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Dissertation

Contributions to port-Hamiltonian systems

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Abstract

In the present thesis we consider dynamical systems which arise from the energy-based modelling of physical systems, namely, so-called port-Hamiltonian systems. Different approaches for this abstraction process exist, resulting in slightly different port-Hamiltonian formulations. In the case of the linear finite-dimensional modelling, we identify two main branches: the geometric approach pioneered by van der Schaft and Maschke [vdSM18] and the linear algebraic approach promoted by Mehl, Mehrmann and Wojtylak [MMW18].

Inspired by these frameworks, we present another view by using the theory of *linear relations*. We show that this allows to elaborate the differences and mutualities of the geometric and linear algebraic views, and we introduce a class of dynamical systems which comprises these two approaches. There is a natural way to associated differential-algebraic equations (DAEs) to these systems, and we study the properties of their matrix pencils.

Moreover, we give sufficient conditions guaranteeing stability of the different port-Hamiltonian formulations by means of generalized Lyapunov inequality for DAEs. We also use the solution of such inequalities to rewrite stable DAEs as port-Hamiltonian systems on the subspace where the solutions evolve. Further, for stabilizable DAEs we construct solutions of generalized algebraic Bernoulli equations which can then be used to rewrite these systems as port-Hamiltonian systems by introducing a suitable output.

For the illustration of this energy-based modelling approach, we consider nonlinear electrical circuits, for which a systematic approach of port-Hamiltonian modelling is established. Each circuit component is modelled as an individual port-Hamiltonian system. The overall circuit model is then derived by considering a port-Hamiltonian interconnection of the components. We further compare this modelling approach with standard formulations of nonlinear electrical circuits. The nonlinearities encompassed in this port-Hamiltonian modelling may produce implicit equations in the dynamics describing the electrical circuit. We briefly discuss the resolution of such implicit equations using a result on the existence of global implicit functions we develop.

Besides implicit equations, we also study different types of algebraic constraints arising in the port-Hamiltonian formalism based on linear relations. We show how to convert

these types of constraints to one another. This conversion process is then applied to link port-Hamiltonian systems to recently established pencils whose coefficients have positive semidefinite Hermitian part.

Zusammenfassung

In der vorliegenden Dissertation betrachten wir dynamische Systeme, die aus der energiebasierten Modellierung physikalischer Systeme entstehen, sogenannte port-Hamiltonsche Systeme. Es existieren verschiedene Ansätze für diesen Abstraktionsprozess, was zu leicht unterschiedlichen port-Hamiltonischen Formulierungen führt. Im Fall der linearen endlichdimensionalen Modellierung identifizieren wir zwei Hauptzweige: den geometrischen Ansatz, den von VAN DER SCHAFT und MASCHKE [vdSM18] entwickelt wurde, und der von MEHL, MEHRMANN und WOJTYLAK [MMW18] untersuchten linear-algebraischen Ansatz.

Von diesen Rahmenwerken inspiriert präsentieren wir eine weitere Sichtweise, indem wir die Theorie der *linearen Relationen* verwenden. Wir zeigen, dass dies erlaubt, die Unterschiede und Gemeinsamkeiten der geometrischen und der linear-algebraischen Sichtweise herauszuarbeiten, und wir führen eine Klasse dynamischer Systeme ein, die diese beiden Ansätze umfasst. Es gibt einen natürlichen Weg, diesen Systemen differential-algebraische Gleichungen (DAEs) zuzuordnen, und wir untersuchen die Eigenschaften ihrer Matrixbüschel.

Darüber hinaus geben wir hinreichende Bedingungen an, die die Stabilität der verschiedenen port-Hamiltonschen Formulierungen mittels verallgemeinerter Lyapunov-Ungleichung für DAEs garantieren. Wir verwenden die Lösung solcher Ungleichungen auch, um stabile DAEs als port-Hamiltonsche Systeme auf dem Unterraum umzuschreiben, in dem sich die Lösungen entwickeln. Weiterhin konstruieren wir für stabilisierbare DAEs Lösungen von verallgemeinerten algebraischen Bernoulli-Gleichungen, die anschließend verwendet werden können, um diese Systeme als port-Hamiltonsche Systeme umzuschreiben, indem wir einen geeigneten Ausgang einführen.

Zur Veranschaulichung dieses energiebasierten Modellierungsansatzes betrachten wir nichtlineare elektrische Schaltungen, für die sich ein systematischer Ansatz der port-Hamiltonschen Modellierung etabliert hat. Jede Schaltungskomponente wird als individuelles port-Hamiltonisches System modelliert. Das Gesamtschaltungsmodell wird dann unter Berücksichtigung einer port-Hamiltonischen Verbindung der Komponenten hergeleitet. Wir vergleichen diesen Modellierungsansatz weiter mit Standardformulierungen nichtlinearer elektrischer Schaltungen. Die in dieser port-Hamiltonischen Modellierung enthaltenen Nichtlinearitäten können implizite Gleichungen in der Dynamik erzeugen, welche die elektrische Schaltung beschreibt. Wir diskutieren kurz die

Auflösung solcher impliziter Gleichungen unter Verwendung eines Ergebnisses über die Existenz globaler impliziter Funktionen, welches wir herleiten.

Neben impliziten Gleichungen untersuchen wir auch verschiedene Arten von algebraischen Nebenbedingungen in dem auf linearen Relationen basierenden port-Hamiltonschen Formalismus. Wir zeigen, wie diese Arten von Nebenbedingungen ineinander überführt werden können. Dieser Konvertierungsprozess wird dann dazu verwendet, port-Hamiltonsche Systeme mit den kürzlich etablierten Büscheln mit Koeffizienten, die einen positiv semidefiniten Hermiteschen Anteil besitzen, zu verknüpfen.

Résumé

Dans la présente thèse, nous considérons des systèmes dynamiques issus de la modélisation énergétique des systèmes physiques, à savoir les systèmes dits hamiltoniens à ports. Différentes approches pour ce processus d'abstraction existent, résultant en des formulations hamiltoniennes à ports légèrement différentes. Dans le cas de la modélisation linéaire en dimension finie, nous identifions deux branches principales : l'approche géométrique initiée par van der Schaft et Maschke [vdSM18] et l'approche algébrique linéaire promue par Mehl, Mehrmann et Wojtylak [MMW18].

Inspirés par ces cadres, nous présentons un autre angle d'approche en utilisant la théorie des *relations linéaires*. Nous montrons que cela permet d'élaborer les différences et mutualités des approches géométriques et algébriques linéaires, et nous introduisons une classe de systèmes dynamiques qui comprend ces deux approches. Il existe une manière naturelle d'associer des équations différentielles-algébriques (EDA) à ces systèmes, et nous étudions les propriétés de leurs faisceaux matriciels.

De plus, nous donnons des conditions suffisantes garantissant la stabilité des différentes formulations hamiltoniennes à ports au moyen de l'inégalité de Lyapunov généralisée pour les EDA. Nous utilisons également la solution de telles inégalités pour réécrire des EDA stables en tant que systèmes hamiltoniens à ports sur le sous-espace où les solutions évoluent. En outre, pour les EDA stabilisables, nous construisons des solutions d'équations de Bernoulli algébriques généralisées. Ces solutions peuvent ensuite être utilisées pour réécrire ces systèmes en tant que systèmes hamiltoniens à ports en introduisant une sortie appropriée.

Pour illustrer cette approche de modélisation énergétique, nous considérons des circuits électriques non linéaires pour lesquels une approche systématique de modélisation hamiltonienne à ports est établie. Chaque composant de circuit est modélisé comme un système hamiltonien à port individuel. Le modèle de circuit global est ensuite dérivé en considérant une interconnexion hamiltonienne à ports des composants. En outre, nous comparons cette approche de modélisation avec des formulations standard de circuits électriques non linéaires. Les non-linéarités englobées dans cette modélisation port-hamiltonienne peuvent produire des équations implicites dans la dynamique décrivant le circuit électrique. Nous discutons brièvement de la résolution de telles équations implicites en utilisant un résultat sur l'existence de fonctions implicites globales que

nous développons.

Un autre aspect de cette thèse concerne l'étude de différentes contraintes algébriques issues du formalisme port-hamiltonien basé sur les relations linéaires. Nous montrons comment convertir ces types de contraintes les uns aux autres. Ce processus de conversion est ensuite utilisé pour relier les systèmes port-hamiltoniens aux récemment étudiés faisceaux aux coefficients à part hermitienne semi-définie positive.

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Contents

Introduction

When HAMILTON elaborated what is today known as the *Hamiltonian formalism* in his seminal paper [Ham35], he considered physical systems 'not disturbed by any foreign force'. This gave rise to the famous equations [Ham35, Eq. A] reading

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\,\eta &= \frac{\partial H}{\partial \varpi}, \\ \tfrac{\mathrm{d}}{\mathrm{dt}}\,\varpi &= -\frac{\partial H}{\partial \eta}, \end{aligned}$$

where H is known as the *Hamiltonian* typically representing the energy of the considered system [Arn89]. Such a foreign force can be a control exerted on the system, and it is only a few decades later with the work [Max68] of MAXWELL that topics related to control were rigorously investigated, giving birth to the field of *control theory*, see [Kan16] and the references therein. Besides a designed control, a foreign force can simply result from the interaction with the environment, that is, due to a different physical system. Nowadays, it is essential to consider both types of foreign forces in applications, such as, for instance, mechanical and electrical systems, where a myriad of subsystems are interconnected and controlled. The port-Hamiltonian modelling philosophy [Bre09] arose as an answer to the increasing complexity found in these applications. Identifying the energy as common denominator of all physical systems and taking the Hamiltonian formalism with its Hamiltonian H representing the energy as a basis, port-Hamiltonian modelling provides a framework allowing for a systematic port-based network modelling of complex systems from various physical domains, where not only the interconnections are based on energy considerations, but controls too [OvdSM$^+$01].

The dynamical systems resulting from the port-Hamiltonian modelling approach are accordingly called *port-Hamiltonian systems* and port-Hamiltonian system models encompass a very large class of nonlinear physical systems [vdSJ14; vdSch17]. On the flip side, since only a certain modelling philosophy is prescribed, there exist different port-Hamiltonian systems frameworks. In the past decades, this modelling approach has gained particularly increased attention from different communities, such as geometric mechanics and mathematical systems theory, from which different definitions of port-Hamiltonian systems emerged, see [JZ12; vdSJ14; vdSch13; BMX$^+$18] for an overview. Surprisingly, little effort has been made so far to thoroughly compare the different resulting dynamical systems.

Content overview

The aim of this thesis can be resumed in the following three points:

- Comparing the linear algebraic port-Hamiltonian approach by Mehl, Mehrmann and Wojtylak [MMW18] with the geometric approach of van der Schaft and Maschke [vdSM18].
- Analyzing the differential-algebraic equations and their corresponding matrix pencil arising from linear port-Hamiltonian modelling approaches.
- Illustrating different aspects of the port-Hamiltonian modelling philosophy.

In Chap. 1, we lay the mathematical foundations for attaining the three abovementioned goals. Afterwards in Chap. 2, we directly introduce the different port-Hamiltonian formulations we aim to compare. In order to compare the aforementioned geometric and linear algebraic approaches, we introduce a third approach based on *linear relations*, a concept which has been treated in several textbooks [BHdS20; Cro98]. The comparison of the introduced formulations is first performed indirectly. More precisely, we show how each formulation extends Hamilton's original equations and give interpretations of the energy of the corresponding system models. Further, we describe the power-conserving interconnection of two systems of a given formulation. We continue by a side-by-side comparison of the different approaches under the assumption that all systems are linear. To enable an even finer comparison between the different port-Hamiltonian frameworks, we restrict us to the case where the systems are 'not disturbed by any foreign force' and show that linear relations formulation of port-Hamiltonian systems can be regarded as the *least common multiple* of the linear algebraic port-Hamiltonian approach by Mehl, Mehrmann and Wojtylak with the geometric approach of van der Schaft and Maschke. In particular, we make use of three facts:

(i) the geometric concept of a Dirac structure translates to the notion of a *skew-adjoint linear relation* in the language of linear relations,

(ii) Lagrangian subspaces correspond to *self-adjoint linear relations*, and

(iii) dissipative matrices can be generalized to *dissipative linear relations*.

With this insight, Chap. 3 is devoted to the analysis of the matrix pencils associated to the differential-algebraic equations arising from the linear relations approach. We are particularly interested in their regularity and the shape of their Kronecker canonical form. This analysis is deepened in Chap. 4, where we examine stability properties of the differential-algebraic equations resulting form the different port-Hamiltonian formulations. The stability analysis relies on Lyapunov inequalities developed in App. A. We also link stable and stabilizable differential-algebraic equations to port-Hamiltonian systems using the aforementioned Lyapunov inequalities and generalized algebraic Bernoulli equations. Chap. 5 is concerned with the same systems studied in Chap. 3 and analyzes algebraic equations arising in these systems. We present certain methods of conversion of these algebraic equations, which enable us to link the linear relations port-Hamiltonian systems to differential-algebraic equations whose pencils have positive semidefinite Hermitian part coefficients [MMW22].

Port-Hamiltonian modelling can be seen as an object-oriented approach. We illustrate this fact with Chap. 6 by presenting a systematic method for the lumped parameter modelling of nonlinear electrical circuits. We compare the resulting dynamical system with well-known formulations of nonlinear electrical circuits like the (charge/flux-oriented) *modified nodal analysis* and the *modified loop analysis*. The nonlinearities we incorporate in this modelling induce implicit equations we discuss by means of a result on global implicit function we develop in App. B.

In the conclusion, we recapitulate our findings and give directions for further research.

Previously published results and joint work

Parts of the present thesis have already been published or submitted for publication as indicated in the following table.

Content	contained in
Chap. 1	[GH21; GHR21; GHR$^+$21] with minor additions.
Chap. 2	new in this form but builds on well-established port-Hamiltonian concepts that were also presented in [GH21; GHR21; GHR$^+$21]
Chap. 3	[GHR21]
Chap. 4	[GH21]
Chap. 5	new
Chap. 6	[GHR$^+$21] with the addition of Sec. 6.4 containing an example from [BH22]
App. A	the author's bachelor's thesis at the Universität Hamburg under the supervision of Timo Reis and Thomas Berger
App. B	[BH22]

Chapter 1

Mathematical toolbox

In this chapter we present all the mathematical tools required to follow the argumentation in the subsequent chapters. The general notation we employ, which is summarized in the *List of notations*, is presented first. Next, we provide insights on matrix pencils and their corresponding differential-algebraic equations. Additionally, we introduce basic definitions and results from the fields of linear relations and differential geometry. Together, they form the framework for the formulation of the port-Hamiltonian systems studied in this thesis.

1.1 Notational preliminaries

General sets

By $\mathbb{N}_0$ we denote the set of natural numbers, while $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ denotes the set of natural numbers excluding zero and $\mathbb{K}$ denotes either the set of real or complex numbers $\mathbb{R}$, $\mathbb{C}$. For the open left complex half-plane we use the notation $\mathbb{C}_-$ while $\mathbb{C}_+$ denotes the open right-half complex plane. For a complex number $z \in \mathbb{C}$ we write $\operatorname{Re} z$ for its real part, $\operatorname{Im} z$ for its imaginary part, $\overline{z}$ for its complex conjugate and $|z|$ for its absolute value with $\imath$ designing the imaginary unit. The closure of a subset S of a topological space is denoted by $\overline{S}$. A ring which is often used in this thesis is the ring of polynomials over $\mathbb{K}$, $\mathbb{K}[s]$ with its quotient field $\mathbb{K}(s)$. For a family of sets

$\{S_i\}_{i\in I}$ we write

$$\prod_{i\in I} S_i = \left\{ f : I \to \bigcup_{i\in I} S_i \;\middle|\; \forall i \in I : f(i) \in S_i \right\}$$

for their Cartesian product. In particular, given sets $S_1, \ldots, S_n$, we write

$$\textstyle\prod_{i=1}^{n} S_i = S_1 \times \ldots \times S_n.$$

For finitely many vector spaces $V_1, \ldots, V_n$ the exterior direct sum is also written as

$$\bigoplus_{i=1}^{n} V_i = V_1 \oplus \ldots \oplus V_n = V_1 \times \ldots \times V_n,$$

with $\mathbb{K}^n = \bigoplus_{i=1}^{n} \mathbb{K}$. For an inner direct sum we write $V_1 \widehat{\oplus} \ldots \widehat{\oplus} V_n$. If the latter sum is additionally orthogonal we write $V_1 \ominus \ldots \ominus V_n$. If we refer to a componentwise sum of finitely many subsets $S_1, \ldots, S_n$ of a vector space S we write

$$S_1 \widehat{+} \ldots \widehat{+} S_n = \left\{ x_1 + \ldots + x_n \;\middle|\; \{x_i\}_{i=1,\ldots n} \in \prod_{i=1}^{n} S_i \right\}.$$

If $V, W \subset \mathbb{K}^n$ fulfill $V \subseteq W$, the *orthogonal minus* is given by $W \widehat{\ominus} V \coloneqq W \cap V^{\perp}$.

Linear maps

Given a ring R and $n, m \in \mathbb{N}$, the set of $m \times n$ matrices with entries in R is denoted by $R^{m\times n}$. For vector spaces $V_1, \ldots, V_k, W$, the set of multilinear maps $f : V_1 \times \ldots \times V_k \to W$ is denotes by $L(V_1, \ldots, V_k; W)$ and we identify $\mathbb{K}^{m\times n} = L(\mathbb{K}^n; \mathbb{K}^m) = \{L : \mathbb{K}^n \to \mathbb{K}^m | L \text{ is a homomorphism}\}$. Note that we allow for m, n to be zero. In this case, we find matrices which are elements of $\mathbb{K}^{0\times q} = \{L : \mathbb{K}^q \to \mathbb{K}^0 | L \text{ is a homomorphism}\}$ or $\mathbb{K}^{q\times 0} = \{L : \mathbb{K}^0 \to \mathbb{K}^q | L \text{ is a homomorphism}\}$ for some $q \in \mathbb{N}_0$. $\mathbb{K}^{0\times q}$ and $\mathbb{K}^{q\times 0}$ have only one element for which we respectively write $0^{0\times q}$ and $0^{q\times 0}$. It is clear that $0^{0\times q}$ and $0^{q\times 0}$ do not have a matrix representation, as $\mathbb{K}^0$ is generated by the empty set, i.e., it has the empty set as basis. The zero elements in $\mathbb{K}^{n\times n}$ and $\mathbb{K}^{m\times n}$ are denoted by 0_n and $0_{m,n}$, respectively and the symbol for the $n \times n$ identity matrix is I_n.
Let $V_1, \ldots V_k$, $W_1, \ldots W_l$ be vector spaces and set $V = \bigoplus_{i=1}^{n} V_i$ and $W = \bigoplus_{i=1}^{m} W_i$. For $A_{ij} \in L(V_j; W_i)$ with $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we introduce the corresponding block operator $A \in L(V; W)$ as

$$A \coloneqq \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{l1} & \cdots & A_{lk} \end{bmatrix} : V \to W$$

$$(x_1,\dots x_k)\mapsto\left(\sum_{i=1}^k A_{1i}x_i,\dots,\sum_{i=1}^k A_{li}x_i\right).$$

For $M_i\in\mathbb{K}^{r_i\times s_i},\ r_i,s_i\in\mathbb{N}_0,\ i=1,\dots l$, the block diagonal operator is

$$\begin{aligned}\operatorname{diag}(M_1,...,M_l):\mathbb{K}^{s_1}\times\ldots\times\mathbb{K}^{s_l}&\to\mathbb{K}^{r_1}\times\ldots\times\mathbb{K}^{r_l},\\ (v_1,...,v_l)&\mapsto(M_1v_1,...,M_lv_l).\end{aligned}$$

If $s_i,r_i\neq 0$ for all $i=1,\dots l$, then

$$\operatorname{diag}(M_1,\dots,M_l)=\begin{bmatrix}M_1&&\\&\ddots&\\&&M_l\end{bmatrix}.$$

In the case that r_i or s_i is 0 for some $i\in\{1,\dots,l\}$ we can find a matrix representation through

$$\begin{aligned}&\operatorname{diag}(0^{0\times1},0^{0\times q})=0^{0\times(q+1)}, &&\operatorname{diag}(0^{1\times0},0^{q\times0})=0^{(q+1)\times0},\\ &\operatorname{diag}(0^{0\times q},M)=\begin{bmatrix}0_{r,q}&M\end{bmatrix}, &&\operatorname{diag}(M,0^{0\times q})=\begin{bmatrix}M&0_{r,q}\end{bmatrix},\\ &\operatorname{diag}(0^{q\times0},M)=\begin{bmatrix}0_{q,s}\\M\end{bmatrix}, &&\operatorname{diag}(M,0^{q\times0})=\begin{bmatrix}M\\0_{q,s}\end{bmatrix},\\ &\operatorname{diag}(0^{q\times0},0^{0\times q})=0_q, &&\operatorname{diag}(0^{0\times q},0^{q\times0})=0_q,\end{aligned}$$

for a matrix $M\in\mathbb{K}^{r\times s}$ with $r,s\in\mathbb{N}$ and $q\in\mathbb{N}_0$.
For $A\in\mathbb{K}^{m\times n}$, $A^\top$ denotes its transpose, $A^*=\overline{A^\top}$ its Hermitian, $A^\dagger$ its Moore–Penrose inverse [Pen55], and A^{-1} its inverse for the case that $n=m$. Further, we write $\operatorname{im}A=\operatorname{ran}A$ for the image/range of A. The set of invertible $n\times n$ matrices over $\mathbb{K}$ is abbreviated by $\mathbf{Gl}_n(\mathbb{K})$. For the set of eigenvalues of a matrix $A\in\mathbb{K}^{n\times n}$, i.e., its spectrum, we write $\sigma(A)\subset\mathbb{C}$. When writing $\|A\|$ without specifying the norm, we always mean the operator norm, i.e.,

$$\|A\|=\sup_{\substack{x\in\mathbb{K}^n\\ \|x\|\le1}}\|Ax\|,$$

and for a vector $x\in\mathbb{K}^n$, $\|x\|$ as in the formula above will always denote the Euclidean norm when not specified otherwise. Throughout this thesis, we assume that $\mathbb{K}^n$ is equipped with the standard scalar product $\langle\cdot,\cdot\rangle:(x,y)\mapsto y^*x$. For square matrices $M,N\in\mathbb{K}^{n\times n}$, we write $M>N$ if $M-N$ is positive definite and $M\geq N$ if $M-N$

is positive semi-definite. If $M + M^* \leq 0$ we say that M is *dissipative*. Further, given a subspace $\mathcal{L} \subseteq \mathbb{K}^n$ and assuming that M and N are symmetric, we write

$$\begin{aligned} M \geq_{\mathcal{L}} M \quad &:\Leftrightarrow \quad x^T M x \geq x^T N x \quad \forall\, x \in \mathcal{L}, \\ M >_{\mathcal{L}} N \quad &:\Leftrightarrow \quad x^T M x > x^T N x \quad \forall\, x \in \mathcal{L} \setminus \{0\}, \end{aligned}$$

which corresponds to the notation of [RV19]. Now, if we are given matrices $M, N \in \mathbb{K}^{m \times n}$ and a subspace $\mathcal{L} \subseteq \mathbb{K}^n$ we write

$$M =_{\mathcal{L}} N \quad :\Leftrightarrow \quad Mx = Nx \quad \forall\, x \in \mathcal{L}.$$

Note that this notation significantly differs from the one used in [RV19] since we do not check $x^* M x = x^* N x$ for all $x \in \mathcal{L}$ nor do we impose symmetry on M and N. Moreover, these conditions might also make no sense since the matrices do not need to be square. If not specified otherwise, P_X denotes the *orthogonal projector* onto a given subspace $X \subset \mathbb{K}^n$. For spaces $Y_1, Y_2, Y_3 \subset \mathbb{K}^n$ with $Y_1 \subset Y_2$ and a linear operator $M : Y_2 \to Y_3$, $M\restriction_{Y_1}$ denotes the restriction of M to the space Y_1 and we write $M(Y_1)$ for the range of $M\restriction_{Y_1}$.

Function spaces

Given any two functions $f : A \to B$, $g : A \to C$ for arbitrary sets A, B, C, we identify the function $h : A \to B \times C$ defined by $x \mapsto (f(x), g(x))$ with the pair (f, g). For open subsets $U \subset \mathbb{K}^n$, $V \subset \mathbb{K}^m$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ we introduce

$$C^k(U, V) = \{\ f : U \to V \ \mid f \text{ is } k\text{-times continuously differentiable } \}$$

and

$$C(X, Y) = \{\ f : X \to Y \ \mid f \text{ is continuous } \}$$

for arbitrary metric spaces X, Y. Next, for $p \in [1, \infty)$, some interval $\mathcal{I} \subset \mathbb{R}$ and a subspace $\mathcal{V} \subset \mathbb{K}^n$ we introduce the *Lebesgue spaces*

$$\begin{aligned} \mathcal{L}^p(\mathcal{I}, \mathcal{V}) &= \left\{ f : \mathcal{I} \to \mathcal{V} \;\middle|\; f \text{ is measurable and } \int_{\mathcal{I}} \|f(\tau)\|^p \, \mathrm{d}\tau < \infty \right\}, \\ \mathcal{L}^p_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) &= \Bigg\{ f : \mathcal{I} \to \mathcal{V} \;\Bigg|\; f \text{ is measurable and } \int_K \|f(\tau)\|^p \, \mathrm{d}\tau < \infty \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all compact } K \subset \mathcal{I} \Bigg\}, \end{aligned}$$

and denoting by O the set of all measurable functions $f : \mathcal{I} \to \mathcal{V}$ such that $f(t) = 0$ for almost all $t \in \mathcal{I}$ with respect to the Lebesgue measure, we introduce the quotient spaces

$$L^p(\mathcal{I}, \mathcal{V}) = \left\{ \ \{f\} \widehat{+} O \ \middle| \ f \in \mathcal{L}^p(\mathcal{I}, \mathcal{V}) \ \right\},$$
$$L^p_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) = \left\{ \ \{f\} \widehat{+} O \ \middle| \ f \in \mathcal{L}^p_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) \ \right\}.$$

With the (well-defined) *p-norm* $\| \cdot \|_{L^p(\mathcal{I},\mathcal{V})}$ for $p \in [1, \infty)$ or simply $\| \cdot \|_p$ for $p \in [1, \infty)$ defined as

$$\|\{f\} \widehat{+} O\|_{L^p(\mathcal{I},\mathcal{V})} = \left(\int_{\mathcal{I}} \|f(\tau)\|^p \, \mathrm{d}\tau \right)^{\frac{1}{p}}$$

$L^p(\mathcal{I}, \mathcal{V})$ becomes a Banach space [AF03]. The *Sobolev spaces* are given by

$$W^{k,p}(\mathcal{I}, \mathcal{V}) = \left\{ \ f \in L^p(\mathcal{I}, \mathcal{V}) \ \middle| \ f^{(l)} \in L^p(\mathcal{I}, \mathcal{V}) \text{ exists for all } l \le k \ \right\},$$
$$W^{k,p}_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) = \left\{ \ f \in L^p_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) \ \middle| \ f^{(l)} \in L^p_{\mathrm{loc}}(\mathcal{I}, \mathcal{V}) \text{ exists for all } l \le k \ \right\},$$

for $k \in \mathbb{N}_0$ and $p \in [1, \infty)$ where $f^{(l)}$ denotes the lth weak derivative of f.

1.2 System theoretic prerequisites

This thesis is mainly concerned with questions originating from the field of mathematical systems theory. For an introduction and an overview of this vast field, we refer the reader to [HP05; Sim17]. The following two sections present certain types of dynamical systems arising in this theory, namely so-called differential-algebraic equations, as well as some tools to study them.

1.2.1 Matrix pencils

The analysis of differential-algebraic equations leads to the study of *matrix pencils*, which are first order matrix polynomials $sE - A \in \mathbb{K}[s]^{m \times n}$ with coefficient matrices $E, A \in \mathbb{K}^{m \times n}$. The following notations allow us to introduce canonical forms for matrix pencils. For $k \in \mathbb{N}$ let $e_i^{[k]} \in \mathbb{K}^k$ (or simply e_i if clear from the context) be the ith canonical unit vector and let $N_k \in \mathbb{K}^{k \times k}$, $K_k, L_k \in \mathbb{K}^{(k-1) \times k}$ be defined by

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \ K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \ L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.$$

Further, for some multi-index $\alpha = (\alpha_1, ..., \alpha_l) \in \mathbb{N}^l$ with $|\alpha| = \alpha_1 + \ldots + \alpha_l$, we introduce

$$\begin{aligned} N_\alpha &= \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{K}^{|\alpha| \times |\alpha|}, \\ K_\alpha &= \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}) \in \mathbb{K}^{(|\alpha|-l) \times |\alpha|}, \\ L_\alpha &= \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{K}^{(|\alpha|-l) \times |\alpha|}, \end{aligned}$$

A Jordan block of size $k \in \mathbb{N}$ at $\lambda \in \mathbb{C}$ corresponds to the matrix

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{K}^{k \times k}.$$

Definition 1.2.1 (Kronecker form, Weierstraß form)**.**
Let $sE - A \in \mathbb{K}[s]^{m \times n}$. We say that $sE - A$ is in *Kronecker form* if

$$E = \operatorname{diag}(I_l, N_\alpha, K_\beta, K_\gamma^\top) \text{ and } A = \operatorname{diag}(J, I_{|\alpha|}, L_\beta, L_\gamma^\top),$$

i.e.,

$$sE - A = \begin{bmatrix} sI_l - J & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{1.1}$$

for some multi-indices $\alpha \in \mathbb{N}^{\ell_\alpha}, \beta \in \mathbb{N}^{\ell_\beta}, \gamma \in \mathbb{N}^{\ell_\gamma}$ with $l, \ell_\alpha, \ell_\beta, \ell_\gamma \in \mathbb{N}_0$ and $J \in \mathbb{K}^{l \times l}$ is in Jordan canonical form over $\mathbb{K}$ (see, e.g., [HJ13, Secs. 3.1 & 3.4]). Further, we say that $sE - A$ is in *Weierstraß form* if $\ell_\beta = \ell_\gamma = 0$, i.e.,

$$sE - A = \begin{bmatrix} sI_l - J & 0 \\ 0 & sN_\alpha - I_{|\alpha|} \end{bmatrix}. \tag{1.2}$$

In this context, the numbers α_i for $i = 1, \ldots, \ell_\alpha$ are referred to as *sizes of the Jordan blocks at* ∞, whereas for $i = 1, \ldots, \ell_\beta$, $j = 1, \ldots, \ell_\gamma$, the numbers $\beta_i - 1$ and $\gamma_j - 1$ are, respectively, called *column* and *row minimal indices*.

Theorem 1.2.2 (Kronecker canonical form [Gan59, Chap. XII], [Ber14, Chap. 2])**.**
Let $sE - A \in \mathbb{K}[s]^{m \times n}$. Then there exist unique $S \in \mathbf{Gl}_m(\mathbb{K}), T \in \mathbf{Gl}_n(\mathbb{K})$ such that $sSET - SAT$ is in Kronecker form (1.1).

By the above result, the following concept is well-defined.

Definition 1.2.3 (Index of a pencil)**.** Let $sE - A \in \mathbb{K}[s]^{m\times n}$ and $S \in \mathbf{Gl}_m(\mathbb{K}), T \in \mathbf{Gl}_n(\mathbb{K})$ such that $sSET - SAT$ is in Kronecker form (1.1). Then the *(Kronecker) index* ν of $sE - A$ is defined as

$$\nu = \max\{\alpha_1, \ldots, \alpha_{\ell_\alpha}, \gamma_1, \ldots, \gamma_{\ell_\gamma}, 0\}, \tag{1.3}$$

where $\alpha \in \mathbb{N}^{\ell_\alpha}$ and $\gamma \in \mathbb{N}^{\ell_\gamma}$ are as in Def. 1.2.1.

Definition 1.2.4 (Eigenvalue of a pencil)**.** A number $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of a pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ if $\operatorname{rk}_{\mathbb{C}}(\lambda E - A) < \operatorname{rk}_{\mathbb{K}(s)}(sE - A)$, and we write

$$\sigma(E, A) := \{\ \lambda \in \mathbb{C} \ \mid \lambda \text{ is an eigenvalue of } sE - A\ \}$$

Lemma 1.2.5. *Let* $sE_i - A_i \in \mathbb{K}[s]^{m_i\times n_i}$ *for* $i = 1, 2$ *and define* $sE - A := s \operatorname{diag}(E_1, E_2) - \operatorname{diag}(A_1, A_2) \in \mathbb{K}[s]^{m\times n} := \mathbb{K}[s]^{(m_1+m_2)\times(n_1+n_2)}$. *Further, let* $S \in \mathbf{Gl}_m(\mathbb{K})$ *and* $T \in \mathbf{Gl}_n(\mathbb{K})$. *Then*

$$\sigma(SET, SAT) = \sigma(E, A) = \sigma(E_1, A_1) \cup \sigma(E_2, A_2).$$

Further, if S, T *are such that* $sSET - SAT$ *is in Kronecker form* (1.1), *then with* J *as in Def. 1.2.1*

$$\sigma(E, A) = \sigma(J).$$

Proof. We only prove the last statement, i.e., $\sigma(E, A) = \sigma(J)$, since the others can directly be deduced from Def. 1.2.4. First note that for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$

$$\begin{aligned}
\operatorname{rk}_{\mathbb{C}}(\lambda K_k - L_k) &= \operatorname{rk}_{\mathbb{K}(s)}(sK_k - L_k) = k - 1,\\
\operatorname{rk}_{\mathbb{C}}(\lambda N_k - I_k) &= \operatorname{rk}_{\mathbb{K}(s)}(sN_k - I_k) = k,
\end{aligned}$$

i.e., (1.1) $\sigma(sK_k - L_l) = \sigma(N_k - I_k) = \emptyset$. Invoking the other two statements of this lemma, we have in the Kronecker form (1.1)

$$\begin{aligned}
\sigma(E, A) =& \sigma(I_l, J) \cup \bigcup_{i=1}^{\ell_\alpha} \sigma(N_{\alpha_i}, I_{\alpha_i}) \cup \bigcup_{i=1}^{\ell_\beta} \sigma(K_{\beta_i}, L_{\beta_i}) \cup \bigcup_{i=1}^{\ell_\gamma} \sigma(K_{\alpha_i}^\top, L_{\alpha_i}^\top)\\
=& \sigma(I_l, J) = \sigma(J).
\end{aligned}$$

$\square$

As for matrices, one can define the concept of semi-simplicity for eigenvalues. Namely, we say that an eigenvalue $\lambda \in \sigma(E, A)$ is *semi-simple* if J in the Kronecker form (1.1) of $sE - A$ has no Jordan blocks of size greater or equal to two at λ. Note that semi-simplicity is well-defined. It is also possible, as for matrices, to introduce it by means of *algebraic* and *geometric multiplicity* for regular pencils (see, e.g., [GT16; GT17]), which are defined as follows.

Definition 1.2.6 (Regular pencil)**.** A pencil $sE - A \in \mathbb{K}[s]^{m \times n}$ is called *regular* if $m = n$ and $\det(sE - A) \neq 0 \in \mathbb{K}[s]$.

Regularity is equivalent to the property that $sE - A$ has no row and column minimal indices in its Kronecker form (1.1), as the following theorem shows.

Theorem 1.2.7 (Weierstraß canonical form [Wei68], [Ber14, Chap. 2])**.** *A pencil $sE - A \in \mathbb{K}^{n \times n}$ is regular if and only if there exist unique $S, T \in \mathbf{Gl}_n(\mathbb{K})$ such that $sSET - SAT$ is in Weierstraß form* (1.2)*.*

For regular matrix pencils the set of eigenvalues fulfils

$$\sigma(E, A) = \{\ \lambda \in \mathbb{C} \ \mid \ \det(\lambda E - A) = 0 \ \}.$$

Note that regularity implies that $sE - A$ is invertible as a matrix with entries in $\mathbb{K}(s)$. In this case, $\sigma(E, A)$ coincides with the set of poles of the entries of $(sE - A)^{-1} \in \mathbb{K}(s)^{n \times n}$.

We state another elementary lemma which can be derived directly from the Weierstraß canonical form for regular matrix pencils. We will characterize the index by means of the growth of the *resolvent* $(sE - A)^{-1}$ on a real half-axis.

Lemma 1.2.8. *Let the pencil $sE - A \in \mathbb{K}[s]^{n \times n}$ be regular. Then the index of $sE - A$ is equal to the smallest number k for which there exists some $M > 0$ and $\omega \in \mathbb{R}$, such that*

$$\forall \lambda > \omega : \quad \|(\lambda E - A)^{-1}\| \leq M|\lambda|^{k-1}.$$

Moreover, the size of the largest Jordan block at an eigenvalue λ of $sE - A$ is equal to the order *(see [Rud87, p. 210]) of λ as a pole of $(sE - A)^{-1} \in \mathbb{K}(s)^{n \times n}$.*

Definition 1.2.9. A rational matrix $G(s) \in \mathbb{K}(s)^{n \times n}$ is called *positive real*, if

(a) $G(s)$ has no poles in the open right complex half-plane.

(b) $G(\lambda) + G(\lambda)^* \geq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$.

It can directly be deduced that a matrix pencil $sE - A \in \mathbb{K}[s]^{n \times n}$ is positive real if and only if $E = E^* \geq 0$ and $A + A^* \leq 0$. We recall some properties of positive real matrix pencils, which result from a combination of [BR14b, Lem. 2.6] and [BR14a, Cor. 2.3].

Lemma 1.2.10. *Let $sE - A \in \mathbb{K}[s]^{n \times n}$ be a positive real pencil. Then the following holds.*

(a) *$sE - A$ is regular if and only if* $\ker E \cap \ker A = \{0\}$.

(b) *The row and column minimal indices are at most zero (if there are any) and their numbers coincide.*

(c) *The eigenvalues of the pencil are contained in the closed left half-plane $\overline{\mathbb{C}_-}$ and the eigenvalues on the imaginary axis are semi-simple.*

(d) *The index of $sE - A$ is at most two.*

1.2.2 Differential-algebraic equations

In this thesis we focus on the linear and time-invariant control systems of differential-algebraic equations (DAEs)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) =& Ax(t) + Bu(t), \\ y(t) =& Cx(t) + Du(t), \end{aligned} \tag{1.4}$$

for some $E, A \in \mathbb{K}^{m \times n}$, $B \in \mathbb{K}^{m \times k}$, $C \in \mathbb{K}^{l \times n}$, $D \in \mathbb{K}^{l \times k}$, and $k, l, m, n \in \mathbb{N}_0$. Here, x denotes the *state*, u the *input* and y the *output* of the system. We write $[E, A, B, C, D] \in \Sigma_{n,m,k,l}$ for such systems. Some special cases deserve their own shortened notation, namely,

- $l = k$: $[E, A, B, C, D] \in \Sigma_{n,m,k}$;
- $l = k$, $n = m$: $[E, A, B, C, D] \in \Sigma_{n,k}$;
- $l = k$, $n = m$, $D = 0$: $[E, A, B, C] \in \Sigma_{n,k}$;
- $l = 0$: $[E, A, B] \in \Sigma_{n,m,k}$;
- $l = 0$ and $n = m$: $[E, A, B] \in \Sigma_{n,k}$;
- $l = k = 0$ and $n = m$: $[E, A] \in \Sigma_n$;
- $l = k = 0$: $[E, A] \in \Sigma_{n,m}$.

Of course, we have to specify what properties the functions in (1.4) have to fulfil. However, depending on the concrete application, different solution concepts are of interest. For example, when considering optimal control problems with quadratic costs, the function space $\mathcal{L}^2$ is a natural candidate, see [Ger15]. Following the behavioral approach from [PW97] and the rather general solution concept found in [BR13], we give the following definition for the solution set of (1.4).

Definition 1.2.11 (Behavior)**.** The *behavior* of a system $[E, A, B, C, D] \in \Sigma_{n,m,k,l}$ is given by

$$\mathfrak{B}_{[E,A,B,C,D]} = \left\{ (x, u, y) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{K}^n \times \mathbb{K}^k \times \mathbb{K}^l) \;\middle|\; Ex \in W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{K}^n),\ \tfrac{\mathrm{d}}{\mathrm{dt}} Ex = Ax + Bu,\ y = Cx + Du \right\}$$

and for $x_0 \in \mathbb{K}^n$ we specify

$$\mathfrak{B}_{[E,A,B,C,D]}(x_0) = \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \;\middle|\; (Ex)(0) = Ex_0 \right\}.$$

Further notions of solution and behavior are discussed and used in, e.g., [Tre13; OSS20]. The next concept is intimately linked to the behavior. It describes the smallest subspace in which all solutions evolve.

Definition 1.2.12 (System space)**.** The *system space* of $[E, A, B, C, D] \in \Sigma_{n,m,k,l}$ is defined as

$$\mathcal{V}^{[E,A,B,C,D]}_{\text{sys}} = \bigcap_{\substack{\mathcal{V} \subset \mathbb{K}^{n+k+l} \text{ subspace} \\ \mathfrak{B}_{[E,A,B,C,D]} \subset L^1_{\text{loc}}(\mathbb{R}, \mathcal{V})}} \mathcal{V}.$$

Further, we have $\mathfrak{B}_{[E,A,B,C,D]} \subset L^1_{\text{loc}}\left(\mathbb{R}, \mathcal{V}^{[E,A,B,C,D]}_{\text{sys}}\right)$.

The system space is well-defined by arguments similar to those used for the proof of [RV19, Prop. 3.2], the difference being that we also consider output systems and allow for complex values. See also [BR13; Ber14] for characterizations of the system space. We further introduce another vector space related to the system space.

Definition 1.2.13 (Space of consistent initial differential variables)**.**
For a system $[E, A, B, C, D] \in \Sigma_{n,m,k,l}$ its *space of consistent initial differential vari-*

ables is defined as

$$\mathcal{V}_{\mathrm{diff}}^{[E,A,B,C,D]} = \left\{ x_0 \in \mathbb{K}^n \;\middle|\; \mathfrak{B}_{[E,A,B,C,D]}(x_0) \neq \emptyset \right\}.$$

Remark 1.2.14. Note that for systems $[E, A, B] \in \Sigma_{n,m,k}$ the concepts of system space and space of consistent initial values are linked by the relation

$$\mathcal{V}_{\mathrm{diff}}^{[E,A,B,C,D]} = P\mathcal{V}_{\mathrm{sys}}^{[E,A,B,C,D]} \mathbin{\widehat{+}} \ker E,$$

where $P : \mathbb{K}^{n+k+l} \to \mathbb{K}^n$ denotes the projection on the first n components, cf. [BR13].

In order to give some intuition on the concepts we just introduced, we present the following proposition recapitulating the findings of [Ber14, Sec. 2.4.2].

Proposition 1.2.15 (Prototypical DAEs)**.** *Let $k, l \in \mathbb{N}$, $A \in \mathbb{K}^{k\times k}$, $B \in \mathbb{K}^{k\times l}$, $B' \in \mathbb{K}^{(k-1)\times l}$, and $x_0 \in \mathbb{K}^k, x_0' \in \mathbb{K}^{k-1}$. Then*

(i) $\mathcal{V}_{\mathrm{sys}}^{[I_k,A,B]} = \mathbb{K}^{k+l}$ *and* $\mathcal{V}_{\mathrm{diff}}^{[I_k,A,B]} = \mathbb{K}^k$ *with*

$$\mathfrak{B}_{[I_k,A,B]}(x_0) = \left\{ (x,u) \;\middle|\; u \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^l),\; x = e^{A\cdot}x_0 + \int_0^{\cdot} e^{A(\cdot-\tau)}Bu(\tau)\,\mathrm{d}\tau \right\}.$$

(ii) it holds

$$\begin{aligned}
\mathcal{V}_{\mathrm{sys}}^{[N_k,I_k,B]} &= \operatorname{ran}\begin{bmatrix} B & N_kB & \cdots & N_k^{k-1}B \\ I & 0 & \cdots & 0 \end{bmatrix},\\
\mathcal{V}_{\mathrm{diff}}^{[N_k,I_k,B]} &= \operatorname{ran}\begin{bmatrix} N_k^0B & \cdots & N_k^{k-1}B & I_k - N_k^\top N_k \end{bmatrix},\\
\mathfrak{B}_{[N_k,I_k,B]}(x_0) &= \Bigg\{ (x,u) \;\Bigg|\; u \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^l),\\
&\quad x = -\sum_{j=0}^{k-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}N_k\right)^j Bu,\; -\sum_{j=0}^{k-2}\left(N_k\left(\tfrac{\mathrm{d}}{\mathrm{dt}}N_k\right)^j Bu\right)(0) = N_kx_0,\\
&\quad \text{and } \left(N_k\left(\tfrac{\mathrm{d}}{\mathrm{dt}}N_k\right)^{j-1} Bu\right)_{j=1,\ldots,k} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^k) \Bigg\}.
\end{aligned}$$

(iii) $\mathcal{V}_{\mathrm{sys}}^{[K_k,L_k,B']} = \mathbb{K}^{k+l}$ *and* $\mathcal{V}_{\mathrm{diff}}^{[K_k,L_k,B']} = \mathbb{K}^k$ *with*

$$\mathfrak{B}_{[K_k,L_k,B']}(x_0) = \left\{ ((\tilde{x},x_k),u) \;\middle|\; (x_k,u) \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^{1+l}),\; (\tilde{x},x_k,u) \in \mathfrak{B}_{[I_{k-1},N_{k-1},[\,e_{k-1}\;B'\,]]}(K_kx_0) \right\}.$$

(iv) *it holds*

$$\begin{aligned}
\mathcal{V}_{\mathrm{sys}}^{[K_k^\top, L_k^\top, B]} &= \Big(\mathrm{gr}\,\big[\, I_{k-1}\ 0 \,\big]\big[\, N_k^0 B\ \cdots\ N_k^{k-1} B \,\big] \\
&\qquad \cap \ker e_k^\top \big[\, N_k^0 B\ \cdots\ N_k^{k-1} B \,\big] \times \mathbb{K}^{k-1}\Big)^{-1}, \\
\mathcal{V}_{\mathrm{diff}}^{[K_k^\top, L_k^\top, B]} &= \big[\, I_{k-1}\ 0 \,\big]\big[\, N_k^0 B\ \cdots\ N_k^{k-1} B \,\big] \ker e_k^\top \big[\, N_k^0 B\ \cdots\ N_k^{k-1} B \,\big], \\
\mathfrak{B}_{[K_k^\top, L_k^\top, B]}(x_0') &= \left\{\ (x,u)\ \middle|\ ((x,0),u) \in \mathfrak{B}_{[N_k, I_k, B]}(K_k^\top x_0')\ \right\}.
\end{aligned}$$

In particular,

(i') $\mathfrak{B}_{[I_k,A]} = \left\{\ t \mapsto e^{At} x_0\ \middle|\ x_0 \in \mathbb{K}^k\ \right\} \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$ *and* $\mathcal{V}_{\mathrm{sys}}^{[I_k,A]} = \mathbb{K}^k$*;*

(ii') $\mathfrak{B}_{[N_k,I_k]} = \{0\} \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$ *and* $\mathcal{V}_{\mathrm{sys}}^{[N_k,I_k]} = \{0\} \subset \mathbb{K}^k$*;*

(iii') $\mathfrak{B}_{[K_k,L_k]} = \left\{ \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-i} f\right)_{i=1,\dots,k} \middle|\, f \in W^{k-1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}) \right\} \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$ *and* $\mathcal{V}_{\mathrm{sys}}^{[K_k,L_k]} = \mathbb{K}^k$*;*

(iv') $\mathfrak{B}_{[K_k^\top,L_k^\top]} = \{0\} \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^{k-1})$ *and* $\mathcal{V}_{\mathrm{sys}}^{[K_k^\top,L_k^\top]} = \{0\} \subset \mathbb{K}^{k-1}$.

The proof techniques of [BR13, Sec. 3.1] can be directly transposed to our slightly more general setting, in which values are taken in $\mathbb{K}$ instead of solely $\mathbb{R}$, yielding the following result.

Proposition 1.2.16 (Solutions under system equivalence)**.**
Let $[E, A, B] \in \Sigma_{n,m,k}$*,* $S \in \mathbf{Gl}_m(\mathbb{K})$ *and* $T \in \mathbf{Gl}_n(\mathbb{K})$ *and* $x_0 \in \mathbb{K}^n$*. Then*

(i) $\mathfrak{B}_{[E,A,B]}(Tx_0) = T\mathfrak{B}_{[SET,SAT,SB]}(x_0)$*;*

(ii) $\mathfrak{B}_{[E,A,B]} = T\mathfrak{B}_{[SET,SAT,SB]}$*;*

(iii) $\mathcal{V}_{\mathrm{sys}}^{[E,A,B]} = \left[\begin{smallmatrix} T & \\ & I \end{smallmatrix}\right] \mathcal{V}_{\mathrm{sys}}^{[SET,SAT,SB]}$*;*

(iv) $\mathcal{V}_{\mathrm{diff}}^{[E,A,B]} = T\mathcal{V}_{\mathrm{diff}}^{[SET,SAT,SB]}$.

The following additional properties are readily seen.

Proposition 1.2.17 (Solutions of block diagonal pencils)**.** *Let* $[E_1, A_1] \in \Sigma_{n_1,m_1}$*,* $[E_2, A_2] \in \Sigma_{n_2,m_2}$ *and* $[E, A] := [\mathrm{diag}\,(E_1, E_2), \mathrm{diag}\,(A_1, A_2)]$*. Then*

(i) $\mathfrak{B}_{[E,A]} = \mathfrak{B}_{[E_1,A_1]} \times \mathfrak{B}_{[E_2,A_2]}$*;*

(ii) $\mathcal{V}_{\mathrm{sys}}^{[E,A]} = \mathcal{V}_{\mathrm{sys}}^{[E_1,A_1]} \times \mathcal{V}_{\mathrm{sys}}^{[E_2,A_2]}$*;*

(iii) $\mathcal{V}_{\mathrm{diff}}^{[E,A]} = \mathcal{V}_{\mathrm{diff}}^{[E_1,A_1]} \times \mathcal{V}_{\mathrm{diff}}^{[E_2,A_2]}$.

Now, Props. 1.2.15–1.2.17 directly yield the following result.

Corollary 1.2.18 (System space in terms of Kronecker form)**.** *Let* $[E,A] \in \Sigma_{n,m}$ *with* $S \in \mathbf{Gl}_m(\mathbb{K})$ *and* $T \in \mathbf{Gl}_n(\mathbb{K})$ *such that* $S(sE-A)T$ *is in Kronecker form* (1.1)*. Then*

$$\mathcal{V}_{\mathrm{sys}}^{[E,A]} = T(\mathbb{R}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \{0\}^{|\gamma|-\ell_\gamma}). \tag{1.5}$$

To close the section, let us introduce stability concepts concerning systems of the form

$$\tfrac{\mathrm{d}}{\mathrm{dt}} Ex(t) = Ax(t) + Bu(t),$$

and we put special emphasis on the case $B = 0$, that is, a system without input.

Definition 1.2.19 (Stability concepts)**.** We say that $[E,A] \in \Sigma_{n,m}$

- is *stable* if

$$\forall x \in \mathfrak{B}_{[E,A]}\ \exists M \geq 0 : \ \operatorname{ess\,sup}_{t\geq 0} \|x(t)\| \leq M$$

  for some $M > 0$.

- is *asymptotically stable* if

$$\forall x \in \mathfrak{B}_{[E,A]} : \ \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau \geq t} \|x(\tau)\| = 0.$$

- has *stable differential variables* if

$$\forall x \in \mathfrak{B}_{[E,A]}\ \exists M \geq 0 : \ \sup_{t\geq 0} \|Ex(t)\| \leq M.$$

- has *asymptotically stable differential variables* if

$$\forall x \in \mathfrak{B}_{[E,A]} : \ \lim_{t\to\infty} \sup_{\tau > t} \|Ex(\tau)\| = 0.$$

Further, $[E,A,B] \in \Sigma_{n,m,k}$

- is *behaviorally stabilizable* if

$$\forall x_0 \in \mathcal{V}_{\mathrm{diff}}^{[E,A,B]}\ \exists (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0)\ \exists M \geq 0 : \operatorname{ess\,sup}_{t\geq 0} \|x(t)\| \leq M.$$

- is *behaviorally asymptotically stabilizable* if
$$\forall x_0 \in \mathcal{V}_{\text{diff}}^{[E,A,B]}\ \exists (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0):\ \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau\geq t} \|x(\tau)\| = 0.$$
- has *behaviorally stablizable differential variables* if
$$\forall x_0 \in \mathcal{V}_{\text{diff}}^{[E,A,B]}\ \exists (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0)\ \exists M \geq 0:\ \sup_{t\geq 0} \|Ex(t)\| \leq M.$$
- has *behaviorally asymptotically stablizable differential variables* if
$$\forall x_0 \in \mathcal{V}_{\text{diff}}^{[E,A,B]}\ \exists (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0):\ \lim_{t\to\infty} \sup_{\tau\geq t} \|Ex(\tau)\| = 0.$$

1.3 Linear relations foundations

We introduce the notion of *linear relation* on $\mathbb{K}^n$ as the subspaces of $\mathbb{K}^n \times \mathbb{K}^n \cong \mathbb{K}^{2n}$. An introduction to linear relations can be found, e.g., in [BHdS20; Cro98]. An important special case of a linear relation is the graph of a square matrix $M \in \mathbb{K}^{n\times n}$, i.e.,
$$\text{gr } M := \{(x, Mx) \mid x \in \mathbb{K}^n\}.$$
This motivates to define the following concepts for linear relations.

Definition 1.3.1 (Concepts and operations on linear relations)**.** Let $n \in \mathbb{N}$, and $\mathcal{L}, \mathcal{M} \subset \mathbb{K}^{2n}$ be linear relations in $\mathbb{K}^n$. The *domain, kernel, range* and *multi-valued part* of a relation $\mathcal{M}$ in $\mathbb{K}^n$ are
$$\begin{aligned} \operatorname{dom} \mathcal{M} &= \{x \in \mathbb{K}^n \mid (x,y) \in \mathcal{M}\}, & \ker \mathcal{M} &= \{x \in \mathbb{K}^n \mid (x,0) \in \mathcal{M}\}, \\ \operatorname{ran} \mathcal{M} &= \{y \in \mathbb{K}^n \mid (x,y) \in \mathcal{M}\}, & \operatorname{mul} \mathcal{M} &= \{y \in \mathbb{K}^n \mid (0,y) \in \mathcal{M}\}, \end{aligned}$$
and *scalar multiplication with* $\alpha \in \mathbb{K}$, *operator-like sum*, *product*, *inverse* and *adjoint* are defined by
$$\begin{aligned} \alpha\mathcal{M} &:= \{(x, \alpha y) \in \mathbb{K}^{2n} \mid (x,y) \in \mathcal{M}\}, \\ \mathcal{M} + \mathcal{L} &:= \{(x, y_1 + y_2) \in \mathbb{K}^{2n} \mid (x, y_1) \in \mathcal{L}, (x, y_2) \in \mathcal{M}\}, \\ \mathcal{M}\mathcal{L} &:= \{(x,z) \in \mathbb{K}^{2n} \mid \exists y \in \mathbb{K}^n \text{ s.t. } (x,y) \in \mathcal{L}, (y,z) \in \mathcal{M}\}, \\ \mathcal{M}^{-1} &:= \{(y,x) \in \mathbb{K}^{2n} \mid (x,y) \in \mathcal{M}\}, \\ \mathcal{M}^* &:= \{(x,y) \in \mathbb{K}^{2n} \mid \langle w, x\rangle = \langle v, y\rangle\ \ \forall\, (v,w) \in \mathcal{M}\}. \end{aligned}$$

A linear relation with $\mathcal{M} \subseteq \mathcal{M}^*$ is called *symmetric*, whereas $\mathcal{M}$ is *self-adjoint*, if $\mathcal{M} = \mathcal{M}^*$. Likewise, $\mathcal{M}$ with $\mathcal{M} \subseteq -\mathcal{M}^*$ is called *skew-symmetric*, and $\mathcal{M}$ is *skew-adjoint*, if it has the property $\mathcal{M} = -\mathcal{M}^*$.

We underline that by writing $(x, y) \in \mathcal{M}$, where $\mathcal{M} \subset \mathbb{K}^{2n}$, we particularly mean that $x, y \in \mathbb{K}^n$. If $\mathbb{K} = \mathbb{C}$ then a linear relation $\mathcal{M}$ is symmetric (self-adjoint) if and only if $\imath\mathcal{M}$ is skew-symmetric (skew-adjoint).
Note that the operator-like sum of two linear relations $\mathcal{L}, \mathcal{M} \subset \mathbb{K}^{2n}$ is <u>not</u> the componentwise sum, which was introduced as

$$\mathcal{L}\widehat{+}\mathcal{M} = \left\{ \ (x_1 + x_2, y_1 + y_2) \in \mathbb{K}^{2n} \ \middle| \ (x_1, y_1) \in \mathcal{L}, \ (x_2, y_2) \in \mathcal{M} \ \right\}.$$

Similarly, with another linear relation $\mathcal{N} \subset \mathbb{K}^{2m}$ we define the *sorted Cartesian product* as

$$\mathcal{L}\widehat{\times}\mathcal{N} = \left\{ \ (x_1, x_2, y_1, y_2) \in \mathbb{K}^{2(n+m)} \ \middle| \ (x_1, y_1) \in \mathcal{M}, \ (x_2, y_2) \in \mathcal{N} \ \right\}.$$

We oftentimes use the identity

$$(-\mathcal{M}^*)^{-1} = \mathcal{M}^{\perp}, \tag{1.6}$$

where $\mathcal{M}^{\perp}$ is the orthogonal complement of $\mathcal{M} \subseteq \mathbb{K}^{2n}$. In particular, we can conclude that

$$2n = \dim \mathcal{M} + \dim \mathcal{M}^{\perp} = \dim \mathcal{M} + \dim(\mathcal{M}^*)^{-1} = \dim \mathcal{M} + \dim \mathcal{M}^*,$$

which gives

$$\dim \mathcal{M}^* = 2n - \dim \mathcal{M}. \tag{1.7}$$

Other well-known identities are

$$\ker \mathcal{M}^* = (\operatorname{ran} \mathcal{M})^{\perp}, \quad (\operatorname{dom} \mathcal{M})^{\perp} = \operatorname{mul} \mathcal{M}^*. \tag{1.8}$$

We will also use the fact that a linear relation $\mathcal{M}$ in $\mathbb{K}^n$ can be written as $\mathcal{M} = \ker[K, L]$ or $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix} \right]$ with matrices $F, G \in \mathbb{K}^{n \times l}$ and $K, L \in \mathbb{K}^{l \times n}$ which we will refer to as *kernel* and *image (or range) representations*. For $\mathbb{K} = \mathbb{C}$, these representations always exist for each choice of $l \in \mathbb{N}$ such that $l \geq \dim \mathcal{M}$, see, e.g., [BTW16, Thm. 3.3]. The proof of the existence of the range representation for $\mathbb{K} = \mathbb{R}$ can also be derived from the proof of [BTW16, Thm. 3.3]. Moreover, the link between two different representations is readily seen.

Proposition 1.3.2. *Let $\mathcal{M}$ be a linear relation in $\mathbb{K}^n$ for which we have two kernel and image representations, i.e.,*

$$\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right] = \ker \left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right] = \ker \left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right]$$

for some matrices $F_1, G_1, K_1^*, L_1^* \in \mathbb{K}^{n\times l_1}$ and $F_2, G_2, K_2^*, L_2^* \in \mathbb{K}^{n\times l_2}$ for some $l_1, l_2 \geq \dim \mathcal{M}$. Then there exist $T_1, T_2^*, S_1^*, S_2 \in \mathbb{K}^{l_1\times l_2}$ with $\operatorname{rk} T_1 = \operatorname{rk} T_2 = \operatorname{rk} S_1 = \operatorname{rk} S_2 = \dim \mathcal{M}$ such that

$$\left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right] T_1 = \left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right] T_2 = \left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right]$$

and

$$S_1 \left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right], \quad S_2 \left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right].$$

Together with (1.6) we have for $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = \ker[K, L]$ that

$$\mathcal{M}^* = \ker[G^*, -F^*] = \operatorname{ran} \left[\begin{smallmatrix} L^* \\ -K^* \end{smallmatrix}\right]. \tag{1.9}$$

In the following result, we characterize symmetry and self-adjointness of a linear relation by means of certain properties of the matrices in the range and kernel representations. The result is well-known, see, e.g., [BHdS20, Cor. 1.10.8] and [vdSM18].

Lemma 1.3.3. *Let $\mathcal{M} \subset \mathbb{K}^{2n}$ be a linear relation. Then $\mathcal{M}$ is symmetric if and only if $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ for some $F, G \in \mathbb{K}^{n\times l}$ with $G^*F = F^*G$. Moreover, the following statements are equivalent, see also [vdSM18],*

(a) *$\mathcal{M}$ is self-adjoint,*

(b) *$\mathcal{M}$ is symmetric and $\dim \mathcal{M} = n$,*

(c) *$\mathcal{M} = \ker[K, L]$ for some $K, L \in \mathbb{K}^{n\times n}$ with $KL^* = LK^*$ and $\operatorname{rk}[K, L] = n$.*

Proof. To prove the first equivalence, assume that $\mathcal{M} \subset \mathbb{K}^{2n}$ is symmetric and let $F, G \in \mathbb{K}^{n\times l}$ such that $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$. The symmetry of $\mathcal{M}$ together with (1.9) now implies that

$$\forall\, z \in \mathbb{K}^n : \; 0 = [G^*, -F^*] \underbrace{\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] z}_{\in \mathcal{M} \subset \mathcal{M}^*} = (G^*F - F^*G)z,$$

whence $G^*F = F^*G$.

Conversely, assume that $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ for some $F, G \in \mathbb{K}^{n\times l}$ with $G^*F = F^*G$. Let

$(x_1, y_1), (x_2, y_2) \in \mathcal{M}$. Then there exists some $z_1, z_2 \in \mathbb{K}^n$ with $x_1 = Fz_1$, $y = Gz_1$, $x_2 = Fz_2$ and $y_2 = Gz_2$. We deduce

$$\langle y_2, x_1\rangle = \langle Gz_2, Fz_1\rangle = \langle z_2, G^*Fz_1\rangle = \langle z_2, F^*Gz_1\rangle = \langle Fz_2, Gz_1\rangle = \langle x_2, y_1\rangle,$$

i.e., $\mathcal{M}$ is symmetric. We now show the equivalences (a)-(c).
"(a)$\Rightarrow$(b)": If $\mathcal{M} \subset \mathbb{K}^{2n}$ is self-adjoint, then, by (1.7),

$$\dim \mathcal{M} = \dim \mathcal{M}^* = 2n - \dim \mathcal{M},$$

which gives $\dim \mathcal{M} = n$.
"(b)$\Rightarrow$(c)": Assume that $\mathcal{M} \subset \mathbb{K}^{2n}$ is symmetric and $\dim \mathcal{M} = n$. By the first equivalence, there exist $F, G \in \mathbb{K}^{n\times n}$ such that $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ and $G^*F = F^*G$. Since $\mathcal{M} = \mathcal{M}^*$, the choices of $K = G^*$ and $L = -F^*$ together with (1.9) lead to $\mathcal{M} = \ker[K, L]$ with $KL^* = LK^*$. Further, we have

$$n = \dim \mathcal{M} = \operatorname{rk} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = \operatorname{rk} [K, L].$$

"(c)$\Rightarrow$(a)": Assume that $\mathcal{M} = \ker[K, L]$ for $K, L \in \mathbb{K}^{n\times n}$ with $\operatorname{rk} [K, L] = n$ and $KL^* = LK^*$. Then, by (1.9), $\mathcal{M}^* = \operatorname{ran} \left[\begin{smallmatrix} L^* \\ -K^* \end{smallmatrix}\right]$. Assume that $(x, y) \in \mathcal{M}^*$. Then there exists some $z \in \mathbb{K}^n$ with $x = L^*z$ and $y = -K^*z$. This yields

$$[K, L] \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = Kx + Ly = KL^*z - LK^*z = 0.$$

Altogether we obtain that $\mathcal{M}^* \subset \mathcal{M}$. On the other hand, we obtain from $\operatorname{rk} [K, L] = n$ that $\dim \mathcal{M} = \dim \ker[K, L] = n$ and $\dim \mathcal{M}^* = \operatorname{rk} \left[\begin{smallmatrix} L^* \\ -K^* \end{smallmatrix}\right] = n$, which, together with $\mathcal{M}^* \subset \mathcal{M}$ leads to $\mathcal{M}^* = \mathcal{M}$. □

Remark 1.3.4. Note that Lem. 1.3.3 can be further modified to characterize skew-adjointness of a linear relation $\mathcal{M}$. In particular, it is analogous to prove the equivalence of the statements, see [vdSM18],

(a) $\mathcal{M}$ is skew-adjoint,

(b) $\mathcal{M}$ is skew-symmetric and $\dim \mathcal{M} = n$,

(c) $\mathcal{M} = \ker[K, L]$ for some $K, L \in \mathbb{K}^{n\times n}$ with $KL^* = -LK^*$ and $\operatorname{rk} [K, L] = n$,

as well as the equivalence of the statements

(d) $\mathcal{M}$ is skew-symmetric,

(e) $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ for some $F, G \in \mathbb{K}^{n\times l}$ with $G^*F = -F^*G$.

Moreover, the equivalence of the statement

(f) $\operatorname{Re}\langle x, y\rangle = 0$ for all $(x, y) \in \mathcal{M}$,

to (d) and (e) follows from considering

$$\operatorname{Re}\langle x, y\rangle = \tfrac{1}{2}(\langle x, y\rangle + \langle y, x\rangle) = z^*(F^*G + G^*F)z,$$

for $(x, y) = (Fz, Gz) \in \mathcal{M}$ with $z \in \mathbb{K}^l$ given by the range representation $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$.

Definition 1.3.5 (dissipative, nonnegative)**.** Let $\mathcal{M} \subset \mathbb{K}^{2n}$ be a linear relation. Then $\mathcal{M}$ is called

(a) *dissipative*, if

$$\operatorname{Re}\langle x, y\rangle \leq 0, \quad \text{for all } (x, y) \in \mathcal{M}.$$

(b) *nonnegative*, denoted by $\mathcal{M} \geq 0$, if $\mathcal{M}$ is symmetric with

$$\langle x, y\rangle \geq 0, \quad \text{for all } (x, y) \in \mathcal{M}$$

and *nonpositive*, denoted by $\mathcal{M} \leq 0$, if $-\mathcal{M}$ is nonnegative.

(c) *maximally dissipative*, if it dissipative, and it is not a proper subspace of any dissipative linear relation in $\mathbb{K}^{2n}$.

(d) *maximally nonnegative*, if it is nonnegative, and it is not a proper subspace of any nonnegative linear relation in $\mathbb{K}^{2n}$.

We would like to remark that other definitions of dissipative linear relations exists in the literature. For example, in [BHdS20, Def. 1.6.1] a linear relation $\mathcal{M} \subseteq \mathbb{C}^{2n}$ is called dissipative if $\operatorname{Im}\langle x, y\rangle \geq 0$ for all $(x, y) \in \mathcal{M}$. However, if $\mathcal{M}$ is dissipative in the sense of Def. 1.3.5 then $-\imath\mathcal{M}$ is dissipative in the aforementioned sense and vice versa.
In the context of port-Hamiltonian systems, see e.g. [vdSM18], a different terminology is custom, and we put a special emphasis on it by giving the following definition.

Definition 1.3.6 (Dirac and Lagrangian subspace)**.** A *Dirac structure* or *Dirac subspace* $\mathcal{D}$ of $\mathbb{K}^n \times \mathbb{K}^n$ is a skew-adjoint linear relation in $\mathbb{K}^n$ and a *Lagrange structure* or *Lagrangian subspace* of $\mathcal{L}$ of $\mathbb{K}^n \times \mathbb{K}^n$ is a self-adjoint linear relation in $\mathbb{K}^n$.

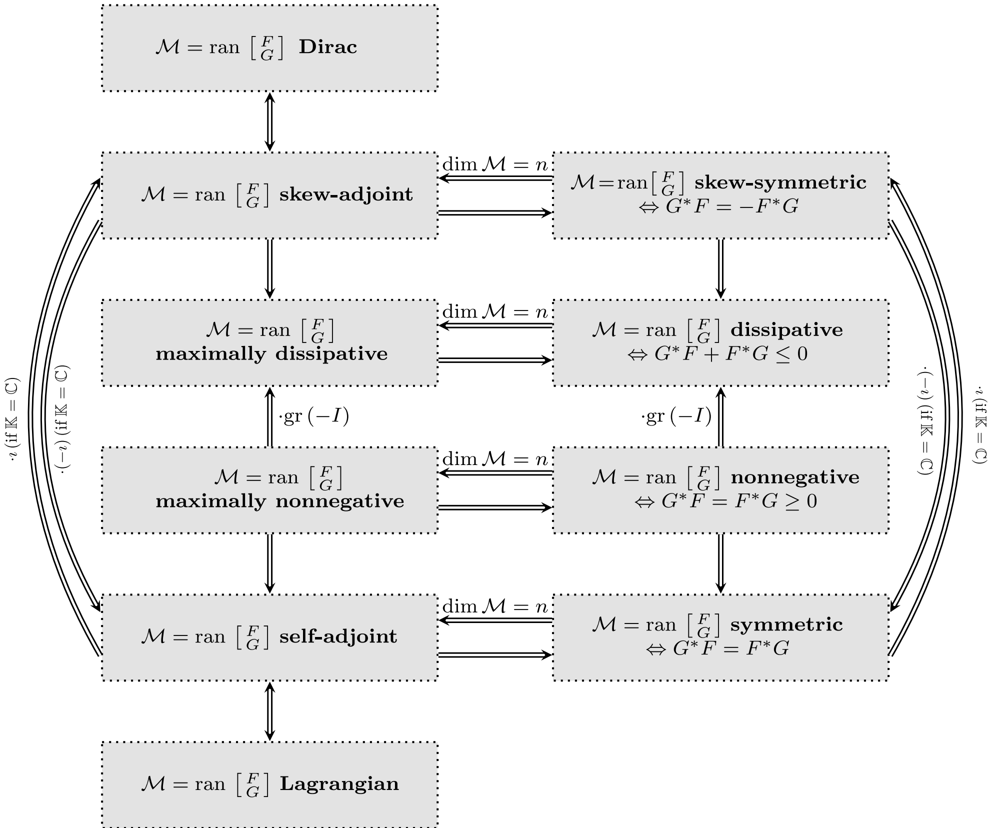

Figure 1.1: An overview of the structural assumptions on the subspace $\mathcal{M}$ in range representation with $F, G \in \mathbb{K}^{n\times n}$.

In particular, Dirac subspaces are maximally dissipative linear relations, and Lagrangian subspaces are maximally nonnegative linear relations, but the converse is not true in general, see Fig. 1.1.

We now collect some basic results on linear relations. As a consequence of Lem. 1.3.3 and Remark 1.3.4, we can characterize nonnegativity and dissipativity as follows.

Lemma 1.3.7. *Let $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ with $F, G \in \mathbb{K}^{n\times l}$ be a linear relation. Then $\mathcal{M}$ is nonnegative if and only if $G^*F = F^*G \ge 0$ and dissipative if and only if $G^*F + F^*G \le 0$. Moreover, the following statements are equivalent.*

(a) *$\mathcal{M}$ is maximally nonnegative.*

(b) $\mathcal{M}$ *is nonnegative and* $\dim \mathcal{M} = n$.

(c) $\mathcal{M}$ *is nonnegative and self-adjoint.*

Further, $\mathcal{M}$ *is maximally dissipative if and only if* $\dim \mathcal{M} = n$ *and* $G^*F + F^*G \leq 0$.

Proof. For the first two equivalences, observe that the range representation yields

$$\langle x, y\rangle \geq 0, \text{ for all } (x,y) \in \mathcal{M} \quad \Longleftrightarrow \quad z^*F^*Gz \geq 0, \text{ for all } z \in \mathbb{K}^n$$

and

$$\operatorname{Re}\langle x, y\rangle \leq 0, \text{ for all } (x,y) \in \mathcal{M} \iff z^*(F^*G + G^*F)z \leq 0, \text{ for all } z \in \mathbb{K}^n.$$

The statements then follows directly from Lem. 1.3.3. We now show the equivalences (a)-(c).
<u>"(a)$\Longrightarrow$(b)":</u> Assume that $\mathcal{M}$ is maximally nonnegative. Then it follows from the definition of nonnegativity that $\mathcal{M}^*$ is nonnegative as well. By the symmetry of $\mathcal{M}$, we further have $\mathcal{M} \subset \mathcal{M}^*$, and maximality leads to $\mathcal{M} = \mathcal{M}^*$. Thus by Lem. 1.3.3, $\dim \mathcal{M} = n$.
<u>"(b)$\Longrightarrow$(a)":</u> Let $\mathcal{M}$ be nonnegative with $\dim \mathcal{M} = n$. Then in particular, $\mathcal{M}$ is symmetric with $\dim \mathcal{M} = n$, whence, by Lem. 1.3.3, it is not a proper subspace of a symmetric relation. In particular, it is not a proper subspace of a nonnegative relation. That is, $\mathcal{M}$ is maximally nonnegative.
<u>"(b)$\Longleftrightarrow$(c)":</u> This equivalence is a direct consequence of the equivalence of the statements (a) and (b) of Lem. 1.3.3.
It remains to prove the last equivalence for dissipative relations. Assume that $\mathcal{M} = \operatorname{ran}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ is dissipative. First note that

$$F^*G + G^*F = \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]^* \left[\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] \leq 0$$

and that $\left[\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right]$ has n positive and n negative eigenvalues. If $\dim \mathcal{M} > n$, then Sylvester's inertia theorem [HJ13, Thm. 4.5.8] yields that $F^*G + G^*F$ has to have at least one positive eigenvalue. Consequently, any n-dimensional dissipative relation is maximal. On the other hand, if $\mathcal{M}$ is dissipative with $\dim \mathcal{M} < n$, we can, again by employing Sylvester's inertia theorem, infer that $\mathcal{M}$ can be further extended to a linear relation which is still dissipative. □

Lemma 1.3.8. *Let* $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ *with* $F, G \in \mathbb{K}^{n\times l}$ *be a dissipative (symmetric) linear relation. Then* $\operatorname{dom} \mathcal{M} \subseteq (\operatorname{mul} \mathcal{M})^\perp$ *and* $\operatorname{ran} \mathcal{M} \subseteq (\ker \mathcal{M})^\perp$*. Furthermore, the following three statements are equivalent.*

(i) $\mathcal{M}$ *is maximally dissipative (self-adjoint).*

(ii) $\mathcal{M}$ *is dissipative (symmetric) and* $\operatorname{dom} \mathcal{M} = (\operatorname{mul} \mathcal{M})^\perp$*.*

(iii) $\mathcal{M}$ *is dissipative (symmetric) and* $\operatorname{ran} \mathcal{M} = (\ker \mathcal{M})^\perp$*.*

Proof. The statement $\operatorname{dom} \mathcal{M} \subseteq (\operatorname{mul} \mathcal{M})^\perp$ as well as the implication "(i)$\Longrightarrow$(ii)" have been proven in [ADW13, Lem. 2.1] for the dissipative case, and they follow from (1.8) for the symmetric case. Further, if $\mathcal{M}$ is dissipative (symmetric), so is $\mathcal{M}^{-1}$ by Lem 1.3.3. Hence, $\ker \mathcal{M} = \operatorname{mul} (\mathcal{M}^{-1}) \subseteq \operatorname{dom} (\mathcal{M}^{-1})^\perp = (\operatorname{ran} \mathcal{M})^\perp$.
"(ii)$\Longrightarrow$(i)": Let $\mathcal{M}$ be dissipative or symmetric and, additionally, assume that $\operatorname{dom} \mathcal{M} = (\operatorname{mul} \mathcal{M})^\perp$. For $k := \dim \operatorname{dom} \mathcal{M}$, let $(x_1, \dots, x_k)$ be a basis of $\operatorname{dom} \mathcal{M}$. Then there exist $y_1, \dots, y_k \in \mathbb{K}^n$ such that $(x_i, y_i) \in \mathcal{M}$ for $i = 1, \dots, k$. Consequently,

$$\operatorname{span} \{(x_1, y_k), \dots, (x_k, y_k)\} \cap (\{0\} \times \operatorname{mul} \mathcal{M}) = \{0\}.$$

Since, further, $\{0\} \times \operatorname{mul} \mathcal{M} \subseteq \mathcal{M}$, we obtain that

$$\operatorname{span} \{(x_1, y_k), \dots, (x_k, y_k)\} \cap (\{0\} \times \operatorname{mul} \mathcal{M}) \subset \mathcal{M},$$

and thus

$$\dim \mathcal{M} \geq \dim \operatorname{dom} \mathcal{M} + \dim \operatorname{mul} \mathcal{M} = \dim(\operatorname{mul} \mathcal{M})^\perp + \dim \operatorname{mul} \mathcal{M} = n.$$

Then Lem. 1.3.7 (resp. Lem. 1.3.3) implies that $\mathcal{M}$ is maximally dissipative (self-adjoint).
"(ii)$\Longleftrightarrow$(iii)": This follows by the already proven equivalence between (i) and (ii), together with $\operatorname{dom} \mathcal{M} = \operatorname{ran} \mathcal{M}^{-1}$, $\operatorname{mul} \mathcal{M} = \ker \mathcal{M}^{-1}$, and the fact that $\mathcal{M}$ is dissipative (maximally dissipative, symmetric, self-adjoint) if and only if the inverse $\mathcal{M}^{-1}$ has the respective property. □

Proposition 1.3.9. *Let* $\mathcal{M} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ *with* $F, G \in \mathbb{K}^{n\times l}$ *be a linear relation with* $\dim \mathcal{M} = n$*. Then* $\mathcal{M} = \operatorname{gr} M$ *for some* $M \in \mathbb{K}^{n\times n}$ *if and only if* $\operatorname{rk} F = n$*.*
In this case, $\mathcal{M}$ *is self-adjoint (skew-adjoint, maximally nonnegative, maximally dissipative) if and only if* M *is Hermitian (skew-Hermitian, positive semi-definite, dissipative).*

Proof. Let $\mathcal{M} = \operatorname{ran}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ with $\dim \mathcal{M} = n$. If $\mathcal{M} = \operatorname{gr} M$ for some $M \in \mathbb{K}^{n\times n}$ then $\operatorname{ran} F = \operatorname{dom} \mathcal{M} = \mathbb{K}^n$ which implies $\operatorname{rk} F = n$. Conversely, let $F \in \mathbb{K}^{n\times l}$ be given with $\operatorname{rk} F = n$. Then $\operatorname{dom} \mathcal{M} = \operatorname{ran} F = \mathbb{K}^n$. Consider the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{K}^n$. Then there exist $x_1, \ldots, x_n$ with $Fx_i = e_i$ for $i = 1, \ldots, n$. Define

$$M = [Gx_1, \ldots, Gx_n] \in \mathbb{K}^{n\times n}.$$

Then, by $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] x_i = \left(\begin{smallmatrix} Fx_i \\ Gx_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_i \\ Me_i \end{smallmatrix}\right) = \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] e_i$, we obtain

$$\operatorname{ran}\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \subset \operatorname{ran}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right].$$

However, since the dimensions of both spaces coincide, we even have equality.
The second part of the result follows from Lem. 1.3.8 and Lem. 1.3.7. □

The following result (cf. [ABJ$^+$09]) enables us to derive classic representations of Dirac and Lagrangian subspaces that can be found in, e.g., [vdSJ14, Sec. 5.2], [DvdS98], and [Cou90, Prop. 1.1.4].

