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# Inclusive charmonium production via $\Upsilon$ decay and break-down of non-relativistic QCD factorization in double quarkonia processes 

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To my father, grandma and grandpa


#### Abstract

In this thesis, we present our theoretical investigations on non-relativistic QCD (NRQCD) factorization for double-quarkonium processes as well as phenomenological study on inclusive charmonium ( $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ ) production via $\Upsilon$ decay. We give two examples to explicitly prove that the NRQCD factorization breaks down at tree-level in double-quarkonium processes. We find a solution to cure this factorization breaking and give a new factorization formalism for double-quarkonium processes. The inclusive charmonia production via $\Upsilon$ decay are calculated in $\Upsilon$ color-singlet (CS) and charmonium CS and color-octet (CO) channels, which includes ${ }^{3} S_{1}^{[1]}$ channel for $\Upsilon$ and ${ }^{1} S_{0}^{[1,8]},{ }^{3} S_{1}^{[1,8]},{ }^{1} P_{1}^{[1,8]},{ }^{3} P_{J}^{[1,8]}$ channels for charmonia. For both CS and CO channels, the computation are done up to $\mathcal{O}\left(\alpha_{s}^{5}\right)$ in strong coupling, where the next-to-leading order (NLO) corrections for $\mathcal{O}\left(\alpha_{s}^{4}\right)$ process are included. Relatively important QED processes are also considered. With our numerical results, we re-fitted the value of the CO long distance matrix element (LDME) $\left\langle\mathcal{O}^{\chi_{c o}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ with fixed value of $\left\langle\mathcal{O}^{\chi_{c o}}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle$ and compare with existed fitted values. For inclusive $J / \psi$ production, all the existing LDME sets can explain the experimental data reasonably well in certain range of choosing renormalization scale. The branching ratio of $h_{c}$ is too small to be measured, while the branching ratio of $\eta_{c}$ is large enough to be measured through current experimental data as long as the $\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ is not too small.


## Zusammenfassung

In dieser Arbeit präsentieren wir theoretische Untersuchungen im Kontext der nicht-relativistischen QCD (NRQCD) Faktorisierung. Unsere phänomenologische Studie bezieht sich auf Doppel-Quarkonium-Erzeugung sowie auf die inklusive Charmonium-Produktion ( $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ ) im Zerfall von $\Upsilon$-Mesonen. Wir geben zwei Beispiele an, um explizit zu belegen, dass die NRQCD-Faktorisierung bei Doppel-Quarkonium Prozessen in führender Ordnung zusammenbricht. Wir finden eine Lösung, um dieses Zerbrechen der Faktorisierung zu heilen, indem wir einen neuen Faktorisierungsformalismus für Doppelquarkoniumprozesse formulieren. Für die inklusive Charmonium-Produktion im $\Upsilon$ Zerfall berücksichtigen wir die $\Upsilon$ Farb-Singulett (CS) und die Charmonium Farb-Singulett (CS) und Farb-Oktett (CO) Kanäle, also den ${ }^{3} S_{1}^{[1]}$ Kanal für $\Upsilon$ und die ${ }^{1} S_{0}^{[1,8]},{ }^{3} S_{1}^{[1,8]},{ }^{1} P_{1}^{[1,8]}$ und ${ }^{3} P_{J}^{[1,8]}$ Kanäle für Charmonia. Für die CS und CO Kanäle wird die Berechnung bis zu $\mathcal{O}\left(\alpha_{s}^{5}\right)$ in der starken Kopplung durchgeführt, welche die Korrekturen in nächsthöherer Ordnung (NLO) der $\mathcal{O}\left(\alpha_{s}^{4}\right)$ Prozesse beinhaltet. Wir berücksichtigen auch die relativ gesehen wichtigen QED-Prozesse. Mit den erhaltenen numerischen Ergebnissen fitteten wir den Wert des CO Langstreckenmatrixelements (LDME) $\left\langle\mathcal{O}^{\chi_{c 0}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ bei einem als fest angenommenen Wert von $\left\langle\mathcal{O}^{\chi}{ }_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle$ und vergleichen die gefitteten Werte mit Werten, die bei früheren Analysen erhalten worden sind. Was inklusive $J / \psi$ Produktion angeht, können unser neuer LDME-Wert ebenso wie alle vorhergehenden LDME Werte die experimentellen Daten in einem bestimmten Bereich der Renormierungsskala verhältnismäßig gut erklären. Das Verzweigungsverhältnis von $h_{c}$ ist zu klein, um gemessen zu werden, während das Verzweigungsverhältnis von $\eta_{c}$ groß genug ist, um durch aktuelle experimentelle Daten gemessen zu werden, solange $\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ nicht zu klein ist.

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## Chapter 1

## Introduction

Heavy quarkonia or simply quarkonia are heavy quark $Q$ and heavy anti-quark $\bar{Q}$ bound states such as $J / \psi$, which was discovered by the experimental groups of Samuel Ting [1] and Burton Richter [2] more than four decades ago. Since then quarkonia have played a crucial role in the establishment and development of the Standard Model of Particle Physics in general, and the Quantum Chromodynamics (QCD) as the theory of the strong interaction in particular. During the past four decades, Quarkonium Physics continuously have attracted much attention of both experimental and theoretical physicists, which can be seen from the active Quarkonium Working Group (QWG). On the experiment side, quarkonium states usually have very clean signature of many observables even when there are only few rare events. This allows the study of both new emergent phenomena in the realm of QCD and new physics beyond the Standard Model. On the theoretic side, the hierarchy of energy scales: $m_{Q} v_{Q}^{2} \ll m_{Q} v_{Q} \ll m_{Q}$ in addition with $\Lambda_{\mathrm{QCD}} \ll m_{Q}$ make Quarkonium Physics serves as an ideal laboratory to study both the pertubative and nonpertubative aspects of QCD. Here $\Lambda_{\mathrm{QCD}}$ is the hadronic scale and $v_{Q}$ is the velocity of the heavy quark in the rest frame of the heavy quarkonium, where for charmonium states $v_{c}^{2} \simeq 0.3$, and for bottomonium states $v_{b}^{2} \simeq 0.1$.

The current default theoretical approach of describing quarkonia production and decay is the non-relativistic QCD (NRQCD) factorization theorem [3], which is based on the NRQCD effective quantum field theory [4]. This theorem states that the theoretical predictions can be separated into process-dependent short-distance coefficients (SDCs) calculated perturbatively as expansion in strong coupling constant $\alpha_{s}$ and supposedly universal long-distance matrix elements (LDMEs) scaling with definite power of $v_{Q}$, which can be obtained through lattice QCD or phenomenological determinations. In such a way, the theoretical calculations are organized as double expansion in $\alpha_{s}$ and $v_{Q}$. The rigorous prove for the factorization for heavy quarkonium inclusive annihilation processes was already given in Ref. [3], while the prove for the factorization for heavy quarkonium inclusive production to all orders in $\alpha_{s}$ is still missing.

Although lacking of rigorous prove, the NRQCD factorization has achieved numerous remarkable successes in describing both production and decay of heavy quarkonium in the past two decades (see Ref. [5,6] and references therein for a review). Currently, the challenges are mainly in understanding charmonium production in particular for $J / \psi$ polarization (namely the $J / \psi$ polarization puzzle). Thanks to the efforts of various groups, the SDCs for most of the phenomenologically relevant inclusive quarkonium production processes are now available for

|  | Butenschön, <br> Kniehl [14, 15] | Ma, Wang, <br> Chao [16] | Gong, Wan, <br> Wang, Zhang [21] | Bodwin, Chung, <br> Kim, Lee [23] |
| :--- | :---: | :---: | :---: | :---: |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle / \mathrm{GeV}^{3}$ | 1.32 |  | 1.16 |  |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }_{0} S_{0}^{[8]}\right)\right\rangle / \mathrm{GeV}^{3}$ | $0.0304 \pm 0.0035$ |  | $0.097 \pm 0.009$ | $0.099 \pm 0.022$ |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle / \mathrm{GeV}^{3}$ | $0.0016 \pm 0.0005$ |  | $-0.0046 \pm 0.0013$ | $0.011 \pm 0.010$ |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle / \mathrm{GeV}^{5}$ | $-0.0091 \pm 0.0016$ |  | $-0.0214 \pm 0.0056$ | $0.011 \pm 0.010$ |
| $\left\langle\mathcal{O}^{\chi c 0}\left({ }_{0}^{3} P_{0}^{[1]}\right)\right\rangle / \mathrm{GeV}^{5}$ |  | 0.107 | 0.107 |  |
| $\left\langle\mathcal{O}^{\chi c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle / \mathrm{GeV}^{3}$ |  | $0.0021 \pm 0.0005$ | $0.0022 \pm 0.0005$ |  |

Table 1.1: Sets of $J / \psi$ and $\chi_{c 0}$ LDMEs determined in Refs. [14-16, 21, 23].

|  | Default set | Set 2 | Set 3 |
| :---: | :---: | :---: | :---: |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle / \mathrm{GeV}^{3}$ | 1.16 | 1.16 | 1.16 |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle / \mathrm{GeV}^{3}$ | $0.089 \pm 0.0098$ | 0 | 0.0146 |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle / \mathrm{GeV}^{3}$ | $0.0030 \pm 0.012$ | 0.014 | 0.0118 |
| $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle / \mathrm{GeV}^{5}$ | $0.0126 \pm 0.0047$ | 0.054 | 0.045 |

Table 1.2: Sets of $J / \psi$ LDMEs determined in Refs. [20,26]. Since an up limit of the CO LDME $\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle=0.0146 \mathrm{Gev}^{3}$ was given in Ref. [26], we replace the fourth column obtained in Ref. [20] with the new ones.
the intermediate states ${ }^{3} S_{1}^{[1 / 8]},{ }^{1} S_{0}^{[8]},{ }^{3} P_{J}^{[1 / 8]}$ at next-to-leading order (NLO) in strong coupling : for the yield [7,8] and polarization [9] in $e^{+} e^{-}$annihilation, yield in two-photon collision [10,11], yield [12] and polarization [13] in photoproduction, yield [17, 18] and polarization [19-22] in hadroproduction, etc. And different sets of LDMEs were obtained by fitting to the experimental data under different considerations (see Table.1.1, 1.2). With these sets of LDMEs, some theoretical predictions are plotted in Fig. 1.1 [6], from which, it can be seen that none of the LDME sets can explain both $J / \psi$ yield and polarization data. This poses a challenge to the universality of LDMEs.

Very recently, $\eta_{c}$ production cross-section in $p p$ collisions was measured by LHCb collaboration [24], followed by theoretical investigations from three groups [25-27]. Despite the consistency of the SDCs obtained by these groups, their conclusions are dramatically different. In [25], the authors conclude that either the universality of the LDMEs is in question or that another important ingredient to current NLO NRQCD analyses has so far been overlooked; while in [26], to describe the $\eta_{c}$ hadroproduction data, an up limit of the $\operatorname{COLDME}\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ was given and related to the CO LDME $\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle$ through HQSS (see the fourth column of Table. 1.2), where similar conclusions are given in Ref. [27].

To clarify the conflicts described above and further test the NRQCD factorization, we choose to study inclusive charmonia production via $\Upsilon(1 S)$ decay. There are mainly four reasons. Firstly, NNLO QCD corrections and non-relativistic correction to NLO QCD corrections for inclusive quarkonium production are far beyond the reach of current techniques. Secondly, as it was pointed out in Ref. [6], because of the much larger errors of theoretical predictions in all models comparing with current experimental measurements errors, it is not higher order which is needed,


Figure 1.1: The first $(a, e, i)$ and second $(b, f, j)$ columns are the predictions of the $J / \psi$ total $e^{+} e^{-}$cross section measured by Belle [28] and the transverse momentum distributions in photoproduction measured by H 1 at HERA $[29,30]$ respectively. The third column $(c, g, k)$ are the predictions in hadroproduction measured by CDF [31] and ATLAS [32]. The fourth column ( $d, h, l$ ) are the predictions of polarization parameter $\lambda_{\theta}$ measured by CDF in Tevatron run II [33]. The predictions are plotted using the values of the CO LDMEs given in [14, 15], [21] and [20] (Default) and listed in Table 1.1, 1.2.
but rather new and diverse production observables. Thirdly, on the experimental side, there are large number ( $102 \times 10^{6}$ ) of $\Upsilon(1 S)$ decay events collected by the Belle detector and more precise values of branching ratios of inclusive $J / \psi, \chi_{c 1}$ production via $\Upsilon(1 S)$ decay were measured very recently [34-36] compared to the previous CLEO [37,38] and ARGUS results [39]. Finally, on the theoretically, the investigations on these processes are far from complete even at leading order and next-to-leading order corrections (virtual and real) are completely missing.

While calculating the processes $\Upsilon\left({ }^{3} P_{J}^{[8]}\right) \rightarrow \chi_{c J}\left({ }^{3} P_{J}^{[1]}\right)+g g$, we find uncanceled infrared divergences, which appear to be quite general in double P -wave quarkonia involved processes. This means the NRQCD factorization breaks down for double quarkonia involved processes and hence put a threat on the application of NRQCD factorization to the double-quarkonium hadron production, which has been a hot topic in recent years due to the huge discrepancies between CMS data [40,41] and NRQCD predictions at leading order in $\alpha_{s}$ [42] as well as being a useful laboratory to investigate the double parton scattering mechanism [43] at hadron colliders.

We cure the NRQCD factorization breaking issue in double quarkonia involved processes through introducing a set of LDMEs for double quarkonia production or single quarkonium production via quarkonium decay, whose QCD corrections can absorb the uncanceled infrared divergences.

This thesis is organized as follows: In Chapter 2 we demonstrate that the current version of NRQCD factorization breaks down for double quarkonia involved processes through two explicit examples. In chapter 3, we give our solutions to absorb the uncanceled infrared divergences as well as new version of NRQCD factorization formula for double quarkonia involved processes. Analytic Born level and NLO calculations for inclusive charmonia production via $\Upsilon(1 S)$ decay are describe in chapter 4 and 5 respectively. The numerical evaluation and phenomenological discussion are done in chapter 6 . We summarize our results and give some outlook in chapter 7. Notations, kinematics, definition of LDMEs, loop integrals, soft integrals and summary of the results of one-loop corrections to the LDMEs can be found in the appendixes.

## Chapter 2

## Breakdown of NRQCD Factorization in Double Quarkonia Processes

NRQCD factorization theorem has been widely applied in describing quarkonium decay and production in variety of processes. The NRQCD factorization theorem has been proven valid at all orders for inclusive quarkonium decay and at next-to-next-to-leading order for most inclusive charmonium production processes in strong coupling [44-46]. In this chapter we will demonstrate that for double quarkonia processes, the NRQCD factorization breaks down at tree-level through two explicit examples.

### 2.1 NRQCD Factorization for Quarkonium Decay and Production

The NRQCD factorization theorem for inclusive decay or production of single quarkonium $H(Q \bar{Q})$ can be written as

$$
\begin{align*}
\Gamma(H \rightarrow X) & =\sum_{n} \hat{\Gamma}(Q \bar{Q}[n] \rightarrow X)\langle\mathcal{O}[n]\rangle_{H}  \tag{2.1}\\
\sigma(a+b \rightarrow H+X) & =\sum_{n} \hat{\sigma}(a+b \rightarrow Q \bar{Q}[n]+X)\left\langle\mathcal{O}^{H}[n]\right\rangle \tag{2.2}
\end{align*}
$$

where $\hat{\Gamma}$, $\hat{\sigma}$ are perturbatively calculable $\mathrm{SDCs},\langle\mathcal{O}[n]\rangle_{H},\left\langle\mathcal{O}^{H}[n]\right\rangle$ are decay and production matrix elements (LDMEs), whose definitions are given in the BBL paper [3] as well as in Appendix B. Here we absorb all the polarization and color factors into the SDCs $(\Gamma, \sigma)$, which means the SDCs are calculated according to

$$
\begin{align*}
d \hat{\Gamma}(Q \bar{Q}[n] \rightarrow X) & =\frac{1}{2 m_{H}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS},  \tag{2.3}\\
d \hat{\sigma}(a+b \rightarrow Q \bar{Q}[n]+X) & =\frac{1}{2 s} \frac{1}{N_{\mathrm{col}} N_{\mathrm{pol}}(n)} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS}, \tag{2.4}
\end{align*}
$$

where $N_{\text {col }}=2 N_{c}, N_{c}^{2}-1$ for CS and CO intermediate states respectively, $s$ is the center mass energy, $d \mathrm{PS}$ is the differential phase space, and $N_{\mathrm{pol}}(n)$ is the number of polarization of freedom, which is given by, in D-dimension,

$$
\begin{align*}
& N_{\mathrm{pol}( }\left({ }^{1} S_{0}\right)=N_{\mathrm{pol}}\left({ }^{3} P_{0}\right)=1, N_{\mathrm{pol}}\left({ }^{3} S_{1}\right)=N_{\mathrm{pol}}\left({ }^{1} P_{1}\right)=D-1,  \tag{2.5}\\
& N_{\mathrm{pol}}\left({ }^{3} P_{1}\right)=\frac{(D-1)(D-2)}{2}, N_{\mathrm{pol}}\left({ }^{3} P_{2}\right)=\frac{(D+1)(D-2)}{2} . \tag{2.6}
\end{align*}
$$

Since we average and summer over the polarization, color degrees for the initial and final states respectively, a factor of $\frac{1}{N_{\text {col }} N_{\text {pol }}(n)}$ is absorbed into the averaged decay amplitude square.

The following heavy quark spin symmetry (HQSS) relations [3] are applied for the ${ }^{3} P_{J}$ and $\chi_{c J}$ LDMEs

$$
\begin{align*}
\left\langle\mathcal{O}\left({ }^{3} P_{J}^{[8]}\right)\right\rangle_{\Upsilon} & =\left\langle\mathcal{O}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle_{\Upsilon}\left(1+\mathcal{O}\left(v_{b}^{2}\right)\right),  \tag{2.7}\\
\left\langle\mathcal{O}^{J / \psi}\left({ }_{3}^{3} P_{J}^{[8]}\right)\right\rangle & =(2 J+1)\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right),  \tag{2.8}\\
\left\langle\mathcal{O}^{\chi_{c J}}\left({ }^{3} S_{1}^{[8]} /{ }^{3} P_{J}^{[1]}\right)\right\rangle & =(2 J+1)\left\langle\mathcal{O}^{\chi_{c 0}}\left({ }^{3} S_{1}^{[8]} /{ }^{3} P_{0}^{[1]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right) . \tag{2.9}
\end{align*}
$$

### 2.2 The Covariant Projection Method

Let us start with considering the processes where a quarkonium in the state $Q \bar{Q}\left({ }^{2 S+1} L_{J}^{[a]}\right)$ is involved. We label the quark pair in the full QCD amplitudes as $Q(p, s, i) \bar{Q}(\bar{p}, \bar{s}, \bar{i})$, where $s, \bar{s}$ are spin indexes, $i, \bar{i}$ are color indexes and $p, \bar{p}$ are 4-momenta.

To calculate the amplitudes $\mathcal{M}$ which include the quarkonium state $Q \bar{Q}\left({ }^{2 S+1} L_{J}^{[a]}\right)$, we apply color, spin and angular momentum projectors on quark pair in the full QCD amplitude $\mathcal{A}$.

The color projectors are given by

$$
\begin{align*}
\mathcal{C}_{1} & =\frac{\delta_{i \bar{i}}}{\sqrt{N_{c}}}  \tag{2.10a}\\
\mathcal{C}_{8}^{a} & =\sqrt{2} T_{\bar{i} i}^{a} \text { (for production) }, \tag{2.10b}
\end{align*}
$$

where $\mathcal{C}_{1}, \mathcal{C}_{8}^{a}$ project out the CS and CO states respectively. For decay of $Q \bar{Q}\left({ }^{2 S+1} L_{J}^{[8]}\right)$ state, the color indexes $\bar{i}, i$ in the color matrix $T_{i i}^{a}$ must interchange.

The spin projectors are given by

$$
\begin{align*}
& \Pi^{0}=\frac{1}{\sqrt{8 m_{Q}^{3}}} \times \begin{cases}\bar{v}^{\bar{s}}(\bar{p}) \gamma_{5} u^{s}(p) & \text { (for production) } \\
\bar{u}^{s}(p) \gamma_{5} v^{\bar{s}}(\bar{p}) & \text { (for decay) }\end{cases}  \tag{2.11}\\
& \Pi^{\alpha}=\frac{1}{\sqrt{8 m_{Q}^{3}}} \times \begin{cases}\bar{v}^{\bar{s}}(\bar{p}) \gamma^{\alpha} u^{s}(p) & \text { (for production) } \\
\bar{u}^{s}(p) \gamma^{\alpha} v^{\bar{s}}(\bar{p}) & \text { (for decay) }\end{cases} \tag{2.12}
\end{align*}
$$

where $\Pi^{0}, \Pi^{\alpha}$ project out spin singlet and triplet states respectively, and $m_{Q}$ is the pole mass of quark $Q$.

Defining

$$
\begin{equation*}
P \equiv p+\bar{p}, q \equiv \frac{p-\bar{p}}{2} \tag{2.13}
\end{equation*}
$$

the S-wave states ( $L=S$ ) can be obtained by simply setting $q=0$, and the P-wave states ( $L=P$ ) can be obtained by differential the amplitudes respecting to $q$ after which $q$ is set to be zero.

Combining all the projectors and the full QCD amplitudes $\mathcal{A}$, we have

$$
\begin{align*}
& \mathcal{M}_{1 S_{0}^{[a]}}=\left.\sum_{s, \bar{s}} \mathcal{C}_{a} \Pi^{0} \mathcal{A}\right|_{q=0}  \tag{2.14a}\\
& \mathcal{M}_{3 S_{1}^{[a]}}=\left.\sum_{s, \bar{s}} \varepsilon_{\alpha}^{*} \mathcal{C}_{a} \Pi^{\alpha} \mathcal{A}\right|_{q=0}  \tag{2.14b}\\
& \mathcal{M}_{1 P_{1}^{[a]}}=\left.\varepsilon_{\alpha}^{*} \frac{\mathrm{~d}}{\mathrm{~d} q_{\alpha}}\left[\sum_{s, \bar{s}} \mathcal{C}_{a} \Pi^{0} \mathcal{A}\right]\right|_{q=0}  \tag{2.14c}\\
& \mathcal{M}_{3 P_{J}^{[a]}}=\left.\varepsilon_{\alpha \beta}^{* J)} \frac{\mathrm{d}}{\mathrm{~d} q_{\beta}}\left[\sum_{s, \bar{s}} \mathcal{C}_{a} \Pi^{\alpha} \mathcal{A}\right]\right|_{q=0}(J=0,1,2), \tag{2.14d}
\end{align*}
$$

where $\varepsilon^{*}$ is the polarization vector or tensor of final $Q \bar{Q}$ pair state and $a=1,8$. If the $Q \bar{Q}$ pair is in the initial state, $\varepsilon^{*}$ must been replaced by $\varepsilon$.

Note: Here our spin projectors Eq.(2.11, 2.12) and formulas Eq.(2.14a-2.14d) are slightly different from the ones in Ref. [47], which are given by

$$
\begin{align*}
& \Pi^{\prime 0}=\frac{1}{\sqrt{8 m_{Q}^{3}}}\left(\frac{\not P}{2}-q q-m_{Q}\right) \gamma_{5}\left(\frac{\not P}{2}+q+m_{Q}\right),  \tag{2.15a}\\
& \Pi^{\prime \alpha}=\frac{1}{\sqrt{8 m_{Q}^{3}}}\left(\frac{P}{2}-q q-m_{Q}\right) \gamma^{\alpha}\left(\frac{P}{2}+q+m_{Q}\right),  \tag{2.15b}\\
& \mathcal{M}_{1 S_{0}^{[a]}}^{\prime[\alpha}=\left.\operatorname{Tr}\left[\mathcal{C}_{a} \Pi^{\prime 0} \mathcal{A}^{\prime}\right]\right|_{q=0},  \tag{2.16}\\
& \mathcal{M}_{3 S_{1}^{[a]}}^{\prime[\alpha}=\left.\varepsilon_{\alpha} \operatorname{Tr}\left[\mathcal{C}_{a} \Pi^{\prime \alpha} \mathcal{A}^{\prime}\right]\right|_{q=0},  \tag{2.17}\\
& \mathcal{M}_{1 P_{1}^{[a]}}^{\prime}=\left.\varepsilon_{\alpha} \frac{\mathrm{d}}{\mathrm{~d} q_{\alpha}} \operatorname{Tr}\left[\mathcal{C}_{a} \Pi^{\prime 0} \mathcal{A}^{\prime}\right]\right|_{q=0},  \tag{2.18}\\
& \mathcal{M}_{3 P_{J}^{[a]}}^{\prime}=\left.\varepsilon_{\alpha \beta}^{(J)} \frac{\mathrm{d}}{\mathrm{~d} q_{\beta}} \operatorname{Tr}\left[\mathcal{C}_{a} \Pi^{\prime \alpha} \mathcal{A}^{\prime}\right]\right|_{q=0} \quad(J=0,1,2), \tag{2.19}
\end{align*}
$$

where $\mathcal{A}^{\prime}$ is the full QCD amplitude with the spinors of $Q \bar{Q}$ pair removed.

The spin projectors given in Ref. [47] are production spin projectors. For the decay spin projectors, the expressions $\frac{p p}{2}-q q-m_{Q}, \frac{p}{2}+q q+m_{Q}$ on the both side of $\gamma_{5}, \gamma_{\alpha}$ should be interchanged. In this way, our projectors Eq. $(2.11,2.12)$ are exactly the same with those in Ref. [47] after fermion spin sum:

$$
\begin{align*}
& \sum_{s} u^{s}(p) \bar{u}^{s}(p)=\not p+m_{Q}=\left(\frac{p}{2}+\not q+m_{Q}\right),  \tag{2.20}\\
& \sum_{\bar{s}} v^{\bar{s}}(\bar{p}) \bar{v}^{\bar{s}}(\bar{p})=\not p-m_{Q}=\left(\frac{p}{2}-\not q-m_{Q}\right) . \tag{2.21}
\end{align*}
$$

However our formulas Eq. $(2.11,2.12)$ are easier to be implemented in FEYNCALC [48], especially for double quarkonia processes as well as open $Q \bar{Q}$ associated processes. For the processes where more than one quarkonia are involved, we just need to apply the projectors onto each quarkonium state.

When calculating the SDCs, we have to square the amplitudes and sum over the polarizations of the external states. For the $Q \bar{Q}\left({ }^{2 S+1} L_{J}^{[1 / 8]}\right)$ states, we have

$$
\begin{align*}
\sum_{\text {pol }} \varepsilon_{\alpha} \varepsilon_{\alpha^{\prime}}^{*} & =\Pi_{\alpha \alpha^{\prime}}  \tag{2.22a}\\
\sum_{\text {pol }} \varepsilon_{\alpha \beta}^{(0)} \varepsilon_{\alpha^{\prime} \beta^{*}}^{*(0)} & =\frac{1}{D-1} \Pi_{\alpha \beta} \Pi_{\alpha^{\prime} \beta^{\prime}},  \tag{2.22b}\\
\sum_{\text {pol }} \varepsilon_{\alpha \beta}^{(1)} \varepsilon_{\alpha^{\prime} \beta^{*}}^{*(1)} & =\frac{1}{2}\left(\Pi_{\alpha \alpha^{\prime}} \Pi_{\beta \beta^{\prime}}-\Pi_{\alpha \beta^{\prime}} \Pi_{\alpha^{\prime} \beta}\right),  \tag{2.22c}\\
\sum_{\text {pol }} \varepsilon_{\alpha \beta}^{(2)} \varepsilon_{\alpha^{\prime} \beta^{*}}^{*(2)} & =\frac{1}{2}\left(\Pi_{\alpha \alpha^{\prime}} \Pi_{\beta \beta^{\prime}}+\Pi_{\alpha \beta^{\prime}} \Pi_{\alpha^{\prime} \beta}\right)-\frac{1}{D-1} \Pi_{\alpha \beta} \Pi_{\alpha^{\prime} \beta^{\prime}}, \tag{2.22d}
\end{align*}
$$

where we define

$$
\begin{equation*}
\Pi_{\alpha \beta}=-g_{\alpha \beta}+\frac{P_{\alpha} P_{\beta}}{4 m_{Q}^{2}}, \tag{2.23}
\end{equation*}
$$

and $D$ is the dimension of space-time, which is set to be $4-2 \epsilon$ to regularize both ultra-violet and infrared divergences in dimensional regularization approach.

### 2.3 Infrared Divergences in Double P-Wave Quarkonia Processes

In this section, we naively apply the NRQCD factorization theorem and covariant projection method in double quarkonia processes. We give two examples $b \bar{b}\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J_{c}}^{[1]}\right)+$ $g g, e^{+} e^{-} \rightarrow c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+c \bar{c}\left({ }^{3} P_{J_{2}}^{[1]}\right)+g$ to demonstrate that there are infrared divergences, which cannot been canceled in current version of NRQCD factorization.

The naive application of NRQCD factorization formula for double-quarkonium processes implies that we use the following calculation formulas:

$$
\begin{align*}
& \Gamma\left(\Upsilon \rightarrow \chi_{c J}+X\right) \\
= & \sum_{n_{1}, n_{2}} \hat{\Gamma}\left(b \bar{b}\left[n_{2}\right] \rightarrow c \bar{c}\left[n_{1}\right]+X\right)\left\langle\mathcal{O}\left[n_{2}\right]\right\rangle \Upsilon\left\langle\mathcal{O}^{\chi_{c J}}\left[n_{1}\right]\right\rangle,  \tag{2.24}\\
& \sigma\left(e^{+} e^{-} \rightarrow J / \psi+\chi_{c J}+X\right) \\
= & \sum_{n_{1}, n_{2}} \hat{\sigma}\left(e^{+} e^{-} \rightarrow c \bar{c}\left[n_{1}\right]+c \bar{c}\left[n_{2}\right]+X\right)\left\langle\mathcal{O}^{J / \psi}\left[n_{1}\right]\right\rangle\left\langle\mathcal{O}^{\chi_{c J}}\left[n_{2}\right]\right\rangle, \tag{2.25}
\end{align*}
$$

where

$$
\begin{align*}
d \hat{\Gamma}\left(b \bar{b}\left[n_{2}\right] \rightarrow c \bar{c}\left[n_{1}\right]+X\right) & =\frac{1}{4 m_{b}} \frac{1}{N_{\mathrm{col}_{1}} N_{\mathrm{pol}}\left(n_{1}\right)} \sum|\mathcal{M}|^{2} d \mathrm{PS}  \tag{2.26}\\
d \hat{\sigma}\left(e^{+} e^{-} \rightarrow c \bar{c}\left[n_{1}\right]+c \bar{c}\left[n_{2}\right]+X\right) & =\frac{1}{2 s} \frac{1}{N_{\mathrm{col}_{2}} N_{\mathrm{pol}}\left(n_{2}\right)} \frac{1}{N_{\mathrm{col}_{1} N_{\mathrm{pol}}\left(n_{1}\right)}^{\sum}|\mathcal{M}|^{2} d \mathrm{PS}} \tag{2.27}
\end{align*}
$$

In our analytical computation, the Feynman diagrams are generated by FEYnARTS [49]; algebraic operations such as color, Dirac algebra, are performed with FEYNCALC [48] and FORM [50].

### 2.3.1 Infrared Divergences in $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g g$

There are 8 Feynman diagrams for the process $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g g$, whose representative ones are shown in Figure.2.1.The rest 6 Feynman diagrams can be obtained through attaching the gluon to the $c$ quark as well as exchanging the two final gluons. For the initial states $b \bar{b}\left({ }^{3} P_{J_{b}}^{[8]}\right)$, we sum over different $J_{b}$ states, leaving the LDMEs to be $\left\langle\mathcal{O}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle_{\Upsilon}$.

Obviously, there exist infrared divergences in the phase-space integration when one of the final state gluon is soft. In the rest of this section we will show how to analytically extract the infrared divergences from the SDCs $\hat{\Gamma}\left(J_{c}\right)$ with the method described in Ref. [51]. We divide the amplitude into two parts $\mathcal{M}_{1}, \mathcal{M}_{2}$, which correspond to the contribution from the diagrams where the gluon with momentum $k_{1}$ attached to charm and bottom quark line respectively (second and first diagrams in Figure.2.1). Thus the amplitude squares are divided into three parts: $\left|\mathcal{M}_{1}\right|^{2},\left|\mathcal{M}_{2}\right|^{2}, 2 \mathcal{M}_{1} \mathcal{M}_{2}^{*}$. Due to the symmetry of identical particles, it is sufficient to consider only the case that $k_{1}$ is soft in extracting the infrared divergences. So the infrared divergent parts of $\hat{\Gamma}\left(J_{c}\right)$ can be expressed as

$$
\begin{equation*}
\hat{\Gamma}^{\mathrm{div}}\left(J_{c}\right)=\hat{\Gamma}_{1}^{\mathrm{div}}+\hat{\Gamma}_{2}^{\mathrm{div}}\left(J_{c}\right)+\hat{\Gamma}_{\mathrm{mix}}^{\mathrm{div}}\left(J_{c}\right) \tag{2.28}
\end{equation*}
$$

where $\hat{\Gamma}_{1}^{\text {div }}, \hat{\Gamma}_{2}^{\text {div }}\left(J_{c}\right), \hat{\Gamma}_{3}^{\text {div }}\left(J_{c}\right)$ are the divergent parts coming from $2\left|\mathcal{M}_{1}\right|^{2}, 2\left|\mathcal{M}_{2}\right|^{2}, 2 \times$ $2 \mathcal{M}_{1} \mathcal{M}_{2}^{*}$ respectively, with $k_{2}$ being soft. The D-dimensional 3-body phase space is given in Eq.(A.10) in Appendix A. Since we only extract the divergent part and drop the finite part, we



Figure 2.1: Representative tree-level Feynman diagrams for the partonic subprocess $b \bar{b}\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g g$.


