On Dirac Generating Operators and Clifford Module Bundles

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Summary

This thesis deals with irreducible Clifford module bundles and Dirac generating operators of Courant algebroids.

Irreducible Clifford module bundles are real vector bundles that carry a representation of a real Clifford algebra bundle in such a way that all the fibrewise representations are of the same 'type', i.e. they are equivalent to a fixed irreducible representation of a real Clifford algebra. As one of the main results of this thesis, we demonstrate a relation between any two irreducible Clifford module bundles of the same type and use this relation to classify all such module bundles. More precisely, given a fixed irreducible Clifford module bundle, we establish a bijection between the isomorphism classes of all irreducible Clifford module bundles of the same type as this fixed bundle and the isomorphism classes of all module bundles that carry the regular representation of the Schur algebra bundle of the fixed module bundle.

Dirac generating operators of Courant algebroids are certain first-order differential operators defined on irreducible Clifford module bundles. The local existence of these operators when the scalar product on the Courant algebroid is of neutral signature (p, p) was shown by Alekseev and Xu in [AX]. We apply our results on Clifford module bundles to extend this result to prove the existence of local Dirac generating operators of Courant algebroids in arbitrary signatures. As a further main result of this thesis we give a description of the set of Dirac generating operators of a Courant algebroid in signature $(p, p+1)$ as an affine space. In particular, the difference of any two such operators is given by Clifford multiplication with a section of the Courant algebroid that satisfies a certain condition. This is the first study of Dirac generating operators that explicitly takes into consideration non-neutral signatures.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit irreduziblen Clifford-Modulbündeln und Dirac-Erzeugendenoperatoren von Courant-Algebroiden.

Irreduzible Clifford-Modulbündel sind reelle Vektorbündel, die eine Darstellung eines reellen Clifford-Algebra-Bündels tragen, sodass die Darstellungen faserweise vom gleichen 'Typ' sind, d.h. sie sind alle äquivalent zu einer festen irreduziblen Darstellung einer reellen Clifford-Algebra. Als eines der Hauptergebnisse dieser Arbeit beweisen wir eine Beziehung zwischen beliebigen irreduziblen Clifford-Modulbündeln des gleichen Typs und verwenden diese Beziehung, um alle derartigen Modulbündel zu klassifizieren. Genauer, gegeben ein festes irreduzibles Clifford-Modulbündel, beweisen wir eine Bijektion zwischen den Isomorphieklassen aller irreduziblen Cliffordmodulbündel vom gleichen Typ wie dieses feste Bündel und den Isomorphieklassen aller Modulbündeln, die die reguläre Darstellung des Schur-Algebra-Bündels des festen Modulbündels tragen.

Dirac-Erzeugendenoperatoren von Courant-Algebroiden sind gewisse Differentialoperatoren erster Ordnung auf irreduziblen Clifford-Modulbündeln. Die lokale Existenz dieser Operatoren, wenn das Skalarprodukt des Courant-Algebroids neutrale Signatur (p, p) hat, wurde von Alekseev und Xu in [AX] bewiesen. Wir nutzen unsere Ergebnisse über Clifford-Modulbündel, um diese Aussage auf die Existenz lokaler Dirac-Erzeugendenoperatoren von Courant-Algebroiden mit beliebigen Signaturen zu verallgemeinern. Ein weiteres Hauptergebnis dieser Arbeit ist die Beschreibung der Menge der Dirac-Erzeugendenoperatoren eines Courant-Algebroids von Signatur $(p, p+1)$ als affiner Raum. Hierbei ist die Differenz zweier solcher Operatoren gegeben durch Clifford-Multiplikation mit einem Schnitt des Courant-Algebroids, der eine bestimmte Bedingung erfüllt. Die vorliegende ist die erste Forschungsarbeit über Dirac-Erzeugendenoperatoren, die nicht-neutrale Signaturen explizit berücksichtigt.

To Amma, my favourite scientist.