Proposition 1.3.10. *Let $\mathcal{M} \subset \mathbb{K}^{2m}$ be a dissipative (symmetric, skew-symmetric, nonnegative) linear relation. Then* $\operatorname{dom} \mathcal{M} \subset (\operatorname{mul} \mathcal{M})^\perp$ *and the operator*

$$M : \operatorname{dom} \mathcal{M} \to (\operatorname{mul} \mathcal{M})^\perp, \quad Mx := P_{(\operatorname{mul} \mathcal{M})^\perp} y, \ \textit{if } (x, y) \in \mathcal{M}, \tag{1.10}$$

is well-defined, dissipative (Hermitian, skew-Hermitian, positive semi-definite) and

$$\mathcal{M} = \operatorname{ran}\begin{bmatrix} P_{\operatorname{dom} \mathcal{M}} \\ MP_{\operatorname{dom} \mathcal{M}} + P_{\operatorname{mul} \mathcal{M}} \end{bmatrix}. \tag{1.11}$$

Further, if $\mathcal{M}$ is maximally dissipative (self-adjoint, skew-adjoint), then $\operatorname{ran} M \subset \operatorname{dom} \mathcal{M}$.

Proof. *Step 1:* We show that the operator M as given by (1.10) is well-defined. To this end, let $(x, y), (x, z) \in \mathcal{M}$. Then $(0, y - z) \in \mathcal{M}$ and consequently $y - z \in \operatorname{mul} \mathcal{M}$, which is equivalent to $P_{(\operatorname{mul} \mathcal{M})^\perp}(y - z) = 0$, i.e., $P_{(\operatorname{mul} \mathcal{M})^\perp} y = Mx = P_{(\operatorname{mul} \mathcal{M})^\perp} z$, as desired.
Step 2: We show that M is dissipative (Hermitian, skew-Hermitian, positive semi-definite) if $\mathcal{M}$ is dissipative (symmetric, skew-symmetric, nonnegative). To this end, let $(x, y) \in \mathcal{M}$. Then the trivial decomposition $y = P_{(\operatorname{mul} \mathcal{M})^\perp} y + P_{\operatorname{mul} \mathcal{M}} y$, together with the range representations of Lem. 1.3.3 and Remark 1.3.4, as well as $\operatorname{dom} \mathcal{M} \subset (\operatorname{mul} \mathcal{M})^\perp$ due to Lem. 1.3.8 yield

$$0 \geq \operatorname{Re}\langle x, y\rangle = \operatorname{Re}\langle x, P_{(\operatorname{mul} \mathcal{M})^\perp} y + P_{\operatorname{mul} \mathcal{M}} y\rangle$$

$$= \mathrm{Re}\langle x, P_{(\mathrm{mul}\ \mathcal{M})^\perp} y\rangle = \mathrm{Re}\langle x, Mx\rangle$$

if $\mathcal{M}$ is dissipative,

$$\begin{aligned} 0 = \mathrm{Re}\langle x, y\rangle &= \mathrm{Re}\langle x, P_{(\mathrm{mul}\ \mathcal{M})^\perp} y + P_{\mathrm{mul}\ \mathcal{M}} y\rangle \\ &= \mathrm{Re}\langle x, P_{(\mathrm{mul}\ \mathcal{M})^\perp} y\rangle = \mathrm{Re}\langle x, Mx\rangle \end{aligned}$$

if $\mathcal{M}$ is skew-symmetric,

$$\begin{aligned} \langle x, Mx\rangle = \langle x, P_{(\mathrm{mul}\ \mathcal{M})^\perp} y\rangle =& \langle x, P_{(\mathrm{mul}\ \mathcal{M})^\perp} y + P_{\mathrm{mul}\ \mathcal{M}} y\rangle \\ =& \langle x, y\rangle = \langle y, x\rangle = \langle Mx, x\rangle \end{aligned}$$

if $\mathcal{M}$ is symmetric, and additionally

$$0 \le \langle x, y\rangle = \langle x, Mx\rangle$$

if $\mathcal{M}$ is nonegative.

Step 3: We prove the range representation (1.11). By $\mathrm{dom}\ \mathcal{M} \subset (\mathrm{mul}\ \mathcal{M})^\perp$ the readily verified decomposition

$$\mathcal{M} = \{\ (x, Mx)\ \mid x \in \mathrm{dom}\ \mathcal{M}\ \} \mathbin{\widehat{\oplus}} (\{0\} \times \mathrm{mul}\ \mathcal{M}) \tag{1.12}$$

is orthogonal. Let $x \in \mathbb{K}^m$. Then (1.12) yields

$$\begin{aligned} &(P_{\mathrm{dom}\ \mathcal{M}} x, MP_{\mathrm{dom}\, M} x + P_{\mathrm{mul}\ \mathcal{M}} x) \\ &= (P_{\mathrm{dom}\ \mathcal{M}} x, MP_{\mathrm{dom}\ \mathcal{M}} x + (0, P_{\mathrm{mul}\ \mathcal{M}} x) \in \mathcal{M}. \end{aligned}$$

this proves the backward inclusion "$\supset$" of (1.11). If $(x, y) \in \mathcal{M}$, then $(x, y) = (x, Mx + z)$ for some $z \in \mathrm{mul}\ \mathcal{M}$. Therefore, there exists some $\hat{z} \in \mathbb{K}^m$ such that $P_{\mathrm{dom}\ \mathcal{M}} \hat{z} = x$ and $P_{(\mathrm{dom}\ \mathcal{M})^\perp} \hat{z} = z$. Combining all of the above, we find

$$\begin{aligned} (x, y) =& (x, Mx + z) = (x, Mx + P_{\mathrm{mul}\ \mathcal{M}} z) \\ =& (P_{\mathrm{dom}\ \mathcal{M}} \hat{z}, MP_{\mathrm{dom}\ \mathcal{M}} \hat{z} + P_{\mathrm{mul}\ \mathcal{M}} P_{(\mathrm{dom}\ \mathcal{M})^\perp} \hat{z}) \\ =& (P_{\mathrm{dom}\ \mathcal{M}} MP_{\mathrm{dom}\ \mathcal{M}} \hat{z} + P_{\mathrm{mul}\ \mathcal{M}} \hat{z}). \end{aligned}$$

This proves the forward inclusion "$\subset$" of (1.11) and thus (1.11) holds.

Step 4: We prove the last statement of the proposition. If $\mathcal{M}$ is maximally dissipative (self-adjoint, skew-adjoint), then by (1.10), Lem. 1.3.3, and Remark 1.3.4 we have $\mathrm{ran}\ M \subset (\mathrm{mul}\ \mathcal{M})^\perp = \mathrm{dom}\ \mathcal{M}$. $\square$

The following result is well-know for Dirac and Lagrange structures, see [vdSM18] and [vdSJ14, Sec. 5.2]. We are also interested in the maximal dissipative case and give the proof for the sake of completeness.

Corollary 1.3.11. *Let $\mathcal{M}$ be a maximally dissipative or self-adjoint linear relation in $\mathbb{K}^n$. Let $\dim\ker\mathcal{M} = k$ and $\dim \operatorname{mul} \mathcal{M} = m$. Then there exist $K, M \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times k}$, and $C \in \mathbb{K}^{n\times m}$ such that*

$$\begin{aligned} \ker K = \ker M &= \operatorname{mul} \mathcal{M} \widehat{\oplus} \ker\mathcal{M}, \\ \operatorname{ran} K \subset \operatorname{ran} \mathcal{M}, &\quad \operatorname{ran} M \subset \operatorname{dom} \mathcal{M}, \\ \operatorname{ran} B = \ker\mathcal{M}, &\quad \operatorname{ran} C = \operatorname{mul} \mathcal{M}, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \mathcal{M} &= \left\{ \ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \middle|\ \exists\lambda \in \mathbb{K}^k : f = Ke + B\lambda \ \wedge B^*e = 0 \ \right\} \\ &= \{ \ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists\lambda \in \mathbb{K}^m : e = Mf + C\lambda \ \wedge \ C^*f = 0 \ \}. \end{aligned} \tag{1.14}$$

Further, if $\mathcal{M}$ is maximally dissipative (skew-adjoint), then K, M are dissipative (skew-Hermitian), whereas if $\mathcal{M}$ is maximally nonnegative (self-adjoint), then K, M are positive semi-definite (symmetric).
Conversely, any linear relation $\mathcal{M}$ with such a representation is a maximally dissipative (skew-adjoint, self-adjoint, maximally nonnegative) relation.

Proof. Step 1: We construct M and C. Let $\mathcal{O} : \operatorname{dom} \mathcal{M} \to (\operatorname{mul} \mathcal{M})^\perp = \operatorname{dom} \mathcal{M}$ be the operator given by (1.10). Let $M \in \mathbb{K}^{n\times n}$ be the matrix representation of $\mathcal{O}P_{\operatorname{dom} \mathcal{M}}$ with respect to the canonical basis. For $x \in \mathbb{K}^n$ it holds

$$\begin{aligned} Mx = 0 &\Longleftrightarrow \mathcal{O}P_{\operatorname{dom} \mathcal{M}}x = 0 \\ &\Longleftrightarrow \big(x \in (\operatorname{dom} \mathcal{M})^\perp\big) \vee (\exists y \in \operatorname{mul} \mathcal{M} : (P_{\operatorname{dom} \mathcal{M}}x, y) \in \mathcal{M}) \\ &\Longleftrightarrow (x \in \operatorname{mul} \mathcal{M}) \vee (P_{\operatorname{dom} \mathcal{M}}x \in \ker\mathcal{M}) \\ &\Longleftrightarrow x \in \operatorname{mul} \mathcal{M} \widehat{\oplus} \ker\mathcal{M}, \end{aligned}$$

and it also follows from Prop. 1.3.10 that $\operatorname{ran} M \subset \operatorname{dom} \mathcal{M}$. Let $C \in \mathbb{K}^{n\times m}$ be a matrix whose columns form a basis of $\operatorname{mul} \mathcal{M}$. Then trivially $\operatorname{ran} C = \operatorname{mul} \mathcal{M}$. Invoking (1.11)

$$\mathcal{M} = \operatorname{ran} \begin{bmatrix} P_{\operatorname{dom} \mathcal{M}} \\ \mathcal{O}P_{\operatorname{dom} \mathcal{M}} + P_{\operatorname{mul} \mathcal{M}} \end{bmatrix}$$

$$
\begin{aligned}
&= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \mid \exists z \in \mathbb{K}^n : f = P_{\text{dom } \mathcal{M}} z, \\
&\qquad\qquad\qquad\qquad\qquad\qquad e = \mathcal{O} P_{\text{dom } \mathcal{M}} z + P_{\text{mul } \mathcal{M}^c} z\ \} \\
&= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \mid \exists z_1 \in \text{dom } \mathcal{M}, z_2 \in (\text{dom } \mathcal{M})^\perp : \\
&\qquad f = P_{\text{dom } \mathcal{M}}(z_1+z_2),\ e = \mathcal{O} P_{\text{dom } \mathcal{M}}(z_1+z_2) + P_{\text{mul } \mathcal{M}}(z_1+z_2)\ \} \\
&= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \mid \exists z_1 \in \text{dom } \mathcal{M},\ z_2 \in \text{mul } \mathcal{M} : \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad f = z_1, e = \mathcal{O} z_1 + z_2\ \} \\
&= \ \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n\ \mid \exists \lambda \in \mathbb{K}^m : C^* f = 0\ \wedge\ e = Mf + C\lambda\ \}.
\end{aligned}
$$

Step 2: We show the properties of M inherited by $\mathcal{M}$. By Prop. 1.3.10, $\mathcal{O}$ is dissipative (Hermitian, skew-Hermitian, positive semi-definite) if $\mathcal{M}$ is maximally dissipative (skew-adjoint, self-adjoint, maximally nonnegative). M being the matrix representation of $\mathcal{O}$ with respect to the canonical basis, it inherits these properties.
Step 3: We construct K and B and show their properties. If $\mathcal{M}$ is maximally dissipative (skew-adjoint, self-adjoint, maximally nonnegative), then so is $\mathcal{M}^{-1}$. Hence, it suffices to apply Steps 1 & 2 to $\mathcal{M}^{-1}$ since mul $\mathcal{M}^{-1} = \ker \mathcal{M}$ and dom $\mathcal{M}^{-1} =$ ran $\mathcal{M}$.
Step 3: We show the reverse statement. Let $\mathcal{M}$ satisfy (1.14) with matrices fulfilling (1.13). Then for all $(f,e) \in \mathcal{M}$ there exist some $\lambda_k \in \mathbb{K}^k$ and $\lambda_m \in \mathbb{K}^m$ such that

$$
\langle f, e\rangle = \langle Ke + B\lambda_k, e\rangle = \langle Ke, e\rangle \quad \text{and} \quad \langle f, e\rangle = \langle f, Mf + C\lambda_m\rangle = \langle f, Mf\rangle.
$$

From this we directly see the properties of dissiptivity, nonnegativity, symmetry, and skew-symmetry of $\mathcal{M}$ inherited by K and M, and we deduce dom $\mathcal{M} \subset (\text{mul } \mathcal{M})^\perp$ and ran $\mathcal{M} \subset (\ker \mathcal{M})^\perp$ by Lem. 1.3.8. Note that

$$
\mathcal{M} = \left[\begin{smallmatrix} K & B \\ I & 0 \end{smallmatrix}\right] \left((\ker \mathcal{M})^\perp \times \mathbb{K}^k\right) = \left[\begin{smallmatrix} I & 0 \\ M & C \end{smallmatrix}\right] \left((\text{mul } \mathcal{M})^\perp \times \mathbb{K}^m\right),
$$

with $\ker \left[\begin{smallmatrix} K & B \\ I & 0 \end{smallmatrix}\right] \cap \left((\ker \mathcal{M})^\perp \times \mathbb{K}^k\right) = \{0\}$ and $\ker \left[\begin{smallmatrix} I & 0 \\ M & C \end{smallmatrix}\right] \cap \left((\text{mul } \mathcal{M})^\perp \times \mathbb{K}^m\right) = \{0\}$ since dom $\mathcal{M} \subset (\text{mul } \mathcal{M})^\perp$ and ran $\mathcal{M} \subset (\ker \mathcal{M})^\perp$. Hence, $\dim \mathcal{M} = (n-k) + k = (n-m) + m = n$, i.e., $\mathcal{M}$ is maximal. □

Besides multiplication, there exists another way to combine two linear relations.

Definition 1.3.12. Let $\mathcal{M}, \mathcal{L}$ be two relations in $\mathbb{K}^{n_1+n_2}$ and $\mathbb{K}^{n_2+n_3}$, respectively, for some n_1, n_2, $n_3 \in \mathbb{N}_0$. We define the *interconnection* of $\mathcal{M}$ and $\mathcal{L}$ with respect to

$\mathbb{K}^{n_2}$ as

$$\mathcal{M} \circ \mathcal{L} := \Big\{ (f_1, f_3, e_1, e_3) \in \mathbb{K}^{n_1+n_3} \times \mathbb{K}^{n_1+n_3} \mid \exists (f_2, e_2) \in \mathbb{K}^{2n_2} : \\ (f_1, f_2, e_1, e_2) \in \mathcal{M} \wedge (-f_2, f_3, e_2, e_3) \in \mathcal{L} \Big\}.$$

Note that this interconnection uses the same notation as the *composition* in [vdSJ14] between a nonpositive relation and a skew-adjoint relation. However, it defines a different object while being conceptually similar. Our notion of interconnection corresponds to the interconnection of two skew-adjoint subspaces, as found in, e.g., [CvdSB07]. In the context of Def. 1.3.12, the sorted Cartesian product $\mathcal{M} \widehat{\times} \mathcal{L}$ can be interpreted as a special case of an interconnection, namely, an interconnection where one interconnects with respect to $\mathbb{K}^{n_2}$ with $n_2 = 0$.
The following result shows that any dissipative relation can be understood as the interconnection of a skew-adjoint relation and a nonpositive relation.

Lemma 1.3.13. *Let $\mathcal{M}$ be a dissipative relation in $\mathbb{K}^n$. Then there exist a skew-adjoint relation $\mathcal{D}$ in $\mathbb{K}^{2n}$ and a nonpositive relation $\mathcal{R}$ in $\mathbb{K}^n$ such that interconnecting with respect to $\mathbb{K}^n$ we have*

$$\mathcal{M} = \mathcal{D} \circ \mathcal{R}.$$

Moreover, if $\mathcal{M}$ is maximal, then $\mathcal{R}$ can be chosen maximal too. In particular, there exist matrices $J = -J^* \in \mathbb{K}^{n\times n}$, $0 \leq R \in \mathbb{K}^{n\times n}$, and $C \in \mathbb{K}^{n\times m}$ with $\dim \mathcal{M} = m$ such that

$$\mathcal{D} = \operatorname{gr} \begin{bmatrix} J & -I_n \\ I_n & 0 \end{bmatrix},$$

$$\mathcal{R} = \{\ (f_R, e_R) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists \lambda \in \mathbb{K}^m : f_R = -Re_R + C\lambda \ \wedge \ C^* e_R = 0\ \}.$$

Proof. By Prop. 1.3.10 there exists a dissipative operator $M : \operatorname{dom} \mathcal{M} \to (\operatorname{mul} \mathcal{M})^\perp$ such that

$$\mathcal{M} = \operatorname{ran} \begin{bmatrix} P_{\operatorname{dom} \mathcal{M}} \\ MP_{\operatorname{dom} \mathcal{M}} + P_{\operatorname{mul} \mathcal{M}} \end{bmatrix}.$$

Extend D (possibly trivially) to a dissipative operator $\bar{M} : \mathbb{K}^n \to \mathbb{K}^n$, identify $\bar{M}$ with its matrix representation with respect to the canonical basis and decompose it as $\bar{M} = J - R$ with $J = -J^* \in \mathbb{K}^{n\times n}$ and $0 \leq R \in \mathbb{K}^{n\times n}$. Then

$$\mathcal{M} = \operatorname{ran} \begin{bmatrix} P_{\operatorname{dom} \mathcal{M}} \\ (J-R)P_{\operatorname{dom} \mathcal{M}} + P_{\operatorname{mul} \mathcal{M}} \end{bmatrix}.$$

Defining

$$\mathcal{D} = \operatorname{gr} \begin{bmatrix} J & -I_n \\ I_n & 0 \end{bmatrix}, \quad \mathcal{R} = \operatorname{ran} \begin{bmatrix} -RP_{\mathrm{dom}\ \mathcal{M}} + P_{\mathrm{mul}\ \mathcal{M}} \\ P_{\mathrm{dom}\ \mathcal{M}} \end{bmatrix},$$

we find

$$\begin{aligned}
\mathcal{D} \circ \mathcal{R} &= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists f_R, e_R \in \mathbb{K}^n : (f, f_R, e, e_R) \in \mathcal{D} \ \wedge \ (-f_R, e_R) \in \mathcal{R}\ \} \\
&= \left\{ \ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \middle|\ \exists f_R, e_R, z \in \mathbb{K}^n : \begin{pmatrix} e \\ e_R \end{pmatrix} = \begin{bmatrix} J & -I_n \\ I_n & 0 \end{bmatrix} \begin{pmatrix} f \\ f_R \end{pmatrix} \ \wedge \right. \\
&\qquad\qquad \left. \begin{pmatrix} -f_R \\ e_R \end{pmatrix} = \begin{bmatrix} -RP_{\mathrm{dom}\ \mathcal{M}} + P_{\mathrm{mul}\ \mathcal{M}} \\ P_{\mathrm{dom}\ \mathcal{M}} \end{bmatrix} z \ \right\} \\
&= \left\{ \ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \middle|\ \exists z \in \mathbb{R}^n : \begin{pmatrix} f \\ e \end{pmatrix} = \begin{bmatrix} P_{\mathrm{dom}\ \mathcal{M}} \\ (J-R)P_{\mathrm{dom}\ \mathcal{M}} + P_{\mathrm{mul}\ \mathcal{M}} \end{bmatrix} z \ \right\} \\
&= \mathcal{M}.
\end{aligned}$$

Further, $\mathcal{D}$ is clearly a skew-adjoint and $\mathcal{R}$ a nonpositive relation. It remains to show that $\mathcal{R}$ can be chosen maximally nonpositive if $\mathcal{M}$ is maximally dissipative. In this case, we can perform a similar construction with Cor. 1.3.11 instead of Prop. 1.3.10. Namely, we know that there exist a dissipative matrix $D \in \mathbb{K}^{n \times n}$ and a matrix $C \in \mathbb{K}^{n \times m}$ where $m = \dim \operatorname{mul} \mathcal{M}$ such that

$$\mathcal{M} = \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists \lambda \in \mathbb{K}^m : e = Mf + C\lambda \ \wedge \ C^* f = 0\ \}.$$

Now decomposing $D = J - R$ with $J = -J^* \in \mathbb{K}^{n \times n}$ and $0 \le R \in \mathbb{K}^{n \times n}$ and defining

$$\begin{aligned}
\mathcal{D} &= \operatorname{gr} \begin{bmatrix} J & -I_n \\ I_n & 0 \end{bmatrix}, \\
\mathcal{R} &= \{\ (f_R, e_R) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists \lambda \in \mathbb{K}^m : f_R = -Re_R + C\lambda \ \wedge \ C^* e_R = 0\ \},
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{D} \circ \mathcal{R} &= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists f_R, e_R \in \mathbb{K}^n : (f, f_R, e, e_R) \in \mathcal{D} \ \wedge \ (-f_R, e_R) \in \mathcal{R}\ \} \\
&= \left\{ \ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \middle|\ \exists f_R, e_R \in \mathbb{K}^n : \begin{pmatrix} e \\ e_R \end{pmatrix} = \begin{bmatrix} J & -I_n \\ I_n & 0 \end{bmatrix} \begin{pmatrix} f \\ f_R \end{pmatrix} \right. \\
&\qquad\qquad \left. \wedge \ \exists \lambda \in \mathbb{K}^m : f_R = Re_R - C\lambda \ \wedge \ C^* e_R = 0 \ \right\} \\
&= \{\ (f,e) \in \mathbb{K}^n \times \mathbb{K}^n \ \mid \exists \lambda \in \mathbb{K}^m : e = (J-R)f + C\lambda \ \wedge \ C^* f = 0\ \}
\end{aligned}$$

$$= \mathcal{M}.$$

Moreover, $\mathcal{D}$ is again readily a skew-adjoint and $-\mathcal{R}$ is maximally nonnegative by Cor. 1.3.11, i.e., $\mathcal{R}$ is maximally nonpositive. □

Next, we show how to obtain the range representation of an interconnection of two linear relations. A similar result to the subsequent proposition was already formulated and proven for skew-adjoint relations in [CvdSB07, Thm. 4]. However, we present the proof for the sake of completeness.

Proposition 1.3.14. *Let $\mathcal{M} = \operatorname{ran} \left[F_1^* \; F_{21}^* \; G_1^* \; G_{21}^* \right]^*$, $\mathcal{L} = \operatorname{ran} \left[F_{22}^* \; F_3^* \; G_{22}^* \; G_3^* \right]^*$ be two relations in $\mathbb{K}^{n_1+n_2}$ and $\mathbb{K}^{n_2+n_3}$, respectively, for some $F_1, G_1 \in \mathbb{K}^{n_1 \times m}$, $F_{21}, G_{21} \in \mathbb{K}^{n_2 \times m}$, $F_{22}, G_{22} \in \mathbb{K}^{n_2 \times l}$ and $F_3, G_3 \in \mathbb{K}^{n_3 \times l}$ with $n_1, n_2, n_3, m, l \in \mathbb{N}_0$. Let $M \in \mathbb{K}^{m \times k}$, $L \in \mathbb{K}^{l \times k}$ with $\operatorname{ran} \left[\begin{smallmatrix} M \\ L \end{smallmatrix} \right] = \ker \left[\begin{smallmatrix} F_{21} & F_{22} \\ G_{21} & -G_{22} \end{smallmatrix} \right]$ for some $k \in \mathbb{N}_0$. Then*

$$\mathcal{M} \circ \mathcal{L} = \operatorname{ran} \begin{bmatrix} F_1 M \\ F_3 L \\ G_1 M \\ G_3 L \end{bmatrix}.$$

Further, the following table shows properties inherited by $\mathcal{M} \circ \mathcal{L}$ from $\mathcal{M}$ and $\mathcal{L}$.

$\mathcal{M}$ \ $\mathcal{L}$	*skew-symmetric*	*dissipative*	*nonpositive*	*symmetric*
skew-symmetric	*skew-symmetric*	*dissipative*	*dissipative*	–
dissipative	*dissipative*	*dissipative*	*dissipative*	–
nonpositive	*dissipative*	*dissipative*	*nonpositive*	*symmetric*
symmetric	–	–	*symmetric*	*symmetric*

Furthermore, if $\mathcal{M} \subset \mathcal{M}^*$ and $\mathcal{L} \subset \mathcal{L}^*$ ($\mathcal{M} \subset -\mathcal{M}^*$ and $\mathcal{L} \subset -\mathcal{L}^*$), then $\mathcal{M} \circ \mathcal{L} \subset (\mathcal{M} \circ \mathcal{L})^*$ ($\mathcal{M} \circ \mathcal{L} \subset -(\mathcal{M} \circ \mathcal{L})^*$).

Proof. The first statement follows by observing

$$\begin{aligned} &(f_1, f_3, e_1, e_3) \in \mathcal{M} \circ \mathcal{L} \\ \iff &\exists (f_2, e_2) \in \mathbb{K}^{2n_2} : (f_1, f_2, e_1, e_2) \in \mathcal{M} \;\wedge\; (-f_2, f_3, e_2, e_3) \in \mathcal{L} \end{aligned}$$

$$
\begin{aligned}
&\iff \exists\lambda\in\mathbb{K}^m\exists\mu\in\mathbb{K}^l : \begin{pmatrix} f_1\\ f_2\\ e_1\\ e_2\end{pmatrix} = \begin{bmatrix} F_1\\ F_{21}\\ G_1\\ G_{21}\end{bmatrix}\lambda \ \wedge\ \begin{pmatrix} -f_2\\ f_3\\ e_2\\ e_3\end{pmatrix} = \begin{bmatrix} F_{22}\\ F_3\\ G_{22}\\ G_3\end{bmatrix}\mu\\
&\iff \exists\lambda\in\mathbb{K}^m\exists\mu\in\mathbb{K}^l : \left(\begin{smallmatrix} f_1\\ e_1\end{smallmatrix}\right) = \left[\begin{smallmatrix} F_1\\ G_1\end{smallmatrix}\right]\lambda \ \wedge\ \left(\begin{smallmatrix} f_3\\ e_3\end{smallmatrix}\right) = \left[\begin{smallmatrix} F_3\\ G_3\end{smallmatrix}\right]\mu \ \wedge\ \left(\begin{smallmatrix} \lambda\\ \mu\end{smallmatrix}\right) \in \ker\left[\begin{smallmatrix} F_{21} & F_{22}\\ G_{21} & -G_{22}\end{smallmatrix}\right]\\
&\iff \exists\nu\in\mathbb{K}^k : \left(\begin{smallmatrix} f_1\\ e_1\end{smallmatrix}\right) = \left[\begin{smallmatrix} F_1\\ G_1\end{smallmatrix}\right]\lambda \ \wedge\ \left(\begin{smallmatrix} f_3\\ e_3\end{smallmatrix}\right) = \left[\begin{smallmatrix} F_3\\ G_3\end{smallmatrix}\right]\mu \ \wedge\ \left(\begin{smallmatrix} \lambda\\ \mu\end{smallmatrix}\right) = \left[\begin{smallmatrix} M\\ L\end{smallmatrix}\right]\nu\\
&\iff \exists\nu\in\mathbb{K}^k : \begin{pmatrix} f_1\\ f_3\\ e_1\\ e_3\end{pmatrix} = \begin{bmatrix} F_1M\\ F_3L\\ G_1M\\ G_3L\end{bmatrix}\nu\\
&\iff (f_1,f_3,e_1,e_3)\in \operatorname{ran}\begin{bmatrix} F_1M\\ F_3L\\ G_1M\\ G_3L\end{bmatrix}.
\end{aligned}
$$

Define

$$
F = \begin{bmatrix} F_1M\\ F_3L\end{bmatrix},\ F_{\mathcal{M}} = \begin{bmatrix} F_1\\ F_{21}\end{bmatrix},\ F_{\mathcal{L}} = \begin{bmatrix} F_{22}\\ F_3\end{bmatrix},
$$

and

$$
G = \begin{bmatrix} G_1M\\ G_3L\end{bmatrix},\ G_{\mathcal{M}} = \begin{bmatrix} G_1\\ G_{21}\end{bmatrix},\ G_{\mathcal{L}} = \begin{bmatrix} G_{22}\\ G_3\end{bmatrix}.
$$

It follows from

$$
\begin{bmatrix} F_{21} & F_{22}\\ G_{21} & -G_{22}\end{bmatrix}\begin{bmatrix} M\\ L\end{bmatrix} = 0
$$

that

$$
M^*G_{21}^*F_{21}M + L^*G_{22}^*F_{22}L = 0,
$$

Hence, we have

$$
\begin{aligned}
G^*F &= \begin{bmatrix} G_1M\\ G_3L\end{bmatrix}^* \begin{bmatrix} F_1M\\ F_3L\end{bmatrix}\\
&= M^*G_1^*F_1M + L^*G_3^*F_3L\\
&= M^*(G_1^*F_1 + G_{21}^*F_{21})M + L^*(G_{22}F_{22} + G_3^*F_3)L\\
&= M^*G_{\mathcal{M}}^*F_{\mathcal{M}}M + L^*G_{\mathcal{L}}^*F_{\mathcal{L}}L.
\end{aligned}
$$

From the latter one easily derives the following table, which is equivalent to the table in the theorem by Lem. 1.3.7.

$\mathcal{M}$ \ $\mathcal{L}$	$G_{\mathcal{L}}^*F_{\mathcal{L}} = -F_{\mathcal{L}}^*G_{\mathcal{L}}$	$G_{\mathcal{L}}^*F_{\mathcal{L}} + F_{\mathcal{L}}^*G_{\mathcal{L}} \le 0$	$G_{\mathcal{L}}^*F_{\mathcal{L}} \le 0$	$G_{\mathcal{L}}^*F_{\mathcal{L}} = F_{\mathcal{L}}^*G_{\mathcal{L}}$
$G_{\mathcal{M}}^*F_{\mathcal{M}} = -F_{\mathcal{M}}^*G_{\mathcal{M}}$	$G^*F = -F^*G$	$G^*F + F^*G \le 0$	$G^*F + F^*G \le 0$	–
$G_{\mathcal{M}}^*F_{\mathcal{M}} + F_{\mathcal{M}}^*G_{\mathcal{M}} \le 0$	$G^*F + F^*G \le 0$	$G^*F + F^*G \le 0$	$G^*F + F^*G \le 0$	–
$G_{\mathcal{M}}^*F_{\mathcal{M}} \le 0$	$G^*F + F^*G \le 0$	$G^*F + F^*G \le 0$	$G^*F \le 0$	$G^*F = F^*G$
$G_{\mathcal{M}}^*F_{\mathcal{M}} = F_{\mathcal{M}}^*G_{\mathcal{M}}$	–	–	$G^*F = F^*G$	$G^*F = F^*G$

For the last statement, that is the inheritance of the maximality, we first consider the case $\mathcal{M} = \pm\mathcal{M}^*$ and $\mathcal{L} = \pm\mathcal{L}^*$ with the same sign, i.e., $\mathcal{M}$ and $\mathcal{L}$ are both maximally nonpositive/symmetric/skew-symmetric. Then

$$\ker\left[\begin{smallmatrix} M^* & L^* \end{smallmatrix}\right] = \left(\operatorname{ran}\left[\begin{smallmatrix} M \\ L \end{smallmatrix}\right]\right)^\perp = \left(\ker\left[\begin{smallmatrix} F_{21} & F_{22} \\ G_{21} & -G_{22} \end{smallmatrix}\right]\right)^\perp = \operatorname{ran}\begin{bmatrix} F_{21}^* & G_{21}^* \\ F_{22}^* & -G_{22} \end{bmatrix}.$$

Further invoking (1.9) we deduce

$$\begin{aligned}
&(f_1, f_3, e_1, e_3) \in \mathcal{M} \circ \mathcal{L} \\
&\iff (f_1, f_3, e_1, e_3) \in \operatorname{ran}\begin{bmatrix} F_1 M \\ F_3 L \\ G_1 M \\ G_3 L \end{bmatrix} \\
&\iff \exists (f_2, e_2) \in \mathbb{K}^{2n_2} : (f_1, f_2, e_1, e_2) \in \operatorname{ran}\begin{bmatrix} F_1 \\ F_{21} \\ G_1 \\ G_{21} \end{bmatrix} \ \wedge\ (-f_2, f_3, e_2, e_3) \in \operatorname{ran}\begin{bmatrix} F_{22} \\ F_3 \\ G_{22} \\ G_3 \end{bmatrix} \\
&\iff \exists (f_2, e_2) \in \mathbb{K}^{2n_2} : (f_1, f_2, e_1, e_2) \in \ker\left[\begin{smallmatrix} G_1^* & G_{21}^* & \pm F_1^* & \pm F_{21}^* \end{smallmatrix}\right] \\
&\qquad\qquad\qquad\qquad\qquad \wedge\ (-f_2, f_3, e_2, e_3) \in \ker\left[\begin{smallmatrix} G_{22}^* & G_3 & \pm F_{22}^* & \pm F_3^* \end{smallmatrix}\right] \\
&\iff \exists (f_2, e_2) \in \mathbb{K}^{2n_2} : \begin{bmatrix} G_1^* & G_{21}^* & 0 & \pm F_1^* & \pm F_{21}^* & 0 \\ 0 & -G_{22}^* & G_3^* & 0 & \pm F_{22}^* & \pm F_3^* \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0 \\
&\iff \exists (f_2, e_2) \in \mathbb{K}^{2n_2} : \begin{bmatrix} G_1^* & 0 & \pm F_1^* & 0 \\ 0 & G_3^* & 0 & \pm F_3^* \end{bmatrix} \begin{pmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{pmatrix} = \begin{bmatrix} F_{21}^* & G_{21}^* \\ F_{22}^* & -G_{22} \end{bmatrix} \begin{pmatrix} -f_2 \\ -e_2 \end{pmatrix} \\
&\iff (f_1, f_3, e_1, e_3) \in \ker\left[\begin{smallmatrix} M^* & L^* \end{smallmatrix}\right] \begin{bmatrix} G_1^* & 0 & \pm F_1^* & 0 \\ 0 & G_3^* & 0 & \pm F_3^* \end{bmatrix} \\
&\iff (f_1, f_3, e_1, e_3) \in \ker\left[\begin{smallmatrix} M^* G_1^* & L^* G_3^* & \pm M^* F_1^* & \pm L^* G_3^* \end{smallmatrix}\right] \\
&\iff (f_1, f_3, e_1, e_3) \in \pm(\mathcal{M} \circ \mathcal{L})^*.
\end{aligned}$$

□

In particular, Prop.1.3.14 states that the interconnection of two maximal linear relations that both are either symmetric or nonpositive is again maximal with the same property. However, the converse is not true as the following example shows, that is two nonmaximal relation can interconnect to a maximal one.

Example 1.3.15. Let $n_1, n_2, n_3 \in \mathbb{N}_0$ and define

$$\mathcal{M} = \mathbb{K}^{n_1} \times \{0\}^{n_2} \times \{0\}^{n_1} \times \{0\}^{n_2}, \quad \mathcal{L} = \{0\}^{n_2} \times \{0\}^{n_3} \times \{0\}^{n_2} \times \mathbb{K}^{n_3}.$$

Then $\mathcal{M}$ and $\mathcal{L}$ are both dissipative and symmetric but not maximal by Lem. 1.3.7 and Lem. 1.3.8. By these results we also deduce that their interconnection with respect to $\mathbb{K}^{n_2}$,

$$\mathcal{M} \circ \mathcal{L} = \mathbb{K}^{n_1} \times \{0\}^{n_3} \times \{0\}^{n_1} \times \mathbb{K}^{n_3},$$

is both maximally dissipative and maximally symmetric.

Observing the results of Prop. 1.3.14 and Ex. 1.3.15 leads to the following conjecture.

Conjecture 1.3.16. *Let $\mathcal{M}$ and $\mathcal{L}$ be maximally dissipative relations in $\mathbb{K}^{n_1+n_2}$ and $\mathbb{K}^{n_2+n_3}$, respectively, for some $n_1, n_2, n_3 \in \mathbb{N}_0$. Then $\mathcal{M} \circ \mathcal{L}$ is a maximally dissipative relation in $\mathbb{K}^{n_1+n_3}$.*

We saw with Lem. 1.3.13 that a maximally dissipative relation can be written as the interconnection of a skew-adjoint relation and a maximally nonpositive relation. Conversely, the question if the interconnection of a skew-adjoint relation and a maximally nonpositive relation yields a maximally dissipative relation is subject to the next conjecture.

Conjecture 1.3.17. *Let $\mathcal{D}$ be a skew-adjoint relation in $\mathbb{K}^{n+m}$ and $\mathcal{R}$ a maximally nonpositive relation in $\mathbb{K}^m$ for some $n, m \in \mathbb{N}_0$. Then $\mathcal{D} \circ \mathcal{R}$ is a maximally dissipative relation in $\mathbb{K}^n$.*

Proposition 1.3.18. *The statements of Con. 1.3.16 and Con. 1.3.17 are equivalent.*

Proof. Clearly, Con. 1.3.17 is a special case of Con. 1.3.16. Conversely, if we are given maximally dissipative relations $\mathcal{M}$ and $\mathcal{L}$ in $\mathbb{K}^{n_1+n_2}$ and $\mathbb{K}^{n_2+n_3}$, respectively, for some $n_1, n_2, n_3 \in \mathbb{N}_0$, then Lem. 1.3.13 shows that there exist skew-adjoint relations $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathbb{K}^{2(n_1+n_2)}$ and $\mathbb{K}^{2(n_2+n_3)}$, respectively, as well as maximally nonpositive relations $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathbb{K}^{n_1+n_2}$ and $\mathbb{K}^{2n_2+n_3}$, respectively, such that

$$\mathcal{M} = \mathcal{D}_1 \circ \mathcal{R}_1, \quad \mathcal{L} = \mathcal{D}_2 \circ \mathcal{R}_2.$$

It is straightforward to deduce

$$\mathcal{M} \circ \mathcal{L} = (\mathcal{D}_1 \circ \mathcal{R}_1) \circ (\mathcal{D}_2 \circ \mathcal{R}_2) = (\mathcal{D}_1 \circ \mathcal{D}_2) \circ (\mathcal{R}_1 \widehat{\times} \mathcal{R}_2).$$

By Prop. 1.3.14, $\mathcal{D}_1 \circ \mathcal{D}_2$ and $\mathcal{R}_1 \widehat{\times} \mathcal{R}_2$ are skew-adjoint and maximally nonpositive, respectiely. Now the statement of Con. 1.3.16 implies that $(\mathcal{D}_1 \circ \mathcal{D}_2) \circ (\mathcal{R}_1 \widehat{\times} \mathcal{R}_2) = \mathcal{M} \circ \mathcal{L}$ is maximally dissipative, i.e., the statement of Con. 1.3.17. □

1.4 Differential geometric notions

For the definition of basic topological concepts used throughout this section, we refer the reader to [Mun00]. When talking about manifolds, most authors implicitly assume a certain degree of smoothness and the term *smooth* is coined differently depending on the mathematical background, cf. [Hir76; Lee12]. Roughly speaking, we will not need the existence of infinitely many derivatives for the scope of this thesis. This motivates the following definition.

Definition 1.4.1 (Manifold)**.** Let $\mathcal{M}$ be a topological space, $r \in \mathbb{N}_0 \cup \{\infty\}$ and $n \in \mathbb{N}_0$. Then $(\mathcal{M}, \{\phi_i, U_i\}_{i \in I})$ (or simply $\mathcal{M}$) is called a *C^r manifold (of dimension n)* if $\mathcal{M}$ is Hausdorff, second-countable and $\{\phi_i, U_i\}_{i \in I}$ is a *maximal C^r atlas* of $\mathcal{M}$ with *compatible C^r charts* $\{\phi_i\}_{i \in I}$, i.e.,

- $\{U_i\}_{i \in I}$ is an open covering of $\mathcal{M}$,
- $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism for all $i \in I$,
- $\phi_i \circ \phi_j^{-1} \in C^r(\phi_j(U_i \cap U_j), \phi_i(U_i \cap U_j))$ for all $i, j \in I$,
- $\{\phi_i, U_i\}_{i \in I}$ is maximal w.r.t. the previous properties under inclusion.

We continue by introducing some notations and recalling some facts, for which we refer the reader to [Hir76; Lee12; KM06] for detailed illustration of the involved concepts. Let $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \Psi)$ be m- and n-dimensional C^r manifolds, respectively, with $r \in \mathbb{N}_0 \cup \{\infty\}$. For $0 \leq s \leq r$, $f : \mathcal{M} \to \mathcal{N}$ is a *C^s map* if for every $p \in \mathcal{M}$ there exist charts $(\phi, U) \in \Phi$ and $(\psi, V) \in \Psi$ such that $p \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} \in C^s(\phi(U), \psi(V))$ and we write $C^s(\mathcal{M}, \mathcal{N})$ for the set of these maps.
A *(vector) bundle (of rank k) over* $\mathcal{M}$ is a pair (E, π) consisting of a topological space E together with a surjective continuous map $\pi : E \to \mathcal{M}$ such that for all $p \in \mathcal{M}$ the *fiber* $E_p = \pi^{-1}(p)$ is a k-dimensional (real) vector space and there exists a neighborhood U of p and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ called *local trivialization* satisfying $\pi_U \circ \varphi = \pi$ with $\pi : U \times \mathbb{R}^k \to U$ being the projection on U and for all $q \in U$, $\varphi\restriction_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism. If U can be chosen as $\mathcal{M}$ for one (and hence all) $p \in \mathcal{M}$, then E is called a *trivial bundle*. If $\mathcal{M}$ and $\mathcal{E}$ both are C^r manifolds and the local trivializations φ can be chosen such that φ, φ^{-1} are C^r maps, then E is called a *C^r bundle*. By (E^*, π^*) we denote the *dual bundle* of (E, π), i.e., the bundle whose fibers are the dual spaces of the fibers of

(E,π). A *subbundle* of (E,π) is a bundle $(\tilde{E},\tilde{\pi})$ with $\tilde{E}$ being a topological subspace of E and $\tilde{\pi} = \pi\restriction_{\tilde{E}}$. A *section* σ of (E,π) is a continuous map $\sigma : \mathcal{M} \to E$ with the property $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$ and we write $\Gamma_s(E)$ for the set of sections σ of E satisfying $\sigma \in C^s(\mathcal{M}, E)$. If the bundle projection π is clear from the context, we simply write E for the bundle instead of (E,π). Given two bundles E_1, E_2 over $\mathcal{M}$ we write $E_1 \oplus E_2$ for their *Whitney sum*, see [Lee12, Ex. 10.7] for more details, and we often identify $(E_1 \oplus E_2)^* = E_1^* \oplus E_2^*$. Let us now introduce some relevant bundles.

For $r \geq 1$ denote by $T_p\mathcal{M}$, $T_p^*\mathcal{M}$ the m-dimensional *tangent* and *cotangent space* of $\mathcal{M}$ at $p \in \mathcal{M}$, respectively, while the $2m$-dimensional C^{r-1} manifolds

$$T\mathcal{M} = \textstyle\coprod_{p\in\mathcal{M}} T_p\mathcal{M}, \qquad\qquad T^*\mathcal{M} = \displaystyle\coprod_{p\in\mathcal{M}} T_p^*\mathcal{M}$$

denote its *tangent* and *cotangent bundle*, respectively. For a C^1 map $f : \mathcal{M} \to \mathcal{N}$

$$\begin{aligned} df : T\mathcal{M} &\to T\mathcal{N} & \qquad df^* : T^*\mathcal{N} &\to T^*\mathcal{M} \\ (p, v) &\mapsto (f(p), df_p v) & (f(p), v) &\mapsto (p, df_p^* v) \end{aligned}$$

denotes the *differential of f* and the *cotangent map of f*, respectively, with $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ being the *differential of f at $p \in \mathcal{M}$* and $df_p^* : T_{f(p)}\mathcal{N} \to T_p\mathcal{M}$ the *pullback of f at $p \in \mathcal{M}$*. $\mathcal{N}$ is called an *(embedded) C^r submanifold of $\mathcal{M}$* if there exists an embedding $\iota \in C^r(\mathcal{N}, \mathcal{M})$ satisfying $\operatorname{rk} d\iota_p = n$ for $r \geq 1$ and all $p \in \mathcal{N}$. A subbundle $(\tilde{E},\tilde{\pi})$ of a bundle (E,π) is a called *C^r subbundle* if it is both a C^r bundle and a C^r submanifold of E. Next, for a real vector space V the set of *contravariant tensors on V of rank k* is given as

$$T^k(V) \coloneqq \textstyle\bigotimes_{i=1}^k V = V \otimes \ldots \otimes V \cong L(\underbrace{V^*, \ldots, V^*}_{k \text{ times}}; \mathbb{R})$$

with $T^k(V^*)$ being the set of *covariant tensors on V of rank k*. A tensor $\alpha \in T^k(V)$ or $T^k(V^*)$ is called *alternating* if for all families $(v_i)_{i=1,\ldots,k}$ in V or V^*, respectively, and all $i \neq j$ it holds

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

The set of *alternating contravariant* and *covariant tensors on V of rank k* are denoted by $\Lambda^k(V)$ and $\Lambda^k(V^*)$, respectively. This enables the introduction of the *bundle of covariant k-tensors on $\mathcal{M}$*

$$T^k T^*\mathcal{M} = \coprod_{p\in\mathcal{M}} T^k(T_p^*\mathcal{M})$$

and the *bundle of alternating covariant k-tensors on $\mathcal{M}$*

$$\Lambda^k T^*\mathcal{M} = \coprod_{p\in\mathcal{M}} \Lambda^k(T_p^*\mathcal{M}).$$

A section of $\Lambda^k T^*\mathcal{M}$ is called a *differential k-form* and the set of C^s differential k-forms is denoted by

$$\Omega_s^k(\mathcal{M}) = \Gamma_s\left(\Lambda^k T^*\mathcal{M}\right).$$

A *symplectic manifold* is a pair $(\mathcal{M}, \omega)$ consisting of a C^1 manifold of even dimension together with a *symplectic form* ω, i.e., an element $\omega \in \Gamma_1\left(\Lambda^2 T^*\mathcal{M}\right)$ which is *closed* ($d\omega = 0$) and *nondegenerate* ($\forall p \in \mathcal{M}\ \forall v \in T_p\mathcal{M} : (\forall w \in T_p\mathcal{M} : \omega_p(v, w) = 0) \Rightarrow (v = 0)$). For example, the *standard symplectic form* on $\mathbb{R}^{2n}$ is the 2-form ω given by $\omega(v_1, v_2) = v_1^\top \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right] v_2$ for $v_1, v_2 \in \mathbb{R}^{2n}$. Similarly, for an n-dimensional C^2 manifold $\mathcal{Q}$, its cotangent bundle $T^*\mathcal{Q}$ becomes a symplectic manifold with the so-called *canonical symplectic form* [Lee12, p.570].

Definition 1.4.2 (Lagrangian submanifold)**.** Let $(\mathcal{M}, \omega)$ be a symplectic manifold of dimension $2n$. A *Lagrangian submanifold* of $\mathcal{M}$ is an n-dimensional submanifold $\mathcal{L}$ of $\mathcal{M}$ such that $T_p\mathcal{L}$ is a Lagrangian subspace of $T_p\mathcal{M}$ with respect to $\omega_p \in \Lambda^2 T_p^*\mathcal{M}$ for all $p \in \mathcal{L}$.

Let $(\mathcal{M}_1, \omega_1)$, $(\mathcal{M}_2, \omega_2)$ be two m-dimensional symplectic manifolds. Then a diffeomorphism $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a *symplectomorphism* if $F^*\omega_2 = \omega_1$ with $F^*\omega_2$ being the *pullback* of ω_2 by F, i.e.,

$$(F^*\omega_2)_p(v_1, \ldots v_m) = (\omega_2)_{F(p)}(dF_p(v_1), \ldots, dF_p(v_m))$$

for all $p \in \mathcal{M}, (v_1, \ldots v_m) \in T_p\mathcal{M}_1$. In the following, we show that gradient fields induce Lagrangian submanifolds.

Proposition 1.4.3. *Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Then the submanifold consisting of the graph of Q, i.e.,*

$$\mathcal{L}_Q := \{(x, Q(x)) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathbb{R}^n\}$$

is a Lagrangian submanifold of $\mathbb{R}^{2n}$ equipped with the standard symplectic form if and only if Q is a gradient field. In other words, there exists some twice continuously differentiable function $H : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla H = Q$.

Proof. Using that $\mathbb{R}^n$ is simply connected, the case of smooth Q follows from [Lee12, Prop. 22.12]. The general case follows by a straightforward modification of the proof of [Lee12, Prop. 22.12]. □

The case where a Lagrangian submanifold is a subspace deserves special attention.

Proposition 1.4.4 ([vdSM18, Prop. 5.2])**.** *A subspace $\mathcal{L} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a Lagrangian submanifold if and only if*

$$\mathcal{L} = \left\{(f, e) \in \mathbb{R}^n \times \mathbb{R}^n \mid S^\top f = P^\top e\right\}$$

for some matrices $S, P \in \mathbb{R}^{n \times n}$ with $S^\top P = P^\top S$ and $\operatorname{rk}[S^\top\ P^\top] = n$*.*

From this proposition, we see that the Lagrangian subspaces are a special case of Lagrangian submanifolds. Likewise, the concept of Dirac subspaces can be extended to the realm of manifolds [Cou90, Def. 2.2.1].

Definition 1.4.5 (Dirac manifold/bundle)**.** Let $\mathcal{M}$ be a manifold. A subbundle $\mathcal{D}$ of $T\mathcal{M} \oplus T^*\mathcal{M}$ is called a *Dirac bundle* or *Dirac manifold* if $\mathcal{D}(x) := \mathcal{D}_x$ is a Dirac subspace of $T_x\mathcal{M} \times T_x^*\mathcal{M}$ for all $x \in \mathcal{M}$.

Often, only Dirac bundles fulfilling additional properties, like being a C^r subbundle for some $r \in \mathbb{N}$ or some *integrability condition* [vdSch06], are of interest. Such assumptions allow for useful representations of Dirac bundles, like generalizations of Cor. 1.3.11, see [DvdS98].
We close this section with yet another concept needed for port-Hamiltonian systems.

Definition 1.4.6 (Resistive relation)**.** A relation $\mathcal{R} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *resistive*, if

$$\forall\ (f, e) \in \mathcal{R}:\ e^\top f \leq 0.$$

The resistive relations in $\mathbb{R}^n$ that are linear, correspond exactly to the dissipative relations in $\mathbb{R}^n$. If a linear resistive relation in $\mathbb{R}^n$ is symmetric, then it is nonpositive. A trivial example for a linear resistive relation that is dissipative but not nonpositive is $\operatorname{gr} -J_2(1)$. Indeed, $\left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left[\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right] \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right\rangle = -x^2 - xy - y^2 \leq -\max\{x^2, y^2\} \leq 0$.

Chapter 2

Port-Hamiltonian formulations

As alluded to in the introduction, it is a peculiarity of port-Hamiltonian modelling that different concepts emerged bearing the same or at least very similar names, without having been thoroughly compared. Each one of these concepts emerged from a common ancestor, which was successively enhanced by different mathematical communities. We propose yet another port-Hamiltonian formulation, but with the intention of using it to build bridges between two well-established formulations. In this chapter, we start by showing where the different port-Hamiltonian formulations originated from and then continue by presenting the current major formulations, for which we elucidate some of their similarities and differences.

2.1 The common ancestor

As the name hints at, taking the equations of Hamilton [Ham35] and extending them by inputs and outputs, one obtains what can be regarded as the original port-Hamiltonian formulation. To be more precise, we consider the *port-Hamiltonian ordinary differential equations (pH-ODE) system* consisting of the equations

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t) &= (J-R)Qx(t) + (G-K)u(t), \\ y(t) &= (G+K)^* Qx(t) + (S+N)u(t), \end{aligned} \tag{2.1a}$$

with $J, Q, R \in \mathbb{K}^{n\times n}$, $G, K \in \mathbb{K}^{n\times k}$ and $S, N \in \mathbb{K}^{k\times k}$ satisfying

$$Q^* = Q, \quad V := \begin{bmatrix} J & G \\ -G^* & N \end{bmatrix} = -V^*, \quad W := \begin{bmatrix} R & K \\ K^* & S \end{bmatrix} \geq 0, \tag{2.1b}$$

for some $n, k \in \mathbb{N}$ and all $t \in \mathbb{R}$. Defining the *Hamiltonian* function $H(x) = \frac{1}{2}x^*Qx$ as the energy function of the system and omitting the dependence on the time t we derive the following *power balance*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} H(x) &= \tfrac{1}{2}\left(\dot{x}^*Qx + x^*Q\dot{x}\right) \\ &= \mathrm{Re}\,\left(x^*Q(J-R)Qx + x^*Q(G-K)u\right) \\ &= \mathrm{Re}\,\left(-x^*QRQx + y^*u - 2x^*QKu - u^*(S+N)u\right) \\ &= \mathrm{Re}\,(y^*u) - \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right)^* W \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right) \\ &\leq \mathrm{Re}\,(y^*u), \end{aligned} \tag{2.2}$$

with the interpretation that the increase in energy amounts at most to what is supplied to it. The need to extend this formulation arises when one wishes to interconnect two pH-ODE systems in a certain *power-conserving* manner. It turns out that this class is not closed under such interconnection, that is, the resulting system is in general not a pH-ODE system any more. A simple example consists of the two scalar pH-ODE systems

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x_1(t) &= -x_1(t) + u_1(t), \qquad & \tfrac{\mathrm{d}}{\mathrm{d}t} x_2(t) &= -x_2(t) + u_2(t), \\ y_1(t) &= -x_1(t), & y_2(t) &= -x_2(t), \end{aligned}$$

where we impose

$$u_1(t) = u_2(t), \quad y_1(t) = -y_2(t).$$

This interconnection is expressing power-conservation because one considers the product of the respective input and outputs to represent, depending on the power flow convention [dJon72], the power going in or leaving the system. Consequently, $u_1y_1 = -u_2y_2$ which implies that the power going out of one system equals the power going into the other. Now the interconnected system reads

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x_1(t) - \tfrac{\mathrm{d}}{\mathrm{d}t} x_2(t) &= -x_1(t) + x_2(t), \\ 0 &= x_1(t) + x_2(t), \end{aligned}$$

which contains an algebraic equation not covered by (2.1a).

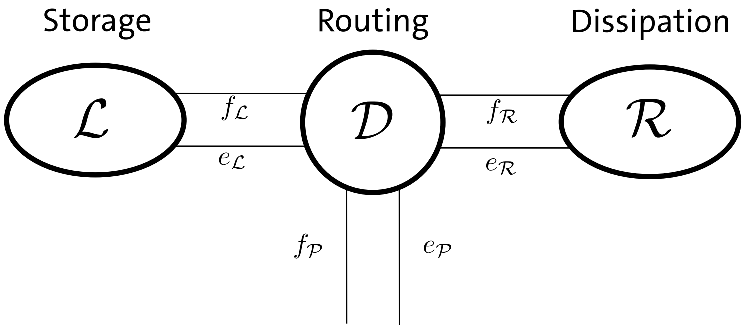

Figure 2.1: Visual representation of a pH-NG system.

2.2 The geometric formulation

The first port-Hamiltonian formulation that is closed under certain power-conserving interconnections we present makes use of the notions introduced in Sec. 1.4. This approach can be traced back [vdSJ14] to Arnol'd and the school of analytical mechanics.

2.2.1 The general nonlinear case

The following definition of a port-Hamiltonian system follows the lines of [vdSM20; BCG+18; BCG+19].

Definition 2.2.1 (Nonlinear geometric port-Hamiltonian system)**.** Let $n_S, n_R, n_P \in \mathbb{N}_0$, $\mathcal{X} \subset \mathbb{R}^{n_S}$ a C^1 submanifold and $\mathcal{B}_R, \mathcal{B}_P$ bundles over $\mathcal{X}$ of dimension n_R and n_P, respectively. Further, let $\mathcal{D}_{T\mathcal{X}\oplus\mathcal{B}_R\oplus\mathcal{B}_P}$ be a Dirac manifold, i.e., $\mathcal{D}(x) \subset (T_x\mathcal{X} \times (\mathcal{B}_R)_x \times (\mathcal{B}_P)_x) \times (T_x^*\mathcal{X} \times (\mathcal{B}_R)_x^* \times (\mathcal{B}_P)_x^*)$ is a Dirac structure for all $x \in \mathcal{X}$. Furthermore, let $\mathcal{L} \subset T^*\mathcal{X}$ be a Lagrangian submanifold equipped with a symplectic form ω. Finally, let $\mathcal{R} \subset \mathcal{B}_R \oplus \mathcal{B}_R^*$ be a resistive bundle, i.e., $\mathcal{R}(x) \subset (\mathcal{B}_R)_x \times (\mathcal{B}_R)_x^*$ is a resistive structure for all $x \in \mathcal{X}$. Then the triple $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ is called a *(nonlinear geometric) port-Hamiltonian system* (pH-NG system). Its *dynamics* are given by the differential inclusion

$$\begin{gathered}(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) \in \mathcal{D}(x(t)),\\ (x(t), e_S(t)) \in \mathcal{L}, \quad (f_R(t), e_R(t)) \in \mathcal{R}(x(t)).\end{gathered} \tag{2.3}$$

When the bundle $T\mathcal{X} \oplus \mathcal{B}_R \oplus \mathcal{B}_P$ is trivial and $\mathcal{D}(x)$ is identical for all $x \in \mathcal{X}$, we say that the Dirac manifold is *constant*, and it is custom in that case to drop the argument $x(t)$ of $\mathcal{D}(x(t))$ when introducing the dynamics (2.3). This special case is exactly what we employ to model electrical circuits in Chap. 6. Note that the letter f in f_R, f_P stands for *flow* and the letter e in e_S, e_R, e_P for *effort*, while S stands for *storage*, R for *resistive*, and P for *(external) ports* representing the interaction with the environment. In the port-Hamiltonian terminology, a *port* is always to be understood as a pair of corresponding flow and efforts, e.g., $(f_S, e_S), (f_R, f_P), (f_P, e_P)$ as well as any combination and/or subdivision of these pairs. The spaces $T_x\mathcal{X}$, $(\mathcal{B}_R)_x$, $(\mathcal{B}_P)_x$ are understood to be *flow spaces*, $T_x^*\mathcal{X}$, $(\mathcal{B}_R)_x$, $(\mathcal{B}_P)_x$ *effort spaces* and their elements *flows* and *efforts*, respectively.

A key property of pH-NG systems is that this class is closed under power-conserving interconnection. Different methods of how to design such interconnections are, for example, elucidated in [BCG+18; CvdSB07; vdSJ14; VvdS10b]. The interconnection we use for the electrical circuits in Chap. 6 follows the ideas presented in [vdSJ14]. Interconnection is based on the assumption that each system has two kinds of external flows and efforts, namely specific and to-be-linked ones, where the latter ones belong to the same space for each Dirac structure.

Definition 2.2.2 (Interconnection of pH-NG systems)**.**
For $i = 1, 2$, let $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$ be two pH-NG systems with

$$\mathcal{D}_i \subset (T\mathcal{X}_i \oplus \mathcal{B}_{Ri} \oplus \mathcal{B}_{Pi} \oplus \mathcal{B}_{\text{link}i}) \oplus (T\mathcal{X}_i \oplus \mathcal{B}_{Ri} \oplus \mathcal{B}_{Pi} \oplus \mathcal{B}_{\text{link}i})^*,$$

where $\mathcal{B}_{\text{link}i} = \mathcal{X}_i \times \mathcal{F}_{\text{link}}$ and $\mathcal{B}^*_{\text{link}i} = \mathcal{X}_i \times \mathcal{E}_{\text{link}}$ are trivial bundles. Here, the external ports are subdivided into a specific external part, and a to-be-linked part. Further, consider the bundles $\mathcal{B}_R$, $\mathcal{B}_P$ over $\mathcal{X}_1 \times \mathcal{X}_2$ given by

$$(\mathcal{B}_R)_{(x_1,x_2)} = (\mathcal{B}_{R1})_{x_1} \times (\mathcal{B}_{R2})_{x_2}, \quad (\mathcal{B}_P)_{(x_1,x_2)} = (\mathcal{B}_{P1})_{x_1} \times (\mathcal{B}_{P2})_{x_2}$$

for $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. The *interconnection* of $(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$,

$$(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \circ (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2) := (\mathcal{D}, \mathcal{L}, \mathcal{R}),$$

with respect to $(\mathcal{F}_{\text{link}}, \mathcal{E}_{\text{link}})$ is defined as the pH-NG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ consisting of the Dirac manifold $\mathcal{D} \subset (T(\mathcal{X}_1 \times \mathcal{X}_2) \oplus \mathcal{B}_R \oplus \mathcal{B}_P) \oplus (T(\mathcal{X}_1 \times \mathcal{X}_2) \oplus \mathcal{B}_R \oplus \mathcal{B}_P)^*$, the Lagrange submanifold $\mathcal{L} \subset T^*(\mathcal{X}_1 \times \mathcal{X}_2)$, and the resistive structure $\mathcal{R} \subset (\mathcal{B}_R) \times (\mathcal{B}_R)^*$

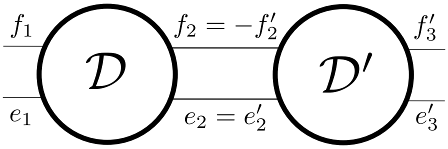

Figure 2.2: Composition of two Dirac structures.

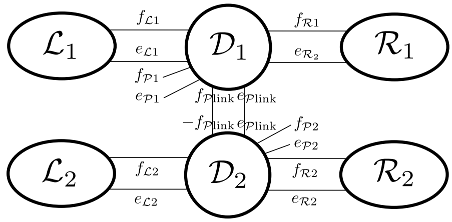

Figure 2.3: Interconnection of two pH-NG systems.

given by

$$\begin{aligned}
\mathcal{D}(x_1,x_2) \coloneqq \Big\{ &\big((f_{S1},f_{S2}),(f_{R1},f_{R2}),(f_{P1},f_{P2}),(e_{S1},e_{S2}),(e_{R1},e_{R2}),(e_{P1},e_{P2})\big) \mid \\
&\qquad \exists (f_{\text{link}},e_{\text{link}}) \in \mathcal{F}_{\text{link}} \times \mathcal{E}_{\text{link}} : \\
&\qquad\quad (f_{S1},f_{R1},f_{P1},f_{\text{link}},e_{S1},e_{R1},e_{P1},e_{\text{link}}) \in \mathcal{D}_1(x_1) \\
&\qquad\quad \wedge\ (f_{S2},f_{R2},f_{P2},-f_{\text{link}},e_{S2},e_{R2},e_{P2},e_{\text{link}}) \in \mathcal{D}_2(x_2) \Big\}, \\
\mathcal{L} \coloneqq \{ &(f_{S1},f_{S2},e_{S1},e_{S2}) \in T^*(\mathcal{X}_1 \times \mathcal{X}_2) \mid (f_{S1},e_{S1}) \in \mathcal{L}_1 \wedge (f_{S2},e_{S2}) \in \mathcal{L}_2 \}, \\
\mathcal{R}(x_1,x_2) \coloneqq \{ &(f_{R1},f_{R2},e_{R1},e_{R2}) \mid (f_{R1},e_{R1}) \in \mathcal{R}_1(x_1) \wedge (f_{R2},e_{R2}) \in \mathcal{R}_2(x_2) \},
\end{aligned}$$

for $(x_1,x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$.

The above constructed set $\mathcal{D}$ is indeed a Dirac structure [vdSJ14, Chap. 6], [DvdS98]. It is readily seen that $\mathcal{L}$ is a Lagrange submanifold and $\mathcal{R}$ is a resistive relation. Hence, the interconnection of pH-NG systems results in a pH-NG system.
Next, we introduce the (sorted Cartesian) product of pH-NG systems, which simply means that several coexisting pH-NG systems are united to one pH-NG system.

Definition 2.2.3 (Product of pH-NG systems)**.** Let $(\mathcal{D}_i,\mathcal{L}_i,\mathcal{R}_i)_{i=1,\dots,n}$ be a finite family of pH-NG systems with $\mathcal{D}_i \subset (T\mathcal{X}_i \oplus \mathcal{B}_{Ri} \oplus \mathcal{B}_{Pi}) \oplus (T\mathcal{X}_i \oplus \mathcal{B}_{Ri} \oplus \mathcal{B}_{Pi})^*$ for $i = 1,\dots,n$. If $n \geq 2$, consider the bundles $\mathcal{B}_R$, $\mathcal{B}_P$ over $\mathcal{X}_1 \times \mathcal{X}_2$ given by

$$(\mathcal{B}_R)_{(x_1,x_2)} = (\mathcal{B}_{R1})_{x_1} \times (\mathcal{B}_{R2})_{x_2}, \quad (\mathcal{B}_P)_{(x_1,x_2)} = (\mathcal{B}_{P1})_{x_1} \times (\mathcal{B}_{P2})_{x_2}$$

for $(x_1,x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. Then the *product* of $(\mathcal{D}_1,\mathcal{L}_1,\mathcal{R}_1)$ and $(\mathcal{D}_2,\mathcal{L}_2,\mathcal{R}_2)$,

$$(\mathcal{D}_1,\mathcal{L}_1,\mathcal{R}_1) \times (\mathcal{D}_2,\mathcal{L}_2,\mathcal{R}_2) \coloneqq (\mathcal{D},\mathcal{L},\mathcal{R}),$$

is defined as the pH-NG system $(\mathcal{D},\mathcal{L},\mathcal{R})$ consisting of the Dirac manifold $\mathcal{D} \subset (T(\mathcal{X}_1 \times \mathcal{X}_2) \oplus \mathcal{B}_R \oplus \mathcal{B}_P) \oplus (T(\mathcal{X}_1 \times \mathcal{X}_2) \oplus \mathcal{B}_R \oplus \mathcal{B}_P)^*$, the Lagrange submanifold

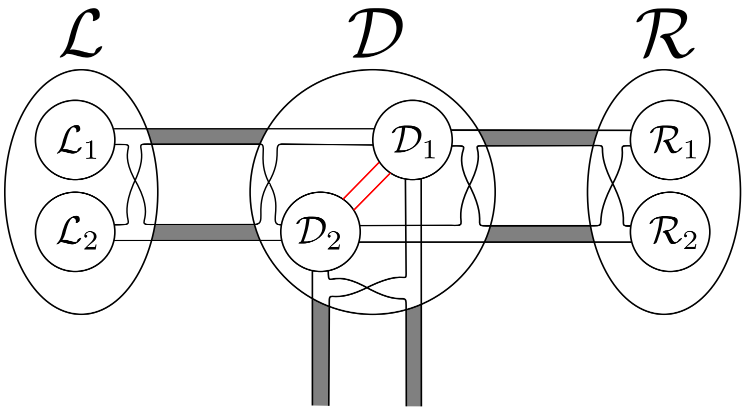

Figure 2.4: Visualization of the sorting of the flow and effort variables arising in the Defs. 2.2.2 & 2.2.3, cf. Figs. 2.1–2.3

$\mathcal{L} \subset T^*(\mathcal{X}_1 \times \mathcal{X}_2)$, and the resistive structure $\mathcal{R} \subset (\mathcal{B}_R) \times (\mathcal{B}_R)^*$ given by

$$\begin{aligned}
\mathcal{D}(x_1, x_2) &\coloneqq \big\{((f_{S1}, f_{S2}), (f_{R1}, f_{R2}), (f_{P1}, f_{P2}), (e_{S1}, e_{S2}), (e_{R1}, e_{R2}), (e_{P1}, e_{P2})) \mid \\
&\qquad (f_{S1}, f_{R1}, f_{P1}, e_{S1}, e_{R1}, e_{P1}) \in \mathcal{D}_1(x_1) \\
&\qquad \wedge\ (f_{S2}, f_{R2}, f_{P2}, e_{S2}, e_{R2}, e_{P2}) \in \mathcal{D}_2(x_2)\big\}, \\
\mathcal{L} &\coloneqq \{\ (f_{S1}, f_{S2}, e_{S1}, e_{S2}) \in T^*(\mathcal{X}_1 \times \mathcal{X}_2)\ \mid (f_{S1}, e_{S1}) \in \mathcal{L}_1\ \wedge\ (f_{S2}, e_{S2}) \in \mathcal{L}_2\ \}, \\
\mathcal{R}(x_1, x_2) &\coloneqq \{\ (f_{R1}, f_{R2}, e_{R1}, e_{R2})\ \mid (f_{R1}, e_{R1}) \in \mathcal{R}_1(x_1)\ \wedge\ (f_{R2}, e_{R2}) \in \mathcal{R}_2(x_2)\ \},
\end{aligned}$$

for $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. We further inductively define

$$\bigtimes_{i=1}^{n} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \coloneqq \left(\bigtimes_{i=1}^{n-1} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right) \times (\mathcal{D}_n, \mathcal{L}_n, \mathcal{R}_n) \tag{2.4a}$$

for $n \geq 2$ with

$$\bigtimes_{i=1}^{1} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \coloneqq (\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1). \tag{2.4b}$$

Remark 2.2.4. Note that the product of two pH-NG systems is well-defined by the same arguments as for the interconnection of two pH-NG systems. In fact, the product of two systems can be seen as an interconnection of two systems. In terms of Def. 2.2.2, it means that several pH-NG systems are interconnected with trivial linking ports. That is, for pH-NG systems $(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$ we add artificial and trivial linking ports $\mathcal{F}_{\text{link}} = \mathcal{E}_{\text{link}} = \{0\}$ (which do not affect the dynamic behavior), i.e., $\mathcal{B}_{Pi}$ gets replaced by $\mathcal{B}_i \oplus \mathcal{B}_{\text{link}i} = \mathcal{B}_{Pi} \oplus (\mathcal{X}_i \times \{0\})$ for $i = 1, 2$, and we interconnect these systems with respect to this trivial port $(\mathcal{F}_{\mathcal{P}\text{link}}, \mathcal{E}_{\mathcal{P}\text{link}})$. This link between the two concepts is illustrated by Fig. 2.4, where the possibly trivial interconnection is represented in red.

As for port-Hamiltonian ODEs (2.1), we can establish a power balance by defining an energy function or Hamiltonian for a pH-NG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$. However, in contrast to the power balance (2.2) for port-Hamiltonian ODEs, we present only a local power balance. To be more precise, combining the results of [Lee12, Thm. 22.13] and [LM87, App. 7 Cor. 1.16] (see also [BCG$^+$19; Fav20; vdSM20]) we know that for each $(x_0, e_0) \in \mathcal{L}$ there exist a neighborhood U of x_0 in $\mathcal{X}$, a neighborhood V of 0 in $\mathbb{R}^k$ for some $k \in \mathbb{N}_0$ together with a C^1 map $F : U \times V \to \mathbb{R}$, $(x, \lambda) \mapsto F(x, \lambda)$ with $D\frac{\partial F(x,\lambda)}{\partial \lambda} \in \mathbb{R}^{(n+k)\times k}$ having rank k on $(\frac{\partial F}{\partial \lambda})^{-1}(0)$ such that

$$\left\{ \left(x, \tfrac{\partial F(x,\lambda)}{\partial x}\right) \;\middle|\; (x, \lambda) \in U \times V \wedge \tfrac{\partial F}{\partial \lambda}(x, \lambda) = 0 \right\}$$

is a neighborhood of (x_0, e_0) in $\mathcal{L}$. Hence, given $\tau \in \mathbb{R}$ and curves satisfying (2.3), there exist some $\epsilon > 0$ and $\lambda \in C^1((\tau - \epsilon, \tau + \epsilon), \mathbb{R}^k)$ such that with the previous considerations we can write

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} F(x(t), \lambda(t)) &= \tfrac{\partial F(x(t),\lambda(t))}{\partial x} \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) + \tfrac{\partial F(x(t),\lambda(t))}{\partial \lambda} \tfrac{\mathrm{d}}{\mathrm{d}t} \lambda(t) \\ &= e_S(t)^\top \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) \\ &= e_R(t)^\top f_R(t) + e_P(t)^\top f_P(t) \\ &\le e_P(t)^\top f_P(t) \end{aligned} \tag{2.5}$$

for $t \in (\tau - \epsilon, \tau + \epsilon)$, i.e., in a neighborhood of $(x(\tau), e_S(\tau))$.

2.2.2 Linear simplifications

When in Def. 2.2.1 the state manifold is a linear space, the Lagrange manifold becomes a linear space, the Dirac manifold becomes constant, and the resulting dynamics are described by linear DAEs. We put an emphasis on the linear case since it will be compared to other port-Hamiltonian formulations. For this reason, we present the following shortened definition.

Definition 2.2.5 (Linear geometric port-Hamiltonian system)**.** Let $n_S, n_R, n_P \in \mathbb{N}_0$, $\mathcal{X} = \mathbb{R}^{n_S}$, $\mathcal{F}_R = \mathbb{R}^{n_R}$, $\mathcal{F}_P = \mathbb{R}^{n_P}$ with $\mathcal{E}_R = \mathcal{F}_R^*$ and $\mathcal{E}_P = \mathcal{F}_P^*$. A triple $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ consisting of a Dirac structure $\mathcal{D} \subset (\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_P) \times (\mathcal{X}^* \times \mathcal{E}_R \times \mathcal{E}_P)$, a Lagrange structure $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ and a linear resistive relation $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ is called a *(linear geometric) port-Hamiltonian system* (pH-LG system). Its *dynamics* are given by the differential inclusion

$$\begin{gathered}(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) \in \mathcal{D}, \\ (x(t), e_S(t)) \in \mathcal{L}, \quad (f_R(t), e_R(t)) \in \mathcal{R}.\end{gathered} \tag{2.6}$$

Remark 2.2.6. For pH-LG systems, the power balance (2.5) can be formulated globally. Given a pH-LG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$, by Cor. 1.3.11 we can write $\mathcal{L}$ as

$$\mathcal{L} = \left\{ \ (x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ \middle|\ \exists \lambda \in \mathbb{R}^k : e = Qx + G\lambda \wedge G^* x = 0 \ \right\}$$

for some real symmetric matrix Q and suitable real matrix G. Further, set $F(x, \lambda) = \frac{1}{2} x^\top Q x + x^\top G \lambda$. Then for all functions x satisfying (2.6) for suitable functions f_R, f_P, e_S, e_R, e_P there exists some function λ such that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{dt}}\, F(x(t), \lambda(t)) &= \tfrac{\mathrm{d}}{\mathrm{dt}}\big(\tfrac{1}{2} x(t)^\top Q x(t) + x(t)^\top G \lambda\big) \\ &= \dot{x}(t)^\top (Qx(t) + G\lambda) + x(t)^\top G \dot{\lambda}(t) \\ &= e_S(t)^\top \tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t) \\ &= e_R(t)^\top f_R(t) + e_P(t)^\top f_P(t) \\ &\le e_P(t)^\top f_P(t).\end{aligned} \tag{2.7}$$

Since complete vector spaces are closed under differentiation it holds $\dot{x} G \lambda = x^\top G \dot{\lambda} = 0$ and we see that we could have defined the 'Hamiltonian' F as a function purely depending on x, namely $F(x) = \frac{1}{2} x^\top Q x$. Now, if $\mathcal{L}$ is given in image representation $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} \tilde{E} \\ \tilde{Q} \end{smallmatrix}\right]$, for functions x, e_S such that $(x(t), e_S(t)) \in \mathcal{L}$ there exists a function z such that $x = \tilde{E} z$ and $e_S = \tilde{Q} z$. Then the Hamiltonian can be reformulated in terms of the variable z as $\frac{1}{2} z^\top \tilde{E}^\top \tilde{Q} z$ since

$$\tfrac{1}{2} z(t)^\top \tilde{E}^\top \tilde{Q} z(t) = \tfrac{1}{2} x(t)^\top e(t) = \tfrac{1}{2} x(t)^\top (Qx(t) + G\lambda(t)) = F(x(t)).$$

Let us now show how port-Hamiltonian ODEs are encompassed by pH-LG systems when $\mathbb{K} = \mathbb{R}$. With the notation from (2.1), let

$$\mathcal{D} = \left(\operatorname{gr} \begin{bmatrix} -J & I_n & 0 & -G \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_k \\ G^* & 0 & I_k & N \end{bmatrix} \right)^{-1}, \quad \mathcal{L} = \operatorname{gr} Q, \quad \mathcal{R} = \operatorname{gr} - \begin{bmatrix} R & K \\ K^* & S \end{bmatrix}. \tag{2.8}$$

Clearly, $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ defines a pH-LG system and its dynamics

$$\begin{aligned}(-\tfrac{\mathrm{d}}{\mathrm{d}t}\, x(t), f_{R1}(t), f_{R2}(t), y(t), e_S(t), e_{R1}(t), e_{R2}(t), u(t)) \in \mathcal{D},\\ (x(t), e_S(t)) \in \mathcal{L}, \quad (f_{R1}(t), f_{R2}(t), e_{R1}(t), e_{R2}(t)) \in \mathcal{R}\end{aligned}$$

imply that, omitting the time-dependency,

$$\begin{aligned}-\dot{x} &= -Je_S + e_{R1} - Gu, & e_{R1} &= -Rf_{R1} - Kf_{R2}, & f_{R1} &= -e_S,\\ y &= Ge_S + e_{R2} + Nu, & e_{R2} &= -K^* f_{R1} - Sf_{R2}, & f_{R2} &= -u, \qquad e_S = Qx,\end{aligned}$$

from which (2.1a) readily follows. Here, we interpreted the output y as an external port flow and the input u as an external port effort, but this is not the only way to interpret a port-Hamiltonian ODE as a pH-LG system. Not only is the opposite possible, cf. [VvdS10b], the external port flows and efforts can also be defined as a combination of y and u. Such a combination is achieved by interconnecting the pH-LG system we just presented with a pH-LG system of the form $(\mathcal{D}', \{0\}, \{0\})$.