Figure 2.2: Representative tree-level Feynman diagrams for the partonic subprocess $b \bar{b}\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g$ and $b \bar{b}\left({ }^{3} S_{1}^{[8]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g$.
set $\epsilon=0$ when it is safe. Then the 3-body phase space is simplified as

$$
\begin{equation*}
d \mathrm{PS}_{1 \rightarrow 3}=\frac{m_{b}^{2}}{32 \pi^{3}}\left[\left(1+r^{2}-x_{1}\right)\left(a_{1}+a_{2}-x_{2}\right)\left(x_{2}+a_{1}-a_{2}\right)\right]^{-\epsilon} d x_{1} d x_{2} \tag{2.29}
\end{equation*}
$$

The phase space integration is divergent at $x_{2}=0\left(k_{1}\right.$ soft $)$, or $x_{3}=0\left(k_{2}\right.$ soft $)$, where $x_{1}=1+r^{2}$ in both cases. The general idea to extract the divergent part at $x_{2}=0$ is that we do some variable transformation to isolate the $\left(1+r^{2}-x_{1}\right)^{-1-2 \epsilon}$ pole where at the same time isolate the pole $x_{2}=0$ rather than $x_{3}=0$. A simple choice is

$$
\begin{equation*}
x_{1}=y_{1}, x_{2}=\frac{1+r^{2}-y_{1}}{1-y_{2}} . \tag{2.30}
\end{equation*}
$$

With the transformation Eq.(2.30), the 3-body phase space is re-expressed as

$$
\begin{equation*}
d \mathrm{PS}_{1 \rightarrow 3}=\frac{m_{b}^{2}\left(1+r^{2}-y_{1}\right)^{1-2 \epsilon}}{32 \pi^{3}\left(1-y_{2}\right)^{2}}\left(\left(a_{1}^{\prime}+a_{2}^{\prime}-x_{2}^{\prime}\right)\left(\frac{1}{1-y_{2}}-\frac{1}{a_{1}^{\prime}+a_{2}^{\prime}}\right)\right)^{-\epsilon} d y_{1} d y_{2} \tag{2.31}
\end{equation*}
$$

where $a_{1}^{\prime}, a_{2}^{\prime}, x_{2}^{\prime}$ are $a_{1}, a_{2}, a_{3}$ expressed in terms of $y_{1}, y_{2}$ rather than $x_{1}, x_{2}$. And the limits of phase space integration become

$$
\begin{equation*}
1+r^{2}>y_{1}>2 r, 1-\left(a_{2}^{\prime}-a_{1}^{\prime}\right)>y_{2}>1-\left(a_{2}^{\prime}+a_{1}^{\prime}\right) \tag{2.32}
\end{equation*}
$$

Then we can extract the expression which contributes to the divergent part according to

$$
\begin{equation*}
d \hat{\Gamma}_{1}^{\mathrm{div}}=\frac{f_{1}\left(1+r^{2}, y_{2}\right)}{\left(1+r^{2}-y_{1}\right)^{1+2 \epsilon}} d y_{1} d y_{2} \tag{2.33}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{1}\left(1+r^{2}, y_{2}\right)=\lim _{y_{1} \rightarrow 1+r^{2}}\left(\left(1+r^{2}-y_{1}\right)^{2} \frac{d \hat{\Gamma}_{1}}{d y_{1} d y_{2}}\right) \tag{2.34}
\end{equation*}
$$

with $d \hat{\Gamma}_{1}$ representing the full expression obtained at $D=4$.
The divergent integration is given by

$$
\begin{equation*}
\int_{2 r}^{1+r^{2}}\left(1+r^{2}-y_{1}\right)^{-1-2 \epsilon} d y_{1}=-\frac{1}{2 \epsilon_{\mathrm{IR}}}+\text { finite terms } \tag{2.35}
\end{equation*}
$$

Therefore, we have

$$
\begin{align*}
\hat{\Gamma}_{1}^{\mathrm{div}} & =-\left.\frac{1}{2 \epsilon_{\mathrm{IR}}} \int_{1-\left(a_{2}^{\prime}+a_{1}^{\prime}\right)}^{1-\left(a_{2}^{\prime}-a_{1}^{\prime}\right)} f_{1}\left(1+r^{2}, y_{2}\right) d y_{2}\right|_{y_{1}=1+r^{2}}  \tag{2.36}\\
& =\frac{-8 \alpha_{s}}{27 \pi m_{c}^{2}} \frac{1}{\epsilon_{\mathrm{IR}}} \times \frac{5 \pi^{2} \alpha_{s}^{3}\left(3 r^{4}+2 r^{2}+7\right)}{72 m_{b}^{7} r^{3}\left(1-r^{2}\right)} . \tag{2.37}
\end{align*}
$$

With the same procedure, we can get

$$
\hat{\Gamma}_{2}^{\text {div }}(J)=\frac{-5 \alpha_{s}}{9 \pi m_{b}^{2}} \frac{1}{\epsilon_{\mathrm{IR}}} \times \begin{cases}\frac{\pi^{2} \alpha_{s}^{3}\left(1-3 r^{2}\right)^{2}}{81 m_{b}^{7} r^{3}\left(1-r^{2}\right)} & J_{c}=0  \tag{2.38}\\ \frac{2 \pi^{2} \alpha_{s}^{3}\left(r^{2}+1\right)}{81 m_{b}^{7} r^{3}\left(1-r^{2}\right)} & J_{c}=1 \\ \frac{2 \pi^{2} \alpha_{s}^{3}\left(6 r^{4}+3 r^{2}+1\right)}{405 m_{b}^{7} r^{3}\left(1-r^{2}\right)} & J_{c}=2\end{cases}
$$

and

$$
\begin{align*}
\hat{\Gamma}_{\text {mix }}^{\mathrm{div}}(0)= & \frac{-10 \pi \alpha_{s}^{4}}{81 m_{b}^{9} r^{3}\left(1-r^{2}\right)^{4} \epsilon_{\mathrm{IR}}}\left(3 r^{4}-10 r^{2}+3\right)\left(r^{4}-4 r^{2} \ln (r)-1\right)  \tag{2.39a}\\
\hat{\Gamma}_{\text {mix }}^{\mathrm{div}}(1)= & \frac{10 \pi \alpha_{s}^{4}}{81 m_{b}^{9} r^{3}\left(1-r^{2}\right)^{4} \epsilon_{\mathrm{IR}}}\left(-r^{6}+9 r^{4}-7 r^{2}\right. \\
& \left.+4 r^{2}\left(r^{4}-3 r^{2}-2\right) \ln (r)-1\right)  \tag{2.39b}\\
\hat{\Gamma}_{\text {mix }}^{\mathrm{div}}(2)= & \frac{2 \pi \alpha_{s}^{4}}{81 m_{b}^{9} r^{3}\left(1-r^{2}\right)^{4} \epsilon_{\mathrm{IR}}}\left(6 r^{8}+23 r^{6}-27 r^{4}+r^{2}\right. \\
& \left.-4 r^{4}\left(9 r^{2}+11\right) \ln (r)-3\right) \tag{2.39c}
\end{align*}
$$

with $\epsilon_{\mathrm{IR}}=\frac{D-4}{2}$ being the infrared regulator in dimensional regularization. In Eq.(2.37) and Eq.(2.38), $\hat{\Gamma}_{1}^{\text {div }}$ and $\hat{\Gamma}_{2}^{\text {div }}\left(J_{c}\right)$ can be factorized as the products of the SDCs of $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow$ $\chi_{c J_{c}}\left({ }^{3} S_{1}^{[8]}\right)+g$ and $\Upsilon\left({ }^{3} S_{1}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g$ (Figure.2.2), with the IR-divergent term that related to the NLO QCD corrections to the LDMEs $\left\langle\mathcal{O}^{\chi_{c J}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ and $\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle_{\Upsilon}$ respectively. This indicates that they will be canceled after taking into account the contribution of $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow$ $\chi_{c J_{c}}\left({ }^{3} S_{1}^{[8]}\right)+g$ and $\Upsilon\left({ }^{3} S_{1}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g$. However, for $\hat{\Gamma}_{\text {mix }}^{\text {div }}\left(J_{c}\right)$ we find that in current version of NRQCD factorization formalism there is no operator to describe such kind of soft gluon effect.







Figure 2.3: Representative tree-level Feynman diagrams for the partonic subprocess $e^{+} e^{-} \rightarrow$ $c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+c \bar{c}\left({ }^{3} P_{J_{2}}^{[1]}\right)+g$.

### 2.3.2 Infrared Divergences in $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+\chi_{c J_{2}}\left({ }^{3} P_{J_{2}}^{[1]}\right)+g$

There are 28 Feynman diagrams for the process $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+\chi_{c J_{2}}\left({ }^{3} P_{J_{2}}^{[1]}\right)+g$ and the representative ones are shown in Figure.2.3. For the final states $c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)$, we sum over different $J_{1}$ states as well, leaving the LDMEs to be $\left\langle{ }^{J / \psi} \mathcal{O}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle$ with the help of heavy quark spin symmetry. We separate the SDCs into three parts, according to where the final state gluon is attached: (1) the squared amplitude of the Feynman diagrams where the final state gluon attaches the ${ }^{3} P_{J_{1}}^{[8]}$ state, (2) the squared amplitude of the Feynman diagrams where the ${ }^{3} P_{J_{2}}^{[1]}$ states are attached with the final state gluon, and (3)the interference terms between the diagrams in the first and second cases. We denote the corresponding divergent parts as $\hat{\sigma}_{1}^{\text {div }}, \hat{\sigma}_{2}^{\text {div }}\left(J_{2}\right), \hat{\sigma}_{\text {mix }}^{\text {div }}\left(J_{2}\right)$, then the divergent part of the SDC of the total cross section $\sigma\left(J_{2}\right)$ can be expressed as

$$
\begin{equation*}
\hat{\sigma}^{\mathrm{div}}\left(J_{2}\right)=\sigma_{1}^{\mathrm{div}}+\sigma_{2}^{\mathrm{div}}\left(J_{2}\right)+\sigma_{\text {mix }}^{\mathrm{div}}\left(J_{2}\right) . \tag{2.40}
\end{equation*}
$$

With similar techniques of extracting infrared divergences analytically explained in the last section, we obtain

$$
\begin{align*}
\hat{\sigma}_{1}^{\mathrm{div}}= & -\frac{8 \alpha_{s}}{27 \pi m_{c}^{2}} \frac{1}{\epsilon_{\mathrm{IR}}} \frac{2^{10} \pi^{3} \alpha^{2} \alpha_{s}^{2} \sqrt{1-4 r_{1}^{2}}}{729 s^{5} r_{1}^{6}} \\
& \times\left(864 r_{1}^{10}-144 r_{1}^{8}-1568 r_{1}^{6}+1224 r_{1}^{4}-130 r_{1}^{2}+27\right), \tag{2.41}
\end{align*}
$$



Figure 2.4: Representative tree-level Feynman diagrams for the partonic subprocess $e^{+} e^{-} \rightarrow$ $c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c}\left({ }^{3} P_{J_{2}}^{[1]}\right)$ and $e^{+} e^{-} \rightarrow c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)$.

$$
\begin{array}{rlr}
\hat{\sigma}_{2}^{\text {div }}\left(J_{2}\right)= & -\frac{4 \alpha_{s}}{3 \pi m_{c}^{2}} \frac{1}{\epsilon_{\mathrm{IR}}} \frac{2^{18} \pi^{3} \alpha^{2} \alpha_{s}^{2} \sqrt{1-4 r_{1}^{2}}}{19683 s^{5} r_{1}^{4}} \\
& \times \begin{cases}\left(144 r_{1}^{8}+152 r_{1}^{6}-428 r_{1}^{4}+182 r_{1}^{2}+1\right) & J_{2}=0 \\
8\left(18 r_{1}^{6}+13 r_{1}^{4}-12 r_{1}^{2}+2\right) & J_{2}=1 \\
\frac{2}{5}\left(360 r_{1}^{8}+308 r_{1}^{6}-188 r_{1}^{4}+20 r_{1}^{2}+1\right) & J_{2}=2\end{cases} \tag{2.42}
\end{array}
$$

and

$$
\begin{align*}
\hat{\sigma}_{\operatorname{mix}}^{\mathrm{div}}(0)= & \frac{2^{19} \pi^{2} \alpha^{2} \alpha_{s}^{3}}{3^{8} s^{6} r_{1}^{4} \epsilon_{\mathrm{IR}}}\left(\left(144 r_{1}^{8}+184 r_{1}^{6}-504 r_{1}^{4}+170 r_{1}^{2}+33\right) \sqrt{1-4 r_{1}^{2}}\right. \\
& \left.+8\left(72 r_{1}^{10}+56 r_{1}^{8}-284 r_{1}^{6}+149 r_{1}^{4}+r_{1}^{2}\right) \tanh ^{-1} \sqrt{1-4 r_{1}^{2}}\right) \tag{2.43a}
\end{align*}
$$

$$
\begin{align*}
\hat{\sigma}_{\text {mix }}^{\mathrm{div}}(1)= & \frac{2^{19} \pi^{2} \alpha^{2} \alpha_{s}^{3}}{3^{8} s^{6} r_{1}^{2} \epsilon_{\mathrm{IR}}}\left(\left(144 r_{1}^{6}+28 r_{1}^{4}-176 r_{1}^{2}+43\right) \sqrt{1-4 r_{1}^{2}}\right. \\
& \left.+\left(576 r_{1}^{10}-176 r_{1}^{8}-792 r_{1}^{6}+424 r_{1}^{4}-48 r_{1}^{2}\right) \tanh ^{-1} \sqrt{1-4 r_{1}^{2}}\right), \\
\hat{\sigma}_{\text {mix }}^{\mathrm{div}}(2)= & \frac{2^{19} \pi^{2} \alpha^{2} \alpha_{s}^{3}}{5 \cdot 3^{8} s^{6} r_{1}^{4} \epsilon_{\mathrm{IR}}}\left(\left(720 r_{1}^{8}+452 r_{1}^{6}-696 r_{1}^{4}+7 r_{1}^{2}-15\right) \sqrt{1-4 r_{1}^{2}}\right.  \tag{2.43b}\\
& \left.+\left(2880 r_{1}^{10}+368 r_{1}^{8}-3560 r_{1}^{6}+1856 r_{1}^{4}-56 r_{1}^{2}\right) \tanh ^{-1} \sqrt{1-4 r_{1}^{2}}\right), \tag{2.43c}
\end{align*}
$$

where $r_{1}=\frac{2 m_{c}}{\sqrt{s}}$, with $\sqrt{s}$ representing the center of mass energy.
It is straightforward to check that $\hat{\sigma}_{1}^{\text {div }}$ and $\hat{\sigma}_{2}^{\text {div }}\left(J_{2}\right)$ will be cancelled after including QCD corrections to the S-wave LDMEs $\left\langle\mathcal{O}^{\chi_{c J}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ of $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+\chi_{c J_{2}}\left({ }^{3} S_{1}^{[8]}\right)$ and $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle$ of $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} S_{1}^{[1]}\right)+\chi_{c J_{2}}\left({ }^{3} P_{J_{2}}^{[1]}\right)$ respectively, where the results of the NLO corrections to these two LDME are listed in Appendix D. The corresponding typical Feynman diagrams for these 2 processes are show in Fig.(2.4). Unfortunately in standard NRQCD calculation there is no term to cancel $\sigma_{\text {mix }}^{\text {div }}\left(J_{2}\right)$, which means the NRQCD factorization does not apply directly to the inclusive $J / \psi+\chi_{c J}$ production at $v_{c}^{4}$ in $e^{+} e^{-}$annihilation neither.

### 2.3.3 NRQCD Factorization Break-down in Other Processes

Since $\hat{\sigma}_{\text {mix }}^{\text {div }}\left(J_{2}\right)$ originate from the interferences between Feynman diagrams with the gluon attaching different P -wave $Q \bar{Q}$ pairs having nothing to do with the initial states $e^{+} e^{-}$nor with the flavors of the heavy quarks, we conclude that there will be similar uncanceled divergences in the NRQCD calculation of double $J / \psi$ and $J / \psi+\Upsilon$ hadroproduction at NLO in $\alpha_{s}$. However in hadroproduction cases the structure of the uncanceled infrared divergences can be much more complicate, since more channels can be involved in. For instance, in $g g \rightarrow c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+$ $b \bar{b}\left({ }^{3} P_{J_{2}}^{[8]}\right)+g$, there will be 6 combinations of four channels $g g \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+b \bar{b}\left({ }^{3} P_{J_{2}}^{[8]}\right), g g \rightarrow$ $c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+b \bar{b}\left({ }^{3} P_{J_{2}}^{[8]}\right), g g \rightarrow c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+b \bar{b}\left({ }^{3} S_{1}^{[8]}\right)$, and $g g \rightarrow c \bar{c}\left({ }^{3} P_{J_{1}}^{[8]}\right)+b \bar{b}\left({ }^{3} S_{1}^{[1]}\right)$ related to the interference terms leading to uncanceled infrared divergences.

Besides, there will be similar uncanceled infrared divergences in the NRQCD calculation of inclusive charmonium production via $\chi_{b J}$ decay. And the uncanceled infrared divergences will appear in the tree level processes such as $\chi_{b J}\left({ }^{3} P_{J_{2}}^{[1]}\right) \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+g g$, which are related to the interferences between $\chi_{b J}\left({ }^{3} S_{1}^{[8]}\right) \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+g$ and $\chi_{b J}\left({ }^{3} P_{J_{2}}^{[1]}\right) \rightarrow J / \psi\left({ }^{3} S_{1}^{[8]}\right)+g$.

## Chapter 3

## Improve the NRQCD Factorization

Conventionally, heavy quark (anti-quark) fields in the long distance matrix elements of heavy quarkonia and non-relativistic QCD lagrangian are expressed in terms of two component Pauli fields. And the long distance matrix elements have to be evaluated non-relativistically in the heavy quark pair rest frame. However, in processes which include two heavy quarkonia, the two heavy quark pairs can not be at rest simultaneously. Thus we express all the relevant fields and operators in covariant form. Our procedures of deriving the covariant form of NRQCD Lagrangian and relevant four-fermion operators are based on the methods described in Ref. [52-56] and references therein.

### 3.1 NRQCD Lagrangian in Covariant Form

We use the timelike (spacetime independent) unit vector $v^{\mu}$ to decompose 4 -vectors and heavy quark fields, which is defined as

$$
\begin{equation*}
v^{\mu} \equiv \frac{P^{\mu}}{m_{H}}=\frac{P^{\mu}}{2 m_{Q}}+\mathcal{O}\left(q^{2} / m_{Q}^{2}\right) \tag{3.1}
\end{equation*}
$$

where $P, q$ are the total and relative momenta of heavy quark pair $Q \bar{Q}$, respectively. Thus a 4 -vector $a^{\mu}$ can be decomposed as

$$
\begin{equation*}
a^{\mu}=v \cdot a v^{\mu}+a_{\mathrm{T}}^{\mu} . \tag{3.2}
\end{equation*}
$$

First, we redefine the heavy quark and anti-quark field through phase redefinition

$$
\begin{equation*}
\Psi_{v}^{(+)}(x) \equiv e^{i m_{Q} v \cdot x} \Psi^{(+)}(x), \quad \Psi_{v}^{(-)}(x) \equiv e^{-i m_{Q} v \cdot x} \Psi^{(-)}(x) \tag{3.3}
\end{equation*}
$$

where $\Psi^{(+)}, \Psi^{(-)}$are heavy quark and anti-quark field, respectively.
Then we split the redefined heavy quark and anti-quark field into large and small components

$$
\begin{align*}
& \psi_{v}(x)=P_{+} \Psi_{v}^{(+)}(x), \quad \psi_{s}(x)=P_{-} \Psi_{v}^{(+)}(x),  \tag{3.4}\\
& \chi_{v}(x)=P_{-} \Psi_{v}^{(-)}(x), \quad \chi_{s}(x)=P_{+} \Psi_{v}^{(-)}(x) \tag{3.5}
\end{align*}
$$

with $P_{+}, P_{-}$being the projectors defined as

$$
\begin{equation*}
P_{+}=\frac{1+\psi}{2}, P_{-}=\frac{1-\psi}{2} . \tag{3.6}
\end{equation*}
$$

From the identity

$$
\begin{equation*}
i \not D \Psi^{( \pm)}(x)=e^{-i m_{Q} v \cdot x}\left(i \not D \pm m_{Q} \psi\right) \Psi_{v}^{( \pm)}(x), \tag{3.7}
\end{equation*}
$$

it can be seen that the phase redefinitions in Eq.(3.3) correspond to removing the dominant part $m_{Q} v$ of the heavy quark and anti-quark momenta by splitting the heavy (anti-) quark momentum according to $p^{\mu}=m v^{\mu}+p_{\top}^{\mu}$, where $p_{\top}^{\mu}$ is the small residual momentum, which is identified as relative momentum of the heavy (anti) quark pair. The projectors $P_{+}, P_{-}$project out the large and small (small and large) components for $\Psi_{v}^{(+)}\left(\Psi_{v}^{(-)}\right)$respectively, which means $\psi_{v}, \chi_{v}$ ( $\psi_{s}, \chi_{s}$ ) are the large (small) component heavy quark and anti-quark effective fields. They satisfy $\psi \psi_{v}=\psi_{v}, \psi \chi_{v}=-\chi_{v}$, and the small component fields are suppressed by the ratio $p_{\top} / m_{Q}$. Inserting $\Psi^{(+)}(x)=e^{-i m_{Q} v \cdot x}\left(\psi_{v}(x)+\psi_{s}(x)\right)$ into the Dirac Equation for heavy quark field,

$$
\begin{equation*}
\left(i \not D-m_{Q}\right) \Psi^{(+)}(x)=0, \tag{3.8}
\end{equation*}
$$

and projecting the resulting equation with $P_{ \pm}$, gives

$$
\begin{equation*}
\psi_{s}(x)=\frac{i \not D_{T}}{i v \cdot D+2 m_{Q}} \psi_{v}(x), \tag{3.9}
\end{equation*}
$$

where $D_{\top}=\not D-\psi v \cdot D$. Similarly, we can get the relation between the small and large component effective field for heavy anti-quark field

$$
\begin{equation*}
\chi_{s}(x)=\frac{i \not D_{T}}{-i v \cdot D+2 m_{Q}} \chi_{v}(x) . \tag{3.10}
\end{equation*}
$$

The same relations Eq. $(3.9,3.10)$ have also been derived in Ref. [53] by integrating out the small component fields $\psi_{s}(x), \chi_{s}(x)$ from the generating functional of QCD Green functions. Finally, the heavy quark and anti-quark field can be re-express as

$$
\begin{align*}
\Psi^{(+)}(x) & =e^{-i m_{Q} v \cdot x}\left(1+\frac{i \not D_{T}}{i v \cdot D+2 m_{Q}}\right) \psi_{v}(x),  \tag{3.11a}\\
\Psi^{(-)}(x) & =e^{i m_{Q} v \cdot x}\left(1+\frac{i \not D_{T}}{-i v \cdot D+2 m_{Q}}\right) \chi_{v}(x) . \tag{3.11b}
\end{align*}
$$

To obtain NRQCD Lagrangian, we have to expanding $\Psi^{( \pm)}(x)$ as power series of $1 / m_{Q}$

$$
\begin{align*}
\Psi^{(+)}(x) & =e^{-i m_{Q} v \cdot x}\left(1+\frac{i \not D_{T}}{2 m_{Q}}+\mathcal{O}\left(1 / m_{Q}^{2}\right)\right) \psi_{v}(x),  \tag{3.12a}\\
\Psi^{(-)}(x) & =e^{i m_{Q} v \cdot x}\left(1+\frac{i \not D_{T}}{2 m_{Q}}+\mathcal{O}\left(1 / m_{Q}^{2}\right)\right) \chi_{v}(x) . \tag{3.12b}
\end{align*}
$$

Thus the heavy quark and anti-quark sectors of full QCD can be expressed as

$$
\begin{align*}
\mathcal{L}_{\Psi^{(+)}}= & \bar{\Psi}^{(+)}\left(i \not D-m_{Q}\right) \Psi^{(+)} \\
= & \bar{\psi}_{v}(x)(i v \cdot D) \psi_{v}(x)+\frac{1}{2 m_{Q}} \bar{\psi}_{v}(x)\left(i D_{T}\right)^{2} \psi_{v}(x) \\
& +\frac{g_{s}}{4 m_{Q}} \bar{\psi}_{v}(x) \sigma_{\mu \nu} G^{\mu \nu} \psi_{v}(x)+\mathcal{O}\left(1 / m_{Q}^{2}\right),  \tag{3.13}\\
\mathcal{L}_{\Psi^{(-)}}= & \bar{\Psi}^{(-)}\left(i \not D-m_{Q}\right) \Psi^{(-)} \\
= & \bar{\chi}_{v}(x)(-i v \cdot D) \chi_{v}(x)+\frac{1}{2 m_{Q}} \bar{\chi}_{v}(x)\left(i D_{T}\right)^{2} \chi_{v}(x) \\
& +\frac{g_{s}}{4 m_{Q}} \bar{\chi}_{v}(x) \sigma_{\mu \nu} G^{\mu \nu} \chi_{v}(x)+\mathcal{O}\left(1 / m_{Q}^{2}\right) . \tag{3.14}
\end{align*}
$$

Eq. $(3.13,3.14)$ are the same with the Lagrangian of heavy quark effective theory (HQET). However, their power counting are different. In NRQCD, the leading order contains the first two terms in Eq. $(3.13,3.14)$, while in HQET, the leading order only contains the first term. Although the terms proportional to the gluon field strength tensor $G^{\mu \nu}$ scale as $1 / m_{Q}$, they describe the interaction of heavy (anti-) quark spin with the gluon field, and hence belong to the relativistic corrections.
Combining Eq. $(3.13,3.14$ ), we get our final result for leading order NRQCD Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{NRQCD}}^{\mathrm{LO}}=\bar{\psi}_{v}\left(i v \cdot D+\frac{\left(i D_{T}\right)^{2}}{2 m_{Q}}\right) \psi_{v}+\bar{\chi}_{v}\left(-i v \cdot D+\frac{\left(i D_{T}\right)^{2}}{2 m_{Q}}\right) \chi_{v} \tag{3.15}
\end{equation*}
$$

where we drop all the terms connecting heavy quark and anti-quark fields, since they are only relevant to creation or annihilation of heavy (anti) quark pairs.
It is also straightforward to check that Eq.(3.15) is identical to

$$
\begin{equation*}
\mathcal{L}_{\text {NRQCD }}^{\text {LO,non-covariant }}=\psi^{\dagger}\left(i D_{t}+\frac{\mathbf{D}^{2}}{2 m_{Q}}\right) \psi+\chi^{\dagger}\left(i D_{t}-\frac{\mathbf{D}^{2}}{2 m_{Q}}\right) \chi, \tag{3.16}
\end{equation*}
$$

with $v=(1,0,0,0)$, where $\psi, \chi$ are two component spinors.


Figure 3.1: NRQCD Feynman rules

### 3.2 NRQCD Feynman Rules

To calculate the non-relativistic scattering of $Q \bar{Q}$ pairs, we must derive the corresponding Feynman rules.
Expanding Eq.(3.15) gives

$$
\begin{align*}
\mathcal{L}_{\text {NRQCD }}^{\mathrm{LO}}= & \bar{\psi}_{v}\left(i v \cdot D+\frac{\left(i D_{T}\right)^{2}}{2 m_{Q}}\right) \psi_{v}+\bar{\chi}_{v}\left(-i v \cdot D+\frac{\left(i D_{T}\right)^{2}}{2 m_{Q}}\right) \chi_{v} \\
= & \bar{\psi}_{v}\left(v \cdot(i \partial)+\frac{\left(i \partial_{\top}\right)^{2}}{2 m_{Q}}-g_{s}\left(A \cdot v+\frac{A \cdot\left(i \partial_{\top}\right)}{m_{Q}}+\frac{\left(i \partial_{\top} \cdot A\right)}{2 m_{Q}}\right)\right) \psi_{v} \\
& +\bar{\chi}_{v}\left(v \cdot(-i \partial)+\frac{\left(i \partial_{\top}\right)^{2}}{2 m_{Q}}-g_{s}\left(-A \cdot v+\frac{A \cdot\left(i \partial_{\top}\right)}{m_{Q}}+\frac{\left(i \partial_{\top} \cdot A\right)}{2 m_{Q}}\right)\right) \chi_{v} \\
& +\mathcal{O}\left(g_{s}^{2}\right) \tag{3.17}
\end{align*}
$$

Then with the definitions in Eq.(3.4-3.10), we can directly read the Feynman rules (Figure 3.1) from Eq.(3.17). For the gluon propagator, we adopt the Coulomb gauge, which is a natural choice in describing bound states such as heavy quarkoium.

### 3.3 Heavy Quark Bilinear Operators in Covariant Form

Here we introduce a set of relevant heavy quark bilinear operators at leading order of $1 / m_{Q}$ expansion [57]

$$
\begin{align*}
\left.\mathcal{K}^{1}{ }^{1} S_{0}^{[1]}\right] & \equiv \bar{\psi}_{v} \gamma_{5} \chi_{v},  \tag{3.18a}\\
\mathcal{K}^{\mu}\left[{ }^{3} S_{1}^{[1]}\right] & \equiv \bar{\psi}_{v} \gamma_{\top}^{\mu} \chi_{v},  \tag{3.18b}\\
\mathcal{K}^{\mu}\left[{ }^{1} P_{1}^{[1]}\right] & \equiv \bar{\psi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{\mu} \gamma_{5} \chi_{v},  \tag{3.18c}\\
\mathcal{K}^{\mu \nu}\left[{ }^{3} P_{J}^{[1]}\right] & \equiv \bar{\psi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{\mu} \gamma_{\top}^{\nu} \chi_{v},  \tag{3.18d}\\
\mathcal{K}^{\mu \nu}\left[{ }^{3} P_{0}^{[1]}\right] & \equiv \frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} \bar{\psi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D D}_{T} \chi_{v},  \tag{3.18e}\\
\mathcal{K}^{\mu \nu}\left[{ }^{3} P_{1}^{[1]}\right] & \equiv \bar{\psi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{T}^{[\mu} \gamma_{\top}^{\nu]} \chi_{v},  \tag{3.18f}\\
\mathcal{K}^{\mu \nu}\left[{ }^{3} P_{2}^{[1]}\right] & \equiv \bar{\psi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{T}^{(\mu} \gamma_{\top}^{\nu)} \chi_{v}, \tag{3.18~g}
\end{align*}
$$

$$
\begin{align*}
\mathcal{K}^{a}\left[{ }^{1} S_{0}^{[8]}\right] & \equiv \bar{\psi}_{v} T^{a} \gamma_{5} \chi_{v},  \tag{3.19a}\\
\mathcal{K}^{a, \mu}\left[{ }^{3} S_{1}^{[8]}\right] & \equiv \bar{\psi}_{v} \gamma_{\top}^{\mu} \chi_{v},  \tag{3.19b}\\
\mathcal{K}^{a, \mu}\left[{ }^{1} P_{1}^{[8]}\right] & \equiv \bar{\psi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{\mu} \gamma_{5} \chi_{v},  \tag{3.19c}\\
\mathcal{K}^{a, \mu \nu}\left[{ }^{3} P_{J}^{[8]}\right] & \equiv \bar{\psi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{\mu} \gamma_{\top}^{\nu} \chi_{v},  \tag{3.19d}\\
\mathcal{K}^{a, \mu \nu}\left[{ }^{3} P_{0}^{[8]}\right] & \equiv \frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} \bar{\psi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D D}_{\top} \chi_{v},  \tag{3.19e}\\
\mathcal{K}^{a, \mu \nu}\left[{ }^{3} P_{1}^{[8]}\right] & \equiv \bar{\psi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{[\mu} \gamma_{\top}^{\nu]} \chi_{v},  \tag{3.19f}\\
\mathcal{K}^{a, \mu \nu}\left[{ }_{2}^{[8]} P_{2}^{[8]}\right] & \equiv \bar{\psi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{T}^{(\mu} \gamma_{\top}^{\nu)} \chi_{v}, \tag{3.19~g}
\end{align*}
$$

where we have used the notation $a^{[\mu} b^{\nu]} \equiv \frac{1}{2}\left(a^{\mu} b^{\nu}-a^{\nu} b^{\mu}\right), a^{(\mu} b^{\nu)} \equiv \frac{1}{2}\left(a^{\mu} b^{\nu}+a^{\nu} b^{\mu}\right)-\frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} a \cdot b$. To obtain the corresponding complex transpose of above heavy quark bilinear operators, we first evaluate the following

$$
\begin{align*}
& \left(\bar{\psi}_{v} \gamma_{5} \chi_{v}\right)^{\dagger}=\chi_{v}^{\dagger}\left(\gamma_{5}\right)^{\dagger}\left(\gamma^{0}\right)^{\dagger} \psi_{v}=-\chi_{v}^{\dagger} \gamma^{0} \gamma_{5} \psi_{v}=-\bar{\chi}_{v} \gamma_{5} \psi_{v},  \tag{3.20}\\
& \left(\bar{\psi}_{v} \gamma_{\top}^{\mu} \chi_{v}\right)^{\dagger}=\chi_{v}^{\dagger}\left(\gamma_{\top}^{\mu}\right)^{\dagger}\left(\gamma^{0}\right)^{\dagger} \psi_{v}=\chi_{v}^{\dagger} \gamma^{0} \gamma_{\top}^{\mu} \psi_{v}=\bar{\chi}_{v} \gamma_{\top}^{\mu} \psi_{v}, \tag{3.21}
\end{align*}
$$

then we have

$$
\begin{align*}
& \mathcal{K}^{\dagger}\left[{ }^{1} S_{0}^{[1]}\right]=-\bar{\chi}_{v} \gamma_{5} \psi_{v},  \tag{3.22a}\\
& \mathcal{K}^{\dagger} \mu\left[{ }^{3} S_{1}^{[1]}\right]=\bar{\chi}_{v} \gamma_{T}^{\mu} \psi_{v},  \tag{3.22b}\\
& \mathcal{K}^{\dagger}{ }^{\mu}\left[{ }^{1} P_{1}^{[1]}\right]=-\bar{\chi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}^{\mu} \gamma_{5} \psi_{v},  \tag{3.22c}\\
& \mathcal{K}^{\dagger}{ }^{\mu \nu}\left[{ }^{3} P_{0}^{[1]}\right]=\frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} \bar{\chi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D D}_{\top} \psi_{v},  \tag{3.22d}\\
& \mathcal{K}^{\dagger} \mu \nu\left[{ }^{3} P_{1}^{[1]}\right]=\bar{\chi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{[\mu} \gamma_{\top}^{\nu]} \psi_{v},  \tag{3.22e}\\
& \mathcal{K}^{\dagger} \mu \nu\left[{ }^{3} P_{2}^{[1]}\right]=\bar{\chi}_{v}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{(\mu} \gamma_{\top}^{\nu)} \psi_{v}  \tag{3.22f}\\
& \mathcal{K}^{\dagger} a\left[{ }^{1} S_{0}^{[8]}\right]=-\bar{\chi}_{v} T^{a} \gamma_{5} \psi_{v},  \tag{3.23a}\\
& \mathcal{K}^{\dagger}{ }^{a, \mu}\left[{ }^{3} S_{1}^{[8]}\right]=\bar{\chi}_{v} T^{a} \gamma_{\top}^{\mu} \psi_{v},  \tag{3.23b}\\
& \mathcal{K}^{\dagger}{ }^{\dagger, \mu}\left[{ }^{1} P_{1}^{[8]}\right]=-\bar{\chi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\mathrm{T}}^{\mu} \gamma_{5} \psi_{v},  \tag{3.23c}\\
& \mathcal{K}^{\dagger} a, \mu \nu\left[{ }^{3} P_{0}^{[8]}\right]=\frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} \bar{\chi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D D}_{T} \psi_{v},  \tag{3.23d}\\
& \mathcal{K}^{\dagger} a, \mu \nu\left[{ }^{3} P_{1}^{[8]}\right]=\bar{\chi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}{ }_{\top}^{[\mu} \gamma_{T}^{\nu]} \psi_{v},  \tag{3.23e}\\
& \mathcal{K}^{\dagger} a, \mu \nu\left[{ }^{3} P_{2}^{[8]}\right]=\bar{\chi}_{v} T^{a}\left(-\frac{i}{2}\right) \overleftrightarrow{D}_{\top}^{(\mu} \gamma_{T}^{\nu} \psi_{v} \tag{3.23f}
\end{align*}
$$

### 3.4 One-Loop Corrections to the Single-quarkonium LDMEs

It is natural to do the matching between perturbative QCD and NRQCD with the same gauge, regularization and renormalization scheme. In the perturbative QCD calculations, the dimensional regularization and on-shell renormalization scheme for heavy quark are applied. Therefore, in the perturbative NRQCD calculation, we use the same regularization and renormalization scheme. As for the choice of gauge, the Feynman gauge is commonly adopted in the perturbative QCD calculations, which is also our choice. However, we will use Coulomb gauge for the gluon propagator in the perturbative NRQCD calculations, where the vanishing of Coulomb gluon contribution in each loop diagram will be explicit and the gauge independent results are guaranteed by the gauge invariance of the LDMEs. Explicit calculations with different covariant gauges confirms the gauge invariance of the LDMEs at one-loop level, where the cancellations between different diagrams also indicate the cancellations between real and virtual corrections in the perturbative QCD calculations with the same gauge.


Figure 3.2: Feynman diagrams for Born-level and one-loop corrections to long distance matrix elements.

The Feynman diagrams for the Born level and loop corrections for $Q\left(p_{1}\right) \bar{Q}\left(p_{2}\right)(n) \rightarrow$ $Q\left(p_{1}^{\prime}\right) \bar{Q}\left(p_{2}^{\prime}\right)(n)$ scattering are shown in Figure.3.2.

At Born level,

$$
\begin{align*}
& \left.\mathcal{M}^{\text {Born }\left[{ }^{1}\right.} S_{0}^{[1,8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) T_{\text {col }} \gamma_{5} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{5} v_{v}\left(p_{2}\right),  \tag{3.24}\\
& \mathcal{M}^{\text {Born }\left[{ }^{3} S_{1}^{[1,8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) T_{\text {col }} \gamma_{\top}{ }_{\mu} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right), ~}  \tag{3.25}\\
& \left.\mathcal{M}^{\text {Born }}{ }^{1} P_{1}^{[1,8]}\right]=-p \cdot p^{\prime} \bar{v}_{v}\left(p_{2}^{\prime}\right) T_{\text {col }} \gamma_{5} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{5} v_{v}\left(p_{2}\right),  \tag{3.26}\\
& \sum_{J=0}^{J=2} \mathcal{M}^{\text {Borr }}\left[{ }^{3} P_{J}^{[1,8]}\right]=-p \cdot p^{\prime} \bar{v}_{v}\left(p_{2}^{\prime}\right) T_{\text {col }} \gamma_{\top} \mu u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right), \tag{3.27}
\end{align*}
$$

where $T_{\text {col }}=1$ for CS LDMEs, $T_{\text {col }}=T^{a}$ for CO LDMEs, and

$$
\begin{align*}
p^{\mu}=\frac{\left(p_{1}-p_{2}\right)^{\mu}}{2} & =p_{1 \top}^{\mu}=-p_{2 \mathrm{~T}}^{\mu}, \quad p^{\prime \mu}=\frac{\left(p_{1}^{\prime}-p_{2}^{\prime}\right)^{\mu}}{2}=p_{1 \top}^{\prime \mu}=-p_{2 \top}^{\prime \mu},  \tag{3.28}\\
u_{v}\left(p_{1}\right) & =P_{+} u\left(p_{1}\right), v_{v}\left(p_{2}\right)=P_{-} v\left(p_{2}\right) . \tag{3.29}
\end{align*}
$$

For the loop contributions, we take $n={ }^{3} S_{1}^{[8]}$ as an example, all the other cases can be obtained in a similar way. Taking $p_{1} \cdot v=p_{1}^{\prime} \cdot v=m_{Q}$ and Eq.(3.28) into account, the contribution of $\mathcal{M}_{1}^{\text {loop }}\left[{ }^{3} S_{1}^{[8]}\right]$ in Figure.3.2 reads

$$
\begin{equation*}
\mathcal{M}_{1}^{\text {loop }}\left[{ }^{3} S_{1}^{[8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) \gamma_{T \mu} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right) T^{b} T^{a} \otimes T^{a} T^{b} I^{\text {single }} \tag{3.30}
\end{equation*}
$$

where

$$
\begin{align*}
I^{\text {single }} & =g_{s}^{2} \mu^{4-D} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{i}{l^{2}}\left[-g_{\rho \sigma}+\frac{l_{\rho} l_{\sigma}-(l \cdot v)\left(l_{\rho} v_{\sigma}+v_{\rho} l_{\sigma}\right)}{l^{2}-(l \cdot v)^{2}}\right] \\
& \times \frac{\left(v^{\rho}+\frac{(2 p+l)_{T}^{\rho}}{2 m_{Q}}\right)\left(v^{\sigma}+\frac{\left(2 p^{\prime}+l\right)_{T}^{\sigma}}{2 m_{Q}}\right)}{\left(l \cdot v+\frac{(p+l)_{T}^{2}}{2 m_{Q}}\right)\left(l \cdot v+\frac{\left(p^{\prime}+l\right)_{T}^{2}}{2 m_{Q}}\right)}, \tag{3.31}
\end{align*}
$$

with $\mu$ representing the renormalization scale and $D=4-2 \epsilon$. Through our whole calculation we use dimensional regularization to regularize both ultra-violet and infrared divergences.
Since NRQCD calculations are only valid in the region $p, p^{\prime}, l \ll m_{Q}$, we must expand the heavy (anti-) quark propagators in $1 / m_{Q}$ before loop integration. Expanding the heavy quark propagators in $1 / m_{Q}$ and keeping terms up to order $1 / m_{Q}^{2}$, we get

$$
\begin{align*}
I^{\text {single }} & =\frac{-i g_{s}^{2} \mu^{4-D}}{m_{Q}^{2}} \int \frac{d^{D} l}{(2 \pi)^{D}}\left[\frac{1}{l^{2}(l \cdot v)^{2}}\left(p \cdot p^{\prime}+\frac{(l \cdot p)\left(l \cdot p^{\prime}\right)}{(l \cdot v)^{2}}\right)\right. \\
& \left.+m_{Q}^{2}\left(\frac{1}{l^{2}(l \cdot v)^{2}}+\frac{1}{l^{2}\left(l^{2}-(l \cdot v)^{2}\right)}\right)\right] \\
& =-\frac{\alpha_{s}}{3 \pi m_{Q}^{2}} \mu^{4-D} p \cdot p^{\prime}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right), \tag{3.32}
\end{align*}
$$

where we have dropped all the terms which vanish after integration (see Appendix D.1.). The term $\int \frac{d^{D} l}{(2 \pi)^{D}}\left(\frac{1}{l^{2}(l \cdot v)^{2}}+\frac{1}{l^{2}\left(l^{2}-(l \cdot v)^{2}\right)}\right)=\int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{(l \cdot v)^{2}\left(l^{2}-(l \cdot v)^{2}\right)}=0$ in Eq. (3.32) corresponds to the Coulomb gluon exchange, which vanishes as expected.