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Contents

1 Introduction

None of this was too painful, it was another of life's perpetual little torments, that was all, nothing when measured against what K. aspired to, he had not come to this place to lead a life of peace and honour.

Franz Kafka, The Castle

This thesis relates objects encountered in two areas of mathematics – spin geometry and generalised geometry.

In spin geometry, we encounter an associative algebra called a Clifford algebra. Such an algebra arises naturally from any pseudo-Euclidean vector space. The Clifford algebra contains the spin group which is the central object of study in spin geometry along with the cohort of its representations and other geometric structures that can be associated to it.

There are at least two approaches to the study of spin representations and more generally Clifford representations at the level of bundles. The classical approach begins by assuming the existence of a spin structure on a pseudo-Riemannian manifold (M, g) [LM]. To every spin structure we can associate a Clifford module bundle. We do this by taking a spin structure and a representation of the corresponding spin group to obtain the so-called spinor bundle S via the associated vector bundle construction. Now, if we denote by $\mathcal{C}\ell(M, g)$ the bundle of Clifford algebras associated to the tangent bundle of the manifold, then we can show that the spinor bundle S

carries a representation of the Clifford algebra bundle $\mathcal{C}\ell(M,g)$ and is therefore in fact a bundle of Clifford modules. In this way, the existence of a spin structure on a pseudo-Riemannian manifold implies the existence of a bundle of Clifford modules. The converse however is not true, one can have Clifford module bundles without spin structure, for examples see [FT, LS1].

An alternative approach to spin geometry starts with the assumption that (M,g) admits a bundle of Clifford modules S over $\mathcal{C}\ell(M,g)$ without necessarily associating it to a specific spin structure. Generally this assumption is weaker than assuming the existence of a spin structure and has been found useful for various applications in mathematics and physics. For a detailed discussion see [FT, LS1].

In this thesis, in the spirit of the latter approach, we begin with a pseudo-Riemannian vector bundle $(E,\langle\cdot,\cdot\rangle)$ of signature (r,s) on a smooth manifold M and consider the Clifford algebra bundle $\mathcal{C}\ell(E)$ associated to it in the natural way. We assume the existence of module bundles S over the Clifford algebra bundle $\mathcal{C}\ell(E)$, here onwards called Clifford module bundles, and ask how any two such irreducible Clifford module bundles S and Σ that are of the 'same type', are related. By 'same type' we mean that each fibre of both Clifford module bundles is equivalent to a fixed Clifford module $\mathbb{R}^N \cong \Sigma_p \cong S_p$ for some $N \in \mathbb{N}$, over the standard Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathcal{C}\ell(E)_p$. This question has been studied thoroughly when the Clifford module bundles are associated to spin structures or spin^c structures [FT, LM, F] but here we consider them in greater generality.

An answer to the above question is relevant in various contexts in which spin geometry plays a role. In particular, it is necessary when a 'parametrisation' of spin structures or spin^c structures is required, such as in Seiberg–Witten theory [N, Sa, FV]. The choice of a spin structure determines the amount of super-symmetry preserved by solutions of supergravity equations [FG], which is another area where the answer to our question is relevant. In this thesis we show that in the context of generalised geometry, when developing a theory of Dirac generating operators of Courant algebroids, an answer to the question becomes important.

In Chapter 2, we collect all the background we need for this thesis. In Chapter 3, we answer how any two irreducible Clifford module bundles of the same type are related and arrive at the first main results of this thesis. We begin with a definition of module bundles over arbitrary algebra bundles. Our algebras are always assumed to be associative and with a unit. For a $\mathcal{C}\ell(E)$-module bundle Σ, the

