each vertex $x_{n_1 n_2 \dots n_k}$ in T_k , since it has infinite degree and $N_G(x_{n_1 n_2 \dots n_k}) \setminus C_{n_1 n_2 \dots n_k} \subseteq N_G(C_{n_1 n_2 \dots n_k})$ is finite by (iii), all but finitely many neighbors of $x_{n_1 n_2 \dots n_k}$ lie in $C_{n_1 n_2 \dots n_k}$. Let z be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k}$ and $B := C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z$. Suppose B is connected. Since $B' := G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), B and B' are the only two components of $G - x_{n_1 n_2 \dots n_k} - z$. By Lemma 4.7(b), B' contains infinitely many neighbors of $x_{n_1 n_2 \dots n_k}$ contradicting $N_G(x_{n_1 n_2 \dots n_k}) \setminus C_{n_1 n_2 \dots n_k} \subseteq N_G(C_{n_1 n_2 \dots n_k})$ which is finite by (iii). Therefore, B is not connected.

Note that at least one component of B is adjacent to $x_{n_1 n_2 \dots n_k}$. If not, then, by the 2-connectedness of G , each component of B has a neighbor in $N_G(C_{n_1 n_2 \dots n_k}) \subseteq G - C_{n_1 n_2 \dots n_k}$. By (iv), this implies $G - x_{n_1 n_2 \dots n_k} - z$ is connected, a contradiction. Suppose there are two components of B , say D and D' , that are both adjacent to $x_{n_1 n_2 \dots n_k}$. Then choose $y_{n_1 n_2 \dots n_k} := z$, $C_{n_1 n_2 \dots n_k 0} := D$ and $C_{n_1 n_2 \dots n_k 1} := D'$. Suppose only one component of B is adjacent to $x_{n_1 n_2 \dots n_k}$, say C . Each component of B other than C is adjacent to z by the connectedness of $C_{n_1 n_2 \dots n_k}$ and has a neighbor in $N_G(C_{n_1 n_2 \dots n_k}) \subseteq G - C_{n_1 n_2 \dots n_k}$ by the 2-connectedness of G . Denote the union of components of B other than C by C' . Since $G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), $C'' := G[(G - C_{n_1 n_2 \dots n_k}) \cup C']$ is connected. Hence, C and C'' are the only two components of $G - x_{n_1 n_2 \dots n_k} - z$ and $N_G(C) = \{x_{n_1 n_2 \dots n_k}, z\}$. Let z' be a neighbor of $x_{n_1 n_2 \dots n_k}$ in C . Then one component D of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$ contains z and C' . Since $G - x_{n_1 n_2 \dots n_k} - z'$ is not connected, D cannot be the only component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$. Let D' be any component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - z'$ other than D . Then D' lies in C and $N_G(D') \subseteq \{x_{n_1 n_2 \dots n_k}, z'\} \cup (N_G(C) - z) = \{x_{n_1 n_2 \dots n_k}, z'\}$. Now, choose $y_{n_1 n_2 \dots n_k} := z'$, $C_{n_1 n_2 \dots n_k 0} := D$ and $C_{n_1 n_2 \dots n_k 1} := D'$.

In both cases, $x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k}$ lies in $C_{n_1 n_2 \dots n_k}$, and $C_{n_1 n_2 \dots n_k 0}$ and $C_{n_1 n_2 \dots n_k 1}$ are two components of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ that are adjacent to $x_{n_1 n_2 \dots n_k}$. Let $x_{n_1 n_2 \dots n_k 0}$ be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k 0}$ and $x_{n_1 n_2 \dots n_k 1}$ be a neighbor of $x_{n_1 n_2 \dots n_k}$ in $C_{n_1 n_2 \dots n_k 1}$. Since $C_{n_1 n_2 \dots n_k 0} \subseteq C_{n_1 n_2 \dots n_k}$ and $C_{n_1 n_2 \dots n_k 1} \subseteq C_{n_1 n_2 \dots n_k}$, $N_G(C_{n_1 n_2 \dots n_k 0}) \subseteq N_G(C_{n_1 n_2 \dots n_k}) \cup \{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\} \subseteq \bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ and $N_G(C_{n_1 n_2 \dots n_k 1}) \subseteq N_G(C_{n_1 n_2 \dots n_k}) \cup \{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\} \subseteq \bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ by (iii). By the connectedness of $C_{n_1 n_2 \dots n_k}$, every component of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ has a neighbor in $\{x_{n_1 n_2 \dots n_k}, y_{n_1 n_2 \dots n_k}\}$. For $n_{k+1} \in \{0, 1\}$, denote the union of the components of $C_{n_1 n_2 \dots n_k} - x_{n_1 n_2 \dots n_k} - y_{n_1 n_2 \dots n_k}$ other than $C_{n_1 n_2 \dots n_k n_{k+1}}$ by $U_{n_{k+1}}$. Then $G[x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k} \cup U_{n_{k+1}}]$ is connected. Since $x_{n_1 n_2 \dots n_{k-1}} \in G - C_{n_1 n_2 \dots n_k}$, $x_{n_1 n_2 \dots n_{k-1}} x_{n_1 n_2 \dots n_k} \in E(G)$ and $G - C_{n_1 n_2 \dots n_k}$ is connected by (iv), $G - C_{n_1 n_2 \dots n_k n_{k+1}} := G[(G - C_{n_1 n_2 \dots n_k}) \cup x_{n_1 n_2 \dots n_k} y_{n_1 n_2 \dots n_k} \cup U_{n_{k+1}}]$ is connected.

Define T_{k+1} where $V(T_{k+1}) := \bigcup_{i=0}^{k+1} \bigcup_{(n_1, n_2, \dots, n_i) \in \{0,1\}^i} x_{n_1 n_2 \dots n_i}$ and

$$E(T_{k+1}) := \bigcup_{i=0}^k \bigcup_{(n_1, n_2, \dots, n_i) \in \{0,1\}^i} \{x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 0}, x_{n_1 n_2 \dots n_i} x_{n_1 n_2 \dots n_i 1}\}.$$

Finally, define $T := \bigcup_{k=0}^{\infty} T_k$. It is easy to see that T satisfies properties (1) through (4). Let z_0 and z_1 be any vertices in $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ respectively. Then z_0 is of the form $x_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i}$ while z_1 is of the form $x_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_j}$. We have $z_0 = x_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i} \in C_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_i} \subseteq C_{n_1 n_2 \dots n_k 0 p_1 p_2 \dots p_{i-1}} \subseteq \dots \subseteq C_{n_1 n_2 \dots n_k 0}$ and $z_1 = x_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots p_j} \in C_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_j} \subseteq C_{n_1 n_2 \dots n_k 1 q_1 q_2 \dots q_{j-1}} \subseteq \dots \subseteq C_{n_1 n_2 \dots n_k 1}$. Therefore, $T_{n_1 n_2 \dots n_k 0} \subseteq C_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1} \subseteq C_{n_1 n_2 \dots n_k 1}$. Since $\bigcup_{j=0}^k \{x_{n_1 n_2 \dots n_j}, y_{n_1 n_2 \dots n_j}\}$ contains both $N_G(C_{n_1 n_2 \dots n_k 0})$ and $N_G(C_{n_1 n_2 \dots n_k 1})$, it separates $C_{n_1 n_2 \dots n_k 0}$ and $C_{n_1 n_2 \dots n_k 1}$ in G and thus separates $T_{n_1 n_2 \dots n_k 0}$ and $T_{n_1 n_2 \dots n_k 1}$ in G . Hence, Property (5) holds for T and the proof is complete. \square

4.4 Subgraph induced by all the contractible edges

In this section, we will extend Theorem 4.1 to any 2-connected locally finite infinite graph G and prove that $\overline{G_C}$ is 2-arc-connected. Note that Lemma 4.5(a) implies that every vertex is incident to at least two contractible edges. Hence, G_C is spanning. Using the following two lemmas, it is easy to see that $\overline{G_C}$ is arc-connected.

Lemma 4.9 (Diestel [9]). *Let G be a locally finite graph. Then a standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of G of which it meets both sides.*

Lemma 4.10 (Diestel and Kühn [12]). *If G is a locally finite graph, then every closed connected subspace of $|G|$ is arc-connected.*

Theorem 4.11. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is arc-connected.*

Proof. Let F be any finite cut of G . By Lemma 4.5(a), F contains at least two edges in G_C . Hence, $\overline{G_C}$ is connected by Lemma 4.9. By Lemma 4.10, $\overline{G_C}$ is arc-connected. \square

Next, we prove that $\overline{G_C}$ is 2-connected.

Lemma 4.12. *Let G be a 2-connected locally finite infinite graph and x be a point of $|G|$. Suppose there is a partition (X, X') of $V(G \setminus x)$ such that $E_G(X, X')$ is non-empty and all edges in $E_G(X, X')$ are non-contractible. Then G contains a subdivision of a 1-way infinite ladder L consisting of two disjoint rays: $R := x_0 P_1 x_1 P_2 x_2 \dots$ and $R' := x'_0 P'_1 x'_1 P'_2 x'_2 \dots$ with the rungs of the ladder being $x_0 x'_0, x_1 x'_1, x_2 x'_2, \dots$, all of which are X - X' edges such that $x \notin \overline{L}$.*

Proof. Since G is 2-connected, $|E_G(X, X')| \geq 2$ unless x is a vertex and $|E_G(X, X')| = 1$. Consider any X - X' edge $x_0 x'_0$ that does not contain x . Let

C be the component of $G - x_0 - x'_0$ containing x and C_1 be a component of $G - x_0 - x'_0$ not containing x .

Suppose we have constructed the finite ladder L_k consisting of two disjoint paths $R_k := x_0 P_1 x_1 P_2 x_2 \dots x_{k-1} P_k x_k$ and $R'_k := x'_0 P'_1 x'_1 P'_2 x'_2 \dots x'_{k-1} P'_k x'_k$ with the rungs of the ladder being $x_0 x'_0, x_1 x'_1, \dots, x_k x'_k$, all of which are X - X' edges such that $L_k \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_k - x_k - x'_k]$ is connected. Let C_{k+1} be a component of $G - x_k - x'_k$ not containing x . Then $C_{k+1} \subseteq C_1$ and $C_{k+1} \cap L_k = \emptyset$. By applying Lemma 4.6 to $E_G(x_k, C_{k+1})$ and $E_G(x'_k, C_{k+1})$, there exist contractible edges $x_k y_{k+1}$ and $x'_k y'_{k+1}$ where $y_{k+1} \in C_{k+1}$ and $y'_{k+1} \in C_{k+1}$. Since $x \notin C_{k+1}$ and all edges in $E_G(X, X')$ are non-contractible, $y_{k+1} \in X$ and $y'_{k+1} \in X'$. Choose a path Q_{k+1} in C_{k+1} between y_{k+1} and y'_{k+1} . Then there exists an X - X' edge $x_{k+1} x'_{k+1}$ on Q_{k+1} such that $V(y_{k+1} Q_{k+1} x_{k+1}) \subseteq X$ and $x'_{k+1} \in X'$. Define $P_{k+1} := x_k y_{k+1} \cup y_{k+1} Q_{k+1} x_{k+1}$, $P'_{k+1} := x'_k y'_{k+1} \cup y'_{k+1} Q_{k+1} x'_{k+1}$, $R_{k+1} := R_k \cup P_{k+1}$, $R'_{k+1} := R'_k \cup P'_{k+1}$ and $L_{k+1} := L_k \cup P_{k+1} \cup P'_{k+1} \cup x_{k+1} x'_{k+1}$. Note that $L_{k+1} \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_{k+1} - x_{k+1} - x'_{k+1}]$ is connected.

Define $R := \bigcup_{k \geq 0} R_k$, $R' := \bigcup_{k \geq 0} R'_k$ and $L := \bigcup_{k \geq 0} L_k$. Then $L \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $x \notin \bar{L}$. \square

Theorem 4.13. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is 2-connected.*

Proof. Suppose $\overline{G_C}$ is not 2-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not connected. Let U and U' be two disjoint non-empty open sets in $|G|$ such that $\overline{G_C} \setminus x \subseteq U \cup U'$, $(\overline{G_C} \setminus x) \cap U \neq \emptyset$ and $(\overline{G_C} \setminus x) \cap U' \neq \emptyset$. Define $X := (\overline{G_C} \setminus x) \cap U \cap V(G)$ and $X' := (\overline{G_C} \setminus x) \cap U' \cap V(G)$. Since G_C is spanning, $X \cup X' = V(G \setminus x)$. Suppose U contains an interior point a of an edge bc of G_C . Then $\overline{G_C} \setminus x$ contains half edges $[ba]$ or $[ca]$ of bc . By the connectedness of half edge, U contains b or c . Suppose U contains an end ω of $|G|$. Then U contains a basic open neighborhood of ω , say $\hat{C}(S, \omega)$, and thus contains infinitely many vertices. The same arguments hold for U' . Therefore, both X and X' are non-empty. Since $G \setminus x$ is connected, $E_G(X, X')$ is non-empty.

Suppose x is a vertex or an end of G . Then all edges in $E_G(X, X')$ are non-contractible and (X, X') is a partition of $V(G \setminus x)$. Suppose x is an interior point of an edge e . Then all edges in $E_G(X, X')$ are non-contractible unless $e \in E_G(X, X') \cap E_C$. Note that, $E_G(X, X') - e$ is non-empty as G is 2-connected and every edge in $E_G(X, X') - e$ is non-contractible. Let $e = yy'$ where $y \in X$ and $y' \in X'$. Suppose $X = \{y\}$. By Lemma 4.5(a), since y is incident to at least two contractible edges, there is a contractible X - X' edge other than e , which is impossible. Therefore, $|X| \geq 2$. Now, all edges in $E_G(X - y, X')$ are non-contractible and $(X - y, X')$ is a partition of $V(G \setminus y)$. In both cases, by Lemma 4.12, G contains a subdivision of a 1-way infinite ladder L such that $x \notin \bar{L}$.

Let ω be the end of $|G|$ containing R and R' . Note that $\omega \neq x$. Since G_C is spanning, $\overline{G_C} \setminus x$ contains all the ends of $|G|$ except possibly x . Without loss of generality, assume $\omega \in U$. Since U is open, there exists a basic open

neighborhood $\hat{C}(S, \omega) \subseteq U$. Since $x'_0, x'_1, x'_2, \dots \in X' \subseteq U'$ converge to ω , all but finitely many of them lie in $\hat{C}(S, \omega)$, contradicting $U \cap U' = \emptyset$. \square

Finally, we prove the main result of this section, namely, $\overline{G_C}$ is 2-arc-connected. This follows from a theorem by Georgakopoulos [7] concerning connected but not path-connected subspaces of locally finite graphs. Note that since $|G|$ is Hausdorff, path-connectedness is equivalent to arc-connectedness.

