# Shape Representations of Digital Sets based on Convexity Properties 

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## Introduction

The only sets which can be handled on computers are discrete or digital sets that means the sets containing a finite number of elements. The discrete nature of digital images makes it necessary to develop suitable systems and methods since a direct use of classical theories is not possible or not adaptable. The dealing with geometrical properties of digital sets is important in many applications of image processing. The topic of digital geometry is to recognize and to describe these properties. Apart from the theoretical foundations, the efficient procedures and techniques play a key role in scientific computation.

A considerable part of books on digital geometry is devoted to convexity (see e.g. [48, Chapter 4.3], or quite recently appeared book by Klette and Rosenfeld [27]). It is a simple observation that convex parts of objects determine visual parts which are of importance, for example, for recognition objects by comparing with given shapes from a database. However, the problem is that many significant parts are not convex since a visual part may have concavities. One is interested in the decomposition of the boundary of a digital set into convex and concave parts. In an earlier paper [34] the decomposition of the boundary was performed by segmenting the boundary into digital line segments. In an other paper [14] it was proposed to define the meaningful parts of the boundary by meaningful parts of the corresponding polygonal representation. The first method is much rougher, however, both techniques have an approximative character. Also, recent publications, whose discussions are related to the considered problem, e.g. [7] is about digital arc segmentation, [5] elucidates new aspects of digital curves and surfaces, shall attract attention.

As a further application, the partition of the boundary of a digital set into meaningful parts, especially, into digital line segments, allows the calculation of the Euclidean perimeter of the set. For this technique it is known that the measured perimeter converges towards the true value if an Euclidean convex region is digitized with increasing grid resolution [30, 44]. In general, this and other methods for the calculation of the perimeter are imprecise [31, 32, 47, 50] and, what is more important, their precision cannot be improved by increasing of the resolution of the digitization. It becomes an interesting question, exists an "exact" decomposition of the boundary of a digital set into convex and concave parts such that the convergence is present.

In digital geometry it is not a simple task to testing convexity of a set [26]. In

1928, Tietze [45] proved that convexity of a set in $\mathbb{R}^{2}$ can be decided locally in a time which is proportional to the length of its boundary. Unfortunately, in digital plane convexity cannot be observed locally [11]. One deals with the problem to decide whether a part of the boundary of a digital set is convex or not by some method which is "as local as possible".

The shape of a two-dimensional object can be represented by its boundary contour. The question is raised: Can a digital set be represented by a polygonal set in the plane $\mathbb{R}^{2}$ whose vertices are elements on the boundary of the digital set such that the representing polygon is a Jordan curve in $\mathbb{R}^{2}$ which contains exactly the points of the given set in its interior. Furthermore, one wants the representing polygon to have the same convexity properties as the digital set. In the case such polygonal set can be found easily one has the advantage of reduction of the data which represents the shape of the digital set.

There is a different aspect which plays a relevant role in the subsequent discussions. In 1987, Scherl proposed a method in the context of document analysis [43] which was based on sets of descriptors. The descriptors were obtained as points of local support with respect to a certain finite number of directions. This approach has practical advantages as well as theoretically appealing properties.

In Euclidean geometry the convexity is certainly such property which sets inherit from their lower-dimensional plane sections. Clearly, digital sets possess finitely many plane sections. On the other hand, the plane sections of digital sets have, in general, different topological structures. In the only situation, where these structures are known, the lower-dimensional theory can be used. This technique is of relevance for testing convexity and efficient convex hull computation of 3D digital objects. Moreover, the extension to higher dimentions is available.

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## Chapter 1

## Digital Space

The study of properties of point configurations is a vast area of the research in geometry whose origins go back at least to the ancient Greeks. The topic of digital geometry is to translate continuous concepts into discrete world. There are basically two possibilities. The first one is defining a discretization mapping

$$
\begin{gathered}
\psi: \mathbb{R}^{d} \rightarrow \mathbb{Z}^{d} \\
\psi\left(S \subseteq \mathbb{R}^{d}\right)=S_{\triangle} \subseteq \mathbb{Z}^{d} .
\end{gathered}
$$

This approach is not canonic, it depends on the used discretization mapping and one has to take care that such mapping is well-defined. The other possible approach is axiomatic way. Here, suitable characteristic properties are translated into digital settings. This approach allows to derive properties of the discrete objects in a rigorous abstract way. In some cases both approaches lead to the same concepts.

### 1.1 Definitions

The digital space $\mathbb{Z}^{d}$ is the set of all points in Euclidean space $\mathbb{R}^{d}$ having integer coordinates. The digital space $\mathbb{Z}^{2}$ is also called digital plane. In image processing digital plane is taken as a mathematical model of digitized black-white images. In this application one usually has a given set, namely, the set $S_{\triangle}$ of black points, and the set $\mathbb{C} S_{\triangle}$ of white points belonging to the complement of $S_{\triangle}$.

The subsets of $\mathbb{Z}^{d}$ are termed digital sets, often they are also called digital objects or digital images. The elements of $\mathbb{Z}^{d}$ are termed grid points. A digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ consisting of grid points which are lying all on a horizontal, vertical or diagonal real line in $\mathbb{R}^{2}$ is called a horizontal, vertical or diagonal grid line, respectively.

The neighborhood structure is a significant concept in the study of digital objects. A neighborhood is defined typically using a distance metric. Assume $x=$ $\left(\eta_{1}, \eta_{2}, \cdots, \eta_{d}\right)$ and $y=\left(\zeta_{1}, \zeta_{2}, \cdots, \zeta_{d}\right)$ are points of $\mathbb{Z}^{d}$. We consider two types of
distances between elements in $\mathbb{Z}^{d}$ :

$$
d_{1}(x, y):=\sum_{i=1}^{d}\left|\eta_{i}-\zeta_{i}\right| \quad \text { and } \quad d_{\infty}:=\max _{i=1, \cdots, d}\left|\eta_{i}-\zeta_{i}\right| .
$$

For $\rho \in\{1, \infty\}$ the points $x$ and $y$ are said to be $d_{\rho}$-adjancent if $d_{\rho}(x, y)=1$. The element $x$ is a $d_{\rho}$-neighbor of $y$ whenever $x$ and $y$ are $d_{\rho}$-adjancent. For the set $\mathcal{N}_{\rho}(x):=\left\{x^{*} \in \mathbb{Z}^{d} \mid d_{\rho}\left(x^{*}, x\right)=1\right\}$ of all neighbors of $x$ holds

$$
\operatorname{Card}\left(\mathcal{N}_{1}(x)\right)=2 d \quad \text { and } \quad \operatorname{Card}\left(\mathcal{N}_{\infty}(x)\right)=3^{d}-1
$$

The $d_{1}$-neighbors of $x$ are called direct neighbors. The $d_{\infty}$-neighbors which are not direct are termed indirect neighbors.

In digital plane $\mathbb{Z}^{2}$ we are able to number the $d_{\infty}$-neighbors of $x$ in the following way:

$$
\begin{array}{ccc}
N_{3}(x) & N_{2}(x) & N_{1}(x) \\
N_{4}(x) & x & N_{0}(x) \\
& & \\
N_{5}(x) & N_{6}(x) & N_{7}(x)
\end{array}
$$

Neighbors with even number are direct or 4-neighbors of $x$, those with odd numbers are indirect neighbors. The 4-neighborhood $\mathfrak{N} \mathcal{N}_{4}(x)$ of $x$ is the set of all direct neighbors of $x$ (excluding $x$ ), the 8-neighborhood $\mathcal{N}_{8}(x)$ of $x$ is the set of all direct and indirect neighbors of $x$ (excluding $x$ ).

In our further considerations we concentrate exclusively on the neighborhood structure corresponding to $\left(d_{\infty}, d_{1}\right)$-adjacency, namely, $d_{\infty}$-adjacency for digital objects and $d_{1}$-adjacency for their complements. Generally, the choice of two different notions of adjacency, one for the object and other for its complement is related to avoiding certain paradoxes [29].

We define for $\rho \in\{1, \infty\}$ :
Definition 1.1 $A$ set $S_{\triangle} \subseteq \mathbb{Z}^{d}$ is termed $d_{\rho}$-connected if for each pair of points $x, y \in S_{\triangle}$ there exists a sequence $x=x_{0}, \cdots, x_{n}=y$ with $x_{i} \in S_{\triangle}$ for all $i=0, \cdots, n$ such that $x_{i}$ and $x_{i+1}$ are $d_{\rho}$-adjacent for $i=0, \cdots, n-1$.

For the sake of simplicity we call the $d_{1}$ - and $d_{\infty}$-connected sets in digital plane 4 -connected and 8-connected, in $\mathbb{Z}^{3}$ we call them 6-connected and 26-connected, respectively. Since connectedness is a typical topological concept we may assume that digital spaces $\mathbb{Z}^{2}$ and $\mathbb{Z}^{3}$ are equipped with 4-topology or 8-topology and 6topology or 26-topology, respectively. In Figure 1.1 the mentioned topologies of digital plane $\mathbb{Z}^{2}$ are shown.


Figure 1.1: The pictures demonstrate 4-topology (left) and 8-topology (right) of $\mathbb{Z}^{2}$. 4-neighborhood and 8-neighborhood of a grid point $\odot$ are indicated.

Definition 1.2 Given an 8-connected digital set $\mathcal{K} \subseteq \mathbb{Z}^{2}$. $\mathcal{K}$ is called a digital 8 -curve whenever each point $x \in \mathcal{K}$ has exactly two 8 -neighbors in $\mathcal{K}$ with the possible exception of at most two points, the so-called end points of the curve, having exactly one neighbor in $\mathcal{K}$.

A curve without end points is termed a closed curve.
Each digital 8-curve can be ordered (and oriented) in a natural manner. Let us consider a finite 8 -curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$, where $\kappa_{i}$ is an 8 -neighbor of $\kappa_{i+1}$ for $i=1,2, \cdots, n-1$. Then the curve $\mathcal{K}$ can be described by means of a simple compact ordered data structure containing the coordinates of $\kappa_{1}$ and a sequence of code numbers in $\{0,1,2,3,4,5,6,7\}$ indicating for each point of $\mathcal{K}$ which of its neighbors will be the next point on the curve. This data structure was proposed by Freeman [18] and is known as the chain code.


Figure 1.2: Chain Code by Freeman

Definition 1.3 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be an ordered digital 8 -curve. For a number $k \in\{0,1, \cdots, 7\}$ the curve $\mathcal{K}$ is called a $(k, k+1(\bmod 8))$-curve whenever the chain code representation of $\mathcal{K}$ consists of only both chain codes $k$ and $k+1(\bmod 8)$.

Since each $(k, k+1(\bmod 8))$-curve is an image by a rotation of some $(0,1)$ curve in the later chapters we may concentrate, without loss of generality, exclusively on $(0,1)$-curves.

The structure of digital 8 -curves has distinguishing features: the horizontal, vertical and diagonal levels.

Definition 1.4 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be an ordered digital 8-curve. For a code number $v \in\{0,1,2,3,4,5,6,7\}$ a level of $\mathcal{K}$ is a maximal subset of the curve whose chain code representation consists only of the code number $v$. The number of successive elements of a level is called the length of the level.

In digital geometry the digital 8-curves are often considered together with a concept called chord property $[11,20,21,41]$. For $x$ and $y$ in $\mathbb{R}^{2}$ the (continuous) line segment joining $x$ and $y$ is the set

$$
[x, y]=\left\{z \in \mathbb{R}^{2} \mid z=\lambda x+(1-\lambda) y, 0 \leq \lambda \leq 1\right\} .
$$

A digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is said to possess the chord property whenever for two points $x, y \in S_{\triangle}$ and for $u \in[x, y]$ there exists $z \in S_{\triangle}$ such that $d_{\infty}(z, u)<1$. It can be shown that a digital set which has the chord property is 8 -connected [41].

In literature a finite digital 8 -curve having the chord property is called digital straight line segment. Rosenfeld [41] has shown that each digital straight line segment can be obtained by digitization of a real straight line segment and that the digitization of a real straight line segment leads to a digital straight line segment.

### 1.2 Boundary of Digital Sets

Definition 1.5 Let $S_{\triangle} \subseteq \mathbb{Z}^{2}$ be a digital set. A point $P \in S_{\triangle}$ is called an interior point of $S_{\triangle}$ whenever all direct neighbors of $P$ belong to $S_{\triangle}$. The interior of $S_{\triangle}$ is the set

$$
\text { int } S_{\triangle}=\bigcup\left\{P \in S_{\triangle} \mid P \text { is an interior point of } S_{\triangle}\right\}
$$

A point $P \in S_{\triangle}$ is called a boundary point of $S_{\triangle}$ whenever $P$ is a 4-neighbor of $\mathbb{C} S_{\triangle}$. The boundary of $S_{\triangle}$ is the set

$$
b d S_{\triangle}=\bigcup\left\{P \in S_{\triangle} \mid P \text { is a boundary point of } S_{\triangle}\right\}
$$

We recall here that the sets under consideration are equipped with 8-topology and their complements with 4-topology. Let $S_{\triangle}$ be an 8-connected set on digital plane and $\mathbb{C} S_{\triangle}$ be its complement.

On the set $b d S_{\triangle}$ we define a binary successor relation: For points $P, Q \in b d S_{\triangle}$ holds $Q$ is successor of $P$ if and only if the following conditions are true:

1. $Q$ is an 8 -neighbor of $P$ with $Q=N_{k}(P)$, and
2. for all points $R \in S_{\triangle}$ which are 8-neighbors of both $P$ and $Q$ (i.e. $R \in S_{\triangle} \cap$ $\mathcal{N}(P) \cap \mathcal{N}(Q))$ is $R=N_{i}(P)$ with $i=k+1(\bmod 8)$ or $i=k+2(\bmod 8)$.

Remark 1.1 The condition 2. means that the elements of $S_{\triangle} \cap \mathcal{N}(P) \cap \mathcal{N}(Q)$ (if exist) are lying on the left hand side of $P$ and $Q$. Furthermore, if holds $S_{\triangle} \cap \mathcal{N}(P) \cap$ $\mathcal{N}(Q)=\emptyset$ then by definition $Q$ is successor of $P$ if and only if $P$ is successor of $Q$. We deduce that the successor relation can be applied only to digital sets which possess some kind of regularity. In this case it is the restriction $S_{\triangle} \cap \mathcal{N}(P) \cap \mathcal{N}(Q) \neq \emptyset$ for all 8-neighbors $P$ and $Q$ on the boundary of $S_{\triangle}$.

Whenever in condition 2. for all $P, Q \in b d S_{\triangle}$ which are 8-neighbors one has $S_{\triangle} \cap \mathcal{N}(P) \cap \mathcal{N}(Q) \neq \emptyset$ the border or oriented boundary of a digital set $S_{\triangle}$ is the boundary of $S_{\triangle}$ equipped with the successor relation. During moving along the border the points of $S_{\triangle}$ are always on the left hand side of the oriented boundary.

Rosenfeld [42] (see also [13]) was able to show that this definition leads to a digital analogon of Jordan's Curve Theorem. If $S_{\triangle}$ is a digital 8 -connected set whose boundary is a closed simple (not self-intersecting) polygonal curve then the boundary of $S_{\triangle}$ separates digital plane into two 4-connected sets. Exactly one of these sets is bounded and is called the interior with respect to $b d S_{\triangle}$ and the other is unbounded and is called the exterior with respect to $b d S_{\triangle}$.

There are configurations on the oriented boundary which by definition of the successor relation cannot appear. In Figure 1.3 demonstrated boundary moves and their rotations by $\frac{k \pi}{2}, k=1,2,3$ are non legal.


Figure 1.3: The demonstrated boundary moves and their rotations by $\frac{k \pi}{2}, k=1,2,3$ are non legal.

Assume an 8-connected digital set $S_{\triangle}$ is given. In the sequel we concentrate on one single connected component of the boundary of $S_{\triangle}$. We are interested in computation the orientation of $b d S_{\triangle}$ started in some point such that this point will be reached again during moving along the boundary. We choose the left hand side orientation. It is clear that the set $b d S_{\triangle}$ is 8 -connected. Moreover, Rosenfeld [42] (see also [12]) has shown that the 8 -connected boundary components of a set can be oriented in such way that each boundary point has exactly one successor and exactly one predecessor on the oriented boundary. By Remark 1.1 the successor relation is useful not for all 8-connected sets of $\mathbb{Z}^{2}$. For example, it cannot be applied to not closed digital 8-curves.

Therefore we introduce a definition of left and right neighbors for pairs of grid points from $\mathbb{Z}^{2}$ which are 8-neighbors:

Definition 1.6 Let $P, Q \in \mathbb{Z}^{2}$ be grid points with the property that $Q=N_{k}(P)$. Then the points $N_{k+i(\bmod 8)}(Q)$ are called left neighbors of $Q$ and $N_{k-i(\bmod 8)}(Q), i=$ $1, \cdots, 4$ are called right neighbors of $Q$ with reference to $P$. Further, the point $N_{k+4(\bmod 8)}(Q)=P$ and $N_{k-4(\bmod 8)}(Q)=P$ is both left and right neighbor of $Q$ with reference to $P$. The element $N_{k}(Q)$ possesses no classification.

We consider two successive points $P, Q \in b d S_{\triangle}$. Assume $Q$ is successor of $P$ with $Q=N_{k}(P)$. The question is which point $R$ on the boundary is the successor of $Q$ in the sense of the left hand side orientation?

We define the modified sets of left and right neighbors of $Q$ with reference to $P$ as follows:

$$
\mathcal{N}^{l}(Q)=\bigcup\left\{R \in \mathcal{N}_{8}(Q) \mid R \text { is a left neighbor of } Q \text { or } R=N_{k}(Q)\right\}
$$

and

$$
\mathcal{N}^{r}(Q)=\bigcup\left\{R \in \mathcal{N}_{8}(Q) \mid R \text { is a right neighbor of } Q \text { and } R \neq N_{k \pm 4(\bmod 8)}(Q)\right\} .
$$

Obviously, the set $\mathcal{N}^{r}(Q)$ cannot possess the element $N_{k-3(\bmod 8)}(Q)$ (see Figure 1.3). Moreover, if $\mathcal{N}^{r}(Q) \neq \emptyset$ then $R$ is the element of $\mathcal{N}^{r}(Q)$ with $R=$ $N_{k-i(\bmod 8)}(Q)$ such that $i$ is maximal. Generally, only situations $i=1$ or $i=2$ are available. Otherwise, whenever $\mathcal{N}^{r}(Q)=\emptyset$ then $R$ is the element of $\mathcal{N}^{l}(Q)$ with $R=N_{k+i(\bmod 8)}(Q)$ such that $i$ is minimal. Here, the situations $0 \leq i \leq 4$ are available. If $i=4$ then the successor of $Q$ is $R=P$. The proposed technique leads to Algorithm Succ.

For orientation of $b d S_{\triangle}$ the most time consuming command is the computation of $\mathcal{N}^{l}(Q)$ and $\mathcal{N}^{r}(Q)$ for every element $Q$ on the boundary. By direct realization it possesses the complexity $O\left(m^{2}\right)$, where $m$ is the number of elements on $b d S_{\triangle}$.

### 1.3 Remarks to Duality

In continuous geometry and topology of the plane there exists a useful duality relation: If we replace a set $S_{\triangle}$ by its complement and if the boundary of $S_{\triangle}$ is a (finite or infinite) Jordan curve then the orientation of the boundary curve is simply inverted. Duality arguments are very convenient for defining convex and concave parts of the boundary of a set. Since it is very simple to agree on a notion of convexity of a boundary part the concave parts are simply defined as convex parts of the complement. The situation is not simple and clear in discrete topology. The set and its complement have different topologies to begin with. Furthermore, unlike in ordinary topology, the boundary of the set and the boundary of the complement may be essentially different. We have in general

$$
b d\left(\mathbb{C} S_{\triangle} \cup b d S_{\triangle}\right) \subseteq b d S_{\triangle}
$$

Since the set $\mathbb{C} S_{\triangle} \cup b d S_{\triangle}$ may possess the configurations from Figure 1.3 the strict inclusion sign is possible.

Algorithm I Succ Successor $R$ of $Q$ on the boundary of $S_{\triangle}$ by given predecessor $P$ with $Q=N_{k}(P)$ (left hand side orientation).

```
\(\mathcal{N}^{l}(Q) \leftarrow \bigcup\left\{R \in \mathcal{N}_{8}(Q) \mid R\right.\) is a left neighbor of \(Q\) or \(\left.R=N_{k}(Q)\right\} ;\)
\(\mathcal{N}^{r}(Q) \leftarrow \bigcup\left\{R \in \mathcal{N}_{8}(Q) \mid R\right.\) is a right neighbor of \(Q\) and \(R \neq\)
\(\left.N_{k \pm 4(\bmod 8)}(Q)\right\} ;\)
if \(\mathcal{N}^{r}(Q) \neq \emptyset\) then
    if \(N_{k-2(\bmod 8)}(Q) \in \mathcal{N}^{r}(Q)\) then \(R \leftarrow N_{k-2(\bmod 8)}(Q)\);
    else \(R \leftarrow N_{k-1(\bmod 8)}(Q)\);
else
    if \(N_{k}(Q) \in \mathcal{N}^{l}(Q)\) then \(R \leftarrow N_{k}(Q)\);
    else
        if \(N_{k+1(\bmod 8)}(Q) \in \mathcal{N}^{l}(Q)\) then \(R \leftarrow N_{k+1(\bmod 8)}(Q)\);
        else
            if \(N_{k+2(\bmod 8)}(Q) \in \mathcal{N}^{l}(Q)\) then \(R \leftarrow N_{k+2(\bmod 8)}(Q)\);
                else
                        if \(N_{k+3(\bmod 8)}(Q) \in \mathcal{N}^{l}(Q)\) then \(R \leftarrow N_{k+3(\bmod 8)}(Q)\);
                else \(R \leftarrow P\);
                end
        end
    end
end
```


## Chapter 2

## Scherl's Descriptors

Scherl [43] proposed in 1987 a method for representing digital sets. This method was developed for document processing applications. An "object" in this context is a connected component $S_{\triangle}$ in a binary document image. Scherl introduced socalled shape descriptors which are boundary points belonging to local extrema of linear functionals corresponding to the main directions in digital plane $\left(0^{\circ}, 45^{\circ}, 90^{\circ}\right.$, $135^{\circ}, 180^{\circ}, 225^{\circ}, 270^{\circ}$ and $315^{\circ}$ ).

Our definition of descriptor points on the boundary of a digital set is partial corresponding with definition from [14] and [15].

Definition 2.1 Let $P_{0}, P_{1}, \cdots, P_{\tau}, P_{\tau+1}, \tau \geq 1$ be successive points on the left hand side oriented boundary of a digital set $S_{\triangle}$ and $k \in\{0,1, \cdots, 7\}$.

For $\tau=1$ and $P_{1}=N_{k}\left(P_{0}\right)$ the point $P_{1}$ is called $T$ - or $S$-descriptor point, respectively, whenever the following condition 1 . is true:

1. $P_{2}=N_{k+l(\bmod 8)}\left(P_{1}\right)$ is a left neighbor of $P_{1}$ with reference to $P_{0}$ such that $2 \leq l \leq 4$,
or
$P_{2}=N_{k-2(\bmod 8)}\left(P_{1}\right)$.
Here, $P_{1}$ is also called $T$-descriptor point of types $k+1(\bmod 8), \cdots, k+l-1(\bmod 8)$ or $S$-descriptor point of type $k-1(\bmod 8)$, respectively.

For $\tau>1, P_{i+1}=N_{k}\left(P_{i}\right), i=1, \cdots, \tau-1$ and $P_{1} \neq N_{k}\left(P_{0}\right), P_{\tau+1} \neq N_{k}\left(P_{\tau}\right)$ the points $P_{1}, \cdots, P_{\tau}$ are called $T$ - or $S$-descriptor points of type $k$, respectively, whenever the following condition 2. is true:
2. $P_{2}$ is a left neighbor of $P_{1}$ with reference to $P_{0}$ and $P_{\tau+1}$ is a left neighbor of $P_{\tau}$ with reference to $P_{\tau-1}$,
or
$P_{2}$ is a right neighbor of $P_{1}$ with reference to $P_{0}$ such that $P_{0} \neq P_{2}$ and $P_{\tau+1}$ is a right neighbor of $P_{\tau}$ with reference to $P_{\tau-1}$ such that $P_{\tau-1} \neq P_{\tau+1}$.
$T$ - and $S$-descriptor points of type $k$ are also termed $T_{k}$ - and $S_{k}$-descriptor points, respectively.
$P$ is called descriptor point of type $k$ whenever it is $T_{k^{-}}$or $S_{k^{-}}$-descriptor point.
For $k \in\{0, \cdots, 7\}$ one defines linear functionals

$$
l_{k}(x, y)=-x \sin \frac{k \pi}{4}+y \cos \frac{k \pi}{4} .
$$

If $P=\left(x_{P}, y_{P}\right)$ is a descriptor point of type $k$ then the descriptor tangent belonging to $P$ is the set

$$
\left\{(x, y) \in \mathbb{R}^{2} \mid l_{k}(x, y)=l_{k}(P)\right\} .
$$

Remark 2.1 When $S_{\triangle}$ consists of one single point it is a $T_{k}$-descriptor point of each type $k=0, \cdots, 7$.

Definition 2.1 means that the points $P_{1}, \cdots, P_{\tau}, \tau \geq 1$ are $T_{k}$-descriptor points whenever $l_{k}\left(P_{1}\right)=\cdots=l_{k}\left(P_{\tau}\right)=\alpha$, the sequence $P_{1}, \cdots, P_{\tau}$ has maximal possible length and the points $P_{0}$ and $P_{\tau+1}$ are lying not on the right side of the hyperplane $\left\{(x, y) \in \mathbb{R}^{2} \mid l_{k}(x, y)=\alpha\right\}$, i.e. $l_{k}\left(P_{0}\right)=\alpha$ and $l_{k}\left(P_{\tau+1}\right)=\alpha$ are possible.

The points $P_{1}, \cdots, P_{\tau}, \tau \geq 1$ are $S_{k}$-descriptor points whenever $l_{k}\left(P_{1}\right)=\cdots=$ $l_{k}\left(P_{\tau}\right)=\alpha$ and $P_{0}$ and $P_{\tau+1}$ are lying exactly on the right side of the hyperplane $\left\{(x, y) \in \mathbb{R}^{2} \mid l_{k}(x, y)=\alpha\right\}$, i.e $l_{k}\left(P_{0}\right) \neq \alpha$ and $l_{k}\left(P_{\tau+1}\right) \neq \alpha$. Under these circumstances the situation $l=3$ in the second part of condition 1. in Definition 2.1 does not appear as a not legal combination on the oriented boundary. Furthermore, for the situation $l=4$ the points are declared to be $T$-descriptor points.

There is here a difference with S-descriptor points defined in [14] and [15]. See Figure 2.1.


Figure 2.1: The set $S_{\triangle}$ is indicated by $\bullet$. The point marked by $\odot$ is a $T$-descriptor point, however, by Definition 2.1 not an $S$-descriptor point.

Unfortunately, there exists no duality between $T$ - and $S$-descriptor points. Generally, $T$ - or $S$-descriptor points cannot be considered as $S$ - or $T$-descriptor points, respectively, if orientation of the boundary is inverted.

Proposition 2.1 Given oriented boundary of a digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. Assume there exist no successive points $P, Q$ and $R$ on the boundary such that $Q=N_{k}(P)$ and $R=N_{k \pm 4(\bmod 8)}(Q)$. If by inversion the orientation of the boundary non legal moves do not appear then $T$-descriptor points become exactly $S$-descriptor points and vice versa.

Proof The proposition can be easily shown by Definition 2.1 of descriptor points.
According to condition 1. in Definition 2.1 for $T$-descriptor points there exists the possibility that more than one descriptor points with different types coincide.

