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Borel chromatic numbers in models of set theory

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Contents

Chapter 1

Introduction

1.1 Preliminaries

Almost sixty years after the invention of forcing by Cohen, 1963 it has become common mathematical knowledge that the value of the continuum cannot be decided within the usual axioms of set theory, ZFC (i.e., it is *independent* from ZFC). The forcing technique is now taught in many set theory courses throughout the world, in several depth levels. This is a very general and flexible technique, and many mathematicians used it to establish independence proofs in a wide variety of mathematical fields, such as topology (e.g., Todorcevic, 1989), functional analysis (e.g., Dales and Woodin, 1987), homological algebra (e.g., Shelah, 1974) etc.

In this work, we shall study set-theoretic independence results for combinatorial statements about definable graphs on Polish spaces, and their consequences for the set theory of the real numbers. Open graphs on Polish spaces were already studied by Abraham, Rubin, and Shelah, 1985; and by Todorcevic, 1989. However, the systematic study of definable graphs started in Kechris, Solecki, and Todorcevic, 1999, as a descriptive set-theoretic approach to concepts and results from graph theory, and this field is nowadays called *descriptive graph combinatorics* (see the survey from Kechris and Marks, 2016). More precisely, we study a *cardinal characteristic of the continuum* related to definable graphs, called *Borel chromatic number* (this is the content of Section 1.2). It should be noted that some of the problems tackled here were independently considered and solved by Zapletal, 2019 as well (see Section 1.1 for details.)

A cardinal characteristic of the continuum is a cardinal number which measures the size of an object related to the continuum. Typically, cardinal characteristics of the continuum lie between $\aleph_1$ and $2^{\aleph_0}$ and can have different values in different models of set theory. Besides Borel chromatic numbers, there will be other cardinal characteristics relevant to this work and we introduce them next:

For $x, y \in \omega^\omega$, we say that y *eventually dominates* x, $x \leq^* y$, if $x(n) \leq y(n)$ for all but finitely many $n \in \omega$.

> *BOUNDING NUMBER.* A set $A \subseteq \omega^\omega$ is *unbounded* iff there exists no single element of ω^ω dominating all elements of A.
>
> The *bounding number*, $\mathfrak{b}$, is the least cardinality of an unbounded set.
>
> *DOMINATING NUMBER.* A set $A \subseteq \omega^\omega$ is *dominating* iff every element of ω^ω is dominated by some element of A.
>
> The *dominating number*, $\mathfrak{d}$, is the least cardinality of dominating set.

For $x, y \in [\omega]^\omega$, we say that x *splits* y iff the sets $y \cap x$ and $y \setminus x$ are both infinite.

$$\begin{tikzcd}
 & & 2^{\aleph_0} & \\
 & \mathfrak{d} \arrow[ur] & & \mathfrak{r} \arrow[ul] \\
\mathfrak{s} \arrow[ur] & & \mathfrak{b} \arrow[ul] \arrow[ur] & \\
 & \mathfrak{h} \arrow[ul] \arrow[ur] & & \\
 & \aleph_1 \arrow[u] & &
\end{tikzcd}$$

FIGURE 1.1: Van Douwen's diagram.

REAPING NUMBER. A family $\mathcal{F} \subseteq \omega^\omega$ is *reaping* iff no single element of $[\omega]^\omega$ splits all elements of $\mathcal{F}$.

The *reaping number*, $\mathfrak{r}$, is the least cardinality of a reaping family.

For $x, y \in [\omega]^\omega$, we say that x *is almost contained in* y, $x \subseteq^* y$, if $y \setminus x$ is finite.

DISTRIBUTIVITY NUMBER. A set $A \subseteq [\omega]^\omega$ is said *dense* iff for every $x \in [\omega]^\omega$ there exists $y \supseteq x$ in A; it is *open* if $y \in A$ and $y' \subseteq^* y$ implies $y' \in A$.

The *distributivity number*, $\mathfrak{h}$, is the least cardinality of a family $\mathcal{F}$ of open dense subsets of $[\omega]^\omega$ such that $\bigcap \mathcal{F} = \emptyset$.

Any two cardinals in this diagram are consistently different (see, for instance, Halbeisen, 2012 for the consistency proofs). A model of set theory witnessing $\mathfrak{c}_1 \neq \mathfrak{c}_2$ is said to *separate* the cardinals $\mathfrak{c}_1$ and $\mathfrak{c}_2$.

A standard way of generating cardinal characteristics of the continuum is by considering σ-ideals of subsets of Polish spaces, with Borel basis: let I be a σ-ideal on a Polish space X.

ADDITIVITY NUMBER. The *additivity number of* I, $\mathrm{add}(I)$, is the least cardinality of a family $\mathcal{F}$, of subsets of X, such that $\bigcup \mathcal{F} \notin I$.

UNIFORMITY NUMBER. The *uniformity number of* I, $\mathrm{non}(I)$, is the least cardinality of a subset $A \subseteq X$ such that $A \notin I$;

COVERING NUMBER. The *covering number number of* I, $\mathrm{cov}(I)$, is the least cardinality of a family $\mathcal{F}$, of elements of I, such that $\bigcup \mathcal{F} = X$.

COFINALITY NUMBER. The *cofinality number of* I, $\mathrm{cof}(I)$ is the least cardinality of a family $\mathcal{F}$ such that, for any $A \in I$, there exists $B \in \mathcal{F}$ such that $A \subseteq B$.

Note that if I contains all singletons, then $\mathrm{add}(I) \geq \aleph_1$; and if I has a Borel basis, then $\mathrm{cof}(I) \leq 2^{\aleph_0}$.

$$\begin{tikzcd}
& & \mathrm{non}(I) \arrow[dr] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(I) \arrow[ur] \arrow[dr] & & \mathrm{cof}(I) \arrow[r] & 2^{\aleph_0} \\
& & \mathrm{cov}(I) \arrow[ur] & &
\end{tikzcd}$$

FIGURE 1.2

For us, the real numbers will always be elements of the *Cantor space*, 2^ω, or the *Baire space*, ω^ω.

> *TOPOLOGY AND MEASURE.* The topology of 2^ω is generated by the clopen sets of the form
> $$[s] = \{x \in 2^\omega \mid s \subseteq x\},$$
> for $s \in 2^{<\omega}$.
> Say that $A \subseteq 2^\omega$ is *meager* iff $A \subseteq \bigcup_{n\in\omega} C_n$, where $(C_n)_{n\in\omega}$ is a sequence of closed sets of empty interior. Let $\mathcal{M}$ denote the σ-ideal of meager sets.
> The Lebesgue measure on 2^ω, μ, is the completion of the product measure satisfying
> $$\mu(\{0\}) = \mu(\{1\}) = 1/2.$$
> Let $\mathcal{N}$ denote the σ-ideal of measure zero sets.

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1.3: Cichon's diagram.

Again, any two cardinals of this diagram are consistently different: a significant portion of Bartoszynski and Judah, 1995 is devoted to the separation of pairs of cardinals in this diagram, respecting the provable inequalities.

1.2 Borel chromatic numbers

One of the oldest and most interesting problems of graph-colorings is the so-called *four-color problem*:

> *Is it possible to color any map in a plane using at most four colors in such a way that regions sharing a common boundary (other than single points) do not share the same color?*[a]
>
> [a]It is accepted that this conjecture was first proposed in 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed (see Wilson, 2013).

This was solved by Appel and Haken, 1976 using a computer-assisted proof which showed that, indeed, only four colors are needed.

Formulating the result from Appel and Haken in a graph-fashion: every planar graph is 4-*colorable*.

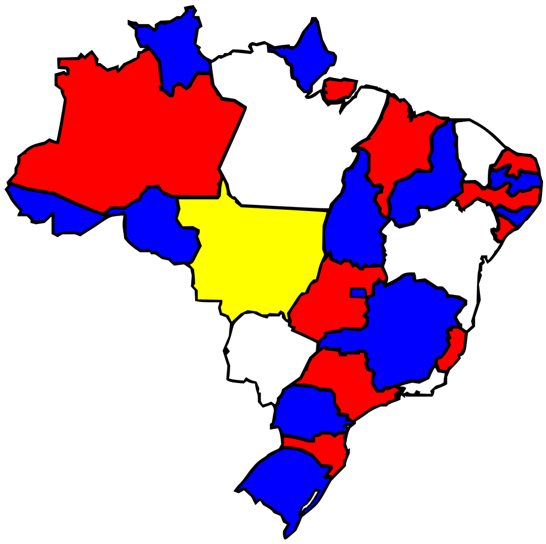

FIGURE 1.4: Map of Brazil colored with four colors

Let G be a graph on a non-empty set X and $\alpha \geq 1$ be an ordinal. We say that a function $c : X \to \alpha$ is an *α-coloring of* G iff $c(x) \neq c(y)$, for all $(x, y) \in G$. The *chromatic number of* G, denoted by $\chi(G)$, is the least $|\alpha|$ for which there exists an α-coloring of G.

Now if X is endowed with a Polish topology, we may study graphs that are *definable*: we say that G is *analytic* (*Borel*, *closed* etc) iff it is analytic (respectively, Borel, closed etc) as a subset of $X^2 \setminus \mathrm{Id}_X$, where Id_X is the *identity function on* X. Furthermore, c is called a *Borel coloring* if, additionally, $c^{-1}(\{\beta\})$ is a Borel set, for every $\beta < \alpha$ (i.e., every *maximally monochromatic* set is Borel). The Borel chromatic number of G, denoted by $\chi_B(G)$, is the least $|\alpha|$ for which there exists a Borel α-coloring of G.

Note that pre-images of colors have the property $c^{-1}(\{\beta\})^2 \cap G = \emptyset$, for every $\beta < \alpha$. The sets $A \subseteq X$ satisfying this property (i.e., $A^2 \cap G = \emptyset$) are said to be *G-independent*, and using this we shall redefine Borel chromatic numbers in a more convenient way:

Let G be an analytic graph on a Polish space X.

> *BOREL CHROMATIC NUMBERS.* The *Borel chromatic number* of G, $\chi_B(G)$, is the least cardinality of a family $\mathcal{F}$, consisting of Borel G-independent sets, such that $\bigcup \mathcal{F} = X$.

Fact 1.2.1 (Miller, 2009, Proposition 2.). *Let G be an analytic graph on Polish space X and A be an analytic G-independent subset of X. Then there exists a Borel G-independent set $B \supseteq A$.*

From this, it follows that for such graphs, the σ-ideal $I(G)$ of all analytic sets $A \subseteq X$ such that $\chi_B(G \cap A^2) \leq \aleph_0$, is *Borel generated* — i.e., for every $A \in I(G)$, there exists a Borel set $B \in I(G)$ such that $B \supseteq A$.

Since we are working only with graphs on Polish spaces, our Borel chromatic numbers are bounded by $2^{\aleph_0}$. We will later see that, when uncountable, Borel chromatic numbers may assume different values in different models of set theory.

One of the most striking features of uncountable Borel chromatic numbers of analytic graphs on Polish spaces is that there exists a smallest one. To state this precisely, we need to define the graph G_0:

THE GRAPH G_0. Fix a family $D \subseteq 2^{<\omega}$ such that

(1) for every $n \in \omega$ there exists a unique $s \in D$ such that $|s| = n$; and

(2) for all $t \in 2^{<\omega}$, there exists $s \in D$ such that $t \subseteq s$.

The graph G_0 is defined on 2^ω by

$$G_0 = \{(s^\frown i^\frown x, s^\frown(1-i)^\frown x) \mid \text{for } s \in D, i < 2 \text{ and } x \in 2^\omega\}.$$

That its Borel chromatic number is uncountable follows from the fact that every Baire measurable G_0-independent set is meager:

Fact 1.2.2 (Kechris, Solecki, and Todorcevic, 1999, Proposition 6.2.)**.** *Every Baire measurable G_0-independent set is meager. In particular, $\chi_B(G_0) \geq \operatorname{cov}(\mathcal{M})$.*

The following is known as the *G_0-dichotomy*:

Fact 1.2.3 (Kechris, Solecki, and Todorcevic, 1999, Theorem 6.3)**.** *Let G be an analytic graph on a Polish space X. Then exactly one of the following holds:*

(1) either $\chi_B(G) \leq \aleph_0$; or

(2) there exists a continuous homomorphism from G_0 to G (i.e., a continuous $\varphi : 2^\omega \to X$ such that $(\varphi(x), \varphi(y)) \in G$, for all $(x, y) \in G_0$).

From this, it readily follows that uncountable Borel chromatic numbers of analytic graphs are at least $\chi_B(G_0)$.

This dichotomy is at the heart of descriptive set theory, as shown by Miller, 2012. There he showed how this dichotomy generalizes several classical descriptive set-theoretic dichotomies, such as the perfect set property, Silver's dichotomy about countable co-analytic equivalence relations etc.

Now we introduce a relative of the graph G_0: the graph G_1. At a first glance, these graphs seem to be too similar. However, it will soon be clear that they may be different in substantial aspects.

THE GRAPH G_1. The graph G_1 is defined on 2^ω by

$$G_1 = \{(x, y) \mid \exists! n (x(n) \neq y(n))\}.$$

First note that a measure version of Fact 1.2.2 holds if one replaces G_0 with G_1 — i.e.:

Fact 1.2.4. *Every Lebesgue measurable G_1-independent set is null. In particular, $\chi_B(G_1) \geq \operatorname{cov}(\mathcal{N})$.*

Effectively, every Lebesgue measurable G_1-independent set has measure zero. To see that:

If $A \subseteq 2^\omega$ is Lebesgue measurable and has positive measure, then from *Lebesgue's density theorem* it follows that there exists a point x of *density* 1 *in* A — i.e.,

$$\delta_A(x) \doteq \inf_{x\in[s]} \frac{\mu(A\cap[s])}{\mu([s])} = 1.$$

Let $s \in 2^{<\omega}$ be such that $\mu(A\cap[s])/\mu([s]) > 1/2$. Then there exists $x \in 2$ such that $s^\frown 0^\frown x \in A$; and that $s^\frown 1^\frown x \in A$ as well. Thus, $A^2 \cap G \neq \emptyset$.

It is not clear, however, whether $\chi_B(G_0) \geq \mathrm{cov}(\mathcal{N})$. In fact, it follows from Miller, 2008, Theorem 3.3 that there exists a G_0-independent positive-measure F_σ subset of 2^ω. It could be, however, that $\chi_B(G_0) \geq \mathrm{cov}(\mathcal{N})$ holds for other reasons (for more, see Section 4.1).

Another important difference between G_0 and G_1 is that G_1 contains 2^n-cycles, for each $n \in \omega$, whereas G_0 contains no cycles whatsoever. Besides this, both graphs contain no odd cycles, which implies that their chromatic number is 2 (see, e.g., Diestel, 2016, Proposition 1.6.1).

An interesting note is that G_0 and G_1 have the same connected components, which are the equivalence classes of *Vitali's equivalence relation*, E_0:

THE RELATION E_0. The relation E_0 is defined on 2^ω by

$$E_0 = \{(x,y) \mid \forall^\infty n(x(n) = y(n))\}.$$

Fact 1.2.5. *The connected components of both G_0 and G_1 are the equivalence classes of E_0.*

That the connected components of G_1 are the equivalence classes of E_0 is clear. As for G_0: first recall that if x and y differ only at 0, then they form a G_0-edge. Now assume that for all $(z,w) \in E_0$ such that $\Delta(z,w) \leq n$ — where $\Delta(z,w)$ is the least bit in which z and w differ —, there exists a G_0-path between z and w. Let $(x,y) \in E_0$ be such that $|x\Delta y| = n+1$; and x', y' be obtained by replacing the first n bits of x, respectively y, by the unique $s \in D$ such that $|s| = n$. Now (x',y') is in G_0 and there are G_0-paths from x to x'; and from y to y', by our assumption. Hence, there is a G_0-path from x to y.

Note that E_0 can be seen as the equivalence relation of rational shifts on the reals (i.e., two real numbers are equivalent if their difference is a rational number). This is the least *non-smooth* Borel equivalence relation defined on a Polish space in the following sense: if E is a Borel equivalence relation on X, then either E is *smooth* — i.e., there exists a Borel map $f : X \to X$ such that $f(x) = f(y)$, whenever $(x,y) \in E$ (it is *Borel reducible to the identity*); or there exists a continuous embedding from E_0 to E. This is the *Glimm-Effros dichotomy* introduced by Harrington, Kechris, and Louveau, 1990 and, in some sense, the G_0-dichotomy may be seen as a graph-analog of this dichotomy.

We defined the Borel chromatic number of E_0 by $\chi_B(E_0) = \chi_B(E_0 \setminus \mathrm{Id}_{2^\omega})$. Note that $(E_0 \setminus \mathrm{Id}_{2^\omega})$-independent sets are partial E_0-selectors and vice-versa.

Our goal is to link the Borel chromatic numbers of G_0, G_1 and E_0 with some familiar cardinals of van Douwen's diagram and draw a cardinal chart. This link is already implicit in the mathematical literature and we will outline it here.

We get a lower and an upper bound for $\chi_B(G_1)$ from van Douwen's diagram. However, in order to get that from the mathematical literature we need the connection between G_1 and *Silver forcing*, defined in the next section (see Fact 1.3.2 b).

Henceforth, we identify Borel G_1-independent sets with Borel v^0-null sets, where v^0 is the σ-ideal of *Silver null* sets, defined in the next section. The lower bound is the distributivity number:

Fact 1.2.6. $\chi_B(G_1) \geq \mathfrak{h}$.