Theorem 4.14 (Georgakopoulos [7]). *Given any locally finite connected graph G , a connected subspace X of $|G|$ is path-connected unless it satisfies the following assertions:*

- (1) X has uncountably many path-components each of which consists of one end only;
- (2) X has infinitely many path-components that contain a vertex; and
- (3) every path-component of X contains an end.

Theorem 4.15. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . Then $\overline{G_C}$ is 2-arc-connected.*

Proof. Suppose $\overline{G_C}$ is not 2-arc-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not arc-connected. Note that $\overline{G_C} \setminus x$ is connected by Theorem 4.13. By Theorem 4.14, $\overline{G_C} \setminus x$ has uncountably many path-components each of which consists of one end only. Let ω and ω' be two such path-components of $\overline{G_C} \setminus x$. Since $\overline{G_C}$ is arc-connected by Theorem 4.11, there exists an arc A joining ω and ω' in $\overline{G_C}$. Now, x must lie in A for otherwise ω and ω' would lie in the same path-component of $\overline{G_C} \setminus x$. But the path-component of $\overline{G_C} \setminus x$ containing ω also contains $[\omega Ax]$, a contradiction. \square

4.5 Outerplanarity of 2-connected locally finite graphs

As mentioned in the introduction, in order to extend Theorem 4.2 to locally finite infinite graphs, we would like to prove that for any 2-connected locally finite infinite graph G , if every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This requires several lemmas listed below.

Lemma 4.16. *Let G be a locally finite graph. Then every arc in $|G|$ whose two endpoints are ends contains a vertex.*

Proof. Suppose A is an arc in $|G|$ whose two endpoints are ends ω_1 and ω_2 . Then there exists a finite set S of vertices such that $\hat{C}(S, \omega_1)$ and $\hat{C}(S, \omega_2)$ are distinct. By the connectedness of A , A contains a vertex of S . \square

Lemma 4.17. *Let G be a locally finite graph and ω be an end in $|G|$. Then every ω -arc A in $|G|$ contains a vertex, say z , and zA contains a ray starting with z .*

Proof. Denote the starting point of A by a . First, we show that A contains a vertex. If a is a vertex, then we are done. If a is an end, then it is true by Lemma 4.16. If a is an interior point of an edge xy , then by the connectedness of A , A contains x or y .

Let z be a vertex in A . By the connectedness of zA , zA contains an interior point of an edge incident to z , say zz_1 . Then the connectedness of zA implies zz_1 lies in zA . Repeat the above argument for z_1A and so on. We obtain a ray that starts with z and lies in zA . \square

Lemma 4.18. *Let G be a locally finite graph and ω be an end in $|G|$. Let A_1 and A_2 be two ω -arcs in $|G|$ that are disjoint except at ω . Then, for all finite subset S of $V(G)$, $\hat{C}(S, \omega)$ contains a subarc $A'_1\omega A'_2$ of $A_1\omega A_2$ and there is an A'_1 - A'_2 path in $C(S, \omega)$.*

Proof. Let x_1 be the last point of A_1 that lies in S and x_2 be the last point of A_2 that lies in S . By Lemma 4.17, x_1A contains a ray R_1 starting with x_1 and x_2A contains a ray R_2 starting with x_2 . Let y_1 be the neighbor of x_1 in R_1 and y_2 be the neighbor of x_2 in R_2 . Then $y_1A_1\omega A_2y_2$ lies in $\hat{C}(S, \omega)$. Also there is a y_1 - y_2 path in $C(S, \omega)$ which automatically contains a y_1A_1 - y_2A_2 path. \square

We also need a result on the characterization of a topological circle in $|G|$ in terms of its vertex and end degrees.

Lemma 4.19 (Bruhn and Stein [4]). *Let C be a subgraph of a locally finite graph G . Then \overline{C} is a circle if and only if \overline{C} is connected and every vertex and end of $|G|$ in \overline{C} has degree two in \overline{C} .*

Now, we can proceed with the proof.

Theorem 4.20. *Let G be a 2-connected locally finite infinite graph and G_C be the subgraph induced by all the contractible edges in G . If every vertex of G is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle.*

Proof. Since G_C is spanning, $\overline{G_C}$ contains all vertices and ends of $|G|$. By Theorem 4.11, $\overline{G_C}$ is arc-connected. Obviously, every vertex of G has degree two in $\overline{G_C}$. Therefore, it remains to prove that every end of $|G|$ has degree two in $\overline{G_C}$.

Claim 4.21. *Let A be an arc in $\overline{G_C}$ and x be a vertex in A . Suppose that both x^- and x^+ exist in A . Let y be any neighbor of x other than x^- and x^+ . Then every x^- - x^+ arc in $|G|$ intersects $\{x, y\}$.*

Proof. Since xx^- and xx^+ are the only contractible edges incident to x , xy is non-contractible. Lemma 4.6 implies that $G - x - y$ has exactly two components, and x^- and x^+ lie in different components. By the connectedness of an arc, every x^- - x^+ arc in $|G|$ intersects $\{x, y\}$. \square

Claim 4.22. *Let ω be an end in $|G|$. Suppose A_1 and A_2 are two edge-disjoint ω -arcs in $\overline{G_C}$. Then A_1 and A_2 can intersect only at the ends of $|G|$ with the only possible exception being that the starting points of A_1 and A_2 are the same vertex.*

Proof. Obviously, A_1 and A_2 cannot intersect at an interior point of an edge. Suppose A_1 and A_2 intersect at a vertex x . If x is not the starting point for both A_1 and A_2 , then the degree of x in $\overline{G_C}$ is at least three, a contradiction. \square

Claim 4.23. *Let ω be an end in $|G|$. Suppose A_1 and A_2 are two edge-disjoint ω -arcs in $\overline{G_C}$ such that the starting points of A_1 and A_2 are distinct vertices. Then there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' unless $A_1 \cap A_2 = \{\omega\}$.*

Proof. Suppose $A_1 \cap A_2 \neq \{\omega\}$. Let ω' be the first point of A_2 that intersects A_1 . By Claim 4.22, ω' is an end in $|G|$ different from ω . Then $A_1\omega'$, $A_2\omega'$ and $\omega A_1\omega'$ are the required three ω' -arcs. \square

Claim 4.24. *Let ω be an end in $|G|$. Suppose there are three edge-disjoint ω -arcs in $\overline{G_C}$. Then there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' .*

Proof. Let A_1, A_2, A_3 be three edge-disjoint ω -arcs in $\overline{G_C}$. By Lemma 4.17, for each $i \in \{1, 2, 3\}$, A_i contains a ray R_i . Denote the first edge of R_i by $x_i y_i$. By Claim 4.22, y_1, y_2, y_3 are all distinct. Therefore, without loss of generality, we can assume that the starting points of A_1, A_2, A_3 are all distinct vertices. Consider A_1 and A_2 . If $A_1 \cap A_2 \neq \{\omega\}$, then the claim follows from Claim 4.23. Suppose $A_1 \cap A_2 = \{\omega\}$. If $A_2 \cap A_3 \neq \{\omega\}$, then again the claim follows from Claim 4.23. Therefore, suppose $A_2 \cap A_3 = \{\omega\}$. But then, A_1, A_2, A_3 are the desired three ω -arcs. \square

Claim 4.25. *For each end ω in $|G|$, $\deg_{G_C}(\omega) \leq 2$.*

Proof. Suppose there are three edge-disjoint ω -arcs in $\overline{G_C}$. By Claim 4.24, there exists an end ω' in $|G|$ such that there are three ω' -arcs in $\overline{G_C}$ that are disjoint except at ω' . Denote these three ω' -arcs by A_1, A_2, A_3 . By Lemma 4.16, without loss of generality, we can assume A_1, A_2, A_3 start with vertices a_1, a_2, a_3 respectively.

By applying Lemma 4.18 to A_1 and A_2 with $S = \{a_1, a_2\}$, we obtain an $a_1^+ A_1 - a_2^+ A_2$ path P . Let $x_1 = P \cap A_1$, $x_2 = P \cap A_2$ and x be the neighbor of x_1 in Q . If P intersects A_3 , then interchange A_2 and A_3 . Therefore, without loss of generality, there is an $a_1^+ A_1 - a_2^+ A_2$ path P that does not intersect A_3 .

Now, apply Lemma 4.18 to $x_2 A_2$ and A_3 with $S = V(P)$. We obtain an $x_2^+ A_2 - A_3$ path Q not intersecting P . Let $y_2 = Q \cap x_2^+ A_2$ and $y_3 = Q \cap A_3$. By Claim 4.21, Q cannot intersect $A_2 x_2^-$, and Q cannot intersect both $A_1 x_1^-$ and $x_1^+ A_1$. Suppose $Q \cap A_1 x_1^- \neq \emptyset$. Let y be the first vertex of Q that lies in $A_1 x_1^-$. Then $x_1^- A_1 y Q y_2 A_2 \omega' A_1 x_1^+$ is an $x_1^- - x_1^+$ arc not intersecting $\{x_1, x\}$,

contradicting Claim 4.21. Suppose $Q \cap x_1^+ A_1 \neq \emptyset$. Then there is an $x_1^+ A_1$ - A_3 subpath in Q not intersecting A_2 , and we interchange A_1 and A_2 . Therefore, without loss of generality, we can assume that there is an $x_2^+ A_2$ - A_3 path Q that does not intersect $P \cup A_1 \cup A_2 x_2^-$. Let u_2 be the neighbor of y_2 in Q and u_3 be the neighbor of y_3 in Q .

Finally, apply Lemma 4.18 to $x_1 A_1$ and $y_2 A_2$ with $S = V(P \cup Q)$. We obtain an $x_1^+ A_1$ - $y_2^+ A_2$ path R not intersecting $P \cup Q$. Let $z_1 = R \cap x_1^+ A_1$ and $z_2 = R \cap y_2^+ A_2$. By Claim 4.21, R cannot intersect $A_1 x_1^-$ and R cannot intersect $A_2 y_2^-$. Also, R cannot intersect both $A_3 y_3^-$ and $y_3^+ A_3$. Suppose $R \cap A_3 y_3^- \neq \emptyset$. Let z be the last vertex of R that lies in $A_3 y_3^-$. Then $y_3^- A_3 z R z_2 A_2 \omega' A_3 y_3^+$ is an y_3^- - y_3^+ arc not intersecting $\{y_3, u_3\}$, contradicting Claim 4.21. Suppose $R \cap y_3^+ A_3 \neq \emptyset$. Let z' be the first vertex of R that lies in $y_3^+ A_3$. Then $y_2^- A_2 x_2 P x_1 A_1 z_1 R z' A_3 \omega' A_2 y_2^+$ is an y_2^- - y_2^+ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 4.21. Therefore, $R \cap (A_1 x_1^- \cup A_2 y_2^- \cup A_3 \cup P \cup Q) = \emptyset$. But, $y_2^- A_2 x_2 P x_1 A_1 z_1 R z_2 A_2 y_2^+$ is an y_2^- - y_2^+ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 4.21. \square

Claim 4.26. *For each end ω in $|G|$, $\deg_{G_C}(\omega) = 2$.*

Proof. Let x be a vertex in G_C . Since $\overline{G_C}$ is arc-connected, there is an ω -arc A in $\overline{G_C}$ joining x to ω . Let y be the neighbor of x in A and a be an interior point of xy . Since $\overline{G_C}$ is 2-connected, $\overline{G_C} \setminus a$ is connected. Suppose $\overline{G_C} - xy$ is not connected. Then there exist two disjoint nonempty open sets U and V in $|G|$ such that $\overline{G_C} - xy \subseteq U \cup V$, $U \cap \overline{G_C} - xy \neq \emptyset$ and $V \cap \overline{G_C} - xy \neq \emptyset$. If $x, y \in U$, then $U \cup [x, a) \cup [y, a)$ and V are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is impossible. If $x \in U$ and $y \in V$, then $U \cup [x, a)$ and $V \cup [y, a)$ are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is also impossible. Therefore, $\overline{G_C} - xy$ is connected and is arc-connected by Lemma 4.10. Let A' be an x - ω arc in $\overline{G_C} - xy$. If $y A \omega \cap A'$ contains a vertex u , then u has degree at least three in G_C , a contradiction. Let ω' be the first point in $y A \omega \cap A'$ which is an end. If $\omega' \neq \omega$, then $\deg_{G_C}(\omega') \geq 3$ contradicting Claim 4.25. Therefore, $\omega' = \omega$ and we have $\deg_{G_C}(\omega) \geq 2$. By Claim 4.25, $\deg_{G_C}(\omega) = 2$. \square

We are now ready to prove the infinite analog of Theorem 4.2.

Theorem 4.27. *Let G be a 2-connected locally finite graph nonisomorphic to K_3 . Then the following are equivalent:*

- (1) *Every vertex of G is incident to exactly two contractible edges.*
- (2) *Every finite bond of G contains exactly two contractible edges.*
- (3) *G is outerplanar.*

Proof.

(2) \Rightarrow (1) Trivial.

(1) \Rightarrow (3) By Theorem 4.2, this is true for finite G . Therefore, assume G is infinite. Suppose every vertex of G is incident to exactly two contractible edges.

By Theorem 4.20, $\overline{G_C}$ is a Hamilton circle. All edges in $E(G) \setminus E_C$ are chords of $\overline{G_C}$ and are non-contractible. Consider any chord xy of $\overline{G_C}$. Since every vertex of G is incident to exactly two contractible edges, by Lemma 4.6, $G - x - y$ consists of exactly two components C_1 and C_2 . Without loss of generality, assume that $x^+ \overline{G_C} y^- \subseteq \overline{C_1}$ and $y^+ \overline{G_C} x^- \subseteq \overline{C_2}$. Then there is no chord of $\overline{G_C}$ between $x^+ \overline{G_C} y^-$ and $y^+ \overline{G_C} x^-$. Hence, no chords of $\overline{G_C}$ are overlapping. Embed $\overline{G_C}$ in a circle of \mathbb{R}^2 and denote the embedding by ϕ . Now, draw every chord xy of $\overline{G_C}$ as a straight line segment between $\phi(x)$ and $\phi(y)$ in \mathbb{R}^2 . This shows that G is outerplanar.

