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Geometric Partial Regularity Results for Functions of Anisotropic Least Gradient and Applications.

vorgelegt von

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Eidesstattliche Erklärung.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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1 *Introduction.*

The problem of minimizing various norms of weak gradients of functions is a very natural one and can easily be imposed in the usual classes of Sobolev and BV functions under Dirichlet boundary conditions.
This thesis will now deal with minimizing the L^1-norm of such gradients and is therefore naturally imposed in the space of functions of bounded variation. The resulting extremals, which are usually referred to as *functions of least gradient*, enjoy the almost unnatural regularity property that ALL of its level sets are bounded by area-minimizing hypersurfaces by a classical result due to E. Bombieri, E. De Giorgi and E. Giusti in [5, Theorem 1].
While the continuity and continuity moduli of functions of least gradient are pretty well-discussed and understood in recent literature, fueled by the fundamental contribution of P. Sternberg, G. Williams and W. Ziemer with [53], questions of and results on higher regularity remain rather sparse to non-existent. As a matter of fact, the only actual contribution to this is due to the last author above and H. Parks [46] from the year 1984.
The very intricate method of [46] employs a kind of geometric solution and stability operator for stationary immersions and then uses the fact that each level set contains at least one such area-stationary immersion to derive a differentiable identity for the function of least gradient. Yet, to apply this solution operator one needs an a priori rigidity condition, which is shown to hold true for certain "nice" reference hypersurfaces. The method of proof has furthermore the a priori restriction on dimensionality between two and seven, but has been extended to one further dimension in [43] for $\mathbb{R}^8$.

Along the lines of this classical theory for functions of least gradient, there has also been a rather recent emergence of interest in *anisotropic formulations of the least gradient problem.* Notable contributions include the work [24] of R. Jerrard, A. Moradifam and A. Nachman, which is somewhat presented as the "anisotropic analogue" to the paper [53], as well as a multitude of smaller publications by A. Moradifam or W. Gorny. The papers [33] and [34] by J. Mazón feature a self-contained derivation of the necessary Euler-Lagrange relations for extremals in the isotropic and anisotropic setting from the realm of convex analysis.
We also emphasize an apparant connection to impedance tomography: Minimizers of anisotropic least gradient problems arise in rather direct fashion from modelling, if one wants to determine the conductivity of a given body, see [39], [40] or also [33]. It is notable that the anisotropies of such minimizers

are so far always given by Riemannian metrics; a further result on continuity for a certain class of such metrics (namely, weighted Euclidean ones) is proven in [60] by A. Zuniga.
The paper [44] by H. Parks from 1988 is moreover probably the first paper to study functions of anisotropic least gradient as an object of interest in itself, but has apparantly been swept under the radar so far.

The purpose of this thesis is now twofold: We will first revisit the approach of the paper [46], impose questions and discuss possible gaps in its structure. Secondly, we will propose solutions (also containing somewhat more detailed proofs) to these problems and discuss implications for functions of anisotropic least gradient hailing from the partial regularity like a certain rigidity of their gradients, boundary regularity or regularity towards fat level sets (implying non-proximity for the bounding area-minimizing hypersurfaces).

We close the introduction with a short overview of the sections and contents of this thesis:

- *The sections 2 and 3* will serve to repeat the necessary mathematical requirements from analysis and geometry and introduce the geometric variational problems that we shall discuss while also containing a first overview of the Parks/Ziemer approach to partial regularity. Two particular details lacking clarity will be reviewed in 3.2 and 3.3.
- After these introductory sections we will move on and investigate and prove results related to the level set boundaries of functions of anisotropic least gradient with respect to their stability and how we may deform them. *Section 4* will deal with each level set boundary seperately, *section 5* will observe how the usual small excess/graph representation theory can be used for stability results which we refine in *section 6* with the aid of elliptic theory, as the graphical representations minimize associated non-parametric integrands.
- The purpose of *section 7* is to conclude for local and partial regularity. After "unlocking" the geometric solution operator by means of the previous three sections, this is more or less similar to [46].
- The following *section 8* moves our local consideration from interior to boundary points. We shall see that the results become conditional, but are nevertheless able to prove at least the existence of some regular points in any case.

- In the last three sections we will finally discuss sufficient conditions for local regularity of an extremal near level sets, as long as the level set of the boundary data is regular. *Section 9* will additionally contain a rigidity result for the gradient of extremals along "good" level sets and *section 10* will subsequently investigate level sets with positive $\mathscr{L}^n$-measure.

- The short *section 11* and will provide a proof that level sets do actually foliate a piece of domain by anisotropically minimal hypersurfaces in case of local regularity.

- In the last *section 12*, which is thematically disjoint from the higher regularity of the preceding sections, our concern is the existence of Lipschitzian extremals where the anisotropic total variation is given by a weighted absolute value. We shall use elliptic quasilinear approximating problems and provide estimates stable enough to pass to the semielliptic case of an anisotropic total variation. The results of this section were furthermore submitted for publication with [47].

- We conclude the thesis in *section 13* with an addendum of some additional remarks and related open questions.

The *main results* of this thesis, in my opinion, are the generic Jacobi nullity rigidity in theorem 6.8, the local and partial regularity results in the theorems 7.1 and 7.2 with the additional list of specific conclusions in corollary 7.6, the characterization and existence of boundary regularity in corollary 8.3 and theorem 8.6 and the gradient definiteness of theorem 9.15 as well as all finer characterizations regarding regularity from the sections 9-11 and the existence result for functions of weighted least gradient from theorem 12.7.

This thesis was written while I was employed as a research assistant/Ph.D. student at the department of mathematics of the University of Hamburg in the research group of Prof. Thomas Schmidt.

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2 *Preliminaries.*

The purpose of the section is to introduce and fix some established theory that we would like to use in what follows. Most of these results will turn out to be somewhat classical but we will have a short repetition without proofs for convenience.

2.1 General Notation.

We will denote by $\mathbb{R}^n$ the usual Euclidean space, where n is a positive integer. We will sometimes put restrictions on the dimension n to make use of certain aspects of regularity theory.
An open ball with radius r and center x will be denoted by $\mathbf{U}(x,r)$. The corresponding closed ball is denoted as $\mathbf{B}(x,r)$. Moreover, we will make use of *an $(n-1)$-dimensional cylinder at x with radius r and height q*; by this we mean the product of an $(n-1)$-dimensional open ball $\mathbf{U}(x,r) \subset \mathbb{R}^{n-1}$ with the interval $(-q,q)$. We will denote this cylinder by $\mathbf{C}(x,r,q)$. Similarly, we will denote by $\mathbf{C}_H(x,r,q)$ the cylinder in normal direction over a fixed hyperplane $H \subset \mathbb{R}^n$. The symbol $\mathbf{Z}(x,r,q)$ (respectively $\mathbf{Z}_H(x,r,q)$) denotes the corresponding closed cylinder.
The standard Euclidean scalar product of two n-dimensional vectors v,w will be denoted by $v \cdot w$ or vw. We will usually equip $\mathbb{R}^n$ with the standard norm by means of this scalar product.

By *a domain*, we will understand a bounded and connected set. The symbol Ω will usually be used to denote the open domain that we work on. Most often, we will also impose some kind of *boundary regularity* on $\partial\Omega$, ie. the topological boundary of Ω, ranging from local Lipschitz to C^k-regularity. We will then say Ω has a Lipschitz or C^k-boundary.

As we are dealing with geometric measure theory, we will make frequent use of *the Lebesgue and Hausdorff measures*, which will be denoted by $\mathscr{L}^n$ and $\mathscr{H}^n$, respectively. Very extensive use will also be made of *the Hausdorff distance* d_H and convergence with respect to it. This distance is defined for all non-empty compact subsets of $\mathbb{R}^n$ via

$$\mathcal{H}(K,L) = \max\left\{\max_{x\in K} d(x,L), \max_{y\in L} d(K,y)\right\}.$$

It can be shown that this distance turns the set of all non-empty compact subsets of $\mathbb{R}^n$ into a complete metric space while an equivalent characterization of the Hausdorff distance sounds

$$\mathcal{H}(K,L) = \inf\{\varepsilon > 0 \mid K \subset \mathbf{e}_\varepsilon(L), L \subset \mathbf{e}_\varepsilon(K)\},$$

where $\mathbf{e}_\varepsilon$ denotes *the ε-expansion* of K, ie.

$$\mathbf{e}_\varepsilon(K) = \{x \in \mathbb{R}^n \mid d(x,K) \leqslant \varepsilon\}.$$

This characterization makes apparant that both sets really lie close to each other at each individual point given a small Hausdorff distance. Note also that the set $\mathbf{e}_\varepsilon(K)$ can be identified with a suitable closed tubular neighborhood in normal direction, in case ε is sufficiently small and K is a submanifold without boundary. We shall also use its open variant as

$$\mathbf{u}_\varepsilon(K) = \{x \in \mathbb{R}^n \mid d(x,K) < \varepsilon\}$$

and define contractions of our open domain $\Omega \subset \mathbb{R}^n$ via

$$\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}, \qquad \mathbf{u}_\varepsilon(\Omega) := \Omega \cup \{x \in \mathbb{R}^n \mid d(x, \partial\Omega) < \varepsilon\}$$

for any $\varepsilon > 0$.

2.1 Remark. The two main characterizations of Hausdorff convergence $K_i \longrightarrow K$ are given by:

- For each $x \in K$ there exists a sequence $x_i \in K_i$ such that $x_i \longrightarrow x$.
- Conversely, given a sequence $x_i \in K_i$ such that $x_i \longrightarrow x$ for any $x \in \mathbb{R}^n$, it follows that $x \in K$.

Note also that $\mathcal{H}(K, L) < \varepsilon$ implies that any $k \in K$ admits a point $l \in L$ such that $d(k, l) < \varepsilon$ and vice versa.

We finally recall the following form of *the Lebesgue covering lemma*: If $V_k \subset \mathbb{R}^n$ denote finitely many open sets and $K \subset \bigcup_k V_k$ with K being compact, then there is $\delta > 0$, *the Lebesgue number of K with respect to the covering $\bigcup_k V_k$*, such that the implication

$$x, y \in K, \quad \|x - y\| < \delta \qquad \Longrightarrow \qquad \text{There exists a } k \text{ with } x, y \in V_k.$$

holds true.

2.2 BV Functions, Sets of Finite Perimeter and Rectifiable Currents.

Here we will recall some results on geometric measure theory in codimension one related to rectifiable currents and sets of finite perimeter. The usual references on this matter include the monographies [11], [50] and [25] for general geometric measure theory and [3], [14] and [12] for sets of finite perimeter/oriented boundaries.

We especially mention the papers [1] for a collection of remarks on parametric integrands (that is, anisotropic total variations) in the setting of oriented boundaries and codimension 1-problems and [24] and [44] for resources on general least gradient problems with anisotropic perimeters.

We generally borrow most of our notation from [11] or [3], including some results on exterior calculus from [11, Chapter 1].

i.) Functions of Bounded Variation. The space of real-valued functions of bounded variation on an open set Ω will be denoted by $\mathrm{BV}(\Omega)$. These are L^1-functions u such that the variation functional for u can be extended to a Radon measure on Ω. By these moreover, we will understand finite inner and outer regular Borel measures on Ω. The Radon-Nikodym derivative of Du by $|Du|$ will be written as V_u, ie. we write $Du = V_u|Du|$ and we note in particular that

$$\int_\Omega u(x) \operatorname{div} \varphi(x)\ dx = -\int_\Omega \varphi(x) \cdot dDu = -\int_\Omega \varphi(x) \cdot V_u(x)\ d|Du| \qquad \text{for all } \varphi \in \mathrm{C}^\infty_c(\Omega, \mathbb{R}^n).$$

Both concepts can be localized by postulating finiteness only on relatively compact contained open sets in Ω.

2.2 Definition (Trace Operator)**.** If Ω has Lipschitz boundary, there is a continuous trace operator

$$\mathrm{T} : \mathrm{BV}(\Omega) \longrightarrow \mathrm{L}^1(\partial\Omega),$$

it moreover holds that T is surjective. We will also denote the trace of u as $u_{\partial\Omega}$. T coincides with the restriction to $\partial\Omega$ for functions $u \in \mathrm{C}(\overline{\Omega})$. We will call T *the inner trace* on $\partial\Omega$, while for $u \in \mathrm{BV}(\mathbb{R}^n \backslash \overline{\Omega})$ we will denote *the outer trace* on $\partial\Omega$ by $\tilde{\mathrm{T}}u$.

As is well-known, we may also extend BV functions to an open superset, eg. to $\mathbb{R}^n$, by prescribing a different BV function on the complement and the gradient measure detects the difference in traces on $\partial\Omega$. If we use the zero function, we receive for the gradient measure of the extended function $\tilde{u}$ that

$$D\tilde{u} = Du + (u_{\partial\Omega} \cdot N_{\partial\Omega}) \cdot \mathscr{H}^{n-1} \llcorner \partial\Omega,$$

where $N_{\partial\Omega}$ is the inner unit normal to $\partial\Omega$. Accordingly, we receive for the total variaton measure of $\tilde{u}$ that

$$|D\tilde{u}| = |Du| + |u_{\partial\Omega}| \cdot \mathscr{H}^{n-1} \llcorner \partial\Omega.$$

Assuming that $u = 1_E$, ie. u is *the characteristic function of a set* $E \subset \Omega$, we will say that E is *a set of finite perimeter* when $u \in \mathrm{BV}(\Omega)$ and write $V_E := V_u$. Sets of finite perimeter in Ω enjoy the *theorem of De Giorgi:* There exists a $\mathscr{H}^{n-1}$-rectifiable set $\partial^* E \subset \Omega$ such that

$$|D1_E| = \mathscr{H}^{n-1} \llcorner \partial^* E.$$

Note the usual difference that $\partial^* E$ need not be $\mathscr{H}^{n-1}$-equivalent to the support of $|D1_E|$, as the support is necessarily closed and may possibly be much larger.

ii.) Currents. To investigate the structure of minimizing level sets, we will also use the interplay between functions of bounded variation and currents. We will only need to do so in the full space $\mathbb{R}^n$ and we will denote by

$$\mathscr{R}_k(\mathbb{R}^n) \qquad \text{and} \qquad \mathscr{I}_k(\mathbb{R}^n)$$

the spaces of rectifiable and integral currents in $\mathbb{R}^n$. Therein, we understand a current T as *rectifiable* if there exists a $\mathscr{H}^k$-rectifiable set $\Sigma_T \subset \mathbb{R}^n$ with finite measure, a measurable unit-norm k-vector field $\vec{T}$, which spans the tangent space to Σ_T almost-everywhere on Σ_T, and an integrable $\mathbb{N}$-valued function θ_T such that

$$T(\omega) = \int_{\Sigma_T} \omega(\vec{T})\theta_T \; d\mathscr{H}^k = \int \omega(\vec{T}) \; d\|T\|$$

holds for all $\omega \in \mathrm{C}_c^\infty(\mathbb{R}^n, \Lambda^k \mathbb{R}^n)$, where the *variation measure* of a rectifiable current T will be denoted by $\|T\|$. We say that T is *integral* if ∂T is also rectifiable. Note moreover that *the mass of* T, which we will write as $\langle \mathscr{A}, T \rangle$, suffices

$$\langle \mathscr{A}, T \rangle = \int_{\Sigma_T} \theta_T \; d\mathscr{H}^k < \infty$$

if T is rectfiable. Assuming that Σ_T and θ_T are only locally rectifiable and integrable, we achieve the concept of locally rectifiable and integral currents and we will denote the corresponding spaces by

$$\mathscr{R}_k^{loc}(\mathbb{R}^n) \qquad \text{and} \qquad \mathscr{I}_k^{loc}(\mathbb{R}^n).$$

Moreover, we let $\mathbf{E}^n \in \mathscr{R}_n^{loc}(\mathbb{R}^n)$ denoting integration over $\mathbb{R}^n$ with the standard orientation vector

$$\vec{\mathbf{E}}^n = e_1 \wedge \dots \wedge e_n, \qquad \text{that is,} \qquad \mathbf{E}^n = \mathscr{L}^n \wedge e_1 \wedge \dots \wedge e_n.$$

Particularily, restricting the current $\mathbf{E}^n$ to a $\mathscr{L}^n$-measurable set $E \subset \mathbb{R}^n$ delivers that

$$\mathbf{E}^n \llcorner 1_E = \mathbf{E}^n \llcorner E \in \mathscr{R}_n^{loc}(\mathbb{R}^n)$$

and

$$\partial(\mathbf{E}^n \llcorner 1_E) = \partial(\mathbf{E}^n \llcorner E)$$

is the oriented boundary of E in $\mathbb{R}^n$. This will be the most important example of rectifiable and integral currents for us in this thesis.

Let us then turn to the connection via duality between codimension one currents and functions of bounded variation.

2.3 Definition (Hodge Star Operators). We define *the Hodge star operator* $\star$ linearly via

$$\star : \Lambda_{n-1}\mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \star e^i := (-1)^{n-i} e_i,$$

respectively,

$$\star : \mathbb{R}^n \longrightarrow \Lambda_{n-1}\mathbb{R}^n, \qquad \star e_i := (-1)^{i-1} e^i,$$

where $e_i \in \mathbb{R}^n$ for $i \in \{1, \dots, n\}$ denotes the standard basis vectors of $\mathbb{R}^n$ and

$$e^i := \bigwedge_{j=1, i\neq j}^{n} e_j.$$

Analogously, we define the Hodge star operator on $\mathbb{R}^n$ and $\Lambda^{n-1}(\mathbb{R}^n)$ via

$$\star : \Lambda^{n-1}\mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \star e_*^i := (-1)^{n-i} e_i,$$

and

$$\star : \mathbb{R}^n \longrightarrow \Lambda^{n-1}\mathbb{R}^n, \qquad \star e_i := (-1)^{i-1} e_*^i$$

where $e_*^i \in \Lambda^{n-1}\mathbb{R}^n$ is dual to $e^i \in \Lambda_{n-1}\mathbb{R}^n$. By its characterizing identity, we see that this is the right definition, the Hodge star operator isometrically identifies both spaces with each other and if

$$\xi = \xi_1 \wedge \dots \wedge \xi_{n-1} \in \Lambda_{n-1}\mathbb{R}^n$$

is a simple vector, then $\star\xi$ is orthogonal to each ξ_k with $k = 1, \dots, n-1$ and

$$|(\star\xi)| = |\xi|.$$

We also note that

$$(\star\star)\xi = (-1)^{n-1}\xi, \qquad (\star\star)v = (-1)^{n-1}v, \qquad (\star\star)\omega = (-1)^{n-1}\omega$$

for any $\xi \in \Lambda_{n-1}\mathbb{R}^n$, $v \in \mathbb{R}^n$ and $\omega \in \Lambda^{n-1}\mathbb{R}^n$.

We proceed to relate between oriented boundaries and total variation measures. A standard consequence in terms of duality with the Hodge star operator and the theorem of De Giorgi (or, equivalently, the boundary rectifiability theorem) ensures that

$$\partial(\mathbf{E}^n \llcorner E) \in \mathscr{R}_{n-1}(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad |D1_E|(\mathbb{R}^n) < \infty$$

and both are related pointwise as distributions by inserting the definitions of ∂ and the gradient measure via

$$\partial(\mathbf{E}^n \llcorner E)(\star v) = -D1_E(v), \qquad \partial(\mathbf{E}^n \llcorner E)(\omega) = (-1)^n D1_E(\star\omega)$$

for $v \in \mathrm{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n),\ \omega \in C_c^\infty(\mathbb{R}^n, \Lambda^{n-1}\mathbb{R}^n)$ while their orientations and mass/variation measures suffice

$$\|\partial(\mathbf{E}^n \llcorner E)\|(A) = |D1_E|(A) \quad \text{for any Borel set } A \subset \mathbb{R}^n, \qquad \star\, \overrightarrow{\partial(\mathbf{E}^n \llcorner E)} = (-1)^n V_E$$

The Hodge star operator hence also translates the formula for the total variation of an extended set of finite perimeter to the dual setting: Given $1_E \in \mathrm{BV}(\Omega)$ with trace $1_\Sigma \in \mathrm{L}^1(\partial\Omega, \mathscr{H}^{n-1})$, we may consider $E \subset \mathbb{R}^n$ by extending through 0 with denoting $\tilde{1_E} \in \mathrm{BV}(\mathbb{R}^n)$ and receive

$$D\left(\tilde{1_E}\right) = D1_E + (1_\Sigma \cdot N_{\partial\Omega}) \cdot \mathscr{H}^{n-1} \llcorner \partial\Omega,$$

which yields in its dualized form that

$$\partial(\mathbf{E}^n \llcorner E) = \partial(\mathbf{E}^n \llcorner E) \llcorner \Omega + [\![\Sigma]\!] = [\![\partial^* E \cap \Omega]\!] + [\![\Sigma]\!],$$

since for $\omega \in \mathrm{C}_c^\infty(\mathbb{R}^n, \Lambda^{n-1}\mathbb{R}^n)$ we have

$$\begin{aligned} D1_E(\star\omega) &= -\int_\Omega V_E \cdot (\star\omega)\ d|D1_E| = \int_\Omega (-1)^n \left(\star\overrightarrow{\partial(\mathbf{E} \llcorner E)} \cdot (\star\omega)\right)\ d\|\partial(\mathbf{E}^n \llcorner E)\| \\ &= (-1)^n \int_\Omega \overrightarrow{\partial(\mathbf{E} \llcorner E)}(\omega)\ d\|\partial(\mathbf{E}^n \llcorner E)\| = (-1)^n (\partial(\mathbf{E}^n \llcorner E) \llcorner \Omega)(\omega) \end{aligned}$$

and the Hodge star maps $N_{\partial\Omega}$ to the (up to sign/orientation) unit $(n-1)$-tangent vector field of $\Sigma \subset \partial\Omega$. Therein, we use the following notation:

2.4 Definition (Associated Currents)**.** Given an oriented submanifold $S \subset \mathbb{R}^n$, we will denote the associated locally rectifiable current by $[\![S]\!]$. We will also use this notation if S is only a locally $\mathscr{H}^k$-rectifiable set in $\mathbb{R}^n$ and suppress its orientation.

In what follows, we shall usually equip a rectifiable current $[\![\Sigma]\!]$ implicitly with the orientation hailing from the interior or exterior trace. We also infer further from $\partial^2 = 0$ that

$$0 = \partial^2(\mathbf{E}^n \llcorner E) = \partial(\partial(\mathbf{E}^n \llcorner E) \llcorner \Omega) + \partial\left([\![\Sigma]\!]\right)$$

and thus

$$\partial(\partial(\mathbf{E}^n \llcorner E) \llcorner \Omega) = -\partial\left([\![\Sigma]\!]\right).$$

This constitutes our main motivation to consider currents as we may seperate the interior from the trace part along a boundary of codimension two.

iii.) Variational Problems for Currents. Finally, we recall some facts about parametric integrands and the setup for geometric variational problems in the class of rectifiable currents. This framework provides a powerful way of investigating properties of minimizing level sets with respect to certain a priori estimates and stability matters. We need only discuss this matter for codimension 1.

2.5 Definition (Integrands). Let $\Omega \subset \mathbb{R}^n$ be open. We will call a continuous function

$$F : \Omega \times \Lambda_{n-1}\mathbb{R}^n \longrightarrow \mathbb{R},$$

which is positively homogeneous, convex and positive on the unit sphere in the second component, *a parametric integrand* (of codimension one). We will say that F is of class C^k (including smooth and analytic) or $\mathrm{C}^{k,\alpha}$, if its restriction to $\Omega \times \Lambda_{n-1}\mathbb{R}^n \backslash \{0\}$ is such. We will say that F is *elliptic*, if there exists a constant $c > 0$ such that

$$p \longmapsto (F - c\mathscr{A})(x, p)$$

is convex on $\Lambda_{n-1}\mathbb{R}^n$ for each $x \in \Omega$. Similarly, a function with the same properties as above

$$\Phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

will be called *an anisotropic total variation* of the corresponding differentiability class respectively elliptic.

We furthermore remind of the properties of an elliptic parametric integrand and that ellipticity is invariant under diffeomorphisms. The *area integrand* $\mathscr{A}$ is trivially elliptic. Finally, we denote *the duality pairing* of a parametric integrand F and a rectifiable current T by $\langle F, T\rangle \in \mathbb{R}$ and *the support* of a locally rectifiable current T by $\operatorname{spt} T$. Similarly, given a total variation Φ we will denote for $u \in \mathrm{BV}(\Omega)$ via Φ_u the Radon measure given by

$$\Phi_u(B) := \int_B \Phi(x, V_u)\ d|Du| \qquad \text{for all Borel sets} \quad B \subset \Omega.$$

If $u = 1_E$ for some set $E \subset \Omega$, then we will write $\Phi_E = \Phi_u$.

2.6 Remark. The Hodge star operators

$$\star : \Lambda_{n-1}\mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \text{and} \qquad \star : \mathbb{R}^n \longrightarrow \Lambda_{n-1}\mathbb{R}^n$$

allow to associate parametric integrands and anisotropic total variations via the formula

$$F(x, p) := \Phi\left(x, (-1)^n \star p\right)$$

and note

$$\Phi(x, v) = F(x, (-1) \star v).$$

We find that this especially implies

$$\langle F, \partial(\mathbf{E}^n \llcorner E) \llcorner \Omega\rangle = \int_{\partial^* E \cap \Omega} \Phi\left((x, (-1)^n \star \left(\overrightarrow{\partial^* E}\right)\right)\ d\mathscr{H}^{n-1} = \int_{\partial^* E \cap \Omega} \Phi(x, V_E)\ d|D1_E| = \Phi_E(\Omega)$$

as

$$(-1)^n \star \left(\overrightarrow{\partial^* E}\right) = V_E \qquad \mathscr{H}^{n-1} \llcorner \partial^* E \quad \text{-almost everywhere.}$$

2.7 Remark (The Parametric Legendre Condition). An important consequence of our definition is the so-called *parametric Legendre condition* for elliptic integrands. Fixing an ellipticity bound $c > 0$ and an elliptic parametric integrand $\mathscr{F}$ of class C^2, we may differentiate twice to get

$$D_2^2\left(\mathscr{F} - c\mathscr{A}\right)(x,p)(q,q) \geqslant 0 \qquad \text{for any} \quad q \in \Lambda_{n-1}\mathbb{R}^n,$$

which is

$$D_2^2\mathscr{F}(x,p)(q,q) \geqslant \frac{c}{|p|}\left(|q|^2 - \left(q \cdot \frac{p}{|p|}\right)^2\right) \qquad \text{for} \quad x \in U,\ 0 \neq p \in \Lambda_{n-1}\mathbb{R}^n.$$

Similarly, in the anisotropic total variation case, we further receive

$$D_2^2\Phi(x,v)(w,w) \geqslant \frac{c}{|v|}\left(|w|^2 - \left(w \cdot \frac{v}{|v|}\right)^2\right) \qquad \text{for} \quad x \in U,\ 0 \neq v \in \mathbb{R}^n.$$

for any $w \in \mathbb{R}^n$. Note also that the Hodge star operators relate and transfer both conditions to associated integrands.

Given open sets $U, V \subset \mathbb{R}^n$ and a diffeomorphism

$$\psi : V \longrightarrow U \qquad \text{with } \psi \text{ of class } \mathrm{C}^k \text{ with } k \geqslant 1,$$

then $\psi^{\#}\mathscr{F}$, which we will refer to as *the pull-back of* $\mathscr{F}$, defines a parametric integrand on $V \times \Lambda_{n-1}\mathbb{R}^n$ of class C^{k-1}. The *push-forward* of a rectifiable current will furthermore be denoted by $\varphi_{\#}T$, whenever this is defined. We will mainly care for *homothetic expansions of* T, that is, when

$$\varphi(x) = \mu_r(x) = \frac{x}{r} \qquad \text{for } r > 0.$$

Without loss of generality shifting a considered point towards the origin, this enables blowup procedures to investigate local properties of rectifiable currents.

Now introducing the variational problems for currents, which we want to consider, we will mostly rely on the concept of almost-minimizing currents with respect to some integrand in the sense of [10].

2.8 Definition (Almost Minimizing Currents)**.** We say that a rectifiable current $T \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ is $(\mathscr{F}, \omega)$-*almost minimizing* in some open set $U \subset \mathbb{R}^n$ if

$$\langle \mathscr{F}, T\rangle \leqslant \langle \mathscr{F}, T + X\rangle + \omega(r)\langle \mathscr{A}, T \llcorner K + X\rangle$$

for all $X \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ with $\partial X = 0$ and $\operatorname{spt} X$ being contained in a compact set $K \subset U$ such that K is contained in some ball of radius $r > 0$ and the function ω is defined for all sufficiently small radii r with $\lim_{r \searrow 0} \omega(r) = 0$.

2.9 Remark. We collect some useful properties on homothetic expansions of almost-minimizing rectifiable currents from [10, Section 1, Lemma 7.1] (cf. also [20, Lemma 3.5] and [11, Theorem 5.4.2]), which are consequences of local flat convergence, minimization and lower mass bounds.

(1) *Scaling properties.* Assume $s, r, t > 0$ and that T is almost-minimizing for $(\mathscr{F}, \omega)$ in $\mathbf{U}(0, s)$. Then $(\mu_r)_{\#}T$ is almost-minimizing in $\mathbf{U}(0, s/r)$ for

$$\mathscr{F}_r(x,p) = \mathscr{F}(rx, p) \qquad \text{and} \qquad \omega_r(t) = \omega(rt).$$

In particular, $\digamma_r \longrightarrow \digamma_0$ locally uniformly, where $\digamma_0(x,p) = \digamma(0,p)$, and $\omega_r \longrightarrow 0$ uniformly near $t = 0$ when $r \to 0$.

(2) *Compactness.* If we have

$$(\mu_i)_\# T \longrightarrow Q \qquad \text{in the local flat norm in } \mathbb{R}^n \text{ as } i \to \infty,$$

for some sequence μ_i of homothetic expansions along a sequence $r_i \to 0$, then Q is locally rectifiable on $\mathbb{R}^n$ and Q is (almost-) minimizing on $\mathbb{R}^n$ with respect to $(\digamma_0, 0)$.

(3) *Proximity of Supports.* If

$$\operatorname{spt} \partial \left((\mu_i)_\# T\right) \subset \operatorname{spt} Q \qquad \text{for all } i \in \mathbb{N},$$

then for all compact sets $H \subset \mathbb{R}^n \backslash \operatorname{spt} Q$ there exists a number $j = j(H) \in \mathbb{N}$ such that

$$\operatorname{spt}((\mu_i)_\# T) \subset \mathbb{R}^n \backslash H$$

for all $i \geqslant j$.

2.3 Least Gradient Problems and Related Minimization Problems.

We now introduce the problems that we are interested in and recap some of the known existence theory for them and their extremals and how they are connected to each other. In all that follows here, we will assume that Φ is an elliptic anisotropic total variation on $\overline{\Omega}$ (or equivalently, on $\mathbb{R}^n$), which is *even*, ie. $\Phi(x,v) = \Phi(x,-v)$ for all $x \in \overline{\Omega}$ and $v \in \mathbb{R}^n$.

i.) The Anisotropic Least Gradient Problem. Let for all that follows Ω be an open domain with Lipschitz boundary. The variational problem that we centrally want to consider in this work is *the anisotropic least gradient problem.* The problem is defined by attaining the number

$$0 \leqslant \inf \left\{ \Phi_u(\Omega) \mid u \in \mathrm{BV}(\Omega),\ u_{\partial\Omega} = f \right\} < \infty$$

for some $f \in \mathrm{L}^1(\partial\Omega)$. A function u which is extremal in the above problem for its trace will be called *a function of anisotropic least gradient.*

In essence, these problems ask for minimizers of an anisotropic L^1-norm under given Dirichlet boundary conditions. Since this norm is however much more badly behaved than its counterparts for $p > 1$, we have to reinterpret it in a generalized setting, namely we have to consider competitors among the functions of bounded variation. This allows for drastically more geometric structure as an extremal might still be $\mathrm{W}^{1,1}$, but the indicator functions of its level sets will obviously at most be BV (in non-trivial cases).

We recall also from [33, Theorem 3.16] that the apparantly weaker minimization property in terms of compactly contained variations

$$\Phi_u(\Omega) \leqslant \Phi_{u+v}(\Omega) \qquad \text{for all} \quad v \in \mathrm{BV}(\Omega),\ \operatorname{spt} v \subset\subset \Omega$$

suffices to conclude that $u \in \mathrm{BV}(\Omega)$ is a function of anistropic least gradient.

2.10 Remark (Existence of Isotropic Extremals). As we are interested in regularity theory for Lipschitzian extremals, we briefly provide a recap of the necessary sufficient conditions. In fact, the two main ones for functions of isotropic least gradient (ie. for $\Phi(x, v) = |v|$) are:

(1) A convex domain Ω and boundary data f satisfying the bounded slope condition (see [41, 4(3) Lemma], where we may drop "smooth" and "uniformly convex" for "bounded slope" and "convex" as one may readily observe from its proof).

(2) A strictly mean convex C^2-domain Ω and boundary data f of class $\mathrm{C}^{1,1}$ (see [53, Theorem 5.9], which is also a special case of our corollary 12.20).

Both assumptions result in the existence of a Lipschitzian solution for the isotropic problem with boundary data f.

2.11 Remark (Existence of Constant Coefficient Extremals). Assuming that our total variation obeys the form $\Phi(x, p) = \Theta(p)$, which we will refer to as *an anisotropic total variation with constant coefficients*, and is of class C^2, then in fact similar conditions work. We have

(1) A convex domain Ω and boundary data f satisfying the bounded slope condition, where the same proof as before works (see [44] or [47, Theorem 1.1]).

(2) A C^2-domain Ω and boundary data f of class $\mathrm{C}^{1,1}$ with a certain parametric boundary curvature condition related to Θ (see [12, Theorem 1.12]).

The results from [12, Theorem 1.12], much like [24], assume explicitly that their anisotropic total variations are even. We did assume this property here mostly for convenience, but it is not clear to me whether this restriction is really essential or whether positive 1-homogeneity would in fact suffice for all results from [12] and [24].

2.12 Remark (Existence of Spatially Varying Extremals). Given an arbitrary total variation Φ the existence of sufficiently regular extremals is not yet fully understood and seems to require additional differential conditions on the dependence on the spatial variable of Φ. The paper [44] provides existence results exploting intricate geometric maximum principles while [47] generates *functions of weighted least gradient* (ie. where $\Phi(x, p) = \mu(x)|p|$)) via quasilinear approximating problems. The results of [47] are contained in section 12 of this thesis.

We will furthermore also consider geometric variational problems for sets of finite perimeter and currents; in particular, we will be interested in their connections.

ii.) Sets of Minimal Anisotropic Perimeter. Let a set $F \subset \mathbb{R}^n$ of finite perimeter be given. We say that $E \subset \mathbb{R}^n$ is *a set of minimal anisotropic perimeter with respect to F in Ω* if E is minimal in

$$\inf\{\Phi_A(\overline{\Omega}) \mid A \subset \mathbb{R}^n,\ 1_A = 1_F \text{ on } \mathbb{R}^n \backslash \overline{\Omega}\}.$$

To pursue the analogy to i.), we will not use the more set-theoretic perimeter notation in this thesis. The problem ii.) corresponds to a relaxed form of i.) which is however only imposed among sets (ie. $\{0, 1\}$-valued functions of bounded variation).
Note though that the interior traces on $\partial\Omega$ need not coincide, ie. it is in general not true that $(1_E)_{\partial\Omega} = (1_F)_{\partial\Omega}$ is valid for any set $E \subset \mathbb{R}^n$ of anisotropic minimal perimeter.

iii.) The Relaxed Anisotropic Least Gradient Problem. Let now $f \in \mathrm{BV}(\mathbb{R}^n)$ be given. The following (fully) generalized problem,

$$\inf\{\Phi_u(\overline{\Omega}) \mid u \in \mathrm{BV}(\mathbb{R}^n),\ u = f \text{ on } \mathbb{R}^n \backslash \overline{\Omega}.\},$$

is known as the *relaxed anisotropic least gradient problem* with respect to f. We remind that

$$\Phi_u(\overline{\Omega}) = \Phi_u(\Omega) + \int_{\partial\Omega} \Phi(x, N_{\partial\Omega})|u_{\partial\Omega} - f|\ d\mathscr{H}^{n-1}$$

and thus, we are prescribing data only on the complement as in ii.). Similarly, we also cannot a priori know whether $u_{\partial\Omega} = f$, but we however do know that a function of anisotropic least gradient with trace f always also solves the problem iii.) by using a traceless extension to $\mathbb{R}^n$ and [33, 3.17 Proposition]. This allows us in particular to vary up to the boundary. Applying the coarea formula for anisotropic total variations from [24, Proposition 2.1] yields the following result:

$$\begin{aligned} & E \text{ is of minimal anisotropic perimeter for } F. \\ \Longrightarrow\ & 1_E \text{ solves the relaxed anisotropic least gradient problem with respect to } 1_F. \end{aligned}$$

iv.) Parametric Obstacle Problems. We say that a current $T \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ solves a *parametric obstacle problem for the parametric integrand* $\digamma$ *and* Ω, if T is minimal in

$$\inf\{\langle \digamma, X\rangle \mid X \in \mathscr{R}_{n-1}(\mathbb{R}^n),\ \operatorname{spt} X \subset \overline{\Omega},\ \partial X = \partial T\}.$$

This amounts to

$$\langle \digamma, T\rangle \leqslant \langle \digamma, T + X\rangle$$

for all $X \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ with $\operatorname{spt} X \subset \overline{\Omega}$ and $\partial X = 0$.

It is clear that the set $\overline{\Omega}$ acts as the obstacle by constraining the supports of comparison currents. In the homological notation of [11], this is written as T being $(\overline{\Omega}, \varnothing)$-minimal (while already using that our $\overline{\Omega}$ is a Lipschitz retract as a domain with Lipschitz boundary). It is moreover known that solutions to such parametric obstacle problems are *locally almost-minimizing* if Ω is a domain of sufficient regularity and hence, the obey the regularity theory for almost-minimizers. We will later prove a precise version of this statement in the the fourth section of this thesis.

2.13 Remark. The existence of extremals for the problems ii.), iii.) and iv.) follows via the direct method under somewhat mild regularity assumptions (in our case, Ω being a domain with Lipschitz boundary suffices).

2.4 Level Sets and Functions of Anisotropic Least Gradient.

The quintessential property of BV extremals in 1-homogeneous variational problems is that their sub- and superlevel sets, which we shall denote by $\{u \geqslant t\}$ or $\{u \leqslant t\}$, also solve variational problems in BV with respect to the same (or a very similar) functional. We recall the following result for Dirichlet problems from [24, Theorem 2.4].

2.14 Theorem (ANISOTROPIC BOMBIERI-DE GIORGI-GIUSTI THEOREM). *Let* $u \in \mathrm{BV}(\Omega)$ *be a function* *of anisotropic least gradient with respect to* Φ *and with trace values* $f \in \mathrm{L}^1(\partial\Omega, \mathscr{H}^{n-1})$. *Then the sets*

$$\{u \geqslant t\}, \{u \leqslant t\} \subset \Omega$$

are of finite perimeter in Ω *and the indicator functions of these sets are functions of anisotropic least gradient with respect to* Φ *and their traces for each* $t \in \mathbb{R}$.

Proof. We shall only adapt some conventions to our setting. Let $u \in \mathrm{BV}(\Omega)$ solve i.) with respect to Φ and f, then choosing an extension of f to $\mathbb{R}^n \backslash \Omega$ in BV, we may extend u to $\mathbb{R}^n$ by f and this extension is a solution to iii.) by [34] and [24, 2.2 Lemma]. Hence, by [24, Theorem 2.4], the sets $\{u \geqslant t\}$ and $\{u \leqslant t\}$ are solutions to ii.) and therefore, their indicator functions as elements of $\mathrm{BV}(\Omega)$ are solutions to the anisotropic least gradient problem i.) with respect to their traces. ■

2.15 Remark. The flexibility in formulating such problems owes to the extension properties of BV functions, the central problem is usually rather to achieve a given trace. Note also that we may again extend each set $\{u \geqslant t\}$ or $\{u \leqslant t\}$ with a traceless extension with values 0 or 1 to $\mathbb{R}^n$ such that the resulting set (respectively, its indicator function) is a set of minimal anisotropic perimeter in $\mathbb{R}^n$.

We will now establish some immediate properties hailing from the geometric minimization in terms of density and proximity.

2.16 Corollary. *Let* $u \in \mathrm{BV}(\Omega)$ *be a function of anisotropic least gradient on* Ω.

(1) We have that $\{u \geqslant t\}, \{u \leqslant t\} \subset \Omega$ *fulfill*

$$\operatorname{spt}|D1_{\{u \geqslant t\}}| \subset \partial\{u \geqslant t\}, \qquad \operatorname{spt}|D1_{\{u \leqslant t\}}| \subset \partial\{u \leqslant t\},$$

and

$$\operatorname{spt}|D1_{\{u \geqslant t\}}| \cap \Omega = \operatorname{clos}(\partial^*\{u \geqslant t\}) \cap \Omega, \qquad \operatorname{spt}|D1_{\{u \leqslant t\}}| \cap \Omega = \operatorname{clos}(\partial^*\{u \leqslant t\}) \cap \Omega$$

for all $t \in \mathbb{R}$ *and both reduced boundaries differ from their closures only by a set of* $\mathscr{H}^{n-1}$*-measure zero.*

(2) If further $u \in \mathrm{C}(\overline{\Omega})$ *and*

$$u^{-1}(t) \cap \Omega = \partial\{u \geqslant t\} \cap \Omega = \partial\{u \leqslant t\} \cap \Omega,$$

then

$$\operatorname{spt}|D1_{\{u \geqslant t\}}| \cap \Omega = \operatorname{spt}|D1_{\{u \leqslant t\}}| \cap \Omega = u^{-1}(t) \cap \Omega.$$

Proof. This follows immediately from the interior lower and upper density bounds by the minimization of theorem 2.14 due to [12, Proposition 12.19], since there exists some $\mu > 0$ such that

$$\mu r^{n-1} \leqslant |D1_E|(\mathbf{U}(x,r)) \leqslant (1-\mu) r^{n-1} \qquad \text{for all} \quad \mathbf{U}(x,r) \subset \Omega$$

with E being either set and $x \in \Omega$ being element of the respective support via [1, I.1(35)] and Federer's theorem [12, Theorem 16.2] about the essential boundary yields the claim. The latter statement follows also from the local characterization of the support from [12, Proposition 12.19] and the fact that $\{u > t\}$ and $\{u < t\}$ are open, since $u \in \mathrm{C}(\overline{\Omega})$. ■

2.17 Lemma (Convergence of Level Sets in Hausdorff Distance)**.** *Assume that*

$$u^{-1}(t) = \partial\{u \geqslant t\} = \partial\{u \leqslant t\}$$

holds. Then we have

$$u^{-1}(s) \longrightarrow u^{-1}(t)$$

in the Hausdorff distance. In case $u^{-1}(t)$ *does not equal its level set boundaries, we still have the one-sided versions*

$$u^{-1}(s) \longrightarrow \partial\{u \geqslant t\} \qquad \text{and} \qquad u^{-1}(s) \longrightarrow \partial\{u \leqslant t\}$$

for $s \nearrow t$ *and* $s \searrow t$ *respectively.*

Proof. We repeat the argument by contradiction from [46, 1.2 Lemma] and set $t = 0$. The one-sided version follows from easy modifications. Assume that

$$\lim_{t\to 0} \mathcal{H}(u^{-1}(t), u^{-1}(0)) \neq 0,$$

then we may find $\varepsilon > 0$, a sequence of non-vanishing real numbers t_i such that $t_i \to 0$ and a sequence of points x_i such that

$$\text{either} \qquad x_i \in \overline{\Omega} \cap u^{-1}(0) \quad \text{and} \quad \mathbf{B}(x_i, \varepsilon) \cap \overline{\Omega} \cap u^{-1}(t_i) = \varnothing \quad \text{for all } i \in \mathbb{N},$$

$$\text{or} \qquad x_i \in \overline{\Omega} \cap u^{-1}(t_i) \quad \text{and} \quad \mathbf{B}(x_i, \varepsilon) \cap \overline{\Omega} \cap u^{-1}(0) = \varnothing \quad \text{for all } i \in \mathbb{N}.$$

We can assume that $x_i \longrightarrow x \in \overline{\Omega}$. Both cases above necessarily imply $u(x) = 0$ by continuity, hence, the second case may be excluded. A simple triangle argument with the first case now yields

$$\mathbf{B}(x, \varepsilon/2) \cap u^{-1}(t_i) = \varnothing \qquad \text{for all } i \text{ large enough}$$

and choosing a subsequence such that $t_i \to 0$ monotonously and without loss of generality from above, we infer a contradiction to

$$x \in \partial\{u \leqslant 0\}$$

and the intermediate value theorem for continuous functions. ∎

2.18 Remark. We recall further that if $u : \overline{\Omega} \longrightarrow \mathbb{R}$ is Lipschitz on $\overline{\Omega}$, then an immediate argument via the coarea formula (see [41, Lemma 6.5(2)]) shows that Du exists $\mathscr{H}^{n-1}$-almost everywhere on $u^{-1}(t) \cap \Omega$ with $Du \neq 0$ on $\mathscr{L}^1$-almost every level set. This clearly leads, for such t, to

$$u^{-1}(t) \cap \Omega = \partial\{u \geqslant t\} \cap \Omega = \partial\{u \leqslant t\} \cap \Omega$$

as a point $x_0 \in u^{-1}(t) \cap \Omega$ with $Du(x_0) \neq 0$ fulfills that $x_0 \in \partial\{u \geqslant t\} \cap \partial\{u \leqslant t\}$ (it may not be locally maximal or minimal) and the remaining points may be approximated.