This follows from the chain of inequalities:

$$\mathfrak{h} \leq \mathrm{cov}(\mathrm{Bor}(2^\omega) \cap r^0) \leq \mathrm{cov}(\mathrm{Bor}(2^\omega) \cap v^0) \leq \chi_B(G_1),$$

where r^0 is the σ-ideal of *completely Ramsey-null sets*, which is the σ-ideal p^0 of Section 1.3, the Mathias forcing, introduced in Section 4.1.

For the inequality $\mathfrak{h} \leq \mathrm{cov}(\mathrm{Bor}(2^\omega) \cap r^0)$, see, e.g., Halbeisen, 2012, Theorem 9.2. The inclusion $\mathrm{Bor}(2^\omega) \cap v^0 \subseteq \mathrm{Bor}(2^\omega) \cap r^0$ follows from the fact that every Borel set is $\mathbb{P}$-*measurable* (see Section 3.3), for $\mathbb{P} \in \{\mathbb{R}, \mathbb{V}\}$[1].

The upper bound for $\chi_B(G_1)$ is the reaping number:

Fact 1.2.7 (Brendle, 1995, Lemma 3)**.** $\chi_B(G_1) \leq \mathfrak{r}$.

One can directly check that the sets $\hat{A}_i$, for $i < 2$, defined in the proof of Lemma 3 of Brendle, 1995 are closed G_1-independent sets and, with the help of a reaping family, they cover 2^ω.

We finally obtain the following diagram:

$$\begin{tikzcd}
 & & 2^{\aleph_0} & \\
 & \chi_B(E_0) \arrow[ur] & & \mathfrak{d} \arrow[ul] \\
 & & \mathfrak{r} \arrow[uu] & \\
 & \chi_B(G_1) \arrow[uu] \arrow[ur] & & \mathfrak{b} \arrow[ul] \arrow[uu] \\
\chi_B(G_0) \arrow[ur] & & \mathfrak{h} \arrow[ul] \arrow[ur] & \\
 & \aleph_1 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

FIGURE 1.5: Borel chromatic numbers added to van Douwen's diagram

In this work we tackle most of the pairs of cardinals in the diagram and show they may be consistently different. There are, however, open inequalities[2] (e.g., the consistency of $\chi_B(G_0) < \mathfrak{h}$).

[1] I would like to thank Yurii Khomskii for pointing this out to me.

[2] The cardinals $\mathrm{cov}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{N})$ are purposely omitted from the diagram, mostly for aesthetic reasons, but also because we could not say anything substantial about "$\mathrm{cov}(\mathcal{N})$ versus $\chi_B(G_0)$". The consistency of $\chi_B(G_0) > \mathrm{cov}(\mathcal{M})$ will follow from the consistency of $\chi_B(G_0) > \mathfrak{d}$.

1.3 Forcing

We may assume familiarity with the modern presentations of the topic that are found in textbooks such as Kunen, 2014 and Jech, 2003. The approach for iterated forcing used here is not new and appeared already in texts such as Baumgartner and Laver, 1979, Newelski and Rosłanowski, 1993 and Geschke and Quickert, 2004.

In order to fix some notation: if $\mathbb{P}$ is a forcing notion and $\alpha \geq 1$ is an ordinal, then $\mathbb{P}_\alpha$ is the *countable support iteration of* $\mathbb{P}$ *of length* α. If V is a model of set theory, the $\mathbb{P}$-V*-model* is the generic extension obtained by forcing with $\mathbb{P}_{\omega_2}$ over V (we shall simply call it "the $\mathbb{P}$*-model*" when V is clear from context).

All forcing notions considered here satisfy *Axiom A*. This axiom implies *properness*, which is known to be a property preserved for countable support iterations and, moreover, it implies that $\aleph_1$ is not collapsed. This way, iteratively forcing with continuum-sized Axiom A notions, using countable support, over a model of CH does not collapse cardinals: it follows from CH that these forcing notions have size $\aleph_1$, which is not collapsed due to properness; moreover, these notions are $\aleph_2$-c.c. and thus preserve all cardinals above $\aleph_1$ as well.

> *AXIOM A.* A forcing notion $\mathbb{P}$ satisfies *Axiom A* if there exists $(\leq_n)_{n\in\omega}$, a $\subseteq$-decreasing sequence of partial orders of $\mathbb{P}$, with the following properties:
>
> (1) if $(p_n)_{n\in\omega}$ is a sequence such that $p_{n+1} \leq_n p_n$, for all $n \in \omega$, then there exists $q \in \mathbb{P}$ such that $q \leq_n p$, for all $n \in \omega$; and
>
> (2) if $A \subseteq \mathbb{P}$, $p \in \mathbb{P}$ and $n \in \omega$, then there exists $q \leq_n p$ compatible with at most countably many elements of A.

Another example of iterable property which has a stronger "Axiom A" counterpart is ω^ω*-boundedness*:

Recall that a forcing notion is ω^ω*-bounding* iff any element of ω^ω, in the generic extension, is eventually dominated by some ground-model element of ω^ω. This readily implies that $\mathfrak{d}$ is bounded by the value of the ground-model continuum — i.e., $\Vdash \mathfrak{d} \leq |2^\omega \cap V|$.

> *STRONG AXIOM A.* A forcing notion $\mathbb{P}$ satisfies *strong Axiom A* if there exists $(\leq_n)_{n\in\omega}$, a non-$\subseteq$-decreasing sequence of partial orders of $\mathbb{P}$ satisfying item 1 of Axiom A, and the following strengthening of item 2:
>
> (2)′ if $A \subseteq \mathbb{P}$, $p \in \mathbb{P}$ and $n \in \omega$, then there exists $q \leq_n p$ compatible with at most *finitely* many elements of A.

Fact 1.3.1 (e.g., Rosłanowski and Shelah, 1999, Theorem 2.1.4.)**.** *Axiom A forcing notions are proper; and strong Axiom A forcing notions are proper and* ω^ω*-bounding.*

The main advantage of working with these (strong) Axiom A forcing notions is due the simplicity of the treatment of their iterations:

Let F be a finite subset of α and $\eta : F \to \omega$. Say that $q \leq_{F,\eta} p$ iff

$$\forall \gamma \in F \left(q \restriction \gamma \Vdash q(\gamma) \leq_{\eta(\gamma)} p(\gamma) \right).$$

FUSION FOR ITERATIONS. A sequence $(p_n, F_n, \eta_n)_{n\in\omega}$ is a fusion sequence iff, for every $n \in \omega$,

(1) $F_n \subseteq \alpha$ is finite;

(2) $F_n \subseteq F_{n+1}$;

(3) $\eta_n : F_n \to \omega$;

(4) $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, for all $\gamma \in F$;

(5) $p_{n+1} \leq_{F_n,\eta_n} p_n$; and

(6) for all $\gamma \in \operatorname{supp}(p_n)$, there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

The fusion q of $(p_n)_{n\in\omega}$ is defined recursively such that

$$\forall \gamma < \alpha\, (q \restriction \gamma \Vdash q(\gamma) \text{ is the fusion of } (p_n(\gamma))_{n\in\omega}).$$

Before introducing the main Axiom A forcing notions used in this work, we need to fix some notation: if $p \subseteq X^{<\omega}$ is a tree, where $X \in \{2, \omega\}$, then

- $\operatorname{st}(p)$, the *stem of* p, is the maximal node $s \in p$ compatible with every other node of p;
- $\operatorname{succ}_s(p)$ is the set of *immediate successors of* $s \in p$ *in* p;
- $\operatorname{spl}(p) = \{s \in p \mid s^\frown 0 \in p \wedge s^\frown 1 \in p\}$ is the set of *splitting nodes of* p; and
- $[p] = \{x \in X^{<\omega} \mid \forall n \in \omega (x \restriction n \in p)\}$ is the *set of branches through* p.

Our Axiom A forcing notions are the following:

SACKS FORCING. A tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* iff it is a *perfect tree* — i.e., for every $t \in p$, there exists $s \supseteq t$ such that $s \in \operatorname{spl}(p)$.
The *Sacks forcing*, $\mathbb{S}$, consists of Sacks trees ordered by direct inclusion.

E_0-FORCING. A tree $p \subseteq 2^{<\omega}$ is an *E_0-tree* iff it is perfect; and for every $s \in \operatorname{spl}(p)$, there are $s_0 \supseteq s^\frown 0$ and $s_1 \supseteq s^\frown 1$, of the same length, such that

$$\left\{x \in 2^\omega \mid s_0^\frown x \in [p]\right\} = \left\{x \in 2^\omega \mid s_1^\frown x \in [p]\right\}.$$

The *E_0-forcing*, $\mathbb{E}_0$, consists of E_0-trees ordered by inclusion.

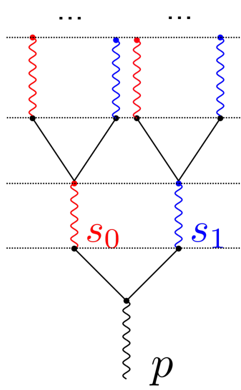

FIGURE 1.6: Illustration of an E_0-tree: after the first splitting node, we choose sequences s_0 (red) and s_1 (blue), of same length; then again two new sequences (red and blue) after the second splitting level, of same length, are chosen to extend both s_0 and s_1, and so on.

SILVER FORCING. A tree $p \subseteq 2^{<\omega}$ is a *Silver tree* iff it is an E_0-tree such that, for every $s \in \mathrm{spl}(p)$,

$$\left\{x \in 2^\omega \mid s^\frown 0^\frown x \in [p]\right\} = \left\{x \in 2^\omega \mid s^\frown 1^\frown x \in [p]\right\}.$$

The *Silver forcing*, $\mathbb{V}$, consists of Silver trees ordered by inclusion.

For a perfect tree $p \subseteq 2^{<\omega}$, there exists a bijection $\sigma \mapsto \sigma^*$ from $2^{<\omega}$ to its set of splitting nodes, $\mathrm{spl}(p)$, described as follows: $\emptyset^* = \mathrm{st}(p)$; and $(\sigma^\frown i)^*$ is the minimal splitting node of p such that $(\sigma^\frown i)^* \supseteq \sigma^{*\frown} i$, for each $i < 2$. Extend $*$ to 2^ω by $a^* = \bigcup_{n \in \omega}(a \restriction n)^*$, for $a \in 2^\omega$.

MILLER FORCING. A tree $p \subseteq \omega^{<\omega}$ is a *Miller tree*, or a *superperfect tree*, iff for every $t \in p$, there exists $s \supseteq t$ such that $\mathrm{succ}_p(s)$ is infinite.

The *Miller forcing*, $\mathbb{M}$, consists of Miller trees ordered by inclusion.

LAVER FORCING. A tree $p \subseteq \omega^{<\omega}$ is a *Laver tree* iff it is a Miller tree such that $\mathrm{succ}_p(t)$ is infinite, for every $t \supseteq \mathrm{st}(p)$.

The *Laver forcing*, $\mathbb{L}$, consists of Laver trees ordered by inclusion.

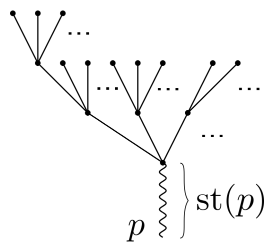

FIGURE 1.7: Illustration of a Laver tree.

Similarly, for a superperfect tree $p \subseteq \omega^{<\omega}$, there exists a bijection $\sigma \mapsto \sigma^*$ from $\omega^{<\omega}$ to $\mathrm{spl}(p)$, described as follows: let $\emptyset^*$ be the stem of p; for each $n \in \omega$, $(\sigma^\frown n)^*$ is the minimal splitting node of p extending the n-th element of $\mathrm{succ}_p(\sigma^*)$. Extend $*$ to ω^ω by $a^* = \bigcup_{n \in \omega}(a \restriction n)^*$, for $a \in \omega^\omega$.

For a Sacks, an E_0, or a Silver tree p, let $L_n(p) = \{\sigma^* \mid \sigma \in 2^n\}$; and for a Miller, or Laver tree p, let $L_n(p) = \{\sigma^* \mid \sigma \in n^n\}$ denote the *n-th splitting level of p*, for each $n \in \omega$.

In all cases, the sequence $(\leq_n)_{n\in\omega}$ is defined as follows: for every $n \in \omega$,

$$q \leq_n p \text{ iff } q \leq p \text{ and } L_n(q) = L_n(p).$$

This way, if $(p_n)_{n\in\omega}$ is a fusion sequence witnessed by $(\leq_n)_{n\in\omega}$, then its fusion is $q = \bigcap_{n\in\omega} p_n$.

To fix a final notation: let $X \in \{2, \omega\}$, $\sigma \in X^{<\omega}$, $\mathbb{P} \in \{\mathbb{S}, \mathbb{E}_0, \mathbb{V}, \mathbb{M}, \mathbb{L}\}$, and $p \in \mathbb{P}$. Let

$$p * \sigma = \{s \in p \mid \sigma^* \subseteq s \text{ or } s \subseteq \sigma^*\},$$

be the *restriction of p to σ*, if $\mathbb{P} \in \{\mathbb{S}, \mathbb{E}_0, \mathbb{V}\}$ and $X = 2$; or $\mathbb{P} \in \{\mathbb{M}, \mathbb{L}\}$ and $X = \omega$.

We have an important ideal of $\mathbb{P}$-null sets, associated with $\mathbb{P}$:

THE IDEAL p^0. The σ-ideal of $\mathbb{P}$-*null* sets is defined by

$$p^0 = \{A \subseteq X^{<\omega} \mid \forall p \in \mathbb{P}\ \exists q \leq p\ ([q] \cap A = \emptyset)\}.$$

That p^0 is a σ-ideal for all forcing notions considered follows form *fusion arguments* (i.e., employing fusion sequences).

It is well-known that $\mathbb{P}$ increases $\mathrm{cov}(p^0)$; $\mathrm{cov}(s^0) \leq 2^{\aleph_0}$; $\mathrm{cov}(m^0) \leq \mathfrak{d}$; and $\mathrm{cov}(\ell^0) \leq \mathfrak{b}$. In fact, these inequalities follow from more general dichotomies about analytic subsets of reals: the *perfect set property*; the *K_σ-regularity* (see Kechris, 1977, Theorem 3.1); and the *ℓ-regularity* (see Goldstern et al., 1995, Lemma 2.3). As for e^0 and v^0:

Fact 1.3.2 (Zapletal, 2004, Lemmas 2.3.29 and 2.3.37)**.** *Let $A \subseteq 2^\omega$ be an analytic set. Then*

- *(a) either $\chi_B(E_0 \cap A^2) \leq \aleph_0$; or there exists an E_0-tree p such that $[p] \subseteq A$; and*
- *(b) either $\chi_B(G_1 \cap A^2) \leq \aleph_0$; or there exists a Silver tree p such that $[p] \subseteq A$.*

It follows that the map $p \mapsto [p]$ is a dense embedding from $\mathbb{E}_0$ to the poset of Borel $I(E_0)$-positive sets; and from $\mathbb{V}$ to the poset of Borel $I(G_1)$-positive sets, which is known to increases $\mathrm{cov}(I)$, for any σ-ideal I (see Zapletal, 2008, Proposition 2.1.2). Moreover, $\mathrm{cov}(e^0) \leq \chi_B(E_0)$; and $\mathrm{cov}(v^0) \leq \chi_B(G_1)$.

Summarizing the comments above:

Forcing	increases
Sacks	$2^{\aleph_0}$
E_0	$\chi_B(E_0)$
Silver	$\chi_B(G_1)$
Miller	$\mathfrak{d}$
Laver	$\mathfrak{b}$

All forcing notions defined in this section add a single real number, the *generic real*, that completely determines the generic filter. That is, if H is a generic filter over the ground model, for any of the forcing notions above, then there exists a unique real number x_H such that $V[H] = V[x_H]$. Naturally, different forcing notions add reals with different properties. For instance, while $\mathbb{M}$ adds unbounded reals, (hence it increases $\mathfrak{d}$), it does not add dominating reals, even using countable support iterations of $\mathbb{M}$ (hence it does not affect $\mathfrak{b}$). Oftentimes it is possible to state an iterable property (i.e., a property preserved for countable support iterations of proper forcing notions) which implies that certain reals are not added, such as ω^ω-boundedness, or the *Laver property*. However, more often than not, we lack such properties and have to provide a direct argument, one which uses the Axiom A structure of the forcing notion. Things turn out to be simpler when the forcing notion does not produce intermediate forcing extensions of the form $V[x]$, for x a real number, between V and its generic extension, $V[H]$.

> *MINIMALITY.* A forcing notion $\mathbb{P}$ is *minimal with respect to the reals* iff for every generic filter H and every $x \in V[H] \cap 2^\omega$, either $V = V[x]$ or $V[x] = V[H]$.

All forcing notions defined in this section are minimal, and this important feature appears in all of our proofs. In fact, it is possible to prove something stronger:

Let $\dot{x}$ be a name for a real and p be a condition witnessing that. Then there exists $q \leq p$ and a continuous $f : [q] \to 2^\omega$, mapping the generic real to $\dot{x}$ — i.e., $q \Vdash f(\dot{x}_{\text{gen}}) = \dot{x}$, where $\dot{x}_{\text{gen}}$ is the name for the generic real.

This property is called *continuous reading of names* and is featured in most of naturally occuring forcing notions, including the ones considered here, and many other non-minimal forcing notions, such as Cohen and random forcings.

Note that the image of the function $f \restriction [q]$ is the set $[T_q(\dot{x})]$, where $T_q(\dot{x})$ is the *tree of q-possibilities for* $\dot{x}$:

> *TREE OF POSSIBILITIES.* For $q \leq p$,
>
> $$T_q(\dot{x}) = \{s \in \omega^{<\omega} \mid \exists r \leq q(r \Vdash s \subseteq \dot{x})\}$$
>
> is the *tree of q-possitibilities for* $\dot{x}$.