The ordered sequence of descriptors on the boundary carries information about the shape of the object and allows a rough reconstruction of it. Each descriptor tangent meets $S_{\triangle}$ in one or more points. All points of the intersection of $S_{\triangle}$ and a descriptor tangent are the descriptor points belonging to the descriptor tangent. An example in Figure 2.2 shows a digital set with both $T$ - and $S$-descriptor points.


Figure 2.2: The points of a digital set $S_{\triangle}$ are marked by $\bullet$. The points $\odot$ are $T$ descriptor points as well as $S$-descriptor points.

Whenever $S_{\triangle}$ is a digitally convex set (see Definition 3.5, p. 22) it has only Tdescriptor points. There are digital sets which possess only $T$-descriptor points and, however, are not digitally convex. It is possible to give characterization of the sets which do not possess $S$-descriptor points (see Section 5.6, p. 69).

According to Definition 2.1 it is simple to constract an algorithm for detection all descriptor points on the oriented boundary. We are mostly interested in the sequence of descriptor points which is ordered in the sense of the given orientation.

Let us consider the oriented boundary of an 8-connected digital set $S_{\triangle}$. Let $\Pi$ be a polygonal curve with vertices $P^{1}, \cdots, P^{n}, n \geq 1$ describing the oriented boundary of $S_{\triangle}$. Detection descriptor points in the cases when $n=1$, i.e. $S_{\triangle}$ consists only of one single point, or $n=2$, i.e. all elements of $S_{\triangle}$ are collinear, is trivial. Otherwise, $\sharp S_{\triangle} \geq 3$ and there are at least three vertices of $\Pi$ which are noncollinear. Assume $V=\left\{P^{1}, \cdots, P^{n}, P^{n+1}=P^{1}, P^{n+2}=P^{2}, P^{n+3}=P^{3}\right\}$ is an extended set of vertices of $\Pi$, between two successive vertices the chain code on the oriented boundary of $S_{\triangle}$ is constant. Without loss of generality we may write $v\left(P^{j}, P^{j+1}\right)$ for the chain code between vetrices $P^{j}$ and $P^{j+1}, j=1, \cdots, n+2$ of $V$.

Algorithms DescTripl and DescQuadr detect when $P^{j}$ is a descriptor point of a triple $P^{j-1}, P^{j}, P^{j+1}$ and when $P^{j}$ and $P^{j+1}$ are both $T$ - or $S$-descriptor points

```
Algorithm II DescTripl Testing when \(P^{j}\) is a descriptor point for the triple of
    vertices \(P^{j-1}, P^{j}, P^{j+1}\) of \(V\) using condition 1 .
\(\mathrm{v}_{1} \leftarrow \mathrm{v}\left(P^{j-1}, P^{j}\right) ; \mathrm{v}_{2} \leftarrow \mathrm{v}\left(P^{j}, P^{j+1}\right) ; / *\) chain codes between successive
vertices on the boundary */
/ * TYPES shows if \(P^{j}\) is a T-descriptor point of one or more types */
if \(v_{1}-2(\bmod 8)=v_{2}\) or \(v_{1}+2(\bmod 8)=v_{2}\) or \(v_{1}+3(\bmod 8)=v_{2}\) or
\(v_{1}+4(\bmod 8)=v_{2}\) then
    if \(v_{1}-2(\bmod 8)=v_{2}\) then
        \(/ * P^{j}\) is an \(S_{v_{1}-1(\bmod 8)^{-}}\)descriptor point \(* /\)
    else
        if \(v_{1}+2(\bmod 8)=v_{2}\) then TYPES \(\leftarrow 1\);
        if \(v_{1}+3(\bmod 8)=v_{2}\) then TYPES \(\leftarrow 2\);
        if \(v_{1}+4(\bmod 8)=v_{2}\) then TYPES \(\leftarrow 3\);
        for \(i \leftarrow 1\) to TYPES do
            \(\mid / * P^{j}\) is a \(T_{\mathrm{v}_{1}+i(\bmod 8)}\)-descriptor point \(* /\)
        end
    end
end
```

of the same type of a quadriple $P^{j-1}, P^{j}, P^{j+1}, P^{j+2}$ of vertices of $V$ for some $j \in$ $\{2, \cdots, n+1\}$, respectively.

Theorem 2.1 Successive testing the condition 1. for triples $P^{j-1}, P^{j}, P^{j+1}$ and the condition 2. for quadriples $P^{j-1}, P^{j}, P^{j+1}, P^{j+2}, j=2, \cdots, n+1$ of vertices of $V$ using Algorithms DescTripl and DescQuadr, respectively, leads to detection all descriptor points ordered in the sense of the given orientation. Detection has a linear time complexity.

Proof It is clear that by the successive testing quadriples $P^{j-1}, P^{j}, P^{j+1}, P^{j+2}$, $j=2, \cdots, n+1$ of vertices of $V$ first when $P^{j}$ is a descriptor point and second when $P^{j}$ and $P^{j+1}$ are both $T$ - or $S$-descriptor points of the same type we detect all of the possible descriptor points in the order given by the orientation. Since one starts detection at each $j$ from 2 until $n+1$ one has the linear time complexity.

We will show that the Algorithm DescTripl detect when $P^{j}$ is a descriptor point described in the condition 1. of Definition 2.1. Since the chain code between each two vertices of $\Pi$ are constant we may assume that $P_{1}=P^{j}, P_{0}$ is the predecessor of $P^{j}$ and $P_{2}$ is the successor of $P^{j}$ on the oriented boundary of $S_{\triangle}$. Obviously, $P_{2}$ is a left or right neighbor of $P_{1}$ with reference to $P_{0}$. Furthermore, the chain codes are $v_{1}=v\left(P^{j-1}, P^{j}\right)=v\left(P_{0}, P_{1}\right)$ and $v_{2}=v\left(P^{j}, P^{j+1}\right)=v\left(P_{1}, P_{2}\right)$. We have $P_{1}=N_{\mathrm{v}_{1}}\left(P_{0}\right)$ and $P_{2}=N_{\mathrm{v}_{2}}\left(P_{1}\right)$. By Definition 2.1 if $\mathrm{v}_{1}+l(\bmod 8)=\mathrm{v}_{2}, 2 \leq l \leq 4$ or $\mathrm{v}_{1}-2(\bmod 8)=\mathrm{v}_{2}$ then $P_{1}$ is a $T$ - or an $S$-descriptor point, respectively. We consider the case for which $P_{1}$ is a $T$-descriptor point, other case is analogous.

> Algorithm III DescQuadr Testing when $P^{j}$ and $P^{j+1}$ are both $T$ - or $S$ descriptor points of the same type for the quadriple of vertices $P^{j-1}, P^{j}, P^{j+1}, P^{j+2}$ of $V$ using condition 2 .
> $v_{1} \leftarrow v\left(P^{j-1}, P^{j}\right) ; v_{2} \leftarrow v\left(P^{j}, P^{j+1}\right) ; v_{3} \leftarrow v\left(P^{j+1}, P^{j+2}\right) ; / *$ chain codes between successive vertices on the boundary $* /$
> if $\left(v_{1}+1(\bmod 8)=v_{2}\right.$ or $v_{1}+2(\bmod 8)=v_{2}$ or $v_{1}+3(\bmod 8)=v_{2}$ or
> $\left.v_{1}+4(\bmod 8)=v_{2}\right)$ and $\left(v_{2}+1(\bmod 8)=v_{3}\right.$ or $v_{2}+2(\bmod 8)=v_{3}$ or
> $v_{2}+3(\bmod 8)=v_{3}$ or $\left.v_{2}+4(\bmod 8)=v_{3}\right)$ then
> end $/ * P^{j}$ and $P^{j+1}$ are $T_{v_{2}}$-descriptor points $* /$
> if $\left(v_{1}-1(\bmod 8)=v_{2}\right.$ or $v_{1}-2(\bmod 8)=v_{2}$ or $\left.v_{1}-3(\bmod 8)=v_{2}\right)$ and
> $\left(v_{2}-1(\bmod 8)=v_{3}\right.$ or $v_{2}-2(\bmod 8)=v_{3}$ or $\left.v_{2}-3(\bmod 8)=v_{3}\right)$ then end $/ * P^{j}$ and $P^{j+1}$ are $S_{v_{2}}-$ descriptor points $* /$

Whenever we have $l=2$ then $P_{1}$ is a descriptor point only of one type $v_{1}+1(\bmod 8)$. Otherwise, $P_{1}$ is a descriptor point of 2 or 3 different types which are $v_{1}+1(\bmod 8)$, $v_{1}+2(\bmod 8)$ or $v_{1}+1(\bmod 8), v_{1}+2(\bmod 8), v_{1}+3(\bmod 8)$, respectively.

To show the second part of the theorem for quadriples we may assume that $P_{1}=$ $P^{j}, P_{\tau}=P^{j+1}$, then the points $P_{0}, P_{2}, P_{\tau-1}, P_{\tau+1}$ are predecessors or successors of $P_{1}$ or $P_{\tau}$ on the oriented boundary. By construction holds $v_{1}=v\left(P^{j-1}, P^{j}\right)=v\left(P_{0}, P_{1}\right)$, $v_{2}=v\left(P^{j}, P^{j+1}\right)=v\left(P_{1}, P_{2}\right)=\cdots=v\left(P_{v-1}, P_{v}\right), v_{3}=v\left(P^{j+1}, P^{j+2}\right)=v\left(P_{\tau}, P_{\tau+1}\right)$ and $v_{1} \neq v_{2}, v_{3} \neq v_{2}$. Algorithm DescQuadr tests using chain codes when $P_{2}$ and $P_{\tau+1}$ are both left or right neighbors of $P_{1}$ and $P_{\tau}$ with reference to $P_{0}$ and $P_{\tau-1}$, respectively. For $T$-descriptor points there are 4 possibilities, for $S$-descriptor points one has 3 possibilities.

A very useful characteristic of descriptor points on the oriented boundary is the fact that their succession is not arbitrarily (see also [15, 43]). This property is shown in the next lemma.

Lemma 2.1 Let P be a descriptor point on the oriented boundary of a digital set $S_{\triangle}$ and let $Q$ be the next descriptor point in the order given by the orientation. Then only following situations are possible:

1. $P$ and $Q$ are both $T_{k^{-}}$or $S_{k}$-descriptor points of the same type $k$. Then the segment between $P$ and $Q$ has the chain code direction $k$.
2. $P$ is a $T_{k}$-descriptor point and $Q$ is a $T_{k+1(\bmod 8)}$-descriptor point. Then either $P=Q$ or the boundary segment between $P$ and $Q$ has only chain code
directions $k$ and $k+1(\bmod 8)$.
3. $P$ is an $S_{k}$-descriptor point and $Q$ is an $S_{k-1(\bmod 8)}$-descriptor point. Then the boundary segment between $P$ and $Q$ has only chain code directions $k$ and $k-$ $1(\bmod 8)$.
4. $P$ is a $T_{k}$-descriptor point and $Q$ is an $S_{k}$-descriptor point. Then the segment between $P$ and $Q$ has only chain code directions $k$ and $k+1(\bmod 8)$.
5. $P$ is an $S_{k}$-descriptor point and $Q$ is a $T_{k}$-descriptor point. Then the segment between $P$ and $Q$ has only chain code directions $k$ and $k-1(\bmod 8)$.

Proof The statement about the chain code directions between two successive descriptor points is shown in [15].

We will show that only 5 situations described in this lemma are available.
Assume $P$ is a $T_{k}$-descriptor point. Then the first possible chain code direction after $P$ is $k+i(\bmod 8), i=1,2,3,4$. If $i=2,3,4$ then two or more descriptor points coincide and the next descriptor point is, obviously, a $T_{k+1(\bmod 8)}$-descriptor point. If the next chain code is $k+1(\bmod 8)$ then by Definition 2.1 it is clear that descriptor points do not appear whenever the chain code directions after $P$ are changing successivelly by $k+1(\bmod 8)$ and $k$. We consider two possible situations. The sequence of $k+1(\bmod 8)$ and $k$ directions endes by $k$ or by $k+1(\bmod 8)$. If the sequence endes by $k+1(\bmod 8)$ then it is possible that the sequence consists only of chain code $k+1(\bmod 8)$, if it endes by $k$ then there exists at least one part of the sequence with the chain code $k+1(\bmod 8)$. Clearly, the next direction on the boundary is different from $k+1(\bmod 8)$ and $k$. Whenever the sequence endes by $k$ or $k+1(\bmod 8)$ we have the following configurations for $k=0,2,4,6$ :

| Next chain code | Sequence endes by $k$ | Sequence endes by $k+1(\bmod 8)$ |
| :---: | :---: | :---: |
| $k$ | not possible | not possible |
| $k+1(\bmod 8)$ | not possible | not possible |
| $k+2(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k+3(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k+4(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k-1(\bmod 8)$ | $S_{k}$ | $S_{k}$ |
| $k-2(\bmod 8)$ | not possible | not possible |
| $k-3(\bmod 8)$ | not possible | $T_{k+1(\bmod 8)}$ |

For $k=1,3,5,7$ one has:

| Next chain code | Sequence endes by $k$ | Sequence endes by $k+1(\bmod 8)$ |
| :---: | :---: | :---: |
| $k$ | not possible | not possible |
| $k+1(\bmod 8)$ | not possible | not possible |
| $k+2(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k+3(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k+4(\bmod 8)$ | $T_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |
| $k-1(\bmod 8)$ | $S_{k}$ | not possible |
| $k-2(\bmod 8)$ | $S_{k}$ | not possible |
| $k-3(\bmod 8)$ | not possible | $T_{k+1(\bmod 8)}$ |

From the assumption that $P$ is an $S_{k}$-descriptor point it can be shown in the anologous way that the next descriptor point is either a $T_{k}$-descriptor point or an $S_{k-1(\bmod 8)}$-descriptor point.

Given an 8-connected digital set $S_{\triangle} \in \mathbb{Z}^{n \times 2}$. According to Lemma 2.1 for two successive descriptor points $P$ and $Q, P \neq Q$ on the oriented boundary of $S_{\triangle}$ we have following possible types $k \in\{0, \cdots, 7\}$ of descriptor points:

|  | Type of $P$ | Type of $Q$ |
| :---: | :---: | :---: |
| 1. | $T_{k}$ | $T_{k}$ |
| 2. | $S_{k}$ | $S_{k}$ |
| 3. | $T_{k}$ | $T_{k+1(\bmod 8)}$ |
| 4. | $T_{k}$ | $S_{k}$ |
| 5. | $S_{k+1(\bmod 8)}$ | $S_{k}$ |
| 6. | $S_{k+1(\bmod 8)}$ | $T_{k+1(\bmod 8)}$ |

In the configurations 3. until 6. the chain code directions between $P$ and $Q$ are $k$ and $k+1(\bmod 8)$. We deduce:

Lemma 2.2 Given an 8-connected digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. Then the boundary of $S_{\triangle}$ can be decomposed into $(k, k+1(\bmod 8))$-curves such that the part on the boundary between two successive curves consists exclusively of descriptor points of the same type.

The part on the boundary between two successive curves from Lemma 2.2 is a segment of a horizontal, vertical or diagonal grid line consisting exclusively of descriptor points of $S_{\triangle}$.

Finely, we consider linear transforms $T^{k} \in \mathbb{Z}^{2 \times 2}, k \in\{0, \cdots, 7\}$

|  | $T^{k}$ | $\left(T^{k}\right)^{-1}$ |
| :--- | :---: | :---: |
| $k=0$ | $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ |
| $k=2$ | $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ |
| $k=3$ |  |  |
| $k=4$ | $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ |
| $\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)$ |  |
| $k=5$ | $\left(\begin{array}{cc}-1 & 0 \\ 0 & -1 \\ -1 & 0\end{array}\right)$ | $\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)$ |
| $\left(\begin{array}{cc}0 & -1 \\ -1 & 0\end{array}\right)$ |  |  |
| $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ | $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$ |  |
| $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ | $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ |  |

such that

$$
T^{k}\left(S_{\triangle}\right)=S_{\triangle} T^{k}
$$

It follows that the segment of the boundary between $P$ and $Q$ under corresponding transform is a $(0,1)$-curve.

Figure 2.3 demonstrates all descriptor points and their types of a digital set.


Figure 2.3: Digital set "Letter A". Descriptor points and their types are indicated.

## Chapter 3

## Digital Convexity

The term of convexity is a central subject of many geometrical investigations. Particularly, in the application oriented disciplines of geometry it plays an important role. The basic constructions of digital geometry are discrete lines, discrete line segments and digitally convex sets. They belong since beginning of the research in digital geometry to the frequently examined objects.

### 3.1 Discrete Lines

We adapt the definition of discrete lines introduced by J.-P. Reveillès [40].
Definition 3.1 $A$ discrete line with $a$ slope $a / b, b \neq 0$ and $\operatorname{pgcd}(a, b)=1$, lower bound $\mu$, arithmetical thickness $\omega$ is the set of grid points which satisfies the double diophantine inequality

$$
\mu \leq a x-b y<\mu+\omega
$$

with all integer parameters.
A (finite or infinite) subsequence of a discrete line is called a discrete line segment.

We denote the preceding discrete line $\mathcal{D}(a, b, \mu, \omega)$. We are mostly interested in naïve lines which verify $\omega=\sup (|a|,|b|)$, we shall denote them $\mathcal{D}(a, b, \mu)$. Without loss of generality we may consider discrete lines under restrictions $a, b>0$ and $a<b$, therefore $\omega=\max (a, b)=b$.

The real straight lines $a x-b y=\mu$ and $a x-b y=\mu+b-1$ are called upper leaning line and lower leaning line of $\mathcal{D}(a, b, \mu)$, respectively. There are no grid points of the complement $\mathbb{C D}(a, b, \mu)$ between the upper and lower leaning lines and $\mathcal{D}(a, b, \mu)$. The grid points satisfying the leaning line equalities are called upper and lower leaning points. We remark that the distinction between lower and upper leaning points depends on the equation, there is here no geometrical invariancy.

It can be shown [40] that a discrete line $\mathcal{D}(a, b, \mu)$ with slope $a / b \leq 1$ has exactly one grid point on each vertical line. If $a / b<1$ then the intersection between
$\mathcal{D}(a, b, \mu)$ and any horizontal line is composed by $[b / a]$ or $[b / a]+1$ successive grid points, where [ ] means the integer part. Thus, according to Proposition 3.2 (see below) and considering discrete line segments with minimal parameters $a$ and $b$, we may denote $U_{F}\left(L_{F}\right)$ the upper (lower) leaning point of a discrete line segment with slope $a / b<1$ whose $x$-coordinate is minimal. In the same way, we denote $U_{L}$ $\left(L_{L}\right)$ the upper (lower) leaning point whose $x$-coordinate is maximal. An example in Figure 3.1 shows a segment of the discrete line $\mathcal{D}(5,8,-4)$ with its leaning points and leaning lines.


Figure 3.1: Segment of discrete line $\mathcal{D}(5,8,-4)$. Dashed lines represent upper and lower leaning lines of the segment. Upper and lower leaning poins are indicated by pale and dark triangles.

A discrete line $\mathcal{D}(a, b, \mu)$, where $0<a<b$, satisfies the chord property and is 8 -connected $[8,40]$. It follows that a discrete line segment is a digital straight line segment in sense of Hübler, Klette, Voss [21]. The finite digital curves with the chord property are discrete line segments. There are infinite digital curves which satisfy the chord property and, however, are not discrete lines [20].

We collect some simple properties of discrete lines. The proof of the following proposition can be found in [40].

Proposition 3.1 A discrete line $\mathcal{D}(a, b, \mu)$ with $0<a<b$ is an 8 -curve.
This result implies that the movement from left to right along a discrete line with $0<a<b$ occurs by using of two translations either $(x, y) \mapsto(x+1, y)$ or $(x, y) \mapsto(x+1, y+1)$. We have shown:

Proposition 3.2 Each discrete line $\mathcal{D}(a, b, \mu)$ with $0<a<b$ is a $(0,1)$-curve.
Since a discrete line $\mathcal{D}(a, b, \mu)$ is an 8 -curve the concept of levels and their lengths of $\mathcal{D}(a, b, \mu)$ are justified, it coincides with the definition for a digital curve.

Proposition 3.3 The lengths of horizontal levels of a discrete line $\mathcal{D}(a, b, \mu)$ with $0<a<b$ are different mostly by one.

Proof The intersection between $\mathcal{D}(a, b, \mu)$ and any horizontal line is composed by $[b / a]$ or $[b / a]+1$ which are possible lengths of the levels.

Proposition 3.4 Let $\mathcal{D}(a, b, \mu)$ with $0<a<b$ be a discrete line. Then the upper and lower leaning points of $\mathcal{D}(a, b, \mu)$ belong to such horizontal levels of the discrete line which possess the maximal lengths.

Proof Trivial.

Clearly this proposition is not true for discrete line segments.
Proposition 3.5 Given a discrete line $\mathcal{D}(a, b, \mu)$ with $0<a<b$. Assume lengths of the horizontal levels of $\mathcal{D}(a, b, \mu)$ are $i$ and $i+1$. Then $\frac{1}{i+1} \leq \frac{a}{b} \leq \frac{1}{i}$.

Proof The slope $\frac{a}{b}$ of the discrete line is at least $\frac{1}{i+1}$ whenever all horizontal levels have lengths $i+1$. Othewise, it can be at most $\frac{1}{i}$ if all levels have lengths $i$.

### 3.2 Digitally Convex Sets

In Euclidean geometry a set in $\mathbb{R}^{d}$ is said to be convex if whenever it contains two points, it also contains the line segment joining them. Already in two-dimensional case there were observed difficulties by direct transfer of this definition into digital circumstances (see e.g. [20]). In the literature there exist different types of digital convexity. The most common of them are studied in [22, 23, 24, 37].

We will introduce some useful concepts from ordinary geometry.
Definition 3.2 A polygonal curve $\Pi=(V, E)$ in $\mathbb{R}^{2}$ consists of a cyclically ordered set of vertices $V=\left(v_{0}, v_{1}, \cdots, v_{n}\right) \subseteq \mathbb{R}^{2}$ and a set of edges $E \subseteq V \times V$. $E$ is the set of all line segments joining $v_{i}$ and $v_{i+1}, i=0,1, \cdots, n-1$.

Usually it is assumed that there are finitely many vertices. Sometimes also infinite edges are allowed.

A polygonal curve $\Pi=(V, E)$ is said to be
bounded if $V$ is a finite set and if there are no infinite edges;
closed if each vertex belongs to exactly two edges;
simple if two edges are either disjoint or meet in a vertex, and if $v_{i} \neq v_{j}$ for $i \neq j$.
A polygonal set $\Pi$ is a finite set of simple closed curves $\Pi_{1}, \Pi_{2}, \cdots, \Pi_{n}$ which are mutually disjoint.

Definition 3.3 Given a polygonal curve $\Pi \subset \mathbb{R}^{2}$ and three successive vertices $\left(x_{1}, y_{1}\right)$, $\left(x_{2}, y_{2}\right)$ and $\left(x_{3}, y_{3}\right)$ on $\Pi$. The point $\left(x_{2}, y_{2}\right)$ is a convex vertex of $\Pi$ if the determinant

$$
\left|\begin{array}{ll}
x_{2}-x_{1} & x_{3}-x_{2} \\
y_{2}-y_{1} & y_{3}-y_{2}
\end{array}\right|
$$

is positive. $\left(x_{2}, y_{2}\right)$ is a concave vertex of $\Pi$ if the determinant is negative. If the determinant vanishes then points $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)$ and $\left(x_{3}, y_{3}\right)$ are collinear.

Definition 3.4 Let $\Pi$ be a polygonal curve. A part of the curve is said to be a maximal convex part if it is consists of a set $P_{1}, P_{2}, \cdots, P_{k}$ of successive vertices of $\Pi$ together with the lines joining them such that $P_{1}$ and $P_{k}$ are concave vertices of $\Pi$ and $P_{2}, P_{3}, \cdots, P_{k-1}$ are convex vertices of $\Pi$. If $\Pi$ has only convex vertices, the only convex part of $\Pi$ is $\Pi$ itself. In this case $\Pi$ is called convex curve.
$A$ maximal concave part of $\Pi$ is defined in the same manner by replacing in the above definition the terms "convex" and "concave".

We here have a perfect duality: If we replace a polygonal set by its complement, the orientation of the curve is reversed and maximal convex parts become maximal concave parts and vice versa. Furthermore, the common part of two successive maximal parts is one single edge of the curve.

Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a segment of a discrete line $\mathcal{D}(a, b, \mu)$. The problem to determine the convex hull of the elements of $\mathcal{K}$ is solved in [9]. The convex hull of $\mathcal{K}$ is a closed polygonal curve which can be subdivided into two polygonal curves joining $\kappa_{1}$ and $\kappa_{n}$ : the lower frontier and upper frontier of the convex hull. It is clear that the lower leaning points of this segment belong to the lower frontier, upper leaning points belong to the upper frontier. How to detect all points of the lower and upper frontier is shown in [9, Proposition 3, p.120] (see also Proposition 4.3, p. 41). Since $\mathcal{K}$ is a segment of a discrete line the intersection of $\mathcal{K}$ and its convex hull consists only of elements of $\mathcal{K}$. Moreover, all vertices of the lower and upper frontier of $\mathcal{K}$ are convex. These facts justify the following concept of digitally convex sets of $\mathbb{Z}^{d}$ :

Let denote $\operatorname{conv} S_{\triangle}$ the convex hull of $S_{\triangle} \subseteq \mathbb{R}^{d}$.
Definition 3.5 A set $S_{\triangle} \subseteq \mathbb{Z}^{d}$ is called digitally convex whenever

$$
S_{\triangle}=\mathbb{Z}^{d} \cap \operatorname{convS}_{\triangle} .
$$

Corollary 3.1 $A$ digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is digitally convex if and only if there exists a convex polygonal set $\Pi \subset \mathbb{R}^{2}$ such that $\Pi \cap \mathbb{Z}^{2}=S_{\triangle}$.

## Proof Trivial.

Segments of discrete lines are simpliest examples for digitally convex sets. In general digitally convex sets are not necessarily 8 -connected, e.g. the set consisting of $(x, y)$ and $(x+2, y+1)$ is digitally convex, but not 8 -connected.

Testing convexity of 8-connected digital sets in $\mathbb{Z}^{2}$ can be restricted to testing convexity of their boundaries which can be subdivided into $(k, k+1(\bmod 8))$-curves (see Lemma 2.2, p. 15). In order to introduce convexity of digital curves we define:

Definition 3.6 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a finite $(0,1)$-curve. $\mathcal{K}$ is said to be lower digitally convex (upper digitally convex) if there is no grid point between $\mathcal{K}$ and the lower (upper) frontier of the convex hull of $\mathcal{K}$.

The algorithm SegConv for testing convexity of digital curves from [9] can be applied for testing convexity of 8-connected sets (see Theorem 3.2, p. 29 and Theorem 3.3, p. 30). This algorithm has the linear time complexity.

### 3.3 Fundamental Segments of 8-curves

We consider a segment $\Sigma$ of a discrete line $\mathcal{D}(a, b, \mu)$, where $0<a<b$, and $\left(x_{1}, y_{1}\right)$ and $\left(x_{n}, y_{n}\right)$ are the first and last points of the discrete segment, respectively. Let us suppose that the point $(x, y)$ with $(x, y)=\left(x_{n}+1, y_{n}\right)$ or $(x, y)=\left(x_{n}+1, y_{n}+1\right)$ is added to $\Sigma$. Is $\bar{\Sigma}=\Sigma \cup(x, y)$ a discrete line segment and, if it is the case, that are its characteristics $\bar{a}, \bar{b}, \bar{\mu}$ ?

Based on Theorem 3.1 (see below) linear Algorithm AddPoint describing solution of this problem was proposed in [8] (see also [9]). For this reason an indicator for each grid point $(x, y)$ called remainder was introduced. The remainder $r(x, y)$ shows the relationship between a point $(x, y)$ and a discrete line $\mathcal{D}(a, b, \mu)$.