Schur algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$ is defined, fibrewise, to be the centraliser of the irreducible representation of $\mathcal{C}\ell(E)_p$ on Σ_p. Then we describe the notion of taking the 'tensor product of module bundles over an algebra bundle' which is the natural generalisation of the algebraic notion of taking the tensor product of modules over an algebra, to fibre bundles. First we show that the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between two Clifford module bundles Σ and S denoted by $L = \mathrm{Hom}_{\mathcal{C}\ell(E)}(\Sigma, S) \longrightarrow M$ is a right $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$-module bundle, i.e., L admits an action of the algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$ from the right. Next we show that Σ admits a natural left action of the algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$. After that, we define in Lemma 3.18 the bundle $\pi : L \otimes_{\mathcal{C}(\Sigma,\Gamma^{\Sigma})} \Sigma \longrightarrow M$ as a quotient of the bundle $L \otimes_{\mathbb{R}} \Sigma \longrightarrow M$ by a subbundle that relates the actions of $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$ on L and on Σ and show that this bundle is isomorphic to S as a $\mathcal{C}\ell(E)$-module bundle. The statement of Theorem 3.20 stated below summarises this result.

Theorem. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be two $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r, s)$ on $\mathbb{R}^N$. Let L be the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the module bundles Σ and S. Then $L \otimes_{\mathcal{C}(\Sigma,\Gamma^{\Sigma})} \Sigma \longrightarrow M$ and $S \longrightarrow M$ are canonically isomorphic as $\mathcal{C}\ell(E)$-module bundles.*

Further, if we fix a $\mathcal{C}\ell(E)$-module bundle of type γ, we obtain a classification of irreducible $\mathcal{C}\ell(E)$-module bundles which establishes a bijection between the set of isomorphism classes of irreducible $\mathcal{C}\ell(E)$-module bundles of type γ and the isomorphism classes of $\mathcal{C}(S, \Gamma^S)$-module bundles of the regular type. By this we mean that the type of the $\mathcal{C}(S, \Gamma^S)$-module bundle is the regular representation of the typical fibre of $\mathcal{C}(S, \Gamma^S)$ on itself by composition from the right. This result, Theorem 3.27, is summarised as follows.

Theorem. *Let $S \longrightarrow M$ be a fixed irreducible $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$. Then there exists a bijection between the following sets:*

$$A \coloneqq \{\textit{isomorphism classes of irreducible } \mathcal{C}\ell(E)\textit{-module bundles of type } [\gamma]\}$$

$$\updownarrow$$

$$B \coloneqq \{\textit{isomorphism classes of right } \mathcal{C}(S, \Gamma^S)\textit{-module bundles of regular type}\}$$

In Chapter 4 we apply the results that we have obtained so far to the study of Dirac generating operators on Courant algebroids in generalised geometry.

Generalised geometry as first developed by Hitchin, Gualtieri, Cavalcanti [Hi, G, CaG] emerged as a way of unifying symplectic geometry and complex geometry into a single framework. The basic idea was to think of symplectic and complex structures not as linear operators on the tangent bundle T of a manifold but on the sum of the tangent and the cotangent bundles $T \oplus T^*$ of the manifold. A Courant algebroid is a generalisation of this notion. Courant and Weinstein first proposed the idea of a Courant algebroid in their work on Dirac bundles [Co]. Liu, Weinstein and Xu in [LWX] systematised the definition of a Courant algebroid in their work generalising the notion of the Drinfeld double to Lie bialgebroids and Roytenberg refined this definition to its current one [R1]. A rich theory has now developed in generalised geometry with the concept of Courant algebroid at its center. A Courant algebroid is a vector bundle $E \longrightarrow M$ endowed with a symmetric bilinear form $\langle \cdot, \cdot \rangle \in \Gamma(\mathrm{Sym}^2 E^*)$, a bracket called the Dorfman backet $[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \longrightarrow \Gamma(E)$ and a bundle map called the anchor map $\pi : E \longrightarrow TM$ which satisfy some compatibility conditions. The Dorfman bracket is interesting in that it is not skew-symmetric but satisfies the Jacobi identity in the Leibniz form i.e., $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for $X, Y, Z \in \Gamma(E)$.