If, additionally, one can always find f which is either constant or injective, then this ensures the forcing is minimal: if f is constant, then $\dot{x}$ is ground-model and $V[x] = V$; and if f is injective, then $f^{-1}(\dot{x}) = \dot{x}_{\text{gen}}$ and $V[x] = V[x_{\text{gen}}]$.

1.4 Summary of results

The task of computing the Borel chromatic numbers of the graphs introduced in Section 1.2 in the models of set theory obtained by iteratively forcing with the notions introduced in Section 1.3 is considerably simpler than targeting for Borel chromatic numbers of graphs that are more general than those ones. The key observation here is that, the preservation of a Borel chromatic number in a certain model of set theory may depend on three factors:

(1) the topology of the space of vertices (e.g., compactness, connectedness etc);

(2) the complexity of the graph (e.g., closed, F_σ, Borel etc); and

(3) on some suitable notion of *smallness* for the graph (e.g., lack of perfect cliques, local countability etc).

There are four notions of smallness which encompass our examples from Section 1.2. For that, fix G a graph on a space X:

NO PERFECT CLIQUES. Say that G *has no perfect cliques* iff there exists no perfect subset $P \subseteq X$ such that $P^2 \subseteq G$.

NO 4*-CYCLES.* Say that G *has no* 4*-cycles* iff there exists no sequence of four elements $x_0, x_1, x_2, x_3 \in X$ such that

$$x_0 G x_1 G x_2 G x_3 G x_0.$$

LOCAL COUNTABILITY. Say that G is *locally countable* iff the set

$$\{y \in X \mid (x, y) \in G\}$$

is countable, for every $x \in X$.

ℓ-UNBOUNDEDNESS. If $(C_n)_{n\in\omega}$ is a cover of G, the *G-locator of* $(C_n)_{n\in\omega}$, $\ell : X^2 \to \omega$, is defined by

$$\ell(x,y) = \begin{cases} \min\{n+1 \mid (x,y) \in C_n\}, & \text{if } (x,y) \in G \\ 0, & \text{if } x = y. \\ \infty, & \text{if } (x,y) \notin G \cup \mathrm{Id}_X. \end{cases}$$

Say that G is *ℓ-unbounded* iff for every $(x, y) \in G$, and n such that $\ell(x, y) = n$, there exists an open set O around x such that $\ell(z, y) > n$, for every $z \in O \setminus \{x\}$.

For example, if G is a closed graph, then ℓ, defined on the connected components of G, is the infinite-valued G-distance on X. Also note that ℓ is identically ∞ only on the (squares of) independent sets.

Clearly, the size of a clique in a graph is a lower bound for its chromatic number, and a perfect clique has the size of the continuum. Therefore "no perfect cliques" is a minimum requirement for a graph to consistently have its Borel chromatic number less than continuum.

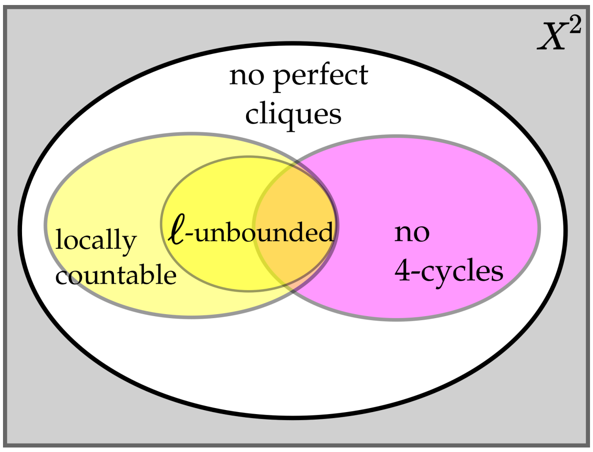

FIGURE 1.8: ℓ-unbounded graphs are locally countable; and neither locally countable nor graphs without 4-cycles have perfect cliques.

The following table summarizes how our graphs fit into these categories:

Graph	complexity	smallness
E_0	F_σ	ℓ-unbounded
G_1	closed	ℓ-unbounded
G_0	closed	ℓ-unbounded + no 4-cycles

Throughout this work, we fix V a model ZFC and, for a forcing notion $\mathbb{P}$, the model obtained by an iteration of $\mathbb{P}$ over V of length ω_2, with countable support, will be called the $\mathbb{P}$-*model*.

In Chapter 2[3] we tackle models obtained by forcing with uniform trees: the E_0-, Silver, and G_0-models. The consistencies of the inequalities $\chi_B(G_1) < \chi_B(E_0)$ and $\chi_B(G_0) < \chi_B(G_1)$ will follow from more general statements about closed graphs without perfect cliques and closed graphs without 4-cycles:

Theorem 2.1.2. *Let G be a closed graph on a Polish space X.*

(a) If G has no perfect cliques then, in the E_0-model, every point in the completion of X is contained in a compact G-independent set coded in the ground model; and

(b) if G has no 4-cycles then, in the Silver model, every point in the completion of X is contained in a compact G-independent set coded in the ground model.

Hence $\chi_B(G) \leq |2^{\aleph_0} \cap V|$ in each of the above cases.

This way, starting from V a model of CH, we obtain:

Corollary 2.1.3. *It is consistent with ZFC that $\chi_B(E_0) < 2^{\aleph_0}$; that $\chi_B(G_1) < \chi_B(E_0)$; and that $\chi_B(G_0) < \chi_B(G_1)$.*

Moreover, we devise a proper forcing notion, the G_0-*forcing*, that increase $\chi_B(G_0)$ and preserves $\mathfrak{d}$:

Theorem 2.1.4. *There exists a strong Axiom A forcing notion, $\mathbb{G}_0$ that adds a real which avoids all Borel G_0-independent sets coded in the ground model. In particular, $\chi_B(G_0) = 2^{\aleph_0}$ in the G_0-model.*

This way, starting from V a model of CH, we obtain:

Corollary 2.1.5. *It is consistent with ZFC that $\chi_B(G_0) > \mathfrak{d}$.*

In Chapter 3[4] we tackle models obtained by forcing with superperfect trees: the Miller and Laver models. As in Chapter 2, the consistencies of the inequalities $\chi_B(E_0) < \mathfrak{d}$ and $\chi_B(E_0) < \mathfrak{b}$ will follow from more general statements about F_σ locally countable graphs and F_σ ℓ-unbounded graphs, this time defined on *totally disconnected*, *compact*, Polish spaces:

Theorem 3.1.1. *Let G be an F_σ graph, with closed cover $(C_n)_{n\in\omega}$, defined on a totally disconnected compact Polish space X.*

(a) If G is locally countable then, in the Miller model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model; and

(b) If G is ℓ-unbounded then, in the Laver model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model.

[3]Joint work with Stefan Geschke.
[4]Joint work with Raiean Banerjee.

Hence $\chi_B(G) \leq |2^{\aleph_0} \cap V|$ *in each of the above cases.*

This way, starting from V a model of CH, we obtain:

Corollary 3.1.2. *It is consistent with ZFC that* $\chi_B(E_0) < \mathfrak{d}$*; and that* $\chi_B(E_0) < \mathfrak{b}$.

As another consequence of our result, we shall be able to solve, with the help of *Ikegami's theorem*, another open problem in the field of *regularity properties*:

Corollary 3.1.3. *In the model obtained by forcing with an* ω_1*-iteration of* $\mathbb{L}$*, with countable support, over* L*, the* constructible universe*: every* $\boldsymbol{\Sigma}^1_2$ *subset of* ω^ω *is* Laver measurable*, but there exists a* $\boldsymbol{\Delta}^1_2$ *subset of* 2^ω *which is not* E_0*-measurable.*

A weaker version of this was asked by, e.g., Fischer, Friedman, and Khomskii, 2014, Question 6.3; Brendle and Löwe, 2011, Fig. 1; and Ikegami, 2010, Fig. 2.1. The following table summarizes our main results:

	Forcing	space of vertices	complexity	smallness
Chapter 2	Sacks	Polish	F_σ	local countability
	Sacks and E_0		closed	no perfect cliques
	Silver		closed	no 4-cycles
	G_0		analytic	countable χ_B
Chapter 3	Miller	tot. discon.	F_σ	local countability
	Laver	compact		ℓ-unboundedness

TABLE 1.1: Summary of the results

Note that due to the G_0-dichotomy, the only Borel chromatic numbers of analytic graphs that are preserved for a forcing increasing $\chi_B(G_0)$ are the countable ones.

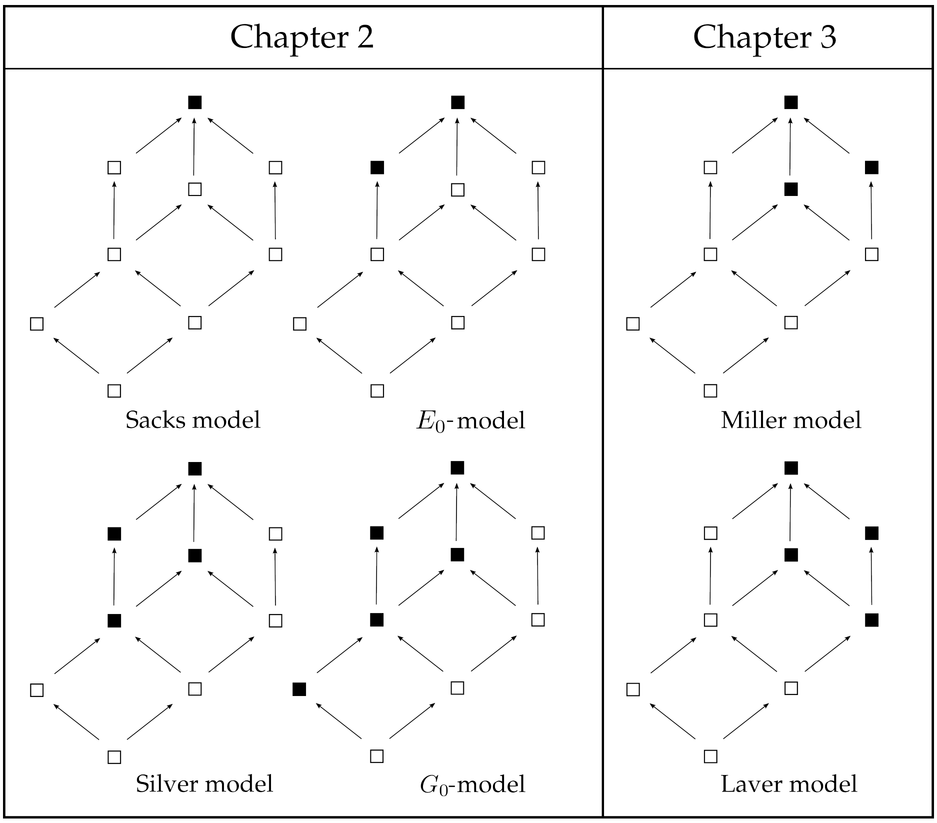

FIGURE 1.9: Chart summarizing the applications of our consistence results for the diagram of Figure 1.5, where □ = $\aleph_1$ and ■ = $\aleph_2$.

Remark. Theorem 2.1.2, item a, was independently solved by Zapletal, 2004, Corollary 3.49, as well as a version of item b for closed graphs (see Zapletal, 2019, Corollary 3.38). Furthermore, a version for closed graphs of Theorem 3.1.1, item a, was also proved by Zapletal, 2019, Example 3.61. The methods used by Zapletal rely on the heavy machinery of his *idealized forcing* (see Zapletal, 2008), as well as iterable properties for "sufficiently definable and homogeneous ideals". The approach we take here is completely different and we will resort only to classical combinatorical arguments of the forcings involved.

Chapter 2

Forcing with uniform trees

2.1 Introduction

In this chapter we deal with the familiar forcing notions of $\mathbb{E}_0$ and Silver trees; and introduce a new proper forcing notion of uniform trees, which may be seen as some type of forcing with "fat" Silver trees. This is necessary since the most natural forcing notion to increase $\chi_B(G_0)$, the poset of Borel $I(G_0)$-positive sets, ordered by inclusion, is not proper. Rather surprisingly, it collapses the continuum to $\aleph_0$ (see Zapletal, 2008, Theorem 4.7.20).

From Geschke, 2011, Theorem 2 we know that there exists a ccc forcing notion that preserves Borel chromatic numbers of closed graphs on Polish spaces, as long as they lack perfect cliques. Moreover, by Geschke, 2011, Lemma 13, there exists a σ-centered forcing notion that preserves Borel chromatic numbers of F_σ locally countable graphs:

Fact 2.1.1 (Geschke, 2011, Corollary 16 and Theorem 2)**.** *Let G be an F_σ graph on a Polish space X.*

(a) If G is locally countable, then there exists a ccc extension of the universe in which every point in the completion of X is contained in a compact G-independent set coded in the ground model; and

(b) if G is closed and has no perfect cliques, then there exists a ccc extension of the universe in which every point in the completion of X is contained in a compact G-independent set coded in the ground model.

Hence $\chi_B(G) \leq |2^{\aleph_0} \cap V|$ in each of the above cases.

Using this, one can build models with arbitrarily large continuum, while preserving Borel chromatic numbers. Furthermore, it follows from *absoluteness*[1] that these statements hold true in the Sacks model as well. Thus $\chi_B(E_0) = |2^{\aleph_0} \cap V|$ in the Sacks model.

We take a different approach from Geschke, 2011 and compute Borel chromatic numbers in the models of set theory introduced in Section 1.3. The disadvantage of this approach is that the forcing notions generating these models are not ccc, so it is harder to tell how Borel chromatic numbers behave for their products. Since we use countable support iterations, this ultimately limits the possible value of the continuum to be at most $\aleph_2$. This is because countable support iterations of non-trivial forcing notions of some length with cofinality $\aleph_1$ collapse the size of the continuum to $\aleph_1$ (see, e.g., Goldstern, 1992, Section 0).

[1] Zapletal, 2008, Theorem 6.1.11 that if a *tame invariant* cardinal characteristic of the continuum is strictly smaller than $\mathrm{cov}(I)$, for some "sufficiently definable" σ-ideal I, then this is also strictly smaller than $\mathrm{cov}(I)$ in any model of $\mathrm{CPA}(I)$, the *Covering Property Axiom for the ideal I*

Theorem 2.1.2. *Let G be a closed graph on a Polish space X.*

(a) If G has no perfect cliques then, in the E_0-model, every point in the completion of X is contained in a compact G-independent set coded in the ground-model; and

(b) if G has no 4-cycles then, in the Silver model, every point in the completion of X is contained in a compact G-independent set coded in the ground-model.

Hence $\chi_B(G) \leq |2^{\aleph_0} \cap V|$ in each of the above cases.

This way, starting from V a model of CH, we obtain:

Corollary 2.1.3. *It is consistent with ZFC that $\chi_B(E_0) < 2^{\aleph_0}$; that $\chi_B(G_1) < \chi_B(E_0)$; and that $\chi_B(G_0) < \chi_B(G_1)$.*

The motivation to devise a strong Axiom A forcing notion that increases $\chi_B(G_0)$ comes from the inequality $\chi_B(G_0) \geq \mathrm{cov}(\mathcal{M})$. Since $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$, a strong Axiom A (hence ω^ω-bounding) forcing that increases $\chi_B(G_0)$ has to preserve $\mathrm{cov}(\mathcal{M})$.

Theorem 2.1.4. *There exists a strong Axiom A forcing notion, $\mathbb{G}_0$ that adds a real which avoids all Borel G_0-independent sets coded in the ground-model. In particular, $\chi_B(G_0) = 2^{\aleph_0}$ in the $\mathbb{G}_0$-model.*

This way, starting from V a model of CH, we obtain:

Corollary 2.1.5. *It is consistent with ZFC that $\chi_B(G_0) > \mathfrak{d}$.*

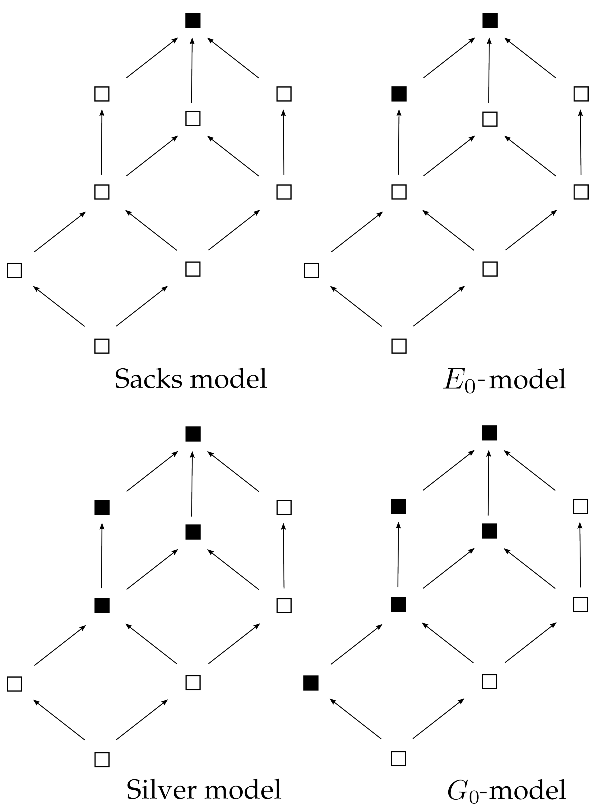

FIGURE 2.1: Chart summarizing the applications of the consistence results of this chapter.

In Section 2.2 we prove Theorem 2.1.2 and it is divided into successor and limit step. The key to unlock the successor step is to combine the minimality of E_0 and

Silver forcings with the notion of *agreeability*. In the iteration part, the first notion of faithfulness is introduced.