The contributions of $\mathcal{M}_{2,3,4}^{\text {loop }}\left[{ }^{3} S_{1}^{[8]}\right]$ in Figure.3.2 are the same with $\mathcal{M}_{1}^{\text {loop }}\left[{ }^{3} S_{1}^{[8]}\right]$ except for color factors, which can be expressed as

$$
\begin{align*}
& \mathcal{M}_{2}^{\text {loop }\left[{ }^{3} S_{1}^{[8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) \gamma_{\top} \mu u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right) T^{a} T^{b} \otimes T^{b} T^{a} I^{\text {single }},}  \tag{3.33}\\
& \mathcal{M}_{3}^{\text {loop }}\left[{ }^{3} S_{1}^{[8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) \gamma_{\top \mu} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right) T^{a} T^{b} \otimes T^{a} T^{b} I^{\text {single }},  \tag{3.34}\\
& \mathcal{M}_{4}^{\text {loop }\left[{ }^{3} S_{1}^{[8]}\right]=\bar{v}_{v}\left(p_{2}^{\prime}\right) \gamma_{\top_{\mu}} u_{v}\left(p_{1}^{\prime}\right) \bar{u}_{v}\left(p_{1}\right) T_{\text {col }} \gamma_{T}^{\mu} v_{v}\left(p_{2}\right) T^{b} T^{a} \otimes T^{b} T^{a} I^{\text {single }} .} \tag{3.35}
\end{align*}
$$

The contributions of the last two diagrams read

$$
\begin{align*}
\left.\mathcal{M}_{5}^{\text {loop }}{ }^{3} S_{1}^{[8]}\right] & =\bar{u}_{v}\left(p_{1}^{\prime}\right) \gamma_{T}^{\mu} v_{v}\left(p_{2}^{\prime}\right) \bar{v}_{v}\left(p_{2}\right) \gamma_{\top} u_{v}\left(p_{1}\right) T^{b} T^{a} T^{b} \otimes T^{a} \\
& \times\left(-i g_{s}^{2} \mu^{4-D}\right) \int \frac{d^{D} l}{(2 \pi)^{D}}\left(\frac{1}{l^{2}(l \cdot v)^{2}}+\frac{1}{l^{2}\left(l^{2}-(l \cdot v)^{2}\right)}\right), \tag{3.36}
\end{align*}
$$

$$
\begin{align*}
\mathcal{M}_{6}^{\text {loop }}\left[^{3} S_{1}^{[8]}\right] & =\bar{u}_{v}\left(p_{1}^{\prime}\right) \gamma_{T}^{\mu} v_{v}\left(p_{2}^{\prime}\right) \bar{v}_{v}\left(p_{2}\right) \gamma_{\top} u_{v}\left(p_{1}\right) T^{a} \otimes T^{b} T^{a} T^{b} \\
& \times\left(-i g_{s}^{2} \mu^{4-D}\right) \int \frac{d^{D} l}{(2 \pi)^{D}}\left(\frac{1}{l^{2}(l \cdot v)^{2}}+\frac{1}{l^{2}\left(l^{2}-(l \cdot v)^{2}\right)}\right), \tag{3.37}
\end{align*}
$$

which represent the Coulomb gluon exchange and vanish as expected.
Note: the $\mathcal{O}\left(v_{Q}^{2}\right)$ corrections are not considered here, namely the terms proportional to $p^{2}$ or $p^{\prime 2}$ are dropped.

Using the identity

$$
\begin{equation*}
T^{a} T^{b}=\frac{\delta^{a b}}{2 C_{A}}+\frac{1}{2}\left(d^{a b c}+i f^{a b c}\right) T^{c} \tag{3.38}
\end{equation*}
$$

we can decompose the following color structure as

$$
\begin{align*}
& T^{a} T^{b} \otimes T^{a} T^{b}=\frac{C_{F}}{2 C_{A}}(\mathbf{1} \otimes \mathbf{1})-\frac{1}{C_{A}}\left(T^{c} \otimes T^{c}\right),  \tag{3.39}\\
& T^{a} T^{b} \otimes T^{b} T^{a}=\frac{C_{F}}{2 C_{A}}(\mathbf{1} \otimes \mathbf{1})+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left(T^{c} \otimes T^{c}\right) . \tag{3.40}
\end{align*}
$$

Adding up the loop contributions Eq.(3.30-3.35), we have

$$
\begin{align*}
\mathcal{M}^{\mathrm{loop}\left[{ }^{3} S_{1}^{[8]}\right]} & =2 \bar{u}_{v}\left(p_{1}^{\prime}\right) \gamma_{\top}^{\mu} v_{v}\left(p_{2}^{\prime}\right) \bar{v}_{v}\left(p_{2}\right) \gamma_{\top \mu} u_{v}\left(p_{1}\right) \\
& \times\left(\frac{C_{F}}{C_{A}}(\mathbf{1} \otimes \mathbf{1})+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left(T^{c} \otimes T^{c}\right)\right) I^{\text {single }} \tag{3.41}
\end{align*}
$$

Comparing with Eq.(3.27), gives

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{NLO}} & =\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{Born}}+\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}} \mu^{4-D}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right) \\
& \times \sum_{J=0}^{J=2}\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle_{\mathrm{Born}}+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle_{\mathrm{Born}}\right] . \tag{3.42}
\end{align*}
$$

The ultraviolet divergences can be removed by renormalization. In order to be consistent with the literature, we adopt $\overline{\mathrm{MS}}$ scheme and define

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{ren}} & =\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{NLO}}-\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{UV}}} \\
& \times \sum_{J=0}^{J=2}\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }_{J}^{3} P_{J}^{[1]}\right]\right\rangle_{\mathrm{Born}}+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }_{J}^{3} P_{J}^{[8]}\right]\right\rangle_{\mathrm{Born}}\right] \tag{3.43}
\end{align*}
$$

where $\mu_{\Lambda}$ is the NRQCD scale and $\gamma_{E}$ is the Euler gamma constant.
Substitute Eq.(3.43) with Eq.(3.42), then we get

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{ren}} & =\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\text {Born }}-\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}} \\
& \times \sum_{J=0}^{J=2}\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle_{\text {Born }}+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle_{\text {Born }}\right] . \tag{3.44}
\end{align*}
$$

As addressed in Ref. [3, 47, 58], the infrared divergences appearing in the SDCs calculated in perturbative QCD must be the same with those in perturbative NRQCD, since these two theories are equivalent at the long-distance regime. This is the so-called matching between two theories in the same regime. Therefore, we have to subtract the long-distance contributions in the SDCs to avoid double counting, which can be done through multiplying the SDCs by the Born level S-wave LDMEs

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\text {Born }} & =\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\text {ren }}+\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}} \\
& \times \sum_{J=0}^{J=2}\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle_{\text {Born }}+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle_{\text {Born }}\right] . \tag{3.45}
\end{align*}
$$

rather than the renormalized ones.

### 3.5 One-loop Corrections to the Un-Decoupled Double-quarkoniumLDMEs

As mentioned in last chapter, the four-fermion operators are not sufficient in the case of double quarkonia involved processes. Therefore we introduce a set of eight-fermion operators and undecoupled LDMEs to describe the double-quarkonium related processes in the first subsection, where the decoupling of the color and Dirac indexes of the un-decoupled LDMEs will be done after one-loop corrections. Detailed calculation of one-loop corrections for one specific undecoupled LDME is shown in the second subsection. In the third subsection we summarize our results of one-loop corrections for the other un-decoupled LDMEs.

### 3.5.1 Definitions of the Un-Decoupled Double-quarkonium LDMEs

First we introduce a set of quasilinear operators which are relevant to two quarkonia $H_{1}, H_{2}$ in the states $n_{1}, n_{2}$ :

$$
\begin{align*}
\mathcal{Q}\left(n_{1}, n_{2}\right) & \equiv \mathcal{K}\left[n_{1}\right] \mathcal{K}\left[n_{2}\right],  \tag{3.46a}\\
\mathcal{Q}\left(n_{1}, n_{2}^{\dagger}\right) & \equiv \mathcal{K}\left[n_{1}\right] \mathcal{K}^{\dagger}\left[n_{2}\right], \tag{3.46b}
\end{align*}
$$



Figure 3.3: Representive Feynman diagrams for one-loop corrections for process including the annihilation and production of 2 heavy quark pairs $Q_{1} \bar{Q}_{1}$ and $Q_{2} \bar{Q}_{2}$, where each diagram represent a type of Feynman diagrams.
where $n_{1}, n_{2}$ can be any states among ${ }^{1} S_{0}^{[1,8]},{ }^{3} S_{1}^{[1,8]},{ }^{1} P_{1}^{[1,8]},{ }^{3} P_{J}^{[1,8]}$, and $\mathcal{K}, \mathcal{K}^{\dagger}$ are defined in Eq.(3.18, 3.19). The corresponding complex conjugate operators are

$$
\begin{align*}
\mathcal{Q}^{\dagger}\left(n_{1}, n_{2}\right) & =\mathcal{K}^{\dagger}\left[n_{1}\right] \mathcal{K}^{\dagger}\left[n_{2}\right],  \tag{3.47a}\\
\mathcal{Q}^{\dagger}\left(n_{1}, n_{2}^{\dagger}\right) & =\mathcal{K}^{\dagger}\left[n_{1}\right] \mathcal{K}\left[n_{2}\right], \tag{3.47b}
\end{align*}
$$

Then the un-decoupled LDMEs for production of two quarkonia $H_{1}, H_{2}$ and inclusive production of quarkonium $H_{1}$ through quarkonium $H_{2}$ decay can be defined as

$$
\begin{align*}
\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle & \equiv\langle 0| \mathcal{Q}^{\dagger}\left(n_{1}^{\prime}, n_{2}^{\prime}\right) \mathcal{P}^{H_{1} H_{2}} \mathcal{Q}\left(n_{1}, n_{2}\right)|0\rangle,  \tag{3.48a}\\
\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle_{H_{2}} & \equiv\left\langle H_{2}\right| \mathcal{Q}^{\dagger}\left(n_{1}^{\prime}, n_{2}^{\dagger}\right) \mathcal{P}^{H_{1}} \mathcal{Q}\left(n_{1}, n_{2}^{\dagger}\right)\left|H_{2}\right\rangle, \tag{3.48b}
\end{align*}
$$

with

$$
\begin{equation*}
\mathcal{P}^{H_{1} H_{2}} \equiv \sum_{X}\left|H_{1} H_{2} X\right\rangle\left\langle H_{1} H_{2} X\right|, \tag{3.49}
\end{equation*}
$$

where the un-decoupled LDMEs carry certain color and Dirac indexes. In addition, $n_{1}$ and $n_{1}^{\prime}\left(n_{2}\right.$ and $\left.n_{2}^{\prime}\right)$ are not necessary identical.

### 3.5.2 One-loop Corrections

In this subsection, we take $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle$ as an example to show how to calculate the one-loop correction for the un-decoupled double-quarkonium LDMEs. All the other cases can be obtained simply by replacing the corresponding color and Dirac structures.

The representative Feynman diagrams for one-loop corrections to $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle$ are shown in Figure.3.3. The first two types of Feynman diagrams actually correspond to the one-loop corrections to $\left\langle\mathcal{O}^{H_{1}}\left(n_{1}\right)\right\rangle,\left\langle\mathcal{O}^{H_{2}}\left(n_{2}\right)\right\rangle$ respectively, which are computed in last section with $n_{1}=n_{1}^{\prime}, n_{2}=n_{2}^{\prime}$. The third type of Feynman diagrams are the interference
between initial $Q_{1} \bar{Q}_{1}$ and final $Q_{2} \bar{Q}_{2}$; the fourth type of Feynman diagrams correspond to the interference between final $Q_{1} \bar{Q}_{1}$ and initial $Q_{2} \bar{Q}_{2}$. For the case of one-loop corrections for $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{[3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle$, the corresponding Feynman diagrams belong to the third type (including 4 Feynman diagrams), whose corresponding short distance coefficients represent the interferences between the amplitudes of production of $Q_{1} \bar{Q}_{1}\left({ }^{3} S_{1}^{[a]}, \varepsilon_{\mu}\right) Q_{2} \bar{Q}_{2}\left({ }^{3} P_{J_{2}}^{[b]}, \varepsilon_{\rho \sigma}\right)$ pairs and the complex conjugate of the amplitudes of production of $Q_{1} \bar{Q}_{1}\left({ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]}, \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*}\right) Q_{2} \bar{Q}_{2}\left({ }^{3} S_{1}^{\left[b^{\prime}\right]}, \varepsilon_{\rho^{\prime}}^{*}\right)$ pairs. Here $a, a^{\prime}, b, b^{\prime}$ are the corresponding color indexes in CO states or 1 in CS cases, $\varepsilon, \varepsilon^{*}$ are the polarization vectors or tensors with explicit Dirac indexes.

With the NRQCD Feynman rules presented in last section (Figure.3.1), we can straightforwardly write the contributions of third type diagrams as

$$
\begin{align*}
\mathcal{M}_{3,1}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}= & C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}} \bar{v}_{v_{2}}\left(p_{2}\right) \gamma_{T_{2}}^{\rho^{\prime}} u_{v_{2}}\left(\bar{p}_{2}\right) \bar{u}_{v_{1}}\left(\bar{p}_{1}\right) \gamma_{T_{1}}^{\mu} v_{v_{1}}\left(p_{1}\right) \\
& \times T^{e} T^{a} \otimes T^{e} T^{b^{\prime}} I^{\text {double }},  \tag{3.50}\\
\mathcal{M}_{3,2}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}= & C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}} \bar{v}_{v_{2}}\left(p_{2}\right) \gamma_{T_{2}}^{\rho^{\prime}} u_{v_{2}}\left(\bar{p}_{2}\right) \bar{u}_{v_{1}}\left(\bar{p}_{1}\right) \gamma_{T_{1}}^{\mu} v_{v_{1}}\left(p_{1}\right)  \tag{3.51}\\
& \times T^{e} T^{a} \otimes T^{b^{\prime}} T^{e} I^{\text {double }},  \tag{3.52}\\
\mathcal{M}_{3,3}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}= & C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}} \bar{v}_{v_{2}}\left(p_{2}\right) \gamma_{T_{2}}^{\rho^{\prime}} u_{v_{2}}\left(\bar{p}_{2}\right) \bar{u}_{v_{1}}\left(\bar{p}_{1}\right) \gamma_{T_{1}}^{\mu} v_{v_{1}}\left(p_{1}\right) \\
& \times T^{a} T^{e} \otimes T^{e} T^{b^{\prime}} I^{\text {double }},  \tag{3.53}\\
\mathcal{M}_{3,4}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}= & C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}} \bar{v}_{v_{2}}\left(p_{2}\right) \gamma_{T_{2}}^{\rho^{\prime}} u_{v_{2}}\left(\bar{p}_{2}\right) \bar{u}_{v_{1}}\left(\bar{p}_{1}\right) \gamma_{T_{1}}^{\mu} v_{v_{1}}\left(p_{1}\right) \\
& \times T^{a} T^{e} \otimes T^{b^{\prime}} T^{e} I^{\text {double }},
\end{align*}
$$

where $C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}}$ is an overall common factor corresponding to the contributions of the P-wave states, and $I^{\text {double }}$ is given by

$$
\begin{align*}
I^{\text {double }} & =g_{s}^{2} \mu^{4-D} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{i}{l^{2}}\left[-g_{\rho \sigma}+\frac{l_{\rho} l_{\sigma}-\left(l \cdot v_{1}\right)\left(l_{\rho} v_{1 \sigma}+v_{1 \rho} l_{\sigma}\right)}{l^{2}-\left(l \cdot v_{1}\right)^{2}}\right] \\
& \times \frac{\left(v_{1}^{\rho}+\frac{\left(2 q_{1}+l\right)_{T_{1}}^{\rho}}{2 m_{1}}\right)\left(v_{2}^{\sigma}+\frac{\left.\left(2 q_{2}+l\right)\right)_{T_{2}}^{\sigma}}{2 m_{2}}\right)}{\left(l \cdot v_{1}+\frac{\left(q_{1}+l\right)_{T_{1}}^{2}}{2 m_{1}}\right)\left(l \cdot v_{2}+\frac{\left(q_{2}+l\right)_{T_{2}}^{2}}{2 m_{2}}\right)}, \tag{3.54}
\end{align*}
$$

with $m_{1}, m_{2}$ representing the masses of heavy quarks $Q_{1}, Q_{2}$, and

$$
\begin{align*}
& q_{1} \equiv \frac{p_{1}-\bar{p}_{1}}{2}, q_{2} \equiv \frac{p_{2}-\bar{p}_{2}}{2},  \tag{3.55}\\
& v_{1} \equiv \frac{p_{1}+\bar{p}_{1}}{2 m_{1}}, v_{2} \equiv \frac{p_{2}+\bar{p}_{2}}{2 m_{2}} \tag{3.56}
\end{align*}
$$

Expanding the heavy quark propagators in $\frac{1}{m_{1}}, \frac{1}{m_{2}}$ and keeping only the terms proportional to $\frac{1}{m_{1} m_{2}}$ gives

$$
\begin{align*}
I^{\text {double }}= & -\frac{i g_{s}^{2} \mu^{4-D}}{m_{1} m_{2}} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}}\left(\frac{q_{1} \cdot q_{2}}{\left(l \cdot v_{1}\right)\left(l \cdot v_{2}\right)}-\frac{\left(l \cdot q_{1}\right)\left(v_{1} \cdot q_{2}\right)}{\left(l \cdot v_{1}\right)^{2}\left(l \cdot v_{2}\right)}\right. \\
& \left.-\frac{\left(l \cdot q_{2}\right)\left(v_{2} \cdot q_{1}\right)}{l \cdot v_{1}\left(l \cdot v_{2}\right)^{2}}+\frac{\left(v_{1} \cdot v_{2}\right)\left(l \cdot q_{1}\right)\left(l \cdot q_{2}\right)}{\left(l \cdot v_{1}\right)^{2}\left(l \cdot v_{2}\right)^{2}}\right) \\
= & \frac{\alpha_{s} \mu^{4-D}}{\pi m_{1} m_{2}}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right)\left(c_{1} q_{1} \cdot q_{2}+c_{2}\left(v_{1} \cdot q_{2}\right)\left(v_{2} \cdot q_{1}\right)\right), \tag{3.57}
\end{align*}
$$

where

$$
\begin{align*}
& c_{1}=\frac{\ln \left(\omega+\sqrt{\omega^{2}-1}\right)-\omega \sqrt{\omega^{2}-1}}{2\left(\omega^{2}-1\right)^{3 / 2}},  \tag{3.58a}\\
& c_{2}=\frac{\left(\omega^{2}+2\right) \sqrt{\omega^{2}-1}-3 \omega \ln \left(\sqrt{\omega^{2}-1}+\omega\right)}{2\left(\omega^{2}-1\right)^{5 / 2}}, \tag{3.58b}
\end{align*}
$$

with $\omega=v_{1} \cdot v_{2}$.
Note: For the double quarkonia case, there are no natural choice of Coulomb gauge for both quarkonia. Here we choose Coulomb gauge in the $v_{1}$ frame, which means the terms proportional to $\frac{1}{m_{1}}$ will be canceled in each diagram while the terms proportional to $\frac{1}{m_{2}}$ will only be totally canceled after adding up all the diagrams. The details of the cancellation, or in other words, the gauge independence, will be discussed elsewhere.

With Eq.(3.38), the color structure $\left(T^{e} T^{a}+T^{a} T^{e}\right) \otimes\left(T^{e} T^{b^{\prime}}+T^{b^{\prime}} T^{e}\right)$ can be decomposed as

$$
\begin{align*}
c_{f} & \equiv\left(T^{e} T^{a}+T^{a} T^{e}\right) \otimes\left(T^{e} T^{b^{\prime}}+T^{b^{\prime}} T^{e}\right) \\
& =\delta^{a b^{\prime}} \frac{2 C_{F}}{C_{A}}(\mathbf{1} \otimes \mathbf{1})+\frac{d^{a b^{\prime} e}}{2 C_{A}}\left(T^{e} \otimes \mathbf{1}+\mathbf{1} \otimes T^{e}\right)+\frac{d^{a e c} d^{b^{\prime} e c^{\prime}}}{4}\left(T^{c} \otimes T^{c^{\prime}}\right) . \tag{3.59}
\end{align*}
$$

Re-expressing $c_{1} q_{1} \cdot q_{2}+c_{2}\left(v_{1} \cdot q_{2}\right)\left(v_{2} \cdot q_{1}\right)$ as $q_{1}^{\nu} q_{2}^{\sigma^{\prime}}\left(c_{1} g_{\nu \sigma^{\prime}}+c_{2} v_{1 \sigma^{\prime}} v_{2 \nu}\right)$ and adding up all the contributions from $\mathcal{M}_{3,(1,2,3,4)}$, we get the total contribution

$$
\begin{equation*}
\mathcal{M}_{3}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}=q_{1}^{\nu} q_{2}^{\sigma^{\prime}} C^{\rho \sigma \mu^{\prime} \nu^{\prime}, b a^{\prime}} \bar{v}_{v_{2}}\left(p_{2}\right) \gamma_{T_{2}}^{\rho^{\prime}} u_{v_{2}}\left(\bar{p}_{2}\right) \bar{u}_{v_{1}}\left(\bar{p}_{1}\right) \gamma_{T_{1}}^{\mu} v_{v_{1}}\left(p_{1}\right) I_{\nu \sigma^{\prime}}^{\text {double }} c_{f} \tag{3.60}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{\nu \sigma^{\prime}}^{\text {double }}=\frac{\alpha_{s} \mu^{4-D}}{\pi m_{1} m_{2}}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right)\left(c_{1} g_{\nu \sigma^{\prime}}+c_{2} v_{1 \sigma^{\prime}} v_{2 \nu}\right) . \tag{3.61}
\end{equation*}
$$

For any given 4-dimensional rank-2 tensor $a^{\mu} b^{\nu}$, we can decompose it as

$$
\begin{equation*}
a^{\mu} b^{\nu}=\frac{g^{\mu \nu}-v^{\mu} v^{\nu}}{3} a \cdot b+a^{[\mu} b^{\nu]}+a^{(\mu} b^{\nu)} \tag{3.62}
\end{equation*}
$$

thus

$$
\begin{align*}
\gamma_{\mathrm{T}_{2}}^{\rho^{\prime}} q_{2}^{\sigma^{\prime}} \otimes \gamma_{\mathrm{T}_{1}}^{\mu} q_{1}^{\nu}= & \left(\frac{g^{\rho^{\prime} \sigma^{\prime}}-v_{2}^{\rho^{\prime}} v_{2}^{\sigma^{\prime}}}{3} q_{2}+\gamma_{T_{2}}^{\left[\rho_{2}^{\prime}\right.} q_{2}^{\left.\sigma^{\prime}\right]}+\gamma_{T_{2}}^{\left(\rho^{\prime}\right.} q_{2}^{\left.\sigma^{\prime}\right)}\right) \\
& \otimes\left(\frac{g^{\mu \nu}-v_{1}^{\mu} v_{1}^{\nu}}{3} q_{1}+\gamma_{T_{1}}^{[\mu} q_{1}^{\nu]}+\gamma_{T_{1}}^{(\mu} q_{1}^{\nu)}\right), \tag{3.63}
\end{align*}
$$

which means $\mathcal{M}_{3}^{\mu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime}, a b a^{\prime} b^{\prime}}$ can be written as summations over $J_{1}, J_{2}^{\prime}$ states. Therefore we have

$$
\begin{align*}
& \left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\mathrm{NLO}} \\
& =\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{[8]}, S_{1} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[8]}\right]\right\rangle_{\text {Born }} \\
& +I_{\nu \sigma^{\prime}}^{\text {double }} \mathcal{P}_{1}^{a b a^{\prime} b^{\prime}, \mu \nu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime} \sigma^{\prime}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right], \tag{3.64}
\end{align*}
$$

with

$$
\begin{align*}
& \mathcal{P}_{1}^{a b a^{\prime} b^{\prime}, \mu \nu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime} \sigma^{\prime}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{[8} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right] \\
& =\sum_{J_{1}, J_{2}^{\prime}}\left(\delta^{a b^{\prime}} \frac{2 C_{F}}{C_{A}}\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}}^{\left[a_{1}^{\prime}\right]},{ }^{3} P_{J_{2}^{\prime}}^{[1]},{ }^{3} P_{J_{1}}^{[1]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }}\right. \\
& +\frac{d^{a b^{\prime} e}}{2 C_{A}}\left(\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} P_{J_{2}^{\prime}}^{[1]},{ }^{3} P_{J_{1}}^{[e]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }}\right. \\
& \left.+\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} P_{J_{2}^{\prime}}^{[e]},{ }^{3} P_{J_{1}}^{[1]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }}\right) \\
& \left.+\frac{d^{a e c} d^{b^{\prime} e c^{\prime}}}{4}\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} P_{J_{2}^{\prime}}^{\left[c^{\prime}\right]},{ }^{3} P_{J_{1}}^{[c]}, P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }}\right), \tag{3.65}
\end{align*}
$$

where the $J_{1}, J_{2}^{\prime}$ states carry $\mu \nu$ and $\rho^{\prime} \sigma^{\prime}$ Dirac indexes respectively. From Eq. (3.64, 3.65), it can be seen that the $S$-wave states flip to be P-wave states and may lead to non-vanishing contributions at NLO.

The ultra-violet divergence in $I_{\nu \sigma^{\prime}}^{\text {double }}$ indicates that the un-decoupled doube-quarkonium LDMEs need renormalization at NLO. Thus we define

$$
\begin{align*}
& \left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }_{3}^{3} S_{1}^{[8]},{ }_{3}^{3} S_{1}^{[8]},{ }^{[8} P_{J_{2}}^{[b]}\right]\right\rangle_{\mathrm{ren}}=\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[3 P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{[3} P_{J_{2}}^{[b]}\right]\right\rangle_{\mathrm{NLO}} \\
& -\frac{\alpha_{s}}{\pi m_{1} m_{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{UV}}}\left(c_{1} g_{\nu \sigma^{\prime}}+c_{2} v_{1 \sigma^{\prime}} v_{2 \nu}\right) \mathcal{P}_{1}^{a b a^{\prime} b^{\prime}, \mu \nu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime} \sigma^{\prime}}\left[{ }^{3} S_{1}^{[8]},{ }_{1}^{[8]} S_{1}^{[8]}\right] . \tag{3.66}
\end{align*}
$$

When $m_{1}=m_{2}$, it is natural to choose $\mu_{\Lambda}=m_{1}$ or $2 m_{1}$. However, when $m_{1} \neq m_{2}$, there is no natural choice of NRQCD scale $\mu_{\Lambda}$. The consequences of scale ambiguity need further
clarification elsewhere.
Substituting Eq. (3.64) with Eq. (3.66), we get our final result

$$
\begin{align*}
& \left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\mathrm{ren}}=\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }_{1}^{3} S_{1}^{[8]},{ }_{1}^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }} \\
& +I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{1}^{\mu \nu \rho \sigma \mu^{\prime} \nu^{\prime} \rho^{\prime} \sigma^{\prime}, a b a^{\prime} b^{\prime}}\left[{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]}\right], \tag{3.67}
\end{align*}
$$

where $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[{ }^{3} P_{J_{1}^{\prime}}^{\left[a^{\prime}\right]},{ }^{3} S_{1}^{[8]},{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{2}}^{[b]}\right]\right\rangle_{\text {Born }}$ vanishes due to symmetric reasons and $I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right)$ is defined as

$$
\begin{equation*}
I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right)=-\frac{\alpha_{s}}{\pi m_{1} m_{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}}\left(c_{1} g_{\nu \sigma^{\prime}}+c_{2} v_{1 \sigma^{\prime}} v_{2 \nu}\right) . \tag{3.68}
\end{equation*}
$$

Up to P-wave, there are other 19 and 20 similar interference results for the one-loop correction to the double-production and decay-production un-decouple LDMEs respectively, which can be obtained in the same manner and expressed as

$$
\begin{align*}
& \left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle_{\text {ren }}=I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right],  \tag{3.69}\\
& \left\langle\mathcal{Q}^{H_{1}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle_{H_{2}, \text { ren }}=I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{2}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right], \tag{3.70}
\end{align*}
$$

where $\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right], \mathcal{P}_{2}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]$ carry certain color and Dirac indexes, whose explicit results for each cases are shown in Appendix D.2.

As for the decoupling of the color and Dirac indexes, they are closely connected with the SDCs and the factorization formulas, therefore we will discuss them in detail in the next section.

### 3.6 NRQCD Factorization Formulas for Double-quarkonium Processes

In order to give correct polarization and color factors for the interference contributions, we sketch the procedures of decoupling the color, Dirac indices (indexes) and the integration over the relative momentum $q$ from short distance and long distance parts in covariant way for single quarkonium involved processes. Here we distinguish indice and index, where for color matrix $T_{i j}^{a}$ and Dirac matrix $\gamma_{i j}^{\mu}, i, j$ are color, Dirac indices while $a, \mu$ are color, Dirac indexes, respectively. Then we apply the same method to double quarkonia involved processes, which leads to our formulas for NRQCD factorization for double-quarkonium processes.

### 3.6.1 Single-quarkonium Processes

At amplitude level, with the validity of factorization theorem, the amplitude of inclusive production or decay of single-quarkonium $H$ can be written as

$$
\begin{align*}
& \mathcal{A}_{1}^{\text {single }}=\sum_{X} \mathcal{T}_{j \bar{j}}^{i \bar{i}}\langle H X| \bar{\Psi}_{j}^{i} \Psi^{i}|0\rangle,  \tag{3.71a}\\
& \mathcal{A}_{2}^{\text {single }}=\sum_{X} \mathcal{T}_{j \bar{j}}^{i \bar{i}}\langle X| \bar{\Psi}_{j}^{i} \Psi_{\bar{j}}^{\bar{i}}|H\rangle, \tag{3.71b}
\end{align*}
$$

where the short distance part $\mathcal{T}_{j \bar{j}}^{i \bar{i}}$ is the matrix element for the inclusive production or decay of the $Q\left(p_{1}\right) \bar{Q}\left(p_{2}\right)$ with the spinors of the heavy quark pair removed, $i, \bar{i}$ and $j, \bar{j}$ are the color and Dirac indices repectively. $\langle H X| \bar{\Psi}_{j}^{i} \Psi^{i}|0\rangle$ and $\langle X| \bar{\Psi}_{j}^{i} \Psi_{\bar{j}}^{\bar{i}}|H\rangle$ are the long distance parts, which represent the amplitudes for production or decay of quarkonium $H$ through intermediate $Q \bar{Q}$ state, and the sum is over all possible extra soft partons.
The decoupling of the color indices can be achieved by using the Fierz rearrangement

$$
\begin{equation*}
\delta_{i i^{\prime}} \delta_{\overline{i i^{\prime}}}=\frac{1}{\sqrt{N_{c}}} \delta_{i \bar{i}} \otimes \frac{1}{\sqrt{N_{c}}} \delta_{i^{\prime} \bar{i}^{\prime}}+\sqrt{2} T_{i \bar{i}}^{a} \otimes \sqrt{2} T_{i^{\prime} \bar{i}^{\prime}}^{a}, \tag{3.72}
\end{equation*}
$$

which implies the color projectors given in Section 2.2.
To decouple the Dirac indices and project the heavy quark pair onto spin singlet and triplet states, we first decouple the large and small components of the heavy (anti-) quark field, which can be done in a similar way with Eq. (3.11) for on-shell heavy quarks at tree level. Thus, for production case, the on-shell heavy quark pair can be expanded as

$$
\begin{align*}
\bar{\Psi}_{j} \Psi_{\bar{j}} & \left.=e^{2 i m_{Q} v \cdot x}\left[\bar{\psi}_{v}\left(1+\frac{\not p_{1 T}}{p_{1} \cdot v+m_{Q}}\right)\right]_{j}\left[\left(1-\frac{\not p_{2 T}}{p_{2} \cdot v+m_{Q}}\right)\right] \chi_{v}\right]_{\bar{j}} \\
& \left.=e^{2 i m_{Q} v \cdot x}\left[\bar{\psi}_{v} P_{+}\left(1+\frac{\not p_{1 T}}{p_{1} \cdot v+m_{Q}}\right)\right]_{j}\left[\left(1-\frac{\not p_{2 T}}{p_{2} \cdot v+m_{Q}}\right)\right] P_{-} \chi_{v}\right]_{\bar{j}} . \tag{3.73}
\end{align*}
$$

Applying the trace formula

$$
\begin{equation*}
P_{+} \Gamma P_{-}=\frac{1}{2} \operatorname{Tr}\left[\Gamma P_{-} \gamma_{5} P_{+}\right] P_{+} \gamma_{5} P_{-}+\frac{1}{2} \operatorname{Tr}\left[\Gamma P_{-} \gamma^{\mu} P_{+}\right] P_{+} \gamma_{\top \mu} P_{-}, \tag{3.74}
\end{equation*}
$$

where $\Gamma$ is any combinations of Dirac matrix, we get

$$
\begin{align*}
\mathcal{T}_{j \bar{j}}\langle H X| \bar{\Psi}_{j} \Psi_{\bar{j}}|0\rangle= & \operatorname{Tr}\left[\mathcal{T}\left(1-\frac{\not p_{2 \top}}{p_{2} \cdot v+m_{Q}}\right) \frac{\gamma_{5} P_{+}}{\sqrt{2}}\left(1+\frac{\not p_{1 \top}}{p_{1} \cdot v+m_{Q}}\right)\right] \frac{\langle H X| \bar{\psi}_{v} \gamma_{5} \chi_{v}|0\rangle}{\sqrt{2}} \\
& +\operatorname{Tr}\left[\mathcal{T}\left(1-\frac{\not p_{2 T}}{p_{2} \cdot v+m_{Q}}\right) \frac{\gamma^{\mu} P_{+}}{\sqrt{2}}\left(1+\frac{\not p_{1 \top}}{p_{1} \cdot v+m_{Q}}\right)\right] \frac{\langle H X| \bar{\psi}_{v} \gamma_{T \mu} \chi_{v}|0\rangle}{\sqrt{2}} . \tag{3.75}
\end{align*}
$$

Here we have dropped the overall phase factor $e^{2 i m_{Q} v \cdot x}$, after all it will be canceled after multiplying with its complex conjugate.
Considering that we start from the full QCD, the normalization of the quarkonium state $H(P)\rangle$ is relativistic:

$$
\begin{equation*}
\left\langle H(P) \mid H\left(P^{\prime}\right)\right\rangle_{\mathrm{QCD}}=(2 \pi)^{3} 2 E_{P} \delta^{3}\left(\mathbf{P}-\mathbf{P}^{\prime}\right), \tag{3.76}
\end{equation*}
$$

where $E_{P}$ is total energy of quarkonium $H$, which is different from the usual NRQCD conventions :

$$
\begin{equation*}
\left\langle H(P) \mid H\left(P^{\prime}\right)\right\rangle_{\mathrm{NRQCD}}=(2 \pi)^{3} \delta^{3}\left(\mathbf{P}-\mathbf{P}^{\prime}\right) . \tag{3.77}
\end{equation*}
$$