(3) \Rightarrow (2) Let B be a finite bond of G between two components X and Y of $G - B$. By Lemma 4.5(a), B contains at least two contractible edges. Suppose B contains three contractible edges x_1y_1, x_2y_2, x_3y_3 such that $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Since X and Y are connected, there exists a path P in X joining x_1 to x_2 and a path Q in Y joining y_1 to y_2 . Let $C := P \cup x_1y_1 \cup Q \cup x_2y_2$. Obviously, $x_3y_3 \notin E(C)$. Take any x_3 - P path P' in X joining x_3 to P at x and any y_3 - Q path Q' in Y joining y_3 to Q at y . Let $R' := P' \cup x_3y_3 \cup Q'$. If $R' = x_3y_3$, then x_3y_3 is a chord of C . Since x_3y_3 is contractible, the two components of $C - x_3 - y_3$ are joined by a path, say R . Then $C \cup R \cup R'$ is a K_4 -subdivision and G is not outerplanar. Suppose $R' \neq x_3y_3$. If both x - y paths in C are not edges, then $C \cup R'$ is a $K_{2,3}$ -subdivision and G is not outerplanar. If one of the two x - y paths in C is an edge, then without loss of generality, assume $x_2 = x$ and $y_2 = y$. Since x_2y_2 is contractible, the two components of $(C \cup R') - x_2 - y_2$ are joined by a path, say R . Then $C \cup R \cup R'$ is a K_4 -subdivision and G is not outerplanar. \square

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Chapter 5

Contractible and removable edges in 3-connected infinite graphs

5.1 Introduction

In 1961, Tutte [15] developed a theory of 3-connected graphs based on the study of essential edges. He proved the famous Tutte's Wheel Theorem: every 3-connected finite graph can be obtained from a wheel by edge additions and vertex splittings. As an immediate corollary, every 3-connected finite graph nonisomorphic to K_4 contains a contractible edge (an edge whose contraction results in a 3-connected graph). Ever since then, much have been known about contractible edges in 3-connected finite graphs. One may consult the survey paper by Kriesell [11] for further details.

Concerning the distribution of contractible edges, Ando, Enomoto and Saito [1] proved that every 3-connected finite graph G nonisomorphic to K_4 contains at least $\frac{|V(G)|}{2}$ contractible edges. Dean [8] proved that for every 3-connected finite graph which is triangle-free or has minimum degree at least 4, the subgraph induced by all the contractible edges is 2-connected.

Theorem 5.1 (Dean [8]). *For every k -connected finite graph ($k \geq 3$) which is triangle-free or has minimum degree at least $\lfloor \frac{3k}{2} \rfloor$, the subgraph induced by all the contractible edges is 2-connected.*

Besides these global results, Kriesell [10] investigated the local structure of contractible edges around a vertex in a 3-connected finite graph. We call a pair of paths (P_1, P_2) in a 2-connected graph H a *border pair* if

- (1) $|V(P_1)| \geq 2$ and $|V(P_2)| \geq 2$.
- (2) all vertices in $P_1 \cup P_2$ have degree 2 in H .

(3) $P_1 \cap (P_2 \cup N_H(P_2)) = \emptyset$ (and hence $P_2 \cap (P_1 \cup N_H(P_1)) = \emptyset$).

(4) $H - V(P_1)$ and $H - V(P_2)$ are 2-connected.

Theorem 5.2 (Kriesell [10]). *Let G be a 3-connected finite graph and x be a vertex of G . If x is not incident to any contractible edges, then $G - x$ is a cycle, or $G - x$ contains a border pair in $G[N_G(x)]$ and all edges in $G - x$ incident to the vertices of the border pair are contractible.*

Traditionally, research on contractible edges have focused on finite graphs because of the proof techniques available such as induction, *reductio ad absurdum*, and the theory of atoms and ends as developed by Mader [18]. Only a few results were known for contractible edges in infinite graphs. For example, Mader [18] proved that every contraction-critical locally finite infinite graph has infinitely many triangles. Therefore, a lot remains to be learned about the distribution and structure of contractible edges in k -connected infinite graphs. As demonstrated below, such investigations can lead to simplifications of proofs, generalizations of results, and better understanding of contractible edges in general. A first step towards this goal was taken by the author in [6] for 2-connected locally finite infinite graphs.

The purpose of this paper is to prove infinite versions of Theorem 5.1 and Theorem 5.2 for 3-connected infinite graphs. Unless otherwise stated, all graphs considered can be finite or infinite. Note that Theorem 5.1 is no longer true for 3-connected locally finite infinite graphs G which is triangle-free or has minimum degree at least 4 as demonstrated by the cartesian product of a double ray and a path of length two, and the cartesian product of a double ray and a triangle respectively. In both cases, the subgraph G_C induced by all the contractible edges consists of three disjoint double rays and is not connected. However, the situation can be remedied by viewing the graph topologically as introduced by Diestel and Kühn [10, 11, 12]. When we add the two ends of G to G_C and form the closure $\overline{G_C}$ in $|G|$ (the Freudenthal compactification of G), $\overline{G_C}$ is topologically 2-connected. In [6], the author proved that for every 2-connected locally finite infinite graph G , $\overline{G_C}$ is topologically 2-connected. Using a similar approach, we will prove an infinite analog of Theorem 5.1 for 3-connected locally finite infinite graphs.

Theorem 5.3. *Let G be a 3-connected locally finite infinite graph which is triangle-free or has minimum degree at least 4, and G_C be the subgraph induced by all the contractible edges. Then $\overline{G_C}$ is topologically 2-connected.*

Kriesell's proof of Theorem 5.2 [10] makes use of the theory of critically connected graphs [18] and is not easily generalizable to infinite graphs. We will use the notion of removable edge (an edge whose deletion results in a subdivision of a 3-connected graph) together with the ideas from Wu's paper [16] to establish an infinite version of Theorem 5.2.

Theorem 5.4. *Let G be a 3-connected graph and x be a vertex of finite degree in G . If x is not incident to any contractible edges, then $G - x$ is a finite cycle,*

or $G - x$ contains a border pair in $G[N(x)]$ and all edges in $G - x$ incident to the vertices of the border pair are contractible.

This immediately leads to the following result concerning the distribution of contractible edges in 3-connected locally finite infinite graphs.

Theorem 5.5. *Every 3-connected locally finite infinite graph has infinitely many contractible edges.*

The concept of removable edge was introduced by Barnette and Gröbaum [2] who proved that every 3-connected finite graph nonisomorphic to K_4 contains a removable edge, and used it to prove Steinitz's theorem on convex 3-polytopes. Holton et al. [9] and Su [14] studied the distribution of removable edges in a 3-connected finite graph. The former proved that every 3-connected finite graph G nonisomorphic to K_4 has at most $\lfloor \frac{4|V(G)|-5}{3} \rfloor$ non-removable edges. The latter proved that except for the wheels W_5 and W_6 , the number of removable edges in a 3-connected finite graph G is at least $\frac{3|V(G)|+18}{7}$ and characterized all the extremal graphs. Here, we will prove an infinite version of Su's result by showing that every 3-connected locally finite infinite graph has infinitely many removable edges.

Theorem 5.6. *Every 3-connected locally finite infinite graph has infinitely many removable edges.*

The paper is organized as follows. After introducing the necessary terminology in Section 2, Theorem 5.6 is proved in Section 3. Section 4 is devoted to the proof of Theorem 5.4 and 5.5. We prove Theorem 5.3 in Section 5.

5.2 Definitions

All basic graph-theoretical terminology can be found in Diestel [9]. A *triangle* is a cycle of length 3 and a *quadrilateral* is a cycle of length 4. A *triad* is a vertex of degree 3 together with all its incident edges and vertices. A *wheel* W_n of order n consists of a cycle of length $n - 1$ and a vertex, called the *center*, which is adjacent to every vertex of the cycle. Let $G = (V, E)$ be a graph and X, Y be two disjoint subsets of V . Denote the set of all edges between X and Y by $E_G(X, Y)$ or simply $E(X, Y)$. Denote the set of all edges incident to a vertex x by $E_G(x)$ and the set of all neighbors of x by $N_G(x)$. Define $N_G(X) := (\bigcup_{x \in X} N_G(x)) \setminus X$. A set S of l vertices is called an *l -separator* if $G - S$ is not connected. For any edge e in a graph G , denote the *deletion* and *contraction* of e by $G - e$ and G/e respectively. An edge of a k -connected graph is said to be *deletable/contractible* if its deletion/contraction results in a k -connected graph. Let $G = (V, E)$ be a 3-connected graph. Denote the set of all contractible edges in G by E_C and the subgraph induced by all the contractible edges by $G_C := (V, E_C)$. Let x be a vertex of degree two. The *suppression* of x involves deleting x and adding an edge between the neighbors of x . Following Holton et al. [9], we define the operation of *edge removal*, $G \ominus e$, as below:

- (1) Delete e from G to get $G - e$.
- (2) If some endvertices of e have degree two in $G - e$, then suppress them.
- (3) If multiple edges occur after (2), then replace them by single edges to make the graph simple.

An edge e is said to be *removable* if $G \ominus e$ is 3-connected. Let $S \subseteq V(G) \setminus V(e)$ with $|S| = 2$. We call (e, S) a *separating pair* of G if $G - e - S$ has exactly two components A and B , both of which have order at least two. In this case, $(e, S; A, B)$ is a *separating group* of G .

Let G be a locally finite graph. A *ray* is a 1-way infinite path, a *double ray* is a 2-way infinite path, and the subrays of a ray or double ray are its *tails*. An *end* is an equivalence class of rays where two rays are *equivalent* if no finite set of vertices separates them. Denote the set of all ends by $\Omega(G)$. We define a topological space, denoted by $|G|$, on G together with its ends, which is known as the *Freudenthal compactification* of G as follows. View G as a 1-complex. Thus, every edge is homeomorphic to the unit interval. The basic open neighborhoods of a vertex consists of a choice of half-edges, one for each incident edge. For an end $\omega \in \Omega(G)$, we take as a basic open neighborhood the set of the form: $\hat{C}(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}(S, \omega)$, where $S \subseteq V$ is a finite set of vertices, $C(S, \omega)$ is the component of $G - S$ in which every ray from ω has a tail, $\Omega(S, \omega)$ is the set of all ends whose rays have a tail in $C(S, \omega)$, and $\mathring{E}(S, \omega)$ is the set of inner points of edges between S and $C(S, \omega)$. Let H be a subgraph of G . Then the closure of H in $|G|$ is called a *standard subspace* and is denoted by \overline{H} . A homeomorphic image of the unit interval $[0, 1]$ in $|G|$ is called an *arc*. A homeomorphic image of the unit circle in $|G|$ is called a *circle*. Let X be a topological space. We say X is *topologically 2-connected* if for all $x \in X$, $X \setminus x$ is connected.

5.3 Removable edges

First, we list several lemmas concerning removable and non-removable edges from Holton et al.'s paper [9]. Although they work primarily on finite graphs, many of their results hold true for infinite graphs as well.

Lemma 5.7 (Holton et al. [9]). *Let G be a 3-connected graph of order at least six and $e \in E(G)$. Then e is non-removable if and only if there is a separating pair (e, S) (or a separating group $(e, S; A, B)$) of G .*

Using Lemma 5.7, we can derive the following fundamental lemma concerning contractible and removable edges in 3-connected graphs.

Lemma 5.8. *Every edge of a 3-connected graph nonisomorphic to K_4 is contractible or removable.*

Combining the above two lemmas, we immediately have:

Lemma 5.9 (Holton et al. [9]). *Let G be a 3-connected graph of order at least six and (xy, S) be a separating pair of G . Then every edge joining S and $\{x, y\}$ or joining the two vertices in S , if exists, is non-contractible and hence removable.*

Lemma 5.9 guarantees the existence of removable edges in a triangle and in a quadrilateral.

Lemma 5.10 (Holton et al. [9]). *Let G be a 3-connected graph nonisomorphic to K_4 . Then every triangle contains at least two removable edges. If $G \not\cong W_5$, then every quadrilateral contains a removable edge.*

Using Lemma 5.7 and 5.9, Holton et al. [9] proved that every cycle in a 3-connected finite graph nonisomorphic to K_4 contains a removable edge or is incident to two removable edges.

Theorem 5.11 (Holton et al. [9]). *Let G be a 3-connected finite graph nonisomorphic to K_4 and C be a cycle of G . Suppose no edges of C are removable. Then there exists an edge xy in C and a vertex a of G such that ax and ay are removable edges, $\deg_G(x) = \deg_G(y) = 3$, and $\deg_G(a) \geq 4$.*

Their proof involves analyzing the separating group $(e, S; A, B)$ in G where e and S are chosen so that $e \in E(C)$ and $\min\{|A|, |B|\}$ is minimal. Here, we generalize the above theorem to 3-connected infinite graphs. We call an edge e *super-non-removable* if there exists a separating group $(e, S; A, B)$ such that both $|A|$ and $|B|$ are infinite. The separating group $(e, S; A, B)$ is then called a *super-separating group*. Super-non-removable edge has the following nice property.

Lemma 5.12. *Let G be a 3-connected infinite graph and e be a super-non-removable edge. Then for all contractible edge $f \neq e$, e is super-non-removable in G/f .*

Proof. Let $(e, S; A, B)$ be a super-separating group in G . Let $e = xy$, $x \in A$ and $y \in B$. Without loss of generality, assume $f \in E(G[A \cup S])$. Let x_f be the vertex corresponding to f in G/f . Note that if f is adjacent to e , then e in G/f is defined to be the edge $x_f y$. If f is not incident to S , then $(e, S; A/f, B)$ is a super-separating group of G/f . If f is incident to S , then $f \in E(A - x, S)$ by Lemma 5.9. Let $f = uv$ where $u \in A - x$ and $v \in S$, and $S = \{v, w\}$. If $A - u$ is not connected, then $\{u, v, w\}$ is a 3-separator contradicting $f = uv$ is contractible. Therefore, $A - u$ is connected and $(e, \{x_f, w\}; A - u, B)$ is a super-separating group of G/f . \square

Lemma 5.13. *Let G be a 3-connected infinite graph and C be a finite cycle in G . Then not all edges in C are super-non-removable.*

Proof. Suppose all edges in C are super-non-removable. By Lemma 5.8 and 5.12, we can keep contracting edges in C until we get a quadrilateral consisting of super-non-removable edges contradicting Lemma 5.10. \square

Lemma 5.13 enables us to apply Holton et al.'s arguments to 3-connected infinite graphs without any changes.

Theorem 5.14. *Let G be a 3-connected graph nonisomorphic to K_4 and C be a finite cycle of G . Suppose no edges of C are removable. Then there exists an edge xy in C and a vertex a of G such that ax and ay are removable edges, $\deg_G(x) = \deg_G(y) = 3$, and $\deg_G(a) \geq 4$.*

As an immediate corollary, every 3-connected graph nonisomorphic to K_4 has a removable edge. Moreover, we can show that every 3-connected locally finite infinite graph has infinitely many removable edges. This will follow from Theorem 5.14 if we can find infinitely many finite cycles that are pairwise far apart.