In Section 2.3 we introduce the G_0*-forcing* and prove a dichotomy for the ideal $I(G_0)$ involving analytic subsets of reals. We first need to replace the ideal $I(G_0)$ to some ideal $I^t(G_0) \supseteq I(G_0)$ such that the poset of Borel $I^t(G_0)$-positive sets is a proper forcing notion.

The content of this chapter is a joint work with Stefan Geschke (Gaspar and Geschke, 2022).

2.2 Separating Borel chromatic numbers

Our strategy is to divide the proof in successor and limit steps of iterations. For this, it will be crucial to tackle the case of adding a single generic real.

It will be slightly more convenient to work with graphs defined on ω^ω rather than arbitrary Polish spaces. This can be done since any perfect Polish space is the continuous injective image of ω^ω. If X is a perfect Polish space, f is a continuous injection such that $f[\omega^\omega] = X$, and G is a closed graph on X, then the *pull-back of G by f*, defined by

$$f^*[G] = \left\{ \left(f^{-1}(x), f^{-1}(y) \right) \in X^2 \mid (x, y) \in G \right\},$$

is a closed graph on X. Moreover:

- G has perfect clique iff $f^*[G]$ has a perfect clique;
- G has a 4-cycle iff $f^*[G]$ has a 4-cycle; and
- $\chi_B(G) = \chi_B(f^*[G])$.

We first will show that the generic real is contained in a compact G-independent set coded in the ground model, for G closed on 2^ω.

Let p be an E_0-tree and G be a closed graph on 2^ω.

AGREEABILITY. Say that $q \leq p$ *agrees with* G iff

$$([q * 0] \times [q * 1]) \cap E_0 \subseteq G.$$

This is similarly defined for Silver trees replacing E_0 with G_1.

Let $D(G)$ be the set of conditions agreeing with G, for either $\mathbb{E}_0$ or $\mathbb{V}$ — i.e.,

$$D(G) = \{q \leq p \mid q \text{ agrees with } G\}.$$

Lemma 2.2.1. *Let G be a closed graph on 2^ω. Considering either E_0 or Silver trees: there exists $q \leq p$ such that $[q]$ is G-independent iff $D(G)$ is not dense below p.*

Proof. Trivially, if $q \leq p$ is such that $[q]$ is G-independent, then no stronger $r \leq q$ agrees with G, which implies that $D(G)$ is not dense below p.

For the converse, assume $D(G)$ is not dense below p. We shall construct a fusion sequence $(p_n)_{n\in\omega}$ such that, for every $n \in \omega$ and every $\sigma \in 2^n$,

(1) no $r \leq p_n$ agrees with G; and

(2) $([p_n * \sigma^\frown 0] \times [p_n * \sigma^\frown 1]) \cap G = \emptyset$.

Let $p_0 \le p$ be such that no stronger condition agrees with G. Assume p_n and let $\{\sigma_0, ..., \sigma_{m-1}\}$ be an enumeration of 2^{n+1}. We define p_{n+1} in 2^{n+1} steps using successive amalgamation:

Let $q_0 = p_n$ and assume q_{j-1} is already defined for all $j < m-1$. By induction, the condition $q_{j-1} * \sigma_j$ does not agree with G. Hence, there exists $(z_0, z_1) \in ([q_{j-1} * \sigma_j^\frown 0] \times [q_{j-1} * \sigma_j^\frown 1]) \cap E_0$ (or G_1, in case of forcing with Silver trees) such that $(z_0, z_1) \notin G$. Since G is a closed graph, let $s_0 \subseteq z_0$ and $s_1 \subseteq z_1$ be such that

$$([s_0] \times [s_1]) \cap G = \emptyset.$$

Let $q_j \le_n q_{j-1}$ be such that $\mathrm{st}(q_j * \sigma_j^\frown 0) \supseteq s_0$, and $\mathrm{st}(q_j * \sigma_j^\frown 1) \supseteq s_1$. Let $p_{n+1} = q_{m-1}$. Then $q = \bigcap_{n\in\omega} p_n$ is a condition such that $[q]$ is a G-independent. $\square$

Lemma 2.2.2. *Let G be a closed graph on 2^ω. Then $D(G)$ is dense below p,*

(a) for the E_0 forcing, if G has a perfect clique; and

(b) for the Silver forcing, if G has a 4-cycle.

Proof. For (a), using the fact that $D(G)$ is dense below p we will construct a fusion sequence $(p_n)_{n\in\omega}$ of perfect trees (but not necessarily E_0-trees) such that, for all $n \in \omega$ and all $\sigma \in 2^n$,

$$[p_n * \sigma^\frown 0] \times [p_n * \sigma^\frown 1] \subseteq G.$$

Then $q = \bigcap_{n\in\omega} p_n$ will be a perfect tree such that $[q]$ is G-clique.

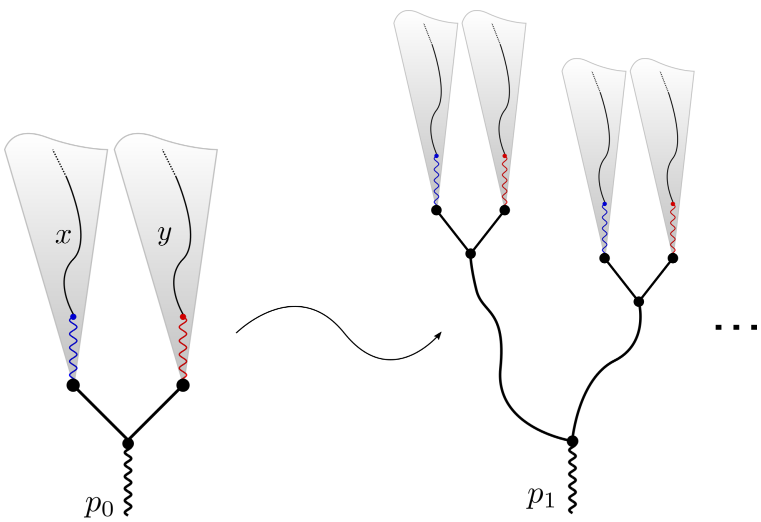

FIGURE 2.2: Depiction of the step from p_0 to p_1 in the fusion argument.

In order to construct such sequence, simply note that if $r \le p$ is a condition that agrees with G, it will follow from the closedness of G that $[r * 0] \times [r * 1] \subseteq G$: for every $(z, w) \in [r * 0] \times [r * 1]$, there exists a sequence $(w_n)_{n\in\omega}$ in $[r * 1]$ such that $(z, w_n) \in E_0$ and $(w_n)_{n\in\omega}$ converges to w. Since r agree with G, then $(z, w_n) \in G$, for every $n \in \omega$ and, since G is a closed graph, $(z, w) \in G$.

For (b), let $r \le p$ be a condition that agrees with G.

For $z \in [r * 0]$, let z' denote the copy of z in $[r * 1]$ — i.e., $(z, z') \in G_1$.

- ▶ Case 1) There exists $(z, w) \in [r * 0]^2 \cap G$ such that $(z', w') \in G$.

Then the set $\{z, z', w, w'\}$ is a 4-G-cycle, since $(z, z'), (w, w') \in ([r * 0] \times [r * 1]) \cap G_1$ and r agrees with G:

$$zGwGw'Gz'Gz.$$

- Case 2) for all $(z, w) \in [r * 0]^2$,

$$(z, w) \in G \leftrightarrow (z', w') \notin G.$$

Let $\left\{z_0, ..., z_{R(4)-1}\right\} \subseteq [r * 0]$ be a set of $R(4)$ vertices, where $R(4)$ denotes the Ramsey number of 4. From Ramsey theorem, either there exists a 4-G-clique in this set (thus, a also a 4-G-cycle); or there exists a G-independent set of size 4, in which case $\left\{z'_0, ..., z'_{R(4)-1}\right\} \subseteq [r * 1]$ contains a 4-G-cycle. □

From Lemmas 2.2.1 and 2.2.2, it follows that the E_0-real is contained in a compact G-independent set coded in the ground model, if G is a closed graph on 2^ω without perfect cliques; and the Silver real is contained in a G-independent set coded in the ground model, if G is a closed graph on 2^ω without 4-cycles. We now need to show that this happens for any other element of ω^ω added by $\mathbb{E}_0$ and $\mathbb{V}$, respectively. This is where the minimality of these forcing notions may play a role:

Fact 2.2.3 (Grigorieff, 1971, Lemma 4.7 and Proposition 4.8). *Let $p \in \mathbb{E}_0$ and $f : [p] \to \omega^\omega$ be a continuous function. Then there exists $q \leq p$ such that $f \upharpoonright [q]$ is either constant or injective. In particular, $\mathbb{E}_0$ adds reals of minimal degree.*

The same holds true replacing $\mathbb{E}_0$ with $\mathbb{V}$.

Only the proof for $\mathbb{V}$ is found in Grigorieff, 1971, since the forcing $\mathbb{E}_0$ was only introduced by Zapletal, 2004, Definition 2.3.28. However, the same proof can be easily adapted to work for $\mathbb{E}_0$ as well.

It will be slightly more convenient to consider graphs that are defined on ω^ω, instead of an arbitrary Polish spaces. This can be done since every perfect Polish space is the continuous injective image of ω^ω. We may use this to pull-back the graph on the Polish space to ω^ω, similarly to how it is done in the proof of the following lemma (see Geschke, 2011, Corollary 3):

Lemma 2.2.4. *Let $\dot{x}$ be a name for an element of ω^ω, witnessed by p, and $f : [p] \to \omega^\omega$ a continuous function in the ground model such that $p \Vdash f(\dot{x}_{\text{gen}}) = \dot{x}$ and f is either constant or injective. If G is a closed graph on ω^ω, then there exists $q \leq p$ such that $f[q]$ is G-independent*

(a) for the E_0-forcing, if G has no perfect cliques; and

(b) for the Silver forcing, if G has no 4-cycles.

Proof. Assume, without loss of generality, that f is injective and let $f^*[G]$ be the the pull-back of G by f. In both cases, (a) and (b), we find $q \leq p$ such that $[q]$ is $f^*[G]$-independent. This implies that $f[q]$ is a G-independent compact set. □

We will tackle the iteration by introducing a suitable notion of *faithfulness*: let $\alpha \geq 1$ be any ordinal and p be an α-iterated ($\mathbb{E}_0$ or $\mathbb{V}$) condition. For $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$, let $p * \sigma$ be defined such that

$$\forall \gamma \in F\, ((p * \sigma) \upharpoonright \gamma \Vdash (p * \sigma)(\gamma) = p(\gamma) * \sigma(\gamma)).$$

Let $\dot{x}$ be a name for an element of ω^ω, which is not added at any proper stage of the iteration, witnessed by p.

FAITHFULNESS 1. A condition $q \leq p$ is *G-(F,η)-faithful* iff

$$([T_{q*\sigma}(\dot{x})] \times [T_{q*\tau}(\dot{x})]) \cap G = \emptyset,$$

for all distinct $\sigma, \tau \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$.

Let $\beta \in F$ and $\eta' : F \to \omega$ be such that $\eta'(\gamma) = \eta(\gamma)$, for all $\gamma \neq \beta$, and $\eta'(\beta) = \eta(\beta) + 1$.

Lemma 2.2.5. *Let G be a closed graph and $q \leq_{F,\eta} p$ be a G-(F,η)-faithful condition. Then there exists a G-(F,η')-faithful condition $r \leq_{F,\eta'} q$,*

(a) for countable support iterations of $\mathbb{E}_0$; if G has no perfect cliques; and

(b) for countable support iterations of $\mathbb{V}$, if G has no 4-cycles.

Proof. The proof works exactly the same both for $\mathbb{E}_0$ and $\mathbb{V}$, using either item (1) or (2) of Lemma 2.2.4, at successor steps. Let $\{\sigma_0, ..., \sigma_{m-1}\}$ be an enumeration of $\prod_{\gamma \in F} 2^{\eta(\gamma)}$.

- Case 1: α is limit.

  In this case, we only need that G is closed and does not have perfect cliques. First we define a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j<m}$, along with names for conditions q_0^σ and q_1^σ, where $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ as follows:

  Suppose we have defined q_{j-1}, for $j < m$. Since $\dot{x}$ is not added in a proper initial stage of the iteration,

$$(q_{j-1} * \sigma_j) \restriction \delta \Vdash T_{q(\delta) * \sigma_j(\delta)^\frown q \restriction (\delta,\alpha)}(\dot{x}) \text{ is a perfect tree.}$$

  Hence, there are names for conditions $q_0^{\sigma_j}$ and $q_1^{\sigma_j}$ such that

$$(q_{j-1} * \sigma_j) \restriction \delta \Vdash q_i^{\sigma_j} \leq (q(\delta) * \sigma_j(\delta)^\frown i)^\frown q \restriction (\delta, \alpha)$$

  and

$$(q_{j-1} * \sigma_j) \restriction \delta \Vdash \left(\left[T_{q_0^{\sigma_j}}(\dot{x}) \right] \times \left[T_{q_1^{\sigma_j}}(\dot{x}) \right] \right) \cap G = \emptyset.$$

  Let $q_j \leq_{F,\eta} q_{j-1}$ be a condition such that $(q_j * \sigma_j) \restriction \delta$ decides all maximal initial segments of $\dot{x}$ that are decided by each $q_i^{\sigma_j}$. Finally, let $r \leq_{F,\eta} q_{m-1}$ be such that $r \restriction \delta = q_{m-1} \restriction \delta$ and, for all $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ and all coordinatewise extensions $\sigma' \in \prod_{\gamma \in F} 2^{\eta'(\gamma)}$ of σ,

$$(r * \sigma') \restriction \delta \Vdash (r * \sigma') \restriction [\delta, \alpha) = q^\sigma_{\sigma'(\eta(\beta))}.$$

  Then r is our desired condition.

- Case 2: $\alpha = \xi + 1$.

  Assume, without loss of generality, $\xi \in F$. Moreover, using Lemma 2.2.4, assume

$$q \restriction \xi \Vdash [T_{q(\xi)}(\dot{x})] \text{ is } G\text{-independent.}$$

We may define a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j\leq m}$ in $\mathbb{V}_\alpha$ as follows:

Suppose we have defined q_{j-1}, for $j < m$ and assume, by shrinking q_{j-1} if necessary, that $(q_{j-1} * \sigma_j) \restriction \xi$ decides all maximal initial segments of $\dot{x}$, that are decided by $q_{j-1}(\xi) * \tau$, for each $\tau \in 2^{\eta(\xi)}$.

Let $q_j \leq_{F,\eta} q_{j-1}$ be such that

$$(q_j * \sigma_j(\beta)^\frown i) \restriction \xi \Vdash q_j(\xi) = \bigcup_{\tau \in 2^{\eta(\xi)}} q_{j-1}(\xi) * \tau^\frown i.$$

Then $r = q_{m-1}$ if the desired condition.

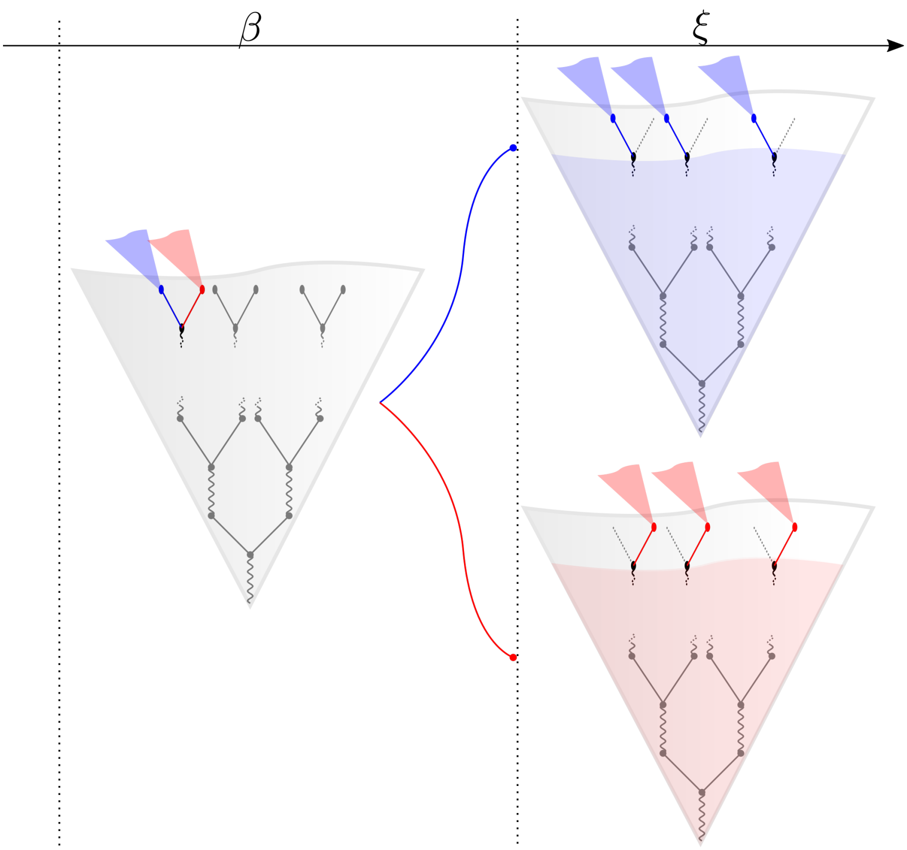

FIGURE 2.3: Depiction of the main iteration argument: restrictions to blue and red at β-coordinate give "instructions" to pick the conditions blue and red, respectively, at ξ-coordinate.

$\square$

Note that in Case 1 we simply used that the graph does not have perfect cliques. In fact reals added at limit stages of iterations of both $\mathbb{E}_0$ and $\mathbb{V}$ are contained in compact G-independent sets, coded in the ground model, as long as G is a closed graph without perfect cliques. It is at successor stages where the difference occurs.