2.19 Proposition (Level sets and Supports)**.** *Let* u *now additionally be Lipschitz on* $\overline{\Omega}$ *and* $t \in \mathbb{R}$ *be an interior point of* $\operatorname{rg} f$*. Then*

$$\operatorname{spt}|D1_{\{u\geqslant t\}}| \cap \Omega = \partial\{u \geqslant t\} \cap \Omega \qquad \text{and} \qquad \operatorname{spt}|D1_{\{u\leqslant t\}}| \cap \Omega = \partial\{u \leqslant t\} \cap \Omega.$$

Proof. Since t is an interior point of the range of f, we preliminarily note that the traces of $1_{\{u\geqslant t\}}$ and

$1_{\{u \leqslant t\}}$ on $\partial\Omega$ may not be identically (equivalent to) 1 or 0 as u is continuous on $\overline{\Omega}$. The constancy theorem for BV functions thus yields that

$$\operatorname{spt}|D1_{\{u \geqslant t\}}| \cap \Omega \neq \varnothing \qquad \text{and} \qquad \operatorname{spt}|D1_{\{u \leqslant t\}}| \cap \Omega \neq \varnothing.$$

We now deal without loss of generality with $\{u \geqslant t\}$ and choose $x_0 \in \partial\{u \geqslant t\} \cap \Omega$ and we are done if $x_0 \in \operatorname{spt}|D1_{\{u \geqslant t\}}|$. We select $\rho > 0$ small enough such that

$$x_0 \in \Omega_\rho \qquad \text{and} \qquad \operatorname{spt}|D1_{\{u \geqslant t\}}| \cap \overline{\Omega}_\rho \neq 0.$$

By the genericity properties of Lipschitzian functions from remark 2.18 and corollary 2.16, we may assume that there is in particular a sequence $(s_i)_{i\in\mathbb{N}}$ such that $s_i \nearrow t$ and

$$\operatorname{spt}|D1_{\{u \geqslant s_i\}}| \cap \Omega = \partial\{u \geqslant s_i\} \cap \Omega = u^{-1}(s_i) \cap \Omega \qquad \text{for all } i \in \mathbb{N}$$

and we may further choose a sequence $(x_{s_i}) \subset \Omega_\rho$ such that $x_{s_i} \longrightarrow x_0$ as $i \to \infty$ with $x_{s_i} \in \partial\{u \geqslant s_i\}$ by one-sided Hausdorff convergence from lemma 2.17 and remark 2.1. Thus especially

$$x_{s_i} \in \operatorname{spt}|D1_{\{u \geqslant s_i\}}| \cap \overline{\Omega}_\rho,$$

and it is a classical consequence of the L^1-convergence $\{u \geqslant s\} \longrightarrow \{u \geqslant t\}$ for $s \nearrow t$ and the fact that each set $\{u \geqslant s\}$ is minimal via Reshetnyak's theorem that also the measures $|D1_{\{u \geqslant s_i\}}| \longrightarrow |D1_{\{u \geqslant t\}}|$ converge in the weak sense locally in Ω. Now the interior lower density bounds imply that any limit of such a sequence (x_{s_i}) can only lie in $\operatorname{spt}|D1_{\{u \geqslant t\}}| \cap \overline{\Omega}_\rho$, by which the proof is completed. ■

2.20 Remark. Note that the above property is, in fact, not true for general continuous functions of bounded variation and "cancellations" may appear. Moreover, the property continues to hold for boundary points of $\operatorname{rg} f$, if their trace on $\partial\Omega$ is non-trivial (as in the above proof).

2.5 Function Spaces and Non-Parametric Partial Differential Equations.

For purposes of notation as well as established results from the regularity theory of elliptic partial differential equations, we will refer to the monograph [13].

We will denote for $0 \leqslant k \leqslant \infty$ via $\mathrm{C}^{k,\alpha}(\Omega)$ *the space of k-times differentiable functions whose k-th derivative fulfills an α-Hölder condition.* Therein, the domain Ω can be open or contain accumulation points of its boundary, in which case we will understand that such a function $u \in \mathrm{C}^{k,\alpha}(\Omega)$ admits an extension to an open superset which is of class $\mathrm{C}^{k,\alpha}$. We will refer to analytic functions as functions of class C^ω. If a function is supposed to have compact support inside Ω, we will add a subscript c to the function space. The norms of such spaces will be denoted via $|\cdot|_{k,\alpha,\Omega}$ or $|\cdot|_{k,\Omega}$ and if the reference domain Ω for the norm is clear from the context, we will omit the subscript.

Generic non-negative constants will be denoted by C. Such constants may depend on data and will change from inequality to inequality, but we will try to make clear what quantities they are independent of if necessary. For describing properties related to small values of our extremal near 0, we will generically use $T > 0$ as a symbol.

Regarding partial differential equations, we will sometimes work with *the associated non-parametric integrand* to some parametric integrand $\mathcal{F}$. This refers to the particular structure of $\mathcal{F}$, if we consider the pairing of $\mathcal{F}$ and some (at least) Lipschitzian graph. As we are only interested in codimension one, we shall use the following dualized construction:

Assume that $\mathcal{F}$ is an elliptic parametric integrand on $\mathbb{R}^n$ and consider the associated total variation Φ. Assuming that

$$v : \mathbf{U}^{n-1}(0,s) \longrightarrow \mathbb{R}, \qquad v \in \mathrm{C}^{0,1}(\mathbf{U}^{n-1}(0,s)) \qquad \text{for some } s > 0,$$

we may evaluate $\mathcal{F}$ on the graph of v (which is a rectifiable current Γ_v of multiplicity one and support inside the cylinder $\mathbf{U}(0,s) \times \mathbb{R}$) to receive

$$\langle \mathcal{F}, \Gamma_v \rangle = \int_{\mathrm{graph}_v} \mathcal{F}(x, \vec{\Gamma}_v)\, d\mathscr{H}^{n-1} = \int_{\mathrm{graph}_v} \Phi(x, N_v)\, d\mathscr{H}^{n-1} = \int_{\mathbf{U}(0,s)} \Phi((z, v(z)), (\nabla - v(z), 1))\, d\mathscr{L}^{n-1}$$

by applying the Hodge star operator, the area formula and choosing adequate orientations. We will denote

$$\Phi^{\S}(z, u, v) := \Phi((z,u), (-v, 1))$$

and call $\Phi^{\S}$ *the associated non-parametric integrand to* $\mathcal{F}$ *and* Φ. Of course, we are not restricted to some ball $\mathbf{U}$ and can choose any open domain. Assuming now that Γ_v is minimizing with respect to $\mathcal{F}$ (or at least stationary with respect to vertical variations fixing $\partial\Gamma_v$), we may especially vary the function v by some perturbation

$$v + t\varphi \qquad \text{for } |t| \text{ small}, \qquad \varphi \in \mathrm{C}_c^\infty(\mathbf{U}^{n-1}(0,s)),$$

to receive by differentiating that v solves

$$0 = \int_{\mathbf{U}(0,s)} D_3\Phi^{\S}(z, v, Dv) \cdot D\varphi + D_2\Phi^{\S}(z, v, Dv)\varphi \; d\mathscr{L}^{n-1} \qquad \text{for all} \quad \varphi \in \mathrm{C}_c^\infty(\mathbf{U}(0,s)),$$

which we will call *the associated non-parametric Euler-Lagrange equation for* $\mathcal{F}$ *and* Φ. Clearly, $\Phi^{\S}$ inherits its regularity from $\mathcal{F}$: If $\mathcal{F}$ is of class $\mathrm{C}^{k,\alpha}$ in its first (respectively, second) variable, then $\Phi^{\S}$ is of class $\mathrm{C}^{k,\alpha}$ in its first and second (respectively, third) variable. Also ellipticity is inherited: We recall from remark 2.7 that, since $\mathcal{F}$ is elliptic, the parametric Legendre condition holds, which is,

$$D_2^2\Phi(x,v)(w,w) \geqslant c|v|^{-1}\left(|w|^2 - \left(w \cdot \frac{v}{|v|}\right)^2\right) \qquad \text{for all } v, w \in \mathbb{R}^n \text{ with } v \neq 0$$

and an ellipticity constant $c > 0$. Applying this to $\Phi^{\S}$, we find

$$
\begin{aligned}
D_3^2\Phi^{\S}(z,u,v)(w,w) &= D_2^2\Phi((z,u),(-v,1))((w,0),(w,0)) \\
&\geqslant \frac{c}{\sqrt{1+v^2}}\left(|w|^2 - \left(\begin{pmatrix} w \\ 0 \end{pmatrix} \cdot \frac{(v,-1)}{\sqrt{1+v^2}}\right)^2\right) \\
&= \frac{c}{\sqrt{1+v^2}}\left(|w|^2 - \frac{(v\cdot w)^2}{1+v^2}\right) \\
&\geqslant c|w|^2\left(\frac{1}{\sqrt{1+v^2}} - \frac{|v|^2}{(1+v^2)^{\frac{3}{2}}}\right) \\
&= \frac{c|w|^2}{(1+|v|^2)^{\frac{3}{2}}} > 0 \qquad \text{for } v,w \in \mathbb{R}^{n-1},
\end{aligned}
$$

and hence, the matrix $D_3^2\Phi^{\S}(z,u,v)$ is pointwise positive definite (though not uniformly). Note also that the latter coefficient is bounded away from zero whenever $|v| \leqslant R$, ie. we have

$$
D_3^2\Phi^{\S}(z,u,v)(w,w) \geqslant C|w|^2 \qquad \text{whenever } v \in \mathbf{B}(0,R),
$$

where $C > 0$ depends on R but not on v.

2.21 Remark. We particularily infer from the above that the associated non-parametric Euler-Lagrange equation is *elliptic* on its domain. Its principal part being *uniformly elliptic* on bounded domains in the gradient variable will furthermore be helpful when we consider appropriate linearizations.

2.22 Remark. The fact that the principal part of the operator is pointwise elliptic is also needed to show that weak and bounded $\mathrm{C}^{1,\beta}$-solutions belong in fact to (locally or globally) $\mathrm{W}^{2,2}$, which is used to linearize and infer higher regularity via Schauder estimates. This opens up higher regularity theory for the graph functions from the measuregeometric regularity theorems (see also [12, Proposition 3.3]).

2.6 Submanifolds, Stationary Immersions and Dependence Theorems.

In this last preliminarial section we want to recall some of the results from [56] (for the area integrand see also [57] and for certain special integrands the last section of [29]) and adjust them to our needs. In particular, we will also fix notation for elementary differential geometry and derive suitable formulas.

We will denote *a submanifold without boundary of dimension* l of $\mathbb{R}^n$ of class $\mathrm{C}^{k,\gamma}$ for $k \geqslant 1$ and $0 < \gamma \leqslant 1$ a locally $\mathbb{R}^l$-embedded set whose local embeddings can be chosen of class $\mathrm{C}^{k,\gamma}$. The *tangent space at* $x \in \Xi$ to a submanifold Ξ will be denoted via $T_x\Xi$. In case $l = n-1$ (which is the relevant case for us), we will abuse notation and denote a choice of unit normal vector field of Ξ as well as the normal space by N_Ξ. The submanifold Ξ is *orientable* if such a normal field exists globally. All declarations extend similarly to *submanifolds with boundary* of $\mathbb{R}^n$. The unit *conormal vector* at boundary points of Ξ (ie. the up to sign unique unit vector which is orthogonal to the boundary tangent space inside the full tangent space) will be denoted by N_Ξ^{co}. Note that we will usually assume that a tangent space $T_{x_0}\Xi \subset \mathbb{R}^n$ is an affine space centered at x_0.

If a submanifold with boundary of sufficient class solves a geometric variational problem, we might consider questions of boundary stability, ie. if we perturb the embedding of $\partial\Xi$ in an adequate $\|\cdot\|_{k,\gamma}$-

norm, we might ask how the submanifold as a whole behaves.

For this purpose, we shall rephrase the setting into a mapping problem.

We let Ξ be an abstract $n-1$-dimensional Riemannian manifold Ξ with boundary which is supposed to be analytic, compact, connected and orientable and we fix an embedding $\iota \in \mathrm{C}^{k,\gamma}(\Xi, \mathbb{R}^n)$ into Euclidean space.

Note that this is no restriction in general, since we may always find a compatible analytic structure in the intrinsic case. We recall also (as in [56, 1.0]) that we may assume Ξ to be isometrically embedded in some high dimensional Euclidean space to simplify computations regarding derivatives.

We fix moreover an elliptic parametric integrand F on $\mathbb{R}^n$ of class $\mathrm{C}^{\mu-1,\alpha}$ with $D_2 F$ also of class $\mathrm{C}^{\mu-1,\alpha}$.

To prepare the application of the regularity theory from [56], we now need to suppose

$$0 < \gamma < \alpha < 1 \qquad \text{and} \qquad 2 \leqslant k \leqslant \mu - 1$$

and in our particular case, we will use the assumption $k = 2$ and choose a $0 < \gamma < 1$ to be fixed later.

Then we let *the parametric integral for the immersion ι with respect to F* be defined via

$$\langle F, \iota \rangle := \int_\Xi F(\iota(x), (\Lambda_{n-1} D\iota(x))(\xi(x)))\ d\mathscr{H}^{n-1},$$

where ξ is a global smooth unit multivector field, which orients Ξ. We write the parametric integral as

$$\langle F, \iota \rangle = \int_\Xi \tilde{F}(x, \iota(x), D\iota(x))\ d\mathscr{H}^{n-1}$$

with

$$\tilde{F}(x, y, t) := F(y, (\Lambda_{n-1} t)(\xi(x))).$$

2.23 Remark. This parametric integral for immersions is a somewhat more general version for arbitrary domains and non-graphical immersions of the associated non-parametric integral of the last section. We will indeed use both versions for our regularity theory and compare properties of both. Note that also the latter two components of $\tilde{F}$ directly inherit their regularity from the respective ones from F.

We denote the *parametric Euler-Lagrange operator* with respect to F by $\mathfrak{H}(\iota)$ and its linearization, *the Jacobi operator*, by $D\mathfrak{H}(\iota)$. We recall the following facts from [56, Sections 2, 3].

2.24 Lemma (The Euler-Lagrange operator)**.** *It holds that*

$$\mathfrak{H} : \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n) \longrightarrow \mathrm{C}^{0,\gamma}(\Xi, \mathbb{R}^n)$$

is a mapping of class $\mathrm{C}^{\mu-2,\alpha-\gamma}_{loc}$. The vector field $\mathfrak{H}(\iota)$ is normal to $\iota(\Xi) \subset \mathbb{R}^n$. We have the explicit formula

$$\mathfrak{H}(\iota) = -\operatorname{div}(D_3 \tilde{F}(x, \iota(x), D\iota(x))) + D_2 \tilde{F}(x, \iota(x), D\iota(x))$$

and the weak integral formula

$$\int_{\Xi} \varphi \cdot \mathfrak{H}(\iota)\ d\mathscr{H}^{n-1} = \int_{\Xi} D_2\tilde{F}(x,\iota(x),D\iota(x))\varphi(x) + D_3\tilde{F}(x,\iota(x),D\iota(x))D\varphi(x)\ d\mathscr{H}^{n-1}$$

for all $\varphi \in \mathrm{C}^1(\Xi,\mathbb{R}^n)$ *such that* $\varphi_{\partial\Xi} = 0$.

Proof. Contained in [56, 1.3 Elliptic Integrands and First Variation]. ■

2.25 Lemma (The Jacobi Operator). *Assume that* ι *is stationary, ie.* $\mathfrak{H}(\iota) = 0$. *The linearization* $D\mathfrak{H}(\iota)$ *of* $\mathfrak{H}$ *is a linear operator*

$$D\mathfrak{H}(\iota) : \mathrm{C}^{2,\gamma}(\Xi,\mathbb{R}^n) \longrightarrow \mathrm{C}^{0,\gamma}(\Xi,\mathbb{R}^n)$$

and we have the weak integral formula

$$\begin{aligned}\int_{\Xi} v_2 \cdot D\mathfrak{H}(\iota)(v_1)\ d\mathscr{H}^{n-1} \quad &= \quad \int_{\Xi} \left(D_2^2\tilde{F}(x,\iota,D\iota)v_1(x) + D_3D_2\tilde{F}(x,\iota,D\iota)Dv_1(x)\right) v_2(x) \\ &\quad + \left(D_2D_3\tilde{F}(x,\iota,D\iota)v_1(x) + D_3^2\tilde{F}(x,\iota,D\iota)Dv_1(x)\right) Dv_2(x)\ d\mathscr{H}^{n-1}\end{aligned}$$

for all $v_1, v_2 \in \mathrm{C}^{2,\gamma}(\Xi,\mathbb{R}^n)$ *such that* $v_{1,\partial\Xi} = v_{2,\partial\Xi} = 0$. *If such a vector field* v_1 *is tangent to* $\iota(\Xi)$, *then* $D\mathfrak{H}(\iota)(v_1) = 0$.

Proof. Set $\varphi = v_2$ in lemma 2.24 and differentiate along a linear curve through ι in direction v_1 (compare also [56, 1.4 Second Variation and the Jacobi Operator]). ■

2.26 Remark. We explicitly note that, as both $\mathfrak{H}$ and $D\mathfrak{H}$ are differential operators, they are local and can of course also be applied to any twice continuously differentiable functions on some open subset of Ξ in their strong form and to any once continuously differentiable function in their weak form.

Let us now spend some words on properties of the Jacobi operator $D\mathfrak{H}(\iota)$ with respect to some immersion ι. Aside from the global formulation, we proceed to derive a convenient expression for the coefficient function of a Jacobi field in a local chart of Ξ. Similar to the last section, the ellipticity and, in particular, the parametric Legendre condition that we imposed on the integrand F will yield ellipticity properties for associated partial differential equations.

2.27 Proposition (Uniform Ellipticity on Normal Fields). *Let* $\iota \in \mathrm{C}^{3,\gamma}(\Xi,\mathbb{R}^n)$. *The weak formulation of the Jacobi operator for* Ξ *and* ι *descends to a weak formulation of a uniformly elliptic partial differential equation for the coefficient function on normal variations in any choice of local coordinates of* Ξ. *Any normal* C^1*-solution to the weak formulation of the Jacobi operator has interior regularity of class* $\mathrm{C}^{2,\gamma}$.

Proof. We choose local coordinates near $x_0 \in \Xi$ and fix an immersion $\iota \in \mathrm{C}^{k,\gamma}(\Xi,\mathbb{R}^n)$ with $k \geqslant 3$ and $0 < \gamma < 1$. The parametric integral in such coordinates reads

$$\begin{aligned}\int_{x(Z)} F(\iota(x),\iota_*(\xi))\ d\mathscr{H}^{n-1} \quad &= \quad \int_Z F\left(\iota(x),\iota_*\left(\frac{\partial_1 x \wedge \dots \wedge \partial_{n-1}x}{|\partial_1 x \wedge \dots \wedge \partial_{n-1}x|}\right)\right)|\partial_1 x \wedge \dots \wedge \partial_{n-1}x|\ d\mathscr{L}^{n-1} \\ &= \quad \int_Z F\left(\iota(x),\iota_*\left(\partial_1 x \wedge \dots \wedge \partial_{n-1}x\right)\right)\ d\mathscr{L}^{n-1}\end{aligned}$$

for some open and bounded coordinate domain $Z \subset \mathbb{R}^{n-1}$ via the area formula and one-homogeneity of F where we thus let

$$\xi = \partial_1 x \wedge ... \wedge \partial_{n-1} x, \qquad \iota_* \xi := (\Lambda_{n-1} D\iota)(\xi) = D\iota(\partial_1 x) \wedge ... \wedge D\iota(\partial_{n-1} x).$$

We now fix normal variations

$$v_1 = \zeta_1 N_{\iota(\Xi)} \qquad \text{and} \qquad v_2 = \zeta_2 N_{\iota(\Xi)} \qquad \text{with } \zeta_1, \zeta_2 \in \mathrm{C}^{2,\gamma}(\Xi),$$

define

$$\iota_t = \iota + t v_1$$

and acknowledge that

$$\frac{d}{dt}_{|t=0} \left(\Lambda_{n-1} D\iota_t\right)(\xi) = \sum_{i=1}^{n-1} D\iota(\partial_1 x) \wedge ... \wedge Dv_1(\partial_i x) \wedge ... \wedge D\iota(\partial_{n-1} x).$$

Since

$$Dv_1(\partial_i x) = \partial_i (v_1 \circ x) = \partial_i \zeta_1 N_{\iota(\Xi)} + \zeta_1 \partial_i N_{\iota(\Xi)}$$

by the chain and product rule, we find

$$\frac{d}{dt}_{|t=0} (\Lambda_{n-1} D\iota_t)(\xi) = \sum_{i=1}^{n-1} D\iota(\partial_1 x) \wedge ... \wedge \left(\partial_i \zeta_1 N_{\iota(\Xi)} + \zeta_1 \partial_i N_{\iota(\Xi)}\right) \wedge ... \wedge D\iota(\partial_{n-1} x).$$

Thus, by varying in both v_1- and v_2-direction and differentiating twice, the second order terms of the weak formulation of the Jacobi operator in such local coordinates x compute to

$$\mathbf{A}(x)(D\zeta_1, D\zeta_2) := D_2^2 F(\iota(x), \iota_*(\xi)) \left(\sum_{i=1}^{n-1} (\partial_i \zeta_1)\, x^i, \sum_{j=1}^{n-1} (\partial_j \zeta_2)\, x^j \right),$$

where we let $x^i \in \Lambda_{n-1}\mathbb{R}^n$ be defined by replacing the i-th factor in $\iota_*\xi$ by $N_{\iota(\Xi)} \in \mathbb{R}^n$. The set $\{x^i \mid i \in \{1, ..., n-1\}\}$ is linearly independent and forms a pointwise basis of $\Lambda_{n-1}\mathbb{R}^n$ with $\iota_*\xi$.

Thus, letting

$$\eta = \sum_{i=1}^{n-1} p_i x^i \in \Lambda_{n-1}\mathbb{R}^n \qquad \text{where} \quad p \in \mathbb{R}^{n-1},$$

we compute for the uniform ellipticity of A with the parametric Legendre condition for F that

$$\mathbf{A}(x)(p,p) \geqslant c\, |\iota_*\xi|^{-1} \left(|\eta|^2 - \left(\eta \cdot \frac{\iota_*\xi}{|\iota_*\xi|} \right)^2 \right) \geqslant C \left(|\eta|^2 - \left(\eta \cdot \frac{\iota_*\xi}{|\iota_*\xi|} \right)^2 \right),$$

as

$$|\iota_*\xi| = \left(\det \left(D(\iota \circ x)^T D(\iota \circ x) \right) \right)^{\frac{1}{2}}$$

may be assumed as bounded from above in local coordinates x. Since furthermore

$$N_{\iota(\Xi)} \cdot D\iota(\partial_i x) = 0 \qquad \text{for all } i = 1, ..., n-1,$$

we infer $\eta \cdot \iota_* \xi = 0$ and hence

$$\mathbf{A}(x)(p,p) \geqslant C|\eta|^2 = C \sum_{i,j=1}^{n-1} p_i \left(x^i \cdot x^j\right) p_j .$$

The latter double sum corresponds to the scalar product of $\Lambda_{n-1}\mathbb{R}^n$ in the basis given by $(x^i)_{i=1,\dots,n-1}$. Denoting $G = (x^i \cdot x^j) \in \mathbb{R}^{(n-1)\times(n-1)}$, this matrix is therefore uniformly positive definite and we conclude

$$\mathbf{A}(x)(p,p) \geqslant p^T G p \geqslant C|p|^2 \qquad \text{with } C > 0 \text{ uniformly in local coordinates } x$$

which finishes the argument. Finally, as the equation is now uniformly elliptic in local coordinates and all involved structure functions are at least of class $\mathrm{C}^{0,\gamma}$, the interior regularity follows by standard interior elliptic regularity theory for linear equations. ■

By completely expressing and rearringing the weak form of the Jacobi operator in local coordinates, we may make the following definition.

2.28 Definition (The Jacobi Equation). We will call the resulting uniformly elliptic partial differential equation *the Jacobi equation* with respect to the given integrand $\mathcal{F}$, the embedding ι and the local coordinates x near some fixed point $x_0 \in \Xi$. It may be written in the divergence form

$$\operatorname{div}(\mathbf{a}^{ij}\zeta_j + \mathbf{b}^i\zeta) + \mathbf{c}^i\zeta_i + \mathbf{d}\zeta = 0 \qquad \text{in local coordinates } x,$$

where the regularity of the coefficients $\mathbf{a}^{ij},\ \mathbf{b}^i,\ \mathbf{c}^i,\ \mathbf{d}$ depends on $\mathcal{F}$ and ι.

As the Jacobi operator is uniformly elliptic on normal variations, the normal field solutions $v \in \mathrm{C}^{2,\gamma}(\Xi, N_\Xi)$ to $D\mathfrak{H}(\iota)(v) = 0$ with homogeneous boundary conditions (ie. $v_{\partial\Omega} = 0$) form a finite dimensional vector space $\mathfrak{K} \subset \mathrm{C}^{2,\gamma}(\Xi, N_\Xi)$ and we will call $\dim \mathfrak{K} \geqslant 0$ *the Jacobi nullity of* $\iota(\Xi)$ (see also [56, 1.4 Proposition]). Note furthermore that a similar proof as for proposition 2.27 may be used to show uniform ellipticity of $D\mathfrak{H}(\iota)$ on variations in any (sufficiently smooth) uniformly non-tangential direction field p.

We move on to recall the central stability result that we want to use for our regularity results.

2.29 Theorem (Special Smooth Dependence Theorem in $\mathrm{C}^{2,\gamma}$). *Assume that $\mathfrak{H}(\iota) = 0$ and $\dim \mathfrak{K} = 0$. Then there exist neighborhoods*

$$U \subset \mathrm{C}^{2,\gamma}(\partial\Xi, \mathbb{R}^n) \quad \textit{of} \quad \iota_{\partial\Xi} \qquad \textit{and} \qquad V \subset \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n) \quad \textit{of} \quad \iota$$

and a map

$$\mathfrak{f} : U \longrightarrow V$$

of class $\mathrm{C}^{\mu-2}$ such that $(\mathfrak{f} \circ r)(\iota) = \iota$ and

$$(r \circ \mathfrak{f})(\kappa) = \kappa \qquad \textit{and} \qquad \mathfrak{H}(\mathfrak{f}(\kappa)) = 0 \qquad \textit{for all } \kappa \in U,$$

where r denotes here the restriction to $\partial\Xi$. Furthermore, for all $\iota^* \in V$ with $\mathfrak{H}(\iota^*) = 0$ (that is, for all stationary $\iota^* \in V$) it follows that $r \circ \iota^* \in U$ and ι^* and $\mathfrak{f} \circ r \circ \iota^*$ can only differ by reparametrization of Ξ (that is, a diffeomorphism of Ξ of at least class C^1).

Proof. This is the content of [56, 3.1]. ■

2.30 Remark. Our usage of $\mathfrak{F}$ for the proof of the later regularity results for functions of anisotropic least gradient will apply the dependence map $\mathfrak{F}$ near a regular value level set for the Dirichlet data, which invokes that the level set structure as a one-parameter family on the boundary $\partial\Omega$ is already well-behaved, and conduct somewhat of a "push-forward" by means of $\mathfrak{F}$ to find a well-behaved one-parameter family of level set boundaries on the full domain $\overline{\Omega}$.

Finally we discuss the regularity of each stationary immersion $\mathfrak{F}(\kappa) \in \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n)$ selected like this. As $\mathfrak{H}(\mathfrak{F}(\kappa)) = 0$, it is somewhat natural to expect higher regularity then $\mathrm{C}^{2,\gamma}$ in case κ and the integrand F are smoother and if we construct our parametrization $\mathfrak{F}(\kappa)$ properly and this does actually follow from some steps in the proof of the dependence theorem.

2.31 Corollary (Higher Regularity). *Let additionally $\iota \in \mathrm{C}^{\nu,\alpha}(\Xi, \mathbb{R}^n)$ and $\kappa \in \mathrm{C}^{m-1,\alpha}(\partial\Xi, \mathbb{R}^n)$. Under the assumptions of the last theorem, we may additionally assume that*

$$\mathfrak{F}(\kappa) \in \mathrm{C}^{\nu,\alpha}(\Xi, \mathbb{R}^n) \qquad \textit{with} \qquad \nu := \min(\mu, m-1).$$

for all $\kappa \in U$.

Proof. We recall that

$$\mathfrak{F}(\kappa) = \phi(\kappa) + \mathfrak{F}_0(\kappa) \qquad \text{with} \qquad (p \circ \mathfrak{H})\,(\phi(\kappa) + \mathfrak{F}_0(\kappa)) = 0$$

from [56, 3.1, proof]. Therein, p is the auxiliary smooth and uniformly non-tangential field, which is used in the construction to replace normal vectors, and $\phi(\kappa)$ is constructed such that

$$\begin{cases} \Delta(\phi(\kappa) - \iota) = 0, \\ (\phi(\kappa) - \iota)_{\partial\Xi} = \kappa - \iota_{\partial\Xi}, \end{cases}$$

ie. such that $(\phi(\kappa) - \iota)$ is the unique harmonic map on Ξ with boundary data $(\kappa - \iota_{\partial\Xi})$, which is thus an n-tuple of scalar valued harmonic maps on Ξ with the respective boundary data component of κ. Hence, it follows

$$\phi(\kappa) = \iota + (\phi(\kappa) - \iota) \in \mathrm{C}^{\nu,\alpha}(\Xi, \mathbb{R}^n)$$

and the p-field $\mathfrak{F}_0(\kappa) \in \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n)$ has by construction with $(\mathfrak{F}_0(\kappa))_{\partial\Xi} = 0$ and we may assume

$$\mathfrak{H}(\phi(\kappa) + \mathfrak{F}_0(\kappa)) = 0,$$

and thus the coefficient of $\mathfrak{F}_0(\kappa)$ obeys quasilinear elliptic regularity theory (by writing the stationarity condition in local coordinates on Ξ and arguing similarly to proposition 2.27 with $(\phi(\kappa), \mathfrak{F}_0(\kappa))$ being close to $(\iota, 0)$ for ellipticity) and is therefore also of class $\mathrm{C}^{\nu,\alpha}$. ■

3 The Jacobi Field Method to Partial Regularity.

The purpose of the section is a quick description and repetition of the original paper [46] for the isotropic case in 3.1 while the later sections 3.2 and 3.3 will discuss two particular problems that the original results contain and for which we will propose an altered approach. We will state results, which appear clear to me from the considerations of [46], in section 3.4.

3.1 A Sketch of the Original Result and Our Assumptions.

Let us now first recall the necessary regularities for our given data.

We will assume for the remainder of this section that $\Omega \subset \mathbb{R}^n$ is an open domain with C^m-boundary and $f \in \mathrm{C}^m(\partial\Omega)$, where $2 \leqslant n \leqslant 7$ and $4 \leqslant m \leqslant \infty$.

The lower bound on continuous differentiability of class C^4 is required for various tools from elliptic regularity theory as well as straightening the boundary. We may now also state the assumed a priori regularity on the extremal.

We will assume that $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ is a function of (isotropic) least gradient with $u_{\partial\Omega} = f$.

The connection of Lipschitz regularity to some bounds on the gradient of u will prove to be essential in the proof. Moreover, we will of course rely on the fact that level sets of u are bounded by nice, minimizing hypersurfaces due to theorem 2.14. As already detailed in remark 2.10, the existence of such Lipschitzian extremals minimizing the standard total variation in a Dirichlet class is well-assured by established theory and as detailed in remark 2.18, the Lipschitzianity already allows for nice genericity properties for level sets of u.

The Main Contents of the Approach. Roughly speaking, the method of [46] can be subdivided into three parts: One is measuregeometric, one is PDE-theoretic and the last relies on the application of the dependence map for minimal surfaces, which is prepared by means of the first two.
A very illustrating example of geometric analysis, all of these considerations rely heavily on local arguments — both in value of the extremal and in points of the level set — and in what sense way we may globalize them to a larger (super-)set. Furthermore, we will be strongly interested in perturbation results of various geometric quantities as we will be fixing a nice level set as our reference one and then see how it varies (and in what sense) as we vary the value of our function of least gradient. These deformations of the reference level set hypersurface will prove to be useful for blowing up — at last we may prove the vanishing Jacobi nullity of the reference hypersurface with it. Finally, with the dependence map at our disposal, we may consider a flow connecting the regular level sets near the boundary of the reference one and transfer it by means of the dependence map to all of the domain.

We now follow up with some more technical descriptions on the three previously named parts:

(α) **The Measuregeometric Part.** After the general setup, one first notes that there are generically "nice" non-fat level sets for a Lipschitzian function of least gradient and we will use one of these as our reference level set to find a neighborhood of it, on which the extremal is of corresponding differentiability class. Those genericity properties together with the minimization properties of the level sets

already imply a lot of structure. Recalling furthermore that all level sets contain at least one minimal hypersurface, the general regularity theory for minimizing currents (more specifically, the geometric a priori estimates) lets us now conclude that the level sets not only approach each other, but that the minimal hypersurfaces may also be regarded as sufficiently differentiable deformations of our reference hypersurace, which converge up to higher order in their function spaces.
These classical, but somewhat ill-recorded, results allow to deform as a *normal graph away from the boundary* while the continuation up to the boundary is a more delicate matter, for which we will use the construction of *almost-minimal diffeomorphisms.*

(β) **The PDE-theoretic Part.** Being able to express nearby minimizing hypersurfaces as normal graphs away from, but arbitrarily close to, the domain boundary, we now want to use this parametrization for blowing-up to prove the existence of *a special Jacobi field.*
This involves heavy usage of local expressions in coordinate charts and geometric partial differential equations, which we derive by means of the vanishing first variation.
Particularily, the mapping in normal direction can be shown to suffice proper estimates such that the blow-ups actually converge and moreover, the converge to a weak solution of the Jacobi equation of the reference hypersurface. This Jacobi field is moreover bounded from below, which is implied by Lipschitzianity here and is the most central part, as a simple argument with elliptic theory now assured that this special Jacobi field eliminates the existence of non-trivial homogeneous Jacobi fields. Hence, the Jacobi nullity of the reference hypersurface vanishes.

(γ) **The Level Set Part.** Due to the previous part, the necessary condition for local smooth dependence on the manifold boundaries for minimizing hypersurfaces near the reference one is granted and the dependence map exists.
Assuming the boundary of the reference hypersurface to be given by a regular level set, small perturbations of the value will still have regular level sets on the boundary and we may parametrize the nearby ones over the reference boundary. Thus, we gain a one-parameter family of embeddings of boundary manifolds, which we shall push foward with the dependence map to their unique minimal hypersurface (close to the reference one) with it as boundary.
By this uniqueness, we subsequently infer that it has to be contained in the appropriate level set of the extremal (as the boundary level set is and at least one minimizing hypersurface is already contained in the whole level set), thus, the extremal evaluated on the one-parameter family of minimizing hypersurfaces returns precisely the value of the parameter. At last, a *local inversion argument* provides the identity for our extremal, which is sufficiently differentiable.

Of course, we will make these rough sketches more precise in the next sections. Revisiting the methods of part (α) will be the content of sections 4 and 5 while section 6 will provide a different approach to part (β) and the conclusions from part (γ) will be conducted in section 7 (with the regular level set flow being introduced already in section 5.3 and the almost-normal diffeomorphisms in section 5.4) for the case of a general anisotropic total variation of sufficient regularity.

3.2 On the Convexity of the Domain.

Here we will discuss the first particular point of the original work that appears like a gap to me or, at least, could use a lot more details.

`Observation.` The original work *does not assume* that Ω is convex.

This apparantly harmless fact is first of in contrast to some earlier works of H. Parks. Indeed, in [41] and [42] we find both times a convex body as our setting. Furthermore, a very essential lemma in [41, 8(1) Lemma] breaks down for non-convex domains. Therein, it is provided that the minimizing hypersurfaces are in fact $(\overline{\Omega}, \varnothing)$-minimizing and not only locally minimizing in Ω. Thus, they are solutions to a parametric obstacle problem and exhibit different regularity properties. (In [41] it is proven for almost every value, while an approximation argument in [42, 3.1(1) Lemma] shows that it is true for *every* value.)

So how is this important? A simple estimate by convexity and properties of the area integrand would now show that a $(\overline{\Omega}, \varnothing)$-minimizing rectifiable current R fulfills

$$\begin{aligned} R \qquad & \textit{area-minimizing on } (\overline{\Omega}, \varnothing) \\ & \Longrightarrow \quad R \qquad \textit{area-minimizing on } \mathbb{R}^n. \end{aligned}$$

This unlocks the boundary regularity theory in the original approach by means of the works of Allard and Hardt/Simon. Yet, all of this is only valid so far on *convex domains*! If the domain Ω is possibly non-convex, there are no details given whether our level sets actually patch together to a manifold with boundary granted the necessary density estimates.
Furthermore, some additional properties like convergence of total masses in [46, 2.1 Lemma, 2.5 Corollary] are not (or, at least, not immediately) justified anymore without minimization with respect to $(\overline{\Omega}, \varnothing)$.

We will propose a solution to this property in section 4 of this thesis. Since those easy arguments by means of convexity are not available anymore, we however have to conjure stronger tools. This will be done in two steps:

(1) Show that locally minimizing hypersurfaces from the BV-setting are solutions to parametric obstacle problems also on general domains.

(2) Show that the (conditional) boundary regularity theory in fact works at every point of the boundary.

While it is somewhat classical that parametric obstacle problems can be interpreted as *almost-minimizing currents*, the boundary regularity theory is still conditional in the sense that we need *half-density estimates.* We shall thus also provide those at each point in question in what follows, mostly hailing from and relying on the works of R. Hardt and F.H. Lin in [20, 29].

3.3 On the Regularity of the Flow of Hypersurfaces.

Now we will discuss the second problem, which is intrinsically tied to the regularity conclusion and the way how boundary manifolds are parametrized and mapped with the dependence map. Most of the notation and content of this subsection is directly taken from the proof in [46, 4.3 Theorem].

Therein, the authors consider the dependence map F between the Banach spaces

$$F : \mathrm{C}^{m-2,\alpha}(f^{-1}(0), \mathbb{R}^n) \longrightarrow \mathrm{C}^{m-2,\alpha}(u^{-1}(0), \mathbb{R}^n),$$

where $0 < \alpha < 1$ is arbitrary, we have fixed the value 0 and additionally assumed that the level set boundaries equal the level set for this value. We denote the inverse of the nearest point projection on $\partial\Omega$ onto $f^{-1}(0)$ by π, which is a map of class C^{m-1}, with

$$\pi : f^{-1}(0) \times I \longrightarrow (\partial\Omega \subset \mathbb{R}^n)\,,$$

where I is some small open interval about 0. The important point is now to consider π as a curve in the above Hölder space via

$$(t \longmapsto \pi_t := \pi(.,t)) \in \mathrm{C}^{m-2,\alpha}(f^{-1}(0); \mathbb{R}^n)$$

Yet, the differentiability of this mapping is somewhat problematic.

`Observation.` Every differentiation in the Hölder norm of class $\mathrm{C}^{m-2,\alpha}$ already involves $(m-2)$ derivatives plus the Hölder constant of the last derivative (eg. $(m-1)$ derivatives by estimating against the Lipschitz constant).

Since π is a map of class C^{m-1} in its two variables, this makes it hard to differentiate the curve $t \mapsto \pi_t$ even once in the corresponding norm of class $\mathrm{C}^{m-2,\alpha}$. The original claim, namely,

$$t \longmapsto \pi_t \quad \textit{is of class } \mathrm{C}^{m-2} \textit{ as a map with values in } \mathrm{C}^{m-2,\alpha}(f^{-1}(0); \mathbb{R}^n),$$

is easily seen to be wrong here in full generality, because already two differentiations of the curve $t \mapsto \pi_t$ in $\mathrm{C}^{m-2,\alpha}$ would involve m differentiations of π, which outreaches the a priori amount of differentiability. But this is precisely the regularity that we need for the conclusion, since we want to consider afterwards

$$g(x,t) = F(\pi_t)(x)$$

as a map of class C^{m-2} in both variables! Our second question here will thus be

Is the result on partial regularity true with a different degree of differentiability?

Clearly, to receive the most regularity for the curve $t \longmapsto \pi_t$, we consequently need to consider the least possible degrees in the Hölder norm to lose the least possible amount of derivatives additionally. Still, we will have to alter the regularity result, as the conclusion will stay true but we will have to subtract an additional degree of differentiability. We will in the process also change the nearest point projection on the boundary for the flow of a rescaled gradient vector field of f to receive more differentiability in the above curve parameter (otherwise we would lose even more regularity in the final result!).

3.4 The Original Results.

We will close this section with a result that I certainly see proven from the original resource [46]. I also strongly need to mention that the only second available resource [58], which was only written by W. Ziemer, uses both of these (actually even stronger) assumptions.

3.1 Theorem (Partial Regularity, *Parks and Ziemer*). *Let $\partial\Omega$ and f be smooth and Ω be convex. Then there exists an open and dense set $\mathscr{O}_u \subset \Omega$ such that $u_{\mathscr{O}_u}$ is smooth.* ■

Observe that this formulation eliminates both problems from the sections 3.2 and 3.3, as convex domains allow for immediate boundary regularity, as we may drop the obstacle and pass to global minimization, plus we do not need to care about losses of derivatives if we work in the smooth category.

3.2 Corollary (Partial Regularity in Eight Dimensions, *Parks*). *Theorem 3.1 still holds while replacing $2 \leqslant n \leqslant 7$ with $n = 8$.*

Proof. The method of choice here is to perturb the (finitely many) isolated singularities away such that we may work with fully regular hypersurfaces by slightly replacing the value, see [43]. Interestingly, in this paper, the boundary regularity result from [22] is in fact referenced, which does need an *absolutely* area minimizing current, and also some other results [43, 2. Lemma] are only formulated for *absolutely* area minimizing currents. ■

The following sections 4,5 and 6 will now serve to complement these results with additional theory to justify results on possibly non-convex domains and only finite degrees of differentiability. In section 7 we will repeat the main steps in the argument to prove local and partial regularity theorems for functions of anisotropic least gradient and prove generalized (partial as well as local) versions of theorem 3.1.

4 Variational Problems for Level Set Boundaries.

Here we begin our discussion of the regularity properties of functions of least gradient and we start with properties of their level sets and level set boundaries.

While the interior regularity of sets of minimal anisotropic perimeter is usually well-established, we also want to discuss matters of *boundary regularity* such that the oriented hypersurfaces become proper submanifolds with boundary. This appears to be somewhat known and does follow from a couple of results from the literature, but proper references are rather unclear and we will spend the first subsection with recapping some details. We will moreover emphasize the correct notion of minimality that sets of minimal anisotropic perimeter fulfill among currents, which will be our starting point here.

The later two subsections will furthermore prepare the stability and deformation results of section 5 by introducing the geometric quantities regarding level sets that we want to consider. We will also include derivations on how to express the converging sequence of level set boundaries *in a non-parametric form* locally uniformly over some tangent space of the limit, which is the essential tool to also connect their geometries up to higher order.

4.1 A General Equivalence Result.

To begin with, we first prove the following equivalence lemma, which subsequently relates sets of minimal anisotropic perimeter and parametric obstacle problems for the associated integrand on currents. The idea of our approach is taken from [29, 0.2] while such relations could to some extent already be found in [42] and [44].

We pursue a proof here under the assumption that our domain Ω has connected boundary to ensure that its complement is connected, which can also be found as an assumption in [24]. Note that this may be enforced with a localization procedure, but it is unclear whether a connected boundary may be globally dropped.

4.1 Proposition (Equivalence of Minimizers). *Suppose we have a bounded Lipschitz domain Ω with connected boundary and a $\mathscr{H}^{n-1}$-measurable set $\Sigma \subset \partial\Omega$. Fix moreover a bounded set of finite perimeter $A \subset \mathbb{R}^n$ such that $\bar{\mathrm{T}}(1_A) = 1_\Sigma$ and $[\![\Sigma]\!]$ is oriented via the outer normal vector and let the parametric integrand F and the anisotropic total variation Φ be associated with each other via the Hodge operator $\star$. Then we have:*

(α) *If $R \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ is F-minimizing with respect to $(\overline{\Omega}, \varnothing)$ with boundary $\partial R = \partial[\![\Sigma]\!]$ (ie. a solution to the parametric obstacle problem), we may associate to R a boundaryless rectifiable current $R^* \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ such that*

$$R^* = \partial(\mathbf{E}^n \llcorner \tilde{E}) = R + (\partial(\mathbf{E}^n \llcorner A)) \llcorner (\mathbb{R}^n \backslash \overline{\Omega})$$

and $\tilde{E} \subset \mathbb{R}^n$ is a set of Φ-minimal anisotropic perimeter in Ω with respect $A \backslash \overline{\Omega}$.

(β) *If $E \subset \mathbb{R}^n$ is a set of Φ-minimal anisotropic perimeter in Ω with respect to $A \backslash \overline{\Omega}$, then the rectifiable current*

$$\tilde{R} = \partial(\mathbf{E}^n \llcorner E) \llcorner \overline{\Omega}$$

is a solution to the parametric obstacle problem for F *with boundary* $\partial \tilde{R} = \partial [\![\Sigma]\!]$.

Proof. We note first that both problems admit extremals and it is no loss of generality to assume existence in both problems at once. Indeed, the existence of sets of minimal perimeter with respect to a given outer trace is clear via the direct method, while $[\![\Sigma]\!] \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ implies that $\partial[\![\Sigma]\!] \in \mathscr{B}_{n-2}(\overline{\Omega})$ and [11, 5.1.6(1)] implies the existence of a solution to the parametric obstacle problem on $\overline{\Omega}$. We will proceed to derive identities for both minimizers.

Recall from section 2.2 that a set of finite perimeter $A \subset \mathbb{R}^n$ with outer trace $\bar{\mathrm{T}}(1_A) = 1_\Sigma$ on $\partial\Omega$ fulfills

$$\begin{aligned} \partial(\mathbf{E}^n \llcorner (A \cap (\mathbb{R}^n \backslash \overline{\Omega}))) &= \partial(\mathbf{E}^n \llcorner A) \llcorner (\mathbb{R}^n \backslash \overline{\Omega}) + [\![\Sigma]\!] \\ &= [\![\partial^* A \cap (\mathbb{R}^n \backslash \overline{\Omega})]\!] + [\![\Sigma]\!] \end{aligned}$$

by dualizing the extension formula for sets of finite perimeter with the Hodge star operator and choosing adequate orientations, which yields

$$\partial\left([\![\partial^* A \cap (\mathbb{R}^n \backslash \overline{\Omega})]\!]\right) = -\partial[\![\Sigma]\!].$$

Now, let $R \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ be as in (α). Then

$$R^* := R + [\![\partial^* A \cap (\mathbb{R}^n \backslash \overline{\Omega})]\!]$$

fulfills $\partial R^* = 0$. Hence, by the isoperimetric theorem in $\mathbb{R}^n$ [25, Theorem 7.9.1], there is a bounded $\mathscr{L}^n$-measurable set $\tilde{E} \subset \mathbb{R}^n$ and a rectifiable current $S \in \mathscr{R}_n(\mathbb{R}^n)$ such that

$$\partial S = R^* \qquad \text{and} \qquad S \text{ is concentrated on } \tilde{E}, \text{ ie. } \|S\|(\mathbb{R}^n \backslash \tilde{E}) = 0.$$

Note that we have

$$(\partial S) \llcorner (\mathbb{R}^n \backslash \overline{\Omega}) = [\![\partial^* A \cap (\mathbb{R}^n \backslash \overline{\Omega})]\!] = \partial(\mathbf{E}^n \llcorner A) \llcorner (\mathbb{R}^n \backslash \overline{\Omega}),$$

hence the difference

$$(S - (\mathbf{E}^n \llcorner A)) \llcorner (\mathbb{R}^n \backslash \overline{\Omega}) \in \mathscr{R}_n(\mathbb{R}^n)$$

equals integrating an integer over $\mathbb{R}^n \backslash \overline{\Omega}$ by the constancy theorem [25, Proposition 7.3.1] and the connectedness of the complement of Ω. As both the sets $\tilde{E}$ and A, on which the respective currents are concentrated, are bounded, we infer that the multiplicity is 0 sufficiently far away from the origin and therefore

$$S \llcorner (\mathbb{R}^n \backslash \overline{\Omega}) = (\mathbf{E}^n \llcorner A) \llcorner (\mathbb{R}^n \backslash \overline{\Omega}),$$

since the integer in question has to be 0. Denoting the integer-valued multiplicity function of S by ϑ, we write

$$S = \left(\mathbf{E}^n \llcorner \tilde{E}\right) \llcorner \vartheta$$

and infer via [11, 4.5.17] that

$$R^* = \sum_{k \in \mathbb{Z}} \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}), \quad \|R^*\| = \sum_{k \in \mathbb{Z}} \|\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\})\|,$$

also yielding $\quad \overrightarrow{R^*} = \overrightarrow{\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\})} \quad \|\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\})\|$ -almost everywhere,

due to the locality of approximate tangent spaces [12, Proposition 10.5] and $\{\vartheta \geqslant l\} \subset \{\vartheta \geqslant k\}$ for $l \geqslant k$.