Lemma 5.15. *Every 2-connected locally finite infinite graph G has an infinite set \mathcal{C} of finite cycles so that the distance between any two cycles in \mathcal{C} is at least two.*

Proof. We construct \mathcal{C} inductively. Obviously, G contains a finite cycle, say C_1 . Define $\mathcal{C}_1 := \{C_1\}$. Suppose we have constructed $\mathcal{C}_n := \{C_1, C_2, \dots, C_n\}$ such that the distance between any two cycles in \mathcal{C}_n is at least two. Define $D_n := \bigcup_{i=1}^n (C_i \cup N_G(C_i))$. Since G is locally finite, D_n is finite. Suppose $F_n := G - D_n$ is a forest. Since G is connected and locally finite, F_n contains finitely many trees. Hence, there exists an infinite tree T as a component in F_n . For any two vertices x, y in T , denote the unique path joining x and y in T by xTy . Define $D'_n := N_G(D_n) \cap V(T)$ and $H_n := \bigcup_{x, y \in D'_n} xTy$. Note that H_n is finite and connected. Let z be a vertex in $T \setminus H_n$. Since T is a tree and H_n is connected, there is a unique path P in T joining z to H_n with $V(P) \cap V(H_n) = \{w\}$. But then w is a cutvertex of G , a contradiction.

Therefore, $G - D_n$ contains a finite cycle, say C_{n+1} , and define $\mathcal{C}_{n+1} := \mathcal{C}_n \cup \{C_{n+1}\}$. Then $\mathcal{C} := \bigcup_{i=1}^{\infty} \mathcal{C}_i$ is the desired infinite set of cycles. \square

Theorem 5.6. *Every 3-connected locally finite infinite graph has infinitely many removable edges.*

Proof. By Lemma 5.15 and Theorem 5.14. \square

5.4 Contractible edges around a vertex

One nice feature of Tutte's theory of 3-connected graphs is that it is true for all 3-connected graphs, finite or infinite. An edge is *essential* if both its deletion and its contraction without removing parallel edges result in a graph that is not simple 3-connected.

Lemma 5.16 (Tutte [15]). *Let G be a 3-connected graph.*

(a) *Every essential edge belongs to a triangle or a triad.*

- (b) *Let xyz be a triangle of G such that xy and xz are essential. Then xy belongs to a triad of G .*
- (c) *Let xa, xb, xc be a triad of G such that xa and xb are essential. Then xa belongs to a triangle of G .*

The following lemma concerns contractible and non-contractible edges incident to a degree-3 vertex in a 3-connected graph and was proved by Ando et al. [1] (see also Wu [16]).

Lemma 5.17 (Ando et al. [1]). *Let G be a 3-connected graph nonisomorphic to K_4 and x be a degree-3 vertex with neighbors a, b, c . If xa and xb are non-contractible, then*

- (a) *both a and b have degree 3 and $ab \in E(G)$, and*
- (b) *each of x, a, b is incident to exactly one contractible edge, the edges forming a matching of G .*

Using the above two lemmas, Wu [16] proved the existence of contractible edges near a vertex of finite degree for both finite and infinite 3-connected graphs.

Lemma 5.18 (Wu [16]). *Let G be a 3-connected graph nonisomorphic to K_4 and x be a vertex of finite degree in G . If x is not incident to any contractible edges, then x has at least four degree-3 neighbors, each of which is incident to exactly two contractible edges.*

Lemma 5.18 implies that every vertex in a contraction-critical 3-connected graph nonisomorphic to K_4 has infinite degree. Kriesell [7] provided a construction of such graphs. On the other hand, if all vertices have finite degree, then there are infinitely many contractible edges.

Theorem 5.5. *Every 3-connected locally finite infinite graph has infinitely many contractible edges.*

Proof. Let G be a 3-connected locally finite infinite graph. Suppose there are only finitely many contractible edges. Then the set S of vertices incident to a contractible edge is finite. By Lemma 5.18, every vertex in $V(G) - S$ is adjacent to a vertex in S . But this is impossible since G is locally finite. \square

The rest of this section is devoted to proving Theorem 5.4 which can be regarded as a generalization of Theorem 5.2 and Lemma 5.18. We need the following two lemmas.

Lemma 5.19. *Let G be a 3-connected graph nonisomorphic to K_4 and x be a vertex in G . If y and z are non-adjacent neighbors of x such that xy and xz are non-contractible in G , then xz is non-contractible in $G \ominus xy$.*

Proof. By Lemma 5.17, $\deg_G(x) \geq 4$. Since xz is non-contractible, there exists a 3-separator $S := \{x, z, w\}$ in G . Suppose $w \neq y$. Let C be the component of $G - S$ containing y . Since y and z are non-adjacent, $|V(C)| \geq 2$. Then S is a 3-separator in $G \ominus xy$ and xz is non-contractible in $G \ominus xy$.

Suppose $w = y$. Let C_1 and C_2 be two components of $G - S$. If $\deg_G(y) \geq 4$, then $G \ominus xy = G - xy$. Hence, $\{x, z, y\}$ is a 3-separator of $G \ominus xy$ and xz is non-contractible in $G \ominus xy$. Suppose $\deg_G(y) = 3$. Let y_1 be the neighbor of y in C_1 and y_2 be the neighbor of y in C_2 . If $|V(C_i)| \geq 2$, then $\{x, z, y_i\}$ is a 3-separator in $G \ominus xy$ and xz is non-contractible in $G \ominus xy$. If $|V(C_1)| = |V(C_2)| = 1$, then G is a wheel of order five with center x . Hence, $G \ominus xy$ is K_4 and xz is non-contractible in $G \ominus xy$. \square

Lemma 5.20. *Let G be a 3-connected graph nonisomorphic to K_4 and x be a vertex of finite degree in G . Suppose all edges incident to x are non-contractible and non-deletable. Then*

- (1) $\deg_G(x) \geq 4$.
- (2) For all $y \in N_G(x)$, $\deg_G(y) = 3$.
- (3) For all $y \in N_G(x)$, y is incident to exactly two contractible edges.
- (4) $G[N_G(x)]$ is not independent.
- (5) If $G[N_G(x)]$ is connected, then $G - x$ is a finite cycle.
- (6) If $G[N_G(x)]$ is not connected, then each component P of $G[N_G(x)]$ is a finite path and $G - V(P)$ is 2-connected. Also, there exist two components of $G[N_G(x)]$ each having order at least 2.

Proof. By Lemma 5.17, $\deg_G(x) \geq 4$. For all $y \in N_G(x)$, since xy is removable by Lemma 5.8 but not deletable, $\deg_G(y) = 3$. By Lemma 5.17, y is incident to exactly two contractible edges in G . This proves (1), (2) and (3).

Suppose $G[N_G(x)]$ is independent. By Lemma 5.8 and 5.19, we can repeatedly remove edges in $E_G(x)$ so that the resulting graph G' is 3-connected nonisomorphic to K_4 , $\deg_{G'}(x) = 3$, and all edges in $E_{G'}(x)$ are non-contractible in G' . But this is impossible by Lemma 5.17. This proves (4).

Suppose $G[N_G(x)]$ is connected. By (2), $G[N_G(x)]$ is a finite path or a finite cycle containing all the vertices of $N_G(x)$. If $G[N_G(x)]$ is a path, then the two endvertices of the path form a 2-separator of G , a contradiction. Therefore, $G[N_G(x)]$ is a cycle. By (2), $G - x = G[N_G(x)]$ and we have (5).

Suppose $G[N_G(x)]$ is not connected. By (2), each component P of $G[N_G(x)]$ is a finite path or a finite cycle. But P is not a cycle for otherwise x is a cutvertex of G . Suppose there is only one component P of $G[N_G(x)]$ having order at least 2 (the existence of P is guaranteed by (4)). By Lemma 5.8 and 5.19, we can repeatedly remove edges in $E_G(x) \setminus E_G(x, P)$. If $|V(P)| \geq 3$, then all edges in $E_G(x) \setminus E_G(x, P)$ are removed. In the resulting 3-connected graph G' , the endvertices of P form a 2-separator of G' , a contradiction. If $|V(P)| = 2$, then

all but one edges in $E_G(x) \setminus E_G(x, P)$ are removed. In the resulting 3-connected graph G' , $\deg_{G'}(x) = 3$ and all edges in $E_{G'}(x)$ are non-contractible in G' . But this is impossible by Lemma 5.17. Hence, there exist two components of $G[N_G(x)]$ each having order at least 2.

Finally, we complete the proof of (6) by showing that $G - V(P)$ is 2-connected for each component P of $G[N_G(x)]$. This is trivial if $|V(P)| = 1$. Suppose $|V(P)| \geq 2$, and let a and b be the endvertices of P . If $G - V(P)$ is not connected, then $\{a, b\}$ is a 2-separator of G , a contradiction. Suppose $G - V(P)$ is connected but not 2-connected. Let w be a cutvertex of $G - V(P)$. If $w = x$, then $\{x, a\}$ is a 2-separator of G , a contradiction. Suppose $w \neq x$. Let C be the component of $G - V(P) - w$ containing x and D be the other component of $G - V(P) - w$. Since G is 3-connected, $N_G(D) = \{a, b, w\}$. But then $\{x, w\}$ is a 2-separator of G unless $C = \{x\}$ which is impossible as there exist two components of $G[N_G(x)]$ each having order at least 2. \square

Theorem 5.4. *Let G be a 3-connected graph nonisomorphic to K_4 and x be a vertex of finite degree in G . If x is not incident to any contractible edges, then $G - x$ is a finite cycle, or $G - x$ contains a border pair in $G[N_G(x)]$ and all edges in $G - x$ incident to the vertices of the border pair are contractible.*

Proof. Let $F \subseteq E_G(x)$ be maximal such that $G' := G - F$ is 3-connected. Then for all $e \in E_{G'}(x)$, e is non-contractible and non-deletable in G' . By Lemma 5.20 (2) and (3), all edges in $G - x$ incident to the vertices of $N_{G'}(x)$ are contractible in G' and thus contractible in G .

If $G'[N_{G'}(x)]$ is connected, then by Lemma 5.20 (5), $G' - x$ is a cycle. Moreover, $G - x = G' - x$ for otherwise x is a cutvertex of G .

If $G'[N_{G'}(x)]$ is not connected, then by Lemma 5.20 (6), each component of $G'[N_{G'}(x)]$ is a path. Also, there exist two components of $G'[N_{G'}(x)]$ each having order at least 2. Denote the set of components of $G'[N_{G'}(x)]$ having order at least 2 by \mathcal{P} . Then $|\mathcal{P}| \geq 2$. We will show that there exist at least two paths $P, P' \in \mathcal{P}$ such that $G' - x - V(P)$ and $G' - x - V(P')$ are 2-connected.

Let P be a path in \mathcal{P} . By Lemma 5.20 (6), $G' - x - V(P)$ is connected. Suppose $G' - x - V(P)$ is not 2-connected. Let w be a cutvertex of $G' - x - V(P)$. Then for all $e \in E_{G'}(x, P)$, $V(e) \cup \{w\}$ is a 3-separator of G' . Therefore, we can remove all but one edges in $E_{G'}(x, P)$ and the remaining edge in $E_{G'}(x, P)$ is non-contractible in the resulting graph G'' . By Lemma 5.19, all edges in $E_{G'}(x) \setminus E_{G'}(x, P)$ are also non-contractible in G'' .

Denote \mathcal{Q} to be the set of paths $Q \in \mathcal{P}$ such that $G' - x - V(Q)$ is 2-connected. Perform the above edge removal procedure for as many $P \in \mathcal{P} \setminus \mathcal{Q}$ as possible and denote the resulting 3-connected graph by H . Suppose $|\mathcal{Q}| = 0$. Then either $\deg_H(x) = 3$ and all edges in $E_H(x)$ are non-contractible in H contradicting Lemma 5.17, or $\deg_H(x) \geq 4$ and $H[N_H(x)]$ is independent contradicting Lemma 5.20 (4). Suppose $|\mathcal{Q}| = 1$. Then $H[N_H(x)]$ has only one component of order at least 2 contradicting Lemma 5.20 (6). Therefore, $|\mathcal{Q}| \geq 2$ and the proof is complete. \square

5.5 Subgraphs induced by all the contractible edges

In this section, we use the method in [6] and prove that for every 3-connected locally finite infinite graph G which is triangle-free or has minimum degree at least 4, $\overline{G_C}$ is topologically 2-connected. This requires the following lemma concerning the existence of contractible edges between a 3-separator and its components.

Lemma 5.21. *Let G be a 3-connected locally finite graph which is triangle-free or has minimum degree at least 4, and xy be a non-contractible edge. Let S be a 3-separator containing xy and C be a component of $G - S$. Then $E_G(x, C)$ contains a contractible edge.*

Proof. Let $S := \{x, y, z\}$. Suppose all edges in $E_G(x, C)$ are non-contractible. Delete a maximal subset F of $E_G(x, C)$ so that $G' := G - F$ is 3-connected. Let $E_{G'}(x, C) = \{xx_1, xx_2, \dots, xx_k\}$. Note that $E_{G'}(x, C)$ is non-empty and all xx_i 's are non-contractible and hence removable in G' .

Suppose $k = 1$. Since G is triangle-free or has minimum degree at least 4, $|V(C)| \geq 2$. But then $\{y, z\}$ is a 2-separator of $G' \ominus xx_1$, a contradiction.

Suppose $k \geq 2$. Then $\deg_{G'}(x) \geq 4$. By the maximality of F , $\deg_{G'}(x_i) = \deg_G(x_i) = 3$ for $1 \leq i \leq k$. Therefore, both G and G' are triangle-free. This implies that $\{x_1, \dots, x_k\}$ are independent in G , $|V(C)| \geq k + 1$, and $|V(G)| \geq k + 6$. Let x, u, v be the neighbors of x_1 and $\{x, x_i, y_i\}$ be a 3-separator of G' for $2 \leq i \leq k$. Since G' is triangle-free, $u \neq y$ and $v \neq y$. Therefore, one of u and v , say u , lies in $C \setminus \{x_1, \dots, x_k\}$. Suppose $y_i \neq x_1$. Let D be the component of $G' - \{x, x_i, y_i\}$ containing x_1 . Since x_1 and x_i are non-adjacent in G' , $|V(D)| \geq 2$ and $\{x, x_i, y_i\}$ is a 3-separator of $G' \ominus xx_1$. If $y_i = x_1$, then $\{x, x_i, u\}$ or $\{x, x_i, v\}$ is a 3-separator of $G' \ominus xx_1$ since $|V(G')| \geq k + 6$. Hence, xx_i is non-contractible in $G' \ominus xx_1$ for $2 \leq i \leq k$. Note that $\{x_2, \dots, x_k\}$ are independent in $G' \ominus xx_1$ and $|V(G' \ominus xx_1)| \geq k + 5$.