Proof of Theorem 2.1.2. Using Lemma 2.2.5 and some faithfulness1, we may construct a fusion sequence $(p_n, F_n, \eta_n)_{n\in\omega}$, such that each p_n is G-(F_n, η_n)-faithful. This way, if q is the fusion of $(p_n)_{n\in\omega}$, then $[T_q(\dot{x})]$ is a G-independent set. □

2.3 Forcing with fat Silver trees

In this section we prove Theorem 2.1.4.

Let G be an analytic graph on a Polish space X. It follows from the G_0-dichotomy (see Fact 1.2.3) that the ideal $I(G)$ has the *inner approximation property* (see Kechris, Louveau, and Woodin, 1987, Subsection 3.2) — i.e., any $I(G)$-positive analytic set contains a compact $I(G)$-positive set.

In fact, for an analytic set A, apply the G_0-dichotomy to the graph $G \cap A^2$. If $\chi_B(G \cap A^2) > \aleph_0$ and φ is a continuous homomorphism from G_0 to G, then $\varphi[2^\omega] \subseteq A$ is the desired compact $I(G)$-positive set.

Surprisingly, the most natural forcing notion to increase $\chi_B(G_0)$ — the forcing notion of Borel $I(G_0)$-positive sets — is *not* proper (see Theorem 4.7.20 of Zapletal, 2008). We solve this problem by introducing a *corrected* forcing notion of perfect trees which will also increase $\chi_B(G_0)$ but is, however, proper.

The main issue with Borel $I(G_0)$-positive sets is that their *translations* may fall inside $I(G_0)$. So, let $I^t(G_0)t$ be the σ-ideal of *translations* of sets in $I(G_0)$ — i.e.,

$$A \in I^t(G_0) \leftrightarrow \exists n \in \omega\ (A + 1_n \in I(G_0)),$$

where $1_n \in 2^\omega$ is the image of the caracteristic function of n, and $+$ denotes the coordinatewise sum mod 2, defined on 2^ω.

Clearly, $I(G_0) \subseteq I^t(G_0)$ and, therefore, $\mathrm{cov}(I^t(G_0)) \leq \chi_B(G_0)$.

We draw inspiration from the fact that the Silver forcing is equivalent to the forcing notion of Borel $I(G_1)$-positive sets (see Fact 1.3.2) and isolate a forcing notion of Silver trees which is equivalent to the forcing notion of Borel $I^t(G_0)$-positive sets.

G_0-FORCING. Say that a Silver tree on $2^{<\omega}$ is a G_0-*tree* iff

$$\forall s \in \mathrm{spl}(p)\ \forall t \in 2^{|s|}\ \ (([p_s]+t) \times ([p_s]+t) \cap G_0 \neq \emptyset)\,.$$

Let $\mathbb{G}_0$ denote the forcing notion of G_0-trees, ordered by inclusion.

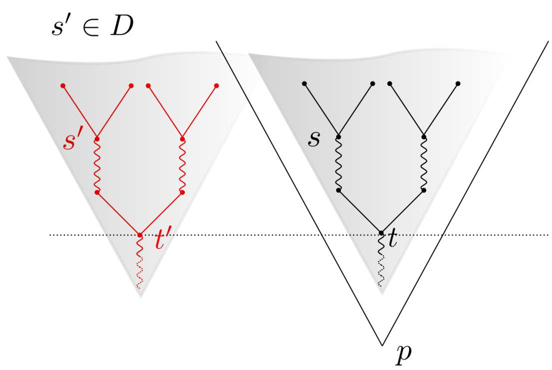

FIGURE 2.4: Depiction of a G_0-tree: if $t \in \mathrm{spl}(p)$, then the translation of p_t by $t' \in 2^{<\omega}$ of length $|t|$, $p_{t'}$, has a splitting node $s' \in D$.

The next claim and lemma are slight modifications of Claim 2.3.31 and Lemma 2.3.29 of Zapletal, 2004, respectively.

Claim 2.3.1. *Let A be an $I^t(G_0)$-positive analytic subset of 2^ω and $s \supseteq \mathrm{st}(A)$. Then for every $t \in 2^{|s|}$, there exists some $n_t \in \omega$ and an $I^t(G_0)$-positive analytic set $A_t \subseteq A$ such that*

(1) $A_t + 1_{n_t} \subseteq A$; and

(2) $s_{n_t} \subseteq \mathrm{st}(A_t + t)$.

Proof. For every $t \in 2^{|s|}$, and every $n \in \omega$, let

$$A_{t,n} \doteq \{x \in A \mid s_n \subseteq x + t \text{ and } x + 1_n \in A\}.$$

Since $A \setminus \bigcup_{t\in 2^{|s|}, n\in\omega} A_{t,n}$ is clearly an $I^t(G_0)$-small analytic set, A is $I^t(G_0)$-positive and I^t is a σ-ideal, there exists n_t such that $A_t = A_{t,n_t}$ is $I^t(G_0)$-positive. It is easy to see that one such set is as required. □

Lemma 2.3.2. *Let A be an analytic subset of 2^ω. Then either $A \in I^t(G_0)$, or it contains the branches of some G_0-tree.*

Proof. Let p, p' be finite uniform trees — i.e., for $s, t \in p$ of same length,

$$s^\frown i \in p \leftrightarrow t^\frown i \in p,$$

for each $i < 2$. Say that p' is a *fat-extension* of p iff for every $t \in 2^{<\omega}$ such that $|t| = \mathrm{ht}(p)$, there exists $t' \in 2^{<\omega}$ such that $s_{n_t} + t' \in \mathrm{spl}(p')$, where n_t is the least n such that $s_{n_t} \supseteq t$.

Let T be a tree on $(2 \times \omega)^{<\omega}$ such that A is the projection of T onto the first coordinate. Inductively we construct sequences $(p^n)_{n\in\omega}$, of finite binary trees, and $(q^n)_{n\in\omega}$, of finite subtrees of $\omega^{<\omega}$, such that

(1) the endnodes of p^n are $t^n_0, ..., t^n_{k_n-1}$, and of q^n are $u^n_0, ..., u^n_{k_n-1}$;

(2) $(t^n_i, u^n_i) \in S$, for $i < k_n$;

(3) p^n is a finite uniform tree fat-extending p^{n-1}; and

(4) $A^n \doteq \bigcap_{i<k_n} \mathrm{proj}[T \restriction (t^n_i, u^n_i)] - t^n_i + t^n_0$ is an $I^t(G_0)$-positive set.

Assume $(t^n_i)_{i\in 2^n}, (u^n_i)_{i<k_n}$ and A^n have been constructed.

Let $m \doteq |t^n_0|$ and complete the level 2^m with $\left\{t^n_{k_n}, ..., t^n_{2^m-1}\right\}$ — i.e., $2^m = \left\{t^n_0, ..., t^n_{k_n-1}, t^n_{k_n}, ..., t^n_{2^m-1}\right\}$. We proceed using successive induction in 2^m steps:

Assume we have defined $n_{j-1} \in \omega$, $A_{j-1} \notin I^t(G_0)$ and, finite trees $p^{n_{j-1}}$ and $q^{n_{j-1}}$, with endnodes $\left\{t^{n_{j-1}}_0, ..., t^{n_{j-1}}_{\ell_n-1}\right\}$ and $\left\{u^{n_{j-1}}_0, ..., u^{n_{j-1}}_{\ell_n-1}\right\}$, respectively, satisfying the four items above.

Let $n_j > n_{j-1}$ and $A_j \subseteq A_{j-1}$ be as in Claim 2.3.1 — i.e., $A_j + 1_{n_j} \subseteq A_{j-1}$; and $s_{n_j} \subseteq \mathrm{st}(A_j + t^n_j)$. For every $i < \ell^{n_{j-1}}_n$, let $A_j * t^{n_{j-1}}_i \subseteq A_{j-1}$ denote the copy of A_j, inside A_{j-1} above $t^{n_{j-1}}_i$ and let $t^{n_j}_{2i} = \mathrm{st}(A_j * t^{n_{j-1}}_i)$ and $t^{n_j}_{2i+1} = \mathrm{st}(A_j * t^{n_{j-1}}_i + 1_{n_j})$. Moreover, for each $i < \ell^{n_{j-1}}_n$, choose $u^{n_{j-1}}_i$ such that $(t^{n_{j-1}}_i, u^{n_{j-1}}_i) \in T$ and the set A^{n_j} is $I^t(G_0)$-positive. Now p^{n_j} is the finite tree generated by downward closure of the nodes $t^{n_j}_i$, for $i < \ell^{n_{j-1}}_n$, and similarly for q^{n_j} (with u's instead of t's).

Let $p^{n+1} = p^{n_{(2^m-1)}}$ and $q^{n+1} = q^{n_{(2^m-1)}}$ and note that p^{n+1} satisfies both G_0-fatness and Silverness. Then

$$p = \bigcup_{n \in \omega} p^n$$

is the desired fat G_0-tree — i.e., $[p] \subseteq A$. $\square$

The proof of the theorem above already hints a proof of properness for $\mathbb{G}_0$ and, in fact, ω^ω-boundedness (which implies the preservation of $\mathfrak{d}$). We soon will see that we are actually not far from proving the *Sacks property* as well.

Lemma 2.3.3. $\mathbb{G}_0$ *satisfies strong Axiom A.*

Proof. First, for each $n \in \omega$, we build a finite subtree p^n of p, with a set of terminal nodes $L_n(p)$, as follows:

Let $p^0 = \{\mathrm{st}(p)\}$ and assume we have defined p^n with terminal nodes $L_n(p) = \{t_0, ..., t_{k_n-1}\}$, where $\mathrm{ht}(p^n) = |t_0| = ... = |t_{k_n} - 1| \doteq \ell_n$. For each $t \in 2^{\ell_n}$, let n_t be the least natural number for which s_{n_t} is a splitting node of $p_{t_0} + t$. Let p^{n+1} be the finite tree generated by the splitting nodes of p^n, together with all the nodes of p of height n_t, for $t \in 2^{\ell_n}$. Note that

$$|L_{n+1}(p)| = |L_n(p)| \cdot 2^{2^{\mathrm{ht}(p^n)}}.$$

Finally, for $q \in \mathbb{G}_0$,

$$q \leq_n p \leftrightarrow q \leq p \text{ and } q^n = p^n.$$

The item (1) of the statement of the strong Axiom A is easily seen to be satisfied. As for (2), let $L_n(p) = \{t_0, ..., t_{k_n-1}\}$. We shall define a finite sequence $(q_i)_{i<k_n}$ using successive amalgamation:

Assume we have defined $q_{i-1} \leq_n p$ such that $q_{i-1} * \sigma_j$ is compatible with at most one element of A, for every for $i < k_n$ and $j < i$. Let $r_i \leq q_{i-1} * \sigma_i$ be a condition compatible with at most one element of A and define q_i to be the amalgamation of r_i into q_{i-1}. Now it is easy to see that $q = q_{k_n-1}$ is such that $q \leq_n p$ and it is compatible with at most k_n elements of A. $\square$

Chapter 3

Forcing with superperfect trees

3.1 Introduction

For forcing notions of superperfect trees, we shall restrict ourselves to *compact* spaces of vertices. The whole idea comes from the proof of minimality for Miller forcing (see, e.g., Groszek, 1987, Theorem 2): in the usual proof, one uses the *pigeonhole principle* to ensure that, for any element of 2^ω of the generic extension, and any Miller tree, either there will be *infinitely many* extensions of the immediate successors of the stem of this tree deciding the same thing about a fixed coordinate of the real, or infinitely many will decide the opposite.

In the case of Laver, this cannot be done so directly (see Groszek, 1987, Theorem 7). However, the compactness of the space appears in the form of *sequential compactness* — i.e., every sequence on the space has a convergent subsequence (see Claim 3.2.2).

Theorem 3.1.1. *Let G be an F_σ graph, with closed cover $(C_n)_{n\in\omega}$, defined on a totally disconnected compact Polish space X.*

(a) If G is locally countable then, in the Miller model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model; and

(b) If G is ℓ-unbounded then, in the Laver model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model.

Hence $\chi_B(G) \leq |2^{\aleph_0} \cap V|$ in each of the above cases.

This way, starting from V a model of CH, we obtain:

Corollary 3.1.2. *It is consistent with ZFC that $\chi_B(E_0) < \mathfrak{d}$; and that $\chi_B(E_0) < \mathfrak{b}$.*

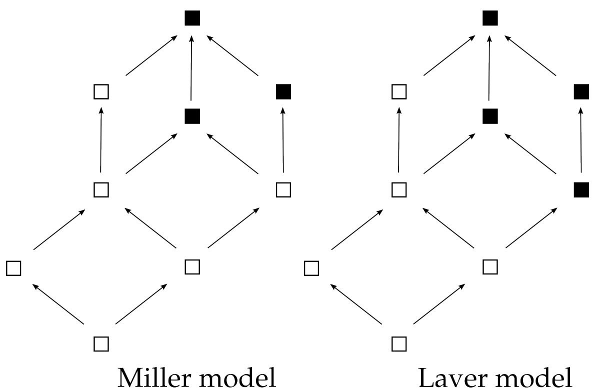

FIGURE 3.1: Chart summarizing the applications of the consistence results of this chapter.

As another consequence of our result, we shall be able to solve an open problem in the field of set theory of the real numbers: in this field, researchers usually study *regularity properties*, i.e., properties of good behaviour of sets of real numbers and whether different properties can be *separated*. If Γ_0 and Γ_1 are classes of sets of reals and P and Q are two regularity properties, we say that a model *separates* $\Gamma_0(P)$ *from* $\Gamma_1(Q)$ if in this model every set in Γ_1 has property Q, but there is a set in Γ_0 that does not have property P. Many of these separation results are known for classes on the second level of the projective hierarchy.

Fischer, Friedman, and Khomskii, 2014, Question 6.3 asked whether it is possible to separate the Silver measurability of all $\mathbf{\Delta}^1_2$ sets from the Laver measurability of all $\mathbf{\Sigma}^1_2$ sets. This question has also been mentioned open by Brendle and Löwe, 2011, Fig. 1 and by Ikegami, 2010, Fig. 2.1.

Mostly through the the work of Ikegami (see Fact 3.3.1), it is now known that these notions of measurability at the second level of the projective hierarchy depend on the amount of generic reals that are added to L, the constructible universe.

It will follow from Theorem 3.1.1, together with Ikegami's theorem, and Fact 3.3.2, that these statements can be separated:

Corollary 3.1.3. *In the model obtained by forcing with an ω_1-iteration of $\mathbb{L}$, with countable support, over L: every $\mathbf{\Sigma}^1_2$ subset of ω^ω is* Laver measurable, *but there exists a $\mathbf{\Delta}^1_2$ subset of 2^ω which is not E_0-measurable.*

Since Silver forcing adds splitting (see, e.g., Brendle, Halbeisen, and Löwe, 2005, Proposition 2.4), Brendle, Halbeisen, and Löwe, 2005, Question 2 asked whether iteratively adding $\aleph_1$ splitting reals over L already implies Silver measurability of all $\mathbf{\Delta}^1_2$ subsets of 2^ω. Since Silver forcing trivially adds E_0-reals as well, we answer this question negatively:

Corollary 3.1.4. *In the model from Corollary 3.1.3 every $\mathbf{\Delta}^1_2$ subset of 2^ω is Silver measurable, even though $\aleph_1$ splitting reals are iteratively added over L.*

In Section 3.2 we prove Theorem 3.1.1. This time we do not need to break the argument into "sucessor vs. limit step". However, we will first prove the single step, and it will be clear how the same combinatorics can be achieved for the iteration, once the key notion of *guiding real* is generalized. There is an important simplification which is done using the minimality of Laver forcing. Notwithstanding, it will be clear how one can "undo" this simplification with the help of *infinitely repeating enumerations*.

In Section 3.3 we discuss the application to regularity properties: we introduce the notion of *measurability* which is defined for all forcing notions introduced here; state Ikegami's theorem; and introduce a diagram of implications between regularities which mirrors the diagram of Figure 1.5.

The content of this chapter is a joint work with Raiean Banerjee (Banerjee and Gaspar, 2022).

3.2 Borel chromatic numbers versus the bounding number

The reason why Theorem 3.1.1 can be proved for totally disconnected compact Polish spaces is because they are the continuous injective image of 2^ω when they lack isolated points: i.e., if X is a compact totally disconnected Polish space without isoalted points, then there exists a continuous injection $f : 2^\omega \to X$ such that $f[2^\omega] = X$. Moreover:

- G is an F_σ graph iff $f^*[G]$ is an F_σ-graph;
- G is locally countable iff $f^*[G]$ is locally countable; and
- if G has closed cover $(C_n)_{n\in\omega}$, then G is ℓ-unbounded iff $f^*[G]$ is $f^*[\ell]$-unbounded, where $f^*[\ell]$ is the $f^*[G]$-locator for the cover $(f^*[C_n])_{n\in\omega}$.

In order to prove Theorem 3.1.1, we first investigate what happens when we add only one generic real to the universe. It turns out that for every real number of the generic extension, there exists a continuous injective function, coded in the ground model, whose image is an independent set and contains this number (Lemma 3.2.4), so it has minimality. We shall go through the important technology that allows us to achieve that; and our result afterwards. This discussion is done for the Laver forcing, since it is the harder one, but the same arguments apply for Miller. When differences between Miller and Laver arise, they will be highlighted.