We now drop negative and absorb higher multiplicities and only consider the oriented boundary

$$\mathbf{E}^n \llcorner \{\vartheta \geqslant 1\} \in \mathscr{R}_n(\mathbb{R}^n)$$

with also

$$\partial\left(\mathbf{E}^n \llcorner \{\vartheta \geqslant 1\}\right) \in \mathscr{R}_{n-1}(\mathbb{R}^n)$$

by means of the boundary rectifiability theorem/De Giorgi's theorem. We may assume

$$\{\vartheta \geqslant 1\}\backslash\overline{\Omega} = A\backslash\overline{\Omega}$$

and our previous identity on $\mathbb{R}^n\backslash\overline{\Omega}$ leads to

$$\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant 1\}) = T + [\![\partial^* A \cap (\mathbb{R}^n\backslash\overline{\Omega})]\!]$$

for some current $T \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ with $\operatorname{spt} T \subset \overline{\Omega}$. Due to $\partial^2 = 0$, we have $\partial T = \partial[\![\Sigma]\!]$. As R solves the parametric obstacle problem for $\partial[\![\Sigma]\!]$ and $\overline{\Omega}$, we get

$$\langle F, R\rangle \leqslant \langle F, T\rangle.$$

The mass-additive decomposition also yields the converse inequality and we receive

$$\langle F, R\rangle = \langle F, T\rangle.$$

This means

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \langle F, \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}) \llcorner \overline{\Omega}\rangle &= \langle F, R^* \llcorner \overline{\Omega}\rangle \\
&= \langle F, R\rangle = \langle F, T\rangle \\
&= \langle F, \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant 1\} \llcorner \overline{\Omega}\rangle,
\end{aligned}$$

which is

$$\|\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\})\|(\overline{\Omega}) = 0 \qquad \text{for all} \quad k \neq 1.$$

Hence,

$$R = (\partial S) \llcorner \overline{\Omega} = \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant 1\}) \llcorner \overline{\Omega} = T$$

and we will subsequently relabel $\{\vartheta \geqslant 1\}$ as $\tilde{E}$.

Moreover, for E as in (β), we let

$$R_\rho = \partial(\mathbf{E}^n \llcorner E) \llcorner \mathbf{u}_\rho(\Omega)$$

and notice that since

$$R_\rho - \tilde{R} = \partial(\mathbf{E}^n \llcorner E) \llcorner (\mathbf{u}_\rho(\Omega)\backslash\overline{\Omega})$$

and

$$\langle \mathscr{A}, R_\rho - \tilde{R} \rangle \longrightarrow 0 \qquad \text{as} \qquad \rho \to 0,$$

we infer that $R_\rho \longrightarrow \tilde{R}$ in the weak-* sense. This already yields

$$\langle F, \tilde{R} \rangle \leqslant \liminf_{\rho \to 0} \left(\langle F, \partial(\mathbf{E}^n \llcorner E) \rangle - \langle F, \partial(\mathbf{E}^n \llcorner E) \llcorner (\mathbb{R}^n \setminus \mathbf{u}_\rho(\Omega)) \rangle \right) = \Phi_E(\mathbb{R}^n) - \Phi_A(\mathbb{R}^n \setminus \overline{\Omega})$$

by lower semicontinuity of the parametric integrand F and duality. Now fix some arbitrary $\rho > 0$. As $E \setminus \overline{\Omega} = A \setminus \overline{\Omega}$, we recall that $\bar{\mathrm{T}}(1_E) = 1_\Sigma$. Thus we find also here that

$$\partial \left(R_\rho - \tilde{R} \right) \llcorner \mathbf{u}_\rho(\Omega) = \partial \left(\partial(\mathbf{E}^n \llcorner E) \llcorner (\mathbf{u}_\rho(\Omega) \setminus \overline{\Omega}) \right) \llcorner \mathbf{u}_\rho(\Omega) = -\partial [\![\Sigma]\!],$$

and thus, also $\tilde{R}$ has the correct distributional boundary, since the boundary of $\tilde{R} + (R_\rho - \tilde{R})$ vanishes in $\mathbf{u}_\rho(\Omega)$ while the support of $\tilde{R}$ has to be contained in $\overline{\Omega}$.

We may finally use these identities and compare both expressions for $R, \tilde{R}$ and $E, \tilde{E}$: It holds

$$\begin{aligned} & \langle F, \tilde{R} \rangle + \Phi_A(\mathbb{R}^n \setminus \overline{\Omega}) \leqslant \Phi_E(\mathbb{R}^n) \\ \leqslant \quad & \Phi_{\tilde{E}}(\mathbb{R}^n) = \langle F, R \rangle + \langle F, \partial(\mathbf{E}^n \llcorner \tilde{E}) \llcorner (\mathbb{R}^n \setminus \overline{\Omega}) \rangle \\ \leqslant \quad & \langle F, \tilde{R} \rangle + \langle F, \partial(\mathbf{E}^n \llcorner A) \llcorner (\mathbb{R}^n \setminus \overline{\Omega}) \rangle, \end{aligned}$$

which proves that we actually have equality, ie.

$$\langle F, R \rangle = \langle F, \tilde{R} \rangle \qquad \text{and} \qquad \Phi_E(\mathbb{R}^n) = \Phi_{\tilde{E}}(\mathbb{R}^n),$$

and thus, $\tilde{E}$ is a set of Φ-minimal perimeter and $\tilde{R}$ solves the parametric obstacle problem with respect to F and $[\![\Sigma]\!]$. ■

4.2 Remark. The necessity of a connected boundary is related to homologically minimizing currents also being absolutely minimizing. While the converse is clear, a connected boundary enforces homologically minimizing currents to also be absolutely minimizing in our setting. Local results on homological minimization of minimizing level set boundaries inside Ω among currents furthermore also appeared in [44, 5.(4) Theorem].

4.2 Interior and Boundary Regularity of Level Set Boundaries.

We now turn back to our Lipschitzian extremal $u : \overline{\Omega} \longrightarrow \mathbb{R}$, which is supposed to be a function of least anisotropic gradient with respect to the anisotropic total variation Φ. The following paragraph fixes what kind of data $(\mathscr{D})$ we shall assume.

ASSUMPTIONS $(\mathscr{D})$: *Let $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ be a function of anisotropic least gradient with respect to Φ with $u_{\partial\Omega} = f$, where $\partial\Omega$ is of class C^m and $f \in \mathrm{C}^m(\partial\Omega)$ with $m \geqslant 4$. We shall furthermore suppose that Φ is elliptic and even and Φ and $D_2\Phi$ are of class $\mathrm{C}^{\mu-1,\alpha}$ for some $\mu \geqslant 3$ and $0 < \alpha < 1$.*

We will now discuss regularity issues of our given level sets by considering them as minimizing currents and applying interior and boundary regularity theory for such. Applying the equivalence lemma to interior points will additionally yield a short proof for interior regularity (recall that F inherits the regularity from Φ). This will also provide somewhat of a reconciliation with the original approach by H. Parks in [41, 42, 46] to use slices of the domain (as a normal current) Ω by the lipschitzian extremal u to describe the level sets.

Let us therefore introduce for $t \in \mathbb{R}$ the currents

$$T_t := \partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner \Omega \qquad \text{and} \qquad S_t := \partial(\mathbf{E}^n \llcorner \{u \leqslant t\}) \llcorner \Omega,$$

which furnish the distributional interpretation of the level set boundaries of u. We have $T_t, S_t \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ by definition.

4.3 Remark. We choose some $t \in \mathbb{R}$ such that $\mathscr{H}^{n-1}(f^{-1}(t)) = 0$, which is true for all except countably many. One may show then (see [19, Lemma 2.12]) that

$$\mathrm{T}(1_{\{u \geqslant t\}}) = 1_{\{f \geqslant t\}} \qquad \text{and} \qquad \mathrm{T}(1_{\{u \leqslant t\}}) = 1_{\{f \leqslant t\}},$$

which implies for the distributional boundary of $\{u \geqslant t\}$ in $\mathbb{R}^n$ that

$$\partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) = \partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner \Omega + [\![\{f \geqslant t\}]\!],$$

and therefore, as $\{u \geqslant t\} \subset \overline{\Omega}$ with $\partial\Omega$ being $\mathscr{L}^n$-negligible,

$$\partial(\Omega \llcorner \{u \geqslant t\}) = \partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner \Omega + [\![\{f \geqslant t\}]\!],$$

where we identify Ω with integration over Ω as a current. Solving yields

$$\partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner \Omega = \partial(\Omega \llcorner \{u \geqslant t\}) - [\![\{f \geqslant t\}]\!] = -\langle \Omega, u, t- \rangle$$

and thus, that our definition of the current T_t (and analougously for S_t) precisely equals (up to sign) the *slice of* Ω *by* u of the original approach. Note in particular that here $[\![\{f \geqslant t\}]\!]$ carries the orientation from the interior unit normal $N_{\partial\Omega}$.

We now apply the equivalence lemma to derive suitable minimization properties for T_t and S_t.

4.4 Corollary (Connected Boundaries). *Let* $(\mathscr{D})$ *hold. If* $\partial\Omega$ *is connected and* $t \in \operatorname{rg} f$ *is a regular value for* f*, then the rectifiable currents* T_t *and* S_t *are solutions to the parametric obstacle problem on* $\overline{\Omega}$ *for* F *with boundary* $\partial[\![\{f \geqslant t\}]\!] = -\partial[\![\{f \leqslant t\}]\!]$ *with orientation via the outer trace.*

Proof. As t is a regular value for f, we note that

$$\partial[\![\{f \geqslant t\}]\!] = -\partial[\![\{f \leqslant t\}]\!] = [\![f^{-1}(t)]\!] \in \mathscr{R}_{n-2}(\mathbb{R}^n).$$

and

$$f^{-1}(t) \subset \operatorname{spt} T_t \cap \operatorname{spt} S_t.$$

We find then that the above currents T_t and S_t can be identified with the current in 4.1(β), as they must be supported in $u^{-1}(t)$ and the level set can only intersect $\partial\Omega$ in an $\mathscr{H}^{n-1}$-negligible set. Furthermore, it is easy to show that

$$\mathrm{T}(1_{\{u \geqslant t\}}) = 1_{\{f \geqslant t\}} \qquad \text{and} \qquad \mathrm{T}(1_{\{u \leqslant t\}}) = 1_{\{f \leqslant t\}},$$

since $f^{-1}(t)$ is $\mathscr{H}^{n-1}$-negligible (either directly or also via [19, Lemma 2.12]). In particular, $\{u \geqslant t\}$ and $\{u \leqslant t\}$ are sets of Φ-minimal anisotropic perimeter with respect to these traces. Thus, by setting $\Sigma = \{f \geqslant t\}$ or $\Sigma = \{f \leqslant t\}$, we may apply Proposition 4.1(β). ■

Furthermore, as convex sets always have connected boundary, our corollary 4.4 also includes the regularity results (at least, for a differentiable boundary) of [41] for a regular level set on the boundary.

If the boundary $\partial\Omega$ is possibly disconnected, we will localize to small balls at boundary points to achieve a similar minimality property.

4.5 Corollary (General Boundaries). *Let ($\mathscr{D}$) hold. If $\partial\Omega$ is not necessarily connected, then for each $x_0 \in f^{-1}(t)$ there exists $r > 0$ (depending on x_0) such that T_t or S_t are solutions to the parametric obstacle problem for F on $\mathbf{B}(x_0, r) \cap \overline{\Omega}$.*

Proof. Let $x_0 \in f^{-1}(t)$. As $\partial\Omega$ is locally bilipschitz to a half-plane, we may assume that $\partial(\mathbf{U}(x_0, r) \cap \Omega)$ is a Lipschitz boundary and connected for $r > 0$ sufficiently small and localize the application of proposition 4.2 to such a ball (where the dependence on x_0 is via the local straightening map).

Assume that $\{u \geqslant t\}$ is continued to a superset of $\overline{\Omega}$ by choosing an extension with the appropriate outer trace. Then $\{u \geqslant t\}$ is in particular a set of minimal perimeter with respect to its inner trace on $\partial(\mathbf{U}(x_0, r) \cap \Omega)$ and we may let the outer trace equal the inner one. Hence the current

$$T_t \llcorner (\mathbf{U}(x_0, r) \cap \Omega) \in \mathscr{R}_{n-1}(\mathbb{R}^n)$$

is a solution to the parametric obstacle problem on $\mathbf{B}(x_0, r) \cap \overline{\Omega}$ by proposition 4.1(β). ■

4.6 Remark (Lesser Regularity and Minimization). The above results are not sharp. In fact, I put quite strong assumptions on everything as we will be restrained to such a regular setting for our regularity theory anyway. Suitable generalizations to Lipschitz boundaries and Lipschitz data and other results on minimization via approximation (as in [42]) should be possible, but I do not know how much use they have aside from possible independent interest.

Let us now discuss the regularity properties of level set boundaries.

4.7 Proposition (Interior Regularity of Level Set Boundaries). *Let F and D_2F be of class $\mathrm{C}^{\mu-1,\alpha}$ for $\mu \geqslant 2$ and $0 < \alpha < 1$, where μ can be ∞ or ω. Then for each $t \in \operatorname{rg} u$ there exists a subset of full measure of*

$$(\partial\{u \geqslant t\} \cup \partial\{u \leqslant t\}) \cap \Omega,$$

which is a submanifold of class $\mathrm{C}^{\mu,\alpha}$ without boundary.

Proof. This is a somewhat classical consequence of interior regularity theory for minimizing currents which we sketch here for the sake of completeness. We first deduce that proposition 4.1 can be locally applied to interior points

$$x_0 \in (\partial\{u \geqslant t\} \cup \partial\{u \leqslant t\}) \cap \Omega$$

without any assumption on boundary and boundary data, ie. for each such x_0 there exists

$$\mathbf{B}(x_0, r) \subset\subset \Omega$$

such that the distributional boundary of the sub-/superlevel set in $\mathbf{B}(x_0, r)$ solves the parametric obstacle problem as a current for $\digamma$ on $\mathbf{B}(x_0, r)$. Writing the level set boundary as a current as X, we may require that the approximate tangent plane H at x_0 exists. The proof of [11, 5.3.16] then shows that the cylindrical excess of X over H at x_0 decreases to 0 and $\operatorname{spt} X$ is by [10, 6.1 Theorem] (or any other interior regularity theorem eg. of [11]) locally a (graphical) submanifold of class $\mathrm{C}^{1,\beta}$ near x_0 for some $0 < \beta < 1$ over H and [12, 3.3 Proposition] shows the higher regularity as the function whose graph constitutes the submanifold solves the Euler-Lagrange equation with respect to the associated non-parametric integrand on H. Since an approximate tangent plane to $(\partial\{u \geqslant t\} \cup \{u \leqslant t\}) \cap \Omega$ exists at $\mathscr{H}^{n-1}$-almost every point, the claim follows. ∎

4.8 Remark. Of course, finer characterizations of *the singular set*, ie. the points x_0 as above where the level set boundary is *not* a submanifold, are possible. As we shall be interested in full regularity, we recall that

$$\begin{array}{ll} & \digamma,\ D_2\digamma \text{ of class } \mathrm{C}^{2,1} \text{ and } n \leqslant 3 \\ \text{or} & \digamma \text{ being a Riemannian metric of class } \mathrm{C}^2 \text{ and } n \leqslant 7 \end{array}$$

suffice by the regularity results from [1, Corollary 3.2] or the existence of normal coordinates and the classical Bernstein theorem (to enforce blowups of our oriented boundaries to be hyperplanes) to achieve full interior regularity of the minimizing level set boundaries.

We are now also prepared to prove that the boundary regularity theory in fact applies at all points of $f^{-1}(t)$ and we will provide a proof in multiple steps.

Each of these steps corresponds to a somewhat known result, but might also be interesting in itself or with respect to other conclusions that one might draw. The general idea is taken from [29, Section 3.1-3.3] and we will provide a complete derivation here. It combines ideas from the theory of small excess and closeness to a (half-)hyperplane from [11, 5.3.16] and [4, Page 128-130] with barrier and maximum principles from [20]. We collect some tools for our setting in the next lemma and then proceed with the proof.

4.9 Lemma. *Let $s > 0$ be given and fix $\mathbf{B}(0, s) \cap \{x_n \leqslant 0\} \subset \mathbb{R}^n$.*

(1) If $R \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ is absolutely minimizing with respect to an elliptic constant coefficient integrand and $(\mathbf{B}(0, s), \varnothing)$ and $\operatorname{spt} \partial R \subset \mathbf{B}(0, s) \cap \{x_n \leqslant 0\}$, then $\operatorname{spt} R \subset \mathbf{B}(0, s) \cap \{x_n \leqslant 0\}$.

(2) Let $A \subset \mathbf{B}(0, s) \cap \{x_n \leqslant 0\}$ be a $\mathscr{L}^n$-measurable set such that $R = \partial(\mathbf{E}^n \llcorner A) \llcorner (\mathbf{U}(0, s) \cap \{x_n < 0\})$ is absolutely minimizing with respect to $(\mathbf{B}(0, s) \cap \{x_n \leqslant 0\}, \varnothing)$. Then there exists $\gamma > 0$ such that

$$\langle \mathscr{A}, R \llcorner \mathbf{U}(0, q)\rangle \leqslant \gamma q^{n-1}$$

holds true for all $0 < q < s$.

(3) If $R \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ is such that

$$(\mu_{r_j})_\# R \longrightarrow Q \quad \textit{and} \quad (\mu_{r_k})_\# Q \longrightarrow Q^* \quad \textit{in the local flat metric in } \mathbb{R}^n,$$

for two sequences $r_j, r_k \to 0$ and $Q, Q^* \in \mathscr{R}^{loc}_{n-1}(\mathbb{R}^n)$, then we may find a new sequence $r_l \to 0$ such that

$$(\mu_{r_l})_\# R \longrightarrow Q^* \quad \textit{in the local flat metric in } \mathbb{R}^n.$$

(4) If $(\mu_{r_j})_{\#}R \longrightarrow Q$ for some $Q \in \mathscr{R}^{loc}_{n-1}(\mathbb{R}^n)$ in the local flat metric and there is some $q > 0$ such that

$$R \llcorner \mathbf{U}(0,q) = \partial(\mathbf{E}^n \llcorner A) \llcorner \mathbf{U}(0,q) \qquad \text{for some } \mathscr{L}^n\text{-measurable set } A \subset \mathbb{R}^n,$$

then there is some $\mathscr{L}^n$-measurable set $B \subset \mathbb{R}^n$ with

$$Q = \partial(\mathbf{E}^n \llcorner B).$$

Proof. Assume that (1) is wrong, then we have

$$R \llcorner \{x_n > 0\} \neq 0$$

and we denote by π the convex projection onto $\{x_n = 0\}$. We may assume by possible translation and slicing that $R \llcorner \{x_n > 0\}$ is an integral current in $\mathbb{R}^n$. Note that

$$\partial\left(\pi_{\#}\left(R \llcorner \{x_n > 0\}\right)\right) = \partial\left(R \llcorner \{x_n > 0\}\right)$$

and that, as $\pi_{\#}\left(R \llcorner \{x_n > 0\}\right) \in \mathscr{R}_{n-1}(\mathbb{R}^n)$, its orientation is almost-everywhere equal to $\pm\vec{\mathbf{E}}^{n-1}$. We may find

$$S \in \mathscr{R}_n(\mathbb{R}^n), \qquad \operatorname{spt} S \subset \mathbf{B}(0,s) \cap \{x_n \geqslant 0\}, \qquad \partial S = R \llcorner \{x_n > 0\} - \pi_{\#}(R \llcorner \{x_n > 0\})$$

by the isoperimetric theorem and the convexity of $\mathbf{B}(0,s) \cap \{x_n \geqslant 0\}$. We decompose with the integer-valued multiplicity function ϑ and write

$$\partial S = \sum_{k \in \mathbb{Z}} \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}), \qquad \|\partial S\| = \sum_{k \in \mathbb{Z}} \|\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\})\|$$

and treat each oriented boundary seperately. First write

$$\partial S = \sum_{k \in \mathbb{Z}} \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}) = \sum_{k \in \mathbb{N}} \partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}) + \partial((-\mathbf{E}^n) \llcorner \{\vartheta \leqslant -k\})$$

such that $\{\vartheta \geqslant k\}, \{\vartheta \leqslant -k\} \subset \mathbf{B}(0,s) \cap \{x_n \geqslant 0\}$ for all $k \in \mathbb{N}$, which of course holds similarly for the masses. By assumption, there is $k \in \mathbb{N}$ such that

$$\partial(\mathbf{E}^n \llcorner \{\vartheta \geqslant k\}) \llcorner \{x_n > 0\} \neq 0 \qquad \text{or} \qquad \partial((-\mathbf{E}^n) \llcorner \{\vartheta \leqslant -k\}) \llcorner \{x_n > 0\} \neq 0.$$

Set the non-vanishing oriented boundary to R_k, denote the integrand as Γ and let $c > 0$ be an ellipticity constant, then $R_k \llcorner \{x_n > 0\}$ is absolutely minimizing with respect to $(\mathbf{B}(0,s), \varnothing)$ and $R_k \llcorner \{x_n = 0\}$ has constant orientation (due to $\{\vartheta \geqslant k\}, \{\vartheta \leqslant k\} \subset \mathbf{B}(0,s) \cap \{x_n \geqslant 0\}$). We find

$$0 \geqslant \langle \Gamma, R_k \llcorner \{x_n > 0\} \rangle - \langle \Gamma, (-R_k) \llcorner \{x_n = 0\} \rangle \geqslant c\left(\langle \mathscr{A}, R_k \llcorner \{x_n > 0\} \rangle - \langle \mathscr{A}, (-R_k) \llcorner \{x_n = 0\}\right) \geqslant 0,$$

by minimization with respect to Γ and $\mathscr{A}$ and ellipticity of Γ, hence, we have equalities. Thus

$$\langle \mathscr{A}, R_k \llcorner \{x_n > 0\} \rangle = \langle \mathscr{A}, (-R_k) \llcorner \{x_n = 0\} \rangle,$$

which means that also $R_k \llcorner \{x_n > 0\}$ is absolutely minimizing with respect to $\mathscr{A}$. This contradicts

[21, Corollary 2], as $\operatorname{spt} R_k \llcorner \{x_n > 0\}$ is not contained in $\{x_n = 0\}$.

To prove that (2) is true, we may compare with the trace of A on the topological boundary $\partial(\mathbf{U}(0,q) \cap \{x_n < 0\})$ and find $\gamma > 0$ independent of $0 < q < s$ by minimization. Item (3) follows from a diagonal argument for the local flat metric and item (4) is proven in [11, 5.4.3(7)]. ■

4.10 Theorem (Boundary Regularity of Level Set Boundaries). *Let $t \in \mathbb{R}$ be a regular value for f. Then the sets $\partial\{u \geqslant t\}$ and $\partial\{u \leqslant t\}$ are $\mathscr{H}^{n-1}$-almost everywhere submanifolds with boundary of class $\mathrm{C}^{\nu,\alpha}$ with*

$$\nu := \min\{\mu, m-1\} \geqslant 3$$

with full boundary regularity and whose geometric boundary is given by $f^{-1}(t)$.

Proof. We recall that the interior almost-everywhere regularity is proved in proposition 4.7 and we will now partition the proof for the complete boundary regularity into the following five steps:

Step 1: Reduction to a locally straight problem. We fix some arbitrary $x_0 \in f^{-1}(t)$ for the rest of this proof and recall $x_0 \in \operatorname{spt} T_t \cap \operatorname{spt} S_t$. Since $\partial\Omega$ is of class C^m, there exists $r > 0$ and a diffeomorphism φ of class C^m such that

$$\varphi : \mathbf{U}(x_0, r) \longrightarrow \mathbf{U}(0, r) \qquad \text{and} \qquad \varphi(\partial\Omega \cap \mathbf{U}(x_0, r)) = \{x_n = 0\} \cap \mathbf{U}(0, r).$$

Without loss of generality we assume that

$$\varphi(\Omega \cap \mathbf{U}(x_0, r)) = \{x_n < 0\} \cap \mathbf{U}(0, r).$$

Straightening furthermore inside $\{x_n = 0\}$, we can also assume

$$\varphi(f^{-1}(t) \cap \mathbf{U}(x_0, r)) = \{x_n = x_{n-1} = 0\} \cap \mathbf{U}(0, r)$$

and

$$\varphi(\{f < t\} \cap \mathbf{U}(x_0, r)) = \{x_n = 0,\ x_{n-1} < 0\} \cap \mathbf{U}(0, r).$$

As T_t and S_t solve a parametric obstacle problem for $\digamma$ on some ball $\mathbf{B} \cap \overline{\Omega}$ near x_0, we pursue a proof without loss of generality for T_t and may arrange that

$$T := \varphi_{\#}\left(T_t \llcorner \mathbf{U}(x_0, r)\right)$$

fulfills

$$\left\langle \left(\varphi^{-1}\right)^{\#} \digamma, T \right\rangle \leqslant \left\langle \left(\varphi^{-1}\right)^{\#} \digamma, T + X \right\rangle$$

for all rectifiable currents X such that $\operatorname{spt} X \subset \{x_n \leqslant 0\} \cap \mathbf{B}(0, s)$ with $0 < s < r$ and $\partial X = 0$.
The pull-back integrand $\left(\varphi^{-1}\right)^{\#} \digamma$ obeys

$$\left(\left(\varphi^{-1}\right)^{\#} \digamma\right)(x, p) := \digamma\left(\varphi^{-1}(x), \left(\Lambda_{n-1} D\left(\varphi^{-1}\right)\right)(p)\right)$$

and thus is of class $\mathrm{C}^{\min(m-1,\mu-1)}$ in x and of class C^{μ} in p and it is elliptic by the invariance of ellipticity under diffeomorphisms.
We assume from now on the straightened configuration without relabeling. Thus we fix some $0 < s < r$

and assume that T solves the parametric obstacle problem for F on $\{x_n \leqslant 0\} \cap \mathbf{B}(0,s)$ and denote by $\lambda > 0$ a Lipschitz bound for F on $\mathbf{B}(0,s)$ with

$$\lambda^{-1}\mathscr{A}(p) \leqslant F(x,p) \leqslant \lambda\mathscr{A}(p) \qquad \text{for all } x \in \mathbf{B}(0,s) \text{ and } p \in \Lambda_{n-1}\mathbb{R}^n.$$

Both F and D_2F are (at least) of class C^2 due to $m \geqslant 4$ and $\mu \geqslant 3$.

Step 2: Half-Space Parametric Obstacle Problems are almost-minimizing. We now want to show that T is almost-minimizing for F in the full ball $\mathbf{B}(0,s)$.
To do so, let us fix a rectifiable current X with $\operatorname{spt} X \subset \mathbf{B}(0,s)$ and $\partial X = 0$. We assume that $\operatorname{spt} X \subset K$ for some compact set $K \subset \mathbf{B}(0,s)$ and K is contained in a ball of radius r^*, hence $\operatorname{diam} K \leqslant 2r^*$. Fix any $x^* \in K$ and recall that there exists $S \in \mathscr{R}_{n-1}(\mathbb{R}^n)$ with

$$\operatorname{spt} S \subset \mathbf{B}(0,s), \qquad \partial S = \partial(T \llcorner K), \qquad \langle F_{x^*}, S\rangle \leqslant \langle F_{x^*}, S + X\rangle$$

for all currents X as above by [11, 5.1.6(1)]. As F_{x^*} has constant coefficients and

$$\operatorname{spt} \partial(T \llcorner K) \subset \mathbf{B}(0,s) \cap \{x_n \leqslant 0\},$$

the barrier principle of lemma 4.9(1) implies $\operatorname{spt} S \subset \mathbf{B}(0,s) \cap \{x_n \leqslant 0\}$. Now we may estimate by the respective minimization properties of T and S and Lipschitzianity of F in the first variable that

$$\begin{aligned} \langle F, T \llcorner K\rangle &\leqslant \langle F, S\rangle \leqslant (1 + 2\lambda^2 r^*)\langle F_{x^*}, S\rangle \\ &\leqslant (1 + 2\lambda^2 r^*)\langle F_{x^*}, T \llcorner K + X\rangle \leqslant \left(1 + \left(4\lambda^2 + 2s\lambda^4\right) r^*\right)\langle F, T \llcorner K + X\rangle, \end{aligned}$$

which proves that T is (F,ω)-almost minimizing in $\mathbf{B}(0,s)$ with

$$\omega(r^*) := \left(4\lambda^3 + 2s\lambda^5\right) r^*.$$

Step 3: Blowups and Regular Tangent Currents. We prove the existence of suitable limits of homothetic expansions.

Note first that we may also consider the set $\{u \geqslant t\} \cap \mathbf{U}(0,r)$ (herein after implicitly straightening by means of φ) such that it holds

$$T = \partial(\mathbf{E}^n \llcorner (\{u \geqslant t\}) \llcorner (\mathbf{U}(0,r) \cap \{x_n < 0\})$$

which yields by lemma 4.9(2) that there is some $\gamma > 0$ with

$$\langle \mathscr{A}, T \llcorner \mathbf{U}(0,q)\rangle \leqslant \gamma q^{n-1},$$

for all $0 < q < s$ and γ does not depend on q. Fixing now $R > 0$, we want to apply the compactness theorem for integral currents in a sequence of nested balls with increasing radii. We may choose $q > 0$ small enough for $0 < qR < s$ and compute

$$\langle \mathscr{A}, ((\mu_q)_\# T) \llcorner \mathbf{U}(0,R)\rangle \leqslant \frac{\langle \mathscr{A}, T \llcorner \mathbf{U}(0,qR)\rangle}{q^{n-1}} \leqslant \gamma R^{n-1}.$$

Moreover, as $(\partial T) \llcorner \mathbf{U}(0,r)$ is an oriented hyperspace of codimension two and multiplicity one, we infer

$$\langle \mathscr{A}, \partial((\mu_q)_\# T) \llcorner \mathbf{U}(0,R) \rangle \leqslant \omega_{n-2} R^{n-2}.$$

In particular, whenever $0 < q < s/R$, the masses in $\mathbf{U}(0,R)$ are uniformly bounded. Thus, we deduce the existence of a sequence $q_i \to 0$ such that the homothetic expansions of T converge in the weak-*-sense (and by uniform local boundedness in mass also in the local flat norm on $\mathbb{R}^n$) in $\mathbb{R}^n$ to some locally rectifiable current $Q \in \mathscr{R}^{loc}_{n-1}(\mathbb{R}^n)$ for $i \to \infty$. It also holds

$$\partial Q = [\![\{x_n = x_{n-1} = 0\}]\!], \qquad \operatorname{spt} Q \subset \{x_n \leqslant 0\}, \qquad 0 \in \operatorname{spt} Q,$$

due to blowing up ∂T and since $\operatorname{spt}(\mu_q)_\# T \subset \{x_n \leqslant 0\}$ for all $q > 0$. Possibly extending T across $\{x_n = 0\}$ to an oriented boundary in $\mathbf{U}(0,s)$, we may also assume that Q is of multiplicity one $\mathscr{H}^{n-1}$-almost everywhere due to lemma 4.9(4).

We want to show now that we may suppose Q to be an oriented half-plane of multiplicity one. By Remark 2.9(2) we have that Q is absolutely minimizing in $\mathbb{R}^n$ for the frozen integrand $\digamma_0$. Taking $\{x_n = 0\}$ as the hyperplane in question, we furthermore use the boundary maximum principle for constant coefficient integrands [20, 4.6 Theorem] to infer that either

$$\Theta_{n-1,*}(x_0, Q) \leqslant 1/2 \qquad \text{or} \qquad \operatorname{spt} Q \cap \mathbf{U}(0,q) \subset \{x_n = 0\} \quad \text{for some } q > 0.$$

The first case implies, by application of eg. [10, Theorem 0.1] as a regularity theorem, that Q is given by integration over a manifold with boundary of class C^1 near x_0. The second case implies, by the constancy theorem and multiplicity one, that $Q \llcorner \mathbf{U}(0,q)$ is a distributional half-ball of multiplicity one. In both cases, the homothetic expansions of Q will converge in the local flat norm to a distributional half-hyperplane of multiplicity one and by lemma 4.9(3), we can realize this limit as a limit of homothetic expansions of T.

<u>*Step 4:*</u> *Excess comparision with a Tangent Half-Plane.* We estimate the lower density by 1/2 in form of small cylindrical excess by means of Step 3 and the local proximity to the tangent half-plane. This step is yet again modelled upon [11, Theorem 5.3.16], where we need to adjust for almost-minimizers, a single converging sequence of homothetic expansions to an oriented plane and the boundary case. Therein, the single sequence and almost-minimization are rather clear, while the flat boundary case follows via additional positional arguments.

By the preceding Step 3, there is a real sequence $q_i \longrightarrow 0$ and a weak-*-limit $Q \in \mathscr{R}^{loc}_{n-1}(\mathbb{R}^n)$ of homothetic expansions of T near $x_0 = 0$, which is a distributional multiplicity one half-plane. As we are only interested in working with the half-plane $\operatorname{spt} Q$ in this step, we choose suitable new coordinates for this proof so that now

$$x = (u,v,w), \quad y = (u,v), \quad u \in \mathbb{R}^{n-2}, \quad v \in \mathbb{R}, \quad w \in \mathbb{R},$$

$$p : \mathbb{R}^n \to \mathbb{R}^n, \quad p(u,v,w) = (u,v,0)$$

$$\operatorname{spt} Q = \{w = 0,\ v \leqslant 0\}, \quad \mathbf{Q} = \{w = 0\} \quad \text{and} \quad \operatorname{spt} \partial T = \{w = v = 0\}.$$

We will set

$$T_i := (\mu_{q_i})_\# T$$

and estimate in this step the quantity

$$\operatorname{Exc}(R, 0, \sigma) := \sigma^{-(n-1)} \left(\langle \mathscr{A}, R \llcorner \{|y| < \sigma\} \rangle - \langle \mathscr{A}, (p_\# R) \llcorner \{|y| < \sigma\} \rangle \right)$$

such that there are sufficiently large $i \in \mathbb{N}$ and sufficiently small $\sigma > 0$ with

$$0 \leqslant \operatorname{Exc}(T_i \llcorner \mathbf{Z_Q}(0, \sigma, \sigma), 0, \sigma) < \varepsilon$$

for arbitarily small $\varepsilon > 0$. Then, applying the small excess boundary regularity theorem for almost-minimizers from [10, Theorem 6.1], we find $0 < c < 1$ such that $\operatorname{spt} T_i \cap \mathbf{C_Q}(0, c\sigma, \sigma)$ is a $\mathrm{C}^{1,\beta}$-graph over $\mathbf{U}(0, c\sigma) \cap \operatorname{spt} Q$ for some $0 < \beta < 1$. Hence, by scaling properties of the homothetic expansions, this yields that $\operatorname{spt} T \cap \mathbf{C_Q}(0, cq_i\sigma, q_i\sigma)$ is a $\mathrm{C}^{1,\beta}$-graph over $\mathbf{U}(0, cq_i\sigma) \cap \operatorname{spt} Q$.

Fix furthermore a suitable cylinder

$$\mathbf{Z_Q}(0, 2\sigma, 2\sigma) \subset \mathbf{U}(0, s) \qquad \text{for } \sigma > 0 \text{ sufficiently small.}$$

Then, by remark 2.9(3), there exists for each $0 < \eta < 1$ some sufficiently large $i \in \mathbb{N}$ such that

$$\operatorname{spt} T_i \cap \mathbf{Z_Q}(0, 2\sigma, 2\sigma) \subset \mathbf{Z_Q}(0, 2\sigma, \eta\sigma).$$

Possibly increasing the natural number i, we infer also by the proximity of remark 2.9(3) that the sets $\operatorname{spt} T_i$ have to avoid the set

$$\mathbf{Z_Q}(0, 2\sigma, 2\sigma) \cap \{v > 0\} \qquad \text{since} \qquad \operatorname{spt} Q = \{w = 0,\ v \leqslant 0\}$$

inside $\mathbf{Z_Q}(0, 2\sigma, 2\sigma)$ and we may therefore arrange to choose

$$z \in \{|y| < \sigma,\ v > 0,\ w = 0\} \qquad \text{and} \qquad \gamma > 0 \qquad \text{such that}$$

$$\mathbf{Z_Q}(z, \gamma, 2\sigma) \subset \mathbf{Z_Q}(0, 2\sigma, 2\sigma) \cap \{v > 0\} \qquad \text{with} \qquad \mathbf{Z_Q}(z, \gamma, 2\sigma) \cap \operatorname{spt} T_i = \varnothing.$$

In particular,

$$p_\# \left(T_i \llcorner \mathbf{Z_Q}(0, 2\sigma, 2\sigma) \right) = [\![\{w = 0, v \leqslant 0\}]\!] \llcorner \mathbf{U}(0, 2\sigma)$$

by the constancy theorem since $(\partial T_i) \llcorner \mathbf{U}(0, r)$ is fixed under $p_\#$ and has multiplicity one and the above disjointedness from $\operatorname{spt} T_i$ shows

$$\left(p_\# \left(T_i \llcorner \mathbf{Z_Q}(0, 2\sigma, 2\sigma) \right) \right) \llcorner \mathbf{Z_Q}(z, \gamma, 2\sigma) = 0.$$

Now, passing from 2σ to a number ρ between σ and 2σ and defining $f(x) = |y| = |p(x)|$, we may assume that

$$\langle \mathscr{A}, \langle T_i \llcorner \mathbf{Z_Q}(0, 2\sigma, 2\sigma), f, \rho+ \rangle \rangle \leqslant 2^{n-1} \mu \rho^{n-2},$$

where $\mu > 0$ is as in [11, 5.3.16, page 611] (as the upper density is finite by lemma 4.9(2)) and

$$T_\rho := (T_i \llcorner \mathbf{Z_Q}(0, 2\sigma, 2\sigma)) \llcorner \{|y| \leqslant \rho\} = T_i \llcorner \mathbf{Z_Q}(0, \rho, \rho) \quad \text{(by choice of } \eta)$$

is hence an integral current in $\mathbb{R}^n$ by the definition of a slice and the boundary rectifiability theorem.

Further, we basically use the same comparison procedure as in [11, page 611, last lines]. Denoting h the affine homotopy from the projection p to the identity, we observe

$$\partial(h_\#([0,1] \times \partial T_\rho) - (T_\rho - p_\# T_\rho)) = 0, \qquad \langle \mathscr{A}, \partial T_\rho \rangle \leqslant (2^{n-1}\mu + \omega_{n-2})\rho^{n-2}$$

together with

$$\langle \mathscr{A}, h_\#([0,1] \times \partial T_\rho) \rangle \leqslant \eta\, \rho \langle \mathscr{A}, \partial T_\rho \rangle$$

by the homotopy formula.

The chain of inequalities is now exactly the same as in [11, page 612] together with

$$\mathbf{Z_Q}(0,\rho,\rho) \subset \mathbf{B}\left(0, 2\sqrt{2}\sigma\right) \qquad \text{by} \quad \sigma \leqslant \rho \leqslant 2\sigma$$

for the almost-minimization. Clearly we have

$$\langle \mathscr{A}, T_i \llcorner \mathbf{C_Q}(0,\rho,\rho) \rangle \leqslant \langle \mathscr{A}, T_i \llcorner \mathbf{Z_Q}(0,\rho,\rho) \rangle$$

by which one can subsequently mimick the steps from [11, page 622] together with almost-minimization in the compact set $\mathbf{Z_Q}(0,\rho,\rho)$, scaling laws for the cylindrical excess and uniform boundedness of the rescaled integrands, for which T_i is almost-minimizing. We receive an upper bound for $\operatorname{Exc}(T_i \llcorner \mathbf{Z_Q}(0,\sigma,\sigma), 0, \sigma)$ which decays to zero with σ and η and finish the proof of step 4 together with T_i projecting down to a half-ball.

Step 5: Higher Regularity Conclusion. We will finally transform back to our original configuration and use elliptic regularity theory for quasilinear equations to prove the claimed boundary regularity.

By the steps 2,3 and 4 and the boundary regularity theorem of [10, Theorem 0.1], there is some open neighborhood V of 0 in $\mathbf{U}(0,r)$ such that $\operatorname{spt} T \cap V$ is given by a submanifold with (flat) boundary of class $\mathrm{C}^{1,\beta}$ for some $\beta \in (0,1)$. Hence, by mapping back with φ^{-1}, we infer that we may write $\partial\{u \geqslant t\}$ near x_0 as the graph of a function v of class $\mathrm{C}^{1,\beta}$, which is defined on a small (half-)ball $\mathbf{B}_-$ in the tangent half-plane to $\partial\{u \geqslant t\}$ at x_0.

Since the interior of $\mathbf{B}_-$ is mapped to the interior of Ω, we infer that v has to solve the associated Euler-Lagrange equation on $\operatorname{int} \mathbf{B}_- = \mathbf{U}_-$ and hence has interior regularity $v \in (\mathrm{C}^{\mu,\alpha} \cap \mathrm{W}^{2,2})(\mathbf{U}_-)$. By assumption, v is of class $\mathrm{C}^{m-1,\alpha}$ on the flat part of $\partial\, \mathbf{B}_-$, hence we may choose a subdomain $D \subset \mathbf{U}_-$ of class $\mathrm{C}^{\nu,\alpha}$, which touches the flat part at x_0 with an open neighborhood of ∂D, and assume

$$v_{\partial D} \in \mathrm{C}^{\nu,\alpha}(\partial D).$$

Linearizing and boundary regularity theory for v on D (in particular, boundary Schauder estimates, eg. [13, Corollary 8.35] with bootstrapping) now yield that

$$v \in \mathrm{C}^{\nu,\alpha}(\overline{D}),$$

which proves the desired regularity for $\partial\{u \geqslant t\}$ near $x_0 \in f^{-1}(t)$ and the proof is finished by noticing arbitrariness. ■

4.11 Remark. Obviously, theorem 4.10 does not need that we are actually dealing with level sets of a function and a proof is pursued somewhat similarly for arbitrary sets of minimal (anisotropic) perimeter. We refer to [29] for more details in the isotropic case and a finer analysis. Moreover, particularily step 4 of its proof can be applied much more general in arbitrary dimension and codimension, but I found it hard to find proper documentation in the literature to refer to.

4.3 Geometric Properties and Local Non-Parametric Representations.

We now prepare the results on the deformation of our reference hypersurface by first discussing how certain quantities related to the minimizing hypersurfaces vary as the hypersurfaces vary and prove preliminary results on uniform non-parametric (ie. graphical) representations on the same tangent space. By using such regularity theory for minimizing currents, this empowers us to vary the value of u and infer that the level sets of u are still controlled in higher order than for a generic Lipschitzian function.

Assumption ($\mathscr{S}$): *From this subsection on, we will assume that the singular set of locally F-minimizing currents in codimension one is empty and that either $\partial\Omega$ is connected or F is independent of the spatial variable x.*

We recall that sufficient conditions for ($\mathscr{S}$) to hold are given in remark 4.8.

Thus, let us first introduce the main class of values that we want to consider in what follows.

4.12 Definition (Mildly Regular Values)**.** We will say that t is *a mildly regular value* when $t \in \mathcal{U}$, where

$$\begin{aligned} \mathcal{U} := \{\, t \in \operatorname{rg} f \quad | \quad & 0 < \mathscr{H}^{n-1}(u^{-1}(t)) < \infty, \\ & Du \text{ exists almost everywhere on } u^{-1}(t) \cap \Omega, \\ & t \text{ is a regular value of } f.\}. \end{aligned}$$

4.13 Remark. (1) Recall from remark 2.18 and [11, 3.2.15] that the coarea formula for lipschitzian functions implies that the first two items are generically (ie. on a set of full $\mathscr{L}^1$-measure) fulfilled in $\operatorname{rg} f$.

(2) Moreover, the second item in the definition of $\mathcal{U}$ especially implies

$$\partial\{u \geqslant t\} = \partial\{u \leqslant t\} = u^{-1}(t)$$

for $t \in \mathcal{U}$ as now also the level set boundaries coincide on the boundary.

(3) Note that we may enforce the third item on a set of full measure with the theorem of Sard.

(4) Any level set $u^{-1}(t) \subset \overline{\Omega}$ with $t \in \mathcal{U}$ is everywhere a submanifold where geometrically interior points allow for local embeddings of class $\mathrm{C}^{\mu,\alpha}$ while boundary points allow for class $\mathrm{C}^{\nu,\alpha}$ due to proposition 4.7 and theorem 4.10.

We will single out one level set of a mildly regular value and our reference hypersurface will without loss of generality be $u^{-1}(0) \subset \overline{\Omega}$ with $0 \in \mathcal{U}$ and we shall write

$$\Xi := u^{-1}(0).$$

The assumption $0 \in \mathcal{U}$ *will pertain through the sections 4-8.*

4.14 Remark. We also recall the following elementary facts from differential topology:

(1) The regularity of boundary level sets close to $f^{-1}(0)$ is *stable* : There exists

$$T > 0 \qquad \text{such that} \qquad f^{-1}(t) \text{ is regular for all } 0 \leqslant |t| \leqslant T.$$

This is due to $f^{-1}(t) = \partial\{f \geqslant t\} = \partial\{f \leqslant t\}$ (relative to $\partial\Omega$) and the Hausdorff convergence of level sets of f in this case (which may be proven by applying lemma 2.17 to $f^{-1}(0)$ on $\partial\Omega$).

(2) We furthermore recall that $f^{-1}(0) \subset \partial\Omega \subset \mathbb{R}^n$ may only have finitely many connected components, as compactness would allow otherwise for an accumulation point, where $f^{-1}(0)$ could not be appropriately diffeomorphic.

We observe now that remark 4.14(2) directly implies the same topological consequence for the level set structure of u by means of a geometric maximum principle.

4.15 Proposition (Geometric Maximum Principles)**.** *The following three statements about oriented boundaries of* $\{u \geqslant t\}$ *and* $\{u \leqslant t\}$ *inside* Ω *hold true for a function* $u \in \mathrm{C}^0(\overline{\Omega})$ *of anisotropic least gradient in any dimension* $n \in \mathbb{N}$*:*

(1) No connected components which are compactly contained in Ω *may exist.*

(2) If t *is a regular value for* f*, then the number of connected components of* $\partial\{u \geqslant t\}$ *and* $\partial\{u \leqslant t\}$ *is finite and each connected component has non-empty geometric boundary.*

(3) We have $\operatorname{rg} u = \operatorname{rg} f$*.*

Proof. This is a consequence of assumption ($\mathscr{S}$) and the boundary regularity of $\partial\{u \geqslant t\}$ and $\partial\{u \leqslant t\}$.