Therefore, we have to rescale $\langle H X| \bar{\psi}_{v} \gamma_{5} \chi_{v}|0\rangle$ and $\langle H X| \bar{\psi}_{v} \gamma_{\top} \mu \chi_{v}|0\rangle$ by a factor of $\sqrt{2 E_{P}}$. Although, we have decoupled the large and small components of the heavy (anti-) quark fields, there are still some relative momentum $q$ dependence in $\psi_{v}, \chi_{v}$ :

$$
\begin{align*}
u(\mathbf{q}) & =\sqrt{\frac{E_{Q}+m_{Q}}{2 E_{Q}}}\binom{\xi}{\frac{\mathbf{q} \cdot \sigma}{E_{Q}+m_{Q}} \xi}  \tag{3.78a}\\
v(-\mathbf{q}) & =\sqrt{\frac{E_{Q}+m_{Q}}{2 E_{Q}}}\binom{\eta}{\frac{-\mathbf{q} \cdot}{E_{Q}+m_{Q}} \eta}, \tag{3.78b}
\end{align*}
$$

which are defined in the rest frame of quarkonium $H$. And the conventional LDMEs are also defined in the quarkonium rest frame, where

$$
\begin{equation*}
E_{P}=2 E_{Q}, p_{1} \cdot v=p_{2} \cdot v=E_{Q}, \not p_{1 T}=-\not p_{2 T}=\not q . \tag{3.79}
\end{equation*}
$$

Consequently, in the quarkonium rest frame,

$$
\begin{align*}
\mathcal{T}_{j \bar{j}}\langle H X| \bar{\Psi}_{j} \Psi_{\bar{j}}|0\rangle= & \frac{\operatorname{Tr}\left[\mathcal{T}\left(\not p_{2}-m_{Q}\right) \gamma_{5} P_{+}\left(\not p_{1}+m_{Q}\right)\right]}{-\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} \frac{\langle H X| \bar{\psi}_{v} \gamma_{5} \chi_{v}|0\rangle}{\sqrt{2}} \\
& +\frac{\operatorname{Tr}\left[\mathcal{T}\left(\not p_{2}-m_{Q}\right) \gamma^{\mu} P_{+}\left(\not p_{1}+m_{Q}\right)\right]}{-\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} \frac{\langle H X| \bar{\psi}_{v} \gamma_{T} \chi_{v}|0\rangle}{\sqrt{2}} \tag{3.80}
\end{align*}
$$

where the covariant spin singlet and triplet production projectors to all order of $q$ expansion are implied

$$
\begin{align*}
& \Pi_{\text {production }}^{0}=-\frac{\left(\not p_{2}-m_{Q}\right) \gamma_{5} P_{+}\left(\not p_{1}+m_{Q}\right)}{\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)}  \tag{3.81a}\\
& \Pi_{\text {production }}^{\mu}=-\frac{\left(\not p_{2}-m_{Q}\right) \gamma^{\mu} P_{+}\left(\not p_{1}+m_{Q}\right)}{\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} \tag{3.81b}
\end{align*}
$$

which are consistent with the ones given in Ref. [58,59]. Up to $\mathcal{O}\left(q^{2}\right)$, the all order projectors in Eq. (3.81) can be simplified to be the ones in Eq. (2.15) multiplied by -1 , where the factor -1 comes from different convention of spinors $\xi, \eta$ [59], which will not change the results of amplitude square. It is straightforward to write down the similar formula with Eq. (3.80) for decay case

$$
\begin{align*}
\mathcal{T}_{j \bar{j}}\langle X| \bar{\Psi}_{j} \Psi_{\bar{j}}|H\rangle= & \frac{\operatorname{Tr}\left[\mathcal{T}\left(\not p_{1}+m_{Q}\right) P_{+} \gamma_{5}\left(\not p_{2}-m_{Q}\right)\right]}{-\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} \frac{\langle X| \bar{\chi}_{v} \gamma_{5} \psi_{v}|H\rangle}{\sqrt{2}} \\
& +\frac{\operatorname{Tr}\left[\mathcal{T}\left(\not p_{1}+m_{Q}\right) P_{+} \gamma^{\mu}\left(\not p_{2}-m_{Q}\right)\right]}{-\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} \frac{\langle X| \bar{\chi}_{v} \gamma_{\top} \psi_{v}|H\rangle}{\sqrt{2}}, \tag{3.82}
\end{align*}
$$

where the covariant spin singlet and triplet decay projectors to all order of $q$ expansion are extracted as

$$
\begin{align*}
& \Pi_{\text {decay }}^{0}=-\frac{\left(\not 1_{1}+m_{Q}\right) P_{+} \gamma_{5}\left(\not p_{2}-m_{Q}\right)}{\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)}  \tag{3.83a}\\
& \Pi_{\text {decay }}^{\mu}=-\frac{\left(\not{ }_{1}+m_{Q}\right) P_{+} \gamma^{\mu}\left(\not p_{2}-m_{Q}\right)}{\sqrt{2 E_{Q}}\left(E_{Q}+m_{Q}\right)} . \tag{3.83b}
\end{align*}
$$

The decoupling of relative momentum $q$ is simply accomplished by expanding the short distance part as power series of $q$. The leading and next-to-leading order of $q$ of expansion correspond to the S -wave and P -wave parts, respectively. Therefore, up to P -wave, the production amplitude is given by

$$
\begin{align*}
\mathcal{A}_{1}^{\text {single }}= & \left.\frac{\langle H X| \mathcal{K}\left[{ }^{1} S_{0}^{[1]}\right]|0\rangle}{\sqrt{2 N_{c}}} \operatorname{rr}\left[\mathcal{T} \mathcal{C}_{1} \Pi_{0}^{\prime}\right]\right|_{q=0}+\left.\frac{\langle H X| \mathcal{K}_{\mu}\left[{ }^{3} S_{1}^{[1]}\right]|0\rangle}{\sqrt{2 N_{c}}} \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{1} \Pi_{1}^{\prime \mu}\right]\right|_{q=0} \\
& +\left.\langle H X| \mathcal{K}^{a}\left[{ }^{1} S_{0}^{[8]}\right]|0\rangle \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{8}^{a} \Pi_{0}^{\prime}\right]\right|_{q=0}+\left.\langle H X| \mathcal{K}_{\mu}^{a}\left[{ }^{3} S_{1}^{[8]}\right]|0\rangle \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{8}^{a} \Pi_{1}^{\prime \mu}\right]\right|_{q=0} \\
& +\left.\frac{\langle H X| \mathcal{K}_{\mu \nu}\left[{ }^{1} P_{1}^{[1]}\right]|0\rangle}{\sqrt{2 N_{c}}}\left[\frac{\mathrm{~d}}{\mathrm{~d} q_{\nu}} \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{1} \Pi_{0}^{\prime}\right]\right]\right|_{q=0} \\
& +\left.\langle H X| \mathcal{K}_{\mu \nu}^{a}\left[{ }^{1} P_{1}^{[8]}\right]|0\rangle\left[\frac{\mathrm{d}}{\mathrm{~d} q_{\nu}} \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{8}^{a} \Pi_{0}^{\prime}\right]\right]\right|_{q=0} \\
& +\left.\frac{\langle H X| \mathcal{K}_{\mu \nu}\left[{ }^{3} P_{J}^{[1]}\right]|0\rangle}{\sqrt{2 N_{c}}}\left[\frac{\mathrm{~d}}{\mathrm{~d} q_{\nu}} \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{1} \Pi_{1}^{\prime \mu}\right]\right]\right|_{q=0} \\
& +\left.\langle H X| \mathcal{K}_{\mu \nu}^{a}\left[{ }^{3} P_{J}^{[8]}\right]|0\rangle\left[\frac{\mathrm{d}}{\mathrm{~d} q_{\nu}} \operatorname{Tr}\left[\mathcal{T} \mathcal{C}_{8}^{a} \Pi_{1}^{\prime \mu}\right]\right]\right|_{q=0}+\mathcal{O}\left(q^{2} / m_{Q}^{2}\right) \tag{3.84}
\end{align*}
$$

where the CS and CO projectors $\mathcal{C}_{1}, \mathcal{C}_{8}^{a}$ are defined in Eq.(2.10), and the spin singlet and triplet projectors $\Pi_{0}^{\prime}, \Pi_{1}^{\prime \alpha}$ are the same with the ones defined in Eq.(2.15). Here and below, we take the production case as an example, similar results for decay case can be obtained in the same way.

Up to now, we have accomplished the decoupling of color, Dirac indices and relative momentum $q$ to the next-to-leading order of $q$ expansion. From Eq.(3.84), it can be seen that there are some color and Dirac indexes are still connected between the long and short distance parts. To calculate the physical unpolarized cross section as well as achieve the decoupling of the remaining un-decoupled color and Dirac indexes, we have to do the amplitude square:

$$
\begin{equation*}
\left|\mathcal{A}_{1}^{\text {single }}\right|^{2}=\sum_{n, n^{\prime}, X} \frac{\langle 0| \mathcal{K}^{\dagger}\left[n^{\prime}\right]|H X\rangle\langle H X| \mathcal{K}[n]|0\rangle}{N_{n} N_{n^{\prime}}} \mathcal{M}^{*}\left[n^{\prime}\right] \mathcal{M}[n], \tag{3.85}
\end{equation*}
$$

where $N_{n, n^{\prime}}=\sqrt{2 N_{c}}, 1$ for CS and CO respectively, $\mathcal{M}[n]$ is the amplitude calculated through the covariant projection method without contracting with polarization vectors or tensors.

Due to the symmetry of color, spatial rotation, parity and charge conjugation, the first nonvanishing interference terms are between the ${ }^{3} S_{1}(L=0)$ and the ${ }^{3} D_{1}(L=2)$ channels, which is already beyond our considerations. Thus for Asingle quarkonium production or decay, there are no non-vanishing interferences between different channels up to P-wave. Consequently, Eq. (3.85) can be expressed as

$$
\begin{align*}
& \left|\mathcal{A}_{1}^{\text {single }}\right|^{2}=\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} S_{0}^{[1]}\right]\right|^{2}+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} S_{0}^{[8]}\right]\right|^{2}+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} S_{1}^{[1]}\right]\right|^{2}+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} S_{1}^{[8]}\right]\right|^{2} \\
& \quad+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} P_{1}^{[1]}\right]\right|^{2}+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} P_{1}^{[8]}\right]\right|^{2}+\sum_{J}\left(\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} P_{J}^{[1]}\right]\right|^{2}+\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} P_{J}^{[8]}\right]\right|^{2}\right) . \tag{3.86}
\end{align*}
$$

Applying the color symmetry

$$
\begin{equation*}
\langle 0| \mathcal{K}^{\dagger b}[n] \mathcal{P}^{H} \mathcal{K}^{a}[n]|0\rangle=\langle 0| \mathcal{K}^{\dagger} c[n] \mathcal{P}^{H} \mathcal{K}^{c}[n]|0\rangle \frac{\delta^{a b}}{N_{c}^{2}-1} \tag{3.87}
\end{equation*}
$$

and the spatial rotation symmetry

$$
\begin{align*}
\langle 0| \mathcal{K}_{\mu^{\prime}}^{\dagger}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu}\left[{ }^{3} S_{1}^{[1]}\right]|0\rangle & =\frac{1}{D-1} \Pi_{\mu \mu^{\prime}}\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[1]}\right]\right\rangle,  \tag{3.88a}\\
\langle 0| \mathcal{K}_{\mu^{\prime}}^{\dagger}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu}\left[{ }^{[ } P_{1}^{[1]}\right]|0\rangle & =\frac{1}{D-1} \Pi_{\mu \mu^{\prime}}\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[1]}\right]\right\rangle,  \tag{3.88b}\\
\langle 0| \mathcal{K}_{\mu^{\prime} \nu^{\prime}}^{\dagger}\left[{ }^{[3} P_{J}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu \nu}\left[{ }^{[ } P_{J}^{[1]}\right]|0\rangle & =\varepsilon_{\mu \nu}^{(J)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*(J)} \frac{\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle}{\left.N_{\mathrm{pol}}{ }^{[3} P_{J}\right)} \tag{3.88c}
\end{align*}
$$

we have

$$
\begin{align*}
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} S_{0}^{[1]}\right]\right|^{2} & =\overline{\left.\sum\left|\mathcal{M}\left[{ }^{1} S_{0}^{[1]}\right]\right|\right|^{2} \frac{\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[1]}\right]\right\rangle}{2 N_{c}},}  \tag{3.89a}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} S_{0}^{[8]}\right]\right|^{2} & =\overline{\sum\left|\mathcal{M}\left[{ }^{1} S_{0}^{[8]}\right]\right|^{2} \frac{\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[8]}\right]\right\rangle}{N_{c}^{2}-1},}  \tag{3.89b}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} S_{1}^{[1]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu} \varepsilon_{\mu^{\prime}}^{*} \mathcal{M}^{* \mu^{\prime}}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{M}^{\mu}\left[{ }^{3} S_{1}^{[1]}\right] \frac{\left\langle\mathcal{O}^{H}\left[3 S_{1}^{[1]}\right]\right\rangle}{2 N_{c}(D-1)}, \tag{3.89c}
\end{align*}
$$

$$
\begin{align*}
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} S_{1}^{[8]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu} \varepsilon_{\mu^{\prime}}^{*} \mathcal{M}^{a, * \mu^{\prime}}\left[{ }^{3} S_{1}^{[8]}\right] \mathcal{M}^{a, \mu}\left[{ }^{3} S_{1}^{[8]}\right] \frac{\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle}{\left(N_{c}^{2}-1\right)(D-1)},  \tag{3.89d}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} P_{1}^{[1]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu} \varepsilon_{\mu^{\prime}}^{*} \mathcal{M}^{* \mu^{\prime}}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{M}^{\mu}\left[{ }^{1} P_{1}^{[1]}\right] \frac{\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[1]}\right]\right\rangle}{2 N_{c}(D-1)},  \tag{3.89e}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{1} P_{1}^{[8]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu} \varepsilon_{\mu^{\prime}}^{*} \mathcal{M}^{a, * \mu^{\prime}}\left[{ }^{1} P_{1}^{[8]}\right] \mathcal{M}^{a, \mu}\left[{ }^{1} P_{1}^{[8]}\right] \frac{\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[8]}\right]\right\rangle}{\left(N_{c}^{2}-1\right)(D-1)},  \tag{3.89f}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} P_{J}^{[1]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu \nu}^{(J)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*(J)} \mathcal{M}^{* \mu^{\prime} \nu^{\prime}}\left[{ }^{3} P_{J}^{[1]}\right] \mathcal{M}^{\mu \nu}\left[{ }^{3} P_{J}^{[1]}\right] \frac{\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle}{\left.2 N_{c} N_{\mathrm{pol}}{ }^{3} P_{J}\right)},  \tag{3.89~g}\\
\left|\mathcal{A}_{1}^{\text {single }}\left[{ }^{3} P_{J}^{[8]}\right]\right|^{2} & =\bar{\sum} \varepsilon_{\mu \nu}^{(J)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*(J)} \mathcal{M}^{a, * \mu^{\prime} \nu^{\prime}}\left[{ }^{3} P_{J}^{[1]}\right] \mathcal{M}^{a, \mu \nu}\left[{ }^{3} P_{J}^{[1]}\right] \frac{\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle}{N_{\mathrm{pol}}\left({ }^{3} P_{J}\right)\left(N_{c}^{2}-1\right)} \tag{3.89h}
\end{align*}
$$

Similar results can be obtained for single-quarkonium decay cases $\left|\mathcal{A}_{2}^{\text {single }}[n]\right|^{2}$ by replacing the production LDMEs with corresponding decay LDMEs and absorbing the polarization and color factors $N_{\text {pol }}(n), 2 N_{c}, N_{c}^{2}-1$ into the average sum $\sum$ for the initial quarkonium state.

Finally, we have accomplished the decoupling of the relative momentum $q$, color and Dirac indices (indexes) completely for single quarkonium production and decay processes. Obviously, Eq.(3.89) and the corresponding decay formulas are identical to the NRQCD factorization formulas (Eq.(2.1-2.4)) for single quarkonium production and decay.

It is worth to note that in the derivation of the NRQCD factorization for single-quarkonium processes, we have chosen the quarkonium rest frame, and expressed the short-distance and long-distance parts in covariant way. This is guaranteed by the Lorentz invariance of the full QCD amplitude square $|\mathcal{A}|^{2}$. After decoupling all these indices (indexes) and relative momentum, the short-distance and long-distance parts are scalars, which can be re-expressed in covariant way. And consequently, we are again free to choose the reference frames for the short-distance and long-distance part.The conventional LDMEs are defined in the quarkonium rest frame and $\sqrt{2 P^{0}}, \sqrt{\left(E_{Q}+m_{Q}\right) /\left(2 E_{Q}\right)}$ in the long-distance parts are absorbed into the SDCs, therefore, the absorbed factors $\sqrt{2 P^{0}}, \sqrt{\left(E_{Q}+m_{Q}\right) /\left(2 E_{Q}\right)}$ in the SDCs must be understood as being defined in the quarkonium rest frame as well.

### 3.6.2 Double-quarkonium Processes

The validity of applying the same method used in deriving the NRQCD factorization formulas for single-quarkonium processes to the double-quarkonium cases must be clarified here, since there is not quarkonium rest frame for both quarkonia, simultaneously. As it was shown in Eq. (3.75), the decoupling of color and Dirac indices can be done in covariant way. It is the time when we decouple the relative momentum $q$ and relate the short-distance part with LDMEs that we are forced to choose the quarkonium rest frame. Actually, up to P-wave, we can decouple the relative momentum $q$ in covariant way as well. Considering the relative momentum dependence in the long-distance part is $\mathcal{O}\left(q^{2}\right)$ or higher orders, we can directly set $q=0$ in the long-distance
part. Therefore, we can decouple relative momentum $q$ with $q$ expansion in the short-distance part. The remaining connection of color and Dirac indexes can be decoupled with color and spatial rotation symmetry given by Eq. $(3.87,3.88)$. Now the long-distance part are two scalars, which can be related to the corresponding conventional NRQCD LDMEs by rescaling a factor of $2 P^{0}=4 m_{Q}$ for each quarkonium. For the short-distance part, it can be identified that the covariant projectors in Eq. (3.81) obtained in the quarkonium rest frame are effectively the same with corresponding part in Eq. (3.75) to all orders in $q$ expansion. In this way, the NRQCD factorization formulas for single-quarkonium processes derived in the last section can be directly extended for double-quarkonium case, at least up to P-wave. Beyond P-wave, a covariant way of relative momentum $q$ expansion of the long-distance part is needed.

For double quarkonia involved processes, at amplitude level, as it was done in single quarkonium case, we express the amplitude as
for inclusive double quarkonia production $H_{1}\left(Q_{1} \bar{Q}_{1}\right), H_{2}\left(Q_{2} \bar{Q}_{2}\right)$ :

$$
\begin{equation*}
\mathcal{A}_{1}^{\text {double }}=\sum_{n_{1}, n_{2}, X} \mathcal{M}\left[n_{1}, n_{2}\right] \frac{\left\langle H_{1} H_{2} X\right| \mathcal{Q}\left[n_{1}, n_{2}\right]|0\rangle}{N_{n_{1}} N_{n_{2}}} \tag{3.90}
\end{equation*}
$$

and for inclusive production of quarkonium $H_{1}$ via decay of quarkonium $H_{2}$ :

$$
\begin{equation*}
\mathcal{A}_{2}^{\text {double }}=\sum_{n_{1}, n_{2}, X} \mathcal{M}\left[n_{1}, n_{2}\right] \frac{\left\langle H_{1} X\right| \mathcal{Q}\left[n_{1}, n_{2}^{\dagger}\right]\left|H_{2}\right\rangle}{N_{n_{1}} N_{n_{2}}} . \tag{3.91}
\end{equation*}
$$

The corresponding amplitude squares are

$$
\begin{align*}
\left|\mathcal{A}_{1}^{\text {double }}\right|^{2}= & \sum_{n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}, X} \frac{\langle 0| \mathcal{Q}^{\dagger}\left[n_{1}^{\prime}, n_{2}^{\prime}\right]\left|H_{1} H_{2} X\right\rangle\left\langle H_{1} H_{2} X\right| \mathcal{Q}\left[n_{1}, n_{2}\right]|0\rangle}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \\
& \times \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right] \\
= & \sum_{n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}} \frac{\left\langle\mathcal{Q}^{H_{1}, H 2}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right],  \tag{3.92}\\
\left|\mathcal{A}_{2}^{\text {double }}\right|^{2}= & \sum_{n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}, X} \frac{\left\langle H_{2}\right| \mathcal{Q}^{\dagger}\left[n_{1}^{\dagger}, n_{2}^{\prime}\right]\left|H_{1} X\right\rangle\left\langle H_{1} X\right| \mathcal{Q}\left[n_{1}, n_{2}^{\dagger}\right]\left|H_{2}\right\rangle}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \\
& \times \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right] \\
= & \sum_{n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}} \frac{\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle_{H_{2}}}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right], \tag{3.93}
\end{align*}
$$

where the eight-fermion operators $\mathcal{Q}\left[n_{1}, n_{2}\right], \mathcal{Q}\left[n_{1}, n_{2}^{\dagger}\right]$ and their conjugate transpose are given in Eq. (3.46, 3.47), the un-decoupled double-production LDMEs $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle$ and
decay-production LDMEs $\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle_{H_{2}}$, are defined in Eq. (3.48). For the same symmetric reasons as single quarkonium case, when $n_{1}, n_{1}^{\prime}$ or $n_{2}, n_{2}^{\prime}$ are not identical, the LDMEs $\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle_{H_{2}},\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime}\right]\right\rangle$ vanish at Born level. When $n_{1}=n_{1}^{\prime}$, $n_{2}=n_{2}^{\prime},\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle_{H_{2}},\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle$ can be related to single quarkonium LDMEs

$$
\begin{align*}
& \frac{\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle}{N_{n_{1}}^{2} N_{n_{2}}^{2}}=\sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle}{N_{\mathrm{pol}}\left(n_{1}\right) N_{\mathrm{pol}}\left(n_{2}\right) N_{\mathrm{col}_{1}} N_{\mathrm{col}_{2}}},  \tag{3.94}\\
& \frac{\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle_{H_{2}}}{N_{n_{1}}^{2} N_{n_{2}}^{2}}=\sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle\left\langle\mathcal{O}\left[n_{2}\right]\right\rangle_{H_{2}}}{N_{\mathrm{pol}}\left(n_{1}\right) N_{\mathrm{pol}}\left(n_{2}\right) N_{\mathrm{col}_{1}} N_{\mathrm{col}_{2}}}, \tag{3.95}
\end{align*}
$$

where we have applied the color and spatial symmetry to decouple the color and Dirac indexes. Thus for symmetric cases, we simply have

$$
\begin{align*}
\left|\mathcal{A}_{1}^{\text {double }}\left[n_{1}, n_{2}\right]\right|^{2} & =\bar{\sum}|\mathcal{M}|^{2} \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle}{N_{\mathrm{pol}}\left(n_{1}\right) N_{\mathrm{pol}}\left(n_{2}\right) N_{\mathrm{col}_{1}} N_{\mathrm{col}_{2}}}  \tag{3.96}\\
\left|\mathcal{A}_{2}^{\text {double }}\left[n_{1}, n_{2}\right]\right|^{2} & =\bar{\sum}|\mathcal{M}|^{2} \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle\left\langle\mathcal{O}\left[n_{2}\right]\right\rangle_{H_{2}}}{N_{\mathrm{pol}}\left(n_{1}\right) N_{\mathrm{col}_{1}}}, \tag{3.97}
\end{align*}
$$

which are same with our naive application of NRQCD factorization for single-quarkonium production and decay in Section 2.3.

An essence difference between $\left|\mathcal{A}_{1,2}^{\text {double }}\right|^{2}$ and $\left|\mathcal{A}_{1,2}^{\text {single }}\right|^{2}$ is that even though we consider $S$-wave and P-wave contributions only, the interferences between different intermediate states in $\left|\mathcal{A}_{1,2}^{\text {double }}\right|^{2}$ may not vanish when the next-leading-order corrections for the LDMEs are included. As it was shown in Appendix D.2, the un-decoupled double-production LDMEs $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle$ and decay-production LDMEs $\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle$ with $n_{1}, n_{2}^{\prime}$ being in $S$-wave states while $n_{1}^{\prime}, n_{2}$ in P-wave states, can evolve to be the combination of two P-wave single-quarkonium LDMEs with some extra coefficients, color and Dirac indexes via soft gluon exchange.

Therefore, with the un-decoupled double-quarkonium LDMEs calculated at one-loop level, the interference contributions can be expressed as

$$
\begin{align*}
& \mathcal{A}_{1}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{1}^{\text {double }}\left[n_{1}, n_{2}\right] \\
= & \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right],  \tag{3.98a}\\
& \mathcal{A}_{2}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{2}^{\text {double }}\left[n_{1}, n_{2}\right] \\
= & \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{2}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right] . \tag{3.98b}
\end{align*}
$$

### 3.7 Interference Contributions and Infrared Divergences

To illustrate Eq.(3.98) in more details and show how to calculate the interference contributions as well as the cancellation of the infrared divergences, we present two examples which are discussed in Chapter 2, namely, $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+\chi_{c J_{2}}\left({ }^{3} S_{1}^{[8]}\right)$ interfering with $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} S_{1}^{[1]}\right)+$ $c \bar{c}\left({ }^{3} P_{J_{2}}^{[1]}\right)$ and $\Upsilon\left({ }^{3} S_{1}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g$ interfering with $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} S_{1}^{[8]}\right)+g$. For above two cases, the interference contributions are given by

$$
\begin{align*}
& \sum_{J_{1}} \mathcal{A}_{1}^{* \text { double }}\left[{ }^{3} P_{J_{1}}^{[8]},{ }^{3} S_{1}^{[8]}\right] \mathcal{A}_{1}^{\text {double }}\left[{ }^{3} S_{1}^{[1]},{ }^{3} P_{J_{2}}^{[1]}\right] \\
& =\sum_{J_{1}} \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{c}, m_{c}\right) \mathcal{P}_{1}\left[{ }^{[ } P_{J_{1}}^{[8]}{ }^{3}{ }^{2} S_{1}^{[8]},{ }^{3} S_{1}^{[1]},{ }^{3} P_{J_{2}}^{[1]}\right]}{2 N_{c}} \\
& \times \mathcal{M}^{* a^{\prime} b^{\prime}, \mu^{\prime} \nu^{\prime} \rho^{\prime}}\left[{ }^{3} P_{J_{1}}^{[8]},{ }^{3} S_{1}^{[8]}\right] \mathcal{M}^{\mu \rho \sigma}\left[{ }^{3} S_{1}^{[1]},{ }^{3} P_{J_{2}}^{[1]}\right] \\
& =\sum_{J_{1}} \frac{2 \delta^{a^{\prime} b^{\prime}}}{C_{A}} \sum_{\text {pol }} \varepsilon_{\mu \nu}^{\left(J_{1}\right)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*\left(J_{1}\right)} \varepsilon_{\rho \sigma}^{\left(J_{2}\right)} \varepsilon_{\rho^{\prime} \sigma^{\prime}}^{\left.* J_{2}\right)} \mathcal{M}^{* a^{\prime} b^{\prime}, \mu^{\prime} \nu^{\prime} \rho^{\prime}}\left[{ }^{3} P_{J_{1}}^{[8]},{ }_{3}^{3} S_{1}^{[8]}\right] \mathcal{M}^{\mu \rho \sigma}\left[{ }^{3} S_{1}^{[1]},{ }^{3} P_{J_{2}}^{[1]}\right] \\
& \times I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{c}, m_{c}\right) \frac{\left.\left\langle\mathcal{O}^{J / \psi}\left[{ }^{3} P_{J_{1}}^{[8]}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{\chi}{ }_{c J_{2}}\left[{ }^{3} P_{J_{2}}^{[1]}\right]\right]\right\rangle_{\text {Born }}}{2 N_{c}\left(N_{c}^{2}-1\right) N_{\text {pol }}\left({ }^{3} P_{J_{1}}\right) N_{\text {pol }}\left({ }^{3} P_{J_{2}}\right)},  \tag{3.99a}\\
& \sum_{J_{b}} \mathcal{A}_{2}^{* \text { double }}\left[{ }^{3} P_{J_{b}}^{[8]},{ }^{3} S_{1}^{[8]}\right] \mathcal{A}_{2}^{\text {double }}\left[{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{c}}^{[1]}\right] \\
& =\sum_{J_{b}} \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{c}, m_{b}\right) \mathcal{P}_{2}\left[{ }^{[ } P_{J_{b}}^{[8]},{ }^{[ } S_{1}^{[8]},{ }^{[3} S_{1}^{[8]},{ }^{3} P_{J_{c}}^{[1]}\right]}{\sqrt{2 N_{c}}} \\
& \times \mathcal{M}^{*}\left[{ }^{3} P_{J_{b}}^{[8]},{ }^{3} S_{1}^{[8]}\right] \mathcal{M}\left[{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{c}}^{[1]}\right] \\
& =\sum_{J_{b}} \frac{d^{a a^{\prime} b^{\prime}}}{C_{A}^{2}} \sum_{\text {pol }} \varepsilon_{\mu \nu}^{\left(J_{c}\right)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*\left(J_{c}\right)} \varepsilon_{\rho \sigma}^{\left(J_{b}\right)} \varepsilon_{\rho^{\prime} \sigma^{\prime}}^{*\left(J_{b}\right)} \mathcal{M}^{* a^{\prime} b^{\prime}, \mu^{\prime} \nu^{\prime} \rho^{\prime}}\left[{ }^{3} P_{J_{b}}^{[8]},{ }^{3} S_{1}^{[8]}\right] \mathcal{M}^{a, \mu \rho \sigma}\left[{ }^{3} S_{1}^{[8]},{ }^{3} P_{J_{c}}^{[1]}\right] \\
& \times I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{c}, m_{b}\right) \frac{\left.\left\langle\mathcal{O}\left[{ }^{3} P_{J_{b}}^{[8]}\right]\right\rangle_{\Upsilon, \text { Born }}\left\langle\mathcal{O} \chi_{c J_{c}}\left[{ }^{3} P_{J_{c}}^{[1]}\right]\right]\right\rangle_{\text {Born }}}{\sqrt{\left.2 N_{c}\left(N_{c}^{2}-1\right) N_{\mathrm{pol}}\left({ }^{3} P_{J_{c}}\right) N_{\mathrm{pol}}{ }^{3} P_{J_{b}}\right)} .} \tag{3.99b}
\end{align*}
$$

From the above two equations, we can see that the color, Dirac indexes are full contracted hence the separation of short-distance and long-distance effects is accomplished. It is also straightforward to check that the above interference contributions indeed reproduce the uncanceled infrared divergences in $e^{+} e^{-} \rightarrow J / \psi\left({ }^{3} P_{J_{1}}^{[8]}\right)+\chi_{c J_{2}}\left({ }^{3} P_{J_{2}}^{[1]}\right)+g$ and $\Upsilon\left({ }^{3} P_{J_{b}}^{[8]}\right) \rightarrow \chi_{c J_{c}}\left({ }^{3} P_{J_{c}}^{[1]}\right)+g g$, which means these infrared divergences can be canceled through matching procedure.

## Chapter 4

## Born Level Calculations

### 4.1 General Discussions

The first investigation on charmonium production via $\Upsilon$ decay was the leading order $\Upsilon \rightarrow$ $\chi_{c J}+X$ calculation in Ref. [60] more than two decades ago, where a small gluon mass was used to regularize the infrared divergences. Since the nowadays calculations are commonly done with dimensional regularization scheme, it is quite necessary to re-calculate $\Upsilon \rightarrow \chi_{c J}+X$ with the same scheme. In Ref. [61], both $b \bar{b}, c \bar{c}$ color-octet leading order contributions for $\Upsilon \rightarrow J / \psi+X$ were considered, where the author concluded that the $b \bar{b}$ color-octet contributions were too small and a branching ratio of $2.5 \times 10^{-4}$ (comparing experimental result $\mathcal{B}(\Upsilon \rightarrow J / \psi+X)=(3.14 \pm$ $0.4) \times 10^{-4}$ (Direct), see Table. 6.2) was contributed from $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ channel with $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=0.014 \mathrm{Gev}^{3}$, which was almost one order larger than the extracted values in varies experimental environment (see Table. (1.1,1.2)). An other branching ratio $2.1 \times 10^{-4}$ was obtained in Ref. [62] through considering the processes $\Upsilon\left({ }^{3} S_{1}^{[1]}\right) \rightarrow J / \psi\left({ }^{1} S_{0}^{[8]} /{ }^{3} P_{J}^{[8]}\right)+g$, where the results strongly depended on the LDME $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle$. As we can see from Table. (1.1,1.2), the extracted values of $\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle$ are dramatically different, even can be negative. In addition, the calculations in Ref. [61,62] are not complete at $\mathcal{O}\left(\alpha_{s}^{5}\right)$, the next-to-leading order corrections to the large contribution channel $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ have not been included. The colorsinglet contributions for $\Upsilon \rightarrow J / \psi+X$ were calculated up to $\mathcal{O}\left(\alpha_{s}^{6}\right)$ in Ref. [63, 64]. However, the color-singlet contributions are much smaller than the experimental data.

Therefore, we calculate all the $c \bar{c} \mathrm{CS}$ and CO channels up to $\mathcal{O}\left(\alpha_{s}^{5}\right)$ as well as some important QED contributions to give a complete analysis. The relevant LDMEs for $\Upsilon$ decay and $\eta_{c}, h_{c}, \chi_{c J}$ production are shown in Table.4.1 with corresponding relative velocity scalings. However, the $b \bar{b}$ color-octet channels are not considered in our work, since CO LDMEs of $\Upsilon$ decay are suppressed by a factor of $v_{b}^{4} \sim 0.01$ comparing with color-singlet LDME $\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{\Upsilon}$, which was confirmed in Ref. [61].

According to NRQCD factorization [3], at leading order $v$ expansion for $\Upsilon, h_{c}, \chi_{c J}$ and next-to-leading order $v$ expansion for $\eta_{c}, J / \psi$ ( the relativistic correction for CS channel are not considered here, although they may have lower order of $v$ expansion compared with CO channels), the inclusive decay width of $\Upsilon$ to $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ can be expressed as

| Relative scaling | Contributing LDMEs |
| :---: | :---: |
| 1 | $\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle,\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle,\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{\Upsilon}$ |
| $v_{c}^{2}$ | $\left\langle\mathcal{O}^{h_{c}}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle,\left\langle\mathcal{O}^{h_{c}}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}^{\chi c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle,\left\langle\mathcal{O}^{\left.\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle}\right.$ |
| $v_{c}^{3}$ or $v_{b}^{3}$ | $\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle \Upsilon$ |
| $v_{c}^{4}$ | $\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} P_{1}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle,\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle$ |
| $v_{b}^{4}$ | $\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle \Upsilon,\left\langle\mathcal{O}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle \Upsilon$ |

Table 4.1: Relative scaling of contributing LDMEs for $\eta_{c}, J / \psi, \chi_{c J}$ production and $\Upsilon$ decay.

$$
\begin{align*}
& \Gamma\left(\Upsilon \rightarrow \eta_{c}+X\right)=\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle \Upsilon\left[\hat{\Gamma}_{1 S_{0}^{[1]}}\left\langle{ }^{\left\langle c_{c}\right.}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle+\hat{\Gamma}_{1 S_{0}^{[8]}}\left\langle\eta_{c}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle\right. \\
& \left.+\hat{\Gamma}_{3 S_{1}^{[8]}}\left\langle{ }^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle+\hat{\Gamma}_{{ }_{1}}{ }_{1}^{[8]}\left({ }^{\left(\eta_{c}\right.}\left({ }^{1} P_{1}^{[8]}\right)\right\rangle\right],  \tag{4.1}\\
& \Gamma(\Upsilon \rightarrow J / \psi+X)=\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle \Upsilon\left[\hat{\Gamma}_{{ }_{3} S_{1}^{[1]}}\left\langle{ }^{J / \psi}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle+\hat{\Gamma}_{{ }_{1} S_{0}^{[8]}}\left\langle{ }^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle\right. \\
& \left.+\hat{\Gamma}_{{ }_{3}}{ }_{1}^{[8]}\left\langle{ }^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle+\sum_{J} \hat{\Gamma}_{3 P_{J}^{[8]}}\left\langle{ }^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle\right],  \tag{4.2}\\
& \Gamma\left(\Upsilon \rightarrow h_{c}+X\right)=\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{\Upsilon}\left[\hat{\Gamma}_{1} S_{0}^{[8]}\left\langle{ }^{\left\langle h_{c}\right.}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle+\hat{\Gamma}_{{ }_{1}} P_{1}^{[1]}\left\langle{ }^{h h_{c}}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle\right],  \tag{4.3}\\
& \Gamma\left(\Upsilon \rightarrow \chi_{c J}+X\right)=\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle \Upsilon\left[\hat{\Gamma}_{{ }_{3} S_{1}^{[8]}}\left\langle\chi_{c J}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle+\hat{\Gamma}_{3_{J}^{[1]}}\left\langle\chi_{c J}\left({ }^{3} P_{J}^{[1]}\right)\right\rangle\right], \tag{4.4}
\end{align*}
$$

where $\hat{\Gamma}_{2 S+1} L_{J}^{[a]}$ are the corresponding SDCs for the processes: $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{2 S+1} L_{J}^{[a]}\right)+X$, with $a=1,8$ representing color-singlet and color-octet respectively.

In QCD theory, $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right)$ can only decay through three gluon channel. Thus the QCD processes start at $\mathcal{O}\left(\alpha_{s}^{4}\right)$ for $c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)$ channel, and $\mathcal{O}\left(\alpha_{s}^{5}\right)$ for all the other S -wave and P-wave
channels. We list all the leading order QCD subprocesses and corresponding order of strong coupling in Table.4.2.

| Born level QCD subprocess | order of $\alpha_{s}$ |
| :---: | :---: |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1]}\right)+g g g / c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[8]}\right)+g / g g g / c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ | $\alpha_{s}^{4}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} P_{1}^{[1]}\right)+c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} P_{1}^{[8]}\right)+g g g / c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1]}\right)+c \bar{c} g$ | $\alpha_{s}^{5}$ |
| $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[8]}\right)+g / c \bar{c} g$ | $\alpha_{s}^{5}$ |

Table 4.2: Leading order subprocesses and orders of strong coupling.