Definition 3.7 The number

$$
r(x, y)=a x-b y
$$

is called remainder at $(x, y)$ with respect to $\mathcal{D}(a, b, \mu)$.
For a discrete line $\mathcal{D}(a, b, 0)$ the remainder $r(x, y)$ represents within multiplicator $\left(a^{2}+b^{2}\right)^{-1 / 2}$ Euclidean distance between $(x, y)$ and the real line $a x-b y=0$.

Theorem 3.1 Let $r(x, y)$ be remainder at $(x, y)$ with respect to $\mathcal{D}(a, b, \mu)$.

1. If $\mu \leq r(x, y)<\mu+b$ then $(x, y) \in \mathcal{D}(a, b, \mu)$ and $\Sigma \cup(x, y)$ is a segment of the discrete line $\mathcal{D}(a, b, \mu)$.
2. If $r(x, y)=\mu-1$ then $\Sigma \cup(x, y)$ is a segment of the discrete line with slope $\vec{U}_{F}(x, y)$.
3. If $r(x, y)=\mu+b$ then $\Sigma \cup(x, y)$ is a segment of the discrete line with slope $\vec{L}_{F}(x, y)$.
4. If $r(x, y)<\mu-1$ or $r(x, y)>\mu+b$ then $\Sigma \cup(x, y)$ is not a segment of a discrete line.
```
Algorithm IV AddPoint adding a point \(P=\left(x_{P}, y_{P}\right)\) to the segment \(\Sigma\) of \(a\)
    discrete line \(\mathcal{D}(a, b, \mu)\).
remainder \(\leftarrow a x_{P}-b y_{P}\);
if \(\mu \leq\) remainder \(<\mu+b\) then
    if remainder \(=\mu\) then \(U_{L} \leftarrow\left(x_{P}, y_{P}\right)\);
    if remainder \(=\mu+b-1\) then \(L_{L} \leftarrow\left(x_{P}, y_{P}\right)\);
else
    if remainder \(=\mu-1\) then
        \(L_{F} \leftarrow L_{L}\);
        \(U_{L} \leftarrow\left(x_{P}, y_{P}\right) ;\)
        \(a \leftarrow\left|y_{P}-y_{U_{F}}\right| ;\)
        \(b \leftarrow\left|x_{P}-x_{U_{F}}\right| ;\)
        \(\mu \leftarrow a x_{P}-b y_{P} ;\)
    else
        if remainder \(=\mu+b\) then
            \(U_{F} \leftarrow U_{L} ;\)
            \(L_{L} \leftarrow\left(x_{P}, y_{P}\right) ;\)
            \(a \leftarrow\left|y_{P}-y_{L_{F}}\right| ;\)
            \(b \leftarrow\left|x_{P}-x_{L_{F}}\right| ;\)
            \(\mu \leftarrow a x_{P}-b y_{P}-b+1 ;\)
        else
            /* the point may not be added to the segment */
        end
    end
end
```

Now we are able to introduce fundamental segments of a $(0,1)$-curve.
Definition 3.8 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a $(0,1)$-curve. Parameters $a$ and $b$ in discrete line segments considered below are assumed to be minimal. A part $\left(\kappa_{i}, \cdots, \kappa_{j}\right)$ is called $a$ fundamental segment of $\mathcal{K}$ whenever one of the following conditions is true:

1. $i=1, j=n$ and $\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ is a segment of $\mathcal{D}(a, b, \mu)$. Then $\mathcal{K}$ consists of one single fundamental segment.
2. $i=1, j<n$ and $\left(\kappa_{1}, \cdots, \kappa_{j}\right)$ is a segment of $\mathcal{D}(a, b, \mu)$ such that $\left(\kappa_{1}, \cdots, \kappa_{j+1}\right)$ is not a segment of any discrete line. Here, $\left(\kappa_{1}, \cdots, \kappa_{j}\right)$ is the first fundamental segment of $\mathcal{K}$.
3. $i>1, j=n$ and $\left(\kappa_{i}, \cdots, \kappa_{n}\right)$ is a segment of $\mathcal{D}(a, b, \mu)$ such that $\left(\kappa_{i-1}, \cdots, \kappa_{n}\right)$ is not a segment of any discrete line. Here, $\left(\kappa_{i}, \cdots, \kappa_{n}\right)$ is the last fundamental segment of $\mathcal{K}$.
4. $i>1, j<n$ and $\left(\kappa_{i}, \cdots, \kappa_{j}\right)$ is a segment of $\mathcal{D}(a, b, \mu)$ such that $\left(\kappa_{i-1}, \cdots, \kappa_{j}\right)$ and $\left(\kappa_{i}, \cdots, \kappa_{j+1}\right)$ are not segments of any discrete line.

The fundamental segment $\left(\kappa_{i}, \cdots, \kappa_{j}\right)$ will be denoted $\mathcal{F}(a, b, \mu)$.
By this definition the convex hull of a fundamental segment of $\mathcal{K}$ and the left or right added point consists at least of one grid point of the complement of $\mathcal{K}[9$, Remark 6]. Hence, fundamental segments are maximal possible subsets of $\mathcal{K}$ belonging to discrete lines. Moreover, fundamental segments do not depend of the orientation of $\mathcal{K}$.

Assume a curve $\mathcal{K}$ possesses $m$ fundamental segments. Then all fundamental segments can be ordered in the sense of the oriented curve, we mark these $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$. It is clear that two successive fundamental segments have allways different slopes and their common part is not empty and it is always a segment of a discrete line. In addition, more than two fundamental segments can possess common parts of $\mathcal{K}$.

Clearly, decomposition of a $(0,1)$-curve into fundamental segments is unique. The problem to find this decomposition is equivalent with the problem to determine subsets of the curve having constant tangents. It can be computed within linear time [17].

In Figure 3.2 fundamental segments of a digital $(0,1)$-curve are indicated.

### 3.4 Convex and Concave Curves

In Euclidean geometry the boundary of a polygonal set $\Pi$ can be easily partitioned into maximal convex and concave parts by means of convex and concave vertices of $\Pi$. The partition of the boundary of a digital set into the meaningful parts is not a simple task. By Tietze's theorem [45] convexity of a set in $\mathbb{R}^{2}$ can be decided locally in a time which is proportional to the length of the boundary of the set. In $\mathbb{Z}^{2}$ one can easily deduce that convexity of a set cannot be decided locally [11]. So, it becomes an interesting question, how far one can decide whether a part of the boundary of a digital set is convex or concave by a method which is "as local as possible". In [14] the idea was suggested to determine a corresponding polygonal representation of a digital set $S_{\triangle}$ using the concept of exposed points of $S_{\triangle}$ (see Definition 4.2, p. 37), the exposed points were defined also for non convex sets. The meaningful parts of the set were defined as meaningful parts of the corresponding


Figure 3.2: Fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, 6$ of a digital curve. The first point of the curve is $(0,0)$. Lower bounds $\mu_{i}, i=1, \cdots, 6$ of fundamental segments are computed with respect to $(0,0)$.
polygonal representation. The obvious and interesting fact is that the parts of the boundary of $S_{\triangle}$ between two successive exposed points are discrete line segments.

In our considerations we use geometry of digital sets, especially, geometry of discrete line segments. We will introduce another approach how to define convex and concave parts of a digital curve using the concept of fundamental segments.

Next definition shows that fundamental segments allow an adaption of the term convex and concave curve from continuous theory.

Definition 3.9 Let $\mathcal{K}$ be a $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. The curve $\mathcal{K}$ is called convex (concave) whenever the sequence of the slopes of fundamental segments is increasing (decreasing), i.e. $\frac{a_{j}}{b_{j}}<\frac{a_{j+1}}{b_{j+1}}$ $\left(\frac{a_{j}}{b_{j}}>\frac{a_{j+1}}{b_{j+1}}\right), 1 \leq j \leq m-1$.

In the case $\mathcal{K}$ is a discrete line segment it is both convex and concave.
Since the sequence of fundamental segments does not depend of the orientation of $\mathcal{K}$ the concave curve is a convex one if the orientation of $\mathcal{K}$ is reversed. In the following considerations only convex case will be proved, the concave case can be formulated analogously and shown by duality.

Lower leaning points of fundamental segments of convex curves are located not arbitrarily. Namely, they appear in the successive order on the curve. This statement is proved in the next proposition.

We mark the $x$ - and $y$-coordinate of a point $P$ as $x_{P}$ and $y_{P}$, respectively.
Proposition 3.6 Let $\mathcal{K}$ be a convex (concave) ( 0,1 )-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=$ $1, \cdots, m$ are fundamental segments of $\mathcal{K}$. Then for lower (upper) leaning points holds

$$
x_{L_{L_{j}}} \leq x_{L_{F_{j+1}}} \quad\left(x_{U_{L_{j}}} \leq x_{U_{F_{j+1}}}\right)
$$

for all $1 \leq j \leq m-1$.
Proof Given fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ of $\mathcal{K}$. Let $\Pi$ be the polygonal set consisting of edges $e_{1}: a_{j} x-b_{j} y=\mu_{j}+b_{j}-1$ and $e_{2}$ : $a_{j+1} x-b_{j+1} y=\mu_{j+1}+b_{j+1}-1$ which are lower leaning lines of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. One single vertex $V=\left(x_{V}, y_{V}\right)$ of $\Pi$ is the intersection point of $e_{1}$ and $e_{2}$. Obviously, $\mathcal{K}$ is above $\Pi$. Since the sequence of the slopes is increasing it holds $x_{L_{L_{j}}} \leq x_{V}$ and $x_{V} \leq x_{L_{F_{j+1}}}$.

In Figure 3.3 a convex curve with different locations of lower leaning points of fundamental segments is represented.


Figure 3.3: Convex curve and its lower leaning points of fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1,2,3$. For lower leaning points holds $x_{L_{L_{1}}}<x_{L_{F_{2}}}$ and $x_{L_{L_{2}}}=x_{L_{F_{3}}}$.

The statement corresponding to Proposition 3.6 about succession of upper leaning points on convex curve is, generally, not true. An example is shown in Figure 3.4.


Figure 3.4: Convex curve and its upper leaning points of fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1,2,3$. For upper leaning points $U_{L_{1}}$ and $U_{F_{2}}$ holds $x_{U_{L_{1}}}<x_{U_{F_{2}}}$, however, for $U_{L_{2}}$ and $U_{F_{3}}$ holds $x_{U_{L_{2}}}>x_{U_{F_{3}}}$.

If for leaning points $L_{L_{j}}$ and $L_{F_{j+1}}$ of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ of a convex curve $\mathcal{K}$ holds $x_{L_{L_{j}}}=x_{L_{F_{j+1}}}$ then, obviously, $L_{L_{j}}=L_{F_{j+1}}$ is a vertex of
the lower frontier of the convex hull of $\mathcal{K}$. Before the situation $x_{L_{L_{j}}}<x_{L_{F_{j+1}}}$ will be examined we introduce a concept of supporting lines.

A real line $L$ is called a lower supporting line in $P \in \mathcal{K}$ (briefly, LSL) if $P \in L$ and there exists a (continuous) neighborhood $N(P)$ of $P$ such that all elements in $\mathcal{K} \cap N(P)$ are lying on or above $L$. A convex curve $\mathcal{K}$ with a fundamental segment $\mathcal{F}(a, b, \mu)$, whose leaning points are $L_{F}$ and $L_{L}$, is lying on or above lower leaning line of $\mathcal{F}(a, b, \mu)$. Hence, $a x-b y=\mu+b-1$ is a LSL in $L_{F}, L_{L}$ and all grid points which belong to $\mathcal{K}$ on the real line segment $\left[L_{L}, L_{F}\right]$. Moreover, there exists no grid point between the segment $\left(L_{F}, \cdots, L_{L}\right)$ of $\mathcal{K}$ and $\left[L_{F}, L_{L}\right]$. If the whole curve $\mathcal{K}$ is on or above a LSL, then the LSL is also called a global lower supporting line (briefly, GLSL).

If an arbitrary $(0,1)$-curve $\mathcal{K}$ with $m$ fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$ and a GLSL in $P \in \mathcal{K}$ such that $P \in\left(L_{F_{1}}, \cdots, L_{L_{m}}\right)$ are given, then $P$ belongs to one of lower leaning lines of $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$. Obviously, there can exist a GLSL in $P$ which is before $L_{F_{1}}$ or after $L_{L_{m}}$, however, this case does not play any role for our further considerations.

Proposition 3.7 Let $\mathcal{K}$ be a convex $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots$, m be fundamental segments of $\mathcal{K}$. Let us assume $x_{L_{L_{j}}}<x_{L_{F_{j+1}}}$ for some $1 \leq j \leq m-$ 1. Then there is no grid point between fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ and the polygonal set with successive edges $e_{1}: a_{j} x-b_{j} y=$ $\mu_{j}+b_{j}-1, e_{2}:$ the real line through $L_{L_{j}}$ and $L_{F_{j+1}}$, and $e_{3}: a_{j+1} x-b_{j+1} y=\mu_{j+1}+$ $b_{j+1}-1$.

Proof Real lines $e_{1}$ and $e_{3}$ describe lower leaning lines of fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. It follows that there is no grid point between the polygonal set with both edges, whose intersection point is the vertex $V$, and these fundamental segments. Thus, we only must show that elements of $\left(L_{L_{j}}, \cdots, L_{F_{j+1}}\right)$ are lying above $e_{2}$.

Let us assume $P \in \mathcal{K}$ is one single point inside the triangle with vertices $L_{L_{j}}$, $L_{F_{j+1}}, V$. Illustration is given in Figure 3.5.


Figure 3.5: Illustration to Proposition 3.7.

We deduce that there exists a GLSL in $P$. Hence, $P$ is on one of lower leaning lines of fundamental segments, but not on $e_{1}$ or $e_{3}$, i.e. there must exists a fundamental segment between $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. It leads to a contradiction that both fundamental segments are successive.

Analogously, the case, where more than one points are inside the triangle, leads to this contradiction.

Now we are able to show the equivalence between convex and lower digitally convex curves.

Theorem 3.2 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a $(0,1)$-curve. The curve $\mathcal{K}$ is convex if and only if $\mathcal{K}$ is lower digitally convex.

Proof Let $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ be fundamental segments of $\mathcal{K}$.

1. Let us assume $\mathcal{K}$ is convex. We consider a polygonal curve $\Pi$ consisting of vertices of the lower frontier of the convex hull of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ before $L_{F_{1}}$, intersection points of lower leaning lines of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right), j=$ $1, \cdots, m-1$ and vertices of the lower frontier of the convex hull of $\mathcal{F}_{m}\left(a_{m}, b_{m}, \mu_{m}\right)$ after $L_{L_{m}}$. Since $\mathcal{K}$ is convex, $\Pi$ possesses increasing slopes and there is no grid point between $\Pi$ and the curve $\mathcal{K}$. By Proposition 3.6, for two successive fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ holds $x_{L_{L_{j}}} \leq x_{L_{F_{j+1}}}$. If $x_{L_{L_{j}}}=x_{L_{F_{j+1}}}$, then $L_{L_{j}}=L_{F_{j+1}}$ is a vertex of $\Pi$. In the case $x_{L_{L_{j}}}<x_{L_{F_{j+1}}}$ for the slope $s$ of the real line segment $\left[L_{L_{j}}, L_{F_{j+1}}\right]$ holds $\frac{a_{j}}{b_{j}}<s<\frac{a_{j+1}}{b_{j+1}}$. According to Proposition 3.7, there is no grid point between $\left(L_{L_{j}}, \cdots, L_{F_{j+1}}\right)$ and $\left[L_{L_{j}}, L_{F_{j+1}}\right]$. In this case, we modify the polygonal curve $\Pi$ by replacing the vertex which is intersection point of lower leaning lines of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ by vertices $L_{L_{j}}$ and $L_{F_{j+1}}$. Hence, modified $\Pi$ has all successive vertices of the lower frontier of the convex hull of $\mathcal{K}$ and there is no grid point between $\mathcal{K}$ and the frontier.
2. We assume that there is no grid point between $\mathcal{K}$ and the lower frontier of the convex hull of $\mathcal{K}$. In the case $m=1$, the statement is, obviously, true. If $m>1$, then the curve $\mathcal{K}$ possesses at least two fundamental segments. It is clear that the points $L_{F_{1}}$ and $L_{L_{m}}$ are always vertices of the lower frontier.

Let us assume $m=2$ and slopes of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ and $\mathcal{F}_{2}\left(a_{2}, b_{2}, \mu_{2}\right)$ are decreasing, i.e. $L_{F_{1}}$ and $L_{L_{2}}$ are vertices of the lower frontier and there exists no other vertex between them. Then, there must be at least one grid point between ( $L_{F_{1}}, \cdots, L_{L_{2}}$ ) and $\left[L_{F_{1}}, L_{L_{2}}\right]$. Otherwise, $\left(L_{F_{1}}, \cdots, L_{L_{2}}\right)$ is a discrete line segment belonging to the curve contradicting the consecutivity of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ and $\mathcal{F}_{2}\left(a_{2}, b_{2}, \mu_{2}\right)$.

For $m>2$ the similar arguments lead to a contradiction when we assume that slopes of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right), 1 \leq j \leq m-1$ are decreasing.

Remark 3.1 From the first part of the proof to Theorem 3.2 we deduce that lower leaning points $\left\{L_{F_{1}}, L_{L_{1}}, L_{F_{2}}, L_{L_{2}}, \cdots, L_{F_{m}}, L_{L_{m}}\right\}$ of a convex curve $\mathcal{K}$ with fundamen-
tal segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are vertices of the lower frontier of the convex hull of $\mathcal{K}$ between $L_{F_{1}}$ and $L_{L_{m}}$.

By duality $\left\{U_{F_{1}}, U_{L_{1}}, U_{F_{2}}, U_{L_{2}}, \cdots, U_{F_{m}}, U_{L_{m}}\right\}$ are vertices of the upper frontier between $U_{F_{1}}$ and $U_{L_{m}}$ of a concave curve.

Let us concentrate on linear transforms $T^{k}$ for $k \in\{0, \cdots, 7\}$ described at the end of Section 2.

Theorem 3.3 Given an 8 -connected digital set $S_{\triangle} \in \mathbb{Z}^{2}$. Assume $S_{\triangle}$ possesses only $T$-descriptor points. Then the set $S_{\triangle}$ is digitally convex if and only if the following conditions are true:

1. The parts of the boundary of $S_{\triangle}$ between descriptor points of type $k$ and $k+1(\bmod 8), k=0,2,4,6$ under the transform $T^{k}$ are convex curves.
2. The parts of the boundary of $S_{\triangle}$ between descriptor points of type $k$ and $k+1(\bmod 8), k=1,3,5,7$ under the transform $T^{k}$ are concave curves.

Proof We consider an 8-connected digital set $S_{\triangle} \in \mathbb{Z}^{2}$. Using Scherl's descriptors it is possible to partition the boundary of $S_{\triangle}$ into $(k, k+1(\bmod 8))$-curves. For $k=0,2,4,6$ the part of the boundary under $T^{k}$ is a ( 0,1 )-curve, the interior of the set $S_{\triangle}$ is on the left side, for $k=1,3,5,7 \mathrm{it}$ is on the right side. In the first situation we must consider the lower frontier of the curve, in the second the upper frontier.

1. Let $S_{\triangle}$ be digitally convex. Then vertices of the lower (upper) frontier of the described above $(0,1)$-curve for $k=0,2,4,6(k=1,3,5,7)$ are vertices of $\operatorname{conv} S_{\triangle}$ and there exists no grid point between the frontier and the curve. By Theorem 3.2 the curve is convex (concave).
2. Assume the $(0,1)$-curves for $k=0,2,4,6(k=1,3,5,7)$ of $S_{\triangle}$ are convex (concave). We consider a polygonal set $\Pi$ having vertices of the lower (upper) frontier of the convex (concave) curves. There is no grid point between curves and the lower (upper) frontier of the convex hull of curves. Thus, we deduce $\Pi \cap \mathbb{Z}^{2}=$ $S_{\triangle}$. Since $S_{\triangle}$ possesses only $T$-descriptor points the polygonal set $\Pi$ is convex. It follows that $S_{\triangle}$ is digitally convex.

### 3.5 Decomposition of Curves into Meaningful Parts

Let us consider a digital $(0,1)$-curve $\mathcal{K}$. By means of fundamental segments of $\mathcal{K}$ and their slopes we are able to define convex and concave parts of $\mathcal{K}$ which are maximal.

Definition 3.10 Let $\mathcal{K}$ be a finite $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. A part consisting of successive fundamental segments $\mathcal{F}_{u}\left(a_{u}, b_{u}, \mu_{u}\right), \cdots, \mathcal{F}_{v}\left(a_{v}, b_{v}, \mu_{v}\right), 1 \leq u \leq v \leq m$ is called a maximal convex part of $\mathcal{K}$ whenever one of the following conditions is true:

1. $u=1, v=m$ and $\frac{a_{j}}{b_{j}}<\frac{a_{j+1}}{b_{j+1}}, 1 \leq j \leq m-1$.
2. $u \neq 1, v \neq m, \frac{a_{u-1}}{b_{u-1}}>\frac{a_{u}}{b_{u}}, \frac{a_{v}}{b_{v}}>\frac{a_{v+1}}{b_{v+1}}$ and $\frac{a_{j}}{b_{j}}<\frac{a_{j+1}}{b_{j+1}}$ for all $u \leq j \leq v-1$.
3. $u=1, v \neq m, \frac{a_{v}}{b_{v}}>\frac{a_{v+1}}{b_{v+1}}$ and $\frac{a_{j}}{b_{j}}<\frac{a_{j+1}}{b_{j+1}}$ for all $1 \leq j \leq v-1$.
4. $u \neq 1, v=m, \frac{a_{u-1}}{b_{u-1}}>\frac{a_{u}}{b_{u}}$ and $\frac{a_{j}}{b_{j}}<\frac{a_{j+1}}{b_{j+1}}$ for all $u \leq j \leq m-1$.

A maximal concave part of $\mathcal{K}$ is defined in the same manner by replacing the signs " $<$ " and " $>$ " in the above definition.

It is clear that a convex (concave) curve consists exactly of one single maximal convex (concave) part. The maximal parts of a curve $\mathcal{K}$ overlap each other. If $\mathcal{K}$ is neither convex nor concave then each its maximal convex and concave part has at least two fundamental segments. The common component of two successive meaningful parts consists of exactly one fundamental segment.

Let us concentrate on Figure 3.6, where the curve from Figure 3.2 is represented again. The slopes of 6 successive fundamental segments are $\frac{a_{i}}{b_{i}}=0.1429,0.4,0.25$, $0.7273,0.6,0.3333$. We deduce that the curve possesses four maximal parts: convex consisting of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ and $\mathcal{F}_{2}\left(a_{2}, b_{2}, \mu_{2}\right)$; concave with $\mathcal{F}_{2}\left(a_{2}, b_{2}, \mu_{2}\right)$ and $\mathcal{F}_{3}\left(a_{3}, b_{3}, \mu_{3}\right)$; convex with $\mathcal{F}_{3}\left(a_{3}, b_{3}, \mu_{2}\right)$ and $\mathcal{F}_{4}\left(a_{4}, b_{4}, \mu_{4}\right)$; concave consisting of $\mathcal{F}_{4}\left(a_{4}, b_{4}, \mu_{4}\right), \mathcal{F}_{5}\left(a_{5}, b_{5}, \mu_{5}\right)$ and $\mathcal{F}_{6}\left(a_{6}, b_{6}, \mu_{6}\right)$. It is an interesting observation that there exists a common point $P$ of more than two maximal convex and concave parts.


Figure 3.6: Maximal convex and concave parts of the curve from Figure 3.2. The point $P$ belonging to each maximal part is indicated.

Given a digital set $S_{\triangle}$ in $\mathbb{Z}^{2}$. We are interested in partitionning the boundary of $S_{\triangle}$ into maximal possible meaningful parts. We are able to decompose the boundary into $(k, k+1(\bmod 8))$-curves using Scherl's descriptors. First we can find the predecomposition in the following manner:


Figure 3.7: Maximal convex and concave parts of "Letter A" indicated by dark and pale triangles, respectively.

1. The segments of the boundary of $S_{\triangle}$ between descriptor points of type $k$ and $k+1(\bmod 8), k=0,2,4,6$ are convex (concave) parts of $S_{\triangle}$ whenever they are convex (concave) parts of the corresponding ( 0,1 )-curve.
2. The segments of the boundary of $S_{\triangle}$ between descriptor points of type $k$ and $k+1(\bmod 8), k=1,3,5,7$ are convex (concave) parts of $S_{\triangle}$ whenever they are concave (convex) parts of the corresponding ( 0,1 )-curve.
Obviously, the pre-decomposition possesses no maximal possible parts. Using the fact that $T$-descriptor points always belong to a convex part and $S$-descriptor points to a concave part we can determine the maximal convex and concave parts of the boundary of $S_{\triangle}$. In the case we can replace $T$ - and $S$-descriptor points by reversing the orientation we have again the perfect duality: the convex parts will become the concave and vice versa. In Figure 3.7 the partition of the boundary of a digital set into convex and concave parts is shown.

### 3.6 Fundamental Polygonal Representations of Digital Curves

In this section we shortly discuss an important application of decomposition curves into fundamental segments. Let $\mathcal{K}$ be a finite $(0,1)$-curve and $\mathcal{F}(a, b, \mu)$ a fundamental segment of $\mathcal{K}$. The (whole) segment $\mathcal{F}(a, b, \mu)$ is located above the lower leaning line $a x-b y=\mu+b-1$ and under the upper leaning line $a x-b y=\mu$. Moreover, there is no grid point between the segment and the leaning lines. We consider two successive fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. The common part of the segments is not empty and located above the both lower leaning lines $a_{j} x-b_{j} y=\mu_{j}+b_{j}-1$ and $a_{j+1} x-b_{j+1} y=\mu_{j+1}+b_{j+1}-1$, also under the both upper leaning lines $a_{j} x-b_{j} y=\mu_{j}$ and $a_{j+1} x-b_{j+1} y=\mu_{j+1}$. Hence, the segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ is above (under) the polygonal curve $\Pi$ with edges given by mentioned real lines and the vertex given by their intersection point, respectively. There exists no grid point between the polygonal curves and the fundamental segments.

These considerations allow to introduce a concept of fundamental polygonal representations:

Definition 3.11 Let $\mathcal{K}$ be a finite $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. A polygonal curve $\Pi$ with edges given by lower (upper) leaning lines of $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$ and vertices given by their intersection points in successive order is called lower (upper) fundamental polygonal representation of $\mathcal{K}$.

Figure 3.8 demonstrates the upper and lower fundamental polygonal representations of the curve from Figure 3.2.

We collect some simple properties of fundamental polygonal representations:


Figure 3.8: Fundamental polygonal representations of the curve from Figure 3.2.

1. There is no grid point between fundamental polygonal representations and the digital curve.
2. Vertices of fundamental polygonal representations are, generally, not grid points.
3. The lower (upper) fundamental polygonal representation of a convex (concave) digital curve possesses only convex (concave) vertices.
4. The fundamental polygonal representations have the same convexity properties as the digital curve.
5. The representations are translations of each other and have the same Euclidean lengths.
6. If for lower (upper) leaning points of fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=$ $1, \cdots, m$ of a convex (concave) curve holds

$$
L_{L_{j}}=L_{F_{j+1}} \quad\left(U_{L_{j}}=U_{F_{j+1}}\right) \quad 1 \leq j \leq m-1
$$

then vertices of the lower (upper) fundamental polygonal representation are vertices of the lower (upper) frontier of the convex hull of $\mathcal{K}$ between $L_{F_{1}}$ and $L_{L_{m}}\left(U_{F_{1}}\right.$ and $\left.U_{F_{m}}\right)$.