If $\partial\Omega$ is connected, then the maximum principle from [24, Lemma 4.2] implies (1). If the total variation Φ has constant coefficients, then assume for a contradiction that there would be some compactly in Ω contained component $S \subset \partial\{u \geqslant t\}$. Compactness ensures that we may find $x_0 \in S \cap \Omega$ such that

$$\alpha := \operatorname{proj}_1(x_0) = \max_{x \in S} \operatorname{proj}_1(x).$$

This implies that the whole connected component is contained in the half-space $\{\operatorname{proj}_1(x) \leqslant \alpha\}$ with $x_0 \in \{\operatorname{proj}_1(x) = \alpha\}$. Now, proposition 4.1 assures that for any sufficiently small $r > 0$ it follows that

$$T := \partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner \mathbf{B}(x_0, r) = [\![\partial\{u \geqslant t\} \cap \mathbf{B}(x_0, r)]\!] \in \mathscr{R}_{n-1}(\mathbb{R}^n)$$

is a solution to the parametric obstacle problem with respect to the associated parametric integrand F in $\mathbf{B}(x_0, r)$ and, in particular, absolutely minimizing inside $\mathbf{U}(x_0, r)$ among rectifiable currents without boundary. Furthermore, the existence of some $\eta > 0$ such that

$$\langle \mathscr{A}, T \llcorner \mathbf{U}(x_0, s) \rangle \leqslant \eta s^{n-1} \qquad \text{for all } 0 < s < r$$

is a consequence of a standard computation (see eg. [1, Equation (33)]) through cutting out/pasting balls. We infer for $R > 0$ and $0 < sR < r$ that

$$\langle \mathscr{A}, T_s \llcorner \mathbf{U}(0,R) \rangle = \frac{\langle \mathscr{A}, T \llcorner \mathbf{U}(x_0, sR)}{s^{n-1}} \leqslant \eta R^{n-1},$$

if

$$T_s := (\tau_{-x_0} \circ \mu_s \circ \tau_{x_0})_{\#} T$$

denotes the homothetic expansion by s^{-1} after shifting x_0 to the origin and shifting back. Thus, by the local finiteness of the mass and the compactness theorem applied to the homothetic expansions with a diagonal argument, we may choose some $s_i \to 0$ and $Q \in \mathscr{R}^{loc}_{n-1}(\mathbb{R}^n)$ such that

$$T_{s_i} \longrightarrow Q \qquad \text{in the weak-*-sense in } \mathbb{R}^n$$

and Q is absolutely minimizing with respect to $\digamma$ in $\mathbb{R}^n$ with $\partial Q = 0$. Moreover, by local Hausdorff convergence of the supports, it must follow that $x_0 \in \operatorname{spt} Q$ and $\operatorname{spt} Q \subset \{\operatorname{proj}_1(x) \leqslant \alpha\}$. Hence, the interior maximum principle [20, 4.4 Lemma] is applicable to imply that there is some $q > 0$ with

$$\operatorname{spt} Q \cap \mathbf{U}(x_0, q) \subset \{\operatorname{proj}_1(x) = \alpha\}.$$

The constancy theorem implies furthermore that

$$\operatorname{spt} Q \cap \mathbf{U}(x_0, q) = \{\operatorname{proj}_1(x) = \alpha\} \cap \mathbf{U}(x_0, q)$$

and, by possibly blowing-up a second time, we may assume that

$$\operatorname{spt} Q = \{\operatorname{proj}_1(x) = \alpha\}.$$

We infer by the regularity theorems (and yet again [11, 5.3.16]) that T is regular near x_0 and, as we may now express $S \subset \partial\{u \geqslant t\}$ as a graph over $\operatorname{spt} Q$ near x_0, it follows from the Harnack inequality and the fact that Φ has constant coefficients that there is also some $q^* > 0$ such that

$$S \cap \mathbf{U}(x_0, q^*) = \{\operatorname{proj}_1(x) = \alpha\} \cap \mathbf{U}(x_0, q^*).$$

It is now easy to see that, using a simple connectedness argument with respect to the hyperspace, also

$$S \supset \{\operatorname{proj}_1(x) = \alpha\}$$

as $S \cap \{\operatorname{proj}_1(x) = \alpha\}$ is both open and closed in $\{\operatorname{proj}_1(x) = \alpha\}$, which is impossible by compactness of $\partial\{u \geqslant t\}$. This proves (1) also for the second case of $(\mathscr{S})$ of constant coefficients.

For item (2) we notice that $\Xi \cap \partial\Omega = f^{-1}(0)$ and hence, Ξ is a submanifold with boundary at each point of its intersection with $\partial\Omega$. Since $f^{-1}(0)$ has only finitely many connected components, there can be only finitely many connected components of Ξ with non-empty geometric boundary. A connected component without geometric boundary would however be compactly contained in Ω, which is impossible by (1).

Item (3) finally is also implied by (1): If $\partial\Omega$ is connected, then connectedness of Ω directly implies the claim while for a total variation with constant coefficients we may not have compactly contained level sets in Ω, which also enforces $\operatorname{rg} u = \operatorname{rg} f$ by continuity of u. ■

4.16 Remark. The conclusions of proposition 4.15 will usually appear as technical tools while dealing with the particular reference level set. The last item is besides of specific importance, since we want to access the level sets of u through the level sets of f.

4.17 Remark. Observe that the proof of proposition 4.15(1) for constant coefficient integrands crucially needs that hyperspaces are local minimizers for a total variation with constant coefficients, as otherwise the maximum principle and the local containment in the hyperspace near x_0 would fail. While this is true as a matter of fact for all constant-coefficient elliptic integrands, the paper [24, Proposition 1.1 ff] provides an example for a weighted integrand with $\partial\Omega$ being disconnected and the conclusion of proposition 4.15(1) fails.

We now proceed to establish local first and higher order properties of the involved hypersurfaces.

To describe these, we will gather some of the regularity results about locally expressing *a converging sequence* of hypersurfaces as graphs over *the same hyperplane*, which will also follow, in essence, from the general small excess and Lipschitz graph approximation theory for minimizing surfaces and currents.

Preparing the next two representation lemmata, we notice for $q > 0$ that if

$$M \cap \mathbf{Z}(0,q,q) = \operatorname{graph} u \qquad \text{for} \qquad u : \mathbf{U}^{n-1}(0,q) \to \mathbb{R} \qquad \text{of class } \mathrm{C}^1,$$

such that $0 \in M$ and $T_0 M = \mathbb{R}^{n-1}$, then we have

$$\nabla u(0) = 0 \qquad \Longrightarrow \qquad \sup_{\mathbf{B}(0,s)} |\nabla u| \longrightarrow 0 \quad \text{as } s \searrow 0.$$

Hence, for all $\varepsilon > 0$ there is $s > 0$ with

$$|u(x)| = |u(x) - u(0)| \leqslant \sup_{\mathbf{B}(0,s)} |\nabla u||x| \leqslant \varepsilon s$$

for all $x \in \mathbf{B}(0,s)$. Geometrically, this yields

$$M \cap \mathbf{Z}(0,s,s) \subset M \cap \mathbf{Z}(0,s,\varepsilon s),$$

which will be applied to the smooth limit hypersurface and holds analogously in the boundary case over half-ball or by rotation and translation for $x_0 \neq 0$ and $T_{x_0} M \neq \mathbb{R}^{n-1}$.

4.18 Lemma (Local Non-Parametric Representations near Interior Points)**.** *For each $x_0 \in \Xi \cap \Omega$ with $H := T_{x_0}\Xi$ there exists $r = r(x_0) > 0$ and $T = T(x_0) > 0$ with the following properties:*

(1) We can write

$$\partial\{u \geqslant t\} \cap \mathbf{C}_H(x_0,r,r) \qquad \textit{and} \qquad \partial\{u \leqslant t\} \cap \mathbf{C}_H(x_0,r,r)$$

locally as graphs over

$$\mathbf{U}(x_0,r) \cap H \qquad \textit{for all} \quad 0 \leqslant |t| \leqslant T.$$

(2) The real-valued graph functions are of class $\mathrm{C}^{\mu,\alpha}$ *on* $\mathbf{U}(x_0, r) \cap H$ *and will be denoted by* u_t^+ *and* u_t^- *respectively. It holds*

$$u_t^+ \longrightarrow u_0 \qquad \text{and} \qquad u_t^- \longrightarrow u_0$$

uniformly on $\mathbf{U}(x_0, r) \cap H$ *as* $t \to 0$.

(3) We may furthermore assume that

$$\|u_t^+ - u_0\|_{\mathrm{C}^{\mu,\gamma}} \longrightarrow 0 \qquad \text{and} \qquad \|u_t^- - u_0\|_{\mathrm{C}^{\mu,\gamma}} \longrightarrow 0$$

on $\mathbf{U}(x_0, r) \cap H$ *for all* $0 < \gamma < \alpha$ *as* $t \to 0$.

Proof. We repeat a direct proof by means the usual regularity theorems for minimizing currents and shift each hypersurface into the same point. Indeed, since Ξ is of class $\mathrm{C}^{\mu,\alpha}$ at x_0 we may choose $s > 0$ small enough to find

$$\Xi \cap \mathbf{Z}_H(x_0, s, s) \subset \mathbf{Z}_H\left(x_0, s, \frac{\varepsilon s}{4}\right)$$

where $\varepsilon > 0$ is to be chosen. By the Hausdorff convergence of lemma 2.17, we may furthermore assume

$$\partial\{u \geqslant t\} \cap \mathbf{Z}_H(x_0, s, s) \subset \mathbf{Z}_H\left(x_0, s, \frac{\varepsilon s}{2}\right)$$

for sufficiently small $|t|$ and there is $x_t \in \partial\{u \geqslant t\}$ with $x_t \longrightarrow x_0$ as $t \to 0$. We may shift each hypersurface by the difference $x_0 - x_t$ and denote the translation map τ_t and thus we get

$$\tau_t\left(\partial\{u \geqslant t\}\right) \cap \mathbf{Z}_H(x_0, s, s) \subset \mathbf{Z}_H(x_0, s, \varepsilon s)$$

for sufficiently small $|t|$ from which we may again as in [11, 5.3.16] prove uniformly small cylindrical excess over H with adequate $\varepsilon > 0$.

Clearly τ_t converges uniformly to the identity and $\tau_t(x_t) = x_0$. On a distributional level, this yields that the supports of all oriented boundaries after pushing-forward with $(\tau_t)_\#$ are close to H (with x_0 being element of their support) in $\mathbf{C}_H(x_0, s, s)$ while each shifted oriented boundary is locally minimizing for the correspondingly shifted integrand. Since clearly all structural constants of the shifted integrands may be assumed uniformly bounded, the regularity theorem [10, Theorem 6.1] implies that *the shifted supports* are graphs over the same ball in H. Shifting back via τ_t^{-1} (that is, $x_t - x_0$) and denoting the convex projection onto H by π, it is immediate from [10, Theorem 6.1] that there are functions

$$u_t : H \cap \mathbf{U}\left(\pi(x_t), \frac{s}{34}\right) \longrightarrow \mathbb{R} \qquad \text{such that} \qquad \operatorname{graph}(u_t) = \partial\{u \geqslant t\} \cap \mathbf{C}_{\tau_t^{-1}(H)}\left(x_t, \frac{s}{34}, \varepsilon s\right).$$

Clearly as

$$\pi(x_t) \longrightarrow x_0 \quad \text{as } t \to 0, \qquad \text{there exists} \qquad r = r(x_0) > 0$$

such that

$$\mathbf{U}(x_0, r) \cap H \subset \mathbf{U}(\pi(x_t), s) \cap H$$

for all sufficiently small $|t|$ and item (1) follows by additionally replacing ε for eg. $\min(\varepsilon, 1/68)$ and choosing r sufficiently large.

Subsequently, we notice that there is $C > 0$ not depending on $|t|$ such that

$$\|u_t\|_{\mathrm{C}^{1,\beta}} \leqslant C \qquad \text{for all sufficiently small } |t| \text{ and some } \beta \in (0,1)$$

due to the regularity theorem and each u_t is again of class $\mathrm{C}^{\mu,\alpha}$ by minimization (see again [12, Theorem 3.3] for the regularity of the graph functions). The standard technique of applying interior Schauder estimates to the partial derivatives of u_t thus leads to

$$\|u_t\|_{\mathrm{C}^{\mu,\gamma'}} \leqslant C \qquad \text{for all sufficiently small } |t| \quad \text{with } \gamma' = \alpha\beta$$

and hence

$$\|u_t\|_{\mathrm{C}^{\mu,\alpha}} \leqslant C \qquad \text{for all sufficiently small } |t|$$

as a fortiori the $\mathrm{C}^{1,\alpha}$-norm of u now will be uniformly bounded. The Hausdorff convergence further yields

$$\|u_t - u_0\|_{\mathrm{C}^0} \longrightarrow 0 \qquad \text{for } t \to 0,$$

and this implies

$$\|u_t - u_0\|_{\mathrm{C}^{\mu,\gamma}} \longrightarrow 0 \qquad \text{for } t \to 0 \quad \text{for all } 0 < \gamma < \alpha,$$

which concludes the proof. ■

Such a uniform representation of the converging sequence of minimizing currents on the tangent space of the limit appears to be essential if one is interested in any kind of stability and deformation result.

We will finally also prove a corresponding statement for boundary points, which is somewhat more specific to our setting as we shall now also use the regular level set structure of the boundary data to nicely rearrange our configuration and prove a uniform representation after applying a straightening diffeomorphism to $\partial\Omega$.

4.19 Lemma (Local Non-Parametric Representations Near The Boundary). *Let $x_0 \in f^{-1}(0) = \partial\Xi$. Then there exist $T = T(x_0) > 0$ and $s = s(x_0) > 0$, a closed half-hyperplane $H \subset \mathbb{R}^n$ with full plane H^* and functions $u_t^{\pm} \in \mathrm{C}^{2,\alpha}(\mathbf{U}(x_0, r) \cap H)$ with*

$$\|u_t^+ - u_0\|_{\mathrm{C}^{2,\gamma}} \longrightarrow 0 \qquad \text{and} \qquad \|u_t^- - u_0\|_{\mathrm{C}^{2,\gamma}} \longrightarrow 0$$

on $\mathbf{U}(x_0, s) \cap H$ for all $0 < \gamma < \alpha$ as $t \to 0$ such that, up to applying a diffeomorphism φ of class C^m and linear shifts along a fixed direction, we have

$$\partial\{u \geqslant t\} \cap \mathbf{C}_{H^*}(x_0, s, s) = \operatorname{graph} u_t^+, \qquad \partial\{u \leqslant t\} \cap \mathbf{C}_{H^*}(x_0, s, s) = \operatorname{graph} u_t^-$$

for all $|t| \leqslant T$.

Proof. Similar to the boundary regularity result in theorem 4.10, we need to rearrange our configuration and we may assume, up to applying a diffeomorphism φ of class C^m, that $x_0 = 0$ and an $r > 0$ that

$$\partial\Omega \cap \mathbf{U}(0, r) = \{x_n = 0\} \qquad \text{and} \qquad \Omega \cap \mathbf{U}(0, r) = \{x_n < 0\}.$$

As $f^{-1}(0)$ is regular, it is a standard consequence that the level sets form a foliation of class C^m near

$f^{-1}(0)$. Thus, we may also assume, up to altering and applying φ, that

$$f^{-1}(t) \cap \mathbf{U}(0,r) = \{x_{n-1} = t,\ x_n = 0\} \cap \mathbf{U}(0,r)$$

for sufficiently small $|t|$. Also, similar to the proof of theorem 4.10, we may assume that the rectifiable currents

$$T_t = \partial(\mathbf{E}^n \llcorner \{u \geqslant t\}) \llcorner (\Omega \cap \mathbf{U}(0,r))$$

are almost-minimizing with respect to the elliptic parametric integrand F', which is the push-forward of F by φ, and some function $\omega(r^*) := Cr^*$. Yet again, we infer that the new integrand F' and D_2F' are at least of class C^2.

Let now $0 < \rho < r/2$ such that

$$\mathbf{Z}_{\mathbf{Q}}(0,\rho,\rho) \subset\subset \mathbf{U}(0,r) \qquad \text{over the hyperplane} \qquad \mathbf{Q} := \{x_{n-1} = 0\}$$

and denote

$$\tau_t(x) := x - te_{n-1} \qquad \text{for } |t| \text{ sufficiently small.}$$

We shift the currents T_t parallel to the level set foliation to consider $(\tau_t)_\# T_t$ and infer

$$\partial((\tau_t)_\# T_t) \llcorner \mathbf{C}_{\mathbf{Q}}(0,\rho,\rho) = [\![\{x_n = x_{n-1} = 0\}]\!] \llcorner \mathbf{C}_{\mathbf{Q}}(0,\rho,\rho)$$

and we may assume that

$$\operatorname{spt}((\tau_t)_\# T_t) \longrightarrow \Xi \qquad \text{for } t \to 0$$

in the Hausdorff distance as the Hausdorff convergence from lemma 2.17 holds still true after diffeomorphically deforming and τ_t converges uniformly to the identity as $t \to 0$. In what follows, we drop the push-forward $(\tau_t)_\#$ from notation.

By theorem 4.10, it holds that Ξ is a submanifold with boundary at x_0 and we may consider its tangent half-plane H at 0 with corresponding full plane H^*. It immediately follows that

$$\{x_{n-1} = x_n = 0\} \subset H \subset \{x_n \leqslant 0\}.$$

Fix $\varepsilon > 0$, then we can choose $\sigma > 0$ small enough to arrange

$$\mathbf{Z}_{H^*}(0,\sigma,\sigma) \subset \mathbf{Z}_{\mathbf{Q}}(0,\rho,\rho) \qquad \text{and} \qquad \Xi \cap \mathbf{Z}_{H^*}(0,\sigma,\sigma) \subset \mathbf{Z}_{H^*}\left(0,\sigma,\frac{\varepsilon\sigma}{2}\right)$$

and therefore we shall assume

$$\operatorname{spt} T_t \cap \mathbf{Z}_{H^*}(0,\sigma,\sigma) \subset \mathbf{Z}_{H^*}(0,\sigma,\varepsilon\sigma) \qquad \text{for all sufficiently small } |t|$$

by Hausdorff convergence. Thus, by arguing as in step 4 of the proof of theorem 4.9 and similar as before for the interior case of non-parametric representations with uniform boundedness of structural constants of the shifts of F' and by choosing $\varepsilon > 0$ sufficiently small, we may write

$$\operatorname{spt} T_t \cap \mathbf{C}_{H^*}(0,s,s) = \operatorname{graph}(u_t)$$

for some sufficiently small $s > 0$ where

$$u_t : \mathbf{U}(0, s) \cap H \longrightarrow \mathbb{R},$$

and the radius $s > 0$ does not depend on t. The family of functions u_t is (at least) of class $\mathrm{C}^{2,\alpha}$ on $\mathrm{int}(\mathbf{U}(0, s) \cap H)$ as they solve the Euler-Lagrange equation for (shifted versions of) $\mathcal{F}'$ due to interior minimization and the higher elliptic regularity theory and we again have

$$\|u_t - u_0\|_0 \longrightarrow 0 \qquad \text{as} \qquad t \to 0.$$

The remainder of the proof now concludes as before as in [12, Proposition 3.3] by possibly choosing a smaller half-ball inside $\mathbf{U}(0, s) \cap H$ and applying the local boundary $\mathrm{C}^{1,\alpha}$-Schauder estimate [13, Corollary 8.36] to the derivatives of u_t for uniform bounds. ■

5 *Convergence of Deformations of Level Set Boundaries.*

This section will now use the information from the local non-parametric representations of section 4.2 to conclude for more general deformation results.

Our objectives are twofold: To get rid of the necessary locality of graphical representations about tangent spaces, we now want to express the converging sequence of oriented boundaries *as a graph in varying normal direction* about Ξ and not locally about some tangent space of Ξ. It will turn out that this is possible at interior points of Ξ and thus, with the aid of a subsequent "patching together"-argument, up to some small distance to $\partial\Omega$.
The second goal will then be to continue the interior representation as a deformation in normal direction up to the boundary. For this matter, we do not require the specific form as a normal graph anymore.

A rather recent account, which we will keep to, on how and what exactly the local expressions as graphs imply can be found in the paper [9] (albeit their setting and variational problems are slightly different and less differentiable). Our treatment here for the global deformations, which appears to be lesser known, will use the very general construction from [9, Section 3] and adapt it to our purposes.

5.1 Deforming as a Normal Graph over a Retract.

The previous non-parametric representations express pieces of the converging hypersurfaces as local graphs about a tangent space, which, in a sense, fix the normal vector in whose direction we deform. To also deform Ξ (and not a tangent space of Ξ), we need to vary the normal vector along Ξ and we will introduce a variant of a *nearest point projection* for this matter. We finally recall that we still assume $0 \in \mathcal{U}$, referring to definition 4.12, in this section and the following ones.

5.1 Definition (A Nearest-Point Projection)**.** We recall that Ξ is a submanifold with boundary of $\mathbb{R}^n$ of class at least $\mathrm{C}^{3,\alpha}$. We choose an open submanifold $\tilde{\Xi} \subset \mathbb{R}^n$ of class at least $\mathrm{C}^{3,\alpha}$ such that $\tilde{\Xi}$ extends Ξ across $\partial\Xi$. Then we denote

$$\Pi : W \longrightarrow \tilde{\Xi}, \qquad \Pi \in \mathrm{C}^{2,\alpha}(W),$$

as *the nearest-point projection onto* Ξ, where $W \subset \mathbb{R}^n$ is a some open neighborhood of Ξ.

The existence of such an open manifold is standard due to the definition of a submanifold with boundary. Furthermore, this definition Π depends on the extension that we chose, when we are sufficiently close to $\partial\Xi$, while it does not, when we are sufficiently far away.

Due to the Hausdorff convergence of lemma 2.17, there exists

$$T > 0 \qquad \text{such that} \qquad \Pi \text{ is well-defined on } u^{-1}(t) \text{ for all } 0 \leqslant |t| \leqslant T$$

and we recall that Π is described via the formula

$$\Pi(x) = x - \delta(x) N_{\tilde{\Xi}}(\Pi(x)) = x - \delta(x) D\delta(x)$$

with δ being *the nearest-point distance* to $\tilde{\Xi}$.

We will use such a projection Π to investigate how normal vectors to level set boundaries vary before and after projection. As the normal vector to Ξ and $T_{x_0}\Xi$ coincides by definition at $x_0 \in \Xi \cap \Omega$ and the minimizing oriented boundaries converge up to higher order due to lemma 4.18(3) near x_0, it is reasonable to expect that we may also invert Π in a graphical manner near x_0.

5.2 Lemma (Invertability of Π Along the Central Ray)**.** *Let $x_0 \in \Xi \cap \Omega$, $H := T_{x_0}\Xi$ and $r(x_0), T(x_0) > 0$ be given via lemma 4.18. Then the normal ray above H from x_0 may only intersect each level set boundary once in $\mathbf{C}_H := \mathbf{C}_H(x_0, r(x_0), r(x_0))$ and there are open neighborhoods $U_t^{\pm}$ of $x_0 + u_t(x_0)N_H$ and V of x_0 in the level set boundary and Ξ respectively such that*

$$\Pi : U_t^{\pm} \longrightarrow V$$

is an diffeomorphism of class $\mathrm{C}^{2,\alpha}$ for all sufficiently small $|t|$.

Proof. Otherwise shrinking $\mathbf{C}_H$ or intersecting with a subset, we shall assume that Π is defined on a neighborhood of $\mathbf{C}_H$. As $\partial\{u \geqslant t\}$ is a graph of $u_t := u_t^+$ inside $\mathbf{C}_H$, it is clear that

$$\mathbf{R} = x_0 + \mathbb{R}N_{\Xi}(x_0)$$

fulfills

$$\mathbf{R} \cap \partial\{u \geqslant t\} \cap \mathbf{C}_H(x_0, r(x_0), r(x_0)) = \{u_t(x_0)\}$$

and thus furthermore

$$\Pi(u_t(x_0)) = x_0 \qquad \text{for all sufficiently small } |t|.$$

Differentiating the formula for Π for $x \notin \Xi$ yields

$$D\Pi(x) = \mathrm{Id} - \left(D\delta(x) \otimes D\delta(x) + \delta(x)D^2\delta(x)\right),$$

which we shall estimate by means of our previous geometric convergences.

We first infer that there is $M > 0$ such that $D^2\delta$ is bounded in norm by M on a sufficiently large neighborhood of $\mathbf{C}_H$, as the signed distance is of class C^2 on such a neighborhood and δ may only differ by sign. Then we let v be a tangent vector at $x_t := u_t(x_0)$ to $\partial\{u \geqslant t\}$ of unit norm and compute

$$\begin{aligned} |D\Pi(x_t)v| &= \left|v - \left((N_{\Xi}(x_0) \cdot v)\, N_{\Xi}(x_0) + \delta(x_t)D^2\delta(x_t)v\right)\right| \\ &\geqslant 1 - (|N_{\Xi}(x_0) \cdot v| + \delta(x_t)M) \\ &= 1 - (\left|(N_{\Xi}(x_0) - N_{\partial\{u \geqslant t\}}(x_t)) \cdot v\right| + \delta(x_t)M) \\ &\geqslant 1 - (|N_{\Xi}(x_0) - N_{\partial\{u \geqslant t\}}(x_t)| + \delta(x_t)M). \end{aligned}$$

Via Hausdorff convergence and convergence of the normal vectors at the central point, we may hence choose $|t|$ small enough to arrange for

$$|D\Pi(x_t)v| \geqslant 1/2 \qquad \text{for all } |t| \text{ small enough and } v \in T_{x_t}\partial\{u \geqslant t\}$$

and therefore a uniform lower bound on the unit sphere of the map

$$D\Pi(u_t(x_0)) : T_{x_t}\partial\{u \geqslant t\} \longrightarrow T_{x_0}\Xi.$$

Thus, using the uniform graphical coordinates on $H \cap \mathbf{C}_H$ and the uniform modulus of continuity of $D\Pi$ on each $\partial\{u \geqslant t\} \cap \mathbf{C}_H$ for all sufficiently small $|t|$ and close to the origin x_0, the (quantitative, see eg. [26, XIV, §1, Lemma 1.3]) inverse function theorem implies the existence of open neighborhoods V, U_t with $x_0 \in V \subset \Xi$ and $x_t \in U_t \subset \partial\{u \geqslant t\}$ such that

$$\Pi : U_t \longrightarrow V \qquad \text{is a diffeomorphism of class } \mathrm{C}^{2,\alpha},$$

which was the claim. ■

5.3 Remark. Note that, as is easily observed by plugging the inverse into the formula for Π, that there are real-valued functions $w_t^{\pm} \in \mathrm{C}^{2,\alpha}(V)$ such that

$$\Pi_{t,\pm}^{-1}(x) = x + w_t^{\pm}(x)N_{\Xi}(y) = (\mathrm{Id} + w_t^{\pm}N_{\Xi})(x),$$

where $\Pi_{t,\pm}^{-1} : V \longrightarrow U_t$ is the local inverse to Π. Of course, the family of functions $w_t^{\pm}$ is possibly of higher class than $\mathrm{C}^{2,\alpha}$, depending on the interior regularity of the level set boundaries and hence on the anisotropic total variation Φ, but this regularity is sufficient for our purposes here.

5.4 Definition. For $x_0 \in \Xi \cap \Omega$ and $s > 0$ sufficiently small, we denote as

$$\mathbf{X}(x_0, s) := \{x \in \mathbb{R}^n \mid x = y + qN_{\Xi}(y),\ \|y - x_0\| < s,\ |q| < s,\ y \in \Xi\} \supset \Xi \cap \mathbf{U}(x_0, s)$$

the *local tubular open neighborhood of* Ξ at x_0.

The following lemma facilitates subsequently the passage from local non-parametric representations about x_0 along the fixed normal direction to local normal graphs on Ξ with correspondingly varying normal direction. We will hence also pass from cylinders (ie. "constant" tubular neighborhoods over the tangent plane) to the varying ones $\mathbf{X}(x_0, s)$.

5.5 Lemma (LOCAL NORMAL GRAPH REPRESENTATIONS)**.** *The following two statements about the family of functions $w_t^{\pm}$ of class (at least) $\mathrm{C}^{2,\alpha}$ are true:*

(1) There are $s = s(x_0) > 0$ and $T = T(x_0) > 0$ such that

$$(\mathrm{Id} + w_t^{+}N_{\Xi})(\mathbf{X}(x_0, s) \cap \Xi) = \partial\{u \geqslant t\} \cap \mathbf{X}(x_0, s)$$

and

$$(\mathrm{Id} + w_t^{-}N_{\Xi})(\mathbf{X}(x_0, s) \cap \Xi) = \partial\{u \leqslant t\} \cap \mathbf{X}(x_0, s)$$

for $|t| \leqslant T$.

(2) We furthermore have

$$\|w_t^{-}\|_0, \|w_t^{+}\|_0 \longrightarrow 0 \qquad \text{as} \qquad t \to 0.$$

Proof. Clearly, $\mathbf{X}(x_0, s) \cap \Xi$ is connected and contained in Ω for $s > 0$ sufficiently small and we may assume by remark 5.3 that the family of functions $w_t^{\pm}$ is well-defined on $\mathbf{X}(x_0, s) \cap \Xi$ for all sufficiently small $|t|$. Considering the case $w_t := w_t^{+}$, we also obviously have

$$(\mathrm{Id} + w_tN_{\Xi})(\mathbf{X}(x_0, s) \cap \Xi) \subset \partial\{u \geqslant t\} \cap \mathbf{X}(x_0, s) \subset \Omega.$$

Now, the Hausdorff convergence $u^{-1}(t) \longrightarrow \Xi$ implies that all oriented boundaries $\partial\{u \geqslant t\}$ and $\partial\{u \leqslant t\}$ have to be fully contained in small neighborhoods of Ξ, yielding

$$w_t \longrightarrow 0 \qquad \text{uniformly on } \mathbf{X}(x_0, s) \cap \Xi \text{ as} \qquad t \to 0.$$

Thus, (1) is proven if we can show that actually equality holds in the above set inclusion, assume this is not the case. In this case, the set $\partial\{u \geqslant t\} \cap \mathbf{X}(x_0, s)$ would necessarily have more than one connected component. We call another connected component Z and infer that Π is also locally invertible at each point of Z. Due to an easy argument with the connectedness of $\Xi \cap \mathbf{X}(x_0, s)$, we find $\Pi(Z) = \Xi \cap \mathbf{X}(x_0, s)$. In particular, at the central point $x_0 \in \Xi$ and due to $\mathbf{X}(x_0, s) \subset \mathbf{C}_H(x_0, r(x_0), r(x_0))$, it follows that

$$\operatorname{card} \mathbf{R} \cap \mathbf{C}_H(x_0, r(x_0), r(x_0)) \cap \partial\{u \geqslant t\} \geqslant 2$$

which contradicts lemma 5.2. ■

The latter lemma 5.5 differs from its non-parametric version inasmuch as we have only proven convergence with respect to the uniform topology. To fix this matter as well as to prepare the subsequent elliptic regularity results, we will now relate both kinds of deformations.

5.6 Remark. We recall some useful formulas from [9, Lemma 4.3, Proof] about the family of normal deformations $\mathrm{Id} + w_t^{\pm} N_\Xi$ expressed in the local non-parametric coordinates on $H := T_{x_0}\Xi$.
Indeed, by first exploiting the usual translation and rotation methods as everything is purely local, we may assume that

$$H = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n, \qquad x_0 = 0, \qquad N_\Xi(x_0) = e_n,$$

and we only vary in direction of the last coordinate. Now, abbreviating $\mathbf{C} := \mathbf{C}_H(x_0, r(x_0), r(x_0))$ and $r := r(x_0)$, we obtain

$$\Xi \cap \mathbf{C} = \{(z, u_0(z)) \mid z \in \mathbf{U}^{n-1}(0, r)\} \qquad \text{and} \qquad \partial\{u \lesseqgtr t\} \cap \mathbf{C} = \{(z, u_t^{\pm}(z)) \mid z \in \mathbf{U}^{n-1}(0, r)\}.$$

Correspondingly, we receive for the normal deformations that

$$(\mathrm{Id} + w_t^{\pm} N\Xi)(z, u_0(z)) = (z, u_0(z)) + w_t^{\pm}(z, u_0(z)) \frac{(-\nabla u_0(z), 1)}{\sqrt{1 + \nabla u_0(z)^2}}$$

for all $z \in \mathbf{U}^{n-1}(0, r)$ close enough to 0 such that $(z, u_0(z)) \in \mathbf{X}(x_0, s)$. We may neglect the dependant coordinate and write without loss of generality $w_t^{\pm}(z) = w_t^{\pm}(z, u_0(z))$. Definining then

$$\mathfrak{W}_t^{\pm}(z, q) = u_t^{\pm}\left(z - q\frac{\nabla u_0(z)}{\sqrt{1 + \nabla u_0(z)^2}}\right) - \left(u_0(z) + \frac{q}{\sqrt{1 + \nabla u_0(z)^2}}\right)$$

on the convex projection of $\mathbf{X}(x_0, s)$ onto H (say, some smaller ball $\mathbf{U} \subset \mathbf{U}^{n-1}(0, r)$) and sufficiently small real numbers, we use

$$(\mathrm{Id} + w_t^{\pm} N_\Xi)(z, u_0(z)) \in \partial\{u \lesseqgtr t\} \cap \mathbf{C}$$

and that the latter set is graphical over $\mathbf{U}$ to easily observe that

$$\mathfrak{W}_t^{\pm}(z, w_t^{\pm}(z)) = 0 \qquad \text{for all} \qquad z \in \mathbf{U}, \quad |t| \text{ sufficiently small},$$

ie. the family of functions $w_t^{\pm}$ solves an implicit equation. We will sometimes refer to the expressions of this remark as *canonical coordinates.*

5.7 Remark. To lighten notation, in what follows we will conventionally drop signs and consider only the level set boundary $\partial\{u \geqslant t\}$ and all associated functions. Clearly, everything will hold similarly for $\partial\{u \leqslant t\}$.

5.8 Corollary (HIGHER ORDER CONVERGENCE OF NORMAL DEFORMATIONS). *We may assume, up to shrinking $s = s(x_0) > 0$ from lemma 5.5, that*

$$\|w_t\|_{2,\gamma,\Xi\cap\mathbf{X}(x_0,s)} \longrightarrow 0 \qquad \text{as} \qquad t \to 0$$

for all $0 < \gamma < \alpha$.

Proof. In view of the uniform convergence to the zero function and the Arzela-Ascoli theorem, it suffices to show that the norms of class $\mathrm{C}^{2,\alpha}$ are bounded uniformly in $|t|$ to show the result and we shall assume canonical coordinates and use remark 5.6. Note also that it suffices to show uniform boundedness for w_t interpreted as a function on $\mathbf{U} \subset \mathbb{R}^{n-1}$.
As $\mu \geqslant 3$, the family of functions $\mathfrak{W}_t$ is of class $\mathrm{C}^{2,\alpha}$ and, owing to the uniform bounds of lemma 4.18(3), we deduce the existence of $C > 0$ with

$$\|\mathfrak{W}_t\|_{2,\alpha} \leqslant C$$

and C is independent of $|t|$. Since each w_t solves an implicit equation defined via $\mathfrak{W}_t$, this furthermore yields (with a possibly different C) that

$$\|w_t\|_{2,\alpha} \leqslant C$$

for all sufficiently small $|t|$. ■

In our last result here we now patch these local representations as normal graphs together via the local uniqueness up to a small distance towards the boundary $\partial\Omega$. This step is more or less taken from [9, Theorem 4.12].

5.9 Proposition (NORMAL GRAPHS OVER A DOMAIN RETRACT). *Let $\rho > 0$ be sufficiently small, then there is $T = T(\rho) > 0$ and functions*

$$w_t : \Xi \cap \Omega_\rho \longrightarrow \mathbb{R}$$

such that

$$\partial\{u \geqslant t\} \cap \Omega_{2\rho} \subset (\mathrm{Id} + w_t N_\Xi)(\Xi \cap \Omega_\rho) \subset \partial\{u \geqslant t\} \cap \Omega_{2^{-1}\rho}$$

for all $|t| \leqslant T$ and with

$$\|w_t\|_{2,\gamma,\Xi\cap\Omega_\rho} \longrightarrow 0 \qquad \text{as} \qquad t \to 0$$

for all $0 < \gamma < \alpha$.

Proof. Here we use the idea that the local normal graph representations must necessarily coincide in intersections of their domains of definition. As $\Xi \cap \overline{\Omega}_\rho$ is compact, we may find finitely many

$$x_i \in \Xi \cap \overline{\Omega}_\rho, \qquad s_i := s(x_i), \qquad w_{t,i} : \mathbf{X}(x_i, s_i) \longrightarrow \mathbb{R}$$

for all $|t|$ sufficiently small as in lemma 5.5 such that

$$\Xi \cap \overline{\Omega}_\rho \subset \bigcup_i \mathbf{X}(x_i, s_i).$$

Applying Hausdorff convergence inside $\overline{\Omega}_\rho$ moreover implies that

$$\partial\{u \geqslant t\} \cap \overline{\Omega}_\rho \subset \left(\bigcup_i \mathbf{X}(x_i, s_i) \right) \cap \{\delta(x) < s\}$$

whenever $|t|$ is small enough, ie. such that additionally the (normal) distance δ to Ξ is bounded by $s := \min_i s_i$. As a consequence of the representation as a normal graph, each normal ray can only intersect each hypersurface of the sequence once inside $\bigcup_i \mathbf{X}(x_i, s_i)$ and, if we choose $|t|$ small enough, we may hence infer that

$$w_{t,i} = w_{t,j} \qquad \text{on} \qquad \mathbf{X}(x_i, s_i) \cap \mathbf{X}(x_j, s_j) \cap \Xi$$

and thus, the family of mappings

$$w_t \qquad \text{via} \qquad w_t = w_{t,i} \quad \text{on} \quad \mathbf{X}(x_i, s_i) \cap \Xi$$

is well-defined and of class $\mathrm{C}^{2,\alpha}$ on $\Xi \cap \Omega_\rho$ and such that

$$\|w_t\|_{2,\gamma,\Xi\cap\Omega_\rho} \longrightarrow 0 \qquad\qquad \text{as} \qquad t \to 0$$

for all $0 < \gamma < \alpha$. We also have

$$(\mathrm{Id} + w_t N_\Xi) \left(\bigcup_i \mathbf{X}(x_i, s_i) \right) = \partial\{u \geqslant t\} \cap \bigcup_i \mathbf{X}(x_i, s_i)$$

for all such $|t|$ small enough. Finally, for the first inclusion, we apply Hausdorff convergence to additionally assume that $\partial\{u \geqslant t\} \cap \overline{\Omega}_{2\rho}$ is contained uniformly in a neighborhood $\mathbf{u}_\lambda(\Xi \cap \overline{\Omega}_{2\rho})$ for sufficiently small $\lambda > 0$ such that $\Pi(\partial\{u \geqslant t\} \cap \overline{\Omega}_{2\rho}) \subset \Xi \cap \Omega_\rho$ and we use $\|w_t\|_{0,\Xi\cap\Omega_\rho} \longrightarrow 0$ to uniformly control the distance to $\partial\Omega$. ■

5.10 Remark. The previous proofs are conceptually somewhat easier than the original strategy in [46, 2.4 Theorem] to construct the normal graph representations, which we are mainly interested in for blowing up the family w_t. However, this method *cannot* as easily be continued up to $\partial\Omega$ due to the need for (geometrically) interior points.

5.2 Uniform Estimates on Normal Vectors.

Our next goal, also with respect to the application of section 2.6 which needs *global deformations*, is to continue the deformation diffeomorphically up to the boundary of Ξ. We begin here with two lemmas regarding the uniformity of normal vectors to our oriented boundaries and we abbreviate for this matter

$$N_{\partial\{u\geqslant t\}} = N_t.$$

5.11 Lemma (UNIFORM BOUNDS FOR NORMAL VECTORS). *There exist $T > 0$ and $L > 0$ such that N_t is uniformly bounded in $\mathrm{C}^{1,\alpha}$, ie.*

$$\|N_t\|_{\mathrm{C}^{1,\alpha}(\partial\{u \geqslant t\})} \leqslant L$$

for each $0 \leqslant |t| \leqslant T$.

Proof. We claim first that the two non-parametric representation lemmata 4.18 and 4.19 imply the existence of finitely many open and bounded convex sets $V_i \subset \mathbb{R}^n$ which cover Ξ and real numbers $L_i > 0$ such that

$$\|N_t\|_{\mathrm{C}^{1,\alpha}(\partial\{u \geqslant t\} \cap V_i)} \leqslant L_i.$$

To prove the claim, let us fix $x_0 \in \Xi$ and choose canonical coordinates at x_0. If u_t again denotes the family of non-parametric representations on some ball about 0 in $\{x_n = 0\}$, then we may define

$$\psi_t(x, y) = u_t(x) - y \qquad \text{where} \qquad x \in \mathbb{R}^{n-1},\ y \in \mathbb{R} \quad \text{are close to } 0.$$

Letting φ denote the diffeomorphism of class C^m which maps a neighborhood of x_0 into canonical coordinates, we find a neighborhood $V \subset \mathbb{R}^n$ of x_0 such that for each $z \in V$ we have, up to neglecting a uniform translation in case of boundary canonical coordinates,

$$z \in \partial\{u \geqslant t\} \cap V \qquad \Longleftrightarrow \qquad (\psi_t \circ \varphi)(z) = 0.$$

Hence, each level set boundary is locally described as the zero set of the function $\psi_t \circ \varphi$ and its unit normal can be computed (up to a choice of sign) by the normalized gradient of $\psi_t \circ \varphi$. The proof of the claim is subsequently finished since φ does not depend on t and ψ_t can be bounded uniformly in $\mathrm{C}^{2,\alpha}$ due to the Schauder estimates from the lemmata 4.18 and 4.19 and by compactness of Ξ, finitely many such neighborhoods V_i and diffeomorphisms φ_i suffice to cover it.

Then, choosing $\lambda > 0$ sufficiently small, we shall assume

$$\mathbf{e}_\lambda(\Xi) \cap \overline{\Omega} \subset \bigcup_i V_i$$

and by Hausdorff convergence, we furthermore arrange for

$$\partial\{u \geqslant t\} \subset \mathbf{e}_\lambda(\Xi) \cap \overline{\Omega} \qquad \text{for all sufficiently small } |t|.$$

Exploiting the Lebesgue covering lemma with respect to the compact set $\mathbf{e}_\lambda(\Xi) \cap \overline{\Omega}$, we find $\delta > 0$ such that the implication

$$x, y \in \partial\{u \geqslant t\},\ \|x - y\| < \delta \qquad \Longrightarrow \qquad \text{There exists an } i \text{ with } x, y \in V_i$$

holds for all sufficiently small $|t|$. Thus, a standard estimate, where we distinguish between distance larger and equal or smaller than δ for the Hölder bound, implies the uniform bound in $\mathrm{C}^{1,\alpha}$ for all such sufficiently small $|t|$. ■

5.12 Lemma (UNIFORM TAYLOR ESTIMATES FOR NORMAL VECTORS). *We may also assume that, by* *possibly refining our choices of $T > 0$ and $L > 0$ from the previous lemma, that*

$$\left| N_x(t) \cdot (y-x) + \frac{1}{2}(y-x)^T DN_x(t)(y-x) \right| \leqslant L\|y-x\|^{2+\alpha}$$

and

$$\|N_y(t) - N_x(t) - DN_x(t) \cdot (y-x)\| \leqslant L\|y-x\|^{1+\alpha}$$

holds for all $x, y \in \partial\{u \geqslant t\}$ and $|t|$ sufficiently small.

Proof. We use the same finite cover $\bigcup_i V_i$ from interior or boundary non-parametric representations with $\Xi \subset \bigcup_i V_i$ as in the last proof together with the maps

$$\Psi_t \equiv \Psi_{t,i} := \psi_t \circ \varphi_i$$

which locally represent the level set boundaries (up to possible translation) as their zero sets. Implicitly fixing some i first, we find

$$(0 \neq D\Psi_t) \in N_t, \qquad |D\Psi_t| \geqslant C > 0$$

uniformly in $|t|$ for some $C > 0$. Then Taylor's formula and the convexity of each V_i immediately implies

$$|D\Psi_t(x)(y-x)| \leqslant C\|y-x\|^{1+\alpha},$$

for all $x, y \in \partial\{u \geqslant t\} \cap V_i$ which leads to

$$|N_x(t)(y-x)| = \left| \frac{D\Psi_t(x)}{|D\Psi_t(x)|}(y-x) \right| \leqslant C\|y-x\|^{1+\alpha},$$

with constants $C > 0$ independent of $|t|$. Computing the derivative of N by means of $D\Psi_t$, we will show the asserted estimates inside V_i with the aid of the above estimate. We first compute

$$D\left(\frac{D\Psi_t}{|D\Psi_t|}\right) = DN_t = -\frac{1}{|D\Psi_t|^3}\left(D^2\Psi_t D\Psi_t\right) \otimes \frac{D\Psi_t}{|D\Phi_t|} + \frac{1}{|D\Psi_t|} D^2\Psi_t$$

and see that the seecond inequality of the lemma follows immediately from the fact that the vector fields $D\Psi_t/|D\Psi_t|$ are of class $\mathrm{C}^{1,\alpha}$ with uniformly bounded norms while

$$\begin{aligned}
\left| N_x(t) \cdot (y-x) + \frac{1}{2}(y-x)^T DN_x(t)(y-x) \right| &\leqslant \left| \left(\frac{D\Psi_t(x)}{|D\Psi_t(x)|} + \frac{1}{2}(y-x)^T \frac{D^2\Psi_t(x)}{|D\Psi_t(x)|} \right)(y-x) \right| \\
&\quad + \left| \frac{1}{2}(y-x)^T \frac{(D^2\Psi_t(x) D\Psi_t(x))}{|D\Psi_t(x)|^3} \frac{D\Psi_t(x)}{|D\Psi_t(x)|} \cdot (y-x) \right| \\
&\leqslant C \left| \left(D\Psi_t(x) + \frac{1}{2}(y-x)^T D^2\Psi_t(x) \right)(y-x) \right| \\
&\quad + C\|y-x\| \left| \frac{D\Psi_t(x)}{|D\Psi_t(x)|}(y-x) \right| \\
&\leqslant C\|y-x\|^{2+\alpha} \qquad \text{for all} \quad x, y \in \partial\{u \geqslant t\} \cap V_i
\end{aligned}$$

holds true where we used the uniform estimate of class $\mathrm{C}^{1,\alpha}$ from above and the uniform Taylor estimate of class $\mathrm{C}^{2,\alpha}$ on the family of functions Ψ_t. This yields the claim locally for all sufficiently small $|t|$ inside each neighborhood V_i. We then conclude this proof similarly to the proof of lemma 5.11 by using the compactness of Ξ with a uniform Lebesgue number argument and the uniform boundedness of normals from lemma 5.11. ■

These additional estimates on the normals will serve as an ingredient to prove the existence of $C^{2,\alpha}$-deformations of Ξ into nearby hypersurfaces. This is due to the fact that [9] only provides the existence of $C^{1,\alpha}$-diffeomorphisms and therefore only uses estimates up to lower order while we will need two times continuously Hölder differentiable for our desired application.

5.13 Remark. In fact, the lemmata 5.11 and 5.12 prove that the normals and their derivatives constitute *a Whitney jet* of class $C^{2,\alpha}$ (see [9, 2.3 Whitney's Extension Theorem]). By compactness and akin to [9, Remark 3.4], it follows rather quickly that each normal obeys such estimates, but we have to walk the extra mile as we are interested *in uniform bounds with respect to t* for the Whitney jets such that we may produce uniformly bounded extensions in $C^{2,\alpha}$.