Remark 2.2.7. The inclusion equation in the dynamics (2.6) of a pH-LG system can be described by DAEs. To obtain such a DAE note that with the help of a kernel representation for $\mathcal{D}$, see Rem. 1.3.4, we can write

$$\begin{bmatrix} K_1 & K_2 & K_3\\ K_4 & K_5 & K_6\\ K_7 & K_8 & K_9 \end{bmatrix} \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t}\, x\\ f_R\\ f_P \end{pmatrix} + \begin{bmatrix} L_1 & L_2 & L_3\\ L_4 & L_5 & L_6\\ L_7 & L_8 & L_9 \end{bmatrix} \begin{pmatrix} e_S\\ e_R\\ e_P \end{pmatrix} = 0$$

for suitable matrices. Invoking the existence of a range representation for $\mathcal{L}$ and $\mathcal{R}$, see [BTW16, Thm. 3.3] and Sec. 1.3, we obtain

$$\begin{bmatrix} K_1 & K_2 & K_3\\ K_4 & K_5 & K_6\\ K_7 & K_8 & K_9 \end{bmatrix} \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t}\, P_{\mathcal{L}} z_1\\ P_{\mathcal{R}} z_2\\ f_P \end{pmatrix} + \begin{bmatrix} L_1 & L_2 & L_3\\ L_4 & L_5 & L_6\\ L_7 & L_8 & L_9 \end{bmatrix} \begin{pmatrix} P_{\mathcal{L}} z_1\\ S_{\mathcal{R}} z_2\\ e_P \end{pmatrix} = 0,$$

for suitable matrices and functions with $x = P_{\mathcal{L}} z_1$, $e_S = P_{\mathcal{L}}$ $f_R = P_{\mathcal{R}}$, and $e_R = S_{\mathcal{R}} z_2$. The latter *implies*

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} K_1 P_{\mathcal{L}} & 0 & 0 & 0\\ K_4 P_{\mathcal{L}} & 0 & 0 & 0\\ K_7 P_{\mathcal{L}} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1\\ z_2\\ f_P\\ e_P \end{pmatrix} = \begin{bmatrix} L_1 P_{\mathcal{L}} & (L_2 + K_2) P_{\mathcal{R}} & K_3 & L_3\\ L_4 P_{\mathcal{L}} & (L_5 + K_5) P_{\mathcal{R}} & K_6 & L_6\\ L_7 P_{\mathcal{L}} & (L_8 + K_8) P_{\mathcal{R}} & K_9 & L_9 \end{bmatrix} \begin{pmatrix} z_1\\ z_2\\ f_P\\ e_P \end{pmatrix}, \tag{2.9}$$

which is a DAE. The difference between the formulation (2.9) and (2.6) is that the DAE may impose less smoothness on $x = P_{\mathcal{L}} z_1$. In order to obtain a system as depicted in (1.4), one intuitively would want to introduce input and output functions u, y by a combination of f_P, e_P since they represent the interaction with the environment. Moreover, it seems quite unnatural to interpret f_P and e_P as states. However, without special knowledge of the matrices involved in the kernel representation of $\mathcal{D}$ it is *a priori* not clear how to achieve this formulation with inputs and outputs.

With these considerations, we can develop a solution theory for pH-LG systems by introducing the notion of behavior.

Definition 2.2.8 (Behavior of pH-LG systems)**.** Let $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ be a pH-LG system and

$$\mathcal{D} = \ker \begin{bmatrix} K & L \end{bmatrix}, \quad \mathcal{L} = \operatorname{ran} \begin{bmatrix} P_{\mathcal{L}} \\ S_{\mathcal{L}} \end{bmatrix}, \quad \mathcal{R} = \operatorname{ran} \begin{bmatrix} P_{\mathcal{R}} \\ S_{\mathcal{R}} \end{bmatrix},$$

for some matrices $K,\ L \in \mathbb{R}^{m_D \times (n_S + n_R + n_P)}$, $P_{\mathcal{L}},\ S_{\mathcal{L}} \in \mathbb{R}^{n_S \times m_S}$ and $P_{\mathcal{R}},\ S_{\mathcal{R}} \in \mathbb{R}^{n_R \times m_R}$. The *behavior* of $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ is defined as

$$\begin{aligned} \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}} = \{ & (P_{\mathcal{L}} z_1, P_{\mathcal{R}} z_2, f_P, S_{\mathcal{L}} z_1, S_{\mathcal{R}} z_2, e_P) \mid \\ & \qquad (z_1, z_2, f_P, e_P) \in \mathfrak{B}_{[E,A]} \wedge P_{\mathcal{L}} z_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_S}) \Big\}, \end{aligned} \tag{2.10}$$

with $E, A \in \mathbb{R}^{m_D \times (m_S + m_R + 2n_P)}$ given by

$$\begin{aligned} E &= \left[K \begin{bmatrix} I_{n_S} & \\ & 0_{n_R+n_P} \end{bmatrix} \; 0_{(n_S+n_R+n_P)\times n_P} \right] \begin{bmatrix} P_{\mathcal{L}} & & \\ & P_{\mathcal{R}} & \\ & & I_{2n_P} \end{bmatrix}, \\ A &= \left(\begin{bmatrix} L & 0_{(n_S+n_R+n_P)\times n_P} \end{bmatrix} \begin{bmatrix} I_{n_S+n_R} & & \\ & & I_{n_P} \\ & I_{n_P} & \end{bmatrix} + \left[K \begin{bmatrix} 0_{n_S} & \\ & I_{n_R+n_P} \end{bmatrix} \; 0 \right] \right) \begin{bmatrix} P_{\mathcal{L}} & & \\ & P_{\mathcal{R}} & \\ & & I_{2n_P} \end{bmatrix}. \end{aligned} \tag{2.11}$$

In order to obtain the most concise DAE representing the dynamics of a pH-LG system, it is beneficial to choose $m_D = \dim \mathcal{D} = n_S + n_R + n_P$, $m_S = \dim \mathcal{L} = n_S$ and $m_R = \dim \mathcal{R}$. In that way, one obtains as few equations and unknowns as possible describing its dynamics, see (2.9). Further, invoking Rem. 2.2.7 shows that $(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$ fulfills the dynamics (2.6) for almost all $t \in \mathbb{R}$.

Proposition 2.2.9. *The behavior $\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$ of a pH-LG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ is well-defined.*

Proof. First note that the kernel and image representaions of $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{R}$ in Def. 2.2.8 always exist as discussed in Sec. 1.3. Next, we show that the behavior $\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$ is invariant under the particular choice of representations. To this end, let

$$\mathcal{D} = \ker\left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right] = \ker\left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right],$$
$$\mathcal{L} = \operatorname{ran}\left[\begin{smallmatrix} P_{\mathcal{L}1} \\ S_{\mathcal{L}1} \end{smallmatrix}\right] = \operatorname{ran}\left[\begin{smallmatrix} P_{\mathcal{L}2} \\ S_{\mathcal{L}2} \end{smallmatrix}\right],$$
$$\mathcal{R} = \operatorname{ran}\left[\begin{smallmatrix} P_{\mathcal{R}1} \\ S_{\mathcal{R}1} \end{smallmatrix}\right] = \operatorname{ran}\left[\begin{smallmatrix} P_{\mathcal{R}2} \\ S_{\mathcal{R}2} \end{smallmatrix}\right]$$

for some matrices $K_1, L_1 \in \mathbb{R}^{m_D \times (n_S+n_R+n_P)}$, $P_{\mathcal{L}1}, S_{\mathcal{L}1} \in \mathbb{R}^{n_S \times m_S}$, $P_{\mathcal{R}1}, S_{\mathcal{R}1} \in \mathbb{R}^{n_R \times m_R}$ and $K_2, L_2 \in \mathbb{R}^{m'_D \times (n_S+n_R+n_P)}$, $P_{\mathcal{L}2}, S_{\mathcal{L}2} \in \mathbb{R}^{n_S \times m'_S}$, $P_{\mathcal{R}2}, S_{\mathcal{R}2} \in \mathbb{R}^{n_R \times m'_R}$. Assume for a moment that $m_D = m'_D = \dim \mathcal{D}$ $(= n_S + n_R + n_P)$, $m_S = m'_S = \dim \mathcal{L}$ $(= n_S)$ and $m_R = m'_R = \dim \mathcal{R}$. Then by Prop. 1.3.2 there exist invertible matrices $S_{\mathcal{D}} \in \mathbf{Gl}_{\dim \mathcal{D}}(\mathbb{R})$, $T_{\mathcal{L}} \in \mathbf{Gl}_{\dim \mathcal{R}}(\mathbb{R})$ and $T_{\mathcal{R}} \in \mathbf{Gl}_{\dim \mathcal{R}}(\mathbb{R})$ such that

$$S_{\mathcal{D}}\left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} P_{\mathcal{L}1} \\ S_{\mathcal{L}1} \end{smallmatrix}\right] S_{\mathcal{L}} = \left[\begin{smallmatrix} P_{\mathcal{L}2} \\ S_{\mathcal{L}2} \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} P_{\mathcal{R}1} \\ S_{\mathcal{R}1} \end{smallmatrix}\right] S_{\mathcal{R}} = \left[\begin{smallmatrix} P_{\mathcal{R}2} \\ S_{\mathcal{R}2} \end{smallmatrix}\right].$$

Then with

$$E_1, A_1 \in \mathbb{R}^{(n_S+n_R+n_P)\times(n_S+\dim \mathcal{R}+2n_P)},$$
$$E_2, A_2 \in \mathbb{R}^{(n_S+n_R+n_P)\times(n_S+\dim \mathcal{R}+2n_P)}$$

as defined by (2.11) for their respective representations and with

$$\bar{T} = \operatorname{diag}(T_{\mathcal{L}}, T_{\mathcal{R}}, I_{2n_P})$$

we have

$$S_{\mathcal{D}}(sE_1 - A)\bar{T} = sE_2 - A_2.$$

By Prop. 1.2.16 and invertibility of $T_{\mathcal{L}}, T_{\mathcal{R}}$ we get that

$$\begin{aligned}
&\Big\{ (P_{\mathcal{L}2}z_1, P_{\mathcal{R}2}z_2, f_P, S_{\mathcal{L}2}z_1, S_{\mathcal{R}2}z_2, e_P) \;\Big|\; \\
&\qquad\qquad (z_1, z_2, f_P, e_P) \in \mathfrak{B}_{[E_2,A_2]} \wedge P_{\mathcal{L}2}z_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_S}) \Big\} \\
&= \Big\{ (P_{\mathcal{L}1}T_{\mathcal{L}}z_1, P_{\mathcal{R}1}T_{\mathcal{R}}z_2, f_P, S_{\mathcal{L}}z_1, S_{\mathcal{R}}z_2, e_P) \;\Big|\; \\
&\qquad\qquad (T_{\mathcal{L}}z_1, T_{\mathcal{R}}z_2, f_P, e_P) \in \mathfrak{B}_{[E_1,A_1]} \wedge P_{\mathcal{L}1}T_{\mathcal{L}}z_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_S}) \Big\} \\
&= \Big\{ (P_{\mathcal{L}1}\tilde{z}_1, P_{\mathcal{R}1}\tilde{z}_2, f_P, S_{\mathcal{L}1}\tilde{z}_1, S_{\mathcal{R}1}\tilde{z}_2, e_P) \;\Big|\; \\
&\qquad\qquad (\tilde{z}_1, \tilde{z}_2, f_P, e_P) \in \mathfrak{B}_{[E_1,A_1]} \wedge P_{\mathcal{L}1}\tilde{z}_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_S}) \Big\},
\end{aligned}$$

proving the desired statement. Now, if we drop the assumption of $m_D = m'_D = \dim \mathcal{D}$, $m_S = m'_S = \dim \mathcal{L}$ and $m_R = m'_R = \dim \mathcal{R}$, then we still get matrices $T_{\mathcal{L}}, T_{\mathcal{R}}$ of rank $\dim \mathcal{L}$ and $\dim \mathcal{R}$, respectively, for which a similar reasoning to that of the proof of Prop. 1.2.16 yields the same result. $\square$

The behavior of a pH-LG system can also be defined though the dynamics of the system.

Proposition 2.2.10. *Let* $(\mathcal{D},\mathcal{L},\mathcal{R})$ *be a pH-LG system with* $\mathcal{D}\subset\mathbb{R}^{2(n_S+n_R+n_P)}$, $\mathcal{L}\subset\mathbb{R}^{2n_S}$ *and* $\mathcal{R}\subset\mathbb{R}^{2n_R}$. *Then*

$$\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}=\Big\{\ (x,f_R,f_P,e_S,e_R,e_P)\in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_S})\times L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{2n_R+2n_P+n_S})\ \Big|\ \textit{F.a.a. } t\in\mathbb{R}:(-\tfrac{\mathrm{d}}{\mathrm{d}t}x(t),f_R(t),f_P(t),e_S(t),e_R(t),e_P(t))\in\mathcal{D},\ (x(t),e_S(t))\in\mathcal{L},\ (f_R(t),e_R(t))\in\mathcal{R}\ \Big\}.$$

Proof. Let $\mathcal{D}=\ker\left[\,K\ L\,\right]$, $\mathcal{L}=\operatorname{ran}\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]$, $\mathcal{R}=\operatorname{ran}\left[\begin{smallmatrix}P_{\mathcal{R}}\\S_{\mathcal{R}}\end{smallmatrix}\right]$ for some matrices $K,L\in\mathbb{R}^{m_D\times(n_S+n_R+n_P)}$, $P_{\mathcal{L}},S_{\mathcal{L}}\in\mathbb{R}^{n_S\times m_S}$, $P_{\mathcal{R}},S_{\mathcal{R}}\in\mathbb{R}^{n_R\times m_R}$ and note that the matrices on the l.h.s and r.h.s of (2.9) are exactly the matrices E,A in (2.11), respectively.
Step 1: We prove the forward inclusion "$\subseteq$". To this end let $(z_1,z_2,f_P,e_P)\in\mathfrak{B}_{[E,A]}$ with $P_{\mathcal{L}}z_1\in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_S})$ and E,A as defined by (2.11), i.e.,

$$(P_{\mathcal{L}}z_1,P_{\mathcal{R}}z_2,f_P,S_{\mathcal{L}}z_1,S_{\mathcal{R}}z_2,e_P)\in\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}.$$

On the one hand, this directly shows

$$(P_{\mathcal{L}}z_1(t),S_{\mathcal{L}}z_1(t))\in\operatorname{ran}\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]=\mathcal{L},\quad (P_{\mathcal{R}}z_2(t),S_{\mathcal{R}}z_2(t))\in\operatorname{ran}\left[\begin{smallmatrix}P_{\mathcal{R}}\\S_{\mathcal{R}}\end{smallmatrix}\right]=\mathcal{R},$$

for almost all $t\in\mathbb{R}$. On the other hand, the procedure of Rem. 2.2.7 shows that

$$(-\tfrac{\mathrm{d}}{\mathrm{d}t}P_{\mathcal{L}}z_1(t),P_{\mathcal{R}}z_2(t),f_P(t),S_{\mathcal{L}}z_1(t),S_{\mathcal{R}}z_2(t),e_P(t))\in\ker\left[\,K\ L\,\right]=\mathcal{D},$$

for almost all $t\in\mathbb{R}$.
Step 2: We prove the backward inclusion "$\supseteq$". To this end, let $(x,f_R,f_P,e_S,e_R,e_P)\in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_S})\times L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{2n_R+2n_P+n_S})$ such that

$$(-\tfrac{\mathrm{d}}{\mathrm{d}t}x(t),f_R(t),f_P(t),e_S(t),e_R(t),e_P(t))\in\mathcal{D},\ (x(t),e_S(t))\in\mathcal{L},\ (f_R(t),e_R(t))\in\mathcal{R},$$

for almost all $t\in\mathbb{R}$. In particular, $(x(t),e_S(t))\in\operatorname{ran}\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]$ for almost all $t\in\mathbb{R}$ implying the existence of a function $z_1(\cdot):\mathbb{R}\to\mathbb{R}^{m_S}$ such that $x=P_{\mathcal{L}}z_1$ and $e_S=P_{\mathcal{L}}z_1$. Without loss of generality we may assume $z_1\in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{m_S})$. Otherwise consider $\tilde{z}_1=\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]^{\dagger}\left(\begin{smallmatrix}x\\e_S\end{smallmatrix}\right)$ since

$$\left(\begin{smallmatrix}x\\e_S\end{smallmatrix}\right)=\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]z_1=\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]^{\dagger}\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]z_1=\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]^{\dagger}\left(\begin{smallmatrix}x\\e_S\end{smallmatrix}\right)=\left[\begin{smallmatrix}P_{\mathcal{L}}\\S_{\mathcal{L}}\end{smallmatrix}\right]\tilde{z}_1.$$

Analogously, we find some function $z_2\in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{m_R})$ such that $f_R=P_{\mathcal{R}}z_2$ and $e_R=P_{\mathcal{R}}z_2$. Repeating the steps of Rem. 2.2.7 shows that $(z_1,z_2,f_P,e_P)\in\mathfrak{B}_{[E,A]}$. Overall, $(x,f_R,f_P,e_S,e_R,e_P)\in\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$. $\square$

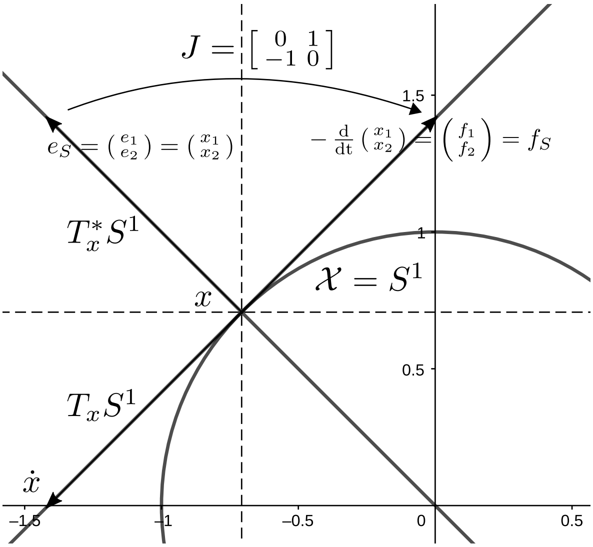

Figure 2.5: Visual representation of the port-Hamiltonian system described in Ex. 2.2.11.

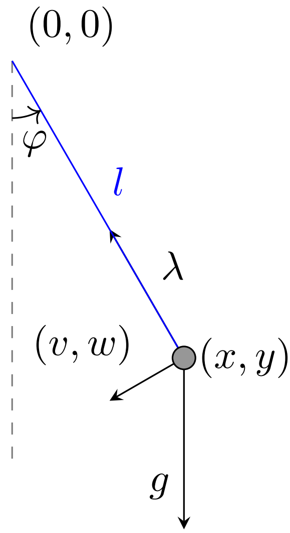

Figure 2.6: The pendulum described in Ex. 2.2.12.

2.2.3 Some illustrative examples

What we have presented so far is a modelling approach, but we have not presented a single example of how a physical system fits into this framework. We start by giving an academic example of a dynamical system on a circle.

Example 2.2.11 (Simple dynamics on a circle)**.** Consider as the state space $\mathcal{X}$ the unit circle S^1 in $\mathbb{R}^2$. Then the tangent space of $\mathcal{X}$ at $x = (x_1, x_2)$ is given by

$$T_x S^1 = \left\{ \, (-\alpha x_2, \alpha x_1) \in \mathbb{R}^2 \;\middle|\; \alpha \in \mathbb{R} \, \right\}$$

and its cotangent space at x is given by

$$T_x^* S^1 = \left\{ \, (\alpha v, \alpha w) \in \mathbb{R}^2 \;\middle|\; v x_2 = w x_1, \ \alpha \in \mathbb{R} \, \right\} = \left\{ \, \alpha x \in \mathbb{R}^2 \;\middle|\; \alpha \in \mathbb{R} \, \right\}.$$

Note that with $J := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ we can write $T_x^* \mathcal{X} = J T_x \mathcal{X}$. We now define a Dirac bundle $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ through

$$\mathcal{D}(x) = \{ \, (f_S, e_S) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \ | \ e_S \in T_x\mathcal{X} \ \wedge \ f_S = J e_S \, \} ,$$

i.e.,

$$\mathcal{D} = \left\{ \, (x_1, x_2, \alpha x_2, -\alpha x_1, \alpha x_1, \alpha x_2) \;\middle|\; (x_1, x_2) \in \mathcal{S}^1 \, \right\}.$$

As should already be clear by the definition of $\mathcal{D}$, we choose $\mathcal{R} = \emptyset$ and do not consider any external ports. Next equip $T^*\mathcal{X}$ with its canonical symplectic form, i.e., for $x \in S^1$ and $v, w \in T_x S^1$ $\omega_x = v^\top J w$. With $H : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto \frac{1}{2}\|x\|^2$, we define the Lagrange submanifold $\mathcal{L} \subset T^*\mathcal{X}$ as

$$\mathcal{L} = \left\{ (x, \nabla H(x)) \in T^*S^1 \;\middle|\; x \in S^1 \right\} = \left\{ (x, x) \in \mathbb{R}^4 \;\middle|\; x \in S^1 \right\}.$$

Now the dynamics which are illustrated by Fig. 2.5 read

$$\begin{aligned} &(-\dot{x}(t), e_S(t)) \in \mathcal{D}(x(t)), \quad (x(t), e_S(t)) \in \mathcal{L} \\ \Leftrightarrow\ &\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -e_2(t) \\ e_1(t) \end{pmatrix} = \begin{pmatrix} -x_2(t) \\ x_1(t) \end{pmatrix} \in \mathcal{S}^1. \end{aligned}$$

and we deduce $x_1(t) = a\cos(t) - b\sin(t)$ and $x_2(t) = a\sin(t) + b\cos(t)$ for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$, that is, we walk on the circle in the positive direction with velocity 1 starting at (a, b) at time $t = 0$.

There is no unique way in modelling physical systems in a port-Hamiltonian fashion. In Ex. 2.2.11 we could have encoded the condition that the solution evolves on the circle in the Lagrangian submanifold instead of the state space. Similarly, a different choice of coordinates is possible.

Example 2.2.12 (Port-Hamiltonian pendulum)**.** We consider a simple pendulum of mass m and length l without friction with $m, l > 0$ as depicted in Fig. 2.6: (x, y) describes the position of the pendulum, (v, w) its velocity, φ the angle of the string relative to $(0, -1)$ and λ the tension force. Let us first describe this system in generalized coordinates with respect to the Cartesian coordinates. To this end, we introduce

- the state $q := (x, y) \in \mathbb{R}^2 \setminus \{0\}$,
- the mass matrix $M \in C(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^{2\times 2})$ with $M(q) > 0$ as $M \equiv mI_2$,
- the generalized momenta $p = M(q)\dot{q} = (mv, mw)$,
- the kinetic energy $K(q, p) := \frac{1}{2}p^\top M(q)^{-1}p = \frac{p_1^2 + p_2^2}{2m} = m\frac{v^2+w^2}{2}$,
- the potential energy $U(q) := mgq_2 = mgy$,
- the Hamiltonian $H(q, p) = \frac{1}{2}p^\top M(q)^{-1}p + U(q)$,
- the holonomic/kinematic constraints matrix $A(q) := 2q$.

Omitting the time-dependency, the system can now be described as

$$\begin{aligned}\dot{q} =& \frac{\partial H}{\partial p}(q,p),\\ \dot{p} =& -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda,\\ 0 =& A^\top(q)\frac{\partial H}{\partial p}(q,p).\end{aligned}$$

One can easily write down (cf. [vdSJ14, Sec. 3.1], [vdSM18, Ex. 2.7]) a pH-NG system whose dynamics are governed by these equations. Namely the pH-NG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ where

$$\mathcal{D}(q,p) = \Big\{(f_q, f_p, e_q, e_p) \in T_{(q,p)}T(\mathbb{R}^2 \setminus \{0\}) \times T^*_{(q,p)}T(\mathbb{R}^2 \setminus \{0\}) \;\Big|\; \exists \lambda \in \mathbb{R}^2 :\ 0 = A^\top(q)\left(\begin{smallmatrix} e_q \\ e_p\end{smallmatrix}\right),\ -\begin{pmatrix} f_q \\ f_p\end{pmatrix} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0\end{smallmatrix}\right]\left(\begin{smallmatrix} e_q \\ e_p\end{smallmatrix}\right) + \left[\begin{smallmatrix} 0 \\ A(q)\end{smallmatrix}\right]\lambda\Big\}$$

for all $(q,p) \in T(\mathbb{R}^2 \setminus \{0\}) =: \mathcal{X}$,

$$\mathcal{L} = \{((q,p), \nabla H(q,p)) \in T^*\mathcal{X}\},$$

and $\mathcal{R} = \{0\}$. By Cor. 1.3.11 and Prop. 1.4.3, this indeed defines a pH-NG system. What if we wanted to use polar coordinates for the description of the pendulum's dynamics? That is, we consider the state $\tilde{q} = (\varphi, l) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+$. How do we find a pH-NG system describing the dynamics of the pendulum in polar coordinates, given a pH-NG system for the Cartesian coordinates? Consider the change of variables

$$\begin{aligned}\psi : \mathbb{R}^2 \setminus \{0\} &\to (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+\\ (x,y) &\mapsto (\varphi, l) := \begin{cases} (\tan^{-1}(-\frac{x}{y}), \|(x,y)\|), & x \neq 0\\ (-\frac{\pi}{2}, \|(x,y)\|), & y = 0 \wedge x < 0\\ (\frac{\pi}{2}, \|(x,y)\|), & y = 0 \wedge x > 0\end{cases}\end{aligned}$$

and identify $d\psi = D\psi$, $d\psi^* = (D\psi)^\top$ with $D\psi$ being the *Jacobian matrix* of ψ. Note that $d\psi_q = D\psi(q) : T_q\mathbb{R}^2 \setminus \{0\} \to T_{\psi(q)}(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+$ is invertible for all $q \in \mathbb{R}^2 \setminus \{0\}$ as a matrix in $\mathbb{R}^{2\times 2}$. That is with $\tilde{\mathcal{X}} = T((\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+)$,

$$\begin{aligned}\Psi : \mathcal{X} &\to \tilde{X},\\ (q,p) &\mapsto (\psi(q), D\psi(q)^{-1}p),\end{aligned}$$

is a diffeomorphism. Consider

- the state $\tilde{q} := (\varphi, l)$,

- the mass matrix $\tilde{M} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+ \to \mathbb{R}^{2\times 2}$ with $\tilde{M}(\tilde{q}) > 0$ as

$$\tilde{M}(\tilde{q}) := D\psi(\psi^{-1}(\tilde{q}))^{-\top} M(\psi^{-1}(\tilde{q})) D\psi(\psi^{-1}(\tilde{q}))^{-1} = \left(\begin{smallmatrix} m\tilde{q}_2^2 & 0 \\ 0 & m \end{smallmatrix}\right),$$

- the generalized momenta $\tilde{p} = \tilde{M}(\tilde{q}) \frac{\mathrm{d}}{\mathrm{dt}} \tilde{q} = (m\tilde{q}_2^2 \frac{\mathrm{d}}{\mathrm{dt}} \tilde{q}_1, m \frac{\mathrm{d}}{\mathrm{dt}} \tilde{q}_2)$,
- the kinetic energy $\tilde{K}(\tilde{q},\tilde{p}) := \frac{1}{2}\tilde{p}^\top \tilde{M}(\tilde{q})^{-1}\tilde{p} = \frac{\tilde{p}_1^2 + (\tilde{q}_2\tilde{p}_2)^2}{2m\tilde{q}_2^2}$,
- the potential energy

$$\tilde{U}(\tilde{q}) := U(\psi^{-1}\tilde{q}) = U(\tilde{q}_2 \sin(\tilde{q}_1), -\tilde{q}_2\cos(\tilde{q}_1)) = -mg\tilde{q}_2\cos(\tilde{q}_1),$$

- the Hamiltonian

$$\tilde{H}(\tilde{q},\tilde{p}) = \tfrac{1}{2}\tilde{p}^\top M(\tilde{q})^{-1}\tilde{p} + \tilde{U}(\tilde{q}) = H(\psi^{-1}(\tilde{q}), D\psi(\psi^{-1}(\tilde{q}))\tilde{p}),$$

- the holonomic/kinematic constraints matrix

$$\tilde{A}(\tilde{q}) := D\psi(\psi^{-1}(\tilde{q}))^{-\top} A(\psi^{-1}(q)) = e_2^{[2]}.$$

Note that by this choice we have conservation of potential and kinetic energy, i.e., $K(q,p) = \tilde{K}(\tilde{q},\tilde{p})$ and $U(q) = \tilde{U}(\tilde{q})$. Further, defining

$$\begin{aligned}\tilde{\mathcal{D}}(\tilde{q},\tilde{p}) = \Big\{ &(f_{\tilde{q}}, f_{\tilde{p}}, e_{\tilde{q}}, e_{\tilde{p}}) \in T_{(\tilde{q},\tilde{p})}\tilde{\mathcal{X}} \times T^*_{(q,p)}\tilde{\mathcal{X}} \ \Big| \\ &\exists \lambda \in \mathbb{R}^2 : \ 0 = \tilde{A}^\top(\tilde{q}) \left(\begin{smallmatrix} e_{\tilde{q}} \\ e_{\tilde{p}} \end{smallmatrix}\right), \ -\left(\begin{smallmatrix} f_{\tilde{q}} \\ f_{\tilde{p}} \end{smallmatrix}\right) = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} e_{\tilde{q}} \\ e_{\tilde{p}} \end{smallmatrix}\right) + \left[\begin{smallmatrix} 0 \\ \tilde{A}(\tilde{q}) \end{smallmatrix}\right] \lambda \Big\}\end{aligned}$$

for all $(\tilde{q},\tilde{p}) \in \tilde{\mathcal{X}}$,

$$\mathcal{L} = \left\{ ((\tilde{q},\tilde{p}), \nabla\tilde{H}(\tilde{q},\tilde{p})) \in T^*\mathcal{X} \right\},$$

and $\tilde{\mathcal{R}} = \{0\}$ we find that

$$\begin{aligned} &\left(-\tfrac{\mathrm{d}}{\mathrm{dt}} \left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right), \left(\begin{smallmatrix} e_q \\ e_p \end{smallmatrix}\right)\right) \in \mathcal{D} \wedge \left(\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right), \left(\begin{smallmatrix} e_q \\ e_p \end{smallmatrix}\right)\right) \in \mathcal{L} \\ \iff \ &\left(-\tfrac{\mathrm{d}}{\mathrm{dt}} \left(\begin{smallmatrix} \tilde{q} \\ \tilde{p} \end{smallmatrix}\right), \left(\begin{smallmatrix} e_{\tilde{q}} \\ e_{\tilde{p}} \end{smallmatrix}\right)\right) \in \tilde{\mathcal{D}} \wedge \left(\left(\begin{smallmatrix} \tilde{q} \\ \tilde{p} \end{smallmatrix}\right), \left(\begin{smallmatrix} e_{\tilde{q}} \\ e_{\tilde{p}} \end{smallmatrix}\right)\right) \in \tilde{\mathcal{L}}. \end{aligned}$$

The procedure of change of coordinates in Ex. 2.2.12 can be formalized for all pH-NG systems.

Proposition 2.2.13. *Let $(\mathcal{D},\mathcal{L},\mathcal{R})$ be a port-Hamiltonian system with $\mathcal{D} \subset (T\mathcal{X} \oplus \mathcal{B}_R \oplus \mathcal{B}_P) \oplus (T\mathcal{X} \oplus \mathcal{B}_R \oplus \mathcal{B}_P)^*$ and $\mathcal{L}$ a Lagrange submanifold of $(T^*\mathcal{X},\omega)$. For a diffeomorphism $\Psi : \mathcal{X} \to \tilde{\mathcal{X}}$ consider the bundles $\tilde{\mathcal{B}}_R$, $\tilde{\mathcal{B}}_P$, and $\tilde{\mathcal{D}} \subset (T\tilde{\mathcal{X}} \oplus \tilde{\mathcal{B}}_R \oplus$*

$$\begin{tikzcd}
\mathcal{L} \subset T^*\mathcal{X} \arrow[r, "p^*_{\mathcal{X}}"] \arrow[d, shift right, "d\Psi^{-*}"'] & \mathcal{X} \arrow[d, shift right, "\Psi"'] & T\mathcal{X} \arrow[l, "p_{\mathcal{X}}"'] \arrow[d, shift right, "d\Psi"'] \\
\tilde{\mathcal{L}} \subset T^*\tilde{\mathcal{X}} \arrow[r, "p^*_{\tilde{\mathcal{X}}}"'] \arrow[u, shift right, "d\Psi^*"'] & \tilde{\mathcal{X}} \arrow[u, shift right, "\Psi^{-1}"'] & T\tilde{\mathcal{X}} \arrow[l, "p_{\tilde{\mathcal{X}}}"] \arrow[u, shift right, "d\Psi^{-1}"']
\end{tikzcd}$$

Figure 2.7: Commutative diagram for the change of coordinates in the proof of Prop. 2.2.13.

$\tilde{\mathcal{B}}_P) \oplus (T\tilde{\mathcal{X}} \oplus \tilde{\mathcal{B}}_R \oplus \tilde{\mathcal{B}}_P)^*$ *over* $\tilde{\mathcal{X}}$ *defined by* $(\tilde{\mathcal{B}}_R)_{\tilde{x}} := (\mathcal{B}_R)_{\Psi^{-1}(\tilde{x})}$, $(\tilde{\mathcal{B}}_P)_{\tilde{x}} := (\mathcal{B}_P)_{\Psi^{-1}(\tilde{x})}$, *and*

$$\tilde{\mathcal{D}}(\tilde{x}) := \Big\{ \ (d\Psi_{\Psi^{-1}(\tilde{x})} f_S, f_R, f_P, d\Psi^{-*}_{\Psi^{-1}(\tilde{x})} e_S, e_R, e_P) \ \Big| \ (f_S, f_R, f_P, e_S, e_R, e_P) \in \mathcal{D}(\Psi^{-1}(\tilde{x})) \ \Big\},$$

for all $\tilde{x} \in \tilde{\mathcal{X}}$*, as well as the subbundle*

$$\tilde{\mathcal{L}} := d\Psi^{-*}(\mathcal{L}) = \left\{ \ (\Psi(x), d\Psi^{-*}_x v) \ \middle| \ (x, v) \in \mathcal{L} \ \right\}$$

of $(T\tilde{\mathcal{X}}, \tilde{\omega})$ *where* $\tilde{\omega}$ *is the pullback of* ω *under* $d\Psi^*$*,* $d\Psi^{**}\omega$*. Further, let* $\tilde{\mathcal{R}} = \mathcal{R}$*. Then* $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \tilde{\mathcal{R}})$ *is a pH-NG system and*

$$\begin{gathered}
(-\tfrac{\mathrm{d}}{\mathrm{dt}} x(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) \in \mathcal{D}(x(t)), \\
(x(t), e_S(t)) \in \mathcal{L}, \quad (f_R(t), e_R(t)) \in \mathcal{R}(x(t))
\end{gathered}$$

if and only if

$$\begin{gathered}
(-\tfrac{\mathrm{d}}{\mathrm{dt}} \Psi(x(t)), f_R(t), f_P(t), d\Psi^{-*}_{x(t)} e_S(t), e_R(t), e_P(t)) \in \tilde{\mathcal{D}}(\Psi(x(t))), \\
(\Psi(x(t)), d\Psi^{-*}_{x(t)} e_S(t)) \in \tilde{\mathcal{L}}, \quad (f_R(t), e_R(t)) \in \tilde{\mathcal{R}}(\Psi(x(t))).
\end{gathered}$$

Proof. With the notation introduced in the proposition, we first check that $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \tilde{\mathcal{R}})$ is a well-defined pH-NG system. Note that $\tilde{\mathcal{D}}$ pointwisely defines a Dirac subspace since for $\tilde{x} \in \tilde{\mathcal{X}}$ and $(\tilde{f}_S, \tilde{f}_R, \tilde{f}_P, \tilde{e}_S, \tilde{e}_R, \tilde{e}_P) \in \tilde{\mathcal{D}}(\tilde{x})$ it holds

$$(d\Psi_{x(t)} f_S, \tilde{f}_R, \tilde{f}_P, d\Psi^{-*}_{x(t)} e_S, \tilde{e}_R, \tilde{e}_P) \in \mathcal{D}(\Psi^{-1}(\tilde{x}))$$

and hence

$$\tilde{f}_S^\top \tilde{e}_S + \tilde{f}_R^\top \tilde{e}_R + \tilde{f}_P^\top \tilde{e}_P = \left(d\Psi_{x(t)} f_S\right)^\top \left(d\Psi^{-*}_{x(t)} e_S\right) + \tilde{f}_R^\top \tilde{e}_R + \tilde{f}_P^\top \tilde{e}_P$$

$$
\begin{aligned}
&= e_S^\top \left(d\Psi^{-1}_{\Psi(x(t))} d\Psi_{x(t)} f_S \right) + \tilde{f}_R^\top \tilde{e}_R + \tilde{f}_P^\top \tilde{e}_P \\
&= e_S^\top \left((d\Psi_{(x(t))})^{-1} d\Psi_{x(t)} f_S \right) + \tilde{f}_R^\top \tilde{e}_R + \tilde{f}_P^\top \tilde{e}_P \\
&= e_S^\top f_S + \tilde{f}_R^\top \tilde{e}_R + \tilde{f}_P^\top \tilde{e}_P = 0.
\end{aligned}
$$

It thus defines pointwisely a skew-symmetric subspace that is maximal with this property by pointwise invertibility of $d\psi$ and $d\psi^{-*}$. Due to $\Psi : \mathcal{X} \to \tilde{\mathcal{X}}$ being a C^2 diffeomorphism, both $d\Psi : T\mathcal{X} \to T\tilde{\mathcal{X}}$ and $d\Psi^* : T^*\tilde{\mathcal{X}} \to T^*\mathcal{X}$ are C^1 diffeomorphisms. The commutative diagram displayed in Fig. 2.7 recapitulates the setting. Since $d(d\Psi^*)$ vanishes nowhere, $d\Psi^{**}\omega$ defines a symplectic form on $T^*\tilde{\mathcal{X}}$. Therefore, we can now check that $\tilde{\mathcal{L}}$ defines a Lagrangian submanifold of $(T^*\tilde{\mathcal{X}}, \tilde{\omega})$. Since it is clear that $\dim \tilde{\mathcal{L}} = \frac{1}{2} \dim T^*\tilde{\mathcal{X}}$, it suffices to show that $\tilde{\omega}\restriction_{\tilde{\mathcal{L}}} = 0$ by Def. 1.4.2 and [Lee12, Prop. 22.5]. Let $\tilde{\iota} : \tilde{\mathcal{L}} \to T^*\tilde{\mathcal{X}}$, $\iota : \mathcal{L} \to T^*\mathcal{X}$ denote the inclusions. Then

$$
\iota^*\omega = \left(d\Psi^* \circ \tilde{\iota} \circ d\Psi^{-*} \right) \omega = d\Psi^{-**} \tilde{\iota}^* d\Psi^{**}\omega = d\Psi^{-**} \tilde{\iota}^* \tilde{\omega} = 0 \Leftrightarrow \tilde{\iota}^* \tilde{\omega} = 0,
$$

i.e., $\mathcal{L}$ is a Lagrangian submanifold if and only if $\tilde{\mathcal{L}}$ is a Lagrangian submanifold. Note that $\tilde{\mathcal{R}}$ is obviously a resistive structure. Overall, $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \tilde{\mathcal{R}})$ is a pH-NG system. The last statement of the proposition follows directly from the construction and the chain rule. □

Remark 2.2.14. By the choice of $\tilde{\omega}$ in Prop. 2.2.13, $d\Psi^*$ becomes a symplectomorphism and as can be observed in Ex. 2.2.12, if ω is the standard symplectic form, then so is $\tilde{\omega}$, see [LM87, Prop. 3.5].

2.3 The linear-algebraic formulation

In this section, we introduce two port-Hamiltonian formulations. The first formulation we present was proposed by the systems theory community and can be found in [BMX⁺18; MMW18]. The second formulation is a relaxation of the first, where we essentially replace the state space $\mathbb{R}^n$ by a subspace of $\mathbb{R}^n$. This restriction conserves all the relevant properties that make port-Hamiltonian formulations so appealing.

The classic definition

The following definition generalizes the pH-ODE systems (2.1) in a straight-forward manner, namely by applying a possibly indefinite matrix E before differentiating the state x in (2.1a) and requiring symmetry of E^*Q instead of Q in (2.1b).

Definition 2.3.1 (Linear-algebraic port-Hamiltonian system)**.**
A system $[E, A, B, C, D] \in \Sigma_{n,m,k}$ is called a *linear algebraic port-Hamiltonian system* (pH-LA system) if there exist matrices $Q \in \mathbb{K}^{m\times n}$, $J, R \in \mathbb{K}^{m\times m}$, $G, K \in \mathbb{K}^{m\times k}$ and $S, N \in \mathbb{K}^{k\times k}$ satisfying

$$Q^*E = E^*Q, \quad V := \begin{bmatrix} J & G \\ -G^* & N \end{bmatrix} = -V^*, \quad W := \begin{bmatrix} R & K \\ K^* & S \end{bmatrix} \geq 0, \tag{2.12}$$

such that

$$A = (J - R)Q, \quad B = G - K, \quad C = (G + K)^*Q, \quad D = S + N,$$

i.e., the system is described by the equations

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\, Ex(t) =&(J - R)Qx(t) + (G - K)u(t), \\ y(t) =&(G + K)^*Qx(t) + (S + N)u(t). \end{aligned} \tag{2.13}$$

Clearly, the systems (2.1) are comprised in this definition and a power balance is achieved in the same manner defining the Hamiltonian $H(x) = \frac{1}{2}x^*Q^*Ex$. Namely for $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\, H(x) =&\tfrac{1}{2}\, \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\, (x^*Q^*Ex) \\ =&\tfrac{1}{2}\, \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\, \big(x^*Q^*EE^\dagger Ex\big) \\ =&\tfrac{1}{2}\, \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\, \big(x^*E^*QE^\dagger Ex\big) \\ =&\tfrac{1}{2} \operatorname{Re}\, \big(\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)^*QE^\dagger Ex + x^*E^*QE^\dagger\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\tfrac{1}{2} \operatorname{Re}\, \big(x^*(E^*E^{\dagger *}Q^*EE^\dagger + Q^*EE^\dagger)\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\tfrac{1}{2} \operatorname{Re}\, \big(x^*(E^*E^{\dagger *}E^*QE^\dagger + Q^*EE^\dagger)\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\tfrac{1}{2} \operatorname{Re}\, \big(x^*(E^*QE^\dagger + Q^*EE^\dagger)\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\tfrac{1}{2} \operatorname{Re}\, \big(x^*(Q^*EE^\dagger + Q^*EE^\dagger)\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\operatorname{Re}\, \big(x^*Q^*\, \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\big) \\ =&\operatorname{Re}\, (x^*Q^*(J - R)Qx + x^*Q^*(G - K)u) \\ =&\operatorname{Re}\, (-x^*Q^*RQx + y^*u - 2x^*Q^*Ku - u^*(S + N)u) \\ =&\operatorname{Re}\,(y^*u) - \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right)^* W \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right) \\ \leq&\operatorname{Re}\,(y^*u). \end{aligned} \tag{2.14}$$

Paying close attention to (2.14), we only need the equalities and inequality in (2.12) to be true on the system space $\mathcal{V}_{\mathrm{sys}}$ and $\left[\begin{smallmatrix} I_n & 0 \end{smallmatrix}\right]\mathcal{V}_{\mathrm{sys}}$, respectively. When considering

DAEs, properties or characterizations are commonly formulated on subspaces such as the system space, the space of consistent differential variables or some related space, see, e.g., App. A or [RV19]. This approach is justified since outside these spaces, no dynamics take place. This motivates the relaxation of Def. 2.3.1 in the subsequent section.

Linear algebraic port-Hamiltonian systems on a subspace

Definition 2.3.2 (Relaxed linear-algebraic port-Hamiltonian system)**.**
A system $[E, A, B, C, D] \in \Sigma_{n,m,k}$ is called a *relaxed linear algebraic port-Hamiltonian system* on $\mathcal{V} \subset \mathbb{K}^{n+k}$ (pH-LAR system) if there exist matrices $Q \in \mathbb{K}^{m\times n}$, $J, R \in \mathbb{K}^{m\times m}$, $G, K \in \mathbb{K}^{m\times k}$ and $S, N \in \mathbb{K}^{k\times k}$ satisfying

$$\begin{aligned} Q^*E &=_{[\,I_n\ 0\,]\mathcal{V}} E^*Q, \\ V &:= \begin{bmatrix} J & G \\ -G^* & N \end{bmatrix} =_{\left[\begin{smallmatrix} E & 0 \\ 0 & I_k \end{smallmatrix}\right]\mathcal{V}} -V^*, \\ W &:= \begin{bmatrix} R & K \\ K^* & S \end{bmatrix} \geq_{\left[\begin{smallmatrix} E & 0 \\ 0 & I_k \end{smallmatrix}\right]\mathcal{V}} 0, \end{aligned} \tag{2.15}$$

such that

$$\begin{bmatrix} A & B \end{bmatrix} =_{\mathcal{V}} \begin{bmatrix} (J-R)Q & G-K \end{bmatrix}, \quad \begin{bmatrix} C & D \end{bmatrix} =_{\mathcal{V}} \begin{bmatrix} (G+K)^*Q & S+N \end{bmatrix}.$$

Now for a pH-LAR system $[E, A, B, C, D]$ on some subspace $\mathcal{V}$ and given some $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$, if $x(t) \in [\,I_n\ 0\,]\,\mathcal{V}$ for almost all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{dt}} Ex \in [\,I_n\ 0\,]\,\mathcal{V}$ for almost all $t \in \mathbb{R}$. Analogously to (2.14) with $H(x) = x^*Q^*Ex$ as Hamiltonian, we obtain the power balance

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}} H(x) &= \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\,(x^*Q^*Ex) \\ &= \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\,\left(x^*Q^*EE^\dagger Ex\right) \\ &= \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\,\left(x^*E^*QE^\dagger Ex\right) \\ &= \tfrac{1}{2} \operatorname{Re}\,\left(\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)^*QE^\dagger Ex + x^*E^*QE^\dagger \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \\ &= \tfrac{1}{2} \operatorname{Re}\,\left(x^*(E^*E^{\dagger *}Q^*EE^\dagger + Q^*EE^\dagger)\,\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \\ &= \tfrac{1}{2} \operatorname{Re}\,\left(x^*(E^*E^{\dagger *}E^*QE^\dagger + Q^*EE^\dagger)\,\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \\ &= \tfrac{1}{2} \operatorname{Re}\,\left(x^*(E^*QE^\dagger + Q^*EE^\dagger)\,\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \\ &= \tfrac{1}{2} \operatorname{Re}\,\left(x^*(Q^*EE^\dagger + Q^*EE^\dagger)\,\tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&= \mathrm{Re}\, \left(x^* Q \tfrac{\mathrm{d}}{\mathrm{dt}}(Ex)\right) \\
&= \mathrm{Re}\, \left(x^* Q^* (Ax + Bu)\right) \\
&= \mathrm{Re}\, \left(x^* Q^* (J - R) Qx + x^* Q^* (G - K) u\right) \\
&= \mathrm{Re}\, \left(-x^* Q^* R Q x + y^* u - 2x^* Q^* K u - u^* (S + N) u\right) \\
&= \mathrm{Re}\, \left((Cx + Du)^* u\right) \\
&= \mathrm{Re}\, (y^* u) - \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right)^* W \left(\begin{smallmatrix} Qx \\ u \end{smallmatrix}\right) \\
&\leq \mathrm{Re}\, (y^* u).
\end{aligned}
$$

A canonical candidate for $\mathcal{V}$ is of course the system space $\mathcal{V}_{\mathrm{sys}}$. However, a smaller subspace can also be of interest, see Sec. 4.2.

2.4 The linear relations formulation

Even if the linear algebraic and the geometric port-Hamiltonian formulations we presented have been studied for decades, they never really have been compared to each other up to now, if not indirectly by showing that both extend (2.1). The next formulation we present arose from the effort to compare the two preceding port-Hamiltonian formulations. We will see in Sec. 2.5 that the following definition can be regarded as a unifying concept.

Definition 2.4.1 (Linear relations port-Hamiltonian system)**.** Let $n, k \in \mathbb{N}_0$, $\mathcal{D}$ be a dissipative and $\mathcal{L}$ a symmetric relation in $\mathbb{K}^{n+k}$. Then the product $\mathcal{D}\mathcal{L}$ defines a *linear relations port-Hamiltonian system* (pH-LR system) whose *dynamics* are given by the differential inclusion

$$
(x(t), e_P(t), \dot{x}(t), f_P(t)) \in \mathcal{D}\mathcal{L}. \tag{2.17}
$$

In order to show that pH-ODEs are encompassed by pH-LR systems, we make use of the approach for pH-LG systems. With the notation from (2.1) and (2.8), $-\mathcal{D}$ is readily skew-adjoint and hence dissipative. It follows by Prop. 1.3.14 that $(-\mathcal{D}) \circ \mathcal{R}$ is a dissipative relation. Further, $\mathcal{L} \widehat{\times} \operatorname{gr} I_k$ is readily a symmetric relation. In this case, for functions x, u, y there holds

$$
(x(t), u(t), \dot{x}(t), -y(t)) \in ((-\mathcal{D}) \circ \mathcal{R})^{-1} (\mathcal{L} \times \operatorname{gr} I_k)
$$

if and only if there exist functions $e_{R1}, e_{R2}, f_{R1}, f_{R2}$ such that

$$\begin{aligned}(-\tfrac{\mathrm{d}}{\mathrm{d}t}\, x(t), f_{R1}(t), f_{R2}(t), y(t), e_S(t), e_{R1}(t), e_{R2}(t), u(t)) \in \mathcal{D},\\ (x(t), e_S(t)) \in \mathcal{L}, \quad (f_{R1}(t), f_{R2}(t), e_{R1}(t), e_{R2}(t)) \in \mathcal{R},\end{aligned}$$

that is, if and only if x, u, y solve (2.1a). When for a pH-LR system the symmetric linear relation can be written as a certain product, it is also possible to formulate a certain power balance and explain interconnections similarly to (2.7). To be more precise, we establish a power balance for pH-LR systems of the form $\mathcal{D}(\mathcal{L} \widehat{\times}\, \mathrm{gr}\, I_k)$ where $\mathcal{L}$ is a symmetric relation in $\mathbb{K}^n$ and k the dimension of the external ports. Let M be the operator defined by Prop. 1.3.10 for $\mathcal{L}$ and let $\iota : (\mathrm{mul}\, \mathcal{L})^\perp \hookrightarrow \mathbb{K}^n$ be the inclusion map. Define $F : \mathbb{K}^n \to \mathbb{R}$ as $F(x) = \frac{1}{2}\,\mathrm{Re}(x^* \iota M P_{\mathrm{dom}\, \mathcal{L}} x)$. Then for $(x, e_P, \frac{\mathrm{d}}{\mathrm{d}t}\, x, f_P)$ satisfying

$$(x(t), e_P(t), \tfrac{\mathrm{d}}{\mathrm{d}t}\, x(t), f_P(t)) \in \mathcal{D}(\mathcal{L} \times \mathrm{gr}\, I_k)$$

there exists some function e_S such that

$$(x(t), e_P(t), e_S(t), e_P(t)) \in \mathcal{L} \times \mathrm{gr}\, I_k, \quad (e_S(t), e_P(t), \tfrac{\mathrm{d}}{\mathrm{d}t}\, x(t), f_P(t)) \in \mathcal{D},$$

and we have $e_S = Mx + \lambda$ for some $\lambda \in \mathrm{mul}\, \mathcal{L}$. Thus,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\, F(x(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\, \tfrac{1}{2}\,\mathrm{Re}(x(t)^* \iota M P_{\mathrm{dom}\, \mathcal{L}} x(t))\\ &= \mathrm{Re}(\dot{x}(t)^* \iota M x(t))\\ &= \mathrm{Re}(\dot{x}(t)^* (\iota M x(t) + \lambda(t)))\\ &= \mathrm{Re}(\dot{x}(t)^* e_S(t))\\ &\le \mathrm{Re}(f_P(t)^* e_P(t)).\end{aligned}$$

As for pH-LG systems, if we have a range representation $\mathcal{L} = \mathrm{ran}\, \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$, then we can reformulate the 'Hamiltonian' F with respect to other variables. Namely, with $(x(t), e_S(t)) \in \mathcal{L}$ we know that there exists some function z such that $x = Ez$ and $e_S = Qz$. Consequently,

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{Re}(z(t)^* E^* Q z(t)) = \tfrac{1}{2}\,\mathrm{Re}(x(t)^* e_S(t)) &= \tfrac{1}{2}\,\mathrm{Re}(x(t)^* M x(t) + x(t)^* \lambda(t))\\ &= \tfrac{1}{2}\,\mathrm{Re}(x(t)^* \iota M P_{\mathrm{dom}\, \mathcal{L}} x(t)) = F(x(t)).\end{aligned}$$

We introduce the notion of behavior for pH-LR systems, which describes our solution concept for them.

Definition 2.4.2 (Behavior of pH-LR system)**.** Let $\mathcal{DL}$ be a pH-LR system with dissipative and symmetric relations $\mathcal{D}, \mathcal{L} \subset \mathbb{K}^{n+k}$, respectively, for some $n, k \in \mathbb{N}_0$. Further, let a range representation

$$\operatorname{ran} \begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix} = \mathcal{DL}$$

be given for some $m \in \mathbb{N}_0$ and $E_1, A_1 \in \mathbb{K}^{n\times m}$, and $E_2, A_2 \in \mathbb{K}^{k\times m}$. We define the *behavior* of $\mathcal{DL}$ as

$$\mathfrak{B}_{\mathcal{DL}} = \left\{ \ (E_1 z, E_2 z, A_2 z) \ \middle| \ z \in \mathfrak{B}_{[E_1, A_2]} \ \right\}.$$

Note that by definition, $(x, e_P, f_P) \in \mathfrak{B}_{\mathcal{DL}}$ fulfills the dynamics (2.17) for almost all $t \in \mathbb{R}$.

Proposition 2.4.3. *The behavior $\mathfrak{B}_{\mathcal{DL}}$ of a pH-LR system $\mathcal{DL}$ is well-defined.*

Proof. Let two range representations

$$\operatorname{ran} \begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix} = \mathcal{DL} = \operatorname{ran} \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}$$

be given for some $m, \tilde{m} \in \mathbb{N}_0$ and $E_1, A_1 \in \mathbb{K}^{n\times m}$, $E_2, A_2 \in \mathbb{K}^{k\times m}$, $\tilde{E}_1, \tilde{A}_1 \in \mathbb{K}^{n\times \tilde{m}}$, and $\tilde{E}_2, \tilde{A}_2 \in \mathbb{K}^{k\times \tilde{m}}$ such that

$$\mathfrak{B}_{\mathcal{DL}} = \left\{ \ (E_1 z, E_2 z, A_2 z) \ \middle| \ z \in \mathfrak{B}_{[E_1, A_2]} \ \right\},$$

and introduce

$$\tilde{\mathfrak{B}}_{\mathcal{DL}} = \left\{ \ (\tilde{E}_1 \tilde{z}, \tilde{E}_2 \tilde{z}, \tilde{A}_2 \tilde{z}) \ \middle| \ \tilde{z} \in \mathfrak{B}_{[\tilde{E}_1, \tilde{A}_2]} \ \right\}.$$

By Prop. 1.3.2 there exist matrices $T, \tilde{T}^* \in \mathbb{K}^{m\times \tilde{m}}$ such that

$$\begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix} T = \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}, \quad \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \tilde{T} = \begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix}.$$

We show $\mathfrak{B}_{\mathcal{DL}} = \tilde{\mathfrak{B}}_{\mathcal{DL}}$. To this end, let $(E_1 z, E_2 z, A_2 z) \in \mathfrak{B}_{\mathcal{DL}}$ for some $z \in \mathfrak{B}_{[E_1, A_2]} = \mathfrak{B}_{[\tilde{E}_1 \tilde{T}, \tilde{A}_2 \tilde{T}]}$. Then $\tilde{T} z \in \tilde{\mathfrak{B}}_{[E_1, A_2]}$ and $(E_1 z, E_2 z, A_2 z) = (\tilde{E}_1 \tilde{z}, \tilde{E}_2 \tilde{z}, \tilde{A}_2 \tilde{z}) \in \tilde{\mathfrak{B}}_{\mathcal{DL}}$. The reverse inclusion is shown analogously by inverting the roles. ◻

The behavior can also be defined through the dynamics of the system.

Proposition 2.4.4. *Let $\mathcal{DL}$ be a pH-LR system with linear relation $\mathcal{D}, \mathcal{L}$ in $\mathbb{K}^{n+k}$. Then*

$$\mathfrak{B}_{\mathcal{DL}} = \Big\{ \ (x, e_P, f_P) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^{2k}) \ \Big| \ (x(t), e_P(t), \dot{x}(t), f_P(t)) \in \mathcal{DL} \text{ f.a.a. } t \in \mathbb{R} \ \Big\}.$$

Proof. Consider any range representation

$$\operatorname{ran} \begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix} = \mathcal{DL}$$

with $E_1, A_1 \in \mathbb{K}^{n\times m}$, $E_2, A_2 \in \mathbb{K}^{k\times m}$ for some $m \in \mathbb{N}_0$ and recall that

$$\mathfrak{B}_{\mathcal{DL}} = \left\{ \ (E_1 z, E_2 z, A_2 z) \ \middle| \ z \in \mathfrak{B}_{[E_1, A_2]} \ \right\}.$$

<u>*Step 1*</u>: We prove the forward inclusion "$\subseteq$". To this end, let $x \in \mathfrak{B}_{\mathcal{DL}}$. Then there exists some $z \in \mathfrak{B}_{[E_1,A_1]}$ such that $x = E_1 z \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n)$ and $e_P = E_2 z, f_P = A_2 z \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$. Hence,

$$(x(t), e_P(t), \dot{x}(t), f_P(t)) \in \mathcal{DL}, \quad \text{f.a.a. } t \in \mathbb{R}.$$

This proves the forward inclusion.
<u>*Step 2*</u>: We prove the backward inclusion "$\supseteq$". To this end, let

$$(x, e_P, f_P) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^{2k})$$

such that

$$(x(t), e_P(t), \dot{x}(t), f_P(t)) \in \mathcal{DL} = \operatorname{ran} \begin{bmatrix} E_1 \\ E_2 \\ A_1 \\ A_2 \end{bmatrix}, \quad \text{f.a.a. } t \in \mathbb{R}.$$

We deduce the existence of a function $z(\cdot) : \mathbb{R} \to \mathbb{K}^m$ such that

$$x(t) = E_1 z(t), \quad e_P(t) = E_2 z(t), \quad \dot{x}(t) = A_1 z(t), \quad f_P(t) = A_2 z(t), \quad \text{f.a.a. } t \in \mathbb{R}.$$

Let $S \in \mathbf{Gl}_n(\mathbb{K})$, $T \in \mathbf{Gl}_m(\mathbb{K})$ be such that $sSE_1T - SA_1T$ is in Kronecker form (1.1). Then with $\tilde{x} := Sx$ and $\tilde{z} := T^{-1}z$ we have

$$L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \ni \tilde{x} = \begin{bmatrix} I_l & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \tilde{z}, \tag{2.18}$$

$$L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n) \ni \tfrac{\mathrm{d}}{\mathrm{dt}}\,\tilde{x} = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \tilde{z}. \tag{2.19}$$

From (2.18) we obtain that the entries of $\tilde{z}$ corresponding to the first block in the Kronecker form (1.1) are in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^l)$. We deduce the same from (2.19) for the entries of $\tilde{z}$ corresponding to the second and last blocks in the Kronecker form (1.1). For the entries corresponding to the third we proceed the same way but rely on both (2.18) and (2.19). Overall, $\tilde{z} \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^m)$ and therefore z too. This gives $z \in \mathfrak{B}_{[E_1,A_2]}$ and overall $(x, e_P, f_P) \in \mathfrak{B}_{\mathcal{DL}}$. □

To close this section, we describe how to interconnect pH-LR systems.

Definition 2.4.5 (Interconnection of pH-LR systems)**.** Let $\mathcal{D}_1(\mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1+k_2})$ and $\mathcal{D}_2(\mathcal{L}_2 \widehat{\times}\, \mathrm{gr}\ I_{k_2+k_3})$ be two pH-LR systems with $\mathcal{L}_1 \subset \mathbb{K}^{2n_1}$ and $\mathcal{L}_2 \subset \mathbb{K}^{2n_2}$. Then the *interconnection* of $\mathcal{D}_1(\mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1+k_2})$ and $\mathcal{D}_2(\mathcal{L}_2 \widehat{\times}\, \mathrm{gr}\ I_{k_2+k_3})$ with respect to $\mathbb{K}^{k_2}$ is defined as the pH-LR system $\mathcal{DL}$ where

$$\mathcal{D} \coloneqq (\mathcal{D}_1^{-1} \circ \mathcal{D}_2^{-1})^{-1}, \quad \mathcal{L} \coloneqq \mathcal{L}_1 \widehat{\times} \mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1+k_3}.$$

By Prop. 1.3.14, the above constructed set $\mathcal{D}$ is indeed a dissipative relation and $\mathcal{L}$ is readily a symmetric relation.

Remark 2.4.6. In the context of Def. 2.4.5 for suitable functions $x_1, x_2, e_{P1}, e_{P2}, e_{P3}$, f_{P1}, f_{P2}, f_{P3} and omitting the dependency on the time, it holds almost everywhere

$$\begin{aligned}
&(x_1, e_{P1}, e_{P2}, \dot{x}_1, f_{P1}, f_{P2}) \in \mathcal{D}_1(\mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1+k_2}) \\
&\wedge\ (x_2, e_{P2}, e_{P3}, \dot{x}_2, -f_{P2}, f_{P3}) \in \mathcal{D}_2(\mathcal{L}_2 \widehat{\times}\, \mathrm{gr}\ I_{k_2+k_3}) \\
&\iff \exists (e_{S1}, e_{S2}) : (e_{S1}, e_{P1}, e_{P2}, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x_1, f_{P1}, f_{P2}) \in \mathcal{D}_1 \\
&\qquad\qquad \wedge\ (e_{S2}, e_{P2}, e_{P3}, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x_2, -f_{P2}, f_{P3}) \in \mathcal{D}_2 \\
&\qquad\qquad \wedge\ (x_1, e_{P1}, e_{P2}, e_{S1}, e_{P1}, e_{P2}) \in \mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1+k_2} \\
&\qquad\qquad \wedge\ (x_2, e_{P2}, e_{P3}, e_{S2}, e_{P2}, e_{P3}) \in \mathcal{L}_2 \widehat{\times}\, \mathrm{gr}\ I_{k_2+k_3} \\
&\iff \exists (e_{S1}, e_{S2}) : (\tfrac{\mathrm{d}}{\mathrm{dt}}\, x_1, f_{P1}, f_{P2}, e_{S1}, e_{P1}, e_{P2}) \in \mathcal{D}_1^{-1} \\
&\qquad\qquad \wedge\ (\tfrac{\mathrm{d}}{\mathrm{dt}}\, x_2, -f_{P2}, f_{P3}, e_{S2}, e_{P2}, e_{P3}) \in \mathcal{D}_2^{-1} \\
&\qquad\qquad \wedge\ (x_1, e_{P1}, e_{S1}, e_{P1}) \in \mathcal{L}_1 \widehat{\times}\, \mathrm{gr}\ I_{k_1} \\
&\qquad\qquad \wedge\ (x_2, e_{P3}, e_{S2}, e_{P3}) \in \mathcal{L}_2 \widehat{\times}\, \mathrm{gr}\ I_{k_3}
\end{aligned}$$

$$\begin{aligned}
\iff\ & \exists (e_{S1}, e_{S2}) : (\tfrac{\mathrm{d}}{\mathrm{dt}}\, x_1, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x_2, f_{P1}, f_{P3}, e_{S1}, e_{S2}, e_{P1}, e_{P3}) \in \mathcal{D}_1^{-1} \circ \mathcal{D}_2^{-1} \\
& \qquad \wedge\ (x_1, x_2, e_{P1}, e_{P3}, e_{S1}, e_{S2}, e_{P1}, e_{P3}) \in \mathcal{L}_1 \widehat{\times} \mathcal{L}_2 \widehat{\times}\ \mathrm{gr}\ I_{k_1+k_3} \\
\iff\ & (x_1, x_2, e_{P1}, e_{P3}, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x_1, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x_2, f_{P1}, f_{P3}) \in (\mathcal{D}_1^{-1} \circ \mathcal{D}_2^{-1})^{-1} (\mathcal{L}_1 \widehat{\times} \mathcal{L}_2 \widehat{\times}\ \mathrm{gr}\ I_{k_1+k_3}).
\end{aligned}$$

Loosely speaking, this means

$$\begin{aligned}
& (x_1, x_2, e_{P1}, e_{P3}, f_{P1}, f_{P3}) \in \mathfrak{B}_{(\mathcal{D}_1^{-1} \circ \mathcal{D}_2^{-1})^{-1} (\mathcal{L}_1 \widehat{\times} \mathcal{L}_2 \widehat{\times}\ \mathrm{gr}\ I_{k_1+k_3})} \\
\iff\ & (x_1, e_{P1}, e_{P2}, f_{P1}, f_{P2}) \in \mathfrak{B}_{\mathcal{D}_1 (\mathcal{L}_1 \widehat{\times}\ \mathrm{gr}\ I_{k_1+k_2})} \\
& \quad \wedge\ (x_2, e_{P2}, e_{P3}, -f_{P2}, f_{P3}) \in \mathfrak{B}_{\mathcal{D}_2 (\mathcal{L}_2 \widehat{\times}\ \mathrm{gr}\ I_{k_2+k_3})}.
\end{aligned}$$

Since the way we showed that pH-ODEs are encompassed by pH-LR systems relied on pH-LG systems, this interconnection should be compatible with the interconnection defined by Def. 2.2.2. We will answer this question in the next section, where we will compare different port-Hamiltonian formulations.

2.5 A comparison of the port-Hamiltonian formulations

For now, we only hinted at similitudes between the different port-Hamiltonian formulations by showing, for example, how they generalize the port-Hamiltonian ODEs or how a power-balance can be realized. In this section, we will take a closer look at how the pH-LG, pH-LA, and pH-LR formulations relate to one another and make use of the intuition on port-Hamiltonian systems we have gathered to this point.

2.5.1 Simple translations between port-Hamiltonian formulations

Here, we present criteria, which allow us the rewriting of either a pH-LA, pH-LG, or pH-LR system as a system of the other types, that is, we give only sufficient conditions enabling the translation from one port-Hamiltonian formulation to another.

From the pH-LA to the pH-LR formulation

Given a pH-LA system (2.12), it is straight-forward to see that

$$\mathcal{D}_1 := \mathrm{gr} \begin{bmatrix} J - R & G - K \\ -(G+K)^* & -(S+N) \end{bmatrix}, \quad \mathcal{L}_1 := \mathrm{ran} \begin{bmatrix} E \\ Q \end{bmatrix}$$

define a dissipative and symmetric relation, respectively, i.e.,

$$\mathcal{D}_1\mathcal{L}_1 = \operatorname{ran}\begin{bmatrix} E & 0 \\ 0 & I \\ (J-R)Q & G-K \\ -(G+K)^*Q & -(S+N) \end{bmatrix}$$

is a pH-LR system. Further, for $(x,u,y) \in \mathfrak{B}_{[E,(J-R)Q,G-K,(G+K)^*Q,S+N]}$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}} Ex(t) =& (J-R)Qx(t) + (G-K)u(t), \\ y(t) =& (G+K)^*Qx(t) + (S+N)u(t), \\ \iff \quad & \begin{pmatrix} Ex \\ u \\ \frac{\mathrm{d}}{\mathrm{dt}} Ex \\ -y \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & I \\ (J-R)Q & G-K \\ -(G+K)^*Q & -(S+N) \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \end{aligned} \tag{2.20}$$

and hence $(Ex, u, \frac{\mathrm{d}}{\mathrm{dt}} Ex, -y) \in \mathcal{D}_1\mathcal{L}_1$. Conversely, for $(x, e_P, \frac{\mathrm{d}}{\mathrm{dt}} x, f_P) \in \mathcal{D}_1\mathcal{L}_1$ there exists some $\mathbb{K}^n$-valued function z such that

$$\begin{pmatrix} x \\ e_P \\ \frac{\mathrm{d}}{\mathrm{dt}} x \\ f_P \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & I \\ (J-R)Q & G-K \\ -(G+K)^*Q & -(S+N) \end{bmatrix} \begin{pmatrix} z \\ e_P \end{pmatrix},$$

which implies

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}} Ez =& (J-R)Qz + (G-K)e_P, \\ -f_P =& (G+K)^*Qz + (S+N)e_P. \end{aligned}$$

Overall we showed that every pH-LA system can be regarded as a pH-LR system.

From the pH-LG to the pH-LR formulation

Similarly, every pH-LG system $(\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$ where $\mathcal{R}_2$ is not only a resistive linear relation but also a nonnegative linear relation can be written down as a pH-LR system. Namely, as $\mathcal{D}_3\mathcal{L}_3$ where

$$\mathcal{D}_3 := (-\mathcal{D} \circ \mathcal{R})^{-1}, \quad \mathcal{L}_3 := \{\ (x, e_P, e_S, e_P') \mid e_P = e_P',\ (x,e) \in \mathcal{L}_2\ \}.$$

Indeed by Prop. 1.3.14, $\mathcal{D}_3$ is a dissipative relation and $\mathcal{L}_3$ readily is a symmetric relation. Further, it is immediate that $(x, e_P, \frac{\mathrm{d}}{\mathrm{dt}} x, -f_P) \in \mathcal{D}_3\mathcal{L}_3$ if and only if there exists some $(f_R, e_R) \in \mathcal{R}_2$ such that $(-\frac{\mathrm{d}}{\mathrm{dt}} x, f_P, x, e_P) \in \mathcal{D}_2$ and $(x,e) \in \mathcal{L}_2$.

From the pH-LA to the pH-LG formualtion

With the additional assumption on a real ($\mathbb{K} = \mathbb{R}$) pH-LA system (2.12) that $\operatorname{ran}\left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ defines a Lagrangian subspace, that is $\dim\operatorname{ran}\left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n$, we can also translate this system to the linear geometric language, cf. [MM19, Thm. 2], by introducing the respectively skew-adjoint, symmetric, and resistive linear relations

$$\mathcal{D}_4 := \left(\operatorname{gr} \begin{bmatrix} -J & I_n & 0 & -G \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_k \\ G^* & 0 & I_k & N \end{bmatrix} \right)^{-1}, \quad \mathcal{L}_4 := \operatorname{ran} \begin{bmatrix} E \\ Q \end{bmatrix}, \quad \mathcal{R}_4 := \operatorname{gr} - \begin{bmatrix} R & K \\ K^* & S \end{bmatrix}. \tag{2.21}$$

Then $(\mathcal{D}_4, \mathcal{L}_4, \mathcal{R}_4)$ defines a pH-LG system and for

$$(x, u, y) \in \mathfrak{B}_{[E,(J-R)Q,G-K,(G+K)^*Q,S+N]}$$

holds

$$\begin{aligned}(-\tfrac{\mathrm{d}}{\mathrm{dt}} Ex(t), -Qx(t), -u(t), y(t), Qx(t),& \\ RQx(t) + Ku(t), K^*Qx(t) + Su(t), u(t)) &\in \mathcal{D}_4, \\ (Ex(t), Qx(t)) &\in \mathcal{L}_4, \\ (-Qx(t), -u(t), RQx(t) + Ku(t), K^*Qx(t) + Su(t)) &\in \mathcal{R}_4.\end{aligned}$$

Conversely, if

$$\begin{aligned}(-\tfrac{\mathrm{d}}{\mathrm{dt}} x(t), f_{R1}(t), f_{R2}(t), f_P(t), e_S(t), e_{R1}(t), e_{R2}(t), e_P(t)) \in \mathcal{D}_4, \\ (x(t), e_S(t)) \in \mathcal{L}_4, \quad (f_{R1}(t), f_{R2}(t), e_{R1}(t), e_{R2}(t)) \in \mathcal{R}_4,\end{aligned}$$

then there exists some $\mathbb{K}^n$-valued function $z(\cdot)$ such that $Ex = z$ and

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{dt}} Ez =& (J - R)Qz + (G - K)(-e_P), \\ f_P =& (G + K)^*Qz + (S + N)(-e_P).\end{aligned}$$

From the pH-LG to the pH-LA formulation

There is an additional assumption enabling us to view a pH-LG system as a pH-LA system. Namely, for a pH-LG system $(\mathcal{D}_5, \mathcal{L}_5, \mathcal{R}_5)$ assume not only that $\mathcal{R}_2$ is a nonnegative linear relation but also that $-\mathcal{D}_5 \circ \mathcal{R}_5$, which is a dissipative linear relation by Prop. 1.3.14, can be written as the graph of a matrix, i.e., $-\mathcal{D}_5 \circ \mathcal{R}_5 = \operatorname{gr} D$ for some dissipative matrix D. The latter can be checked with Prop. 1.3.9. Next, we decompose D as $D = V - W$ with $V = -V^*$ and $W \geq 0$. Further, we partition the matrices V, W with respect to the state and external ports dimensions as

$$V = \begin{bmatrix} J & G \\ -G^* & N \end{bmatrix}, \quad W = \begin{bmatrix} R & K \\ K^* & S \end{bmatrix}.$$

Choosing any range representation $\mathcal{L}_5 = \operatorname{gr} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix} \right]$, we obtain the pH-LA system

$$[E, (J-R)Q, G-K, (G+K)^*Q, S+N].$$

Then for any functions x, f_P, e_p there exist functions f_R, e_R such that

$$\begin{aligned} (-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) &\in \mathcal{D}_5, \\ (x(t), e_S(t)) \in \mathcal{L}_5, \quad (f_R(t), e_R(t)) &\in \mathcal{R}_5, \end{aligned}$$

if and only if there exists some function z such that $x = Ez$ and

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\, Ez =& (J-R)Qz + (G-K)(-e_P), \\ f_P =& (G+K)^*Qz + (S+N)(-e_P). \end{aligned}$$

From the pH-LR to the pH-LA formulation

The same technique as for the translation from a pH-LG to a pH-LA system can be used to translate a pH-LR system $\mathcal{D}_6\mathcal{L}_6$ to the pH-LA setting. Here we can directly check whether $\mathcal{D}_6$ is the graph of a dissipative matrix and proceed as before to translate all such pH-LR systems into pH-LA systems.

From the pH-LR to the pH-LG formulation

A possibility to write a real ($\mathbb{K} = \mathbb{R}$) pH-LR system $\mathcal{D}_6\mathcal{L}_6$ as pH-LG system is the following. First, we check whether $\mathcal{L}_6 = \mathcal{L}_7 \widehat{\times} \operatorname{gr} I_{n_P}$ for a self-adjoint relation $\mathcal{L}_7$ in $\mathbb{R}^{n_S}$. If this condition is fulfilled, we invoke Lem. 1.3.13 and find a Dirac structure $\mathcal{D}_7$ and a resistive structure $\mathcal{R}_7$ such that $\mathcal{D}_6 = \mathcal{D}_7 \circ \mathcal{R}_7$. Then $(-\mathcal{D}_7^{-1}, \mathcal{L}_7, \mathcal{R}_7)$ is a pH-LG system. Further, $(x, e_P, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x, -f_P) \in \mathcal{D}_6\mathcal{L}_6$ if and only if there exists some $(f_R, e_R) \in \mathcal{R}_7$ such that $(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x, f_P, f_R, x, e_P, e_R) \in \mathcal{D}_7$, $(x, e) \in \mathcal{L}_7$, $(f_R, e_R) \in \mathcal{R}_7$. Here, compared to Def. 2.2.5, the order in which the external and resistive ports appear in the inclusion with $\mathcal{D}_7$ is inverted. This does however not pose a loss of generality.