Dirac generating operators (DGOs) were first defined by Alekseev and Xu in an unpublished manuscript [AX] in an approach to understand Courant algebroids and as analogues of Kostant's cubic Dirac type operators [Ko]. Alekseev and Xu defined Dirac generating operators to be first-order odd differential operators. These operators were defined on irreducible module bundles of Clifford algebra bundles associated to pseudo-Riemannian vector bundles $(E, \langle \cdot, \cdot \rangle)$ of neutral signature (n, n) and satisfied the properties in Remark 4.2. In their work, they showed that with a Dirac generating operator, it is possible to define a Dorfman bracket and an anchor map on E such that the pseudo-Riemannian vector bundle $(E, \langle \cdot, \cdot \rangle)$ becomes a Courant algebroid. Since Courant algebroids are, in particular, pseudo-Riemannian vector bundles, DGOs can be defined on Courant algebroids. In such an instance, if the Dorfman bracket and the anchor map induced by the DGO coincide with the Dorfman bracket and the anchor map of the underlying Courant algebroid, then the DGO is called a 'Dirac generating operator of a Courant algebroid'. Note that DGOs of a Courant algebroid are not unique.

Alekseev and Xu proved that DGOs exist locally on every $\mathcal{C}\ell(E)$-module bundle and showed that any two local DGOs $\not{d}$ and $\not{d}'$ differ by the Clifford action of a section of E. Cortés and David in [CD1] developed a refined approach to Dirac generating operators by taking into account the structure of regular Courant algebroids with scalar product of neutral signature. They use a dissection of a regular Courant algebroid to arrive at an explicit formula for the standard form of the globally defined canonical DGO in terms of data encoding the regular Courant algebroid. They then applied their results to the study of integrability of generalised almost Hermitian structures.

Following the work of Alekseev and Xu in [AX], the notion of a DGO has found relevance in various contexts that involve a Courant algebroid. In the work by Chen and Stiénon [CS], it is shown that given a pair of Lie algebroid structures on a vector bundle A and its dual A^*, and given a line bundle $\mathcal{L}$ of a specified kind, the condition for the pair (A, A^*) to be a Lie bi-algebroid is for the sum of two naturally defined differential operators associated to A and $\mathcal{L}$, to be a Dirac generating operator. In a recent extension of this work Cai, Chen, Honglei, Lang and Xiang in [CCLX] show that the square of a Dirac generating operator gives rise to an invariant of the split Courant algebroid to which it corresponds. Gruetzmann, Michel and Xu in [GMX] show that the square of a DGO is an invariant in the case of more general Courant algebroids. DGOs have also been employed in gauge fixing Weyl degrees of freedom in the space of generalised connections with torsion T in the work of Garcia-Fernandez [GF]. They have also found application in the context of generalised scalar curvature [GFS].

An alternative approach to Courant algebroids has been taken from the perspective of graded symplectic supermanifolds by Weinstein, Roytenberg, Ševera [R1, R2, Š] and others. Dirac generating operators have also been studied within this approach [Š, ŠV, GMX].

In [CD2], Cortés and David develop a theory of T-duality for transitive Courant algebroids with scalar product of neutral signature. They recognise that canonical Dirac generating operators $\not{d}_E$ and $\not{d}_{\widetilde{E}}$ intertwine the map induced by T-duality between the spaces of sections of the canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\widetilde{E}}$ of the T-dual Courant algebroids E and $\widetilde{E}$ respectively. In order to extend their theory to more general Courant algebroids such as, for example, odd exact Courant algebroids [Ru], which are not of neutral signature and are of odd rank, they suggested that it would be useful to develop a theory of Dirac generating operators for such Courant algebroids. This was our primary motivation to study DGOs of Courant algebroids in arbitrary signatures.

The classification of real Clifford algebras tells us that real Clifford algebras are sensitive to the signature of the underlying pseudo-Euclidean vector space. Since a Courant algebroid structure is built on top of a pseudo-Riemannian vector bundle, we approach DGOs by keeping the signature of the Courant algebroid at the center of our enquiry. In doing that we are able to extend the result of Alekseev and Xu [AX] and Cortés and David [CD1] on the existence of local DGOs of regular Courant algebroids in neutral signature to arbitrary signatures. This is the first of our main results, Theorem 4.4, in Chapter 4.