5.3 The Regular Level Set Flow of the Dirichlet Data.

In this section we introduce the tools to work with the regular level set structure on the boundary more precisely. We will first recall some facts from elementary differential topology on how to relate regular level sets and afterwards consider the corresponding immersions in appropriate function spaces. These deformations of the boundary manifolds $f^{-1}(t) \subset \partial\Omega$ are also the last ingredient for subsequently producing *global* diffeomorphisms between level set boundaries.

We assume that $0 \in \mathrm{rg}(f)$ is a regular value of f. To relate with the dependence map $\mathfrak{F}$ in a closed form, we need to find an appropriate one-parameter family of parametrizations of level sets "near" $f^{-1}(0)$ in the appropriate topology. We shall consider the rescaled gradient vector field

$$X := \frac{\mathrm{grad}^{\partial\Omega} f}{\left|\mathrm{grad}^{\partial\Omega} f\right|^2},$$

where this gradient is understood as the tangential gradient on $\partial\Omega$ with respect to the restriction of the Euclidean metric. Clearly, X is well-defined near $f^{-1}(0)$ and we may consider its local flow κ near $f^{-1}(0)$. Thus, there is some open neighborhood $W \subset \partial\Omega$ of $f^{-1}(0)$ and

$$T > 0 \qquad \text{such that the flow} \qquad \kappa : W \times (-T, T) \longrightarrow \partial\Omega$$

of X is well defined. Fix some $x \in W$, then we compute

$$\frac{d}{dt} f(\kappa(x,t)) = Df(\kappa(x,t)) \left(\frac{\mathrm{grad}^{\partial\Omega} f}{\left|\mathrm{grad}^{\partial\Omega} f\right|^2} \right) = \mathrm{grad}^{\partial\Omega} f \cdot \frac{\mathrm{grad}^{\partial\Omega} f}{\left|\mathrm{grad}^{\partial\Omega} f\right|^2} = 1,$$

which leads, by the fundamental theorem of calculus, to

$$f(\kappa(x,t)) - f(\kappa(x,0)) = \int_0^t d\tau = t$$

and which is

$$f(\kappa(x,t)) = t + f(x).$$

Hence, $\kappa(x,t) \in f^{-1}(t)$ for all $x \in f^{-1}(0)$ and thus, by standard flow properties, $\kappa_t := \kappa(.,t)$ diffeomorphicly maps $f^{-1}(0)$ into the level set $f^{-1}(t)$ for all sufficiently small $|t|$. We record the following result:

5.14 Lemma (Diffeomorphicity of Regular Level Sets)**.** *There is $T > 0$ such that we have that*

$$\kappa_t : \left(f^{-1}(0) \subset \mathbb{R}^n\right) \longrightarrow \mathbb{R}^n$$

is a diffeomorphism onto $f^{-1}(t)$ of class C^{m-1} for all $0 \leqslant |t| \leqslant T$ with

$$\|\kappa_t - \kappa_0\|_{2,\gamma,f^{-1}(0)} \longrightarrow 0$$

for all $0 < \gamma \leqslant 1$ as $t \to 0$.

Proof. Invertability follows from the above while it remains to notice that X is a vector field of class C^{m-1} near $f^{-1}(0)$ and κ is at least of class C^3 in both its variables. ∎

5.15 Remark. (1) We furthermore recall for any fixed $x \in W$ and small $|t|$ that the mapping

$$t \longmapsto \kappa(x,t)$$

is *actually m times continuously differentiable in t*, since the flow equation yields

$$\partial_t \kappa = X \circ \kappa.$$

(2) The flow κ is a diffeomorphism when restricted to $f^{-1}(0) \times (-T,T)$, since an inverse is given by the map

$$z \longmapsto (\kappa(z, -f(z)), f(z))$$

and it hence embeds sufficiently small neighborhoods of $f^{-1}(0) \times \{0\}$ into $W \subset \partial\Omega$.

We will now consider the one-parameter family of diffeomorphisms κ_t as a curve in appropriate function spaces on $f^{-1}(0)$.

As sketched before, we need to consider the function κ, which is a function of two variables, as a parametrized curve of one variable in a Hölder space of the other variable. Thus, we will consider

$$(t \in (-T,T)) \longmapsto \kappa_t.$$

Since each κ_t is of class C^{m-1}, one can consider this as a curve in each Hölder space up to class $\mathrm{C}^{m-2,\alpha}$ for any $0 < \alpha < 1$. To find the most convenient one, we shall choose an arbitrary α and suppose

$$t \longmapsto \left(\kappa_t \in \mathrm{C}^{2,\alpha}(f^{-1}(0); \mathbb{R}^n)\right).$$

Note that we moreover have that

$$\|\cdot\|_{2,\alpha,f^{-1}(0)} \leqslant C\|\cdot\|_{3,f^{-1}(0)}$$

due to $f^{-1}(0) \subset \mathbb{R}^n$ being compact and we recall from uniform continuity (up to class C^3) that $t \longmapsto \kappa_t$ is a continuous curve in $C^{2,\alpha}(f^{-1}(0); \mathbb{R}^n)$. Choosing one particular $s \in (-T,T)$, we can furthermore estimate

$$\left\| \frac{\kappa_{s+h} - \kappa_s}{h} - \partial_t \kappa_s \right\|_{2,\alpha} \leqslant C \left\| \frac{\kappa_{s+h} - \kappa_s}{h} - \partial_t \kappa_s \right\|_3 \longrightarrow 0 \quad as \quad h \to 0,$$

because both κ and $\partial_t \kappa$ are of class C^{m-1} with $m-1 \geqslant 3$ and we can infer the convergence of difference quotients to the derivative holds uniformly due to uniform continuity and the fact that we may consider the difference quotient as an evaluation of $\partial_t \kappa$ by the one-dimensional real mean value theorem. Indeed, fixing $\varepsilon > 0$ we may find by uniform continuity $\delta > 0$ such that

$$|\partial_t \kappa_s(x) - \partial_t \kappa_{s'}(x)| = |\partial_t \kappa(x,s) - \partial_t \kappa(x,s')| < \varepsilon$$

for all $x \in f^{-1}(0)$ whenever $|s - s'| < \delta$. By the mean value theorem, we may infer

$$\frac{\kappa(x, s+h) - \kappa(x,s)}{h} = \partial_t \kappa(x, s + \theta_x h)$$

with $\theta_x \in (0,1)$ for each $x \in f^{-1}(0)$ and small enough $|h| < \delta$. Hence, we receive

$$\left| \frac{\kappa(x, s+h) - \kappa(x,s)}{h} - \partial_t \kappa(x,t) \right| < \varepsilon$$

for all $x \in f^{-1}(0)$ and all derivatives can be handled in essentially the same way by uniform continuity. Since additionally

$$s \longmapsto \partial_t \kappa_s$$

is continuous in the norm of class $C^{2,\alpha}$, we deduce that the curve

$$s \longmapsto \kappa_s$$

is of class C^1 in the Hölder space $C^{2,\alpha}(f^{-1}(0), \mathbb{R}^n)$.

We can iterate this step until we run out of regularity, employing also remark 5.15(1), to subsequently prove that:

5.16 Lemma (Regular Level Set Flow on The Boundary). *There exists $T > 0$ such that it holds*

$$(t \longmapsto \kappa_t) \in C^{m-3}\left((-T,T); C^{2,\alpha}(f^{-1}(0); \mathbb{R}^n).\right)$$

for any $0 < \alpha < 1$ and $m \geqslant 4$. ■

5.17 Remark. Note moreover that the case $m = \infty$ is clearly also covered and in this case we can infer that

$$(t \longmapsto \kappa_t) \in C^{\infty}\left((-T,T); C^{2,\alpha}(f^{-1}(0); \mathbb{R}^n)\right).$$

It is not clear to me whether the case $m = \omega$ is also included like that.

At last we record a technical lemma about the co-normals of level set boundaries at boundary points which also readily follows from the uniform non-parametric representations after reverting the straightening.

5.18 Lemma (Uniform Convergence of Co-Normals)**.** *We have*

$$\|N_t^{co} \circ \kappa_t - N_\Xi^{co}\|_{0,f^{-1}(0)} \longrightarrow 0 \qquad as \qquad t \to 0$$

on $f^{-1}(0)$.

Proof. We fix $x_0 \in f^{-1}(0)$ and first assume boundary canonical coordinates in the form of lemma 4.19. Thus, all oriented boundaries are given as translated graphs of some functions u_t about some small half-ball in the half-hyperplane $\{z_n = 0,\ z_{n-1} \leqslant 0\}$. We recall that the restriction of the graph to $\{z_n = z_{n-1} = 0\}$ parametrizes each boundary manifold up to translation, but since each boundary is flat in canonical coordinates, we infer

$$\partial_i u_t \equiv 0 \qquad \text{on } \{z_n = z_{n-1} = 0\} \qquad \text{for } i \neq n-1.$$

Thus, by reverting the straightening via the diffeomorphism φ^{-1}, we acknowledge that

$$T_y f^{-1}(t) = D\left(\varphi^{-1}\right)\left(\varphi(y)\right)(\mathbb{R}^{n-2} \times \{0\}^2),$$

which is hence independent of $|t|$. We may apply the standard Gram-Schmidt orthogonalization procedure to find a orthonormal basis of $T_y f^{-1}(t)$, which only depends on the first $n-2$ partial derivatives of φ. As the full tangent space at y to $\partial\{u \geqslant t\}$ is the direct sum of the boundary tangent space and the co-normal space, we now map the last partial derivative back with $D\varphi(\varphi^{-1}(y))$ and complete the orthogonalization. Hence the co-normal $N_t^{co}(y)$ depends in such a representation only on $\partial_{n-1} u_t$ and thus, by uniform convergence in C^1 of the graph functions, we deduce for $\varepsilon > 0$ the existence of a neighborhood $V_{x_0} \subset \mathbb{R}^n$ such that

$$\|N_t^{co}(y) - N_\Xi^{co}(x)\| < \varepsilon \qquad \text{for any } y \in \partial\{u \geqslant t\} \cap V_{x_0} \text{ and } x \in \Xi \cap V_{x_0}.$$

We may choose finitely many such x_0, which we denote as x_i, and arrange

$$f^{-1}(0) \subset \bigcup_i V_{x_i}.$$

Choosing $\lambda > 0$ small enough, we shall assume

$$f^{-1}(0) \subset \mathbf{e}_\lambda\left(f^{-1}(0)\right) \subset \bigcup_i V_{x_i}$$

and use the Hausdorff convergence $f^{-1}(t) \longrightarrow f^{-1}(0)$ to infer

$$f^{-1}(t) \subset \mathbf{e}_\lambda\left(f^{-1}(0)\right) \subset \bigcup_i V_{x_i} \qquad \text{for all sufficiently small } |t|.$$

By uniform convergence $\kappa_t \longrightarrow \kappa_0$, we may additionally set

$$\|\kappa_t - \kappa_0\|_0 < \delta \qquad \text{for all sufficiently small } |t|$$

and we may let $\delta > 0$ equal the Lebesgue number of $\mathbf{e}_\lambda\left(f^{-1}(0)\right)$ with respect to $\bigcup_i V_{x_i}$. Thus, for each $x \in f^{-1}(0)$, we find that $x = \kappa_0(x)$ and $\kappa_t(x)$ are contained in the same member of the union, implying

$$\|(N_t^{co} \circ \kappa_t)(x) - N_\Xi^{co}(x)\| < \varepsilon \qquad \text{for all sufficiently small } |t|$$

and $x \in f^{-1}(0)$. ∎

5.4 Global Deformations of Level Set Boundaries.

We may now prove that all level sets near Ξ are actually even diffeomorphic to Ξ via the construction that is called *an almost-normal diffeomorphism* in [9, Sections 1-3] and that we may regard nearby hypersurfaces as such deformations of Ξ with norms tending to zero.

We prepare the application of [9] with the following lemma on the constancy of components of the converging oriented boundaries to reduce to the connected case:

5.19 Lemma (Stability of Connected Components)**.** *There exists $T > 0$ such that the number of connected components of $\partial\{u \geqslant t\}$ equals the number of connected components of Ξ for all $0 \leqslant |t| \leqslant T$.*

Proof. As Ξ is closed and can only have finitely many connected components, all such components must be at pairwise positive distance. Choosing an infinitesimal smaller than the minimum of these distances and employing the Hausdorff convergence $u^{-1}(t) \longrightarrow \Xi$, we deduce that all connected components of $\partial\{u \geqslant t\}$ must be close to exactly one component of Ξ, hence, we assume directly that Ξ is connected.

Now write

$$\partial\Xi = f^{-1}(0) = \bigcup_{i=1}^{N} \Theta_i \qquad \text{with a natural number} \qquad 0 < N < \infty,$$

where each $\Theta_i \subset f^{-1}(0)$ is a connected component of $f^{-1}(0)$. The diffeomorphism κ_t descends to a diffeomorphism on each Θ_i such that

$$f^{-1}(t) = \kappa_t(f^{-1}(0)) = \kappa_t\left(\bigcup_{i=1}^{N} \Theta_i\right) = \bigcup_{i=1}^{N} \kappa_t(\Theta_i)$$

and we find that the number of connected components of boundary data $f^{-1}(t)$ is constant if $|t|$ is sufficiently small. Fixing also $\rho > 0$ sufficiently small, there exists

$$w_t : \Xi \cap \Omega_\rho \longrightarrow \mathbb{R}$$

with

$$\partial\{u \geqslant t\} \cap \Omega_{2\rho} \subset (\mathrm{Id} + w_t N_\Xi)(\Xi \cap \Omega_\rho) \subset \partial\{u \geqslant t\} \cap \Omega_{2^{-1}\rho}$$

for all $|t| \leqslant T(\rho)$ by proposition 5.9. Choosing additionally $|t| \leqslant \min\big(T(\rho), T(2^{-1}\rho)\big)$, we may as well assume that w_t is well-defined on $\Xi \cap \Omega_{2^{-1}\rho}$ for all such $|t|$. By pushing $f^{-1}(0)$ inwards along Ξ at $\partial\Xi$ and possibly decreasing $\rho > 0$, we may find a connected set $\Xi_\rho \subset \Xi$ with

$$\Xi \cap \Omega_\rho \subset \Xi_\rho \subset \Xi \cap \Omega_{2^{-1}\rho} \qquad \Longrightarrow \qquad (\mathrm{Id} + w_t N_\Xi)(\Xi_\rho) \quad \text{is connected for all } |t| \text{ small enough.}$$

Thus we also find that

$$\partial\{u \geqslant t\} \cap \Omega_{2\rho} \subset (\mathrm{Id} + w_t N_\Xi)(\Xi_\rho),$$

where the latter set is connected, and therefore there is one connected component of $\partial\{u \geqslant t\}$ containing $\partial\{u \geqslant t\} \cap \Omega_{2\rho}$ for all such $|t|$, which we denote Z_t.

Choose some $i \in \{1, ..., N\}$ and $x_0 \in \Theta_i$, then the boundary non-parametric representation lemma 4.19 implies that there is an open neighborhood $V_i \subset \mathbb{R}^n$ of x_0 such that $\partial\{u \geqslant t\} \cap V_i$ is connected for all sufficiently small $|t|$. When ρ is small enough, we infer

$$(\partial\{u \geqslant t\} \cap \Omega_{2\rho}) \cap V_i \neq \varnothing \qquad \text{for all sufficiently small } |t|,$$

and therefore, in particular that $\partial\{u \geqslant t\} \cap V_i$ has to be contained in Z_t. Since also $\kappa_t(\Theta_i) \cap V_i \neq \varnothing$, we infer that this also has to be true for $\kappa_t(\Theta_i) \subset f^{-1}(t)$ and iterating this procedure for each connected component Θ_i of $f^{-1}(0)$ and adjusting ρ appropriately, we find that $\bigcup \kappa_t(\Theta_i) = f^{-1}(t)$ has to be fully contained in the geometric boundary of Z_t for all sufficiently small $|t|$. As furthermore no connected component of $\partial\{u \geqslant t\}$ without geometric boundary can exist by lemma 4.15(2), the lemma is proven. ■

5.20 Theorem (Global Diffeomorphicity of Level Set Boundaries). *There exists $T > 0$ such that there are diffeomorphisms ι_t of class $\mathrm{C}^{2,\alpha}$ with*

$$\iota_t : \Xi \longrightarrow \partial\{u \geqslant t\}$$

for each $0 \leqslant |t| \leqslant T$ such that

$$\|\iota_t - \mathrm{Id}_\Xi\|_{\mathrm{C}^{2,\gamma}(\Xi;\mathbb{R}^n)} \longrightarrow 0 \qquad \textit{as} \quad t \to 0$$

for all $0 < \gamma < \alpha$.

Proof. By lemma 5.19, we may immediately assume that Ξ and $\partial\{u \geqslant t\}$ are both connected as we could otherwise treat each connected component seperately and find precisely one connected component of $\partial\{u \geqslant t\}$, which is close in Hausdorff distance, for all sufficiently small $|t|$.

Step 1: It is our goal in this proof to apply the general construction of almost-normal diffeomorphisms from [9, Theorem 3.1] with uniform bounds in $\mathrm{C}^{2,\alpha}$. We recall from definition 5.1 that we chose an open submanifold $\tilde{\Xi} \subset \mathbb{R}^n$ of dimension $n-1$ such that $\Xi \subset \tilde{\Xi}$ and with

$$\|N_{\tilde{\Xi}}\|_{\mathrm{C}^{2,\alpha}(\tilde{\Xi};\mathbb{R}^n)} \leqslant L \qquad \text{for some large enough } L > 0,$$

which is again possible as Ξ is at least of class $\mathrm{C}^{3,\alpha}$ as a submanifold with boundary. We shall now check the hypotheses of [9, Theorem 3.1] and note immediately that hypothesis (a) on the domain submanifold Ξ is fulfilled (see also the remarks from [9, Remark 3.4]) as Ξ is in particular of class $\mathrm{C}^{2,1}$ and it remains to check that we may choose $L > 0$ large enough for hypothesis (b) on the target submanifolds $\partial\{u \geqslant t\}$ with boundary to hold uniformly in small $|t|$.

If $n = 2$, it is clear that (b,i) holds immediately owing to lemma 5.16 and 5.18 uniformly in small $|t|$. If $n > 2$, we note first that clearly the family of mappings

$$\kappa_t : f^{-1}(0) \longrightarrow f^{-1}(t)$$

is uniformly bounded in $\mathrm{C}^{2,\alpha}(f^{-1}(0), \mathbb{R}^n)$ and moreover, the three quantities

$$\|\kappa_t - \mathrm{Id}_{f^{-1}(0)}\|_{\mathrm{C}^1(f^{-1}(0),\mathbb{R}^n)}, \qquad \|(N_t \circ \kappa_t) - N_\Xi\|_{\mathrm{C}^0(f^{-1}(0),\mathbb{R}^n)}, \qquad \|(N_t^{co} \circ \kappa_t) - N_\Xi^{co}\|_{\mathrm{C}^0(f^{-1}(0),\mathbb{R}^n)},$$

can be uniformly chosen as small as we desire owing again to lemma 5.16 and 5.18 and the boundary non-parametric representation lemma 4.19. Therein, we again use the notation

$$(N_t \circ \kappa_t)(x) := N_t(\kappa_t(x)) = N_{\partial\{u \geqslant t\}}(\kappa_t(x)) \qquad \text{and} \qquad (N_t^{co} \circ \kappa_t)(x) := N_t^{co}(\kappa_t(x)) = N^{co}_{\partial\{u \geqslant t\}}(\kappa_t(x))$$

to compare normal and co-normal vectors along the boundary diffeomorphisms κ_t. This yields hypothesis (b,i) also for the higher dimensional case. Further, for both $n = 2$ and $n > 2$, we may find for each $\rho > 0$ by proposition 5.9 a number $T = T(\rho) > 0$ such that

$$w_t : \Xi \cap \Omega_\rho \longrightarrow \mathbb{R} \qquad \text{with} \qquad \partial\{u \geqslant t\} \cap \Omega_{2\rho} \subset (\mathrm{Id} + w_t N_\Xi)(\Xi \cap \Omega) \subset \partial\{u \geqslant t\} \cap \Omega_{2^{-1}\rho}$$

for all $|t| \leqslant T$. We recall from (the proofs of) corollary 5.8 and proposition 5.9 that the normal deformations $w_t N_\Xi$ are uniformly bounded in $\mathrm{C}^{2,\alpha}(\Xi \cap \Omega_\rho)$ while the uniform convergence to zero of w_t implies that also the quantity

$$\|w_t N_\Xi\|_{\mathrm{C}^1(\Xi\cap\Omega_\rho,\mathbb{R}^n)}$$

may be chosen as small as we desire by decreasing $T(\rho) > 0$. Finally, we shall prove the containment relation from [9, Theorem 3.1, (3.10)] via the two Hausdorff convergences

$$u^{-1}(t) \longrightarrow \Xi \qquad \text{and} \qquad f^{-1}(t) \longrightarrow f^{-1}(0) \quad \text{for} \quad t \to 0,$$

by choosing $\tilde{\rho} > 0$ and $\rho > 0$ such that

$$\Xi \setminus \mathbf{u}_{\tilde{\rho}}\left(f^{-1}(0)\right) \subset \Xi \cap \Omega_\rho$$

to have the family of functions w_t defined for sufficiently small $|t|$ and then estimating the distance to its image space geometric boundaries $f^{-1}(t) \subset \partial\{u \geqslant t\}$ uniformly via Hausdorff convergence. Indeed, for each $x \in \partial\{u \geqslant t\} \cap \mathbf{u}_{2\tilde{\rho}}\left(f^{-1}(0)\right)$ there are

$$x^* \in f^{-1}(0) \quad \text{with} \quad d(x, x^*) < 2\tilde{\rho} \qquad \text{and} \qquad x^{**} \in f^{-1}(t) \quad \text{with} \quad d(x^*, x^{**}) \leqslant \tilde{\rho}$$

hence,

$$d(x, f^{-1}(t)) < 3\tilde{\rho} \qquad \Longrightarrow \qquad \left(\partial\{u \geqslant t\}\setminus \mathbf{u}_{3\tilde{\rho}}\left(f^{-1}(t)\right)\right) \subset \partial\{u \geqslant t\}\setminus \mathbf{u}_{2\tilde{\rho}}(f^{-1}(0))$$

for such sufficiently small $|t|$. Using Hausdorff convergence and choosing $\lambda > 0$ small, we may then arrange

$$\partial\{u \geqslant t\}\setminus \mathbf{u}_{2\tilde{\rho}}(f^{-1}(0)) \subset \mathbf{u}_\lambda\left(\Xi\setminus \mathbf{u}_{2\tilde{\rho}}(f^{-1}(0))\right),$$

with

$$\Pi\left(\partial\{u \geqslant t\}\setminus \mathbf{u}_{2\tilde{\rho}}(f^{-1}(0))\right) \subset \Xi\setminus \mathbf{u}_{\tilde{\rho}}(f^{-1}(0))$$

for all sufficiently small $|t|$ and $\lambda > 0$. In particular, we contain

$$\partial\{u \geqslant t\}\setminus \mathbf{u}_{3\tilde{\rho}}\left(f^{-1}(t)\right) \subset (\mathrm{Id} + w_t N_\Xi)(\Xi\setminus \mathbf{u}_{\tilde{\rho}}\left(f^{-1}(0)\right)$$

for all sufficiently small $|t|$, which is assumption [9, (3.10)] adapted to our setting.

This shows that also (b,ii) is fulfilled for appropriately small $\tilde{\rho} > 0$ (which we shall use as the parameter ρ from [9]) which we will fix and it holds uniformly for $|t| \leqslant T$ for an appropriate $T = T(\rho) > 0$.

Step 2: Now applying [9, Theorem 3.1], we infer that there are diffeomorphisms

$$\iota_t : \Xi \longrightarrow \partial\{u \geqslant t\} \qquad \text{for all} \quad |t| \leqslant T(\rho), \qquad \iota_t \in \mathrm{C}^{1,\alpha}(\Xi, \mathbb{R}^n),$$

which fulfill the estimate

$$\|\iota_t - \iota_\Xi\|_{\mathrm{C}^0(\Xi,\mathbb{R}^n)} \leqslant C\left(\mathcal{H}(\Xi, \partial\{u \geqslant t\}) + \|\kappa_t - \mathrm{Id}\,\|_{\mathrm{C}^1(f^{-1}(0),\mathbb{R}^n)} + \|w_t N_\Xi\|_{\mathrm{C}^0(\Xi\cap\Omega_\rho,\mathbb{R}^n)}\right)$$

with a constant $C > 0$ independent of $|t|$. We find in particular that

$$\|\iota_t - \iota_\Xi\|_{\mathrm{C}^0(\Xi,\mathbb{R}^n)} \longrightarrow 0 \qquad \text{as} \qquad t \to 0$$

and that the family of functions ι_t is uniformly bounded in $\mathrm{C}^{1,\alpha}$. If we may now show that the functions ι_t are also uniformly bounded in $\mathrm{C}^{2,\alpha}$, we find that

$$\|\iota_t - \iota_\Xi\|_{\mathrm{C}^{2,\gamma}(\Xi,\mathbb{R}^n)} \longrightarrow 0 \qquad \text{as} \qquad t \to 0$$

for each $0 < \gamma < \alpha < 1$ and the proof is finished.

This is however clear from the construction in [9, Theorem 3.1] via the implicit function theorem (see the definition of the boundary deformation in [9, Claim 3.5] with equations [9, (3.22-24)] and its construction in [9, (3.52)] via solving [9, (3.50-51)]): The family of functions $w_t N_\Xi$ enjoys uniform bounds in $\mathrm{C}^{2,\alpha}$ while the construction of the diffeomorphisms ι_t near $\partial\Xi$ only depends on the extension $\tilde{\Xi}$ of Ξ and extensions of $\partial\{u \geqslant t\}$ which we may also construct in $\mathrm{C}^{2,\alpha}$ due to the lemmata 5.11 and 5.12. In particular, these lemmata imply that the functions whose zero sets characterize the extensions of $\partial\{u \geqslant t\}$ have uniformly bounded norms of class $\mathrm{C}^{2,\alpha}$ and our proof concludes. ■

5.21 Remark. Also some results on *global* deformations have to have been used in [46], but except for uniform Hölder estimates and uniform convergence under a nearest point-projection Π for normal vectors, there was no justification given.

5.22 Remark. For purposes of our regularity results, it will especially be important that we may deform from above and below, ie. that t may be larger or smaller than 0. Note also that all results on convergence and deformations of section 5 also hold with the same proofs for $\partial\{u \leqslant t\}$ for all sufficiently small $|t|$.

6 Elliptic Regularity Theory of Local Deformations.

In the first part of this section we will aim at a different derivation of the important Harnack-Type inequality, which is used to construct the "special" Jacobi field with desirable properties in [46, Sections 3,4].

The original proof was direct, employing the first variation in local coordinates to rewrite the stationarity condition of each $\partial\{u \geqslant t\}$ as some form of local geometric partial differential equation for the family of functions w_t, whose structural constants may be bounded independently of $t \geqslant 0$.

Here we alter our approach to compare the deformations in varying normal direction w_t with the deformations in "straight" vertical direction above the tangent planes $u_t - u_0$, which are easily implied to solve a linear elliptic equation with uniformly bounded coefficients in the deformation parameter $t \geqslant 0$. Thus, the vertical deformations allow immediately for applications of linear elliptic regularity theory which we shall seek to "carry over" to the normal deformations.

Finally, we include the limiting blow-up argument to construct the "special" Jacobi field by blowing up the vanishing first variation in global coordinates.

6.1 Comparing Deformations and the Harnack-Type Inequality.

We begin with two preparatory lemmata for the uniform Harnack-Type inequality for the family of functions w_t.

6.1 Lemma (Pointwise Comparisons of Deformations)**.** *Let $x_0 \in \Xi \cap \Omega$. Then there are $r > 0$, $T > 0$ and $C = C(x_0) > 0$ such that*

$$(u_t - u_0)(z) \leqslant C\, w_t((z, u_0(z))) \qquad \text{and} \qquad w_t((z, u_0(z))) \leqslant C\,(u_t - u_0)(z)$$

for all $0 \leqslant t \leqslant T$ and $z \in \mathbf{U}(x_0, r) \cap T_{x_0}\Xi$, where C only depends on the point x_0 and the (uniformly bounded) supremum on $[0, T]$ of the C^1-norms of u_t.

Proof. By choosing appropriate coordinates as usual, we may assume that

$$x_0 = 0, \qquad \left(T_{x_0}\Xi = \mathbb{R}^{n-1}\right) \subset \mathbb{R}^n, \qquad N_\Xi(x_0) = e_n.$$

Using the deformations from lemma 4.18 und lemma 5.5, we furthermore shall choose $r > 0$ sufficiently small such that $u_t : \mathbf{U}^{n-1}(0, 2r) \longrightarrow \mathbb{R}$ is well-defined and, by abuse of notation, that $w_t : \mathbf{U}^{n-1}(0, r) \longrightarrow \mathbb{R}$ is also well-defined for all sufficiently small $t \geqslant 0$.

We recall form remark 5.6 that the family of functions w_t solves the implicit equation

$$\mathfrak{W}_t(z, w_t(z)) = 0 \qquad \text{on} \quad \mathbf{U}^{n-1}(0, r),$$

whenever $|t|$ is sufficiently small, where

$$\mathfrak{W}_t(z, q) = u_t\left(z - q\frac{\nabla u_0(z)}{\sqrt{1 + \nabla u_0(z)^2}}\right) - \left(u_0(z) + \frac{q}{\sqrt{1 + \nabla u_0(z)^2}}\right),$$

from which we may derive the implicit formula

$$w_t(z) = \sqrt{1+\nabla u_0(z)^2}\left(u_t\left(z - w_t(z)\frac{\nabla u_0(z)}{\sqrt{1+\nabla u_0(z)^2}}\right) - u_0(z)\right)$$

for all $z \in \mathbf{U}(0,r)$ and $t \geqslant 0$ small enough.

Hence we compute for an arbitrary point $z \in \mathbf{U}(0,r)$ that

$$\begin{aligned} w_t(z) &\leqslant C\left(u_t\left(z - w_t(z)\frac{\nabla u_0(z)}{\sqrt{1+\nabla u_0(z)^2}}\right) \pm u_t(z) - u_0(z)\right) \\ &\leqslant C\left(\|u_t\|_{1,\mathbf{U}(0,2r)}\, w_t(z) + (u_t - u_0)(z)\right), \end{aligned}$$

where the constant only depends on the gradient of u_0 through Lipschitz estimates. As $\nabla u_0(0) = 0$ and we may assume that $u_t \longrightarrow u_0$ in $\mathrm{C}^1(\mathbf{U}(0,2r))$, we may possibly decrease r to infer that

$$1 - C\|u_t\|_{1,\mathbf{U}(0,2r)} \geqslant 1/2$$

for all sufficiently small $t \geqslant 0$ and deduce

$$w_t(z) \leqslant C(u_t - u_0)(z),$$

where the constant only depends on the C^1-norm of the family of functions u_t, which is uniformly bounded as a consequence of the interior non-parametric representation lemma 4.19(3).

Conversely, we also find

$$\begin{aligned} (u_t - u_0)(z) &= (u_t - u_0)(z) \pm u_t\left(z - w_t(z)\frac{\nabla u_0(z)}{\sqrt{1+\nabla u_0(z)^2}}\right) \\ &\leqslant \|u_t\|_{1,\mathbf{U}(0,2r)} w_t(z) + w_t(z) \\ &\leqslant C w_t(z), \end{aligned}$$

where yet again C only depends on some bound of the C^1-norm of the family u_t. ■

6.2 Lemma (Uniform Linear PDE Estimates)**.** *Each member of the family of functions*

$$u_t - u_0 : \mathbf{U}(0,2r) \longrightarrow \mathbb{R}$$

solves a linear uniformly elliptic partial differential equation of second order

$$\operatorname{div}\left(\mathbf{a}_t \cdot \nabla(u_t - u_0) + \mathbf{b}_t(u_t - u_0)\right) + \mathbf{c}_t \cdot \nabla(u_t - u_0) + \mathbf{d}_t(u_t - u_0) = 0 \qquad \text{on } \mathbf{U}(0,2r)$$

depending on F and t with coefficients of (at least) class $\mathrm{C}^{0,\alpha}$ such that all structural constants may be bounded independently of $0 < t \leqslant T$. Furthermore, we have

$$\|\mathbf{a}_t\|_{\mathrm{C}^{0,\alpha}} + \|\mathbf{b}_t\|_{\mathrm{C}^{0,\alpha}} + \|\mathbf{c}_t\|_{\mathrm{C}^{0,\alpha}} + \|\mathbf{d}_t\|_{\mathrm{C}^{0,\alpha}} \leqslant C$$

with a constant $C > 0$ independent of t.

Proof. We again assume canonical coordinates and denote by $\Phi^{\S}$ the non-parametric integrand with variables (z, u, p) associated to the anisotropic total variation Φ and recall that each u_t solves, up to possibly changing the orientation of the graph, the quasilinear Euler-Lagrange equation in weak form

$$0 = \int_{\mathbf{U}(0,2r)} D_p\Phi^{\S}(z, u_t(z), \nabla u_t(z)) \cdot \nabla\varphi + D_u\Phi^{\S}(z, u_t(z), \nabla u_t(z))\varphi \; d\mathscr{L}^{n-1}$$

with $\varphi \in \mathrm{C}^{\infty}_c(\mathbf{U}(0, 2r))$. Hence for each fixed t and $s \in [0, 1]$ we may set $u_{t,s} = (1 - s)u_0 + su_t$ and calculate for the difference $u_t - u_0$ that

$$D_p\Phi^{\S}(z, u_t(z), \nabla u_t(z)) - D_p\Phi^{\S}(z, u_0(z), \nabla u_0(z)) = \int_0^1 \frac{d}{ds} D_p\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds$$

and

$$D_u\Phi^{\S}(z, u_t(z), \nabla u_t(z)) - D_u\Phi^{\S}(z, u_0(z), \nabla u_0(z)) = \int_0^1 \frac{d}{ds} D_u\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds.$$

Computing those derivatives yields

$$\frac{d}{ds} D_p\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s}) = D_{up}\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s})(u_t - u_0) + D^2_p\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s}) \cdot \nabla(u_t - u_0)$$

and

$$\frac{d}{ds} D_u\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s}) = D^2_u\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s})(u_t - u_0) + D_{pu}\Phi^{\S}(z, u_{t,s}, \nabla u_{t,s}) \cdot \nabla(u_t - u_0).$$

Thus, subtracting the weak Euler-Lagrange equation for u_0 from the one for u_t, we infer that

$$\begin{aligned} 0 \quad = \quad & \int_{\mathbf{U}(0,2r)} \nabla\varphi \cdot (\mathbf{a}_t(z) \cdot \nabla(u_t - u_0)(z) + \mathbf{b}_t(z)(u_t - u_0)(z)) \\ & \qquad + \varphi\,(\mathbf{c}_t(z) \cdot \nabla(u_t - u_0)(z) + \mathbf{d}_t(z)(u_t - u_0)(z)) \; d\mathscr{L}^{n-1}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{a}_t(z) &= \int_0^1 D^2_p\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds, \quad & \mathbf{b}_t(z) &= \int_0^1 D_{up}\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds, \\ \mathbf{c}_t(z) &= \int_0^1 D_{pu}\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds, \quad & \mathbf{d}_t(z) &= \int_0^1 D^2_u\Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \; ds. \end{aligned}$$

Using the Schauder estimates for the family of functions u_t, we see that there exist a constant $C > 0$ such that

$$\|\mathbf{a}_t(z)\| \leqslant C \qquad \text{and} \qquad \|\mathbf{b}_t(z)\| + \|\mathbf{c}_t(z)\| + |\mathbf{d}_t(z)| \leqslant C.$$

are uniformly bounded in sufficiently small $t > 0$ for all $z \in \mathbf{U}(0, 2r)$. Moreover, we infer from the parametric Legendre condition on F with an ellipticity bound $c > 0$ and section 2.4 that

$$\begin{aligned}\xi^T \mathbf{a}_t(z)\xi &= \int_0^1 \xi^T \cdot D_p^2 \Phi^{\S}(z, u_{t,s}(z), \nabla u_{t,s}(z)) \cdot \xi \, ds \\ &\geqslant c \int_0^1 \frac{|\xi|^2}{\sqrt{1 + |\nabla u_{t,s}(z)|^2}} \left(1 - \frac{|\nabla u_{t,s}(z)|}{\sqrt{1 + |\nabla u_{t,s}(z)|^2}}\right) ds \\ &= c \int_0^1 \frac{|\xi|^2}{(1 + |\nabla u_{t,s}(z)|^2)^{\frac{3}{2}}} \, ds.\end{aligned}$$

Yet again, by means of the Schauder estimates for u_t, we may bound all terms by their suprema in $z \in \mathbf{U}(0, 2r)$ and t small and reciprocally only decrease further. In particular, there exists $\lambda > 0$ such that

$$c \int_0^1 \frac{|\xi|^2}{(1 + |\nabla u_{t,s}(z)|^2)^{\frac{3}{2}}} \, ds \geqslant \lambda |\xi|^2 > 0 \qquad \text{for all } z \in \mathbf{U}(0, 2r) \text{ and } \xi \in \mathbb{R}^{n-1}$$

and for all sufficiently small $t > 0$. Thus, the functions $u_t - u_0$ solve linear uniformly elliptic partial differential equations whose coefficients and ellipticity constant may be bounded independently of small $t > 0$. Note subsequently that all involved derivatives of $\Phi^{\S}$ are at least of class $\mathrm{C}^{0,\alpha}$ while the functions $u_{t,s}$ and $\nabla u_{t,s}$ are uniformly Lipschitz in t and s due to the Schauder estimates up to second order. This proves that also the Hölder constants of the coefficients may be uniformly bounded. ■

Combining lemma 6.1 with lemma 6.2, we may now feasibly conclude for a Harnack-Type inequality for the family of normal deformations w_t via the standard one applied to the family $u_t - u_0$.

6.3 Theorem (Uniform Harnack-type Estimates). *Fix $x_0 \in \Xi \cap \Omega$ and let $K \subset \Xi \cap \Omega$ be compact and connected. Then we have:*

*(1) (*Local Harnack-Type Inequality*) There exist $r = r(x_0) > 0$ and $T = T(x_0) > 0$ with $C = C(x_0) > 0$ such that*

$$\sup\{w_t(x) \mid x \in \mathbf{U}(x_0, r) \cap \Xi\} \leqslant C(x_0) \ \inf\{w_t(x) \mid x \in \mathbf{U}(x_0, r) \cap \Xi\}$$

for all $0 \leqslant t \leqslant T$.

*(2) (*Harnack-Type Inequality on Retracts*) There exists $T = T(K) > 0$ and $C = C(K) > 0$ such that*

$$\sup\{w_t(x) \mid x \in K\} \leqslant C(K) \ \inf\{w_t(x) \mid x \in K\}$$

for all $0 \leqslant t \leqslant T$.

Proof. We again choose standard coordinates and observe that we may assume that

$$(u_t - u_0) \geqslant 0 \qquad \text{on} \qquad \mathbf{U}^{n-1}(0, 2r)$$

for all sufficiently small $t \geqslant 0$, where r is as in the proof of lemma 6.1. We acknowledge this as follows: It is a topological fact that the graph of u_0 over $\mathbf{U}^{n-1}(0, 2r)$ seperates sufficiently small cylinders into precisely two open and connected components above and below the graph, which we call A and B, and u does not vanish on A and B. Assuming that

$$x \in A \cap \{u > 0\} \qquad \text{and} \qquad y \in A \cap \{u < 0\},$$

we find a continuous path $\gamma : [0,1] \to A$ such that

$$\gamma(0) = x, \qquad \gamma(1) = y$$

and hence some $t \in (0,1)$ with $\gamma(t) = 0$, which contradicts that γ maps to A. This implies that both A and B are contained in one of the above sets and due to the level set Ξ equaling its sub- and superlevel set boundaries, they cannot be contained in the same set. A uniform sign condition for $u_t - u_0$ follows by choosing a correct configuration.

Thus, by lemma 6.2 and standard linear elliptic regularity theory, we may invoke eg. [13, Theorem 8.20] to assume that there is a constant $C > 0$ such that

$$\sup_{\mathbf{U}^{n-1}(0,r)} (u_t - u_0) \leqslant C \inf_{\mathbf{U}^{n-1}(0,r)} (u_t - u_0) \qquad \text{with} \qquad C \neq C(t)$$

for all sufficiently small $t \geqslant 0$. Note that C does not depend on t as the coefficients of the respective linear equations may be uniformly bounded uniformly in small $t \geqslant 0$.

Further, we find by lemma 6.1 also that

$$\sup_{\mathbf{U}^{n-1}(0,r)} w_t \leqslant C \sup_{\mathbf{U}^{n-1}(0,r)} (u_t - u_0) \qquad \text{and} \qquad \inf_{\mathbf{U}^{n-1}(0,r)} (u_t - u_0) \leqslant C \inf_{\mathbf{U}^{n-1}(0,r)} w_t,$$

from which 6.3(1) may be easily derived by readjusting to a smaller r.

For the second statement it suffices to treat cover the compact and connected set K with finitely many balls where 6.3(1) holds and apply a simple continuation argument across their intersections. ■

6.4 Remark. The latter theorem 6.3 provides a valuable alternative route to acquiring the Harnack-Type inequality of [46, 3.1 Theorem]. In fact, providing a similar geometric PDE in local coordinates as in the proof in [46] for general parametric integrands/anisotropic total variations seemed to me like a very peculiar thing to carry out, while our strategy here is able to conclude directly from the linearization of the non-parametric integrand with some further elementary manipulations.

6.2 Higher Order Convergence of Normal Blowups and the Jacobi Nullity.

Since our approach is meant to forsake the complicated local expressions, we will now proceed to introduce more lemmata to "carry over" more asymptotic information from elliptic regularity estimates for the horizontal deformations $u_t - u_0$ to the normal deformations w_t.

Also, in this part, our assumptions on $\mathcal{U}$ and the regularity as lipschitzian level sets will now enter.

6.5 Lemma. *Assuming canonical coordinates at $x_0 \in \Xi \cap \Omega$, there exist $C > 0$, $T > 0$ and $r > 0$ such that*

$$\left\| \frac{u_t - u_0}{t} \right\|_{2,\alpha,\mathbf{U}(0,r)} \leqslant C$$

for all $0 < t \leqslant T$, where C depends additionally on the choice of some $x^* \in \Xi \cap \Omega$ in the connected component of Ξ of x_0 where $Du(x^*) \neq 0$.

Proof. We note first that each of the finitely many connected components of Ξ admits points $x^* \in \Xi \cap \Omega$ with $Du(x^*) \neq 0$ since $0 \in \mathcal{U}$. We proceed to first uniformly bound the supremum.

By lemma 6.1, we find $r > 0$ and $T > 0$ to achieve

$$\left\| \frac{u_t - u_0}{t} \right\|_{0,\mathbf{U}^{n-1}(0,r)} \leqslant C \left\| \frac{w_t}{t} \right\|_{0,\mathbf{U}(0,r)}$$

for all $0 < t \leqslant T$ and we may choose a compact and connected set $K \subset \Xi \cap \Omega$ such that

$$x^* \in K \qquad \text{and} \qquad (\mathrm{Id}(\mathbf{U}^{n-1}(0,r)), u_0(\mathbf{U}^{n-1}(0,r)) \subset K.$$

Theorem 6.4(2) now yields a constant $C(K) > 0$ such that

$$\left\| \frac{w_t}{t} \right\|_{0,\mathbf{U}(0,r)} \leqslant \left\| \frac{w_t}{t} \right\|_{0,K} \leqslant C(K) \inf_K \left(\frac{w_t}{t} \right) \leqslant C(K) \frac{w_t(x^*)}{t}$$

as $x^* \in K$ for all sufficiently small $t > 0$.

Further, since $Du(x^*) \neq 0$, there are $\delta > 0$ and $m^* > 0$ such that it follows

$$0 < m^* \leqslant \frac{|u(x^* + hN_\Xi(x^*))|}{|h|} \qquad \text{for all} \quad |h| \leqslant \delta, \quad h \neq 0,$$

and therefore

$$\frac{|h|}{|u(x^* + hN_\Xi(x^*))|} \leqslant \frac{1}{m^*} < \infty$$

for all such h. Since

$$(\mathrm{Id} + w_t N_\Xi)(x^*) \in u^{-1}(t) \qquad \text{and} \qquad w_t(x^*) \longrightarrow 0 \quad \text{as} \quad t \to 0,$$

we may arrange $t > 0$ small enough to set $h_t = w_t(x^*)$ and deduce

$$\frac{w_t(x^*)}{t} = \frac{|h_t|}{|u(x^* + h_t N_\Xi(x^*))|} \leqslant \frac{1}{m^*}$$

for all sufficiently small $|t|$. Concatenating all three inequalities, we receive

$$\left\| \frac{u_t - u_0}{t} \right\|_{0,\mathbf{U}(0,r)} \leqslant C \qquad \text{for all sufficiently small } t > 0.$$

We recall now from lemma 6.2 that all coefficients $\mathbf{a}_t, \mathbf{b}_t, \mathbf{c}_t, \mathbf{d}_t$ may be bounded in $\mathrm{C}^{0,\alpha}$ independently of sufficiently small $t \geqslant 0$. Subsequently, as the supremum of $t^{-1}(u_t - u_0)$ may be uniformly bounded, we may conclude via linear elliptic regularity theory and interior Schauder estimates (eg. [13, Theorem 6.2]) that we may also, by possibly readjusting $r > 0$ and with a constant C independent of t, assume that

$$\|u_t - u_0\|_{2,\alpha,\mathbf{U}(0,r)} \leqslant C \|u_t - u_0\|_{0,\mathbf{U}(0,r)}$$

which yields

$$\left\| \frac{u_t - u_0}{t} \right\|_{2,\alpha,\mathbf{U}(0,r)} \leqslant C \qquad \text{for all sufficiently small } t > 0,$$

finishing the proof. ■

As we are however interested mainly in the family of normal deformations w_t, the next lemma will relate also the derivatives of w_t to the ones of $u_t - u_0$. Then the uniform bound in $\mathrm{C}^{2,\alpha}$ will then immediately allow us to generate a partial limit for the blowups of w_t.

We therefore have to make use of some rather unappealing calculations involving the implicit equation for w_t, which relates normal and vertical deformations. Afterwards, we may prove that the partial local limits in fact solve the linearization of the Euler-Lagrange operator in its weak form and patch together to solution on the geometric interior of Ξ.

6.6 Lemma (Uniform Convergence of Normal Blowups)**.** *There exists an open neighborhood $V \subset \Xi \cap \Omega$ of $x_0 \in \Xi \cap \Omega$ and a subsequence $t_j \longrightarrow 0$ as $j \to \infty$ such that*

$$\frac{w_{t_j}}{t_j} \longrightarrow \zeta_{x_0} \qquad \text{in} \qquad \mathrm{C}^{1,\gamma}(V) \qquad \text{for all} \qquad 0 < \gamma < \alpha.$$

Moreover, we have $\zeta \geqslant K^{-1} > 0$ on V where $K > 0$ is any Lipschitz constant of u.