Remark 2.5.1 (On the interconnection of port-Hamiltonian systems)**.**
Now that we have seen how pH-LG systems can be interpreted as pH-LR systems,

we check that the interconnection introduced for pH-LR systems is compatible with interconnection defined by Def. 2.2.2. To this end, for $i = 1, 2$ let $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$ be a pH-LG system with $\mathcal{D}_i$ a skew-adjoint relation in $\mathbb{R}^{n_{Si}+n_{Ri}+n_{Pi}+n_I}$, $\mathcal{L}_i$ a self-adjoint relation in $\mathbb{R}^{n_{Si}}$, and $\mathcal{R}_i$ a nonpositive relation in $\mathbb{R}^{n_{Ri}}$. Consider their interconnection with respect to $\mathbb{R}^{n_I}$

$$(\mathcal{D}_3, \mathcal{L}_3, \mathcal{R}_3) := (\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \circ (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2),$$

where in particular,

$$\mathcal{D}_3 = \mathcal{D}_1 \circ \mathcal{D}_2, \quad \mathcal{L}_3 = \mathcal{L}_1 \widehat{\times} \mathcal{L}_2, \quad \mathcal{R}_3 = \mathcal{R}_1 \widehat{\times} \mathcal{R}_2.$$

We arrive at the pH-LR system

$$(-\mathcal{D}_3 \circ \mathcal{R}_3)^{-1}(\mathcal{L}_3 \widehat{\times} \operatorname{gr} I_{n_{P1}+n_{P2}}).$$

Further, consider the pH-LR systems

$$(-\mathcal{D}_i \circ \mathcal{R}_i)^{-1}(\mathcal{L}_i \widehat{\times} \operatorname{gr} I_{n_{Pi}+n_I}),$$

$i = 1, 2$, as well as their interconnection $\mathcal{D}_4\mathcal{L}_4$ given by

$$\begin{aligned}
\mathcal{D}_4 &= \left(((-\mathcal{D}_1 \circ \mathcal{R}_1)^{-1})^{-1} \circ ((-\mathcal{D}_2 \circ \mathcal{R}_2)^{-1})^{-1}\right)^{-1} = ((-\mathcal{D}_1 \circ \mathcal{R}_1) \circ (-\mathcal{D}_2 \circ \mathcal{R}_2))^{-1}, \\
\mathcal{L}_4 &= \mathcal{L}_1 \widehat{\times} \mathcal{L}_2 \widehat{\times} \operatorname{gr} I_{n_{P1}+n_{P2}} = \mathcal{L}_3 \widehat{\times} \operatorname{gr} I_{n_{P1}+n_{P2}}.
\end{aligned}$$

Our goal is to compare the pH-LR systems $(-\mathcal{D}_3 \circ \mathcal{R}_3)^{-1}(\mathcal{L}_3 \widehat{\times} \operatorname{gr} I_{n_{P1}+n_{P2}})$ and $\mathcal{D}_4\mathcal{L}_4$. To this end, note the following equivalences.

$$\begin{aligned}
&(-\mathcal{D}_3 \circ \mathcal{R}_3)^{-1} = \mathcal{D}_4 \\
\iff &-\mathcal{D}_3 \circ \mathcal{R}_3 = (-\mathcal{D}_1 \circ \mathcal{R}_1) \circ (-\mathcal{D}_2 \circ \mathcal{R}_2) \\
\iff &-(\mathcal{D}_1 \circ \mathcal{D}_2) \circ (\mathcal{R}_1 \widehat{\times} \mathcal{R}_2) = (-\mathcal{D}_1 \circ \mathcal{R}_1) \circ (-\mathcal{D}_2 \circ \mathcal{R}_2).
\end{aligned}$$

To assert the latter, consider

$$\begin{aligned}
&(f_1, f_2, f_{P1}, f_{P2}, e_1, e_2, e_{P1}, e_{P2}) \in -(\mathcal{D}_1 \circ \mathcal{D}_2) \circ (\mathcal{R}_1 \widehat{\times} \mathcal{R}_2) \\
\iff &\exists (f_{R1}, f_{R2}, e_{R1}, e_{R2}) \in \mathcal{R}_1 \widehat{\times} \mathcal{R}_2 : \\
&(f_1, f_2, -f_{R1}, -f_{R2}, f_{P1}, f_{P2}, e_1, e_2, e_{R1}, e_{R2}, e_{P1}, e_{P2}) \in -(\mathcal{D}_1 \circ \mathcal{D}_2) \\
\iff &\exists (f_{R1}, e_{R1}) \in \mathcal{R}_1,\ (f_{R2}, e_{R2}) \in \mathcal{R}_2 : \\
&(f_1, f_2, -f_{R1}, -f_{R2}, f_{P1}, f_{P2}, -e_1, -e_2, -e_{R1}, -e_{R2}, -e_{P1}, -e_{P2}) \in \mathcal{D}_1 \circ \mathcal{D}_2
\end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \exists (f_{R1}, e_{R1}) \in \mathcal{R}_1,\, (f_{R2}, e_{R2}) \in \mathcal{R}_2,\, (f_I, e_I) \in \mathbb{R}^{2n_I} :\\
&\qquad (-f_1, f_{R1}, -f_{P1}, f_I, e_1, e_{R1}, e_{P1}, e_I) \in \mathcal{D}_1\\
&\qquad \wedge\ (-f_2, f_{R2}, -f_{P2}, -f_I, e_2, e_{R2}, e_{P2}, e_I) \in \mathcal{D}_2\\
&\Longleftrightarrow \exists (f_I, e_I) \in \mathbb{R}^{2n_I} : (f_1, f_{P1}, -f_I, e_1, e_{P1}, e_I) \in -\mathcal{D}_1 \circ \mathcal{R}_1\\
&\qquad \wedge\ \in (f_2, f_{P2}, f_I, e_2, e_{P2}, e_I) - \mathcal{D}_2 \circ \mathcal{R}_2\\
&\Longleftrightarrow (f_1, f_2, f_{P1}, f_{P2}, e_1, e_2, e_{P1}, e_{P2}) \in (-\mathcal{D}_1 \circ \mathcal{R}_1) \circ (-\mathcal{D}_2 \circ \mathcal{R}_2).
\end{aligned}$$

In particular with $\mathcal{D}_4 = \mathcal{L}_3 \widehat{\times}\, \mathrm{gr}\ I_{n_{P1}+n_{P2}}$, we deduce

$$(-\mathcal{D}_3 \circ \mathcal{R}_3)^{-1}(\mathcal{L}_3 \widehat{\times}\, \mathrm{gr}\ I_{n_{P1}+n_{P2}}) = \mathcal{D}_4 \mathcal{L}_4,$$

i.e., interconnecting two pH-LG systems and converting the interconnection to a pH-LR system leads to the same result as converting the pH-LG systems to pH-LR systems first and then interconnecting them.
Closing the remark on interconnections, recall that we did not define the concept of interconnection for pH-LA or pH-LAR systems. The reasons are twofold. For one, we saw that pH-LR systems can always be interpreted as pH-LA systems, for which we defined the concept. This is exactly how interconnections for pH-LA systems are often introduced, see, e.g., [BMX+18] and [VvdS10a] for the pH-ODE case. Secondly, no real alternatives are discussed. Of course, for two pH-LA systems one could allow as interconnection any linear relation between their inputs and outputs such that one can prove a power-balance and such that the resulting system is again pH-LA [MM19]. However, the author is not aware of any characterization of such a concept and in [MM19] one resorts to the interconnection induced by Dirac subspaces, which is comprised in this abstract concept. From this one basic geometric interconnection one can build others and in fact all interconnections of practical relevance [DvdS98; VvdS10b].

We can also relate the behavior of a pH-LG systems to that of a pH-LR system.

Proposition 2.5.2. *Let $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ be a pH-LG system with $\mathcal{D} \subset \mathbb{R}^{(n_S+n_R+n_P)} \times \mathbb{R}^{(n_S+n_R+n_P)}$, $\mathcal{L} \subset \mathbb{R}^{n_S} \times \mathbb{R}^{n_S}$ and $\mathcal{R} \subset \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$. If $(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$, then it holds $(x, e_P, f_P) \in \mathfrak{B}_{(-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\hat{\times}\,\mathrm{gr}\ I_{n_P})}$. Conversely, if $(x, e_P, f_P) \in \mathfrak{B}_{(-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\hat{\times}\,\mathrm{gr}\ I_{n_P})}$, then there exist $f_R, e_R \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_R})$ and $e_S \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_S})$ such that $(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$. That is loosely speaking,*

$$(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}} \Longleftrightarrow (x, e_P, -f_P) \in \mathfrak{B}_{(-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\hat{\times}\,\mathrm{gr}\ I_{n_P})}.$$

Proof. First note that $(-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P})$ is a well-defined pH-LR system by the previous results of this section. Now let $(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$. Invoking Props. 2.2.10 & 2.4.4 we have

$$\begin{aligned}
&(x, f_R, f_P, e_S, e_R, e_P) \in \mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}\\
\iff &(-\dot{x}(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) \in \mathcal{D}\\
&\wedge\ (x(t), e_S(t)) \in \mathcal{L}\ \wedge\ (f_R(t), e_R(t)) \in \mathcal{R},\ \text{f.a.a. } t\in\mathbb{R}\\
\implies &(e_S(t), e_P(t), \dot{x}(t), -f_P(t)) \in (-\mathcal{D}\circ\mathcal{R})^{-1}\\
&\wedge\ (x(t), e_P(t), e_S(t), e_P(t)) \in \mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P},\ \text{f.a.a. } t\in\mathbb{R}\\
\implies &(x(t), e_P(t), \dot{x}(t), -f_P(t)) \in (-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P}),\ \text{f.a.a. } t\in\mathbb{R}\\
\iff &(x, e_P, -f_P) \in \mathfrak{B}_{(-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\hat{\times}\,\mathrm{gr}\; I_{n_P})}.
\end{aligned}$$

Conversely, let $(x(t), e_P(t), \dot{x}(t), -f_P(t)) \in (-\mathcal{D}\circ\mathcal{R})^{-1}(\mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P})$ for almost all $t\in\mathbb{R}$. That is for any range representations $\mathrm{ran}\left[\begin{smallmatrix} F_1^* & F_2^* & F_3^* & F_4^*\end{smallmatrix}\right]^* \in \mathbb{K}^{2(n_S+n_R)\times m_1}$ of $(-\mathcal{D}\circ\mathcal{R})^{-1}$ and $\mathrm{ran}\left[\begin{smallmatrix} G_1^* & G_2^* & G_3^* & G_4^*\end{smallmatrix}\right]^* \in \mathbb{K}^{2(n_S+n_R)\times m_2}$ of $\mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P}$, we find a function $(z_1, z_2) : \mathbb{R}\to\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$ such that

$$\begin{pmatrix} x\\ e_P\\ 0\\ 0\\ \dot{x}\\ -f_P\end{pmatrix} = \begin{bmatrix} F_1 & 0\\ F_2 & 0\\ F_3 & -G_1\\ F_4 & -G_2\\ 0 & G_3\\ 0 & G_4\end{bmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix}.$$

Without loss of generality $(z_1, z_2) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{m_1+m_2})$. Otherwise, choose

$$\begin{pmatrix} \tilde{z}_1\\ \tilde{z}_2\end{pmatrix} = \begin{bmatrix} F_1 & 0\\ F_2 & 0\\ F_3 & -G_1\\ F_4 & -G_2\\ 0 & G_3\\ 0 & G_4\end{bmatrix}^\dagger \begin{pmatrix} x\\ e_P\\ 0\\ 0\\ \dot{x}\\ -f_P\end{pmatrix}.$$

With $e_S := F_3 z_1 = G_1 z_2 \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_s})$ we have

$$\begin{aligned}
&(e_S(t), e_P(t), \dot{x}(t), -f_P(t)) \in (-\mathcal{D}\circ\mathcal{R})^{-1}\\
&\qquad\qquad \wedge\ (x(t), e_P(t), e_S(t), e_P(t)) \in \mathcal{L}\widehat{\times}\,\mathrm{gr}\; I_{n_P},\ \text{f.a.a. } t\in\mathbb{R}.
\end{aligned}$$

Similarly for any range representations $\left[\begin{smallmatrix} \tilde{F}_1^* & \tilde{F}_2^* & \tilde{F}_3^* & \tilde{F}_4^* & \tilde{F}_5^* & \tilde{F}_6^*\end{smallmatrix}\right]^* \in \mathbb{R}^{2(n_S+n_R+n_P)\times\tilde{m}_1}$ of $-\mathcal{D}$ and $\left[\begin{smallmatrix} \tilde{G}_1^* & \tilde{G}_2^*\end{smallmatrix}\right]^* \in \mathbb{K}^{2n_R\times\tilde{m}_2}$ of $\mathcal{R}$ we now find a function $(\tilde{z}_1, \tilde{z}_2) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\tilde{m}_1}\times\mathbb{R}^{\tilde{m}_2})$ such that

$$\begin{pmatrix} \dot{x}\\ 0\\ -f_P\\ e_S\\ 0\\ e_P\end{pmatrix} = \begin{bmatrix} F_1 & 0\\ F_2 & G_1\\ F_3 & 0\\ F_4 & 0\\ F_5 & -G_2\\ F_6 & 0\end{bmatrix}\begin{pmatrix} \tilde{z}_1\\ \tilde{z}_2\end{pmatrix},$$

and in turn function $f_R, e_R \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n_R})$ such that

$$(-\dot{x}(t), f_R(t), f_P(t), e_S(t), e_R(t), e_P(t)) \in \mathcal{D} \wedge (x(t), e_S(t)) \in \mathcal{L} \wedge (f_R(t), e_R(t)) \in \mathcal{R},$$

for almost all $t \in \mathbb{R}$, completing the proof. □

Remark 2.5.3. In this section we did not compare pH-LAR systems with the other types of port-Hamiltonian systems as it is beyond the scope of this thesis. However, we point out how a comparison could be achieved. Just as the pH-LAR formulation was obtained by restricting the constitutive equations of the pH-LA formulation on subspaces, it is possible to formulate pH-LR systems where the linear relations are only defined on a subspace of $\mathbb{K}^n$. In fact, most of the literature on linear relations, see, e.g., [BHdS20, Chap. 1], considers linear relations as subspaces of 'abstract' vector spaces and we could have formulated Sec. 1.3 completely on finite vector spaces without specifying $\mathbb{K}^n$ for some $n \in \mathbb{N}_0$, enabling a comparison with the pH-LAR formulation.

2.5.2 Dissipative-Hamiltonian pencil

In Sec. 2.5.1 we saw how pH-LR systems encompass pH-LA and pH-LG systems. To enable a finer comparison between the presented linear port-Hamiltonian formulations, we will focus on the case where no external ports are present, i.e., there is no interaction with the environment. Recall that in this case, the dynamics of a pH-LR system $\mathcal{DL}$ with $\mathcal{D}, \mathcal{L} \subset \mathbb{K}^n \times \mathbb{K}^n$ are given by $(x, \dot{x}) \in \mathcal{DL}$. Defining the set-valued map $F \rightrightarrows \mathbb{K}^n,\ x \mapsto \{\ y \in \mathbb{K}^n \mid (x, y) \in \mathcal{DL}\ \}$, we can rewrite the dynamics as the standard differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad t \in \mathbb{R}.$$

Although solution theories for solving such differential-inclusions exist, see, e.g., [AF90], we saw with Def. 2.4.2 and Prop.2.4.4 how the problem of solving this differential inclusion can be reduced to solving a DAE. In fact, there exist different approaches to convert this differential inclusion to a DAE, of which four are presented in Fig. 2.8. Let us briefly motivate these cases. A base observation is that $(x, \dot{x}) \in \mathcal{DL}$ is pointwisely equivalent to the existence of some $e \in \mathbb{K}^n$ such that $(x, e) \in \mathcal{L}$ and $(e, \dot{x}) \in \mathcal{D}$. Now if

(i) $\mathcal{L} = \ker\left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right]$ and $\mathcal{D} = \ker\left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right]$, then $K_1 x + L_1 e = 0$ and $K_2 e + L_2 \dot{x} = 0$. With invertible matrices U, V such that $U(sK_1 - L_1) = \left[\begin{smallmatrix} sK_{11} - L_{11} \\ sK_{12} \end{smallmatrix}\right]$ and

$V(sK_2 - L_2) = \left[\begin{smallmatrix} sL_{21} \\ sK_{22}-L_{22} \end{smallmatrix}\right]$ for some full column rank matrices L_{11}, K_{22} we obtain

$$K_1x + L_1e = 0 \iff \left[\begin{smallmatrix} K_{11} \\ K_{12} \end{smallmatrix}\right] x = -\left[\begin{smallmatrix} L_{11} \\ 0 \end{smallmatrix}\right] e \iff (L_{11}^\dagger K_{11}x = -e \ \wedge \ K_{12}x = 0),$$

and

$$K_2e + L_2\dot{x} = 0 \iff \left[\begin{smallmatrix} 0 \\ K_{22} \end{smallmatrix}\right] e = -\left[\begin{smallmatrix} L_{21} \\ L_{22} \end{smallmatrix}\right] \dot{x} \iff (K_{22}^\dagger L_{22}\dot{x} = -e \ \wedge \ L_{21}\dot{x} = 0).$$

Combining these equivalences yields,

$$\begin{bmatrix} K_{22}^\dagger L_{22} \\ L_{21} \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} L_{11}^\dagger K_{11} \\ 0 \\ K_{12} \end{bmatrix} x.$$

(ii) $\mathcal{L} = \ker\left[\begin{smallmatrix} K_1 & L_1 \end{smallmatrix}\right]$ and $\mathcal{D} = \operatorname{ran}\left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right]$, then $K_1x + L_1e = 0$ and there exists some $z \in \operatorname{dom} F_2 = \operatorname{dom} G_2$ such that $\left(\begin{smallmatrix} e \\ \dot{x} \end{smallmatrix}\right) = \left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right] z$. Integrating $\dot{x} = G_2z$, we write $x = \int G_2z$. Consequently, $L_1F_2z = -K_1\int G_2z$, i.e.,

$$\tfrac{\mathrm{d}}{\mathrm{dt}} L_1F_2z = -K_1G_2z.$$

(iii) $\mathcal{L} = \operatorname{ran}\left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right]$ and $\mathcal{D} = \ker\left[\begin{smallmatrix} K_2 & L_2 \end{smallmatrix}\right]$, then there exists some $z \in \operatorname{dom} F_1 = \operatorname{dom} G_1$ such that $\left(\begin{smallmatrix} x \\ e \end{smallmatrix}\right) = \left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right] z$ and $K_2e + L_2\dot{x} = 0$. Combining both yields

$$L_2 \tfrac{\mathrm{d}}{\mathrm{dt}} F_1z = -K_2G_1z.$$

(iv) $\mathcal{L} = \operatorname{ran}\left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right]$ and $\mathcal{D} = \operatorname{ran}\left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right]$, then there exist $z_1 \in \operatorname{dom} F_1 = \operatorname{dom} G_1$ and $z_2 \in \operatorname{dom} F_2 = \operatorname{dom} G_2$ such that $\left(\begin{smallmatrix} x \\ e \end{smallmatrix}\right) = \left[\begin{smallmatrix} F_1 \\ G_1 \end{smallmatrix}\right] z_1$ and $\left(\begin{smallmatrix} e \\ \dot{x} \end{smallmatrix}\right) = \left[\begin{smallmatrix} F_2 \\ G_2 \end{smallmatrix}\right] z_2$. In particular, $G_1z_1 = e = F_2z_2$, i.e., $\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in \ker\left[\begin{smallmatrix} G_1 & -F_2 \end{smallmatrix}\right]$. Taking inspiration from the proof of Prop. 1.3.14 we find that $\mathcal{DL} = \operatorname{ran}\left[\begin{smallmatrix} F_1X \\ G_2Y \end{smallmatrix}\right]$ where X, Y are matrices fulfilling $\operatorname{ran}\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] = \ker\left[\begin{smallmatrix} G_1 & -F_2 \end{smallmatrix}\right]$. From this observation we derive the existence of some $z \in \operatorname{dom} F_1X = \operatorname{dom} G_2Y$ such that $\left(\begin{smallmatrix} x \\ \dot{x} \end{smallmatrix}\right) = \left(\begin{smallmatrix} F_1X \\ G_2Y \end{smallmatrix}\right) z$, that is

$$\tfrac{\mathrm{d}}{\mathrm{dt}} G_2Yz = F_1Xz.$$

In the following, we will focus on the case (iv), that is both $\mathcal{D}$ and $\mathcal{L}$ are given in range representations. Note that this case corresponds to the statement of Prop. 2.4.4, while the bottom-left case in Fig. 2.8 follows the lines of Prop. 2.2.10. Two main reasons motivate this choice.

1. We will see in Chap. 3 that Prop. 1.3.10 enables us to derive properties of the underlying pencil of the DAE when $\mathcal{D}$ and $\mathcal{L}$ are given in range representations.

<table>
<tr><td>$(x,\dot x)\in\mathcal{DL}\iff$</td><td>$\mathcal{D}=\ker\left[\begin{smallmatrix}K_2 & L_2\end{smallmatrix}\right]$</td><td>$\mathcal{D}=\operatorname{ran}\left[\begin{smallmatrix}F_2\\ G_2\end{smallmatrix}\right]$</td></tr>
<tr><td>$\mathcal{L}=\ker\left[\begin{smallmatrix}K_1 & L_1\end{smallmatrix}\right]$</td><td>$\begin{bmatrix}K_{22}^\dagger L_{22}\\ L_{21}\\ 0\end{bmatrix}\dot x=\begin{bmatrix}L_{11}^\dagger K_{11}\\ 0\\ K_{12}\end{bmatrix}x,$
where
$U(sK_1-L_1)=\left[\begin{smallmatrix}sK_{11}-L_{11}\\ sK_{12}\end{smallmatrix}\right],$
$V(sK_2-L_2)=\left[\begin{smallmatrix}sL_{21}\\ sK_{22}-L_{22}\end{smallmatrix}\right]$
with invertible U,V and
full column rank
matrices L_{11},K_{22}</td><td>$\dot x=G_2z,$
$\frac{\mathrm{d}}{\mathrm{dt}}L_1F_2z=-K_1G_2z$</td></tr>
<tr><td>$\mathcal{L}=\operatorname{ran}\left[\begin{smallmatrix}F_1\\ G_1\end{smallmatrix}\right]$</td><td>$x=F_1z,$
$L_2\frac{\mathrm{d}}{\mathrm{dt}}F_1z=-K_2G_1z$</td><td>$x=F_1Xz,$
$\frac{\mathrm{d}}{\mathrm{dt}}F_1Xz=G_2Yz,$
where
$\operatorname{ran}\left[\begin{smallmatrix}X\\ Y\end{smallmatrix}\right]=\ker\left[\begin{smallmatrix}G_1 & -F_2\end{smallmatrix}\right]$</td></tr>
</table>

Figure 2.8: How to solve the differential inclusion $(x,\dot x)\in\mathcal{DL}$ by means of DAEs.

2. In the author's opinion, the derivation is performed at the right place when comparing the four resulting DAEs displayed in Fig. 2.8. While this is also true for the case where $\mathcal{D}$ is in range and $\mathcal{L}$ is in kernel representation, one can relate initial values of the differential inclusion to the consistent initial differential variables of the resulting DAE when both $\mathcal{D}$ and $\mathcal{L}$ are given in range representations.

Now back in the context of (iv), let $\mathcal{L}=\operatorname{ran}\left[\begin{smallmatrix}F_1\\ G_1\end{smallmatrix}\right]$ and $\mathcal{D}=\operatorname{ran}\left[\begin{smallmatrix}F_2\\ G_2\end{smallmatrix}\right]$ with $F_1,F_2,G_1,G_2\in\mathbb{K}^{n\times n}$ and $\mathcal{DL}=\operatorname{ran}\left[\begin{smallmatrix}F_1X\\ G_2Y\end{smallmatrix}\right]$ where $X,Y\in\mathbb{K}^{n\times m}$ are matrices fulfilling $\operatorname{ran}\left[\begin{smallmatrix}X\\ Y\end{smallmatrix}\right]=\ker\left[\begin{smallmatrix}G_1 & -F_2\end{smallmatrix}\right]$. Then denoting $E=F_1X$ and $A=G_2Y$, the dynamics read

$$\forall\, t\in\mathbb{R}:\quad \left(\begin{smallmatrix}x(t)\\ \dot x(t)\end{smallmatrix}\right)\in\mathcal{DL}=\operatorname{ran}\left[\begin{smallmatrix}E\\ A\end{smallmatrix}\right].$$

This leads to the existence of some function $z(\cdot):\mathbb{R}\to\mathbb{K}^m$ with

$$\left(\begin{smallmatrix}x(t)\\ \dot x(t)\end{smallmatrix}\right)=\left[\begin{smallmatrix}E\\ A\end{smallmatrix}\right]z(t),$$

and thus

$$\forall\, t\in\mathbb{R}:\quad \tfrac{\mathrm{d}}{\mathrm{dt}}Ez(t)=\dot x(t)=Az(t).$$

Conversely assuming that a $\mathbb{K}^m$-valued function $z(\cdot)$ solves the DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ez(t) = Az(t)$ on $\mathbb{R}$, we obtain that $x(\cdot) \coloneqq Ez(\cdot)$ fulfills

$$\forall\, t \in \mathbb{R}: \quad \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} Ez(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}Ez(t) \end{pmatrix} = \begin{pmatrix} Ez(t) \\ Az(t) \end{pmatrix} = \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] z(t) \in \operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

Similarly, one can relate the dynamics of a pH-LG system without external flows and efforts to a DAE induced by a matrix pencil. This evidently remains a true statement if we consider a pH-LA system without input and output. These considerations lead to the following definition.

Definition 2.5.4 (Dissipative-Hamiltonian matrix pencil)**.**
We call a matrix pencil $sE - A \in \mathbb{K}[s]^{n \times m}$

(i) *ordinary dissipative-Hamiltonian (dH-ODE)*, if

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

holds for some linear relation $\mathcal{D} = \operatorname{gr} D \subset \mathbb{K}^{2n}$ with dissipative matrix $D \in \mathbb{K}^{n \times n}$ and some linear relation $\mathcal{L} = \operatorname{gr} Q \subset \mathbb{K}^{2n}$ with symmetric matrix $Q \in \mathbb{K}^{n \times n}$.

(ii) *linear algebraic dissipative-Hamiltonian (dH-LA)*, if

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

holds for some linear relation $\mathcal{D} = \operatorname{gr} D \subset \mathbb{K}^{2n}$ with dissipative matrix $D \in \mathbb{K}^{n \times n}$ and some symmetric linear relation $\mathcal{L} \subset \mathbb{K}^{2n}$.

(iii) *linear geometric dissipative-Hamiltonian (dH-LG)*, if

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

holds for some skew-adjoint linear relation $\mathcal{D} \subset \mathbb{K}^{2n}$ and some self-adjoint linear relation $\mathcal{L} \subset \mathbb{K}^{2n}$.

(iv) *linear relations dissipative-Hamiltonian (dH-LR)*, if

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

holds for some dissipative linear relation $\mathcal{D} \subset \mathbb{K}^{2n}$ and some symmetric linear relation $\mathcal{L} \subset \mathbb{K}^{2n}$.

For dH-LA pencils, there exists an equivalent definition linking them in a more direct way to the previously used terminology.

Proposition 2.5.5. *A pencil $sE - A \in \mathbb{K}[s]^{n\times m}$ is dH-LA if and only if the system*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$$

is pH-LA.

Proof. Assume that $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ is pH-LA. Then there exists a dissipative matrix $D \in \mathbb{K}^{n\times n}$ together with a matrix $Q \in \mathbb{K}^{n\times m}$ such that $E^*Q = Q^*E$ and $A = DQ$. Then $\mathcal{L} \coloneqq \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ is a symmetric linear relation and with $\mathcal{D} \coloneqq \operatorname{gr} D$ we obtain

$$\begin{aligned}
\mathcal{D}\mathcal{L} &= \{(x_1,x_2) \in \mathbb{K}^{2n} \mid \exists y \in \mathbb{K}^n \text{ s.t. } (x_1,y) \in \mathcal{L} \wedge (y,z_2) \in \mathcal{D}\} \\
&= \{(x_1,x_2) \in \mathbb{K}^{2n} \mid \exists z,y \in \mathbb{K}^n \text{ s.t. } (x_1,y) = (Ez,Qz) \in \mathcal{L} \wedge x_2 = Dy\} \\
&= \{(x_1,x_2) \in \mathbb{K}^{2n} \mid \exists z \in \mathbb{K}^n \text{ s.t. } x_1 = Ez \wedge x_2 = DQz\} \\
&= \operatorname{ran} \left[\begin{smallmatrix} E \\ DQ \end{smallmatrix}\right],
\end{aligned}$$

i.e., $sE - A$ is dH-LA.

Now, assume that $sE-A$ is dH-LA. Then $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ for some relations $\mathcal{D},\mathcal{L}$ in $\mathbb{K}^n$ which are respectively dissipative and symmetric with $\mathcal{D} = \operatorname{gr} D$ for some dissipative matrix $D \in \mathbb{K}^{n\times n}$. Let $\operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ \hat{Q} \end{smallmatrix}\right]$ be any range representation of $\mathcal{L}$ with $\hat{E},\hat{Q} \in \mathbb{K}^{n\times\hat{m}}$. Then

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} I \\ D \end{smallmatrix}\right] \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ \hat{Q} \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ D\hat{Q} \end{smallmatrix}\right].$$

Consequently by Prop. 1.3.2, there exists some $\tilde{Q} \in \mathbb{K}^{n\times m}$ such that

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ D\hat{Q} \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} E \\ D\tilde{Q} \end{smallmatrix}\right]. \tag{2.22}$$

Let $(v_i)_{i=1,\dots k}$ be a basis of $\ker(A - D\tilde{Q})^\perp$. Then by (2.22) there exists a family $(w_i)_{i=1,\dots k}$ in $\mathbb{K}^m$ such that for $i = 1,\dots,k$

$$Av_i = D\tilde{Q}w_i \quad \wedge \quad Ev_i = Ew_i.$$

Now let $T \in \mathbb{K}^{n\times n}$ be the matrix linearly extending the mapping rule

$$x \mapsto \begin{cases} v_i, & x = w_i, \\ x, & x \in \ker(A - D\tilde{Q}). \end{cases}$$

Note that by construction T is well-defined and $A = D\tilde{Q}T$ as well as $ET = T$. Defining $Q \coloneqq \tilde{Q}T$ we obtain $A = DQ$ and $E^*Q = T^*E^*\tilde{Q}T = T^*\tilde{Q}^*ET = Q^*E$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ is pH-LA. ∎

Remark 2.5.6. The following justifies the terminology given in Def. 2.5.4.

(i) Similarly to the statement in Prop. 2.5.5, the dH-ODE pencils correspond exactly to the pH-ODE systems where the size of the input and output vectors is zero since with the notation of (2.1a) and identifying $J = \frac{D-D^*}{2}$, $R = \frac{D+D^*}{2}$ we see

$$(x, \dot{x}) \in (\mathrm{gr}\ D)(\mathrm{gr}\ Q) \Leftrightarrow (x, \dot{x}) \in \mathrm{ran} \left[\begin{smallmatrix} I \\ DQ \end{smallmatrix}\right] \Leftrightarrow \tfrac{\mathrm{d}}{\mathrm{dt}}\, x = (J - R)Qx.$$

(ii) Properties of pencils associated to *dissipative Hamiltonian descriptor system* of the form

$$E\dot{x} = DQx,$$

where D is dissipative and $E^*Q = Q^*E$, were rigorously investigated in [MMW18]. In the context of Prop. 2.5.5, it is only suitable to call such pencils $sE - DQ$ (linear algebraic) dissipative-Hamiltonian.

(iii) Any real ($\mathbb{K} = \mathbb{R}$) dH-LG pencil $sE - A$ corresponds to some pH-LG system without external ports, namely to $(-\mathcal{D}^{-1}, \mathcal{L}, \{0\})$. This is due to the fact that in Sec. 2.5.1 we saw

$$(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x, e_S) \in -\mathcal{D}^{-1} \wedge (e_S, x) \in \mathcal{L} \Longleftrightarrow (x, \tfrac{\mathrm{d}}{\mathrm{dt}}\, x) \in \mathcal{D}\mathcal{L},$$

from which we deduce the existence of some function z with $x = Ez$ and $\frac{\mathrm{d}}{\mathrm{dt}}\, Ez = Az$. Note that the reverse is not true. That is, a pH-LG system without external ports does not always correspond to a dH-LG in the discussed sense. This is precisely the case when $\mathcal{R} \neq \{0\}$ since it leads to considering the interconnection $-\mathcal{D} \circ \mathcal{R}$, which is not skew-adjoint anymore. We exclude this case on purpose as one can extend resistive relations to the complex case $\mathbb{K} = \mathbb{C}$ in two ways. In the context of Def. 1.4.6, the condition $e^\top f \leq 0$ can be replaced by either $\langle f, e \rangle \leq 0$ or $\mathrm{Re}\langle f, e \rangle \leq 0$. In the first case, the extended definition would fall between Def.1.3.5 (a)&(b) since no symmetry is assumed. In the latter case the extended definition would correspond exactly to Def. 1.3.5 (a), i.e., to dissipative linear relations and the classes of dH-LG and dH-LR pencils would match since the interconnection of a Dirac structure and a dissipative relation yields a dissipative relation. More importantly for our choice, for pH-LG systems $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ without external flow and effort variables, the case $\mathcal{R} = \{0\}$ describes a significant class in the literature, namely the so-called *generalized pH DAE systems* introduced in [vdSM18], which we also extend to the complex case $\mathbb{K}=\mathbb{C}$ with Def. 2.5.4. We also point out that in [vdSM18] a different terminology was preferred. Namely, $\mathcal{D},\mathcal{L}$ were assumed to be a Dirac and a Lagrangian subspace, respectively, which in the language of linear relations means that $\mathcal{D}$ is skew-adjoint and $\mathcal{L}$ is self-adjoint, see Fig. 1.1.

From Def. 2.5.4 we can already deduce that the dH-LA and the dH-LG pencils are special cases of dH-LR pencils, of which we will analyze properties such as regularity and the shape of their Kronecker form. Before we start with such investigations, we discuss the relationship between the different concepts presented in Def. 2.5.4.
The difference between dH-LG and dH-LR pencils is that instead of skew-adjoint and self-adjoint linear relations, we allow for dissipative $\mathcal{D}$, whereas $\mathcal{L}$ is allowed to be only symmetric. This is a generalization in two respects: First, the relations $\mathcal{D}$ and $\mathcal{L}$ may have a dimension less than n and, second, we allow for relations $\mathcal{D}$ with $\operatorname{Re}\langle x,y\rangle\leq 0$ instead of $\operatorname{Re}\langle x,y\rangle=0$ for all $(x,y)\in\mathcal{D}$. When comparing the dH-LA with the dH-LG pencils the dissipativity of $D\in\mathbb{K}^{n\times n}$ leads, via Lem. 1.3.7, to the maximal dissipativity of $\mathcal{D}$, whereas, by Lem. 1.3.3, $\mathcal{L}$ is symmetric but not necessarily self-adjoint. Summarizing from the previous findings, the differences between dH-LA and dH-LG pencils, which by Rem. 2.5.6 correspond exactly to the approaches to pH-DAEs in [MMW18] and [vdSM18], respectively, are recapitulated in Fig. 2.9.
The question arises if these different definitions actually generate different classes of pencils. We will elucidate the relationship between the different port-Hamiltonian pencils and start with presenting a system in which (i) in Fig. 2.9 effectively prevents a dH-LA pencil from being dH-LG.

Example 2.5.7. Let $E=Q=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $A=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $D=\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$. Then $A=DQ$ and $Q^*E=1=E^*Q$, i.e., the pencil $sE-A$ is dH-LA.
Next we show that it is not dH-LG. Seeking for a contradiction, assume that $\mathcal{D},\mathcal{L}\subseteq\mathbb{K}^4$ be skew-adjoint and self-adjoint subspaces such that

$$\operatorname{ran}\begin{bmatrix}E\\A\end{bmatrix}=\operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\\0\\1\end{smallmatrix}\right)\right\}=\mathcal{D}\mathcal{L}. \tag{2.23}$$

Then we see that $\operatorname{mul}\mathcal{D}\mathcal{L}=\ker\mathcal{D}\mathcal{L}=\{0\}$, which gives $\operatorname{mul}\mathcal{D}=\ker\mathcal{L}=\{0\}$. This together with Lem. 1.3.8 yields, by invoking $\operatorname{ran}\mathcal{L}=\operatorname{dom}\mathcal{L}^{-1}$, that $\operatorname{dom}\mathcal{D}=\operatorname{ran}\mathcal{L}=\mathbb{K}^2$, and we infer from Prop. 1.3.9 that $\mathcal{D}=\operatorname{gr}\hat{D}$ and $\mathcal{L}=(\operatorname{gr}E)^{-1}$ for some skew-Hermitian $\hat{D}\in\mathbb{K}^{2\times 2}$ and some Hermitian $E\in\mathbb{K}^{2\times 2}$. Hence, we can rewrite

(i) ran $\left[\begin{smallmatrix} Q \\ E \end{smallmatrix}\right]$ needs to be n-dimensional for dH-LG pencils, whereas for dH-LA pencils, it might have a smaller dimension.

(ii) the relation $\mathcal{D}$ needs to be a graph of a matrix for dH-LA pencils, whereas for dH-LG pencils, $\mathcal{D}$ might have a multi-valued part.

(iii) the relation $\mathcal{D}$ is skew-adjoint for dH-LG pencils, whereas for pH-LA pencils, $\mathcal{D}$ might be dissipative.

Figure 2.9: Differences between the port-Hamiltonian approaches in [vdSM18] and [MMW18]

(2.23) as

$$\operatorname{span}\left\{\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right)\right\} = \operatorname{ran}\begin{bmatrix} E \\ \hat{D} \end{bmatrix}, \tag{2.24}$$

from which we deduce

$$\operatorname{ran} E = \operatorname{span}\left\{e_1^{[2]}\right\}, \qquad \operatorname{ran}(\hat{D}^*) = \operatorname{ran} \hat{D} = \operatorname{span}\left\{e_2^{[2]}\right\}.$$

Since the space on the left-hand side in (2.24) is one-dimensional, we obtain $\ker E \cap \ker \hat{D} \neq \{0\}$. On the other hand, (2.24), $E = E^*$ and $\hat{D} = -\hat{D}^*$ leads to

$$\ker E = (\operatorname{ran} E^*)^\perp = \operatorname{span}\left\{e_2^{[2]}\right\}, \qquad \ker \hat{D} = (\operatorname{ran} \hat{D}^*)^\perp = \operatorname{span}\left\{e_1^{[2]}\right\}.$$

This implies $\ker E \cap \ker \hat{D} = \{0\}$, which is a contradiction to the already proven fact that $\ker E \cap \ker \hat{D}$ is a non-trivial space. Consequently, the pencil $sE - A$ cannot be dH-LG.

Based on (ii) in Fig. 2.9, our second example presents a dH-LG pencil which is not a dH-LA pencil.

Example 2.5.8. Consider

$$\mathcal{D} = \operatorname{ran} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \subseteq \mathbb{K}^4, \quad \mathcal{L} = \operatorname{ran} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathbb{K}^4.$$

Then, by using Lem. 1.3.3 and Rem. 1.3.4, we see that $\mathcal{D}$ is skew-adjoint and $\mathcal{L}$ is self-adjoint. It can be seen that both $\operatorname{mul}\mathcal{D}$ and $\ker\mathcal{L}$ are spanned by the second canonical unit vector, and

$$\mathcal{D}\mathcal{L} = \operatorname{ran} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Assume that $\mathcal{D}\mathcal{L} = (\operatorname{gr} \hat{D})\hat{\mathcal{L}}$ with $\hat{D} \in \mathbb{K}^{2\times 2}$ and some symmetric $\hat{\mathcal{L}} \subset \mathbb{K}^4$. The symmetry of $\hat{\mathcal{L}}$ yields

$$3 = \dim \mathcal{D}\mathcal{L} = \dim(\operatorname{gr} \hat{D})\hat{\mathcal{L}} \leq \dim \hat{\mathcal{L}} \leq 2,$$

which is a contradiction. Hence, factorizing $\mathcal{D}\mathcal{L} = (\operatorname{gr} \hat{D})\hat{\mathcal{L}}$ is not possible, whence $sE - A$ with $E = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ is dH-LG but not dH-LA.

Combining the previous two (probably minimal in terms of dimension) counter-examples, we present a dH-LR system that is neither dH-LA nor dH-LG.

Example 2.5.9. Given

$$\mathcal{D} = \operatorname{ran} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \subseteq \mathbb{K}^8, \qquad \mathcal{L} = \operatorname{ran} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathbb{K}^8,$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

one deduces

$$\mathcal{D}\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} E \\ A \end{bmatrix}.$$

Invoking Lem. 1.3.3 and Rem. 1.3.4 we obtain that $\mathcal{D}$ is a dissipative and $\mathcal{L}$ a symmetric relation, i.e., $sE - A$ is dH-LR. Assume that there exist a dissipative matrix $\hat{D} \in \mathbb{K}^{4\times 4}$ and a symmetric relation $\hat{\mathcal{L}}$ in $\mathbb{K}^4$ such that $\mathcal{D}\mathcal{L} = (\operatorname{gr} \hat{D})\hat{\mathcal{L}}$. The symmetry of $\hat{\mathcal{L}}$ yields

$$5 = \dim \mathcal{D}\mathcal{L} = \dim(\operatorname{gr} \hat{D})\hat{\mathcal{L}} \leq \dim \hat{\mathcal{L}} \leq 4;$$

a contradiction, and we conclude that $sE - A$ is not dH-LA. Now assume that $\mathcal{D}\mathcal{L} = \tilde{\mathcal{D}}\tilde{\mathcal{L}}$ for some skew-adjoint relation $\tilde{\mathcal{D}}$ and some self-adjoint $\tilde{\mathcal{L}}$. Observing that mul $\tilde{\mathcal{D}}\tilde{\mathcal{L}} = \ker \tilde{\mathcal{D}}\tilde{\mathcal{L}} = \operatorname{span}\left\{e_3^{[4]}, e_4^{[4]}\right\}$ we deduce mul $\tilde{\mathcal{D}} = \operatorname{mul} \tilde{\mathcal{L}} = \operatorname{span}\left\{e_3^{[4]}, e_4^{[4]}\right\}$. This together with Lem. 1.3.8 yields, by invoking ran $\tilde{\mathcal{L}} = \operatorname{dom} \tilde{\mathcal{L}}^{-1}$, that dom $\tilde{\mathcal{D}} = \operatorname{ran} \tilde{\mathcal{L}} = \operatorname{span}\left\{e_1^{[4]}, e_2^{[4]}\right\}$. Regarding Prop. 1.3.10, let

$$\begin{aligned}\tilde{L} &: \operatorname{ran} \tilde{\mathcal{L}} = \mathbb{K}^2 \times \{0\}^2 \to (\ker \tilde{\mathcal{L}})^\perp = \mathbb{K}^2 \times \{0\}^2,\\ \tilde{D} &: \operatorname{dom} \tilde{\mathcal{D}} = \mathbb{K}^2 \times \{0\}^2 \to (\operatorname{mul} \tilde{\mathcal{D}})^\perp = \mathbb{K}^2 \times \{0\}^2\end{aligned}$$

be the operators given by (1.10) for $\tilde{\mathcal{L}}^{-1}$ and $\tilde{\mathcal{D}}$, respectively. Now with (1.11) we can write

$$\tilde{\mathcal{D}}\tilde{\mathcal{L}} = \operatorname{ran} \begin{bmatrix} \tilde{L} \\ \tilde{D} \end{bmatrix} \oplus (\ker \tilde{\mathcal{L}} \times \{0\}^4) \oplus (\{0\}^4 \times \operatorname{mul} \tilde{\mathcal{D}})$$

from which follows that

$$\operatorname{span}\left\{\begin{pmatrix} 1\\0\\0\\0\\0\\1\\0\\0 \end{pmatrix}\right\} = \operatorname{ran} \begin{bmatrix} \tilde{L} \\ \tilde{D} \end{bmatrix}, \tag{2.25}$$

with ran $\tilde{L} = \operatorname{span}\left\{e_1^{[4]}\right\}$ and ran $\tilde{D} = \operatorname{span}\left\{e_2^{[4]}\right\}$. Since the space on the left-hand side in (2.25) is one-dimensional, we obtain $\ker \tilde{L} \cap \ker \tilde{D} \neq \{0\}$. On the other hand, (2.25), $\tilde{L} = \tilde{L}^*$ and $\tilde{D} = -\tilde{D}^*$ lead to

$$\ker \tilde{L} = (\operatorname{ran} \tilde{L}^*)^\perp = \operatorname{span}\left\{e_2^{[4]}\right\}, \qquad \ker \tilde{D} = (\operatorname{ran} \tilde{D}^*)^\perp = \operatorname{span}\left\{e_1^{[4]}\right\},$$

with orthogonal complements taken in $\mathbb{K}^2 \times \{0\}^2$. This implies $\ker \tilde{L} \cap \ker \tilde{D} = \{0\}$, contradicting the already proven fact $\ker E \cap \ker \hat{D} \neq \{0\}$. Consequently, the pencil $sE - A$ cannot be dH-LG.

The next example shows that there are nontrivial pencils which are dH-LA as well as dH-LG but not dH-ODE.

Example 2.5.10. With $E = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, $\mathcal{D} = \operatorname{gr} A$, and $\mathcal{L} = (\operatorname{gr} E)^{-1}$ we see $\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$. It is further readily seen that $sE - A$ satisfies the condition in (ii) and (iii) of Def. 2.5.4, i.e., $sE - A$ is both dH-LA and dH-LG. However, dom $\mathcal{D}\mathcal{L} = \operatorname{span}\{e_1^{[2]}\}$ and as we saw in Rem.2.5.6, dom $\mathcal{D}\mathcal{L}$ needs to equal $\mathbb{R}^2$ in order for $sE - A$ to be a dH-ODE pencil.

The examples presented so far justify the Euler diagram of Fig. 2.10 in which each region is nonempty.

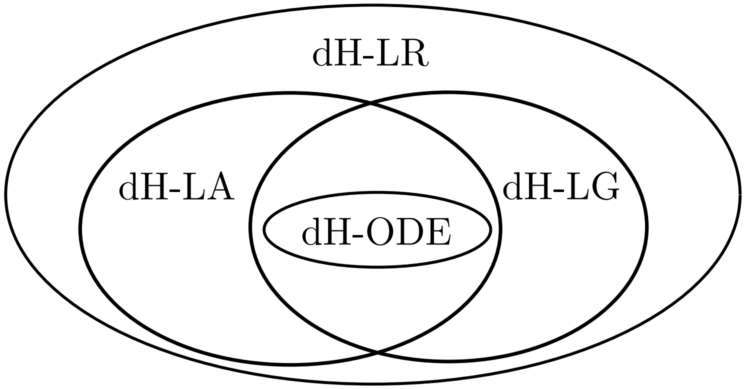

Figure 2.10: A visual representation of the relationship between the different dissipative-Hamiltonian pencil classes presented in Def. 2.5.4.

Our last example aims to illustrate that the missing incorporation of dissipation in the formulation of dH-LG systems, as pointed out in (iii) of Fig. 2.9, effectively lets the intersection of the classes of dH-LG and dH-LA systems shrink.

Example 2.5.11. Let $E = Q = -D = -A = 1 \in \mathbb{R}^{1\times 1}$. Then, clearly, $A = DQ$ and $Q^*E = 1 = E^*Q$, i.e., $sE - A$ is port-Hamiltonian in the sense of [MMW18], and

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \operatorname{span}\left\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\right\}. \tag{2.26}$$

Now assume that $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ holds for some skew-symmetric linear relation $\mathcal{D} \subset \mathbb{R}^2$ and symmetric $\mathcal{L} \subset \mathbb{R}^2$. As $\mathcal{D} \subset \mathbb{R}^2$ is skew-symmetric, we immediately obtain that it is either trivial, or it is spanned by the first or second canonical unit vector in $\mathbb{R}^2$. In the first two cases $\mathcal{D} = \{0\}$ and $\mathcal{D} = \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$, we have $y = 0$ for all $(x, y) \in \mathcal{D}\mathcal{L}$, which contradicts (2.26). On the other hand, if $\mathcal{D} = \operatorname{span}\{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$, we have $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \mathcal{D}\mathcal{L}$, which is again a contradiction to (2.26).

Remark 2.5.12. We briefly characterize the pencils generated by pH-LG systems $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ without external ports when $\mathcal{R} \neq \{0\}$. Since a linear resistive relation is a dissipative relation, $-\mathcal{D} \circ \mathcal{R}$ is a dissipative relation too and any range representation $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right]$ of $(-\mathcal{D} \circ \mathcal{R})^{-1}\mathcal{L}$, cf. Sec. 2.5.1, defines a dH-LR pencil $sE - A$. For example, the dH-LA pencil of Ex. 2.5.11 is induced this way by choosing $\mathcal{D} = \{0\}$, $\mathcal{L} = \operatorname{gr} I_1$ and $\mathcal{R} = -\operatorname{gr} I_1$.

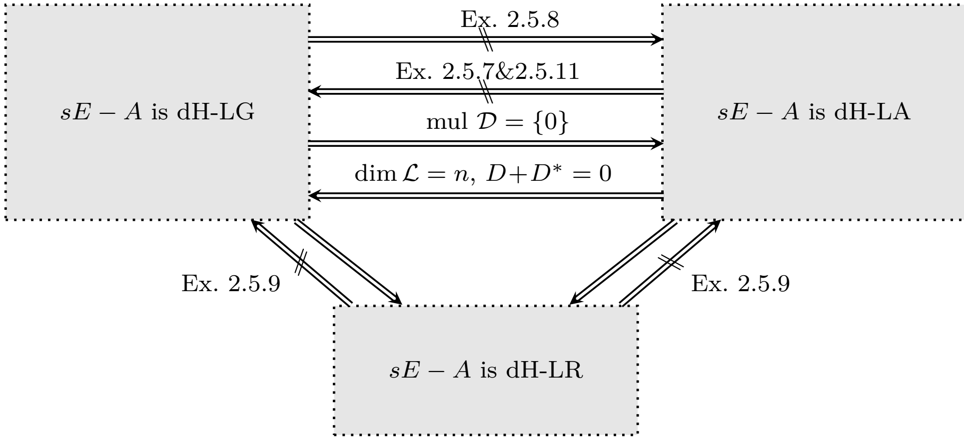

Figure 2.11: Relations between the port-Hamiltonian concepts from Def. 2.5.4, with matrices $E, A \in \mathbb{K}^{n\times m}$, $D \in \mathbb{K}^{n\times n}$ and subspaces $\mathcal{D}, \mathcal{L} \subset \mathbb{K}^{2n}$.

Until this point we highlighted the differences between dH-LG and dH-LG pencils, i.e., the approaches of [vdSM18] and [MMW18], we now analyse their mutualities. That is, we give sufficient conditions on a matrix pencil which is dH-LG to be dH-LA, and vice-versa.

Proposition 2.5.13. *Assume that $sE - A \in \mathbb{K}[s]^{n\times m}$ is dH-LA. Let $A = DQ$ for some dissipative $D \in \mathbb{K}^{n\times n}$ and $Q \in \mathbb{K}^{n\times m}$, which exist according to Prop. 2.5.5. If, additionally $D + D^* = 0$ and $\dim \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n$, then $sE - A$ is pH-LG and, in particular, $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ holds for the self-adjoint relation $\mathcal{L} := \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ and the skew-adjoint relation $\mathcal{D} := \operatorname{gr} D$.*

Proof. Assume that $E, A, Q \in \mathbb{K}^{n\times m}$ fulfil $A = DQ$, $D + D^* = 0$, $E^*Q = Q^*E$ and $\dim \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n$. Then, by $\operatorname{Re}\langle x, Dx\rangle = 0$ for all $x \in \mathbb{K}^n$, we have that $\mathcal{D} := \operatorname{gr} D$ is skew-symmetric. Further, since $\dim \operatorname{gr} D = n$, Lem. 1.3.7 implies that $\mathcal{D}$ is even skew-adjoint. Moreover, by using Lem. 1.3.3, $\dim \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n$ and $E^*Q = Q^*E$ imply that $\mathcal{L} := \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ is self-adjoint. Then, the result follows since for $A = DQ$ it holds

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} E \\ DQ \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} I \\ D \end{smallmatrix}\right] \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

□

Proposition 2.5.14. *Assume that $sE - A \in \mathbb{K}[s]^{n\times m}$ is dH-LG, i.e.,* $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ *holds for some skew-adjoint $\mathcal{D} \subset \mathbb{K}^{2n}$ and some self-adjoint $\mathcal{L} \subset \mathbb{K}^{2n}$. If additionally* $\operatorname{mul}\mathcal{D} = \{0\}$*, then $sE - A$ is dH-LA. Namely, there exist some $Q \in \mathbb{K}^{n\times m}$ and some skew-Hermitian $D \in \mathbb{K}^{n\times n}$ such that $A = DQ$ and $E^*Q = Q^*E$. Hence,* $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \operatorname{gr} D \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$*. Further, these matrices fulfil $\mathcal{D} = \operatorname{gr} D$ and $\mathcal{L} \supseteq \operatorname{ran}\left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$.*

Proof. Assume that $sE - A \in \mathbb{K}[s]^{n\times m}$ fulfills $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ for some skew-adjoint $\mathcal{D} \subset \mathbb{K}^{2n}$ with $\operatorname{mul}\mathcal{D} = \{0\}$, and some $\mathcal{L} \subset \mathbb{K}^{2n}$. Then, by Rem. 1.3.4, $\dim\mathcal{D} = n$, whence there exist $F, G \in \mathbb{K}^{n\times n}$, such that $\mathcal{D} = \operatorname{ran}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$. The property $\operatorname{mul}\mathcal{D} = \{0\}$ further leads to $\ker F = \{0\}$, whence, by Prop. 1.3.9, $\mathcal{D} = \operatorname{gr} D$ for some skew-Hermitian $D \in \mathbb{K}^{n\times n}$. Further, the self-adjointness of $\mathcal{L}$ leads, by using Lem. 1.3.3, to the existence of some $E_1, Q_1 \in \mathbb{K}^{n\times n}$ with $E_1^*Q_1 = Q_1^*E_1$ and $\mathcal{L} = \operatorname{ran}\left[\begin{smallmatrix} E_1 \\ Q_1 \end{smallmatrix}\right]$. The latter matrix has moreover full column rank since self-adjointness of $\mathcal{L}$ implies, by Lem. 1.3.3, that $\dim\mathcal{L} = n$. Now we obtain

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L} = \operatorname{gr} D \operatorname{ran}\begin{bmatrix} E_1 \\ Q_1 \end{bmatrix} = \operatorname{ran}\begin{bmatrix} E_1 \\ DQ_1 \end{bmatrix}.$$

Consequently, there exists some $T \in \mathbb{K}^{n\times m}$ with

$$\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \begin{bmatrix} E_1 \\ DQ_1 \end{bmatrix} T = \begin{bmatrix} E_1T \\ DQ_1T \end{bmatrix},$$

which implies that $A = DQ$ for $Q = Q_1T$, and

$$\mathcal{L} = \operatorname{ran}\begin{bmatrix} E_1 \\ Q_1 \end{bmatrix} \supseteq \operatorname{ran}\begin{bmatrix} E_1 \\ Q_1 \end{bmatrix} T = \operatorname{ran}\left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right].$$

Invoking $E = E_1T$, we obtain that

$$E^*Q = T^*E_1^*Q_1T = T^*Q_1^*E_1T = Q^*E$$

and the desired statement follows. □

Before analyzing the properties of dissipative-Hamiltonian pencils in the next chapter, we point out that the observations made in this section concerning the relationship of the different pencil classes are concisely recapitulated in Figs. 2.9–2.11.

Chapter 3

Properties of dissipative-Hamiltonian pencils

Having now introduced the different classes of dissipative-Hamiltonian pencils, we now turn our attention to their properties. We first ask when port-Hamiltonian pencils are regular and then investigate how the Kronecker form (1.1) of possibly singular pencils looks like. Having already shown in Sec. 2.5.2 that dH-LR pencils encompass the dH-LA and dH-LG pencils, that is the approaches of [MMW18] and [vdSM18], we focus on this formulation for our analysis. Namely, we consider pencils $sE - A \in \mathbb{K}[s]^{n\times m}$ for which there exist a dissipative relation $\mathcal{D} \subset \mathbb{K}^{2n}$ and a symmetric relation $\mathcal{L} \subset \mathbb{K}^{2n}$ such that

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

We put a special emphasis on the case where $\mathcal{L}$ is a nonnegative relation. In this case, for an image representation $\operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ of $\mathcal{L}$ we have

$$G^*F \geq 0$$

and the *Hamiltonian* defined by $\frac{1}{2}\operatorname{Re}(z^*G^*Fz) = \frac{1}{2}z^*G^*Fz$, see Sec. 2.4, is positive semi-definite and can be regarded as an energy function for the system. Since properties of special cases of dH-LR pencils, namely dH-LA and dH-LG pencils, have been studied in [MMW18] and [vdSM18], we will successively compare their results with ours.

3.1 Regularity of dissipative-Hamiltonian pencils

In this section, we study regularity of square dH-LR pencils $sE - A \in \mathbb{K}[s]^{n\times n}$. The following technical result presents a certain range representation of such a product of relations, of which we show properties under additional assumptions reflecting the conditions found in Def. 2.5.4.

Lemma 3.1.1. *Let $\mathcal{D} \subseteq \mathbb{K}^{2n}$ be a dissipative and $\mathcal{L} \subseteq \mathbb{K}^{2n}$ be a symmetric linear relation, and assume that $\ker \mathcal{L} \cap \operatorname{mul} \mathcal{D} = \{0\}$. Let $n_1 = \dim(\operatorname{ran} \mathcal{L} \cap \operatorname{dom} \mathcal{D})$ and $n_2 = n - n_1$. Then there exists some unitary matrix $U \in \mathbb{K}^{n\times n}$, such that the product of $\mathcal{D}$ and $\mathcal{L}$ has a representation*

$$\mathcal{D}\mathcal{L} = \operatorname{ran} \operatorname{diag}(U, U) \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix} \tag{3.1}$$

for some matrices $L_{ij}, D_{ij} \in \mathbb{K}^{n_i \times n_j}$ with

$$L_{11} = L_{11}^*, \qquad D_{11} + D_{11}^* \leq 0, \tag{3.2}$$

$$L_{22} = L_{22}^2 = L_{22}^*, \qquad -D_{22} = D_{22}^2 = -D_{22}^*, \qquad \operatorname{ran} L_{22} \cap \operatorname{ran} D_{22} = \{0\}. \tag{3.3}$$

Moreover, the following holds:

(i) *If $\mathcal{L}$ is nonnegative then L_{11} is positive semi-definite. If, additionally, $\mathcal{L}$ is maximal then $\ker L_{11} \subset \ker L_{21}$.*

(ii) *If $\mathcal{D}$ is skew-symmetric then D_{11} is skew-Hermitian.*

(iii) $\ker L_{22} \cap \ker D_{22} = \{0\}$ *if and only if*

$$\operatorname{mul} \mathcal{D} \widehat{\oplus} \ker \mathcal{L} = (\operatorname{ran} \mathcal{L})^\perp \widehat{+} (\operatorname{dom} \mathcal{D})^\perp.$$

(iv) *If, additionally, $\mathcal{D} = \operatorname{gr} D$ for some dissipative $D \in \mathbb{K}^{n\times n}$ and $\mathcal{L}$ is self-adjoint, then $L_{21} = D_{22} = 0$ and $L_{22} = I_{n_2}$. Furthermore, we have*

$$\begin{aligned} \ker L_{11} \times \{0\} &= U^* \operatorname{mul} \mathcal{L}, \\ \ker D_{11} \times \{0\} &= U^* \{x \in \operatorname{ran} \mathcal{L} \mid Dx \in \ker \mathcal{L}\}. \end{aligned}$$

(v) *If, additionally, $\mathcal{D}$ is maximally dissipative and $\mathcal{L} = (\mathrm{gr}\ L)^{-1}$ for some $L \in \mathbb{K}^{n\times n}$, then L is Hermitian, and $D_{22} = -I_{n_2}$, $D_{21} = L_{22} = 0$. Furthermore, we have*

$$\begin{aligned} \ker L_{11} \times \{0\} &= U^* \{x \in \mathrm{dom}\ \mathcal{D} \mid Lx \in \mathrm{mul}\ \mathcal{D}\}, \\ \ker D_{11} \times \{0\} &= U^* \ker \mathcal{D}. \end{aligned}$$

Proof. Step 1: We show that there exist orthogonal decompositions, see also [ABJ+09],

$$\mathcal{D} = \{(x, Dx)\} \oplus (\{0\} \times \mathrm{mul}\ \mathcal{D}), \quad \mathcal{L} = \{(Lx, x)\} \oplus (\ker \mathcal{L} \times \{0\}) \tag{3.4}$$

for *some* linear operators $D : \mathrm{dom}\ \mathcal{D} \to (\mathrm{mul}\ \mathcal{D})^\perp$ and $L : \mathrm{ran}\ \mathcal{L} \to (\ker \mathcal{L})^\perp$. We obtain such a decomposition by applying Prop. 1.3.10 on $\mathcal{D}$ and $\mathcal{L}^{-1}$.
Step 2: We show that

$$\mathcal{D}\mathcal{L} = \left(\begin{bmatrix} L \\ D \end{bmatrix} (\mathrm{dom}\ \mathcal{D} \cap \mathrm{ran}\ \mathcal{L}) \right) \oplus (\ker \mathcal{L} \times \{0\}) \oplus (\{0\} \times \mathrm{mul}\ \mathcal{D}). \tag{3.5}$$

To prove "$\subseteq$", let $(x, z) \in \mathcal{D}\mathcal{L}$. Then there exists some $y \in \mathbb{K}^n$ such that $(x, y) \in \mathcal{L}$ and $(y, z) \in \mathcal{D}$. Therefore, $y \in \mathrm{ran}\ \mathcal{L} \cap \mathrm{dom}\ \mathcal{D}$. This implies with (3.4) that $x = Ly + v_L$ and $z = Dy + v_D$ for some $v_L \in \ker \mathcal{L}$ and $v_D \in \mathrm{mul}\ \mathcal{D}$. Hence,

$$(x, z) \in \left(\begin{bmatrix} L \\ D \end{bmatrix} (\mathrm{dom}\ \mathcal{D} \cap \mathrm{ran}\ \mathcal{L}) \right) \oplus (\ker \mathcal{L} \times \{0\}) \oplus (\{0\} \times \mathrm{mul}\ \mathcal{D}).$$

To prove "$\supseteq$", let $(Ly + v_L, Dy + v_D) \in \mathbb{K}^{2n}$ with $y \in \mathrm{ran}\ \mathcal{L} \cap \mathrm{dom}\ \mathcal{D}$, $v_L \in \ker \mathcal{L}$, and $v_D \in \mathrm{mul}\ \mathcal{D}$. This implies $(Ly, y) \in \mathcal{L}$, $(y, Dy) \in \mathcal{D}$ and hence $(Ly, Dy) \in \mathcal{D}\mathcal{L}$. Then $(0, 0) \in \mathcal{D}$ and $(0, 0) \in \mathcal{L}$ further lead to $(v_L, 0), (0, v_D) \in \mathcal{D}\mathcal{L}$, and thus $(Ly + v_L, Dy + v_D) \in \mathcal{D}\mathcal{L}$.
Step 3: Consider the orthogonal decomposition $\mathbb{K}^n = X_1 \oplus X_2$ with

$$X_1 := \mathrm{ran}\ \mathcal{L} \cap \mathrm{dom}\ \mathcal{D}, \quad X_2 := (\mathrm{ran}\ \mathcal{L} \cap \mathrm{dom}\ \mathcal{D})^\perp = (\mathrm{ran}\ \mathcal{L})^\perp \widehat{+} (\mathrm{dom}\ \mathcal{D})^\perp. \tag{3.6}$$

Our next objective is to show

$$(\ker \mathcal{L} \times \{0\}) \oplus (\{0\} \times \mathrm{mul}\ \mathcal{D}) = \begin{bmatrix} P_{\ker \mathcal{L}} \\ -P_{\mathrm{mul}\ \mathcal{D}} \end{bmatrix} (\ker \mathcal{L} \widehat{\oplus}\, \mathrm{mul}\ \mathcal{D}). \tag{3.7}$$

The inclusion "$\supseteq$" in (3.7) is immediate. To prove "$\subseteq$", it suffices to show that both spaces $\ker \mathcal{L} \times \{0\}$ and $\{0\} \times \mathrm{mul}\ \mathcal{D}$ are contained in the set on the right-hand side of

(3.7). Consider the space $X_3 := \ker \mathcal{L} \widehat{\oplus} \operatorname{mul} \mathcal{D}$. Then by Lem. 1.3.8 we have $\ker \mathcal{L} \subseteq (\operatorname{ran} \mathcal{L})^\perp$ and $\operatorname{mul} \mathcal{D} \subseteq (\operatorname{dom} \mathcal{D})^\perp$, whence $X_3 \subseteq X_2$. Since $X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D} \subset \ker \mathcal{L}$, we have $(\ker \mathcal{L})^\perp \cap (X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D}) = \{0\}$, we have that $P_{\ker \mathcal{L}}|_{X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D}}$ is injective. This together with $\dim(X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D}) = \dim \ker \mathcal{L}$ gives $\ker \mathcal{L} = P_{\ker \mathcal{L}}(X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D})$. Hence, for each $(v_L, 0) \in \ker \mathcal{L} \times \{0\}$ there exists $x \in X_3 \widehat{\ominus} \operatorname{mul} \mathcal{D}$ with $P_{\ker \mathcal{L}} x = v_L$ and $P_{\operatorname{mul} \mathcal{D}} x = 0$ and therefore $(v_L, 0) \in \left[\begin{smallmatrix} P_{\operatorname{mul} \mathcal{L}} \\ -P_{\operatorname{mul} \mathcal{D}} \end{smallmatrix}\right](X_3)$. Analogously, we can show that $\{0\} \times \operatorname{mul} \mathcal{D} \subseteq \left[\begin{smallmatrix} P_{\operatorname{mul} \mathcal{L}} \\ -P_{\operatorname{mul} \mathcal{D}} \end{smallmatrix}\right](X_3)$, which altogether shows (3.7).

Step 4: Based on the space decomposition $\mathbb{K}^n = X_1 \oplus X_2$ as in (3.6), we define

$$\hat{L}_{11} := P_{X_1} L|_{X_1}, \quad \hat{L}_{21} := P_{X_2} L|_{X_1}, \quad \hat{L}_{22} := P_{\ker \mathcal{L}} : X_2 \to X_2 \tag{3.8}$$

and

$$\hat{D}_{11} := P_{X_1} D|_{X_1}, \quad \hat{D}_{21} := P_{X_2} D|_{X_1}, \quad \hat{D}_{22} := -P_{\operatorname{mul} \mathcal{D}} : X_2 \to X_2.$$

Let $n_i := \dim X_i$, $i = 1, 2$, and

$$U_1 := [u_1, \ldots, u_{n_1}] \in \mathbb{K}^{n \times n_1}, \quad U_2 := [u_{n_1+1}, \ldots, u_n] \in \mathbb{K}^{n \times n_2},$$

where the columns are an orthonormal basis of X_1 and X_2, respectively. Then $U = [U_1, U_2] \in \mathbb{K}^{n \times n}$ is unitary and

$$L_{ij} := U_i^* \hat{L}_{ij} U_j, \quad D_{ij} := U_i^* \hat{D}_{ij} U_j, \quad i, j = 1, 2. \tag{3.9}$$

Combining (3.5) and (3.7), we obtain

$$\begin{aligned}
\mathcal{D}\mathcal{L} &= \left(\left[\begin{smallmatrix} L \\ D \end{smallmatrix}\right](X_1)\right) \oplus (\ker \mathcal{L} \times \{0\}) \oplus (\{0\} \times \operatorname{mul} \mathcal{D}) \\
&= \begin{bmatrix} \hat{L}_{11} \\ \hat{L}_{21} \\ \hat{D}_{11} \\ \hat{D}_{21} \end{bmatrix} (X_1) \oplus \begin{bmatrix} 0 \\ \hat{L}_{22} \\ 0 \\ \hat{D}_{22} \end{bmatrix} (X_2) \\
&= \operatorname{diag}(U, U) \left(\begin{bmatrix} L_{11} & 0 \\ L_{21} & 0 \\ D_{11} & 0 \\ D_{21} & 0 \end{bmatrix} \underbrace{(U^* X_1)}_{= \mathbb{K}^{n_1} \times \{0\}} \oplus \begin{bmatrix} 0 & 0 \\ 0 & L_{22} \\ 0 & 0 \\ 0 & D_{22} \end{bmatrix} \underbrace{(U^* X_2)}_{= \{0\} \times \mathbb{K}^{n_2}} \right) \\
&= \operatorname{diag}(U, U) \ \operatorname{ran} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}.
\end{aligned}$$

This completes the proof of (3.1).

Step 5: We show that (3.2) and (3.3) hold. Let $(y, x) \in \mathcal{L}$. Then $y = Lx + v_L$ for some $v_L \in \ker \mathcal{L} \subseteq (\operatorname{ran} \mathcal{L})^\perp$ and some $x \in X_1$. Consequently,

$$\langle \hat{L}_{11} x, x \rangle = \langle P_{X_1} L x, x \rangle = \langle L x, x \rangle = \langle L x + v_L, x \rangle = \langle y, x \rangle = \langle x, y \rangle = \langle x, \hat{L}_{11} x \rangle, \tag{3.10}$$

where in the second last equation the symmetry of $\mathcal{L}$ was used and the last equation follows from a repetition of the first steps in the second component of the inner product. This implies that $\hat{L}_{11}$ is Hermitian. Consequently, $L_{11} = U_1^* \hat{L}_{11} U_1$ is Hermitian. Similarly, one can show that if $\mathcal{D}$ is dissipative then D_{11} is dissipative, whence (3.2) holds. Since $L_{22} = U_2^* \hat{L}_{22} U_2$ and $D_{22} = U_2^* \hat{D}_{22} U_2$ with orthogonal projectors $\hat{L}_{22} = P_{\ker \mathcal{L}}$ and $-\hat{D}_{22} = P_{\mathrm{mul}\, \mathcal{D}}$ we have

$$L_{22} = U_2^* \hat{L}_{22} U_2 = U_2^* \hat{L}_{22}^2 U_2 = U_2^* \hat{L}_{22} U_2 U_2^* \hat{L}_{22} U_2 = L_{22}^2 = L_{22}^*,$$
$$-D_{22} = U_2^* \hat{D}_{22} U_2 = U_2^* \hat{D}_{22}^2 U_2 = U_2^* \hat{D}_{22} U_2 U_2^* \hat{D}_{22} U_2 = D_{22}^2 = -D_{22}^*.$$

Furthermore,

$$\mathrm{ran}\ D_{22} \cap \mathrm{ran}\ L_{22} = U_2^*(\mathrm{ran}\ P_{\mathrm{mul}\, \mathcal{D}} \cap \mathrm{ran}\ P_{\ker \mathcal{L}}) = U_2^*(\mathrm{mul}\ \mathcal{D} \cap \ker \mathcal{L}) = \{0\},$$

which implies $\mathrm{mul}\ \mathcal{D} \cap \ker \mathcal{L} = \{0\}$ and hence (3.3).
Step 6: We prove (i)-(iii). If $\mathcal{L}$ is nonnegative, then $\langle y, x\rangle \geq 0$ for all $(x, y) \in \mathcal{L}$ which implies, by using (3.10), that $\langle \hat{L}_{11} x, x\rangle \geq 0$ for all $x \in X_1$ and thus $L_{11} = U_1^* \hat{L}_{11} U_1$ is positive semi-definite. Next we show that $\ker L_{11} \subset \ker L_{21}$, if $\mathcal{L}$ is maximal. From the maximality we have $(\ker \mathcal{L})^\perp = \mathrm{ran}\ \mathcal{L}$ and thus the operator $L : \mathrm{ran}\ \mathcal{L} \to \mathrm{ran}\ \mathcal{L}$ from Step 1 can be decomposed as

$$L = \begin{bmatrix} \hat{L}_{11} & \tilde{L}_{21}^* \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \mathrm{ran}\ \mathcal{L} = (\mathrm{dom}\ \mathcal{D} \cap \mathrm{ran}\ \mathcal{L}) \oplus (\mathrm{ran}\ \mathcal{L} \widehat{\ominus} (\mathrm{dom}\ \mathcal{D} \cap \mathrm{ran}\ \mathcal{L})),$$

and L is nonnegative, i.e., $\langle Lx, x\rangle \geq 0$ for all $x \in \mathrm{ran}\ \mathcal{L}$. We show that $\ker \hat{L}_{11} \subset \ker \tilde{L}_{21}$. Assume that there exists some $x \in \ker \hat{L}_{11}$ with $z := -\tilde{L}_{21} x \neq 0$. Since $L \geq 0$ we have for all $\alpha \in \mathbb{R}$

$$0 \leq \left\langle L \begin{pmatrix} \alpha x \\ z \end{pmatrix}, \begin{pmatrix} \alpha x \\ z \end{pmatrix} \right\rangle = \left\langle \begin{bmatrix} \hat{L}_{11} & \tilde{L}_{21}^* \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} \begin{pmatrix} \alpha x \\ z \end{pmatrix}, \begin{pmatrix} \alpha x \\ z \end{pmatrix} \right\rangle = -2\alpha \|z\|^2 + \|\tilde{L}_{22} z\|^2.$$

Choosing α sufficiently large, we obtain a contradiction. Hence $\ker \hat{L}_{11} \subset \ker \tilde{L}_{21}$. Further, decompose $X_2 = (X_2 \cap \mathrm{ran}\ \mathcal{L}) \oplus (X_2 \cap (\mathrm{ran}\ \mathcal{L})^\perp)$ and, without restriction, assume that the vectors $u_{n_1+1}, \ldots, u_{n_1+\hat{k}}$ for some $\hat{k} \geq 1$ are an orthonormal basis of $X_2 \cap \mathrm{ran}\ \mathcal{L}$. Then

$$\hat{L}_{21} = P_{X_2} L|_{X_1} = P_{X_2 \cap \mathrm{ran}\ \mathcal{L}} L|_{X_1} + P_{X_2 \cap (\mathrm{ran}\ \mathcal{L})^\perp} L|_{X_1} = P_{X_2 \cap \mathrm{ran}\ \mathcal{L}} L|_{X_1} = \tilde{L}_{21}$$

and this implies

$$\ker L_{11} = \ker U_1^* \hat{L}_{11} U_1 = U_1^* \ker \hat{L}_{11} \subset U_1^* \ker \tilde{L}_{21}$$
$$= \ker U_1^* \hat{L}_{21} = \ker U_2^* \hat{L}_{21} U_1 = \ker L_{21}.$$

The assertion (ii) can be proven analogously to (i). To show (iii), first assume that $\ker L_{22} \cap \ker D_{22} = \{0\}$. Then

$$\ker \hat{L}_{22} \cap \ker \hat{D}_{22} = U_2(\ker L_{22} \cap \ker D_{22}) = \{0\} \tag{3.11}$$

and taking orthogonal complements in X_2, we obtain

$$X_2 = (\ker \hat{L}_{22} \cap \ker \hat{D}_{22})^\perp = \operatorname{ran} \hat{L}_{22} + \operatorname{ran} \hat{D}_{22} = \ker \mathcal{L} \widehat{\oplus} \operatorname{mul} \mathcal{D}.$$

Conversely, assume that $X_2 = \ker \mathcal{L} \widehat{\oplus} \operatorname{mul} \mathcal{D}$. Then, again by taking orthogonal complements in X_2,

$$\ker \hat{L}_{22} \cap \ker \hat{D}_{22} = (\ker \mathcal{L} \widehat{\oplus} \operatorname{mul} \mathcal{D})^\perp = X_2^\perp = \{0\}.$$

Now invoking (3.11) and the injectivity of U_2, we obtain $\ker L_{22} \cap \ker D_{22} = \{0\}$.
<u>*Step 7:*</u> We prove (iv). Assume that $\mathcal{L}$ is self-adjoint and $\mathcal{D} = \operatorname{gr} D$ for some dissipative $D \in \mathbb{K}^{n \times n}$. Then we have that $\operatorname{mul} \mathcal{D} = \{0\} = (\operatorname{dom} \mathcal{D})^\perp$ and $\ker \mathcal{L} = (\operatorname{ran} \mathcal{L})^\perp$. Hence, $X_1 = \operatorname{ran} \mathcal{L} = X_2^\perp$. This implies that $\hat{L}_{21} = \hat{D}_{22} = 0$ and thus $L_{21} = D_{22} = 0$. Invoking (iii), we have $\ker L_{22} = \ker L_{22} \cap \ker D_{22} = \{0\}$ which implies $L_{22} = I_{n_2}$. Furthermore, $\operatorname{mul} \mathcal{L} = \ker \hat{L}_{11} = U(\ker L_{11} \times \{0\})$ and together with $(\operatorname{ran} \mathcal{L})^\perp = \ker \mathcal{L}$ we obtain

$$\left\{\, x \in \operatorname{ran} \mathcal{L} \;\middle|\; Dx \in (\operatorname{ran} \mathcal{L})^\perp \,\right\} = \ker(P_{\operatorname{ran} \mathcal{L}} D|_{\operatorname{ran} \mathcal{L}}) = \ker \hat{D}_{11} = U(\ker D_{11} \times \{0\}).$$

The proof of (v) is analogous to the proof of (iv) and is therefore omitted. □

Remark 3.1.2. In the context of Lem. 3.1.1 if $\ker \mathcal{L} \cap \operatorname{mul} \mathcal{D} \neq \{0\}$, still a certain form similar to (3.1) can be achieved, see Thm. 3.2.7 which contains two additional block columns representing all column minimal indices equal to two. As a consequence, the matrix pencil $sE - A$ given by $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ turns out to be singular which is discussed in Sec. 3.1.
Furthermore, a converse result to Lem. 3.1.1 holds. If a linear relation is given by the right-hand side of (3.1) then we can define $\mathcal{D}$ and $\mathcal{L}$ satisfying (3.1) as follows:

$$\mathcal{L} = \{ \left(U \left(\begin{smallmatrix} L_{11} x_1 \\ L_{21} x_1 + L_{22} x_2 \end{smallmatrix} \right), U \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right) \right) \;:\; x_i \in \mathbb{K}^{n_i}, i = 1, 2 \},$$
$$\mathcal{D} = \{ \left(U \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right), U \left(\begin{smallmatrix} D_{11} x_1 \\ D_{21} x_1 + D_{22} x_2 \end{smallmatrix} \right) \right) \;:\; x_i \in \mathbb{K}^{n_i}, i = 1, 2 \}.$$

From (3.2) and (3.3) it is straightforward to show that $\mathcal{L}$ and $\mathcal{D}$ are symmetric and dissipative, respectively.