Theorem. *Let E be a regular Courant algebroid with scalar product of arbitrary signature. Every $\mathcal{C}\ell(E)$-module bundle S of type $[\gamma]$, where γ is an irreducible representation, admits locally a Dirac generating operator.*

One of the difficulties in generalising the result of [AX] and [CD1] to Courant algebroids E of arbitrary signatures lay in the problem of constructing an E-connection D^S over any irreducible $\mathcal{C}\ell(E)$-module bundle S for a given generalised connection D on E such that D^S is compatible with the Clifford action. This in turn relies on the answer to the question of how two irreducible Clifford module bundles S and Σ of the 'same type', are related. Thus we are led back to our theory of Clifford module bundles in Chapter 3 and see that the theorems and various lemmata proven there find an application here.

If we have two DGOs $\not{d}$ and $\not{d}'$ of a Courant algebroid of signature $(p, p+1)$, then what is $\not{d} - \not{d}'$? This is the question we ask next. In answering that we find that the space of local DGOs in signature $(p, p+1)$ is also an affine space, analogous to the space of local DGOs in neutral signature [CD1]. Note that as per our sign conventions, we have $\mathcal{C}\ell(p, p+1) \cong \mathbb{R}[2^p] \oplus \mathbb{R}[2^p]$. The challenge in arriving at the final main result of this thesis came from having to make do with not having injectivity of real irreducible Clifford representations $\rho : \mathcal{C}\ell(p, p+1) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2^p})$. To address this issue we prove Lemma 4.16:

Lemma. *Let $\mathcal{C}\ell(E)$ be a Clifford algebra whose underlying pseudo-Euclidean vector space has signature $(p, p+1)$ and let (S, γ) be an irreducible $\mathcal{C}\ell(E)$-module. Let*

$A \in \mathrm{End}(S)$ and assume that for all $v \in E$ we have $\{A, \gamma_v\} = \lambda \, \mathrm{Id}_S$ for some $\lambda \in \mathbb{R}$. Then $A = \gamma_u$ for some $u \in E$.

Our proof to this lemma is combinatorial in nature. Before giving the abstract proof in full generality, we explore many examples and even prove this lemma for the Clifford algebra $\mathcal{C}\ell(2,3)$. Finally as the last result of this thesis we prove Theorem 4.10, the proof of which constitutes the remaining part of Chapter 4:

Theorem. *Suppose there exists a DGO $\not{d}$ of a Courant algebroid $E \longrightarrow M$ of signature $(p, p+1)$ on an irreducible $\mathcal{C}\ell(E)$-module bundle (S, γ). Then the set of DGOs of E on (S, γ) has the structure of an affine space modelled on*

$$V_{\not{d}} := \{e \in \Gamma(E) \mid \{\not{d}, \gamma_e\} \in \mathcal{C}^\infty(M)\}.$$

In particular, $V_{\not{d}}$ is independent of the choice of the DGO $\not{d}$.

In future work, one of the obvious next open questions is to determine the space of local DGOs in all signatures. This will provide us with the base work necessary to construct a canonical i.e., globally defined, DGO and determine a standard expression for it in all signatures. With the results we have obtained in this thesis, the immediate next open problem is to construct a canonical DGO and its standard form in the $(p, p+1)$-signature case. In order to further develop the theory of Dirac generating operators, it would be interesting to integrate the approach to study DGOs using deformation quantisation of graded Poisson algebras as in [Š, ŠV, GMX] with our approach to DGOs in this thesis. Another interesting open problem is to clarify the relation between Kostant's cubic Dirac type operators [Ko] and Dirac generating operators.

2
Preliminaries

They say 'selbstständig', implying that you stand on your own. But who actually stands on their own? We are all, if we stand, supported by any number of things.