## $4.2 \quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$

There are 6 Feynman diagrams for the subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$, which are shown in Figure.4.1. This is the only subprocess that contributes at $\mathcal{O}\left(\alpha_{s}^{4}\right)$ and was calculated in Ref. [61], which is consistent analytically and numerically with our result. The virtual and real corrections for this subprocess will be discussed in detail in Chapter.5. The D-dimensional Born level amplitude square is given by


Figure 4.1: Representative tree-level Feynman diagrams for the partonic subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$.

$$
\begin{align*}
& \bar{\sum}\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right)\right|^{2} \\
= & \frac{80 g_{s}^{4}}{27(D-1) m_{c}^{3} m_{b}\left(s-4 m_{b}^{2}\right)^{2}\left(u-4 m_{b}^{2}\right)^{2}\left(s+u-8 m_{c}^{2}\right)^{2}} \\
& \times\left(512(5-D) m_{b}^{8} m_{c}^{2}-s u m_{c}^{2}\left[(2-D) s u+4 m_{c}^{2}(3 D-8)\left(s+u-4 m_{c}^{2}\right)\right]\right. \\
& +16 m_{b}^{6}\left[(D-2)^{2}\left(s^{2}+u^{2}\right)+(5 D-16) s u-4 m_{c}^{2}\left(2 D^{2}-7 D+12\right)(s+u)\right. \\
& \left.+16 m_{c}^{4}\left(2 D^{2}+D-8\right)\right]-4 m_{b}^{4}\left[(s+u)\left(2(D-2)^{2}\left(s^{2}+u^{2}\right)+(5 D-16) s u\right)\right. \\
& -4 m_{c}^{2}\left(3\left(2 D^{2}-7 D+6\right)\left(s^{2}+u^{2}\right)+\left(8 D^{2}-31 D+28\right) s u\right) \\
& \left.+64 m_{c}^{4}\left(2 D^{2}-5 D+1\right)(s+u)-64 m_{c}^{6} D(4 D-9)\right] \\
& +m_{b}^{2}\left[(D-2)^{2}\left(s^{2}+s u+u^{2}\right)^{2}-8 m_{c}^{2}(s+u)\left(2(D-2)^{2}\left(s^{2}+u^{2}\right)\right.\right. \\
& \left.+\left(3 D^{2}-17 D+24\right) s u\right)+16 m_{c}^{4}\left(\left(36-34 D+8 D^{2}\right) s^{2}\right. \\
& \left.+\left(92-77 D+16 D^{2}\right) s u+2\left(18-17 D+4 D^{2}\right) u^{2}\right) \\
& \left.\left.-64 m_{c}^{6}\left(8 D^{2}-39 D+48\right)(s+u)+256 m_{c}^{8}\left(24-17 D+3 D^{2}\right)\right]\right) . \tag{4.5}
\end{align*}
$$

## $4.3 \quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} /{ }^{1} P_{1}^{[8]}\right)+g g g$

No previous investigations exist for the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} /{ }^{1} P_{1}^{[8]}\right)+g g g$. In addition, due to symmetric reasons, the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} P_{1}^{[1]}\right)+g g g$ vanishes.

There are 36 Feynman diagrams for each of these subprocesses, where the representative ones are shown in Figure.4.2. The SDCs of the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$ contain infrared divergences, we discuss these subprocesses in detail together with real corrections in next chapter. The analytic results for these subprocesses are two lengthy to be presented here.
$4.4 \quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[8]} /{ }^{3} P_{J}^{[8]}\right)+g$
These subprocesses have been studied in Ref. [62]. Similar subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow q \bar{q}\left({ }^{1} S_{0}^{[1]} /{ }^{3} P_{J}^{[1]}\right)+$ $\gamma$ have also been calculated in Ref. [65]. Their analytical results are consistent with each other.



Figure 4.2: Representative tree-level Feynman diagrams for the partonic subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} /{ }^{1} P_{1}^{[8]} /{ }^{3} P_{J}^{[1,8]}\right)+g g g$. All the other diagrams can be obtained through exchanging the final state gluon.


Figure 4.3: Tree-level Feynman diagrams for the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[8]} /{ }^{3} P_{J}^{[8]}\right)+g$.


Figure 4.4: Tree-level Feynman diagrams for the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} / 3 S_{1}^{[1]} /\right.$ $\left.{ }^{1} P_{1}^{[1,8]} /{ }^{3} P_{J}^{[1,8]}\right)+c \bar{c} g$.

There are 6 Feynman diagrams for each of the leading order subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{1} S_{0}^{[8]} /{ }^{3} P_{J}^{[8]}\right)+g$, which are shown in Figure.4.3. Due to symmetric reasons, the contribution of the leading order subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]} /{ }^{1} P_{1}^{[8]}\right)+g$ vanish. These leading order subprocesses contribute at one-loop level. We adopt the helicity projector method described in Ref. [65] to calculate the one-loop amplitude. For the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[8]}\right)+g$, we obtain the same analytical result with Ref. [62]. For the P-wave intermediate states subprocesses, we use similar method of calculating the virtual corrections described in the next chapter. Our numerical results are consistent with those in Ref. [62].
$4.5 \quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} /{ }^{3} S_{1}^{[1]} /{ }^{1} P_{1}^{[1,8]} /{ }^{3} P_{J}^{[1,8]}\right)+c \bar{c} g$
There are 6 Feynman diagrams for each of the leading order subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{1} S_{0}^{[1,8]} /\right.$ $\left.{ }^{3} S_{1}^{[1]} /{ }^{1} P_{1}^{[1,8]} /{ }^{3} P_{J}^{[1,8]}\right)+c \bar{c} g$, which are shown in Figure.4.4. One of the $c \bar{c}$ associated subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c} g$ was calculated in Ref. [63]. Using the input parameters in Ref. [63], we can reproduce their numerical results. All the other $c \bar{c}$ associated subprocesses are new contributions. The analytical results are also too complicated to be listed here.


Figure 4.5: Representative tree-level Feynman diagrams for the partonic subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+c \bar{c} g$.


Figure 4.6: Tree-level Feynman diagrams for the subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow \gamma^{*} \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c} g$.
4.6 $\quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+c \bar{c} g$

Comparing with the $c \bar{c}$ associated subprocesses discussed in the last subsection, there are 6 extra Feynman diagrams that contribute to the subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+c \bar{c} g$ due to gluon fragmentation $g^{*} \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)$. The corresponding Feynman diagrams are shown in Figure. 4.5.

### 4.7 QED processes

As mentioned in Ref. [63], the dominate color-singlet QED contribution for inclusive $J / \psi$ production via $\Upsilon(1 S)$ decay comes from the photon fragmentation $\gamma^{*} \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)$ (the last diagram in Figure. 4.7), whose branching ratio is about $1.5 \times 10^{-5}$. The same order subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow \gamma^{*} \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c} g$ (Figure. 4.6) only makes contribution of $1.14 \times 10^{-6}$ for




Figure 4.7: Representative tree-level Feynman diagrams for the partonic subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+g g$.
the branching ratio. Our calculation for the $\eta_{c}$ case indicates that all the other QED processes contribute at order of $10^{-6}$ or smaller for the branching ratio. Therefore, for simplicity, we consider only these two QED subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow \gamma^{*} \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+c \bar{c} g$ and $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{3} S_{1}^{[1]}\right)+g g$.

## Chapter 5

## Next-to-leading Order Corrections

### 5.1 Virtual Corrections

Typical Feynman diagrams of virtual corrections are shown in Fig.5.1. They fall into four groups, which include self-energy diagrams, vertex correction diagrams, counter term diagrams, and diagrams that generated from tree-level diagrams of Fig.4.1 by attaching one virtual gluon line in all other possible ways.

Labeling the tree-level amplitude and the amplitude of virtual corrections as $\mathcal{M}_{\text {Born }}$ and $\mathcal{M}_{\text {virtual }}$ respectively, the SDC of virtual corrections can be evaluated as

$$
\begin{equation*}
d \hat{\Gamma}^{\mathrm{VC}}=\frac{1}{4 m_{b}} \frac{1}{\left(N_{c}^{2}-1\right)(D-1)} d \mathrm{PS}_{1 \rightarrow 3} \sum 2 \operatorname{Re}\left(\mathcal{M}_{\text {Born }}^{*} \mathcal{M}_{\text {virtual }}\right), \tag{5.1}
\end{equation*}
$$

where $d \mathrm{PS}_{1 \rightarrow 3}$ is the three body phase-space and $\bar{\sum}$ implies average over the sum of color and polarization degrees. The identical particle factor $1 / 2$ (two identical gluons) is included in $\sum 2 \operatorname{Re}\left(\mathcal{M}_{\text {Born }}^{*} \mathcal{M}_{\text {virtual }}\right)$.

### 5.1.1 Renormalization Scheme

The UV divergences are removed through renormalization. We adopt a mixed renormalization scheme [10], in which for the heavy quark field $\psi_{Q}$, quark mass $m_{Q}$ and gluon field $A_{\mu}^{a}$ the corresponding renormalization constants $Z_{2}, Z_{m}$, and $Z_{3}$ are defined in the on-shell (OS) scheme,



Figure 5.1: Representative Feynman diagrams for virtual corrections of subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$.
while for the strong coupling $g_{s}$ its renormalization constant $Z_{g}$ is defined in modified-minimalsubtraction ( $\overline{\mathrm{MS}}$ ) scheme. At one-loop level the counter terms then read

$$
\begin{align*}
\delta Z_{g}^{\overline{\mathrm{MS}}} & =-\frac{\beta_{0}}{2} \frac{\alpha_{s}}{4 \pi} C_{\epsilon} \frac{1}{\epsilon_{\mathrm{UV}}},  \tag{5.2}\\
\delta Z_{2}^{\mathrm{OS}} & =-C_{F} \frac{\alpha_{s}}{4 \pi} C_{\epsilon}\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\frac{2}{\epsilon_{\mathrm{IR}}}+3 \ln \frac{\mu^{2}}{m^{2}}+4\right],  \tag{5.3}\\
\delta Z_{3}^{\mathrm{OS}} & =\frac{\alpha_{s}}{4 \pi} C_{\epsilon}\left(\beta_{0}-2 C_{A}\right)\left[\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right],  \tag{5.4}\\
\delta Z_{m}^{\mathrm{OS}} & =-3 C_{F} \frac{\alpha_{s}}{4 \pi} C_{\epsilon}\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\ln \frac{\mu^{2}}{m^{2}}+\frac{4}{3}\right], \tag{5.5}
\end{align*}
$$

where $C_{\epsilon}=\left(4 \pi e^{-\gamma_{E}}\right)^{\epsilon}$ and $\mu$ is the renormalization scale, which implies that the dimensional regularization with $D=4-2 \epsilon$ is employed in our calculation. $\beta_{0}=(11 / 3) C_{A}-(4 / 3) T_{F} n_{f}$ is the one-loop coefficient of the QCD beta function with $n_{f}=3$ active quark flavors in our calculation. In Eq. $(5.3,5.5)$, the mass $m$ should be substituted by $m_{c}$ and $m_{b}$ for charm and bottom quark, respectively.

### 5.1.2 Strategies of Analytical Calculations

As mentioned previously, the Feynman diagrams are generated by FEYNARTS [49], algebraic operations such as color, Dirac algebra, are performed with FEYNCALC [48] and FORM [50].

We apply the covariant projection method before loop integrations, which means the relative momentum $q$ is set to 0 before loop integration. Thus we have one less mass scale in the loop integrals, Coulomb singularities do not exist neither. However, the loop integrals become non-standard because of the linear dependent propagators in the loop integrals. Therefore, we use Mathematica package \$Apart [66] to get linear independent propagators. With the \$Apart package, the linear dependent propagators such as $\frac{1}{l^{2}\left(l^{2}-a\right)\left(l^{2}-b\right)\left(l^{2}-a_{1} a-a_{2} b\right)}$ can be split as

$$
\begin{align*}
& \text { \$Apart }\left[\frac{1}{l^{2}\left(l^{2}-a\right)\left(l^{2}-b\right)\left(l^{2}-a_{1} a-a_{2} b\right)}\right] \\
= & \left(\frac{a_{2}}{a_{1}+a_{2}-1}\right) \frac{1}{\left(l^{2}\right)^{2}\left(l^{2}-a\right)\left(l^{2}-a_{1} a-a_{2} b\right)} \\
& +\left(\frac{a_{1}}{a_{1}+a_{2}-1}\right) \frac{1}{\left(l^{2}\right)^{2}\left(l^{2}-b\right)\left(l^{2}-a_{1} a-a_{2} b\right)} \\
& -\left(\frac{1}{a_{1}+a_{2}-1}\right) \frac{1}{\left(l^{2}\right)^{2}\left(l^{2}-a\right)\left(l^{2}-b\right)} . \tag{5.6}
\end{align*}
$$

More complicated cases such as loop integrals containing negative or higher power of propagators can also be done in the same manner. This means this method can apply with P -wave intermediate states of quarkonium and no tensor reduction is needed since the FIRE [67] package is able to dealing with negative power of propagators as long as all the propagators are linear independent. A simple example would be

$$
\begin{align*}
& \operatorname{FIRE}\left[\int \frac{d^{D} l}{(2 \pi)^{D}} \frac{P_{1} \cdot l}{l^{2}\left(l^{2}+P_{2} \cdot l\right)}\right] \\
& =\frac{(s+u)}{4} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}\left(l^{2}+P_{2} \cdot l\right)} \\
&  \tag{5.7}\\
& \quad+\frac{(s+u)}{16 m_{c}^{2}} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l^{2}+P_{2} \cdot l\right)} .
\end{align*}
$$

At the end of the reduction of scalar integrals done with FIRE [67], the analytical results of virtual corrections are expressed in terms of some master integrals. Among them, the coefficients of some integrals have extra pole $\frac{1}{\epsilon}$. For instance, the scalar integral $\int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}\left(l^{2}-2 k_{3} l l\right)\left(l^{2}+2 k_{4} \cdot l\right)}$ will be further reduced by FIRE according to

$$
\begin{align*}
& \operatorname{FIRE}\left[\int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}\left(l^{2}-2 k_{3} \cdot l\right)\left(l^{2}-2 k_{4} \cdot l\right)}\right] \\
= & \frac{24(D-3)\left(4 m_{b}^{2}+20 m_{c}^{2}+s+5 u\right)}{(4-D)\left(16 m_{c}^{2}\left(-12 m_{b}^{2}+s-3 u\right)+64 m_{c}^{4}+(s+3 u)^{2}\right)} \\
& \times \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l^{2}-2 k_{3} \cdot l\right)\left(l^{2}+2 k_{4} \cdot l\right)} \\
= & \frac{24(D-3)\left(4 m_{b}^{2}+20 m_{c}^{2}+s+5 u\right)}{(4-D)\left(16 m_{c}^{2}\left(-12 m_{b}^{2}+s-3 u\right)+64 m_{c}^{4}+(s+3 u)^{2}\right)} \\
& \times \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}\left(l+k_{3}+k_{4}\right)^{2}}, \tag{5.8}
\end{align*}
$$

where the extra pole $\frac{1}{4-D}=\frac{1}{2 \epsilon}$ arises, which means we must calculate these combined integrals up to $\mathcal{O}(\epsilon)$. Fortunately, only some tadpole and bubble integrals are combined with extra poles, whose analytical results up to $\mathcal{O}(\epsilon)$ are given in Appendix D.

After above procedures, the analytical results are simplified with Mathematica, and transformed to be $\mathrm{C}++$ code for further numerical evaluations.

We extract the divergences analytically and find that they cancel with the divergences in the real corrections. This can be served as a check point of our analytical computations.

### 5.2 Real Corrections and $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$

Since the real corrections and the two processes $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$ share the same kinematic settings (see Appendix A.) and their infrared divergences are subtracted with the same method, we discuss these processes together in this section.

In our calculation of the real corrections, we choose Feynman gauge with the polarization sum of the gluon as

$$
\begin{equation*}
\sum_{\mathrm{pol}^{*}} \epsilon_{\mu} \epsilon_{\nu}^{*}=-g_{\mu \nu} \tag{5.9}
\end{equation*}
$$

Consequently, the non-physical degree of freedom of final state gluon should be subtracted by gluon ghost contribution:

$$
\begin{align*}
& \quad \sum_{\text {pol }}\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g\right)\right|^{2} \\
& =\quad \\
& \sum_{\mathrm{pol}^{*}}\left[\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g\right)\right|^{2}\right. \\
& -2\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g u_{g} \bar{u}_{g}\right)\right|^{2} \\
&  \tag{5.10}\\
& -2\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+u_{g} g \bar{u}_{g}\right)\right|^{2} \\
& \\
& \left.-2\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+u_{g} \bar{u}_{g} g\right)\right|^{2}\right],
\end{align*}
$$

where $u_{g}, \bar{u}_{g}$ stand for ghost and anti-ghost.
In this way, we have to calculate (Fig.5.2) $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g$ (96 Feynman diagrams) and $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g u_{g} \bar{u}_{g} / u_{g} g \bar{u}_{g} / u_{g} \bar{u}_{g} g / q \bar{q} g$ (6 Feynman diagrams for each subprocess).

### 5.2.1 The Two Cutoff Phase Space Slicing Method

Soft, collinear, and soft-collinear divergences are encountered in the final state phase-space integrations in the real corrections. And soft divergences also present in the two P-wave processes


Figure 5.2: Representative Feynman diagrams for real corrections of subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$.
$b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$. These divergences are subtracted with the phase space slicing method [68].

The general idea of the two cutoff phase space slicing method is decomposing the phase space into two regions which are named as soft (S) and hard (H):

$$
\begin{align*}
\hat{\Gamma} & =\frac{1}{2 M} \frac{1}{N_{\mathrm{pol}}(n) N_{\mathrm{col}}} \int \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS} \\
& =\frac{1}{2 M} \frac{1}{N_{\mathrm{pol}}(n) N_{\mathrm{col}}} \int_{\mathrm{S}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS}+\frac{1}{2 M} \frac{1}{N_{\mathrm{pol}}(n) N_{\mathrm{col}}} \int_{\mathrm{H}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS} \tag{5.11}
\end{align*}
$$

The decomposition is sufficient for calculating the processes $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$. As for the real corrections, there are collinear divergences, thus the hard region is further decomposed into hard collinear (HC) and hard non-collinear (HNC) regions:

$$
\begin{equation*}
\int_{\mathrm{H}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS}=\int_{\mathrm{HC}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS}+\int_{\mathrm{HNC}} \bar{\sum}|\mathcal{M}|^{2} d \mathrm{PS} . \tag{5.12}
\end{equation*}
$$

For the processes $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+q \bar{q} g$, there are only collinear divergences (no soft divergences), therefore decomposing the phase space into collinear and non-collinear regions is sufficient.

We introduce two slicing parameters $\delta_{s}$ and $\delta_{c}$, with the help of which, the soft regions are defined as

$$
\begin{equation*}
\frac{E_{3}}{m_{c}}<\delta_{s} \text { or } \frac{E_{4}}{m_{c}}<\delta_{s} \text { or } \frac{E_{5}}{m_{c}}<\delta_{s}, \tag{5.13}
\end{equation*}
$$

and the collinear regions are defined as

$$
\begin{equation*}
s_{34}<\delta_{c} m_{c}^{2} \text { or } s_{35}<\delta_{c} m_{c}^{2} \text { or } s_{45}<\delta_{c} m_{c}^{2} \tag{5.14}
\end{equation*}
$$

where $E_{3}, E_{4}, E_{5}$ are the energy of soft gluons with momentum $k_{3}, k_{4}, k_{5}$, respectively and $s_{34}=\left(k_{3}+k_{4}\right)^{2}, s_{35}=\left(k_{3}+k_{5}\right)^{2}, s_{45}=\left(k_{4}+k_{5}\right)^{2}$. In our case, three identical gluons are in the final state, we subtract the soft and collinear divergences in one of the soft regions Eq.(5.13) and collinear regions Eq.(5.14), which means we must choose a reference frame which are symmetric under exchanging of final state gluons.

### 5.2.2 Soft region

As an example, let us consider the case when $k_{5}$ is soft, which indicates $E_{5}<\delta_{s} m_{c}$. In the limit that gluon is soft, the amplitude of corresponding diagram factorizes into the Born amplitude without the soft gluon and a eikonal factor, namely the eikonal approximation, which can be summarized as $[69,70]$

$$
\begin{equation*}
\left.\left.\mid k_{5} \text { soft }\right\rangle \left.=g_{s} \frac{p_{i} \cdot \varepsilon^{*}\left(k_{5}\right)}{p_{i} \cdot k_{5}} \mathbf{T}_{i} \right\rvert\, \text { Born }\right\rangle, \tag{5.15}
\end{equation*}
$$

where $\mid k_{5}$ soft $\rangle$ is the amplitude in the limit that $k_{5} \rightarrow 0, \mid$ Born $\rangle$ is the Born level amplitude without the soft gluon, $p_{i}$ is the four momentum of the parton $i$ with which the soft gluon attached, $\mathbf{T}_{i}$ is a color operator acting on the Bron amplitude which is defined as

$$
\mathbf{T}_{i}= \begin{cases}T^{a} & \text { if the parton } i \text { is an incoming anti-quark or outgoing quark, }  \tag{5.16}\\ -T^{a} & \text { if the parton } i \text { is an incoming quark or outgoing anti-quark, } \\ i f^{a b c} & \text { if the parton } \mathrm{i} \text { is a gluon. }\end{cases}
$$

The corresponding soft region contribution of real corrections is given by

$$
\begin{equation*}
d \hat{\Gamma}_{k_{5} \text { soft }}^{\mathrm{RC}}=\frac{1}{4 m_{b}} \frac{1}{\left(N_{c}^{2}-1\right)(D-1)} d \mathrm{PS}_{1 \rightarrow 3} \int_{\text {soft }} d \mathrm{PS}_{k_{5}} \bar{\sum}\left|\mathcal{M}_{k_{5} \text { soft }}\right|^{2}, \tag{5.17}
\end{equation*}
$$

where $\mathcal{M}_{k_{5} \text { soft }}$ is obtained by attaching the soft gluon to the Born level amplitude according to Eq.(5.15), and the phase-space of the $k_{5}$ soft region $\int_{\text {soft }} d \mathrm{PS}_{k_{5}}$ is given by

$$
\begin{align*}
\int_{\text {soft }} d \mathrm{PS}_{k_{5}} & =\int_{\text {soft }} \frac{\mu^{4-D} d^{D-1} k_{5}}{2(2 \pi)^{D-1} E_{5}} \\
& =\frac{\left(\pi \mu^{2}\right)^{\epsilon} \Gamma(1-\epsilon)}{(2 \pi)^{3} \Gamma(1-2 \epsilon)} \int_{0}^{\delta_{s} m_{c}} E_{5}^{1-2 \epsilon} d E_{5} \int_{0}^{\pi} \sin \theta_{1}{ }^{1-2 \epsilon} d \theta_{1} \int_{0}^{\pi} \sin \theta_{2}{ }^{-2 \epsilon} d \theta_{2} . \tag{5.18}
\end{align*}
$$

Therefore, in the $k_{5} \rightarrow 0$ limit, we have

$$
\begin{align*}
d \hat{\Gamma}_{k_{5} \text { soft }}^{\mathrm{RC}}= & d \hat{\Gamma}^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \\
& \times \frac{3}{2} \int_{\text {soft }} d \mathrm{PS}_{k_{5}}\left[\frac{t}{\left(k_{3} \cdot k_{5}\right)\left(k_{4} \cdot k_{5}\right)}-\frac{8 m_{c}^{2}}{\left(P_{2} \cdot k_{5}\right)^{2}}\right. \\
& \left.+\frac{u_{c}}{\left(k_{4} \cdot k_{5}\right)\left(P_{2} \cdot k_{5}\right)}+\frac{s_{c}}{\left(k_{3} \cdot k_{5}\right)\left(P_{2} \cdot k_{5}\right)}\right], \tag{5.19}
\end{align*}
$$

with $\hat{\Gamma}^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right)$ calculated in D-dimension.
To calculate the soft integrals in Eq.(5.19), we have to parameterize the momenta in certain frame which must be symmetric under exchange of final state gluons since we only calculate the $k_{5}$ soft limit. We choose the center of mass frame of $P_{1},-P_{2}$, which at the same time is the center of mass frame of $k_{3}, k_{4}$ in the $k_{5} \rightarrow 0$ limit:

$$
\begin{align*}
P_{1} & =\left(E_{P_{1}}, 0,|\boldsymbol{p}| \sin \theta,|\boldsymbol{p}| \cos \theta\right)  \tag{5.20}\\
P_{2} & =\left(E_{P_{2}}, 0,|\boldsymbol{p}| \sin \theta,|\boldsymbol{p}| \cos \theta\right)  \tag{5.21}\\
k_{3} & =E_{3}(1,0,0,1)  \tag{5.22}\\
k_{4} & =E_{4}(1,0,0,-1)  \tag{5.23}\\
k_{5} & =E_{5}\left(1, \sin \theta_{1} \sin \theta_{2}, \sin \theta_{1} \cos \theta_{2}, \cos \theta_{1}\right) \tag{5.24}
\end{align*}
$$

where

$$
\begin{equation*}
E_{P_{1}}=\frac{s_{b}+u_{b}}{2 \sqrt{t}}, E_{P_{2}}=\frac{s_{c}+u_{c}}{2 \sqrt{t}}, E_{3}=E_{4}=\frac{\sqrt{t}}{2},|\boldsymbol{p}|=\frac{a}{2 \sqrt{t}}, \cos \theta=\frac{u-s}{a}, \tag{5.25}
\end{equation*}
$$

with $a=\sqrt{(s+u)^{2}-64 m_{c}^{2} m_{b}^{2}}$.
Using above parameterization, the results of the soft integrals in Eq.(5.19) are listed in Appendix $F$. Then the final result of $d \hat{\Gamma}_{k_{5} \text { soft }}^{\mathrm{RC}}$ reads

$$
\begin{align*}
d \hat{\Gamma}_{k_{5} \text { soft }}^{\mathrm{RC}}= & d \hat{\Gamma}^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \\
& \times \frac{3 C_{\epsilon}}{8 \pi^{2}}\left[\frac{2}{\epsilon^{2}}+\frac{1}{\epsilon}\left(\ln \left(\frac{\mu^{4} t}{4 m_{c}^{2} s_{c} u_{c} \delta_{s}^{4}}\right)-1\right)-\frac{s_{c}+u_{c}}{a} \ln \left(\frac{s_{c}+u_{c}+a}{s_{c}+u_{c}-a}\right)\right. \\
& +\ln \left(\frac{\mu^{2}}{4 m_{c}^{2} \delta_{s}^{2}}\right)\left(\ln \left(\frac{\mu^{2}}{4 m_{c}^{2} \delta_{s}^{2}}\right)-\ln \left(\frac{s_{c} u_{c}}{4 m_{c}^{2} t}\right)-1\right) \\
& +\frac{1}{2} \ln ^{2}\left(\frac{s_{c}+u_{c}+a}{s_{c}+u_{c}-a}\right)-\frac{\pi^{2}}{2}+\left(\frac{1}{2} \ln ^{2}\left(\frac{s_{c}+u_{c}-a}{2 s_{c}}\right)\right. \\
& \left.\left.+\operatorname{Li}_{2}\left(-\frac{s-u+a}{s_{c}+u_{c}-a}\right)-\mathrm{Li}_{2}\left(\frac{s-u-a}{2 s_{1}}\right)+(s \leftrightarrow u)\right)\right] . \tag{5.26}
\end{align*}
$$

### 5.2.3 Hard-collinear region

Assuming $k_{4}$ is collinear to $k_{5}$, which means $s_{45}<\delta_{c} m_{c}^{2}$, and the momenta $k_{4}, k_{5}$ can be parameterized as

$$
\begin{align*}
& k_{4}=\left(z P_{2}+\frac{k_{\perp}^{2}}{2 z P_{2}}, k_{\perp}, z P_{2}\right)  \tag{5.27}\\
& k_{5}=\left((1-z) P_{2}+\frac{k_{\perp}^{2}}{2(1-z) P_{2}},-k_{\perp},(1-z) P_{2}\right), \tag{5.28}
\end{align*}
$$

where $k_{\perp}$ is the small transverse momentum. Then, at leading order in $k_{\perp}$, the $1 \rightarrow 4$ phase-space factorizes according to

$$
\begin{equation*}
d \mathrm{PS}_{1 \rightarrow 4}=d \mathrm{PS}_{1 \rightarrow 3} \times d \mathrm{PS}_{c}^{45}, \tag{5.29}
\end{equation*}
$$

with

$$
\begin{equation*}
d \mathbf{P S}_{c}^{45}=\frac{\left(4 \pi \mu^{2}\right)^{\epsilon}}{16 \pi^{2} \Gamma(1-\epsilon)}\left(z(1-z) s_{45}\right)^{-\epsilon} d z d s_{45} . \tag{5.30}
\end{equation*}
$$

At the same time, the squared matrix element factorizes as

$$
\begin{align*}
& \bar{\sum}\left|\mathcal{M}_{4^{\prime} \rightarrow 45}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g / g q \bar{q}\right)\right|^{2} \\
=\quad & \frac{2 g_{s}^{2}}{s_{45}} P_{44^{\prime}}(z, \epsilon) \bar{\sum}\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right)\right|^{2} . \tag{5.31}
\end{align*}
$$

Consequently, the SDC in the $k_{4}| | k_{5}$ limit factorizes as

$$
\begin{align*}
& d \hat{\Gamma}_{4^{\prime} \rightarrow 45, \text { hard }}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g / g q \bar{q}\right) \\
= & \left(1-\frac{\delta_{4,5}}{2}\right) d \Gamma^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \frac{g_{s}^{2}}{8 \pi^{2}} \\
& \times \frac{\left(4 \pi \mu^{2}\right)^{\epsilon}}{\Gamma(1-\epsilon)} \int_{0}^{\delta_{c} m_{c}^{2}} s_{45}^{-1-\epsilon} d s_{45} \int_{z_{\min }}^{z_{\max }}(z(1-z))^{-\epsilon} P_{44^{\prime}} d z \\
= & \left(1-\frac{\delta_{4,5}}{2}\right) d \Gamma^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \\
& \times \frac{g_{s}^{2}}{8 \pi^{2}}\left(\frac{4 \pi \mu^{2}}{\delta_{c} m_{c}^{2}}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)}\left(-\frac{1}{\epsilon}\right) \int_{z_{\text {min }}}^{z_{\max }}(z(1-z))^{-\epsilon} P_{44^{\prime}} d z, \tag{5.32}
\end{align*}
$$

with the Altarelli-Parisi splitting function $P_{44^{\prime}}[?]$ and the integration limits $z_{\max }, z_{\min }$ given by

$$
\begin{align*}
& \delta_{4,5}=1, z_{\min }=\frac{2 \delta_{s} m_{c}}{\sqrt{t}}, z_{\max }=1-\frac{2 \delta_{s} m_{c}}{\sqrt{t}}  \tag{5.33}\\
& P_{44 \prime}(z, \epsilon)=2 C_{A}\left[\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right] \tag{5.34}
\end{align*}
$$

for $g \rightarrow g g$ splitting, and

$$
\begin{align*}
& \delta_{4,5}=0, z_{\min }=0, z_{\max }=1  \tag{5.35}\\
& P_{44 \prime}(z, \epsilon)=\frac{1}{2}\left(z^{2}+(1-z)^{2}\right)-z(1-z) \epsilon \tag{5.36}
\end{align*}
$$

for $g \rightarrow q \bar{q}$ splitting.
In the case of $g \rightarrow g g$ splitting, hard conditions for the splitting gluons are applied to avoid double counting of the soft-collinear region.
Thus we get our final results for hard-collinear region:

$$
\begin{align*}
& d \hat{\Gamma}_{4^{\prime} \rightarrow 45, \text { hard }}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g\right) \\
= & d \hat{\Gamma}^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \frac{3 g_{s}^{2}}{8 \pi^{2}}\left(\frac{4 \pi \mu^{2} e^{-\gamma_{E}}}{m_{c}^{2}}\right)^{\epsilon} \\
& \times\left[\frac{1}{\epsilon}\left(2 \ln \left(\frac{2 \delta_{s} m_{c}}{\sqrt{t}}\right)+\frac{11}{6}\right)-\left(2 \ln \left(\frac{2 \delta_{s} m_{c}}{\sqrt{t}}\right)+\frac{11}{6}\right) \ln \left(\delta_{c}\right)\right. \\
& \left.-\ln ^{2}\left(\frac{2 \delta_{s} m_{c}}{\sqrt{t}}\right)+\frac{67}{18}-\frac{\pi^{2}}{3}\right] \tag{5.37}
\end{align*}
$$

for $g \rightarrow g g$ splitting, and

$$
\begin{align*}
& d \hat{\Gamma}_{4^{\prime} \rightarrow 45, \text { hard }}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g+q \bar{q}\right) \\
= & d \hat{\Gamma}^{\mathrm{LO}}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g\right) \\
& \times \frac{g_{s}^{2}}{24 \pi^{2}}\left(\frac{4 \pi \mu^{2} e^{-\gamma_{E}}}{m_{c}^{2}}\right)^{\epsilon}\left[-\frac{1}{\epsilon}+\ln \left(\delta_{c}\right)-\frac{5}{3}\right] \tag{5.38}
\end{align*}
$$

for $g \rightarrow q \bar{q}$ splitting.

### 5.2.4 Hard-non-collinear region

In the hard-non-collinear region, the final state phase-space integration is finite, thus we directly perform the numerical integration in four-dimension. In the rest frame of $P_{1},-P_{2}$, the hard-noncollinear region conditions are given by

$$
\begin{align*}
& E_{3}=\frac{s_{1}+u_{1}+u_{2}-t_{2}-4 m_{b}^{2}}{2 \sqrt{s_{1}}}>\delta_{s} m_{c},  \tag{5.39a}\\
& E_{4}=\frac{s_{1}-s_{2}}{2 \sqrt{s_{1}}>\delta_{s} m_{c},}  \tag{5.39b}\\
& E_{5}=\frac{t_{2}+s_{2}-u_{1}-u_{2}+4 m_{b}^{2}}{2 \sqrt{s_{1}}}>\delta_{s} m_{c},  \tag{5.39c}\\
& s_{34}=s_{1}+u_{1}+u_{2}-t_{2}-s_{2}-4 m_{b}^{2}>\delta_{c} m_{c}^{2},  \tag{5.39d}\\
& s_{35}=s_{2}>\delta_{c} m_{c}^{2},  \tag{5.39e}\\
& s_{45}=4 m_{b}^{2}+t_{2}-u_{1}-u_{2}>\delta_{c} m_{c}^{2} . \tag{5.39f}
\end{align*}
$$

We express the four body phase-space in covariant form [78] . Therefore, these 6 hard-noncollinear region conditions can be easily implemented in the numerical integration of the final state phase-space.

## $5.3 \quad b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$

There are 36 Feynman diagrams (Fig.4.2) for the tree-level subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+$ ggg. As expected, infrared divergences appear in the final state phase-space integration. In NRQCD factorization, such kind of infrared divergences are absorbed into NLO corrections to the the color-octet matrix element $\left\langle\chi_{c J}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$. The color-singlet subprocesses $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{3} P_{J}^{[1]}\right)+g g g$ were studied in Ref. [60], where a small mass cut-off regularization was adopted.