Under the assumptions of Lem. 3.1.1, that is the multi-valued part of $\mathcal{D}$ and the kernel of $\mathcal{L}$ intersect trivially, we present a first characterization of regularity.

Proposition 3.1.3. *Let $sE - A \in \mathbb{K}[s]^{n\times n}$ be a dH-LR pencil, i.e.,*

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

for some dissipative relation $\mathcal{D} \subset \mathbb{K}^{2n}$ and some symmetric relation $\mathcal{L} \subset \mathbb{K}^{2n}$. If $\operatorname{mul}\mathcal{D} \cap \ker \mathcal{L} = \{0\}$*, then there exists a unitary matrix $U \in \mathbb{K}^{n\times n}$ and an invertible matrix $T \in \mathbb{K}^{n\times n}$, such that, for some $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 = n$,*

$$U^*(sE - A)T = \begin{bmatrix} sL_{11} - D_{11} & 0 \\ sL_{21} - D_{21} & sL_{22} - D_{22} \end{bmatrix} \tag{3.12}$$

with $L_{ij}, D_{ij} \in \mathbb{K}^{n_i \times n_j}$, $i, j = 1, 2$, satisfying

$$L_{11} = L_{11}^*, \qquad D_{11} + D_{11}^* \leq 0, \tag{3.13}$$

$$L_{22} = L_{22}^2 = L_{22}^*, \qquad -D_{22} = D_{22}^2 = -D_{22}^*, \qquad \operatorname{ran} L_{22} \cap \operatorname{ran} D_{22} = \{0\}. \tag{3.14}$$

Moreover, $sE - A$ is regular if and only if the following two conditions hold.

(i) *$sL_{11} - D_{11}$ is regular, and*

(ii) *$sL_{22} - D_{22}$ is regular.*

Furthermore, (ii) is equivalent to $\ker \mathcal{L} \widehat{+} \operatorname{mul}\mathcal{D} = (\operatorname{ran}\mathcal{L})^\perp \widehat{+} (\operatorname{dom}\mathcal{D})^\perp$.

Proof. By Lem. 3.1.1, there exists a unitary matrix $U \in \mathbb{K}^{n\times n}$, such that

$$\operatorname{ran}\begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{D}\mathcal{L} = \operatorname{ran}\operatorname{diag}(U,U)\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}$$

with $L_{ij}, D_{ij} \in \mathbb{K}^{n_i\times n_j}$ satisfying (3.13) & (3.14). Hence there exists some invertible $T \in \mathbb{K}^{n\times n}$, such that

$$\begin{bmatrix} E \\ A \end{bmatrix} T = \operatorname{diag}(U,U)\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix},$$

which shows (3.12). For the proof of the remaining statement, we make use of the identity

$$\det(sE - A) = \det(T)^{-1}\det(U)\det(sL_{11} - D_{11})\det(sL_{22} - D_{22}). \tag{3.15}$$

Hence the regularity of $sE-A$ is equivalent to (i) and (ii). Using $L_{22}=L_{22}^2=L_{22}^*$ and $-D_{22}=D_{22}^2=-D_{22}^*$, the pencil $sL_{22}-D_{22}$ is positive real. Therefore, by Lem. 1.2.10, condition (ii) is equivalent to $\ker L_{22}\cap\ker D_{22}=\{0\}$ and invoking Lem. 3.1.1 (iii) proves the claim. $\square$

Note that the assumption $\operatorname{mul}\mathcal{D}\cap\ker\mathcal{L}=\{0\}$ in Prop. 3.1.3 is required to ensure the regularity of the pencil since a nontrivial element of this intersection implies a column minimal index equal to one.

Corollary 3.1.4. *Let $sE-A\in\mathbb{K}[s]^{n\times n}$ such that there exist dissipative and nonnegative relations $\mathcal{D}=\operatorname{ran}\left[\begin{smallmatrix}D_1\\D_2\end{smallmatrix}\right]$ and $\mathcal{L}=\operatorname{ran}\left[\begin{smallmatrix}L_1\\L_2\end{smallmatrix}\right]$, respectively, with $L_1,L_2,D_1,D_2\in\mathbb{K}^{n\times n}$ such that*

$$\operatorname{ran}\left[\begin{smallmatrix}E\\A\end{smallmatrix}\right]=\mathcal{D}\mathcal{L}.$$

Then with $\mathcal{X}:=\operatorname{ran}D_1\cap\operatorname{ran}L_2$, $sE-A$ is regular if and only if the following conditions are fulfilled.

(i) $\ker\mathcal{L}\cap\operatorname{mul}\mathcal{D}=\{0\}$*;*

(ii) $\ker\mathcal{L}\,\widehat{\oplus}\,\operatorname{mul}\mathcal{D}=(\operatorname{ran}\mathcal{L})^\perp\widehat{+}(\operatorname{dom}\mathcal{D})^\perp$*;*

(iii) $\ker P_{\mathcal{X}}L_1\restriction_{\mathcal{X}}\cap\ker P_{\mathcal{X}}D_2\restriction_{\mathcal{X}}=\{0\}$.

Proof. We first assume that $sE-A$ is regular. Let $x\in\operatorname{mul}\mathcal{D}\cap\ker\mathcal{L}$, i.e., $(0,x)\in\mathcal{D}$ and $(x,0)\in\mathcal{L}$. Then also $(0,x),(x,0)\in\mathcal{D}\mathcal{L}=\operatorname{ran}\left[\begin{smallmatrix}E\\A\end{smallmatrix}\right]$. By [BTW16, Thm. 4.5], $sE-A$ has a column minimal index equal to one if $x\neq 0$ in the case $\mathbb{K}=\mathbb{C}$. The case $\mathbb{K}=\mathbb{R}$ is proven analogously, see also [BTW16, Ex. 4.4 (iii)]. By regularity of $sE-A$ we deduce $x=0$, i.e., $\operatorname{mul}\mathcal{D}\cap\ker\mathcal{L}=\{0\}$ and we can apply Lem. 3.1.1 and Prop. 3.1.3. Employing their notation, we obtain $U,T\in\mathbf{Gl}_n(\mathbb{K})$ such that (3.12) holds with its upper left entry being a positive real pencil by nonnegativity of $\mathcal{L}$ and by the choices of (3.8) & (3.9), (ii) & (iii) are equivalent to Prop. 3.1.3 (i) & (ii), respectively. Hence, (ii) & (iii) hold by regularity of $sE-A$.
Assume now that (i)–(iii) hold. We can then apply Prop. 3.1.3 and since (ii) & (iii) are equivalent to Prop. 3.1.3 (i) & (ii), $sE-A$ is regular. $\square$

Further, we can conclude from Rem. 3.1.2 that a certain converse to Prop. 3.1.3 is also true: each pencil $sE-A$ given by (3.12) is pH in our sense with $\operatorname{mul}\mathcal{D}\cap\ker\mathcal{L}=\{0\}$. Let us elaborate this fact with the following proposition.

Proposition 3.1.5. *Let $n_1, n_2 \in \mathbb{N}_0$ with $L_{ij}, D_{ij} \in \mathbb{K}^{n_i \times n_j}$, $i, j = 1, 2$, satisfying*

$$\begin{aligned} &L_{11} = L_{11}^*, && D_{11} + D_{11}^* \leq 0, && \\ &L_{22} = L_{22}^2 = L_{22}^*, && -D_{22} = D_{22}^2 = -D_{22}^*, && \operatorname{ran} L_{22} \cap \operatorname{ran} D_{22} = \{0\}. \end{aligned}$$

Then

$$\mathcal{D} = \operatorname{ran} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}, \quad \mathcal{L} = \operatorname{ran} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$$

are dissipative and symmetric relations, respectively, and $\ker \mathcal{L} \cap \operatorname{mul} \mathcal{D} = \{0\}$. *Further,*

$$\mathcal{D}\mathcal{L} = \operatorname{ran} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}.$$

Proof. The symmetry of $\mathcal{L}$ and the dissipativity of $\mathcal{D}$ follow from Lem. 1.3.3 and Lem. 1.3.7, respectively, noting that

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} L_{11}^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}^* \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}^* \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11}^* + D_{11} & 0 \\ 0 & 0 \end{bmatrix} \leq 0.$$

It is readily seen that

$$\begin{aligned} \mathcal{D}\mathcal{L} = \{\ &(L_{11}x_1, L_{21}x_1 + L_{22}x_{21}, D_{11}x_1, D_{21}x_1 + D_{22}x_{22}) \mid \\ &x_1 \in \mathbb{K}^{n_1},\ x_{21}, x_{22} \in \mathbb{K}^{n_2}\ \} \supseteq \operatorname{ran} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}. \end{aligned}$$

To prove the reverse inclusion, let $(L_{11}x_1, L_{21}x_1 + L_{22}x_{21}, D_{11}x_1, D_{21}x_1 + D_{22}x_{22}) \in \mathcal{D}\mathcal{L}$ with $x_1 \in \mathbb{K}^{n_1}$, $x_{21}, x_{22} \in \mathbb{K}^{n_2}$. Then defining $x_2 = L_{22}x_{21} + D_{22}x_{22}$ and decomposing

$$L_{22}x_{21} = \underbrace{L_{22}x_{211}}_{\in \operatorname{ran} D_{22}} + \underbrace{L_{22}x_{212}}_{\in (\operatorname{ran} D_{22})^\perp}, \quad D_{22}x_{22} = \underbrace{D_{22}x_{221}}_{\in \operatorname{ran} L_{22}} + \underbrace{D_{22}x_{222}}_{\in (\operatorname{ran} L_{22})^\perp},$$

we derive $L_{22}x_{211} = 0$ and $D_{22}x_{221} = 0$ since $\operatorname{ran} L_{22} \cap \operatorname{ran} D_{22} = \{0\}$. Further,

$$\begin{pmatrix} L_{22}x_2 \\ D_{22}x_2 \end{pmatrix} = \begin{pmatrix} L_{22}^2 x_{21} \\ D_{22}^2 x_{22} \end{pmatrix} + \begin{pmatrix} L_{22}D_{22}x_{222} \\ D_{22}L_{22}x_{212} \end{pmatrix} = \begin{pmatrix} L_{22}x_{21} \\ D_{22}x_{22} \end{pmatrix} + \begin{pmatrix} L_{22}^* D_{22}x_{222} \\ D_{22}^* L_{22}x_{212} \end{pmatrix} = \begin{pmatrix} L_{22}x_{21} \\ D_{22}x_{22} \end{pmatrix},$$

by virtue of L_{22} and $-D_{22}$ being self-adjoint orthogonal projections. To complete the proof, note that $\mathrm{mul}\ \mathcal{D} = \{0\}^{n_1} \times \mathrm{ran}\ D_{22}$ and $\ker \mathcal{L} = \{0\}^{n_1} \times \mathrm{ran}\ L_{22}$ and therefore $\ker \mathcal{L} \cap \mathrm{mul}\ \mathcal{D} = \{0\}$. □

We apply Prop. 3.1.3 to the special case that $\mathcal{D} = \mathrm{gr}\ D$ from some dissipative $D \in \mathbb{K}^{n\times n}$.

Corollary 3.1.6. *Let $E, D, Q \in \mathbb{K}^{n\times n}$ with $Q^*E = E^*Q$ and $D + D^* \leq 0$. Consider the following three statements.*

(i) *$sE - DQ$ is a regular pencil;*

(ii) *$sE - Q$ is a regular pencil;*

(iii) *For $\mathcal{L} = \mathrm{ran}\ \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$, it holds $\dim \mathcal{L} = n$, i.e., $\mathcal{L}$ is a self-adjoint linear relation.*

Then

$$\text{(i)} \implies \text{(ii)} \iff \text{(iii)}.$$

If additionally, $Q^*E \geq 0$ and

$$\ker E \cap \ker(Q^*DQ) = \{0\}, \tag{3.16}$$

then (ii) $\implies$ (i).

Proof. Note that for $A = DQ$, $\mathcal{D} = \mathrm{gr}\ D$ and $\mathcal{L} = \mathrm{ran}\ \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ we have

$$\mathrm{ran}\ \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathrm{ran}\ \left[\begin{smallmatrix} E \\ DQ \end{smallmatrix}\right] = \mathrm{ran}\ \left[\begin{smallmatrix} I \\ D \end{smallmatrix}\right] \mathrm{ran}\ \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = \mathrm{gr}\ D\, \mathrm{ran}\ \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

with dissipative $\mathcal{D}$ and symmetric $\mathcal{L}$ by Rem. 1.3.4 and Lem. 1.3.3.
"(i) ⇒ (iii)": Assume that $sE - DQ$ is regular. The multi-valued part of $\mathcal{D} = \mathrm{gr}\ D$ is trivial, whence $\mathrm{mul}\ \mathcal{D} \cap \ker \mathcal{L} = \{0\}$. Thus we can apply Prop. 3.1.3 (ii), which gives

$$\ker \mathcal{L} = \ker \mathcal{L} \widehat{\oplus}\, \mathrm{mul}\ \mathcal{D} = (\ker \mathcal{L})^\perp \widehat{+} (\mathrm{dom}\ \mathcal{D})^\perp = (\ker \mathcal{L})^\perp.$$

Then Lem. 1.3.3 yields that $\mathcal{L}$ is self-adjoint.
"(iii) ⇒ (ii)": Let $\mathcal{L}$ be self-adjoint. Then Lem. 3.1.1 (iv) with $\mathcal{D} = -\mathrm{gr}\ I_n$ implies that there exist unitary matrix U and a Hermitian matrix L_{11} with

$$\mathrm{ran}\ \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = \mathcal{L} = \mathrm{ran}\ \mathrm{diag}\,(U, U) \begin{bmatrix} L_{11} & 0 \\ 0 & I_{n-n_1} \\ D_{11} & 0 \\ D_{21} & 0 \end{bmatrix} \tag{3.17}$$

for some Hermitian $D_{11}, L_{11} \in \mathbb{K}^{n_1 \times n_1}$ and $D_{21} \in \mathbb{K}^{n_2 \times n_1}$ with $D_{11} + D_{11}^* \leq 0$. Moreover, by Lem. 3.1.1 (iv), we further have

$$\ker D_{11} \times \{0\} = \{x \in \operatorname{ran}\, \mathcal{L} \mid Dx \in \ker \mathcal{L}\}.$$

Since, by Lem. 1.3.8, $\operatorname{ran}\, \mathcal{L} = (\ker \mathcal{L})^{\perp}$, we obtain that the latter space is trivial. Therefore, D_{11} is invertible. Further, by using (3.17), we obtain that there exists some invertible $T \in \mathbb{K}^{n \times n}$ with

$$\left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] T = \operatorname{diag}(U, U) \begin{bmatrix} L_{11} & 0 \\ 0 & I_{n-n_1} \\ D_{11} & 0 \\ D_{21} & 0 \end{bmatrix}.$$

This gives $\det(sE - Q) = \det(UT^{-1}) \det(sL_{11} - D_{11}) \cdot s^{n-n_1}$. The polynomial $\det(sL_{11} - D_{11})$ is nonzero, since the invertibility of D_{11} yields that it does not vanish at the origin. Therefore, $\det(sE - Q)$ is a product of nonzero polynomials, whence the pencil $sE - Q$ is regular.

"(ii) $\Rightarrow$ (iii)": If $sE - Q$ is regular, then $\ker E \cap \ker Q = \{0\}$, and the dimension formula gives

$$\dim \mathcal{L} = \dim \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n.$$

It remains to prove that "(ii) $\Rightarrow$ (i)" holds under the additional assumptions $Q^*E \geq 0$ and (3.16). As we have already shown that (ii) implies (iii), we can further use that $\mathcal{L}$ is self-adjoint. By using $\mathcal{D} = \operatorname{gr}\, D$, we can apply Lem. 3.1.1 (iv) to see that there exists a unitary matrix $U \in \mathbb{K}^{n \times n}$, such that

$$\operatorname{ran}\, \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L} = \operatorname{ran}\, \operatorname{diag}(U, U) \begin{bmatrix} L_{11} & 0 \\ 0 & I_{n_2} \\ D_{11} & 0 \\ D_{21} & 0 \end{bmatrix}$$

with $n_1 = \dim \operatorname{ran}\, \mathcal{L} = \operatorname{rk}\, Q$, $n_2 = n - n_1$, and matrices $L_{ij}, D_{ij} \in \mathbb{K}^{n_i \times n_j}$ with $L_{11} = L_{11}^*$ and $D_{11} + D_{11}^* \leq 0$. Invoking (3.16), Lem. 3.1.1 (iv) further yields

$$\begin{aligned} (\ker L_{11} \cap \ker D_{11}) \times \{0\} &= (\ker L_{11} \times \{0\}) \cap (\ker D_{11} \times \{0\}) \\ &= U^* \left((Q \ker E) \cap \{x \in \operatorname{ran}\, Q \mid Dx \in (\operatorname{ran}\, Q)^{\perp}\} \right). \end{aligned} \tag{3.18}$$

Since $(\operatorname{ran}\, Q)^{\perp} = \ker Q^*$ we obtain from (3.16) that

$$(Q \ker E) \cap \{x \in \operatorname{ran}\, Q \mid Dx \in \ker Q^*\} = \{0\}. \tag{3.19}$$

Indeed, let x be an element of (3.19) then $x = Qy$, $Dx \in \ker Q^*$ and $y \in \ker E$. This implies $y \in \ker E \cap \ker(Q^*DQ)$ and hence $y = 0$. Thus, (3.19) holds and combined

(3.18) this results in $\ker L_{11} \cap \ker D_{11} = \{0\}$. On the other hand, the assumption $Q^*E \geq 0$ implies, by using Lem. 1.3.7, that $\mathcal{L}$ is nonnegative. Then Lem. 3.1.1 (i) implies that $L_{11} \geq 0$. Thus, $sL_{11} - D_{11}$ is positive real, and Lem. 1.2.10 together with the already proven identity $\ker L_{11} \cap \ker D_{11} = \{0\}$ yields that $sL_{11} - D_{11}$ is regular. Further, by Lem 1.3.8 together with the self-adjointness of $\mathcal{L}$, we have $\ker \mathcal{L} = (\operatorname{ran} \mathcal{L})^{\perp}$. Additionally, invoking $\operatorname{dom} \mathcal{D} = \mathbb{K}^n$ and $\operatorname{mul} \mathcal{D} = \{0\}$, we see that $\ker \mathcal{L} \widehat{+} \operatorname{mul} \mathcal{D} = (\operatorname{ran} \mathcal{L})^{\perp} \widehat{+} (\operatorname{dom} \mathcal{D})^{\perp}$. This means that (i) and (ii) in Prop. 3.1.3 hold, implying that $sE - A$ is regular. $\square$

Note that the statement "(i) $\Rightarrow$ (ii)" has already been obtained in [MMW18, Prop. 4.1]. The implication "(ii)$\Rightarrow$(i)" does not hold in general, see [MMW18, Ex. 4.7].

For convenience, we discuss how the lower-triangular form (3.12) can be obtained from the port-Hamiltonian formulations from [MMW18] and [vdSM18], i.e., for dH-LA and dH-LG pencils, respectively. Invoking Prop. 2.5.5, let $sE - DQ \in \mathbb{K}[s]^{n \times n}$ be a dH-LA pencil with $E, D, Q \in \mathbb{K}^{n \times n}$ satisfying $Q^*E = E^*Q$ and $D + D^* \leq 0$. Then we consider $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ and $\mathcal{D} = \operatorname{gr} D$, see Sec. 2.5. This implies $\operatorname{mul} \mathcal{D} \cap \ker \mathcal{L} = \{0\}$ and since $\mathcal{L}$ is symmetric and hence $E \ker Q = \ker \mathcal{L} \subseteq \operatorname{ran} \mathcal{L}^{\perp} = \ker Q^*$. If we choose $C_i \in \mathbb{K}^{n \times d_i}$, $d_i \in \mathbb{N}$, $i = 1, 2, 3$, as matrices whose columns consist of an orthonormal basis of $\operatorname{ran} Q$, $\ker Q^*$ and $E \ker Q$, respectively, then $U = [C_1, C_2]$ is unitary and the matrices in (3.12) are given by

$$\begin{aligned} L_{11} &= C_1^* E C_1, & L_{21} &= C_2^* E C_1 = 0, & L_{22} &= C_3^* C_2, \\ D_{11} &= C_1^* D C_1, & D_{21} &= C_2^* D C_1, & D_{22} &= 0. \end{aligned}$$

The additional assumption $E \ker Q = \ker Q^* = \{0\}$ implies that the second block row and columns in (3.12) vanish implying that the pH pencil is then equivalent to a positive real-pencil.

Let now a dH-LG pencil $sE - A$ be given with $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ for some skew-adjoint $\mathcal{D}$ and self-adjoint $\mathcal{L}$. Here the important assumption $\operatorname{mul} \mathcal{D} \cap \ker \mathcal{L} = \{0\}$ to guarantee regularity is not trivially fulfilled, but it is not hard to see that $\operatorname{mul} \mathcal{D} \cap \ker \mathcal{L} \neq \{0\}$ is equivalent to the existence of a non-vanishing function $x(\cdot)$ such that

$$\forall t \geq 0: \quad (0, \tfrac{\mathrm{d}}{\mathrm{d}t} x(t)) \in \mathcal{D}, \quad (x(t), 0) \in \mathcal{L},\ e(t) = 0. \tag{3.20}$$

With this assumption we consider the range representations $\mathcal{D} = \operatorname{ran} \left[\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right]$ and $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and matrices $C_1, C_2, C_3, C_4 \in \mathbb{K}^{n \times d_i}$ whose columns are orthonormal bases of $\operatorname{ran} D_1 \cap \operatorname{ran} L_2$, $\ker D_1^* \widehat{+} \ker L_2^*$, $D_2 \ker D_1$, and $L_1 \ker L_2$. Then the lower-triangular form (3.12) is given by

$$L_{11} = C_1^* L_1 C_1, \qquad L_{21} = C_2^* L_1 C_1, \qquad L_{22} = C_4 C_4^*, \tag{3.21}$$

$$D_{11} = C_1^* D_2 C_1, \qquad D_{21} = C_2^* D_2 C_1, \qquad D_{22} = C_3 C_3^*. \tag{3.22}$$

Furthermore, the maximality of $\mathcal{D}$ and $\mathcal{L}$ implies $\ker D_1^* = \operatorname{ran} D_1^\perp = \operatorname{mul} \mathcal{D} = D_2 \ker D_1$ and $\ker L_2^* = L_1 \ker L_2$. Hence the regularity of the pencil $sL_{22} - D_{22}$ is by Prop. 3.1.3 automatically fulfilled.
The following example highlights that regularity can fail arbitrarily badly in terms of size of the row and column minimal indices even if we consider a pencil $sE - DQ$ that is both dH-LA and dH-LG.

Example 3.1.7. Let $k \in \mathbb{N}$, $Q = I_{2k+1}$ and let $E, D \in \mathbb{K}^{2(k+1)\times 2(k+1)}$ be defined by

$$sE - DQ = sE - D = \begin{bmatrix} 0 & sK_{k+1}^\top - L_{k+1}^\top \\ sK_{k+1} + L_{k+1} & 0 \end{bmatrix} \in \mathbb{K}[s]^{(2k+1)\times(2k+1)}.$$

Then $sE - DQ$ is readily singular and we immediately see that $\mathcal{D} = \operatorname{gr} D$ defines a skew-adjoint and $\mathcal{L} = (\operatorname{gr} E)^{-1}$ a self-adjoint subspace. In particular, the pencil is both dH-LA and dH-LG, and has one row and one column minimal index which are both equal to k.

3.2 Kronecker form of dissipative-Hamiltonian pencils

Our goal in this section is to describe the Kronecker form of dissipative-Hamiltonian pencils. We first stick to our previous restriction of square pencils. As Ex. 3.1.7 suggests, even with this restriction this task fails without the right assumptions. From the lower triangular form (3.12), we derive some structural properties of regular dH-LR pencils $sE - A$. Besides an index analysis, we will further present some results on the location of the eigenvalues of $sE - A$. We show that $sE - A$ does not have eigenvalues with positive real part and, except for a possible eigenvalue at the origin of higher order, and the purely imaginary eigenvalues are proven to be semi-simple. An overview of the results of this section can be found in Fig. 3.1.

Proposition 3.2.1. *Let $E, A \in \mathbb{K}^{n\times n}$ such that* $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ *for some dissipative relation $\mathcal{D} \subset \mathbb{K}^{2n}$ and a nonnegative relation $\mathcal{L} \subset \mathbb{K}^{2n}$. If $sE - A$ is regular, then the following holds:*

(a) $\sigma(E,A) \subseteq \overline{\mathbb{C}_-}$ *and the non-zero eigenvalues on the imaginary axis are semi-simple. The size of the Jordan blocks at* 0 *is at most two.*

(b) *The size of the Jordan blocks at* ∞*, i.e., the index, is at most three.*

(c) *If additionally* $\mathcal{D}$ *is maximally dissipative and* $\mathcal{L} = (\mathrm{gr}\ L)^{-1}$ *for some positive definite* $L \in \mathbb{K}^{n\times n}$*, then* $sE - A$ *has index at most one and the eigenvalue zero is semi-simple.*

Proof. Since $sE-A$ is regular, Prop. 3.1.3 yields that there exist invertible $S, T \in \mathbb{K}^{n\times n}$, such that

$$S(sE-A)T = \begin{bmatrix} sL_{11} - D_{11} & 0 \\ sL_{21} - D_{21} & sL_{22} - D_{22} \end{bmatrix} \in \mathbb{K}[s]^{n\times n} \tag{3.23}$$

with $L_{ij}, D_{ij} \in \mathbb{K}^{n_i\times n_j}$ for some $n_1, n_2 \in \mathbb{N}$ with $n_1+n_2 = n$ and, using Lem. 3.1.1 (i), we have

$$L_{11} = L_{11}^* \geq 0, \quad D_{11} + D_{11}^* \leq 0, \quad L_{22} = L_{22}^2 = L_{22}^*, \quad -D_{22} = D_{22}^2 = -D_{22}^*, \tag{3.24}$$

and $\mathrm{ran}\ L_{22} \cap \mathrm{ran}\ D_{22} = \{0\}$. It follows from [RS11, Thm. 4.1] that

$$\sigma(L_{22}, D_{22}) \subseteq \{0\}, \tag{3.25}$$

and, moreover, the possible eigenvalue zero is semi-simple and the index of $sL_{22} - D_{22}$ is at most one.

Further, since $L_{11} \geq 0$ and $D_{11} + D_{11}^* \leq 0$ implies that $sL_{11} - D_{11}$ is positive real, we have by Lem. 1.2.10, (3.25) and (3.23) that

$$\sigma(E,A) = \sigma(L_{11}, D_{11}) \cup \sigma(L_{22}, D_{22}) \subseteq \overline{\mathbb{C}_-}.$$

Next we prove (a): As we have already shown that each eigenvalue of $sE-A$ has a nonpositive real part, it remains to prove the statements on the sizes of the Jordan blocks of $sE-A$ at $\lambda \in \sigma(E,A) \cap \imath\mathbb{R}$. Let $\lambda \in \sigma(E,A) \cap \imath\mathbb{R}$. By Lem. 1.2.8 we have to show that the order of λ as a pole of $(sE-A)^{-1}$ is equal to one, if $\lambda \neq 0$, and at most two if $\lambda = 0$. We have from (3.23) that

$$(sE-A)^{-1} = T \begin{bmatrix} sL_{11} - D_{11} & 0 \\ sL_{21} - D_{21} & sL_{22} - D_{22} \end{bmatrix}^{-1} S$$

$$=T\begin{bmatrix} (sL_{11}-D_{11})^{-1} & 0 \\ -(sL_{22}-D_{22})^{-1}(sL_{21}-D_{21})(sL_{11}-D_{11})^{-1} & (sL_{22}-D_{22})^{-1} \end{bmatrix} S, \tag{3.26}$$

implying that the order of λ as a pole of $(sE-A)^{-1}$ is equal to the maximal order of λ as a pole of the block entries

$$(sL_{ii}-D_{ii})^{-1},\ i=1,2, \quad \text{and} \quad (sL_{22}-D_{22})^{-1}(sL_{21}-D_{21})(sL_{11}-D_{11})^{-1}. \tag{3.27}$$

Since $sL_{11}-D_{11}$ is positive real, the order of λ as a pole of $(sL_{11}-D_{11})^{-1}$ is at most one by Lem. 1.2.10. Moreover, by (3.25), the only possible pole of $(sL_{22}-D_{22})^{-1}$ might be at $\lambda=0$ and this pole is of order one. In summary, this shows that the pole order of (3.27) and thus of (3.26) at $\lambda=0$ is at most two and the pole order of (3.26) at $\lambda\in \imath\mathbb{R}\setminus\{0\}$ is at most one. This completes the proof of (a).
We prove (b). Since $sL_{11}-D_{11}$ is positive real, its index is at most two and hence, by Lem. 1.2.8 there exist some $M_1,\omega_1>0$ such that

$$\forall\,\lambda>\omega_1: \quad \|(\lambda L_{11}-D_{11})^{-1}\|\leq M_1\lambda. \tag{3.28}$$

As we have previously shown, the index of $sL_{22}-D_{22}$ is at most one, i.e., there exist some $M_2,\omega_2>0$ such that

$$\forall\,\lambda>\omega_2: \quad \|(\lambda L_{22}-D_{22})^{-1}\|\leq M_2. \tag{3.29}$$

A combination of (3.28) and (3.29) yields for all $\lambda>\max\{\omega_1,\omega_2\}$

$$\begin{aligned} &\|(\lambda L_{22}-D_{22})^{-1}(\lambda L_{21}-D_{21})(\lambda L_{11}-D_{11})^{-1}\| \\ \leq{}& \|(\lambda L_{22}-D_{22})^{-1}\|\|(\lambda L_{21}-D_{21})\|\|(\lambda L_{11}-D_{11})^{-1}\| \\ \leq{}& M_1M_2(\|L_{21}\|+\|D_{21}\|)\lambda^2. \end{aligned} \tag{3.30}$$

Let $M:=\|S^{-1}\|\|T^{-1}\|M_1M_2(\|L_{21}\|+\|D_{21}\|)|$ and $\omega:=\max\{\omega_1,\omega_2\}$, then (3.30) implies with (3.26) that

$$\forall\,\lambda>\omega: \quad \|(\lambda E-A)^{-1}\|\leq M\lambda^{k-1}, \tag{3.31}$$

with $k=3$ and thus, by Lem. 1.2.8, the index of $sE-A$ is at most three.
It remains to prove (c). To this end, assume that $\mathcal{D}$ is maximally dissipative and that $\mathcal{L}=(\mathrm{gr}\ L)^{-1}$ for some positive definite $L\in\mathbb{K}^{n\times n}$. To show that $sE-A$ has at most

index one, we have to verify (3.31) with $k = 1$. Since L is positive definite, Lem. 3.1.1 (i) & (v) gives $L_{11} \geq 0$ and $\ker L_{11} = \{0\}$. That is, L_{11} is positive definite as well. Hence, we can use [RS11, Thm. 4.1] to infer that there exists some $M_3 > 0$ with

$$\forall\, \lambda > 0: \quad \|(\lambda L_{11} - D_{11})^{-1}\| \leq \frac{M_3}{\lambda}. \tag{3.32}$$

Using (3.32), there exists some $M_4 := M_2 M_3(\|L_{21}\| + \|D_{21}\|)$ and $\omega_4 := \max\{0, \omega_3, \omega_2\}$ such that for all $\lambda > \omega_4$ it holds

$$\begin{aligned} &\|(\lambda L_{22} - D_{22})^{-1}(\lambda L_{21} - D_{21})(\lambda L_{11} - D_{11})^{-1}\| \\ &\leq \|(\lambda L_{22} - D_{22})^{-1}\| \|(\lambda L_{21} - D_{21})\| \|(\lambda L_{11} - D_{11})^{-1}\| \\ &\leq M_2 M_3(\|L_{21}\| + \|D_{21}\|) \\ &= M_4. \end{aligned}$$

Thus, by Lem. 1.2.8, $sE - A$ has index at most one. To conclude that zero is a semi-simple eigenvalue, recall from Lem. 3.1.1 (v) that $D_{22} = -I_{n_2}$, $L_{22} = 0$. Consequently, the pole order of (3.27) and whence of (3.26) at $\lambda = 0$ is at most one. As a result of Lem. 1.2.8, the eigenvalue $\lambda = 0$ is semi-simple. $\square$

Note that Prop. 3.2.1 (c) was conjected in [vdSM18, Rem. 2.6] for skew-adjoint $\mathcal{D}$. The following example shows that without maximality assumptions on the subspaces $\mathcal{D}$ and $\mathcal{L}$ an index of $sE - A$ equal to three is possible.

Example 3.2.2. Using the canonical unit vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ we consider the relations

$$\mathcal{D} = \operatorname{ran} \begin{bmatrix} E_D \\ A_D \end{bmatrix} = \operatorname{ran} \begin{bmatrix} e_1 & e_2 & 0 \\ -e_2 & e_1 & e_3 \end{bmatrix}, \quad \mathcal{L} = \operatorname{ran} \begin{bmatrix} E_L \\ A_L \end{bmatrix} = \operatorname{ran} \begin{bmatrix} e_1 & e_3 \\ e_1 & e_2 \end{bmatrix}.$$

Since

$$0 = A_D^* E_D + E_D^* A_D \leq 0, \quad A_L^* E_L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

we have that $\mathcal{D}$ is dissipative, and $\mathcal{L}$ is nonnegative. It can be further seen that the product of $\mathcal{D}$ and $\mathcal{L}$ reads

$$\mathcal{D}\mathcal{L} = \operatorname{span}\{(0, e_3), (e_3, e_1), (e_1, -e_2)\},$$

and we obtain the range representation $\mathcal{D}\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right]$ with

$$E := \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right], \quad A := \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{smallmatrix}\right].$$

Since $A^{-1}E$ is nilpotent with $(A^{-1}E)^2 \neq 0$, we have that the Kronecker form of $sE - A$ is consisting of exactly one Jordan block at ∞ with size 3. In particular, the index of $sE - A$ is equal to three.

Next we show that for a square the pencil $sE-A$ induced by $\mathcal{DL}$ is already regular with index one under the assumption of Prop. 3.2.1 (c). This result was previously obtained in [vdSch13, Prop. 4.1] for the special case where $\mathcal{D}$ is a skew-adjoint subspace.

Corollary 3.2.3. *Let $sE-A$ be a matrix pencil with $E, A \in \mathbb{K}^{n\times n}$ and* $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{DL}$ *and let $\mathcal{D} \subseteq \mathbb{K}^{2n}$ be maximally dissipative and $\mathcal{L} = (\mathrm{gr}\ Q)^{-1}$ for some positive definite $Q \in \mathbb{K}^{n\times n}$. Then $sE - A$ is regular and has index at most one.*

Proof. Since $\mathcal{L} = (\mathrm{gr}\ Q)^{-1} = \mathrm{gr}\,(Q^{-1})$ we have $\mathrm{mul}\ \mathcal{D} \cap \ker \mathcal{L} = \mathrm{mul}\ \mathcal{D} \cap \{0\} = \{0\}$ and by Lem. 3.1.1 (v) there exist unitary $U, X \in \mathbb{K}^{n\times n}$ such that

$$U^*(sE - A)X = \left[\begin{smallmatrix} sL_{11}-D_{11} & 0 \\ sL_{21} & I_n \end{smallmatrix}\right], \tag{3.33}$$

with $sL_{11} - D_{11}$ positive real and $\ker L_{11} \times \{0\} = U^* \{x \in \mathrm{dom}\ \mathcal{D} \mid Qx \in \mathrm{mul}\ \mathcal{D}\}$. Hence, if $x \in \ker L_{11} \times \{0\}$, then $x \in \mathrm{dom}\ \mathcal{D}$ with $Qx \in \mathrm{mul}\ \mathcal{D}$. In virtue of Lem. 1.3.8, we have $\mathrm{mul}\ \mathcal{D} = (\mathrm{dom}\ \mathcal{D})^{\perp}$ and hence $\langle Qx, x\rangle = 0$, and the positive definiteness of Q leads to $x = 0$. Consequently, the kernel of L_{11} is trivial, and we obtain $\ker L_{11} \cap \ker D_{11} = \{0\} \cap \ker D_{11} = \{0\}$. Now invoking Lem. 1.2.10 (a), we obtain that $sL_{11} - D_{11}$ is regular and thus, by (3.33), $sE - A$ is regular, too. Moreover, the index is at most one by Prop. 3.2.1 (c). $\square$

We have seen in Ex. 3.1.7 that square dH-LR pencils may have arbitrarily large row and column indices. Moreover, the following two examples show that the index and the size of the Jordan blocks on the imaginary axis may be arbitrarily large as well. Note that these examples present dH-LR pencils which are both dH-LA and dH-LG, i.e., they fall within the scope of both [MMW18] and [vdSM18].

Example 3.2.4. For $k \in \mathbb{N}$, consider the pencil

$$sL - D = \begin{bmatrix} & & & & & -1 \\ & & & & \iddots & s \\ & & & -1 & \iddots & \\ & & 1 & s & & \\ & \iddots & \iddots & & & \\ 1 & s & & & & \end{bmatrix} \in \mathbb{K}[s]^{2k\times 2k}.$$

Then $L \in \mathbb{K}^{2k\times 2k}$ is Hermitian and $D \in \mathbb{K}^{2k\times 2k}$ is skew-Hermitian. Hence, the relation $\mathcal{D} = \mathrm{gr}\ D$ is skew-adjoint (in particular dissipative), and $\mathcal{L} = (\mathrm{gr}\ L)^{-1}$ is self-adjoint. Then for $E = L$ and $A = D$, $\mathcal{DL} = [\begin{smallmatrix} E \\ A \end{smallmatrix}]$. It can be seen that $E^{-1}A$ is nilpotent with $(E^{-1}A)^{2k-1} \neq 0$. Consequently, the Kronecker form (1.1) of $sE - A$ is consisting of exactly one Jordan block at the eigenvalue ∞ with size $2k$. Therefore, the index of $sE - A$ reads $2k$.

Example 3.2.5. For $k \in \mathbb{N}$, consider the pencil

$$sL - D = \begin{bmatrix} & & & & s \\ & & & \iddots & -1 \\ & & s & -1 & \iddots \\ & \iddots & 1 & & \\ s & 1 & \iddots & & \end{bmatrix} \in \mathbb{K}[s]^{(2k+1)\times(2k+1)},$$

which is consisting of the Hermitian matrix $L \in \mathbb{K}^{(2k+1)\times(2k+1)}$ and the skew-Hermitian matrix $D \in \mathbb{K}^{(2k+1)\times(2k+1)}$. As in the previous example, the choices $\mathcal{D} = \operatorname{gr} D$, $\mathcal{L} = (\operatorname{gr} L)^{-1}$ lead to the pH pencil $sE - A := sL - D$. It can be seen that $A^{-1}E$ is nilpotent with $(A^{-1}E)^{2k} \neq 0$. Consequently, the Kronecker form (1.1) of $sE - A$ is consisting of exactly one Jordan block at the eigenvalue 0 with size $2k + 1$.

The preceding examples show that additional assumptions on $\mathcal{D}$ and $\mathcal{L}$ are required for a further specification of the Kronecker form of pH pencils. We first present a decompostion which will be advantageous to prove such statements and turn our attention to possibly singular square pencils which do not need to be square anymore.

Proposition 3.2.6. *Let $\mathcal{D} \subseteq \mathbb{K}^{2n}$ be maximally dissipative and $\mathcal{L} \subseteq \mathbb{K}^{2n}$ be maximally nonnegative. Further, let $E, A \in \mathbb{K}^{n\times m}$ be such that* $\operatorname{ran} [\begin{smallmatrix} E \\ A \end{smallmatrix}] = \mathcal{DL}$. *Then there exist some invertible $S \in \mathbb{K}^{n\times n}$, $T \in \mathbb{K}^{m\times m}$ and $n_i \in \mathbb{N}, i = 1, 2, 3, 4$, such that*

$$S(sE - A)T = \begin{bmatrix} sL_{11} - D_{11} & 0 & 0 & 0 & 0 & 0 \\ D_{21} & sI_{n_2} & 0 & 0 & 0 & 0 \\ sL_{21} & 0 & I_{n_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & sI_{n_4} & -I_{n_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.34}$$

where $sL_{11} - D_{11} \in \mathbb{K}[s]^{n_1\times n_1}$ is regular and positive real, and $\ker L_{11} \subset \ker L_{21}$.

Proof. The proof consists of two steps. In the first step we derive a certain range representation for $\mathcal{DL}$. In second step, (3.34) is obtained from the resulting range representation.

<u>*Step 1:*</u> We show that there exists some $\hat{m} \in \mathbb{N}$ and an invertible matrix $S \in \mathbb{K}^{n \times n}$ and some $n_1, n_2, n_3, n_4 \in \mathbb{N}$, such that

$$\mathcal{DL} = \operatorname{diag}(S, S) \ \operatorname{ran} \left[\begin{smallmatrix} L \\ D \end{smallmatrix}\right], \quad sL - D = \begin{bmatrix} sL_{11}-D_{11} & 0 & 0 & 0 & 0 & 0 \\ D_{21} & sI_{n_2} & 0 & 0 & 0 & 0 \\ sL_{21} & 0 & I_{n_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & sI_{n_4} & -I_{n_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{K}[s]^{n \times \hat{m}} \tag{3.35}$$

for some positive real and regular pencil $sL_{11} - D_{11} \in \mathbb{K}[s]^{n_1 \times n_1}$, $D_{21} \in \mathbb{K}^{n_2 \times n_1}$, $L_{21} \in \mathbb{K}^{n_3 \times n_1}$.
Consider the space $X := \operatorname{mul} \mathcal{D} \cap \ker \mathcal{L}$, and the relations

$$\hat{\mathcal{D}} := \mathcal{D} \widehat{\ominus} (\{0\} \times X), \quad \hat{\mathcal{L}} := \mathcal{L} \widehat{\ominus} (X \times \{0\}).$$

Then we obtain an orthogonal decomposition

$$\mathcal{DL} = \hat{\mathcal{D}}\hat{\mathcal{L}} \oplus (\{0\} \times X) \oplus (X \times \{0\}) \tag{3.36}$$

and

$$\operatorname{mul} \hat{\mathcal{D}} = \operatorname{mul} \mathcal{D} \widehat{\ominus} X, \quad \ker \hat{\mathcal{L}} = \ker \mathcal{L} \widehat{\ominus} X.$$

This implies $\operatorname{mul} \hat{\mathcal{D}} \cap \ker \hat{\mathcal{L}} = \{0\}$. It can be further seen that $\hat{\mathcal{D}}$ is dissipative and $\hat{\mathcal{L}}$ is nonnegative. Further, define

$$\mathcal{V} := \mathbb{K}^{2n} \widehat{\ominus} (\{0\} \times X) \widehat{\ominus} (X \times \{0\}) = (\{0\} \times X)^{\perp} \cap (X \times \{0\})^{\perp}.$$

The previous considerations show that both $\hat{\mathcal{D}}$ and $\hat{\mathcal{L}}$ are subsets of $\mathcal{V}$. Moreover, set $k_X := \dim X$ and let $\iota : \mathcal{V} \to \mathbb{K}^{2(n-k_X)} = \mathbb{K}^{\dim \mathcal{V}}$ be a vector space isometry. It follows that

$$\tilde{\mathcal{D}} := \iota(\hat{\mathcal{D}}), \quad \tilde{\mathcal{L}} := \iota(\hat{\mathcal{L}}) \tag{3.37}$$

are maximally dissipative and maximally nonnegative linear relations in $\mathbb{K}^{2(n-k_X)}$, respectively, satisfying $\operatorname{mul} \tilde{\mathcal{D}} \cap \ker \tilde{\mathcal{L}} = \{0\}$ and note that

$$\tilde{\mathcal{D}}\tilde{\mathcal{L}} = \iota(\hat{\mathcal{D}}\hat{\mathcal{L}}). \tag{3.38}$$

Then Lem. 3.1.1 implies the existence of some unitary $\tilde{U} \in \mathbb{K}^{(n-k_X) \times (n-k_X)}$, such that, with $k_1 := \dim(\operatorname{ran} \tilde{\mathcal{L}} \cap \operatorname{dom} \tilde{\mathcal{D}})$, $k_2 := n - k_X - k_1$,

$$\tilde{\mathcal{D}}\tilde{\mathcal{L}} = \operatorname{ran} \operatorname{diag}(\tilde{U}, \tilde{U}) \begin{bmatrix} \tilde{L}_{11} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} \\ \tilde{D}_{11} & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \tag{3.39}$$

for some matrices $\tilde{L}_{ij}, \tilde{D}_{ij} \in \mathbb{K}^{k_i \times k_j}$ with $\tilde{L}_{11} \geq 0$, $\ker \tilde{L}_{11} \subseteq \ker \tilde{L}_{21}$, $\tilde{D}_{11} + \tilde{D}_{11}^* \leq 0$ and

$$\tilde{D}_{22}^2 = -\tilde{D}_{22} = -\tilde{D}_{22}^*, \quad \tilde{L}_{22}^2 = \tilde{L}_{22} = \tilde{L}_{22}^*, \quad \ker \tilde{D}_{22} \widehat{\oplus} \ker \tilde{L}_{22} = \mathbb{K}^{k_2}. \tag{3.40}$$

Invoking (3.36)–(3.39) and

$$\mathcal{V} \ominus (\{0\} \times X) \ominus (X \times \{0\}) \cong \mathbb{K}^{2(n-k_X)} \times \mathbb{K}^{k_X} \times \mathbb{K}^{k_X}$$

yields the existence of a unitary matrix $\hat{U} \in \mathbb{K}^{n \times n}$ such that

$$\mathcal{DL} = \operatorname{diag}(\hat{U}, \hat{U}) \ \operatorname{ran} \begin{bmatrix} \tilde{L}_{11} & 0 & 0 & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} & 0 & 0 \\ 0 & 0 & I_{k_X} & 0 \\ \tilde{D}_{11} & 0 & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} \end{bmatrix}.$$

Lem. 1.2.10 (b) implies that $s\tilde{L}_{11} - \tilde{D}_{11}$ has only column and row minimal indices equal to zero and their number coincides. Hence there exist invertible $S_1, T_1 \in \mathbb{K}^{k_1 \times k_1}$ and some $n_1 \in \mathbb{N}$, such that

$$S_1(s\tilde{L}_{11} - \tilde{D}_{11})T_1 = \begin{bmatrix} sL_{11} - D_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

for some positive real and regular pencil $sL_{11} - D_{11} \in \mathbb{K}[s]^{n_1 \times n_1}$. Since $\tilde{\mathcal{L}}$ is maximally nonnegative, Lem. 3.1.1 (i) yields

$$\ker L_{11} \times \mathbb{K}^{k_1 - n_1} = \ker \left[\begin{smallmatrix} L_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] = \ker \tilde{L}_{11} T_1 = T_1^{-1} \ker \tilde{L}_{11} \subset T_1^{-1} \ker \tilde{L}_{21} = \ker \tilde{L}_{21} T_1.$$

Consequently, for some $L_{21}^{(1)} \in \mathbb{K}^{k_2 \times n_1}$

$$\tilde{L}_{21} T_1 = \left[L_{21}^{(1)}, 0_{k_2 \times (k_1 - n_1)} \right] \quad \text{and} \quad \ker L_{11} \subseteq \ker L_{21}^{(1)}.$$

Further, by using $[D_{21}^{(1)}, D_{21}^{(2)}] := \tilde{D}_{21} T_1$, $D_{21}^{(1)} \in \mathbb{K}^{k_2 \times n_1}$, $D_{21}^{(2)} \in \mathbb{K}^{k_2 \times (k_1 - n_1)}$, we find

$$\begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & I_{k_2 + k_X} & 0 & 0 \\ 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & I_{k_2 + k_X} \end{bmatrix} \operatorname{ran} \begin{bmatrix} \tilde{L}_{11} & 0 & 0 & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} & 0 & 0 \\ 0 & 0 & I_{k_X} & 0 \\ \tilde{D}_{11} & 0 & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} \end{bmatrix}$$

$$= \operatorname{ran} \begin{bmatrix} S_1 \tilde{L}_{11} T_1 & 0 & 0 & 0 \\ \tilde{L}_{21} T_1 & \tilde{L}_{22} & 0 & 0 \\ 0 & 0 & I_{k_X} & 0 \\ S_1 \tilde{D}_{11} T_1 & 0 & 0 & 0 \\ \tilde{D}_{21} T_1 & \tilde{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ L_{21}^{(1)} & 0 & \tilde{L}_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} & 0 \\ D_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1)} & D_{21}^{(2)} & \tilde{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix}.$$

Denoting $k_3 := \dim\ker\tilde{D}_{22}$, $n_3 := \dim\ker\tilde{L}_{22}$, (3.40) implies that $k_2 = k_3 + n_3$. Let $\tilde{S} \in \mathbb{K}^{k_2\times k_2}$ be a matrix whose first k_3 columns form a basis of $\tilde{D}_{22}$ and whose last n_3 columns form a basis of $\tilde{L}_{22}$. Then $\tilde{S}^*(s\tilde{L}_{22} - \tilde{D}_{22})\tilde{S} = \operatorname{diag}(s\hat{L}_{22}, \hat{D}_{22})$ for some $\hat{L}_{22} \in \mathbb{K}^{k_3\times k_3}$, $\hat{D}_{22} \in \mathbb{K}^{n_3\times n_3}$, which are positive definite by (3.40). Then, by taking a suitable block congruence transformation, we obtain that there exists some invertible $S_2 \in \mathbb{K}^{k_2\times k_2}$ such that the Weierstraß form is given by

$$S_2(s\tilde{L}_{22} - \tilde{D}_{22})S_2^* = \begin{bmatrix} sI_{k_3} & 0 \\ 0 & I_{n_3} \end{bmatrix}.$$

Hence, with $\left[\begin{smallmatrix} L_{21}^{(1,1)} \\ L_{21} \end{smallmatrix}\right] := S_2 L_{21}^{(1)}$ for some $L_{21}^{(1,1)} \in \mathbb{K}^{n_3\times n_1}$ and $L_{21} \in \mathbb{K}^{n_3\times n_1}$ which implies

$$\ker L_{11} \subseteq \ker L_{21}^{(1)} = \ker S_2 L_{21}^{(1)} \subseteq \ker L_{21}.$$

Further, decomposing

$$[S_2 D_{21}^{(1)}, S_2 D_{21}^{(2)}] = \left[\begin{smallmatrix} D_{21}^{(1,1)} & D_{21}^{(2,1)} \\ D_{21}^{(1,2)} & D_{21}^{(2,2)} \end{smallmatrix}\right] \in \mathbb{K}^{(k_3+n_3)\times(n_1+(k_1-n_1))}$$

leads to

$$\begin{bmatrix} I_{k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_X} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{k_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ L_{21}^{(1)} & 0 & \tilde{L}_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} & 0 \\ D_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1)} & D_{21}^{(2)} & \tilde{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix}$$

$$= \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ S_2 L_{21}^{(1)} & 0 & S_2\tilde{L}_{22}T_2 & 0 & 0 \\ 0 & 0 & 0 & I_{k_X} & 0 \\ D_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ S_2 D_{21}^{(1)} & S_2 D_{21}^{(2)} & S_2\tilde{D}_{22}T_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 & 0 & 0 \\ L_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} & 0 \\ D_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1,1)} & D_{21}^{(2,1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix}. \tag{3.41}$$

Now let $S_3 \in \mathbb{K}^{k_3\times k_3}$, $T_3 \in \mathbb{K}^{(k_1-n_1)\times(k_1-n_1)}$ be invertible with $S_3 D_{21}^{(2,1)} T_3 = \left[\begin{smallmatrix} I_{k_5} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then setting $n_2 := k_3 - k_5$ and using

$$\left[\begin{smallmatrix} D_{21}^{(1,1,1)} \\ -D_{21} \end{smallmatrix}\right] := S_3 D_{21}^{(1,1)}, \quad D_{21}^{(1,1,1)} \in \mathbb{K}^{k_5\times n_1},\ D_{21} \in \mathbb{K}^{n_2\times n_1},$$

we find for the lower five block rows in (3.41)

$$\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1-k_1} & 0 & 0 & 0 \\ 0 & 0 & S_3 & 0 & 0 \\ 0 & 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} \operatorname{ran} \begin{bmatrix} D_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1,1)} & D_{21}^{(2,1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix}$$
$$= \operatorname{ran} \begin{bmatrix} D_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1,1,1)} & I_{k_5} & 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1,1,2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{n_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} D_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{k_5} & 0 & 0 & 0 & 0 & 0 & 0 \\ -D_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{n_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} \tag{3.42}$$

and for the upper five block rows in (3.41)

$$\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1-k_1} & 0 & 0 & 0 \\ 0 & 0 & S_3 & 0 & 0 \\ 0 & 0 & 0 & I_{k_4} & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} \end{bmatrix} \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 & 0 & 0 \\ L_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} & 0 \end{bmatrix} \tag{3.43}$$
$$= \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 & 0 & 0 \\ L_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_X} & 0 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{k_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ L_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{k_X} & 0 \end{bmatrix}.$$

Then the form (3.35) is achieved by setting $n_4 := k_5 + k_X$, $\hat{m} := n + 2k_X$, and performing a joint permutation of block rows of the form $2 \to 6 \to 5 \to 3 \to 4 \to 2$ and block columns $(3 \to 8 \to 7 \to 5 \to 2 \to 6 \to 3)$ of the matrices on the right-hand side in (3.42) and (3.43). Combining all of the so far transformations leads to an invertible $S \in \mathbb{K}^{n\times n}$ with (3.35).

<u>*Step 2:*</u> Let $E, A \in \mathbb{K}^{n\times m}$ be such that $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{DL}$ for some maximally dissipative relation $\mathcal{D} \subseteq \mathbb{K}^{2n}$ and some maximally nonnegative relation $\mathcal{L} \subseteq \mathbb{K}^{2n}$. Then the result from Step 1 gives

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{DL} = \operatorname{diag}(S^{-1}, S^{-1}) \operatorname{ran} \left[\begin{smallmatrix} L \\ D \end{smallmatrix}\right] \tag{3.44}$$

with matrices $L, D \in \mathbb{K}^{n\times \hat{m}}$ as in (3.35). If $m \geq \hat{m}$ then there exists some invertible $T \in \mathbb{K}^{m\times m}$ such that

$$\left[\begin{smallmatrix} SE \\ SA \end{smallmatrix}\right] T = \left[\begin{smallmatrix} L & 0 \\ D & 0 \end{smallmatrix}\right].$$

Hence (3.34) follows from (3.35). If $m < \hat{m}$ then the block structure in (3.35) implies that $d := \dim \mathcal{DL} = \dim \operatorname{ran} \left[\begin{smallmatrix} L \\ D \end{smallmatrix}\right] = n_1 + n_2 + n_3 + 2n_4$ and that the first d columns in $\left[\begin{smallmatrix} L \\ D \end{smallmatrix}\right]$ are linearly independent. Since $m \geq d$, we can remove $\hat{m} - m$ zero columns from L and D which leads to matrices $\hat{L}, \hat{D} \in \mathbb{K}^{n\times m}$ which are still of the form (3.35). Observe that (3.44) still holds after replacing L with $\hat{L}$ and D with $\hat{D}$. Hence, there exists some invertible $T \in \mathbb{K}^{m\times m}$ such that $S(sE - A)T = s\hat{L} - \hat{D}$ which implies (3.34). ☐

The main result on the Kronecker form of dissipative-Hamiltonian DAEs is given below. Here, we additionally assume the maximality of the underlying subspaces. This can be viewed as a refinement or extension of the lower-triangular form which is given for regular pencils in Prop. 3.1.3. The main difference is that the invertible left transformation can no longer be chosen to be unitary.

Theorem 3.2.7. *Let $E, A \in \mathbb{K}^{n\times m}$ such that* $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ *for some maximally dissipative relation $\mathcal{D} \subseteq \mathbb{K}^{2n}$ and a maximally nonnegative relation $\mathcal{L} \subseteq \mathbb{K}^{2n}$. Then the Kronecker form of $sE - A$ has the following properties:*

(a) *The column minimal indices are at most one (if there are any).*

(b) *The row minimal indices are zero (if there are any).*

(c) *We have $\sigma(E, A) \subseteq \overline{\mathbb{C}_-}$. Furthermore, the non-zero eigenvalues on the imaginary axis are semi-simple. The Jordan blocks at ∞ and at zero have size at most two, i.e., the index is at most two.*

Proof. Invoking Prop. 3.2.6, we assume without loss of generality that $sE - A$ is in the block diagonal decomposition (3.34) with positive real $s\tilde{L}_{11} - \tilde{D}_{11} \in \mathbb{K}[s]^{n_1\times n_1}$ and $\ker \tilde{L}_{11} \subset \ker \tilde{L}_{21}$. First observe that the block lower-triangular pencil

$$sE_r - A_r := \begin{bmatrix} s\tilde{L}_{11} - \tilde{D}_{11} & 0 & 0 \\ \tilde{D}_{21} & sI_{n_2} & 0 \\ s\tilde{L}_{21} & 0 & I_{n_3} \end{bmatrix} \tag{3.45}$$

obtained from (3.34) is regular. Since, moreover, a simple column permutation yields that the Kronecker form of $[sI_{n_4}, -I_{n_4}]$ is given by $\operatorname{diag}(sK_2 - L_2, \ldots, sK_2 - L_2) \in \mathbb{K}[s]^{n_4\times 2n_4}$, we obtain that the column minimal indices of $sE - A$ are one (if there are any) and the row minimal indices of $sE - A$ are at most zero (if there are any). This proves (a) & (b).

We continue with the proof of (c). Considering (3.34), (3.45) and invoking Lem. 1.2.10 (c) yields

$$\sigma(E, A) = \sigma(E_r, A_r) \subseteq \sigma(\tilde{L}_{11}, \tilde{D}_{11}) \cup \{0\} \subseteq \overline{\mathbb{C}_-}.$$

It remains to show the statements on the index and the sizes of the Jordan blocks to eigenvalues on the imaginary axis. Here we proceed as in the proof of Prop. 3.2.1 by using the resolvent of (3.45) which is given by

$$\begin{bmatrix} s\tilde{L}_{11}-\tilde{D}_{11} & 0 & 0 \\ \tilde{D}_{21} & sI_{n_2} & 0 \\ s\tilde{L}_{21} & 0 & I_{n_3} \end{bmatrix}^{-1} = \begin{bmatrix} (s\tilde{L}_{11}-\tilde{D}_{11})^{-1} & 0 & 0 \\ -s^{-1}\tilde{D}_{21}(s\tilde{L}_{11}-\tilde{D}_{11})^{-1} & s^{-1}I_{n_2} & 0 \\ -s\tilde{L}_{21}(s\tilde{L}_{11}-\tilde{D}_{11})^{-1} & 0 & I_{n_3} \end{bmatrix}. \quad (3.46)$$

Regarding Lem. 1.2.8, the pole order of (3.46) at $\lambda \in \sigma(E,A)$ is equal to the size of the largest Jordan block of (3.45) at λ. Since $s\tilde{L}_{11}-\tilde{D}_{11}$ is positive real, the pole order of (3.46) at the non-zero eigenvalues on the imaginary axis is at most one and hence these eigenvalues are semi-simple. The pole order of $(sE_r-A_r)^{-1}$ at $\lambda=0$ is at most two and hence the size of the Jordan blocks at 0 in the Kronecker form of $sE-A$ is at most two, by Lem. 1.2.8.

We finally show that the index of $sE-A$ as in (1.3) is at most two. Since the index is invariant under pencil equivalence of sE_r-A_r, we can assume without restriction that $s\tilde{L}_{11}-\tilde{D}_{11}$ is already given in Weierstraß canonical form. Further, $s\tilde{L}_{11}-\tilde{D}_{11}$ is positive real and hence its the index is by Lem. 1.2.10 (d) at most two. Altogether, we obtain for some $k_1,k_2\in\mathbb{N}$ and $\tilde{J}\in\mathbb{K}^{k_2\times k_2}$ in Jordan canonical form that

$$s\tilde{L}_{11}-\tilde{D}_{11} = \operatorname{diag}\left(\left[\begin{smallmatrix} -1 & s \\ 0 & -1 \end{smallmatrix}\right],\ldots,\left[\begin{smallmatrix} -1 & s \\ 0 & -1 \end{smallmatrix}\right], -I_{k_1}, sI_{k_2}-\tilde{J}\right). \quad (3.47)$$

Consequently, there exist $M_1,\omega_1>0$ such that

$$\forall \lambda>\omega_1: \quad \|(\lambda\tilde{L}_{11}-\tilde{D}_{11})^{-1}\| \le M_1\lambda. \quad (3.48)$$

Looking at the block entries of (3.46), we continue to show the existence of some $M_2,\omega_2>0$ satisfying

$$\forall \lambda>\omega_2: \quad \|\lambda\tilde{L}_{21}(\lambda\tilde{L}_{11}-\tilde{D}_{11})^{-1}\| \le M_2\lambda. \quad (3.49)$$

Invoking the block diagonality of $s\tilde{L}_{11}-\tilde{D}_{11}$ and the structure of the blocks in (3.47) it suffices to show that (3.49) holds for $s\tilde{L}_{11}-\tilde{D}_{11} = \left[\begin{smallmatrix} -1 & s \\ 0 & -1 \end{smallmatrix}\right]$. Prop. 3.2.6 yields $\ker\tilde{L}_{11}\subset\ker\tilde{L}_{21}$, which implies with $\ker\tilde{L}_{11}=\{\alpha e_1 \mid \alpha\in\mathbb{K}\}$ for $x=\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\in\mathbb{K}^2$ and for all $\lambda>0$ and $M_2 := \|\tilde{L}_{21}e_1\|$ that

$$\begin{aligned} \|\lambda\tilde{L}_{21}(\lambda\tilde{L}_{11}-\tilde{D}_{11})^{-1}x\| &= \left\|\lambda\tilde{L}_{21}\begin{bmatrix} -1 & -\lambda \\ 0 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| \\ &= \left\|\lambda\tilde{L}_{21}\begin{pmatrix} -x_1-\lambda x_2 \\ -x_2 \end{pmatrix}\right\| \end{aligned}$$

$$= \| - \lambda \tilde{L}_{21} e_2 x_2 \|$$
$$\leq M_2 \lambda \|x\|.$$

This proves (3.49). Further, one can directly conclude from (3.48) that there exist $M_3, \omega_3 > 0$ such that

$$\forall \lambda > \omega_3 : \quad \| - \lambda^{-1} \tilde{D}_{21} (\lambda \tilde{L}_{11} - \tilde{D}_{11})^{-1} \| \leq M_3 \lambda, \tag{3.50}$$

and trivially

$$\forall \lambda > 1 : \quad \|\lambda^{-1} I_{n_2}\|, \|I_{n_3}\| \leq 1. \tag{3.51}$$

Overall, we see with (3.46) and (3.48)-(3.51) that there exist some $M, \omega > 0$ with

$$\forall \lambda > \omega : \quad \|(\lambda E_r - A_r)^{-1}\| \leq M \lambda. \tag{3.52}$$

This means by Lem. 1.2.8 that $\alpha_i \leq 2$ for all $i = 1, \dots, \ell_\alpha$. Furthermore, the block structure in (3.34) implies $\gamma_i \leq 1$ for all $i = 1, \dots, \ell_\gamma$ and hence the index of $sE - A$ as in (1.3) is at most two. □

The following example from [MMW18] shows that without the maximality assumption on $\mathcal{L}$, arbitrarily large row minimal indices might occur.

Example 3.2.8. For some $n \in \mathbb{N}$, let $\mathcal{D} = \operatorname{gr} D$ with $D = J_n(0) - J_n(0)^*$ and $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ for $E = Q = [I_{n-1} \, , \, 0_{(n-1)\times 1}]^*$. Then $\mathcal{D}$ is skew-adjoint and $\mathcal{L}$ is nonnegative but not maximal. Hence for $A = DQ$, $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$, and it is shown in [MMW18] that the pencil $sE - A$ has one row minimal index equal to $n - 1$.

Previously, the Kronecker form was described in [MMW18] for dH-LA pencils where $\mathcal{L}$ is nonegative with the usual physical interpretation in terms of energy functionals. We give a brief comparison of Thm. 3.2.7 with [MMW18, Thm. 4.3].

Remark 3.2.9.

(i) As [MMW18, Thm. 4.3] treats dH-LA pencils, it employs the assumption that $\operatorname{mul} \mathcal{D} = \{0\}$.

(ii) [MMW18, Thm. 4.3] shows that dH-LA pencils have the property that all its eigenvalues have nonpositive real part. Further, the nonzero imaginary eigenvalues are semi-simple. A statement on the sizes of the Jordan blocks corresponding to the eigenvalue zero is not contained.

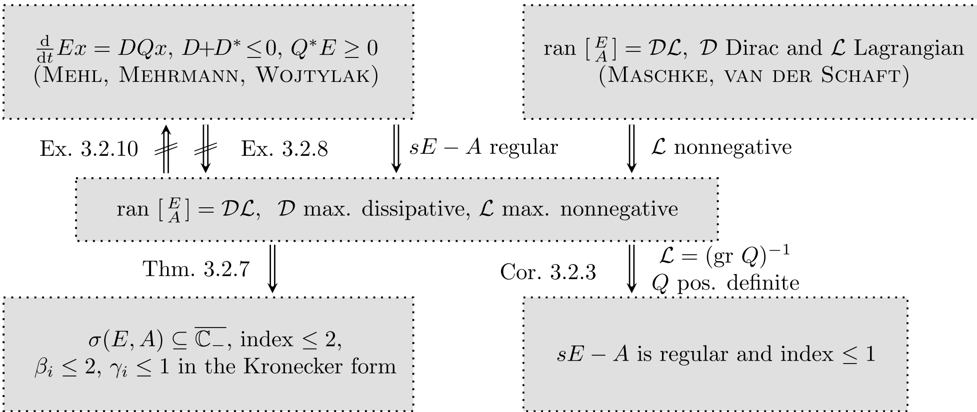

Figure 3.1: Properties of matrix pencils arising in port-Hamiltonian formulations.

(iii) Instead of our assumption of maximality of the nonnegative relation $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$, the weaker assumption that all row minimal indices of $sE - Q$ are zero has been used in [MMW18, Thm. 4.3] to describe the Kronecker form of dH-LA pencils.

We present an example of a pencil which is subject of Thm. 3.2.7 but it cannot be represented as a pencil which is subject of [MMW18, Thm. 4.3].

Example 3.2.10. Let $E = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and consider

$$\mathcal{D} = \operatorname{ran} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad \mathcal{L} = \left(\mathrm{gr}\ \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right)^{-1}.$$

Then $\mathcal{D}$ is maximally dissipative, $\mathcal{L}$ is maximally nonnegative, and $\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$. Therefore, the pencil $sE - A$ meets the assumptions of Thm. 3.2.7.
We show in the following that it is not possible to rewrite $\mathcal{D}\mathcal{L} = (\mathrm{gr}\ D)\hat{\mathcal{L}}$ for some dissipative matrix $D \in \mathbb{K}^{2\times 2}$ and a nonnegative relation $\hat{\mathcal{L}} \subset \mathbb{K}^4$. To this end, let $\hat{\mathcal{L}} = \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ \hat{Q} \end{smallmatrix}\right]$ with $\hat{Q}^*\hat{E} \geq 0$. Then

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = (\mathrm{gr}\ D)\ \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ \hat{Q} \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} \hat{E} \\ D\hat{Q} \end{smallmatrix}\right]$$

and hence there exists some invertible $T \in \mathbb{K}^{2\times 2}$ with $\hat{E}T = E$ and $D\hat{Q}T = A$. Thus, $D\hat{Q}T = -I_2$ and hence $\hat{Q}T = -D^{-1}$. With $\hat{Q}T = \left[\begin{smallmatrix} q_1 & q_2 \\ q_3 & q_4 \end{smallmatrix}\right]$ we have $T^*\hat{Q}^*E = \left[\begin{smallmatrix} q_1+q_3 & 0 \\ q_2+q_4 & 0 \end{smallmatrix}\right] \geq 0$ and hence $q_1 + q_3 \geq 0$ and $q_2 + q_4 = 0$. Since D is dissipative, $\hat{Q}T$ is also dissipative and therefore

$$0 \geq \langle \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), (D + D^*) \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \rangle = 2 \operatorname{Re} \langle \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), D \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \rangle = \operatorname{Re} (q_1 + q_2 + q_3 + q_4) = q_1 + q_3 \geq 0.$$

This implies $q_1 + q_3 = 0$ and hence $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \in \ker(\hat{Q}T)^* = \ker \hat{Q}^*$, which contradicts the invertibility of $\hat{Q}$.

Chapter 4

Stability and port-Hamiltonian systems

Under the light of Prop. A.0.3, Prop. 3.2.1 and Thm. 3.2.7 gave first results towards stability properties of DAEs with dissipative-Hamiltonian pencils. In this chapter, we first continue our analysis of dissipative-Hamiltonian pencils in that regard, exploiting results and proof techniques of Chap. 3 under the assumption that the symmetric linear relation is nonnegative. In the process, we will discover a second crucial assumption for stability, namely the triviality of the kernel of the symmetric relation in Def. 2.5.4. Afterwards, we explore how stable and stabilizable DAEs can be interpreted as relaxed linear-algebraic port-Hamiltonian systems.

4.1 Stability of dissipative-Hamiltonian pencils

To ensure stability of $[E, A] \in \Sigma_n$ given by a dissipative-Hamiltonian pencil $sE - A$ with

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

for a dissipative relation $\mathcal{D}$ and a symmetric relation $\mathcal{L}$, we need the additional assumption that $\mathcal{L} = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ is *nonnegative*, i.e., $L_1^\top L_2 \geq 0$. In the case that $\mathcal{L}$ is self-adjoint, one can use a suitable CS-decomposition, see [MMW18, Prop. 3.1], so that L_1 and L_2 can be chosen in such a way that $L_1 \geq 0$. Similarly, for maximally dissipative subspaces $\mathcal{D} = \operatorname{ran} \left[\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right]$ one can choose D_1, D_2 such that $D_2 + D_2^\top \leq 0$.

As a main result of this section we provide sufficient conditions for the stability of DAEs induced by square dissipative-Hamiltonian pencils. This result of the results and proof techniques of Chap. 3.

Proposition 4.1.1 (Stability of square dH-LR pencils)**.** *Let $sE - A \in \mathbb{K}[s]^{n\times n}$ such that there exist a maximally dissipative $\mathcal{D} = \operatorname{ran}\left[\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right]$ and a maximally nonegative $\mathcal{L} = \operatorname{ran}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ with $\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$ and $L_1, L_2, D_1, D_2 \in \mathbb{K}^{n\times n}$. Then with $\mathcal{X} := \operatorname{ran} D_1 \cap \operatorname{ran} L_2$, $[E, A]$ is stable if the following hold.*

(i) $\ker P_{\mathcal{X}} L_1\restriction_{\mathcal{X}} \cap \ker P_{\mathcal{X}} D_2\restriction_{\mathcal{X}} = \{0\}$*;*

(ii) $\ker \mathcal{L} = \{0\}$*.*

Proof. Since both $\mathcal{D}$ and $\mathcal{L}$ are maximal by Lem. 1.3.8 we have $\operatorname{mul} \mathcal{D} = (\operatorname{dom} \mathcal{D})^\perp$, $\ker \mathcal{L} = (\operatorname{ran} \mathcal{L})^\perp$ and with $\ker \mathcal{L} = \{0\}$ we deduce

$$\operatorname{mul} \mathcal{D} \mathbin{\widehat{\oplus}} \ker \mathcal{L} = (\operatorname{ran} \mathcal{L})^\perp \widehat{\oplus} (\operatorname{dom} \mathcal{D})^\perp$$

and trivially

$$\ker \mathcal{L} \cap \operatorname{mul} \mathcal{D} = \{0\}.$$

Additionally, with (i) and the nonnegativity of $\mathcal{L}$, Cor. 3.1.4 implies that $sE - A$ is regular. Now, invoking Prop. 3.2.1 yields that $\sigma(E, A) \subseteq \overline{\mathbb{C}_-}$ with semi-simple non-zero eigenvalues on $\imath\mathbb{R}$. By Cor. A.0.4, it remains to show that also the (possible) eigenvalue 0 is semi-simple to deduce stability of $[E, A]$. To this end, we revisit the proof of Prop. 3.2.1. In this context, recall that the order of λ as a pole of $(sE - A)^{-1}$ is equal to the maximal order of λ as a pole of (3.27). Since $\ker \mathcal{L} = \{0\}$ we deduce $L_{22} = 0$ and $D_{22} = -I$. Moreover, $sL_{11} - D_{11}$ is positive real by nonnegativity of $\mathcal{L}$ and hence by [RS11, Thm. 4.1] the order of 0 as a pole of (3.27) and therefore $(sE - A)^{-1}$ is at most 1. Invoking Lem. 1.2.8 completes the proof. □

This result is directly applicable to dH-LG pencils, i.e., to the framework of [vdSM18].

Corollary 4.1.2. *Let $sE - A \in \mathbb{K}^{n\times n}$ be a dH-LG pencil, i.e., there exist a Dirac subspace $\mathcal{D}$ and a Lagrangian subspace $\mathcal{L}$ such that*

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

Then $[E, A]$ is stable if the following conditions are satisfied

(i) $\mathcal{L}$ is nonnegative;

(ii) $\ker \mathcal{L} = \{0\}$*;*

(iii) $sE - A$ *is regular.*

Proof. In contrast to Prop. 4.1.1, except reorganizing the statement, Prop. 4.1.1 (i) has been replaced by condition (iii), which is justified by Cor. 3.1.4. □

Note that in the context of Cor. 4.1.2, the conditions (i) and (iii) alone are not sufficient to guarantee stability, as the following example underlines.

Example 4.1.3. Let $E = I_2$, $D = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, and $Q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Define $A = DQ = J_2(0)$. Then $sE - A$ is a dH-LR pencil with $\mathcal{D} = \operatorname{gr} D$ and $\mathcal{L} = \operatorname{gr} Q$. In particular, $sE - A$ is a dH-ODE pencil and therefore regular. Further, note that $\mathcal{D}$ and $\mathcal{L}$ are dissipative and nonnegative relations, respectively, which are maximal in this respect by virtue of being the graph of a matrix. Overall, we find that conditions (i) and (iii) of Cor.4.1.2 are satisfied. However, the eigenvalue 0 of A is readily not semi-simple. By Prop. A.0.3, $[E, A]$ is not stable.

We are also able to formulate a result for dH-LA pencils, i.e., for the framework of [MMW18]. To this end, let us first highlight the following fact.