Similarly, we can repeatedly remove edges $xx_1, xx_2, \dots, xx_{k-1}$ from G' so that $G'' := G' \ominus xx_1 \ominus \dots \ominus xx_{k-1}$ is 3-connected and xx_k is non-contractible in G'' . However, $\{y, z\}$ is a 2-separator of $G'' \ominus xx_k$, a contradiction. \square

Corollary 5.22. *Let G be a 3-connected locally finite infinite graph which is triangle-free or has minimum degree at least 4. Then every vertex is incident to at least two contractible edges.*

Another lemma we will need is the following well-known result for locally finite infinite graphs.

Lemma 5.23 (Diestel [9]). *Let U be an infinite set of vertices in a connected locally finite infinite graph. Then there exists a ray R such that there are infinitely many disjoint U - R paths.*

Lemma 5.24. *Let G be a 3-connected locally finite infinite graph which is triangle-free or has minimum degree at least 4 and x be a point of $|G|$. Suppose*

there is a partition (X, X') of $V(G \setminus x)$ such that $E(X, X')$ is non-empty and all edges in $E(X, X')$ are non-contractible. Then G contains infinitely many disjoint X - X' edges: $x_0x'_0, x_1x'_1, \dots$ such that $x_ix'_i$ converges to an end ω of G with $\omega \neq x$.

Proof. Let $x_0x'_0$ be a X - X' edge and S_0 be a 3-separator containing x_0 and x'_0 . Let C_0 be a component of $G - S_0$ such that $x \notin \overline{C_0}$ and C'_0 be a component of $G - S_0$ such that $x \in \overline{C'_0} \cup S_0$. By Lemma 5.21, there exists contractible edges x_0y_0 and $x'_0y'_0$ with $y_0 \in X \cap C_0$ and $y'_0 \in X' \cap C_0$. Note that $y_0 \neq y'_0$. Choose a path Q_0 in C_0 between y_0 and y'_0 . Then there exists a X - X' edge $x_1x'_1$ in Q_0 .

Let S_1 be a 3-separator containing x_1 and x'_1 . Let C'_1 be the component of $G - S_1$ so that $C'_1 \cap \{x_0, x'_0\} \neq \emptyset$ and C_1 be any component of $G - S_1$ other than C'_1 . Then $\{x_0, x'_0\} \subseteq C'_1 \cup S_1$. By Lemma 5.21, there exists contractible edges x_1y_1 and $x'_1y'_1$ with $y_1 \in X \cap C_1$ and $y'_1 \in X' \cap C_1$. Choose a path Q_1 in C_1 between y_1 and y'_1 . Then there exists a X - X' edge $x_2x'_2$ in Q_1 . Note that $x_2x'_2$ is disjoint from $x_0x'_0$ and $x_1x'_1$. Let $z_0^1 \in C'_1 \cap \{x_0, x'_0\}$. Then any two internally disjoint paths between z_0^1 and x_2 must meet $\{x_1, x'_1\}$.

Let S_2 be a 3-separator containing x_2 and x'_2 . Let C'_2 be the component of $G - S_2$ so that $C'_2 \cap \{x_1, x'_1\} \neq \emptyset$ and C_2 be any component of $G - S_2$ other than C'_2 . Then $\{x_1, x'_1\} \subseteq C'_2 \cup S_2$. Since G is 3-connected, $G[C_2 \cup \{x_2, x'_2\}]$ is 2-connected. Because any two internally disjoint paths between z_0^1 and x_2 must meet $\{x_1, x'_1\}$, $z_0^1 \notin C_2$. If $z_0^1 \in C'_2$, then $\{x_0, x'_0\} \subseteq C'_2 \cup S_2$. If $z_0^1 \in S_2$, then there can only be one $z_0^1 - C_2$ edge for otherwise there will be two internally disjoint paths between z_0^1 and x_2 not intersecting $\{x_1, x'_1\}$. This $z_0^1 - C_2$ edge is contractible by Lemma 5.21. Since $x_0x'_0$ is non-contractible, $\{x_0, x'_0\} \subseteq C'_2 \cup S_2$. By Lemma 5.21, there exists contractible edges x_2y_2 and $x'_2y'_2$ with $y_2 \in X \cap C_2$ and $y'_2 \in X' \cap C_2$. Choose a path Q_2 in C_2 between y_2 and y'_2 . Then there exists a X - X' edge $x_3x'_3$ in Q_2 . Note that $x_3x'_3$ is disjoint from $x_0x'_0$, $x_1x'_1$ and $x_2x'_2$. Let $z_0^2 \in C'_2 \cap \{x_0, x'_0\}$ and $z_1^2 \in C'_2 \cap \{x_1, x'_1\}$. Then for $i = 0, 1$, any two internally disjoint paths between z_i^2 and x_3 must meet $\{x_2, x'_2\}$.

Suppose we have constructed pairwise disjoint X - X' edges $x_0x'_0, x_1x'_1, \dots, x_nx'_n$ together with:

1. a 3-separator S_{n-1} containing x_{n-1} and x'_{n-1} ,
2. a component C'_{n-1} of $G - S_{n-1}$ such that $x_0, x'_0, x_1, x'_1, \dots, x_{n-2}, x'_{n-2} \in C'_{n-1} \cup S_{n-1}$,
3. a component C_{n-1} of $G - S_{n-1}$ other than C'_{n-1} such that $x_n, x'_n \in C_{n-1}$,
4. vertices y_{n-1} and y'_{n-1} in C_{n-1} such that $x_{n-1}y_{n-1}$ and $x'_{n-1}y'_{n-1}$ are contractible edges,
5. a path Q_{n-1} in C_{n-1} between y_{n-1} and y'_{n-1} containing an X - X' edge $x_nx'_n$, and
6. vertices $z_0^{n-1}, z_1^{n-1}, \dots, z_{n-2}^{n-1}$ such that $z_i^{n-1} \in C'_{n-1} \cap \{x_i, x'_i\}$ for $i = 0, 1, \dots, n-2$.

Let S_n be a 3-separator containing x_n and x'_n . Let C'_n be the component of $G - S_n$ so that $C'_n \cap \{x_{n-1}, x'_{n-1}\} \neq \emptyset$ and C_n be any component of $G - S_n$ other than C'_n . Then $\{x_{n-1}, x'_{n-1}\} \subseteq C'_n \cup S_n$. Since G is 3-connected, $G[C_n \cup \{x_n, x'_n\}]$ is 2-connected. For each $i = 0, 1, \dots, n-2$, because any two internally disjoint paths between z_i^{n-1} and x_n must meet $\{x_{n-1}, x'_{n-1}\}$, $z_i^{n-1} \notin C_n$. If $z_i^{n-1} \in C'_n$, then $\{x_i, x'_i\} \subseteq C'_n \cup S_n$. If $z_i^{n-1} \in S_n$, then $\{x_{n-1}, x'_{n-1}\} \subseteq C'_n$. If there are at least two $z_i^{n-1} - C_n$ edges, then since $G[C_n \cup \{x_n, x'_n\}]$ is 2-connected, we can find two internally disjoint paths between z_i^{n-1} and x_n not meeting $\{x_{n-1}, x'_{n-1}\}$, a contradiction. Thus, there can only be one $z_i^{n-1} - C_n$ edge which is contractible by Lemma 5.21. Since $x_i x'_i$ is non-contractible, $\{x_i, x'_i\} \subseteq C'_n \cup S_n$. By Lemma 5.21, there exists contractible edges $x_n y_n$ and $x'_n y'_n$ with $y_n \in X \cap C_n$ and $y'_n \in X' \cap C_n$. Choose a path Q_n in C_n between y_n and y'_n . Then there exists a X - X' edge $x_{n+1} x'_{n+1}$ in Q_n . Note that $x_{n+1} x'_{n+1}$ is disjoint from $x_0 x'_0, x_1 x'_1, \dots, x_n x'_n$. For each $i = 0, 1, \dots, n-1$, since $\{x_i, x'_i\} \subseteq C'_n \cup S_n$, we can choose $z_i^n \in C'_n \cap \{x_i, x'_i\}$.

By applying Lemma 7.17 to $U := \{x_0, x_1, \dots\}$, there exists a ray R such that there are infinitely many disjoint U - R paths. Let ω be the end containing R . Without loss of generality, we may assume $x_0 x'_0, x_1 x'_1, \dots$ converges to ω . If x is not an end, then clearly, $\omega \neq x$. Suppose x is an end. Then $x \in \overline{C'_0}$. By construction, $x_0, x'_0, x_1, x'_1 \in C_0 \cup S_0$. Consider $Q'_1 := x_1 y_1 \cup Q_1 \cup x'_1 y'_1$. Since $\{x_0, x'_0\} \subseteq C'_1 \cup S_1$, $Q'_1 \cap \{x_0, x'_0\} = \emptyset$. Therefore, $\{x_2, x'_2\} \subseteq C_0 \cup S_0$. Suppose we have shown that $\{x_n, x'_n\} \subseteq C_0 \cup S_0$. Consider $Q'_n := x_n y_n \cup Q_n \cup x'_n y'_n$. Since $\{x_0, x'_0\} \subseteq C'_n \cup S_n$, $Q'_n \cap \{x_0, x'_0\} = \emptyset$. Therefore, $\{x_{n+1}, x'_{n+1}\} \subseteq C_0 \cup S_0$. By induction, $\{x_i, x'_i\} \subseteq C_0 \cup S_0$ for all $i = 0, 1, 2, \dots$. Hence, $\omega \in \overline{C_0}$ and $\omega \neq x$. \square

We are now ready for the proof of Theorem 5.3.

Theorem 5.3. *Let G be a 3-connected locally finite infinite graph which is triangle-free or has minimum degree at least 4, and G_C be the subgraph induced by all the contractible edges. Then $\overline{G_C}$ is topologically 2-connected.*

Proof. Suppose $\overline{G_C}$ is not topologically 2-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not connected. Let U and U' be two disjoint non-empty open sets in $|G|$ such that $\overline{G_C} \setminus x \subseteq U \cup U'$, $(\overline{G_C} \setminus x) \cap U \neq \emptyset$ and $(\overline{G_C} \setminus x) \cap U' \neq \emptyset$. Define $X := (\overline{G_C} \setminus x) \cap U \cap V(G)$ and $X' := (\overline{G_C} \setminus x) \cap U' \cap V(G)$. Since G_C is a spanning subgraph of G by Corollary 5.22, $X \cup X' = V(G \setminus x)$. By the connectedness of an edge and the definition of a basic open neighborhood of an end, both X and X' are non-empty. Since $G \setminus x$ is topologically connected, $E(X, X')$ is non-empty. Suppose x is a vertex or an end of G . By the connectedness of an edge, all edges in $E(X, X')$ are non-contractible. Suppose x is an interior point of an edge e . Then all edges in $E(X, X')$ are non-contractible unless $e \in E(X, X') \cap E_C$. Note that in this case, $E(X, X') - e$ is non-empty as G is 2-connected, and every edge in $E(X, X') - e$ is non-contractible. Let $e = yy'$ where $y \in X$ and $y' \in X'$. Suppose $X = \{y\}$. By

Corollary 5.22, y is incident to at least two contractible edges, a contradiction. Therefore, we can use y instead of x and $(X - y, X')$ as the required partition of $V(G \setminus y)$ in Lemma 5.24.

By Lemma 5.24, G contains infinitely many disjoint X - X' edges: $x_0x'_0, x_1x'_1, \dots$ such that $x_ix'_i$ converges to an end ω of G with $\omega \neq x$. Since G_C is spanning, $\overline{G_C} \setminus x$ contains all the ends of $|G|$ except possibly x . Without loss of generality, assume $\omega \in U$. Since U is open, there exists a basic open neighborhood $\hat{C}(S, \omega) \subseteq U$. Since $x'_0, x'_1, x'_2, \dots \in U'$ converge to ω , infinitely many of them lie in $\hat{C}(S, \omega)$ contradicting $U \cap U' = \emptyset$. \square

Note that Theorem 5.3 is best possible as demonstrated by the cartesian product of a ray and a triangle which contains a triangle and has minimum degree 3. Here, $\overline{G_C}$ contains 3 disjoint rays together with their common end, and is connected but not topologically 2-connected.

Lemma 5.25 (Diestel and Kühn [12]). *Let G be a locally finite graph. Then every closed connected subspace of $|G|$ is arc-connected.*

Corollary 5.26. *Let G be a 3-connected locally finite infinite graph which is triangle-free or has minimum degree at least 4. Then every contractible edge of G lies in a circle consisting of contractible edges.*

Proof. Let xy be a contractible edge in G . Since $\overline{G_C}$ is topologically 2-connected, $\overline{G_C} - xy$ is connected. By Lemma 5.25, $\overline{G_C} - xy$ is arc-connected and there is an arc A between x and y in $\overline{G_C} - xy$. Then $A \cup xy$ is a circle consisting of contractible edges. \square

Unfortunately, the arguments used in the proof of Theorem 5.3 cannot be generalized directly to k -connected locally finite infinite graphs for $k \geq 4$. We end this paper with the following conjecture.

Conjecture 5.27. *Let G be a k -connected locally finite infinite graph ($k \geq 4$) which is triangle-free or has minimum degree greater than $\frac{3}{2}(k-1)$, and G_C be the subgraph induced by all the contractible edges. Then $\overline{G_C}$ is topologically 2-connected.*

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Chapter 6

Contraction-critical 4-connected locally finite infinite graphs

6.1 Introduction

It is well-known that the only contraction-critical 2-connected and 3-connected finite graphs are K_3 and K_4 respectively. For 4-connected graphs, Fontet [6] and Martinov [8] independently proved that every contraction-critical 4-connected finite graph is either the square of a cycle or the line graph of a cyclically 4-edge-connected cubic graph. This is equivalent to the characterization that a 4-connected finite graph is contraction-critical if and only if it is 4-regular and every edge lies in a triangle (see also Martinov [9]).

Recently, Ando and Egawa [1] proved that for any vertex of degree greater than four in a 4-connected finite graph, there exists a contractible edge at distance one or less from that vertex. They [2] also showed that for every non-contractible edge not lying in a triangle, there exists a contractible edge at distance one or less from that edge.

For infinite graphs, Kriesell [7] gave a construction of contraction-critical k -connected graphs for $k \geq 2$ such that all vertices have infinite degree. On the other hand, the author [6, 7] showed that for $k = 2, 3$, every k -connected locally finite infinite graph contains infinitely many contractible edges. In this paper, we modify the results in Ando and Egawa's paper [1] slightly and extend Fontet and Martinov's result to locally finite infinite graphs.