Here, we use the same approach in the computation of the real corrections of $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g g$ (only one slicing parameter $\delta_{s}$ is needed in this case) to extract the infrared divergences. In the $k_{5}$ soft limit (as an example) and choosing axial gauge for the soft gluon

$$
\begin{align*}
& \sum_{\text {pol }} \epsilon_{\beta^{\prime}}\left(k_{5}\right) \epsilon_{\beta}^{*}\left(k_{5}\right) \\
= & -g_{\beta^{\prime} \beta}+\frac{P_{2 \beta^{\prime}} k_{5 \beta}+P_{2 \beta} k_{5 \beta^{\prime}}}{P_{2} \cdot k_{5}}-\frac{P_{2}^{2} k_{5 \beta^{\prime}} k_{5 \beta}}{\left(P_{2} \cdot k_{5}\right)^{2}}, \tag{5.40}
\end{align*}
$$

the squared matrix element can be factorized as

$$
\begin{align*}
& \left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1 / 8]}\right)+g g g\right)\right|_{k_{5} \text { soft }}^{2} \\
= & 4 g_{s}^{2} \frac{\epsilon^{\beta^{\prime}}\left(k_{5}\right) \epsilon^{* \beta}\left(k_{5}\right) \varepsilon_{\alpha \beta}^{(J) *}\left(P_{2}\right) \varepsilon_{\alpha^{\prime} \beta^{\prime}}^{(J)}\left(P_{2}\right)}{\left(P_{2} \cdot k_{5}\right)^{2}} \\
& \times \mathcal{M}_{\text {Born }}^{\alpha}\left(T_{c}-T_{\bar{c}}\right)\left(T_{c}-T_{\bar{c}}\right) \mathcal{M}_{\mathrm{Born}}^{* \alpha^{\prime}}, \tag{5.41}
\end{align*}
$$

where $\epsilon^{* \beta}\left(k_{5}\right), \varepsilon_{\alpha \beta}^{(J) *}\left(P_{2}\right)$ are polarization vector and tensor for the soft gluon and $c \bar{c}$ pair respectively. $T_{c}$ and $T_{\bar{c}}$ are color matrices that corresponding soft gluon attaching to charm and anti-charm quark line respectively. And $\mathcal{M}_{\text {Born }}^{\alpha}$ represents the amplitude for the Born-level subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ with the Lorentz index of polarization vector of $c \bar{c}$ pair labeled as $\alpha$ and color projectors adopted according to the color states of $c \bar{c}\left({ }^{3} P_{J}^{[1 / 8]}\right)$. Therefore, soft region contribution in the case of $k_{5}$ soft then reads

$$
\begin{align*}
d \hat{\Gamma}_{k 5 \text { soft }}= & \frac{1}{4 m_{b}} \frac{1}{\left.N_{\mathrm{col}} N_{\mathrm{pol}}{ }^{3} P_{J}\right)} d \mathrm{PS}_{1 \rightarrow 3} \int_{\mathrm{soft}} d \mathrm{PS}_{k_{5}} \\
& \times \overline{\sum \sum}\left|\mathcal{M}\left(b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1 / 8]}\right)+g g g\right)\right|_{k_{5} \mathrm{soft}}^{2} \\
= & \frac{1}{4 m_{b}} \frac{1}{N_{\mathrm{col}} N_{\mathrm{pol}}\left({ }^{3} P_{J}\right)} d \mathrm{PS}_{1 \rightarrow 3} \int_{\mathrm{soft}} d \mathrm{PS}_{k_{5}} \frac{4 g_{s}^{2}}{\left(P_{2} \cdot k_{5}\right)^{2}} \\
& \times\left(-g_{\beta^{\prime} \beta}+\frac{P_{2 \beta^{\prime}} k_{5 \beta}+P_{2 \beta} k_{5 \beta^{\prime}}}{P_{2} \cdot k_{5}}-\frac{P_{2}^{2} k_{5 \beta^{\prime}} k_{5 \beta}}{\left(P_{2} \cdot k_{5}\right)^{2}}\right) \\
& \times \overline{\sum \varepsilon_{\alpha \beta}^{(J) *} \varepsilon_{\alpha^{\prime} \beta^{\prime}}^{(J)} \mathcal{M}_{\mathrm{Born}}^{\alpha}\left(T_{c}-T_{\bar{c}}\right)\left(T_{c}-T_{\bar{c}}\right) \mathcal{M}_{\mathrm{Born}}^{* \alpha^{\prime}} .} \tag{5.42}
\end{align*}
$$

Now we confront with new type of tensor integral $\int_{\text {soft }} d \mathrm{PS}_{k_{5}} \frac{k_{5} k_{5 \beta^{\prime}}}{\left(P_{2} \cdot k_{5}\right)^{4}}$. This tensor integral cannot be reduced to scalar integrals through conventional tensor reduction procedure, since it is not Lorentz covariant due to the cut-off of soft gluon energy. Therefore, we explicitly evaluate these tensor integrals such as $\int_{\text {soft }} d \mathrm{PS}_{k_{5}} \frac{\left(k_{3} \cdot k_{5}\right)\left(k_{4} \cdot k_{5}\right)}{\left(P_{2} \cdot k_{5}\right)^{4}}$. As a result, the D-dimensional Born-level squared matrix element of subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ cannot be factorized out from the D-dimensional squared matrix elements of the subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} P_{J}^{[1,8]}\right)+g g g$ in the soft limit, which is different from the case of the real corrections. Nevertheless, this does not affect the cancellation of infrared divergences, since the divergences appear to be proportional to the 4-dimensional Born-level squared matrix element of subprocess $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$.

## Chapter 6

## Numerical Evaluation and Phenomenological Results

We are now in the position to present our numerical analysis of inclusive $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ production in $\Upsilon(1 S)$ decay at $\mathcal{O}\left(\alpha_{s}^{5}\right)$ ( the results of $\mathcal{O}\left(\alpha_{s}^{6}\right) J / \psi$ color-singlet processes are included [64] ) in the NRQCD factorization framework. The phase-space integrations are performed numerically with the help of the CUBA [71] package. The numerical values of the one-loop master integrals are computed using $\mathrm{C}++$ package QCDLOOP [72]

Since there are several natural scales $2 m_{b}, m_{b}, 2 m_{c}, m_{c}$ involved in our evaluation, and no unbiased choices of renormalization scale which can truly represent the scale of the processes, we adopt the fastest apparent convergence (FAC) scheme [73] to fix the renormalization scale $\mu_{\mathrm{FAC}}$ and then explore the renormalization scale dependence in the range $\mu_{\mathrm{FAC}} / 2<\mu<2 \mu_{\mathrm{FAC}}$. In the first section, we describe our strategies of extracting the $\mu$ dependence, obtaining $\mu_{\mathrm{FAC}}$ and reducing the uncertainty due to high order of $\alpha_{s}(\mu)\left(\alpha_{s}^{5}\right)$. Our parameter settings and the experimental data are presented in the second section. And in the next 3 sections, our final results of inclusive $\chi_{c J}, J / \psi$ and $h_{c}, \eta_{c}$ production via $\Upsilon(1 S)$ decay will be presented separately.

### 6.1 Approaches of Exploring Renormalization Scale Dependence

To investigate the renormalization scale dependence, we express the SDCs of the relevant processes (Table.4.2) as

$$
\begin{align*}
& \hat{\Gamma}_{{ }_{1} S_{0}^{\prime 1]}}^{g g g / c \bar{c} g}=f_{{ }_{1} S_{0}^{[1]}}^{g g / \bar{c} g} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-5},  \tag{6.1a}\\
& \hat{\Gamma}_{{ }_{1} S_{0}^{8]}}^{g / g g g / c \bar{c} g}=f_{{ }_{1} S_{0}^{8]}}^{g / g g / c \bar{c} g} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-5}, \tag{6.1b}
\end{align*}
$$

$$
\begin{align*}
& \hat{\Gamma}_{{ }_{3} S_{1}^{\text {I1] }}}^{g g / g g g}=f_{{ }_{3} S_{1}^{[1]}}^{g g / \text { IIgg }} \alpha_{s}^{6}(\mu) \mathrm{GeV}^{-5},  \tag{6.1d}\\
& \hat{\Gamma}_{{ }_{3} S_{1}^{1,8]}}^{c \bar{c} g}=f_{S_{1}}^{c \bar{c} g} S_{1}^{1,8]} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-5},  \tag{6.1e}\\
& \hat{\Gamma}_{{ }^{3} S_{1}^{[8]}}^{g g}=\left(f_{3 S_{1}^{8]}}^{g g, \mathrm{LO}}+f_{{ }_{3} S_{1}^{[8]}}^{g g \text { corr }} \alpha_{s}(\mu)\right) \alpha_{s}^{4}(\mu) \mathrm{GeV}^{-5},  \tag{6.1f}\\
& \hat{\Gamma}_{{ }_{1} P_{1}^{[1,8]}}^{g g g / c \bar{c} g}=f_{{ }_{1} P_{1}^{(1)}}^{g g g / c \bar{c} g} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-7},  \tag{6.1g}\\
& \hat{\Gamma}_{{ }_{3} P_{J}^{(1])}}^{g g g / c \bar{c} g}=f_{3_{J} P_{J}^{(1)}}^{g g g / c \bar{c} g} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-7},  \tag{6.1h}\\
& \hat{\Gamma}_{{ }_{3} P_{J}^{8]}}^{g / g g / c \bar{c} g}=f_{3_{J}{ }_{P}^{8]}}^{g / g g / c \bar{c} g} \alpha_{s}^{5}(\mu) \mathrm{GeV}^{-7}, \tag{6.1i}
\end{align*}
$$

where the label "corr" means the combined contributions of the real and virtual corrections, the subscripts and superscripts of $\hat{\Gamma}, f$ label the $c \bar{c}$ fock states and extra partons $X$, respectively. Thus all the coefficients $(f s)$ are dimensionless numbers after numerical phase-space integrations. Here the $\mathcal{O}\left(\alpha_{s}^{6}\right)$ coefficients are taken from Ref. [64], and $\hat{\Gamma}_{{ }_{3} P_{J}^{[1,8]}}^{g g g}$ represent the SDCs after subtracting the infrared divergences.

Since $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ QCD production through $\Upsilon$ CS channels can be viewed as $\Upsilon \rightarrow$ $g g g(g)$ followed by $g(g) \rightarrow \eta_{c} / J / \psi / h_{c} / \chi_{c J}+X$, the large uncertainties due to $\alpha_{s}$ and $m_{b}$ choosing can be largely reduced by normalize the partial decay width to the decay width of $\Upsilon \rightarrow g g g[63,64]$, which have been calculated up to $\mathcal{O}\left(\alpha_{s}^{4}\right)$ and is given by [74]

$$
\begin{equation*}
\Gamma(\Upsilon \rightarrow g g g)=\hat{\Gamma}_{g g g}\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{\Upsilon}, \tag{6.2}
\end{equation*}
$$

with

$$
\begin{equation*}
\hat{\Gamma}_{g g g}=\frac{20 \alpha_{s}^{3}(\mu)}{243 m_{b}^{2}}\left(\pi^{2}-9\right)\left(1+\frac{\alpha_{s}(\mu)}{\pi}\left(-19.4+\frac{3 \beta_{0}}{2}\left(1.161+\ln \left(\frac{\mu}{m_{b}}\right)\right)\right)\right) \tag{6.3}
\end{equation*}
$$

In this way, we express the branching ratios as

$$
\begin{equation*}
\mathcal{B}_{\eta_{c} / J / \psi / h_{c} / \chi_{c J}}=\Gamma_{\mathrm{Nor}}\left(\Upsilon \rightarrow \eta_{c} / J / \psi / h_{c} / \chi_{c J}+X\right) \times \mathcal{B}(\Upsilon \rightarrow g g g), \tag{6.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{\mathrm{Nor}}\left(\Upsilon \rightarrow \eta_{c} / J / \psi / h_{c} / \chi_{c J}+X\right)=\frac{\Gamma\left(\Upsilon \rightarrow \eta_{c} / J / \psi / h_{c} / \chi_{c J}+X\right)}{\Gamma(\Upsilon \rightarrow g g g)} \tag{6.5}
\end{equation*}
$$

and $\mathcal{B}(\Upsilon \rightarrow g g g) \equiv \mathcal{B}_{\text {ggg }}=81.7 \%$ [35]. Note that now the theoretical predictions do not depend on the CS LDME $\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{\Upsilon}$.

Consequently, we express the branching ratios as (we drop all the units of Gev for simplicity, thus each term of the expressions below should be understood as compensated by certain powers of Gev to make the branching ratios dimensionless)

$$
\begin{align*}
& \mathcal{B}_{\eta_{c}}=\left[\left(f_{{ }_{1} S_{0}^{11]}}^{g g}+f_{{ }_{1} S_{0}^{(1]}}^{c \bar{c} g}\right)\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle\right. \\
& +\left(f_{1 S_{0}^{[8]}}^{g g g}+f_{{ }_{1} S_{0}^{[8]}}^{c \bar{c} g}+f_{1 S_{0}^{[8]}}^{g}\right)\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle \\
& +\left(f_{3_{S}}^{\mathrm{LO}, g g} / \alpha_{s}^{[8]}(\mu)+f_{{ }_{3} S_{1}^{[8]}}^{\mathrm{corr}, g g}+f_{{ }_{3} S_{1}^{[8]}}^{c \bar{c} g}\right)\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle \\
& \left.+\left(f_{1 P_{1}^{[8]}}^{g g}+f_{1 P_{1}^{[8]}}^{c \bar{c} g}\right)\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} P_{1}^{[8]}\right)\right\rangle\right] \alpha_{s}^{5}(\mu) \mathcal{B}_{g g g} / \hat{\Gamma}_{g g g},  \tag{6.6}\\
& \mathcal{B}_{J / \psi}=\left[\left(f_{S_{1}^{[1]}}^{c \bar{c} g}+f_{{ }_{3} S_{1}^{[1]}}^{\mathrm{QED}} / \alpha_{s}^{3}(\mu)+f_{3}^{g g} S_{1}^{[1]} \alpha_{s}(\mu)+f_{{ }_{3}}^{g g g g} \alpha_{1}^{[1]}(\mu)\right)\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle\right. \\
& +\left(f_{1 S_{0}^{[8]}}^{g g g}+f_{1 S_{0}^{[8]}}^{c \bar{c} g}+f_{1}^{g} S_{0}^{[8]}\right)\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle \\
& +\left(f_{{ }_{3} S_{1}^{8]}}^{\mathrm{LO}, g g} / \alpha_{s}(\mu)+f_{S_{1}^{[8]}}^{\text {corr,gg }}+f_{{ }_{3} S_{1}^{[8]}}^{c \bar{q} g}\right)\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle \\
& \left.+\sum_{J}\left(f_{{ }_{3} P_{J}^{[8]}}^{g}+f_{{ }_{3} P_{J}^{[8]}}^{g g g}+f_{{ }_{3} P_{J}^{[8]}}^{c \bar{c} g}\right)\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle\right] \alpha_{s}^{5}(\mu) \mathcal{B}_{g g g} / \hat{\Gamma}_{g g g},  \tag{6.7}\\
& \mathcal{B}_{h_{c}}=\left[\left(f_{{ }_{1} P_{1}^{1]}}^{g g g}+f_{1_{1}^{11]}}^{c \bar{c} g}\right)\left\langle\mathcal{O}^{h_{c}}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle\right. \\
& \left.+\left(f_{{ }_{1} S_{0}^{[8]}}^{g g g}+f_{{ }_{1} S_{0}^{[8]}}^{c \bar{c}}+f_{{ }_{1} S_{0}^{[8]}}^{g}\right)\left\langle\mathcal{O}^{h_{c}}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle\right] \alpha_{s}^{5}(\mu) \mathcal{B}_{g g g} / \hat{\Gamma}_{g g g},  \tag{6.8}\\
& \mathcal{B}_{\chi_{c J}}=\left[\left(f_{3 P_{J}^{[1]}}^{c \bar{c} g}+f_{3 P_{J}^{[1]}}^{g g g}\right)\left\langle\mathcal{O}^{\chi_{c J}}\left({ }^{3} P_{J}^{[1]}\right)\right\rangle\right. \\
& \left.+\left(f_{S_{1}^{[8]}}^{\mathrm{LO}, g g} / \alpha_{s}(\mu)+f_{{ }_{3} S_{1}^{[8]}}^{\mathrm{corr}, g g}+f_{\left.{ }_{3} S_{1}^{[8]}\right]}^{c \bar{c} \bar{g}}\right)\left\langle\mathcal{O}^{\chi}{ }_{c J}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle\right] \alpha_{s}^{5}(\mu) \mathcal{B}_{g g g} / \hat{\Gamma}_{g g g} . \tag{6.9}
\end{align*}
$$

As NLO QCD corrections are included in our calculation, to study the $\mu$ dependence, we need to employ the one-loop and two-loop formulas for $\alpha_{s}(\mu)$ running,

$$
\begin{equation*}
\frac{\alpha_{s}^{(1)}(\mu)}{4 \pi}=\frac{1}{\beta_{0} L}, \frac{\alpha_{s}^{(2)}(\mu)}{4 \pi}=\frac{1}{\beta_{0} L}-\frac{\beta_{1} \ln L}{\beta_{0}^{3} L^{2}}, \tag{6.10}
\end{equation*}
$$

where $L=\ln \left(\frac{\mu^{2}}{\Lambda_{\mathrm{CcD}}^{2}}\right)$, and $\beta_{1}=(34 / 3) C_{A}^{2}-4 C_{F} T_{F} n_{f}-(20 / 3) C_{A} T_{F} n_{f}$ is the two-loop coefficient of the QCD beta function. Here $\Lambda_{\mathrm{QCD}}$ has different values at one-loop and two-loop .


Figure 6.1: $\mu$ dependence of K-factor, LO, virtual plus real corrections, NLO SDCs from $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ subprocess.

We adopt the fastest apparent convergence (FAC) scheme [73] to fix the renormalization scale $\mu_{\mathrm{FAC}}$ and then explore the renormalization scale dependence in the range $\mu_{\mathrm{FAC}} / 2<\mu<2 \mu_{\mathrm{FAC}}$. The value of $\mu_{\mathrm{FAC}}$ can be fixed by the requirement that the LO SDC of the process $\Upsilon\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)+X$ equals to the NLO ones:

$$
\begin{equation*}
\text { K-factor }=\frac{f_{8, g g}^{\mathrm{LO}}\left(\alpha_{s}^{(1)}(\mu)\right)^{4}}{\left(f_{8, g g}^{\mathrm{LO}}+f_{8, g g}^{\mathrm{corr}} \alpha_{s}^{(2)}(\mu)\right)\left(\alpha_{s}^{(2)}(\mu)\right)^{4}}=1, \tag{6.11}
\end{equation*}
$$

which gives $\mu_{\mathrm{FAC}}=6.2 \mathrm{Gev}$ (Figure.6.1).

### 6.2 Parameter Settings and Experimental Data

In our numerical analysis, we adopt the values

$$
\begin{align*}
m_{c} & =\frac{m_{J / \psi}}{2}=1.5 \mathrm{GeV}  \tag{6.12a}\\
m_{b} & =\frac{m_{\Upsilon}}{2}=4.75 \mathrm{GeV}  \tag{6.12b}\\
\alpha & =1 / 128,  \tag{6.12c}\\
\mu_{\Lambda} & =m_{c}=1.5 \mathrm{GeV}  \tag{6.12d}\\
\Lambda_{\mathrm{QCD}} & =249(389) \mathrm{MeV}, n_{f}=3, \text { for one-loop (two-loop). } \tag{6.12e}
\end{align*}
$$

Our choices of the LDMEs are given in Table.(1.1, 1.2).
The experimental data are summarized in Table.(6.1, 6.2).
For the real corrections and two infrared divergent P -wave processes, certain soft and collinear cut conditions Eq.(5.39) are applied for the hard non-collinear region, which means the numerical results depend on the soft and collinear cuts $\delta_{s}, \delta_{c}$. However the combined numerical results of

| $\psi(2 S) \rightarrow J / \psi+X$ | $(59.5 \pm 0.8) \%$ |
| :---: | :---: |
| $\psi(2 S) \rightarrow \chi_{c 1}+\gamma$ | $(9.2 \pm 0.4) \%$ |
| $\psi(2 S) \rightarrow \chi_{c 2}+\gamma$ | $(8.72 \pm 0.34) \%$ |
| $\chi_{c 1} \rightarrow J / \psi+\gamma$ | $(34.4 \pm 1.5) \%$ |
| $\chi_{c 2} \rightarrow J / \psi+\gamma$ | $(19.5 \pm 0.8) \%$ |
| $h_{c} \rightarrow \eta_{c}+\gamma$ | $(51 \pm 6) \%$ |

Table 6.1: Significant feed-down channels

| $J / \psi+X$ | $(5.4 \pm 0.4) \times 10^{-4}$ | $(3.41 \pm 0.4) \times 10^{-4}$ (Direct) |
| :---: | :---: | :---: |
| $\psi(2 S)+X$ | $(1.23 \pm 0.20) \times 10^{-4}$ |  |
| $\chi_{c 0}+X$ | $<4 \times 10^{-3}, \mathrm{CL}=90 \%$ |  |
| $\chi_{c 1}+X$ | $(1.90 \pm 0.35) \times 10^{-4}$ | $(1.78 \pm 0.35) \times 10^{-4}$ (Direct) |
| $\chi_{c 2}+X$ | $(2.8 \pm 0.8) \times 10^{-4}$ | $(2.69 \pm 0.8) \times 10^{-4}$ (Direct) |

Table 6.2: Experimental data on branching ratios of inclusive $J / \psi, \psi(2 S), \chi_{c J}$ production via $\Upsilon$ decay. The direct contributions are obtained through subtracting the central values of the feed-down contributions in Table. 6.1 from the total branching ratios.
hard non-collinear and soft-collinear regions approach constants as $\delta_{s}, \delta_{c}$ becoming smaller. As addressed in Ref. [68], the collinear cut $\delta_{c}$ must be much smaller than the soft cut $\delta_{s}$ due to the requirement that contributions proportional to $\ln \left(\delta_{c}\right) \ln \left(\delta_{s}\right)-\mathrm{Li}_{2}\left(\delta_{c} / \delta_{s}\right)$ vanish. In our numerical analysis we choose $\delta_{c}=\delta_{s} / 100$ and start with $\delta_{s}=0.1$. In order to achieve the accuracy of $0.1 \%$ of numerical int egration, we have to limit ourself that $\delta_{s}>10^{-4}$. Because of the cancellation of hard non-collinear and soft-collinear regions, the final accuracy of our numerical analysis is about $1 \%$, which is sufficient in the current required accuracy of QCD calculations. The results of hard non-collinear and soft-collinear regions as well as the combined results as functions of soft cut $\delta_{s}$ are plotted in Figure. $(6.2,6.3$ ) for the NLO corrections and infrared divergent CO P-wave process. From Figure. (6.2, 6.3), it can be seen that the combined results indeed converge to constants for each process, which is served as a check point of our calculations.

With above numerical settings, we obtain our results of $f s$ and summarize them in Table 6.3.

| ${ }_{{ }_{1}^{1} S_{0}^{11]}}^{g g g}\left(10^{-6}\right)$ | 1.89 | $f_{1 S_{0}^{11]}}^{c \bar{c} g}\left(10^{-6}\right)$ | 1.07 |
| :---: | :---: | :---: | :---: |
| ${ }_{1_{1} S_{0}^{88]}}^{g g g}\left(10^{-5}\right)$ | 1.50 | $f_{1}^{1} S_{0}^{\text {¢8] }}$ [8] $\left(10^{-6}\right)$ | 2.14 |
| $f_{1}^{1} S_{0}^{\text {[8] }}$ ( $\left.10^{-4}\right)$ | 1.15 |  |  |
| $f^{3}{ }_{3}^{\text {SED }}$ (1] $\left(10^{-8}\right)$ | 1.76 | $f_{3 S_{1}^{11]}}^{c \bar{c} g}\left(10^{-6}\right)$ | 1.32 |
| ${ }^{3}{ }^{3} S_{1}^{[1]}\left(10^{-6}\right)$ | 8.46 | $f_{{ }_{3} S_{1}^{[1]}}^{g g g g}\left(10^{-8}\right)$ | 3.4 |
| $f_{3 S_{1}^{8]}}^{\mathrm{LD}, g}\left(10^{-4}\right)$ | 2.38 | $f_{{ }_{3} S_{1}^{[8]}}^{\text {corr, } 8 g}\left(10^{-4}\right)$ | $4.85+13.62 \ln \left(\frac{\mu}{m_{b}}\right)$ |
|  | 1.23 | $f_{1 P_{1}^{11]}}^{c \bar{c} g}\left(10^{-7}\right)$ | 1.5 |
| ${ }_{1_{1} P_{1}^{\text {P }}}^{\text {ggg }}\left(10^{-7}\right)$ | 6.4 | $f_{{ }_{1} P_{1}^{[8]}}^{c \bar{c} g}\left(10^{-7}\right)$ | 3.0 |
| $\sum_{J} f_{{ }_{3} P_{J}^{[8]}}^{g g g}\left(10^{-4}\right)$ | -2.78 | $\sum_{J} f_{\left.{ }_{3} P_{J}^{[8]}\right]}^{c \bar{c} g}\left(10^{-6}\right)$ | 1.3 |
| $\sum_{J} f_{{ }^{3} P_{J}^{[8]}}^{g}\left(10^{-4}\right)$ | 1.97 |  |  |
| $f_{3_{0} P_{0}^{11]}}^{g g g}\left(10^{-5}\right)$ | -4.18 | $f_{3}^{c \bar{c} g}{ }_{0}^{\text {[1] }}\left(10^{-7}\right)$ | 1.73 |
|  | -2.06 | $f_{{ }_{3} P_{1}^{[1]}}^{c \bar{c} g}\left(10^{-7}\right)$ | 1.04 |
| $f_{3 P_{2}{ }^{11]}}^{g g g}\left(10^{-5}\right)$ | -2.65 | $f_{3 P_{2}{ }^{11]}}^{\text {çg }}\left(10^{-7}\right)$ | 0.35 |

Table 6.3: The numerical results of $f s$ introduced in Eq.(6.1).


Figure 6.2: Numerical dependences of the phase-space slicing parameter $\delta_{s}$ in the NLO corrections to $b \bar{b}\left({ }^{3} S_{1}^{1}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ with $\mu=m_{b}, \delta_{c}=\delta_{s} / 100$.


Figure 6.3: Numerical dependences of the phase-space slicing parameter $\delta_{s}$ for processes $b \bar{b}\left({ }^{3} S_{1}^{1}\right) \rightarrow \sum_{J} c \bar{c}\left({ }^{3} P_{J}^{[8]}\right)+g g g$ with $\mu_{\Lambda}=m_{c}$.


Figure 6.4: $\mu$ dependence of branching ratios for $\chi_{c 1}, \chi_{c 2}$ with $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$, $\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=0.0027 \mathrm{GeV}^{3}$ (Chao et al.) and $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.0794 \mathrm{GeV}^{5},\left\langle{ }^{\chi_{c 0}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=$ $0.00574 \mathrm{GeV}^{3}$ (Bodwin et al.), to be compared with experimental data.

## $6.3 \chi_{c J}$

From the numerical results listed in Table. 6.3, it can be seen that the contributions of $c \bar{c}$ associated processes are tiny and the contributions from the dominant CS channel $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow$ $c \bar{c}\left({ }^{3} P_{J}^{[1]}\right)+g g g$ are negative with $\mu_{\Lambda}=m_{c}$. Thus the numerical results come out from the cancellation of color-singlet $\chi_{c J}\left({ }^{3} P_{J}^{[1]}\right)$ and color-octet $\chi_{c J}\left({ }^{3} S_{1}^{[8]}\right)$ contributions. Consequently, the final results strongly depend on the values of the LDMEs $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle,\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$.

The are two dramatically different fits of the LDMEs of $\chi_{c J}$ production. In Ref. [16,21], the CS LDME $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle$ was related to the first derivative of the wave function at origin $\left|R_{1 P}^{\prime}(0)\right|=0.075 \mathrm{GeV}^{5}$, which was obtained using the Buchmüller-Tye potential and gave $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$ [75]. With the fixed CS LDME $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle$, slightly different values for the CO LDMEs $\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=\left(2.2_{-0.3}^{+0.5}\right) \times 10^{-3} \mathrm{GeV}^{3}[16]$ and $\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=(2.21 \pm 0.12) \times$ $10^{-3} \mathrm{GeV}^{3}$ [21] were obtained by fitting $\chi_{c J}$ hadroproduction data (see Table. 1.1). In Ref. [76], an alternative set of $\chi_{c J}$ production LDMEs is given through fitting to ATLAS cross-section data [77] of $\chi_{c 1}, \chi_{c 2}$ production with the combined NLO SDCs in $\alpha_{s}$ and the LP-fragmentation contributions, where

$$
\begin{align*}
\left\langle\mathcal{O}^{\chi c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle & =(5.74 \pm 1.31) \times 10^{-3} \mathrm{Gev}^{3},  \tag{6.13}\\
\frac{\left\langle\mathcal{O}^{\chi c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle}{m_{c}^{2}} & =(3.53 \pm 1.08) \times 10^{-2} \mathrm{Gev}^{3} . \tag{6.14}
\end{align*}
$$



Figure 6.5: $\mu$ dependence of refitting value for $\left\langle{ }^{\chi_{c 0}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ with $\left\langle{ }^{\chi_{c 0}}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$ and $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.0794 \mathrm{GeV}^{5}$.

Combing the CS LDME $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$, upper bound value of CO LDMEs $\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=2.7 \times 10^{-3} \mathrm{GeV}^{3}$ given in Ref. [16,21] and the central values $\left\langle{ }^{\chi_{c 0}}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=$ $0.0794 \mathrm{GeV}^{5},\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=5.74 \times 10^{-3} \mathrm{GeV}^{3}$ given in Ref. [76] with our numerical values of $f s$, we plot the renormalization scale dependence of $\mathcal{B}\left(\Upsilon \rightarrow \chi_{c J}+X\right)(J=1,2)$ in Fig.6.4, from which we find that the theoretical predictions with the LDMEs given in Ref. [16, 21] underestimate the experimental data; while the theoretical predictions with the LDMEs given in Ref. [76] overshoot the experimental data.

We therefore refit the value of $\left\langle\chi^{c o}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ to $\Upsilon$ decays to $\chi_{c 1}$ and $\chi_{c 2}$ with two different values of the CS LDME $\left\langle\chi_{c o}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$ and $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.0794 \mathrm{GeV}^{5}$. With the condition that $\mathcal{B}\left(\Upsilon \rightarrow \chi_{c 0}+X\right)>0$, we thus exclude the range $\mu<3.7 \mathrm{GeV}$ with $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$. It can be seen from Figure. 6.5 that the value of $\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ increases as $\mu$ increasing, but is still relative stable. While $\mu$ changes from 3.7 Gev to $2 \mu_{\mathrm{FAC}},\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ goes from $(3.7 \pm 0.28) \times 10^{-3} \mathrm{GeV}^{3}$ to $(4.71 \pm 0.65) \times 10^{-3} \mathrm{GeV}^{3}$ with $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.107 \mathrm{GeV}^{5}$ and from $(3.23 \pm 0.28) \times 10^{-3} \mathrm{GeV}^{3}$ to $(4.52 \pm 0.65) \times 10^{-3} \mathrm{GeV}^{3}$ with $\left\langle\chi_{c 0}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.0794 \mathrm{GeV}^{5}$, where the corresponding values at $\mu=\mu_{\mathrm{FAC}}=6.2 \mathrm{Gev}$ are $(4.04 \pm 0.47) \times 10^{-3} \mathrm{GeV}^{3}$ and $(3.74 \pm 0.47) \times 10^{-3} \mathrm{GeV}^{3}$, respectively, which is about 2 times larger than the central value obtained in Ref. [16,21] and roughly agree with the lower bound value given in Ref. [76].

## $6.4 J / \psi$

From Table. 6.1, we can see that all $J / \psi$ intermediate states ${ }^{3} S_{1}^{[1 / 8]},{ }^{1} S_{0}^{[8]},{ }^{3} P_{J}^{[8]}$ for $J / \psi$ give significant contributions as long as their LDMEs are not too small. Especially, there are also strong cancellations between $c \bar{c}\left({ }^{3} P_{J}^{[8]}\right)+g g g$ and $c \bar{c}\left({ }^{3} P_{J}^{[8]}\right)+g$ channels, where the former channel is not considered in previous investigations in the literature. Inserting the numerical results in Table. 6.3 into Eq.(6.7), we plot the renormalization scale $\mu$ dependence of the branching ratio of inclusive $J / \psi$ production via $\Upsilon$ decay with six different sets of relevant LDMEs in Table. (1.1, 1.2) (see Figure. 6.6). Here we have applied the conclusion $\mu>3.7 \mathrm{Gev}$ obtained in last section.


Figure 6.6: $\mu$ dependence $J / \psi$ branching ratio with different LDME sets from four groups.

From Figure. 6.6, it can be seen that all the six sets of LDMEs can reasonably describe the experimental data in certain regions of renormalization scale $\mu$. With the condition that the theoretical predictions are exactly the same with the central value of experimental data, we summarize the fitted renormalization scale from $J / \psi$ production in Table. 6.4

| Set of LDMEs | Renormalization Scale $\mu$ (Gev) |
| :---: | :---: |
| Butenschön et al. | 3.76 |
| Gong et al. | 3.87 |
| Bodwin et al. | 7.33 |
| Chao et al. (Default) | 4.56 |
| Chao et al. (Set 2) | 5.09 |
| Chao et al. (Set 3) | 4.84 |

Table 6.4: The fitted renormalization scale $\mu$ using different sets of LDMEs with the experimental data.


Figure 6.7: $\mu$ dependence $h_{c}$ branching ratio with $\left\langle{ }^{h_{c}}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle=3\left\langle\chi_{c o}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle=0.321 \mathrm{GeV}^{5}$ and $\left\langle h_{c}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle=3\left\langle\chi_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle=0.015 \mathrm{GeV}^{3}$ (here we used the upper bound of our fitted value for $\left.\left\langle{ }^{\chi_{c o}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle\right)$.

## $6.5 h_{c}, \eta_{c}$

The LDMEs for $\eta_{c}, h_{c}$ production are determined through HQSS relations between the $\eta_{c}$ and $J / \psi\left(h_{c}\right.$ and $\left.\chi_{c 0}\right)$, which are given by

$$
\begin{align*}
\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{1} S_{0}^{[1]} /{ }^{1} S_{0}^{[8]}\right)\right\rangle & =\frac{1}{3}\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} S_{1}^{[1]} /{ }^{3} S_{1}^{[8]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right),  \tag{6.15}\\
\left\langle\mathcal{O}^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle & =\left\langle\mathcal{O}^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right),  \tag{6.16}\\
\left\langle\mathcal{O}_{c}^{\eta_{c}}\left({ }^{1} P_{1}^{[8]}\right)\right\rangle & =3\left\langle\mathcal{O}^{J / \psi}\left({ }^{3} P_{0}^{[8]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right),  \tag{6.17}\\
\left\langle\mathcal{O}^{h_{c}}\left({ }^{1} P_{1}^{[1]} /{ }^{1} S_{0}^{[8]}\right)\right\rangle & =3\left\langle\mathcal{O}^{\chi_{c 0}}\left({ }^{3} P_{0}^{[1]} /{ }^{3} S_{1}^{[8]}\right)\right\rangle\left(1+\mathcal{O}\left(v_{c}^{2}\right)\right), \tag{6.18}
\end{align*}
$$

where the production LDMEs for $J / \psi, \chi_{c 0}$ are already given in Table. (1.1, 1.2).
With the numerical results listed in Table. 6.3, we plot our theoretical predictions for the renormalization scale dependence of $\mathcal{B}\left(\Upsilon \rightarrow h_{c}+X\right)$ ( Figure. 6.7) and $\mathcal{B}\left(\Upsilon \rightarrow \eta_{c}+X\right)$ (Figure. 6.8), where we have added the feed-down contributions from $h_{c}$ to the branching ratios for $\eta_{c}$ production.

Obviously, the branching ratio for inclusive $h_{c}$ production via $\Upsilon$ decay is too small to be observed in current accumulated experimental data.

As for the branching ratio for inclusive $\eta_{c}$ production via $\Upsilon$ decay, the results highly depend on the value of the LDME $\left\langle{ }^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$. Therefore, we also separate the contributions from intermediate state $\eta_{c}\left({ }^{3} S_{1}^{[8]}\right)$ (see Fig. 6.9), from which we can conclude that the contributions


Figure 6.8: $\mu$ dependence $\eta_{c}$ branching ratio with different LDME sets from four groups.
from $\eta_{c}\left({ }^{1} S_{0}^{[1,8]}\right), \eta_{c}\left({ }^{1} P_{1}^{[8]}\right)$ intermediate states as well as feed-down from $h_{c}$ are small, and the large total branching ratios in Figure. $(6.9,6.10)$ almost entirely come from the $\eta_{c}\left({ }^{3} S_{1}^{[8]}\right)$ intermediate state as long as $\left\langle{ }^{\eta_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ is not too small.

To be concrete, here we give some combined predictions at fitted renormalization scale $\mu$ as well as fastest apparent convergence scale $\mu_{\mathbf{F A C}}$ from different LDME sets in Table.

On the experimental side, there are $102 \times 10^{6} \Upsilon(1 S)$ decay events collected by the Belle detector and $\eta_{c}$ can decay into stable hadrons with the fraction of few percent, which means the inclusive $\eta_{c}$ production via $\Upsilon$ decay can be measured if its branching ratio is in the magnitude of $10^{-3}$. On the theoretical side, the measurement of $\mathcal{B}\left(\Upsilon \rightarrow \eta_{c}+X\right)$ can largely reduce the uncertainty of the LDMEs $\left\langle{ }^{n_{c}}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ as well as $\left\langle{ }^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle$ with the validity of HQSS. Thus the discrepancy among those LDME sets in Table. $(1.1,1.2)$ can be further clarified.


Figure 6.9: $\mu$ dependence of $\eta_{c}$ branching ratio with or without contributions from $\eta_{c}\left({ }^{3} S_{1}^{[8]}\right)$ intermediate state using the different LDME sets.


Figure 6.10: Dependence of $\eta_{c}$ branching ratio on the value of the LDME $\left\langle\eta_{c}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$ with three different renormalization scales.

| Set of LDMEs | $\mu=\mu_{\mathrm{FAC}}=6.2 \mathrm{Gev}$ | fix $\mu$ by fitting $J / \psi$ data |
| :---: | :---: | :---: |
| Butenschön et al. | $7.1 \times 10^{-4}$ | $1.0 \times 10^{-3}$ |
| Gong et al. | $2.0 \times 10^{-3}$ | $3.0 \times 10^{-3}$ |
| Bodwin et al. | $2.1 \times 10^{-3}$ | $1.8 \times 10^{-3}$ |
| Chao et al. (Default) | $1.9 \times 10^{-3}$ | $2.3 \times 10^{-3}$ |
| Chao et al. (Set 2) | $1.1 \times 10^{-4}$ | $3.8 \times 10^{-5}$ |
| Chao et al. (Set 3) | $4.0 \times 10^{-4}$ | $3.9 \times 10^{-4}$ |

Table 6.5: Predictions of the branching ratios of inclusive $\eta_{c}$ production via $\Upsilon(1 S)$ decay with different sets of LDMEs at the fitted renormalization scale in $J / \psi$ branching ratio as well as $\mu=\mu_{\mathrm{FAC}}=6.2 \mathrm{Gev}$.