Lemma 4.1.4. *Let* $J, R \in \mathbb{K}^{n\times n}$ *with* $J = -J^*$ *and* $R \geq 0$*. Then* $\ker(J - R) = \ker J \cap \ker R$*.*

Proof. If $x \in \ker(J - R)$ then $x^*(J - R)x = 0$ and this implies $0 = x^*(J^* - R)x = -x^*(J + R)x$. Adding both trivial expression yields $0 = -2x^*Rx$. By positive semi-definiteness of R, we have $x \in \ker R$ [TW09, Lem. 12.3.1]. Therefore, $0 = (J - R)x = Jx$, i.e., $x \in \ker J$. The reverse inclusion is trivial. □

Corollary 4.1.5. *Let* $E, J, R, Q \in \mathbb{K}^{n\times n}$ *such that* $Q^*E \geq 0$*,* $J = -J^*$*, and* $R \geq 0$*, i.e.,* $sE - (J - R)Q$ *is a dH-LA pencil with* $\mathcal{D} = \operatorname{gr}(J - R)$ *and* $\mathcal{L} = \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$*. Then* $sE - (J - R)Q \in \mathbb{K}[s]^{n\times n}$ *is regular if and only if* $\ker E \cap \ker(Q^\top JQ) \cap \ker(Q^\top RQ) = \{0\}$*. Furthermore,* $[E, (J - R)Q]$ *is stable if additionally* $\ker Q \subseteq \ker E$*.*

Proof. Assume first that $sE - (J - R)Q$ is regular. Then by Cor. 3.1.6 $sE - Q$ is regular and since regularity is invariant under pencil equivalence we can infer by [MMW18, Prop. 3.1] that $Q = Q^* = Q^2$, $E = E^* = E^2$, and $E + Q = I_n$ without loss of generality. Moreover by regularity of $sE - Q$ $\ker E \cap \ker Q = \{0\}$, which implies $\operatorname{rk} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right] = n$, i.e., $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ is Lagrangian. By Prop. 3.1.4 (iii) with

$\mathcal{X} = \operatorname{ran} I_n \cap \operatorname{ran} Q = (\ker Q^*)^\perp$ the regularity of $sE - (J-R)Q$ implies

$$\begin{aligned}\{0\} &= \ker P_{\operatorname{ran} Q} E\restriction_{\operatorname{ran} Q} \cap \ker P_{\operatorname{ran} Q}(J-R)\restriction_{\operatorname{ran} Q} \\ &= \{\ Qx \ \mid Q^*EQx = Q^*(J-R)Qx = 0\ \} \\ &= Q(\ker Q^*EQ \cap \ker Q^*(J-RQ)). \end{aligned} \tag{4.1}$$

Note that $QEQ = QQE = QE = EQ$ and therefore $\ker Q \widehat{\oplus} \ker E \subseteq \ker QEQ$. On the other hand, if $QEx = EQx = 0$, then $Ex \in \ker Q$ and $Qx \in \ker E$. Hence, $x = Ex + Qx \in \ker Q \widehat{\oplus} \ker E$. Overall,

$$\ker Q \widehat{\oplus} \ker E = \ker QEQ.$$

We can use this equality to infer that (4.1) is equivalent to

$$\begin{aligned} & Q(\ker QEQ \cap \ker Q(J-R)Q) = \{0\} \\ \iff & (\ker Q \widehat{\oplus} \ker E) \cap \ker Q(J-R)Q \subseteq \ker Q \\ \iff & \ker E \cap \ker Q(J-R)Q = \{0\}. \end{aligned}$$

Furthermore, Lem. 4.1.4 implies

$$\ker Q(J-R)Q = \ker(QJQ - QRQ) = \ker(QJQ) \cap \ker(QRQ),$$

which proves $\ker E \cap \ker Q^*JQ \cap \ker Q^*RQ = \{0\}$. Conversely, if this intersection is trivial then in particular $\ker E \cap \ker Q = \{0\}$ and hence $\dim \mathcal{L} = n$ as before. Now invoking by Cor. 3.1.6 yields the regularity of $sE - A$. To prove stability we apply Prop. 4.1.1 by recalling that Prop. 4.1.1 (i) is satisfied by our previous considerations and observing that Prop. 4.1.1 (ii) is equivalent to $\ker Q \subseteq \ker E$. □

The above characterization of regularity was also obtained recently in [SPF⁺21]. However, Prop. 4.1.1 characterizes the regularity and the stability for a larger class of DAEs.

The results presented so far indicate that two properties are key to guarantee stability of a system induced by a dissipative-Hamiltonian pencil. The first is the nonnegativity of the symmetric relation, which was already highlighted in Chap. 3 with its interpretation of existence of an energy function. The second one is that the kernel of the symmetric relation is trivial. To highlight this property in turn, let us consider a dH-LA pencil $sE - (J-R)Q$ subject to Cor. 4.1.5. Then with $\mathcal{L} = \operatorname{ran} \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$ we have

$$\ker \mathcal{L} = \{0\} \iff \ker Q \subseteq \ker E.$$

Now let $z(\cdot)$ satisfy $\frac{\mathrm{d}}{\mathrm{dt}} Ez(t) = (J-R)Qz(t)$ for $t \in \mathbb{R}$. Then with $x = Ez$, $e = Qz$, and $\mathcal{D} = \mathrm{gr}\,(J-R)$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}} Ez = (J-R)Qx \iff \left(\begin{smallmatrix} Ez \\ Qz \end{smallmatrix}\right) \in \mathcal{L} \;\wedge\; \begin{pmatrix} Qz \\ \frac{\mathrm{d}}{\mathrm{dt}} Ez \end{pmatrix} \in \mathcal{D}$$
$$\iff \left(\begin{smallmatrix} x \\ e \end{smallmatrix}\right) \in \mathcal{L} \wedge \left(\begin{smallmatrix} e \\ \dot{x} \end{smallmatrix}\right) \in \mathcal{D} \iff (x, \dot{x}) \in \mathcal{D}\mathcal{L}.$$

If $\ker Q \subseteq \ker E$, then $e = Qz = 0$ implies $x = Ez = 0$. Conversely, Let a function $x(\cdot)$ be given such that $(x, \dot{x}) \in \mathcal{D}\mathcal{L}$. Then there exists a function $e(\cdot)$ such that $(x, e) \in \mathcal{L}$ and $(e, \dot{x}) \in \mathcal{D}$. The range representation of $\mathcal{L}$ guarantees the existence of a function $z(\cdot)$ such that $x = Ez$ and $e = Qz$. Additionally, the range representation of $\mathcal{D}$ gives $\dot{x} = (J-R)e$. Overall, $\frac{\mathrm{d}}{\mathrm{dt}} Ez = (J-R)Qz$. And if $e = 0$, then $\ker \mathcal{L} = \{0\}$ again yields $Ez = 0$. Recapitulating, in the terminology of Chap. 1 & 2 this means that differential variables Ez having vanishing effort e are trivial. Next, we present another result for dH-LA pencils where the symmetric relation is nonnegative and has trivial kernel.

Proposition 4.1.6. *Let $E, J, R, Q \in \mathbb{K}^{n\times n}$ such that $Q^*E \geq 0$, $J = -J^*$, and $R \geq 0$, i.e., $sE - (J-R)Q$ is a dH-LA pencil with $\mathcal{D} = \mathrm{gr}\,(J-R)$ and $\mathcal{L} = \left[\begin{smallmatrix} E \\ Q \end{smallmatrix}\right]$. Further assume $\ker Q \subseteq \ker E$. Then the following hold.*

(a) *If $sE - Q$ is regular then Q is invertible.*

(b) *Let Q be invertible. Then $[E, (J-R)Q]$ is stable if and only if $\ker J \cap \ker R \cap (Q \ker E) = \{0\}$.*

Proof. If $sE - Q$ is regular then $\ker E \cap \ker Q = \{0\}$. This together with $\ker Q \subseteq \ker E$ implies $\ker Q = \{0\}$ and hence Q is invertible. This proves (a).
Next, observe that by Lem. 4.1.4 $\ker(J-R) = \ker J \cap \ker R$. To characterize the stability of $[E, (J-R)Q]$, we use Cor. A.0.4. If $[E, (J-R)Q]$ is stable then $sE - (J-R)Q$ is regular. Hence, using the invertibility of Q and $\ker Q \subseteq \ker E$ we find

$$\{0\} = \ker E \cap \ker(J-R)Q = (Q \ker E) \cap \ker J \cap \ker R. \tag{4.2}$$

Conversely, assume that (4.2) holds. Then

$$\{0\} = (Q \ker E) \cap \ker J \cap \ker R = \ker E \cap \ker(J-R)Q$$
$$= \ker E \cap \ker(Q^\top JQ) \cap \ker(Q^\top RQ).$$

By Cor. 4.1.5 $[E, (J-R)Q]$ is stable. □

If Q is invertible then the conditions in Prop. 4.1.1 and Prop. 4.1.6 (b) coincide.
The following example from [MMW18] shows that not every DAE given by a dH-LA pencil is stable.

Example 4.1.7. Consider

$$sE - JQ = \left[\begin{smallmatrix} s & 1 \\ 0 & s \end{smallmatrix}\right], \quad J := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \quad Q := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right],$$

which is a dH-LA pencil with Jordan block of size 2 at zero. It is therefore unstable. The above example is an ordinary differential equation and $x \mapsto x^\top Q^\top E x = x^\top Q x$ is not a Lyapunov function.

For now, the dissipative-Hamiltonian pencils we considered were square and therefore regular if the corresponding DAE was stable. How to formulate similar results concerning the stability of DAEs induced by nonsquare dissipative-Hamiltonian pencils is discussed in the following remark.

Remark 4.1.8 (Stability of nonsquare dH-LR pencils)**.** In order to derive a similar result to Prop. 4.1.1 for nonsquare pencils, the obvious approach is to imitate the proof technique using Prop. 3.2.6 instead of Prop. 3.1.3. However, this technique allows to deduce a similar statement to that of Prop. 4.1.1 for nonsquare pencils $sE - A \in \mathbb{K}^{n \times m}$ only if $n = m$. If $n < m$, it is readily seen that a nontrivial column minimal index must exist in its Kronecker form (1.1) and hence by Prop. A.0.3 it is unstable. Further, the case $n > m$ leads to a contradiction under the assumptions of Prop. 4.1.1, for which we sketch how to obtain it. Inspecting how (3.34) was obtained in the proof of Prop. 3.2.6 we see that in this context the assumptions successively lead to $k_X = 0$, $k_2 = n - k_1$, $n_1 = k_1$, $k_2 = n_3$, $k_3 = 0$, $n_2 = 0$, $k_5 = 0$, $n_4 = 0$, and $\hat{m} = n$, that is for some $S \in \mathbf{Gl}_n(\mathbb{K})$ we have

$$\operatorname{diag}(S, S) \operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \operatorname{ran} \begin{bmatrix} L_{11} & 0 \\ L_{21} & 0 \\ D_{11} & 0 \\ 0 & -I_{n_3} \end{bmatrix}. \tag{4.3}$$

However, see also (3.45), the pencil $s \left[\begin{smallmatrix} L_{11} & 0 \\ L_{21} & 0 \end{smallmatrix}\right] - \begin{bmatrix} D_{11} & 0 \\ 0 & -I_{n_3} \end{bmatrix}$ is regular. A dimension count shows that (4.3) leads to a contradiction if $n > m$.
One can of course refine the conditions (i) and (ii) of Prop. 4.1.1 by only demanding that certain block matrices arising in the proof of Prop. 3.2.6 and causing a nonsemisimple eigenvalue 0 or a column minimal index are trivial. However, such conditions do not possess obvious algebraic or geometric interpretations.

Remark 4.1.9 (Stability in terms of the geometric behavior)**.**
Given a dH-LR pencil $sE - A$ with ran $\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{DL}$, we know that $\mathfrak{B}_{\mathcal{DL}} = E\mathfrak{B}_{[E,A]}$. Therefore, asking about the boundedness of the trajectories $x \in \mathfrak{B}_{\mathcal{DL}}$ is equivalent to asking about the boundedness of the differential variables Ez for all solutions $z \in \mathfrak{B}_{[E,A]}$. Results on the stability of the differential variables of $[E, A]$ can be derived analogously to the results for the stability of $[E, A]$ presented in this section. To be more precise, one invokes the statements of Prop. A.0.3 involving the Lyapunov inequality A.3 instead of A.1. Similarly, if we are interested in the asymptotic behavior of solutions, one invokes the statements of Prop. A.0.3 involving the Lyapunov inequalities (A.2) and (A.4).

4.2 Stable and stabilizable DAEs as pH-LAR systems

The aim of this section is to show that stable DAEs $[E, A] \in \Sigma_{n,m}$ and stabilizable DAEs $[E, A, B] \in \Sigma_{n,m,k}$ can be rewritten as a pH-LAR systems on a suitable subspace. The following result shows that, in the case of stable DAEs, one can choose the system space as this subspace.

Proposition 4.2.1. *Let $[E, A] \in \Sigma_{n,m}$ be stable and let $X \in \mathbb{K}^{m\times m}$ with $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ be a solution of* (A.1) *given by Prop. A.0.3. Then with the choice of $Q := XE$ and*

$$J := \tfrac{1}{2}(AQ^\dagger - (AQ^\dagger)^*), \quad R := -\tfrac{1}{2}(AQ^\dagger + (AQ^\dagger)^*),$$

$[E, A]$ becomes a pH-LAR system on $\mathcal{V}_{\mathrm{sys}}^{[E,A]}$ with

$$\begin{aligned} J &=_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} -J^*, & R &\geq_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0, \\ A &=_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} (J - R)Q, & Q^*E &>_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0. \end{aligned}$$

Further, $\mathcal{V}_{\mathrm{sys}}^{[E,A]} \subseteq \mathcal{V}_{\mathrm{sys}}^{[E,(J-R)Q]}$.

Proof. Let $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ be a solution of (A.1) given by Prop. A.0.3. Since $\ell_\beta = 0$ by Prop. A.0.3, considering (1.5) we deduce $XE\mathcal{V}_{\mathrm{sys}}^{[E,A]} = E\mathcal{V}_{\mathrm{sys}}^{[E,A]} = \mathcal{V}_{\mathrm{sys}}^{[E,A]}$. Then

$$Q^\dagger Q =_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} E^\dagger E =_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} I_n,$$

and thus $(J-R)Q = AQ^\dagger Q =_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} A$. Since $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$, we have for all $x \in \mathcal{V}_{\mathrm{sys}}^{[E,A]} \setminus \{0\}$

$$x^*Q^*Ex = x^*E^*XEx > 0.$$

With $E\mathcal{V}_{\mathrm{sys}}^{[E,A]} = Q\mathcal{V}_{\mathrm{sys}}^{[E,A]}$ we find that $R \geq_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ is equivalent to

$$\begin{aligned} 0 \leq_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} Q^*RQ &= -Q^*(AQ^\dagger + (Q^\dagger)^*A^*)Q \\ &= -Q^*A - A^*Q \\ &=_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} -E^*XA - A^*XE, \end{aligned}$$

which holds by virtue of X. Moreover, $J =_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} J^*$ holds trivially. Now let $x \in \mathfrak{B}_{[E,A]}$. Since x evolves in $\mathcal{V}_{\mathrm{sys}}^{[E,A]}$

$$Ex(t) = Ax(t) = (J-R)Qx(t),$$

i.e., $x \in \mathfrak{B}_{[E,(J-R)Q]}$ from which $\mathcal{V}_{\mathrm{sys}}^{[E,A]} \subseteq \mathcal{V}_{\mathrm{sys}}^{[E,(J-R)Q]}$ follows. $\square$

In the following remark, we investigate some properties of the approach to rewriting a stable DAE as port-Hamiltonian system.

Remark 4.2.2.

1. In the context of Prop. 4.2.1, we saw that $\mathcal{V}_{\mathrm{sys}}^{[E,A]} \subseteq \mathcal{V}_{\mathrm{sys}}^{[E,(J-R)Q]}$. The question arises when the reverse inclusion holds true as well. For the construction we presented, this can be answered. It turns out that equality holds if in addition to being stable there are no Jordan blocks at ∞ in the Kronecker form (1.1), i.e., $\ell_\alpha = 0$. We sketch how to show this by considering the case where $sE - A$ is already in Kronecker canonical form (1.1). Note that by Rem. A.0.5 we are free in extending the solution X to (A.1) given by Prop. A.0.3 on $(\mathcal{V}_{\mathrm{sys}}^{[E,A]})^\perp$. For the reverse inclusion, we need to show $AQ^\dagger Q = A$ on $(\mathcal{V}_{\mathrm{sys}}^{[E,A]})^\perp$. To this end, it is beneficial to extend X as the identity on the orthogonal complement of the system space since this equation boils down to $A = AE^\dagger E$ on $(\mathcal{V}_{\mathrm{sys}}^{[E,A]})^\perp$. The overdetermined blocks are preserved since $L_k^\top (K_k^\top)^\dagger K_k^\top = L_k^\top K_k K_k^\top = L_k^\top I_{k-1} = L_k^\top$ for $k \in \mathbb{N}$. Unfortunately, the same cannot be said of the blocks at infinity since $I_k N_k^\dagger N_k = N_k^\top N_k = \mathrm{diag}\,(I_{k-1}, 0)$, i.e., a block $sN_k - I_k$ gets replaced by $\mathrm{diag}\,(sK_k^\top - L_k^\top, sK_1 - L_1)$, resulting in an unstable DAE with (the same) stable differential variables.

2. If $[E, A] \in \Sigma_{n,m}$ is stable with index at most one then we can redefine Q in Prop. 4.2.1 in such a way that the relations therein hold on $\mathbb{K}^n$ instead of $\mathcal{V}_{\text{sys}}^{[E,A]}$. Here we assume for simplicity again that E and A are already given in Kronecker form (1.1), i.e.,

$$sE - A = \begin{bmatrix} sI_{n_0} - A_0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 \\ 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}.$$

Let $X_0 > 0$ satisfy $A_0^* X_0 + X_0 A_0 \leq 0$ by virtue of the spectrum of A_0. Then defining

$$\hat{Q} := \begin{bmatrix} X_0 & 0 & \\ 0 & I_{|\alpha|} & 0 \\ 0 & 0 & K_\gamma^\top \end{bmatrix}$$

and $J := \frac{1}{2}(A\hat{Q}^\dagger - (A\hat{Q}^\dagger)^*)$, $R := -\frac{1}{2}(A\hat{Q}^\dagger + (A\hat{Q}^\dagger)^*)$ leads to

$$\hat{Q}^* E = \begin{bmatrix} X_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\gamma - \ell_\gamma} \end{bmatrix} \geq 0$$

and $A = (J - R)\hat{Q}$. For comparison, the approach of Prop. 4.2.1 would yield

$$Q = \begin{bmatrix} X_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top \end{bmatrix}$$

when extending a solution X to (A.1) as the identity on the orthogonal complement of the system space. If $sE - A$ has index greater than one then this extension of Q is still possible but leads to a nonsymmetric $\hat{Q}^* E$.

Similar considerations can be done for systems which are stable *backwards in time*, i.e., $[-E, A]$ is stable. Note that the Kronecker form (1.1) of $sE - A$ and $s(-E) - A$ differ only in the sign of the matrix in Jordan normal form. This motivates the following result.

Corollary 4.2.3. *Let $[E, A] \in \Sigma_{n,m}$. If $sE - A$ neither has nonsemi-simple eigenvalues on the imaginary axis nor column minimal index in its Kronecker form, then $[E, A]$ is a pH-LAR system on its system space. To be more precise, there exist matrices $Q \in \mathbb{K}^{m \times n}$, $J, R \in \mathbb{K}^{m \times m}$ such that*

$$J =_{E\mathcal{V}_{\text{sys}}^{[E,A]}} -J^*, \qquad R \geq_{E\mathcal{V}_{\text{sys}}^{[E,A]}} 0,$$
$$A =_{\mathcal{V}_{\text{sys}}^{[E,A]}} (J - R)Q, \qquad Q^* E = E^* Q.$$

Further, $\mathcal{V}_{\text{sys}}^{[E,A]} \subseteq \mathcal{V}_{\text{sys}}^{[E,(J-R)Q]}$ and $Q^* E\restriction_{\mathcal{V}_{\text{sys}}^{[E,A]}}$ is invertible.

Proof. By virtue of the properties of $sE-A$ there exist invertible $S \in \mathbb{K}^{m\times m}$ and $T \in \mathbb{K}^{n\times n}$ such that $S(sE-A)T = \operatorname{diag}(sE_1-A_1, -sE_2-A_2)$ with $\sigma(E_1,A_1), \sigma(E_2,A_2) \subseteq \overline{\mathbb{C}_-}$ both with semi-simple eigenvalues on $\imath\mathbb{R}$. By Prop. A.0.3 and Prop. 4.2.1 there exist matrices $J_1, J_2, R_1, R_2, Q_1, Q_2$ of appropriate size such that

$$\begin{aligned} J_1 &=_{E_1\mathcal{V}_{\text{sys}}^{[E_1,A_1]}} -J_1^*, & R_1 &\geq_{E_1\mathcal{V}_{\text{sys}}^{[E_1,A_1]}} 0, \\ A_1 &=_{\mathcal{V}_{\text{sys}}^{[E_1,A_1]}} (J_1-R_1)Q_1, & Q_1^*E_1 &>_{\mathcal{V}_{\text{sys}}^{[E_1,A_1]}} 0, \\ J_2 &=_{E_2\mathcal{V}_{\text{sys}}^{[E_2,A_2]}} -J_2^*, & R_2 &\geq_{E_2\mathcal{V}_{\text{sys}}^{[E_2,A_2]}} 0, \\ A_2 &=_{\mathcal{V}_{\text{sys}}^{[E_2,A_2]}} (J_2-R_2)Q_2, & Q_2^*E_2 &>_{\mathcal{V}_{\text{sys}}^{[E_2,A_2]}} 0. \end{aligned}$$

Further, $\mathcal{V}_{\text{sys}}^{[E_1,A_1]} \subseteq \mathcal{V}_{\text{sys}}^{[E_1,(J_1-R_1)Q_1]}$ and $\mathcal{V}_{\text{sys}}^{[E_2,A_2]} \subseteq \mathcal{V}_{\text{sys}}^{[E_2,(J_2-R_2)Q_2]}$. Invoking Props. 1.2.16 & 1.2.17 and $\mathcal{V}_{\text{sys}}^{[E_2,A_2]} = \mathcal{V}_{\text{sys}}^{[-E_2,A_2]}$, the statement of the corollary follows by setting

$$\begin{aligned} J &:= S^{-1}\operatorname{diag}(J_1,J_2)S^{-*}, \\ R &:= S^{-1}\operatorname{diag}(R_1,R_2)S^{-*}, \\ Q &:= S^{*}\operatorname{diag}(Q_1,-Q_2)T^{-1}. \end{aligned}$$

Note that in comparison to Prop. 4.2.1 we lose the definiteness of Q^*E on $\mathcal{V}_{\text{sys}}^{[E,A]}$ but keep the invertibility on this space. □

A similar approach allows to rewrite certain behaviorally stabilizable systems $[E,A,B] \in \Sigma_{n,k}$, i.e.,

$$\tfrac{\mathrm{d}}{\mathrm{dt}} Ex(t) = Ax(t) + Bu(t),$$

as a pH-LAR system after introducing a suitable output. To be more precise, we consider the class $\Sigma_{n,k}^s$ consisting of behaviorally stabilizable systems $[E,A,B] \in \Sigma_{n,k}$ for which $sE-A$ is regular and only has semi-simple eigenvalues on the imaginary axis.

If $[E,A,B] \in \Sigma_{n,k}^s$, then by applying a Jordan decomposition on the first block entry in the Weierstraß form (1.2) we see that there exists some invertible $S,T \in \mathbb{K}^{n\times n}$, $n_1, n_2 \in \mathbb{N}$ with $n_1+n_2 = n_0$ such that

$$\begin{aligned} S(sE-A)T &= \begin{bmatrix} sI_{n_1}-A_1 & 0 & 0 \\ 0 & sI_{n_2}-A_2 & 0 \\ 0 & 0 & sN_\alpha - I_{|\alpha|} \end{bmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ SB &= \begin{bmatrix} B_1 \\ B_2 \\ B_\alpha \end{bmatrix}, \quad \sigma(A_1) \subseteq \mathbb{C}_+, \quad [I_{n_2}, A_2] \text{ is stable.} \end{aligned} \tag{4.4}$$

Furthermore, $[I_{n_1}, A_1, B_1]$ is controllable.

Lemma 4.2.4. *If $[E, A, B] \in \Sigma^s_{n,k}$ fulfills (4.4) with $S = T = I_n$, then a stabilizing feedback is given by $u(t) = -B_1^* P_1 x_1(t)$ with $P_1 > 0$ being the unique solution of*

$$A_1^* P_1 + P_1 A_1 = P_1 B_1 B_1^* P_1, \tag{4.5}$$

in the sense that for all $x_0 \in \mathcal{V}^{[E,A,B]}_{\mathrm{diff}}$ there exist $(x, u) \in \mathfrak{B}_{[E,A,B]}(x_0)$ and $M \geq 0$ such that $\operatorname{ess\,sup}_{t\geq 0} \|x(t)\| \leq M$ *and $(x, u) = (x_1, x_2, x_\alpha, -B_1^* P_1 x_1)$.*

Proof. Clearly, (4.5) has a unique positive definite solution if and only if

$$(-A_1^*)^* P_1^{-1} + P_1^{-1}(-A_1^*) = -B_1 B_1^*$$

has a unique positive definite solution. Since $[I_{n_1}, A_1, B_1]$ is controllable and $\sigma(-A_1^*) \subset \mathbb{C}_-$ such a solution exists by [TSH01, Thm. 3.28] (if $\mathbb{K} = \mathbb{R}$, the case $\mathbb{K} = \mathbb{C}$ being shown analogously). It remains to show that $u(t) = -B_1^* P_1 x_1(t)$ is a stabilizing feedback. Note that $[I_{n_1}, A_1 - B_1 B_1^T P_1, B_1]$ is also controllable and that (4.5) is also equivalent to

$$(A_1^* - P_1^* B_1 B_1^*)^* P_1^{-1} + P_1^{-1}(A_1^* - P_1^* B_1 B_1^*) = -B_1 B_1^*.$$

Now invoking [TSH01, Thm. 3.28] again shows $\sigma(A_1 - B_1 B_1^* P_1) \subseteq \mathbb{C}_-$. Moreover, with

$$x_1(t) = e^{(A_1 - B_1 B_1^* P_1)t} x_1(0), \quad \forall t \geq 0,$$

there exist $M, \beta > 0$ such that for all $k \geq 0$ which are smaller than the index of $[E, A]$

$$\|x_1^{(k)}(t)\| \leq M e^{-\beta t}, \quad \forall t \geq 0.$$

Therefore, we have for some $\hat{M} > 0$ such that for all $k \geq 0$ which are smaller then the index of $[E, A]$

$$\|u^{(k)}(t)\| \leq \hat{M} e^{-\beta t}, \quad \forall t \geq 0. \tag{4.6}$$

Since $[I_{n_2}, A_2]$ is stable, the variation of constants formula implies that the solution x_2 of $\dot{x}_2 = A_2 x_2(t) + B_2 u(t)$ are bounded. Next, we only consider the first block-entry (if any) of $sN_\alpha - I_{|\alpha|}$, $sN_{\alpha_1} - I_{\alpha_1}$, since the others are treated analogously. With $B_\alpha = \left[\begin{smallmatrix} B_{\alpha_1}^\top & \cdots & B_{\alpha_{\ell(\alpha)}}^\top \end{smallmatrix}\right]^\top$ the solution $x_{\alpha_1}(t) = (x_{\alpha_1,1}(t), \ldots, x_{\alpha_1,\alpha_1}(t))^\top$ of $[N_{\alpha_1}, I_{\alpha_1}]$ fulfills

$$\begin{pmatrix} \dot{x}_{\alpha_1,2} \\ \vdots \\ \dot{x}_{\alpha_1,\alpha_1} \\ 0 \end{pmatrix} = \tfrac{\mathrm{d}}{\mathrm{dt}} N_{\alpha_1} \begin{pmatrix} x_{\alpha_1,1} \\ \vdots \\ x_{\alpha_1,\alpha_1-1} \\ x_{\alpha_1,n_{\alpha_1}} \end{pmatrix} = \begin{pmatrix} x_{\alpha_1,1} \\ \vdots \\ x_{\alpha_1,\alpha_1-1} \\ x_{\alpha_1,n_{\alpha_1}} \end{pmatrix} + B_{\alpha_1} u,$$

and inspecting the last row leads to

$$\|x_{\alpha_1,\alpha_1}(t)\| = \|\alpha_1 u(t)\| \leq \|B_{\alpha_1}\|\hat{M}e^{-\beta t},$$

which tends to zero as $t \to \infty$. Similarly, the penultimate row leads to

$$x_{\alpha_1,\alpha_1-1}(t) = \dot{x}_{\alpha_1,\alpha_1}(t) - e_{\alpha_1-1}^\top B_{\alpha_1} u(t) = B_{\alpha_1}\dot{u}(t) - e_{\alpha_1-1}^\top B_{\alpha_1} u(t),$$

which is again exponentially bounded by (4.6). Repeating the last step with the remaining rows starting with the $\alpha_1 - 2$th row shows that $\|x_{\alpha_1}(t)\| \to 0$ as $t \to \infty$ which completes the proof that u stabilizes the solution of $[E, A, B]$. □

Below, we exploit the behavioral stabilizability to obtain solutions to certain matrix equalities on the system space which allows us to reformulate the system in a port-Hamiltonian way. Similar equations for stabilization of DAEs have been studied under the name *generalized algebraic Bernoulli equation* in [BBQ07].

Lemma 4.2.5. *Let $[E, A, B] \in \Sigma_{n,k}^s$. Then there exist some $X_1, X_2 \geq_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ such that the following holds.*

$$A^*X_1E + E^*X_1A =_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} E^*X_1BB^*X_1E, \tag{4.7}$$

$$A^*X_2E + E^*X_2A \leq_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0, \tag{4.8}$$

with $(X_2 \pm X_1)E\mathcal{V}_{\mathrm{sys}}^{[E,A]} = E\mathcal{V}_{\mathrm{sys}}^{[E,A]}$ and $X_1 + X_2 >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$.

Proof. All conditions are invariant under transformations of the form $[E, A, B] \to [SET, SAT, SB]$ for all invertible $S, T \in \mathbb{K}^{n\times n}$. Hence we can assume without restriction that $[E, A, B]$ is already given in the block diagonal form on the right-hand side of (4.4). Introduce $\tilde{E} = \operatorname{diag}(I_{n_2}, N_\alpha)$, $\tilde{A} = \operatorname{diag}(A_2, I_{|\alpha|})$ and then we set $X_1 := \operatorname{diag}(P_1, 0_{n_2+|\alpha|})$, where $P_1 > 0$ is a solution of (4.5) given by Lem. 4.2.4 and $X_2 := \operatorname{diag}(0_{n_1}, P_2^{-1}, 0_{|\alpha|})$ where $P_2 > 0$ is a solution of the Lyapunov inequality $A_2^*P_2 + P_2A_2 \leq 0$. Clearly, $X_1, X_2 \geq_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ and they satisfy (4.7) and (4.8), respectively. Furthermore, $E\mathcal{V}_{\mathrm{sys}}^{[E,A]} = \mathbb{K}^{n_1+n_2} \times \{0\}^{|\alpha|}$ and hence $X_2 \pm X_1 = \operatorname{diag}(\pm P_1, P_2^{-1}, 0_{|\alpha|})$ map $E\mathcal{V}_{\mathrm{sys}}^{[E,A]}$ into itself with $X_1 + X_2 >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$. □

Based on this result, we show how to interpret a stabilizable system as a pH-LAR system, which can be viewed as an analogue to Prop. 4.2.1.

Proposition 4.2.6. *Let $[E,A,B] \in \Sigma^s_{n,k}$. Then there exist $X_1, X_2 \geq_{E\mathcal{V}^{[E,A]}_{\mathrm{sys}}} 0$ such that with the choices of*

$$\begin{aligned} Q &\coloneqq (X_2 - X_1)E, \\ J &\coloneqq \tfrac{1}{2}(AQ^\dagger - (AQ^\dagger)^*), \\ R &\coloneqq -\tfrac{1}{2}(AQ^\dagger + (AQ^\dagger)^*), \end{aligned} \tag{4.9}$$

the system $[E,A,B,C,D]$ with $C = B^*Q$ and $D = I_k$, i.e.,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= B^*Qx(t) + u(t), \end{aligned} \tag{4.10}$$

is a pH-LAR system on on $\mathcal{V}^{[E,A]}_{\mathrm{sys}} \times \mathbb{K}^k$. To be more precise,

$$\begin{aligned} \left[\begin{smallmatrix} J & B \\ -B^* & 0 \end{smallmatrix}\right] &=_{E\mathcal{V}^{[E,A]}_{\mathrm{sys}} \times \mathbb{K}^k} - \left[\begin{smallmatrix} J & B \\ -B^* & 0 \end{smallmatrix}\right]^*, & \left[\begin{smallmatrix} R & 0 \\ 0 & I \end{smallmatrix}\right] &\geq_{E\mathcal{V}^{[E,A]}_{\mathrm{sys}} \times \mathbb{K}^k} 0, \\ A &=_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} (J-R)Q, & E^\top Q &=_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} Q^*E. \end{aligned} \tag{4.11}$$

Proof. The proof is completely analogous to the proof of Prop. 4.2.1 except that we use (4.7) and (4.8) instead of (A.1) to prove the inequality in (4.11). To be more precise, let X_1, X_2 be the solutions of (4.7) and (4.8), respectively, given by Lem. 4.2.5. Then with $(X_1 - X_2)E\mathcal{V}^{[E,A]}_{\mathrm{sys}} = E\mathcal{V}^{[E,A]}_{\mathrm{sys}} = \mathcal{V}^{[E,A]}_{\mathrm{sys}}$ we have

$$R \geq_{E\mathcal{V}^{[E,A]}_{\mathrm{sys}}} 0 \Leftrightarrow 2Q^*RQ \geq_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} 0$$

and

$$\begin{aligned} 2Q^*RQ &=_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} -Q^*A - A^*Q \\ &= -E^*(X_2 - X_1)A - A^*(X_2 - X_1)E \\ &\geq_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} E^*X_1BB^*X_1E \geq_{\mathcal{V}^{[E,A]}_{\mathrm{sys}}} 0. \end{aligned}$$

□

Remark 4.2.7. In the context of Prop. 4.2.6, one can show that the solutions (x,u) of $[E, A - BB^*(X_2 - X_1)E]$ correspond to the solutions (x,u,y) of (4.10) when imposing $y = 0$. This restriction corresponds to an *interconnection* with respect to a Dirac structure of the form $\operatorname{ran} \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$. Furthermore, if $sE - A$ has index one then $N = 0$ in (4.4) and $E\mathcal{V}^{[E,A]}_{\mathrm{sys}} \times \mathbb{K}^k$ coincides with $\left[\begin{smallmatrix} E & 0 \\ 0 & I_k \end{smallmatrix}\right] \mathcal{V}^{[E,A,B]}_{\mathrm{sys}}$. If E is invertible then $\mathcal{V}^{[E,A,B]}_{\mathrm{sys}} = \mathbb{K}^{n+k}$, $E\mathcal{V}^{[E,A]}_{\mathrm{sys}} = \mathbb{K}^n$ and $Q = X_1 + X_2 > 0$.

Remark 4.2.8. In Prop. 4.2.6 we defined a suitable output to obtain a pH system. More generally, in [GS18, Thm. 3.6] it was shown that descriptor systems $[E, A, B]$ with output $y(t) = Cx(t) + Du(t)$ can be rewritten as a pH-LA system if there exists an invertible solution X of a linear matrix inequality (typically referred to as Kalman–Yakubovich–Popov inequality (KYP)). Moreover, one would have $X = Q$ in the context of (2.12). Such invertible solutions can be obtained by restricting the system (4.10) to the space $\mathcal{V}_{\mathrm{sys}}^{[E,A]} \times \mathbb{K}^k$. In this case Q in (4.9) can be replaced by the invertible $\hat{Q} := (X_2 - X_1)E\restriction_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}}$.

Chapter 5

Dirac and Lagrange constraints in dissipative-Hamiltonian pencils

As we saw in Sec. 2.5.2, there exist dH-LG pencils, that is pencils corresponding to the framework of [vdSM18], that are not dH-LA, i.e., they do not fit in the framework of [MMW18]. Nevertheless, in this chapter we show how the behavior of a DAE induced by a dH-LG pencil can be associated to the behavior of a DAE induced by a dH-LA pencil. The approach we present originates from [vdSM18] and was called 'From Dirac to Lagrange constraints, and back'. We extend this technique to the case where the maximally dissipative relation is not Dirac, i.e., dissipation is actually present in the system, as well as to two other types of constraints. The extension to other constraints enables us to link dissipative-Hamiltonian pencils with nonnegative symmetric relation to a certain type of pencils recently discussed in [MMW22].

5.1 Constraints in dissipative-Hamiltonian pencils

Consider a dH-LR pencil $sE - A$ with respectively maximally dissipative and maximally symmetric relations $\mathcal{D}$ and $\mathcal{L}$ with

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

We are interested in constraints affecting trajectories in the 'geometric' behavior of the pencil, that is

$$x \in \mathfrak{B}_{\mathcal{D}\mathcal{L}}.$$

Following the terminology of [vdSM18], $\mathcal{D}\mathcal{L}$ is said to have *Lagrange constraints* if $\operatorname{mul} \mathcal{L} \neq \{0\}$ and *Dirac constraints* if $\operatorname{mul} \mathcal{D} \neq \{0\}$. To understand this terminology, let functions $x(\cdot), e(\cdot)$ be given with $(x, e) \in \mathcal{L}$ and $(e, \dot{x}) \in \mathcal{D}$, i.e., $(x, \dot{x}) \in \mathcal{D}\mathcal{L}$. By maximality of $\mathcal{D}$ and $\mathcal{L}$ we have $\operatorname{dom} \mathcal{D} = (\operatorname{mul} \mathcal{D})^{\perp} \neq \mathbb{K}^n$ and $\operatorname{dom} \mathcal{L} = (\operatorname{mul} \mathcal{L})^{\perp} \neq \mathbb{K}^n$. Then Dirac and Lagrange algebraic constraints respectively arise as

$$e \in \operatorname{dom} \mathcal{D}, \quad x \in \operatorname{dom} \mathcal{L}.$$

Analogously, one can formulate the constraints

$$e \in \operatorname{ran} \mathcal{L}, \quad \dot{x} \in \operatorname{ran} \mathcal{D},$$

which arise if $\ker \mathcal{L} \neq \{0\}$ and $\ker \mathcal{D} \neq \{0\}$, respectively. In the terminology of [vdSJ14; DvdS98] the (inclusion) constraint $\dot{x} \in \operatorname{ran} \mathcal{D}$ is associated to so-called *conserved quantity* rather than *algebraic constraints*. Namely, if $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n)$, then one can choose $x_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \operatorname{ran} \mathcal{D})$ and $x_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, (\operatorname{ran} \mathcal{D})^{\perp})$ such that $x = x_1 + x_2$. In particular, x, x_1, x_2 are differentiable almost everywhere. Since $\operatorname{ran} \mathcal{D}$ and $(\operatorname{ran} \mathcal{D})^{\perp}$ are closed under classical differentiation, $\dot{x}_1 \in \operatorname{ran} \mathcal{D}$ and $\dot{x}_2 \in (\operatorname{ran} \mathcal{D})^{\perp}$. Then also $\dot{x}_2 = \dot{x} - \dot{x}_1 \in \operatorname{ran} \mathcal{D}$. Hence,

$$\dot{x}_2 = 0.$$

In this case, the *conserved quantity* is x_2. Similarly, combining two constraints we derive

$$\dot{x} \in \operatorname{dom} \mathcal{L} \cap \operatorname{ran} \mathcal{D}.$$

Definition 5.1.1. Let $\mathcal{D}$ and $\mathcal{L}$ be dissipative and symmetric relations with

$$\operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}.$$

We say that the dH-LR pencil $sE - A$ has *constraints of type*

$(D1)$ if $\operatorname{mul} \mathcal{D} \neq \{0\}$; $(D2)$ if $\ker \mathcal{D} \neq \{0\}$;

$(L1)$ if $\operatorname{mul} \mathcal{L} \neq \{0\}$; $(L2)$ if $\ker \mathcal{L} \neq \{0\}$.

Such constraints can cause practical problems. It was stated in [vdSM18, Rem. 2.6] that *Lagrange algebraic constraints typically have index* 2 *or higher*. This fact is underlined in Ex. 3.2.4, where for arbitrary large $k \in \mathbb{N}_0$ we have $\operatorname{mul} \mathcal{L} = \mathbb{K} \times \{0\}^{2k-1}$, and the index of the pencil reads $2k$. Hence, being able to rewrite the system without certain constraints can be desirable.
Note that the Dirac constraints are not present in dH-LA pencils, which correspond to the framework of [MMW18]. Being able to eliminate Dirac constraints for a dH-LG system given by a Dirac structure $\mathcal{D}$ and a Lagrange structure $\mathcal{L}$, that is a system in the framework of [vdSM18], would link both frameworks. More generally, one can ask when a product $\mathcal{DL}$ with $\mathcal{D}$ dissipative and $\mathcal{L}$ symmetric can be rewritten as $\mathcal{DL} = (\operatorname{gr} D)\mathcal{L}$ or even as $\mathcal{DL} = (\operatorname{gr} D)\hat{\mathcal{L}}$ with a different symmetric linear relation $\hat{\mathcal{L}}$. However, this fails in general by Ex. 2.5.8, which shows that there exist dH-LG pencils that are not dH-LA pencils.

5.2 Extension of dissipative-Hamiltonian pencils

Instead of rewriting the product $\mathcal{DL}$ directly with different 'factors', we will present procedures such that one finds a different pH-LR system $\tilde{\mathcal{D}}\tilde{\mathcal{L}}$ where one type of constraints presented in Def. 5.1.1 is not present anymore and for which a one-to-one correspondence between its behavior and the behavior of the original system can be established in the following sense.

Definition 5.2.1. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be two dissipative relations in $\mathbb{K}^{n_1}$, $\mathbb{K}^{n_2}$, respectively, and $\mathcal{L}_1$, $\mathcal{L}_2$ be two symmetric relations in $\mathbb{K}^{n_1}$, $\mathbb{K}^{n_2}$, respectively, with $n_1 < n_2$. We say that the pH-LR system given by $\mathcal{D}_1\mathcal{L}_1$ *extends* the pH-LR system given by $\mathcal{D}_2\mathcal{L}_2$ if

(i) $x \in \mathfrak{B}_{\mathcal{D}_1\mathcal{L}_1} \Longrightarrow \exists\, \Lambda : \mathbb{R} \to \mathbb{K}^{n_2-n_1}$ s.t. $(x, \Lambda) \in \mathfrak{B}_{\mathcal{D}_2\mathcal{L}_2}$;

(ii) $(x, \Lambda) \in \mathfrak{B}_{\mathcal{D}_2\mathcal{L}_2} \Longrightarrow x \in \mathfrak{B}_{\mathcal{D}_1\mathcal{L}_1}$.

We achieve such extensions of the behavior by extending the linear relations of the system as follows.

Definition 5.2.2. Let $\mathcal{N}$ be a linear relation in $\mathbb{K}^n$. We say that $\mathcal{N}$ *extends* to the linear relation $\mathcal{M}$ in $\mathbb{K}^{n+k}$ with $k \in \mathbb{N}$ if

(i) $\forall (x,y) \in \mathcal{N}\ \exists! v \in \mathbb{K}^k : (x,v,y,0) \in \mathcal{M}$;

(ii) $(x,v,y,0) \in \mathcal{M} \Longrightarrow (x,y) \in \mathcal{N}$.

Then next result describes how we extend maximally dissipative and maximally relations to graphs and inverses of graphs.

Lemma 5.2.3. *Let $\mathcal{M}$ be a maximally dissipative or self-adjoint linear relation in $\mathbb{K}^n$. Let* $\dim\ker\mathcal{M} = k$ *and* $\dim \operatorname{mul} \mathcal{M} = m$. *Further let $K, M \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times k}$, and $C \in \mathbb{K}^{n\times m}$ given by Cor. 1.3.11 such that*

$$\begin{aligned}\mathcal{M} &= \left\{ \ (Ke + B\lambda, e) \ \middle|\ \lambda \in \mathbb{K}^k \ \wedge B^*e = 0 \ \right\} \\ &= \left\{ \ (x, Mx + C\lambda) \ \mid \lambda \in \mathbb{K}^m \ \wedge \ C^*x = 0 \ \right\}.\end{aligned}$$

Then with

$$K^d_{\text{ext}} := \begin{bmatrix} K & B \\ -B^* & 0 \end{bmatrix}, \qquad M^d_{\text{ext}} := \begin{bmatrix} M & C \\ -C^* & 0 \end{bmatrix},$$
$$K^s_{\text{ext}} := \begin{bmatrix} K & B \\ B^* & 0 \end{bmatrix}, \qquad M^s_{\text{ext}} := \begin{bmatrix} M & C \\ C^* & 0 \end{bmatrix},$$

$\mathcal{M}$ extends to both gr M^d_{ext} *and* gr M^s_{ext}, *and $\mathcal{M}^{-1}$ extends to both* $(\text{gr } K^d_{\text{ext}})^{-1}$ *and* $(\text{gr } K^s_{\text{ext}})^{-1}$. *Further, if $\mathcal{M}$ is maximally dissipative (skew-adjoint), then* $(\text{gr } K^d_{\text{ext}})^{-1}$ *and* gr M^d_{ext} *are maximally dissipative (skew-adjoint), whereas if $\mathcal{M}$ is maximally nonnegative, then* $(\text{gr } -K^d_{\text{ext}})^{-1}$ *and* gr $-M^d_{\text{ext}}$ *are maximally dissipative.*

Proof. Step 1: We show that $\mathcal{M}$ extends to gr $\left[\begin{smallmatrix} M & C \\ \pm C^* & 0 \end{smallmatrix}\right]$. Let $(x,y) \in \mathcal{M}$. Then there exists some $\lambda \in \mathbb{K}^m$ such that $y = Mx + C\lambda$. Note that λ is unique since by Cor. 1.3.11 the columns of C form a basis of mul $\mathcal{M}$ and ran $M \subset$ dom $\mathcal{M} = (\text{mul } \mathcal{M})^\perp$. Further, $\pm C^*x = 0$. Overall,

$$\begin{pmatrix} Mx + C\lambda \\ 0 \end{pmatrix} = \begin{bmatrix} M & C \\ \pm C^* & 0 \end{bmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \tag{5.1}$$

i.e., $(x, \lambda, y, 0) \in$ gr $M^d_{\text{ext}} \cap$ gr M^s_{ext}. Conversely, let $(x, v, y, 0) \in$ gr $\left[\begin{smallmatrix} M & C \\ \pm C^* & 0 \end{smallmatrix}\right]$ or equivalently $(x, v, y, 0) \in$ gr $M^d_{\text{ext}} \cup$ gr M^s_{ext}. Then with (5.1) we directly see $(x,y) \in \mathcal{M}$.
Step 2: We show the properties of gr M^d_{ext}, gr M^s_{ext} that are inherited by $\mathcal{M}$. On the one hand, if $\mathcal{M}$ is maximally dissipative (skew-adjoint), then by Cor. 1.3.11 M is

dissipative (skew-Hermitian) and by construction M_{ext}^d too. Consequently, gr M_{ext}^d is maximally dissipative (skew-adjoint). On the other hand, if $\mathcal{M}$ is maximally non-negative (self-adjoint), then by Cor. 1.3.11 M is positive semi-definite (Hermitian), i.e., $-M$ dissipative and by construction $-M_{\text{ext}}^d$ is dissipative too. Consequently, gr $-M_{\text{ext}}^s$ is maximally nonnegative (self-adjoint).
Step 3: We prove the statements concerning (gr $K_{\text{ext}}^d)^{-1}$ and (gr $K_{\text{ext}}^s)^{-1}$. If $\mathcal{M}$ is maximally dissipative (skew-adjoint, self-adjoint, maximally nonnegative), then so is $\mathcal{M}^{-1}$. Hence, it suffices to apply Steps 1 & 2 to $\mathcal{M}^{-1}$, since mul $\mathcal{M}^{-1} = \ker \mathcal{M}$ and dom $\mathcal{M}^{-1} = $ ran $\mathcal{M}$. $\square$

5.3 Conversion of constraints in dH-pencils

With the four matrices constructed in Lem. 5.2.3, we can extend a pH-LR system $\mathcal{DL}$ to four different systems, each with one type of constraint removed, provided that the relation corresponding to the type of constraint is maximal. The first result of this kind we present enables us in particular to associate the behavior of a pH-LR system induced by a dH-LR pencil to the behavior of a pH-LA system induced by a dH-LA pencil, despite Ex. 2.5.8.

Proposition 5.3.1 (From (D1) to (L1) constraints). *Let $\mathcal{D}$ be a maximally dissipative relation in $\mathbb{K}^n$ with* $\dim \operatorname{mul} \mathcal{D} = m$ *and $\mathcal{L}$ a symmetric relation in $\mathbb{K}^n$. Let $M_{\text{ext}}^d \in \mathbb{K}^{(n+m)\times(n+m)}$ be the matrix given by Prop. 5.2.3 such that $\mathcal{D}$ extends to* gr M_{ext}^d. *Then*

$$\mathfrak{B}_{\mathcal{DL}} \times \{0\} = \mathfrak{B}_{(\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})}.$$

In particular, $\mathcal{DL}$ extends to (gr $M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$.

Proof. Step 1: We show that $(x,y) \in \mathcal{DL} \iff (x,0,y,0) \in (\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$. To this end, let $(x,y) \in \mathcal{DL}$. Then there exists some $e \in \mathbb{K}^n$ such that $(x,e) \in \mathcal{L}$ and $(e,y) \in \mathcal{D}$. By Prop. 5.2.3 there exits a unique $\lambda \in \mathbb{K}^m$ such that $(e,\lambda,y,0) \in \text{gr } M_{\text{ext}}^d$. Moreover, $(x,0,e,\lambda) \in \mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1}$. Hence, $(x,0,y,0) \in (\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$. Conversely, let now $(x,0,y,0) \in (\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$ be given. Then there exists $(e,\lambda) \in \mathbb{K}^{n+m}$ such that $(x,0,e,\lambda) \in (\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$ and $(e,\lambda,y,0) \in \text{gr } M_{\text{ext}}^d$. We directly deduce $(x,e) \in \mathcal{L}$ and since $\mathcal{D}$ extends to gr M_{ext}^d also $(e,y) \in \mathcal{D}$. Combining both we obtain $(x,y) \in \mathcal{DL}$.
Step 2: We show the forward inclusion "$\subseteq$". To this end let $x \in \mathfrak{B}_{\mathcal{DL}}$. Then by

Prop. 2.4.4,

$$(x(t), \dot{x}(t)) \in \mathcal{DL},$$

for almost all $t \in \mathbb{R}$. Setting $\Lambda \equiv 0$ we obtain with Step 1

$$(x(t), \Lambda(t), \dot{x}(t), \dot{\Lambda}(t)) \in (\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1}).$$

Invoking Prop. 2.4.4 yields $(x, \Lambda) \in \mathfrak{B}_{(\text{gr } M_{\text{ext}}^d)(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})}$.
<u>*Step 3*</u>: We show the backward inclusion "$\supseteq$". Let $(x, \Lambda) \in \mathfrak{B}_{\text{gr } M_{\text{ext}}^d(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})}$ be given. Then there exist functions $e_S(\cdot) : \mathbb{R} \to \mathbb{K}^n$, $e_m(\cdot) : \mathbb{R} \to \mathbb{K}^m$ such that

$$(x(t), \Lambda(t), e_S(t), e_m(t)) \in \mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1} \quad \text{and} \quad (e_S(t), e_m(t), \dot{x}(t), \dot{\Lambda}(t)) \in \text{gr } M_{ext}^d,$$

for almost all $t \in \mathbb{R}$. From the latter we deduce $\Lambda(t) = 0$ and hence $\dot{\Lambda}(t) = 0$ for almost all $t \in \mathbb{R}$. From Step 1 we know

$$(x(t), \dot{x}(t)) \in \mathcal{DL},$$

i.e., $x \in \mathfrak{B}_{\mathcal{DL}}$ by Prop. 2.4.4, completing this step. □

Proposition 5.3.2 (From (D2) to (L2) constraints)**.** *Let $\mathcal{D}$ be a maximally dissipative relation in $\mathbb{K}^n$ with $\dim\ker \mathcal{D} = k$ and $\mathcal{L}$ a symmetric relation in $\mathbb{K}^n$. Let $K_{\text{ext}}^d \in \mathbb{K}^{(n+k)\times(n+k)}$ be the matrix given by Prop. 5.2.3 such that $\mathcal{D}^{-1}$ extends to $\text{gr } K_{\text{ext}}^d$. Then $\mathcal{DL}$ extends to $(\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$.*

Proof. <u>*Step 1*</u>: We show that for all $(x, y) \in \mathcal{DL}$ the exists a unique $\lambda \in \mathbb{K}^k$ such that there exists a $\mu \in \mathbb{K}^k$ with $(x, \mu, y, \lambda) \in (\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$. Further, $(x, \mu', y, \lambda) \in (\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$ for all $\mu \in \mathbb{K}^k$. In order to show this, let $(x, y) \in \mathcal{DL}$. Then there exists some $e \in \mathbb{K}^n$ such that $(x, e) \in \mathcal{L}$ and $(e, y) \in \mathcal{D}$. Since $\mathcal{D}^{-1}$ extends to $\text{gr } K_{\text{ext}}^d$, there exists a unique $\lambda \in \mathbb{K}^k$ such that $(e, 0, y, \lambda)$ and it is readily seen that $(x, \mu, e, 0) \in \mathcal{L}\widehat{\times} \text{ gr } 0_k$ for all $\mu \in \mathbb{K}^k$. Hence, $(x, \mu, y, \lambda) \in (\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$, showing the desired statements of this step.
<u>*Step 2*</u>: We show that for all $(x, \mu, y, \lambda) \in (\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$ holds $(x, y) \in \mathcal{DL}$. Given $(x, \mu, y, \lambda) \in (x, \Lambda_k) \in (\text{gr } K_{\text{ext}}^d)^{-1}(\mathcal{L}\widehat{\times} \text{ gr } 0_k)$, there exists $(e_1, e_2) \in \mathbb{K}^{n+k}$ such that

$$(e_1, e_2, y, \lambda) \in (x, \Lambda_k) \in (\text{gr } K_{\text{ext}}^d)^{-1} \quad \text{and} \quad (x, \mu, e_1, e_2) \in \mathcal{L}\widehat{\times} \text{ gr } 0_k.$$

We directly deduce $e_2 = 0$ with $(x, e_1) \in \mathcal{L}$ and since $\mathcal{D}^{-1}$ extends to $\text{gr } K_{\text{ext}}^d$, $(e_1, y) \in \mathcal{D}$. Combining both we obtain $(x, y) \in \mathcal{DL}$.

Step 3: We show the forward implication "$\Longrightarrow$" of the statement. To this end, let $x \in \mathfrak{B}_{\mathcal{DL}}$. By Prop. 2.4.4

$$(x(t), \dot{x}(t)) \in \mathcal{DL},$$

for almost all $t \in \mathbb{R}$. Let range representations

$$\mathcal{L} = \operatorname{ran} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \operatorname{ran} \begin{bmatrix} D_1 \\ L_1 \end{bmatrix}.$$

be given. Then stetting

$$\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix} \right) := \begin{bmatrix} L_1 & 0 \\ L_2 & -D_1 \\ 0 & D_2 \end{bmatrix}^\dagger \begin{pmatrix} x \\ 0 \\ \dot{x} \end{pmatrix}$$

and $e_S := L_2 z_1 = D_1 z_2 \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n)$, we obtain

$$(x(t), e_S(t)) \in \mathcal{L} \quad \text{and} \quad (e_S(t), \dot{x}(t)) \in \mathcal{D},$$

for almost all $t \in \mathbb{R}$. Since $\mathcal{D}^{-1}$ extends to gr K^d_{ext}, there exists a unique function $\lambda(\cdot) : \mathbb{R} \to \mathbb{K}^k$ such that

$$(e_S(t), 0, y(t), \lambda(t)) \in (\text{gr } K^d_{\mathrm{ext}})^{-1},$$

from which follows that $\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$. Hence defining $\Lambda_k : \mathbb{R} \to \mathbb{K}^k$ by $t \mapsto \int_0^t \lambda(\tau)\, \mathrm{d}\tau$ gives $\Lambda_k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^k)$. Step 1 enables us to write

$$(x(t), \Lambda_k(t), e_S(t), 0) \in \mathcal{L} \widehat{\times} \text{ gr } 0_k \quad \text{and} \quad (e_S(t), 0, \dot{x}(t), \dot{\Lambda}_k(t)) \in (\text{gr } K^d_{\mathrm{ext}})^{-1},$$

i.e.,

$$(x(t), \Lambda_k(t), \dot{x}(t), \dot{\Lambda}_k(t)) \in (\text{gr } K^d_{\mathrm{ext}})^{-1}(\mathcal{L} \widehat{\times} \text{ gr } 0_k)$$

for almost all $t \in \mathbb{R}$. Now Prop. 2.4.4 implies $(x, \Lambda_k) \in \mathfrak{B}_{(\text{gr } K^d_{\mathrm{ext}})^{-1}(\mathcal{L} \widehat{\times} \text{ gr } 0_k)}$.

Step 4: We show the backward implication "$\Longleftarrow$" of the statement. To this end let $(x, \Lambda_k) \in \mathfrak{B}_{(\text{gr } K^d_{\mathrm{ext}})^{-1}(\mathcal{L} \widehat{\times} \text{ gr } 0_k)}$. Then

$$(x(t), \Lambda_k(t), \dot{x}(t), \dot{\Lambda}_k(t)) \in (\text{gr } K^d_{\mathrm{ext}})^{-1}(\mathcal{L} \widehat{\times} \text{ gr } 0_k),$$

for almost all $t \in \mathbb{R}$ and hence

$$(x(t), \dot{x}(t)) = \mathcal{DL},$$

for almost $t \in \mathbb{R}$ by Step 2. Invoking Prop. 2.4.4 shows $x \in \mathfrak{B}_{\mathcal{DL}}$, completing the proof. $\square$

Proposition 5.3.3 (From (L1) to (D1) constraints)**.** *Let $\mathcal{D}$ be a dissipative relation in $\mathbb{K}^n$ and $\mathcal{L}$ a self-adjoint relation in $\mathbb{K}^n$ with* $\dim \operatorname{mul} \mathcal{L} = m$*. Let $M^s_{\text{ext}} \in \mathbb{K}^{(n+m)\times(n+m)}$ be the matrix given by Prop. 5.2.3 such that $\mathcal{L}$ extends to* $\operatorname{gr} M^s_{\text{ext}}$*. Then $\mathcal{D}\mathcal{L}$ extends to $(\mathcal{D}\widehat{\times}(\operatorname{gr} 0_k)^{-1})(\operatorname{gr} M^s_{\text{ext}})$.*

Proof. The proof is completely analogous to the proof of Prop. 5.3.2. □

Proposition 5.3.4 (From (L2) to (D2) constraints)**.** *Let $\mathcal{D}$ be a dissipative relation in $\mathbb{K}^n$ and $\mathcal{L}$ a self-adjoint relation in $\mathbb{K}^n$ with* $\ker \operatorname{mul} \mathcal{D} = k$*. Let $K^s_{\text{ext}} \in \mathbb{K}^{(n+m)\times(n+m)}$ be the matrix given by Prop. 5.2.3 such that $\mathcal{L}^{-1}$ extends to* $\operatorname{gr} K^s_{\text{ext}}$*. Then*

$$\mathfrak{B}_{\mathcal{D}\mathcal{L}} \times \{0\} = \mathfrak{B}_{(\mathcal{D}\widehat{\times} \operatorname{gr} 0_k)(\operatorname{gr} K^s_{\text{ext}})^{-1}}.$$

In particular, $\mathcal{D}\mathcal{L}$ extends to $(\mathcal{D}\widehat{\times} \operatorname{gr} 0_k)(\operatorname{gr} K^s_{\text{ext}})^{-1}$.

Proof. The proof is completely analogous to the proof of Prop. 5.3.1. □

The titles of Props. 5.3.1–5.3.4 are fitting. If we call $\mathcal{D}\mathcal{L}$ the original system and $\tilde{\mathcal{D}}\tilde{\mathcal{L}}$ the resulting system of these propositions, then in Prop. 5.3.1 $\dim \operatorname{mul} \tilde{\mathcal{D}} = 0$ and $\dim \operatorname{mul} \tilde{\mathcal{L}} = \dim \operatorname{mul} \mathcal{L} + \dim \operatorname{mul} \mathcal{D}$, in Prop. 5.3.2 $\dim \ker \tilde{\mathcal{D}} = 0$ and $\dim \ker \tilde{\mathcal{L}} = \dim \ker \mathcal{L} + \dim \ker \mathcal{D}$, in Prop. 5.3.3 $\dim \operatorname{mul} \tilde{\mathcal{L}} = 0$ and $\dim \operatorname{mul} \tilde{\mathcal{D}} = \dim \operatorname{mul} \mathcal{D} + \dim \operatorname{mul} \mathcal{L}$, and in Prop. 5.3.1 $\dim \ker \tilde{\mathcal{L}} = 0$ and $\dim \ker \tilde{\mathcal{D}} = \dim \ker \mathcal{D} + \dim \ker \mathcal{L}$.

Remark 5.3.5. Although Props. 5.3.1 & 5.3.3 are obtained similarly to the results found in [vdSM18, Sec. 3.1], there is a notable difference. In [vdSM18] the state variable of the original was extended by the 'Lagrange multiplier' λ, which represents the multivalued part of either $\mathcal{D}$ or $\mathcal{L}$ depending on what type of constaints is being replaced. By doing so, λ is formally differentiated in the formulation of the system's dynamics, which a priori may impose additional constraints to the solutions of the original system.
In Prop. 5.3.1, we do not make use of this Lagrange multiplier in order to extend the state of the original system, although we extend the state of the original system by the same length as λ. However, this variable is always trivial in the dynamics of the augmented system, which is not the case of λ.
In Prop. 5.3.3, we make use of this Lagrange multiplier in order to extend the system variable, but differently. As can be taken from the proof of Prop. 5.3.3, instead of directly taking λ as additional state variable, we integrate it and extend the state of the original system by this primitive Λ.

It is clear from the definition of our notion of an extension (Def. 5.2.1) that no trajectory of the original system is lost and no additional obtained in the augmented system.

Note that while gr M^d_{ext} and (gr $K^d_{\text{ext}})^{-1}$ become skew-adjoint if $\mathcal{D}$ is skew-adjoint in the context of Lem. 5.2.3, gr M^s_{ext} and (gr $K^s_{\text{ext}})^{-1}$ do not become nonnegative if $\mathcal{L}$ is nonnegative as M^s_{ext} and K^s_{ext} are indefinite. This leads to the observation that gr $-M^d_{\text{ext}}$ and (gr $-K^d_{\text{ext}})^{-1}$ are dissipative if $\mathcal{L}$ is nonnegative. Hence, extending $\mathcal{L}$ with gr $-M^d_{\text{ext}}$ and (gr $-K^d_{\text{ext}})^{-1}$ instead of gr M^s_{ext} and (gr $K^s_{\text{ext}})^{-1}$ does not yield a dissipative-Hamiltonian system in this case. In turns out that gr M^s_{ext} and (gr $K^s_{\text{ext}})^{-1}$ and using this particular extension, we can link dissipative-Hamiltonian pencils to certain pencils recently discussed in [MMW22].

Definition 5.3.6. A pencil $sE - A \in \mathbb{K}^{n\times n}$ is said to have *positive semidefinite Hermitian part coefficients* (posH pencil) if $-E$ and A are dissipative.

Corollary 5.3.7 (From a dH-LR pencil to a posH pencil)**.**
Let $sE - A$ be a dH-LR pencil with

$$\operatorname{ran} \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

for some maximally dissipative and maximally nonnegative relations $\mathcal{D}$ and $\mathcal{L}$, respectively. Further, let $\dim \operatorname{mul} \mathcal{D} = m$ *and* $\dim \ker \mathcal{L} = k$ *and let $K, M \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times k}$, and $C \in \mathbb{K}^{n\times m}$ given by Cor. 1.3.11 such that*

$$\begin{aligned}
\mathcal{L} &= \left\{ \ (Ke + B\lambda, e) \ \middle| \ \lambda \in \mathbb{K}^k \ \wedge B^* e = 0 \ \right\}, \\
\mathcal{D} &= \{ \ (x, Mx + C\lambda) \ \mid \lambda \in \mathbb{K}^m \ \wedge \ C^* x = 0 \ \}.
\end{aligned}$$

Define the pair $(L, D) \in \mathbb{K}^{(n+m+k)\times(n+m+k)} \times \mathbb{K}^{(n+m+k)\times(n+m+k)}$ as either

$$\left(\begin{bmatrix} K & 0 & B \\ 0 & 0 & 0 \\ -B^* & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & C & 0 \\ -C^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \quad \text{or} \quad \left(\begin{bmatrix} K & B & 0 \\ -B^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & 0 & C \\ 0 & 0 & 0 \\ -C^* & 0 & 0 \end{bmatrix} \right).$$

Then $sL - D$ is a posH pencil and

$$\mathfrak{B}_{\mathcal{D}\mathcal{L}} \times \{0\}^{m+k} = L\mathfrak{B}_{[L,D]}.$$

In particular for $x \in \mathfrak{B}_{\mathcal{D}\mathcal{L}}$ we have

$$(x, 0, 0, \dot{x}, 0, 0) \in (\text{gr } D)(\text{gr } L)^{-1} = \operatorname{ran} \begin{bmatrix} L \\ D \end{bmatrix}.$$

Further, if $\mathcal{D}$ is skew-adjoint, then gr D *is skew-adjoint too.*

Proof. With the notation of the statement, let

$$\tilde{L} = \begin{bmatrix} K & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} K & 0 & B \\ 0 & 0 & 0 \\ -B^* & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} M & C & 0 \\ -C^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

<u>*Step 1*</u>: We extend $\mathcal{DL}$ with Prop. 5.3.1. To this end, let $M^d_{\text{ext}} \in \mathbb{K}^{(n+m)\times(n+m)}$ be the matrix given by Lem. 5.2.3 such that $\mathcal{D}$ extends to gr M^d_{ext}. Then by Prop. 5.3.1

$$\mathfrak{B}_{\mathcal{DL}} \times \{0\} = \mathfrak{B}_{(\text{gr } M^d_{\text{ext}})(\mathcal{L} \widehat{\times} (\text{gr } 0_m)^{-1})}.$$

In particular, $\mathcal{DL}$ extends to $(\text{gr } M^d_{\text{ext}})(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$.
<u>*Step 2*</u>: We extend $(\text{gr } M^d_{\text{ext}})(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$ with Prop. 5.3.4. To this end, note that $\dim\ker\mathcal{L} = \dim\ker\mathcal{L}\widehat{\times}\text{ gr } 0_m = k$ with

$$\mathcal{L}\widehat{\times}\text{ gr } 0_m = \left\{ \left(\left[\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} e \\ x \end{smallmatrix}\right) + \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right] \lambda, \left(\begin{smallmatrix} e \\ x \end{smallmatrix}\right) \right) \;\middle|\; \lambda \in \mathbb{K}^k \;\wedge\; \left[\begin{smallmatrix} B^* & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} e \\ x \end{smallmatrix}\right) = 0 \right\}.$$

Further, $\text{gr } M^d_{\text{ext}} \widehat{\times} \text{ gr } 0_k = \text{gr } D$. Now Prop. 5.3.4 shows that

$$\mathfrak{B}_{(\text{gr } M^d_{\text{ext}})(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})} \times \{0\} = \mathfrak{B}_{(\text{gr } D)(\text{gr } \tilde{L})^{-1}}.$$

In particular, $(\text{gr } M^d_{\text{ext}})(\mathcal{L}\widehat{\times}(\text{gr } 0_m)^{-1})$ extends to $(\text{gr } D)(\text{gr } \tilde{L})^{-1}$.
<u>*Step 3*</u>: We derive $\mathfrak{B}_{\mathcal{DL}} \times \{0\}^{n+k} = L\mathfrak{B}_{[L,D]}$. The two previous steps show that $\mathfrak{B}_{\mathcal{DL}} \times \{0\}^{n+k} = \tilde{L}\mathfrak{B}_{[\tilde{L},D]}$. It therefore suffices to show $\tilde{L}\mathfrak{B}_{[\tilde{L},D]} = L\mathfrak{B}_{[L,D]}$. Consider

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} K & 0 & B \\ 0 & 0 & 0 \\ \pm B^* & 0 & 0 \end{bmatrix} z = \begin{bmatrix} M & C & 0 \\ -C^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z,$$

i.e., the constitutive equations of $[\tilde{L}, D]$ and $[L, D]$, depending on the sign, and it is readily seen that $\mathfrak{B}_{[\tilde{L},D]} = \mathfrak{B}_{[L,D]}$. Moreover, we see that for $z \in \mathfrak{B}_{[L,D]}$ holds $\left[\begin{smallmatrix} 0_{n+m} \\ I_k \end{smallmatrix}\right] z = 0$. Combining both, we obtain

$$\mathfrak{B}_{\mathcal{DL}} \times \{0\}^{n+k} = \tilde{L}\mathfrak{B}_{[\tilde{L},D]} = \left[\begin{smallmatrix} I_{n+m} & 0 \\ 0 & -I_k \end{smallmatrix}\right] L\mathfrak{B}_{[L,D]} = L\mathfrak{B}_{[L,D]}.$$

Now for $x \in \mathfrak{B}_{[\mathcal{DL}]}$ there exists some $z \in \mathfrak{B}_{[L,D]}$ such that

$$\left(\begin{smallmatrix} x \\ 0 \\ 0 \end{smallmatrix}\right) = Lz \text{ and hence } \left(\begin{smallmatrix} \dot{x} \\ 0 \\ 0 \end{smallmatrix}\right) = Dz,$$

i.e.,

$$(x, 0, 0, \dot{x}, 0, 0) \in (\text{gr } D)(\text{gr } L)^{-1} = \text{ran} \begin{bmatrix} L \\ D \end{bmatrix}.$$

Step 4: We show the properties of L and D. Cor. 1.3.11 shows that K is positive semi-definite if $\tilde{L}$ is maximally nonnegative and M dissipative (skew-Hermitian) if $\mathcal{D}$ is maximally dissipative (skew-adjoint). Hence by construction, $-L$ and D are dissipative. Further, if $\mathcal{D}$ is skew-adjoint then D is skew-Hermitian and gr D is skew-adjoint.

Step 5: The statement is proven for the choices of

$$\tilde{L} = \begin{bmatrix} K & B & 0 \\ B^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} K & B & 0 \\ -B^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} M & 0 & C \\ 0 & 0 & 0 \\ -C^* & 0 & 0 \end{bmatrix}.$$

It suffices to reproduce the steps 1–4 but inverting the order in which Props. 5.3.1 & 5.3.4 are invoked. □

Remark 5.3.8. The proof technique of Cor. 5.3.7 shows that if $\mathcal{D}$ and $\mathcal{L}$ are maximally dissipative and maximally symmetric relations, respectively, then $\mathcal{D}\mathcal{L}$ can be extended to a system $\tilde{\mathcal{D}}\tilde{\mathcal{L}}$ with neither (D1) nor (L2) constraints and to a system $\hat{\mathcal{D}}\hat{\mathcal{L}}$ with neither (D2) nor (L1) constraints.

Chapter 6

Electrical circuits and port-Hamiltonian systems

Tremendous progress has been recently made in port-Hamiltonian modelling of constrained dynamical systems, which leads to differential-algebraic equations [BMX$^+$18; MMW18; vdSM18; vdSM20; vdSch13]. This enables to apply the framework to modelling of multibody systems with holonomic and non-holonomic constraints as well as electrical circuits. Examples of the latter class has been considered from a port-Hamiltonian point of view in [vdSch10; vdSJ14; vdSch13; VvdS10a]. However, an approach to electrical circuits has been only made for the case where the circuit contains only capacitances and inductances [BMvdS95]. The recent progress in port-Hamiltonian differential-algebraic equations however allows to treat a by far wider class of electrical circuits, see also [NPS22]. This is exactly the purpose of this chapter, where we consider a variety of electrical components, such as resistances, capacitances, inductances, diodes, transformers, transistors, current sources and voltage sources from a port-Hamiltonian perspective. To be more precise, we present pH-NG systems describing physical models for these electrical components governed by nonlinear equations. Thereafter, we introduce a pH-NG system representing the circuit interconnection structure by utilizing the underlying graph of the given electrical circuit. This gives rise to a port-Hamiltonian model, which only incorporates the Kirchhoff laws. Finally, the port-Hamiltonian model of the electrical circuit is obtained by an interconnection with the individual port-Hamiltonian systems representing the components. The resulting system will be compared to other classical formulations of electrical circuits.

6.1 Port-Hamiltonian systems on graphs

We present some basic graph theoretical notions from [Die17] required to formulate port-Hamiltonian systems on graphs [MvdS13].

Definition 6.1.1 (Graphs and subgraphs)**.** A *directed graph* is a quadruple $\mathcal{G} = (V, E, \operatorname{init}, \operatorname{ter})$ consisting of a *vertex set* V, an *edge set* E and two maps $\operatorname{init}, \operatorname{ter} : E \to V$ assigning to each edge e an *initial vertex* $\operatorname{init}(e)$ and a *terminal vertex* $\operatorname{ter}(e)$. The edge e is said to be *directed from* $\operatorname{init}(e)$ *to* $\operatorname{ter}(e)$. $\mathcal{G}$ is said to be *loop-free*, if $\operatorname{init}(e) \neq \operatorname{ter}(e)$ for all $e \in E$. Let $V' \subset V$ and $E' \subset E$ with

$$E' \subset E\restriction_{V'} := \{e \in E : \operatorname{init}(e) \in V' \wedge \operatorname{ter}(e) \in V'\}.$$

Then the triple $(V', E', \operatorname{init}\restriction_{E'}, \operatorname{ter}\restriction_{E'})$ is called a *subgraph of* $\mathcal{G}$. If $E' = E\restriction'_V$, then the subgraph is called the *induced subgraph* on V'. If $V' = V$, then the subgraph is called *spanning*. Additionally, a *proper subgraph* is one where $E' \neq E$. $\mathcal{G}$ is called *finite*graph!finite, if V and E are finite.

The notion of a *path* in a directed graph $\mathcal{G} = (V, E, \operatorname{init}, \operatorname{ter})$ is quite descriptive. However, since a path may also go through an edge in reverse direction, we define for each $e \in E$ an additional edge $-e \notin E$ with $\operatorname{init}(-e) = \operatorname{ter}(e)$ and $\operatorname{ter}(-e) = \operatorname{init}(e)$.

Definition 6.1.2 (Paths, connectivity, cycles, forests and trees)**.**
Let $\mathcal{G} = (V, E, \operatorname{init}, \operatorname{ter})$ be a directed loop-free finite graph. For $r \in \mathbb{N}$, an r-tuple $e = (e_1, \ldots, e_r) \in E^r$ is called a *path from* v *to* w, if there exists an r-tuple $(n_1, \ldots, n_r) \in \{0, 1\}^r$ such that

$$\begin{aligned}
&\left|\bigcup_{i=1}^{r} \{\operatorname{init}(e_i), \operatorname{ter}(e_i)\}\right| = r + 1,\\
&\operatorname{ter}((-1)^{n_i} e_i) = \operatorname{init}((-1)^{n_{i+1}} e_{i+1}) \ \ \forall i \in \{1, \ldots, r-1\},\\
&\operatorname{init}((-1)^{n_1} e_1) = v \wedge \operatorname{ter}((-1)^{n_r} e_r) = w.
\end{aligned}$$

For $r \geq 2$, an $r - tuple$ $(e_1, \ldots, e_r)$ is a *cycle* if $(e_1, \ldots, e_{r-1})$ is a path from v to w and e_r is a path from w to v for some $v, w \in V$. Two vertices v, w are *connected*, if there is a path from v to w. This gives an equivalence relation on the vertex set. The induced subgraph on an equivalence class of connected vertices gives a *component* of the graph. A graph is called *connected*, if there is only one component. A subgraph $\mathcal{K} = (V, E', \mathrm{init}\restriction_{E'}, \mathrm{ter}\restriction_{E'})$ of a directed graph $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ is called a *spanning forest* in $\mathcal{G}$, if $\mathcal{K}$ does not contain any cycles and is maximal with this property, that is, $\mathcal{K}$ is not a proper subgraph of a subgraph of $\mathcal{G}$ which does not contain any cycles. A subgraph $\mathcal{K}$ is called *tree*, if it does not contain any cycles and is connected.

In the context of electrical circuits, finite and loop-free directed graphs are of major importance and associate a special matrix to these graphs [And91, Sec. 3.2].

Definition 6.1.3 (Incidence matrix)**.** Let $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ be a finite and loop-free directed graph. Let $E = \{e_1, \ldots, e_m\}$ and $V = \{v_1, \ldots, v_n\}$. Then the *incidence matrix* is $A_0 \in \mathbb{R}^{n\times m}$ of $\mathcal{G}$ is defined entry-wise through

$$a_{jk} = \begin{cases} 1 & \mathrm{init}(e_k) = v_j, \\ -1 & \mathrm{ter}(e_k) = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{G}$ has $k \in \mathbb{N}$ components if and only if $\mathrm{rk}\, A_0 = n - k$ [And91, p. 140]. This allows to remove up to k rows from A_0 such that a matrix with same rank is obtained. The choice of these to-be-deleted rows has to be done in a special way: One has to choose a row set, which corresponds to a vertex set S that contains at most one vertex per component of $\mathcal{G}$. This deletion plays a crucial role in the definition of the notions of *Kirchhoff-Dirac structure* and *Kirchhoff-Lagrange structure*. Later we will show that interconnections with these structures correspond to the Kirchhoff laws in electrical circuits.