## Chapter 4

## Polygonal Representations of Digital Sets

In Section 3.2 we introduced such concepts of Euclidean Geometry and Convexity Theory like polygonal curves, polygonal sets, convex and concave vertices of polygonal curves. In the plane $\mathbb{R}^{2}$ the boundary of a polygonal set can be decomposed into convex and concave parts in an obvious way. On the other hand, we have shown how to partition the boundary of a digital set in $\mathbb{Z}^{2}$ into meaningful parts by means of Scherl's descriptors and fundamental segments. However, in spite of the precision, proposed decomposition has also different disadvantages, e.g. before the set can be partitioned into convex and concave parts one needs information about slopes of fundamental segments, more than two parts can possess common elements, the decomposition is less suitable for our visual system. Therefore, the aim of this chapter is to describe digital sets by means of polygonal sets in $\mathbb{R}^{2}$ with corresponding convexity properties.

We state some important characteristics of polygonal sets in the plane $\mathbb{R}^{2}$.

1. For a polygonal set we can define in an obvious way such topological concepts like connected components, simply connected polygonal sets or holes of the set.
2. The famous Jordan Curve Theorem states that any (bounded) simple closed polygonal curve $\Pi$ separates the plane into the interior and the exterior with respect to the curve. More formally: The set $\mathbb{R}^{2} \backslash \Pi$ consists of exactly two disjoint connected components. One of them is declared as the interior and the other as the exterior with respect to $\Pi$.
3. In the definition of convex and concave parts of a polygonal curve we have a perfect duality: If we replace a polygonal set by its complement, the orientation of the boundary is reversed and convex parts become concave parts and vice versa.

Definition 4.1 Given a digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. A polygonal representation of $S_{\triangle}$ is a polygonal set $\Pi=(V, E)$ with vertices $V$ and edges $E$ such that

$$
x \in S_{\triangle} \quad \Longleftrightarrow \quad x \in \Pi \cap \mathbb{Z}^{2}
$$

A polygonal set representing a digital set is not unique. We are free to require a number of additional characteristics. The polygonal representation $\Pi$ of a digital set $S_{\triangle}$ is said to be
discrete if all vertices of $\Pi$ are in $\mathbb{Z}^{2}$;
faithful if convex parts of the boundary of $S_{\triangle}$ correspond to convex parts of the boundary of $\Pi$, and if the same is true for concave parts.

Remark 4.1 A boundary point of a digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ may belong to more than two maximal parts. To be more precise, a representation $\Pi$ is faithful if the succession of convex and concave parts of the boundary of $S_{\triangle}$ corresponds to the succession of convex and concave parts of the boundary of $\Pi$.

The chain code representation of the boundary of a digital set is a discrete polygonal representation having a maximal number of vertices. It is, however, not faithful. Using fundamental polygonal representations of the boundary of a digital set $S_{\triangle}$ (see Section 3.6) we are able to determine a faithful polygonal representation of $S_{\triangle}$. Generally, this representation is not discrete. In our further considerations we are mostly interested in polygonal representations of digital sets which are both discrete and faithful. We distinguish between two situations: first, the polygonal representations of digitally convex, and, second, of non convex sets. The first case is far better studied in $[9,14,21]$. Evidently, the unique discrete and faithful polygonal representation of a digitally convex set is the convex hull of the set. Linear algorithms for determination the convex hull of a digital set and testing convexity are developed [9, 21].

We introduce another implementation which is based on the concept of fundamental segments of a digital curve. Since in the literature it is more usual to consider lower polygonal representations, in Section 4.1 convex curves and their lower polygonal representations are studied. In Section 4.2 upper representations of general curves are considered (in this case upper representations are convenient for computation). Furthermore, we will show the existence of a linear algorithm which detects discrete polygonal representation of a digital set with "only few" uncorresponding parts.

### 4.1 Convex Case

We need concepts from ordinary convexity theory. Assume $\mathbb{R}^{d}$ is equipped with Euclidean geometry, i.e. to an arbitrary pair of vectors $x=\left(\eta_{1}, \eta_{2}, \cdots, \eta_{d}\right)$ and
$y=\left(\zeta_{1}, \zeta_{2}, \cdots, \zeta_{d}\right)$ the inner product

$$
\langle x, y\rangle=\sum_{j=1}^{d} \eta_{j} \zeta_{j}
$$

is assignet.
Definition 4.2 Given a convex set $S \subseteq \mathbb{R}^{d}$. A point $x_{0} \in S$ is an exposed point of $S$ if there exists a nonzero vector $x^{*}$ such that $\left\langle x_{0}, x^{*}\right\rangle>\left\langle x, x^{*}\right\rangle$ for all $x \in S \backslash\left\{x_{0}\right\}$.

If $S_{\triangle} \subseteq \mathbb{Z}^{d}$ is digitally convex then $x_{0}$ is an exposed point of $S_{\triangle}$ if it is an exposed point of convS $S_{\triangle}$.

It is well known from convexity theory that the convex hull of a polygonal set is the convex hull of its exposed points [46]. On the other hand vertices of a convex polygonal set are exactly its exposed points.

For digitally convex sets there exists a simple characterization of exposed points (the following lemma is proved in [15]):

Lemma 4.1 Let $S_{\triangle} \subseteq \mathbb{Z}^{2}$ be a digitally convex set. A point $P \in S_{\triangle}$ is an exposed point of $S_{\triangle}$ if and only if $P$ is a vertex of each discrete polygonal representation $\Pi$ of $S_{\triangle}$.

Lemma 4.1 can be shown also for the digitally convex sets from $\mathbb{Z}^{d}$. The discrete polygonal representation must be understood as a polyhedral representation with all vertices in $\mathbb{Z}^{d}$.
Proof 1. Assume given a digitally convex set $S_{\triangle} \subseteq \mathbb{Z}^{d}$ and $x_{0}$ is a vertex of each discrete polyhedral representation of $S_{\triangle}$. Since $\operatorname{conv} S_{\triangle}$ is a discrete polyhedral representation of $S_{\triangle}$ so $x_{0}$ is also a vertex of $\operatorname{conv} S_{\triangle}$. The exposed points of a convex set are exactly its vertices. Hence, $x_{0}$ is an exposed point of $\operatorname{conv} S_{\triangle}$ and of $S_{\triangle}$ as well.
2. Let $\Pi$ be a discrete polyhedral representation of $S_{\triangle}$. If $x_{0}$ is an exposed point of $S_{\triangle}$ then there exists a nonzero vector $x^{*}$ such that $\left\langle x_{0}, x^{*}\right\rangle>\left\langle x, x^{*}\right\rangle$ for all $x \in S_{\triangle} \backslash\left\{x_{0}\right\}$, also for all $x \in \Pi \cap \mathbb{Z}^{d} \backslash\left\{x_{0}\right\}$. For

$$
H:=\left\{x \in \mathbb{R}^{d} \mid\left\langle x_{0}, x^{*}\right\rangle \geq\left\langle x, x^{*}\right\rangle\right\}
$$

holds $\Pi \subseteq H$. The point $x_{0}$ is a (topological) boundary point of $H$ and, consequently, it is also a boundary point of $\Pi$. In $\mathbb{R}^{d}$ this implies that $x_{0}$ belongs to a facet of $\Pi$ which is $d-1$ face of $\Pi$ (definitions see Section 5.5). The point $x_{0}$ is exposed, it must belong to the boundary of the facet, thus, to the $d-2$ face. In all cases $x_{0}$ belongs to the boundary of $i$-face, $i=1, \cdots, d-1$. We deduce that $x_{0}$ is a vertex of П.

These considerations justify the following definition for exposed points of digital sets which are not necessarily digitally convex.

Definition 4.3 Given a digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. A point $P \in S_{\triangle}$ is called an exposed point of $S_{\triangle}$ whenever a discrete and faithful polygonal representation $\Pi$ of $S_{\triangle}$ exists and $P$ is a vertex of $\Pi$.

It is clear that exposed points in Definitions 4.2 and Definition 4.3 in the case when $S_{\triangle}$ is digitally convex are the same points. Thus, exposed points of a digitally convex set $S_{\triangle}$ are exactly vertices of the convex hull of $S_{\triangle}$.

Our goal is to detect all successivelly ordered exposed points on the boundary of an 8-connected digitally convex set.

Assume, $S_{\triangle}$ is 8-connected and digitally convex, the boundary of $S_{\triangle}$ is oriented. Then there exists the unique polygonal representation which is discrete and faithful. This representation is the convex hull $\operatorname{conv} S_{\triangle}$ of $S_{\triangle}$. Since the boundary of a digitally convex set possesses only $T$-descriptor points and can be decomposed into convex and concave ( 0,1 )-curves (see Theorem 3.3, p. 30), we may restrict our considerations to convex $(0,1)$-curves. The polygonal representation of a $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ can be subdivided into two polygonal curves between $\kappa_{1}$ and $\kappa_{n}$ : lower and upper polygonal representation. Thus, computation exposed points of a digitally convex set can be restricted to computation vertices of discrete and faithful lower polygonal representations of convex curves. Obviously, the discrete and faithful lower polygonal representation of a curve $\mathcal{K}$ is lower frontier of the convex hull of $\mathcal{K}$.

Assume $\mathscr{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$ are fundamental segments of $\mathcal{K}$ with lower leaning points $L_{F_{i}}$ and $L_{L_{i}}, i=1, \cdots, m$. Then by Remark 3.1, p. 30 vertices of the discrete and faithful polygonal representation between $L_{F_{1}}$ and $L_{L_{m}}$ coincide with leaning points $\left\{L_{F_{1}}, L_{L_{1}}, L_{F_{2}}, L_{L_{2}}, \cdots, L_{F_{m}}, L_{L_{m}}\right\}$ of fundamental segments of $\mathcal{K}$. However, for computation vertices of the lower polygonal representation it is not recommendable to determine all fundamental segments of $\mathcal{K}$ and their leaning points. In spite of linearity, this technique is applicable only to convex curves. For an arbitrary curve such algorithm is not suitable. We will show how to find leaning points of fundamental segments (within linear time) without computation fundamental segments.

In convex as well as in general case we are mostly interested in leaning points of fundamental segments which are last (or, by duality, first) points of horizontal levels of $\mathcal{K}$.

Proposition 4.1 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a convex (concave) $(0,1)$-curve which is not a segment of a horizontal or diagonal grid line and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. Then the lower (upper) leaning points $L_{F_{i}}$ and $L_{L_{i}}$ ( $U_{F_{i}}$ and $U_{L_{i}}$ ) for all $1 \leq i \leq m$ are last (first) points of horizontal levels of $\mathcal{K}$.

Proof Since the lower leaning line of a fundamental segment is a GLSL (global lower supporting line) of $\mathcal{K}$, the proposition is trivially true.

The linear algorithm for detection vertices of the lower polygonal representation of a convex $(0,1)$-curve $\mathcal{K}$ with fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$
(not known) can be construct on the basis of Algorithm AddPoint. We apply Algorithm AddPoint to $\mathcal{K}$ and if the situation "the point may not be added to the segment" appears then the algorithm will be restarted at the last leaning point $L_{L}$. This procedure is presented in Algorithm MainLoop. A similar algorithm was proposed in [9] for computing the convex hull of a digitally convex set.

```
Algorithm V MainLoop computing (some) vertices of the lower polygonal
                                    representation of a \((0,1)\)-curve \(\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)\).
\(/ * V\) vertices of the lower polygonal representation \(* /\)
START \(\leftarrow \kappa_{1} ; / *\) start point \(* /\)
repeat
    Determine the segment \(\left(S T A R T, \cdots, \kappa_{j}\right)\) of a discrete line \(\mathcal{D}(a, b, \mu)\) such
    that \(j=n\) or (START, \(\left.\cdots, \kappa_{j+1}\right)\) is not a segment of any discrete line;
    \(V \leftarrow V \cup L_{F} \cup L_{L} ;\)
    if (START, \(\left.\cdots, \kappa_{j+1}\right)\) is not a segment of any discrete line then
        \(S T A R T \leftarrow L_{L} ;\)
    else
        STOP;
    end
until STOP;
```

The algorithm can be used for computation upper polygonal representations of concave curves when the lower leaning points are replaced by upper leaning points, i.e. following lines have to be changed:

$$
V \leftarrow V \cup L_{F} \cup L_{L} ; \quad \text { by } \quad V \leftarrow V \cup U_{F} \cup U_{L} ;
$$

and

$$
S T A R T \leftarrow L_{L} ; \quad \text { by } \quad S T A R T \leftarrow U_{L} ;
$$

The described technique is similar to method for segmentation digital curves into discrete line segments by Debled-Rennesson and Reveillès [8]. The latter algorithm is linear, the both algorithms differ only in the points, where they are restarted. We deduce that Algorithm MainLoop is linear, too.

The next proposition shows the principle the algorithm MainLoop works if the given $(0,1)$-curve is convex.

Proposition 4.2 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a convex $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$, $i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. Then $L_{L_{j}} \in \mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ for all $1 \leq j \leq m-1$.

If $\mathcal{K}$ is concave then $U_{L_{j}} \in \mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ for all $1 \leq j \leq m-1$.
Proof By Proposition 3.6 one has $x_{L_{L_{j}}} \leq x_{L_{F_{j+1}}}$ for all $1 \leq j \leq m-1$. In the case $x_{L_{L_{j}}}=x_{L_{F_{j+1}}}$ the statement of the proposition is trivially true. We concentrate on the situation $x_{L_{L_{j}}}<x_{L_{F_{j+1}}}$. Assume $L_{L_{j}} \notin \mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ for some
$1 \leq j \leq m-1$. Then there exists a grid point $P \in \mathbb{C} \mathcal{K}$ between lower leaning line of $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ and the segment $\left(L_{L_{j}}, \cdots, L_{L_{j+1}}\right)$ (see sketch in Figure 4.1).


Figure 4.1: Illustration to Proposition 4.2
By the symmetry there is a grid point $Q \in \mathbb{C} \mathcal{K}$ between lower leaning line of $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and the segment $\left(L_{F_{j}}, \cdots, L_{F_{j+1}}\right)$. We deduce that $\left(L_{F_{j}}, \cdots, L_{F_{j+1}}\right)$ and ( $L_{L_{j}}, \cdots, L_{L_{j+1}}$ ) are not segments of any discrete line. Then there exists a fundamental segment $\mathcal{F}(a, b, \mu)$ of $\mathcal{K}$ such that $\mathcal{F}(a, b, \mu) \subseteq\left(L_{F_{j}+1}, \cdots, L_{L_{j+1}-1}\right)$, i.e. $\mathcal{F}(a, b, \mu)$ is a fundamental segment whose first point is lying after $L_{F_{j}}$ and last point is lying before $L_{L_{j+1}}$. It leads to a contradiction that $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ are successive.

Remark 4.2 By symmetry it holds $L_{L_{j+1}} \in \mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ for convex and $U_{L_{j+1}} \in$ $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ for all $1 \leq j \leq m-1$ for concave curves.

Assume Algorithm MainLoop is applied to a convex $(0,1)$-curve $\mathcal{K}$ and the leaning points $L_{F_{j}}$ and $L_{L_{j}}$ are detected in some repeat-loop. According to Proposition 4.2 one has $L_{L_{j}} \in \mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. Since $x_{L_{L_{j}}} \leq x_{L_{F_{j+1}}}$ the algorithm detects by starting at $L_{L_{j}}$ either the leaning points $L_{F_{j+1}}$ and $L_{L_{j+1}}$, or $L_{F_{j+2}}$ and $L_{L_{j+2}}$ is also possible. An example (here, the dual case for concave curve and its upper polygonal representation), where the latter situation appears, is demonstrated in Figure 4.2. In such situations Algorithm MainLoop fails.

The solution of this problem is described in [9]. Let $r(x, y)=a x-b y$ be remainder at $(x, y)$ with respect to a discrete line $\mathcal{D}(a, b, \mu)$. We consider a $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ which is a segment of the discrete line $\mathcal{D}(a, b, \mu)$. One has, obviously, $r\left(L_{F}\right)=r\left(L_{L}\right)$ and $r\left(U_{F}\right)=r\left(U_{L}\right)$. The lower and upper leaning lines are global lower and upper supporting lines of $\mathcal{K}$. Hence, for all $\kappa_{i}, i=1, \cdots, n$ holds $r\left(L_{F}\right)=r\left(L_{L}\right) \leq r\left(\kappa_{i}\right) \leq r\left(U_{F}\right)=r\left(U_{L}\right)$.

Proposition 4.3 Given a $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ which is a segment of a discrete line. Then the vertices of the lower (upper) frontier of $\mathcal{K}$ are following points:

1. $\kappa_{1}, \kappa_{n}$ and lower (upper) leaning poins $L_{F}$ and $L_{L}\left(U_{F}\right.$ and $\left.U_{L}\right)$ of the curve $\mathcal{K}$;


Figure 4.2: An example, where Algorithm MainLoop fails. The vertex $U_{F_{2}}=U_{L_{2}}$ of the discrete and faithful upper polygonal representation of a concave curve will be not detected.
2. between $\kappa_{1}$ and $L_{F}\left(\kappa_{1}\right.$ and $\left.U_{F}\right)$ : the maximal sequence of the points $\left\{P_{i}\right\}$, $i=1, \cdots, k$ such that $r\left(\kappa_{1}\right)<r\left(P_{1}\right)<\cdots<r\left(P_{k}\right)<r\left(L_{F}\right)\left(r\left(\kappa_{1}\right)>r\left(P_{1}\right)>\right.$ $\left.\cdots>r\left(P_{k}\right)>r\left(U_{F}\right)\right)$;
3. between $L_{L}$ and $\kappa_{n}\left(U_{L}\right.$ and $\left.\kappa_{n}\right)$ : the maximal sequence of the points $\left\{P_{i}\right\}$, $i=1, \cdots, k$ such that $r\left(\kappa_{n}\right)<r\left(P_{1}\right)<\cdots<r\left(P_{k}\right)<r\left(L_{L}\right)\left(r\left(\kappa_{n}\right)>r\left(P_{1}\right)>\right.$ $\left.\cdots>r\left(P_{k}\right)>r\left(U_{L}\right)\right)$.

Situations described in this proposition are visible in Figure 4.3. Points indicated in bold style justify Proposition 4.3.

We modify Algorithm MainLoop for concave curves and their discrete and faithful upper polygonal representations (in general case the upper polygonal representation is convenient for computation, see Section 4.2, Remark 4.3). In each repeatloop we test if elements between start-point and the detected first upper leaning point $U_{F}$ satisfying Proposition 4.3 exist. In this way we will also detect vertices of the polygonal representation before $U_{F_{1}}$. Furthermore, we must test if some vertices of the polygonal representation after $U_{L_{m}}$ exist.

Algorithm UpPolRep (on p. 43) detects all vertices of the discrete and faithful upper polygonal representation of a concave $(0,1)$-curve. Obviously, the algorithm has the complexity $O(n)$.

The described technique is similar to the method for testing convexity from [9]. There is another proposal how to detect all exposed points of a digitally convex set introduced in [14]. It leads also to a linear algorithm.

### 4.2 General Case

By digitally convex sets the existence of exposed points is guaranted. Moreover, exposed points are uniquelly given. In the general case the digital sets must be tested if a faithful and discrete polygonal representation really exists. On the other hand, as


Figure 4.3: Convex hull of a segment of the discrete line from Figure 3.1. The remainder of each point is represented. The dark voxels show upper and lower leaning points. Convex hull of the segment is convex hull of points indicated in bold style with following remainders $[0,2,3,3,2,-1,-4,-4,-3,0]$ (beginning by first point of the segment in the order given by left hand side orientation).
it will be demonstrated below, some sets which are not digitally convex can possess more than one of such representations. It means that in Definition 4.3 exposed points of a digital set depend on a discrete and faithful polygonal representation of the set. Exposed points of some representation are not necessarily exposed points of the other.

Let us consider a convex (concave) $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$. There exists the discrete and faithful lower (upper) polygonal representation of $\mathcal{K}$, it is lower (upper) frontier of the convex hull of $\mathcal{K}$ with only convex (concave) vertices. Exists an upper (lower) polygonal representation which is discrete and faithful of a convex (concave) curve, i.e. a discrete representation having only convex (concave) vertices? The existence of a faithful upper (lower) polygonal representation was shown in Section 3.6. It is upper (lower) fundamental polygonal representation of $\mathcal{K}$. However, this representation is in general not discrete.

The following proposition shows that the curves which do not possess disrete and faithful polygonal representations are not exceptional.

Proposition 4.4 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a convex (concave) $(0,1)$-curve which is not a segment of a horizontal or diagonal grid line. If for the chain code direction holds

1. $v\left(\kappa_{1}, \kappa_{2}\right)=1\left(v\left(\kappa_{1}, \kappa_{2}\right)=0\right)$, or
2. $\mathrm{v}\left(\kappa_{n-1}, \kappa_{n}\right)=0\left(\mathrm{v}\left(\kappa_{n-1}, \kappa_{n}\right)=1\right)$
```
AlgorithmVI UpPolRep computing vertices of the upper polygonal
    representation of a \((0,1)\)-curve \(\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)\).
\(/ * V\) vertices of the upper polygonal representation \(* /\)
START \(\leftarrow \kappa_{1} ; / *\) start point \(* /\)
repeat
    Determine the segment \(\left(S T A R T, \cdots, \kappa_{j}\right)\) of a discrete line such that \(j=n\)
    or (START, \(\cdots, \kappa_{j+1}\) ) is not a segment of any discrete line;
    \(V \leftarrow V \cup\) maximal sequence \(\left\{P_{i=1, \cdots, k}\right\}\) between START and \(U_{F}\)
    satisfying Proposition 4.3;
    \(V \leftarrow V \cup U_{F} \cup U_{L}\);
    if (START, \(\left.\cdots, \kappa_{j+1}\right)\) is not a segment of any discrete line then
        \(S T A R T \leftarrow U_{L} ;\)
    else
        STOP;
    end
until STOP;
\(V \leftarrow V \cup\) maximal sequence \(\left\{P_{i=1, \cdots, k}\right\}\) between \(U_{L}\) and \(\kappa_{n}\) satisfying
Proposition 4.3;
```

then there exists no discrete and faithful upper (lower) polygonal representation of the curve $\mathcal{K}$.

Proof Assume the curve is convex. If the chain code direction $v\left(\kappa_{1}, \kappa_{2}\right)=1$ then, obviously, there exists an element $\kappa_{i}, i \geq 2$ which belongs to each upper discrete polygonal representation of $\mathcal{K}$, since the slope through $\kappa_{1}$ and $\kappa_{i}$ is 1 . It follows that the vertex $\kappa_{i}$ is always concave. An example is demonstrated in Figure 4.4 (see p. 46). Here, the chain code is $v\left(\kappa_{1}, \kappa_{2}\right)=1$ and $i=2$. The point $\kappa_{2}$ is a concave vertex of each upper discrete polygonal representation.

Other cases can be shown analogously.
To show more examples, where curves do not possess discrete and faithful polygonal representations, we consider the first $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ and last $\mathcal{F}_{m}\left(a_{m}, b_{m}, \mu_{m}\right)$ fundamental segments of a convex (concave) curve $\mathcal{K}$. It is possible that the situations described in Proposition 4.3 appear. It follows that the sequences of points $\left\{P_{i}\right\}$ between $\kappa_{1}$ and $U_{F_{1}}$ or $U_{L_{m}}$ and $\kappa_{n}$ (between $\kappa_{1}$ and $L_{F_{1}}$ or $L_{L_{m}}$ and $\kappa_{n}$ ), if not empty, belongs to each discrete upper (lower) polygonal representation of $\mathcal{K}$ and vertices $\left\{P_{i}\right\}$ are concave (convex). Hence, also here there exists no discrete and faithful upper (lower) polygonal representation of the curve $\mathcal{K}$. For example, the vertices described in Proposition 4.4 belong to the sequence $\left\{P_{i}\right\}$. The elements of $\left\{P_{i}\right\}$ are vertices of each discrete polygonal representation and by Definition 4.3 are not exposed, however, they are exposed points in sense of [14].

These facts lead to the idea of defining exposed points locally, that means between first leaning point of first fundamental segment and last leaning point of last
fundamental segment:
Definition 4.4 Given a $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$. Let $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ be fundamental segments of $\mathcal{K}$. A point $P \in \mathcal{K}$ is called a locally exposed point of $\mathcal{K}$ whenever one of the following conditions is true:

1. The point $P$ is a vertex of each upper (lower) discrete polygonal representation.
2. There exists a discrete upper (lower) polygonal representation $\Pi$ such that between $U_{F_{1}}$ and $U_{L_{m}}\left(L_{F_{1}}\right.$ and $\left.L_{L_{m}}\right)$ the convex and concave parts of $\Pi$ correspond to the convex and concave parts of $\mathcal{K}$, and $P$ is a vertex of $\Pi$.
3. There exists no upper (lower) discrete and faithful polygonal representation between $U_{F_{1}}$ and $U_{L_{m}}\left(L_{F_{1}}\right.$ and $\left.L_{L_{m}}\right)$, and $P$ is a vertex of a discrete upper (lower) polygonal representation $\Pi$ of $\mathcal{K}$ such that the number of uncorresponding parts in $\Pi$ is minimal.

Unlike the convex case, where we considered only the leaning points of fundamental segments, in the general case there can exist exposed points which are not necesserily leaning points of fundamental segments. Moreover, we have again to treat different locations of upper and lower leaning points of fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ of a curve $\mathcal{K}$. It is possible that points $U_{F_{i}}$ or $L_{L_{i}}$ are lying inside horizontal levels of $\mathcal{K}$. However, it is always true that the first upper leaning point $U_{F_{1}}$ of the first fundamental segment is the first point of a horizontal level, the last lower leaning point $L_{L_{m}}$ of the last fundamental segment is the last point of a horizontal level.

Proposition 4.5 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a $(0,1)$-curve which is not a segment of a horizontal or diagonal grid line and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, m$ are fundamental segments of $\mathcal{K}$. Then upper (lower) leaning points $U_{L_{i}}\left(L_{F_{i}}\right)$ for all $i=1, \cdots, m$ are first (last) points of horizontal levels of $\mathcal{K}$, i.e. points of levels with minimal (maximal) x-coordinate.

Proof In general, the first and last element of a fundamental segment can be located inside horizontal levels of $\mathcal{K}$. Since the whole segment is lying under the upper leaning line, it is possible that $U_{F_{i}}, i=2, \cdots, m$ is the first element of the segment, hence, is lying inside a horizontal level. However, $U_{L_{i}}, i=1, \cdots, m$ must be the first point of a horizontal level. Analogously, the mentioned segment is above the lower leaning line. Then $L_{L_{i}}, i=1, \cdots, m-1$ can be located inside a horizontal level, however, $L_{F_{i}}, i=\cdots, m$ is allways the last point of a horizontal level of $\mathcal{K}$.

Remark 4.3 A problem in the general case is the fact that leaning points of successive fundamental segments can be successive elements of $\mathcal{K}$ or not (see Figure 3.4, p. 27), and may be located inside horizontal levels.

The situation, where some leaning points of fundamental segments lying inside horizontal levels of $\mathcal{K}$, becomes much simplificated if we use the fact that fundamental segments of $a(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ do not depend of the orientation and by Proposition 4.5 leaning points $U_{L_{i}}$ and $L_{F_{i}}, i=1, \cdots, m$ are first and last points of horizontal levels, respectively. A lower polygonal representation of $\mathcal{K}$ is one from all upper polygonal representations of the $(0,1)$-curve $\overline{\mathcal{K}}$ with reversed orientation defined as follows:

$$
\overline{\mathcal{K}}=\left(\kappa_{n}, \cdots, \kappa_{1}\right) \cdot\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right) .
$$

Hence, our considerations can be restricted to upper polygonal representations of digital curves that is convenient for computation.

We will show that the existence of a polygonal representation which is discrete and faithful depends on the locations of leaning points of successive fundamental segments.

Lemma 4.2 Given a convex $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$. Let $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=$ $1, \cdots, m, m \geq 2$ be fundamental segments of $\mathcal{K}$. If upper leaning points offundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ and $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ for some $1 \leq j \leq m-1$ satisfy $x_{U_{L_{j}}}<x_{U_{F_{j+1}}}$ then there exists no discrete and faithful upper polygonal representation of the curve $\mathcal{K}$. Moreover, there exists at least one point $P \in\left(U_{L_{j}}, \cdots, U_{F_{j+1}}\right)$ which is a concave vertex of each discrete polygonal representation.