Proof. Assuming canonical coordinates at x_0 and recalling that the family of functions w_t solves an implicit equation given via the function $\mathfrak{W}$, we shall now write

$$w_t(z) = \mathfrak{A}(z)\left(u_t\left(z - w_t(z)\mathfrak{V}(z)\right) - u_0(z)\right)$$

with

$$\mathfrak{A}(z) = \sqrt{1 + \nabla u_0(z)^2}, \qquad \mathfrak{V}(z) = \frac{\nabla u_0(z)}{\sqrt{1+\nabla u_0(z)^2}} = \frac{\nabla u_0(z)}{\mathfrak{A}(z)},$$

for all $z \in \mathbf{U}(0,r)$ and all sufficiently small $t > 0$. Then differentiation in i-th direction directly verifies that

$$\begin{aligned} \partial_i w_t(z)\left(1 + \nabla u_t\left(z - w_t(z)\mathfrak{V}(z)\right)\cdot \mathfrak{V}(z)\right) &= \partial_i \mathfrak{A}(z)\left(u_t\left(z - w_t(z)\mathfrak{V}(z)\right) - u_0(z)\right) \\ +\quad & \mathfrak{A}(z)\left(\nabla u_t\left(z - w_t(z)\mathfrak{V}(z)\right)\cdot\left(e_i - w_t(z)\partial_i\mathfrak{V}(z)\right) - \partial_i u_0(z)\right). \end{aligned}$$

Since $|\mathfrak{V}(z)| < 1$, we may choose $t > 0$ and $r > 0$ uniformly small enough to arrange for the factor of $\partial_i w_t$ to be always positive (say, bounded from below by 1/2). Hence, we conclude for

$$\left|\frac{\partial_i w(z)}{t}\right| \leqslant C\left(\frac{w_t(z)}{t} + \frac{(u_t-u_0)(z)}{t} + \frac{|\partial_i(u_t-u_0)(z)|}{t}\right) \leqslant C$$

for all sufficiently small $t > 0$ and $i = 1, ..., n$, where C here depends on the supremum of $\|u_t\|_{2,\mathbf{U}(0,2r)}$ through Lipschitz estimates for the gradient of u_t over small enough $t > 0$. Thus

$$\left\|\frac{w_t}{t}\right\|_{1,\mathbf{U}(0,r)} \leqslant C \qquad \text{uniformly in small enough } t > 0.$$

Subsequent differentiation in j-th direction analogously verifies that

$$\left\| \frac{w_t}{t} \right\|_{2,\mathbf{U}(0,r)} \leqslant C \qquad\qquad \text{uniformly in small enough } t > 0,$$

where C here similarly depends on the supremum of $\|u_t\|_{3,\mathbf{U}(0,2r)}$ over suffciently small $t > 0$. Clearly, the radius r may be chosen by lemma 4.18(3) so that $\|u_t\|_{3,\mathbf{U}(0,2r)}$ may be bounded independently of small $t > 0$ by uniform interior Schauder estimates. We especially find that

$$\left\| \frac{w_t}{t} \right\|_{1,\alpha,\mathbf{U}(0,r)} \quad \text{is uniformly bounded, hence} \qquad \frac{w_t}{t} \longrightarrow \zeta_{x_0} \in \mathrm{C}^{1,\gamma}(\mathbf{U}^{n-1}(0,r))$$

without relabeling the subsequence for all $0 < \gamma < \alpha$. Finally, we recall for any $x = (z, u_0(z)) \in \Xi \cap \Omega$ that

$$\frac{w_t(x)}{t} = \frac{|(\mathrm{Id} + w_t N_\Xi)(x) - x|}{|u((\mathrm{Id} + w_t N_\Xi)(x)) - u(x)|} \geqslant K^{-1} > 0,$$

which passes to such a local limit ζ_{x_0} by uniform convergence. ■

Finally, with lemma 6.6 at our disposal, we are in position to show that the local limits ζ_{x_0} can be patched together to construct the desirable special Jacobi field.

6.7 Proposition (Existence of The Special Jacobi Field)**.** *There exists a normal vector field*

$$\zeta_0 N_\Xi \in \mathrm{C}^{2,\alpha}_{loc}(\Xi \cap \Omega, \mathbb{R}^n),$$

which suffices

$$D\mathfrak{H}(\zeta_0 N_\Xi) \equiv 0 \qquad\qquad \text{on} \quad \Xi \cap \Omega$$

and $\zeta_0 \geqslant K^{-1} > 0$ on $\Xi \cap \Omega$, where $K > 0$ is any Lipschitz constant of u.

Proof. We fix $\rho > 0$ sufficiently small and receive $T(2^{-1}\rho) > 0$ by proposition 5.9 such that

$$w_t : \Xi \cap \Omega_{2^{-1}\rho} \longrightarrow \mathbb{R}$$

is well-defined for all $0 < t \leqslant T(2^{-1}\rho)$, and choose some $x_0 \in \Xi \cap \overline{\Omega}_\rho$ together with $v \in \mathrm{C}^1(\Xi, \mathbb{R}^n)$ such that $\operatorname{spt} v$ is compactly contained in $\Xi \cap \Omega_{2^{-1}\rho}$. We denote for this proof $\iota = \iota_\Xi \in \mathrm{C}^{\nu,\alpha}(\Xi, \mathbb{R}^n)$ as the canonical inclusion of Ξ into $\mathbb{R}^n$ and acknowledge that the regularity and minimality of Ξ among rectifiable currents implies

$$\mathfrak{H}(\iota) = 0$$

while the minimality of each $\partial\{u \geqslant t\}$ for sufficiently small $t > 0$ implies that

$$\int_\Xi \mathfrak{H}((\mathrm{Id} + w_t N_\Xi) \circ \iota) \cdot v \; d\mathscr{H}^{n-1} = 0$$

by considering variations of the respective embeddings as rectifiable currents with fixed boundary. Note that the latter Euler-Lagrange operator is well-defined as the support of v is sufficiently restricted. Together we infer

$$\int_\Xi t^{-1} \left((\mathfrak{H}((\mathrm{Id} + w_t N_\Xi) \circ \iota) - \mathfrak{H}(\iota)) \cdot v \right) \; d\mathscr{H}^{n-1} = 0$$

for all such vector fields v of class C^1. By lemma 6.6, we may assume that there is a neighborhood

$V \subset \Xi \cap \Omega$ of x_0 and a function $\zeta_{x_0} \in \mathrm{C}^{1,\alpha}(V)$ such that

$$\frac{w_t}{t} \longrightarrow \zeta_{x_0} \qquad \text{in the topology of } \mathrm{C}^{1,\gamma}(V) \text{ for each } 0 < \gamma < \alpha.$$

We moreover require v to have compact support in V. Now, writing

$$\iota_t := (\mathrm{Id} + w_t N_\Xi) \circ \iota = \iota + (w_t N_\Xi) \circ \iota,$$

and expressing the above equation in weak form we have

$$\begin{aligned} 0 &= \int_\Xi t^{-1} \left(\mathfrak{H}((\mathrm{Id} + w_t N_\Xi) \circ \iota) - \mathfrak{H}(\iota)\right) \cdot v \, d\mathscr{H}^{n-1} \\ &= \int_\Xi t^{-1} \left((D_3\tilde{F}(x, \iota_t, D\iota_t) - D_3\tilde{F}(x, \iota, D\iota))Dv\right) + (D_2\tilde{F}(x, \iota_t, D\iota_t) - D_2\tilde{F}(x, \iota, D\iota))v \, d\mathscr{H}^{n-1}. \end{aligned}$$

Applying Taylor's theorem to $D_2\tilde{F}$ and $D_3\tilde{F}$, we may let $t \to 0$ to infer that ζ_{x_0} is a weak solution of class C^1 to the Jacobi equation in local coordinates about x_0. Thus it holds $\zeta_{x_0} \in \mathrm{C}^{2,\alpha}(V)$ by proposition 2.27 and

$$\int_\Xi v \cdot D\mathfrak{H}(\zeta_{x_0} N_\Xi) \, d\mathscr{H}^{n-1} = 0$$

for all $v \in \mathrm{C}^{2,\alpha}(\Xi, \mathbb{R}^n)$ such that $\operatorname{spt} v \subset\subset V$. It immediately follows that

$$D\mathfrak{H}(\zeta_{x_0} N_\Xi) = 0 \qquad \text{on } V.$$

Now, eg. covering $\Xi \cap \overline{\Omega}_\rho$ by finitely many such local domains V and succesively choosing subsequences, we may infer that there is a sequence $t_i(\rho) \to 0$ and

$$\zeta_\rho \in \mathrm{C}^{2,\alpha}(\Xi \cap \Omega_\rho)$$

such that

$$\frac{w_{t_i(\rho)}}{t_i(\rho)} \longrightarrow \zeta_\rho \qquad \text{in the topology of } \mathrm{C}^{1,\gamma}(\Xi \cap \Omega_\rho) \qquad \text{and} \qquad D\mathfrak{H}(\zeta_\rho N_\Xi) = 0 \quad \text{on } \Xi \cap \Omega_\rho.$$

Finally, diagonalizing with respect to $\rho > 0$ and $t > 0$ as ρ approaches 0 shows the existence of the desired function ζ_0 such that

$$\zeta_0 \in \mathrm{C}^{2,\alpha}_{loc}(\Xi \cap \Omega), \qquad \zeta_0 \geqslant K^{-1} > 0, \qquad D\mathfrak{H}(\zeta_0 N_\Xi) = 0 \qquad \text{on} \quad \Xi \cap \Omega,$$

where we have used lemma 6.6 and uniform convergence for the globally positive lower bound. ■

Now the simple argument by local comparison from [46, 4.2 Corollary], which relies solely on the fact that the Jacobi operator descends to a linear uniformly elliptic partial differential equation for the coefficient function and elliptic regularity theory, can be employed to conclude the non-existence of homogeneous Jacobi fields also in the anisotropic setting. We will now collect the final results on the Jacobi nullity that we will need.

6.8 Theorem (THE JACOBI NULLITY OF GENERIC LEVEL SETS). *Let $\zeta N_\Xi \in \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n)$ be a Jacobi field with $0 < \gamma < \alpha < 1$ on Ξ with $\zeta_{\partial\Xi} = 0$. Then $\zeta \equiv 0$ or equivalently, $\dim \mathfrak{K} = 0$. Moreover, almost* *every level set of u does not allow non-trivial homogeneous Jacobi fields.*

Proof. Let $\zeta N_\Xi \in \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n)$ be a homogeneous Jacobi field on Ξ. As the Jacobi operator is linear, we infer that

$$D\mathfrak{H}\left((\zeta_0 - c\zeta)N_\Xi\right) = 0 \qquad \text{on} \quad \Xi \cap \Omega$$

for each $c \in \mathbb{R}$. By choosing an appropriate sign and $|c|$ small enough, we may assume that

$$\zeta_0 - c\zeta \geqslant 0 \qquad \text{on} \quad \Xi \cap \Omega.$$

If $\zeta \not\equiv 0$ on Ξ, then we might find a point $x^* \in \Xi \cap \Omega$ such that we may as well assume that

$$(\zeta_0 - c\zeta)(x^*) = 0.$$

By proposition 2.27 and definition 2.28, we recall that the coefficient function of a Jacobi field solves a uniformly elliptic partial differential equation in local coordinates on Ξ and hence, the Harnack inequality is available for solutions in such local charts. Exploiting the same simple covering argument as in the proof of theorem 6.3(2), the zero at $x^* \in \Xi \cap \Omega$ implies

$$\zeta_0 - c\zeta \equiv 0 \qquad \text{on the connected component of } x^* \text{ in } \Xi,$$

which is a contradiction, as $\zeta_0 - c\zeta$ is bounded away from 0 near $\partial\Xi$. We deduce that necessarily $\zeta \equiv 0$ on Ξ, hence $\dim \mathfrak{K} = 0$.

The last statement of the corollary follows from the fact that level sets of u which fulfill $Du \neq 0$ almost everywhere on the level set constitute almost all level sets and this suffices to construct the special Jacobi field ζ from proposition 6.7 on such a level set (which hence also equals its level set boundaries). ∎

7 *Local and Partial Regularity of a Function of Anisotropic Least Gradient.*

In this section we approach the original conclusion on partial regularity from the perspective of our setup and discuss the questions arisen in section 3.3.

7.1 The Construction and Proofs.

With the last two sections at hand, we are at the point to give a proof of the partial regularity result with a fixed regularity in the flow variable (which moreover, as we have seen in section 5.3, is precisely the value of the extremal u by means of its boundary values).

We repeat the full proofs here and begin with the (value-wise) local result. Its idea is roughly to subsequently transfer the flow regularity with the dependence map (that we are allowed to apply via theorem 6.8) from boundary to full domain and to exploit stationarity properties in conjunction with elliptic theory to find a differentiable identity for the extremal near the level set.

7.1 Theorem (Local Differentiability)**.** *Assume* $0 \in \mathcal{U}$*, ie.* 0 *is a mildly regular value of* u *and set* $\mathrm{M} = \min(\mu - 2, m - 3)$*. Then there exists an open set* $\mathscr{O}_0$ *with*

$$\Xi \cap \Omega \subset \mathscr{O}_0 \subset \Omega$$

and u *is of class* C^{M} *on* $\mathscr{O}_0$*, where* $\mathrm{M} > 0$ *can be* ∞*.*

Proof. As $\Xi = u^{-1}(0)$ can only have finitely many connected components by proposition 4.15(2), we may directly assume that Ξ is connected and, furthermore, that all level set boundaries $\partial\{u \geqslant t\} \subset \overline{\Omega}$ are diffeomorphic to Ξ for all sufficiently small $|t|$ via the almost-normal diffeomorphism construction from theorem 5.20.

Step 1: As Ξ is a submanifold with boundary of $\mathbb{R}^n$ of class at least $\mathrm{C}^{3,\alpha}$ by theorem 4.10, where $0 < \alpha < 1$ is fixed here through the regularity of $\mathscr{F}$, we observe that in particular

$$\iota_\Xi \in \mathrm{C}^{2,\alpha}(\Xi, \mathbb{R}^n), \qquad \text{where } \iota_\Xi \text{ denotes the canonical inclusion into } \mathbb{R}^n.$$

Since $0 \in \mathcal{U}$, we may apply theorem 6.8 to infer that $\dim \mathfrak{K} = 0$. We may hence by theorem 2.29 conclude that the dependence map

$$\mathfrak{F} : \mathrm{C}^{2,\gamma}(\partial\Xi, \mathbb{R}^n) \longrightarrow \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n) \qquad \text{for any } 0 < \gamma < \alpha < 1$$

for the Hölder space of class $\mathrm{C}^{2,\gamma}$ is well-defined and apply it to the $\mathscr{F}$-stationary canonical inclusion ι_Ξ of Ξ into $\mathbb{R}^n$. We will now use the local flow κ as defined in section 5.3 and lemma 5.14. We recall that

$$\kappa_0 = \iota_{f^{-1}(0)} = \iota_{\Xi,\partial\Xi} \qquad \text{and} \qquad \mathfrak{F}(\kappa_0) = \iota_\Xi.$$

Possibly decreasing our maximal value $T > 0$, we may consider the function

$$\mathrm{K} : \Xi \times (-T, T) \longrightarrow \mathbb{R}^n, \qquad \mathrm{K}(x,t) = \mathfrak{F}(\kappa_t)(x).$$

Since $\mathfrak{F}$ is of class $\mathrm{C}^{\mu-2}$ by theorem 2.29 and the regularity of $\mathscr{F}$, we find that K is of class C^{M} in its second variable due to lemma 5.16 with, as before,

$$\mathrm{M} = \min(\mu - 2, m - 3) \geqslant 1.$$

Since every $\mathfrak{F}(\kappa_t) \in \mathrm{C}^{2,\gamma}(\Xi, \mathbb{R}^n)$, we infer that K is of class $\mathrm{C}^{2,\gamma}$ in its first variable and by the higher regularity corollary 2.31, we have

$$\mathfrak{F}(\kappa_t) \in \mathrm{C}^{\nu,\gamma}(\Xi, \mathbb{R}^n) \qquad \text{for all sufficiently small } |t| \geqslant 0,$$

which leads to K being of class C^{M}, since $\nu = \min(\mu, m-1)$ whence $\nu \geqslant \mathrm{M}$.

Step 2: We claim now that:

There is $T > 0$ such that $\mathrm{K}(x,t) \in u^{-1}(t)$ whenever $0 \leqslant |t| \leqslant T$.

Indeed, by theorem 5.20, we may find diffeomorphisms ι_t of class $\mathrm{C}^{2,\alpha}$ with

$$\iota_t : \Xi \longrightarrow \partial\{u \geqslant t\} \qquad \text{with} \qquad \|\iota_t - \iota_\Xi\|_{2,\gamma} \longrightarrow 0 \quad \text{for} \quad t \to 0$$

and $0 < \gamma < \alpha < 1$. We find also that each ι_t is stationary with respect to Ξ and F, ie.

$$\mathfrak{H}(\iota_t) = 0 \qquad\qquad \text{for all sufficiently small } |t| \geqslant 0$$

since each level set boundary is locally minimizing inside Ω as a current. Moreover, by lemma 5.14, we recall that

$$\kappa_t : f^{-1}(0) \longrightarrow f^{-1}(t) \qquad \text{with} \qquad \|\kappa_t - \kappa_0\|_{2,\gamma} \longrightarrow 0 \quad \text{for} \quad t \to 0.$$

Hence, for all sufficiently small $t > 0$, we conclude that $\mathfrak{F}$ is defined on κ_t while ι_t lies in the image of $\mathfrak{F}$. However, as the geometric boundaries of the submanifolds $\partial\{u \geqslant t\}$ and $\mathrm{K}(\Xi, t) = (\mathfrak{F}(\kappa_t))(\Xi)$ coincide, the uniqueness properties of $\mathfrak{F}$ from theorem 2.29 now assure that

$$\mathrm{K}(\Xi, t) = \partial\{u \geqslant t\} \subset u^{-1}(t)$$

and the claim is proved.

Thus, by evaluating with u, the above claim yields

$$u(\mathrm{K}(x,t)) = t, \qquad \text{that is} \qquad u \circ \mathrm{K} = \mathrm{proj}_2$$

for all $|t| \leqslant T$ and $x \in \Xi$. We will show finally that we may locally invert K to finish the proof.

Step 3: We may decompose

$$\partial_t \mathrm{K}_{|t=0} = V + W$$

with

$$V(x) \in T_x\Xi \qquad \text{and} \qquad W(x) \in N_\Xi(x)$$

for all $x \in \Xi$. Set $\zeta = W \cdot N_\Xi$ and, for a contradiction, let $x_0 \in \Xi \cap \Omega$ be such that $\zeta(x_0) = 0$. Possibly decreasing $T > 0$, we may replace K by another one-parameter family

$$\tilde{\mathrm{K}}(x,t) := \mathrm{K}(\tilde{\kappa}_t(x),t) \qquad \text{for } x \in \Xi \text{ and } t \in (-T,T),$$

where $\tilde{\kappa}_t(x) := \tilde{\kappa}(x,t)$ is constructed with the flow of the negative tangential projection of $\partial_t \mathrm{K}_{|t=0}$ near x_0 and suitably cut off towards $\partial\Xi$. We may assume that $\tilde{\kappa}$ is of class C^2 and as $\tilde{\mathrm{K}}$ only differs by a t-dependent reparametrization of Ξ, it embeds the same images $\partial\{u \geqslant t\}$. Thus we find

$$\partial_t\tilde{\mathrm{K}}_{|t=0} = W = \zeta N_\Xi \qquad \text{on a neighborhood of } x_0 \text{ in } \Xi \cap \Omega, \text{ hence} \qquad \partial_t\tilde{\mathrm{K}}_{|t=0}(x_0) = 0,$$

but on the other hand, we find for all $x \in \Xi$ and $h \in (-T,T)$ that

$$|h|^{-1}|\tilde{\mathrm{K}}(x,h) - \tilde{\mathrm{K}}(x,0)| = \frac{|\tilde{\mathrm{K}}(x,h) - \tilde{\mathrm{K}}(x,0)|}{|u(\tilde{\mathrm{K}}(x,h)) - u(\tilde{\mathrm{K}}(x,0))|}$$

such that

$$\left|\partial_t\tilde{\mathrm{K}}(x)_{|t=0}\right| = \lim_{h\to 0} \frac{|\tilde{\mathrm{K}}(x,h) - \tilde{\mathrm{K}}(x,0)|}{|u(\tilde{\mathrm{K}}(x,h)) - u(\tilde{\mathrm{K}}(x,0))|} \geqslant K^{-1},$$

where $0 < K < \infty$ is any Lipschitz constant of u.

Since ι_Ξ is an immersion of Ξ and, by the above, $\partial_t \mathrm{K}(x_0, 0)$ is complementary for any $x_0 \in \Xi \cap \Omega$, we deduce that the differential of K at $(x_0, 0)$ is of full rank for any $x_0 \in \Xi \cap \Omega$. Thus, we may apply the inverse function theorem near x_0 and we find a local inverse to K on some small open neighborhood of x_0 in Ω.

We are thus in position to invert K to receive

$$u = \mathrm{proj}_2 \circ \mathrm{K}^{-1} \qquad \text{on some open neighborhood of } x_0,$$

where the latter function is of class C^{M} on its domain of definition. We subsequently prove the proposition by covering $\Xi \cap \Omega$ up to the boundary of the domain Ω. ■

Finally, we also derive a result on an open and dense set in the same fashion, for which we repeat the full set of data assumptions ($\mathscr{D}$).

7.2 Theorem (Partial Regularity for Anisotropic Functions of Least Gradient). *Let $\Omega \subset \mathbb{R}^n$ be an open domain of class C^m and assume $f \in \mathrm{C}^m(\partial\Omega)$ where $4 \leqslant m \leqslant \infty$. Let $\Phi : \overline{\Omega}\times\mathbb{R}^n \to \mathbb{R}$ be an elliptic even anisotropic total variation such that Φ and $D_2\Phi$ are of class $\mathrm{C}^{\mu-1,\alpha}$ with $3 \leqslant \mu \leqslant \infty$ and $0 < \alpha < 1$. Suppose $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ is a function of anisotropic least gradient with respect to Φ with trace values $u_{\partial\Omega} = f$, let $m \geqslant n - 1$ and the singularity assumption ($\mathscr{S}$) hold.*

Given these assumptions, it follows that there exists an open and dense set $\mathscr{O}_u \subset \Omega$ such that

$$u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_u)$$

holds, where again $\mathrm{M} = \min(\mu - 2, m - 3)$.

Proof. We rephrase the proof by using mildly regular values. Owing to the lower bound on m and the theorem of Sard, we deduce that

$$\mathscr{L}^1([a,b]\backslash\mathcal{U}) = 0,$$

ie. almost every value of u is mildly regular. In particular, almost every value t allows for an open neighborhood $\mathscr{O}_t \subset \Omega$ as in theorem 7.1. Let us additionally define the open set $\mathbf{O} \subset \Omega$ as the union of all open balls inside Ω such that u is constant on one such ball. Then we may set

$$\mathscr{O}_u := \bigcup_{t\in\mathcal{U}} \mathscr{O}_t \cup \mathbf{O},$$

which is clearly open in Ω. One can see that it is also dense as follows. Assume that $\mathscr{O}_u$ would not be dense, then we could find an open ball $\mathbf{U}^* \subset \Omega$ such that $\mathbf{U}^*$ is not contained in $\mathscr{O}_u$. Since u cannot be constant on $\mathbf{U}^*$, its image $u(\mathbf{U}^*)$ is necessarily an (non-degenerated) interval J. Then however $J \subset [a,b]\backslash\mathcal{U}$, which contradicts the full measure of $\mathcal{U}$ in $[a,b]$. ■

It is not clear to what extent the foregoing theorem turns out to be sharp. Additional sufficient conditions for better regularity properties of u remain to be investigated. I also want to highlight that we do not know whether

$$\mathscr{L}^n(\Omega\backslash\mathscr{O}_u) = 0 \qquad \text{or} \qquad \mathscr{L}^n(\Omega\backslash\mathscr{O}_u) > 0$$

holds true for a general function of least gradient u.

On the other hand, it is immediate that the maximum regularity we may expect *in full generality* is class C^m, since the planar function

$$U(x,y) = j\left(\frac{y}{x}\right) \qquad \text{for } x,y \in \mathbb{R},\ j \in \mathrm{C}^m(\mathbb{R})$$

is of least gradient for suitable domains, say, some closed ball $\mathbf{B} \subset \mathbb{R}^2\backslash\{x=0\}$, while possibly not being C^{m+1}.

We will now use the occasion of this section to recall the very enlightening example by John E. Brothers from [46, 0. Introduction].

7.3 Example (Brothers)**.** Let $\mathbf{B} \subset \mathbb{R}^2$ denote the closed standard unit ball of $\mathbb{R}^2$ and denote by $\mathbf{S}$ its boundary. We let $f((x,y)(\varphi)) = \cos(2\varphi) \in \mathrm{C}^\omega(\mathbf{S})$. Then the Lipschitzian function u with

$$u(x,y) = \begin{cases} 2x^2 - 1 & \text{if } |x| \geqslant 1/\sqrt{2},\ |y| \leqslant 1/\sqrt{2}, \\ 0 & \text{if } |x| \leqslant 1/\sqrt{2},\ |y| \leqslant 1/\sqrt{2}, \\ 1 - 2y^2 & \text{if } |x| \geqslant 1/\sqrt{2},\ |y| \geqslant 1/\sqrt{2}, \end{cases}$$

on the closed unit ball $\mathbf{B}$ is the unique function of isotropic least gradient for f.

7.4 Remark. The classical purpose of this example was to illustrate that functions of least gradient might not be everywhere regular, but still allow for a "large" part of the domain to be. Notice also that example 7.3 uses analytic boundary data and achieves analytic partial regularity. We remark also that the function u from example 7.3 fulfills $\mathscr{L}^2(\mathbf{B}\setminus\mathscr{O}_u) = 0$.

We consider now an apropriately tweaked version of example 7.3.

7.5 Example (INTERPOLATING π). Let us define a sequence of boundary data $f_n \in \mathrm{C}^\omega(\mathbf{S})$ via

$$f_n((x,y)(\varphi)) = \cos(2n\varphi) \qquad \text{such that} \qquad \#\left(f_n^{-1}(0)\right) = 4n.$$

Then there exists a unique Lipschitzian function $u_n \in \mathrm{C}^{0,1}(\overline{\Omega})$, which is of isotropic least gradient with boundary data f_n. One readily verifies (eg. by "drawing" level sets) that $u_n^{-1}(0) \subset \mathbf{B}$ is the closed $4n$-gon with vertices composed of $f_n^{-1}(0)$. In particular, we find

$$\mathscr{L}^2(\mathbf{B} \setminus u_n^{-1}(0)) \longrightarrow 0 \qquad \text{as } n \to \infty,$$

which shows that the subset of $\mathscr{O}_u \subset \Omega$, on which an extremal is non-constant, can yet become arbitrarily small in $\mathscr{L}^n$-measure.

7.2 Some Specific Regularity Theorems.

We close this section with a collection of results for specific assumptions on differentiability of the involved data which our theorems 7.1 and 7.2 allow.

7.6 Corollary (SOME IMPLICATIONS AND REGULARITY THEOREMS). *Assume that Ω is an open domain of class C^m and $f \in \mathrm{C}^m(\partial\Omega)$ with $m \geqslant 4$. Let $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ be a function of anisotropic least gradient with respect to the elliptic total variation Φ with $u_{\partial\Omega}$ and let $0 \in \mathcal{U}$. Assume that assumption $(\mathscr{S})$ holds.*

(α) *If $\Phi = \Theta$ has constant coefficients and Θ is at least of class $\mathrm{C}^{3,\alpha}$ for some $0 < \alpha < 1$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^1 on $\mathscr{O}_0$.*

(β) *If $\Phi = \mu\Theta$ where Θ has constant coefficients and μ is a positive weight function on $\overline{\Omega}$ such that Θ is at least of class $\mathrm{C}^{3,\alpha}$ and μ is at least of class $\mathrm{C}^{2,\alpha}$ for some $0 < \alpha < 1$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^1 on $\mathscr{O}_0$.*

(γ) *The above item holds in particular for the weighted isotropic case, ie. when*

$$\Phi(x,v) = \mu(x)|v| \qquad \textit{such that} \qquad \mu \in \mathrm{C}^{2,\alpha}(\overline{\Omega})$$

for some $0 < \alpha < 1$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^1 on $\mathscr{O}_0$.

(δ) *More generally, if Φ is a Riemannian total variation, ie. when*

$$\Phi(x,v) = \sqrt{v^T \mathcal{G}(x) v} \qquad \textit{where} \qquad \mathcal{G} \in \mathrm{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^{n\times n})$$

for some $0 < \alpha < 1$ and the matrix field $\mathcal{G}$ is uniformly positive definite on $\overline{\Omega}$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^1 on $\mathscr{O}_0$.

Let us for now assume that $m = \infty$ (or $m = \omega$).

(ϵ) *If $\Phi = \Theta$ has constant coefficients and Θ is of class $\mathrm{C}^{\varsigma,\alpha}$ for some $3 \leqslant \varsigma \leqslant \infty$ and $0 < \alpha < 1$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class $\mathrm{C}^{\varsigma-2}$ on $\mathscr{O}_0$.*

(ζ) *If Φ is a Riemannian total variation and $\mathcal{G}$ is of class $\mathrm{C}^{\varsigma,\alpha}$ for some $2 \leqslant \varsigma \leqslant \infty$ and $0 < \alpha < 1$ on $\overline{\Omega}$, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class $\mathrm{C}^{\varsigma-1}$ on $\mathscr{O}_0$.*

We shall now assume that Φ is smooth or analytic. Then:

(η) *If $m \geqslant 4$ and $\partial\Omega$ and f are of class C^m, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^{m-3} on $\mathscr{O}_0$.*

At last, we choose a fixed $l \in \mathbb{N}$.

(θ) *If Φ and $D_2\Phi$ are of class $\mathrm{C}^{l+1,\alpha}$ for some $0 < \alpha < 1$ with $\partial\Omega, f$ being of class C^{l+3}, then there exists an open neighborhood $\mathscr{O}_0 \subset \Omega$ such that $u^{-1}(0) \subset \mathscr{O}_0$ and u is of class C^l on $\mathscr{O}_0$.*

If $m \geqslant n - 1$, then all of the preceding regularity conclusions also hold on an open and dense set $\mathscr{O}_u \subset \Omega$.

7.7 Remark (The Isotropic Case). We furthermore recover the claimed results from [46, 4.4,4.5] with corollary 7.6(η) but we unfortunately lose another degree of differentiability owing to the different choice of function spaces.

7.8 Remark (The Weighted Case). In view of the existence results of [47], where C^2-regularity for $\partial\Omega$ and f suffices to construct Lipschitzian functions of weighted isotropic least gradient (together with additional assumptions independent of the differentiability, of course), higher interior regularity for weights of class $\mathrm{C}^{2,\alpha}$ from corollary 7.6(γ) appears plausible.

8 *Partial Boundary Regularity of Extremals.*

The remaining sections 8 to 11 will now investigate additional structural properties, which functions of anisotropic least gradient enjoy in the presence of local differentiability and partial regularity and which also hail to some extent from example 7.3. We shall assume throughout that the assumptions ($\mathscr{D}$) and ($\mathscr{S}$) are still in effect.

In this section, we will continue the investigation of section 7 to include partial regularity results up to the boundary. The results we achieve will yet stay conditional, ie. even if we may apply theorem 7.1 near a given level set $u^{-1}(t)$ for a value $t \in \mathcal{U}$, we do not necessarily achieve the corresponding boundary regularity at each $f^{-1}(t)$ of u.

However, much like for the classical boundary regularity results for minimizing currents, we are able to show the existence of *at least finitely many points* where boundary regularity must hold true.

8.1 The Key Identity at Geometric Boundary Points.

Without loss of generality, we assume again that $0 \in \mathcal{U}$ is our value of the extremal function $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ of anisotropic least gradient of interest. We recall first from section 7, and more particularially the proof of theorem 7.1, the existence of $T > 0$ such that

$$\mathrm{K}(x,t) := \mathfrak{F}(\kappa_t)(x) \qquad \text{for} \qquad x \in \Xi \quad \text{and} \quad 0 \leqslant |t| \leqslant T,$$

is well-defined and of class C^{M} where

$$\mathrm{M} = \min(\mu - 2, m - 3).$$

While the proof of theorem 7.1 chose a point in $\Xi \cap \Omega$, we will now place our investigation at

$$x_0 \in f^{-1}(0) \qquad \text{where} \qquad f^{-1}(0) = \partial\Xi = \Xi \cap \partial\Omega.$$

By definition of $\mathfrak{F}$, this delivers

$$\mathrm{K}(x_0, t) = \kappa_t(x_0) \qquad \text{for all} \quad 0 \leqslant |t| \leqslant T$$

and hence

$$\partial_t \mathrm{K}(x_0, t) = X(\kappa(x_0, t)) = \frac{\operatorname{grad}^{\partial\Omega} f(\kappa(x_0, t)}{|\operatorname{grad}^{\partial\Omega} f(\kappa(x_0, t)|^2}$$

due to the construction of κ as the local flow of level sets of f in section 5.3.

8.1 Lemma. *Let $v \in \mathbb{R}^n$ be tangential to $f^{-1}(0)$ at x_0. Then $\partial_t \mathrm{K}(x_0, 0) \in T_{x_0}\partial\Omega$ and*

$$\partial_t \mathrm{K}(x_0, 0) \cdot v = 0.$$

Proof. This is clear, as

$$Y := \operatorname{grad}^{\partial\Omega} f(\kappa(x_0, 0)) = \operatorname{grad}^{\partial\Omega} f(x_0) \neq 0$$

spans the normal space to $f^{-1}(0)$ at x_0 inside $T_{x_0}\partial\Omega$ and X is just a rescaled version. ■

The latter lemma especially implies the orthogonal decomposition

$$T_{x_0}\partial\Omega = T_{x_0}f^{-1}(0) \quad \ominus \quad \mathbb{R}\cdot X(x_0)$$

and we record the following property of $D\,\mathrm{K}$ at boundary points x_0.

8.2 Lemma (Rank $(n-1)$-Property of the Derivative). *For each $x_0 \in f^{-1}(0)$ we have that*

$$\operatorname{rank} D_x\,\mathrm{K}(x_0,0) = n-1 \qquad \textit{and} \qquad \operatorname{rank} D\,\mathrm{K}(x_0,0) \geqslant n-1.$$

Proof. As we know that

$$\mathrm{K}(x_0,0) = \iota_{\Xi}(x_0,0)$$

and Ξ is a submanifold with boundary of class at least $\mathrm{C}^{3,\alpha}$, the derivative $D_x\,\mathrm{K}$ simply embeds the abstract tangent space of Ξ into the euclidean space $\mathbb{R}^n$ and hence has full rank $n-1$, from which the lemma immediately follows. ∎

These assessments all follow rather immediately and correspond roughly to the fact that Ξ admits an extension across $\partial\Xi$ (a fact which we also already used before) and the full tangent space at boundary points is the tangent space to that extension. For our purposes it is however necessary to actually achieve

$$\operatorname{rank} D\,\mathrm{K}(x_0,0) = n,$$

since we are interested in inverting K. We will now formally prove that the full rank property from the interior case also leads to local boundary regularity, while we will discuss sufficiencies for this matter in section 8.2.

8.3 Corollary (Partial Regularity and Partial Boundary Regularity). *Assume that $D\,\mathrm{K}(x_0,0)$ is invertible at $x_0 \in f^{-1}(0)$. Then there exists an open neighborhood $\mathscr{O}_{x_0} \subset \overline{\Omega}$ of x_0 such that*

$$u_{\mathscr{O}_{x_0}} \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_{x_0}).$$

where $\mathrm{M} = \min(\mu-2, m-3)$.

Proof. We recall by boundary regularity that Ξ is a submanifold with boundary of class $\mathrm{C}^{\nu,\alpha}$, where $\nu = \min(\mu, m-1) \geqslant 3$ and we shall extend Ξ across $\partial\Xi$ to an open submanifold $\tilde{\Xi} \subset \mathbb{R}^n$ of corresponding class $\mathrm{C}^{\nu,\alpha}$. Since

$$\mathrm{K} : \Xi \times (-T,T) \longrightarrow \mathbb{R}^n$$

and we consider $x_0 \in \partial\Xi$, we may consider x_0 as an interior point of $\tilde{\Xi}$ and extend K to a function $\tilde{\mathrm{K}}$ such that

$$\tilde{\mathrm{K}} : \tilde{\Xi} \times (-T,T) \longrightarrow \mathbb{R}^n$$

is again of class C^{M} (since $\nu \geqslant \mathrm{M}$) in both its variables. Applying the inverse function theorem to $\tilde{\mathrm{K}}$ at $(x_0,0)$ leads to connected open neighborhoods $U \subset \tilde{\Xi}$ of x_0 and $V \subset \mathbb{R}^n$ of x_0 as well as $T > 0$ such that

$$\tilde{\mathrm{K}} : U \times (-T,T) \longrightarrow V$$

is a diffeomorphism of class C^{M}. As

$$\tilde{\mathrm{K}}_{(U\cap\Xi)\times(-T,T)} = \mathrm{K},$$

we infer that

$$\tilde{\mathrm{K}}(x,t) \in \partial\{u \geqslant t\} \qquad \text{whenever} \qquad x \in U \cap \Xi, \quad 0 \leqslant |t| < T$$

and thus

$$u(\tilde{\mathrm{K}}(x,t)) = \mathrm{proj}_2(x,t) \qquad \text{whenever} \qquad x \in U \cap \Xi, \quad 0 \leqslant |t| < T.$$

The function $\mathrm{proj}_2 \circ \tilde{\mathrm{K}}^{-1}$ is further of class C^{M} on V. As

$$\tilde{\mathrm{K}}(x,t) = \kappa(x,t) \qquad \text{for all} \quad x \in \partial\Xi \cap U, \quad 0 \leqslant |t| < T,$$

it follows by remark 5.15(2) that we may choose U such that $(U \backslash \partial\Xi) \times (-T,T)$ has exactly two connected components and that

$$\tilde{\mathrm{K}}((\partial\Xi \cap U) \times (-T,T)) = V \cap \partial\Omega.$$

Especially, also $V\backslash\partial\Omega$ has exactly two connected components by diffeomorphicity and hence, they have to be contained in Ω or in $\mathbb{R}^n\backslash\overline{\Omega}$. From this we deduce that

$$\tilde{\mathrm{K}}((U \cap \Xi) \times (-T,T)) = V \cap \overline{\Omega} =: \mathscr{O}_{x_0},$$

with

$$u = \mathrm{proj}_2 \circ \tilde{\mathrm{K}}^{-1} \qquad \text{on} \quad \mathscr{O}_{x_0} \subset \overline{\Omega}$$

and the proof is finished. ∎

8.2 Sufficient Local Conditions for Partial Boundary Regularity.

Now we shall try to find sufficient conditions to enable corollary 8.3 and prove partial boundary regularity. Note that "partial" here is somewhat doubly referred to, as first not each level set admits regularity at all, but even if we are dealing with eg. mildly regular values, then not each boundary point might admit boundary regularity.

We begin this section by using the lemmas 8.1 and 8.2 to reformulate the full rank condition in some first order properties of the involved hypersurfaces.

8.4 Proposition (TRANSVERSALITY AND PARTIAL BOUNDARY REGULARITY). *Let* $0 \in \mathcal{U}$ *and* $x_0 \in f^{-1}(0)$. *Then it holds*

$$\mathrm{rank}\, D\,\mathrm{K}(x_0,0) = n \qquad \Longleftrightarrow \qquad \Xi \textit{ intersects } \partial\Omega \textit{ transversely at } x_0.$$

Proof. We recall first that two submanifolds intersect transversely at x_0 if their tangent spaces at x_0 span $\mathbb{R}^n$, ie. here

$$T_{x_0}\partial\Omega + T_{x_0}\Xi = \mathbb{R}^n.$$

Let us first assume that $\mathrm{rank}\, D\,\mathrm{K}(x_0,0) = n$. This means

$$\mathrm{im}\, D_x\,\mathrm{K}(x_0,0) + \mathbb{R}\cdot\partial_t\,\mathrm{K}(x_0,0) = T_{x_0}\Xi + \mathbb{R}\cdot\partial_t\,\mathrm{K}(x_0,0) = \mathbb{R}^n.$$

Since in particular $\partial_t\,\mathrm{K}(x_0,0) \in T_{x_0}\partial\Omega$ by lemma 8.1, we deduce that intersection is transverse at x_0.

For the contrary, let the intersection at x_0 be transverse. By construction, we have

$$T_{x_0} f^{-1}(0) \subset T_{x_0} \partial\Omega \qquad \text{and} \qquad T_{x_0} f^{-1}(0) \subset T_{x_0} \Xi$$

and $\partial_t \mathrm{K}(x_0, 0)$ is orthogonal to $T_{x_0} f^{-1}(0)$ by lemma 8.1. Hence, if $\partial_t \mathrm{K}(x_0, 0) \in T_{x_0}\Xi$ would hold, then

$$T_{x_0} \partial\Omega = T_{x_0} f^{-1}(0) \quad \oplus \quad \mathbb{R} \cdot \partial_t \mathrm{K}(x_0, 0) = T_{x_0}\Xi,$$

which contradicts transversality at x_0. Thus, $\partial_t \mathrm{K}(x_0, 0) \notin T_{x_0}\Xi$ and we infer with the aid of lemma 8.2 that $\operatorname{rank} D\,\mathrm{K}(x_0, 0) = n$. ■

8.5 Remark. Proposition 8.4 allows for a different condition to check if information about the local geometry is available. Note that this theorem does not claim that a transverse intersection is necessary for partial boundary regularity of u, but provides a reformulation for our sufficient condition. Also, I do not know to what genericity such transverse intersection is fulfilled.

Even though we equivalently expressed our condition in geometric terms, we still do not know whether any such point actually always exist. The following argument will prove the existence of at least some points.

8.6 Theorem (Existence of Boundary Regular Points.). *Let $0 \in \mathcal{U}$. Then each connected component of Ξ admits at least one point $x_0 \in \partial\Xi$ such that there is an open neighborhood $\mathscr{O}_{x_0} \subset \overline{\Omega}$ of x_0 where*

$$u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_{x_0}) \qquad \textit{for} \quad \mathrm{M} = \min(\mu - 2, m - 3).$$

Proof. As every connected component of Ξ is a submanifold with boundary of the corresponding regularity of Ξ, it is furthermore true that each connected component does not admit non-trivial Jacobi fields. Thus we directly assume Ξ to be connected without relabeling and prove the existence of at least one regular point $x_0 \in \partial\Xi$.

We decompose again like in the proof of theorem 7.1 for

$$\partial_t \mathrm{K}_{|t=0} = V + W \qquad \text{on } \Xi$$

such that

$$V(x) \in T_x\Xi \qquad \text{and} \qquad W(x) \in N_\Xi(x)$$

and claim that

$$\zeta = W \cdot N_\Xi = \partial_t \mathrm{K}_{|t=0} \cdot N_\Xi$$

is a solution to the Jacobi equation from definition 2.28 on Ξ. To this end, we recall by construction

$$\mathfrak{H}(\boldsymbol{\mathfrak{F}}(\kappa_t)) = 0 \qquad\qquad \text{for all sufficiently small } |t| \geqslant 0,$$

from which we infer that

$$h^{-1}\left(\mathfrak{H}(\boldsymbol{\mathfrak{F}}(\kappa_h)) - \mathfrak{H}(\boldsymbol{\mathfrak{F}}(\kappa_0)\right) = h^{-1}\left(\mathfrak{H}(\boldsymbol{\mathfrak{F}}(\kappa_h)) - \mathfrak{H}(\iota_\Xi)\right) = 0 \qquad \text{for all sufficiently small } |h| > 0$$

and thus, since $t \longmapsto \mathfrak{F}(\kappa_t)$ is at least of class C^1 for $|t|$ near 0, we find

$$D\mathfrak{H}(\iota_\Xi)\left(\frac{d}{dt}_{|t=0}\mathfrak{F}(\kappa_t)\right) = 0.$$

We deduce that $\partial_t \mathrm{K}_{|t=0}$ is of class $\mathrm{C}^{2,\gamma}$ and it is a Jacobi field on Ξ with respect to ι_Ξ and F. We furthermore recall from step 3 of the proof of theorem 7.1 that $\zeta > 0$ on $\Xi \cap \Omega$.

Assume then for a contradiction that Ξ and $\partial\Omega$ intersect nowhere transversely, which we may equivalently express as

$$N_\Xi(x) = N_{\partial\Omega}(x) \qquad \text{for all} \qquad x \in f^{-1}(0).$$

Since we have by lemma 8.1 that

$$\partial_t \, \mathrm{K}_{|t=0}(x) \in T_x\partial\Omega \qquad \text{for all} \qquad x \in f^{-1}(0),$$

this yields

$$\zeta = \partial_t \, \mathrm{K}_{|t=0} \cdot N_\Xi = \partial_t \, \mathrm{K}_{|t=0} \cdot N_{\partial\Omega} \equiv 0 \qquad \text{on} \qquad \partial\Xi.$$

Consequently, as W is now a homogeneous Jacobi field on Ξ, ie. $W \in \mathfrak{K}$, we infer $\zeta \equiv 0$ on Ξ by theorem 6.8 via $\dim \mathfrak{K} = 0$, which is a contradiction.

Hence, we have shown that there must at least be one $x_0 \in \partial\Omega$ such that Ξ and $\partial\Omega$ intersect transversely at x_0 and we may subsequently apply proposition 8.4 to get the desired neighborhood $\mathscr{O}_{x_0} \subset \overline{\Omega}$ with $u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_{x_0})$. ■

9 *Fine Properties of Lipschitzian Functions of Least Gradient and Regularity.*

Here we will review the a priori assumptions that we have imposed on the values of u that we want to consider. While values hailing from the set $\mathcal{U}$, the mildly regular values, were sufficient to induce partial regularity for the extremal, we may actually even induce local regularity (value-wise) by weaker assumptions. Furthermore, we may prove that the gradient actually behaves rather rigid on such level sets and translates seamlessly to the smooth setting.
These strategies were (yet again) somewhat referenced in [46], but the actual application was without details or was only sketched in the introduction of that paper. We shall of course, additionally, prove these results directly for the anisotropic case as before.

In what follows, we will mainly deal with values and properties of the extremal on the level set for this value. Therefore, we propose to refer to these properties as "fine" properties of the function of least gradient (to some extent in the spirit of the paper [8]) and its level set structure. The main change in this section is to drop the assumption on almost-everywhere existence of Du on some level set with respect to $\mathscr{H}^{n-1}$.

9.1 Rectifiable Level Sets and Jacobi Admissibility.

Let us first observe the following lemma on the level set structure, which will prove to be useful.