Definition 6.1.4 (Kirchhoff-Dirac structure, Kirchhoff-Lagrange structure)**.**
Assume that $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ is a finite and loop-free directed graph with incidence matrix $A_0 \in \mathbb{R}^{n\times m}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be the components of $\mathcal{G}$ and let $V_1, \ldots, V_k \subset V$ be the corresponding vertex sets. Let $S \subset V$ such that S contains at most one vertex from each component, that is

$$\forall\, v, v' \in S,\ i \le k: \quad (v, v' \in V_i) \Longrightarrow (v = v'). \tag{6.1}$$

Let $A \in \mathbb{R}^{(n-k)\times m}$ be constructed from $A_0 \in \mathbb{R}^{n\times m}$ by deleting the rows corresponding to the vertices from S. The *Kirchhoff-Dirac structure* of $\mathcal{G}$ is

$$\mathcal{D}_K^S(\mathcal{G}) := \Bigg\{ (j, i, \phi, u) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \;\Bigg|\; \begin{bmatrix} I & A \\ 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ i \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ A^\top & -I \end{bmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = 0 \Bigg\}. \tag{6.2}$$

Assume that $S = \{v_1, ..., v_{|S|}\}$ (which is - by a reordering of the vertices - no loss of generality). Then the *Kirchhoff-Lagrange structure* of $\mathcal{G}$ with respect to S is

$$\mathcal{L}_K^S(\mathcal{G}) := \{0\} \times \mathbb{R}^{n-|S|} \subset \mathbb{R}^{n-|S|} \times \mathbb{R}^{n-|S|}. \tag{6.3}$$

Remark 6.1.5. By Rem. 1.3.4, $\mathcal{D}_K^S(\mathcal{G})$ in (6.2) is a Dirac structure, whereas Prop. 1.4.4 implies that $\mathcal{L}_K^S(\mathcal{G})$ in (6.3) is a Lagrangian submanifold of $\mathbb{R}^{n-|S|} \times \mathbb{R}^{n-|S|}$. The concepts of Def. 6.1.4 allow to introduce the pH-NG system $(\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\})$ with dynamics

$$(-\tfrac{\mathrm{d}}{\mathrm{d}t}\, q(t), i(t), \phi(t), u(t)) \in \mathcal{D}_K^S(\mathcal{G}), \quad (q(t), \phi(t)) \in \mathcal{L}_K^S(\mathcal{G}). \tag{6.4}$$

Then, by the equivalence of $(q(t), \phi(t)) \in \mathcal{L}_K^S(\mathcal{G})$ to $q(t) = 0$ and $\phi(t) \in \mathbb{R}^{n-|S|}$, we see that (6.4) holds if and only if

$$q(t) = 0 \,\wedge\, Ai(t) = 0 \,\wedge\, A^\top \phi_2(t) = -u(t).$$

In particular, $i(t) \in \ker A$ and $u(t) \in \operatorname{im} A^\top$. In the context of electrical circuits, this will indeed represent Kirchhoff's current and voltage law [Rei14, Thm. 4.5 & Thm. 4.6]. The choice of S can be interpreted as the set of *grounded vertices*. The quantities q, i, ϕ and u can respectively be thought as the *vertex charges, the edge currents, the vertex potentials,* and the *edge voltages*.

Note that (6.4) is indeed a pH-NG system. However, this system is of rather pathological nature, since it does not contain any 'true dynamics', as the differential variable q is nulled by the Lagrangian submanifold. Note that these 'true dynamics' come into play later on, when we interconnect with dynamic circuit elements like capacitances and inductances.

In the terminology of [MvdS13], $\mathcal{D}_K^S(\mathcal{G})$ corresponds to the Kirchhoff-Dirac structure of a graph when $|S| = \emptyset$. Moreover, a Dirac structure similar to (6.2) has been used in [vdSch10], with the main difference being that in our present case all nodes are considered to be 'boundary nodes' in the nomenclature of [vdSch10].

We briefly present an alternative (slightly less straight-forward) construction of pH-NG systems on graphs, namely by means of cycles instead to vertices. For a given spanning forest $\mathcal{T}$ of a loop-free directed graph $\mathcal{G}$ with n edges, m vertices and k connected components, the minimality property yields that the incorporation of any edge of $\mathcal{G}$ not belonging to $\mathcal{T}$ (called *chord*) results in a subgraph with exactly one cycle. Consequently, the set of edges in the complement of $\mathcal{T}$ in $\mathcal{G}$ leads to a set $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_{m-n+k}\}$ of cycles, the so-called *fundamental cycles* (see [And91, p. 148] & [Die17, p. 26]). We equip each fundamental cycle with the orientation of its corresponding chord [And91, p. 148] and consider the associated *fundamental cycle matrix* $B \in \mathbb{R}^{(m-n+k)\times m}$ which is defined entrywise by (cf. [And91, Sec. 3.3])

$$b_{jl} = \begin{cases} 1 & e_l \in C_j \text{ and the orientations agree,} \\ -1 & e_l \in C_j \text{ and the orientations do not agree,} \\ 0 & \text{otherwise.} \end{cases}$$

This enables us to introduce the following Dirac structure and Lagrangian submanifold

$$\begin{aligned} \mathcal{D}'_K(\mathcal{G}) &:= \Bigg\{ (\varphi, u, \iota, i) \in \mathbb{R}^{m-n+k} \times \mathbb{R}^m \times \mathbb{R}^{m-n+k} \times \mathbb{R}^m \;\Bigg|\; \\ &\qquad \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \varphi \\ u \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ B^\top & -I \end{bmatrix} \begin{pmatrix} \iota \\ i \end{pmatrix} = 0 \Bigg\}, \\ \mathcal{L}'_K(\mathcal{G}) &:= \{0\} \times \mathbb{R}^{n-m+k}, \end{aligned} \tag{6.5}$$

which form the pH-NG system $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ with dynamics

$$(-\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t), i(t), \iota(t), u(t)) \in \mathcal{D}'_K(\mathcal{G}), \quad (\psi(t), \iota(t)) \in \mathcal{L}'_K(\mathcal{G}), \tag{6.6}$$

from which, analogous to Rem. 6.1.5, one can derive that (6.6) is equivalent to $\psi(t) = 0$, $Bu(t) = 0$ and $i(t) = B^\top \iota$. Since $\operatorname{im} B = \ker A^\top$ [Rei14, Thm. 4.4], the relations $u(t) \in \ker B = 0$ and $i(t) \in \operatorname{im} B^\top$ respectively represent Kirchhoff's voltage and current law. The quantities ψ, u, ι and i can respectively be thought as the *cycle fluxes, the edge voltages, the cycle currents* and the *edge currents.*

6.2 Electrical circuits as port-Hamiltonian systems

Our essential idea to port-Hamiltonian modelling of electrical circuits is to extend the tuple of voltages across and currents through the edges - in the case where we

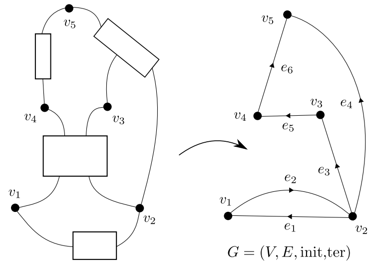

Figure 6.1: Obtaining the underlying graph of the electrical circuit.

consider a vertex-based formulation of the Kirchhoff laws - by vertex charges and potentials, and - in the case where we consider a loop-based formulation of the Kirchhoff laws - by cycle fluxes and cycle currents, along with an accordant modelling of the graph interconnection structure by means of the approach in the preceding section. The electrical components are modelled by separate pH-NG systems, and thereafter coupled with the one representing the interconnection structure.

The circuits may be composed of *two-terminal* and *multi-terminal* components. We will speak of ℓ_t*-terminal components*, with $\ell_t \in \mathbb{N}$ denoting the number of terminals [Wil10]. Each ℓ_t-terminal component connects ℓ_t vertices of the electrical circuit through its terminals. For instance, a resistance has two terminals, whereas a transistor has three terminals, and a transformer has four terminals. To regard an electrical circuit as a graph (see Fig. 6.1), we need to replace the ℓ_t-terminal components by ℓ_p edges between the vertices they connect, for some $\ell_p \in \mathbb{N}$, which we call the number of ports. Such a device is also called a ℓ_p*-port component.* This replacement is displayed in Fig. 6.1. The direction assigned to each edge is not a physical restriction but rather a definition of the *positive direction* of the corresponding voltage and current [Rei14]. The physical properties of the electrical components will be reflected by port-Hamiltonian dynamics on these edges. The replacement of an ℓ_t-terminal component by ℓ_p edges between vertices, i.e., by a graph, is subject to physical modelling. For further details on terminals, ports and their relation, we refer to [Wil10].

To be more precise, for $\ell_p, \ell_t \in \mathbb{N}$, an ℓ_t*-terminal component on* ℓ_p *edges* will be regarded as a pH-NG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$, where $\mathcal{D} \subset \mathbb{R}^{n_S+n_R+\ell_p} \times \mathbb{R}^{n_S+n_R+\ell_p}$, with $\ell_p = n_S + n_R$ for some $n_S, n_R \in \mathbb{N}_0$. We associate to $\mathcal{D}$ a graph $\mathcal{G} = (V, E, \text{init}, \text{ter})$ with $|V| = \ell_t$ and $|E| = \ell_p$ (cf. Fig. 6.3). The external flow and effort variables will always represent the current through [Rei14, Def. 3.2] and the voltage along [Rei14, Def. 3.6] the corresponding edges, respectively.

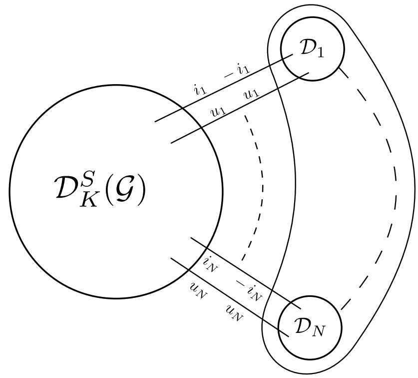

Figure 6.2: Visual representation of the Dirac structure $\mathcal{D}$ resulting from the interconnection (6.7).

6.2.1 Electrical circuits as interconnections of port-Hamiltonian systems

Let an electrical circuit consisting of N electrical components $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)_{i\in\{1,\ldots,N\}}$, each with $\ell_{p,i}$ ports, be given, with $N \in \mathbb{N}$ and let

$$(\mathcal{G}_i)_{i\in\{1,\ldots,N\}} = (V_i, E_i, \mathrm{init}_i, \mathrm{ter}_i)_{i\in\{1,\ldots,N\}}$$

be the respective graphs resulting from the physical modelling of the $\ell_{p,i}$-port components (see Fig. 6.3), where we assume that the edge sets $E_1, \ldots, E_N$ are disjoint. We define the *underlying graph of the circuit* $\mathcal{G}$ (see Fig. 6.1) as

$$\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter}) \coloneqq \left(\bigcup_{i=1}^{N} V_i, \bigcup_{i=1}^{N} E_i, \mathrm{init}, \mathrm{ter} \right),$$

with $\mathrm{init}(e) = \mathrm{init}_i(e)$ and $\mathrm{ter}(e) = \mathrm{ter}_i(e)$ if $e \in E_i$ for some $i \in \{1, ..., N\}$ and let $V = \{v_1, \ldots, v_n\}$, $E = \{e_1, \ldots, e_m\}$ for some $n, m \in \mathbb{N}$. Further, let $A_0 \in \mathbb{R}^{n\times m}$ be the incidence matrix associated to $\mathcal{G}$ and let $S \subset V$ with property (6.1) represent the vertices grounded in the circuit. We model the dynamics of the electrical circuits as the dynamics of the pH-NG system

$$(\mathcal{D}, \mathcal{L}, \mathcal{R}) \coloneqq (\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\}) \circ \left(\bigtimes_{i=1}^{N} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right), \tag{6.7}$$

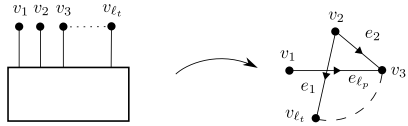

Figure 6.3: Replacing an ℓ_t-terminal component by a graph with ℓ_p edges.

Figure 6.4: Deriving the underlying graph of a capacitance, conductance, ideal diode, PN-junction diode, inductance, resistance, or sources.

where the interconnection is performed with respect to the flow and effort spaces

$$(\mathcal{F}_{\text{link}}, \mathcal{E}_{\text{link}}) = \left(\prod_{i=1}^{N} \mathbb{R}^{m_{P_i}}, \prod_{i=1}^{N} \mathbb{R}^{m_{P_i}} \right) = (\mathbb{R}^m, \mathbb{R}^m)$$

corresponding to the port variables associated to the currents and voltages of the ℓ_p-port components.

6.2.2 Physical modelling of circuit components as port-Hamiltonian systems

We present a couple of ‘prominent’ electrical components from a port-Hamiltonian viewpoint; among them are capacitances, inductances, resistances, diodes, transformers, transistors and sources. Note that this list is by no means complete. In principle, our approach also allows to incorporate components which are modelled by partial differential equations, such as transmission lines and refined models of semiconductor devices. This involves a further generalization of pH-NG systems on infinite-dimensional spaces and particularly leads to the notion of *Stokes-Dirac structure*, see [BKv+10; Mv04a; Mv04b; Rei21].

Throughout this section, i will denote currents and u will denote voltages. An oftentimes used Dirac structure will be, for $\ell_p \in \mathbb{N}$,

$$\mathcal{D}_{\ell_p} = \left\{ \begin{pmatrix} -i \\ i \\ u \\ u \end{pmatrix} \in \mathbb{R}^{4\ell_p} \;\middle|\; i, u \in \mathbb{R}^{\ell_p} \right\}. \tag{6.8}$$

It can easily verified that this is indeed a Dirac structure. The variable i stands for the vector of currents, whereas u is the vector of voltages in the component. Note that a copy of the voltage and negative of the current vector is required, since it is later on eliminated by the interconnection according to Def. 2.2.2.

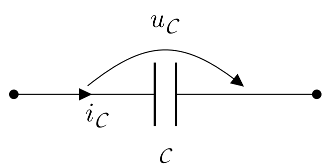

Figure 6.5: Capacitance: circuit symbol.

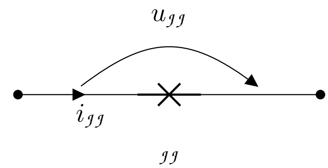

Figure 6.6: Josephson junction: circuit symbol.

Capacitances

Let $H_{\mathcal{C}} \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$. A capacitance with ℓ_p ports is modelled as a pH-NG system $(\mathcal{D}_{\mathcal{C}}, \mathcal{L}_{\mathcal{C}}, \mathcal{R}_{\mathcal{C}})$, where $\mathcal{D}_{\mathcal{C}} = \mathcal{D}_{\ell_p}$ with $\mathcal{D}_{\ell_p}$ as in (6.8), $\mathcal{R}_{\mathcal{C}} = \{0\}$, and

$$\mathcal{L}_{\mathcal{C}} = \left\{ (u_{\mathcal{C}}, q_{\mathcal{C}}) \in \mathbb{R}^{2\ell_p} \middle|\ q_{\mathcal{C}} = \nabla H_{\mathcal{C}}(u_{\mathcal{C}}) \right\}.$$

The dynamics consequently read

$$(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, q_{\mathcal{C}}(t), i_{\mathcal{C}}(t), u_{\mathcal{C}}(t), u_{\mathcal{C}}(t)) \in \mathcal{D}_{\mathcal{C}}, \quad (q_{\mathcal{C}}(t), u_{\mathcal{C}}(t)) \in \mathcal{L}_{\mathcal{C}}.$$

Here, $q_{\mathcal{C}}$ represents the *charge* of the capacitance and the *Hamiltonian* $H_{\mathcal{C}}$ represents the *energy storage function* of the system. From this pH-NG system, one can derive

$$i_{\mathcal{C}}(t) = \tfrac{\mathrm{d}}{\mathrm{dt}}\, q_{\mathcal{C}}(t), \quad u_{\mathcal{C}}(t) = \nabla H_{\mathcal{C}}(q_{\mathcal{C}}(t)).$$

If the capacitance has two terminals, then we obtain a conventional capacitance with one port as in Fig. 6.4.

Inductances

Let $H_{\mathcal{L}} \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$. An inductance with ℓ_p ports is modelled as a pH-NG system $(\mathcal{D}_{\mathcal{L}}, \mathcal{L}_{\mathcal{L}}, \mathcal{R}_{\mathcal{L}})$ with

$$\mathcal{D}_{\mathcal{L}} = \left\{ \begin{pmatrix} -u_{\mathcal{L}} \\ i_{\mathcal{L}} \\ i_{\mathcal{L}} \\ u_{\mathcal{L}} \end{pmatrix} \in \mathbb{R}^{4\ell_p} \;\middle|\; u_{\mathcal{L}}, i_{\mathcal{L}} \in \mathbb{R}^{\ell_p} \right\}$$

and

$$\mathcal{L}_{\mathcal{L}} = \left\{ (\psi_{\mathcal{L}}, i_{\mathcal{L}}) \in \mathbb{R}^{2\ell_p} \middle|\ i_{\mathcal{L}} = \nabla H_{\mathcal{L}}(\psi_{\mathcal{L}}) \right\}, \quad \mathcal{R}_{\mathcal{L}} = \{0\}.$$

The dynamics are now given by

$$(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, \psi_{\mathcal{L}}(t), i_{\mathcal{L}}(t), i_{\mathcal{L}}(t), u_{\mathcal{L}}(t)) \in \mathcal{D}_{\mathcal{L}}, \quad (\psi_{\mathcal{L}}(t), i_{\mathcal{L}}(t)) \in \mathcal{L}_{\mathcal{L}}.$$

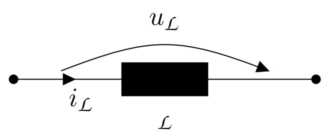

Figure 6.7: Inductance: circuit symbol.

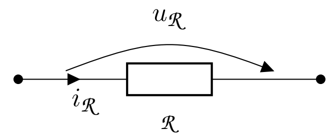

Figure 6.8: Resistance/conductance: circuit symbol.

Here, $\psi_{\mathcal{L}}$ represents the *magnetic flux* of the inductance and the *Hamiltonian* $H_{\mathcal{L}} \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$ represents the *energy storage function* of the system. From this pH-NG system, one can derive

$$u_{\mathcal{L}}(t) = \tfrac{\mathrm{d}}{\mathrm{dt}}\, \psi_{\mathcal{L}}(t), \quad i_{\mathcal{L}}(t) = \nabla H_{\mathcal{L}}(\psi_{\mathcal{L}}(t)).$$

If the inductance has two terminals, we obtain a conventional inductance with one port as in Fig. 6.4. A prominent example of a nonlinear inductance is the *Josephson junction* Fig. 6.6 for which the Hamiltonian reads $H_{\mathcal{JJ}}(\psi_{\mathcal{JJ}}) = a(1 - \cos(b\psi_{\mathcal{JJ}}))$ for some constants a, b [FN19, p. 709].

Conductances and resistances

Let $\mathcal{R}_{\mathcal{R}} \subset \mathbb{R}^{\ell_p} \times \mathbb{R}^{\ell_p}$ be a resistive relation. Consider the pH-NG system $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$, where $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\ell_p}$ with $\mathcal{D}_{\ell_p}$ as in (6.8), $\mathcal{L}_{\mathcal{R}} = \{0\}$. The dynamics are specified by

$$(-i_{\mathcal{R}}(t), i_{\mathcal{R}}(t), u_{\mathcal{R}}(t), u_{\mathcal{R}}(t)) \in \mathcal{D}_{\mathcal{R}}, \quad (-i_{\mathcal{R}}(t), u_{\mathcal{R}}(t)) \in \mathcal{R}_{\mathcal{R}}. \tag{6.9}$$

If, for some accretive function $g : \mathbb{R}^{\ell_p} \to \mathbb{R}^{\ell_p}$ (that is, $\phi_{\mathcal{R}}^\top g(\phi_{\mathcal{R}}) \geq 0$ for all $\phi_{\mathcal{R}} \in \mathbb{R}^{\ell_p}$), $\mathcal{R}_{\mathcal{R}}$ reads

$$\mathcal{R}_{\mathcal{R}} = \left\{(-i_{\mathcal{R}}, u_{\mathcal{R}}) \in \mathbb{R}^{2\ell_p} | i_{\mathcal{R}} = g(u_{\mathcal{R}})\right\},$$

then (6.9) leads to $i_{\mathcal{R}}(t) = g(u_{\mathcal{R}}(t))$. That is, $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$ describes a conductance with ℓ_p ports. On the other hand, if for some accretive function $r : \mathbb{R}^{\ell_p} \to \mathbb{R}^{\ell_p}$,

$$\mathcal{R}_{\mathcal{R}} = \left\{(-i_{\mathcal{R}}, u_{\mathcal{R}}) \in \mathbb{R}^{2\ell_p} | u_{\mathcal{R}} = r(i_{\mathcal{R}})\right\},$$

then (6.9) leads to $u_{\mathcal{R}}(t) = r(i_{\mathcal{R}}(t))$, i.e., $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$ models a resistance with ℓ_p ports.

If the conductance/resistance has two terminals, then we obtain a conventional conductance/resistance with one port as in Fig. 6.8.

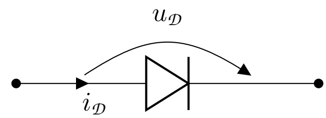

Figure 6.9: Circuit symbol of a diode.

Remark 6.2.1. Resistances form a pathological case of a pH-NG system, since the underlying Lagrangian submanifold is trivial (cf. Rem. 6.1.5). Therefore, the 'dynamics' of the pH-NG system are actually 'statics'. The same holds for the models diodes, transformers and transistors which are discussed in the following.

Ideal and PN-junction diodes

An *ideal diode* is modelled as a two-terminal component $(\mathcal{D}_{\mathfrak{D}}, \mathcal{L}_{\mathfrak{D}}, \mathcal{R}_{\mathfrak{D}})$ with one port (see Fig. 6.4), and dynamics

$$(-i_{\mathfrak{D}}(t), i_{\mathfrak{D}}(t), u_{\mathfrak{D}}(t), u_{\mathfrak{D}}(t)) \in \mathcal{D}_{\mathfrak{D}},\ \ (j_{\mathfrak{D}}(t), \phi_{\mathfrak{D}}(t)) \in \mathcal{R}_{\mathfrak{D}},$$

where $\mathcal{D}_{\mathfrak{D}} = \mathcal{D}_1$ with $\mathcal{D}_1$ as defined in (6.8), $\mathcal{L}_{\mathfrak{D}} = \{0\}$ and

$$\mathcal{R}_{\mathfrak{D}} = \left\{(-i_{\mathfrak{D}}, u_{\mathfrak{D}}) \in \mathbb{R}^2 \mid i_{\mathfrak{D}} u_{\mathfrak{D}} = 0 \ \wedge\ i_{\mathfrak{D}} \leq 0 \ \wedge\ u_{\mathfrak{D}} \leq 0\right\}.$$

From this pH-NG system, one can derive that

$$(i_{\mathfrak{D}}(t), u_{\mathfrak{D}}(t)) \in (\{0\} \times \mathbb{R}_{\leq 0}) \cup (\mathbb{R}_{\geq 0} \times \{0\}).$$

A *PN-junction diode* is modelled as a one-port component $(\mathcal{D}_{\mathfrak{D}}, \mathcal{L}_{\mathfrak{D}}, \mathcal{R}_{\mathfrak{D}})$ with $\mathcal{D}_{\mathfrak{D}}$ and $\mathcal{L}_{\mathfrak{D}}$ as for the ideal diode, and the resistive relation is, for some constants $a, b > 0$, given by

$$\mathcal{R}_{\mathfrak{D}} = \left\{(-i_{\mathfrak{D}}, u_{\mathfrak{D}}) \in \mathbb{R}^2 \mid i_{\mathfrak{D}} = a\left(e^{\frac{u_{\mathfrak{D}}}{b}} - 1\right)\right\}.$$

From the dynamics of this pH-NG system, one can derive the characteristic equation [MR17, Eq. (39.46)]

$$i_{\mathfrak{D}}(t) = a\left(e^{\frac{u_{\mathfrak{D}}(t)}{b}} - 1\right).$$

The PN-junction diode serves as an approximation for an ideal diode. In a certain sense, the behavior of a PN-junction diode indeed tends to that of the ideal diode, if $b \to 0$.

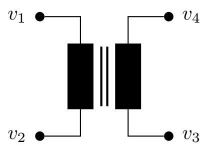

Figure 6.10: Circuit symbol of a transformer.

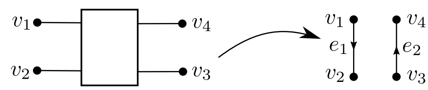

Figure 6.11: Deriving the underlying graph of a transformer.

Transformers

A *transformer* is modelled as a four-terminal component with two ports, see Fig. 6.11. It is described by the pH-NG system $(\mathcal{D}_\mathcal{T}, \mathcal{L}_\mathcal{T}, \mathcal{R}_\mathcal{T})$, where we use the Dirac structure $\mathcal{D}_\mathcal{T} = \mathcal{D}_2$ with $\mathcal{D}_2$ as defined in (6.8) and trivial Lagrangian submanifold $\mathcal{L}_\mathcal{T} = \{0\}$. The dynamics are given by

$$\begin{aligned}(-i_{\mathcal{T}1}(t), -i_{\mathcal{T}2}(t), i_{\mathcal{T}1}(t), i_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t)) &\in \mathcal{D}_\mathcal{T}, \\ (-i_{\mathcal{T}1}(t), -i_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t)) &\in \mathcal{R}_\mathcal{T},\end{aligned}$$

with, for some $T \in \mathbb{R}$,

$$\mathcal{R}_\mathcal{T} = \left\{(-i_{\mathcal{T}1}, -i_{\mathcal{T}2}, u_{\mathcal{T}1}, u_{\mathcal{T}2}) \in \mathbb{R}^4 \mid T i_{\mathcal{T}1} = -i_{\mathcal{T}2},\ u_{\mathcal{T}1} = T u_{\mathcal{T}2}\right\}.$$

From this pH-NG system, one can derive $T i_{\mathcal{T}1}(t) = -i_{\mathcal{T}2}(t)$ and $u_{\mathcal{T}1}(t) = T u_{\mathcal{T}2}(t)$, which means that a transformer is a power-conserving component.

NPN transistors

A *transistor* is a component with three terminals, which are called *emitter*, *basis* and *collector*. We replace this by a graph with two edges, which are respectively located are between basis and collector, and basis and emitter, see Fig. 6.13. The behavior of a transistor *of type NPN* is often modelled by the *Ebers-Moll model* [SS04, Eqs. (5.26) & (5.27)], which can, in a certain voltage and current range around zero, be summarized by the equations

$$\begin{aligned} i_C(t) &= i_S\left(e^{\frac{u_{BE}(t)}{V_T}} - 1\right) - \tfrac{i_S}{\alpha_R}\left(e^{\frac{u_{BC}(t)}{V_T}} - 1\right), \\ i_E(t) &= \tfrac{i_S}{\alpha_F}\left(e^{\frac{u_{BE}(t)}{V_T}} - 1\right) - i_S\left(e^{\frac{u_{BC}(t)}{V_T}} - 1\right), \end{aligned} \tag{6.10}$$

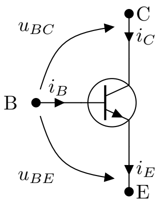

Figure 6.12: Circuit symbol of a NPN transistor.

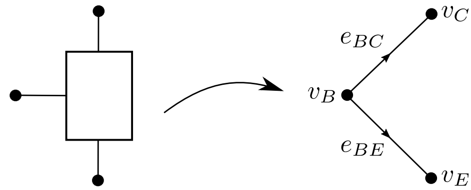

Figure 6.13: Deriving the underlying graph of an NPN transistor.

for some constants $\alpha_F \in \left[\frac{50}{51}, \frac{1000}{1001}\right]$, $\alpha_R \in \left[\frac{1}{100}, \frac{1}{2}\right]$, $i_S \in [10^{-15}, 10^{-12}]$, $V_T \approx \frac{1}{40}$ [SS04, pp. 382-394]. Hereby, $i_C(t)$, $i_E(t)$, $u_{BE}(t)$, $u_{BC}(t)$ respectively denote the collector current, emitter current, basis-emitter voltage and basis collector voltage. Note that, by the Kirchhoff laws, the basis current fulfills $i_B(t) = i_E(t) - i_C(t)$ and the collector emitter voltage is given by $u_{CE}(t) = u_{BE}(t) - u_{BC}(t)$. We model an NPN transistor as a 'resistive' two-port component $(\mathcal{D}_{\mathcal{N}}, \mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}})$ on two edges, where $\mathcal{D}_{\mathcal{N}} = \mathcal{D}_2$ with $\mathcal{D}_2$ as defined in (6.8), $\mathcal{L}_{\mathcal{N}} = \{0\}$ and

$$\mathcal{R}_{\mathcal{N}} = \left\{ (i_C, -i_E, u_{BC}, u_{BE}) \in \mathbb{R}^4 \,\middle|\, \begin{array}{rl} i_C = & i_S \left(e^{\frac{u_{BE}}{V_T}} - 1 \right) - \frac{i_S}{\alpha_R} \left(e^{\frac{u_{BC}}{V_T}} - 1 \right), \\ i_E = & \frac{i_S}{\alpha_F} \left(e^{\frac{u_{BE}}{V_T}} - 1 \right) - i_S \left(e^{\frac{u_{BC}}{V_T}} - 1 \right) \end{array} \right\} \cap U_0,$$

where $U_0 \subset \mathbb{R}^4$ is a neighborhood of the origin. The dynamics of the system read

$$\begin{aligned} (i_C(t), -i_E(t), -i_C(t), i_E(t), u_{BC}(t), u_{BE}(t), u_{BC}(t), u_{BE}(t)) &\in \mathcal{D}_{\mathcal{N}}, \\ (i_C(t), -i_E(t), u_{BC}(t), u_{BE}(t)) &\in \mathcal{R}_{\mathcal{N}}, \end{aligned}$$

which implies (6.10), at least as long as $(i_C(t), -i_E(t), u_{BC}(t), u_{BE}(t)) \in U_0$. Note that we have provided the collector current $i_C(t)$ with another sign, since it is — in contrast to the emitter current and the basis-emitter current — directed contrarily to the basis-collector current.

Note that, if we choose $U_0 = \mathbb{R}^4$, then the relation $\mathcal{R}_{\mathcal{N}}$ is *not* resistive, since there may exist quadruples $(i_C, -i_E, u_{BC}, u_{BE}) \in \mathcal{R}_{\mathcal{N}}$ for which it holds $i_C u_{BC} - i_E u_{BE} > 0$. However, we can show that $\mathcal{R}_{\mathcal{N}}$ is resistive for a suitable neighborhood $U_0 \subset \mathbb{R}^4$ of

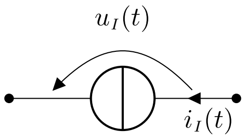

Figure 6.14: Circuit symbol of a current source.

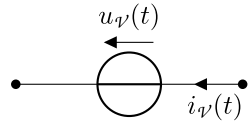

Figure 6.15: Circuit symbol of a voltage source.

the origin. This can be seen as follows: Since for $(u_{BC}, u_{BE}) \in (\mathbb{R} \setminus \{0\})^2$ holds

$$\begin{aligned}
&\begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}^\top \begin{pmatrix} -\frac{i_S}{\alpha_R}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) + i_S\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \\ i_S\left(e^{\frac{u_{BC}}{V_T}} - 1\right) - \frac{i_S}{\alpha_F}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \end{pmatrix} \\
&= \begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}^\top \underbrace{\begin{bmatrix} -\frac{i_S}{\alpha_R u_{BC}}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) & \frac{i_S}{u_{BE}}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \\ \frac{i_S}{u_{BC}}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) & -\frac{i_S}{\alpha_F u_{BE}}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \end{bmatrix}}_{=:A(u_{BC}, u_{BE})} \begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}.
\end{aligned}$$

Namely, by using that $A(\cdot, \cdot)$ has a continuous extension to $\mathbb{R}^2$ with

$$A(0,0) = \frac{i_S}{V_T} \cdot \begin{bmatrix} -\frac{1}{\alpha_R} & 1 \\ 1 & -\frac{1}{\alpha_F} \end{bmatrix}.$$

By $\alpha_F \in \left[\frac{50}{51}, \frac{1000}{1001}\right]$, $\alpha_R \in \left[\frac{1}{100}, \frac{1}{2}\right]$, we have $\alpha_F \cdot \alpha_R < 1$, which leads to negative definiteness of $A(0,0) = \frac{1}{2}(A(0,0) + A(0,0)^\top)$. The continuity of $(u_{BC}, u_{BE}) \mapsto \frac{1}{2}(A(u_{BC}, u_{BE}) + A(u_{BC}, u_{BE})^\top)$ implies that there exists some neighborhood $U_0 \subset \mathbb{R}^4$ such that this function takes values in the cone of negative definite matrices on U_0. This consequences that, by taking this neighborhood U_0, $\mathcal{R}_{\mathcal{N}}$ is a resistive relation.

Current and voltage sources

The sources of the electrical circuit represent the ports of the system, i.e., points at which physical interaction of the electrical circuit with the environment happens. We may distinguish two types of sources: *current sources* and *voltage sources*, see Fig. 6.14 and Fig. 6.15. The name indicates which physical variable is controlled or influenced by the environment. This variable is also denoted as *input*, while the other is denoted as *output*. However, this distinction is not relevant for the *geometrical*

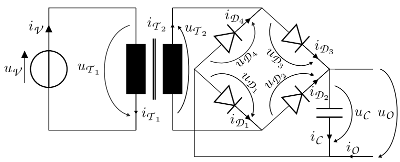

Figure 6.16: AC/DC converter circuit.

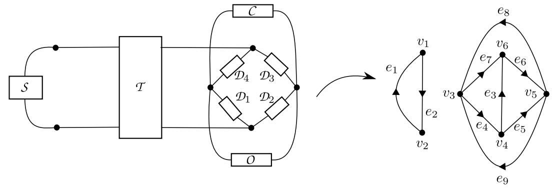

Figure 6.17: Obtaining the underlying graph of the AC/DC converter.

formulation of pH-NG systems (cf. [MMW18]). We unite both classes under the term *sources*. These have two terminals, and, consequently, one port (see Fig. 6.4). Sources are modelled as a pH-NG system $(\mathcal{D}_{\mathcal{S}}, \mathcal{L}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}})$, where the Dirac structure is $\mathcal{D}_{\mathcal{S}} = \mathcal{D}_1$ with $\mathcal{D}_1$ as defined in (6.8), and the Lagrangian submanifold and resistive relation are trivial, i.e., $\mathcal{L}_{\mathcal{S}} = \mathcal{R}_{\mathcal{S}} = \{0\}$. The dynamics are

$$(-i_{\mathcal{S}}(t), i_{\mathcal{S}}(t), u_{\mathcal{S}}(t), u_{\mathcal{S}}(t)) \in \mathcal{D}_{\mathcal{S}}.$$

Example 6.2.2 (AC/DC converter)**.** We illustrate our methodology by considering an AC/DC converter, which we model by the electrical circuit shown in Fig. 6.16. The AC/DC converter consists of a source $\mathcal{S} = (\mathcal{D}_{\mathcal{S}}, \mathcal{L}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}})$, a transformer $\mathcal{T} = (\mathcal{D}_{\mathcal{T}}, \mathcal{L}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}})$, four PN-junction diodes $\mathcal{J}_i = (\mathcal{D}_{\mathcal{J}_i}, \mathcal{L}_{\mathcal{J}_i}, \mathcal{R}_{\mathcal{J}_i})$ for $i \in \{1, ..., 4\}$, a capacitor $\mathcal{C} = (\mathcal{D}_{\mathcal{C}}, \mathcal{L}_{\mathcal{C}}, \mathcal{R}_{\mathcal{C}})$, and a 'sink' $\mathcal{O} = (\mathcal{D}_{\mathcal{O}}, \mathcal{L}_{\mathcal{O}}, \mathcal{R}_{\mathcal{O}})$ (modelled like a source), which are connected by the vertices $v_1, ..., v_6$ as shown in Fig. 6.17. The circuit graph $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ with $V = \{v_1, ..., v_6\}$ and $E = \{e_1, ..., e_9\}$ has two components, and we ground the nodes below the voltage source and the capacitance, i.e., we choose $S = \{v_2, v_3\}$. Let $A \in \mathbb{R}^{4\times 9}$ be obtained from the incidence matrix of $\mathcal{G}$ by deleting the rows corresponding to the grounded nodes. We arrive at a pH-NG system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ as in (6.7), whose dynamics read

$$\left(\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} -q_1 \\ -q_4 \\ -q_5 \\ -q_6 \\ -q_{\mathcal{C}} \end{pmatrix}, \begin{pmatrix} -i_{\mathcal{T}1} \\ -i_{\mathcal{T}2} \\ -i_{\mathcal{D}1} \\ -i_{\mathcal{D}2} \\ -i_{\mathcal{D}3} \\ -i_{\mathcal{D}4} \end{pmatrix}, \begin{pmatrix} -i_{\mathcal{V}} \\ -i_{\mathcal{O}} \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ u_{\mathcal{C}} \end{pmatrix}, \begin{pmatrix} u_{\mathcal{T}1} \\ u_{\mathcal{T}2} \\ u_{\mathcal{D}1} \\ u_{\mathcal{D}2} \\ u_{\mathcal{D}3} \\ u_{\mathcal{D}4} \end{pmatrix}, \begin{pmatrix} u_{\mathcal{V}} \\ u_{\mathcal{O}} \end{pmatrix} \right) \in \mathcal{D},$$

$$\left(\begin{pmatrix}-q_1\\-q_4\\-q_5\\-q_6\\-q_{\mathcal{C}}\end{pmatrix},\begin{pmatrix}\phi_1\\\phi_4\\\phi_5\\\phi_6\\u_{\mathcal{C}}\end{pmatrix}\right)\in\mathcal{L},\qquad\left(\begin{pmatrix}-i_{\mathcal{T}1}\\-i_{\mathcal{T}2}\\-i_{\mathcal{D}1}\\-i_{\mathcal{D}2}\\-i_{\mathcal{D}3}\\-i_{\mathcal{D}4}\end{pmatrix},\begin{pmatrix}u_{\mathcal{T}1}\\u_{\mathcal{T}2}\\u_{\mathcal{D}1}\\u_{\mathcal{D}2}\\u_{\mathcal{D}3}\\u_{\mathcal{D}4}\end{pmatrix}\right)\in\mathcal{R}.$$

6.3 Comparison with other formulations of electrical circuits

With the attention electrical circuits attracted over the past decades, quite a bunch of 'standard formulations' of the dynamics have emerged, see, e.g., [Rei01]. An overview of popular models in the context of DAEs is found in [Ria13]. We compare for certain electrical circuits the dynamics of our port-Hamiltonian modelling (6.7) with other equations used in the modelling of electrical circuits.

6.3.1 The (charge/flux-oriented) modified nodal analysis

Let an electrical circuit consisting of conductances, inductances, capacitances and sources be given. Let

$$\begin{aligned}&(\mathcal{D}_{\mathcal{R}_i},\mathcal{L}_{\mathcal{R}_i},\mathcal{R}_{\mathcal{R}_i})_{i\in\{1,\dots,l_{\mathcal{R}}\}},\quad&&(\mathcal{D}_{\mathcal{L}_i},\mathcal{L}_{\mathcal{L}_i},\mathcal{R}_{\mathcal{L}_i})_{i\in\{1,\dots,l_{\mathcal{L}}\}},\\&(\mathcal{D}_{\mathcal{C}_i},\mathcal{L}_{\mathcal{C}_i},\mathcal{R}_{\mathcal{C}_i})_{i\in\{1,\dots,l_{\mathcal{C}}\}},\quad&&(\mathcal{D}_{\mathcal{S}_i},\mathcal{L}_{\mathcal{S}_i},\mathcal{R}_{\mathcal{S}_i})_{i\in\{1,\dots,l_{\mathcal{S}}\}}.\end{aligned}$$

be the pH-NG systems modelling the components as derived in Section 6.2.1. Let $\ell_{p,\mathcal{R}i}$ be the number of ports of the component modelled by $(\mathcal{D}_{\mathcal{R}_i},\mathcal{L}_{\mathcal{R}_i},\mathcal{R}_{\mathcal{R}_i})$, and let $\ell_{p,\mathcal{L}i}$ and $\ell_{p,\mathcal{C}i}$ be analogously defined. Moreover, let

$$m_{\mathcal{R}}=\sum_{i=1}^{l_{\mathcal{R}}}\ell_{p,\mathcal{R}i},\quad m_{\mathcal{L}}=\sum_{i=1}^{l_{\mathcal{R}}}\ell_{p,\mathcal{L}i},\quad m_{\mathcal{C}}=\sum_{i=1}^{l_{\mathcal{C}}}\ell_{p,\mathcal{R}i},\quad m_{\mathcal{S}}=l_{\mathcal{S}},$$

and introduce

$$i_{\mathcal{R}}=\begin{pmatrix}i_{\mathcal{R}1}\\\vdots\\i_{\mathcal{R}m_{\mathcal{R}}}\end{pmatrix},\ i_{\mathcal{L}}=\begin{pmatrix}i_{\mathcal{L}1}\\\vdots\\i_{\mathcal{L}m_{\mathcal{L}}}\end{pmatrix},\ i_{\mathcal{C}}=\begin{pmatrix}i_{\mathcal{C}1}\\\vdots\\i_{\mathcal{C}m_{\mathcal{C}}}\end{pmatrix},\ i_{\mathcal{S}}=\begin{pmatrix}i_{\mathcal{S}1}\\\vdots\\i_{\mathcal{S}m_{\mathcal{S}}}\end{pmatrix},\ i=\begin{pmatrix}i_{\mathcal{R}}\\i_{\mathcal{L}}\\i_{\mathcal{C}}\\i_{\mathcal{S}}\end{pmatrix},$$

and analogous notations for $u_{\mathcal{R}}$, $u_{\mathcal{L}}$, $u_{\mathcal{C}}$, $u_{\mathcal{S}}$, u, as well as

$$q_{\mathcal{C}} = \begin{pmatrix} q_{\mathcal{C}1} \\ \vdots \\ q_{\mathcal{C}m_{\mathcal{C}}} \end{pmatrix}, \quad \psi_{\mathcal{L}} = \begin{pmatrix} \psi_{\mathcal{L}1} \\ \vdots \\ \psi_{\mathcal{L}m_{\mathcal{L}}} \end{pmatrix}, \quad g(u_{\mathcal{R}}) = \begin{pmatrix} g_1(u_{\mathcal{R}1}) \\ \vdots \\ g_{m_{\mathcal{R}}}(u_{\mathcal{R}m_{\mathcal{R}}}) \end{pmatrix},$$

$$H_{\mathcal{C}}(q_{\mathcal{C}}) = \sum_{i=1}^{m_{\mathcal{C}}} H_{\mathcal{C}i}(q_{\mathcal{C}i}), \quad H_{\mathcal{L}}(\psi_{\mathcal{L}}) = \sum_{i=1}^{m_{\mathcal{L}}} H_{\mathcal{L}i}(\psi_{\mathcal{L}i}).$$

Further, let $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ be the graph induced by the electrical circuit with $|V| = n$ and $|E| = m$. Let S be the set of grounded vertices (cf. Def. 6.1.4), and let $A \in \mathbb{R}^{(n-|S|)\times m}$ be obtained from the incidence matrix of $\mathcal{G}$ by deleting the rows corresponding to the vertices in S. By a suitable reordering, we may sort into edges to the specific components, i.e.,

$$A = \begin{bmatrix} A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{C}} & A_{\mathcal{S}} \end{bmatrix},$$

where the columns of $A_{\mathcal{R}} \in \mathbb{R}^{(n-|S|)\times m_{\mathcal{R}}}$, $A_{\mathcal{L}} \in \mathbb{R}^{(n-|S|)\times m_{\mathcal{L}}}$, $A_{\mathcal{C}} \in \mathbb{R}^{(n-|S|)\times m_{\mathcal{C}}}$ and $A_{\mathcal{S}} \in \mathbb{R}^{(n-|S|)\times m_{\mathcal{S}}}$ respectively represent the edges corresponding to conductances, inductances, capacitances and sources. For the representation of the port-Hamiltonian dynamics of the electrical circuit, first note that the Dirac structure of the pH-NG system

$$\bigtimes_{i=1}^{m_{\mathcal{R}}} (\mathcal{D}_{\mathcal{R}_i}, \mathcal{L}_{\mathcal{R}_i}, \mathcal{R}_{\mathcal{R}_i}) \times \bigtimes_{i=1}^{m_{\mathcal{L}}} (\mathcal{D}_{\mathcal{L}_i}, \mathcal{L}_{\mathcal{L}_i}, \mathcal{R}_{\mathcal{L}_i}) \times \bigtimes_{i=1}^{m_{\mathcal{C}}} (\mathcal{D}_{\mathcal{C}_i}, \mathcal{L}_{\mathcal{C}_i}, \mathcal{R}_{\mathcal{C}_i}) \times \bigtimes_{i=1}^{m_{\mathcal{S}}} (\mathcal{D}_{\mathcal{S}_i}, \mathcal{L}_{\mathcal{S}_i}, \mathcal{R}_{\mathcal{S}_i})$$

is given by

$$\mathcal{D}_{\mathrm{prod}} = \Big\{ \big(-i_{\mathcal{L}}, -i_{\mathcal{C}}, -i_{\mathcal{R}}, i_{\mathcal{R}}, i_{\mathcal{L}}, i_{\mathcal{C}}, -i_{\mathcal{S}}, i_{\mathcal{S}}, i_{\mathcal{L}}, u_{\mathcal{C}}, u_{\mathcal{R}}, u_{\mathcal{R}}, u_{\mathcal{L}}, u_{\mathcal{C}}, u_{\mathcal{S}}, u_{\mathcal{S}}\big) \in \mathbb{R}^{2m} \times \mathbb{R}^{2m} \;\Big|\; i_{\mathcal{L}}, u_{\mathcal{L}} \in \mathbb{R}^{m_{\mathcal{L}}},\ i_{\mathcal{C}}, u_{\mathcal{C}} \in \mathbb{R}^{m_{\mathcal{C}}},\ i_{\mathcal{R}}, u_{\mathcal{R}} \in \mathbb{R}^{m_{\mathcal{R}}},\ i_{\mathcal{S}}, u_{\mathcal{S}} \in \mathbb{R}^{m_{\mathcal{S}}} \Big\}$$

and

$$\mathcal{D}_K^S(\mathcal{G}) = \Bigg\{ \big(j, i_{\mathcal{R}}, i_{\mathcal{L}}, i_{\mathcal{C}}, i_{\mathcal{S}}, \phi, u_{\mathcal{R}}, u_{\mathcal{L}}, u_{\mathcal{C}}, u_{\mathcal{S}}\big) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \;\Bigg|\; \begin{bmatrix} I & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{C}} & A_{\mathcal{S}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ i_{\mathcal{R}} \\ i_{\mathcal{L}} \\ i_{\mathcal{C}} \\ i_{\mathcal{S}} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & I & 0 & 0 & 0 \\ -A_{\mathcal{L}}^\top & 0 & I & 0 & 0 \\ -A_{\mathcal{C}}^\top & 0 & 0 & I & 0 \\ -A_{\mathcal{S}}^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi \\ u_{\mathcal{R}} \\ u_{\mathcal{L}} \\ u_{\mathcal{C}} \\ u_{\mathcal{S}} \end{pmatrix} = 0 \Bigg\}.$$

It follows that the Dirac structure of

$$(\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\}) \circ \left(\bigtimes_{i=1}^{N} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right)$$

is given by

$$\mathcal{D} = \left\{ \left(j, -u_{\mathcal{L}}, -i_{\mathcal{C}}, -i_{\mathcal{R}}, -i_{\mathcal{S}}, \phi, i_{\mathcal{L}}, u_{\mathcal{C}}, u_{\mathcal{R}}, u_{\mathcal{S}}\right) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \;\middle|\; \begin{bmatrix} I & 0 & A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{S}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ -u_{\mathcal{L}} \\ -i_{\mathcal{C}} \\ -i_{\mathcal{R}} \\ -i_{\mathcal{S}} \end{pmatrix} + \begin{bmatrix} 0 & -A_{\mathcal{L}} & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & 0 & 0 & I & 0 \\ -A_{\mathcal{L}}^\top & 0 & 0 & 0 & 0 \\ -A_{\mathcal{C}}^\top & 0 & I & 0 & 0 \\ -A_{\mathcal{S}}^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi \\ i_{\mathcal{L}} \\ u_{\mathcal{C}} \\ u_{\mathcal{R}} \\ u_{\mathcal{S}} \end{pmatrix} = 0 \right\}, \tag{6.11a}$$

whereas the Lagrangian submanifold and resistive relation read

$$\mathcal{L} = \left\{ \left(q, \psi_{\mathcal{L}}, q_{\mathcal{C}}, \phi, i_{\mathcal{L}}, u_{\mathcal{C}}\right) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^{m_{\mathcal{L}}} \times \mathbb{R}^{m_{\mathcal{C}}} \times \mathbb{R}^{n-|S|} \times \mathbb{R}^{m_{\mathcal{L}}} \times \mathbb{R}^{m_{\mathcal{C}}} \;\middle|\; q = 0 \wedge i_{\mathcal{L}} = \nabla H_{\mathcal{L}}(\psi_{\mathcal{L}}) \wedge u_{\mathcal{C}} = \nabla H_{\mathcal{C}}(q_{\mathcal{C}}) \right\}, \tag{6.11b}$$

$$\mathcal{R} = \left\{(-i_{\mathcal{R}}, u_{\mathcal{R}}) \in \mathbb{R}^{m_{\mathcal{R}}} \times \mathbb{R}^{m_{\mathcal{R}}} \mid i_{\mathcal{R}} = g(u_{\mathcal{R}})\right\}. \tag{6.11c}$$

The triple $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ with $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{R}$ as in (6.11) is the port-Hamiltonian representation of a circuit with conductances, inductances, capacitances and sources in a compact form. The dynamics of $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ read

$$(-\tfrac{\mathrm{d}}{\mathrm{d}t}\, q(t), -\tfrac{\mathrm{d}}{\mathrm{d}t}\, \psi_{\mathcal{L}}(t), -\tfrac{\mathrm{d}}{\mathrm{d}t}\, q_{\mathcal{C}}(t), -i_{\mathcal{R}}(t), -i_{\mathcal{S}}(t), \phi(t), i_{\mathcal{L}}(t), u_{\mathcal{C}}(t), u_{\mathcal{R}}(t), u_{\mathcal{S}}(t)) \in \mathcal{D},$$
$$(q(t), \psi_{\mathcal{L}}(t), q_{\mathcal{C}}(t), \phi(t), i_{\mathcal{L}}(t), u_{\mathcal{C}}(t)) \in \mathcal{L}, \quad (-i_{\mathcal{R}}(t), e_{\mathcal{R}}(t)) \in \mathcal{R},$$

which is equivalent to

$$\begin{bmatrix} I & 0 & A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{S}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} -\tfrac{\mathrm{d}}{\mathrm{d}t}\, q(t) \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\, \psi_{\mathcal{L}}(t) \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\, q_{\mathcal{C}}(t) \\ -i_{\mathcal{R}}(t) \\ -i_{\mathcal{S}}(t) \end{pmatrix} + \begin{bmatrix} 0 & -A_{\mathcal{L}} & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & 0 & 0 & I & 0 \\ -A_{\mathcal{L}}^\top & 0 & 0 & 0 & 0 \\ -A_{\mathcal{C}}^\top & 0 & I & 0 & 0 \\ -A_{\mathcal{S}}^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi(t) \\ i_{\mathcal{L}}(t) \\ u_{\mathcal{C}}(t) \\ u_{\mathcal{R}}(t) \\ u_{\mathcal{S}}(t) \end{pmatrix} = 0,$$

$$q(t) = 0, \;\; i_{\mathcal{L}}(t) = \nabla H_{\mathcal{L}}(\psi_{\mathcal{L}}(t)), \;\; u_{\mathcal{C}}(t) = \nabla H_{\mathcal{C}}(q_{\mathcal{C}}(t)), \;\; i_{\mathcal{R}}(t) = g(u_{\mathcal{R}}(t)).$$

Plugging in the latter relations, we obtain

$$\begin{aligned} A_{\mathcal{C}} \tfrac{\mathrm{d}}{\mathrm{dt}} q_{\mathcal{C}}(t) + A_{\mathcal{R}} g(A_{\mathcal{R}}^\top \phi(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_{\mathcal{S}} i_{\mathcal{S}}(t) &= 0, \\ -A_{\mathcal{L}}^\top \phi(t) + \tfrac{\mathrm{d}}{\mathrm{dt}} \psi_{\mathcal{L}}(t) &= 0, \\ -A_{\mathcal{S}}^\top \phi(t) + u_{\mathcal{S}}(t) &= 0, \\ A_{\mathcal{C}}^\top \phi(t) - \nabla H_{\mathcal{C}}(q_{\mathcal{C}}(t)) &= 0, \\ i_{\mathcal{L}}(t) - \nabla H_{\mathcal{L}}(\psi(t)) &= 0. \end{aligned} \tag{6.12}$$

If we additionally assume that $\nabla H_{\mathcal{C}} \in C^1(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$, $\nabla H_{\mathcal{L}} \in C^1(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$ are homeomorphisms, we can introduce the inverse functions $Q_{\mathcal{C}} \coloneqq (\nabla H_{\mathcal{C}})^{-1} \in C(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$, $\Psi_{\mathcal{L}} \coloneqq (\nabla H_{\mathcal{L}})^{-1} \in C(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$. Then (6.12) leads to $q_{\mathcal{C}}(t) = Q_{\mathcal{C}}(u_{\mathcal{C}}(t))$ and $\psi_{\mathcal{L}}(t) = \Psi_{\mathcal{L}}(i_{\mathcal{L}}(t))$. Further, decomposing

$$A_{\mathcal{S}} = \begin{bmatrix} A_I & A_{\mathcal{V}} \end{bmatrix}, \quad u_{\mathcal{S}} = \begin{pmatrix} u_I \\ u_{\mathcal{V}} \end{pmatrix}, \quad i_{\mathcal{S}} = \begin{pmatrix} i_I \\ i_{\mathcal{V}} \end{pmatrix}$$

into edges, voltages and currents to current and voltage sources, we see that (6.12) leads to the so-called *charge/flux-oriented modified nodal analysis* [Bäc07, Eq. (3.21)]

$$\begin{aligned} A_{\mathcal{C}} \tfrac{\mathrm{d}}{\mathrm{dt}} q_{\mathcal{C}}(t) + A_{\mathcal{R}} g(u_{\mathcal{R}}(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_I i_I(t) + A_{\mathcal{V}} i_{\mathcal{V}}(t) &= 0, \\ -A_{\mathcal{L}}^\top \phi(t) + \tfrac{\mathrm{d}}{\mathrm{dt}} \psi_{\mathcal{L}}(t) &= 0, \\ -A_{\mathcal{V}}^\top \phi(t) + u_{\mathcal{V}}(t) &= 0, \\ q_{\mathcal{C}}(t) - Q_{\mathcal{C}}(A_{\mathcal{C}}^\top \phi(t)) &= 0, \\ \psi_{\mathcal{L}}(t) - \Psi_{\mathcal{L}}(i_{\mathcal{L}}(t)) &= 0. \end{aligned} \tag{MNA c/f}$$

If we additionally assume that $Q_{\mathcal{C}} \in C(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$ and $\Psi_{\mathcal{L}} \in C^1(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$, then we can, by denoting the Jacobians by $\mathcal{C}(u_{\mathcal{C}}) = \frac{\mathrm{d}}{\mathrm{d}u_{\mathcal{C}}} Q_{\mathcal{C}}(u_{\mathcal{C}})$ and $\mathcal{L}(i_{\mathcal{L}}) = \frac{\mathrm{d}}{\mathrm{d}i_{\mathcal{L}}} \Psi_{\mathcal{L}}(i_{\mathcal{L}})$, reformulate (MNA c/f) to obtain the *modified nodal analysis* [Rei14, Eq. (52)]

$$\begin{aligned} A_{\mathcal{C}} \mathcal{C}(A_{\mathcal{C}}^\top \phi(t)) A_{\mathcal{C}}^\top \tfrac{\mathrm{d}}{\mathrm{dt}} \phi(t) + A_{\mathcal{R}} g(A_{\mathcal{R}}^\top \phi(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_I i_I(t) + A_{\mathcal{V}} i_{\mathcal{V}}(t) &= 0, \\ -A_{\mathcal{L}}^\top \phi(t) + \mathcal{L}(i_{\mathcal{L}}(t)) \tfrac{\mathrm{d}}{\mathrm{dt}} i_{\mathcal{L}}(t) &= 0, \\ -A_{\mathcal{V}}^\top \phi(t) + u_{\mathcal{V}}(t) &= 0. \end{aligned} \tag{MNA}$$

Note that, if $H_{\mathcal{C}} \in C^2(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R})$, $H_{\mathcal{L}} \in C^2(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R})$, then $\mathcal{C}(u_{\mathcal{C}})$ and $\mathcal{L}(i_{\mathcal{L}})$ are, respectively, the inverses of the Hessians of $H_{\mathcal{C}}$ and $H_{\mathcal{L}}$ at $Q_{\mathcal{C}}(u_{\mathcal{C}})$ and $\Psi_{\mathcal{L}}(i_{\mathcal{L}})$.

6.3.2 The (charge/flux-oriented) modified loop analysis

We present an alternative modelling involving the pH-NG system $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ with $\mathcal{D}'_K(\mathcal{G})$ and $\mathcal{L}'_K(\mathcal{G})$ as in (6.5). That is, the loops in the underlying graph structure are now taken to model the Kirchhoff laws. First note that the external flows and efforts variables in the pH-NG system $(\mathcal{D}^S_K(\mathcal{G}), \mathcal{L}^S_K(\mathcal{G}), \{0\})$ in Rem. 6.1.5 are, respectively, the current and the voltage of the components, while the external flow and effort variables in $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ are, respectively, the voltage and the current of the components. This means that in order to obtain a pH-NG system $(\mathcal{D}', \mathcal{L}', \mathcal{R}')$ describing the circuit dynamics by performing an interconnection of $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ with $N \in \mathbb{N}$ electrical components $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)_{i\in\{1,\dots,N\}}$, i.e.,

$$(\mathcal{D}', \mathcal{L}', \mathcal{R}') \coloneqq (\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\}) \circ \left(\bigtimes_{i=1}^{N} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right),$$

we have to adjust the definition of the components by interchanging the role of the effort and flow variables, which is possible by an argument similar to one in Rem. 6.1.5. Given an electrical circuit consisting of resistances, inductances, capacitances and sources, it can, completely analogous to Section 6.3.1, be shown that the dynamics of $(\mathcal{D}', \mathcal{L}', \mathcal{R}')$ lead, under certain additional invertibility and smoothness assumptions on the functions representing capacitances and inductances, to the *modified loop analysis* [Rei14, Eq. (53)]

$$\begin{aligned}
B_{\mathcal{L}} \mathcal{L}(B_{\mathcal{L}}^\top \iota(t)) B_{\mathcal{L}}^\top \tfrac{\mathrm{d}}{\mathrm{dt}}\, \iota(t) + B_{\mathcal{R}} r(B_{\mathcal{R}}^\top \iota(t)) + B_{\mathcal{C}} u_{\mathcal{C}}(t) + B_I u_I(t) + B_{\mathcal{V}} u_{\mathcal{V}}(t) &= 0, \\
-B_{\mathcal{C}}^\top \iota(t) + \mathcal{C}(u_{\mathcal{C}}(t)) \tfrac{\mathrm{d}}{\mathrm{dt}}\, u_{\mathcal{C}}(t) &= 0, \\
-B_I^\top \iota(t) + i_I(t) &= 0.
\end{aligned}$$

6.4 Implicit equations in port-Hamiltonian systems

When we introduced the different port-Hamiltonian formulations in Chap. 2 we were only focussed on developping a solution theory for the port-Hamiltonian systems inducing linear equations. This focus comes naturally from the fact that we only compared linear port-Hamiltonian theories with each other but also because nonlinear implicit differential equations are inherently more difficult to solve. A powerful tool classically used to solve nonlinear implicit differential equations is the implicit function theorem, see, e.g., the classical textbook on DAEs [KM06]. The authors of [vdSM20] show how to locally resolve algebraic constraints for nonlinear port-Hamiltonian systems. However, it is not always possible to do this globally, as the following example shows.

Example 6.4.1 (Port-Hamiltonian system on the circle).
Let $\mathcal{X} = S^1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\}$ and let $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ be defined through

$$\mathcal{D}(x) = \left\{ (f,g) \in T_x\mathcal{X} \oplus T_x^*\mathcal{X} \;\middle|\; f = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] g \right\} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

for $x \in \mathcal{X}$. Further, define $\mathcal{L} \subset T^*\mathcal{X}$ as

$$\mathcal{L} = \{ (x,y) \mid x \in (-1,1),\; y = \nabla H(x) \} = \left\{ (x,x) \;\middle|\; x \in S^1 \right\}$$

with $H : S^1 \to \mathbb{R}$, $x \mapsto \frac{\|x\|^2}{2}$. Clearly, $\mathcal{D}$ is a Dirac structure and $\mathcal{L}$ is a Lagrangian submanifold of $T^*\mathcal{X}$ and hence $(\mathcal{D}, \mathcal{L}, \{0\})$ is a port-Hamiltonian system whose dynamics read

$$\begin{aligned} &(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, x(t), e(t)) \in \mathcal{D}(x(t)), \quad (x(t), e(t)) \in \mathcal{L} \\ \iff\; &\tfrac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) \end{pmatrix}. \end{aligned}$$

These dynamics are explicit in the sense that one can write them as

$$\dot{x} = g(x).$$

However, one could formulate this setting differently. Let $\tilde{X} = \mathbb{R}^2$, $\tilde{\mathcal{D}} = \left(\mathrm{gr}\,\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right)^{-1}$, $\tilde{\mathcal{L}} = \left\{ (x,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; \|x\|^2 = 1 \right\}$. Then $\tilde{\mathcal{D}}$ is a Dirac structure, $\tilde{\mathcal{L}}$ a Lagrangian submanifold of $\mathbb{R}^2 \times \mathbb{R}^2$ and $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \{0\})$ defines again a port-Hamiltonian system whose dynamics read

$$\begin{aligned} &(-\tfrac{\mathrm{d}}{\mathrm{dt}}\, \tilde{x}(t), \tilde{e}(t)) \in \tilde{\mathcal{D}}(x(t)), \quad (\tilde{x}(t), \tilde{e}(t)) \in \tilde{\mathcal{L}} \\ \Leftrightarrow\; &\tfrac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{x}_2(t) \\ -\tilde{x}_1(t) \end{pmatrix} \wedge \tilde{x}_1(t)^2 + \tilde{x}_1(t)^2 = 1. \end{aligned}$$

Thus we see that the dynamics of $(\mathcal{D}, \mathcal{L}, \{0\})$ and $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \{0\})$ admit exactly the same solutions. However, the equations describing the dynamics of $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \{0\})$ can be rendered explicit, at least not globally. For $(\mathcal{D}, \mathcal{L}, \{0\})$ the information $x_1^2 + x_2^2$ was directly encoded in the state manifold, i.e., in the possible initial values, while it was incorporated as *Lagrange algebraic constraints* (cf. Chap. 5 and see [vdSM20]) for $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}, \{0\})$.

When modelling physical systems in a port-Hamiltonian fashion it might a priori not be clear how to proceed in order to obtain simpler equations. Note that in both cases we have $\mathcal{R} = \{0\}$. Even more complex implicit equations can be introduced by resistive structures since the assumptions on them are quite minimal. Nevertheless, one can ask when implicit equations generated by a port-Hamiltonian system can globally be rewritten as explicit equations. Such questions led to the formulation of Thm. B.1.1 for which we provide the following practical example occurring in the port-Hamiltonian modelling of electrical circuits.

Example 6.4.2. Consider two PN-junction diodes $(\mathcal{D}_{\mathcal{D}1}, \mathcal{L}_{\mathcal{D}1}, \mathcal{R}_{\mathcal{D}1})$, $(\mathcal{D}_{\mathcal{D}2}, \mathcal{L}_{\mathcal{D}2}, \mathcal{R}_{\mathcal{D}2})$ with

$$\begin{aligned} \mathcal{R}_{\mathcal{D}1} &= \left\{(-i_{\mathcal{D}1}, u_{\mathcal{D}1}) \in \mathbb{R}^2 \mid i_{\mathcal{D}1} = a_1 \left(e^{\frac{u_{\mathcal{D}1}}{b_1}} - 1\right) \ \wedge \ u_{\mathcal{D}1} \in (u_1^{\min}, u_1^{\max})\right\}, \\ \mathcal{R}_{\mathcal{D}2} &= \left\{(-i_{\mathcal{D}2}, u_{\mathcal{D}2}) \in \mathbb{R}^2 \mid i_{\mathcal{D}2} = a_2 \left(e^{\frac{u_{\mathcal{D}2}}{b_2}} - 1\right) \ \wedge \ u_{\mathcal{D}2} \in (u_2^{\min}, u_2^{\max})\right\}, \end{aligned} \tag{6.13}$$

for some constants $a_1, a_2, b_1, b_2 > 0$ and $u_1^{\min} < u_1^{\max}, u_2^{\min} < u_2^{\max} \in \mathbb{R}$. In comparison to the diodes presented in Sec. 6.2.2, the additional restrictions

$$u_{\mathcal{D}1} \in (u_1^{\min}, u_1^{\max}) =: Y_1, \quad u_{\mathcal{D}2} \in (u_2^{\min}, u_2^{\max}) =: Y_2, \tag{6.14}$$

reflect physical properties, e.g., the regions of operation of the corresponding devices that is being modelled. Note that $\mathcal{R}_{\mathcal{D}1}, \mathcal{R}_{\mathcal{D}2}$ still define resistive structures. From (6.13) and (6.14) we can also derive restrictions

$$i_{\mathcal{D}1} \in (i_1^{\min}, i_1^{\max}) =: X_1, \quad i_{\mathcal{D}2} \in (i_2^{\min}, i_2^{\max}) =: X_2. \tag{6.15}$$

Next, consider a current source $(\mathcal{D}_{\mathcal{S}}, \mathcal{L}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}})$. We consider a parallel connection of the two diodes and the current source with current $i_{\mathcal{S}}$ and voltage $u_{\mathcal{S}}$ as depicted in Fig. 6.18. Choosing $S = \{w\}$ we have,

$$\begin{aligned} \mathcal{D}_K^S(\mathcal{G}) &= \left\{ \left(\begin{pmatrix} j_v \\ i_{\mathcal{D}_1} \\ i_{\mathcal{D}_2} \\ i_{\mathcal{S}} \end{pmatrix}, \begin{pmatrix} \phi_v \\ u_{\mathcal{D}_1} \\ u_{\mathcal{D}_2} \\ u_{\mathcal{S}} \end{pmatrix} \right) \in \mathbb{R}^8 \ \middle| \ \begin{array}{l} j_v + i_{\mathcal{D}1} + i_{\mathcal{D}2} - i_{\mathcal{S}} = 0, \\ \phi_v = -u_{\mathcal{D}1} = -u_{\mathcal{D}2} = u_{\mathcal{S}} \end{array} \right\}, \\ \mathcal{L}_K^S(\mathcal{G}) &= \{0\} \times \mathbb{R}. \end{aligned}$$

Applying the procedure from Sec. 6.2.1, we arrive at the port-Hamiltonian system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ as in (6.7) modelling the circuit depicted in Fig. 6.18. From its dynamics we derive

$$\begin{aligned} i_{\mathcal{D}1}(t) + i_{\mathcal{D}2}(t) &= i_{\mathcal{S}}(t), \\ u_{\mathcal{D}1}(t) = u_{\mathcal{D}2}(t) &= -u_{\mathcal{S}}(t), \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{6.16}$$

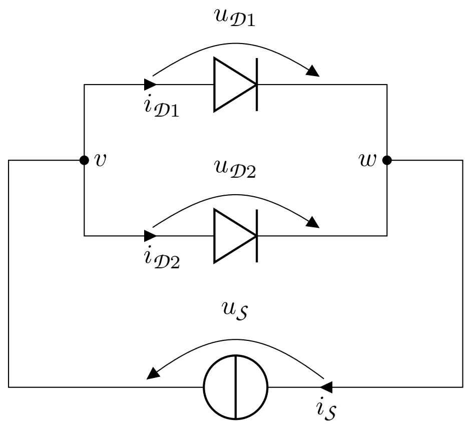

Figure 6.18: Circuit containing two diodes and a current source.

From the constitutive relations (6.13) and (6.16), it is clear that we can describe $i_{\mathcal{D}1}, i_{\mathcal{D}2}$ and hence $i_{\mathcal{S}}$ as a function of $u_{\mathcal{S}}$. Since we have a current source, one might to want to establish the converse. However, it is not evident how $u_{\mathcal{S}}$ is given in terms of the current $i_{\mathcal{S}}$. Invoking (6.13) and (6.16), $i_{\mathcal{S}}$ and $u_{\mathcal{S}}$ satisfy the relation

$$i_{\mathcal{S}} = a_1 \left(e^{\frac{-u_{\mathcal{S}}}{b_1}} - 1 \right) + a_2 \left(e^{\frac{-u_{\mathcal{S}}}{b_2}} - 1 \right) =: f(u_{\mathcal{S}}).$$

Further, by (6.14) and (6.16), $u_{\mathcal{S}}$ has to satisfy $u_{\mathcal{S}} \in -Y_1 \cap -Y_2 =: Y$, whereas $i_{\mathcal{S}}$ has to satisfy $u_{\mathcal{S}} \in X_1 + X_2 = (i_1^{\min} + i_2^{\min}, i_1^{\max} + i_2^{\max}) =: X$ by (6.15) and (6.16). Note that both X and Y are open intervals, and we exclude the trivial case that Y is empty. Recapitulating, with $F : X \times Y \to \mathbb{R}, (i_{\mathcal{S}}, u_{\mathcal{S}}) \mapsto i_{\mathcal{S}} - f(u_{\mathcal{S}})$ and

$$Z := \{ \ (i_{\mathcal{S}}, u_{\mathcal{S}}) \in X \times Y \ \mid F(i_{\mathcal{S}}, u_{\mathcal{S}}) = 0 \ \},$$

we seek the existence of a function $g \in C(\pi_1(Z), Y)$ such that

$$\{ \ (i_{\mathcal{S}}, g(i_{\mathcal{S}})) \ \mid i_{\mathcal{S}} \in \pi_1(Z) \ \} = Z.$$

The existence of such a function is obviously equivalent to the invertibility of f on $\pi_2(Z) \subseteq Y$, which holds true since its derivative is strictly negative. Nevertheless, we check the assumptions (i)-(iii) of Thm.B.1.1 in order to illustrate it. Since Z is the zero set of a continuous function, it is relatively closed in $X \times Y$, i.e., (i) holds. For (ii), note that

$$D_{u_{\mathcal{S}}} F(i_{\mathcal{S}}, u_{\mathcal{S}}) = -f'(u_{\mathcal{S}}) = \tfrac{a_1}{b_1} \exp\left(\tfrac{-u_{\mathcal{S}}}{b_1}\right) + \tfrac{a_2}{b_2} \exp\left(\tfrac{-u_{\mathcal{S}}}{b_2}\right) > 0.$$

It remains to find diffeomorphisms $\phi : \pi_1(Z) \to \mathbb{R}$, $\psi : \pi_2(Z) \to \mathbb{R}$ and a continuous weight $\omega : [0, \infty) \to (0, \infty)$ such that the growth bound in (iii) is satisfied. Choose any diffeomorphism $\phi : \pi_1(Z) \to \mathbb{R}$, which exists since $\pi_1(Z) = X$ is an open interval. Define $\psi := \phi \circ f$, which is a diffeomorphism since f is invertible on $\pi_2(Z)$. Then $\psi' = (\phi' \circ f) \cdot f'$. Further, let $\omega(t) = t + 1$ for $t \in [0, \infty)$ and note that $S(i_{\mathcal{S}}, u_{\mathcal{S}}) = \big(D_{u_{\mathcal{S}}} F(i_{\mathcal{S}}, u_{\mathcal{S}})\big)^{-1} = \big(-f'(u_{\mathcal{S}})\big)^{-1}$ for $(i_{\mathcal{S}}, u_{\mathcal{S}}) \in Z$. Recalling that $i_{\mathcal{S}} = f(u_{\mathcal{S}})$ for all $(i_{\mathcal{S}}, u_{\mathcal{S}}) \in Z$ we find

$$\begin{aligned}
&\|D\psi(u_{\mathcal{S}}) \cdot S(i_{\mathcal{S}}, u_{\mathcal{S}})\| \cdot \left\| D_{i_{\mathcal{S}}} F(i_{\mathcal{S}}, u_{\mathcal{S}}) \cdot \big(D\phi(i_{\mathcal{S}})\big)^{-1} \right\| \\
&= \left| \phi'(i_{\mathcal{S}}) \cdot f'(u_{\mathcal{S}}) \cdot \big(-f'(u_{\mathcal{S}})\big)^{-1} \right| \cdot \left| \phi'(i_{\mathcal{S}})^{-1} \right| = \frac{|\phi'(i_{\mathcal{S}})|}{|\phi'(i_{\mathcal{S}})|} = 1 \\
&\le |\psi(u_{\mathcal{S}})| + 1 = \omega(\|\psi(u_{\mathcal{S}})\|),
\end{aligned}$$

for all $(i_{\mathcal{S}}, u_{\mathcal{S}}) \in Z$, proving (iii).

Conclusions

At the beginning of this thesis, we had set three main objectives. The first objective, namely, comparing the linear algebraic port-Hamiltonian approach by Mehl, Mehrmann and Wojtylak [MMW18] with the geometric approach of van der Schaft and Maschke [vdSM18] was successfully achieved by providing a framework for port-Hamiltonian DAEs in Chap. 2 based on linear relations, which comprise both of the previous approaches by [vdSM18] and [MMW18]. This setting is more general than [vdSM18] since it does not assume maximality of the involved linear relations, and it is more general than [MMW18], since $\mathcal{D}$ could possibly be multi-valued. A detailed comparison of both approaches is shown in Figs. 2.9–2.11, where one can see that in general none of the approaches from [vdSM18; MMW18] implies the other. One of the reasons is that after considering [MMW18] in the language of linear relations, the symmetric subspaces $\mathcal{L}$ are not required to be maximal.

In particular, we have introduced DAEs by means of a product of dissipative linear relation $\mathcal{D}$ and symmetric linear relation $\mathcal{L}$. For our second objective of analyzing the DAEs and their corresponding matrix pencil arising from linear port-Hamiltonian modelling approaches, we have analyzed the Kronecker canonical of these matrix pencils in Chap. 3. Special emphasis has been placed on the case where the relation $\mathcal{L}$ is nonnegative. In particular, we have given statements of the eigenvalue locations and bounds on the index and minimal indices in Thm. 3.2.7. This condition played also a central role in the analysis conducted in Chap. 4 of stability properties of the corresponding DAEs. We saw that additionally assuming regularity and a trivial kernel of $\mathcal{L}$ are key to guarantee the stability of these DAEs. Furthermore, we showed in Prop. 4.2.1 that stable DAEs can always be reformulated as port-Hamiltonian systems if we restrict the coefficients of the underlying equations to the system space. This restriction is not necessary if we consider index one DAEs, see Rem. 4.2.2, a corresponding symmetric relation $\mathcal{L}$ can be obtained from a solution of a generalized Lyapunov inequality presented in App. A. Similarly, we showed in Prop. 4.2.6 that stabilizable systems can be reformulated as port-Hamiltonian systems using the solutions of generalized algebraic Bernoulli equations (4.7) and (4.8). The analysis of port-Hamiltonian pencils was concluded in Chap. 5, where constraints arising in the dynamics of port-Hamiltonian systems were investigated and a link to pencils with positive semidefinite Hermitian part was established.

The previous described results naturally also contributed to the last goal of illustrating different aspects of the port-Hamiltonian modelling philosophy. Additionally, we gave academic examples in Secs. 2.2.3 & 6.4 illustrating certain phenomena but also presented an application of the port-Hamiltonian modelling approach in Chap. 6, where we modelled nonlinear electrical circuits in a port-Hamiltonian fashion and showed that one can derive well-established formulations of electrical circuits from the resulting model.

Open questions and future works

An open question that arose early on in this thesis is whether the interconnection of two maximally dissipative relations yields a maximally dissipative relation, see Con. 1.3.16. If it fails, under the light of Con. 1.3.17 and Prop. 1.3.18, intuition tells us a counterexample should exist when the nonpositive relation is the graph of a negative definite matrix, since a maximal nonpositive relation that is purely multivalued or consists only of a kernel is in particular a Dirac structure and interconnecting it with a Dirac structure again gives a Dirac structure. Note that the interconnection $\mathcal{D} \circ \imath\mathcal{R}$ for a Dirac structure $\mathcal{D}$ and a maximally nonpositive relation $\mathcal{R}$ is a Dirac structure since $\imath\mathcal{R}$ becomes a Dirac structure. On the other hand, several routes to investigate this problem are available. The ansatz corresponding to the proof methodology for the corresponding statement in Prop. 1.3.14 would be to verify $\operatorname{mul}(\mathcal{D} \circ \mathcal{R})^{\perp} \subset \operatorname{dom} \mathcal{D} \circ \mathcal{R}$. Another possibility of proving Con. 1.3.16 would be to show that $\mathcal{M}_1 \circ \mathcal{M}_2$ admits a representation as shown in Cor. 1.3.11 for two maximally dissipative relations $\mathcal{M}_1, \mathcal{M}_2$. Here, one can start with the representation given by Lem. 1.3.13 and proceed as in the proof of Prop. 1.3.18. The main challenge is to derive the correct matrix C in the context of Cor. 1.3.11.

Given a dissipative Hamiltonian pencil $sE - A$, there exist different suitable linear relations such that $\mathcal{DL} = \operatorname{ran}\left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right]$, e.g., $(\alpha^{-1}\mathcal{D})(\alpha\mathcal{L}) = \mathcal{DL}$ for $\alpha > 0$. Different linear relations might exhibit different properties, potentially inhibiting us to derive structural properties of the pencil based on our results. A detailed analysis could help determine more precisely when our findings can be applied. Further, we saw in Chap. 2 that the energy of the system can be interpreted in terms of the symmetric relation. This leads to the question of what can be considered to be the correct energy of a system modelled by a DAE with a dissipative-Hamiltonian pencil, since different representations exist. It seems to be advisable to start with the linear relations first when modelling a physical system and to only derive the underlying DAE later on through a range representation of their product. Speaking of the energy of a port-Hamiltonian system, Ex. 4.1.3 shows that a gap between the state having finite energy and the state being bounded exists, which might be worthwhile characterizing.