Recall the *pure decision property* for Laver forcing:

Fact 3.2.1. *Let $p \in \mathbb{L}$ and φ be a formula of the forcing language. Then there exists a stem-preserving extension $q \leq p$ such that q decides φ.*

Let $\dot{x}$ be a name for an element of 2^ω and p be a condition forcing it. Roughly speaking, the backbone of $\dot{x}$ is composed of a sequence of ground model reals (the guiding reals) that approximate $\dot{x}$ in a helpful manner.

Claim 3.2.2. *There exists a stem-preserving extension $q \leq p$ with the following property: for every $\sigma \in \omega^{<\omega}$, there exists a ground model real $x_\sigma \in 2^\omega$ such that, for all $k \in \omega$,*

$$q * \sigma^\frown k \Vdash \dot{x} \upharpoonright (|\sigma| + k) = x_\sigma \upharpoonright (|\sigma| + k).$$

Proof. Note that for every $r \leq p$ and $\sigma \in \omega^{<\omega}$, one may find a stem-preserving extension $r_k \leq r * \sigma^\frown k$ that decides $\dot{x} \upharpoonright (|\sigma| + k)$, for each $k \in \omega$. Fix $(x_k)_{k\in\omega}$ a sequence such that $x_k \in [\dot{x} \upharpoonright (|\sigma| + k)]$, for every $k \in \omega$. Since the space 2^ω is compact, there exists $I \in [\omega]^\omega$ such that $(x_k)_{k\in I}$ converges and we let $x_\sigma = \lim_{k\in I} x_k$, which is defined for the condition $r' = \bigcup_{k\in I} r_k$.

From this observation, one may construct a fusion sequence $(p_n)_{n\in\omega}$ such that, for each $n, k \in \omega$, and $\sigma \in n^n$, there exists $x_\sigma \in 2^\omega$ such that

$$p_n * \sigma^\frown k \Vdash \dot{x} \upharpoonright (|\sigma| + k) = x_\sigma \upharpoonright (|\sigma| + k).$$

Then $q = \bigcap_{n\in\omega} p_n$ is our desired condition. □

The real x_σ is called the *σ-guiding real*. This automatically gives us a continuous ground model function $f : [q] \to 2^\omega$, defined by

$$f(a^*) = \lim x_{a\upharpoonright n},$$

for each $a \in \omega^\omega$ (hence $a^* \in [p]$), such that $q \Vdash f(x_{\text{gen}}) = \dot{x}$ (this shows that Laver forcing has the continuous reading of names).

For this reason, let us assume that p is already chosen so that x_σ is defined for p, for every $\sigma \in \omega^{<\omega}$. It turns out that $\dot{x}$ is ground model iff $\dot{x} = x_\sigma$, for some $\sigma \in \omega^{<\omega}$. If this is not the case, then it is possible to define a p-rank on $\omega^{<\omega}$ as follows: for $\sigma \in \omega^{<\omega}$,

$$\begin{cases} r(\sigma) = 0 & \leftrightarrow \exists^\infty n \in \omega \ \ (x_\sigma \neq x_{\sigma^\frown n}) \\ r(\sigma) = k > 0 & \leftrightarrow \neg\, r(\sigma) < k \wedge \exists^\infty n \in \omega \ \ \big(r(\sigma^\frown n) < k\big). \end{cases}$$

There may be no levels of $\omega^{<\omega}$ for which every node at this level has p-rank zero. However, there will be *frontiers* with this property:

Say that an antichain $A \subseteq p$ is a *frontier of p* iff every branch through p has exactly one initial segment in A — i.e., for every $x \in [p]$, there exists a unique $n \in \omega$ such that $x \restriction n \in A$. Say that $A \subseteq \omega^{<\omega}$ is a p-frontier iff $A^* = \{\sigma^* \mid \sigma \in \omega^{<\omega}\}$ is a frontier of p. A sequence $(A_n)_{n\in\omega}$ of p-frontiers forms a *p-chain* iff for all $\sigma \in A_{n+1}$, there exists a unique $\tau \in A_n$ such that $\tau \subsetneq \sigma$.

Fact 3.2.3 (Brendle, Khomskii, and Wohofsky, 2016, Theorem 16)**.** *There exists a p-chain $(A_n)_{n\in\omega}$, each consisting of rank zero nodes, such that $x_{\tau\restriction m} = x_\sigma$, for all $\sigma \subsetneq \tau$ with $\sigma \in \operatorname{succ}(A_n)$ and $\tau \in \operatorname{succ}(A_{n+1})$; and all $m \in \omega$ with $|\sigma| \leq m < |\tau|$.*

Without loss of generality, assume p is the Laver tree generated by the initial segments of these frontiers. From this, we already have that $f \restriction [p]$ is injective. Moreover, by identifying the nodes of p having the same guiding real, *assume that every $\sigma \in \omega^{<\omega}$ has p-rank zero*. This can actually be easily done for Miller forcing. As for Laver forcing, this assumption is justified with the following argument:

Let $\sim$ be the equivalence relation on $\omega^{<\omega}$ defined by

$$\sigma \sim \tau \leftrightarrow x_\sigma = x_\tau,$$

for each $\sigma, \tau \in \omega^{<\omega}$; let $p/\sim$ be the set of equivalence classes, and $\pi : \omega^\omega \to p/\sim$ be the projection $\sigma \mapsto [\sigma]_\sim$, for each $\sigma \in \omega^{<\omega}$.

Say that $\boldsymbol{a} \in p/\sim$ is an immediate successor of some different class $\boldsymbol{b}$ iff there exist $\sigma \in \boldsymbol{a}, \tau \in \boldsymbol{b}$ such that σ is an immediate successor of τ. Note that, since $\dot{x}$ is not a ground-model real, then every node of $p/\sim$ has infinitely many immediate successors. From this, let j be a bijection identifying the nodes of $\omega^{<\omega}$ with $p/\sim$.

Now, for every equivalence class $\boldsymbol{a}$, we define $\mathcal{M}(\boldsymbol{a})$ to be the set of all maximal nodes of $\boldsymbol{a}$ (i.e., $\sigma \in \mathcal{M}(\boldsymbol{a})$ iff there is no $\tau \supsetneq \sigma$ in $\boldsymbol{a}$). Let $G_{\boldsymbol{a}} : \omega \to \mathcal{M}(\boldsymbol{a})$ be an enumeration of $\mathcal{M}(\boldsymbol{a})$ such that $G_{\boldsymbol{a}}^{-1}(\{\sigma\})$ is infinite, for every $\sigma \in \mathcal{M}(\boldsymbol{a})$ (i.e., it enumerates $\mathcal{M}(\boldsymbol{a})$ with infinitely many repetitions).

Note that if $i : \omega^{<\omega} \to \omega^{<\omega}$ is such that $i(\emptyset) = \emptyset$; and $j(i(\sigma^\frown n)) = [\tau]_\sim$, for some τ immediate successor of $G_{[j(i(\sigma)\restriction|\sigma|)]}(n)$, then $(\pi^{-1}[\operatorname{ran}(j \circ i)])^*$ is a stem-preserving Laver subtree of p. In our proofs, a function i with this property is constructed and, with this, we know how to pull-back from the equivalence classes to a Laver subtree of p. For this reason, from now on we shall always assume that p is defined to have rank 0 on every node extending the stem, and convey that it is always possible to run the argument above using frontiers along with infinitely repeating enumerations of their nodes. This simplification allows us to highlight the relevant techniques, which are common to both forcing notions, as well as pinpoint the relevant combinatorial differences.

Now we have all necessary technology to tackle the single case:

Lemma 3.2.4. *Let G be an F_σ graph on 2^ω, with closed cover $(C_n)_{n\in\omega}$. Then there exists a stem-preserving extension $q \leq p$ such that $f[q]$ is G-independent*

(a) for the Miller forcing, if G is locally countable; and

(b) for the Laver forcing, if G is ℓ-unbounded.

Proof. We first define an order-preserving injection $i : \omega^{<\omega} \to \omega^{<\omega}$, and a strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers such that for all $\sigma, \tau \in n^{\leq n}$,

(1) $\ell\left(\left[x_{i(\sigma)} \restriction |i(\sigma)| + k_n\right], \left[x_{i(\tau)} \restriction |i(\tau)| + k_n\right]\right) \geq |\sigma| - |\tau|$, if $\tau \subseteq \sigma$; and

(2) $\ell\left(\left[x_{i(\sigma)} \restriction |i(\sigma)| + k_n\right], \left[x_{i(\tau)} \restriction |i(\tau)| + k_n\right]\right) \geq |\sigma| + |\tau| - 2|\Delta(\sigma,\tau)|$, if σ and τ are distinct; where $\Delta(\sigma,\tau)$ denotes the longest common initial segment between σ and τ; and

(3) $i(\sigma)$ is the least k_{n-1}-th immediate successor of $i(\sigma \restriction |\sigma| - 1)$, if $\sigma \notin (n-1)^{\leq n-1}$.

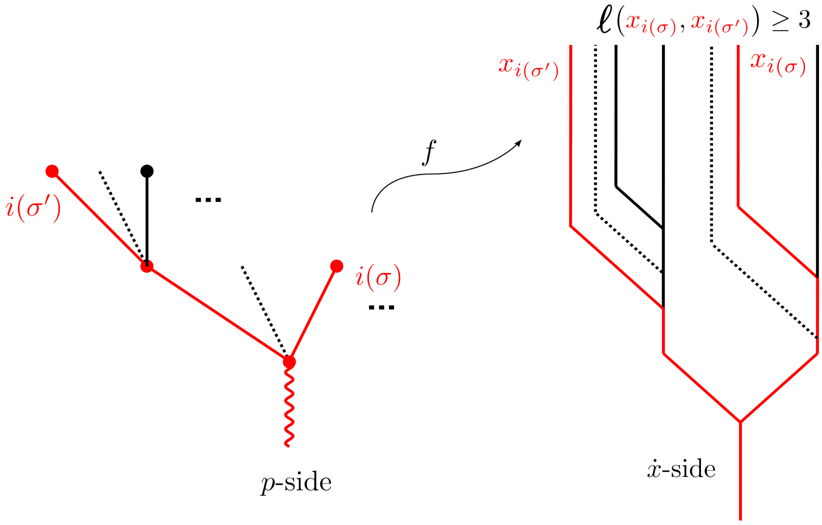

FIGURE 3.2: Depiction of item 2: for incompatible nodes σ, σ' of heights 1 and 2, respectively, the locator of $\left(x_{i(\sigma)}, x_{i(\sigma')}\right)$ has value at least $1 + 2 = 3$.

Once this is done, we get that $q = \operatorname{ran}(i)^* = \{i(\sigma)^* \mid \sigma \in \omega^{<\omega}\}$ is our desired condition (i.e., a Miller or a Laver tree, depending which case we are considering).

From this it easily follows that, if $a, b \in [q]$ are distinct, then $f(a)$ and $f(b)$ do not form an edge: in fact, for every $n \in \omega$, there exists $\sigma_{a,n}, \sigma_{b,n}$ such that $|\sigma_{a,n}| = |\sigma_{b,n}| = n+1$, $i(\sigma_{a,n})^* \subseteq a$ and $i(\sigma_{b,n})^* \subseteq b$. Then

$$\ell(f(a), f(b)) \geq \ell\left(x_{i(\sigma_{a,n})}, x_{i(\sigma_{b,n})}\right) \geq 2(n + 1 - |\Delta(\sigma_{a,n}, \sigma_{b,n})|);$$

and the sequence $|\Delta(\sigma_{a,n}, \sigma_{b,n})|$ is constant. Hence, $\ell(f(a), f(b)) = \infty$.

Assume $i \restriction n^{\leq n}$ has been defined and let $\prec$ denote the lexicographic order on $\omega^{<\omega}$. By induction on σ, also assume $i(\tau)$ has been defined, for all $\tau \prec \sigma$.

From local countability, there exists a real $a_0 \supseteq i(\sigma \restriction |\sigma| - 1)$ such that

$$\ell\left(f(a_0^*), x_{i(\tau)}\right) > \begin{cases} n - |\tau|, & \text{if } i(\tau) \subseteq a_0\text{; and} \\ n + |\tau| - 2|\Delta(\sigma,\tau)|, & \text{if } i(\tau) \text{ and } a_0 \text{ are incompatible.} \end{cases}$$

From the closedness of C_n, there exists $i(\sigma) \supseteq i(\sigma \restriction |\sigma| - 1) \restriction (|\sigma| - 1)$, an initial segment of a_0 and a natural number k_{n+1}, such that

$$\ell\left(\left[x_{i(\sigma)} \restriction |i(\sigma)| + k_{n+1}\right], \left[x_{i(\tau)} \restriction |i(\tau)| + k_{n+1}\right]\right) \geq \ell\left(f(a_0^*), x_{i(\tau)}\right). \qquad \square$$

Our goal now is to prove some version of Lemma 3.2.4 for countable support iterations of the Laver forcing. The proof of this lemma will be useful to justify the proof of Claim 3.2.6.

For an ordinal $\alpha \geq 1$, let $\mathbb{L}_\alpha$ denote the countable support iteration of $\mathbb{L}$.
Let F be a finite subset of α and $\eta : F \to \omega$. Say that $q \leq_{F,\eta} p$ iff

$$\forall \gamma \in F \left(q \restriction \gamma \Vdash q(\gamma) \leq_{\eta(\gamma)} p(\gamma) \right).$$

Now if $\dot{x}$ is a name for an element of 2^ω not added at proper stage of the iteration and p is a condition forcing it, then we may define an iterated version of the guiding reals:

Claim 3.2.5. *For every $\gamma < \alpha$ and $\sigma \in \omega^{<\omega}$, there exists an $\mathbb{L}_\alpha$-condition $p^\gamma_\sigma \leq p$, and an $\mathbb{L}_\gamma$-name for a real x^γ_σ, such that $p^\gamma_\sigma \restriction \gamma$ forces that $p^\gamma_\sigma(\gamma) \leq_0 p(\gamma)$; and*

$$p^\gamma_\sigma(\gamma) * (\sigma^\frown k)^\frown p^\gamma_\sigma \restriction (\gamma, \alpha) \Vdash \dot{x} \restriction (|\sigma| + k) = x^\gamma_\sigma \restriction (|\sigma| + k),$$

for all $k \in \omega$.

Proof. Note that if φ is a formula of the forcing language and $q \leq p$, then by the virtue of the pure decision property there exists $r \leq q$ such that

$$r \restriction \gamma \Vdash r(\gamma) \leq_0 q(\gamma) \text{ and } r^\frown[\gamma, \alpha) \text{ decides } \varphi.$$

This way we get $p^\gamma_\sigma \leq p$ such that, for every $k \in \omega$:

$$p^\gamma_\sigma \restriction \gamma \Vdash p^\gamma_\sigma(\gamma) * (\sigma^\frown k)^\frown p^\gamma_\sigma \restriction (\gamma, \alpha) \text{ decides } \dot{x} \restriction (|\sigma| + k).$$

The definition of the $\mathbb{L}_\gamma$ now follows from compactness of 2^ω, as in the proof of the single case (Claim 3.2.2). □

Moreover, one may construct a sequence $(p^\gamma_\sigma)_{\sigma \in \omega^{<\omega}}$ such that each p^γ_σ is as above, and $p^\gamma_{\sigma^\frown n} \leq p^\gamma_\sigma$ for all $n \in \omega$ and $\sigma \in \omega^{<\omega}$. From this it is possible to define an *iterated rank at coordinate* γ: for $\sigma \in \omega^{<\omega}$,

$$r^\gamma(\sigma) = 0 \leftrightarrow \exists^\infty k \in \omega \left(p^\gamma_{\sigma^\frown k} \restriction \gamma \Vdash x^\gamma_\sigma \neq x^\gamma_{\sigma^\frown k} \right).$$

Positive ranks are defined similarly to the single case.

We will tackle the iteration by introducing a second notion of faithfulness: let $\alpha \geq 1$ be any ordinal, $p \in \mathbb{L}_\alpha$, $F \subseteq \alpha$, $\eta : F \to \omega$, and $\sigma \in \prod_{\gamma \in F} \eta(\gamma)^{\eta(\gamma)}$. We define $p * \sigma$ such that

$$\forall \gamma \in F \left((p * \sigma) \restriction \gamma \Vdash (p * \sigma)(\gamma) = p(\gamma) * \sigma(\gamma) \right).$$

Let $\dot{x}$ be a name for an element of ω^ω, which is not added at any proper stage of the iteration, witnessed by p.

> *FAITHFULNESS 2.* A condition $q \leq p$ is *G-(F, η)-faithful* iff
>
> $$\ell\left([\dot{x}_{q*\sigma}], [\dot{x}_{q*\tau}]\right) \geq \ell_{\max} \doteq \max_{\gamma \in F}\{|\sigma(\gamma)| + |\tau(\gamma)| - 2|\Delta(\sigma(\gamma), \tau(\gamma))|\},$$
>
> for all distinct $\sigma, \tau \in \prod_{\gamma \in F} \eta(\gamma)^{\eta(\gamma)}$.

Let $\beta \in F$ and $\eta' : F \to \omega$ be such that $\eta'(\gamma) = \eta(\gamma)$, for all $\gamma \notin \{\beta, \bar{\gamma}\}$; $\eta'(\beta) = \eta(\beta) + 1$ if $\bar{\gamma} \neq \beta$; and $\eta'(\bar{\gamma}) = \eta_{\max} + \ell_{\max} + 1$, where $\eta_{\max} = \max_{\gamma \in F} \eta(\gamma)$.