Proof We treat two possibilities $y_{U_{L_{j}}}=y_{U_{F_{j+1}}}$ and $y_{U_{L_{j}}}<y_{U_{F_{j+1}}}\left(y_{U_{L_{j}}}>y_{U_{F_{j+1}}}\right.$ is not possible) and show that in the both cases the discrete and faithful polygonal representation does not exist.

1. The restriction $y_{U_{L_{j}}}=y_{U_{F_{j+1}}}$ means that the leaning points $U_{L_{j}}$ and $U_{F_{j+1}}$ belong to the same horizontal level of $\mathcal{K}$. Obviously, the predecessor of $U_{F_{j+1}}$ is located above the upper leaning line $a_{j+1} x-b_{j+1} y=\mu_{j+1}$ of the fundamental segment $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. It follows that the first element of this fundamental segment is also the first leaning point $U_{F_{j+1}}$.

We check the possible discrete polygonal representations. The slope of the real line through $U_{L_{j}}$ and $U_{F_{j+1}}$ is 0 . The representation with vertices $U_{F_{j}}, U_{L_{j}}, U_{F_{j+1}}, U_{L_{j+1}}$ is not faithful, such one with vertices $U_{F_{j}}, U_{L_{j}}, U_{L_{j+1}}$ is not allowed. Furthermore, it is clear that the element $U_{L_{j}}$, which is by Proposition 4.5 the first point of a horizontal level of $\mathcal{K}$, is a concave vertex of each discrete upper polygonal representation.
2. Assume $y_{U_{L_{j}}}<y_{U_{F_{j+1}}}$. We distinguish between different locations of the intersection point $(x, y)$ of upper leaning lines $a_{j} x-b_{j} y=\mu_{j}$ and $a_{j+1} x-b_{j+1} y=$ $\mu_{j+1}$.

If $x<x_{U_{L_{j}}}$ (an example see Figure 4.4, the intersection point $(x, y)$ of upper leaning lines is indicated) then $U_{F_{j+1}} \notin \mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$. We deduce that $U_{F_{j+1}}$ cannot be the first point of the fundamental segment $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. It follows that
$U_{F_{j+1}}$ is the first point of a horizontal level and $U_{F_{j+1}}$ or the point $P$ between $U_{L_{j}}$ and $U_{F_{j+1}}$ satisfying Proposition 4.3 is a concave vertex of each upper discrete polygonal representation.


Figure 4.4: Dashed line represents the upper leaning line of $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$. Polygonal set with vertices $U_{F_{j}}, U_{L_{j}}, U_{F_{j+1}}$ is given in solid style. Upper leaning points $U_{F_{j}}, U_{L_{j}}, U_{F_{j+1}}$ of a convex curve cannot be successive vertices of a discrete upper polygonal representation since $P \in \mathcal{K}$ is lying above $\left[U_{L_{j}}, U_{F_{j+1}}\right]$.

Let us assume that $x>x_{U_{F_{j+1}}}$. Analogously to the case above we deduce $U_{L_{j}} \notin$ $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ and there is an element $P \in\left(U_{L_{j}}, \cdots, U_{F_{j+1}}\right)$ which is a concave vertex of each upper discrete polygonal representation.

Otherwise, let $x_{U_{L_{j}}}<x<x_{U_{F_{j+1}}}$ (an example see Figure 4.5). It holds $U_{L_{j}} \notin$ $\mathcal{F}_{j+1}\left(a_{j+1}, b_{j+1}, \mu_{j+1}\right)$ and $U_{F_{j+1}} \notin \mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$. Hence, the set with vertices $U_{F_{j}}$, $U_{L_{j}}, U_{F_{j+1}}, U_{L_{j+1}}$ is not a polygonal representation. The points $U_{L_{j}}$ and $U_{F_{j+1}}$ are concave vertices of each upper discrete polygonal representation.


Figure 4.5: Upper leaning points $U_{F_{j}}, U_{L_{j}}, U_{F_{j+1}}, U_{L_{j+1}}$ of a convex curve cannot be successive vertices of a discrete upper polygonal representation since $P \in \mathbb{C K}$ is lying on $\left[U_{L_{j}}, U_{F_{j+1}}\right]$.

In the next proposition we examine situations, where discrete and faithful polygonal representations exist locally.

Proposition 4.6 Given a convex $(0,1)$-curve $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$. Let $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$,
$i=1, \cdots m, m \geq 2$ be fundamental segments of $\mathcal{K}$. Assume for some $1 \leq j \leq m-1$ one of the following conditions is true:

1. $x_{U_{F_{j+1}}}=x_{U_{L_{j}}}$,
2. $x_{U_{F_{j+1}}}<x_{U_{L_{j}}}$ and there is no element $\kappa \in \mathcal{K}$ such that $x_{U_{L_{j}}}<x_{\kappa}<x_{U_{L_{j+1}}}$ and $\kappa$ is lying above the real line through $U_{L_{j}}$ and $U_{L_{j+1}}$,
3. $x_{U_{F_{j+1}}}<x_{U_{L_{j}}}$ and there is no element $\kappa \in \mathcal{K}$ such that $x_{U_{F_{j}}}<x_{\kappa}<x_{U_{F_{j+1}}}$ and $\kappa$ is lying above the real line through $U_{F_{j}}$ and $U_{F_{j+1}}$,
then there exists a discrete and faithful upper polygonal representation of the segment of $\mathcal{K}$ between $U_{F_{j}}$ and $U_{L_{j+1}}$.

Proof 1. If condition 1. is true then, obviously, the set with vertices $U_{F_{j}}, U_{F_{j+1}}=U_{L_{j}}$ and $U_{L_{j+1}}$ is a discrete and faithful polygonal representation.
2. It is clear that $U_{F_{j}}, U_{L_{j}}$ and $U_{L_{j+1}}$ are successive vertices of a discrete polygonal representation. Without loss of generality we may assume that the leaning points do not coincide. The point $U_{F_{j+1}}$ is lying not above the upper leaning line $a_{j} x-b_{j} y=\mu_{j}$. Since the slope $s_{j+1}$ of the upper leaning line $a_{j+1} x-b_{j+1} y=\mu_{j+1}$ is greater than $s_{j}$ of $a_{j} x-b_{j} y=\mu_{j}$ and $x_{U_{L_{j}}}>x_{U_{F_{j+1}}}$ it follows that the intersection point of these real lines is on the left side of $U_{L_{j}}^{j+1}$. For the slope $s$ through $U_{L_{j}}$, $U_{L_{j+1}}$ holds $s_{j}<s_{j+1} \leq s$ and the segment between $U_{L_{j}}$ and $U_{L_{j+1}}$ located under the polygonal representation (an example see Figure 4.6).


Figure 4.6: The set with vertices $U_{F_{j}}, U_{L_{j}}, U_{L_{j+1}}$ (sketched) is a discrete and faithful upper polygonal representation of the convex segment $\left(U_{F_{j}}, \cdots, U_{L_{j+1}}\right)$. In addition, the set with vertices $U_{F_{j}}, U_{F_{j+1}}, U_{L_{j+1}}$ (not sketched) is a discrete and faithful polygonal representation of this segment, too.
3. Analogously to condition 2 . we can show that the set with vertices $U_{F_{j}}, U_{F_{j+1}}$ and $U_{L_{j+1}}$ is a discrete and faithful polygonal representation.

An example in Figure 4.7 shows the invalidity of the second part of the condition 2. from Proposition 4.6.


Figure 4.7: Polygonal set with vertices $U_{F_{j}}, U_{L_{j}}, U_{L_{j+1}}$ is given in solid style, one with vertices $U_{F_{j}}, U_{F_{j+1}}, U_{L_{j+1}}$ is represented by dashed lines. Since points $P, Q \in \mathcal{K}$ of a convex curve are lying above $\left[U_{L_{j}}, U_{L_{j+1}}\right]$ the second part of the condition 2. from Proposition 4.6 is injured, however, not of the condition 3.

Proposition 4.7 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a segment of a discrete line $\mathcal{D}(a, b, \mu)$. If $\kappa_{1}=U_{F}$ then there exists no element $\kappa \in \mathcal{K}$ such that $\kappa$ is lying above the real line through $U_{F}$ and $U_{L}$.

Proof Trivial.
Using Proposition 4.6 we are able to show following lemma:
Lemma 4.3 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a convex $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=$ $1, \cdots, m, m \geq 2$ are fundamental segments of $\mathcal{K}$. For each fundamental segment $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right), j=1, \cdots, m-1$ the sequence of first leaning points $\left\{U_{F_{j+i}}\right\}, i \geq 1$, $j+i \leq m$ such that $x_{U_{F_{j+i}}} \leq x_{U_{L_{j}}}$ is not empty. Assume for the sequence $\left\{U_{F_{j+i}}\right\}$ the index $i$ is maximal and the following condition is true:
there is no element $\kappa \in \mathcal{K}$ such that $x_{U_{L_{j}}}<x_{\kappa}<x_{U_{L_{j+i}}}$ and $\kappa$ is lying above the real line through $U_{L_{j}}$ and $U_{L_{j+i}}$,
then there exists a discrete and faithful upper polygonal representation of the segment of $\mathcal{K}$ between $U_{F_{1}}$ and $U_{L_{m}}$.

Proof We start with $U_{F_{1}}$ and $U_{L_{1}}$. By assumption we can find the maximal sequence of the first leaning points for $j=1, \cdots, m-1$ such that $x_{U_{F_{2}}} \leq \cdots \leq x_{U_{F_{j+i}}} \leq x_{U_{L_{1}}}$ with $j+i \leq m$. If $j+i=m$ then by Proposition 4.6 the point $U_{L_{1}}$ is a convex vertex of a discrete and faithful upper polygonal representation with vertices $U_{F_{1}}, U_{L_{1}}, U_{L_{m}}$ and we can stop.

If $j+i<m$ then the set with vertices $U_{F_{1}}, U_{L_{1}}, U_{L_{j+i}}$ is a discrete and faithful polygonal representation between $U_{F_{1}}$ and $U_{L_{j+i}}$. We set $j+i=k$. By constraction $x_{U_{F_{k+1}}}>x_{U_{L_{1}}}$. The maximal sequence $\left\{U_{F_{k+i}}\right\}, i \geq 1, k+i \leq m$ such that $x_{U_{F_{k+i}}} \leq x_{U_{k_{j}}}$ is not empty. Moreover, all elements of this sequence are not above
the real line through $U_{L_{1}}, U_{L_{k}}$. It follows the point $U_{L_{k}}$ is a convex vertex of the set with vertices $U_{F_{1}}, U_{L_{1}}, U_{L_{k}}$ and $U_{L_{k+i}}$. The described procedure leads to a discrete and faithful upper polygonal representation between $U_{F_{1}}$ and $U_{L_{m}}$. In this situation the last leaning points of special fundamental segments $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right), j>1$ are vertices of the representation. An example of a suitable convex curve is shown in Figure 4.8.


Figure 4.8: Convex curve with fundamental segments $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=1, \cdots, 7$ satisfying Lemma 4.3 is demonstrated. First and last upper leaning points of fundamental segments are marked by pale triangles. Vertices $\left\{U_{F_{1}}, U_{L_{1}}, U_{L_{4}}, U_{L_{6}}, U_{L_{7}}\right\}$ of a discrete and faithful upper polygonal representation between $U_{F_{1}}$ and $U_{L_{7}}$ are indicated by $\odot$. The slopes of the representation are $0.2308,0.3636,0.4118,0.5000$.

In Lemma 4.3 conditions 1. and 2. of Proposition 4.6 are required. In the similar way, we are able to formulate a lemma, where conditions 1 . and 3 . of this proposition can be used.

Lemma 4.4 Let $\mathcal{K}=\left(\kappa_{1}, \cdots, \kappa_{n}\right)$ be a convex $(0,1)$-curve and $\mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right), i=$ $1, \cdots, m, m \geq 2$ are fundamental segments of $\mathcal{K}$. For each fundamental segment $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right), j=2, \cdots, m$ the sequence of last leaning points $\left\{U_{L_{j-i}}\right\}, i \geq 1, j-i \geq$ 1 such that $x_{U_{F_{j}}} \leq x_{U_{L_{j-i}}}$ is not empty. Assume for the sequence $\left\{U_{L_{j-i}}\right\}$ the index $i$ is maximal and the following condition is true:
there is no element $\kappa \in \mathcal{K}$ such that $x_{U_{F_{j}}}<x_{\kappa}<x_{U_{F_{j-i}}}$ and $\kappa$ is lying above the real line through $U_{F_{j}}$ and $U_{F_{j-i}}$,
then there exists a discrete and faithful upper polygonal representation of the segment of $\mathcal{K}$ between $U_{F_{1}}$ and $U_{L_{m}}$.

Proof The lemma can be shown in the same manner as Lemma 4.3. We must start with $U_{L_{m}}$ and $U_{F_{m}}$ and run the whole curve in the opposite direction until the point $U_{F_{1}}$ is reached.

In the both last lemmas for all fundamental segments the inequality $x_{U_{F_{j+1}}} \leq x_{U_{L_{j}}}$ for all $j=1, \cdots, m-1$ must be fulfilled.

We know that fundamental segments of a curve do not depend of the orientation, i.e. Proposition 4.6, Lemma 4.2, Lemma 4.3 and Lemma 4.4 can be easily formulated for concave curves and their lower polygonal representations. From numerical point of view, since the first upper and the last lower leaning points of fundamental segments can be located inside a horizontal level, it follows that the Lemma 4.3 is much useful for upper polygonal representations. The reformulation of Lemma 4.4 is much useful for lower polygonal representations.

### 4.3 Numerical Implementation

We remind:

- Each convex and concave part of a curve consists at least of two fundamental segments.
- The common segment of successive convex and concave parts is exactly a fundamental segment.
- Upper leaning points of fundamental segments of a concave part $\left(\mathcal{F}_{i}, \cdots, \mathcal{F}_{j}\right)$ are successive elements on the curve, i.e. $x_{U_{L_{i}}} \leq x_{U_{F_{i+1}}} \cdots, x_{U_{L_{j-1}}} \leq x_{U_{F_{j}}}$.

Following simple statements are useful:

- $U_{F_{1}}, U_{L_{1}}$ and $U_{L_{m}}$ are always vertices of the polygonal representation computed by Algorithm UpPolRep.
- The curve is concave if and only if the algorithm detects all concave vertices.
- On a concave curve the algorithm does not stop twice at the same point.
- If on a convex curve conditions of Lemma 4.3 are satisfied then the algorithm detects discrete and faithful representation (between $U_{F_{1}}$ and $U_{L_{m}}$ ).
- Algorithm UpPolRep calculates a convex vertex $\Longrightarrow$ the curve possesses at least one convex part.
- On a convex part only situations described in Lemma 4.2 and Lemma 4.3 (with or without the last condition) are available, i.e. for upper leaning points of successive fundamental segments holds one of the following conditions:

1. $x_{U_{L_{j}}}<x_{U_{F_{j+1}}}$ for a fundamental segment $\mathcal{F}\left(a_{j}, b_{j}, \mu_{j}\right)$ of the part;
2. $x_{U_{F_{j+1}}} \leq x_{U_{L_{j}}}$ for all fundamental segments $\mathcal{F}\left(a_{j}, b_{j}, \mu_{j}\right)$, further, for the sequence $\left\{U_{F_{j+i}}\right\}$ with $x_{U_{F_{j+i}}} \leq x_{U_{L_{j}}}$ and the maximal index $i$ one of the conditions is true:
a) there is no element $\kappa \in \mathcal{K}$ such that $x_{U_{L_{j}}}<x_{\kappa}<x_{U_{L_{j+i}}}$ and $\kappa$ is lying above the real line through $U_{L_{j}}$ and $U_{L_{j+i}}$;
b) there is at least one element $\kappa \in \mathcal{K}$ such that $x_{U_{L_{j}}}<x_{\kappa}<x_{U_{L_{j+i}}}$ and $\kappa$ is lying above the real line through $U_{L_{j}}$ and $U_{L_{j+i}}$.

Assume Algorithm UpPolRep is applied to an arbitrary $(0,1)$-curve $\mathcal{K}$ which has at least two fundamental segments. The algorithm computes $U_{F}=U_{F_{1}}$ and $U_{L}=U_{L_{1}}$ of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$. Following situations are available: $U_{L_{1}} \in \mathcal{F}_{i}\left(a_{i}, b_{i}, \mu_{i}\right)$, $i \geq 2$ or $U_{L_{1}} \notin \mathcal{F}_{2}\left(a_{2}, b_{2}, \mu_{2}\right)$. The first situation is clear: the algorithm detects in the next loop $U_{L_{i+j}} \in \mathcal{F}_{i+j}\left(a_{i+j}, b_{i+j}, \mu_{i+j}\right)$ for some $j \geq 0$. In the second situation the algorithm starts by $U_{L_{1}}$ and computes leaning points of the part of $\mathcal{F}_{1}\left(a_{1}, b_{1}, \mu_{1}\right)$ behind $U_{L_{1}}$. Here, fundamental segments $\mathcal{F}\left(a_{1}, b_{1}, \mu_{1}\right)$ and $\mathcal{F}\left(a_{2}, b_{2}, \mu_{2}\right)$ belong to a maximal convex part (otherwise, it contradicts Proposition 3.6) and the vertex $U_{L}=U_{L_{1}}$ is concave. Moreover, the algorithm stops at least twice at the same point. We deduce

- If the algorithm does not stop twice at the same point then it computes $U_{L}=$ $U_{L_{i}}, i \geq 2$ of some fundamental segment.
- On a convex curve Algorithm UpPolRep may stop twice at the same point.

Let $\left(\mathcal{F}_{i}, \cdots, \mathcal{F}_{j}\right)$ be a maximal convex part and $\left(\mathcal{F}_{j}, \cdots, \mathcal{F}_{k}\right)$ a maximal concave part. Assume $U_{F_{l}} \neq U_{L_{l}}$ for all $i \leq l \leq k$ (the case when $U_{F_{l}}=U_{L_{l}}$ for some $l$ can be considered similar). Since for $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ holds $x_{U_{F_{j}}}<x_{U_{L_{j}}}$, Algorithm UpPolRep computes the vertex $U_{L_{j}}$ which is concave. Moreover, vertices $U_{F_{j+1}}, U_{L_{j+1}}, \cdots$, $U_{F_{k-1}}, U_{L_{k-1}}, U_{F_{k}}$ are concave, too. We deduce that to a maximal concave part the algorithm calculates a corresponding part of a polygonal representation with only concave vertices.

Let $\left(\mathcal{F}_{i}, \cdots, \mathcal{F}_{j}\right)$ and $\left(\mathcal{F}_{j}, \cdots, \mathcal{F}_{k}\right)$ be successive maximal concave and convex parts of the curve. From considerations above follows $U_{F_{j}}$ is a concave vertex of the representation. Furthermore, $U_{L_{j}}$ will be computed. Since on $\left(\mathcal{F}_{j}, \cdots, \mathcal{F}_{k}\right)$ only the situations 1., 2.a) or 2.b) described above are possible, it follows that either $U_{L_{j}}$ is convex (examples are demonstrated in Figure 4.4, Figure 4.7, and Figure 4.8 for $j=1$ ), or $U_{L_{j}}$ is concave, however, in the next step the algorithm detects a convex vertex (see Figure 4.5). Hence, the algorithm detects at least one convex vertex which is either $U_{L_{j}}$, or an element on $\mathcal{F}_{j}\left(a_{j}, b_{j}, \mu_{j}\right)$ behind $U_{L_{j}}$. Thereafter, a corresponding to $\left(\mathcal{F}_{j}, \cdots, \mathcal{F}_{k}\right)$ part computed by Algorithm UpPolRep is faithful (the situation 2.a), or may possess concave vertices (situations 1. and 2.b). In the situation 2.b) the curve may possess a discrete and faithful polygonal representation
as it shown in Figure 4.7. However, this representation will not be computed by the algorithm. It is the only one situation where Algorithm UpPolRep fails.

From both considered cases we deduce that the polygonal representation may have "only few" uncorresponding parts, where "only few" means that the curve possesses no discrete and faithful polygonal representation, or a situation similar to one from Figure 4.7 appears. Thus, the polygonal representation computed by Algorithm UpPolRep is a good approximation of a representation with the same convexity properties. Faithfulness of the calculated discrete representation can be decided from the fundamental polygonal representation whose edges are leaning lines of fundamental segments. The fundamental polygonal representation can be computed within linear time. If the succession of maximal convex and concave parts of the fundamental polygonal representation corresponds to the succession of maximal parts of the representation calculated by Algorithm UpPolRep then it is faithful.

An example, where Algorithm UpPolRep is applied to the curve from Figure 3.2 (on p. 26), is demonstrated in Figure 4.9. This example shows that, in general, the existence of a discrete and faithful polygonal representation may not be decided locally.


Figure 4.9: An upper discrete and faithful polygonal representation between $U_{F_{1}}$ and $U_{L_{6}}$ of the curve from Figure 3.2 computed by Algorithm UpPolRep. Vertices of the representation are marked by $\odot$, slopes of edges are indicated. The part consisting of $\mathcal{F}_{3}\left(a_{3}, b_{3}, \mu_{3}\right)$ and $\mathcal{F}_{4}\left(a_{4}, b_{4}, \mu_{4}\right)$ represented by dark voxels is maximal convex, however, between $U_{F_{3}}$ and $U_{L_{4}}$ Algorithm UpperPolRep does not compute a polygonal representation which is faithful.

For upper polygonal representations of a discrete curve we used the fact that the last upper leaning points of all fundamental segments are first points of the horizontal levels of the curve. Moreover, if we start at the first point of a horizontal level then the both upper leaning points of the detected discrete line will always be first points of horizontal levels. Mostly, a lower polygonal representation of a curve
is different from an upper. Thus, there is another possibility to detect a discrete polygonal representation which is faithful. By construction of the lower polygonal representation we must allow that the last lower leaning points of fundamental segments can be located inside of horizontal levels of a curve. In such situation, Remark 4.4 can be helpful, however, not in each case.

Remark 4.4 In situations, where the last point of a fundamental segment $\mathcal{F}(a, b, \mu)$ of a curve $\mathcal{K}$ is the last lower leaning point and is lying inside a horizontal level of $\mathcal{K}$, the lower leaning point which is the last point of a horizontal level can be find by subtraction of the vector $(b, a)$, i.e. $L_{L}-(b, a)$. Here, the case $L_{F}=L_{L}-(b, a)$ must be studied deeply.

Polygonal representation of "Letter A" is demonstrated in Figure 4.10. $T$ - and $S$ descriptor points are assumed to be always vertices of the polygonal representation. Considering non empty boundary parts between successive descriptors

1. $T_{k}$ and $S_{k}$;
2. $S_{k}$ and $S_{k-1(\bmod 8)}$;
3. $S_{k-1(\bmod 8)}$ and $T_{k-1(\bmod 8)}$,
we deduce that by Proposition 4.4 the first vertex: in first case before $S_{k}$-descriptor points is convex, in second case before $S_{k^{-}}$and after $S_{k-1(\bmod 8)^{-}}$-descriptor points is convex, in third case after $S_{k-1(\bmod 8)}$-descriptor points is convex.


Figure 4.10: Polygonal representation of "Letter A" computed by Algorithm UpPolRep (with the left hand side orientation). Vertices of the representation are marked by $\odot$.

## Chapter 5

## Some Applications in higher Dimension

At the beginning of this chapter we give theoretical foundations of characteristics which sets inherit from their lower-dimensional plane sections. The theorems in Sections 5.1 and 5.2 are considered without proofs since they hold consistently from ordinary theory.

The studying of plane sections of digital sets and plane sections of sets which are transformed using some affine mapping shows that geometrical and topological structures of these can be described by lower-dimensional theory. All algorithms introduced before (in some cases with few modifications) are applicably. Furthermore, we propose an idea for efficient construction of the convex hull of a digital set from $\mathbb{Z}^{3}$. This method can be extended also to higher dimension. Since in ordinary theory convex hulls of point sets define polytopes they are equivalent to geometrical figures representing convex hulls of digital sets. For this reason we give a short introduction into polytope theory.

Finely, we introduce the concept of $d$-convexity. $D$-convex sets in $\mathbb{Z}^{2}$ and $\mathbb{Z}^{3}$ are digitally convex sets which have specific properties. We will show how to construct the oriented boundary of the $d$-convex hull of an arbitrary set from $\mathbb{Z}^{2}$ and the $d$ convex hull of a set from $\mathbb{Z}^{3}$.

### 5.1 Convexity and Plane Sections

What are the properties which sets inherit from their lower-dimensional plane sections? In the Euclidean geometry the convexity is certainly one of them. We will observe that one often obtains strong conclusions from weak assumptions on plane sections.

We need some definitions from ordinary theory.
Definition 5.1 Given a set $S \subseteq \mathbb{R}^{d}$ and a vector $\bar{x} \in \mathbb{R}^{d}$. A hyperplane $H$ is the set

$$
H=\left\{x \in \mathbb{R}^{d} \mid\langle x, \bar{x}\rangle=\alpha\right\},
$$

where $\langle x, \bar{x}\rangle$ designates the inner product.
The hyperplane $H$ is called supporting hyperplane of $S$, if $\alpha=\sup \{\langle x, \bar{x}\rangle \mid x \in S\}$.

Definition 5.2 Given a set $S \subseteq \mathbb{R}^{d}$. The intersection $S \cap H$ of $S$ with a hyperplane $H$ is called a plane section of $S$.

The following theorem is a restriction to $\mathbb{R}^{3}$ of a theorem of Aumann [1] for sets in $\mathbb{R}^{d}$.

Theorem 5.1 If $S$ is a compact set in $\mathbb{R}^{3}$, and if each plane section of $S$ is a simply connected compact set, then $S$ is convex.

The reverse of Theorem 5.1 is obviously true. The theorem of Aumann was generalized by Liberman [33]. See also Mathematical Reviews, vol. 6, p. 184, 1945 . We state it for $\mathbb{R}^{3}$.

Theorem 5.2 Let $S$ be a compact set in $\mathbb{R}^{3}$. Suppose the intersection of $S$ with every supporting hyperplane of $S$ is a simply connected compact set. Then the boundary of $S$ is the boundary of a convex set.

The proofs of Theorems 5.1 and 5.2 can be found in [46]. If in addition to the hypothesis of Theorem 5.2 one also assumes that $S$ does not bound any residual bounded holes (it means that $S$ should be simply connected) then $S$ will be convex.

In digital space we have the advantage that there exist finitely many plane sections of a digital set which are not empty. However, plane sections of a digital set $S_{\triangle}$ possess, generally, different topological structures. Thus, there is no possibility for the direct translation of these theorems into digital world.

In Theorems 5.1 and 5.2 one needs topological assumptions (connectedness, $S$ is required to be compact). The simply connectedness in digital plane can be easily defined by means of the connectedness of the complement. Here, it will be used the validity of the Jordan's curve theorem for 8 -connected sets and their 4-connected complements. We may define: Given an 8-connected set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. $S_{\triangle}$ is said to be 8 -simply connected if the complement $\mathbb{C} S_{\triangle}$ is a 4 -connected set. Obviously, a digitally convex set which is 8 -connected is 8 -simply connected as well.

For the sake of simplicity we designate the $Y Z$-coordinate plane as $X$-plane, $X Z$ coordinate plane as $Y$-plane and $X Y$-coordinate plane as $Z$-plane. The following statement is trivially true.

Proposition 5.1 Let $S_{\triangle} \subseteq \mathbb{Z}^{3}$ be a 26-connected digitally convex set. Then a non empty intersection of $S_{\triangle}$ with a plane which is parallel to $X$-, $Y$ - or Z-coordinate plane is a digitally convex set of $\mathbb{Z}^{2}$.

Remark 5.1 Plane sections of 26-connected digitally convex sets from Proposition 5.1 are not necesserily 8-connected. An example is shown in Figure 5.1.