Judith Butler

2.1 Clifford Algebras and Spin Geometry

In this section we describe an algebra, called Clifford algebra, that can naturally be associated to a vector space equipped with a quadratic form. We will begin with a short note on vector spaces equipped with quadratic forms.

2.1.1 Quadratic vector spaces

Let $(V, \mathbb{K})$ be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with a symmetric bilinear form $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{K}$. Then to this bilinear form can be associated a map $q : V \longrightarrow \mathbb{K}$ called the **associated quadratic form** which maps $v \mapsto q(v) := \langle v, v \rangle$. The symmetric bilinear form can be recovered from the associated quadratic form by polarization as follows:

$$\langle x, y \rangle = \frac{1}{2}(q(x+y) - q(x) - q(y)).$$

Conversely, given a vector space $(V, \mathbb{K})$ over a field $\mathbb{K}$, a **quadratic form** is a map $q : V \longrightarrow \mathbb{K}$ such that $q(av) = a^2 q(v)$ for $a \in \mathbb{K}, v \in V$ and the function $(u, v) \mapsto$

$q(u+v) - q(u) - q(v)$ is bilinear. A symmetric bilinear form called the **associated symmetric bilinear form** can be naturally associated to a quadratic form by defining $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{K}$ as $(u, v) \mapsto \langle u, v \rangle := \frac{1}{2}(q(u+v) - q(u) - q(v))$.

If we fix a vector space $(V, \mathbb{K})$, then set of all symmetric bilinear forms over V are in one-to-one correspondence to the set of all quadratic forms over V. That is, the notion of having a quadratic form over a vector space is equivalent to the notion of having a symmetric bilinear form on it. In our work we will use these notions interchangeably depending on the context and a vector space equipped with either a symmetric bilinear form or a quadratic form will be called a quadratic vector space.

2.1.2 Clifford Algebra: Definition

Henceforth we will assume that all our vector spaces are over $\mathbb{R}$ unless stated otherwise. Given a vector space V we can associate to it, naturally, an algebra called the exterior algebra $\Lambda^\bullet V$ and this association carries over naturally to vector bundles. In a similar way, a vector space V equipped with a quadratic form q has a naturally associated algebra called the Clifford algebra $\mathcal{C}\ell(V, q)$ and this association also carries over naturally to quadratic vector bundles. Furthermore, the exterior algebra bundle $\Lambda^\bullet E \longrightarrow M$ associated to a vector bundle $E \longrightarrow M$ and the Clifford algebra bundle associated to the same quadratic vector bundle $E \longrightarrow M$ are isomorphic as vector bundles (but not as algebra bundles). We will now define Clifford algebra in a few equivalent ways. For details and proofs, the reader is invited to look at standard texts like Spin Geometry by Lawson and Michelsohn [LM].

Definition 2.1. The Clifford algebra of a quadratic vector space (V, q) is the quotient of the tensor algebra $\mathcal{T}(V) := \sum_{r=0}^{\infty} \bigotimes^r V$ of V by the ideal $\mathcal{I}_q(V)$ generated by all elements of the form $v \otimes v + q(v)1$, where $v \in V$, i.e.,

$$\mathcal{C}\ell(V, q) := \mathcal{T}(V)/\mathcal{I}_q(V).$$

There is a natural injection $V \hookrightarrow \mathcal{C}\ell(V, q)$ which is the image of $V = \bigotimes^1 V$ under the canonical projection $\pi_q : \mathcal{T}(V) \longrightarrow \mathcal{C}\ell(V, q)$.