9.1 Lemma. *Let $t \in \operatorname{int} \operatorname{rg}(f)$ be such that*

$$\mathscr{L}^n(u^{-1}(t)) = 0 \qquad \textit{and} \qquad t \textit{ is a regular value for } f.$$

Then

$$u^{-1}(t) = \partial\{u \geqslant t\} = \partial\{u \leqslant t\}.$$

Proof. We treat the case $u^{-1}(t) = \partial\{u \geqslant t\}$. By continuity, we have that $\partial\{u \geqslant t\} \subset u^{-1}(t)$ and we assume for a contradiction that $x_0 \in u^{-1}(t)$ with $x_0 \notin \partial\{u \geqslant t\}$. By t being a regular value for f, it holds $x_0 \in \Omega$. If $x_0 \notin \partial\{u \leqslant t\}$, then there is an open neighborhood around x_0, where $u \equiv t$ and a contradiction is immediate. Hence $x_0 \in \partial\{u \leqslant t\} \cap \Omega$. As $x_0 \in \operatorname{spt} |D1_{\{u \leqslant t\}}|$, it holds

$$\mathscr{L}^n(\{u \leqslant t\} \cap \mathbf{U}(x_0, \rho)) > 0 \qquad \text{for all } \rho > 0.$$

On the other hand, since $x_0 \notin \partial\{u \geqslant t\}$, there is by continuity some $\rho > 0$ with

$$\{u < t\} \cap \mathbf{U}(x_0, \rho) = \varnothing \qquad \Longrightarrow \qquad \{u \leqslant t\} \cap \mathbf{U}(x_0, \rho) = u^{-1}(t) \cap \mathbf{U}(x_0, \rho).$$

Together we receive

$$\mathscr{L}^n(u^{-1}(t) \cap \mathbf{U}(x_0, \rho)) > 0,$$

which is again a contradiction. ■

9.2 Remark. Note that this proof works as long as the level set of f equals it sub- and superlevel set boundaries on $\partial\Omega$, eg. also in rectifiable cases. In [41] we furthermore find an interpretation in the distributional setting.

We recall that we may assume that

$$\operatorname{rg} f = \operatorname{rg} u = [a, b] \qquad \text{for some real numbers } a < b$$

by assumption ($\mathscr{S}$) and our geometric maximum principles. Lemma 9.1 motivates the next class of values that we want to investigate.

9.3 Definition (RECTIFIABLE VALUES)**.** We will say that t is *a rectifiable value* when $t \in \mathcal{V}$, where

$$\mathcal{V} := \{\, t \in (a, b) \quad | \quad \mathscr{L}^n(u^{-1}(t)) = 0, \\ t \text{ is a regular value of } f \,\}.$$

Yet again, t being regular for f of course excludes a and b, while lemma 9.1 shows that the boundaries equal the level set and thus the level sets themselves are in particular rectifiable. In this section, we will work with the class $\mathcal{V}$, while we will also drop the last condition in the next section to investigate fat level sets.

We now want to find out how behavior of the gradient on a rectifiable level set precisely matters for the regularity theory.

9.4 Remark. This point is somewhat hidden, but existent, in the original paper. Indeed, in [46, 4.5 Theorem], the authors induce partial regularity by what I would call mildly regular values, while the original assumptions throughout for the regularity near a fixed level set only need the existence of a certain sequence related to the gradient of u (cf. [46, p. 510, 1.1(5,v)]). Roughly speaking, we need difference quotients at one point to be bounded away from zero (while we may compare then for each other point via the Harnack-Type inequality). I do however think that [46, p. 522, 4.2 Theorem, proof] needs an additional argument to compare if the gradient need not exist (there is also no notion of the point x^* from [46, 1.1(5,v)] in that proof).

Let us again set $t = 0$ and denote $u^{-1}(0) =: \Xi$. As $0 \in \mathcal{V}$, we have again that Ξ is a submanifold with boundary of class $\mathrm{C}^{\nu,\alpha}$ and we recall from proposition 4.15(2) that Ξ can only have finitely many connected components such that each one has non-empty geometric boundary contained in $f^{-1}(0) \subset \partial\Omega$.

Comparing with $0 \in \mathcal{U}$, where derivatives have existed on each connected component, we now single out one (of the finitely many) connected components of Ξ which we denote

$$\Sigma \subset \Xi.$$

The hypersurface Σ inherits regularity from Ξ and we set

$$\Theta := \Sigma \cap \partial\Omega \subset f^{-1}(0).$$

Thus, the set $\Sigma \subset \mathbb{R}^n$ is a submanifold with boundary of $\mathbb{R}^n$ which is of class $\mathrm{C}^{\nu,\alpha}$. The boundary submanifold $\Theta \subset \mathbb{R}^n$ is of class C^m.

We recall the assumption from [46] and provide a clarifying lemma to refer more easily to such a property.

9.5 Definition (JACOBI ADMISSIBILITY)**.** We say that the connected component Σ of Ξ is *Jacobi* *admissible* if there exists some $x^* \in \Sigma \cap \Omega$ and sequences

$$x_i^* \longrightarrow x^* \qquad \text{and} \qquad t_i^* \longrightarrow 0$$

as $i \to \infty$ such that

$$t_i^* > 0 \qquad \text{and} \qquad x_i^* \in \partial\{u \geqslant t_i^*\} \cap \Omega$$

and

$$0 < \lim_{i\to\infty} \frac{|u(x_i^*) - u(x^*)|}{|x_i^* - x^*|} < \infty.$$

9.6 Lemma (Characterization of Jacobi Admissibility). *It holds, up to possibly changing the sign of u:*

$$\begin{aligned} & \Sigma \textit{ is Jacobi admissible.} \\ \Longleftrightarrow \quad & \textit{There exists } x^* \in \Sigma \cap \Omega \textit{ such that } Du(x^*) \neq 0 \textit{ or } Du \textit{ does not exist at } x^*. \end{aligned}$$

Proof. Clearly, when Σ is Jacobi admissible, the gradient may not exist and vanish at the distinguished point $x^* \in \Sigma \cap \Omega$ and the implication holds.

Conversely, let an $x^* \in \Sigma \cap \Omega$ be given such that $Du(x^*)$ does not exist and vanish. Hence, either $Du(x^*) \neq 0$ or Du does not exist at x^*.

If $Du(x^*) \neq 0$, we may decompose

$$\mathbb{R}^n = T_{x^*}\Sigma \quad \ominus \quad \mathbb{R}\, N_\Sigma(x^*)$$

and notice, since $u \equiv 0$ on Σ, that

$$Du(x^*) = \partial_{N_\Sigma(x^*)} u(x^*) N_\Sigma(x^*) \qquad \text{with} \quad \partial_{N_\Sigma(x^*)} u(x^*) \neq 0.$$

By definition of the directional derivative, we find

$$0 \neq \partial_{N_\Sigma(x^*)} u(x^*) = \lim_{h\to 0} h^{-1}(u(x^* + hN_\Sigma(x^*)) - u(x^*)),$$

which yields a suitable sequence with appropriately bounded difference quotients along the normal ray. Finally, the intermediate value theorem for continuous functions assures that we may choose such points in the superlevel set boundaries.

Assume now further that Du does not exist at $x^* \in \Sigma \cap \Omega$. Thus, we can find at least one sequence $z_i \longrightarrow x^*$ such that

$$0 < \lim_{i\to\infty} \frac{|u(z_i) - u(x^*)|}{|z_i - x^*|} < \infty$$

holds true (as otherwise Du would exist and vanish at x^*). We may assume $z_i \notin u^{-1}(0)$ and select a subsequence to arrange that

$$(z_i)_{i\in\mathbb{N}} \subset \{u < 0\} \qquad \text{or} \qquad (z_i)_{i\in\mathbb{N}} \subset \{u > 0\}.$$

A possible passage from u to $-u$ allows us to choose the second alternative, while the function $-u \in \mathrm{BV}(\Omega)$ is again a function of anisotropic least gradient. It remains to show that we may always choose a point in the corresponding superlevel set boundary.
Consider the straight line from x^* to z_i. As u is continuous, this line has to intersect the superlevel set boundary $\partial\{u \geqslant u(z_i)\} \cap \Omega$ in some point $x_i^* \in \Omega$. Thus

$$|x^* - x_i^*| \leqslant |x^* - z_i|,$$

implying

$$\frac{|u(z_i) - u(x^*)|}{|z_i - x^*|} \leqslant \frac{|u(x_i^*) - u(x^*)|}{|x_i^* - x^*|} < \infty$$

and the claim follows by taking the $\liminf$ and passing to the subsequence of (x_i^*) realizing it. ■

9.7 Remark (The Sign of the Extremal). The slight abuse of notation of the sign of u without loss of generality is also referenced in [46, 1.1(5,(vi-vii)]. Without it, we could proceed all arguments regarding Jacobi admissibility with

$$t_i^* < 0 \qquad \text{and} \qquad x_i^* \in \partial\{u \leqslant t_i^*\} \cap \Omega,$$

which is possible as $0 \in \mathcal{V}$ implies that Ξ equals its superlevel and sublevel set boundaries by lemma 9.1 and we can approximate from both sides value-wise. Also, if we pass from u to $-u$ and the anisotropic total variation is not even in the second variable, then we also have to pass to

$$\Phi^*(x, v) := \Phi(x, -v)$$

when changing the sign, and $-u$ is a function of anisotropic least gradient with respect to Φ^*.

9.2 Regularity near Rectifiable Level Sets.

We can now return to the regularity theory and our approach is to deal with local regularity and rigidity results near rectifiable level sets.

Obviously, there is

$$0 \in \mathcal{U} \qquad \Longrightarrow \qquad 0 \in \mathcal{V}$$

and each of the finitely many connected components of a mildly regular zero level set Ξ is Jacobi admissible. The results of the preceding sections 7 and 8 hence have to follow as special cases of what is possible for a general rectifiable level set.

9.8 Remark (On the Necessity of the Non-Vanishing Gradient). We recall for $0 \in \mathcal{U}$ from section 6 and more particularily lemma 6.5 that the assumption that there is some $x^* \in \Xi \cap \Omega$ with $Du(x^*) \neq 0$ has entered for the first time in that particular result and the following construction of the "special" Jacobi field ζ_0 with positive lower bound. Regarding the remaining results, everything was independent of the behavior of the gradient on the level set. We recall from the preceding sections, if only $0 \in \mathcal{V}$, that we also have:

(1) The Hausdorff convergence $\mathcal{H}(\Xi, u^{-1}(t)) \longrightarrow 0$ and the locally uniform expressions about tangent spaces of Ξ as in section 2.4 and 4.3.

(2) The existence of a one-parameter family of normal graph representations w_t on boundary retracts Ω_ρ as in proposition 5.9 with Harnack-type inequalities as in section 6.1 and theorem 6.3.

(3) The existence of a one-parameter family of global diffeomorphic deformations ι_t as in section 5.4 and theorem 5.20.

We will use these results freely in the remainder of section 9 and suitable one-sided versions in section 10 in our study of fat level sets.

We can hence directly begin to construct the special Jacobi field and investigate the Jacobi nullity of connected components Σ of Ξ. To do so, let us again assume and fix $0 \in \mathcal{V}$. The next lemma relates the distinguished sequence from the definition of Jacobi admissibility to the family of normal deformations w_t in accordance with remark 9.4.

9.9 Lemma (Jacobi Admissibility and Difference Quotients in Normal Direction)**.** *Let $\Sigma \subset \Xi$ be a connected component and assume that Σ is Jacobi admissible with distinguished point $x^* \in \Sigma \cap \Omega$. Then there exist a sequence $t_i^* \to 0$ and $m^* > 0$ such that*

$$0 < m^* \leqslant \frac{\left|u\left((\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*)\right) - u(x^*)\right|}{\left|(\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*) - x^*\right|} < \infty.$$

for all $i \in \mathbb{N}$.

Proof. Recall that we have

$$(\mathrm{Id} + w_t N_\Xi)(x^*) \in \partial\{u \geqslant t\} \cap \Omega \qquad \text{and} \qquad (\mathrm{Id} + w_t N_\Xi)(x^*) \longrightarrow x^*$$

for all sufficiently small $t \geqslant 0$ such that w_t is defined near $x^* \in \Sigma \cap \Omega$. Applying theorem 6.3(1) at x^* for the local Harnack-type inequality, there are a neighborhood $V \subset \Sigma \cap \Omega$ of x^* and constants $C = C(x^*) > 0$ and $T = T(x^*) > 0$ such that

$$\sup\{w_t(x) \mid x \in V\} \leqslant C \inf\{w_t(x) \mid x \in V\}$$

for all $0 \leqslant t \leqslant T(x^*)$. By definition of Jacobi admissibility, there is some $m > 0$ and a sequence

$$x_i^* \longrightarrow x^* \qquad \text{with} \qquad u(x_i^*) > 0$$

for all $i \in \mathbb{N}$ such that we have

$$\frac{|u(x_i^*) - u(x^*)|}{|x_i^* - x^*|} \geqslant m > 0$$

for all sufficiently large i. Note also that we have

$$u\left((\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*)\right) = u(x_i^*) \qquad \text{where} \qquad t_i^* = u(x_i^*).$$

By exploting $x_i^* \longrightarrow x^*$, it follows that

$$|\Pi(x_i^*) - x_i^*| \leqslant |x^* - x_i^*| \qquad \text{and} \qquad |\Pi(x_i^*) - x_i^*| = w_{t_i^*}(\Pi(x_i^*))$$

by the nearest-point property and the bijectivity of Π near x^*. Assuming furthermore via $\Pi(x_i^*) \longrightarrow$

$\Pi(x^*)$ that

$$\Pi(x_i^*) \in V \qquad \text{and} \qquad t_i^* \leqslant T \qquad \text{for all } i \in \mathbb{N},$$

we infer

$$|x^* - x_i^*| \geqslant w_{t_i^*}(\Pi(x_i^*)) \geqslant C^{-1}\, w_{t_i^*}(x^*) = C^{-1}\, |x^* - (\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*)|,$$

which leads by taking reciprocals to

$$\begin{aligned}\infty > \frac{\left|u\left((\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*)\right) - u(x^*)\right|}{\left|(\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*) - x^*\right|} &= \frac{|u(x_i^*) - u(x^*)|}{\left|(\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*) - x^*\right|} \\ &\geqslant C^{-1} \frac{|u(x_i^*) - u(x^*)|}{|x_i^* - x^*|} \\ &\geqslant mC^{-1} =: m^* > 0\end{aligned}$$

for all large $i \in \mathbb{N}$. ■

9.10 Remark. Notice that the property above is immediate in case $Du(x^*) \neq 0$, as the boundedness away from zero holds as a matter of fact for all sequences $x^* + tN_\Xi$ approaching $x^* \in \Sigma$. If, however, the boundedness away from zero holds only true for difference quotients along *one certain sequence* approaching x^*, I do see a need to compare the two sequences of difference quotients, which is not detailed in [46], as we need a positive lower bound in normal direction to execute the blow-up along w_t.

With lemma 9.9 at hand, we may proceed with the remaining regularity theory as in the sections 7 and 8. We record the following result.

9.11 Proposition (Regularity near Jacobi Admissible Components)**.** *Let $0 \in \mathcal{V}$ and let $\Sigma \subset \Xi$ be a Jacobi admissible connected component. Then there exists an open neighborhood $\mathscr{O}_\Sigma \subset \Omega$ of Σ such that*

$$u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_\Sigma)$$

holds with $\mathrm{M} = \min(\mu - 2, m - 3) \geqslant 1$. Furthermore, there exists at least one $x_0 \in \Theta = \Sigma \cap \partial\Omega$ and an open neighborhood $\mathscr{O}_{x_0}$ of x_0 in $\overline{\Omega}$ such that

$$u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_{x_0})$$

is valid.

Proof. The only necessary essential change to be made is to show why the Jacobi nullity of Σ vanishes. Fixing the real sequence $t_i^* > 0$ with $t_i \to 0$ as $i \to \infty$ and $m^* > 0$ from lemma 9.9, we find

$$\frac{w_{t_i^*}(x^*)}{t_i^*} = \frac{\left|(\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*) - x^*\right|}{\left|u\left((\mathrm{Id} + w_{t_i^*} N_\Xi)(x^*)\right) - u(x^*)\right|} \leqslant \frac{1}{m^*}.$$

Thus, by choosing $0 < \rho < \operatorname{dist}(x^*, \partial\Omega)$, we may proceed on the connected component $\Sigma \subset \Xi$ as in lemma 6.5 and section 6.2 to use the Harnack-type inequality on retracts of theorem 6.3(2) to compare with the point $x^* \in \Sigma \cap \Omega$ for uniform bounds up to class $\mathrm{C}^{1,\alpha}$ for the family of functions w_{t_i}/t_i for

$i \in \mathbb{N}$ large enough. In particular, as in proposition 6.7, there exists

$$\zeta_0 N_\Xi \in \mathrm{C}^{2,\alpha}_{loc}(\Sigma \cap \Omega, \mathbb{R}^n)$$

with

$$D\mathfrak{H}(\zeta_0 N_\Xi) \equiv 0 \qquad \text{on} \quad \Sigma \cap \Omega \qquad \text{and} \qquad \zeta_0 \geqslant K^{-1} > 0$$

for a Lipschitz constant $K > 0$ of u and the comparison argument of theorem 6.8 shows that $\dim \mathfrak{K} = 0$ on Σ. As the proofs of theorem 7.1 and theorem 8.6 now translate without change by using the dependence map $\mathfrak{f}$ associated with the canonical inclusion ι_Σ of Σ into $\mathbb{R}^n$, our result follows. ■

9.3 Definiteness of the Gradient of an Extremal.

With the regularity criterion from section 9.2 at hand, in the final part of this section we will proceed in characterizing the behavior of the gradient on rectifiable level sets. To do so, we will draw some rather immediate further conclusions from the regularity theory and introduce a new argument related to, in a sense, the "propagation" of the gradient of u along a rectifiable level set. We find especially that Du behaves exactly as for a "usual" regular level set in the differentiable sense.

Let us first begin by disjointly decomposing the level set $\Xi := u^{-1}(0)$ for $0 \in \mathcal{V}$ such that

$$\Xi = \bigcup_{i=1}^{N} \Sigma_i = \bigcup_{j=1}^{M} \Sigma_j \cup \bigcup_{k=1}^{K} \Sigma_k, \qquad 1 < N < \infty, \qquad M + K = N,$$

where each Σ_i is a connected component and each connected component Σ_j is Jacobi admissible while each connected component Σ_k is not.

As Jacobi admissibility was particularily implied by possible non-existence of the gradient of u, we straightforwardly obtain the next proposition.

9.12 Proposition (Existence of Du)**.** *Let $0 \in \mathcal{V}$. Then the derivative Du exists everywhere on $\Xi \cap \Omega$.*

Proof. Using the above decomposition of Ξ, we find by proposition 9.11 that there is an open neighborhood $\mathscr{O}_{\Sigma_j} \subset \Omega$ such that

$$\Sigma_j \cap \Omega \subset \mathscr{O}_{\Sigma_j} \qquad \text{and} \qquad u \in \mathrm{C}^1(\mathscr{O}_{\Sigma_j})$$

for all $j = 1, ..., M$. If Du would further not exist at some $x \in \Sigma_k \cap \Omega$, then Σ_k would be Jacobi admissible. ■

Clearly, we do not know any continuity of Du transverse to each Σ_k, but we may infer by the very definition that $Du = 0$ on each $\Sigma_k \cap \Omega$. Fixing one $\Sigma := \Sigma_k$ and denoting its geometric boundary by Θ, we observe that we still have

$$\| \operatorname{grad}^{\partial\Omega} f \| \neq 0 \qquad \text{on} \qquad \Theta \subset \partial\Omega$$

as Θ is composed of finitely many connected components of the regular level set $f^{-1}(0) \subset \partial\Omega$. The next result is henceforth immediately clear.

9.13 Proposition (Lack of Boundary Regularity)**.** *Let* $0 \in \mathcal{V}$*. If* $\Sigma \subset \Xi$ *is a connected component of* Ξ *with geometric boundary* Θ *which is not Jacobi admissible, then no* $x_0 \in \Theta$ *admits an open neighborhood* $\mathscr{O}_{x_0} \subset \overline{\Omega}$ *such that*

$$Du \in \mathrm{C}(\mathscr{O}_{x_0}, \mathbb{R}^n)$$

holds. ■

9.14 Remark. The propositions 9.12 and 9.13 illustrate that connected components of rectifiable level sets which are not Jacobi admissible account for rather singular behavior of Du. It is not clear to me to what extent in genericity these components appear for functions of anisotropic least gradient (or whether possibly all connected components Σ are in fact Jacobi admissible).

We are now going to sharpen the results of proposition 9.12 in the case of Jacobi admissibility. Given a Jacobi admissible connected component $\Sigma := \Sigma_j \subset \Xi$, we can infer by proposition 9.12 that there is

$$x^* \in \Sigma \cap \Omega \qquad \text{such that} \qquad Du(x^*) \neq 0,$$

as there has to be at least one sequence of difference quotients being in absolute value bounded away from 0 at x^* while $Du(x^*)$ exists. Clearly, this however a priori does not exclude the possibility of some different

$$\bar{x} \in \Sigma \cap \Omega \qquad \text{such that} \qquad Du(\bar{x}) = 0.$$

Realizing this assertion is the content of our next theorem.

9.15 Theorem (Definiteness on Rectifiable Level Sets)**.** *Let* $0 \in \mathcal{V}$*. Then for any connected component* $\Sigma \subset \Xi$ *it holds:*

$$\textit{Either} \qquad Du \neq 0 \quad \textit{on} \quad \Sigma \cap \Omega \qquad \textit{or} \qquad Du \equiv 0 \quad \textit{on} \quad \Sigma \cap \Omega.$$

Proof. We already know by proposition 9.12 that Du necessarily exists everywhere on $\Xi \cap \Omega$ and we may assume that the connected component Σ is Jacobi admissible. Hence, it will suffice to assume that there are two points, $x^*, \bar{x} \in \Sigma \cap \Omega$, such that

$$Du(x^*) \neq 0 \qquad \text{and} \qquad Du(\bar{x}) = 0$$

and we must derive a contradiction from this.

We can find $\rho > 0$ sufficiently small such that $x^*, \bar{x} \in \Omega_\rho$ and fix $\varepsilon > 0$. Due to the gradient vanishing at $\bar{x}$, there is also $\bar{\delta} > 0$ such that

$$\frac{|u(\bar{x}) - u(y)|}{|\bar{x} - y|} < \varepsilon$$

whenever $|\bar{x} - y| < \bar{\delta}$. This corresponds to

$$\varepsilon^{-1} < \frac{|\bar{x} - y|}{|u(y)|}$$

for all such y with the aid of $\bar{x} \in \Xi$. Since we can assume that we may invert the nearest-point projection on nearby level set boundaries near $\bar{x} \in \Xi \cap \Omega_\rho$, we may choose

$$y = (\mathrm{Id} + w_t N_\Xi)(\bar{x}) \in \partial\{u \geqslant t\} \cap \Omega$$

for appropriately small $t > 0$ and receive

$$\varepsilon^{-1} < \frac{|\bar{x} - (\mathrm{Id} + w_t N_\Xi)(\bar{x})|}{t} = \frac{w_t(\bar{x})}{t}.$$

Choosing a connected and compact K with $\Sigma \cap \overline{\Omega}_\rho \subset K \subset \Sigma$ and by the Harnack-type inequality from theorem 6.3(2), there exist constants $C > 0$ and $T > 0$ such that

$$\sup\{w_t(x) \mid x \in K\} \leqslant C \inf\{w_t(x) \mid x \in \Sigma \cap K\}$$

holds uniformly in $0 \leqslant t \leqslant T$ when $\rho > 0$ is small enough.

On the other hand, as $Du(x^*) \neq 0$, we may also find $\delta^* > 0$ and $m^* > 0$ with

$$m^* \leqslant \frac{|u(x^* + hN_\Xi) - u(x^*)|}{h}$$

whenever $0 < |h| < \delta^*$ and using an orthogonal decomposition of $\mathbb{R}^n$ in terms of tangent and normal space at $x^* \in \Sigma \cap \Omega$. Assuming $t > 0$ small enough such that $w_t(x^*)$ is defined with $w_t(x^*) < \delta^*$, we can hence choose

$$h = w_t(x^*) \qquad \text{such that} \qquad (\mathrm{Id} + w_t N_\Xi)(x^*) \in \partial\{u \geqslant t\} \cap \Omega$$

in the above inequality for sufficiently small $t > 0$ and rearrange for

$$\frac{|(\mathrm{Id} + w_t N_\Xi)(x^*) - x^*|}{|u\left((\mathrm{Id} + w_t N_\Xi)(x^*)\right) - u(x^*)|} = \frac{|(\mathrm{Id} + w_t N_\Xi)(x^*) - x^*|}{t} \leqslant \frac{1}{m^*},$$

which of course again implies

$$\frac{w_t(x)}{t} \leqslant \frac{C}{m^*}$$

for all suffciently small $t > 0$ and $x \in K$.

Set now $x = \bar{x}$, choose $\varepsilon = \frac{m^*}{C}$ and conclude by comparing with the uniform Harnack-type inequality that, whenever $t > 0$ is small enough,

$$\varepsilon^{-1} = \frac{C}{m^*} < \frac{|(\mathrm{Id} + w_t N_\Xi)(\bar{x}) - \bar{x}|}{t} = \frac{w_t(\bar{x})}{t} \leqslant \frac{C}{m^*},$$

which is the sought contradiction. ■

Applying the foregoing theorem to mildly regular and rectifiable values, we also obtain the following immediate consequences on the rigidity of Du.

9.16 Corollary (Mildly Regular Values Are Regular). *Let $0 \in \mathcal{U}$. Then 0 is a regular value for u_Ω.*

Proof. The extremal u is at least of class C^1 near $\Xi \cap \Omega$ due to theorem 7.1 and the continuous gradient of u may not vanish on $\Xi \cap \Omega$ due to theorem 9.15. ■

9.17 Corollary. *Let $0 \in \mathcal{V}$ and $\Sigma \subset \Xi$. If Σ is Jacobi admissible, then $Du \neq 0$ on $\Sigma \cap \Omega$.*

Proof. The last proof localized to the connected component $\Sigma \subset \Xi$. ■

10 The Behavior of an Extremal near a Fat Level Set.

In this section of the thesis we turn to a further characterization of the regularity of functions of anisotropic least gradient which will now also drop the assumption that the level set equals its sub- and superlevel set boundaries.

More concretely, we precisely want to find out "how" functions of anisotropic least gradient approach their fat level sets, by which we understand that a level set has positive top dimensional Lebesgue measure.
As is easily observed for topological reasons and because the level set is fat, we may choose some point in one of the level set boundaries of a (locally) fat level set and locally partition a small neighborhood into two parts: One where the function is constant and one where it varies. Obviously the levels will converge from the varying side by continuity, but we a priori do not know anything about the behavior of higher derivatives. Our results here will show that, in fact, also the higher derivatives allow for continuous extensions across such level set boundaries.

10.1 The Topology of Boundary Regular Level Sets.

Recalling the definition of the classes of values $\mathcal{U} \subset \operatorname{rg} u$ and $\mathcal{V} \subset \operatorname{rg} u$, we consequently drop one further assumption and introduce the following definition.

10.1 Definition (Boundary Regular Values)**.** We will say that t is a boundary regular value when $t \in \mathcal{W}$, where

$$\mathcal{W} := \{ \ t \in (a,b) \quad | \quad t \text{ is a regular value of } f \quad \}.$$

Our purpose now is therefore of course to study also fat level sets (or respectively, fat components of level sets) and how regularity from the boundary data f can propagate here to the full domain. Since

$$t \in \mathcal{W} \quad \text{and} \quad \mathscr{L}^n(u^{-1}(t)) = 0 \qquad \Longrightarrow \qquad t \in \mathcal{V},$$

the "slim" case is already covered and we shall further assume without loss of generality that $t \in \mathcal{W}$ with

$$\mathscr{L}^n(u^{-1}(t)) > 0 \qquad \text{for the remainder of section 10.}$$

We again assume and set

$$(t = 0) \in \mathcal{W} \qquad \text{and} \qquad \Xi := u^{-1}(0).$$

Note that the set Ξ here is bestowed with a different meaning than before as Ξ is now not necessarily globally a manifold anymore. Therefore we shall again start with some elementary observations on the topological structure of such a level set.

10.2 Lemma. *Let $0 \in \mathcal{W}$. Then Ξ has only finitely many connected components.*

Proof. Assume we had infinitely many connected components. Then each connected component of $\Xi = u^{-1}(0)$ contains connected components of $\partial\{u \geqslant 0\}$ and $\partial\{u \leqslant 0\}$ and we hence have infinitely many, which is not possible by proposition 4.15(2). ∎

Owing to lemma 10.2, we will decompose

$$\Xi = \bigcup_{i=1}^{N} \Sigma_i \qquad 1 \leqslant N < \infty,$$

for the union of connected components of Ξ and subsequently deduce the following structural result with the aid of some more terminology.

10.3 Definition. Let $0 \in \mathcal{W}$. Then a connected component $\Sigma \subset \Xi$ of the zero level set Ξ is called *rectifiable* if

$$\Sigma \subset \partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\}$$

and it is called *fat* if

$$\mathscr{L}^n(\Sigma) > 0.$$

10.4 Lemma (Boundary Regular Decomposition). *If $0 \in \mathcal{W}$, then Ξ can be decomposed into finitely many connected components which are either rectifiable or fat.*

Proof. It remains to show that any connected component Σ of Ξ is either rectifiable or fat. Assuming that $\mathscr{L}^n(\Sigma) > 0$, then Σ is fat and cannot be contained in $\partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\}$, as the latter set has $\mathscr{L}^n$ measure zero. Thus Σ is not rectifiable.

When $\mathscr{L}^n(\Sigma) = 0$, then Σ is not fat. As moreover

$$\Sigma \cap \partial\Omega \subset f^{-1}(0) \subset \partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\},$$

we assume for a contradiction that there exists some point

$$x_0 \in \Sigma \cap \Omega \qquad \text{with} \qquad x_0 \notin \partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\}.$$

If x_0 is contained in neither level set boundary, then we could find some open ball $\mathbf{U}$ about x_0 on which $u \equiv 0$ and which is, by maximality of the connected component of x_0, included in Σ, which immediately contradicts $\mathscr{L}^n(\Sigma) = 0$. Hence we assume further without loss of generality that

$$x_0 \in \partial\{u \geqslant 0\} \qquad \text{and} \qquad x_0 \notin \partial\{u \leqslant 0\}.$$

Now, as in particular $x_0 \in \operatorname{spt}|D1_{\{u \geqslant 0\}}| \cap \Omega$, basically the same argument as in lemma 9.1 leads to the existence of some $r > 0$ with $\mathscr{L}^n(u^{-1}(0) \cap \mathbf{U}(x_0, r)) > 0$ with also $u^{-1}(0) \cap \mathbf{U}(x_0, r)$ being connected. Hence, again by maximality of the connected component, we find the contradiction $\mathscr{L}^n(\Sigma) > 0$. ∎

10.5 Remark. We especially highlight the following fact from the last proof and the proof of lemma 9.1: As soon as there is some zero $x_0 \in \Omega$, which is contained in precisely one of the level set boundaries, the connected component of this point x_0 inside $u^{-1}(0)$ is fat and we may locally partition small balls $\mathbf{U}$ about this point into a "varying" and a "constant" side in terms of connected components of $\mathbf{U}$ without the level set boundary.

Decomposing further with the aid of lemma 10.4, we shall write

$$\Xi = \bigcup_{k=1}^{N_r} \Sigma_k \cup \bigcup_{l=1}^{N_f} \Sigma_l, \qquad \text{with} \qquad 1 \leqslant N_r + N_f = N < \infty,$$

where the connected components $\Sigma_k \subset \Xi \subset \overline{\Omega}$ are rectifiable and the connected components $\Sigma_l \subset \Xi \subset \overline{\Omega}$ are fat.

10.2 General Deformation Results and Regularity for Rectifiable Components.

We will now recall suitable one-sided versions of our previous results to be able to work with a general level set whose upper and lower level set boundaries need not necessarily coincide. Most of the previous results will follow verbatim but we use the occasion to recall some details. Afterwards we will acknowledge that rectifiable components of a general boundary regular level set allow for the same regularity theory as developed in section 9.

We shall thus consider without loss of generality the upper level set boundary $\partial\{u \leqslant 0\}$ and provide one-sided results for $t \searrow 0$ as before. Note that, of course, the other case can also be reduced to this form by passing from u to $-u$ and remaining of anisotropic least gradient.

10.6 Remark. All remaining results from the sections 2.4, 4,5 and 6.1 yet again follow similarly (compare also to remark 9.8) with the only change that we are now bound to consider $t > 0$ in each of those results. In particular, we recall:

(1) We have the one-sided Hausdorff convergence $u^{-1}(t) \longrightarrow \partial\{u \leqslant 0\}$ for $t \searrow 0$.

(2) We have the existence of global diffeomorphisms $\iota_t : \partial\{u \leqslant 0\} \to \partial\{u \geqslant t\}$ of class $\mathrm{C}^{2,\alpha}$, which allow us to appropriately deform level sets into close ones.

(3) We have the existence of normal deformations w_t on retracts and local and uniform Harnack-type inequalities on connected components of $\partial\{u \leqslant 0\}$.

In case we choose a rectfiable connected component Σ of Ξ we will now quickly show that the results of section 9 may be localized to such a connected component. In particular, all regularity results remain valid near Σ.

10.7 Proposition (Regularity and Rectifiable Components). *Let $0 \in \mathcal{W}$ and $\Sigma \subset \Xi$ be a rectifiable connected component of Ξ. Then all regularity results of section 9 remain true localized at the connected component Σ.*

Proof. In view of section 9.2, all that remains is to show that all deformation results can be extended to two-sided versions near $\Sigma \subset \Xi$. We infer from the definition of a rectifiable component that Σ is a connected component for both $\partial\{u \geqslant 0\}$ and $\partial\{u \leqslant 0\}$ and applying the one-sided results iteratively will thus near Σ yield the argument, since, in case of Jacobi admissibility, the one-parameter family $\mathfrak{F}(\kappa_t)$ again parametrizes the correct level set boundaries near Σ. ■

10.3 Regularity near the Level Set Boundary of a Fat Component.

With the case of a rectifiable connected component covered by slight modification of the results of section 9, we will now turn to the case of an actually fat connected component $\Sigma \subset \Xi$. Such a fat component may contain more than one connected component of the level set boundaries.

10.8 Example (Brothers' Fat Level Set). We may allude to the example of Brothers to illustrate. In this case, the extremal has precisely one connected fat level set,

$$\Xi = u^{-1}(0) \qquad \text{with} \qquad 0 \in \mathcal{W},$$

and $\Xi \cap \{\partial\{u \geqslant 0\}$ and $\Xi \cap \partial\{u \leqslant 0\}$ have each precisely 2 connected components, which we denote as $\Sigma_1, ..., \Sigma_4$ (corresponding to the 4 straight parts of the boundary of the square), that we may choose. Quickly checking the solution formula from example 7.3 yields that the derivative exists nowhere on $(\Sigma_1 \cup ... \cup \Sigma_4) \cap \Omega$.

Thus, we proceed to fix a connected component $\Sigma \subset \Xi$ such that $\mathscr{L}^n(\Sigma) > 0$ and we will decompose it into the following finite unions of connected components

$$\Sigma \cap \partial\{u \geqslant 0\} = \bigcup_{k=1}^{N_l} \Sigma_k \qquad \text{and} \qquad \Sigma \cap \partial\{u \leqslant 0\} = \bigcup_{l=1}^{N_u} \Sigma_l,$$

where $1 \leqslant N_l, N_u < \infty$. The next result combines differentiable strong maximum principles, which we have already derived and used before in the context of local graph representations with sign conditions, with topological arguments to rule out the possibility of intersections of the upper and lower level set boundaries inside the domain.

10.9 Proposition. *Let $0 \in \mathcal{W}$ and consider a fat connected component $\Sigma \subset \Xi$. Then*

$$\partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\} \cap \Sigma \cap \Omega = \varnothing.$$

Proof. Assume there was $x_0 \in \partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\} \cap \Sigma \cap \Omega$. As

$$\partial\{u \geqslant 0\} = \partial\{u < 0\}$$

as sets in $\overline{\Omega}$, we find

$$\{u < 0\} \subset \{u \leqslant 0\} \qquad \text{and} \qquad x_0 \in \partial\{u < 0\} \cap \partial\{u \leqslant 0\}.$$

As both oriented boundaries are smooth and minimizing, standard arguments show that both boundaries may be written locally at x_0 as a graph over the same hyperplane. Further, the above set inclusion is yielding a non-negative sign condition for the graph functions. Hence, the same linearization and linear elliptic partial differential equation argument an in section 6.1 (see eg. also [24, Lemma 4.4]) for the difference of non-parametric representations implies

$$\partial\{u \geqslant 0\} \cap U = \partial\{u \leqslant 0\} \cap U \qquad \text{for some open neighborhood } U \subset \Omega \text{ of } x_0.$$

We let $\Sigma^\pm \subset \Sigma$ denote the connected components of x_0 in respectively $\partial\{u \geqslant 0\}$ and $\partial\{u \leqslant 0\}$. A standard topological argument exploiting the connectedness, the local geometric maximum principles for the graph functions and the boundary regularity of Σ then yields that

$$\Sigma^+ \cap \Omega = \Sigma^- \cap \Omega$$

where hence

$$\Sigma^+ = \Sigma^- =: \Sigma^* \qquad \text{with} \qquad \Sigma^* \subset \partial\{u \geqslant 0\} \cap \partial\{u \leqslant 0\} \cap \Sigma.$$

We subsequently reach a contradiction to $\mathscr{L}^n(\Sigma) > 0$ if we may show that $\Sigma^* = \Sigma$. To do so, by connectedness of Σ^*, note that Σ^* divides sufficiently small connected neighborhoods $V \subset \overline{\Omega}$ of Σ^* into two connected components, which are contained in $\{u > 0\}$ and $\{u < 0\}$. Thus $V \cap \Sigma = \Sigma^*$ and therefore $\Sigma = \Sigma^*$ by maximality of Σ as a connected component of $u^{-1}(0)$. ■

10.10 Corollary. *Each point*

$$x_0 \in (\partial\{u \geqslant 0\} \cup \partial\{u \leqslant 0\}) \cap \Sigma \cap \Omega$$

admits an $r = r(x_0) > 0$ *such that* $\mathbf{U}(x_0, r) \backslash (\partial\{u \geqslant 0\} \cup \partial\{u \leqslant 0\})$ *has exactly two connected components and* $u \equiv 0$ *on one of these components.*

Proof. Any such $x_0 \in \Sigma \cap \Omega$ can at most be contained in one level set boundary by proposition 10.9 and the claim follows by remark 10.5. ∎

Let us now investigate the regularity of the extremal u as we approach the level set boundaries of such a fat connected component.

10.11 Example (Regularity and Brothers' Fat Level Set)**.** Recall the 4 connected components $\Sigma_1,...,\Sigma_4 \subset \Xi$ of the level set boundaries of the zero level of Brothers' example with $0 \in \mathcal{W}$ and choose any

$$x_0 \in (\Sigma_1 \cup ... \cup \Sigma_4) \cap \Omega.$$

As the solution is constructed from polynomials except for the zero level set, we may fix a small ball $\mathbf{U}(x_0, r) \subset \Omega$ and extend as u as

$$\tilde{u} \in \mathrm{C}^\omega(\mathbf{U}(x_0, r)) \qquad \text{such that} \qquad \tilde{u}_{\mathbf{U}(x_0,r)\backslash u^{-1}(0)} = u$$

by continuing across $\partial(u^{-1}(0)) \cap \Omega$ locally with the adequate polynomial function. In particular, there locally exists an analytic extension of u across the level set boundaries inside Ω, which is non-constant.

We will now investigate this phenomenon for the level set boundaries associated with fat components of general functions of anisotropic least gradient. As the first preparing step, we prove a definiteness-type result analogous to section 9.3 for the fat case.

In what follows we shall fix some connected component

$$\Sigma^* \subset \Sigma \cap \partial\{u \leqslant 0\}$$

to investigate for our considerations. We recall for the sake of completeness that Σ^* is hence a submanifold of $\mathbb{R}^n$ of class $\mathrm{C}^{\mu,\alpha}$ with boundary of class $\mathrm{C}^{\nu,\alpha}$.

10.12 Lemma (Definiteness on Fat Components)**.** *Let* $0 \in \mathcal{W}$. *Then* Du *either exists and vanishes everywhere on* $\Sigma^* \cap \Omega$ *or* Du *exists nowhere on* $\Sigma^* \cap \Omega$.

Proof. Let $x_0 \in \Sigma^* \cap \Omega$, then by corollary 10.10 there exists $r > 0$ such that u has, in particular, a minimum in x_0 on $\mathbf{U}(x_0, r)$. Thus, if $Du(x_0)$ exists, then $Du(x_0) = 0$.

Assume now that there would be $x^*, \bar{x} \in \Sigma^* \cap \Omega$ such that Du does not exist at x^* and $Du(\bar{x}) = 0$. Using the same strategy as in the characterization from lemma 9.6, we find a sequence

$$x_i^* \longrightarrow x^* \qquad \text{and} \qquad t_i^* \searrow 0$$

as $i \to \infty$ such that

$$t_i^* > 0 \qquad \text{and} \qquad x_i^* \in \partial\{u \geqslant t_i^*\} \cap \Omega$$

and $m > 0$ with the property that

$$0 < m \leqslant \frac{|u(x_i^*) - u(x^*)|}{|x_i^* - x^*|} = \frac{t_i^*}{|x_i^* - x^*|} \qquad \text{for all } i \in \mathbb{N}.$$

Possibly choosing i large enough and using the family of functions w_t associated with Σ^* via remark 10.6, we may as well assume that

$$\frac{w_{t_i^*}(x^*)}{t_i^*} \leqslant \frac{1}{m^*}$$

for some $m^* > 0$ and all $i \in \mathbb{N}$ by following along the lines of lemma 9.9. Now the same argument as in theorem 9.15 delivers the contradiction together with the Harnack-type inequality from theorem 6.3(2), as the sequence of functions $w_{t_i^*}$ must blow up at $\bar{x}$ while it is uniformly bounded at x^*. ■

Assuming the case of non-existence of the derivative on some connected component of the level set boundary of a fat level set, our previous methods also readily allow for a result on the Jacobi nullity in that new case.

10.13 Proposition (The Jacobi Nullity on Fat Components)**.** *Let* $0 \in \mathcal{W}$ *and assume that* Du *does not exist on* $\Sigma^* \cap \Omega$*. Then* $\dim \mathfrak{K} = 0$.

Proof. Basically the same proof as the two times before, which is conducted here only on the connected component $\Sigma^* \subset \Sigma \cap \partial\{u \leqslant 0\}$ with the family of functions w_t/t for $t > 0$. As in the last proof, the non-existence of Du at any (and thus, every by lemma 10.12) $x^* \in \Sigma^* \cap \Omega$ yields the uniform boundedness of w_t/t along a subsequence and the blow up argument can be performed as already done before, eg. in the proof of proposition 9.11. ■

10.14 Remark. One may also somewhat reasonably conjecture that in fact *all boundary regular connected fat level sets do not admit an existing gradient on their level set boundary in* Ω, which would imply that proposition 10.13 becomes unconditional.

At last, we turn to the continuous extendability of derivatives of u near fat level set from the varying side and we may prove the following theorem. Let us remark that, while we will immediately provide regularity *up to the* fat level set boundary, actually also regularity *near* the fat level set boundary was not even clear before, but our construction in terms of dependence on boundary level sets can also be applied here.

10.15 Theorem (Regularity near Fat Level Sets)**.** *Let* $0 \in \mathcal{W}$*, let* $\Sigma \subset \Xi$ *be a fat connected component of* Ξ *and let* Σ^* *be a connected component of* $\partial\{u \leqslant 0\} \cap \Sigma$ *such that* Du *does not exist on* $\Sigma^* \cap \Omega$*. Then for all* $x_0 \in \Sigma^* \cap \Omega$ *there exists some ball* $\mathbf{U}(x_0, r) \subset \Omega$ *about* x_0 *with* $r = r(x_0) > 0$ *and a function* $\tilde{u} \in \mathrm{C}^{\mathrm{M}}(\mathbf{U}(x_0, r))$ *such that*

$$\tilde{u}_{\mathbf{U}(x_0,r)\setminus u^{-1}(0)} = u \qquad \textit{with} \qquad \mathrm{M} = \min(\mu - 2, m - 3) \geqslant 1.$$

In particular, u *is of class* C^{M} *up to the boundary component* $\Sigma^* \subset \Sigma$ *inside* Ω *from both its sides.*

Proof. Following along remark 10.6, there are embeddings

$$\iota_t : \Sigma^* \longrightarrow \partial\{u \geqslant t\}, \qquad \iota_t \in \mathrm{C}^{2,\alpha}(\Sigma^*, \mathbb{R}^n)$$

for $t > 0$ sufficiently small whose restriction to $\partial\Sigma^*$ is the restriction of the regular level set flow κ_t to $\partial\Sigma^*$. Proposition 10.14 allows to use the dependence map $\mathfrak{f}$ on the space $\mathrm{C}^{2,\gamma}(\partial\Sigma^*, \mathbb{R}^n)$ for any $0 < \gamma < \alpha < 1$ to define our usual one-parameter family

$$\mathrm{K}(x,t) := \mathfrak{f}(\kappa_t)(x) \qquad \text{for} \quad x \in \Sigma^*,\ |t| \text{ sufficiently small}$$

and observe again that K is of class C^{M}. Clearly, $D_1\mathrm{K}(x_0, 0)$ again immerses the tangent space to Σ^* into $\mathbb{R}^n$ at x_0 for any $x_0 \in \Sigma^* \cap \Omega$ and, similar to our previous regularity results, we infer that $D\mathrm{K}$ has full rank at $(x_0, 0)$ by using difference quotients with positive real h. The inverse function theorem thus again yields connected neighborhoods $U \subset \Sigma^*$ of x_0 and $V \subset \mathbb{R}^n$ of x_0 and a real number $T > 0$ such that

$$\mathrm{K} : U \times (-T, T) \longrightarrow V$$

is a diffeomorphism of class C^{M}. As again, by choosing $t > 0$ sufficiently small, we find

$$u \circ \mathrm{K} = \mathrm{proj}_2 \qquad \text{on} \qquad U \times (0, T),$$

since now the images $\mathrm{K}(\Sigma^*, t) = \iota_t(\Sigma^*)$ only coincide for non-negative t, and hence

$$u = \mathrm{proj}_2 \circ \mathrm{K}^{-1} \qquad \text{on} \qquad \mathrm{K}\left(U \times (0, T)\right).$$

We may assume that $U \times [(-T, 0) \cup (0, T)]$ and $V \backslash \Sigma^*$ have precisely two connected components and that the components of $V \backslash \Sigma^*$ are contained in $\{u > 0\}$ and $\{u = 0\}$ respectively. As K preserves connected components (as a diffeomorphism on $U \times (-T, T)$) and $\mathrm{K}(x_0, t) \in u^{-1}(t)$ for $t > 0$, we observe that

$$\mathrm{K}(U \times (0, T)) = V \cap \{u > 0\}$$

and choosing a small ball about x_0 inside V proves the theorem. ■

10.16 Remark. Of course, u also always allows for a constant extension from the constant side of the local subdivision.

11 *Local Foliations by Minimal Hypersurfaces.*

We want to collect in this last section of the regularity theory some consequences of the preceding sections on how the level sets fill out the domain, ie. to what extent they provide a foliation of the domain by (anisotropic) area-minimizing hypersurfaces.