Figure 5.1: The represented 3-dimensional digital set $S_{\triangle} \subseteq \mathbb{Z}^{3}$ is digitally convex. The plane section $\{z=1\} \cap S_{\triangle}$ consisting of elements $(2,1,1)$ and $(1,3,1)$ is not 8 -connected.

```
Algorithm VII PlaneSec Parallel to \(X\)-coordinate plane sections of a finite
                                    point set \(S_{\triangle} \subseteq \mathbb{Z}^{3}\).
\(/ * S_{\triangle}=\left(x_{S_{\Delta}}, y_{S_{\Delta}}, z_{S_{\Delta}}\right) \in \mathbb{Z}^{n \times 3} * /\)
/* PlaneSec \(\{\cdot\}\) is one single plane section \(* /\)
for \(i \leftarrow \min \left(x_{S_{\Delta}}\right)\) to \(\max \left(x_{S_{\Delta}}\right)\) do
    PlaneSec \(\{i\} \leftarrow \emptyset\);
    for \(j \leftarrow 1\) to \(n\) do
            if \(x_{S_{\triangle}}(j)=i\) then
            PlaneSec \(\{i\} \leftarrow \operatorname{PlaneSec}\{i\} \cup\left(y_{S_{\Delta}}(j), z_{S_{\Delta}}(j)\right) ;\)
            end
    end
end
```

It is very easy to find algorithmically all plane sections of a finite point set $S_{\triangle} \subseteq \mathbb{Z}^{3}$ which are parallel to $X$-, $Y$ - or Z-coordinate plane. Algorithm PlaneSec describes this procedure. The complexity of Algorithm PlaneSec is $O(\mathrm{~nm})$, where $n$ is the number of elements of the set $S_{\triangle}$ and $m=\max \left(x_{S_{\Delta}}\right)-\min \left(x_{S_{\Delta}}\right)+1$.

The parallel to $X$-, $Y$ - or $Z$-coordinate plane sections of a 26 -connected set from $\mathbb{Z}^{3}$ can be treated in the same manner as sets from $\mathbb{Z}^{2}$ using 8-topology. On such plane sections we are able to define 8 -connected sets, interior and closure of a set, boundary of a set, etc.

### 5.2 Convexity and Affine Transformations

The affine hull aff $S$ of $S \subseteq \mathbb{R}^{d}$ is the set $\left\{\sum_{i=1}^{n} \lambda_{i} x_{i} \mid x_{1}, \cdots, x_{n} \in S, \sum_{i=1}^{n} \lambda_{i}=1\right\}$ of all affine combinations of points of $S$. The set $S$ is termed affine set whenever aff $S=S$. Obviously, affine sets are convex.

A mapping $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ is called an affine transformation if it preserves affine structure, in the sense that it maps every affine combination of points of $\mathbb{R}^{d}$ onto the same affine combination of their images. Such transformations map affine sets onto affine sets.

Hence, a mapping $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ is an affine transformation whenever

$$
T\left(\sum_{i=1}^{n} \lambda_{i} x_{i}\right)=\sum_{i=1}^{n} \lambda_{i} T\left(x_{i}\right)
$$

for any finite set of points $x_{i} \in \mathbb{R}^{d}, i=1, \cdots, n$ and $\sum_{i=1}^{n} \lambda_{i}=1$. For each vector $q \in \mathbb{R}^{d}$, the mapping $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ defined by the equation $T(x)=x+q$ is an affine transformation called the translation of $\mathbb{R}^{d}$ through $q$.

It is easy to show that the composite of affine transformations is an affine transformation. Moreover, affine transformations preserve parallelism between affine sets and if $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ is an injective (surjective,bijective) affine transformation then $l \geq d(l \leq d, l=d)[49]$.

Clearly every linear transformation from $\mathbb{R}^{d}$ to $\mathbb{R}^{l}$ is also an affine one. That not every affine transformation is linear follows from the observation that it does not need map the zero vector of $\mathbb{R}^{d}$ onto the zero vector of $\mathbb{R}^{l}$.

The following theorem is well known from ordinary theory [49].
Theorem 5.3 The affine transformations $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ are precisely those mappings which can be expressed in the form $T(x)=Q(x)+q$, for some real $l \times d$ matrix $Q$ and some real $l \times 1$ matrix $q$. If $T$ is non singular then inverse $T^{-1}$ is an affine transformation.

We note that the affine transformation $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ determines the matrices $Q$ and $q$ uniquely.

Affine transformations preserve convex sets. Moreover, if $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}(d \geq$ 2 ) is an injective mapping which preserves convex sets then $f$ must be an affine transformation [49]. This result was proved in 1971 by W. Meyer and D. C. Kay. The affine transformations play a central role in the study of convexity-preserving mappings. Notice that not every convexity-preserving mapping is affine. To see that this is so, we note that every continuous mapping from $\mathbb{R}^{d}$ into $\mathbb{R}$ is convexitypreserving.
Theorem 5.4 Let $S \subseteq \mathbb{R}^{d}$ be convex and $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{l}$ be an affine transformation. Then $T(S)$ is convex.

### 5.3 Transformations of Point Sets on 2D and 3D Lattices

In this section we will study properties of point sets on lattices.
Definition 5.3 A lattice in $\mathbb{R}^{d}$ is a subgroup $\Lambda$ of $\mathbb{R}^{d}$ which satisfies the following property: there exists a basis $\mathcal{B}=\left(b_{1}, b_{2}, \cdots, b_{d}\right)$ of $\mathbb{R}^{d}$ such that $\Lambda$ is the set of all $\mathbb{Z}$-linear combinations of the $b_{i}$. In this case $\mathcal{B}$ is called $a$ basis of the lattice.

Recall, a basis $\mathcal{B}=\left(b_{1}, \cdots, b_{d}\right)$ is called orthogonal if $\left\langle b_{i}, b_{j}\right\rangle=0$ for $i \neq j$ and orthonormal if moreover $\left\|b_{i}\right\|=1$ for all $i$. We refer to [35] for a comprehensive view of lattice theory. The space $\mathbb{Z}^{d}$ is an example of the lattice in $\mathbb{R}^{d}$ with an orthonormal basis. It is clear that on an arbitrary lattice in $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$ which possesses an orthonormal basis we are able to define 8 - or 26-topology, respectively.

Our further considerations we may restrict without loss of generality to non singular affine transformations $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ with $T(x)=Q(x)+q$ such that $q=0$, i.e. transformations are linear.

A linear transformation $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ (we use the same symbol for the matrix and the transformation defined by it) is given by a non singular $2 \times 2$ matrix

$$
T=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

From ordinary theory it is well known that each non singular linear transformation is bijective and vice versa. If $T$ possesses only integer entries and the determinant of $T$ is $\pm 1$ then the inverse matrix $T^{-1}$ of $T$ has only integer elements as well. Hence, in this case $T$ and $T^{-1}$ are bijective mappings from $\mathbb{Z}^{2}$ into $\mathbb{Z}^{2}$. A matrix $T$ having integer elements and the determinant $\pm 1$ is called unimodular matrix. It is immediately clear that the set of all unimodular matrices is a group with respect to matrix multiplication.

Given a linear transformation $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ defined by a non singular $2 \times 2$ matrix. Let $T^{-1}$ be the inverse matrix of $T$. We introduce the $T_{8}$-neighborhood of an element $P \in \mathbb{R}^{2}$ with $T^{-1}(P)=Q$ as follows

$$
N_{0}^{T}(P)=T\left(N_{0}(Q)\right), \cdots, N_{7}^{T}(P)=T\left(N_{7}(Q)\right) .
$$

Since $T$ preserve parallelism and is bijective it follows that an element of $\mathbb{R}^{2}$ and its $T_{8}$-neigthborhood belong to some lattice on $\mathbb{R}^{2}$ which depends on the transformation $T$.

We are able to define the $T_{8}$-connected and $T_{8}$-convex sets, $T_{8}$-interior, $T_{8}$-closure and $T_{8}$-boundary on a lattice of $\mathbb{R}^{2}$ in the same manner as by 8 -topology for digital sets on the lattice $\mathbb{Z}^{2}$. The interior, closure, boundary and complement with respect to $T_{8}$-topology will be marked as $\operatorname{int}_{T_{8}}, c l_{T_{8}}, b d_{T_{8}}$ and $\mathbb{C}_{T_{8}}$, respectively. The $T_{8}$-neighborhood of an element $x \in \mathbb{R}^{2}$ will be denoted as $\mathfrak{N}_{T_{8}}(x)$.

Obviously, the 8-topology on an arbitrary lattice with an orthonormal basis is available if we choose the identity matrix as the transformation $T$.

Lemma 5.1 Given a finite point set $S$ on a lattice of $\mathbb{R}^{2}$ with an orthonormal basis and a non singular linear transformation $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$. Assume the interior, closure and boundary with respect to $T_{8}$-topology are int $T_{8}, c l_{T_{8}}$ and $b d_{T_{8}}$, respectively. Then

1. The set $S$ is 8 -connected if and only if $T(S)$ is $T_{8}$-connected.
2. The set $S$ is digitally convex if and only if $T(S)$ is $T_{8}$-convex.

Furthermore, if $S$ is 8 -connected then

$$
\begin{aligned}
T(\operatorname{int}(S)) & =\operatorname{int}_{T_{8}}(T(S)), \\
T(c l(S)) & =\operatorname{cl}_{T_{8}}(T(S)), \\
T(b d(S)) & =b d_{T_{8}}(T(S)) .
\end{aligned}
$$

Proof The transformation $T$ and its inverse $T^{-1}$ are bijective.

1. The first statement holds from the fact that $y \in \mathcal{N}(x) \Leftrightarrow T(y) \in \mathcal{N}_{T_{8}}(T(x))$.
2. Since $T$ and $T^{-1}$ preserve convexity the second statement is obviously true.

The rest follows by definition of neighbors $N_{k}^{T}(P), k=0, \cdots, 7$ for $P \in S$.
The previous considerations can be extended to 3D case. Given a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ defined by a $3 \times 3$ matrix

$$
T=\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right) .
$$

The $T_{26}$-neighborhood of an element $P \in \mathbb{R}^{3}$ with $T^{-1}(P)=Q$ is defined as

$$
N_{0}^{T}(P)=T\left(N_{0}(Q)\right), \cdots, N_{25}^{T}(P)=T\left(N_{25}(Q)\right) .
$$

A point of $\mathbb{R}^{3}$ and its $T_{26}$-neigthborhood form a lattice in $\mathbb{R}^{3}$ which depends on the transformation $T$. The 26-topology on an arbitrary lattice in $\mathbb{R}^{3}$ with an orthonormal basis is available whenever the transformation $T=i d$. Lemma 5.1 can be easily translated into 3D space.

Lemma 5.2 Given a finite point set $S$ on a lattice of $\mathbb{R}^{3}$ with an orthonormal basis and a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$. Assume the interior, closure and boundary with respect to $T_{26}$-topology are int $T_{26}, c l_{T_{26}}$ and $b d_{T_{26}}$, respectively. Then

1. The set $S$ is 26 -connected if and only if $T(S)$ is $T_{26}$-connected.
2. The set $S$ is digitally convex if and only if $T(S)$ is $T_{26}$-convex.

Furthermore, if S is 26-connected then

$$
\begin{aligned}
T(\operatorname{int}(S)) & =\operatorname{int}_{T_{26}}(T(S)), \\
T(c l(S)) & =\operatorname{cl}_{T_{26}}(T(S)), \\
T(b d(S)) & =b d_{T_{26}}(T(S)) .
\end{aligned}
$$

Proof This lemma is an extension of Lemma 5.1 to 3D case.

### 5.4 Plane Sections of Transformed 3D Sets

Since affine transformations preserve parallelism and convexity the structure of the plane sections of transformed 3D sets is not arbitrary. We consider a transformation $T$ given by a $3 \times 3$ matrix $T=\left(a_{1}, a_{2}, a_{3}\right)=\left(a^{1}, a^{2}, a^{3}\right)$, where $a_{i} \in \mathbb{R}^{1 \times 3}$ are row vectors, $a^{i} \in \mathbb{R}^{3 \times 1}$ are column vectors, $i=1,2,3$. We note that in general $a_{1} \neq a^{1}$, i.e. $a_{1}^{T}$ and $a^{1}$ are different vectors.

The transformed by $T$ coordinate planes $X=\{(0, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}, Y=\{(\alpha, 0, \beta) \mid$ $\alpha, \beta \in \mathbb{R}\}$ and $Z=\{(\alpha, \beta, 0) \mid \alpha, \beta \in \mathbb{R}\}$ will be denoted as $X_{T^{-}}, Y_{T^{-}}$and $Z_{T^{-}}$-planes, hence, $X_{T}=T(X), Y_{T}=T(Y)$ and $Z_{T}=T(Z)$. Obviously, for hyperplanes $X, Y, Z$ or an arbitrary hyperplane $H$ and a non singular linear transformation $T$ the transformed sets $X_{T}, Y_{T}, Z_{T}$ and $T(H)$ are hyperplanes, too.

Proposition 5.2 Given arbitrary sets $X_{1}, X_{2} \subseteq \mathbb{R}^{3}$ and a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$. Then

$$
\begin{aligned}
T\left(X_{1} \cup X_{2}\right) & =T\left(X_{1}\right) \cup T\left(X_{2}\right), \\
T\left(X_{1} \cap X_{2}\right) & =T\left(X_{1}\right) \cap T\left(X_{2}\right), \\
T\left(X_{1} \backslash X_{2}\right) & =T\left(X_{1}\right) \backslash T\left(X_{2}\right), \\
T\left(\mathbb{C} X_{1}\right) & =\mathbb{C}\left(T\left(X_{1}\right)\right),
\end{aligned}
$$

where $\mathbb{C}$ means the set theoretical complement and $X_{1} \backslash X_{2}=X_{1} \cap \mathbb{C} X_{2}$.
Proof The transformation $T$ is non singular, it follows that it is bijective. Thus, for an arbitrary set $X \subseteq \mathbb{R}^{3}$ holds

$$
T(X)=T\left(\left\{x \in \mathbb{R}^{3} \mid x \in X\right\}\right)=\left\{y \in \mathbb{R}^{3} \mid y \in T(X)\right\}
$$

From the previous proposition we deduce that transformed plane sections of the original set are plane sections of the transformed set.

Proposition 5.3 Given a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$. The hyperplanes $H_{X}, H_{Y}$ and $H_{Z} \subseteq \mathbb{R}^{3}$ are parallel to $X$-, $Y$ - and $Z$-plane, respectively, if and only if $T\left(H_{X}\right), T\left(H_{Y}\right)$ and $T\left(H_{Z}\right)$ are parallel to $X_{T^{-}}, Y_{T^{-}}$and $Z_{T^{-}}$-plane, respectively.

Proof The inverse $T^{-1}$ of $T$ is a linear transformation, the both are affine transformations. Furthermore, affine transformations preserve parallelism. Thus,

$$
H_{X}\left\|X \quad \Leftrightarrow \quad T\left(H_{X}\right)\right\| T(X)=X_{T}
$$

Proposition 5.4 Given a point set $S \subseteq \mathbb{R}^{3}$ and a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ with $T=\left(a_{1}, a_{2}, a_{3}\right)=\left(a^{1}, a^{2}, a^{3}\right)$. Assume $H_{X}, H_{Y}$ and $H_{Z}$ are hyperplanes which are parallel to $X$-, $Y$ - and Z-plane, respectively. Iffor the matrix Tholds

1. $a^{1}=a_{1}^{T}=\alpha e^{1}$ then $T\left(S \cap H_{X}\right)$ is parallel to $X$-plane,
2. $a^{2}=a_{2}^{T}=\beta e^{2}$ then $T\left(S \cap H_{Y}\right)$ is parallel to $Y$-plane,
3. $a^{3}=a_{3}^{T}=\gamma e^{3}$ then $T\left(S \cap H_{Z}\right)$ is parallel to $Z$-plane, where $\left\{e^{1}, e^{2}, e^{3}\right\}$ are unit vectors of $\mathbb{R}^{3}$ and $\alpha, \beta, \gamma \in \mathbb{R} \backslash\{0\}$.

Proof We will prove only the first statement, the others follow similar.
Assume for the matrix $T$ holds $a_{1}=a^{1}=\alpha e^{1}$. Then for the $x$-coordinate of an element $h$ of $H_{X}$ and $x$-coordinate of $T(h)$ one has $\alpha x_{h}=x_{T(h)}$. The $y$ - and $z$-coordinates $y_{T(h)}$ and $z_{T(h)}$ of $T(h)$ do not depend of $x_{h}$. If we assume that $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ is the matrix $T$ without the first row and first column and $\left(\begin{array}{l}x \\ y \\ z\end{array}\right) \in H_{X}$ then the hyperplane $T\left(H_{X}\right)$ has the form

$$
T\left(H_{X}\right)=\left\{\left.\left(\begin{array}{c}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{array}\right) \in \mathbb{R}^{3} \right\rvert\, \bar{x}=\alpha x=\text { const }, \bar{y}=a y+b z, \bar{z}=c y+d z\right\} .
$$

The hyperplanes $H_{X}$ and $T\left(H_{X}\right)$ are parallel and from $H_{X} \| X$ and $T\left(H_{X}\right) \| X_{T}$ follows $X_{T} \| X$. It holds $T\left(S \cap H_{X}\right)=T(S) \cap T\left(H_{X}\right)$ is parallel to $X_{T}$, hence, it is parallel to $X$, too.

One designates the matrices $T_{i}$ and $T^{i}$ as follows

$$
T_{i}=T \backslash a_{i}, \quad T^{i}=T \backslash a^{i},
$$

which are the matrix $T=\left(a_{1}, a_{2}, a_{3}\right)=\left(a^{1}, a^{2}, a^{3}\right)$ without the row $i$ or column $i$, respectively. The matrix $T$ without a row $i$ and column $j$ is a $2 \times 2$ matrix and will be marked $T_{j}^{i}$.

Proposition 5.5 Given a point set $S$ on a lattice of $\mathbb{R}^{3}$ with an orthonormal basis. Assume $H_{X}, H_{Y}$ and $H_{Z}$ are hyperplanes which are parallel to $X$-, $Y$ - and $Z$-plane, respectively.

1. If $S \cap H_{X}$ is 8 -connected (digitally convex) then there exists a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ such that $T\left(S \cap H_{X}\right)$ is $\left(T_{1}^{1}\right)_{8}$-connected (convex).
2. If $S \cap H_{Y}$ is 8-connected (digitally convex) then $T\left(S \cap H_{Y}\right)$ is $\left(T_{2}^{2}\right)_{8}$-connected (convex).
3. If $S \cap H_{Z}$ is 8 -connected (digitally convex) then $T\left(S \cap H_{Z}\right)$ is $\left(T_{3}^{3}\right)_{8}$-connected (convex).

Proof We prove the first statement, the second and third follow analogously.
Let us consider a linear transformation $T=\left(a_{1}, a_{2}, a_{3}\right)=\left(a^{1}, a^{2}, a^{3}\right)$ given by a non singular $3 \times 3$ matrix such that $a^{1}=a_{1}^{T}=\alpha e^{1}$. By Proposition 5.4 it follows that $T\left(S \cap H_{X}\right)$ is parallel to $X$-plane. For $\left(\begin{array}{l}x \\ y \\ z\end{array}\right) \in S$ we have

$$
T\left(S \cap H_{X}\right)=\left\{\left.\left(\begin{array}{c}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{array}\right) \in \mathbb{R}^{3} \right\rvert\, \bar{x}=\alpha x=\text { const, } \bar{y}=a y+b z, \bar{z}=c y+d z\right\}
$$

where $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)=T_{1}^{1}$. If $S \cap H_{X}$ is 8 -connected (digitally convex) then by Lemma 5.1 we deduce $T\left(S \cap H_{X}\right)$ is $\left(T_{1}^{1}\right)_{8}$-connected (convex).

The extension of Proposition 5.5 is given in the next lemma.
Lemma 5.3 Given a point set $S$ on a lattice of $\mathbb{R}^{3}$ with an orthonormal basis and a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ with $T=\left(a_{1}, a_{2}, a_{3}\right)=$ $\left(a^{1}, a^{2}, a^{3}\right)$. Assume $H_{X}, H_{Y}$ and $H_{Z}$ are hyperplanes which are parallel to $X$-, $Y$ and Z-plane, respectively. Let op be an operator op: op $=$ int, op $=c l$ or $o p=b d$. If $a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}$ then

1. $S \cap H_{X}$ is 8-connected (digitally convex) if and only if $T\left(S \cap H_{X}\right)$ is $\left(T_{1}^{1}\right)_{8}$ connected (convex).
2. If $S \cap H_{X}$ is 8-connected then

$$
T\left(o p\left(S \cap H_{X}\right)\right)=o p_{\left(T_{1}^{1}\right)_{8}}\left(T\left(S \cap H_{X}\right)\right)
$$

If $a^{2}=a_{2}^{T}=\beta e^{2}, \beta \in \mathbb{R} \backslash\{0\}$ then

1. $S \cap H_{Y}$ is 8-connected (digitally convex) if and only if $T\left(S \cap H_{Y}\right)$ is $\left(T_{2}^{2}\right)_{8-}$ connected (convex).
2. If $S \cap H_{Y}$ is 8 -connected then

$$
T\left(o p\left(S \cap H_{Y}\right)\right)=o p_{\left(T_{2}^{2}\right)_{8}}\left(T\left(S \cap H_{Y}\right)\right)
$$

If $a^{3}=a_{3}^{T}=\gamma e^{3}, \gamma \in \mathbb{R} \backslash\{0\}$ then

1. $S \cap H_{Z}$ is 8-connected (digitally convex) if and only if $T\left(S \cap H_{Z}\right)$ is $\left(T_{3}^{3}\right)_{8}$ connected (convex).
2. If $S \cap H_{Z}$ is 8 -connected then

$$
T\left(o p\left(S \cap H_{Z}\right)\right)=o p_{\left(T_{3}^{3}\right)_{8}}\left(T\left(S \cap H_{Z}\right)\right) .
$$

Proof We will prove only the first part of the lemma, others are similar.

1. Assume $a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}$ and $S \cap H_{X}$ is 8 -connected (digitally convex) then by Proposition 5.5 it holds $T\left(S \cap H_{X}\right)$ is $\left(T_{1}^{1}\right)_{8}$-connected (convex).

Otherwise, assume $a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}$ and $T\left(S \cap H_{X}\right)$ is $\left(T_{1}^{1}\right)_{8}$-connected (convex). Then $T_{1}^{1}$ is a linear transformation represented by a $2 \times 2$ non singular matrix and it holds

$$
T^{-1}=\left(\begin{array}{ccc}
\frac{1}{\alpha} & 0 & 0 \\
0 & \left(\mathbf{T}_{\mathbf{1}}^{\mathbf{1}}\right)^{-\mathbf{1}}
\end{array}\right)
$$

The sets $S \cap H_{X}$ and $T\left(S \cap H_{X}\right)$ are both parallel to $X$-coordinate plane, thus, without loss of generality we may consider only $y$ - and $z$-coordinates of these sets, which are $\left(S \cap H_{X}\right)^{1}$ and $T\left(S \cap H_{X}\right)^{1}$, respectively, (or $\left(S \cap H_{X}\right)_{1}$ and $T\left(S \cap H_{X}\right)_{1}$, respectively). One has $T\left(S \cap H_{X}\right)^{1}=T_{1}^{1}\left(\left(S \cap H_{X}\right)^{1}\right)$ (or $\left.T\left(S \cap H_{X}\right)_{1}=T_{1}^{1}\left(\left(S \cap H_{X}\right)_{1}\right)\right)$, from Lemma 5.1 it follows that $\left(S \cap H_{X}\right)^{1}$ (or $\left.\left(S \cap H_{X}\right)_{1}\right)$ is 8-connected (digitally convex).
2. The statement holds by Lemma 5.1.

Proposition 5.6 Given a point set $S$ on an arbitrary lattice of $\mathbb{R}^{3}$, non singular linear transformations $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ and $U: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ and a hyperplane $H$. If $S \cap H$ is $U_{8}$-connected (convex) then there exists a non singular linear transformation $V$ : $\mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ such that $T(S \cap H)$ is $V_{8}$-connected (convex).

Proof It holds $\operatorname{dim}(S \cap H)=\operatorname{dim}(T(S \cap H))=2, T$ is bijective and preserve parallelism. According to Lemma 5.1 the statement of the proposition is true.

In Proposition 5.5 and Lemma 5.3 we examined only special plane sections of transformed sets. Generally, the linear transformations given by a $3 \times 3$ non singular matrix with $a^{i}=a_{i}^{T}=\alpha e^{i}, \alpha \neq 0, i=1,2,3$ have a very useful property: all plane sections of a transformed 3D set which are parallel to $X$-, $Y$ - and $Z$-coordinate plane may be considered with some $T_{8}$-topology which we are able to determine. The following lemma is an extension of Lemma 5.3.

Lemma 5.4 Given a point set $S$ on a lattice of $\mathbb{R}^{3}$ with an orthonormal basis and a non singular linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ with $T=\left(a_{1}, a_{2}, a_{3}\right)=$ $\left(a^{1}, a^{2}, a^{3}\right)$ and $T_{i}^{i}=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ for some $i=1,2,3$. Assume $U: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ is a non singular linear transformation.

$$
\text { If } a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}, i=1 \text { and }
$$

1. If $S \cap T^{-1}(Y)$ is $U_{8}$-connected (convex) and $a \neq 0$ then for the non singular linear transformation $V=\left(T^{-1}\right)_{2}^{2}$ holds $T(S) \cap Y$ is $V_{8}^{-1}$-connected (convex).
2. If $S \cap T^{-1}(Z)$ is $U_{8}$-connected (convex) and $d \neq 0$ then for $V=\left(T^{-1}\right)_{3}^{3}$ holds $T(S) \cap Z$ is $V_{8}^{-1}$-connected (convex).
If $a^{2}=a_{2}^{T}=\alpha e^{2}, \alpha \in \mathbb{R} \backslash\{0\}, i=2$ and
3. If $S \cap T^{-1}(X)$ is $U_{8}$-connected (convex) and $a \neq 0$ then for $V=\left(T^{-1}\right)_{1}^{1}$ holds $T(S) \cap X$ is $V_{8}^{-1}$-connected (convex).
4. If $S \cap T^{-1}(Z)$ is $U_{8}$-connected (convex) and $d \neq 0$ then for $V=\left(T^{-1}\right)_{3}^{3}$ holds $T(S) \cap Z$ is $V_{8}^{-1}$-connected (convex).

If $a^{3}=a_{3}^{T}=\alpha e^{3}, \alpha \in \mathbb{R} \backslash\{0\}, i=3$ and

1. If $S \cap T^{-1}(X)$ is $U_{8}$-connected (convex) and $a \neq 0$ then for $V=\left(T^{-1}\right)_{1}^{1}$ holds $T(S) \cap X$ is $V_{8}^{-1}$-connected (convex).
2. If $S \cap T^{-1}(Y)$ is $U_{8}$-connected (convex) and $d \neq 0$ then for $V=\left(T^{-1}\right)_{2}^{2}$ holds $T(S) \cap Y$ is $V_{8}^{-1}$-connected (convex).

Proof Assume $a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}$ and $S \cap T^{-1}(Y)$ is $U_{8}$-connected (convex). It holds $T\left(S \cap T^{-1}(Y)\right)=T(S) \cap Y$ (illustration see Figure 5.2).

According to Proposition 5.6 there exists a linear transformation $V^{-1}$ such that $T(S) \cap Y$ is $V_{8}^{-1}$-connected (convex).