Theorem 6.1. *A 4-connected locally finite graph is contraction-critical if and only if it is 4-regular and every edge lies in a triangle.*

Theorem 6.2. *A 4-connected locally finite infinite graph is contraction-critical if and only if it is the line graph of a 3-edge-connected and cyclically 4-edge-connected cubic graph.*

Moreover, a theorem of Ando and Egawa [1] on the lower bound of contractible edges can be generalized to 4-connected locally finite graphs.

Theorem 6.3. *Every 4-connected locally finite graph G has at least $|V_{\geq 5}(G)|$ contractible edges.*

6.2 Definitions

All graph-theoretical terminology not defined here can be found in Diestel [9]. For any graph G , denote the set of degree i vertices in G by $V_i(G)$ and the set of vertices of degree at least i in G by $V_{\geq i}(G)$. A subset S of $V(G)$ is called a k -separator if $|S| = k$ and $G - S$ is not connected. Now suppose G is k -connected. An edge e in G is *contractible* if the graph obtained by contracting e , denoted by G/e , is k -connected. Otherwise, e is *non-contractible*. Let x be a vertex in G . Denote the neighbors of x by $N_G(x)$ and the set of contractible edges incident to x by $E_C(x)$. A k -connected graph is *contraction-critical* if all of its edges are non-contractible. A graph G is *cyclically k -edge-connected* if for any edge cut F with $|F| < k$, at least one of the components of $G - F$ contains no cycle. Let H be a graph. The *square* of H , denoted by H^2 , is the graph on $V(H)$ such that two vertices are adjacent if and only if they have distance at most two in H . We call a subgraph H of G *squareable* if $H^2 \subseteq G$. A *ray* is a 1-way infinite path.

6.3 Contraction-critical 4-connected locally finite infinite graphs

First, we need two results from Ando and Egawa's paper [1] which can be generalized to infinite graphs by essentially the same proofs (see Appendix for details).

Lemma 6.4 (Ando and Egawa [1]). *Let G be a 4-connected graph and x be a vertex of finite degree such that $E_C(x) = \emptyset$. Then $G[N_G(x)]$ contains a subgraph $H_1 \cup H_2$ such that $V(H_1) \cap V(H_2) = \emptyset$, $H_i \cong K_2$ for $i = 1, 2$, and $V(H_i) \cap V_4(G) \neq \emptyset$ for $i = 1, 2$.*

Lemma 6.5 (Ando and Egawa [1]). *Let G be a 4-connected graph and x be a vertex of finite degree greater than 4. Then there is a contractible edge whose distance from x is one or less. Moreover, if $G[N_G(x) \cap V_4(G)] \not\cong P_4$, then there are at least two contractible edges whose distance from x is one or less.*

By combining Lemma 6.4 and 6.5, we immediately obtain Theorem 6.1.

Theorem 6.1. *A 4-connected locally finite graph is contraction-critical if and only if it is 4-regular and every edge lies in a triangle.*

Proof. (\Leftarrow) Obvious. (\Rightarrow) Lemma 6.5 implies that every vertex of G has degree 4. By Lemma 6.4, every edge of G lies in a triangle. \square

Now, we are ready to prove Theorem 6.2. The proof follows closely as that of Martinov [8]. The only difference is that in his proof, Martinov used the fact that a cubic finite graph is cyclically 4-edge-connected if and only if its line graph is 4-connected. This is no longer true for infinite graphs as demonstrated by the cubic infinite tree. For arbitrary cubic graphs, we have the following analogous results.

Lemma 6.6. *Let G be a cubic graph. Then G is 3-edge-connected and cyclically 4-edge-connected if and only if $L(G)$ is 4-connected.*

Proof. (\Leftarrow) Suppose G is not 3-edge-connected. Let F be an edge cut with $|F| < 3$. Since G is cubic, every component of $G - F$ contains an edge. Hence, $L(G)$ is not 3-connected. Suppose G is not cyclically 4-edge-connected. There exists an edge cut F with $|F| < 4$ such that every component of $G - F$ contains a cycle. Then $L(G)$ is not 4-connected.

(\Rightarrow) Suppose $L(G)$ is not 4-connected. Let S be a separator of $L(G)$ with $|S| < 4$ and F be the set of edges in G corresponding to S in $L(G)$. Then F is an edge cut of G . If G is not 3-edge-connected, then we are done. Suppose G is 3-edge-connected. Consider any component C of $G - F$. If C is not 2-edge-connected, then C has a cut edge, say e . Let C_1 and C_2 be the two components of $C - e$. Since $|F| < 4$, one of C_1 and C_2 is incident to less than two edges in F , contradicting the 3-edge-connectedness of G . Therefore, every component of $G - F$ is 2-edge-connected and contains a cycle. Hence, G is not cyclically 4-edge-connected. \square

Lemma 6.7. *Let G be a 3-edge-connected and cyclically 4-edge-connected cubic graph. Then $L(G)$ is contraction-critical 4-connected.*

Proof. By Lemma 6.6, $L(G)$ is 4-connected. Since G is cubic, every vertex of $L(G)$ has degree 4 and every edge of $L(G)$ lies in a triangle. By Theorem 6.1, $L(G)$ is contraction-critical 4-connected. \square

Lemma 6.8. *Let G be a contraction-critical 4-connected locally finite infinite graph. Then no edge of G lies in two triangles.*

Proof. Suppose G has an edge that lies in two triangles. Then G contains a squareable finite path of length at least three, say P . Let $P = x_1x_2 \dots x_k$ where $k \geq 4$. By Theorem 6.1, every vertex of G has degree 4. Let x be the neighbor of x_2 other than x_1, x_3, x_4 . If $x \in P$, then $x = x_{k-1}$ or $x = x_k$. Suppose $x = x_{k-1}$. Then $k \geq 6$ and $\{x_1, x_k\}$ is a 2-separator of G , a contradiction. Suppose $x = x_k$. Then $k \geq 5$ and $\{x_1, x_{k-1}, x_k\}$ is a 3-separator of G or $G = C_k^2$, a contradiction. Hence, $x \notin P$.

By Theorem 6.1, xx_2 lies in a triangle. Therefore, x is adjacent to x_1, x_3 or x_4 . If x is adjacent to x_1 , then $xx_1x_2 \dots x_k$ is squareable. If x is adjacent to x_3 , then $k = 4$ for otherwise $\deg_G(x_3) > 4$. Now, $\{x, x_1, x_4\}$ is a 3-separator of

G or $G = K_5$, a contradiction. If x is adjacent to x_4 , then $k = 4$ or $k = 5$ for otherwise $\deg_G(x_4) > 4$. For $k = 4$, $x_1x_3x_2x_4x$ is a squareable path. For $k = 5$, $\{x, x_1, x_5\}$ is a 3-separator of G or $G = C_6^2$, a contradiction.

From the arguments above, we see that G contains a squareable ray, say $R := y_1y_2y_3 \dots$. But then $\{y_2, y_3\}$ separates y_1 and y_4R in G which is impossible. \square

Lemma 6.9. *Let G be a contraction-critical 4-connected locally finite infinite graph. Then G is the line graph of a 3-edge-connected and cyclically 4-edge-connected cubic graph.*

Proof. By Theorem 6.1, G is 4-regular and every edge of G lies in a triangle. Consider any vertex x of G . Since no edge of G lies in two triangles by Lemma 6.8, $G[x \cup N_G(x)]$ consists of two triangles intersecting at x . It follows that G is the line graph of a cubic graph, say H . Since $L(H)$ is 4-connected, by Lemma 6.6, H is 3-edge-connected and cyclically 4-edge-connected. \square

Theorem 6.2 follows from Lemma 6.7 and 6.9.

6.4 Appendix

Here we give the necessary modifications to Ando and Egawa's paper [1] for proving Lemma 6.4 and 6.5. Let G be a k -connected graph and S be a k -separator of G . A union of at least one but not all components of $G - S$ is called a *fragment*. For any fragment A , denote $\bar{A} := G - A - N_G(A)$. For an edge e of G , a fragment A of G is said to be a *fragment with respect to e* if $V(e) \subseteq N_G(A)$. For a subset F of $E(G)$, we say that A is a *fragment with respect to F* if A is a fragment with respect to some $e \in F$. For a subset U of $V(G)$, a fragment A of G is said to be *U -opposite* if $U \subseteq \bar{A}$. For a vertex x of G , a subset $U \subseteq V(G) \setminus \{x\}$ is said to be *x -admissible* if (i) $E_C(x) \subseteq E(x, U)$ and (ii) U is contained in a fragment with respect to $E(x)$.

First, we need the following modification of Lemma 3 of Ando and Egawa [1] whose proof is exactly the same.

Lemma 6.10 (Ando and Egawa [1]). *Let x be a vertex of finite degree of a 4-connected graph G and let U be an x -admissible subset of G . Let A be a U -opposite fragment with respect to $E(x)$ such that $|A \cap N_G(x)|$ is minimum, and let $y \in N_G(x) \cap A$. Then $N_G(y) \cap U = \emptyset$, and there exists $z \in N_G(x) \cap N_G(A)$ such that (i) $U \cap \{y, z\} = \emptyset$, (ii) $yz \in E(G)$ and (iii) $\{y, z\} \cap V_4(G) \neq \emptyset$.*

Using the above lemma, we can easily prove Lemma 6.4 which corresponds to Lemma 4 of Ando and Egawa [1].

Lemma 6.4 (Ando and Egawa [1]). *Let G be a 4-connected graph and x be a vertex of finite degree such that $E_C(x) = \emptyset$. Then $G[N_G(x)]$ contains a subgraph $H_1 \cup H_2$ such that $V(H_1) \cap V(H_2) = \emptyset$, $H_i \cong K_2$ for $i = 1, 2$, and*

$V(H_i) \cap V_4(G) \neq \emptyset$ for $i = 1, 2$.

Proof. The proof is basically the same as that of Lemma 4 of Ando and Egawa [1] except that we consider a fragment A with respect to $\tilde{E}(x) := E(x) - xz$ such that $|A \cap N_G(x)|$ is minimum. Also, the proof of Lemma 4 of Ando and Egawa [1] on P.107 is modified as follows.

If $|B| = 1$ or $|\bar{B}| = 1$, then without loss of generality, assume $|B| = 1$. Suppose $B \subseteq A$. We have $S \cap B = \emptyset$ and $|(S \cap B) \cup (S \cap T) \cup (T \cap A)| = 4$. But then B is a fragment with respect to $\tilde{E}(x)$ such that $|B \cap N_G(x)| < |A \cap N_G(x)|$, a contradiction. Suppose $B \subseteq \bar{A}$. Then $S \cap B = \emptyset$ and $|(S \cap B) \cup (S \cap T) \cup (T \cap \bar{A})| \leq 3$, which is impossible. Therefore, $B \subseteq S$. Notice that $|B \cap N_G(x)| = 1$ implies $|A \cap N_G(x)| = 1$.

If $|S \cap T| = 3$, then $S \cap \bar{B} = T \cap \bar{A} = \emptyset$. Since $|(S \cap \bar{B}) \cup (S \cap T) \cup (T \cap \bar{A})| = 3$, $\bar{A} \cap \bar{B} = \emptyset$. But then $\bar{A} = \emptyset$ which is absurd. If $|S \cap T| = 2$, $|A \cap T| = 2$ and $|S \cap \bar{B}| = 1$, then $T \cap \bar{A} = \emptyset$. Since $|(S \cap \bar{B}) \cup (S \cap T) \cup (T \cap \bar{A})| = 3$, $\bar{A} \cap \bar{B} = \emptyset$. But then $\bar{A} = \emptyset$ which is absurd. Therefore, $|S \cap T| = 2$ and $|A \cap T| = |S \cap \bar{B}| = 1$. Now, $\bar{B} \cap A = \emptyset$ for otherwise $|A \cap N_G(x)| \geq 2$. Hence, $|A| = 1$. The rest of the proof proceeds as in [1]. \square

As a result, we can generalize Proposition 1, 2 and 3 of Ando and Egawa [1] to arbitrary 4-connected graphs.

Proposition 6.11 (Ando and Egawa [1]). *Let G be a 4-connected graph and let xy be a non-contractible edge such that $x, y \in V_4(G)$. If both $E_C(x)$ and $E_C(y)$ are empty, then $N_G(x) \cap N_G(y) \cap V_4(G) \neq \emptyset$ and $N_G(x) \cap N_G(y) \cap V_{\geq 5}(G) = \emptyset$.*

Proposition 6.12 (Ando and Egawa [1]). *Let G be a 4-connected graph and let xy be a non-contractible edge such that $x, y \in V_4(G)$, $|E_C(x)| \leq 1$ and $|E_C(y)| \leq 1$. If $N_G(x) \cap N_G(y) \cap V_{\geq 5}(G) \neq \emptyset$, then $|N_G(x) \cap N_G(y)| \geq 2$.*

Proof. The proof is the same as that of Proposition 2 of Ando and Egawa [1] except that we choose a fragment A so that $z \in A$ and $|A \cap (N_G(x) \cup N_G(y))|$ is minimum. \square

Proposition 6.13 (Ando and Egawa [1]). *Let G be a 4-connected graph and let xy be a non-contractible edge such that $x, y \in V_4(G)$, $E_C(x) = \emptyset$ and $|E_C(y)| \leq 1$. If $N_G(x) \cap N_G(y) \cap V_{\geq 5}(G) \neq \emptyset$, then $N_G(x) \cap N_G(y) \cap V_4(G) \neq \emptyset$.*

Finally, we can prove Lemma 6.5 corresponding to Theorem 1 of Ando and Egawa [1].

Lemma 6.5 (Ando and Egawa [1]). *Let G be a 4-connected graph and x be a vertex of finite degree greater than 4. Then there is a contractible edge whose distance from x is one or less. Moreover, if $G[N_G(x) \cap V_4(G)] \not\cong P_4$, then there are at least two contractible edges whose distance from x is one or less.*

Proof. The proof follows that of Theorem 1 of Ando and Egawa [1] except that we consider a U -opposite fragment A with respect to $E(x)$ so that $|A \cap N_G(x)|$ is

minimum. Note that for such fragment A , $|N_G(y) \cap A| = 2$. The second last sentence of the proof of Theorem 1 of Ando and Egawa [1] on P.115 is replaced by the following. By the minimality of $|A \cap N_G(x)|$, $|A \cap B \cap N_G(x)| = |A \cap N_G(x)|$. However, this implies $|N_G(y) \cap A \cap B| = 2$ contradicting $N_G(y) \cap (A \cap B) = \{v\}$. \square

Based on the above results, Theorem 2 of Ando and Egawa [1] can be generalized to 4-connected locally finite graphs which is Theorem 6.3 in the Introduction.