Coming back to the underlying DAE of a port-Hamiltonian system, recall that for given dissipative and symmetric linear relations $\mathcal{D}$, $\mathcal{L}$, there exist different methods of associating a DAE to the dynamics of the corresponding port-Hamiltonian system, as depicted in Fig. 2.8. For such a system, the different pencils exhibit in general different properties. Take for example the dissipative-Hamiltonian pencil derived in Ex. 3.2.2 where both $\mathcal{D}$ and $\mathcal{L}$ were given in range representation. This pencil is regular. However, if $\mathcal{D}$ was given in range representation, then the corresponding pencil $sE - A$ in Fig. 2.8 would be singular. This comes from the fact that $\mathcal{L}$ in Ex. 3.2.2 is not maximal, i.e., the matrices involved in its range representation have a common kernel. This implies that E and A have a common kernel according to Fig. 2.8, making $sE - A$ singular. This opens the door for further research of the other DAEs derived in Fig. 2.8, as the different methods might suite different applications better depending on their properties.

There is the possibility to extend the techniques applied in Chap. 4 to stable and stabilizable systems also to systems where solutions to the so-called Kalman–Yakubovich–Popov inequality exist. This has already been done for certain systems in [BMvD19; GS18]. However, these results do not rely on the KYP presented in [RRV15; RV16], which is closely related to our setting as the inequalities therein are considered on a subspace. The main difficulty here is to determine when and where the solutions to these relaxed inequalities are invertible.

One can also think of a possible extension of Chap. 5 in the realm of nonlinear port-Hamiltonian systems. This has already been achieved for two of the four constraints studied in Chap. 5 in [vdSM20], namely, for the Dirac and Lagrange constraints introduced in [vdSM18]. It remains to be determined if the other two constraints we introduced can be transposed to the nonlinear case and possibly determine a nonlinear version of systems corresponding to posH pencils.

Recently, [NPS22] presented a port-Hamiltonian formulation of electrical circuits with desirable index properties. An interesting task would be to compare it with the modelling presented in this thesis and examine its index properties. Since we can derive the MNA and MLA equations from it under certain assumptions, the procedure should rely on techniques from the proof of [Rei14, Thm. 6.6]. The results presented in this thesis are not expected to yield better index characterizations than [Rei14] in the case of linear MNA equations, since [Rei14] additionally incorporates several structural properties of the electrical circuits for the index analysis.

As we saw in Chap. 6, Josephson junctions are incorporated in our modelling framework. Hence, after quantizing our electrical circuit model [FN19, Sec. 17.2], we should be able to model qubits [FN19, Sec. 17.3]. It would be interesting to see how this fits into the port-Hamiltonian modelling of quantum circuits proposed by [Mox20].

Finally, we want to point out that the comparison between 'geometric' and 'linear-algebraic' approaches as done here for lumped parameter linear systems can also be performed for distributed parameter systems based on the same port-Hamiltonian modelling philosophy, e.g., [Rei21; Siu11; JZ12; RCvdS^{+}20].

Appendices

Appendix A

Lyapunov inequalities for differential-algebraic equations

In mathematical systems theory, key properties of linear systems can be characterised by means of algebraic criteria. For ordinary differential equations classical results such as the Kalman rank conditions for controllability and observability [Son98, pp. 89, 271] come to mind, which have been generalized in a rather straight-forward manner for differential-algebraic equations [BR13, Rem. 4.6], [BR17, Rem. 6.5]. In contrast to the results for ordinary-differential equations, other criteria like the Kalman–Yakubovich–Popov inequality [RRV15] and the corresponding Lur'e equation [RV19] were generalized in a special way: these criteria are restricted to certain relevant subspaces.
In the case of stability properties, generalizations of the classical Lyapunov equation, see [Son98, Thm. 5.7.18], for differential-algebraic equations were previously given, e.g., in [TMK94; IT02; Sty02] and recently in [AAM21, Thm. 4]. The aim of this appendix is to present matrix inequalities restricted to subspaces linked to stability properties of differential-algebraic equations. Results going in a similar direction are presented in [LT12, Thm. 2.7] for switched systems and [Ber10, Thm. 3.6.2] for time-varying systems. Here, we characterise stability properties of systems $[E, A] \in \Sigma_{n,m}$ by means of certain solutions of the *Lyapunov inequalities*

$$A^* XE + E^* XA \leq_{\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0, \qquad XE\mathcal{V}_{\mathrm{sys}}^{[E,A]} = E\mathcal{V}_{\mathrm{sys}}^{[E,A]}, \tag{A.1}$$

$$A^*XE + E^*XA <_{\mathcal{V}_{\text{sys}}^{[E,A]}} 0, \qquad XE\mathcal{V}_{\text{sys}}^{[E,A]} = E\mathcal{V}_{\text{sys}}^{[E,A]}, \tag{A.2}$$

$$A^*XE + E^*XA \leq_{E^\dagger E\mathcal{V}_{\text{sys}}^{[E,A]}} 0, \qquad XE\mathcal{V}_{\text{sys}}^{[E,A]} = E\mathcal{V}_{\text{sys}}^{[E,A]}, \tag{A.3}$$

$$A^*XE + E^*XA <_{E^\dagger E\mathcal{V}_{\text{sys}}^{[E,A]}} 0, \qquad XE\mathcal{V}_{\text{sys}}^{[E,A]} = E\mathcal{V}_{\text{sys}}^{[E,A]}. \tag{A.4}$$

To be more precise, we say that for $[E, A]$ there exists a solution to either (A.1)–(A.4) if there exists some $X \in \mathbb{K}^{m\times m}$ with $X >_{E\mathcal{V}_{\text{sys}}^{[E,A]}} 0$ for which the corresponding Lyapunov inequality holds.

Proposition A.0.1. *Let $k \in \mathbb{N}$ and $A \in \mathbb{K}^{k\times k}$. Then the following statements hold.*

	$[I_k, A]$	$[N_k, I_k]$	$[K_k, L_k]$	$[K_k^\top, L_k^\top]$
there exists a solution $X >_{E\mathcal{V}_{\text{sys}}} 0$ *to* (A.1)	$\Leftrightarrow \sigma(A_l) \subset \overline{\mathbb{C}_-}$ *with semi-simple eigenvalues on* $\imath\mathbb{R}$	✓	✗	✓
there exists a solution $X >_{E\mathcal{V}_{\text{sys}}} 0$ *to* (A.2)	$\Leftrightarrow \sigma(A_l) \subset \mathbb{C}_-$	✓	✗	✓
there exists a solution $X >_{E\mathcal{V}_{\text{sys}}} 0$ *to* (A.3)	$\Leftrightarrow \sigma(A_l) \subset \overline{\mathbb{C}_-}$ *with semi-simple eigenvalues on* $\imath\mathbb{R}$	✓	$\Leftrightarrow k = 1$	✓
there exists a solution $X >_{E\mathcal{V}_{\text{sys}}} 0$ *to* (A.4)	$\Leftrightarrow \sigma(A_l) \subset \mathbb{C}_-$	✓	$\Leftrightarrow k = 1$	✓
stable	$\Leftrightarrow \sigma(A_l) \subset \overline{\mathbb{C}_-}$ *with semi-simple eigenvalues on* $\imath\mathbb{R}$	✓	✗	✓
asymptotically stable	$\Leftrightarrow \sigma(A_l) \subset \mathbb{C}_-$	✓	✗	✓
stable differential variables	$\Leftrightarrow \sigma(A_l) \subset \overline{\mathbb{C}_-}$ *with semi-simple eigenvalues on* $\imath\mathbb{R}$	✓	$\Leftrightarrow k = 1$	✓
asymptotically stable differential variables	$\Leftrightarrow \sigma(A_l) \subset \mathbb{C}_-$	✓	$\Leftrightarrow k = 1$	✓

Proof. We discuss each prototipical DAE separately.
Step 1: First consider $[I_k, A]$ and note that the concepts of stability and stable differential variables as well as asymptotic stability and asymptotically stable differential variables coincide for this system. The proof of the statements contained in the table are standard and can be found in, e.g., [HP05, Thm. 3.3.20, Thm. 3.3.49], [Wal98, Chap. VII] and [Lya48, Chap. II].
Step 2: Now consider $[N_k, I_k]$. Since 0 is the only solution to the corresponding equation and $\mathcal{V}_{\text{sys}}^{[N_k,I_k]} = \{0\}$ by Prop. 1.2.15, the statements trivially hold true and one

may always choose $X = I_k$.

<u>*Step 3:*</u> Next, we turn our attention to $[K_k, L_k]$. Let $X = X^* \in \mathbb{K}^{m\times m}$. Then

$$L_k^* X K_k + K_k^* X L_k = \begin{bmatrix} & 0_1 \\ X & \end{bmatrix} + \begin{bmatrix} & X \\ 0_1 & \end{bmatrix} = \begin{bmatrix} * & * \\ * & 0_1 \end{bmatrix} \in \mathbb{K}^{(k-1)\times(k-1)},$$

and with $\mathcal{V}_{\mathrm{sys}}^{[K_k,L_k]} = \mathbb{K}^k$ by Prop. 1.2.15 one has

$$K_k^\dagger K_k \mathcal{V}_{\mathrm{sys}}^{[K_k,L_k]} = \begin{bmatrix} I_{k-1} & \\ & 0_1 \end{bmatrix} \mathbb{K}^k = \mathbb{K}^{k-1} \times \{0\}.$$

From these equations one easily deduces the statements concerning the Lyapunov equations and may choose $X = 1$ when $k = 1$. Moreover, by Prop. 1.2.15, $x \in \mathfrak{B}_{[K_k,L_k]}$ if and only if there exists some $f \in W^{k-1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{K})$ such that $x = \left(\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{k-i} f\right)_{i=1,\ldots,k}$. Consequently, $K_k x = \left(\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{k-i} f\right)_{i=1,\ldots,k-1}$, from which the remaining statements of this case follow.

<u>*Step 4:*</u> Finally, the argumentation for $[K_k^\top, L_k^\top]$ is the same as for $[N_k, I_k]$ and one may always choose $X = I_{k-1}$, i.e., $X = 0_{0\times 0}$ when $k = 1$. $\square$

Proposition A.0.2. *Let $[E_1, A_1] \in \Sigma_{n_1,m_1}, [E_2, A_2] \in \Sigma_{n_2,m_2}$ and define*

$$[E, A] := [\mathrm{diag}\,(E_1, E_2), \mathrm{diag}\,(A_1, A_2)] \in \Sigma_{n,m} := \Sigma_{n_1+n_2,m_1+m_2}.$$

Further, let $S \in \mathbf{Gl}_m(\mathbb{K})$ and $T \in \mathbf{Gl}_n(\mathbb{K})$. Then the following statements hold.

$[E, A]$ has the property that	*if and only if*	*if and only if*
there exists a solution $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ to (A.1)	*there exist solutions $X_i >_{E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}} 0$ to* (A.1) *for $[E_i, A_i]$ with $i = 1, 2$.*	*there exists a solution $\tilde{X} >_{SET\mathcal{V}_{\mathrm{sys}}^{[SET,SAT]}} 0$ to* (A.1)
there exists a solution $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ to (A.2)	*there exist solutions $X_i >_{E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}} 0$ to* (A.2) *for $[E_i, A_i]$ with $i = 1, 2$.*	*there exists a solution $\tilde{X} >_{SET\mathcal{V}_{\mathrm{sys}}^{[SET,SAT]}} 0$ to* (A.2)
there exists a solution $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ to (A.3)	*there exist solutions $X_i >_{E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}} 0$ to* (A.3) *for $[E_i, A_i]$ with $i = 1, 2$.*	*there exists a solution $\tilde{X} >_{SET\mathcal{V}_{\mathrm{sys}}^{[SET,SAT]}} 0$ to* (A.3)
there exists a solution $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ to (A.4)	*there exist solutions $X_i >_{E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}} 0$ to* (A.4) *for $[E_i, A_i]$ with $i = 1, 2$.*	*there exists a solution $\tilde{X} >_{SET\mathcal{V}_{\mathrm{sys}}^{[SET,SAT]}} 0$ to* (A.4)
it is stable	*both $[E_1, A_1]$ and $[E_2, A_2]$ are stable*	*$[SET, SAT]$ is stable*
it is asymptotically stable	*both $[E_1, A_1]$ and $[E_2, A_2]$ are asymptotically stable*	*$[SET, SAT]$ is asymptotically stable*
it has stable differential variables	*both $[E_1, A_1]$ and $[E_2, A_2]$ have stable differential variables*	*$[SET, SAT]$ has stable differential variables*
it has asymptotically stable differential variables	*both $[E_1, A_1]$ and $[E_2, A_2]$ have asymptotically stable differential variables*	*$[SET, SAT]$ has asymptotically stable differential variables*

Proof. Step 1: The eight statements contained in the bottom half of the table follow directly from Props. 1.2.16 & 1.2.17 and Def. 1.2.19.

Step 2: For the four statements in the upper left quarter of the table, we first show that if $X := \begin{bmatrix} X_1 & X_3 \\ X_3^* & X_2 \end{bmatrix}$ is a solution for $[E, A]$ to either (A.1)–(A.4) with $X_i \in \mathbb{K}^{m_i \times m_i}$ and $i = 1, 2, 3$, then $\operatorname{diag}(X_1, X_2)$ is a solution too. It is readily seen that the inequality containing the term $A^* \operatorname{diag}(X_1, X_2)E + E^* \operatorname{diag}(X_1, X_2)A$ is satisfied with $\operatorname{diag}(X_1, X_2) >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$. For the condition $\operatorname{diag}(X_1, X_2)E\mathcal{V}_{\mathrm{sys}}^{[E,A]} = E\mathcal{V}_{\mathrm{sys}}^{[E,A]}$, let $v_i \in E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}$ for $i = 1, 2$. Then $(v_1, 0), (0, v_2) \in E\mathcal{V}_{\mathrm{sys}}^{[E,A]}$. Hence,

$$\begin{bmatrix} X_1 & X_3 \\ X_3^* & X_2 \end{bmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} X_1 v_1 \\ X_3^* v_1 \end{pmatrix} \in E\mathcal{V}_{\mathrm{sys}}^{[E,A]} \text{ and } \begin{bmatrix} X_1 & X_3 \\ X_3^* & X_2 \end{bmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} X_3 v_2 \\ X_2^* v_2 \end{pmatrix} \in E\mathcal{V}_{\mathrm{sys}}^{[E,A]}.$$

Consequently, $X_i v_i \in \mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}$ for $i = 1, 2$ and since in particular $X_i >_{E\mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}} 0$ we deduce $X_i E_i \mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]} = E_i \mathcal{V}_{\mathrm{sys}}^{[E_i,A_i]}$.

Conversely, if for $i = 1, 2$ X_i are given as solutions for $[E_i, A_i]$ to either (A.1)–(A.4), respectively, then it is evident that $\operatorname{diag}(X_1, X_2)$ is a solution for $[E, A]$ to the corresponding Lyapunov equation.

Step 3: For the four statements in the upper right quarter of the table, note that $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ is a solution for $[E, A]$ to either (A.1)–(A.4) if and only

$$S^{-*} X S^{-1} >_{SET\mathcal{V}_{\mathrm{sys}}^{[SET,SAT]}} 0$$

is a solution to the corresponding Lyapunov equation for $[SET, SAT]$ by virtue of Prop. 1.2.16. □

Proposition A.0.3. *Let $[E, A] \in \Sigma_{n,m}$. Then the following statements hold.*

$[E, A]$	*if and only if*	*if and only if*
is stable	*there exists a solution* $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ *to* (A.1)	$\sigma(E, A) \subset \overline{\mathbb{C}_-}$ *with semisimple eigenvalues on* $\imath\mathbb{R}$ *and* $\ell(\beta) = 0$ *in the Kronecker form* (1.1)
is asymptotically stable	*there exists a solution* $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ *to* (A.2)	$\sigma(E, A) \subset \mathbb{C}_-$ *and* $\ell(\beta) = 0$ *in the Kronecker form* (1.1)
has stable differential variables	*there exists a solution* $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ *to* (A.3)	$\sigma(E, A) \subset \overline{\mathbb{C}_-}$ *with semisimple eigenvalues on* $\imath\mathbb{R}$ *and* $\ell(\beta) = \lvert\beta\rvert$ *in the Kronecker form* (1.1)
has asymptotically stable differential variables	*there exists a solution* $X >_{E\mathcal{V}_{\mathrm{sys}}^{[E,A]}} 0$ *to* (A.4)	$\sigma(E, A) \subset \mathbb{C}_-$ *and* $\ell(\beta) = \lvert\beta\rvert$ *in the Kronecker form* (1.1)

Proof. This result is a combination of Lem. 1.2.5, Thm. 1.2.2, and Props. A.0.1 & A.0.2. □

Corollary A.0.4. *Let* $[E, A] \in \Sigma_n$. *Then it is asymptotically stable (stable) if and only if* $sE - A$ *is regular and* $\sigma(E, A) \subseteq \mathbb{C}_-$ ($\overline{\mathbb{C}_-}$ *with semisimple eigenvalues on* $\imath\mathbb{R}$).

Proof. This follows directly from Prop. A.0.3 noticing that $\ell(\beta) = 0$ implies $\ell(\gamma) = 0$ for square pencils in the Kronecker form (1.1) of $[E, A]$, that is $sE - A$ is regular by Thm. 1.2.7. □

Remark A.0.5. Note that the solutions to the Lyapunov inequalities (A.1)–(A.4) are not unique. This is already the case for an ODE $[I, A]$ with, e.g., $A = -I$ and $A = 0$. Further, one can freely choose to linearly extend the solution on the complement of the system space. Further, it is strictly speaking not necessary to include the condition $XE\mathcal{V}_{\text{sys}}^{[E,A]} = \mathcal{V}_{\text{sys}}^{[E,A]}$ in (A.1)–(A.4) to derive the results presented in this section. Conversely, if solutions exist, then there exist solutions satisfying $XE\mathcal{V}_{\text{sys}}^{[E,A]} = \mathcal{V}_{\text{sys}}^{[E,A]}$.

Appendix B

An implicit function theorem

When dealing with nonlinear dynamical systems with constraints, i.e., implicit differential equations of the form

$$F\big(x(t), \dot{x}(t)\big) = 0,$$

where the number of equations does not match the number of variables, it is often necessary to solve this equation for $\dot{x}(t)$, preferably globally in the form $\dot{x}(t) = g\big(x(t)\big)$. This means to find a global implicit function of the equation $F(x, y) = 0$. Numerous results on global implicit function theorems exist, and we mention the relevant literature. However, most results involve conditions which are not easy to check in practice. In this second appendix, we provide a novel extension of the global implicit function theorem under conditions which can easily be verified.

In the following, we summarize some results on global implicit functions, tailored to be applicable in our framework. We consider equations of the form $F(x, y) = 0$ for which we want to find a unique maximal solution $y(x)$. There are several approaches available in the literature which provide a solution to this problem, see, e.g., [Rhe69] for an early result. Most works concentrate on the case that the partial derivative $\frac{\partial F}{\partial y}(x, y)$ is invertible for all (x, y), i.e., $F(x, y) = 0$ is locally solvable for $y(x)$ in a neighborhood of every point (a, b) such that $F(a, b) = 0$. We discuss some important work:

- For $F : X \times Y \to \mathbb{R}^l$, where $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open and X is convex, Sandberg [San81] provides necessary and sufficient conditions for the existence of a unique $g \in C(X, Y)$ such that $F^{-1}(0) = \{\ (x, g(x)) \ \mid x \in X\ \}$. However, the conditions are not easy to check; in particular, it needs to be guaranteed that

$$\begin{aligned}&\text{for some } x_0 \in X \text{ there exists exactly one}\\&y_0 \in Y \text{ such that } F(x_0, y_0) = 0.\end{aligned} \tag{B.1}$$

Furthermore, the result of Sandberg is not applicable in the case that the maximal solution g is not defined on all of X.

- Using the theory of covering maps, Ichiraku [Ich85] improves the characterization of Sandberg. Nevertheless, the condition (B.1) is still present and the results are only applicable in the case of globally defined g. However, in [Ich85, Thm. 5] it is shown that in the case $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $l = m$ for the existence of a unique solution $g \in C(\mathbb{R}^m, \mathbb{R}^n)$ it is sufficient that $\frac{\partial F}{\partial y}(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, condition (B.1) holds and

$$\forall\, (x, y) \in F^{-1}(0) : \quad \left\| \left(\frac{\partial F}{\partial y}(x, y) \right)^{-1} \right\| \cdot \left\| \frac{\partial F}{\partial x}(x, y) \right\| \le M \tag{B.2}$$

for some $M \ge 0$.

- The above result of Ichiraku has in turn been improved by Gutú and Jaramillo [GJ07, Cor. 5.3], who showed that the condition (B.1) can be replaced by the intuitive condition "$F^{-1}(0)$ is connected" and in the condition (B.2) the constant M can be replaced by the term $\omega(\|y\|)$, where $\omega : [0, \infty) \to (0, \infty)$ is a continuous *weight*, which means that ω is nondecreasing and

$$\int_0^\infty \frac{\mathrm{d}t}{\omega(t)} = \infty.$$

These conditions are indeed easy to check. The only drawback is that F needs to be defined on all of $\mathbb{R}^m \times \mathbb{R}^n$ and the solution g is defined on all of $\mathbb{R}^m$.

- A result which is similar to that of Gutú and Jaramillo, but holds for some $X \subseteq \mathbb{R}^m$ which is open, connected and starlike with respect to some $a \in X$ such that $F(a, b) = 0$ for some $b \in Y = \mathbb{R}^n$, has been derived by Cristea [Cri07]. The assumption of connectedness of $F^{-1}(0)$ is not needed. However, a version of assumption (B.2) (with $M = \omega(\|y\|)$) is required to hold on all of $X \times \mathbb{R}^n$.

- Yet another approach has been pursued by Zhang and Ge [ZG06] who show that for existence of a unique solution $g \in C(\mathbb{R}^m, \mathbb{R}^n)$ it is sufficient that the element-wise absolute value of $\frac{\partial F}{\partial y}$ is uniformly strictly diagonally dominant in the sense that there exists $d > 0$ such that

$$\left|\left(\frac{\partial F}{\partial y}(x,y)\right)_{ii}\right| - \sum_{j \neq i} \left|\left(\frac{\partial F}{\partial y}(x,y)\right)_{ij}\right| \geq d$$

for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and all $i = 1, \ldots, n$. While this condition is easy to check, it is very restrictive as it already excludes a lot of linear equations $Ax + By = 0$ where B is not strictly diagonally dominant, but invertible.

As discussed above, typical limitations of the approaches are that F needs to be defined on all of $\mathbb{R}^m \times \mathbb{R}^n$ or the solution g is required to be globally defined. In [Blo91] these limitations are resolved as X and Y are assumed to be open and X is connected, and maximal solutions of $F(x, y) = 0$ are considered in every connected component of $F^{-1}(0)$. Assuming that $Z := F^{-1}(0)$ is connected we may then find a solution $g \in C(\pi_1(Z), Y)$, where $\pi_1 : X \times Y \to X,\ (x, y) \mapsto x$ is the projection onto the first component, provided that $\pi_1(Z)$ is open and simply connected and $\pi_1 : Z \to \pi_1(Z)$ "lifts lines" (for a precise definition see [Pla74, Def. 1.1]). This result can be extended in a straightforward way to the case where $l \geq m$ and rk $\frac{\partial F}{\partial y}(x, y) = n$ for all $(x, y) \in X \times Y$ since it is only necessary to show that π_1 is locally a homeomorphism, which replaces the condition that $F(x, y) = 0$ is locally solvable for $y(x)$ as in [Blo91, Thm. 4]; then [Blo91, Lem. 1] can still be applied to $\pi_1 : Z \to \pi_1(Z)$. The drawback of this result is that the condition "$\pi_1 : Z \to \pi_1(Z)$ lifts lines" is not easy to check.

In the present paper, we provide a generalization of [GJ07, Cor. 5.3] to the case of functions defined only on open subsets and where the partial derivative $\frac{\partial F}{\partial y}$ is only required to have a left inverse instead of being invertible. The crucial assumption is that the projections $\pi_i(Z)$ on the ith component, $i = 1, 2$, are diffeomorphic to some Banach spaces and the transformation of the equation $F(x, y) = 0$ satisfies a generalized version of (B.2). We stress that this assumption in particular implies that $\pi_i(Z)$ must be open and simply connected. The main result is presented in Sec. B.1 and a discussion together with some illustrative examples is given in Sec. B.2.

B.1 Main result

In this section we state and prove the following main result of this appendix.

Theorem B.1.1. *Let $X \subseteq \mathcal{U}$, $Y \subseteq \mathcal{V}$ be open sets, $\mathcal{U}, \mathcal{V}, \mathcal{Z}$ be Banach spaces, $F \in C^1(X \times Y, \mathcal{Z})$ and*

$$Z \subseteq \{ \ (x, y) \in X \times Y \ \mid F(x, y) = 0 \ \}$$

be such that

1. *Z is path-connected and closed in $X \times Y$;*
2. *$\forall\, (x, y) \in Z \ \exists\, S(x, y) \in \mathcal{L}(\mathcal{Z}, \mathcal{V}) : \ S(x, y) D_y F(x, y) = \mathrm{id}_{\mathcal{V}}$;*[1]
3. *for the projections $\pi_i(p_1, p_2) = p_i$ with $i \in \{1, 2\}$, $(p_1, p_2) \in \mathcal{U} \times \mathcal{V}$, there exist diffeomorphisms $\phi : \pi_1(Z) \to \mathcal{X}$, $\psi : \pi_2(Z) \to \mathcal{Y}$ for some Banach spaces $\mathcal{X}, \mathcal{Y}$, and a continuous weight $\omega : [0, \infty) \to (0, \infty)$ such that for all $(x, y) \in Z$ we have*

$$\|D\psi(y) \cdot S(x, y)\|_{\mathcal{L}(\mathcal{Z}, \mathcal{Y})} \cdot \left\| D_x F(x, y) \cdot \big(D\phi(x)\big)^{-1} \right\|_{\mathcal{L}(\mathcal{X}, \mathcal{Z})} \leq \omega(\|\psi(y)\|_{\mathcal{Y}}).$$

Then there exists a unique $g \in C(\pi_1(Z), Y)$ such that

$$\{ \ (x, g(x)) \ \mid x \in \pi_1(Z) \ \} = Z,$$

and g is Fréchet-differentiable at every $x \in \pi_1(Z)$.

The proof of Thm.B.1.1 requires us to recall the following concepts, which can be found in [GJ07, pp. 77–80].

Definition B.1.2. Let Z be a metric space, and let $\mathcal{P}$ be a family of continuous paths in Z. We say that Z is *$\mathcal{P}$-connected*, if the following conditions hold:

(1) If the path $p : [a, b] \to Z$ belongs to $\mathcal{P}$, then the reverse path $\overline{p}$, defined by $\overline{p}(t) = p(a - t + b)$, also belongs to $\mathcal{P}$;

(2) Every two points in Z can be joined by a path in $\mathcal{P}$.

We say that Z is *locally $\mathcal{P}$-contractible* if every point $z_0 \in Z$ has an open neighborhood U which is *$\mathcal{P}$-contractible*, in the sense that there exists a homotopy $H : U \times [0, 1] \to U$ between the constant function $U \ni z \mapsto z_0$ and the identity id_U, which satisfies

[1]Here $\mathcal{L}(\mathcal{Z}, \mathcal{V})$ denotes the Banach space of all bounded linear operators $A : \mathcal{Z} \to \mathcal{V}$ and $\mathrm{id}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}, \ v \mapsto v$ is the identity operator on $\mathcal{V}$.

(a) $H(z_0, t) = z_0$, for all $t \in [0,1]$,

(b) for every $z \in U$, the path $p_z \coloneqq H(z,t)$ belongs to $\mathcal{P}$.

Further, let Z' also be a metric space and $p : [0,1] \to Z'$ be a path in Z'. We say that a continuous map $f : Z \to Z'$ has the *continuation property for* p, if for every $b \in (0,1]$ and every path $q \in C([0,b), Z)$ such that $f \circ q = p\restriction_{[0,b)}$, there exists a sequence $\{t_n\}$ in $[0,b)$ convergent to b and such that $\{q(t_n)\}$ converges in Z. Furthermore, a continuous map $f : Z \to Z'$ is called a *covering map*, if every $z' \in Z'$ has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open subsets of Z each of which is mapped homeomorphically into U by f.

Proof of Theorem B.1.1. We proceed in several steps.

Step 1: We first reduce the original problem to a simpler case. By the existence of ϕ, ψ in assumption (iii) it follows that, for $i \in \{1,2\}$, $\pi_i(Z)$ are open sets in $\mathcal{U}, \mathcal{V}$, resp., and since $Z \subseteq \pi_1(Z) \times \pi_2(Z)$, it is no loss of generality to assume $X \times Y = \pi_1(Z) \times \pi_2(Z)$. That is, we search for an implicit function for the restriction $F : \pi_1(Z) \times \pi_2(Z) \to \mathcal{Z}$ instead of $F : X \times Y \to \mathcal{Z}$. Next, we argue that it suffices to prove the theorem for cases in which (i)–(iii) are satisfied with $\phi = \mathrm{id}_{\mathcal{X}}$ and $\psi = \mathrm{id}_{\mathcal{Y}}$. Note that these assumptions imply $\mathcal{U} = X = \pi_1(Z) = \mathcal{X}$ and $\mathcal{V} = Y = \pi_2(Z) = \mathcal{Y}$ since X and Y are open subspaces of $\mathcal{U}$ and $\mathcal{V}$, respectively. Having proved this case, we can conclude the general case by considering the function $\tilde{F} = F \circ (\phi^{-1}, \psi^{-1})$ with $\tilde{F} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. Next we translate the conditions (i)–(iii) on F to conditions on $\tilde{F}$ for $\tilde{Z} \coloneqq (\phi, \psi)(Z)$.

(i)' We have that $\tilde{Z}$ is path-connected and closed in $\mathcal{X} \times \mathcal{Y}$ if, and only if, Z is path-connected and closed in $X \times Y$.

(ii)' Define

$$\tilde{S} : \tilde{Z} \to \mathcal{Y},\ (\tilde{x}, \tilde{y}) \mapsto \left(D(\psi^{-1})(\tilde{y})\right)^{-1} \cdot S(\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y})).$$

With the identification $(x,y) = (\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y}))$ we obtain for all $(x,y) \in Z$ that

$$\begin{aligned} D_y F(x,y) &= D_y\left(\tilde{F}(\phi(x), \psi(y))\right) \\ &= (D_{\tilde{y}}\tilde{F})(\phi(x), \psi(y)) \cdot D\psi(y) \\ &= D_{\tilde{y}}\tilde{F}(\tilde{x}, \tilde{y}) \left(D(\psi^{-1})(\tilde{y})\right)^{-1}, \end{aligned}$$

where the latter equality is a consequence of the inverse function theorem. Using this we find that

$$\begin{aligned} & S(x,y)D_yF(x,y) = \mathrm{id}_{\mathcal{V}} \\ \iff\ & S\big(\phi^{-1}(\tilde{x}),\psi^{-1}(\tilde{y})\big)\cdot\big(D_{\tilde{y}}\tilde{F}\big)(\tilde{x},\tilde{y})\cdot\big(D(\psi^{-1})(\tilde{y})\big)^{-1} = \mathrm{id}_{\mathcal{V}} \\ \iff\ & \underbrace{\big(D(\psi^{-1})(\tilde{y})\big)^{-1}\cdot S\big(\phi^{-1}(\tilde{x}),\psi^{-1}(\tilde{y})\big)}_{=\tilde{S}(\tilde{x},\tilde{y})}\cdot D_{\tilde{y}}\tilde{F}(\tilde{x},\tilde{y}) = \mathrm{id}_{\mathcal{Y}}. \end{aligned}$$

(iii)' Similar to the computations above we obtain that, for $(x,y)\in Z$,

$$D_{\tilde{x}}\tilde{F}(\tilde{x},\tilde{y}) = D_xF(x,y)\cdot\big(D\phi(x)\big)^{-1}.$$

Therefore, we have for all continuous weights $\omega:[0,\infty)\to(0,\infty)$ and all $(x,y)\in Z$ that, omitting the spaces in the subscripts of the norms,

$$\begin{aligned} & \|D\psi(y)\cdot S(x,y)\|\cdot\left\|D_xF(x,y)\cdot\big(D\phi(x)\big)^{-1}\right\| \le \omega(\|\psi(y)\|) \\ \iff\ & \left\|\tilde{S}(\tilde{x},\tilde{y})\right\|\cdot\left\|D_{\tilde{x}}\tilde{F}(\tilde{x},\tilde{y})\right\| \le \omega(\|\tilde{y}\|). \end{aligned}$$

Now, define the projections $\tilde{\pi}_i(p_1,p_2)=p_i$ with $i\in\{1,2\}$, $(p_1,p_2)\in\mathcal{X}\times\mathcal{Y}$. We recapitulate the situation with the following commuting diagram:

$$\begin{tikzcd} & X=\pi_1(Z) \arrow[r, "\phi"] & \tilde{\pi}_1(\tilde{Z})\subseteq\mathcal{X} & \\ X\times Y\supseteq Z \arrow[ur, "\pi_1"] \arrow[rrr, "(\phi,\psi)"] \arrow[dr, "\pi_2"'] & & & \tilde{Z}\subseteq\mathcal{X}\times\mathcal{Y} \arrow[ul, "\tilde{\pi}_1"'] \arrow[dl, "\tilde{\pi}_2"] \\ & Y=\pi_2(Z) \arrow[r, "\psi"] & \tilde{\pi}_2(\tilde{Z})\subseteq\mathcal{Y} & \end{tikzcd}$$

For the conclusion, consider $\tilde{g}\coloneqq\psi\circ g\circ\phi^{-1}$ together with the equality $\pi_1(Z)=\phi^{-1}(\tilde{\pi}_1(\tilde{Z}))$.

Step 2: By Step 1, in the following we assume that $X=\pi_1(Z)$ and $Y=\pi_2(Z)$ as well as $\phi=\mathrm{id}_{\mathcal{X}}$ and $\psi=\mathrm{id}_{\mathcal{Y}}$. We show that $\pi_1:Z\to\pi_1(Z)$ is a local homeomorphism between connected metric spaces. Clearly, Z and $\pi_1(Z)$ are metric spaces and since Z is path-connected, $\pi_1(Z)$ is path-connected as well. To show that $\pi_1:Z\to\pi_1(Z)$ is a local homeomorphism, let $(a,b)\in Z$, i.e., $F(a,b)=0$. Applying the implicit function theorem, see, e.g., [Die69, Thm. 10.2.1], yields open neighborhoods $U \subseteq X$ of a, $V \subseteq Y$ of b, and $g \in C^1(U, V)$ such that

$$\{ \ (x, g(x)) \ | \ x \in U \ \} = \{ \ (x, y) \in U \times V \ | \ S(a, b)F(x, y) = 0 \ \}.$$

Consider the restriction $\hat{\pi}_1 : Z \cap (U \times V) \to \pi_1\big(Z \cap (U \times V)\big)$. Then $\hat{\pi}_1$ is injective since $\hat{\pi}_1(x_1, y_1) = \hat{\pi}_1(x_2, y_2)$ for some $(x_1, y_1), (x_2, y_2) \in Z \cap (U \times V)$ gives $x_1 = x_2$ and $S(a, b)F(x_1, y_1) = S(a, b)F(x_2, y_2)$, thus $y_1 = g(x_1) = g(x_2) = y_2$. Therefore, $\hat{\pi}_1$ is bijective and continuous. Furthermore, it is easy to see that $\hat{\pi}_1$ is an open map, and hence it is a homeomorphism.

Step 3: Let

$$\mathcal{P} := C^1\big([0, 1], \pi_1(Z)\big)$$

and observe that $\pi_1(Z)$ is $\mathcal{P}$-connected and locally $\mathcal{P}$-contractible since it is open. We show that π_1 has the continuation property for every path in $\mathcal{P}$, that is, for all $q_1 \in \mathcal{P}$, all $b \in (0, 1]$ and all $q_2 \in C([0, b), Y)$ such that $(q_1(t), q_2(t)) \in Z$ for all $t \in [0, b)$ there exists a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0, b)$ with $\lim_{n\to\infty} t_n = b$ such that $\big(q_2(t_n)\big)_{n\in\mathbb{N}}$ converges and

$$\lim_{n\to\infty} (q_1(t_n), q_2(t_n)) \in Z.$$

First note that q_2 is differentiable at any $t \in [0, b)$, since there exists a local implicit function as in Step 2, so that $q_2(s) = g(q_1(s))$ for all s in a neighborhood of t. Since g and q_1 are differentiable we obtain $\dot{q}_2(t) = Dg(q_1(t))\dot{q}_1(t)$. Moreover, it can be seen that the derivative is continuous at each point in $[0, b)$. Then, using property (iii), it can be proved by only a slight modification of the proof of [GJ07, Cor. 5.3] that for any sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0, b)$ with $\lim_{n\to\infty} t_n = b$ the sequence $\big(q_2(t_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges in $Y = \mathcal{V}$. Since Z is closed in $X \times Y = \mathcal{U} \times \mathcal{V}$ by (i) we thus obtain $\lim_{n\to\infty}(q_1(t_n), q_2(t_n)) \in Z$.

Step 4: We show that $\pi_1 : Z \to \pi_1(Z)$ is a homeomorphism. By [GJ07, Thm. 2.6] and Step 3 we may infer that π_1 is a covering map. Since $\pi_1(Z) = \phi^{-1}(\mathcal{X})$ is in particular simply connected by (iii) it follows from [Lee12, Prop. A.79] that $\pi_1 : Z \to \pi_1(Z)$ is a homeomorphism.

Step 5: By Step 4 we have $\big(x \mapsto (x, g(x)) = \pi_1^{-1}(x)\big) \in C(\pi_1(Z), Z)$ which uniquely defines the desired function $g \in C(\pi_1(Z), Y)$. Since $\pi_1(Z)$ is in particular open by condition (iii), for all $x \in \pi_1(Z)$ we have that g coincides with any solution provided by the implicit function theorem as in Step 1 in a neighborhood of x. The implicit function theorem provides Fréchet-differentiability of the local solution, thus g is Fréchet-differentiable at x. □

We like to emphasize that Z in Theorem B.1.1 may only be a subset of the zero set $F^{-1}(0)$. This allows to exclude points (x, y) in $F^{-1}(0)$ at which $D_y F(x, y)$ has no left inverse, or, actually, to exclude open sets containing such points so that Z is closed (alternatively, one may restrict the sets X and Y). Then a global implicit function may still exist in each connected component of Z, provided the growth bound in (iii) is satisfied.

Remark B.1.3. An important question that arises is whether the growth bound in condition (iii) in Theorem B.1.1 is independent of the choice of the diffeomorphisms ϕ and ψ. Use the notation from Theorem B.1.1, assume that conditions (i)–(iii) are satisfied and let $\hat{\phi} : \pi_1(Z) \to \hat{\mathcal{X}}$ and $\hat{\psi} : \pi_2(Z) \to \hat{\mathcal{Y}}$ be diffeomorphisms for some Banach spaces $\hat{\mathcal{X}}, \hat{\mathcal{Y}}$. Then, omitting the subscripts indicating the spaces corresponding to the norms, we have the estimate

$$\begin{aligned}
&\left\| D\hat{\psi}(y) \cdot S(x, y) \right\| \cdot \left\| D_x F(x, y) \cdot \left(D\hat{\phi}(x) \right)^{-1} \right\| \\
&\leq \| D\psi(y) \cdot S(x, y) \| \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y) \right)^{-1} \right\| \\
&\quad \cdot \left\| D_x F(x, y) \cdot \left(D\phi(x) \right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x) \right)^{-1} \right\| \\
&\leq \omega(\|\psi(y)\|_{\mathcal{Y}}) \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y) \right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x) \right)^{-1} \right\|
\end{aligned}$$

for all $(x, y) \in Z$. If the last term satisfies

$$\omega(\|\psi(y)\|_{\mathcal{Y}}) \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y) \right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x) \right)^{-1} \right\| \leq \hat{\omega}(\|\hat{\psi}(y)\|_{\hat{\mathcal{Y}}})$$

for all $(x, y) \in Z$ and some continuous weight $\hat{\omega}$, then the growth bound in condition (iii) would indeed be independent of ϕ and ψ. However, it is still an open problem whether this is true (or a counterexample exists) and remains for future research.

B.2 Illustrative examples

In this section we discuss the assumptions in Theorem B.1.1 and provide some illustrative examples.

We like to highlight that none of the assumptions (i)–(iii) in Theorem B.1.1 can be omitted in general. It is clear that connectedness of Z in (i) and local solvability guaranteed by (ii) are indispensable. Counterexamples in finite dimension are constructed for (iii) in the following examples. Condition (iii) basically consists of two parts. The first one is to check whether $\pi_i(Z)$, $i = 1, 2$, are diffeomorphic to some Banach spaces. The second part is the growth bound involving the diffeomorphisms, the partial derivative $D_x F$ and the left inverse of $D_y F$.

First, we like to discuss why we chose the projections of Z as the domains of the diffeomorphisms in Theorem B.1.1, whereas intuitively one could consider the open sets X and Y as the domains.

Remark B.2.1. In a possible different formulation of Theorem B.1.1 one could choose diffeomorphisms $\tilde{\phi} : X \to \mathcal{X}$ and $\tilde{\psi} : Y \to \mathcal{Y}$ and then consider, *mutatis mutandis*, the corresponding growth bound in condition (iii). This would relax the assumptions on the projections $\pi_i(Z)$, which would then not necessarily need to be open and simply connected. However, for the proof technique to be feasible we need to additionally require that $\pi_1(Z)$ is simply connected. Indeed, the proof is analogous, but the modified theorem does not cover basic examples.

For $F : \mathbb{R} \times (-1, 1) \to \mathbb{R}$, $(x, y) \mapsto x - y$ and $Z := F^{-1}(0)$ it is easy to check that conditions (i) and (ii) are satisfied. The growth bound in condition (iii) reads

$$\begin{aligned} &\forall\, (x, y) \in Z: \ |\tilde{\psi}'(y)| \cdot |\tilde{\phi}'(x)^{-1}| \leq \omega(|\tilde{\psi}(y)|) \\ \iff \quad &\forall\, y \in (-1, 1): \ |\tilde{\phi}'(y)| \geq \frac{|\tilde{\psi}'(y)|}{\omega(|\tilde{\psi}(y)|)} \end{aligned}$$

for some continuous weight ω. Note that $\tilde{\phi}'((-1, 1))$ is bounded and $\tilde{\phi}(y) \neq 0$ for all $y \in \mathbb{R}$, hence

$$\int_{-1}^{1} |\tilde{\phi}'(y)| \mathrm{d}y = \left| \int_{-1}^{1} \tilde{\phi}'(y) \mathrm{d}y \right| < \infty.$$

Then the change of variables theorem with the substitution $t = \tilde{\psi}(y)$ together with the inverse function theorem yields that

$$\begin{aligned} \infty > \int_{-1}^{1} |\tilde{\phi}'(y)| \mathrm{d}y \geq \int_{-1}^{1} \frac{|\tilde{\psi}'(y)|}{\omega(|\tilde{\psi}(y)|)} \mathrm{d}y = \\ \int_{-\infty}^{\infty} \frac{\left|\tilde{\psi}'\left(\tilde{\psi}^{-1}(t)\right)\right|}{\omega(|t|)} \left|(\tilde{\psi}^{-1})'(t)\right| \mathrm{d}t = 2 \int_{0}^{\infty} \frac{1}{\omega(t)} \mathrm{d}t = \infty, \end{aligned}$$

a contradiction.

Nevertheless, a global implicit function obviously exists. The assumptions of Theorem B.1.1 are satisfied since $\pi_1(Z) = \pi_2(Z) = (-1, 1)$ and we may choose $\phi = \psi$, with which the growth bound holds true.

We continue by presenting an example where in assumption (iii) it is not possible to find suitable diffeomorphisms and, at the same time, a global implicit function does not exist.

Example B.2.2. Consider

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,\ (x_1, x_2, y) \mapsto \begin{pmatrix} x_1 - \cos y \\ x_2 - \sin y \end{pmatrix}, \quad Z \coloneqq F^{-1}(0).$$

Then assumptions (i) and (ii) in Theorem B.1.1 are satisfied. Since $\pi_1(Z) = S^1$, the unit circle in $\mathbb{R}^2$, there is no Banach space $\mathcal{X}$ such that $\pi_1(Z)$ is diffeomorphic to $\mathcal{X}$. Indeed, no global implicit function can exist, since $y \mapsto (\cos y, \sin y)$ is not injective on $\mathbb{R}$.

In the next example the growth bound in condition (iii) is not satisfied for any suitable choice of diffeomorphisms and, at the same time, a global implicit function does not exist.

Example B.2.3. We choose $F : X \times Y \to \mathbb{R}^2 = \mathcal{Z}$ as a function of the type $F(x, y) = x - \tilde{F}(y)$ and $Z \coloneqq F^{-1}(0)$. This means that the existence of a global implicit function is equivalent to $\tilde{F}$ being injective. We further set $X \times Y = \mathbb{R}^2 \times (0, 1)^2$ and construct $\tilde{F}$ by successively defining its restrictions $\tilde{F}\restriction_{(0,\delta)\times(0,1)} = \tilde{F}_1$ and $\tilde{F}\restriction_{(\delta,1)\times(0,1)} = \tilde{F}_2$ for some $0 < \delta \le \frac{1}{2}$. Choose $\varepsilon, \alpha > 0$, and consider the non-injective function

$$\tilde{F}_1 : (0, \delta) \times (0, 1) \to \mathbb{R}^2,\ (y_1, y_2) \mapsto \begin{pmatrix} (\alpha + y_2) \sin\left(\frac{2\pi}{\delta}(1 + \varepsilon) y_1\right) \\ (\alpha + y_2) \cos\left(\frac{2\pi}{\delta}(1 + \varepsilon) y_1\right) \end{pmatrix}.$$

Observe that im $\tilde{F}_1 = B_{\alpha+1}(0) \setminus \overline{B_\alpha(0)}$, where $B_\alpha(z)$ denotes the open ball with radius α around $z \in \mathbb{R}^2$, i.e., the image of $\tilde{F}_1$ is an annulus. Next, define $\tilde{F}_2$ similarly to $\tilde{F}_1$ using elementary functions such that the hole of the annulus is filled as displayed in Fig. B.1, i.e., $\overline{B_\alpha(0)} \subset$ im $\tilde{F}_2 \subset B_{\alpha+1}(0)$. Note that $\tilde{F}_2$ can be chosen such that the resulting composition $\tilde{F}$ is differentiable everywhere. Overall, we have constructed a *non-injective* function $\tilde{F}$.

Observe that im $\tilde{F} = \pi_1(Z) = B_{\alpha+1}(0)$ and $\pi_2(Z) = (0, 1)^2$. Then the three conditions on F translate to $\tilde{F}$ as follows:

(i') the graph of $\tilde{F}$ is connected;

(ii') $\forall\, y \in (0, 1)^2 :\ \text{rk}\ D\tilde{F}(y) = 2\,;$

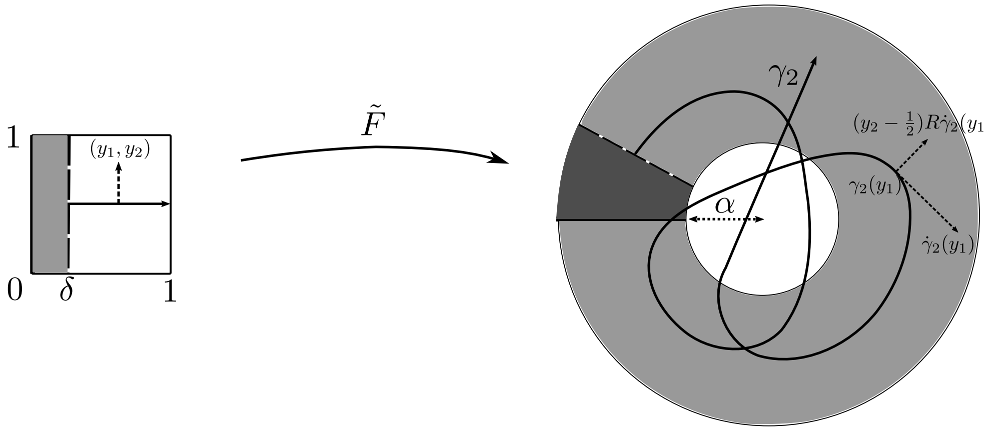

Figure B.1: Illustration of the construction of $\tilde{F}$.

(iii') there exist diffeomorphisms $\phi : B_{\alpha+1}(0) \to \mathcal{X}, \psi : (0,1)^2 \to \mathcal{Y}$ for some Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, and a continuous weight $\omega : [0,\infty) \to (0,\infty)$ such that for all $y \in (0,1)^2$ we have

$$\left\|D\psi(y) \cdot D\tilde{F}_1(y)^{-1}\right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\|\left(D\phi(\tilde{F}_1(y))\right)^{-1}\right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \omega(\|\psi(y)\|_{\mathcal{Y}}).$$

Note that (i') is guaranteed by our choice of $Y = (0,1)^2$ and the continuity of $\tilde{F}$, which holds by construction. For (ii') note that

$$D\tilde{F}_1(y_1, y_2) = \begin{bmatrix} (\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon)\cos\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) & \sin\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \\ -(\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon)\sin\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) & \cos\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \end{bmatrix},$$

and $\det\left(\tilde{F}_1(y_1, y_2)\right) = (\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon) \neq 0$. Further, the use of the constant α (large enough) guarantees that rk $D\tilde{F}(y) = 2$ when filling $\overline{B_\alpha(0)}$ as displayed in Fig B.1. Hence, (ii') is satisfied. Next, we show that (iii') is not satisfied, although, obviously, both $\pi_1(Z) = B_{\alpha+1}(0)$ and $\pi_2(Z) = (0,1)^2$ are diffeomorphic to some Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$. We show that condition (iii') is not satisfied for any diffeomorphisms $\phi : B_{\alpha+1}(0) \to \mathbb{R}^2$ and $\psi : (0,1)^2 \to \mathbb{R}^2$ by considering two cases. Let $(\hat{y}_1, \hat{y}_2) := \psi^{-1}(0,0) \in (0,1)^2$.
Case 1: Assume that $\hat{y}_1 \leq \delta$. We show that the growth bound fails for $\tilde{F}_1$. Seeking a contradiction, assume that we have

$$\left\|D\psi(y) \cdot D\tilde{F}_1(y)^{-1}\right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\|\left(D\phi(\tilde{F}_1(y))\right)^{-1}\right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \omega(\|\psi(y)\|_{\mathcal{Y}})$$

for all $y \in (0, \delta) \times (0, 1)$ and some weight ω. Although we did not specify the norms on $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, using the weight property and the equivalence of all norms on $\mathbb{R}^{n^2}, \mathbb{R}^n$, respectively, guarantees the existence of positive constants c_1, c_2 such that

$$\begin{aligned}
&c_1 \left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_F \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \\
&\leq \left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \\
&\leq \omega(\|\psi(y)\|_{\mathcal{Y}}) \leq \omega(c_2 \|\psi(y)\|_2),
\end{aligned}$$

where $\|\cdot\|_F$ is the Frobenius norm. Observing that $\tilde{\omega}(\cdot) \coloneqq c_1^{-1}\omega(c_2 \,\cdot)$ again defines a weight, we obtain

$$\left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_F \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \tilde{\omega}(\|\psi(y)\|_2)$$

for all $y \in (0, \delta) \times (0, 1)$. In order to simplify the computations, choose $\delta = \frac{1}{2}$ and $\alpha = \varepsilon = 1$. Since $\tilde{F}_1\left((0, \frac{1}{2}) \times \{\hat{y}_2\}\right) = (1 + \hat{y}_2)S^1$ is a compact subset of $B_2(0)$ and $B_2(0) \ni z \mapsto \|D\phi(z)\|$ is a continuous mapping we have

$$\exists\, \gamma > 0 \;\forall\, y \in (0, \tfrac{1}{2}) \times \{\hat{y}_2\} : \; \left\| D\phi(\tilde{F}_1(y)) \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \gamma.$$

This gives $\left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \geq \gamma^{-1}$ for all $y \in (0, \frac{1}{2}) \times \{\hat{y}_2\}$. Accordingly, we may calculate that for all $y_1 \in (0, \frac{1}{2})$ we have

$$D\tilde{F}_1(y_1, \hat{y}_2)^{-1} = \begin{bmatrix} \frac{1}{8\pi(1+\hat{y}_2)} \cos(8\pi y_1) & -\frac{1}{8\pi(1+\hat{y}_2)} \sin(8\pi y_1) \\ \sin(8\pi y_1) & \cos(8\pi y_1) \end{bmatrix}$$

and hence

$$\begin{aligned}
\left\| D\psi(y_1, \hat{y}_2) \cdot D\tilde{F}_1(y_1, \hat{y}_2)^{-1} \right\|_F &= \sqrt{\tfrac{1}{64\pi^2(1+\hat{y}_2)^2} \left(\frac{\partial \psi_1}{\partial y_1}^2 + \frac{\partial \psi_2}{\partial y_1}^2 \right) + \frac{\partial \psi_1}{\partial y_2}^2 + \frac{\partial \psi_2}{\partial y_2}^2} \\
&\geq \tfrac{1}{8\pi(1+\hat{y}_2)} \left\| \frac{\partial \psi}{\partial y_1}(y_1, \hat{y}_2) \right\|_2 .
\end{aligned}$$

Note that for all $y_1 \in (0, \hat{y}_1)$ we have that $\|\psi(y_1, \hat{y}_2)\|_2 > 0$ and, because

$$\lim_{y_1 \to 0} \|\psi(y_1, \hat{y}_2)\|_2 = \infty,$$

the set

$$\mathcal{S} \coloneqq \left\{ \; y_1 \in (0, \hat{y}_1) \;\middle|\; \tfrac{\partial}{\partial y_1} \|\psi(y_1, \hat{y}_2)\|_2^2 < 0 \; \right\}$$

has compact complement $(0,\hat{y}_1)\setminus\mathcal{S}$. Furthermore, for all $y_1\in(0,\hat{y}_1)$ we have that

$$\begin{aligned}
\tfrac{1}{2}\left|\tfrac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right| &= \left|\psi(y_1,\hat{y}_2)^\top\tfrac{\partial\psi}{\partial y_1}(y_1,\hat{y}_2)\right| \\
&\le \|\psi(y_1,\hat{y}_2)\|_2\|\tfrac{\partial\psi}{\partial y_1}(y_1,\hat{y}_2)\|_2 \\
&\le 8\pi(1+\hat{y}_2)\|\psi(y_1,\hat{y}_2)\|_2\left\|D\psi(y_1,\hat{y}_2)\cdot D\tilde{F}_1(y_1,\hat{y}_2)^{-1}\right\|_F \\
&\le 8\pi(1+\hat{y}_2)\gamma\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}(\|\psi(y_1,\hat{y}_2)\|_2).
\end{aligned}$$

With

$$\xi := \int_{(0,\hat{y}_1)\setminus\mathcal{S}} \frac{\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 < \infty$$

and the substitutions $t=\|\psi(y_1,\hat{y}_2)\|_2^2$ and $u=\sqrt{t}$ we may then derive

$$\begin{aligned}
16\pi\gamma\hat{y}_1 &\ge \int_0^{\hat{y}_1} \frac{\left|\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right|}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 \\
&\ge \int_{\mathcal{S}} \frac{\left|\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right|}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 \\
&= \int_{\mathcal{S}} \frac{-\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 - \xi + \xi \\
&= \int_0^{\hat{y}_1} \frac{-\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 + \xi \\
&= \int_{\|\psi(\hat{y}_1,\hat{y}_2)\|_2^2}^{\infty} \frac{1}{\sqrt{t}\tilde{\omega}(\sqrt{t})}\,\mathrm{d}t + \xi \\
&= 2\int_0^{\infty} \frac{1}{\tilde{\omega}(u)}\,\mathrm{d}u + \xi \\
&= \infty,
\end{aligned}$$

a contradiction.

Case 2: Assume that $\hat{y}_1 > \delta$. We show that the growth bound fails for $\tilde{F}_2$, which is similar to Case 1. To this end, we render the definition of $\tilde{F}_2$ more precisely. First define the curve

$$\gamma_1 : (0,\delta) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} (\alpha+\frac{1}{2})\sin\left(\frac{2\pi}{\delta}(1+\varepsilon)t\right) \\ (\alpha+\frac{1}{2})\cos\left(\frac{2\pi}{\delta}(1+\varepsilon)t\right) \end{pmatrix}$$

and the (rotation) matrix $R := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\tilde{F}_1$ can alternatively be written as

$$\tilde{F}_1(y_1,y_2) = \gamma_1(y_1) + (y_2 - \tfrac{1}{2})R\dot{\gamma}_1(y_1)$$

for all $(y_1, y_2) \in (0, \delta) \times (0, 1)$. In view of this, we may choose a curve $\gamma_2 : (\delta, 1) \to \mathbb{R}^2$ as depicted in Fig. B.1 such that

$$\tilde{F}_2(y_1, y_2) = \gamma_2(y_1) + (y_2 - \tfrac{1}{2})R\dot{\gamma}_2(y_1)$$

for all $(y_1, y_2) \in (\delta, 1) \times (0, 1)$ and, as mentioned before, $\overline{B_\alpha(0)} \subset \operatorname{im} \tilde{F}_2 \subset B_{\alpha+1}(0)$ and $D\tilde{F}_2(y_1, y_2)$ is invertible for all $(y_1, y_2) \in (\delta, 1) \times (0, 1)$. Omitting the details, the same arguments as in Case 1 may now be applied to arrive at a contradiction. In particular, fixing $y_2 = \hat{y}_2$ leads to a curve $\tilde{\gamma}_2(y_1) = \gamma_2(y_1) + (\hat{y}_2 - \frac{1}{2})R\dot{\gamma}_2(y_1)$ along which the growth bound is violated.

Remarks B.2.4. Finally, we like to point out that, while Theorem B.1.1 is already quite general, still it does not cover all relevant cases. Consider

$$F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3,\ (x_1, x_2, y_1, y_2) \mapsto \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix},$$

then

$$Z := F^{-1}(0) = \left\{ \ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \ \middle|\ x_1^2 + x_2^2 = 1,\ y_1 = x_1,\ y_2 = x_2 \ \right\}$$

and $\pi_1(Z), \pi_2(Z)$ are both the unit circle in $\mathbb{R}^2$, i.e., closed subsets which are not simply connected, for which it is not possible to satisfy condition (iii). However, a global implicit function obviously exists. Further research is necessary to cover examples of this type.

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List of notations

Acronyms

DAE	differential-algebraic equation – p. 13
dH-LA	linear algebraic dissipative-Hamiltonian – p. 76
dH-LG	linear geometric dissipative-Hamiltonian – p. 76
dH-LR	linear relations dissipative-Hamiltonian – p. 76
dH-ODE	ordinary dissipative-Hamiltonian – p. 76
KYP	Kalman–Yakubovich–Popov inequality – p. 128
MNA	modified nodal analysis – p. 159
MNA c/f	charge/flux-oriented modified nodal analysis – p. 159
pH-LA	linear algebraic port-Hamiltonian – p. 59
pH-LAR	relaxed linear algebraic port-Hamiltonian – p. 60
pH-LG	linear geometric port-Hamiltonian – p. 48
pH-LR	linear relations port-Hamiltonian – p. 61
pH-NG	nonlinear geometric port-Hamiltonian – p. 43
pH-ODE	port-Hamiltonian ordinary differential-equation – p. 41
posH	positive semidefinite Hermitian part coefficients – p. 137

General numbers, sets and spaces

$\mathbb{C}$	set of complex numbers – p. 5
$\mathbb{C}_-$	open left complex half-plane – p. 5
$\mathbb{C}_+$	open right complex half-plane – p. 5
df^*	cotangent map of $f \in C^1(\mathcal{M}, \mathcal{N})$ – p. 37
df	differential of a map $f \in C^1(\mathcal{M}, \mathcal{N})$ – p. 37
$D\psi$	Jacobian matrix of ψ also indentified with its Fréchet derivative – p. 55
$D_xF(x, y)$	partial Fréchet-derivative with respect to x of a function $F : X \times Y \to \mathcal{Z}$ with $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ Banach spaces – p. 180
$D_yF(x, y)$	partial Fréchet-derivative with respect to y of a function $F : X \times Y \to \mathcal{Z}$ with $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ Banach spaces – p. 180
$\frac{\partial F}{\partial x}(x, y)$	partial derivative with respect to x of a function $F : X \times Y \to \mathbb{R}^l$ with $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ – p. 178
$\frac{\partial F}{\partial y}(x, y)$	partial derivative with respect to y of a function $F : X \times Y \to \mathbb{R}^l$ with $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ – p. 177
df_p	differential of a map $f \in C^1(\mathcal{M}, \mathcal{N})$ at $p \in \mathcal{M}$ – p. 37
df_p^*	pullback of a map $f \in C^1(\mathcal{M}, \mathcal{N})$ at $p \in \mathcal{M}$ – p. 37
$(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \circ (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$	interconnection of two pH-NG systems $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$, $i = 1, 2$ – p. 44
$(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \times (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$	product of two pH-NG systems $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$, $i = 1, 2$ – p. 45
$E_1 \oplus E_2$	Whitney sum of two bundle E_1, E_2 – p. 37
$[E, A, B, C, D] \in \Sigma_{n,m,k,l}$	special class of DAEs – p. 13
$[E, A, B, C, D] \in \Sigma_{n,m,k}$	special class of DAEs – p. 13
$[E, A, B, C, D] \in \Sigma_{n,k}$	special class of DAEs – p. 13

$[E, A, B, C] \in \Sigma_{n,k}$	special class of DAEs – p. 13
$[E, A, B] \in \Sigma_{n,m,k}$	special class of DAEs – p. 13
$[E, A, B] \in \Sigma_{n,k}$	special class of DAEs – p. 13
$[E, A, B] \in \Sigma_{n,k}^{s}$	special class of DAEs – p. 124
$[E, A] \in \Sigma_{n,m}$	special class of DAEs – p. 13
$[E, A] \in \Sigma_{n}$	special class of DAEs – p. 13
$f^{(l)}$	lth weak derivative of f – p. 9
$\imath$	imaginary unit – p. 5
$\mathrm{Im}\ z$	imaginary part of $z \in \mathbb{C}$ – p. 5
init	initial vertex map – p. 142
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$ – p. 5
$\mathbb{K}^n$	space of n-dimensional vectors over $\mathbb{K}$ – p. 6
$\mathbb{K}[s]$	ring of polynomials over $\mathbb{K}$ – p. 5
$\mathbb{K}(s)$	quotient field of $\mathbb{K}[s]$ – p. 5
$\mathbb{N}$	set of natural numbers excluding 0 – p. 5
$\mathbb{N}_0$	set of natural numbers – p. 5
$\\| \cdot \\|_{L^p(\mathcal{I},\mathcal{V})}$	or simply $\\| \cdot \\|_p$, p-norm for $p \in [1, \infty)$ – p. 9
$\coprod$	symbol for disjoint unions – p. 37
$\bigotimes$	symbol for tensor products – p. 37
$\prod$	symbol for Cartesian products – p. 6
$S_1 \times \ldots \times S_n$	Cartesian product of a finite family of sets $\{S_i\}_{i=1,\ldots n}$ – p. 6
$\upharpoonright$	symbol for the restriction of a function to a subset of its domain – p. 8
$\mathbb{R}$	set of real numbers – p. 5
$\mathrm{Re}\ z$	real part of $z \in \mathbb{C}$ – p. 5
$R^{m\times n}$	space of $m \times n$-matrices with entries in a ring R – p. 6

$\overline{S}$	closure of a subset S of a topological space – p. 5
$\langle \cdot, \cdot \rangle$	denotes the standard scalar product in $\mathbb{K}^n$ – p. 7
$\bigoplus_{i=1}^n V_i$	exterior direct sum of finitely many vector spaces $V_1, \dots V_n$ – p. 6
$S_1 \widehat{+} \dots \widehat{+} S_n$	componentwise sum of finitely many subsets $S_1, \dots, S_n$ of a vector space S – p. 6
$V_1 \oplus \dots \oplus V_n$	exterior direct sum of finitely many vector spaces $V_1, \dots V_n$ – p. 6
$V_1 \widehat{\oplus} \dots \widehat{\oplus} V_n$	inner direct sum of finitely many vector spaces $V_1, \dots V_n$ – p. 6
$W \widehat{\ominus} V$	orthogonal minus with $V \subseteq W \subseteq \mathbb{K}^n$ – p. 6
$V_1 \oplus \dots \oplus V_n$	orthogonal inner direct sum of finitely many vector spaces $V_1, \dots V_n$ – p. 6
ter	terminal vertex map – p. 142
$T\mathcal{M}$	tangent bundle of a manifold $\mathcal{M}$ – p. 37
$T^*\mathcal{M}$	cotangent bundle of a manifold $\mathcal{M}$ – p. 37
$T_p\mathcal{M}$	tangent space of a manifold $\mathcal{M}$ at $p \in \mathcal{M}$ – p. 37
$T_p^*\mathcal{M}$	cotangent space of a manifold $\mathcal{M}$ at $p \in \mathcal{M}$ – p. 37
$\mathcal{V}_{\mathrm{diff}}^{[E,A,B,C,D]}$	space of consistent initial differential variables of a system $[E, A, B, C, D]$ – p. 15
$\mathcal{V}_{\mathrm{sys}}^{[E,A,B,C,D]}$	system space of a system $[E, A, B, C, D]$ – p. 14
$\|x\|$	usually denotes the Euclidean norm of $x \in \mathbb{K}^n$ – p. 7
$\times_{i=1}^n (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$	product of n pH-NG systems $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$, $i = 1, \dots, n$ – p. 46
$\lvert z\rvert$	absolute value of $z \in \mathbb{C}$ – p. 5
$\overline{z}$	complex conjugate of $z \in \mathbb{C}$ – p. 5

Function sets and spaces

$\mathfrak{B}_{\mathcal{DL}}$	behavior of pH-LR system $\mathcal{DL}$ – p. 63
$\mathfrak{B}_{\mathcal{D},\mathcal{L},\mathcal{R}}$	behavior of pH-LG system $(\mathcal{D},\mathcal{L},\mathcal{R})$ – p. 50
$\mathfrak{B}_{[E,A,B,C,D]}$	behavior of a system $[E,A,B,C,D]$ – p. 14
$\mathfrak{B}_{[E,A,B,C,D]}(x_0)$	behavior of a system $[E,A,B,C,D]$ with initial value x_0 – p. 14
$C^1(X \times Y, \mathcal{Z})$	set of Fréchet-differentiable functions $f : X \times Y \to \mathcal{Z}$ with $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, and Banach spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ – p. 180
$C^k(U,V)$	set of k-times continuously differentiable functions $f : U \to V$ with open subsets $U \subset \mathbb{K}^n$, $V \subset \mathbb{K}^m$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ – p. 8
$C^s(\mathcal{M},\mathcal{N})$	set of C^s maps $f : \mathcal{M} \to \mathcal{N}$ with C^s manifolds $\mathcal{M}$, $\mathcal{N}$ – p. 36
$C(X,Y)$	set of continuous functions $f : X \to Y$ with metric spaces X, Y – p. 8
$\Gamma_s(E)$	set of C^s sections of a bundle E – p. 37
$\mathrm{id}_{\mathcal{V}}$	identity map on a set $\mathcal{V}$ – p. 180
$L(V_1,\ldots,V_k;W)$	set of multilinear maps $f : V_1 \times \ldots \times V_k \to W$ for vector spaces $V_1,\ldots,V_k,W$ – p. 6
$\Lambda^k T^*\mathcal{M}$	bundle of alternating covariant k-tensors on $\mathcal{M}$ – p. 38
$\Lambda^k(V^*)$	subset of $T^k(V^*)$ consisting of all alternating tensors – p. 37
$\Lambda^k(V)$	subset of $T^k(V)$ consisting of all alternating tensors – p. 37

$\mathcal{L}^p(\mathcal{I},\mathcal{V})$	space of measurable functions $f:\mathcal{I}\to\mathcal{V}$ for which the pth power of its norm is integrable over the interval $\mathcal{I}\subset\mathbb{R}$ with $p\in[1,\infty)$ and a subspace $\mathcal{V}\subset\mathbb{K}^n$ – p. 8
$\mathcal{L}^p_{\text{loc}}(\mathcal{I},\mathcal{V})$	set of functions that are locally in $\mathcal{L}^p(\mathcal{I},\mathcal{V})$ – p. 8
$L^p(\mathcal{I},\mathcal{V})$	quotient space of $\mathcal{L}^p(\mathcal{I},\mathcal{V})$ – p. 9
$L^p_{\text{loc}}(\mathcal{I},\mathcal{V})$	quotient space of $\mathcal{L}^p_{\text{loc}}(\mathcal{I},\mathcal{V})$ – p. 9
$\mathcal{L}(\mathcal{Z},\mathcal{V})$	Banach space of all bounded linear operators $A:\mathcal{Z}\to\mathcal{V}$ with $\mathcal{V},\mathcal{Z}$ Banach spaces – p. 180
$\Omega^k_s(\mathcal{M})$	set of C^s differential $k-$forms on $\mathcal{M}$ – p. 38
$T^kT^*\mathcal{M}$	bundle of covariant k-tensors on $\mathcal{M}$ – p. 37
$T^k(V^*)$	set of covariant tensors on V of rank k – p. 37
$T^k(V)$	set of contravariant tensors on V of rank k – p. 37
$W^{k,p}(\mathcal{I},\mathcal{V})$	set of functions in $L^p(\mathcal{I},\mathcal{V})$ for which $f^{(l)}\in L^p(\mathcal{I},\mathcal{V})$ exists for all $l\le k$ with $k\in\mathbb{N}_0$ – p. 9
$W^{k,p}_{\text{loc}}(\mathcal{I},\mathcal{V})$	set of functions in $L^p_{\text{loc}}(\mathcal{I},\mathcal{V})$ for which $f^{(l)}\in L^p_{\text{loc}}(\mathcal{I},\mathcal{V})$ exists for all $l\le k$ with $k\in\mathbb{N}_0$ – p. 9

Matrices

$0^{0\times q}$	zero element of $\mathbb{K}^{0\times q}$, $q\in\mathbb{N}_0$ – p. 6
$0_{m,n}$	zero element of $\mathbb{K}^{n\times n}$, $m,n>0$ – p. 6
0_n	zero element of $\mathbb{K}^{n\times n}$, $n>0$ – p. 6
$0^{q\times 0}$	zero element of $\mathbb{K}^{q\times 0}$, $q\in\mathbb{N}_0$ – p. 6
$A^\dagger$	Moore-Penrose inverse of a matrix $A\in\mathbb{K}^{m\times n}$ – p. 7
A^{-1}	inverse of a matrix $A\in\mathbf{Gl}_n(\mathbb{K})$ – p. 7
A^*	Hermitian of a matrix $A\in\mathbb{K}^{m\times n}$ – p. 7
$A^\top$	transpose of a matrix $A\in\mathbb{K}^{m\times n}$ – p. 7
$\|A\|$	usually denotes the operator norm of $A\in\mathbb{K}^{m\times n}$ – p. 7

diag	block diagonal operator – p. 7
$e_i^{[k]}$	ith canonical unit vector in $\mathbb{K}^n$ – p. 9
e_i	ith canonical unit vector – p. 9
$\mathbf{Gl}_n(\mathbb{K})$	set of invertible matrices in $\mathbb{K}^{n\times n}$ – p. 7
I_n	$n \times n$ identity matrix – p. 6
$J_k(\lambda)$	Jordan block of size $k \in \mathbb{N}$ at $\lambda \in \mathbb{C}$ – p. 10
K_α	special matrix for a multi-index α – p. 10
K_k	special matrix for $k \in \mathbb{N}$ – p. 9
L_α	special matrix for a multi-index α – p. 10
L_k	special matrix for $k \in \mathbb{N}$ – p. 9
$M(Y_1)$	range of $M\restriction_{Y_1}$ for a linear operator $M : Y_2 \to Y_3$ with $Y_1 \subset Y_2$ – p. 8
$M > N$	$M - N$ is positive definite for square matrices M, N – p. 7
$M \geq N$	$M - N$ is positive semi-definite for square matrices M, N – p. 7
$M >_{\mathcal{L}} N$	$M - N$ is positive definite on a subspace $\mathcal{L}$ for square matrices M, N – p. 8
$M \geq_{\mathcal{L}} M$	$M - N$ is positive semi-definite on a subspace $\mathcal{L}$ for square matrices M, N – p. 8
$M =_{\mathcal{L}} N$	M equals N on a subspace $\mathcal{L}$ for square matrices M, N – p. 8
N_α	special matrix for a multi-index α – p. 10
N_k	special matrix for $k \in \mathbb{N}$ – p. 9

P_X	usually denotes the orthogonal projector onto a subspace X – p. 8
$sE - A$	first order polynomial in $\mathbb{K}[s]^{m\times n}$ – p. 9
$\sigma(E, A)$	spectrum of a square matrix pencil $sE - A$ – p. 11
$\sigma(A)$	spectrum of a square matrix A – p. 7

Linear relations

$\alpha\mathcal{M}$	scalar multiplication of a linear relation $\mathcal{M}$ wth $\alpha \in \mathbb{K}$ – p. 18
$\mathcal{D}'_K(\mathcal{G})$	special Dirac structure induced by a graph $\mathcal{G}$ – p. 145
$\operatorname{dom}\mathcal{M}$	domain of a linear relation $\mathcal{M}$ – p. 18
$\mathcal{D}^S_K(\mathcal{G})$	Kirchhoff-Dirac structure of a graph $\mathcal{G}$ with set of grounded vertices S – p. 144
$\ker\mathcal{M}$	kernel of a linear relation $\mathcal{M}$ – p. 18
$\mathcal{L}'_K(\mathcal{G})$	special Lagrange structure induced by a graph $\mathcal{G}$ – p. 145
$\mathcal{L}\widehat{\times}\mathcal{N}$	sorted Cartesian product of two linear relation $\mathcal{L}$ and $\mathcal{N}$ – p. 19
$\mathcal{L}^S_K(\mathcal{G})$	Kirchhoff-Lagrange structure of a graph $\mathcal{G}$ with set of grounded vertices S – p. 144
$\mathcal{M}^{-1}$	inverse of a linear relation $\mathcal{M}$ – p. 18
$\mathcal{M}\circ\mathcal{L}$	interconnection of two linear relation $\mathcal{L}$ and $\mathcal{N}$ – p. 30
$\mathcal{M}\mathcal{L}$	product of two linear relation $\mathcal{M}$ and $\mathcal{L}$ – p. 18
$\mathcal{M}+\mathcal{L}$	operator-like sum of two linear relation $\mathcal{M}$ and $\mathcal{L}$ – p. 18
$\mathcal{M}^*$	adjoint of a linear relation $\mathcal{M}$ – p. 18

$\operatorname{mul} \mathcal{M}$	multi-valued part of a linear relation $\mathcal{M}$ – p. 18
$\operatorname{ran} \mathcal{M}$	range of a linear relation $\mathcal{M}$ – p. 18

Index

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Erklärung zum Eigenanteil bei der Zusammenarbeit

Aus dieser Dissertation hervorgegange Veröffentlichungen: [GHR+21; GHR21; GH21; BH22].

Kapitel 1 besteht hauptsächlich aus bekannten Ergebnissen, die größtenteils aus den Grundlagen für [GHR21; GH21; GHR+21] stammen. Diese wurden für dieses Kapitel zusammengetragen und in der Darstellung vereinheitlicht und — vor allem mit den Seiten 22–29 — erweitert.

Kapitel 2 präsentiert teils bekannte port-Hamiltonische Formulierungen, die bereits auch in [GHR21; GH21; GHR+21] präsentiert wurden. Diese wurden für diese Disseration von mir vereinheitlicht und mit einigen Ergebnissen, wie z. B. Abschnitt 2.2.3, der Betrachtung von Lösungsmengen linearer port-Hamiltonischer Systeme, oder genaueren Vergleichen verschiedener Formulierungen, erweitert.

Kapitel 3 beruht auf [GHR21]. Die Idee zu dieser Arbeit entspringt Gesprächen zwischen Dr. Hannes Gernandt und mir am Anfang meines Forschungsaufenthaltes an der TU Ilmenau. Gemeinsam haben wir diesen Gedanken unter Betreuung von Univ.-Prof. Dr. Timo Reis umgesetzt.

Kapitel 4 basiert auf [GH21] in Kooperation mit Dr. Hannes Gernandt, bei welcher zu gleichen Teilen beigetragen wurde.

Kapitel 5 entstammt keiner Kooperation.

Kapitel 6 basiert hauptsächlich auf [GHR+21]. Die ursprüngliche Idee zu [GHR+21] entspringt einer privaten Korrespondenz zwischen Prof. Arjan van der Schaft und Univ.-Prof. Timo Reis, welche mir während meines Forschungsaufenthaltes an der TU Ilmenau nahegebracht wurde. Dort habe ich diese mit Dr. Hannes Gernandt zu gleichen Teilen umgesetzt.

Appendix A besteht aus unveröffentlichten und erweiterten Ergenissen meiner Bachelorarbeit, welche von Univ.-Prof. Timo Reis und Jun.-Prof. Thomas Berger betreut wurde.

Appendix B basiert auf der mit Jun.-Prof. Thomas Berger geschriebenen Arbeit [BH22]. Dabei habe ich vor allem das Hauptergebnis auf offene Teilmengen unendlichdimensionaler Räume erweitern können und relevante Beispiele beitragen können.

Ort, Datum Frédéric Enrico Haller
Hamburg, 9. April 2022