## Chapter 7

## Summary and Outlook

More than four decades have past since the discovery of the $J / \psi$ particle, the mechanism of quarkonium production somehow still remains mysterious. Despite the NRQCD factorization have achieved tremendous success in describing both quarkonium decay and production, none of the CO LDME sets extracted from various experimental environments based on complete NLO calculations by different groups is able to describe all of the studied $J / \psi$ yield and polarization data, which challenges the universality of LDMEs.

While there exists giant obstacles in performing higher order (NNLO and NLO with relativistic corrections), more production observables are proposed to explore the mechanism of quarkonium production, among which the double quarkonia hadroproduction attracts much attention. However, we find that for double quarkonia involved processes, the current version of NRQCD factorization formalism breaks down even at tree level, which is proven by two explicit examples. To solve this problem, we

1. reformulate NRQCD factorization in covariant formalism;
2. introduce a set of four-fermion operators and double quarkonium LDMEs to describe double quarkonia production or single quarkonium production via heavier quarkonium decay;
3. calculate the loop corrections for the newly introduced double quarkonium LDMEs and relate them to the single quarkonium LDMEs;
4. give new NRQCD factorization formulas for double quarkonia involved processes;
5. show how to calculate the contributions from the non-trivial interferences between different initial and final intermediate states.

The new NRQCD factorization formulas for double quarkonia involved processes are much more complicated than the NRQCD factorization for single quarkonim involved processes due to the interferences between different initial and final intermediate states of these two quarkonia.

We have also systematically investigated inclusive production of charmonium ( $\eta_{c}, J / \psi, h_{c}, \chi_{c J}$ ) production via $\Upsilon(1 S)$ decay. Our calculation of the SDCs includes all QCD contributions from ${ }^{1} S_{0}^{[1,8]},{ }^{3} S_{1}^{[1,8]},{ }^{1} P_{1}^{[1,8]},{ }^{3} P_{J}^{[1,8]}$ intermediate states for charmonia up to $\mathcal{O}\left(\alpha_{s}^{5}\right)$ as well as some important QED contributions, where the real and virtual corrections for the $\mathcal{O}\left(\alpha_{s}^{4}\right)$ process $b \bar{b}\left({ }^{3} S_{1}^{[1]}\right) \rightarrow c \bar{c}\left({ }^{3} S_{1}^{[8]}\right)+g g$ are calculated. For the intermediate states of $\Upsilon$ decay, only the CS
channel is considered since the CO contributions are suppressed by a factor of $v_{b}^{4} \simeq 0.01$. With our numerical results, we
6. find that the existing fitted CO LDME of $\chi_{c J}$ production from hadroproduction either under estimates or overshoots the experimental data of $\mathcal{B}\left(\Upsilon \rightarrow \chi_{c 1}+X\right)$ and $\mathcal{B}\left(\Upsilon \rightarrow \chi_{c 2}+X\right)$;
7. refit the CO LDME $\left\langle{ }^{\chi}{ }_{c 0}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$, which lies in the intermediate value between existing fitted values;
8. find that for inclusive $J / \psi$ production via $\Upsilon$ decay, the experimental data can be reasonably well described with all different sets of LDME in certain region of renormalization scale;
9. obtain the upper bound $\left(<6 \times 10^{-5}\right)$ of the branching ratio for $h_{c}$ production, which indicates that it is too small to be observed in current accumulated $\Upsilon$ decay events;
10. find that the branching ratio of inclusive $\eta_{c}$ production almost completely depends on the value of the LDME $\left\langle\eta_{c}\left({ }^{3} S_{1}^{[8]}\right)\right\rangle$, where most of the LDME sets predict the branching ratio can be in order of $10^{-3}$ provided that the HQSS are valid, which may indicate that the branching ratio can be measured. In other words, the measured branching ratio of inclusive $\eta_{c}$ production via $\Upsilon(1 S)$ decay may give significant constraint to the LDME $\left\langle{ }^{J / \psi}\left({ }^{1} S_{0}^{[8]}\right)\right\rangle$.

For the phenomenological consequences of interference contributions, there are many processes can be investigated such as inclusive $J / \psi+J / \psi, J / \psi+\chi_{c J}, J / \psi+\Upsilon$ hadron production at next-to-leading order and inclusive $J / \psi, \chi_{c J}$ production via $\chi_{b J}$ decay. All of these processes are phenomenologically important. For the former case, double-quarkonium hadron production have been continuously measured in the last few years, for the later case, due to the large accumulation of $\Upsilon(n S)$ data in $B$ factories, the branching ratios of $\chi_{b J} \rightarrow J / \psi / \chi_{c J}+X$ might be measured.

## Appendix A

## Notations and Kinematics

## A. 1 Notations

Through out our calculation and discussion, the notation $v$ or $v_{i}$ is frequently used. In some cases $v$ represents a time-like unit vector, which is connected with the total momentum $P_{i}$ of a quark pair $Q_{i} \bar{Q}_{i}$ through the definition $v_{i}^{\mu} \equiv \frac{P_{i}^{\mu}}{m_{H_{i}}}$, where $m_{H_{i}}$ is the mass of quark pair $Q_{i} \bar{Q}_{i}$. In the other cases, $v$ or $v_{Q}$ represents the relative velocity of quark $Q$ and anti-quark $\bar{Q}$ in the rest frame of $Q \bar{Q}$ pair. We specify the meaning of $v$ where some confusion may exist.

The polarization, color factors $N_{\text {pol }}(n), N_{n}, N_{\text {col }}$ for intermediate state $n$ are also frequently used, which are defined as

$$
\begin{align*}
N_{\mathrm{pol}}\left({ }^{1} S_{0}\right) & =N_{\mathrm{pol}}\left({ }^{3} P_{0}\right)=1, N_{\mathrm{pol}}\left({ }^{3} S_{1}\right)=N_{\mathrm{pol}}\left({ }^{1} P_{1}\right)=D-1,  \tag{A.1}\\
N_{\mathrm{pol}}\left({ }^{3} P_{1}\right) & =\frac{(D-1)(D-2)}{2}, N_{\mathrm{pol}}\left({ }^{3} P_{2}\right)=\frac{(D+1)(D-2)}{2},  \tag{A.2}\\
N_{n} & =\sqrt{2 N_{c}}, 1 \text { for CS states and CO states, respectively, }  \tag{A.3}\\
N_{\mathrm{col}} & =1, N_{c}^{2}-1 \text { for CS states and CO states, respectively. } \tag{A.4}
\end{align*}
$$

For the Dirac matrices $\gamma^{\mu}$, $\gamma^{5}$, we choose the Dirac basis, where

$$
\gamma^{0}=\left(\begin{array}{cc}
1 & 0  \tag{A.5}\\
0 & -1
\end{array}\right), \gamma^{i}=\left(\begin{array}{cc}
0 & \sigma^{i} \\
-\sigma^{i} & 0
\end{array}\right), \gamma_{5}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right),
$$

We distinguish indice and index, where for color matrix $T_{i j}^{a}$ and Dirac matrix $\gamma_{i j}^{\mu}, i, j$ are color, Dirac indices while $a, \mu$ are color, Dirac indexes, respectively.

## A. 2 Kinematics

In our calculation of inclusive charmonia production via $\Upsilon(1 S)$ decay, the total momenta of $b \bar{b}, c \bar{c}$ pairs are always labeled as $P_{1}, P_{2}$, respectively. The mass of $b, c$ quarks are labeled as $m_{b}, m_{c}$, and we define $r=\frac{m_{c}}{m_{b}}$. And the kinematics of multi-body decay are summarized as follows:

1. For 2-body decay $b \bar{b}\left(P_{1}\right) \rightarrow c \bar{c}\left(P_{2}\right)+g / \gamma\left(k_{3}\right)$, we have the 2-body phase space

$$
\begin{equation*}
\mathrm{PS}_{1 \rightarrow 2}=\frac{m_{b}^{2}-m_{c}^{2}}{8 \pi m_{b}^{2}} \tag{A.6}
\end{equation*}
$$

2. For 3-body decay $b \bar{b}\left(P_{1}\right) \rightarrow c \bar{c}\left(P_{2}\right)+g\left(k_{3}\right)+g\left(k_{4}\right)$, we define

$$
\begin{equation*}
s \equiv\left(P_{1}-k_{4}\right)^{2}, t \equiv\left(P_{1}-P_{2}\right)^{2}, u \equiv\left(P_{1}-k_{3}\right)^{2}, \tag{A.7}
\end{equation*}
$$

so that $s+u+t=4 m_{b}^{2}+4 m_{c}^{2}$. For convenience, we also introduce

$$
\begin{align*}
& s_{c} \equiv s-4 m_{c}^{2}, \quad s_{b} \equiv 4 m_{b}^{2}-s=2 P_{1} \cdot k_{4}=4 m_{b}^{2} x_{3} \\
& u_{c} \equiv u-4 m_{c}^{2}, \quad u_{b} \equiv 4 m_{b}^{2}-u=2 P_{1} \cdot k_{3}=4 m_{b}^{2} x_{2}, \\
& t^{\prime} \equiv 4 m_{b}^{2}+4 m_{c}^{2}-t=s+u=2 P_{1} \cdot P_{2}=4 m_{b}^{2} x_{1} . \tag{A.8}
\end{align*}
$$

Then in $D=4-2 \epsilon$ dimensions, the 3-body phase space is given by

$$
\begin{align*}
d \mathrm{PS}_{1 \rightarrow 3}= & \frac{\left(8 \pi m_{b}\right)^{2 \epsilon}}{512 \pi^{3} m_{b}^{2} \Gamma(2-2 \epsilon)}\left[\left(4 m_{b}^{2}-4 m_{c}^{2}-s_{b}-u_{b}\right)\right. \\
& \left.\times\left(4 m_{b}^{2}\left(s_{b}+u_{b}+4 m_{c}^{2}\right)-s_{b} u_{b}-16 m_{b}^{4}\right)\right]^{-\epsilon} d s_{b} d u_{b}  \tag{A.9}\\
= & \frac{\pi^{2 \epsilon} m_{b}^{2-4 \epsilon}}{32 \pi^{3} \Gamma(2-2 \epsilon)}\left[\left(1+r^{2}-x_{1}\right)\left(a_{1}+a_{2}-x_{2}\right)\left(x_{2}+a_{1}-a_{2}\right)\right]^{-\epsilon} d x_{1} d x_{2}, \tag{A.10}
\end{align*}
$$

with

$$
\begin{equation*}
a_{1}=\frac{\sqrt{x_{1}^{2}-4 r^{2}}}{2}, a_{2}=\frac{2-x_{1}}{2} . \tag{A.11}
\end{equation*}
$$

And the corresponding limits of the phase space integration are given by

$$
\begin{equation*}
1+r^{2}>x_{1}>2 r, a_{2}+a_{1}>x_{2}>a_{2}-a_{1} \tag{A.12}
\end{equation*}
$$

3. The phase space for a general 4-body decay process $P(M) \rightarrow k_{1}\left(m_{1}\right)+k_{2}\left(m_{2}\right)+$ $k_{3}\left(m_{3}\right)+k_{4}\left(m_{4}\right)$ can be written in covariant form [78]:

$$
\begin{align*}
d \mathrm{PS}_{1 \rightarrow 4}= & \frac{1}{(2 \pi)^{8}} \frac{\pi^{2}}{4} \int_{\left(m_{2}+m_{3}+m_{4}\right)^{2}}^{\left(M-m_{1}\right)^{2}} d s_{1} \int_{\left(m_{3}+m_{4}\right)^{2}}^{\left(\sqrt{\left.s_{1}-m_{2}\right)^{2}}\right.} d s_{2} \\
& \times \int_{u_{1-}}^{u_{1+}} \frac{d u_{1}}{\left[\lambda\left(M^{2}, s_{2}, s_{2}^{\prime}\right) \lambda\left(M^{2}, m_{2}^{2}, u_{1}\right)\right]^{1 / 2}\left(1-\xi_{2}^{2}\right)^{1 / 2}} \\
& \times \int_{u_{2-}}^{u_{2+}} \frac{d u_{2}}{\left[\lambda\left(M^{2}, m_{3}^{2}, u_{2}\right)\right]^{1 / 2}\left(1-\eta_{2}^{2}\right)^{1 / 2}} \int_{t_{2-}}^{t_{2+}} \frac{d t_{2}}{\left(1-\zeta_{2}^{2}\right)^{1 / 2}}, \tag{A.13}
\end{align*}
$$

with

$$
\begin{align*}
& s_{1}=\left(P-k_{1}\right)^{2}, s_{2}=\left(P-k_{1}-k_{3}\right)^{2}, u_{1}=\left(P-k_{3}\right)^{2}, u_{2}=\left(P-k_{4}\right)^{2}, \\
& t_{2}=\left(P-k_{3}-k_{4}\right)^{2}, s_{1}^{\prime}=m_{1}^{2}, s_{2}^{\prime}=s_{2}+M^{2}+m_{1}^{2}+m_{2}^{2}-s_{1}-u_{1}, \\
& t_{1}^{\prime}=m_{2}^{2}, t_{2}^{\prime}=t_{2}+M^{2}+m_{2}^{2}+m_{3}^{2}-u_{1}-u_{2}, \tag{A.14}
\end{align*}
$$

and

$$
\begin{align*}
& \lambda(a, b, c)=a^{2}+b^{2}+c^{2}-2 a b-2 a c-2 b c  \tag{A.15}\\
& \xi_{2}=\frac{\left(M^{2}+s_{2}^{\prime}-s_{2}\right)\left(M^{2}+t_{1}^{\prime}-t_{1}\right)-2 M^{2}\left(s_{2}^{\prime}+t_{1}^{\prime}-m_{1}^{2}\right)}{\left[\lambda\left(M^{2}, s_{2}, s_{2}^{\prime}\right) \lambda\left(M^{2}, t_{1}^{\prime}, t_{1}\right)\right]^{1 / 2}},  \tag{A.16}\\
& \eta_{2}=\frac{2 M^{2}\left(s_{2}+m_{3}^{2}-m_{4}^{2}\right)-\left(M^{2}-s_{2}^{\prime}+s_{2}\right)\left(M^{2}+m_{3}^{2}-u_{2}\right)}{\left[\lambda\left(M^{2}, s_{2}, s_{2}^{\prime}\right) \lambda\left(M^{2}, m_{3}^{2}, u_{2}\right)\right]^{1 / 2}},  \tag{A.17}\\
& \omega_{2}=\frac{2 M^{2}\left(t_{1}+m_{3}^{2}-t_{2}\right)-\left(M^{2}+t_{1}-t_{1}^{\prime}\right)\left(M^{2}+m_{3}^{2}-u_{2}\right)}{\left[\lambda\left(M^{2}, m_{3}^{2}, u_{2}\right) \lambda\left(M^{2}, t_{1}^{\prime}, t_{1}\right)\right]^{1 / 2}},  \tag{A.18}\\
& \zeta_{2}=\left(\omega_{2}-\xi_{2} \eta_{2}\right)\left[\left(1-\xi_{2}^{2}\right)\left(1-\eta_{2}^{2}\right)\right]^{-1 / 2} \tag{A.19}
\end{align*}
$$

The limits of the integrations variables are given by

$$
\begin{align*}
u_{1 \pm}= & M^{2}+m_{2}^{2}-\frac{\left.\left(s_{1}+m_{2}^{2}-s_{2}\right)\left(M^{2}+s_{1}-s_{1}^{\prime}\right)\right)}{2 s_{1}} \\
& \pm \frac{\left[\lambda\left(M^{2}, s_{1}, s_{1}^{\prime}\right) \lambda\left(s_{1}, m_{2}^{2}, s_{2}\right)\right]^{1 / 2}}{2 s_{1}},  \tag{A.20}\\
u_{2 \pm}= & M^{2}+m_{3}^{2}-\frac{\left.\left(s_{2}+m_{3}^{2}-m_{4}^{2}\right)\left(M^{2}+s_{2}-s_{2}^{\prime}\right)\right)}{2 s_{2}} \\
& \pm \frac{\left[\lambda\left(M^{2}, s_{2}, s_{2}^{\prime}\right) \lambda\left(s_{2}, m_{3}^{2}, m_{4}^{2}\right)\right]^{1 / 2}}{2 s_{2}},  \tag{A.21}\\
t_{2 \pm}= & t_{1}+m_{3}^{2}-\frac{\left(M^{2}+m_{3}^{2}-u_{2}\right)\left(M^{2}+t_{1}-t_{1}^{\prime}\right)}{2 M^{2}} \\
& +\frac{\left[\lambda\left(M^{2}, m_{3}^{2}, u_{2}\right) \lambda\left(M^{2}, t_{1}^{\prime}, t_{1}\right)\right]^{1 / 2}}{2 M^{2}}\left\{-\xi_{2} \eta_{2} \pm\left[\left(1-\xi_{2}^{2}\right)\left(1-\eta_{2}^{2}\right)\right]^{1 / 2}\right\} . \tag{A.22}
\end{align*}
$$

In the case of process $b \bar{b}\left(P_{1}\right) \rightarrow c \bar{c}\left(P_{2}\right)+g\left(k_{3}\right)+g\left(k_{4}\right)+g\left(k_{5}\right), M=2 m_{b}, m_{1}=2 m_{c}, m_{3}=$ $m_{4}=m_{5}=0$, then we can simplify above expressions as:

$$
\begin{align*}
d \mathrm{PS}_{1 \rightarrow 4}= & \frac{1}{(2 \pi)^{8}} \frac{\pi^{2}}{4} \int_{0}^{\left(2 m_{b}-2 m_{c}\right)^{2}} d s_{1} \int_{0}^{s_{1}} d s_{2} \\
& \times \int_{u_{1-}}^{u_{1+}} \frac{d u_{1}}{\left[\lambda\left(4 m_{b}^{2}, s_{2}, s_{2}^{\prime}\right)\right]^{1 / 2}\left(1-\xi_{2}^{2}\right)^{1 / 2}\left(4 m_{b}^{2}-u_{1}\right)} \\
& \times \int_{u_{2-}}^{u_{2+}} \frac{d u_{2}}{\left(1-\eta_{2}^{2}\right)^{1 / 2}\left(4 m_{b}^{2}-u_{2}\right)} \int_{t_{2-}}^{t_{2+}} \frac{d t_{2}}{\left(1-\zeta_{2}^{2}\right)^{1 / 2}} \tag{A.23}
\end{align*}
$$

with

$$
\begin{align*}
& s_{1}=\left(P_{1}-P_{2}\right)^{2}, s_{2}=\left(P_{1}-P_{2}-k_{4}\right)^{2}, u_{1}=\left(P_{1}-k_{4}\right)^{2}, \\
& u_{2}=\left(P_{1}-k_{5}\right)^{2}, t_{2}=\left(P_{1}-k_{4}-k_{5}\right)^{2}, s_{1}^{\prime}=4 m_{c}^{2} \\
& s_{2}^{\prime}=4 m_{b}^{2}+4 m_{c}^{2}+s_{2}-s_{1}-u_{1}, t_{2}^{\prime}=4 m_{b}^{2}+t_{2}-u_{1}-u_{2}, \tag{A.24}
\end{align*}
$$

and

$$
\begin{align*}
\xi_{2} & =\frac{\left(4 m_{b}^{2}+s_{2}^{\prime}-s_{2}\right)\left(4 m_{b}^{2}-u_{1}\right)-8 m_{b}^{2}\left(s_{2}^{\prime}-4 m_{c}^{2}\right)}{\left[\lambda\left(4 m_{b}^{2}, s_{2}, s_{2}^{\prime}\right)\right]^{1 / 2}\left(4 m_{b}^{2}-u_{1}\right)}  \tag{A.25}\\
\eta_{2} & =\frac{8 m_{b}^{2} s_{2}-\left(4 m_{b}^{2}-s_{2}^{\prime}+s_{2}\right)\left(4 m_{b}^{2}-u_{2}\right)}{\left[\lambda\left(4 m_{b}^{2}, s_{2}, s_{2}^{\prime}\right)\right]^{1 / 2}\left(4 m_{b}^{2}-u_{2}\right)}  \tag{A.26}\\
\omega_{2} & =\frac{8 m_{b}^{2}\left(u_{1}-t_{2}\right)-\left(4 m_{b}^{2}+u_{1}\right)\left(4 m_{b}^{2}-u_{2}\right)}{\left(4 m_{b}^{2}-u_{1}\right)\left(4 m_{b}^{2}-u_{2}\right)}  \tag{A.27}\\
\zeta_{2} & =\left(\omega_{2}-\xi_{2} \eta_{2}\right)\left[\left(1-\xi_{2}^{2}\right)\left(1-\eta_{2}^{2}\right)\right]^{-1 / 2} \tag{A.28}
\end{align*}
$$

The limits of the integrations variables are given by

$$
\begin{align*}
u_{1 \pm}= & 4 m_{b}^{2}-\frac{\left(s_{1}-s_{2}\right)\left(4 m_{b}^{2}+s_{1}-4 m_{c}^{2}\right)}{2 s_{1}} \pm \frac{\left[\lambda\left(4 m_{b}^{2}, s_{1}, 4 m_{c}^{2}\right)\right]^{1 / 2}\left(s_{1}-s_{2}\right)}{2 s_{1}}  \tag{A.29}\\
u_{2 \pm}= & 4 m_{b}^{2}-\frac{\left(4 m_{b}^{2}+s_{2}-s_{2}^{\prime}\right)}{2} \pm \frac{\left[\lambda\left(4 m_{b}^{2}, s_{2}, s_{2}^{\prime}\right)\right]^{1 / 2}}{2},  \tag{A.30}\\
t_{2 \pm}= & u_{1}-\frac{\left(4 m_{b}^{2}-u_{2}\right)\left(4 m_{b}^{2}+u_{1}\right)}{8 m_{b}^{2}}+\frac{\left(4 m_{b}^{2}-u_{1}\right)\left(4 m_{b}^{2}-u_{2}\right)}{8 m_{b}^{2}} \\
& \times\left\{-\xi_{2} \eta_{2} \pm\left[\left(1-\xi_{2}^{2}\right)\left(1-\eta_{2}^{2}\right)\right]^{1 / 2}\right\} . \tag{A.31}
\end{align*}
$$

## Appendix B

## Definitions of the-Single Quarkonium LDMEs

## B. 1 Single-Quarkonium LDMEs in Non-Covariant Form

For the production of quarkonium $H$ :

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle & =\langle 0| \chi^{\dagger} \psi \mathcal{P}^{H} \psi^{\dagger} \chi|0\rangle,  \tag{B.1a}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle & =\langle 0| \chi^{\dagger} \sigma^{i} \psi \mathcal{P}^{H} \psi^{\dagger} \sigma^{i} \chi|0\rangle,  \tag{B.1b}\\
\left\langle\mathcal{O}^{H}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle & =\langle 0| \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right)^{i} \psi \mathcal{P}^{H} \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right)^{i} \chi|0\rangle,  \tag{B.1c}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle & =\frac{1}{3}\langle 0| \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \sigma\right) \psi \mathcal{P}^{H} \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \sigma\right) \chi|0\rangle,  \tag{B.1d}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{1}^{[1]}\right)\right\rangle & =\frac{1}{2}\langle 0| \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \times \sigma\right)^{i} \psi \mathcal{P}^{H} \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \times \sigma\right)^{i} \chi|0\rangle,  \tag{B.1e}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{2}^{[1]}\right)\right\rangle & =\langle 0| \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}{ }^{(i} \sigma^{j)}\right) \psi \mathcal{P}^{H} \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\left(i \sigma^{j)}\right) \chi|0\rangle,\right. \tag{B.1f}
\end{align*}
$$

and for the decay of quarkonium $H$ :

$$
\begin{align*}
\left\langle\mathcal{O}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle_{H} & =\langle H| \psi^{\dagger} \chi \chi^{\dagger} \psi|H\rangle,  \tag{B.2a}\\
\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{H} & =\langle H| \psi^{\dagger} \sigma \chi \cdot \chi^{\dagger} \sigma \psi|H\rangle,  \tag{B.2b}\\
\left\langle\mathcal{O}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle_{H} & =\langle H| \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right) \chi \cdot \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right) \psi|H\rangle,  \tag{B.2c}\\
\left\langle\mathcal{O}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle_{H} & =\frac{1}{3}\langle H| \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \sigma\right) \chi \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \sigma\right) \psi|H\rangle,  \tag{B.2d}\\
\left\langle\mathcal{O}\left({ }^{3} P_{1}^{[1]}\right)\right\rangle_{H} & =\frac{1}{2}\langle H| \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \times \sigma\right) \chi \cdot \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \times \sigma\right) \psi|H\rangle,  \tag{B.2e}\\
\left\langle\mathcal{O}\left({ }^{3} P_{2}^{[1]}\right)\right\rangle_{H} & =\langle H| \psi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}{ }^{(i} \sigma^{j)}\right) \chi \chi^{\dagger}\left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}^{(i}} \sigma^{j)}\right) \psi|H\rangle, \tag{B.2f}
\end{align*}
$$

where $\psi$ is the two component Pauli spinor field that annihilates a heavy quark, $\chi$ is the two component Pauli spinor field that creates a heavy anti-quark, $\sigma$ is the Pauli and color matrix, $\overleftrightarrow{\mathrm{D}}$
is the difference between the covariant derivative acting on the spinor to the right and on the spinor to the left: $\psi^{\dagger} \overleftrightarrow{\mathbf{D}} \chi=\psi^{\dagger}(\mathbf{D} \chi)-(\mathbf{D} \psi)^{\dagger} \chi$, and $\overleftrightarrow{\mathbf{D}}^{i} \sigma^{j)}=\left(\overleftrightarrow{\mathbf{D}}^{i} \sigma^{j}+\overleftrightarrow{\mathbf{D}}^{j} \sigma^{i}\right) / 2-\frac{\overleftrightarrow{\mathbf{D}} \cdot \sigma}{3}$ and

$$
\begin{equation*}
\mathcal{P}^{H}=\sum_{X}|H X\rangle\langle X H| . \tag{B.3}
\end{equation*}
$$

The CO production and decay LDMEs are defined simply by inserting two identical color matrices $T^{a}$ between the bilinear operators to the corresponding CS ones.

## B. 2 Single Quarkonium LDMEs in Covariant Form

For the production of quarkonium $H$ :

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle & =\langle 0| \mathcal{K}^{\dagger}\left[{ }^{1} S_{0}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}\left[{ }^{1} S_{0}^{[1]}\right]|0\rangle,  \tag{B.4a}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle & =\sum \varepsilon_{\mu^{\prime}} \varepsilon_{\mu}^{*}\langle 0| \mathcal{K}^{\dagger} \mu^{\prime}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu}\left[{ }^{3} S_{1}^{[1]}\right]|0\rangle \\
& =\langle 0| \mathcal{K}^{\dagger} \mu\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu}\left[{ }^{3} S_{1}^{[1]}\right]|0\rangle,  \tag{B.4b}\\
\left\langle\mathcal{O}^{H}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle & =\sum \varepsilon_{\mu^{\prime}} \varepsilon_{\mu}^{*}\langle 0| \mathcal{K}^{\dagger} \mu^{\prime}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu}\left[{ }^{1} P_{1}^{[1]}\right]|0\rangle \\
& =\langle 0| \mathcal{K}^{\dagger} \mu\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu}\left[{ }^{1} P_{1}^{[1]}\right]|0\rangle,  \tag{B.4c}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}} \varepsilon_{\mu \nu}^{*(0)}\langle 0| \mathcal{K}^{\dagger} \mu^{\prime} \nu^{\prime}\left[{ }^{3} P_{0}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{0}^{[1]}\right]|0\rangle \\
& =\langle 0| \mathcal{K}^{\dagger \mu \nu}\left[{ }^{3} P_{0}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu \nu}\left[{ }^{3} P_{0}^{[1]}\right]|0\rangle,  \tag{B.4d}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{1}^{[1]}\right)\right\rangle & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}, \varepsilon_{\mu \nu}^{*(1)}\langle 0| \mathcal{K}^{\dagger} \mu^{\prime} \nu^{\prime}\left[{ }^{3} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{1}^{[1]}\right]|0\rangle} \\
& =\langle 0| \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu \nu}\left[{ }^{3} P_{1}^{[1]}\right]|0\rangle,  \tag{B.4e}\\
\left\langle\mathcal{O}^{H}\left({ }^{3} P_{2}^{[1]}\right)\right\rangle & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}} \varepsilon_{\mu \nu}^{*(2)}\langle 0| \mathcal{K}^{\dagger} \mu^{\prime} \nu^{\prime}\left[{ }^{3} P_{2}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{2}^{[1]}\right]|0\rangle \\
& =\langle 0| \mathcal{K}^{\dagger \mu \nu}\left[{ }^{3} P_{2}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu \nu}\left[{ }^{3} P_{2}^{[1]}\right]|0\rangle ; \tag{B.4f}
\end{align*}
$$

and for the decay of quarkonium $H$ :

$$
\begin{align*}
\left\langle\mathcal{O}\left({ }^{1} S_{0}^{[1]}\right)\right\rangle_{H} & =\langle H| \mathcal{K}\left[{ }^{1} S_{0}^{[1]}\right] \mathcal{K}^{\dagger}\left[{ }^{1} S_{0}^{[1]}\right]|H\rangle,  \tag{B.5a}\\
\left\langle\mathcal{O}\left({ }^{3} S_{1}^{[1]}\right)\right\rangle_{H} & =\sum \varepsilon_{\mu^{\prime}} \varepsilon_{\mu}^{*}\langle H| \mathcal{K}^{\mu^{\prime}}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{K}^{\dagger} \mu\left[{ }^{3} S_{1}^{[1]}\right]|H\rangle \\
& =\langle H| \mathcal{K}_{\mu}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{K}_{\mu}^{\dagger}\left[{ }^{3} S_{1}^{[1]}\right]|H\rangle,  \tag{B.5b}\\
\left\langle\mathcal{O}\left({ }^{1} P_{1}^{[1]}\right)\right\rangle_{H} & =\sum \varepsilon_{\mu^{\prime}} \varepsilon_{\mu}^{*}\langle H| \mathcal{K}^{\mu^{\prime}}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{K}^{\dagger} \mu\left[{ }^{1} P_{1}^{[1]}\right]|H\rangle \\
& =\langle H| \mathcal{K}_{\mu}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{K}_{\mu}^{\dagger}\left[{ }^{1} P_{1}^{[1]}\right]|H\rangle,  \tag{B.5c}\\
\left\langle\mathcal{O}\left({ }^{3} P_{0}^{[1]}\right)\right\rangle_{H} & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}}^{(0)} \varepsilon_{\mu \nu}^{*(0)}\langle H| \mathcal{K}^{\mu^{\prime} \nu^{\prime}\left[{ }^{3} P_{0}^{[1]}\right] \mathcal{K}^{\dagger \mu \nu}\left[{ }^{3} P_{0}^{[1]}\right]|H\rangle} \\
& =\langle H| \mathcal{K}^{\mu \nu}\left[{ }^{[ } P_{0}^{[1]}\right] \mathcal{K}_{\mu \nu}^{\dagger}\left[{ }^{3} P_{0}^{[1]}\right]|H\rangle,  \tag{B.5d}\\
\left\langle\mathcal{O}\left({ }^{3} P_{1}^{[1]}\right)\right\rangle_{H} & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}}^{(1)} \varepsilon_{\mu \nu}^{*(1)}\langle H| \mathcal{K}^{\mu^{\prime} \nu^{\prime}}\left[{ }^{3} P_{1}^{[1]}\right] \mathcal{K}^{\dagger} \mu \nu\left[{ }^{3} P_{1}^{[1]}\right]|H\rangle \\
& =\langle H| \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{1}^{[1]}\right] \mathcal{K}_{\mu \nu}^{\dagger}\left[{ }^{3} P_{1}^{[1]}\right]|H\rangle,  \tag{B.5e}\\
\left\langle\mathcal{O}\left({ }^{3} P_{2}^{[1]}\right)\right\rangle_{H} & =\sum \varepsilon_{\mu^{\prime} \nu^{\prime}, \varepsilon_{\mu \nu}^{* 2}\langle H| \mathcal{K}^{\mu^{\prime} \nu^{\prime}\left[{ }^{3} P_{2}^{[1]}\right] \mathcal{K}^{\dagger \mu \nu}\left[{ }^{3} P_{2}^{[1]}\right]|H\rangle}} \\
& =\langle H| \mathcal{K}^{\mu \nu}\left[{ }^{3} P_{2}^{[1]}\right] \mathcal{K}_{\mu \nu}^{\dagger}\left[{ }^{3} P_{2}^{[1]}\right]|H\rangle . \tag{B.5f}
\end{align*}
$$

The CO production and decay LDMEs are obtained by replacing the CS heavy-quark bilinear $\mathcal{K}$ with the corresponding CO ones $\mathcal{K}^{a}$, which are defined in Section 3.2.

By choosing $v=(1,0,0$,$) , it can be directly proven that the LDMEs define in Eq.(B.4, B.5)$ are exactly the same with those defined in Eq.(B.1, B.2).

## Appendix C

## Loop Integrals in the Virtual Corrections of LDMEs

## Vanishing Loop Integrals

Since all denominators in the loop integrals presented here are homogeneous about the loop momentum $l$, all the loop integrals containing odd powers of $l$ vanish.

There are certain type of loop integrals which do not contain the denominator $\frac{1}{l^{2}}$, where the Feynman parametrization cannot applied explicitly. However, all these integrals are covariant, so we can freely choose the reference frames without changing the results. For this type of integrals, we keep $i \varepsilon$ explicitly and perform contour integration to integrate out the energy component. These integrals appear to be zero since we always can choose the contour which avoids all the poles.