11.1 Remark. We directly remark that such a foliation property is not guaranteed a priori, as our boundary regularity proof is only conditional and we can, for appropriate level sets, only guarantee the existence of an open neighborhood

$$\mathscr{O}_t \subset \Omega \qquad \text{such that} \qquad u^{-1}(t) \cap \Omega \subset \mathscr{O}_t\,.$$

As these neighborhoods might become smaller and smaller towards $\partial\Omega$, it does *not* follow that they need contain any $u^{-1}(s)$ with $t \neq s$ and hence, that level sets near $u^{-1}(t)$ actually foliate an open superset.

Now, we combine the local differentiability of at least class C^1 in the interior of Ω from our regularity theory with the definiteness of the gradient on nice hypersurfaces to conclude for an actual foliation. We provide a proof for the case of a mildly regular value.

11.2 Theorem (Stability near Mildly Regular Level Sets)**.** *Let $t \in \mathcal{U}$ be a mildly regular value. Then there exists an open neighborhood $\mathscr{P}_t \subset \Omega$ of $u^{-1}(t) \cap \Omega$ and $T > 0$ such that*

$$u^{-1}(s) \subset \mathscr{P}_t \qquad \textit{for all} \quad 0 \leqslant |s-t| \leqslant T \qquad \textit{and} \qquad u \in \mathrm{C}^{\mathrm{M}}(\mathscr{P}_t).$$

Proof. We may assume that $u^{-1}(t)$ is connected. From theorem 7.1 we have the existence of an open neighborhood $\mathscr{O}_t \subset \Omega$ of $\Xi \cap \Omega$ such that $u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_t)$. Hence there is $x_0 \in u^{-1}(t) \cap \Omega$ and $r > 0$ such that

$$Du \neq 0 \qquad \text{on} \quad \mathbf{U}(x_0, r) \subset \mathscr{O}_t \subset \Omega.$$

Thus, $u(\mathbf{U}(x_0, r)) \subset \mathbb{R}$ is not a singleton and $u(x_0)$ an interior point. Fixing $T > 0$ such that

$$[t-T, t+T] \subset u(\mathbf{U}(x_0, r)),$$

we infer that the level sets of levels in $[t-T, t+T]$ fulfill

$$u^{-1}(s) \cap \mathbf{U}(x_0, r) = \partial\{u \leqslant s\} \cap \mathbf{U}(x_0, r) = \partial\{u \geqslant s\} \cap \mathbf{U}(x_0, r).$$

Possibly readjusting $T > 0$, we may also by lemma 5.19 assume that

$$\partial\{u \geqslant s\},\ \partial\{u \leqslant s\} \quad \text{are connected} \quad \text{for } s \in [t-T, t+T].$$

Using the differentiable versions of geometric maximum principles with $\{u < s\} \cap \Omega \subset \{u \leqslant s\} \cap \Omega$ and a simple connectedness argument, we therefore receive

$$\partial\{u \geqslant s\} \cap \Omega = \partial\{u \leqslant s\} \cap \Omega \qquad \text{for} \quad s \in [t-T, t+T],$$

which yields, by additionally assuming that $f^{-1}(s)$ is regular for all $s \in [t-T, t+T]$, that

$$\partial\{u \geqslant s\} = \partial\{u \leqslant s\} \qquad \text{for} \quad s \in [t-T, t+T].$$

Let us assume for a contradiction that there is some

$$x_0 \in u^{-1}(s)\backslash\partial\{u \geqslant s\} = u^{-1}(s)\backslash\partial\{u \leqslant s\}$$

for such s. Then $x_0 \in \Omega$ and $\mathscr{L}^n(u^{-1}(s)) > 0$, which yields that the connected component Σ of x_0 in $u^{-1}(s)$ is fat, but proposition 10.9 would imply that $\partial\{u \geqslant s\}$ and $\partial\{u \leqslant s\}$ would be disjoint in Σ inside Ω. We hence find

$$u^{-1}(s) = \partial\{u \geqslant s\} = \partial\{u \leqslant s\} \qquad \text{for} \quad s \in [t-T, t+T].$$

This proves that $s \in \mathcal{V}$ for all $s \in [t-T, t+T]$ and we recall that each such s allows for some point $x_s \in \mathbf{U}(x_0, r) \subset \Omega$ such that $Du(x_s) \neq 0$. We conclude that each level set $u^{-1}(s)$ is Jacobi admissible and hence each such real s allows for a neighborhood $\mathscr{O}_s \subset \Omega$ with $u^{-1}(s) \cap \Omega \subset \mathscr{O}_s$ and $u \in \mathrm{C}^{\mathrm{M}}(\mathscr{O}_s)$ by proposition 9.11. We define

$$\mathscr{P}_t := \bigcup_{s\in[t-T,t+T]} \mathscr{O}_s \subset \Omega,$$

then $\mathscr{P}_t$ is clearly open as a union of open sets and we notice

$$u^{-1}(s) \subset \mathscr{P}_t \qquad \text{for all} \quad s \in [t-T, t+T], \qquad u \in \mathrm{C}^{\mathrm{M}}(\mathscr{P}_t),$$

which finishes the proof. ■

11.3 Remark. These stability criteria provide a definitive answer for the mildly regular case. Of course, by corollary 9.16, each mildly regular value is moreover classically regular, but the direct applicability of corollary 9.16 falls short by the behavior of a single neighborhood $\mathscr{O}_t \subset \Omega$ described in remark 11.1.

We further fix an implication from the last proof to conclude that mildly regular values behave much like their regular counterparts.

11.4 Corollary (Stability of Mildly Regular Values and Local Foliations)**.** *Let $t \in \mathcal{U}$, then there is $T > 0$ such that*

$$s \in \mathcal{U} \qquad \textit{whenever} \quad |s-t| \leqslant T.$$

Moreover, the level sets of u foliate an open neighborhood of $u^{-1}(t)$ in Ω of class C^{M} with $\mathrm{M} = \min(\mu-2, m-3)$.

Proof. Using the set $\mathscr{P}_t \subset \Omega$ from theorem 11.2, we first find that all level sets $u^{-1}(s)$ for s close enough to t are contained in $\mathscr{P}_t$. Choosing finitely many points in each connected component of $u^{-1}(t)$ with $Du \neq 0$ implies that we may also find points in each connected component of $u^{-1}(s)$ where $Du \neq 0$ by continuity of the gradient. Thus, theorem 9.15 yields that $Du \neq 0$ on $u^{-1}(s) \cap \Omega$ and hence, there is in particular $T > 0$ with $s \in \mathcal{U}$ for all such real s with $|s-t| \leqslant T$. The uniform continuity of u on $\overline{\Omega}$ finally yields on open neighborhood $V \subset \overline{\Omega}$ of $u^{-1}(t)$ such that

$$|u(x) - t| \leqslant T \qquad \text{for all } x \in V$$

from which we infer the foliation property for $V \cap \Omega$ by standard results about regular level sets with $Du \neq 0$. ■

11.5 Remark. Recalling the interpolation example 7.5, we also acknowledge that there is no lower bound what percentage of the domain Ω will be foliated and arbitarily small foliations (in $\mathscr{L}^n$-measure) are possible.

12 *Existence Results for Lipschitzian Functions of Weighted Least Gradient.*

This section is disjoint from the higher partial regularity theory that we have discussed before, but will deal with its a priori assumptions: Namely, we shall find sufficient conditions for the existence of Lipschitzian functions of anisotropic least gradient for total variations

$$\Phi(x, v) = \mu(x)|v|$$

for sufficiently regular weights μ, which we shall call *functions of weighted least gradient.*

Our method of choice to do so will involve elliptic quasilinear problems with (strict) boundary curvature estimates, which we will introduce in the first subsection, and linearly scaled boundary data yielding convergence to functions of weighted least gradient. The scaling method used here hails from [41] and the results of this section are contained in the paper [47]. For the remainder of this section, the following assumptions shall hold true:

(1) *Let us assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected with $\partial\Omega$ is class C^2 and with $f \in \mathrm{C}^2(\partial\Omega)$ and $\mu : \overline{\Omega} \to \mathbb{R}$ of class C^2.*

(2) *We will moreover fix some $\alpha > 0$ such that we may estimate $\alpha \leqslant \mu \leqslant \alpha^{-1}$ and let d denote the signed distance to $\partial\Omega$.*

(3) *We will choose signs for section 12 such that d is positive on Ω and $N_{\partial\Omega}$ is the inner unit normal vector. The mean curvature H of $\partial\Omega$ will be non-negative for convex sets.*

(4) *We will further often sum over repeated indices in this section.*

12.1 Elliptic Quasilinear Approximating Problems.

We are in the following interested in solutions u_δ to the Dirichlet problem

$$\begin{cases} Qu := a^{ij}(x, Du)D_{ij}u + b(x, Du) = 0, \\ u_{\partial\Omega} = f/\delta, \end{cases} \tag{P(δ)}$$

on the domain Ω with Dirichlet data f and where $\delta > 0$ is small and with the quasilinear differential operator Q given by

$$a^{ij}(x, p) := \mu(x)((1 + |p|^2)\delta_{ij} - p_i p_j), \qquad b(x, p) = (1 + |p|^2)(D\mu(x) \cdot p).$$

We may usually assume that f can be extended to $\overline{\Omega}$ or $\mathbb{R}^n$. Let us now recall that $\lambda = \lambda(x, z, p)$ and $\Lambda = \Lambda(x, z, p)$ denote the *smallest* and *largest* eigenvalues of the matrix $\mathfrak{a} = (a^{ij})$ such that

$$\lambda|\xi|^2 \leqslant \xi^T \mathfrak{a} \xi \leqslant \Lambda|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^n$$

and the *Bernstein-$\mathscr{E}$-function* is defined via

$$\mathscr{E}(x, z, p) := p^T \mathfrak{a}(x, z, p)p.$$

We first collect some technical necessities related to decompositions for the differential operators.

12.1 Lemma. *We have*

$$\lambda(x,z,p) = \mu(x), \qquad \Lambda(x,z,p) = \mu(x)(1+|p|^2),$$

and

$$\mathscr{E}(x,z,p) = \mu(x)|p|^2.$$

In particular, the problem $\mathrm{P}(\delta)$ *is elliptic on* $\overline{\Omega} \times \mathbb{R}^n$.

Proof. Simple computations based on the minimal surface/prescribed mean curvature equation case (eg. [13, Section 10, Example (iii)]) and the fact that μ is uniformly bounded from below. ■

In what follows, we will at first denote by u_δ a sufficiently regular solution to $\mathrm{P}(\delta)$. We will now try to establish the existence of such solutions via the continuity method. In view of the a priori bounds for the interpolated problems with $\sigma \in [0,1]$, we shall already assume inequality ($\mathscr{C}$) for boundary gradient estimates, which is stated in the later theorem 12.5 and inequality ($\mathscr{G}$) from proposition 12.6 for an easy gradient maximum principle. We resume:

12.2 Proposition (Existence of Solutions)**.** *There exists a unique solution* $u_\delta \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$ *to* $\mathrm{P}(\delta)$ *if the boundary curvature inequality* ($\mathscr{C}$) *and the differential inequality* ($\mathscr{G}$) *for the weight* μ *holds.*

Proof. We shall set $\delta = 1$ without loss of generality, replace for f and $\partial\Omega$ the regularity C^2 by $\mathrm{C}^{2,\alpha}$ for some $0 < \alpha < 1$ and recall the following cut-off homotopy construction from [49, Page 419, Remark] to exploit the boundary curvature estimates in the optimal fashion:

Let a smooth $\gamma : \mathbb{R} \longrightarrow [0,1]$ be such that

$$\gamma \equiv 0 \quad \text{for} \quad x \leqslant 0, \qquad \gamma \equiv 1 \quad \text{for} \quad x \geqslant 1/2$$

and set $\Gamma(p) := \gamma(|p|^2 - 1/2)$. Then for any $\sigma \in [0,1]$ we define

$$b(x,p;\sigma) := (\Gamma(p) + \gamma(\sigma)(1-\Gamma(p)))\, b(x,p)$$

and we consider a solution to the quasilinear elliptic partial differential equation

$$a^{ij}(x,Du)u_{ij} + b(x,Du;\sigma) = 0$$

with boundary values $u_{\partial\Omega} = \sigma f$. Observe that

$$b(x,p;1) = b(x,p) \quad \text{and} \quad b(x,p;0) \equiv 0 \quad \text{whenever } 0 \leqslant |p| \leqslant 1/\sqrt{2}$$

as well as

$$b(x,p;\sigma) = b(x,p) \quad \text{for } |p| \geqslant 1 \text{ and all } \sigma \in [0,1].$$

Thus, the remark [49, Page 419, Remark] applies with simple modifications (eg. cutting off $b(x,p;0)$ at radius $1/\sqrt{2}$ instead of radius 1 to make the linear problem well-defined, cf. also [13, Theorem 11.6] for the fixed point theorem) and it only remains to find a priori bounds in C^1 independent of $\sigma \in [0,1]$ for solutions to the above σ-dependent problem.

Therein, a uniform maximum principle is clear by [13, Theorem 10.1] as constants solve the quasilinear problem. Next, we note

$$b(x,p;\sigma) = b(x,p) \qquad \text{for all } \sigma \in [0,1] \text{ when } |p| \geqslant 1$$

and find that the second summand vanishes whenever $|p| \geqslant 1$ for all $\sigma \in [0,1]$ while the first is independent of $\sigma \in [0,1]$. In particular, also the boundary gradient estimates may be conducted independently of σ, as we need only work with the first summand. Finally, the differential inequality ($\mathscr{G}$) continues to hold by equality for eg. $|p| \geqslant 2$ for all $\sigma \in [0,1]$, by which we infer a uniform gradient bound via proposition 12.6 and the fact only large values of $|p|$ need to be considered (see eg. the discussion after [13, 15.1 Theorem]). Finally, we may pass from $C^{2,\alpha}$ back to C^2-regularity by approximating the boundary and the boundary data and exploting uniform interior second order and global first order Hölder bounds. ■

12.3 Remark. The structure of the above construction is taken from [49, Page 456] while we have adapted slight changes for simplicity (eg. Serrin's original cutoff function vanishes up to $|p| = 1$ and is only smooth away from $p = 0$).

12.4 Remark (The Divergence Structure of $\mathrm{P}(\delta)$). By rescaling with the positive factor $(1+|p|^2)^{\frac{3}{2}}$ one may easily observe that a sufficiently smooth function u_δ solves $\mathrm{P}(\delta)$ if and only if

$$\operatorname{div} \mathbf{A}(x, Du_\delta) = 0, \quad \text{where} \quad \mathbf{A}(x,p) := \mu(x)\frac{p}{\sqrt{1+|p|^2}}.$$

Writing the quasilinear Dirichlet problem in this form, we also recognize the equation as a weighted non-parametric minimal surface equation. Note furthermore that any such solution u_ε is in fact, by convexity, a minimizer for the linear growth functional

$$\mathscr{F}_\mu(v) := \int_\Omega \mu(x)\sqrt{1+|Dv(x)|^2}\, d\mathscr{L}^n$$

among $v \in W^{1,1}(\Omega)$ with coinciding trace values f/δ on $\partial\Omega$.

We will now turn to the boundary gradient estimates via [13, Chapter 14.3] and use differential conditions on applied to the distance d to $\partial\Omega$ or, equivalently, conditions on the boundary curvature. Let us first repeat some results on the particular structure of our differential operator, hailing directly from the structure of the minimal surface operator.

Recall that we may decompose the main part $\mathfrak{a} = (a^{ij})$ into

$$a^{ij} = \Lambda a_\infty^{ij} + a_0^{ij},$$

where

$$a_\infty^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{|p|^2} \qquad \text{and} \qquad a_0^{ij}(x,p) = \mu(x)\frac{p_i p_j}{|p|^2}.$$

This follows immediately from such decomposition for the minimal surface operator (see eg. [13, p. 342]). Note that a_∞^{ij} does actually not depend on the weight function μ. In particular, if we denote via

$\mathscr{K}^{\pm}$ *the boundary curvature operators* with

$$\mathscr{K}^{\pm}(x) = -\sum_{i,j=1}^{n} a_{\infty}^{ij}(x, \pm Dd(x)),$$

then $\mathscr{K}^{+} = \mathscr{K}^{-} = \mathscr{K}$ and we obtain

$$\mathscr{K}(x) = -\sum_{i,j=1}^{n} a_{\infty}^{ij}(x, \pm Dd(x)) = -\Delta d(x) = (n-1)H(x)$$

for $x \in \partial\Omega$ as for the minimal surface case. As our operators however do now also depend on the position x, we furthermore write

$$b = |p|\Lambda b_{\infty}$$

with

$$b_{\infty}(x,p) := \frac{D\mu(x)}{\mu(x)} \cdot \frac{p}{|p|}.$$

The central idea is now that we want to prove that, as we scale our boundary data linearly from f to f/δ, our estimates also correspondingly scale with $1/\delta$ and we shall show that the strict boundary curvature estimates from the theory of quasilinear elliptic equations are eligible for such bounds.

12.5 Theorem (Scalable Boundary Gradient Estimate)**.** *Assume that $\partial\Omega$ fulfills*

$$H(x) > \frac{D\mu(x) \cdot N_{\partial\Omega}(x)}{(n-1)\mu(x)} \qquad \textit{for all } x \in \partial\Omega. \tag{$\mathscr{C}$}$$

Then there exists $K > 0$, which does not depend on $0 < \delta \leqslant 1$, such that

$$\sup_{\partial\Omega} |Du_{\delta}| \leqslant K/\delta + \|Df/\delta\|_{\mathrm{C}^0(\overline{\Omega})}$$

for the solution $u_{\delta} \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$ to $P(\delta)$.

Proof. We follow along the lines of [13, Theorem 14.9], but track details in an explicit fashion. Let us first set $\delta = 1$ and write $u = u_{\delta}$. Recall that we chose the signs of d and $N_{\partial\Omega}$ such that

$$Dd = N_{\partial\Omega} \qquad \text{on} \quad \partial\Omega.$$

Then we receive

$$b_{\infty}(x, Dd(x)) = \frac{D\mu(x)}{\mu(x)} \cdot N_{\partial\Omega} = -b_{\infty}(x, -Dd(x)),$$

hence, our assumption ($\mathscr{C}$) becomes

$$\mathscr{K}(x) > b_{\infty}(x, Dd(x)) \qquad \text{for all} \quad x \in \partial\Omega.$$

To include non-zero boundary values $f \in \mathrm{C}^2(\partial\Omega)$, we transform our differential equation according to [13, (14.5)], ie. we consider the differential operator

$$\tilde{Q}v := a^{ij}(x, D(v+f))D_{ij}(v+f) + b(x, D(v+f))$$

and note that

$$\tilde{Q}(u-f) = Qu = 0 \qquad \text{while} \qquad (u-f)_{\partial\Omega} = 0.$$

Let us now set $w = kd$ for some positive real k to compare via the operator $\tilde{Q}$ and estimate by means of the decomposition of a^{ij} and b that

$$\begin{aligned} &\tilde{Q}w = (\Lambda a^{ij}_\infty + a^{ij}_0)D_{ij}w + (\Lambda a^{ij}_\infty + a^{ij}_0)D_{ij}f + \Lambda|Dw + Df|b_\infty \\ = \;& \Lambda\Big(k(a^{ij}_\infty D_{ij}d + |Dd + Df/k|b_\infty) + a^{ij}_\infty D_{ij}f + (ka^{ij}_0 D_{ij}d + a^{ij}_0 D_{ij}f)/\Lambda\Big), \end{aligned}$$

where all structure functions have arguments are x and $Dw + Df$. We recall and note that our assumptions on the boundary curvature imply the existence of $k_0, \chi, \kappa > 0$ such that

$$a^{ij}_\infty(Dd + Df/k)D_{ij}d + |Dd + Df/k|b_\infty(y, Dd + Df/k) \leqslant -\chi < 0$$

for all $y \in \{\operatorname{dist}(y,\partial\Omega) \leqslant \kappa\} \cap \overline{\Omega}$ and $k \geqslant k_0 > 0$.

As $a^{ij}_\infty D_{ij}f$ only depends on Dd and f in second order, we immediatly observe

$$\left|a^{ij}_\infty D_{ij}f\right| \leqslant 2\|D^2 f\|_{\mathrm{C}^0} \qquad \text{on} \quad \overline{\Omega}$$

while we compute

$$\left|\frac{a^{ij}_0(x,p)}{\Lambda(x,p)}\right| = \frac{|p_i p_j|}{(1+|p|^2)|p|^2} \leqslant \frac{|p_i p_j|}{|p|^4} \longrightarrow 0 \qquad \text{as} \quad |p| \to \infty.$$

Summing up, we infer

$$\tilde{Q}w \leqslant \Lambda k\left(-\chi + \frac{a^{ij}_\infty D_{ij}f}{k} + \frac{a^{ij}_0 D_{ij}(d + f/k)}{\Lambda}\right)$$

and we may rearrange the real positive k_0 (since $|D(kd) + Df| \longrightarrow \infty$ as $k \to \infty$) such that also

$$\left(-\chi + \frac{a^{ij}_\infty D_{ij}f}{k} + \frac{a^{ij}_0 D_{ij}(d + k^{-1}f)}{\Lambda}\right) < 0 \qquad \text{for all} \quad k \geqslant k_0.$$

Furthermore, we observe for $k \geqslant k_0$ that,

$$\Lambda(x, Dw + Df)k = \mu(x)(1 + |Dw + Df|^2)k \geqslant \alpha k|Dw + Df|^2 > 0,$$

which leads to

$$\tilde{Q}w < 0 \qquad\qquad \text{for} \quad k \geqslant k_0.$$

By the maximum principle, we may increase k_0 to arrange for

$$\sup_\Omega |u - f| \leqslant 2\sup_\Omega |f| \leqslant k\kappa \qquad \text{for} \quad k \geqslant k_0$$

such that w is eligible as an upper barrier on the closed set with class C^2-boundary

$$\mathscr{N} := \{y \in \overline{\Omega} \mid \operatorname{dist}(y,\partial\Omega) \leqslant \kappa\}$$

due to

$$\tilde{Q}w < 0 = \tilde{Q}(u-f) = Qu \qquad \text{on} \quad \operatorname{int}\mathscr{N}.$$

We may apply the comparison principle [13, Theorem 10.1] via the operator $\tilde{Q}$. Replacing w by $-w$, we construct similarly a uniform lower barrier and the boundary gradient estimate follows by standard considerations about normal derivatives in terms of

$$\sup_{\partial\Omega}|Du| \leqslant \sup_{\partial\Omega}|D(u-f)| + |Df| \leqslant k + \|Df\|_{C^0(\overline{\Omega})}.$$

We now make the following essential claim:

Replacing f by f/δ for $0 < \delta \leqslant 1$, the inequality holds with $k_\delta = K/\delta$, where $K \geqslant k_0$ is fixed as above.

To prove the claim without loss of generality for upper barriers, we expand terms and write

$$\begin{aligned}
\tilde{Q}w &\leqslant \Lambda k\left(-\chi + k^{-1}a_\infty^{ij}D_{ij}f + k^{-1}a_0^{ij}D_{ij}f/\Lambda + a_0^{ij}D_{ij}d/\Lambda\right)\\
&= \Lambda k\left(-\chi + k^{-1}a_\infty^{ij}f_{ij}\right)\\
&\quad +\mu\Lambda k\left(\frac{k^{-1}(kd+f)_i(kd+f)_j f_{ij}}{|Dw+Df|^2\Lambda} + \frac{(kd+f)_i(kd+f)_j d_{ij}}{|Dw+Df|^2\Lambda}\right).
\end{aligned}$$

We consider each term in the factors seperately and replace f by f/δ and k by k/δ, thus finding for the second term

$$|(k/\delta)^{-1}a_\infty^{ij}(f/\delta)_{ij}| = k^{-1}|a_\infty^{ij}f_{ij}|.$$

and moreover, for the third term,

$$\begin{aligned}
&\left|\frac{(k/\delta)^{-1}((k/\delta)d+(f/\delta))_i((k/\delta)d+(f/\delta))_j(f/\delta)_{ij}}{|D((w/\delta)+(f/\delta))|^2\Lambda(x,D(w/\delta+f/\delta))}\right|\\
=\ &\left|\frac{k^{-1}(w+f)_i(w+f)_j f_{ij}}{|Dw+Df|^2\Lambda(x,D(w/\delta+f/\delta))}\right| \leqslant \frac{k^{-1}|(w+f)_i(w+f)_j f_{ij}|}{\mu(x)|Dw+Df|^4},
\end{aligned}$$

since

$$\Lambda(x, Dw/\delta + Df/\delta) = \delta^{-2}\mu(x)(\delta^2 + |Dw+Df|^2) \geqslant \delta^{-2}\mu(x)|Dw+Df|^2,$$

and by replacing f_{ij} by d_{ij} and ignoring the factor k^{-1}, we bound the modulus of the fourth term similarly.

Thus, if $K > 0$ is chosen as before and large enough with $K \geqslant k_0 > 0$ so to dominate

$$\left|\frac{a_0^{ij}(x,p)}{\Lambda(x,p)}\right| \leqslant \frac{|p_ip_j|}{|p|^4} \qquad \text{sufficiently small with } p = D(kd) + Df,$$

then the same will hold in fact for all $0 < \delta \leqslant 1$ and the modulus of the sum will strictly not exceed χ for all $0 < \delta \leqslant 1$. Hence,

$$\tilde{Q}_\delta\left(\frac{w}{\delta}\right) < 0 \qquad \text{for all } 0 < \delta \leqslant 1$$

and the proof of the claim is readily finished. ■

12.2 Convergence to Functions of Weighted Least Gradient and Examples.

In this subsection we will now use the scalable a priori bounds on the family $\mathrm{P}(\delta)$ of quasilinear problems to generate a function of weighted least gradient. To be able to proceed to do so, it remains to show that a scalable maximum principle for the gradient of solutions is available, which seems to be the largest caveat in this approach and which results in more conditions that the total variation (and in our case especially, the weight) has to fulfill. We use the following proposition, whose usage is somewhat referred to in [44].

12.6 Proposition. *Assume that the weight function $\mu \in \mathrm{C}^2(\overline{\Omega})$ fulfills the inequality*

$$p^T \cdot D^2\mu \cdot p \leqslant \frac{n+1}{n} \frac{(D\mu \cdot p)^2}{\mu} \qquad \textit{for any} \quad p \in \mathbb{R}^n \quad \textit{on } \overline{\Omega}. \tag{$\mathscr{G}$}$$

Then the solution $u_\delta \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$ to $\mathrm{P}(\delta)$ fulfills

$$\max_{\overline{\Omega}} |Du_\delta| \leqslant \max_{\partial\Omega} |Du_\delta|.$$

Proof. Dividing the equation $Qu_\delta = 0$ by the weight function $\mu > 0$ and by the positive factor $1 + |p|^2$, we see that we may write the problem $\mathrm{P}(\delta)$ in the form

$$\tilde{a}^{ij}(x,p) = \delta_{ij} - \frac{1}{2}\left(p_i \frac{p_j}{1+|p|^2} + \frac{p_i}{1+|p|^2} p_j \right), \quad \tilde{b}(x,p) = \frac{D\mu(x) \cdot p}{\mu(x)}.$$

Hence, employing the differential operator $\mathscr{D} = D_z + |p|^{-2} p_i D_{x_i}$, we compute that the inequality $(\mathscr{G})$ is precisely

$$|p|^2 \mathscr{D}\tilde{b} \leqslant \frac{\tilde{b}^2}{n}.$$

The assertion is a now a consequence of the gradient maximum principle of [13, 15.1]. ■

Such further restrictions as the inequality $(\mathscr{G})$ seem to appear naturally in this approach while dealing with x-dependencies.

Exploiting the scalable bounds of theorem 12.5 with proposition 12.6, we can now prove the following existence theorem via a regularization procedure.

12.7 Theorem (Existence of Functions of Weighted Least Gradient)**.** *Let Ω be a bounded open and connected set of class C^2 with $f \in \mathrm{C}^2(\partial\Omega)$ and let $\mu \in \mathrm{C}^2(\overline{\Omega})$. Assume that the boundary curvature inequality $(\mathscr{C})$ and the differential inequality $(\mathscr{G})$ hold. Then there exists a function of weighted least gradient u with respect to the weight μ and f with $u \in \mathrm{C}^{0,1}(\overline{\Omega})$.*

Proof. Let $u_\delta \in \mathrm{C}^2(\overline{\Omega})$ denote the solution to the problem $\mathrm{P}(\delta)$, which exists due to proposition 12.2, and define

$$\mathscr{U}_\delta := \delta u_\delta \qquad \text{for} \quad 0 < \delta \leqslant 1.$$

It is evident that $\mathscr{U}_\delta = f$ on $\partial\Omega$ while we estimate for its gradient that

$$\max_{\overline{\Omega}} |D\mathscr{U}_\delta| \leqslant \delta \left(K/\delta + \|Df/\delta\|_{\mathrm{C}^0(\Omega)} \right),$$

where C does not depend on δ. Hence,

$$\max_{\overline{\Omega}} |D\mathscr{U}_\delta| \leqslant \text{const} < \infty \qquad \text{independently of } 0 < \delta \leqslant 1.$$

We will not relabel and assume, by compactness with respect to the uniform topology, that $\mathscr{U}_\delta \longrightarrow u$ as $\delta \to 0$ where u is a Lipschitzian function on $\overline{\Omega}$.

To show that u is actually of weighted least gradient on Ω, let us fix any

$$v \in \mathrm{C}^{0,1}(\overline{\Omega}) \qquad \text{with} \qquad v_{\partial\Omega} = f,$$

to compare and estimate

$$\begin{aligned}
\mu|Du|(\Omega) &\leqslant \liminf_{\delta\to 0} \mu|\delta Du_\delta|(\Omega) \\
&\leqslant \liminf_{\delta\to 0} \delta \int_\Omega \mu(x)\sqrt{1+|Du_\delta|^2}\ dx \\
&\leqslant \liminf_{\delta\to 0} \delta \int_\Omega \mu(x)\sqrt{1+|D(v/\delta)|^2}\ dx \\
&\leqslant \liminf_{\delta\to 0} \delta \left(\int_\Omega \mu(x)\ dx + \int_\Omega \delta^{-1}\mu(x)|Dv|\ dx\right) \\
&= \mu|Dv|(\Omega)
\end{aligned}$$

due to lower semicontinuity of the total variation and minimization of u_δ for the functional $\mathscr{F}_\mu$ by remark 12.4. A simple approximation argument akin to [41, 4.(iii) Lemma] shows that actually

$$\mu|Du|(\Omega) \leqslant \mu|D(u+v)|(\Omega) \qquad \text{for all} \quad v \in \mathrm{BV}(\Omega)$$

such that v has compact support in Ω, which suffices to assure u being a function of weighted least gradient by [33, Proposition 3.16]. ■

12.8 Remark. It appears to be not clear which approximation is the "correct" one for generating functions of (anisotropic) least gradient by means of quasilinear elliptic theory. Pursuing these questions of existence and regularity via the functional $\mathscr{F}_\mu$ in BV with regularity of a minimizer up to the boundary however seems to be quite non-trivial.

12.9 Remark (Neumann Problems). A similar approximation method has also been applied to total variation problems by A. Porretta in [48]. As that paper deals with variational problems *without* Dirichlet boundary, which translate to approximating problems *with Neumann boundary conditions*, the author is able to exploit the Neumann boundary conditions for uniform gradient estimates, making direct arguments possible. Since we though also arrive at some term of the form

$$\int_\Omega \mu(x)\sqrt{\delta^2+|Dv|^2}\ dx \quad \text{approximating} \quad \int_\Omega \mu(x)|Dv|\ dx,$$

it appears that for the case of Dirichlet problems (and also without the lower order term here) it is sometimes convenient to "hide" the viscosity in terms of scaled boundary data, if one may rely on aedequately scaled estimates.

Applying stronger assumptions like convexity, we will find that theorem 12.7 yields in particular a sufficient condition and quantification of how much the weight is allowed to vary to still generate Lipschitzian functions of weighted least gradient. We recall that a domain is called *uniformly convex* if there is some *curvature radius* $R > 0$ such that each point in its boundary lies on the surface of a ball of radius $R > 0$ containing $\overline{\Omega}$.

12.10 Corollary (Existence on Uniformly Convex Domains). *Let Ω be uniformly convex with curvature radius $R > 0$ and $f \in \mathrm{C}^2(\partial\Omega)$. Then if*

$$\frac{\|D\mu\|_{\mathrm{C}^0}}{(n-1)\alpha} < R^{-1} \qquad \textit{and inequality } (\mathscr{G}) \textit{ holds,}$$

there exists a function $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ of weighted least gradient with $u_{\partial\Omega} = f$.

Proof. This follows immediately from

$$\frac{D\mu \cdot N_{\partial\Omega}}{(n-1)\mu} \leqslant \frac{\|D\mu\|_{\mathrm{C}^0}}{(n-1)\alpha} < R^{-1} \leqslant H,$$

as it is a classical fact that all principal curvatures on a uniformly convex domain are bounded from below by R^{-1}. ∎

12.11 Remark. (1) We observe that the condition

$$\sup_{\partial\Omega} \frac{|D\mu|}{(n-1)\mu} < R^{-1}$$

would be sufficient for corollary 12.10 to hold. Also, if Ω is only *strictly mean convex* in the sense that $H \geqslant H_0 > 0$, we may replace R^{-1} by H_0 for the same conclusion.

(2) The isotropic case corresponds to $D\mu \equiv 0$ and we hence recover the existence of isotropic Lipschitzian least gradient functions on uniformly or strictly mean convex domains with C^2-boundary data from [53]. If the weight is allowed to vary, we must however impose a smallness condition such that it is not allowed to vary too much.

(3) If Ω is only assumed to be convex, it follows that all principal curvatures are non-negative, but the mean curvature may obviously vanish, and the growth of μ has to account for singular or flat regions of $\partial\Omega$. In particular, we may penalize such regions with the weight μ to still make existence of Lipschitzian extremals possible for each $f \in \mathrm{C}^2(\partial\Omega)$.

(4) Note that corollary 12.10 recovers and generalizes the original remark from [44] (at least for C^2-domains), since linear barriers $\pi \equiv \pi(x_0)$ can be constructed by means of Df in this case (see eg. [15, Theorem 1.1]) and they require the (stronger) sign condition

$$D\mu \cdot N_{\partial\Omega} < 0 \qquad \text{on } \partial\Omega$$

to derive the uniform sign for $Q(\pi)$.

We will now discuss some conditions implying the validity of our existence theorem.

12.12 Example. (1) Let μ be affine-linear and given by

$$\mu(x) = \beta x + \gamma, \qquad \beta \in \mathbb{R}^n, \quad \gamma \in \mathbb{R}.$$

Recall that the weight μ needs to fulfill

$$\min_{\overline{\Omega}} \beta x + \gamma > 0,$$

thus we might always increase γ to dominate the fraction while the position of Ω in space implies additional degrees of freedom. Furthermore, the inequality ($\mathscr{G}$) for the gradient maximum principle trivially holds and the curvature inequality ($\mathscr{C}$) becomes

$$H(x) > \frac{\beta \cdot N_{\partial\Omega}(x)}{(n-1)\mu(x)} \qquad \text{for all} \quad x \in \partial\Omega.$$

We infer that, up to positive factor, the mean curvature needs to be positive where β and $N_{\partial\Omega}$ point in the same direction while even flat or negatively curved regions are allowed where β and $N_{\partial\Omega}$ point in opposite directions.

(2) Even if μ is not linear, we may always add positive constants to the weight μ to make μ^{-1} as small as we desire and fit a given positive threshold of $H_0 > 0$. Hence, each weight $\mu \in \mathrm{C}^2(\overline{\Omega})$ on a strictly mean convex domain admits $\gamma(\mu) > 0$ such that the weight $\tilde{\mu} = \mu + \gamma(\mu)$ admits a Lipschitz minimizer.

(3) More generally, if μ is locally concave on Ω, then inequality ($\mathscr{G}$) also clearly holds, and we moreover need to require that the growth of μ is sufficiently small while its modulus is sufficiently large to make the curvature inequality ($\mathscr{C}$) work. If Ω is strictly mean convex, then in particular concave pertubations of the form

$$\Phi(x, v) := (1 + \nu(x))|v| \qquad \text{with} \quad \nu \in \mathrm{C}^2_c(\Omega)$$

are eligible.

(4) We observe that the boundary curvature condition is fulfilled at $x_0 \in \partial\Omega$ with $H(x_0) > 0$ if μ is non-decreasing on some neighborhood on the outward normal ray, as then clearly $D\mu(x_0) \cdot N_{\partial\Omega}(x_0) \leqslant 0$. If Ω is only mean-convex, then we may require the strict inequality $D\mu(x_0) \cdot N_{\partial\Omega}(x_0) < 0$ (implying of course that μ strictly increases along that ray).

(5) In general, regions of low or negative mean curvature are tolerated where μ grows strongly while regions of high mean curvature have to account for areas where μ decreases across $\partial\Omega$.

Let us finally recall the different ways to impose differential inequalities on the distance to the boundary d.

Our method has imposed a differential condition on the boundary curvature by means of the approximating problem $\mathrm{P}(\delta)$ to assure existence and scaled gradient bounds for this problem, while the paper [44, 1(5), last inequality] employs a differential condition directly by means of the weighted integrand $\Phi(x,p) = \mu(x)|p|$. Indeed, in defining the quasilinear differential operator

$$\begin{aligned}\left(Q'u\right)(x) \quad &:= \quad \sum_{i,j=1}^{n} D_{p_i,p_j}\Phi(x,Du)u_{ij} + \sum_{k=1}^{n} D_{x_k,p_k}\Phi(x,Du)\\ &= \quad \operatorname{div}\left(D_p\Phi(x,Du)\right),\end{aligned}$$

which is well-defined as long as $Du \neq 0$, we shall assume

$$(Q'd)(x) < 0 \qquad \text{for all } x \in \partial\Omega. \tag{$\mathscr{C}'$}$$

Given such a condition on d by means of Φ, we repeat from [44, 4(5)] the existence of a function of weighted least gradient w, which is shown to be essentially bounded in Ω and obeys a Lipschitz estimate if at least one point is constrained to the boundary, but otherwise not even necessarily continuous.

A simple computation shows that both conditions actually coincide and we may conclude:

12.13 Proposition (CONSISTENCY)**.** *Assuming that Φ is the weighted total variation*

$$\Phi(x,p) = \mu(x)|p|, \qquad\qquad \mu \in \mathrm{C}^2(\overline{\Omega}),$$

and the inequalities ($\mathscr{C}'$) and ($\mathscr{G}$) hold on a domain Ω of class C^2, then there exists a function of weighted least gradient $u \in \mathrm{C}^{0,1}(\overline{\Omega})$ with $u_{\partial\Omega} = f$ for any $f \in \mathrm{C}^2(\partial\Omega)$.

Proof. To apply our Lipschitz existence theorem, we only need to show that both conditions on boundary curvature actually coincide. We therefore compute

$$D_{x_k,p_k}\Phi(x,p) = \mu_{x_k}(x)\frac{p_k}{|p|}, \qquad D_{p_i,p_j}\Phi(x,p) = \mu(x)\left(\frac{\delta_{ij}}{|p|} - \frac{p_ip_j}{|p|^3}\right),$$

and find

$$Q'd = \mu(x)(\delta_{ij} - d_id_j)d_{ij} + D\mu\cdot N_{\partial\Omega}.$$

Now we notice that

$$(\delta_{ij} - d_id_j)d_{ij} = \Delta d = -(n-1)H,$$

which leads to

$$Q'd = \mu\Delta d + D\mu\cdot N_{\partial\Omega} < 0 \quad \text{if and only if} \quad \frac{D\mu\cdot N_{\partial\Omega}}{(n-1)\mu} < H$$

on $\partial\Omega$ and the proof is finished. ∎

12.14 Remark. Note also that we may reduce $\mu \in \mathrm{C}^4(\overline{\Omega})$ from [44, Preliminaries 1(3)] to $\mu \in \mathrm{C}^2(\overline{\Omega})$ in proposition 12.13 as we need not deal with the minimizing level set boundaries like in [44, Lipschitz Regularity 6.(1)(α)] by assuming the inequality ($\mathscr{G}$).

12.15 Remark. The curvature condition ($\mathscr{C}'$) is in particular a strict and C^2-differentiable version of the barrier condition from [24] and we observe that our approximating problems $\mathrm{P}(\delta)$ appear to be somewhat compatible.

We finally prove a slight sharpening of theorem 12.7, which coincides with the results of [53, Theorem 5.9].

12.16 Corollary (Dirichlet Data of class $\mathrm{C}^{1,1}$). *Let the assumptions of theorem 12.7 hold with $f \in \mathrm{C}^2(\partial\Omega)$ replaced by $f \in \mathrm{C}^{1,1}(\partial\Omega)$. Then theorem 12.7 still holds.*

Proof. We denote by $u_{j,\delta} \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$ the unique solution to $\mathrm{P}(\delta)$ with f replaced by $f_j \in \mathrm{C}^2(\overline{\Omega})$, where

$$f_j \longrightarrow f \qquad \text{in } \mathrm{C}^1 \quad \text{and} \quad \sup_{j\in\mathbb{N}} \|D^2 f_j\|_{\mathrm{C}^0} < \infty.$$

Inspecting the proof of the boundary gradient estimate yields that we may find some $L > 0$ such that

$$\max_{\overline{\Omega}} |Du_{j,\delta}| \leqslant L/\delta + \|Df_j/\delta\|_{\mathrm{C}^0}$$

and the constant $L > 0$ is in fact *independent* of $j \in \mathbb{N}$. Thus, for each fixed $0 < \delta \leqslant 1$, there exists a uniformly converging subsequence, which we do not relabel, such that

$$u_{j,\delta} \longrightarrow u_\delta, \qquad \mathrm{Lip}(u_\delta) \leqslant L/\delta + \|Df/\delta\|_{\mathrm{C}^0}, \qquad u_{\delta,\partial\Omega} = f/\delta,$$

where $\mathrm{Lip}(u)$ denotes the Lipschitz constant of u on $\overline{\Omega}$. Using interior gradient Hölder estimates (eg. [13, Theorem 13.1]) and the weak form of $\mathrm{P}(\delta)$, we infer that $u_\delta \in \mathrm{C}^1(\Omega)$ and, by uniform convergence in C^1, it is a weak solution to $\mathrm{P}(\delta)$ with boundary data f, ie.

$$\int_\Omega \mathbf{A}(x, Du_\delta(x)) \cdot D\varphi(x)\; d\mathscr{L}^n = 0 \qquad \text{for all} \quad \varphi \in \mathrm{C}_c^\infty(\Omega).$$

This suffices to conclude that u_δ minimizes the functional $\mathscr{F}_\mu$ in the Dirichlet class for f/δ and we may subsequently proceed as in the proof of theorem 12.7 by choosing a convergent subsequence of $\mathscr{U}_\delta = \delta u_\delta$. ∎

13 *Further Discussion and Related Open Questions.*

The Regularity of Extremals.

(1) Even though nearly all mappings used in our regularity construction suffice Hölder estimates, the late regularity theorems 7.1/7.2 do only provide a conclusion on differentiability. It should arguably be possible to chase Hölder regularities and include local estimates in the end.

(2) All applications of the later sections involved somewhat generic a priori assumptions on the level set. I did not investigate whether some of them might in fact hold always and not only generically (eg. whether it is even possible that the derivative vanishes uniformly on some connected component of a level set boundary).

(3) This method for inducing regularity could in theory also be applied to functions minimizing an anisotropic total variation together with some volume term (think of Rudin-Osher-Fatemi for example). The main drawback here is however the little investigation that has happened on Dirichlet problems for this category and I am only aware of some one-dimensional deductions by BREZIS in [6].

(4) Questions and results regarding the stability and deformations of level sets were to some extent also involved in the more applied theory regarding the usage of anisotropic least gradient problems in conductivity imaging in [30, 31]. The deformation theory of eg. section 5 might also be applied there (for whatever possible good or worse).

(5) All results from sections 4,5,6 do not explicity need absolute 1-homogeneity (except of course in quotations of eg. [24]). Most likely this assumption is more technical than essential.

The Existence of Extremals.

(6) In view of the results by uniform strict boundary curvature estimates from section 12, one might ask why we restrained ourselves to Lipschitz regularity and did not cover Hölder estimates with respect to less regular Dirichlet data.
Apparantly, such lower regularity is also less understood even for quasilinear elliptic equations (with only some papers by LIEBERMAN and SIMON in [27, 28, 51] available to my knowledge). It appears that some form of new idea might be necessary to prove the global Hölder estimates to generate/pass to a limit. Also, PARKS' method of generating Lipschitzian extremals from [44] *might* be able to also generate Hölderian extremals for less regular data.

(7) More general, it remains unclear whether such a viscosity/scaling approach via a priori estimates for quasilinear elliptic problems is also feasible to handle total variations $\Phi = \mu\Theta$ of product form, or even arbitrary form, or whether the equations become too unwieldy to argue and conclude. Furthermore, as already remarked in the introduction of [48], also the case of the ROF functional could be investigated for more general anisotropies with respect to Lipschitz/Hölder/Sobolev estimates.

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Abstract.

This thesis deals with a certain regularity theory für differentialbility of higher order (that is, differentiability of class $C^{k,\alpha}$) for anisotropic functions of least gradient. The approach, which hails from a pioneering paper by H. R. Parks and W. P. Ziemer, is geometric and analytic and uses in an essential manner the property that all level sets of such functions are bounded by surfaces which solve a geometric variational problem. In addition to generalizing to anisotropic problems we fill a gap in the proof, make convergence properties for the minimizing hypersurfaces more precise and introduce a new and simplified proof for a Harnack type inequality of central importance. After the main regularity theorem for local and partial regularity we investigate sufficient conditions for local regularity and prove a definiteness theorem for the gradient which has only been sketched before. Subsequently we seek criteria for local boundary regularity and local regularity near a singular ("fat") level set.

Concluding the thesis, we also provide a new existence theorem for regular extremals to a weighted least gradient problem.

Kurzzusammenfassung.

Diese Doktorarbeit beschäftigt sich mit einer Regularitätstheorie für höhere Differenzierbarkeit (dh. der Klassen $C^{k,\alpha}$) für anisotrope Least-Gradient-Funktionen. Die Heransgehensweise, welche aus einem innovativen Paper von H. R. Parks und W. P. Ziemer für den isotropen Fall entstammt, ist geometrisch und analytisch und benutzt in essentiellem Maße die Eigenschaft, dass alle Niveaumengen solcher Funktionen durch Flächen berandet werden, die ein geometrisches Minimierungsproblem lösen. Neben der Verallgemeinerung auf anisotrope Probleme schließen wir eine Lücke im Beweis, präzisieren Konvergenzresultate für die minimierenden Flächen und geben einen neuen vereinfachten Beweis für eine Ungleichung vom Harnack-Typ von zentraler Wichtigkeit. Nach dem hauptsächlichen Regularitätssatz für lokale und partielle Regularität untersuchen wir hinreichende Bedingungen für lokale Regularität und beweisen einen vormals nur skizzierten Definitheitssatz für den Gradienten. Schließlich geben wir Kriterien für lokale Randregularität und lokale Regularität nahe einer singulären ("fetten") Niveaumenge.

Am Ende der Arbeit beschäftigen wir uns zudem mit einem neuen Existenzsatz für reguläre Extrempunkte eines gewichteten Least-Gradient-Problems.

Declaration of Pre-publications.

The only publication hailing from this thesis at the time of submission is the paper [47], which is submitted and not published yet and basically comprises of the full section 12.