Lemma 3.2.6. *Let G be an F_σ graph on 2^ω, with closed cover $(C_n)_{n \in \omega}$, and $q \leq_{F,\eta} p$ be a G-(F, η)-faithful condition. Then there exists a G-(F, η')-faithful condition $r \leq_{F,\eta} q$,*

(a) *for countable support iterations of $\mathbb{M}$; if G is locally countable; and*

(b) *for countable support iterations of $\mathbb{L}$, if G is ℓ-unbounded.*

Proof. Let $\{\sigma_0, ..., \sigma_{m-1}\}$ be an enumeration of $\prod_{\gamma \in F \setminus \{\bar{\gamma}\}} \eta(\gamma)^{\eta(\gamma)}$. We define a $\leq_{F,\eta}$-decreasing sequence $(p_j)_{j<m}$, as follows:

Assume we have constructed p_{j-1}. Since $\dot{x}$ is not added at any proper stage of the iteration, there exists $q_j \leq_{F,\eta} p_{j-1}$ such that all $\tau \in \omega^{\leq \eta_{\max}+\ell_{\max}+1}$ have $\bar{\gamma}$-rank zero for the condition $q_j * \sigma_j$.

Using ideas from the proof of Lemma 3.2.4, we define an order-preserving injection i on all the set of all τ's as above; a strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers; and a condition $p_j \leq_{F,\eta} q_j$ such that $(p_j * \sigma_j) \restriction \bar{\gamma}$ forces that

(1) $\ell\left(\left[x^{\bar{\gamma}}_{i(\tau)} \restriction |i(\tau)| + k_n\right], \left[x^{\bar{\gamma}}_{i(\tau')} \restriction |i(\tau')| + k_n\right]\right) \geq |\tau| - |\tau'|$, if $\tau' \subseteq \tau$; and

(2) $\ell\left(\left[x^{\bar{\gamma}}_{i(\tau)} \restriction |i(\tau)| + k_n\right], \left[x^{\bar{\gamma}}_{i(\tau')} \restriction |i(\tau')| + k_n\right]\right) \geq |\tau| + |\tau'| - 2|\Delta(\tau, \tau')|$, if τ and τ' are incompatible.

(3) $i(\tau)$ is the least k_{n-1}-th immediate successor of $i(\tau \restriction |\tau| - 1)$, if $\sigma \notin (n-1)^{\leq n-1}$, for all $n \leq \eta_{\max} + \ell_{\max} + 1$,

for all $\tau, \tau' \in \mathrm{dom}(i)$. In particular,

$$\ell\left(\left[x^{\bar{\gamma}}_{i(\tau)} \restriction |i(\tau)| + k_{\bar{n}}\right], \left[x^{\bar{\gamma}}_{i(\tau')} \restriction |i(\tau')| + k_{\bar{n}}\right]\right) \geq \ell_{\max} + 1$$

when $|\tau| = |\tau'| = \lceil \eta_{\max} + (\ell_{\max} + 1)/2 \rceil$, and $|\Delta(\tau, \tau')| \leq \eta_{\max}$, where $\bar{n} = \eta_{\max} + \ell_{\max} + 1$.

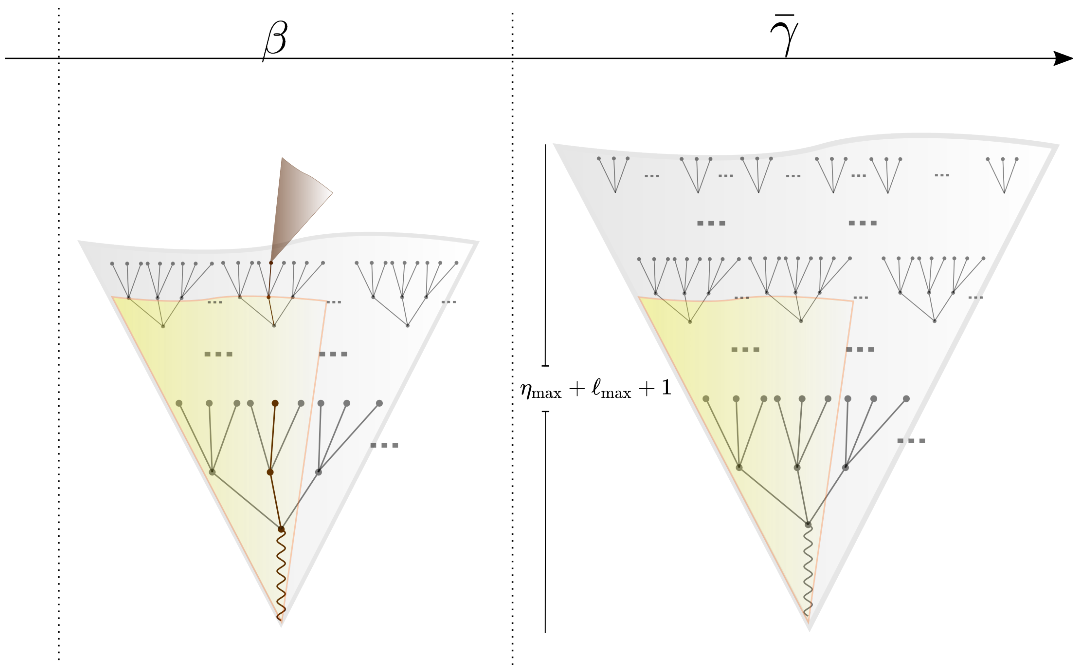

FIGURE 3.3: First we prune at $\bar{\gamma}$-coordinate, with corresponding tails, in order to make sure that we have a reservoir of only $\bar{\gamma}$-rank zero nodes; and items 1 and 2 above are satisfied.

If $\beta = \bar{\gamma}$, simply let $r = p_{m-1}$; if $\beta \neq \bar{\gamma}$, let $\left\{ I_\tau \mid \tau \in \eta'(\beta)^{\eta'(\beta)} \right\}$ denote a partition of ω into finitely many infinite pieces.

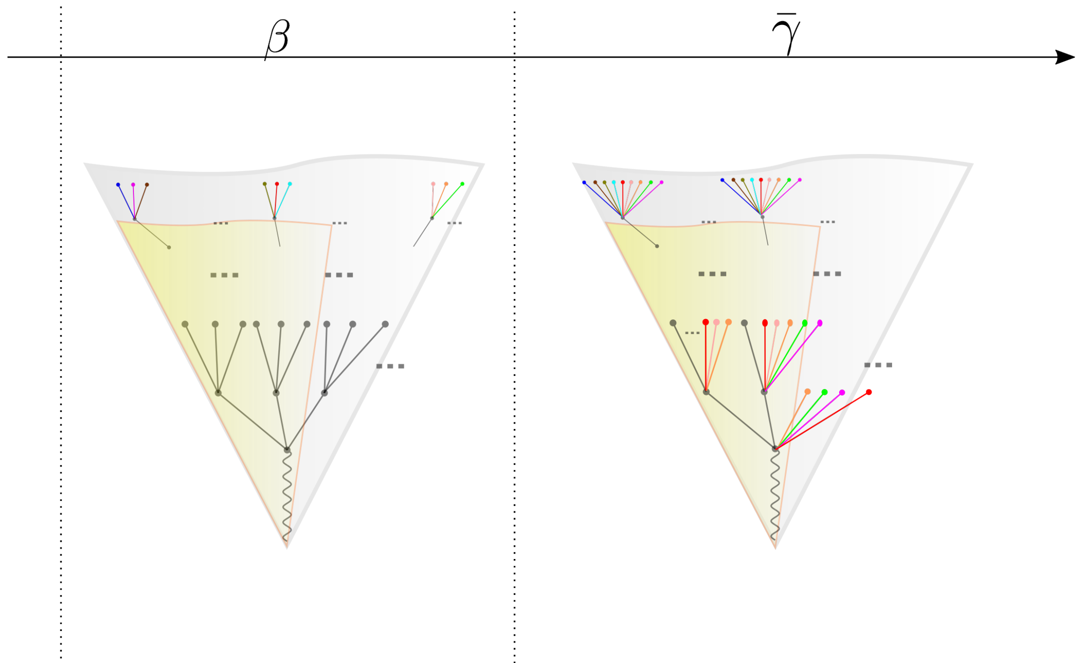

FIGURE 3.4: To each node $\tau \in \eta'(\beta)^{\eta'(\beta)}$ we assign a color. Then at $\bar{\gamma}$-coordinate, we color each set $\mathrm{succ}(\mathrm{st}(p(\bar{\gamma})) * \tau)$, for $\tau \in \eta(\bar{\gamma})^{\leq \eta(\bar{\gamma})}$ — excluding the nodes of $\eta(\bar{\gamma})^{\leq \eta(\bar{\gamma})}$ — in a way such that each of the $\eta'(\beta)^{\eta'(\beta)}$ colors appears infinitely often.

Then $r \leq_{F,\eta} p_{m-1}$ is defined such that

(1) $r \restriction \bar{\gamma} = p_{m-1} \restriction \bar{\gamma}$;

(2) for all coordinatewise extensions $\sigma' \in \prod_{\gamma \in F \setminus \{\bar{\gamma}\}} \eta'(\gamma)^{\eta'(\gamma)}$, of the restricted product of nodes $\sigma \in \prod_{\gamma \in F \setminus \{\bar{\gamma}\}} \eta(\gamma)^{\eta(\gamma)}$, for all $\bar{\sigma} \in \eta(\bar{\gamma})^{<\eta(\bar{\gamma})}$,

$$(r * \sigma') \restriction \bar{\gamma} \Vdash \mathrm{succ}(\mathrm{st}(r(\bar{\gamma}) * \bar{\sigma}) \setminus \{0, ..., \eta(\bar{\gamma}) - 1\}^* = I^*_{\sigma'(\beta)},$$

where $\{0, ..., k-1\}^*$ denotes the first k immediate successors of the stem of the restriction of $r(\bar{\gamma})$ to $\bar{\sigma}$, $r(\bar{\gamma}) * \bar{\sigma}$; for all $\bar{\sigma} \in \eta(\bar{\gamma})^{\eta(\bar{\gamma})}$,

$$(r * \sigma') \restriction \bar{\gamma} \Vdash \mathrm{succ}(\mathrm{st}(r(\bar{\gamma}) * \bar{\sigma}) = I^*_{\sigma'(\beta)};$$

(3) and $r \restriction (\bar{\gamma} + 1) \Vdash r \restriction (\bar{\gamma}, \alpha) = p_{m-1} \restriction (\bar{\gamma}, \alpha)$.

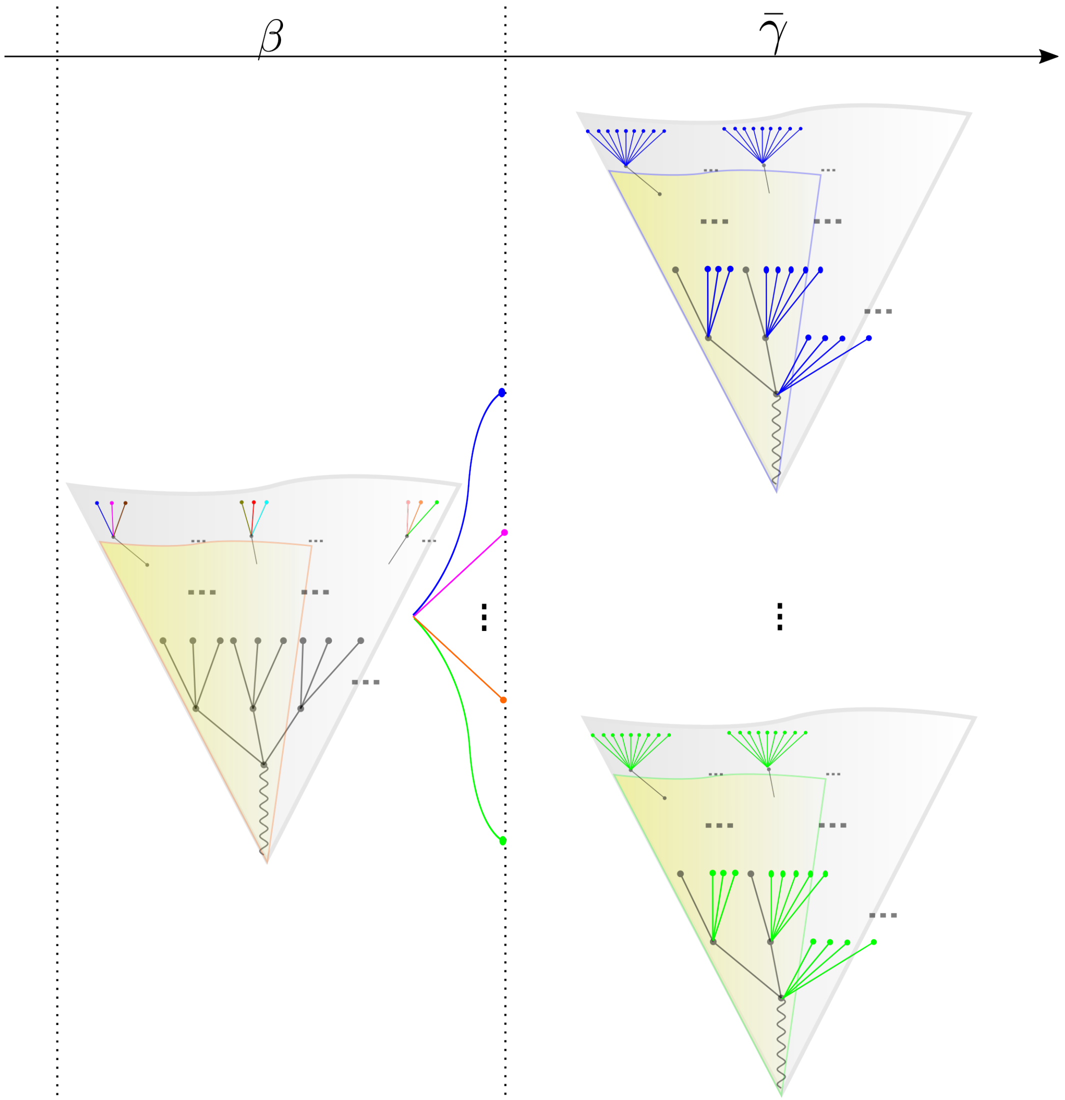

FIGURE 3.5: Similarly to the argument depicted in Figure 2.2, restrictions to a node of a certain color at β-coordinate give instructions to pick the conditions of the correspondent color at $\bar{\gamma}$-coordinate.

□

Proof of Theorem 3.1.1. Using Lemma 3.2.6 and some bookkeeping, we may construct a fusion sequence $(p_n, F_n, \eta_n)_{n\in\omega}$ such that each p_n is G-(F_n, η_n)-faithful. Let $q \in \mathbb{L}_\alpha$ be fusion of $(p_n)_{n\in\omega}$; and $(x(\gamma))_{\gamma\in\mathrm{supp}(q)}$ be a sequence in $(\omega^\omega)^{\mathrm{supp}(q)}$. Define a function f by

$$f\left(\left(x(\gamma)_{\gamma\in\mathrm{supp}(q)}\right)\right) = \bigcup_{n\in\omega} \dot{x}_{q*(x(\gamma)\restriction\eta_n(\gamma))_{\gamma\in F_n}}.$$

This is a ground model continuous injection $f : (\omega^\omega)^{\mathrm{supp}(q)} \to 2^\omega$ mapping the generic sequence to $\dot{x}$ — i.e., $q \Vdash f(x_{\mathrm{gen}}(\gamma))_{\gamma\in\mathrm{supp}(q)} = \dot{x}$. Due to the above property, we have $\ell(f(x), f(y)) = \infty$, for all distinct $x, y \in (\omega^\omega)^{\mathrm{supp}(q)}$. Hence, $f\left[(\omega^\omega)^{\mathrm{supp}(q)}\right]$ is a ground model Borel G-independent set. □

3.3 An application to regularity properties

Regularity properties emerged as early as the discipline of descriptive set theory in the works of Borel (e.g., Borel, 1898), Baire (e.g., Baire, 1899), Lebesgue (e.g., Lebesgue, 1905) etc[1]. The two main examples of this type of good behavior (i.e., regularity) of sets of reals are the notions of Lebesgue and Baire measurabilities.

From the close relationship between these measurability notions and the random and the Cohen forcings, various new notions of measurability emerged. Namely, if $\mathbb{P}$ is a forcing notion of perfect subtrees, either of $2^{<\omega}$ or $\omega^{<\omega}$:

$\mathbb{P}$-*MEASURABILITY.* Say that a set A is $\mathbb{P}$-*measurable* if

$$\forall p \in \mathbb{P}\ \exists q \leq p\ \ ([q] \subseteq A \text{ or } [q] \cap A = \emptyset)\,.$$

Note that in case we always have that $[q] \cap A = \emptyset$, we obtain the σ-ideal of $\mathbb{P}$-null sets defined in Section 1.3.

If $\mathbb{P}$ is the random forcing, $\mathbb{P}$-measurability is equivalent to the Lebesgue measurability, and if $\mathbb{P}$ is the Cohen forcing, then $\mathbb{P}$-measurability is equivalent to the Baire measurability. Moreover, through the works of Solovay, 1969 and Ihoda and Shelah, 1989, we know that Lebesgue and Baire measurabilities on the $\boldsymbol{\Delta}^1_2$ and $\boldsymbol{\Sigma}^1_2$ levels may depend on the amount of random and, respectively, Cohen, reals that are added over L.

Ikegami's theorem gives us characterizations for other proper forcing notions, such as the ones considered here. In order to state it, we need the definition of *quasi-generic real*: say that a real number x is $\mathbb{P}$-*quasi-generic over* M, a model of ZF, iff $x \notin B$, for all Borel set $B \in p^0 \cap M$ (i.e., Borel null sets *coded* in M).

Clearly, generic reals are quasi-generic reals. Moreover, these notions coincide when the forcing notion is ccc (so, Cohen and random generics are the same as quasi-generics). For non-ccc proper forcing notions, however, this might not be the case: rather, a Sacks-quasi-generic real over M is a new real over M; a Miller quasi-generic real over M is an unbounded real over M; a Laver quasi-generic real over M is a dominating real over M etc.