One has

$$
T^{-1}=\frac{1}{\operatorname{det} T_{1}^{1}}\left(\begin{array}{ccc}
\frac{\operatorname{det} T_{1}^{1}}{\alpha} & 0 & 0 \\
0 & d & -b \\
0 & -c & a
\end{array}\right), \quad \text { i.e. } \quad\left(T^{-1}\right)_{2}^{2}=\left(\begin{array}{cc}
\frac{1}{\alpha} & 0 \\
0 & \frac{a}{\operatorname{det} T_{1}^{1}}
\end{array}\right) .
$$



Figure 5.2: Illustration to Lemma 5.4.
Then for $\left(\begin{array}{l}x \\ y \\ z\end{array}\right) \in S$ the set $T(S) \cap Y$ has the form

$$
T(S) \cap Y=\left\{\left.\left(\begin{array}{c}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{array}\right) \in \mathbb{R}^{3} \right\rvert\, \bar{x}=\alpha x, \bar{y}=a y+b z=0, \bar{z}=c y+d z\right\} .
$$

It follows

$$
T(S) \cap Y=\left\{\left.\left(\begin{array}{c}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{array}\right) \in \mathbb{R}^{3} \right\rvert\, \bar{x}=\alpha x, \bar{y}=0, \bar{z}=\frac{\operatorname{det} T_{1}^{1}}{a} z\right\} .
$$

Thus, the statement of the lemma is true.
The case where $a^{1}=a_{1}^{T}=\alpha e^{1}, \alpha \in \mathbb{R} \backslash\{0\}$ and $S \cap T^{-1}(Z)$ is $U_{8}$-connected (convex) with $d \neq 0$ follows similar.

The second and third parts of the lemma can be shown in the same manner as the first.

From ordinary theory it is well known that by using elementary matrix transformations each $T$ can be represented as a matrix with $a^{1}=a_{1}^{T}=\alpha e^{1}, a^{2}=a_{2}^{T}=\beta e^{2}$ or $a^{3}=a_{3}^{T}=\gamma e^{3}$. Moreover, if the elements of $T$ are integers then this representation can be implemented in such way that it has only integers, too. Furthermore, Algorithm PlaneSec from Section 5.1 (see p. 57) with few modifications can be applied.

Theoretically, plane sections of 3D sets described in Lemma 5.1, Lemma 5.3 and Lemma 5.4 can be treated with 8 -topology. However, the inverse of $2 \times 2$
matrices from these lemmas possess, generally, non integer elements. So, if in the praxis the inverse has non integer elements which are also non computer numbers then the plane sections can be treated with $T_{8}$-topology. The "useful" elements of a transformed plane section are some elements in the original plane which can be obtained if the transformed and original elements ordered in the same manner. Thus, the inverse transformation of $T$ is not really required.

The special plane sections of a digital set $S_{\triangle} \subseteq \mathbb{Z}^{3}$ which are supporting hyperplanes of $S_{\triangle}$ can be determined by described transformations. Using 2D theory we are able to obtain the convex hull of elements of $S_{\triangle}$ belonging to supporting hyperplanes. It is well known that the convex hull of a 2D set can be computed within linear time. Hence, the described technique leads to detection of the convex hull of $S_{\triangle}$ whenever we apply many enough transformations.

### 5.5 Polytopes in $\mathbb{R}^{d}$

Convex hulls of digital sets can be considered as convex hulls of point sets. The latter objects are called convex polytopes. Polytopes have been investigated since antiquity. They were amongst the first convex sets ever to be studied. Points, line segments, polygons, tetrahedra, cubes, octahedra, dodecahedra, and icosahedra are all discussed in Euclid's Elements, written around 300 вс. Polytopes are of big importance for many mathematical subjects: integration theory, algebraic topology and geometry, linear and combinatorial optimization.

In this chapter we give an introduction into polytope theory. We refer to Grünbaum [19] and McMullen, Shephard [36] for a comprehensive view of polytope theory.

Definition 5.4 The convex hull of a finite set $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ of points in $\mathbb{R}^{d}$ :

$$
P=\operatorname{conv}(X):=\left\{\sum_{i=1}^{n} \lambda_{i} x_{i} \mid \lambda_{i} \geq 0, \sum_{i=1}^{n} \lambda_{i}=1\right\}
$$

is called V-polytope.
A bounded solution set of a finite system of linear inequalities:

$$
P=P(A, b):=\left\{x \in \mathbb{R}^{d} \mid a_{i}^{T} x \leq b_{i}, 1 \leq i \leq m\right\},
$$

where $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^{m}$, is called H -polytope.
$A$ subset $P \subseteq \mathbb{R}^{d}$ that can be represented as a $V$-polytope or (equivalently, by the main theorem below!) as an H-polytope is called polytope.

The dimension of a polytope $P$ is defined as the dimension of its affine hull: $\operatorname{dim}(P):=\operatorname{dim}(\operatorname{aff}(P))$.

The empty set and $P$ itself are improper faces of dimension -1 and $d$, respectively, of the polytope $P$. A proper face (or briefly face) of $P$ is the (non empty) intersection of $P$ with a supporting hyperplane. Face of $P$ of dimension $0,1, i$, or $d-1$ is called vertex, edge, $i$-face or facet, respectively.

A polytope with all of its vertices in $\mathbb{Z}^{d}$ is called integral polytope or lattice polytope.

By definition polytopes are compact sets of $\mathbb{R}^{d}$. Moreover, by the main theorem of the polytope theorie holds:

Theorem 5.5 The definitions of $V$-polytopes and of $H$-polytopes are equivalent. That is, every V-polytope has a description by a finite system of inequalities, and every $H$-polytope can be obtained as the convex hull of a finite set of points (its vertices).

To see the main theorem at work, consider the following two statements: the first one is easy to see for V-polytope, but not for H-polytope, and for the second statement we have the opposite effect.

1. Projections: Every image of a polytope $P$ under affine map $T: x \mapsto A x+b$ is a polytope.
2. Intersections: Intersection of a polytope with an affine subspace is a polytope.

However, the computational step from one of the main theorem's descriptons of polytopes to the other - a convex hull computation - is far from trivial. Essentially, there are three types of algorithm available: inductive algorithms (inserting vertices, using a so-called beneath-beyond technique), projection, respectively, intersection algorithm (known as Fourier-Motzkin elimination [3, 4, 28], respectively, double description algorithm [38, 39]), and reverse search methods (as introduced by Avis and Fukuda [2]).

A convex polytope $P$ can be described in many ways. In our context the most important descriptions are those listed below.

1. Vertex description: The set of all vertices of $P$ specified by their coordinates.
2. Facet description: The set of all facets of $P$ specified by their defining linear inequalities.
3. Double description: The set of vertices of $P$, the set of all facets of $P$, and the incidence relation between the vertices and the facets specified by an incidence matrix.
4. Lattice description: The face lattice of $P$ specified by its Hasse diagram (a directed graph of an order relation that joins nodes $a$ to $b$ iff $a \leq b$ and there are no elements between $a$ and $b$ in the sense that $a \leq c \leq b$ then either $c=a$
or $c=b$. For the face lattice the order relation is contaiment), with vertex and facet nodes augmented by coordinates and defining linear inequalities, respectively.
5. Boundary description: A triangulation of the boundary of $P$ specified by a simplical complex, with vertices and maximal simplices augmented by coordinates and defining normalized linear inequalities, respectively.

These five descriptions make explicit to varying degrees the geometric information carried by polytope $P$ and the combinatorial information of its facial structure. The vertex description and the facet description each carry only rudimentary geometric information about $P$. They therefore are called purely geometric descriptions. The other three descriptions are called combinatorial since they also carry more or less complete combinatorial information about the face structure of $P$.

The following problem is known as irredundancy problem: Given a set $S$ of $n$ points in $\mathbb{R}^{d}$, compute the vertex description of $P=\operatorname{conv} S$. Let $\lambda(n, d)$ be the time to solve a linear programming problem in $d$ variables with $n$ constraints. $O(n)$ for fixed $d$. This problem seeks to compute all points in $S$ that are irredundant, in the sense that they cannot be represented as a convex combination of the remaining point in $S$. Testing whether a point $s \in S$ is irredundant amounts to solving a linear programming problem in $d$ variables with $n-1$ constraints. The straightforward method of successively testing points for irredundancy results in an algorithm with running time $O(n \lambda(n-1, d))$, which for fixed dimension $d$ is $O\left(n^{2}\right)$. Clarkson [6] has ingeniously improved this method so that every linear program involves only at most $m$ constraints, where $m$ is the number of vertices of $P$, i.e. the output size. The resulting running time is $O(n \lambda(d, m)$ ), which for fixed $d$ is $O(n m)$. We note that for the case $d=3$ there are even algorithms with running time $O(n \log m)$ (see [10]), which can be shown to be asymptotically worst-case optimal [25].

### 5.6 D-Connected Sets in Digital Plane

In Chapter 2 we introduced $T$ - and $S$-descriptor points of an 8 -connected digital set $S_{\triangle}$. Whenever $S_{\triangle}$ is digitally convex and 8 -connected it has only $T$-descriptor points. It is possible to give a characterization of 8 -connected sets without $S$ descriptor points. In order to do this we need further definition.

Definition 5.5 A digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is said to be $d$-connected (directional connected) if it is 8-connected and if all intersections of $S_{\triangle}$ with horizontal, vertical or diagonal grid lines are 8-connected.

Generally, for slopes $s_{1}, \cdots, s_{k}$ of real straight lines which are given arbitrarily, it is possible to define $d$-connected sets in digital plane. In this case the definition must be modificated in the manner such that the intersection of each real line with
the slope $s_{i}, i=1, \cdots, k$ and a digital set does not possess any holes which are grid points lying on this line.

Remark 5.2 In Definition 5.5 the assumption of 8 -connectedness is not unnecessary as Figure 5.3 shows.


Figure 5.3: The set $S_{\triangle}$ (points $\left.\bullet\right)$ is digitally convex and not 8-connected. The intersections of $S_{\triangle}$ with horizontal, vertical and diagonal grid lines is 8-connected, however, $S_{\triangle}$ is not d-connected.

Obviously, every 8 -connected digitally convex set is $d$-connected, but not vice versa. The Figure 5.4 shows a conterexample. Moreover, 8 -connected digitally convex sets are $d$-connected for an arbitrary set of slopes.


Figure 5.4: Example of a d-connected set which is not digitally convex. The point $\odot$ belongs to the convex hull of the set.

The theoretical characterisation of $d$-connected sets is given in the following lemma (proof see [15]).

Lemma 5.5 Let $S_{\triangle}$ be an 8-connected digital set. The set $S_{\triangle}$ is d-connected if and only if all its descriptor points are T-descriptor points.

In the case a digital set is 8 -connected the algorithms for boundary orientation and detection Scherl's descriptors from previous sections can be used for testing $d$ connectedness. In general the $d$-connected sets can be easily recognized by means of special linear transformations.

|  | $m-2$ | $m-1$ | $m$ | $m+1$ | $m+2$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $n+2$ |  |  | $T\left(N_{3}(P)\right)$ |  |  |
| $n+1$ |  | $T\left(N_{4}(P)\right)$ |  | $T\left(N_{2}(P)\right)$ |  |
| $n$ | $T\left(N_{5}(P)\right)$ |  | $T(P)$ |  | $T\left(N_{1}(P)\right)$ |
| $n-1$ |  | $T\left(N_{6}(P)\right)$ |  | $T\left(N_{0}(P)\right)$ |  |
| $n-2$ |  |  | $T\left(N_{7}(P)\right)$ |  |  |

Table 5.1: The table shows changing 8-topology of $\mathbb{Z}^{2}$ using the linear transformation $T=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$.

Assume for a transformation $T=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ holds $a=b=d=1$ and $c=-1$, thus, we have

$$
T=\left(\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right)
$$

$T$ is rotation by $45^{\circ}$ with $-\frac{1}{4} \cdot T^{4}=i d$ (id denotes the identity matrix) and $\operatorname{det} T=2$. The transformation $T$ with

$$
P \in \mathbb{Z}^{2} \mapsto T(P) \in \mathbb{Z}^{2}
$$

changes 8-topology of $\mathbb{Z}^{2}$ in the way described in Table 5.1. $T$ is no unimodular matrix, however, it is a bijective mapping from $\mathbb{R}^{2}$ into $\mathbb{R}^{2}$ and its inverse matrix is

$$
T^{-1}=\frac{1}{2} \cdot\left(\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right)
$$

Hence, the structures of $T$ and $T^{-1}$ will not make any difficulties by numerical implementation.

Proposition 5.7 The set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is a horizontal or vertical grid line if and only if the set $T\left(S_{\triangle}\right)$, where $T=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$, is a diagonal grid line.

## Proof Trivial.

Let $S_{\triangle}=\left(x_{S_{\triangle}}, y_{S_{\Delta}}\right)$ be an arbitrary digital set from $\mathbb{Z}^{2}$ having $n$ elements. We will check connectedness of the intersections of horizontal grid lines and $S_{\triangle}$. All
horizontal grid lines whose intersections with $S_{\triangle}$ are non empty are given by following equations:

$$
\begin{aligned}
y & =\min \left(y_{S_{\Delta}}\right) \\
y & =\min \left(y_{S_{\Delta}}\right)+1, \\
& \vdots \\
y & =\max \left(y_{S_{\triangle}}\right) .
\end{aligned}
$$

We sort elements of $S_{\triangle}$ with $y$-coordinates mentioned above. Hence, we receive $I=\max \left(y_{S_{\Delta}}\right)-\min \left(y_{S_{\Delta}}\right)+1$ sets which we mark by $Y_{i}, i=1, \cdots, I$. All elements from the sets $Y_{i}$ have the same $y$-coordinates. We denote them by $y_{i}$ and consider the set $x_{Y_{i}}$ which is the vector corresponding to the $x$-coordinate of $Y_{i}$. If all elements

$$
\left(\begin{array}{c}
\min \left(x_{Y_{i}}\right), y_{i} \\
\min \left(x_{Y_{i}}\right)+1, y_{i} \\
\vdots \\
\max \left(x_{Y_{i}}\right), y_{i}
\end{array}\right)
$$

belong to $Y_{i}$ then the intersection of the horizontal line $y=y_{i}$ with $S_{\triangle}$ is obviously 8 -connected. In this manner we can check connectedness of the intersections of horizontal grid lines and the set $S_{\triangle}$.

The presented considerations are given in Algorithm Intersec. For realization of this algorithm we do not need the condition of 8-connectedness for the given set $S_{\triangle}$. The complexity of the algortithm is $O(n m)$, where $n$ is the number of elements of $S_{\triangle}$ and $m=\max \left(y_{S_{\Delta}}\right)-\min \left(y_{S_{\Delta}}\right)+1$.

All vertical grid lines such that their intersections with $S_{\triangle}$ are non empty are given by equations

$$
\begin{aligned}
x & =\min \left(x_{S_{\Delta}}\right), \\
x & =\min \left(x_{S_{\Delta}}\right)+1, \\
& \vdots \\
x & =\max \left(x_{S_{\Delta}}\right) .
\end{aligned}
$$

We receive $J=\max \left(x_{S_{\Delta}}\right)-\min \left(x_{S_{\Delta}}\right)+1$ sets which we mark by $X_{j}, j=1, \cdots, J$. All elements from $X_{j}$ have the same $x$-coordinate, they will be denoted as $x_{j}$. If all elements

$$
\left(\begin{array}{c}
x_{j}, \min \left(y_{X_{j}}\right) \\
x_{j}, \min \left(y_{X_{j}}\right)+1 \\
\vdots \\
x_{j}, \max \left(y_{X_{j}}\right)
\end{array}\right)
$$

belong to $X_{j}$ then the intersection of the vertical line $x=x_{j}$ with $S_{\triangle}$ is an 8connected set. In this manner we can find intersections of vertical grid lines with $S_{\triangle}$ and decide if they are connected.

```
Algorithm VIII Intersec Connectedness of the intersections of horizontal grid lines with \(S_{\triangle} \subseteq \mathbb{Z}^{2}\).
```

```
\(/ * S_{\triangle}=\left(x_{S_{\Delta}}, y_{S_{\Delta}}\right) \in \mathbb{Z}^{n \times 2} * /\)
\(I \leftarrow \max \left(y_{S_{\Delta}}\right)-\min \left(y_{S_{\Delta}}\right)+1 ; / *\) number of horizontal grid lines such that
their intersections with \(S_{\triangle}\) is non empty */
for \(i \leftarrow 1\) to \(I\) do
    \(Y_{i} \leftarrow \emptyset\); /* sets of points belonging to intersections of \(S_{\triangle}\) with horizontal
    grid lines */
end
\(y \leftarrow \emptyset\); / * vector with \(y\)-coordinates of \(Y_{1}, \cdots, Y_{I} * /\)
for \(i \leftarrow 1\) to \(I\) do
    \(y \leftarrow y \cup\left(\min \left(y_{S_{\Delta}}\right)+i-1\right) ;\)
end
for \(i \leftarrow 1\) to \(n\) do
        for \(j \leftarrow 1\) to \(I\) do
        if \(y_{S_{\Delta}}(i)=y(j)\) then
            \(Y_{j} \leftarrow Y_{j} \cup\left(x_{S_{\Delta}}(i), y_{S_{\Delta}}(i)\right) ;\)
        end
    end
end
/* size \(\left(Y_{i}\right)\) is number of elements of \(Y_{i} * /\)
for \(i \leftarrow 1\) to \(I\) do
    if size \(\left(Y_{i}\right)=\max \left(x_{Y_{i}}\right)-\min \left(x_{Y_{i}}\right)+1\) then
        \(/ * Y_{i}\) is 8 -connected \(* /\)
    end
end
```

The intersections of $S_{\triangle}$ with diagonal grid lines are more complicated to find. For this purpose we use the linear transformation $T=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$.

Proposition 5.8 The set $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is a diagonal grid line if and only if the set $T\left(S_{\triangle}\right)$, where $T=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$, is a subset $L$ of a horizontal grid line $\bar{y}=$ const $\in \mathbb{Z}$ with

$$
L=\left\{(x, y) \in \mathbb{Z}^{2} \mid y=\bar{y}, x=2 k\right\} \text { or } L=\left\{(x, y) \in \mathbb{Z}^{2} \mid y=\bar{y}, x=2 k+1\right\},
$$

or a subset $L$ of a vertical grid line $\bar{x}=$ const $\in \mathbb{Z}$ with

$$
L=\left\{(x, y) \in \mathbb{Z}^{2} \mid x=\bar{x}, y=2 k\right\} \text { or } L=\left\{(x, y) \in \mathbb{Z}^{2} \mid x=\bar{x}, y=2 k+1\right\} .
$$

## Proof See Table 5.1.

It follows that with respect to the topology induced by $T$ we are able to check connectedness of intersections of mentioned subsets of horizontal and vertical grid lines and the set $T\left(S_{\triangle}\right)$.

Thus, the non empty intersections of $T\left(S_{\triangle}\right)$ with subsets of horizontal digital lines described in Proposition 5.8 can be determined from equations

$$
\begin{aligned}
y & =\min \left(T\left(y_{S_{\Delta}}\right)\right), \\
y & =\min \left(T\left(y_{S_{\Delta}}\right)\right)+2, \\
y & =\min \left(T\left(y_{S_{\Delta}}\right)\right)+4, \\
& \vdots \\
y & =\max \left(T\left(y_{S_{\Delta}}\right)\right) .
\end{aligned}
$$

Here, we receive $I=\frac{\max \left(T\left(y_{S_{\Delta}}\right)\right)-\min \left(T\left(y_{S_{\Delta}}\right)\right)}{2}+1$ sets which will be denoted $Y_{i}, i=$ $1, \cdots, I$. All elements from $Y_{i}$ have the same $y$-coordinates $y_{i}$. If all elements

$$
\left(\begin{array}{c}
\min \left(x_{Y_{i}}\right), y_{i} \\
\min \left(x_{Y_{i}}\right)+2, y_{i} \\
\min \left(x_{Y_{i}}\right)+4, y_{i} \\
\vdots \\
\max \left(x_{Y_{i}}\right), y_{i}
\end{array}\right)
$$

belong to $Y_{i}$ then the intersection is 8 -connected.
Analogously, the intersections of $T\left(S_{\triangle}\right)$ with subsets of vertical digital lines from Proposition 5.8 can be determined from

$$
\begin{aligned}
x & =\min \left(T\left(x_{S_{\Delta}}\right)\right), \\
x & =\min \left(T\left(x_{S_{\Delta}}\right)\right)+2, \\
x & =\min \left(T\left(x_{S_{\Delta}}\right)\right)+4, \\
& \vdots \\
x & =\max \left(T\left(x_{S_{\Delta}}\right)\right) .
\end{aligned}
$$

We have $J=\frac{\max \left(T\left(x_{S_{\triangle}}\right)\right)-\min \left(T\left(x_{S_{\triangle}}\right)\right)}{2}+1$ sets which we mark $X_{j}, j=1, \cdots, J$. All elements from $X_{j}$ have the same $x$-coordinates $x_{j}$. If all the elements

$$
\left(\begin{array}{c}
x_{j}, \min \left(y_{X_{j}}\right) \\
x_{j}, \min \left(y_{X_{j}}\right)+2 \\
x_{j}, \min \left(y_{X_{j}}\right)+4 \\
\vdots \\
x_{j}, \max \left(y_{X_{j}}\right)
\end{array}\right)
$$

belong to $X_{j}$ then the intersection is 8 -connected. Whenever the described property is valid for all sets $Y_{1}, \cdots, Y_{I}$ and $X_{1}, \cdots, X_{J}$ then it follows that the non empty intersections of $S_{\triangle}$ with diagonal grid lines are 8-connected.

### 5.7 D-Convex Hull in 2D

Let $S \subseteq \mathbb{R}^{2}$ be a convex polygonal set. The convex set $S$ is called $d$-convex (directional convex) if the edges of $S$ are segments of real straight lines

$$
L(\alpha)(x, y): \quad \rho=-x \sin \frac{\alpha \pi}{4}+y \cos \frac{\alpha \pi}{4},
$$

with 4 main directions $\alpha=0,1,2,3$ (considering of the directions $4,5,6,7$ is not required since segments with directions $0,1,2,3$ are parallel to such with $4,5,6,7$, respectively). We denote $L(\alpha)(x, y)$ briefly $L(\alpha)$.

Analogously to $d$-connected sets, it is possible to define $d_{\beta}$-convex sets whenever real straight lines $L(\beta): \rho=-x \sin \frac{\beta \pi}{4}+y \cos \frac{\beta \pi}{4}$ with arbitrary directions $\beta$ are given. Clearly, the intersection of $d$-convex sets is $d$-convex. The $d$-convex hull of $S$ (will be designated by $\operatorname{dconv}(S)$ ) is the smallest $d$-convex set containing $S$, i.e.

$$
\operatorname{dconv}(S)=\bigcap\left\{\bar{S} \subseteq \mathbb{R}^{2} \mid S \subseteq \bar{S}, \bar{S} \text { is } d \text {-convex }\right\} .
$$

It holds $S \subseteq \operatorname{conv}(S) \subseteq d \operatorname{conv}(S)$.
Remark 5.3 In general, affine transformations do not preserve d-convexity. However, for transformations $T$ which are rotations of a set $S$ by $\frac{k \pi}{4}, k=0, \cdots, 7$ obviously nolds: $S$ is $d$-convex if and only if $T(S)$ is $d$-convex.

Let us consider directions $\beta$ such that $\beta \subseteq \alpha=0,1,2,3$. If a set is $d_{\beta}$-convex then it is $d$-convex, too. It follows

$$
d \operatorname{conv}(S) \subseteq \operatorname{dconv}_{\beta}(S)
$$

Proposition 5.9 Given $S \subseteq \mathbb{R}^{2}$. Let $\beta=0,2$ and $\gamma=1,3$ be directions of real straight lines $L(\beta): \rho=-x \sin \frac{\beta \pi}{4}+y \cos \frac{\beta \pi}{4}$ and $L(\gamma): \rho=-x \sin \frac{\gamma \pi}{4}+y \cos \frac{\gamma \pi}{4}$, respectively. Then

$$
\operatorname{dconv}(S)=\operatorname{dconv}_{\beta}(S) \cap \operatorname{dconv} v_{\gamma}(S)
$$

Proof It holds $\beta \subseteq \alpha$ and $\gamma \subseteq \alpha$, thus,

$$
d \operatorname{conv}(S) \subseteq \operatorname{dconv}_{\beta}(S) \quad \text { and } \quad d \operatorname{conv}(S) \subseteq \operatorname{dconv}_{\gamma}(S)
$$

Hence, we have

$$
\operatorname{dconv}(S) \subseteq \operatorname{dconv}_{\beta}(S) \cap \operatorname{dconv} v_{\gamma}(S)
$$

Since $\beta \cup \gamma=\alpha$ the statement of the proposition is true.
We are able to introduce the concept of $d$-convexity on digital plane $\mathbb{Z}^{2}$.

Definition 5.6 Given an arbitrary digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$. $S_{\triangle}$ is said to be d-convex whenever

$$
S_{\triangle}=\operatorname{dconv}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}
$$

Proposition 5.10 Let $S_{\triangle}$ be d-convex. Then $S_{\triangle}$ is 8-connected.
Proof The proposition follows from the fact that $d \operatorname{conv}\left(S_{\triangle}\right)$ is 8 -connected.
The $d$-convex sets are digitally convex and $d$-connected, the reverse statement is not true. In Figure 5.5 the $d$-convex hull of the digital set from Figure 5.4 which is $d$-connected but not $d$-convex is shown.


Figure 5.5: d-convex hull of the digital set from Figure 5.4. Points marked by o belong to the d-convex hull but not to the set.

The following lemma shows a simple theoretical characterisation of $d$-convex digital sets using Scherl's descriptors.

Lemma 5.6 Let $S_{\triangle} \in \mathbb{Z}^{2}$ be an 8-connected digital set. The set $S_{\triangle}$ is d-convex if and only if each boundary point of $S_{\triangle}$ is a $T$-descriptor point.

Proof 1. It is clear that each boundary point of a $d$-convex digital set is a descriptor point. Since $S_{\triangle}$ is $d$-convex it follows that $S_{\triangle}$ is $d$-connected. By Lemma 5.5 the set $S_{\triangle}$ has only $T$-descriptor points.
2. Assume $S_{\triangle} \subseteq \mathbb{Z}^{2}$ is an 8-connected digital set and each boundary point of $S_{\triangle}$ is a $T$-descriptor point. It follows that $S_{\triangle}$ is 8 -simply connected. The polygonal set which describes the boundary is local convex at each point, hence, it is convex [46]. Then $S_{\triangle}$ is digitally convex. Furthermore, all boundary points are lying on $L(\alpha)$ with 4 main directions $\alpha=0,1,2,3$. Hence, $S_{\triangle}$ is $d$-convex.

In the case an 8 -connected digital set is given testing $d$-convexity is possible using Lemma 5.6. Otherwise, Proposition 5.9 can be easily applied on an arbitrary digital set for construction its $d$-convex hull.