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Chapter 7

Contractible edges in k -connected infinite graphs

7.1 Introduction

Thomassen [20] proved the existence of contractible edges in any k -connected triangle-free finite graphs. Later, Egawa, Enomoto and Saito [14] generalized Thomassen's result and proved that every k -connected triangle-free finite graph G contains at least $\min\{|V(G)| + \frac{3}{2}k^2 - 3k, |E(G)|\}$ contractible edges. Dean [8] studied the distribution of contractible edges and proved that for every k -connected finite graph ($k \geq 3$) which is triangle-free or has minimum degree at least $\lfloor \frac{3k}{2} \rfloor$, the subgraph induced by all the contractible edges is 2-connected. Other sufficient conditions for the existence of contractible edges in k -connected finite graphs are: minimum degree at least $\lfloor \frac{5k}{4} \rfloor$ (Egawa [13]), k odd and K_4^- -free (Kawarabayashi [15]), and bowtie-free (Ando et al. [3]). One may consult the survey papers by Kriesell [16] and Ando [1] for further details.

For infinite graphs, Mader [18] proved that every contraction-critical k -connected locally finite infinite graph has infinitely many triangles. This implies that every k -connected triangle-free locally finite infinite graph contains a contractible edge. For 2-connected locally finite infinite graphs, it is easy to see that every vertex is incident to at least two contractible edges. The author [7] proved that every 3-connected locally finite infinite graph contains infinitely many contractible edges. This paper investigates various sufficient conditions that guarantee the existence of contractible edges in k -connected locally finite infinite graphs. In Section 3 and 4, we prove the existence of contractible edges in any k -connected triangle-free locally finite graphs and in any k -connected locally finite graphs with minimum degree greater than $\frac{3}{2}(k-1)$ respectively.

Theorem 7.1. *Let G be a k -connected triangle-free graph and x be a vertex of finite degree. Suppose S is a k -separator containing x and an edge. Then for any component C of $G - S$, there is a contractible edge joining x and C .*

Theorem 7.2. *Let G be a k -connected graph with minimum degree greater than $\frac{3}{2}(k-1)$ and x be a vertex of finite degree. Suppose S is a k -separator containing x . Then for any component C of $G - S$, there is a contractible edge joining x and C .*

In Section 5, we prove the existence of contractible edges in k -connected locally finite graphs with no adjacent triangles.

Theorem 7.3. *Let G be a k -connected graph ($k \geq 3$) with no adjacent triangles and x be a vertex of finite degree. Then x is incident to a contractible edge.*

In Section 6, we investigate k -connected locally finite infinite graphs with minimum end vertex-degree greater than k , and generalize Egawa's [13] and Dean's [8] results mentioned above.

Theorem 7.4. *Every k -connected locally finite infinite graph such that the minimum degree is at least $\lfloor \frac{5k}{4} \rfloor$ and all ends have vertex-degree greater than k contains a contractible edge.*

Theorem 7.5. *Let G be a k -connected locally finite infinite graph ($k \geq 4$) which is triangle-free or has minimum degree at least $\lfloor \frac{3k}{2} \rfloor$, and all ends have vertex-degree greater than k . Suppose G_C is the subgraph induced by all the contractible edges. Then the closure of G_C in the Freudenthal compactification of G is topologically 2-connected.*

7.2 Definitions

All graph-theoretical terminology not defined here can be found in Diestel [9]. Denote a cycle of length n by C_n and a complete graph with n vertices by K_n . The complete graph K_3 is called a *triangle*. Denote K_4^- to be the graph obtained by deleting an edge from K_4 . A *bowtie* is the graph formed by two triangles with exactly one vertex in common. We say two triangles are *adjacent* if they have a vertex or an edge in common. Therefore, a graph with no adjacent triangles is K_4^- - and bowtie-free. For two disjoint graphs H and H' , denote by $H * H'$ the graph obtained from $H \cup H'$ by joining all the vertices of H to all the vertices of H' . Define the *expansion* of H by H' , $H \triangleleft H'$, to be the graph G such that $V(G) := V(H) \times V(H')$ and $E(G) := \{((x, x'), (y, y')) : x = y \text{ and } x'y' \in E(H'); \text{ or } x \neq y \text{ and } xy \in E(H)\}$.

Let G be a k -connected graph. An edge e in G is *contractible* if the graph obtained by contracting e , denoted by G/e , is k -connected. Denote the set of contractible edges by E_C and the set of non-contractible edges by E_{NC} . We say G is *contraction-critical* if every edge of G is non-contractible. A subset S of $V(G)$ is called a *separator* if $G - S$ is not connected. If $|S| = l$, then S is called a *l -separator*. Denote the set of all k -separators of G by \mathcal{S} . We say S *contains* an edge if $G[S]$ contains an edge. Denote the connectivity of G by $\kappa(G)$. A vertex x of G is *critical* if $\kappa(G - x) = \kappa(G) - 1$. Define $Cr(G) := \{x \in V(G) : x \text{ is critical}\}$.

Let $S \in \mathcal{S}$. A union of at least one but not all components of $G - S$ is called a *fragment* of S . For any fragment F , define $\tilde{F} := G - F - N(F)$ which is also a fragment. Let \mathcal{X} be a set of subsets of $V(G)$. Define $\mathcal{S}_{\mathcal{X}} := \{S \in \mathcal{S} : \exists X \in \mathcal{X} \text{ with } X \subseteq S\}$. A \mathcal{X} -*fragment* is a fragment of $S \in \mathcal{S}_{\mathcal{X}}$. An \mathcal{X} -*end* is an inclusion-minimal \mathcal{X} -fragment while an \mathcal{X} -*atom* is a \mathcal{X} -fragment of the minimum size. If $\mathcal{X} = \{\{x\}\}$, then we simply write x -fragment, x -end and x -atom. We say G is \mathcal{X} -*critical* if (1) for all $X \in \mathcal{X}$, there exists $S \in \mathcal{S}$ such that $X \subseteq S$, and (2) for all \mathcal{X} -fragment F , there exists $X \in \mathcal{X}$ and $S \in \mathcal{S}$ such that $S \cap F \neq \emptyset$ and $X \subseteq S \setminus \tilde{F}$. Note that if $\mathcal{X} := \{V(e) : e \in E(G)\}$, then being \mathcal{X} -critical is the same as being contraction-critical. If $\mathcal{X} := \{\emptyset\}$, then a \mathcal{X} -critical graph is called *almost critical*. Equivalently, G is almost critical if and only if every fragment of G intersects some k -separator.

Mader [18] studied the properties of ends and atoms extensively, and proved the following useful tools for k -connected graphs.

Lemma 7.6 (Mader [18]). *Let B be a \mathcal{X} -end of G such that $|B| > \frac{\kappa(G)}{2}$ and $|\tilde{B}| \geq \frac{\kappa(G)}{2}$. If there exists a k -separator T such that $T \cap B \neq \emptyset$ and $T \cap (B \cup N(B))$ contains an $X \in \mathcal{X}$, then there exists a fragment F of T such that $F \subseteq N(B)$ and $|F| < \frac{\kappa(G)}{2}$.*

Lemma 7.7 (Mader [18]). *Let A be a \mathcal{X} -atom of G . If there exists a k -separator T such that $T \cap A \neq \emptyset$ and $T \cap (A \cup N(A))$ contains an $X \in \mathcal{X}$, then $A \subseteq T$ and $|A| \leq \frac{1}{2}|T - N(A)|$.*

7.3 Contractible edges in k -connected triangle-free graphs

First, we state a lemma for k -connected triangle-free graphs whose proof can be found in [14, 20].

Lemma 7.8. *Let G be a k -connected triangle-free graph, S be a k -separator containing an edge, and C be a component of $G - S$. Then $|C| \geq k$.*

Now, we combine the ideas and techniques used in Egawa et al. [14]’s and Mader [18]’s papers to prove the following useful lemma.

Lemma 7.9. *Let G be a k -connected graph ($k \geq 3$) and x be a vertex of finite degree. Let \mathcal{X} be any one of the following three sets of subsets of $V(G)$: $\{\{x\}\}$, $\{V(e) : e \in E(G) \text{ and } x \in V(e)\}$ or $\{\{x\} \cup V(e) : e \in E(G)\}$. Suppose there exists a \mathcal{X} -fragment F such that $E(x, F) \subseteq E_{NC}$. Then there exists a finite \mathcal{X} -fragment F' such that*

- (1) F' lies in a k -separator containing x ,
- (2) $|F'| < k - 1$, and
- (3) $E(x, F') \subseteq E_{NC}$.

Proof. We need only to prove (1) and (2) since (1) implies (3). Choose a \mathcal{X} -fragment F such that $E(x, F) \subseteq E_{NC}$ and $|F \cap N(x)|$ is minimum. Denote $S := N(F)$. Let y be a neighbor of x in F , and T be a k -separator containing x and y . Let C be a fragment of T and $\tilde{C} = G - T - C$.

If $F \setminus T = \emptyset$, then $F \subseteq T$ and $|F| \leq k - 1$. Suppose $|F| = k - 1$. Then $S \cap T = \{x\}$ and $\tilde{F} \cap T = \emptyset$. If $C \cap S = \emptyset$, then $C \cap \tilde{F} = \emptyset$ and $C = \emptyset$, a contradiction. Therefore, $C \cap S \neq \emptyset$. Similarly, $\tilde{C} \cap S \neq \emptyset$. But now, $C \cap \tilde{F} = \emptyset$ and $\tilde{C} \cap \tilde{F} = \emptyset$ implying $\tilde{F} = \emptyset$, a contradiction. Hence, $|F| < k - 1$ and F is the desired \mathcal{X} -fragment.

Suppose $F \setminus T \neq \emptyset$. Without loss of generality, assume $C \cap F \neq \emptyset$. Since $y \in F \cap N(x)$ and $y \notin C$, $|C \cap F \cap N(x)| < |F \cap N(x)|$. Therefore, $C \cap F$ is not a \mathcal{X} -fragment and $|(S \cap C) \cup (S \cap T) \cup (F \cap T)| > k$. This implies $|(S \cap \tilde{C}) \cup (S \cap T) \cup (\tilde{F} \cap T)| < k$ and $\tilde{C} \cap \tilde{F} = \emptyset$. If $\tilde{C} \cap F \neq \emptyset$, then by similar arguments, $C \cap \tilde{F} = \emptyset$ and \tilde{F} is the desired \mathcal{X} -fragment. If $\tilde{C} \cap F = \emptyset$, then $\tilde{C} \subsetneq S$. Since $|(S \cap \tilde{C}) \cup (S \cap T) \cup (\tilde{F} \cap T)| < k$ and $x \in S \cap T$, $|\tilde{C}| < k - 1$ and \tilde{C} is the desired \mathcal{X} -fragment. \square

Using the above two lemmas, we can easily prove Theorem 7.1. Note that $k = 1$ is trivial, and $k = 2$ was proved by the author in [6] for any 2-connected graphs nonisomorphic to K_3 .

Theorem 7.1. *Let G be a k -connected triangle-free graph and x be a vertex of finite degree. Suppose S is a k -separator containing x and an edge. Then for any component C of $G - S$, there is a contractible edge joining x and C .*

Proof. Suppose $E(x, C) \subseteq E_{NC}$. Let $\mathcal{X} = \{\{x\} \cup V(e) : e \in E(G)\}$. Then C is a \mathcal{X} -fragment. By Lemma 7.9, there exists a finite \mathcal{X} -fragment F with $|F| < k - 1$, contradicting Lemma 7.8. \square

Corollary 7.10. *Let G be a k -connected triangle-free graph ($k \geq 2$) and x be a vertex of finite degree. Then x is incident to at least two contractible edges.*

7.4 Contractible edges in k -connected graphs with minimum degree greater than $\frac{3}{2}(k - 1)$

Lemma 7.11. *Let G be a k -connected graph with minimum degree greater than $\frac{3}{2}(k - 1)$, S be a k -separator, and C be a component of $G - S$. Then $|C| > \frac{k-1}{2}$ or equivalently, $|C| \geq \frac{k}{2}$.*

Proof. We have $|C| + k \geq 1 + \delta(G) > \frac{3k-1}{2}$ implying $|C| > \frac{k-1}{2}$ which is equivalent to $|C| \geq \frac{k}{2}$. \square

Lemma 7.12. *Let G be a k -connected graph with minimum degree greater than $\frac{3}{2}(k - 1)$ and x be a vertex of finite degree. If A is an x -atom of G , then $E(x, A) \subseteq E_C$.*

Proof. Suppose $E(x, A) \not\subseteq E_C$. Then there exists a neighbor y of x in A such that xy is non-contractible. Let T be a k -separator containing x and y . By Lemma 7.7, $|A| \leq \frac{k-1}{2}$. But this contradicts Lemma 7.11. Hence, $E(x, A) \subseteq E_C$. \square

Theorem 7.2. *Let G be a k -connected graph with minimum degree greater than $\frac{3}{2}(k-1)$ and x be a vertex of finite degree. Suppose S is a k -separator containing x . Then for any component C of $G - S$, there is a contractible edge joining x and C .*

Proof. Suppose there exists a x -fragment F such that $E(x, F) \subseteq E_{NC}$. By Lemma 7.9, there exists a finite x -fragment F' that lies in a k -separator containing x and $E(x, F') \subseteq E_{NC}$. Without loss of generality, we can assume that F' is an x -end. By Lemma 7.12, F' is not an x -atom. Therefore, by Lemma 7.11, $|F'| > \frac{k}{2}$. Let y be a neighbor of x in F' and T be a k -separator containing x and y . Then $y \in T \cap F'$ and $x \in T \cap N(F')$. By Lemma 7.6, there exists a fragment of T with size less than $\frac{k}{2}$ contradicting Lemma 7.11. \square

Corollary 7.13. *Let G be a k -connected graph ($k \geq 2$) with minimum degree greater than $\frac{3}{2}(k-1)$ and x be a vertex of finite degree. Then x is incident to at least two contractible edges.*

Since an integer which is at least $\lfloor \frac{3k}{2} \rfloor$ is greater than $\frac{3}{2}(k-1)$ but not vice versa, Theorem 7.2 is slightly stronger than the corresponding result in Dean's paper [8].

The bound $\frac{3}{2}(k-1)$ is best possible as demonstrated by the following examples. For $k = 2l + 1$, define $G := (C_4 \triangleleft K_l) * K_1$ and let x be the vertex in K_1 . For $k = 2l$, define $G := (C_4 \triangleleft K_{l-1}) * K_2$ and let x be one of the vertices in K_2 . In both cases, G is k -connected, $\delta(G) \leq \frac{3}{2}(k-1)$, and all edges incident to x are non-contractible.