Fact 3.3.1 (Ikegami, 2006, Theorem 1.3)**.** *For $\mathbb{P}$ satisfying a stronger form of properness, and additional definability requirements (see Ikegami, 2006, Definitions 2.3 and 2.4):*

(a) every $\boldsymbol{\Delta}^1_2$ set of reals is $\mathbb{P}$-measurable if, and only if,

$$\{r \mid r \text{ is } \mathbb{P}\text{-quasi-generic over } L[x]\} \neq \emptyset,$$

for each $x \in \omega^\omega$; and

(b) every $\boldsymbol{\Sigma}^1_2$ set of reals is $\mathbb{P}$-measurable if, and only if,

$$\{r \mid r \text{ is } \mathbb{P}\text{-quasi-generic over } L[x]\} \in p^1,$$

for each $x \in \omega^\omega$; where p^1 simply is the set of complements of sets in p^0.

There are various implications between regularity properties (see, e.g., Brendle and Khomskii, 2012, Section 4, pg. 1350), as well as consistent separations.

Fact 3.3.2 (Brendle and Löwe, 1999)**.** *For $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{R}$[2]$\}$: every $\boldsymbol{\Delta}^1_2$ set is $\mathbb{P}$-measurable if, and only if, every $\boldsymbol{\Sigma}^1_2$ set is $\mathbb{P}$-measurable.*

[1]See Kanamori, 1995 for a historical overview on the topic

[2]$\mathbb{R}$ is *Mathias forcing* defined in Section 4.1.

$$
\begin{tikzcd}
 & & & & \mathbf{\Sigma}^1_2(\mathbb{S}) & & \\
 & & \mathbf{\Delta}^1_2(\mathbb{E}_0) \arrow[urr] & & & & \mathbf{\Sigma}^1_2(\mathbb{M}) \arrow[ull] \\
 & & & & \mathbf{\Delta}^1_2(\mathbb{SP}) \arrow[uu] & & \\
 & & \mathbf{\Delta}^1_2(\mathbb{V}) \arrow[uu] \arrow[urr] & & & & \mathbf{\Sigma}^1_2(\mathbb{L}) \arrow[uu] \arrow[ull] \\
\mathbf{\Delta}^1_2(\mathbb{G}_0) \arrow[urr] & & & & \mathbf{\Sigma}^1_2(\mathbb{R}) \arrow[ull] \arrow[urr] & & \\
 & & \forall x\ (\aleph_1^{L[x]} < \aleph_1) \arrow[ull] \arrow[urr] & & & &
\end{tikzcd}
$$

FIGURE 3.6: Regularity counterpart of diagram in 1.5.

Proof of Corollary 3.1.3. It follows from from Theorem 3.1.1 that Laver forcing does not add E_0-quasi-generic reals over any $L[x]$, for $x \in \omega^\omega$. Now Ikegami's theorem, together with , gives us the desired statement. $\square$

Chapter 4

Questions

4.1 Minimality

As said at the end of Section 1.2, the consistencies of the inequalities $\chi_B(G_0) < \mathrm{cov}(\mathcal{N})$ and $\chi_B(G_0) < \mathfrak{h}$ are open.

We need to first identify a Silver tree p with a partial function $\hat{p} : \omega \to 2$ with infinite codomain — i.e., $|\omega \setminus \mathrm{dom}(p)|$ is infinite.

> *MATHIAS FORCING.* A tree $p \subseteq 2^{<\omega}$ is a *Mathias tree* iff it is a Silver tree such that $\hat{p}^{-1}(\{0\})$ is finite.
>
> The *Mathias forcing*, $\mathbb{R}$, consists of Mathias trees ordered by direct inclusion.

> *RANDOM FORCING.* A tree $p \subseteq 2^{<\omega}$ is a *random tree* iff it is perfect such that $\mu([p * \sigma]) > 0$, for every $\sigma \in 2^{<\omega}$.
>
> The *random forcing*, $\mathbb{B}$, consists of random trees ordered by inclusion.

It is well-known that Mathias forcing increases $\mathfrak{h}$; and that the random forcing increases $\mathrm{cov}(\mathcal{N})$. Moreover, exactly the same argument presented in the proof of Lemma 2.2.2, item b, works for $\mathbb{R}$ to show that the Mathias real is contained in a compact G-independent set, if G is a closed graph without 4-cycles. As for the random real, we can say this about the graph G_0:

Let $p \in \mathbb{B}$; and recursively construct a fusion sequence $(p_n)_{n\in\omega}$ such that

(1) $([p_n * \sigma^\frown 0] \times [p_n * \sigma^\frown 1]) \cap G_0 = \emptyset$; and

(2) $\mu([p_n]) = \prod_{k=0}^{n} \left(1 - 1/2^{k+1}\right) \mu([p])$.

The recipe is basically keeping the tree intact above splitting nodes that are not in the sparse dense set D, and throwing half of the measure away from the tree above a splitting node in D.

This way, if $q = \bigcap_n p_n$, then

$$\mu([q]) = \prod_{n=0}^{\infty} \left(1 - 1/2^{k+1}\right) \mu([p]) = \Phi(1/2)\mu([p])$$

where $\Phi(x) = \prod_{n=0}^{\infty} \left(1 - x^{k+1}\right)$ is the *real-valued Euler function*, which converges to a value larger than 0 when $|x| < 1$, due to *Euler's pentagonal-number theorem* (see, e.g., Apostol, 1998, Theorem 14.3). It follows that $q \in \mathbb{B}$ and $[q]$ is G_0-independent.

However, since none of these forcing notions are minimal, we cannot use Lemma 2.2.4.

Question 1. *Is it consistent with ZFC that $\chi_B(G_0) < \mathrm{cov}(\mathcal{N})$? What about $\chi_B(G_0) < \mathfrak{h}$?*

Another interesting problem on minimality is whether the forcing $\mathbb{G}_0$ is minimal. It is unclear whether the methods from Grigorieff, 1971 may be adapted to encompass this forcing notion.

In fact, the proof of Fact 2.2.3 in Grigorieff, 1971 shows that $\mathbb{V}$ has the stronger 2-*localization property*:

> *2-LOCALIZATION PROPERTY.* A forcing notion $\mathbb{P}$ has the 2-*localization property* if for every $\mathbb{P}$-name $\dot{x}$ for an element of ω^ω is contained in a ground-model *binary tree* — i.e., a tree on ω^ω such that every node has at most two immediate successors.

This property implies the *Sacks property*, the conjunction between ω^ω-boundedness and the Laver property. It is known that the Sacks property implies that $\mathrm{cof}(\mathcal{N})$ is preserved for countable support iterations.

Question 2. *Does* $\mathbb{G}_0$ *add reals of minimal degree? Does it have the* 2-*localization property?*

4.2 Other Borel graphs

In this section we introduce some interesting graphs that, in some sense, do not fall into the categories considered in this work. They are the *geometric graphs*, the *Turing graph*, and the *Li-Yorke graphs*.

Let X the space of vertices be either $\mathbb{R}^n$, the euclidean space of dimension $n \geq 1$; or S^1, the unit circle in $\mathbb{R}^2$. Let $D = (\varepsilon_n)_{n\in\omega}$ be a sequence of positive real numbers.

> *GEOMETRIC GRAPHS.* The *D-geometric graph*, G_D, is defined by
>
> $$G_D = \left\{(x,y) \in X^2 \mid \exists n \in \omega(d(x,y) = \varepsilon_n)\right\},$$
>
> where d is the euclidean distance in $\mathbb{R}^n$; or the arc-length distance in S^1.

It is easy to see that the geometric graph is an F_σ graph. If $X = \mathbb{R}^2$, and $(\varepsilon_n)_{n\in\omega}$ is constant and equal to 1, then G_D is also known as the *unit distance graph on the plane*. This graph has chromatic number between 5 and 7[1], and the 7-coloring of G_D is trivially Borel (see figure below).

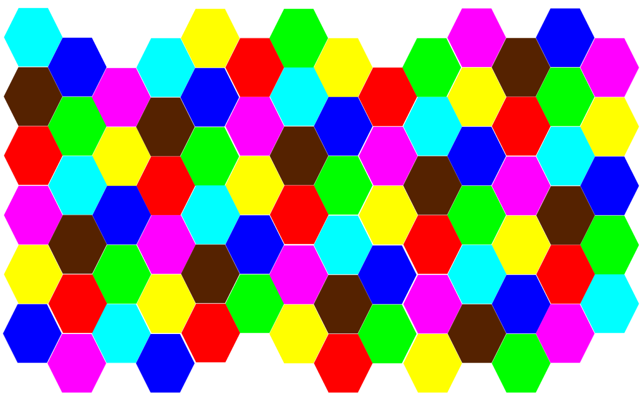

FIGURE 4.1: Tile the plane with hexagons of diameter less than 1 and bigger than 0.76. Proceeding by coloring these hexagons as in this pattern, we see that 7 colors are sufficient to color the unit distance graph on the plane.

[1] Moser and Moser, 1961 showed it is not 3; and De Grey, 2018 showed, with a computer-assisted proof, that it is not 4.

The exact value of $\chi(G_D)$ and $\chi_B(G_D)$ are, however, still open and this is known as the *Hadwiger-Nelson Problem*.

We have a condition that ensures that $\chi_B(G_D)$ is uncountable:

Fact 4.2.1 (Szekely, 1984, Theorem 2.1)**.** *If $D = (\varepsilon)_{n\in\omega}$ converges to 0, then $\chi_\mu(G_D) \geq \mathrm{cov}(\mathcal{N})$. Moreover, if $X = \mathbb{R}$ and the coordinates of D consist of linearly independent reals over the rationals, then $\chi(G_D) = 2$ — i.e., G_D is bipartite.*

Setting $\varepsilon_n = (1/\pi)^n$ gives us G_D satisfying both conditions of the above fact.

From Fact 4.2.1, together with the G_0-dichotomy, we have that

$$\chi_B(G_D) \geq \max\{\chi_B(G_0), \mathrm{cov}(\mathcal{N})\}.$$

Note that G_D is locally countable iff $X \in \{\mathbb{R}, S^1\}$.

For the next example, we need to recall the notion of *Turing reducibility*:

Let M be an oracle Turing machine and $x, y \in 2^\omega$. We say that x *is Turing-reducible to y via M* if M equipped with the oracle y decides x[2].

Consider the partial function f_M that maps every y to the unique x such that $x \leq_M y$ (if such an x exists). It may happen that M equipped with the oracle y does not halt on every input — i.e., f_M can be partial. However, the domain of f_M is G_δ since for every input n the set $\{y \in 2^\omega \mid M \text{ with oracle } y \text{ halts for } n\}$ is open, because every computation is finite and thus only uses some finite part of the oracle. For the same reason (i.e., finiteness of computations) f_M is continuous (see, e.g., Abraham and Geschke, 2004, Section 6). If $f_M = 2^\omega$ we say that M is a *total Turing machine*.

> *THE (TOTAL) TURING GRAPH.* The *Turing graph*, G_T, is the graph defined on a G_δ subset of 2^ω by
>
> $$G_T = \left\{(x, y) \in (2^\omega)^2 \mid \exists M(x \leq_M y \vee y \leq_M x\right\}.$$
>
> If we restrict ourselves only to total Turing machines, we have the *total Turing graph*, G_T^{total}:
>
> $$G_T^{\text{total}} = \left\{(x, y) \in (2^\omega)^2 \mid \exists M \text{ total } (x \leq_M y \vee y \leq_M x\right\}.$$

Fact 4.2.2 (Geschke, communicated)**.**

(a) The Turing graph G_T is a $G_{\delta\sigma}$ graph, which is not G_δ; and the total Turing graph G_T^{total}, is an F_σ graph, which is not closed. Moreover:

(b) Both graphs G_T and G_T^{total} have cliques of size $\aleph_1$, but no perfect cliques.

Like the geometric graphs defined on $\mathbf{R}^n$, for $n \geq 2$, with distances converging to 0, the (total) Turing graph is an example of non-locally countable graph, with complexity higher than "closed".

The following fact shows that item b of Fact 4.2.2 may not be possible for graphs of different complexity:

Fact 4.2.3 (Kubiś, 2003)**.** *If G is a G_δ graph on a Polish space X, then G has an uncountable clique if, and only if, it has a perfect clique.*

Sometimes the existence of uncountable cliques implies the existence of perfect cliques even for $G_{\delta\sigma}$ graphs, as in the case with Li-Yorke graphs:

[2]Here we are identifying the elements of 2^ω with subsets of ω.

Let X be a compact Polish space, with metric d_X; and $f : X \to X$ be a continuous function. We say that (X, f) is a *topological dynamical system*. For $x, y \in X$, say that they are *proximal* if $\liminf d_X(f^n(x), f^n(y)) = 0$; and that x and y are *asymptotic* if $\limsup d_X(f^n(x), f^n(y)) = 0$.

> *LI-YORKE GRAPHS.* The (X, f)*-Lee-Yorke graph*, G_f^{LY}, is the graph defined on X by
>
> $$G_f^{LY} = \{(x, y) \in X^2 \mid x \text{ and } y \text{ are proximal but not asymptotic}\}.$$

Li-Yorke graphs are essentially $G_{\delta\sigma}$ graphs. Say that (X, f) is *chaotic* if G_f^{LY} has an uncountable clique [3].

It has always been the case that proofs of chaos for dynamical systems yielded perfect cliques in the corresponding Li-Yorke graphs. One example is the construction of Blanchard et al., 2002 of perfect cliques for Li-Yorke graphs of systems of positive topological entropy.

Fact 4.2.4 (Geschke, Grebík, and Miller, 2020)**.** *Li-Yorke graphs of chaotic dynamical systems have perfect cliques.*

Question 3. *Let $D = (\varepsilon)_{n\in\omega}$ be a sequence of positive real numbers and G_D be the geometric graph on $\mathbb{R}^n$, for $n \geq 2$; and (X, f) be a non-chaotic dynamical system. What are the values of $\chi_B(G_D)$, $\chi_B(G_T)$ and $\chi_B(G_f^{LY})$ in the Sacks model?*

Assume that the geometric graph, G_D, is defined either on $\mathbb{R}$ or S^1. In this case, G_D is F_σ locally countable and $\chi_B(G_D) \leq |2^{\aleph_0} \cap V|$ in the Sacks model. Now since Theorem 3.1.1 is proved only for totally disconnected compact spaces, we can use the real line $\mathbb{R}$ to test the role of compactness and connectedness. In order to test the role of connectedness only, one can use S^1 instead.

Question 4. *What is the value of $\chi_B(G_D)$ in the Laver model? What about $\chi_B(G_f^{LY})$ for non-chaotic systems (X, f)?*

It would also be interesting to find models that separate the Borel chromatic numbers of the three graphs:

Question 5. *What are the consistently possible strict inequalities between $\chi_B(G_D)$, $\chi_B(G_T)$ and $\chi_B(G_f^{LY})$?*

[3] For example, Li and Yorke, 2004 showed that every dynamical system on the unit interval with a point of period three is chaotic.

Appendix A

Summary

Borel chromatic number in models of set theory, by Michel GASPAR.

In this work we study the behavior of definable graphs on Polish spaces in various models of set theory. More specifically, we investigate their *Borel chromatic numbers*, one of the so-called *cardinal characteristics of the continuum*.

We show that the statement "the Borel chromatic number of a graph is bounded by the continuum of the ground model" may be forced, depending on (1) the topology of the space of vertices; (2) the complexity of the graph (e.g., analytic, closed etc); and on (3) some suitable notion of "smallness" which may be satisfied for the graph (e.g., local countability, inexistence of perfect cliques etc). For that, we use countable support iterations of Axiom A forcing notions.

Furthermore, from the results of Chapter 3 we are also able to solve a relatively old problem about *regularity properties*, showing that *Silver* and *Laver measurability* may be separated on the second level of the projective hierarchy.

The content of Chapter 2 is a joint work with Stefan Geschke; and the content of Chapter 3 is a joint work with Raiean Banerjee.

Appendix B

Zusammenfassung

Borel-chromatischen Zahlen in Modellen der Mengenlehre, von Michel GASPAR.

In dieser Arbeit untersuchen wir das Verhalten von definierbaren Graphen auf polnischen Räumen in verschiedenen Modellen der Mengenlehre. Genauer gesagt untersuchen wir ihre Borel-chromatischen Zahlen, eine der so genannten *Kardinalcharakteristiken des Kontinuums*.

Wir zeigen, dass die Aussage „die Borel-chromatische Zahl eines Graphen ist durch das Kontinuum des Grundmodells begrenzt" mit der Forcing-Methode erzwungen werden kann, abhängig von (1) der Topologie des Raumes der Ecken; (2) der Komplexität des Graphen (z.B. analytisch, abgeschlossen usw.); und von (3) einem geeigneten Begriff der „Schmalheit", der für den Graphen erfüllt sein kann (z.B. lokale Abzählbarkeit, das Nichtvorhandensein von perfekten Cliquen usw.). Hierfür verwenden wir eine Iteration von Forcings, die Axiom A erfüllen, mit abzählbarem Träger.

Darüber hinaus lösen wir in Kapitel 3 ein relativ altes Problem über *Regularitätseigenschaften*, indem wir zeigen, dass *Silver-* und *Laver- Messbarkeit* auf der zweiten Stufe der projektiven Hierarchie voneinander getrennt werden können.

Der Inhalt von Kapitel 2 ist eine gemeinsame Arbeit mit Stefan Geschke und der Inhalt von Kapitel 3 ist eine gemeinsame Arbeit mit Raiean Banerjee.

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