We are interested in computation the oriented boundary of the $d$-convex hull. Given an arbitrary digital set $S_{\triangle}=\left(x_{S_{\triangle}}, y_{S_{\Delta}}\right) \subseteq \mathbb{Z}^{2}$. Let $\beta=0,2$ and $\gamma=1,3$ be directions of $L(\beta)$ and $L(\gamma)$, respectively. Construction the boundary of the set $\operatorname{dconv}_{\beta}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ depends on boundary points which belong to extrema of linear functionals $L(\beta)$ corresponding to directions $\beta=0$ and $\beta=2$. We have four possibilities: points belong to the minima and maxima of linear functionals corresponding to $\beta=0$ and minima and maxima corresponding to $\beta=2$. Hence, the oriented boundary of the set $\operatorname{dconv}_{\beta}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ is given by following elements:

$$
\begin{gathered}
\left(\min \left(x_{S_{\Delta}}\right), \min \left(y_{S_{\Delta}}\right)\right) \\
\left(\min \left(x_{S_{\Delta}}\right)+1, \min \left(y_{S_{\Delta}}\right)\right) \\
\vdots \\
\left(\max \left(x_{S_{\Delta}}\right), \min \left(y_{S_{\Delta}}\right)\right) \\
\left(\max \left(x_{S_{\Delta}}\right), \min \left(y_{S_{\Delta}}\right)+1\right) \\
\vdots \\
\left(\max \left(x_{S_{S}}\right), \max \left(y_{S_{\Delta}}\right)\right) \\
\left(\max \left(x_{S_{\Delta}}\right)-1, \max \left(y_{S_{\Delta}}\right)\right) \\
\vdots \\
\left(\min \left(x_{S_{\Delta}}\right), \max \left(y_{S_{\Delta}}\right)\right) \\
\left(\min \left(x_{S_{\Delta}}\right), \max \left(y_{S_{\Delta}}\right)-1\right) \\
\vdots \\
\left(\min \left(x_{S_{\Delta}}\right), \min \left(y_{S_{\Delta}}\right)+1\right)
\end{gathered}
$$

The oriented boundary of the set $d \operatorname{con} v_{\gamma}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ for directions $\gamma=1$ and $\gamma=3$ can be determined using the linear transformation $T=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$. In the manner described above, we construct the oriented boundary of the set $d \operatorname{conv}_{\beta}\left(T\left(S_{\triangle}\right)\right) \cap \mathbb{Z}^{2}$. Altogether we have 8 ordered digital subsets which are 8 -connected segments of grid lines. Obviously, the set $\operatorname{dconv}_{\beta}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ and $T^{-1}\left(\operatorname{dconv}_{\beta}\left(T\left(S_{\triangle}\right)\right) \cap \mathbb{Z}^{2}\right)$ have common points which are vertices of the $d$-convex hull of $S_{\triangle}$. Furthermore, the oriented boundary of the set $d \operatorname{conv} v_{\gamma}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ can be determined if we use the fact that diagonal grid lines can possess only 4 steps $(x, y) \mapsto(x+1, y+1)$, $(x, y) \mapsto(x+1, y-1),(x, y) \mapsto(x-1, y+1)$ and $(x, y) \mapsto(x-1, y-1)$. Thus, from segments of horizontal and vertical grid lines of $d \operatorname{conv}_{\beta}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ and all possible diagonal grid lines of $d$ conv $v_{\gamma}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ the boundary of the $d$-convex hull of $S_{\triangle}$ can be easily computed. A part of the described procedure is implemented in Algorithm DHull2D.

We discuss the complexity of Algorithm DHull2D. Clearly, the complexity of for-loops are $O(m)$, where $m=\max (\lambda)-\min (\lambda)+1$ and $\lambda=x_{S_{\Delta}}$ for the first and last loops, $\lambda=y_{T\left(S_{\Delta)}\right)}$ for the second, and $\lambda=y_{S_{\Delta}}$ for the third loop. The determination of the common point of two sets cosumes the time $O\left(m^{2}\right)$, where $m$ is maxi-

## Algorithm IX DHull2D Part of the oriented boundary of the set

 $d \operatorname{conv}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}$ for an arbitrary digital set $S_{\triangle} \subseteq \mathbb{Z}^{2}$.```
\(/ * S_{\triangle}=\left(x_{S_{\Delta}}, y_{S_{\Delta}}\right) * /\)
\(\operatorname{MIN}\{0\} \leftarrow \emptyset ; / *\) ordered elements of \(\operatorname{dconv}_{\beta}\left(S_{\triangle}\right) \cap \mathbb{Z}^{2}\) belonging to the
minima of \(L(\beta)(x, y)\) for \(\beta=0 * /\)
\(\operatorname{MIN}\{1\} \leftarrow \emptyset ; / *\) ordered elements of \(T^{-1}\left(\operatorname{dconv}_{\beta}\left(T\left(S_{\triangle}\right)\right) \cap \mathbb{Z}^{2}\right)\) on the
(continuous) boundary of dconv \(\left(S_{\triangle}\right)\) belonging to the minima of \(L(\gamma)(x, y)\)
for \(\gamma=1 * /\)
\(\operatorname{MIN}\{2\} \leftarrow \emptyset ; / *\) the same as \(\operatorname{MIN}\{0\}\) for \(\beta=2 * /\)
PART \(\leftarrow \emptyset ; / *\) part of the oriented boundary \(* /\)
for \(i \leftarrow \min \left(x_{S_{\Delta}}\right)\) to \(\max \left(x_{S_{\Delta}}\right)\) do
    \(\operatorname{MIN}\{0\} \leftarrow \operatorname{MIN}\{0\} \cup\left(i, \min \left(y_{S_{\Delta}}\right)\right) ;\)
end
for \(i \leftarrow \min \left(y_{T\left(S_{\Delta}\right)}\right)\) to \(\max \left(y_{T\left(S_{\Delta}\right)}\right)\) do
    \(\operatorname{MIN}\{1\} \leftarrow \operatorname{MIN}\{1\} \cup T^{-1}\left(\max \left(x_{T\left(S_{\triangle}\right)}\right), i\right) ;\)
end
for \(i \leftarrow \min \left(y_{S_{\Delta}}\right)\) to \(\max \left(y_{S_{\Delta}}\right)\) do
    \(\operatorname{MIN}\{2\} \leftarrow \operatorname{MIN}\{2\} \cup\left(\max \left(x_{S_{\Delta}}\right), i\right) ;\)
end
\(/ *\) find the common point \(I=\left(x_{I}, y_{I}\right)\) of \(\operatorname{MIN}\{0\}\) and \(\operatorname{MIN}\{1\} * / ;\)
\(/ *\) find the common point \(J=\left(x_{J}, y_{J}\right)\) of \(\operatorname{MIN}\{1\}\) and \(\operatorname{MIN}\{2\} * /\);
for \(i \leftarrow 0\) to \(x_{J}-x_{I}\) do
    \(P A R T \leftarrow P A R T \cup\left(x_{I}+i, y_{I}+i\right) ;\)
end
```

mum of both numbers of elements of sets, i.e. we have $m=\max _{\lambda \in\left\{x_{S_{\triangle}}, y_{T\left(S_{\Delta}\right)}\right\}}(\max (\lambda)-$ $\min (\lambda)+1)$ and $m=\max _{\lambda \in\left\{y_{S_{\triangle}}, y_{T\left(S_{\Delta}\right)}\right\}}(\max (\lambda)-\min (\lambda)+1)$.

### 5.8 D-Convex Hull in 3D

We extend the considerations from Section 5.7 to 3D case. Given following planes:

$$
\begin{array}{ll}
P_{1}(\alpha)(x, y, z): & \rho=-x \sin \frac{\alpha \pi}{4}+y \cos \frac{\alpha \pi}{4} \\
P_{2}(\alpha)(x, y, z): & \rho=-y \sin \frac{\alpha \pi}{4}+z \cos \frac{\alpha \pi}{4} \\
P_{3}(\alpha)(x, y, z): & \rho=-z \sin \frac{\alpha \pi}{4}+x \cos \frac{\alpha \pi}{4}
\end{array}
$$

with 4 main directions $\alpha=0,1,2,3$. We denote $P_{i}(\alpha)(x, y, z), i=1,2,3$ briefly $P_{i}(\alpha)$.

Proposition 5.11 For planes $P_{1}(\alpha), P_{2}(\alpha)$ and $P_{3}(\alpha), \alpha=0,1,2,3$ holds

$$
\begin{aligned}
& P_{1}(0)=-P_{2}(2), \\
& P_{2}(0)=-P_{3}(2), \\
& P_{3}(0)=-P_{1}(2) .
\end{aligned}
$$

## Proof Trivial.

Let $S \subseteq \mathbb{R}^{3}$ be a convex polytope. The convex set $S$ is called $d$-convex (directional convex) if the faces of $S$ can be represented by planes $P_{1}(\alpha), P_{2}(\alpha)$ and $P_{3}(\alpha)$, $\alpha=0,1,2,3$. According to Proposition 5.11 each $d$-convex set can possess maximal 18 faces. In general, it is possible to define $d_{\beta}$-convex sets in $\mathbb{R}^{3}$ whenever at least three non parallel planes $P(\beta)$ are given.

We consider planes $P_{1}(0), P_{2}(0), P_{3}(0)$ (or $\left.P_{1}(2), P_{2}(2), P_{3}(2)\right)$. The corresponding to these planes $d$-convex hull will be denoted as $d \operatorname{conv}_{\beta}$. Then, obviously, for every set $S \subseteq \mathbb{R}^{3}$ holds

$$
d \operatorname{conv}(S) \subseteq \operatorname{dconv}_{\beta}(S)
$$

For planes $P_{3}(0), P_{2}(1), P_{2}(3)$ the corresponding $d$-convex hull will be denoted as $d \operatorname{conv}_{\beta_{1}}$, for $P_{1}(0), P_{3}(1), P_{3}(3)$ as $d \operatorname{conv}_{\beta_{2}}$, and for $P_{2}(0), P_{1}(1), P_{1}(3)$ as $d \operatorname{conv}_{\beta_{3}}$. It follows

$$
\operatorname{dconv}(S) \subseteq \operatorname{dconv}_{\beta}(S) \cap \bigcap_{i=1,2,3} \operatorname{dconv}_{\beta_{i}}(S)
$$

Proposition 5.12 Let $S \subseteq \mathbb{R}^{3}$ be an arbitrary set. Then

$$
\operatorname{dconv}(S)=\operatorname{dconv}_{\beta}(S) \cap \bigcap_{i=1,2,3} \operatorname{dconv}_{\beta_{i}}(S)
$$

Proof The polytope $\operatorname{dconv}_{\beta}(S)$ has at most 6 different faces. Also the polytopes $\operatorname{dconv}_{\beta_{i}}(S)$ possess at most 6 different faces. Two faces of them are represented by $P_{i}(0), i=1,2,3$. Thus, we have altogether 18 faces which represent $d$-convex hull of the set $S$.

We note that Proposition 5.12 can be considered as an extention of Proposition 5.9 to 3D case.

Lemma 5.7 Assume $S \subseteq \mathbb{R}^{3}$ is $d$-convex. Then each plane section $S \cap H$, where $H$ is a hyperplane which is $P_{1}(\alpha), P_{2}(\alpha)$ or $P_{3}(\alpha), \alpha=0,1,2,3$, is $d$-convex.

Proof Assume $S$ is $d$-convex, hence, it is convex, too. From ordinary theory follows that each plane section $S \cap H=\operatorname{dconv}(S) \cap H$ is convex. Without loss of generality we may consider the plane $P_{1}(2): \rho=-x$, others follow in the same manner.

Let $H$ be the hyperplane $x=$ const. The polygonal convex set $S \cap H$ has maximal 8 edges which are segments of real lines $L(\alpha)$ with directions $\alpha=0,1,2,3$. We deduce that $S \cap H$ is $d$-convex.

We introduce $d$-convexity in digital space $\mathbb{Z}^{3}$.
Definition 5.7 Given a digital set $S_{\triangle} \subseteq \mathbb{Z}^{3}$. $S_{\triangle}$ is said to be $d$-convex whenever

$$
S_{\triangle}=\operatorname{dconv}\left(S_{\triangle}\right) \cap \mathbb{Z}^{3} .
$$

We are interested in computation $d$-convex hull of an arbitrary finite digital set $S_{\triangle}=\left(x_{S_{\Delta}}, y_{S_{\Delta}}, z_{S_{\Delta}}\right)$ in $\mathbb{Z}^{n \times 3}$.

Linear functionals $P_{i}(\alpha), i=1,2,3$ with directions $\alpha=0,1,2,3$ possess on the compact set $S_{\triangle}$ maximum and minimum. It follows planes $P_{i}(\alpha)$ with $\rho=$ $\max _{(x, y, z) \in S_{\triangle}} P_{i}(\alpha)(x, y, z)$ and $\rho=\min _{(x, y, z) \in S_{\triangle}} P_{i}(\alpha)(x, y, z)$ are supporting hyperplanes of $S_{\triangle}$. Hence, one has 18 supporting hyperplanes $H_{j}, j=1, \cdots, 18$. Let $\bar{H}_{j}$ be the halfspace bounded by $H_{j}$ such that $S \subseteq \bar{H}_{j}$ for all $j=1, T \cdots, 18$. Then holds

$$
\operatorname{dconv}\left(S_{\triangle}\right)=\bigcap_{j=1}^{18} \bar{H}_{j}
$$

Given linear transformations $T_{X}, T_{Y}, T_{Z}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ by non singular $3 \times 3$ matrices

$$
T_{X}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & a & b \\
0 & c & d
\end{array}\right), \quad T_{Y}=\left(\begin{array}{ccc}
a & 0 & b \\
0 & 1 & 0 \\
c & 0 & d
\end{array}\right), \quad T_{Z}=\left(\begin{array}{ccc}
a & b & 0 \\
c & d & 0 \\
0 & 0 & 1
\end{array}\right),
$$

such that for the set $S_{\triangle}$ we have

$$
T_{X}\left(S_{\triangle}\right)=S_{\triangle} T_{X}
$$

$T_{Y}\left(S_{\triangle}\right), T_{Z}\left(S_{\triangle}\right)$ are defined in the same manner.
If the elements $a, b, c$ and $d$ are integers then the transformations map $\mathbb{Z}^{3}$ into $\mathbb{Z}^{3} . T_{X}, T_{Y}$ and $T_{Z}$ do not change the $x$-, $y$ - and $z$-coordinate of $S_{\triangle}$, respectively, so they are rotations around the $X$-, $Y$ - and $Z$-axis.

If we choose $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)=T_{1}^{1}=T_{2}^{2}=T_{3}^{3}$ then $T_{X}, T_{Y}$ and $T_{Z}$ are rotations by $45^{\circ}$.

The set $d \operatorname{conv}_{\beta}\left(S_{\triangle}\right)$ from Proposition 5.12 is computed as shown below. The complexity of this procedure is $O\left(m^{3}\right)$, where
$m=\max _{\lambda \in\{x, y, z\}}\left(\max \left(\lambda_{S_{\Delta}}\right)-\min \left(\lambda_{S_{\Delta}}\right)+1\right)$.

```
\(\operatorname{donv}_{\beta} \leftarrow \emptyset ;\)
for \(i \leftarrow \min \left(x_{S_{\Delta}}\right)\) to \(\max \left(x_{S_{\Delta}}\right)\) do
    for \(j \leftarrow \min \left(y_{S_{\Delta}}\right)\) to \(\max \left(y_{S_{\Delta}}\right)\) do
        for \(k \leftarrow \min \left(z_{S_{\Delta}}\right)\) to \(\max \left(z s_{\Delta}\right)\) do
                \(\operatorname{dconv}_{\beta} \leftarrow \operatorname{deonv}_{\beta} \cup(i, j, k) ;\)
        end
    end
end
```

Further, the sets $\operatorname{dconv}_{\beta_{i}}$ from Proposition 5.12 have to be determined. Then the elements of $d$-convex hull of $S_{\triangle}$ can be sorted out from sets $d \operatorname{conv}_{\beta}\left(S_{\Delta}\right)$ and $d \operatorname{conv} \hat{\beta}_{i}\left(S_{\triangle}\right)$. For numerical implementation we recommend to calculate an extended set $\overline{d \operatorname{conv}}_{\beta_{1}}\left(\overline{d \operatorname{conv}}_{\beta_{2}}\right.$ and $\overline{\operatorname{dconv}}_{\beta_{3}}$, analogously) as follows:

```
\(\overline{\operatorname{dconv}}_{\beta_{1}} \leftarrow \emptyset\);
\(T \leftarrow S_{\triangle} T_{X} ;\)
for \(i \leftarrow \min \left(x_{S_{\Delta}}\right)\) to \(\max \left(x_{S_{\Delta}}\right)\) do
    for \(j \leftarrow \min \left(y_{T}\right)\) to \(\max \left(y_{T}\right)\) do
        for \(k \leftarrow \min \left(z_{T}\right)\) to \(\max \left(z_{T}\right)\) do
                \(\overline{\operatorname{dconv}}_{\beta_{1}} \leftarrow \overline{d \operatorname{conv}}_{\beta_{1}} \cup T_{X}^{-1}(i, j, k) ;\)
        end
    end
end
```

It holds $d \operatorname{conv}_{\beta_{i}} \subseteq \overline{\operatorname{dconv}}_{\beta_{i}}$, however, elements which do not belong to $d \operatorname{conv} v_{\beta_{i}}$ will not be choosen by the command:

```
\(/ * \operatorname{dconv}_{\beta}(j)\) means \(j\)-th element of the set dconv\({ }_{\beta} * /\)
if \(\operatorname{dconv}_{\beta}(j) \in \bigcap_{i=1,2,3} \overline{\operatorname{donv}}_{\beta_{i}}\) then
```

This command is the most time consuming one with the complexity $O\left(m^{4}\right)$ and $m=\max _{\lambda \in\left\{\beta, \beta_{i=1,2,3}\right\}}$ size $\left(d \operatorname{conv} v_{\lambda}\right)$, where size means the number of elements of a set.

The discussed procedures are demonstrated in Algorithm DHull3D (see p. 86) which computes $d$-convex hull of an arbitrary set $S_{\triangle} \in \mathbb{Z}^{n \times 3}$. Probable the $d$-convex hull of a 26 -connected set (with $T_{8}$-connected plane sections) can be computed by an algorithm with a more-optimal time complexity.

Figures $5.6-5.9$ show an example of a 3D digital set and its $d$-convex hull given in different representations.


Figure 5.6: 26-connected digital set in voxel representation. The set is neither digitally nor d-convex.


Figure 5.7: $d$-convex hull of the set from Figure 5.6.


Figure 5.8: Digital set from Figure 5.6 represented as set of grid points.


Figure 5.9: $d$-convex hull of the point set from Figure 5.8 as polytope.

Algorithm X DHull3D D-convex hull of an arbitrary digital set $S_{\triangle} \subseteq \mathbb{Z}^{3}$.

```
\(/ * S_{\triangle}=\left(x_{S_{\triangle}}, y_{S_{\triangle}}, z_{S_{\Delta}}\right) \in \mathbb{Z}^{n \times 3} * /\)
\(d^{\operatorname{conv}}{ }_{\beta} \leftarrow \emptyset\);
for \(i \leftarrow 1\) to 3 do \(\overline{d \operatorname{conv}}_{\beta_{i}} \leftarrow \emptyset ;\)
\(T \leftarrow S_{\triangle} T_{X}\);
for \(i \leftarrow \min \left(x_{S_{\Delta}}\right)\) to \(\max \left(x_{S_{\Delta}}\right)\) do
    for \(j \leftarrow \min \left(y_{S_{\Delta}}\right)\) to \(\max \left(y_{S_{\Delta}}\right)\) do
        for \(k \leftarrow \min \left(z_{S_{\Delta}}\right)\) to \(\max \left(z_{S_{\Delta}}\right)\) do
                \(\operatorname{dconv}_{\beta} \leftarrow \operatorname{dconv}_{\beta} \cup(i, j, k) ;\)
        end
    end
    for \(j \leftarrow \min \left(y_{T}\right)\) to \(\max \left(y_{T}\right)\) do
        for \(k \leftarrow \min \left(z_{T}\right)\) to \(\max \left(z_{T}\right)\) do
            \(\overline{\operatorname{dconv}}_{\beta_{1}} \leftarrow \overline{\operatorname{conv}}_{\beta_{1}} \cup T_{X}^{-1}(i, j, k) ;\)
        end
    end
end
\(T \leftarrow S_{\triangle} T_{Y} ;\)
for \(i \leftarrow \min \left(x_{T}\right)\) to \(\max \left(x_{T}\right)\) do
    for \(j \leftarrow \min \left(y_{S_{\Delta}}\right)\) to \(\max \left(y_{S_{\Delta}}\right)\) do
        for \(k \leftarrow \min \left(z_{T}\right)\) to \(\max \left(z_{T}\right)\) do
                \(\overline{\operatorname{dconv}}_{\beta_{2}} \leftarrow \overline{\operatorname{conv}}_{\beta_{2}} \cup T_{Y}^{-1}(i, j, k) ;\)
        end
    end
end
\(T \leftarrow S_{\triangle} T_{Z} ;\)
for \(i \leftarrow \min \left(x_{T}\right)\) to \(\max \left(x_{T}\right)\) do
    for \(j \leftarrow \min \left(y_{T}\right)\) to \(\max \left(y_{T}\right)\) do
        for \(k \leftarrow \min \left(z S_{\Delta}\right)\) to \(\max \left(z s_{\Delta}\right)\) do
            \(\overline{\operatorname{donv}}_{\beta_{3}} \leftarrow \overline{d \operatorname{conv}}_{\beta_{3}} \cup T_{Z}^{-1}(i, j, k) ;\)
        end
    end
end
dconv \(\leftarrow \emptyset\);
\(/ *\) size \(\left(\right.\) dconv \(\left._{\beta}\right)\) is the number of elements of \(\operatorname{dconv}_{\beta} * /\)
for \(j \leftarrow 1\) to size \(\left(\operatorname{dconv}_{\beta}\right)\) do
        \(/ * \operatorname{dconv}_{\beta}(j)\) means \(j\)-th element of the set \(\operatorname{dconv}_{\beta} * /\)
        if \(\operatorname{dconv}_{\beta}(j) \in \bigcap_{i=1,2,3} \overline{\operatorname{dconv}}_{\beta_{i}}\) then
        \(d\) conv \(\leftarrow d \operatorname{conv} \cup \operatorname{dconv}_{\beta}(j) ;\)
    end
end
```


## Conclusions

Discrete lines, discrete line segments and digitally convex sets are basic constructs of digital geometry. The boundary of a digital set can be decomposed into convex and concave parts by the method proposed here which is exact. This technique is related to the characterization of discrete lines by Debled-Rennesson and Reveillès [8]. For the decomposition we can easily find a polygonal set which represents the shape of the set and has the same convexity properties. However, in spite of the precision the disadvantage of the presented decomposition is the fact that the corresponding polygonal representation can possess vertices whose coordinates are not integers.

An alternative possibility would be polygonal representations of digital sets which are no longer faithful. For the boundary of a digital set we can find a polygonal representation which is discrete and possesses "only few" uncorresponding parts. This representation can be performed in the time proportional to the length of the boundary.

In both metods the polygonal representation of a set can be used as a basis for further simplification of the representing polygonal set by discrete evolution [34].

The concept of Scherl's descriptors can be generalized whenever more linear funcionals with different directions are considered. In a more general setting we are able to define descriptors for boundary curves in $\mathbb{R}^{2}$ [16]. In the context of digital boundary one can make use of the property that there is only a finite number of possible tangent directions.

It is a well-known fact that sets inherit convexity from their lower dimensional plane sections. Plane sections of digital sets and sections of sets which are transformed using some affine mapping are investigated. We were able to show that the geometrical and topological structures of the sections can be described by lower dimensional theory. Furthermore, the concept of $d$-convexity is introduced and studied here. A way how to construct the $d$-convex hull of an arbitrary set from $\mathbb{Z}^{3}$ is shown.

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Name: Helene Dörksen
Titel: Shape Representations of Digital Sets based on Convexity Properties
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Abstract
The only sets which can be handled on computers are discrete or digital sets that means the sets containing a finite number of elements. The discrete nature of digital images makes it necessary to develop suitable systems and methods since a direct use of classical theories is not possible or not adaptable.

The dealing with geometrical properties of digital sets is important in many applications of image processing. The topic of digital geometry is to recognize and to describe these properties. Apart from the theoretical foundations, the efficient procedures and techniques play a key role in scientific computation.

In digital geometry it is not a simple task to testing convexity of a set ${ }^{1}$. In 1928 , Tietze ${ }^{2}$ proved that convexity of a set in $\mathbb{R}^{2}$ can be decided locally in a time which is proportional to the length of its boundary. Unfortunately, in digital plane $\mathbb{Z}^{2}$ convexity cannot be observed locally ${ }^{3}$. One deals with the problem to decide whether a part of the boundary of a digital set is convex or not by some method which is "as local as possible".

Discrete lines, discrete line segments and digitally convex sets are basic constructs of digital geometry. The boundary of a digital set on $\mathbb{Z}^{2}$ can be decomposed into convex and concave parts by the method proposed in this paper. This technique is related to the characterization of discrete lines by Debled-Rennesson and Reveillès ${ }^{4}$. For this decomposition we can easily find a (continuous) polygonal set which represents the shape of the digital set and has the same convexity properties, i.e. it is faithful. However, the disadvantage of the presented decomposition is the fact that the corresponding polygonal representation can possess vertices whose coordinates are not integers.

An alternative possibility would be polygonal representations of digital sets which are no longer faithful. For the boundary of a digital set we can find a polygonal representation which is discrete and possesses "only few" uncorresponding parts. The both representations, i.e. a faithful one and a representation with "only few" uncorresponding parts, can be performed in the time proportional to the length of the boundary.

It is a well-known fact that sets inherit convexity from their lower dimensional plane sections. Plane sections of digital sets and sections of sets which are transformed using some affine mapping are investigated. We were able to show that the geometrical and topological structures of the sections can be described by lower dimensional theory. Furthermore, the concept of $d$-convexity is introduced and studied here. A way how to construct the $d$-convex hull of an arbitrary set from $\mathbb{Z}^{3}$ is shown.

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## Zusammenfassung

Digitale Objekte sind als Mengen mit ganzzahligen Koordinaten interpretierbar. Die diskrete Natur digitaler Bilder macht es erforderlich, angepasste Begriffsysteme und Methoden zu entwickeln, denn eine direkte Verwendung klassischer Theorien ist nicht möglich. In zahlreichen Anwendungen ist es von großer Bedeutung, Bildinhalte unter Berücksichtigung der geometrischen Eigenschaften zu erkennen und zu beschreiben. Neben der theoretischen Analyse spielt die Entwicklung effizienter Algorithmen eine sehr wichtige Rolle.

Die Verifikation der Konvexität einer Menge ist im digitalen Kontext eine nichttriviale Aufgabe ${ }^{1}$. Im Jahre 1928 zeigte Tietze ${ }^{2}$, dass die Konvexität einer Menge aus $\mathbb{R}^{2}$ (sogar allgemein in $d$ Dimensionen) lokal entschieden werden kann mit einem Aufwand, der proportional ist zur Länge ihres Randes. In der digitalen Ebene $\mathbb{Z}^{2}$ (und auch im $\mathbb{Z}^{d}$ ) ist dagegen eine lokale Entscheidung nicht möglich ${ }^{3}$. Man befasst sich daher mit dem Problem, zu entscheiden, ob ein Teil des Randes einer digitalen Menge konvex oder konkav ist, indem man ein Verfahren benutzt, welches "so lokal wie möglich" ist.

Diskrete Geraden, Strecken und digitale konvexe Mengen sind Grunddefinitionen der digitalen Geometrie. In dieser Arbeit wurde eine Methode entwickelt für die Zerlegung des Randes einer Menge aus $\mathbb{Z}^{2}$ in konvexe und konkave Teile. Die Methode basiert auf Eigenschaften von digitalen Geraden nach DebledRennesson und Reveillès ${ }^{4}$. Zu dieser Zerlegung kann eine (kontinuierliche) polygonale Menge bestimmt werden, die die From der digitalen Menge repräsentiert, das heisst, dass ihr Rand gleiche Konvexitätseigenschaften besitzt, wie die digitale Ausgangsmenge. Man bezeichnet eine solche Darstellung als treu. Nachteilig ist hierbei die Tatsache, dass eine treue Darstellung Ecken mit nicht ganzzahligen Koordinaten haben kann.

Eine alternative Möglichkeit stellen polygonale Darstellungen dar, die "fast treu" sind und deren Ecken ganzzahlige Koordinaten haben. Sowohl treue als auch "fast treue" Darstellungen können mit einem Aufwand bestimmt werden, der proportinal zur Randlänge ist.

Es ist wohlbekannt, dass Mengen die Konvexität von ihren ebenen Schnitten erben. Ebene Schnitte digitale Mengen sowie Schnitte digitaler Mengen unter affinen Transformationen werden hier untersucht. Es wird gezeigt, wie deren geometrische und topologische Strukturen zu beschreiben sind. Ein Algorithmus zur Berechnung der $d$-konvexen Hülle einer Menge aus $\mathbb{Z}^{3}$ beschließt die Arbeit.

[^1]
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