7.5 Contractible edges in k -connected graphs with no adjacent triangles

As shown in Section 3, every k -connected triangle-free locally finite infinite graph contains infinitely many contractible edges. Here we generalize this result to k -connected locally finite infinite graphs that may contain triangles but have no adjacent triangles (i.e. K_4^- -free and bowtie-free).

Lemma 7.14 (Ando et al. [2]). *Let G be a k -connected K_4^- -free graph and e be a non-contractible edge not contained in a triangle. Let S be a k -separator containing e and C be a component of $G - S$. Then $|C| \geq k - 1$.*

Lemma 7.15. *Let G be a k -connected graph ($k \geq 3$) and x be a vertex of finite degree. Let $\mathcal{X}' := \{V(e) : e \in E(G), x \in V(e) \text{ and } e \text{ does not lie in a triangle}\}$. Suppose there exists a \mathcal{X}' -fragment F such that for any edge $e \in E(x, F)$, e*

is non-contractible and e does not lie in a triangle. Then there exists a finite \mathcal{X}' -fragment F' such that

- (1) F' lies in a k -separator containing x ,
- (2) $|F'| < k - 1$, and
- (3) $E(x, F') \subseteq E_{NC}$.

Proof. The proof is the same as that of Lemma 7.9 and is omitted. Note that some edges in $E(x, F')$ may lie in a triangle. \square

Lemma 7.16. *Let G be a k -connected K_4^- -free graph ($k \geq 3$). Suppose x is a vertex of finite degree and is contained in at most one triangle. Then x is incident to a contractible edge.*

Proof. Suppose all edges incident to x are non-contractible. Since x is contained in at most one triangle, there exists a \mathcal{X}' -fragment F such that for any edge $e \in E(x, F)$, e is non-contractible and e does not lie in a triangle. By Lemma 7.15, there exists a finite \mathcal{X}' -fragment F' such that $|F'| < k - 1$. But this contradicts Lemma 7.14. \square

This immediately implies Theorem 7.3.

Theorem 7.3. *Let G be a k -connected graph ($k \geq 4$) with no adjacent triangles and x be a vertex of finite degree. Then x is incident to a contractible edge.*

7.6 Generalization of Egawa's and Dean's results

In this section, an *end* refers to an equivalence class of rays where two rays are *equivalent* if no finite set of vertices separates them. Basic results concerning the *Freudenthal compactification* of G can be found in the papers by Diestel and Kühn [10, 11, 12]. Following Bruhn and Stein [4], we define the *vertex-degree* of an end ω in G as the supremum (in fact, this is a maximum) over the cardinalities of sets of disjoint rays in ω . For k -connected locally finite graphs, Bruhn and Stein [5] (see also Stein [19]) proved that the vertex-degree of any end is at least k .

As we saw in the Introduction, Egawa [13] proved that every k -connected finite graph with minimum degree at least $\lfloor \frac{5k}{4} \rfloor$ contains a contractible edge. Here the same conclusion holds for locally finite infinite graphs if we also require that every end has vertex-degree greater than k . The key is Lemma 7.18 which guarantees the existence of a minimal fragment within each fragment.

Lemma 7.17. *Let G be a connected locally finite infinite graph and U be an infinite set of vertices in G . Then G contains a ray R with infinitely many disjoint $U - R$ paths.*

Lemma 7.18. *Let G be a k -connected locally finite infinite graph and \mathcal{X} be a set of subsets of $V(G)$. If every end has vertex-degree greater than k , then every \mathcal{X} -fragment of G contains a minimal \mathcal{X} -fragment.*

Proof. Suppose F_0 is a \mathcal{X} -fragment of G not containing a minimal \mathcal{X} -fragment. Then there exists a strictly decreasing infinite sequence of \mathcal{X} -fragments of G : $F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq \dots$. Denote $S_i := N(F_i)$ and $F'_i := G[F_i \cup S_i]$ for $i \in \mathbb{N}$. Obviously, $F'_0 \supsetneq F'_1 \supsetneq F'_2 \supsetneq \dots$ and all F'_i are connected. Note that for all $j \geq i$, $S_i \cap F_j = \emptyset$. Define $S := \bigcup_{i=0}^{\infty} S_i$. Since all S_i 's are distinct, S contains infinitely many vertices. By Lemma 7.17, there exists a ray R in F'_0 such that there are infinitely many disjoint $S - R$ paths in F'_0 . Let ω be the end in G containing R . Note that for all $i \in \mathbb{N}$, F_i contains a subray of R . Otherwise, there exists a subray R' of R that lies in $G - S_i - F_i$ contradicting the fact that there are infinitely many disjoint paths between $\bigcup_{j>i} S_j$ and R' . Hence, $\omega \in \bigcap_{i=0}^{\infty} \overline{F_i}$.

Consider any ray Q in ω . There are infinitely many disjoint $S - Q$ paths in F'_0 . Denote these paths by $x_i P_i y_i$ for $i \in \mathbb{N}$ where $x_i \in S$ and $y_i \in Q$. Without loss of generality, we can assume that $x_i \in S_i$, $P_i \cap S = \{x_i\}$, and y_0, y_1, y_2, \dots appear in that order in Q .

Suppose $Q \cap S = \emptyset$. Then for all $i \in \mathbb{N}$, $Q \subseteq F_i$. For all $j \geq i$, $Q \subseteq F_j$ implies that $x_i \in S_j \cup F_j$. Since $x_i \in S_i$ and $S_i \cap F_j = \emptyset$, $x_i \in S_j$. In particular, $x_0, x_1, \dots, x_k \in S_{k+1}$ which is impossible. Therefore, $Q \cap S \neq \emptyset$.

Suppose $Q \cap S_i \neq \emptyset$ for some $i \geq 0$. Let z be a vertex in $Q \cap S_i$. Consider any $j \geq i$. If $z \in S_j$, then $z \in Q \cap S_j$. If $z \notin S_j$, then since $S_i \cap F_j = \emptyset$, $z \in G - S_j - F_j$. Because F_j contains a subray of Q , $zQ \cap S_j \neq \emptyset$. In both cases, $Q \cap S_j \neq \emptyset$.

Consider any $k+1$ disjoint rays R_0, R_1, \dots, R_k in ω . For $0 \leq i \leq k$, let n_i be the smallest integer such that $R_i \cap S_{n_i} \neq \emptyset$. Take $m = \max\{n_0, n_1, \dots, n_k\}$. Then, $R_i \cap S_m \neq \emptyset$ for all $0 \leq i \leq k$ which is impossible. \square

Using Lemma 7.18, we can proceed as in Mader's proof [18] and prove that the cardinality of an atom in a contraction-critical k -connected locally finite infinite graph with minimum end vertex-degree greater than k is at most $\frac{k}{4}$.

Lemma 7.19 (Mader [18]). *Let G be a k -connected graph and \mathcal{X} be a set of subsets of $V(G)$. Suppose A is an \mathcal{X} -atom of G and F is a \mathcal{X} -fragment of G such that $A \subseteq N(F)$. Then $|F \cap N(A)| \geq |A|$.*

Lemma 7.20 (Mader [18]). *Let A be an \mathcal{X} -atom of a \mathcal{X} -critical graph G . If for all $x \in N_G(A)$, there is a $X \in \mathcal{X}$ containing x such that $X \cup A \neq \emptyset$ and $X \subseteq A \cap N_G(A)$, then $G - A$ is almost critical, $\kappa(G - A) = \kappa(G) - |A|$, and $N_G(A) \subseteq Cr(G - A)$.*

Theorem 7.21 (Mader [18]). *Every almost critical k -connected locally finite infinite graph G with minimum end vertex-degree greater than k has fragments F_1, F_2, F_3, F_4 such that $F_1, F_2, F_3, F_4 \cap Cr(G)$ are pairwise disjoint.*

Proof. The proof goes through as in Mader's paper [18] for finite graphs (see also Mader [17]) because Lemma 7.18 guarantees that every fragment contains a minimal fragment. \square

Theorem 7.22. *Let G be a contraction-critical k -connected locally finite infinite graph with minimum end vertex-degree greater than k . Then G contains an atom of cardinality at most $\frac{k}{4}$.*

Proof. The proof is implicit in Mader's paper [18] and is included here for completeness. Let $\mathcal{X} := \{V(e) : e \in E(G)\}$. Then G is \mathcal{X} -critical. By Lemma 7.9, G has a \mathcal{X} -atom A . By Lemma 7.20, $G - A$ is almost critical, $\kappa(G - A) = \kappa(G) - |A|$, and $N_G(A) \subseteq Cr(G - A)$. By Theorem 7.21, $G - A$ has fragments F_1, F_2, F_3, F_4 such that $F_1, F_2, F_3, F_4 \cap Cr(G - A)$ are disjoint. Note that F_1, F_2, F_3, F_4 are fragments of G and $N_G(F_i) = N_{G-A}(F_i) \cup A$. Since $\kappa(G - A) = \kappa(G) - |A|$, $A \subseteq N_G(F_i)$. By Lemma 7.19, $|F_i \cap N_G(A)| \geq |A|$. Since $N_G(A) \subseteq Cr(G - A)$, all $F_i \cap N_G(A)$ are disjoint. Therefore, $k = |N_G(A)| \geq \sum_{i=1}^4 |F_i \cap N_G(A)| \geq 4|A|$ and $|A| \leq \frac{k}{4}$. \square

As a corollary, we obtain Theorem 7.4.

Theorem 7.4. *Every k -connected locally finite infinite graph such that the minimum degree is at least $\lfloor \frac{5k}{4} \rfloor$ and all ends have vertex-degree greater than k contains a contractible edge.*

Finally, we will generalize Dean's result [8] for locally finite infinite graphs with minimum end vertex-degree greater than k as mentioned in the Introduction. This partially answered a conjecture raised in [7]. Note that the author proved the cases when $k = 2$ [6] and $k = 3$ [7] without the end vertex-degree condition. First, we need the following lemma due to Dean [8].

Lemma 7.23 (Dean [8]). *Let G be a k -connected graph and F be a subset of $E(G)$. Define $\mathcal{X} := \{V(e) : e \in F\}$. Suppose every \mathcal{X} -fragment contains at least $\lfloor \frac{k}{2} \rfloor + 1$ vertices. Let B be a minimal \mathcal{X} -fragment. Then every edge of F incident to a vertex in B is contractible.*

Proof. Suppose there is a non-contractible edge $e \in F$ such that $V(e) \cap B \neq \emptyset$. Note that $V(e) \subseteq B \cup N(B)$. Let T be a k -separator containing e . Then $T \cap B \supseteq V(e) \cap B \neq \emptyset$ and $V(e) \subseteq T \cap (B \cup N(B))$. By Lemma 7.6, there exists a fragment F of T such that $|F| < \frac{\kappa(G)}{2}$, a contradiction. \square

Theorem 7.5. *Let G be a k -connected locally finite infinite graph ($k \geq 4$) which is triangle-free or has minimum degree at least $\lfloor \frac{3k}{2} \rfloor$, and all ends have vertex-degree greater than k . Suppose G_C is the subgraph induced by all the contractible edges. Then the closure of G_C in the Freudenthal compactification of G is topologically 2-connected.*

Proof. Denote the closure of G_C in the Freudenthal compactification of G by $\overline{G_C}$. Suppose $\overline{G_C}$ is not topologically 2-connected. Then there exists a point x in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not connected. Let U and U' be two disjoint non-empty open sets in $|G|$ such that $\overline{G_C} \setminus x \subseteq U \cup U'$, $(\overline{G_C} \setminus x) \cap U \neq \emptyset$ and $(\overline{G_C} \setminus x) \cap U' \neq \emptyset$. Define $X := (\overline{G_C} \setminus x) \cap U \cap V(G)$ and $X' := (\overline{G_C} \setminus x) \cap$

$U' \cap V(G)$. Since G_C is a spanning subgraph of G by Corollary 7.10 and 7.13, $X \cup X' = V(G \setminus x)$. By the connectedness of an edge and the definition of a basic open neighborhood of an end, both X and X' are non-empty. Since $G \setminus x$ is topologically connected, $E(X, X')$ is non-empty. Suppose x is a vertex or an end of G . By the connectedness of an edge, all edges in $E(X, X')$ are non-contractible. Define $Z := X$, $Z' := X'$, $F := E(Z, Z')$, $\mathcal{X} := \{V(e) : e \in F\}$ and $w := x$. Suppose x is an interior point of an edge e . Then all edges in $E(X, X')$ are non-contractible unless $e \in E(X, X') \cap E_C$. Note that in this case, $E(X, X') - e$ is non-empty as G is 2-connected, and every edge in $E(X, X') - e$ is non-contractible. Let $e = yy'$ where $y \in X$ and $y' \in X'$. By Corollary 7.10 and 7.13, $|X| \geq 2$ and $|X'| \geq 2$. Define $Z := X - y$, $Z' := X'$, $F := E(Z, Z')$, $\mathcal{X} := \{V(e) : e \in F\}$ and $w := y$. Note that all edges in $E(Z, Z')$ are non-contractible.

By Lemma 7.18, there is a minimal \mathcal{X} -fragment not containing w , say B . Let zz' be an Z - Z' edge in $N_G(B)$. By Theorem 7.1 and 7.2, there are two contractible edges za and $z'a'$ such that $a, a' \in B$. Since B does not contain w and all edges in $E(Z, Z')$ are non-contractible, we have $a \in X$ and $a' \in X'$. As B is connected, there exists an a - a' path P in B . This implies that B contains a non-contractible edge in $F = E(Z, Z')$ contradicting Lemma 7.23. \square

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Chapter 8

Summary

Zusammenfassung

In der vorliegenden Dissertation werden die Verteilung der kontrahierbaren Kanten in bestimmten Arten von Untergraphen und die Eckenüberdeckungen kontrahierbaren Kanten in 2-zusammenhängenden endlichen Graphen untersucht. Verschiedene Ergebnisse über der Existenz und Verteilung der kontrahierbaren Kanten in k -zusammenhängenden endlichen Graphen zu unendlichen, lokal endlichen Graphen verallgemeinern werden.

Summary

In this dissertation, the distribution of contractible edges in certain types of subgraphs and vertex covers of contractible edges in 2-connected finite graphs are investigated. Various results concerning the existence and distribution of contractible edges in k -connected finite graphs are generalized to locally finite infinite graphs.

Publications derived from the dissertation

T. Chan, *Contractible edges in 2-connected locally finite graphs*, Electron. J. Comb. **22** (2015), #P2.47.

T. Chan, *Contractible and removable edges in 3-connected infinite graphs*, Graphs Comb. **31** (2015), 871-883.

Eidesstattliche Versicherung
Declaration on oath

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Hamburg, 09.08.2016