Choosing $v=(1,0,0,0)$, we have

$$
\begin{align*}
& \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{(l \cdot v+i \varepsilon)^{\alpha}} \\
= & \int \frac{d^{D-1} \mathbf{l}}{(2 \pi)^{D}} \int d l_{0} \frac{1}{\left(l_{0}+i \varepsilon\right)^{\alpha}} \\
= & 0 . \tag{C.1}
\end{align*}
$$

Second, we consider $\int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l \cdot v_{1}+i \varepsilon\right)^{\alpha}\left(l \cdot v_{2}+i \varepsilon\right)^{\beta}}$, with $\alpha, \beta$ being positive integers and $v_{1} \neq v_{2}$. Choosing $v_{1}=(1,0,0,0)$, then $l \cdot v_{2}$ can be written as $\lambda l_{0}+a$, with $\lambda>0$ and $a$ being nonzero and real.

$$
\begin{align*}
& \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l \cdot v_{1}+i \varepsilon\right)^{\alpha}\left(l \cdot v_{2}+i \varepsilon\right)^{\beta}} \\
= & \int \frac{d^{D-1} \mathbf{l}}{(2 \pi)^{D}} \int d l_{0} \frac{1}{\left(l_{0}+i \varepsilon\right)^{\alpha}\left(\lambda l_{0}+a+i \varepsilon\right)^{\beta}} \\
= & 0 . \tag{C.2}
\end{align*}
$$

With Eq. (C.1, C.2), it can be easily seen that the following two integrals also vanish:

$$
\begin{align*}
& \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l^{2}-(l \cdot v)^{2}\right)(l \cdot v+i \varepsilon)^{2}} \\
= & \int \frac{d^{D-1} \mathbf{l}}{(2 \pi)^{D}} \int d l_{0} \frac{1}{\left(l_{0}+i \varepsilon\right)^{2}\left(-\mathbf{l}^{2}\right)} \\
= & 0 .  \tag{C.3}\\
& \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{\left(l^{2}-\left(l \cdot v_{1}\right)^{2}\right)\left(l \cdot v_{1}+i \varepsilon\right)\left(l \cdot v_{2}+i \varepsilon\right)} \\
= & \int \frac{d^{D-1} \mathbf{l}}{(2 \pi)^{D}} \int d l_{0} \frac{1}{\left(l_{0}+i \varepsilon\right)\left(\lambda l_{0}+a+i \varepsilon\right)\left(-\mathbf{l}^{2}\right)} \\
= & 0 . \tag{C.4}
\end{align*}
$$

## Non-vanishing Loop Integrals

Let us consider the following integration

$$
\begin{equation*}
\int_{0}^{1} d x x^{-1+2 \epsilon}(1-x)^{-1-2 \epsilon} \tag{C.5}
\end{equation*}
$$

which is both ultra-violet $(\epsilon>0)$ and infrared $(\epsilon<0)$ divergent. We then split the integral as

$$
\begin{align*}
& \int_{0}^{1} d x x^{-1+2 \epsilon}(1-x)^{-1-2 \epsilon} \\
= & \int_{0}^{a} d x x^{-1+2 \epsilon}(1-x)^{-1-2 \epsilon}+\int_{a}^{1} d x x^{-1+2 \epsilon}(1-x)^{-1-2 \epsilon} \\
= & \frac{\left(\frac{1}{a}-1\right)^{-2 \epsilon_{\mathrm{UV}}}}{2 \epsilon_{\mathrm{UV}}}-\frac{\left(\frac{1}{a}-1\right)^{-2 \epsilon_{\mathrm{IR}}}}{2 \epsilon_{\mathrm{IR}}} \\
= & \frac{1}{2}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right) . \tag{C.6}
\end{align*}
$$

Thus we have ( define $\omega=v_{1} \cdot v_{2}$ )

$$
\begin{align*}
I_{v} & \equiv \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}(l \cdot v)^{2}} \\
& =\int_{0}^{1} d x \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{4 \mu^{2} \Gamma(3)(1-x)}{\left[x l^{2}+2 \mu(1-x) l \cdot v\right]^{3}} \\
& =\frac{-4 i \Gamma(1+\epsilon)}{(4 \pi)^{\frac{D}{2}} \mu^{2 \epsilon}} \int_{0}^{1} d x x^{-1+2 \epsilon}(1-x)^{-1-2 \epsilon} \\
& =\frac{-i}{8 \pi^{2}}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right) \tag{C.7}
\end{align*}
$$

and

$$
\begin{align*}
I_{v_{1} v_{2}} \equiv & \equiv \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{1}{l^{2}\left(l \cdot v_{1}\right)\left(l \cdot v_{2}\right)} \\
= & \int_{0}^{1} d x_{1} \int_{0}^{1} d x_{2} \int_{0}^{1} d x_{3} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{4 \mu^{2} \Gamma(3) \delta\left(1-x_{1}-x_{2}-x_{3}\right)}{\left[x_{1} l^{2}+2 \mu\left(x_{2} l \cdot v_{1}+x_{3} l \cdot v_{2}\right)\right]^{3}} \\
= & \int_{0}^{1} d x_{1} \int_{0}^{1} d x_{2} \int_{0}^{1} d x_{3} \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{4 \mu^{2} \Gamma(3) \delta\left(1-x_{1}-x_{2}-x_{3}\right)}{x_{1}^{3}\left[l^{2}-\mu^{2}\left(x_{2}^{2}+x_{3}^{2}+2 x_{2} x_{3} \omega\right) / x_{1}^{2}\right]^{3}} \\
= & \frac{-i 2^{1-\epsilon} \Gamma(1+\epsilon)}{(4 \pi)^{\frac{D}{2}} \mu^{2 \epsilon}(1-\omega)^{1+\epsilon}} \int_{0}^{1} d x_{1} \int_{\frac{x_{1}-1}{2}}^{\frac{1-x_{1}}{2}} d x_{2} x_{1}^{2 \epsilon-1}\left[x_{2}^{2}+\frac{1+\omega}{1-\omega} \frac{\left(1-x_{1}\right)^{2}}{4}\right]^{-1-\epsilon} \\
= & \frac{i 4 \Gamma(1+\epsilon)}{(4 \pi)^{\frac{D}{2}} \mu^{2 \epsilon}\left(\omega^{2}-1\right)(\epsilon-1) \epsilon}[(\omega+1)(\omega(\epsilon-1)-\epsilon+2) \\
& \left.-2_{2} F_{1}\left(1, \frac{1}{2}-\epsilon ;-\frac{1}{2} ; \frac{\omega-1}{\omega+1}\right)\right] \int_{0}^{1} d x_{1} x_{1}^{-1+2 \epsilon}\left(1-x_{1}\right)^{-1-2 \epsilon} \\
= & \frac{-i \ln \left(\sqrt{\omega^{2}-1}+\omega\right)}{8 \pi^{2}} \frac{1}{\sqrt{\omega^{2}-1}}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right) . \tag{C.8}
\end{align*}
$$

We decompose the following tensor integrals as

$$
\begin{align*}
I_{v_{1}^{2} v_{2}}^{\mu} & \equiv \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{l^{\mu}}{l^{2}\left(l \cdot v_{1}\right)^{2}\left(l \cdot v_{2}\right)}=v_{1}^{\mu} I_{v_{1}^{2} v_{2}}^{v_{1}}+v_{2}^{\mu} I_{v_{1} v_{2}}^{v_{2}},  \tag{C.9}\\
I_{v_{1} v_{2}^{2}}^{\mu} & \equiv \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{l^{\mu}}{l^{2}\left(l \cdot v_{1}\right)\left(l \cdot v_{2}\right)^{2}}=v_{1}^{\mu} I_{v_{1} v_{2}^{2}}^{v_{1}}+v_{2}^{\mu} I_{v_{1} v_{2}^{2}}^{v_{2}}, \tag{C.10}
\end{align*}
$$

$$
\begin{align*}
I_{v_{1}^{2} v_{2}^{2}}^{\mu} & \equiv \int \frac{d^{D} l}{(2 \pi)^{D}} \frac{l^{\mu} l^{\nu}}{l^{2}\left(l \cdot v_{1}\right)^{2}\left(l \cdot v_{2}\right)^{2}} \\
& =g^{\mu \nu} I_{v_{1}^{2} v_{2}^{2}}^{g}+\left(v_{1}^{\mu} v_{2}^{\nu}+v_{1}^{\nu} v_{2}^{\mu}\right) I_{v_{1}^{1} v_{2}^{2}}^{v_{1} v_{2}}+v_{1}^{\mu} v_{1}^{\nu} I_{v_{1}^{2} v_{2}^{2}}^{v_{1}^{2}}+v_{2}^{\mu} v_{2}^{\nu} I_{v_{1}^{2} v_{2}^{2}}^{v_{2}^{2}} \tag{C.11}
\end{align*}
$$

Contracting with $v_{1}, v_{2}, g_{\mu \nu}$, results in

$$
\begin{align*}
& \left\{\begin{array}{l}
I_{v_{1} v_{2}}=I_{v_{1}^{2}}^{v_{1}}+\omega I_{v_{1}^{2}}^{v_{2}} v_{2} \\
I_{v}=I_{v_{1}^{2}}^{v_{2}}+\omega I_{v_{1}^{2}}^{v_{1}} v_{2}
\end{array}\right.  \tag{C.12}\\
& \left\{\begin{array}{l}
I_{v_{1} v_{2}}=I_{v_{1} v_{2}^{2}}^{v_{2}}+\omega I_{v_{1} v_{2}^{2}}^{v_{1}} \\
I_{v}=I_{v_{1} v_{2}^{2}}^{v_{1}}+\omega I_{v_{1} v_{2}^{2}}^{v_{2}},
\end{array}\right. \tag{C.13}
\end{align*}
$$

Solve Eq.(C.12, C.13, C.14), then we get

$$
\begin{align*}
& I_{v_{1}^{2} v_{2}}^{v_{1}}=I_{v_{1} v_{2}^{2}}^{v_{2}}=\frac{\omega I_{v}-I_{v_{1} v_{2}}}{\omega^{2}-1}  \tag{C.15}\\
& I_{v_{1}^{2} v_{2}}^{v_{2}}=I_{v_{1} v_{2}^{2}}^{v_{1}}=-I_{v_{1}^{2} v_{2}^{2}}^{g}=\frac{\omega I_{v_{1} v_{2}}-I_{v}}{\omega^{2}-1}  \tag{C.16}\\
& I_{v_{1}^{2} v_{2}^{2}}^{v_{2}^{2}}=I_{v_{1}^{2} v_{2}^{2}}^{v_{2}^{2}}=\frac{\left(\omega^{2}+2\right) I_{v}-3 \omega I_{v_{1} v_{2}}}{\left(\omega^{2}-1\right)^{2}}  \tag{C.17}\\
& I_{v_{1}^{2} v_{2}^{2} v_{2}^{2}}^{v_{2}}=\frac{\left(2 \omega^{2}+1\right) I_{v_{1} v_{2}-3 \omega I_{v}}^{\left(\omega^{2}-1\right)^{2}}}{} . \tag{C.18}
\end{align*}
$$

## Appendix D

## Summary of One-loop Corrections to the LDMEs

Since the one-loop corrections to all the LDMEs only differ in color and Dirac structures, we summarize the color and Dirac algebra we have applied.

## Color and Dirac Algebra

The following color algebra

$$
\begin{equation*}
T^{a} T^{e}=\frac{\delta^{a e}}{2 C_{A}}+\frac{1}{2}\left(d^{a e c}+i f^{a e c}\right) T^{c}=\frac{\delta^{a e}}{2 C_{A}}+2 \operatorname{Tr}\left(T^{a} T^{e} T^{c}\right) T^{c} \tag{D.1}
\end{equation*}
$$

where

$$
\begin{equation*}
d^{a e c}=2 \operatorname{Tr}\left(T^{a} T^{e} T^{c}+T^{e} T^{a} T^{c}\right), \quad i f^{a e c}=2 \operatorname{Tr}\left(T^{a} T^{e} T^{c}-T^{e} T^{a} T^{c}\right), \tag{D.2}
\end{equation*}
$$

color symmetry

$$
\begin{equation*}
\langle 0| \mathcal{K}^{\dagger}[n] \mathcal{P}^{H} \mathcal{K}^{a}[n]|0\rangle=\langle 0| \mathcal{K}^{\dagger}[n] \mathcal{P}^{H} \mathcal{K}^{c}[n]|0\rangle \frac{\delta^{a b}}{N_{c}^{2}-1}, \tag{D.3}
\end{equation*}
$$

and the spatial rotation symmetry

$$
\begin{align*}
\langle 0| \mathcal{K}_{\mu^{\prime}}^{\dagger}\left[{ }^{3} S_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu}\left[{ }^{3} S_{1}^{[1]}\right]|0\rangle & =\frac{1}{D-1} \Pi_{\mu \mu^{\prime}}\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[1]}\right]\right\rangle,  \tag{D.4a}\\
\langle 0| \mathcal{K}_{\mu^{\prime}}^{\dagger}\left[{ }^{1} P_{1}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu}\left[{ }^{1} P_{1}^{[1]}\right]|0\rangle & =\frac{1}{D-1} \Pi_{\mu \mu^{\prime}}\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[1]}\right]\right\rangle,  \tag{D.4b}\\
\langle 0| \mathcal{K}_{\mu^{\prime} \nu^{\prime}}^{\dagger}\left[{ }^{3} P_{J}^{[1]}\right] \mathcal{P}^{H} \mathcal{K}_{\mu \nu}\left[{ }^{3} P_{J}^{[1]}\right]|0\rangle & =\varepsilon_{\mu \nu}^{(J)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*(J)} \frac{\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle}{N_{J}}, \tag{D.4c}
\end{align*}
$$

are frequently applied in the decoupling of the color and Dirac indexes.

When both S-wave states are in CS , which includes CS single-quarkonium and CS-CS double-quarknium cases, the color structure is simply given by

$$
\begin{equation*}
T^{e} \otimes T^{e} \tag{D.5}
\end{equation*}
$$

When at least one of the S -wave states are in CO, we can decompose the following color structures as

$$
\begin{align*}
& T^{a} T^{e} \otimes T^{e}=\frac{1}{2 C_{A}}\left(\mathbf{1} \otimes T^{a}\right)+2 \operatorname{Tr}\left(T^{a} T^{e} T^{c}\right)\left(T^{c} \otimes T^{e}\right), \\
& T^{e} T^{a} \otimes T^{e}=\frac{1}{2 C_{A}}\left(\mathbf{1} \otimes T^{a}\right)+2 \operatorname{Tr}\left(T^{e} T^{a} T^{c}\right)\left(T^{c} \otimes T^{e}\right), \\
& T^{a} T^{b} \otimes T^{a} T^{b}=\frac{C_{F}}{2 C_{A}}(\mathbf{1} \otimes \mathbf{1})-\frac{1}{C_{A}}\left(T^{c} \otimes T^{c}\right), \\
& T^{a} T^{b} \otimes T^{b} T^{a}=\frac{C_{F}}{2 C_{A}}(\mathbf{1} \otimes \mathbf{1})+\left(\frac{C_{A}}{2}-\frac{1}{C_{A}}\right)\left(T^{c} \otimes T^{c}\right), \\
& T^{a} T^{e} \otimes T^{b^{\prime}} T^{e}=\frac{\delta^{a b^{\prime}}}{\left(2 C_{A}\right)^{2}}(\mathbf{1} \otimes \mathbf{1})+\frac{\operatorname{Tr}\left(T^{a} T^{b^{\prime}} T^{c}\right)}{C_{A}}\left(T^{c} \otimes \mathbf{1}\right)+\frac{\operatorname{Tr}\left(T^{b^{\prime}} T^{a} T^{c}\right)}{C_{A}}\left(\mathbf{1} \otimes T^{c}\right) \\
& +4 \operatorname{Tr}\left(T^{a} T^{e} T^{c}\right) \operatorname{Tr}\left(T^{b^{\prime}} T^{e} T^{c^{\prime}}\right)\left(T^{c} \otimes T^{c^{\prime}}\right),  \tag{D.8a}\\
& T^{a} T^{e} \otimes T^{e} T^{b^{\prime}}=\frac{\delta^{a b^{\prime}}}{\left(2 C_{A}\right)^{2}}(\mathbf{1} \otimes \mathbf{1})+\frac{\operatorname{Tr}\left(T^{a} T^{b^{\prime}} T^{c}\right)}{C_{A}}\left(T^{c} \otimes \mathbf{1}\right)+\frac{\operatorname{Tr}\left(T^{a} T^{b^{\prime}} T^{c}\right)}{C_{A}}\left(\mathbf{1} \otimes T^{c}\right) \\
& +4 \operatorname{Tr}\left(T^{a} T^{e} T^{c}\right) \operatorname{Tr}\left(T^{e} T^{b^{\prime}} T^{c^{\prime}}\right)\left(T^{c} \otimes T^{c^{\prime}}\right),  \tag{D.8b}\\
& T^{e} T^{a} \otimes T^{b^{\prime}} T^{e}=\frac{\delta^{a b^{\prime}}}{\left(2 C_{A}\right)^{2}}(\mathbf{1} \otimes \mathbf{1})+\frac{\operatorname{Tr}\left(T^{b^{\prime}} T^{a} T^{c}\right)}{C_{A}}\left(T^{c} \otimes \mathbf{1}\right)+\frac{\operatorname{Tr}\left(T^{b^{\prime}} T^{a} T^{c}\right)}{C_{A}}\left(\mathbf{1} \otimes T^{c}\right) \\
& +4 \operatorname{Tr}\left(T^{e} T^{a} T^{c}\right) \operatorname{Tr}\left(T^{b^{\prime}} T^{e} T^{c^{\prime}}\right)\left(T^{c} \otimes T^{c^{\prime}}\right),  \tag{D.8c}\\
& T^{e} T^{a} \otimes T^{e} T^{b^{\prime}}=\frac{\delta^{a b^{\prime}}}{\left(2 C_{A}\right)^{2}}(\mathbf{1} \otimes \mathbf{1})+\frac{\operatorname{Tr}\left(T^{b^{\prime}} T^{a} T^{c}\right)}{C_{A}}\left(T^{c} \otimes \mathbf{1}\right)+\frac{\operatorname{Tr}\left(T^{a} T^{b^{\prime}} T^{c}\right)}{C_{A}}\left(\mathbf{1} \otimes T^{c}\right) \\
& +4 \operatorname{Tr}\left(T^{e} T^{a} T^{c}\right) \operatorname{Tr}\left(T^{e} T^{b^{\prime}} T^{c^{\prime}}\right)\left(T^{c} \otimes T^{c^{\prime}}\right) . \tag{D.8d}
\end{align*}
$$

Therefore, we have

1. for CS single-quarkonium and CS-CS double-quarkonium LDMEs

$$
\begin{equation*}
4 \times\left(T^{e} \otimes T^{e}\right) \tag{D.9}
\end{equation*}
$$

2. for CO single-quarkonium LDMEs

$$
\begin{equation*}
\left(T^{a} T^{b}+T^{b} T^{a}\right) \otimes\left(T^{a} T^{b}+T^{b} T^{a}\right)=2\left[\frac{C_{F}}{C_{A}}(\mathbf{1} \otimes \mathbf{1})+\left(\frac{C_{A}}{2}-\frac{2}{C_{A}}\right)\left(T^{a} \otimes T^{a}\right)\right] \tag{D.10}
\end{equation*}
$$

3. for CO-CS and CS-CO double-quarkonium LDMEs

$$
\begin{align*}
& \left(T^{e} T^{a}+T^{a} T^{e}\right) \otimes\left(2 T^{e}\right)=\frac{2}{C_{A}}\left(\mathbf{1} \otimes T^{a}\right)+2 d^{a e c}\left(T^{c} \otimes T^{e}\right)  \tag{D.11}\\
& \left(2 T^{e}\right) \otimes\left(T^{e} T^{a}+T^{a} T^{e}\right)=\frac{2}{C_{A}}\left(T^{a} \otimes \mathbf{1}\right)+2 d^{a e c}\left(T^{e} \otimes T^{c}\right) \tag{D.12}
\end{align*}
$$

4. for CO-CO double-quarkonium LDMEs

$$
\begin{align*}
\left(T^{a} T^{e}+T^{e} T^{a}\right) \otimes\left(T^{e} T^{b^{\prime}}+T^{b^{\prime}} T^{e}\right)= & \frac{\delta^{a b^{\prime}}}{C_{A}^{2}}(\mathbf{1} \otimes \mathbf{1})+\frac{d^{a b^{\prime} c}}{C_{A}}\left(T^{c} \otimes \mathbf{1}+\mathbf{1} \otimes T^{c}\right) \\
& +d^{a e c} d^{b^{\prime} \cdot c^{\prime}}\left(T^{c} \otimes T^{c^{\prime}}\right) \tag{D.13}
\end{align*}
$$

## D. 1 Single-Quarkonium and Symmetric Double-quarkonium LDMEs

Although the one-loop corrections to the single-quarkonium LDMEs were well known long time ago, we list these results of production LDMEs here to make our discussion more complete. All the results of single-quarkonium decay LDMEs can be obtained simply by replacing corresponding production LDMEs with decay LDMEs.

$$
\begin{align*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\mathrm{ren}} & =\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[8]}\right]\right\rangle_{\text {Born }}-\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}} \\
& \times \sum_{J=0}^{J=2}\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[1]}\right]\right\rangle_{\text {Born }}+\left(\frac{C_{A}}{2}-\frac{2}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle_{\text {Born }}\right] \tag{D.14}
\end{align*}
$$

$$
\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[8]}\right]\right\rangle_{\text {ren }}=\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[8]}\right]\right\rangle_{\text {Born }}-\frac{2 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}}
$$

$$
\begin{equation*}
\times\left[\frac{C_{F}}{C_{A}}\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[1]}\right]\right\rangle_{\mathrm{Born}}+\left(\frac{C_{A}}{2}-\frac{2}{C_{A}}\right)\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[8]}\right]\right\rangle_{\mathrm{Born}}\right] \tag{D.15}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[1]}\right]\right\rangle_{\text {ren }}=\left\langle\mathcal{O}^{H}\left[{ }^{3} S_{1}^{[1]}\right]\right\rangle_{\text {Born }}-\frac{4 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}} \sum_{J=0}^{J=2}\left\langle\mathcal{O}^{H}\left[{ }^{3} P_{J}^{[8]}\right]\right\rangle_{\text {Born }} \tag{D.16}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[1]}\right]\right\rangle_{\text {ren }}=\left\langle\mathcal{O}^{H}\left[{ }^{1} S_{0}^{[1]}\right]\right\rangle_{\text {Born }}-\frac{4 \alpha_{s}}{3 \pi m_{Q}^{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}}\left\langle\mathcal{O}^{H}\left[{ }^{1} P_{1}^{[8]}\right]\right\rangle_{\text {Born }} . \tag{D.17}
\end{equation*}
$$

For the symmetric double-quarkonium LDMEs $\left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle$, with $n_{1}=n_{1}^{\prime}, n_{2}=$ $n_{2}^{\prime}$ and $\left\langle\mathcal{Q}^{H_{1}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle$, we have

$$
\begin{align*}
& \left\langle\mathcal{O}^{H_{1}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle_{H_{2}, \text { ren }} \\
= & \left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle_{\text {ren }}\left\langle\mathcal{O}\left[n_{2}\right]\right\rangle_{H_{2}, \text { Born }}+\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}\left[n_{2}\right]\right\rangle_{H_{2}, \text { ren }},  \tag{D.18}\\
& \left\langle\mathcal{O}^{H_{1}, H_{2}}\left[n_{1}, n_{2}, n_{1}, n_{2}\right]\right\rangle_{\text {ren }} \\
= & \left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle_{\text {ren }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}+\left\langle\mathcal{O}^{H_{1}}\left[n_{1}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {ren }}, \tag{D.19}
\end{align*}
$$

whose one-loop corrections are given by the one-loop correction of one of the single-quarkonium LDME times the Born level LDME of the other.

## D. 2 Interference Contributions of Double-quarkonium LDMEs

For the un-decoupled double-quarkonium LDMEs defined in Subsection 3.5.1, whose one-loop corrections have been calculated in Subsection 3.5.2, we can express the results as

$$
\begin{align*}
& \left\langle\mathcal{Q}^{H_{1}, H_{2}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle_{\text {ren }}=I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right],  \tag{D.20}\\
& \left\langle\mathcal{Q}^{H_{1}}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]\right\rangle_{H_{2}, \text { ren }}=I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{2}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right], \tag{D.21}
\end{align*}
$$

where

$$
\begin{equation*}
I_{\nu \sigma^{\prime}}^{\text {doble }}\left(m_{1}, m_{2}\right)=-\frac{\alpha_{s}}{\pi m_{1} m_{2}}\left(\frac{4 \pi \mu^{2}}{\mu_{\Lambda}^{2}} e^{-\gamma_{E}}\right)^{\epsilon} \frac{1}{\epsilon_{\mathrm{IR}}}\left(c_{1} g_{\nu \sigma^{\prime}}+c_{2} v_{1 \sigma^{\prime}} v_{2 \nu}\right), \tag{D.22}
\end{equation*}
$$

with $c_{1}, c_{2}$ given in Eq. (3.58).
And consequently, the interference contributions can be expressed as

$$
\begin{align*}
& \mathcal{A}_{1}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{1}^{\text {double }}\left[n_{1}, n_{2}\right] \\
= & \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right],  \tag{D.23a}\\
& \mathcal{A}_{2}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{2}^{* \text { double }}\left[n_{1}, n_{2}\right] \\
= & \frac{I_{\nu \sigma^{\prime}}^{\text {double }}\left(m_{1}, m_{2}\right) \mathcal{P}_{2}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]}{N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}} \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{M}\left[n_{1}, n_{2}\right], \tag{D.23b}
\end{align*}
$$

with

$$
\begin{align*}
\sigma\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right] & =\frac{1}{2 s} \int \mathcal{A}_{1}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{1}^{* \text { double }}\left[n_{1}, n_{2}\right] d \mathrm{PS}  \tag{D.24a}\\
\Gamma\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right] & =\frac{1}{2 m_{H}} \int \mathcal{A}_{2}^{* \text { double }}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] \mathcal{A}_{2}^{* \text { double }}\left[n_{1}, n_{2}\right] d \mathrm{PS} \tag{D.24b}
\end{align*}
$$

Note: Here in our discussion, $n_{1}, n_{2}^{\prime}$ are always S -wave states, while $n_{1}^{\prime}, n_{2}$ are the corresponding P-wave states that $n_{1}, n_{2}^{\prime}$ changed to be. For instance, ${ }^{1} S_{0}^{[1]}$ flips to be ${ }^{1} P_{1}^{[8]}$ and ${ }^{3} S_{1}^{[8]}$ flips to be ${ }^{3} P_{J}^{[1 / 8]}$. The color and Dirac indexes $a, \mu$ belong to the state $n_{1} ; b, \rho \sigma$ belong to the state $n_{2} ; a^{\prime}, \mu^{\prime} \nu^{\prime}$ belong to the state $n_{1}^{\prime} ; b^{\prime}, \rho^{\prime}$ belong to the state $n_{2}^{\prime} ; e, \nu \sigma^{\prime}$ are common indexes that two quarkonium share due to soft gluon exchange. The short-distance part is $\mathcal{M}\left[n_{1}, n_{2}\right] \mathcal{M}^{*}\left[n_{1}^{\prime}, n_{2}^{\prime}\right] /\left(N_{n_{1}} N_{n_{2}} N_{n_{1}^{\prime}} N_{n_{2}^{\prime}}\right)$ and carries the same color, Dirac indexes with the un-decoupled double-quarkonium LDME.
$\mathbf{C O}\left(n_{1}^{\prime}\right)-\mathbf{C S}\left(n_{2}^{\prime}\right)-\mathbf{C S}\left(n_{1}\right)-\mathbf{C O}\left(n_{2}\right)$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=4 \delta^{a^{\prime} b} \sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}}{\left(N_{c}^{2}-1\right)^{2} N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} . \tag{D.25}
\end{equation*}
$$

Here and below, the polarization sum must be done for $n_{1}^{\prime}$ and $n_{2}$ states, which means

$$
\varepsilon_{1} \varepsilon_{1}^{*}= \begin{cases}\varepsilon_{\nu} \varepsilon_{\varepsilon^{\prime}}^{*}, & n_{1}^{\prime}={ }^{1} P_{1}^{[1 / 8]}  \tag{D.26}\\ \varepsilon_{\mu \nu}^{\left(J_{\nu}\right)} \varepsilon_{\mu^{\prime} \nu^{\prime}}^{*\left(J_{1}\right)}, & n_{1}^{\prime}={ }^{3} P_{J_{1}}^{[1 / 8]}\end{cases}
$$

and

$$
\varepsilon_{2} \varepsilon_{2}^{*}=\left\{\begin{array}{ll}
\varepsilon_{\sigma} \varepsilon_{\sigma^{\prime}}^{*}, & n_{2}={ }^{1} P_{1}^{[1 / 8]}  \tag{D.27}\\
\left.\varepsilon_{\rho \sigma} \rho_{2}\right) & \varepsilon_{\rho^{\prime} \sigma^{\prime}}^{*\left(J_{2}\right)},
\end{array} n_{2}={ }^{3} P_{J_{2}}^{[1 / 8]} .\right.
$$

$\mathrm{CS}-\mathrm{CS}-\mathrm{CO}-\mathrm{CO}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=\frac{2 \delta^{a b}}{C_{A}} \sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\mathrm{Born}}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\mathrm{Born}}}{\left(N_{c}^{2}-1\right) N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} . \tag{D.28}
\end{equation*}
$$

$\mathrm{CO}-\mathrm{CS}-\mathrm{CO}-\mathrm{CO}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=2 d^{a a^{\prime} b} \sum_{\text {pol }}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}}{\left(N_{c}^{2}-1\right)^{2} N_{J_{1}} N_{J_{2}}} \tag{D.29}
\end{equation*}
$$

## $\mathrm{CO}-\mathrm{CO}-\mathrm{CS}-\mathrm{CS}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=\frac{2 \delta^{\alpha^{\prime} b^{\prime}}}{C_{A}} \sum_{\text {pol }}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Bor }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}}{\left(N_{c}^{2}-1\right) N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} . \tag{D.30}
\end{equation*}
$$

## $\mathrm{CO}-\mathrm{CO}-\mathrm{CS}-\mathrm{CO}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=2 d^{a^{\prime} b b^{\prime}} \sum_{\text {pol }}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Borm }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Borm }}}{\left(N_{c}^{2}-1\right)^{2} N_{\text {pol }}\left(n_{1}^{\prime}\right) N_{\text {pol }}\left(n_{2}\right)} . \tag{D.31}
\end{equation*}
$$

$\mathrm{CS}-\mathrm{CO}-\mathrm{CO}-\mathrm{CS}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=\frac{\delta^{a b^{\prime}}}{C_{A}^{2}} \sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}}{N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} . \tag{D.32}
\end{equation*}
$$

## $\mathrm{CO}-\mathrm{CO}-\mathrm{CO}-\mathrm{CS}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=\frac{d^{a a^{\prime} b^{\prime}}}{C_{A}} \sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Born }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Born }}}{\left(N_{c}^{2}-1\right) N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} . \tag{D.33}
\end{equation*}
$$

## $\mathrm{CS}-\mathrm{CO}-\mathrm{CO}-\mathrm{CO}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=\frac{d^{a b b^{\prime}}}{C_{A}} \sum_{\mathrm{pol}}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\mathrm{Born}}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\mathrm{Born}}}{\left(N_{c}^{2}-1\right) N_{\mathrm{pol}}\left(n_{1}^{\prime}\right) N_{\mathrm{pol}}\left(n_{2}\right)} \tag{D.34}
\end{equation*}
$$

$\mathrm{CO}-\mathrm{CO}-\mathrm{CO}-\mathrm{CO}$

$$
\begin{equation*}
\mathcal{P}_{1}\left[n_{1}^{\prime}, n_{2}^{\prime}, n_{1}, n_{2}\right]=d^{a e a^{\prime}} d^{b e b^{\prime}} \sum_{\text {pol }}\left(\varepsilon_{1} \varepsilon_{1}^{*}\right)\left(\varepsilon_{2} \varepsilon_{2}^{*}\right) \frac{\left\langle\mathcal{O}^{H_{1}}\left[n_{1}^{\prime}\right]\right\rangle_{\text {Bor }}\left\langle\mathcal{O}^{H_{2}}\left[n_{2}\right]\right\rangle_{\text {Bor }}}{\left(N_{c}^{2}-1\right)^{2} N_{\text {pol }}\left(n_{1}^{\prime}\right) N_{\text {pol }}\left(n_{2}\right)} . \tag{D.35}
\end{equation*}
$$

## Appendix E

## Master Integrals in Virtual Corrections at $\mathcal{O}(\epsilon)$

Note: we neglect all imaginary parts, since only the real parts contribute to the NLO corrections.
We follow the notation of Ref. [79]

$$
\begin{align*}
I_{1}^{D}\left(m_{1}^{2}\right) & =\frac{\mu^{4-D}}{i \pi^{\frac{D}{2}} r_{\Gamma}} \int d^{D} l \frac{1}{\left(l^{2}-m_{1}^{2}+i \varepsilon\right)},  \tag{E.1}\\
I_{2}^{D}\left(p_{1}^{2} ; m_{1}^{2}, m_{2}^{2}\right) & =\frac{\mu^{4-D}}{i \pi^{\frac{D}{2}} r_{\Gamma}} \int d^{D} l \frac{1}{\left(l^{2}-m_{1}^{2}+i \varepsilon\right)\left(\left(l+q_{1}\right)^{2}-m_{2}^{2}+i \varepsilon\right)}, \tag{E.2}
\end{align*}
$$

where $r_{\Gamma} \equiv \frac{\Gamma^{2}(1-\epsilon) \Gamma(1+\epsilon)}{\Gamma(1-2 \epsilon)}=1-\epsilon \gamma_{E}+\epsilon^{2}\left[\frac{\gamma_{E}^{2}}{2}-\frac{\pi^{2}}{12}\right]+\mathcal{O}\left(\epsilon^{3}\right)$.
The tadpole integral reads

$$
\begin{equation*}
I_{1}^{D}\left(m^{2}\right)=m^{2}\left(\frac{\mu^{2}}{m^{2}}\right)^{\epsilon}\left(\frac{1}{\epsilon}+1+\epsilon\left(1+\frac{\pi^{2}}{6}\right)\right) . \tag{E.3}
\end{equation*}
$$

The bubble integral with two vanishing masses reads

$$
\begin{equation*}
I_{2}^{D}(s ; 0,0)=\left(\frac{\mu^{2}}{s}\right)^{\epsilon}\left(\frac{1}{\epsilon}+2+\epsilon\left(4-\frac{\pi^{2}}{2}\right)\right) . \tag{E.4}
\end{equation*}
$$

The bubble integral with one vanishing mass is given by

$$
\begin{equation*}
I_{2}^{D}\left(s ; 0, m^{2}\right)=\left(\frac{\mu^{2}}{m^{2}}\right)^{\epsilon}\left(\frac{1}{\epsilon}-d_{1}+\epsilon\left(\frac{d_{2}}{2}+\frac{\pi^{2}}{6}\right)\right) \tag{E.5}
\end{equation*}
$$

where

$$
\begin{equation*}
d_{1}=\frac{s-m^{2}}{s} \ln \left(\frac{\left|m^{2}-s\right|}{m^{2}}\right)-2 \tag{E.6}
\end{equation*}
$$

and

$$
\begin{align*}
d_{2}= & 2\left(\frac{s-m^{2}}{s}\right)\left(\ln \left(\frac{m^{2}-s}{m^{2}}\right) \ln \left(\frac{m^{2}-s}{|s|}\right)-2 \ln \left(\frac{m^{2}-s}{m^{2}}\right)\right. \\
& \left.-\operatorname{Li}_{2}\left(\frac{m^{2}-s}{m^{2}}\right)+\frac{\pi^{2}}{6}\right)+8 \tag{E.7}
\end{align*}
$$

for $s<m^{2}$;

$$
\begin{align*}
d_{2}= & \left(\frac{s-m^{2}}{s}\right)\left(\ln ^{2}\left(\frac{s-m^{2}}{m^{2}}\right)+\ln ^{2}\left(\frac{s-m^{2}}{s}\right)-4 \ln \left(\frac{s-m^{2}}{m^{2}}\right)\right. \\
& \left.+2 \operatorname{Li}_{2}\left(\frac{s-m^{2}}{s}\right)-\frac{5 \pi^{2}}{3}\right)+8 \tag{E.8}
\end{align*}
$$

for $s>m^{2}$.
The bubble integral with two different masses is given by

$$
\begin{equation*}
I_{2}^{D}\left(t / 4 ; m_{c}^{2}, m_{b}^{2}\right)=\left(\frac{\mu^{2}}{(t / 4)}\right)^{\epsilon}\left(\frac{1}{\epsilon}-c_{1}+\epsilon\left(\frac{c_{2}}{2}+\frac{\pi^{2}}{6}\right)\right) \tag{E.9}
\end{equation*}
$$

with

$$
\left.\begin{array}{rl}
c_{1}=- & \frac{1}{t}(
\end{array}-a \ln \left(\frac{s+u-a}{8 m_{c} m_{b}}\right)+2\left(m_{b}^{2}-m_{c}^{2}\right) \ln \left(\frac{m_{c}^{2}}{m_{b}^{2}}\right)+t \ln \left(\frac{t}{4 m_{c} m_{b}}\right)\right)-2, ~\left(\ln \left(\frac{a-s_{b}-u_{b}}{2 a}\right)+\ln \left(\frac{t}{4 m_{b}^{2}}\right)+2\right) .
$$

## Appendix $F$

## Soft Integrals

To extract the infrared divergences in $k_{5} \rightarrow 0$ limit in real corrections, we need the results of following integrals

$$
\left.\begin{array}{rl}
\int_{\text {soft }} \frac{d \mathrm{PS}_{k_{5}}}{\left(P_{2} \cdot k_{5}\right)^{2}}= & \frac{C_{\epsilon}}{32 \pi^{2} m_{c}^{2}}\left[-\frac{1}{\epsilon}-\frac{s_{c}+u_{c}}{a} \ln \left(\frac{s_{c}+u_{c}+a}{s_{c}+u_{c}-a}\right)-\ln \left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)\right], \\
\int_{\text {soft }} \frac{d \mathrm{PS}_{k_{5}}}{\left(k_{3} \cdot k_{5}\right)\left(k_{4} \cdot k_{5}\right)}= & \frac{C_{\epsilon}}{4 \pi^{2} t}\left[\frac{1}{\epsilon^{2}}+\frac{1}{\epsilon} \ln \left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)+\frac{1}{2} \ln ^{2}\left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)-\frac{\pi^{2}}{4}\right], \\
\int_{\text {soft }} \frac{d \mathrm{PS}_{k_{5}}}{\left(P_{2} \cdot k_{5}\right)\left(k_{3} \cdot k_{5}\right)}= & \frac{C_{\epsilon}}{8 \pi^{2} s_{c}}\left[\frac{1}{\epsilon^{2}}+\frac{1}{\epsilon} \ln \left(\frac{\mu^{2} t}{s_{c}^{2} \delta_{s}^{2}}\right)+\ln ^{2}\left(\frac{s_{c}+u_{c}-a}{2 s_{c}}\right)+\frac{1}{2} \ln ^{2}\left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)\right. \\
& -\ln \left(\frac{s_{c}^{2}}{4 m_{c}^{2} t}\right) \ln \left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)-\frac{1}{2} \ln ^{2}\left(\frac{s_{c}+u_{c}+a}{s_{c}+u_{c}-a}\right) \\
& \left.+2 \operatorname{Li}_{2}\left(-\frac{s-u+a}{s_{c}+u_{c}-a}\right)-2 \operatorname{Li}_{2}\left(\frac{s-u-a}{2 s_{1}}\right)-\frac{\pi^{2}}{4}\right], \\
\int_{\text {soft }} \frac{d \mathrm{PS}_{k_{5}}}{\left(P_{2} \cdot k_{5}\right)\left(k_{4} \cdot k_{5}\right)}= & \frac{C_{\epsilon}}{8 \pi^{2} u_{c}}\left[\frac{1}{\epsilon^{2}}+\frac{1}{\epsilon} \ln \left(\frac{\mu^{2} t}{u_{c}^{2} \delta_{s}^{2}}\right)+\ln ^{2}\left(\frac{s_{c}+u_{c}-a}{2 u_{c}}\right)+\frac{1}{2} \ln ^{2}\left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)\right. \\
& -\ln \left(\frac{u_{c}^{2}}{4 m_{c}^{2} t}\right) \ln \left(\frac{\mu^{2}}{4 \delta_{s}^{2} m_{c}^{2}}\right)-\frac{1}{2} \ln ^{2}\left(\frac{s_{c}+u_{c}+a}{s_{c}+u_{c}-a}\right) \\
& \left.+2 \operatorname{Li}_{2}\left(-\frac{u-s+a}{s_{c}+u_{c}-a}\right)-2 \operatorname{Li}_{2}\left(\frac{u-s-a}{2 u_{c}}\right)-\frac{\pi^{2}}{4}\right] . \tag{F.4}
\end{array} \quad \text { (F.4) }\right)
$$

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## Eidesstattliche Versicherung / Declaration on oath

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

Die eingereichte schriftliche Fassung entspricht der auf dem elektronischen Spei chermedium.

Die Dissertation wurde in der vorgelegten oder einer ähnlichen Formnicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

Hamburg, den 09.07.2018