A Clifford algebra can also be seen as generated by the vector space $V \subset \mathcal{C}\ell(V, q)$ subject to the relations:

$$v \cdot v = -q(v)1 \tag{2.1}$$

for $v \in V$. For all $v, w \in V$ the relation is

$$v \cdot w + w \cdot v = -2\langle v, w\rangle \tag{2.2}$$

where $2\langle v, w\rangle = q(v + w) - q(v) - q(w)$ is the polarisation of q. A Clifford algebra can therefore be seen to be fully characterised by the following universal property:

Proposition 2.2. *[LM] Let $f : V \longrightarrow A$ be an injective linear map into a real associative algebra with unit such that*

$$f(v) \cdot f(v) = -q(v)1$$

for all $v \in V$. Then f extends uniquely to a real algebra homomorphism $\widetilde{f} : \mathcal{C}\ell(V, q) \longrightarrow A$. Furthermore, $\mathcal{C}\ell(V, q)$ is the unique real associative algebra with this property. This can be summarised by the commutative diagram

$$\begin{tikzcd} (V,q) \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \mathcal{C}\ell(V,q) \arrow[dl, "\widetilde{f}"] \\ & A & \end{tikzcd}$$

This universal property of Clifford algebras gives rise to a functor between the category of real quadratic vector spaces and the category of real associative algebras with units.

2.1.2.1 Relation between the exterior algebra and Clifford algebra

If the quadratic form is identically zero, $q = 0$, then observe that $\mathcal{C}\ell(V, 0) \cong \Lambda^\bullet V$. The vector space underlying the exterior algebra $\Lambda^\bullet V$ and the Clifford algebra are canonically isomorphic as is summarised by the below proposition.

Proposition 2.3. *[LM] There is a canonical vector space isomorphism $\Lambda^\bullet V \xrightarrow{\cong} \mathcal{C}\ell(V, q)$.*

Note that the above isomorphism is not an isomorphism of algebras in general and is only so if $q = 0$. Furthermore, since this map is canonical, it makes sense to speak of embeddings in the sense of

$$\Lambda^r V \subset \mathcal{C}\ell(V, q) \quad \text{for all } r \geq 0. \tag{2.3}$$

We can deduce from the above proposition that $\dim \mathcal{C}\ell(V, q) = 2^{\dim(V)}$.

2.1.3 $\mathbb{Z}_2$-grading of Clifford algebras and Clifford modules

Consider the unique automorphism that extends the map $\alpha(v) = -v$ for $v \in V$,

$$\alpha : \mathcal{C}\ell(V, q) \longrightarrow \mathcal{C}\ell(V, q), \tag{2.4}$$

Since $\alpha^2 = \mathrm{Id}$, there is a decomposition of $\mathcal{C}\ell(V, q)$ into eigenspaces of α,

$$\mathcal{C}\ell(V, q) = \mathcal{C}\ell^0(V, q) \oplus \mathcal{C}\ell^1(V, q), \tag{2.5}$$

where $\mathcal{C}\ell^i(V, q) \coloneqq \{x \in \mathcal{C}\ell(V, q) \mid \alpha(x) = (-1)^i x\}$ are the eigenspaces of α. The eigenspaces satisfy the below property where the indices are taken modulo 2,

$$\mathcal{C}\ell^i(V, q) \cdot \mathcal{C}\ell^j(V, q) \subseteq \mathcal{C}\ell^{i+j}(V, q). \tag{2.6}$$

An algebra that satisfies properties (2.5) and (2.6) is called a $\mathbb{Z}_2$-graded algebra. $\mathcal{C}\ell^0(V, q)$ is called the even part of $\mathcal{C}\ell(V, q)$ and is a subalgebra. However, $\mathcal{C}\ell^1(V, q)$ is only a linear subspace of $\mathcal{C}\ell(V, q)$ and is called the odd part. This grading is called the **canonical $\mathbb{Z}_2$-grading** and has important consequences in the theory and application of Clifford algebras.

2.1.3.1 Volume element and volume grading of Clifford algebras

We will now consider the Clifford algebra $\mathcal{C}\ell(r, s) := \mathcal{C}\ell(V, q)$ where $V = \mathbb{R}^{r+s}$ and

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2. \tag{2.7}$$

$\mathcal{C}\ell(r, s)$ will be called **standard Clifford algebra**. We begin with the below definition of volume element.

Definition 2.4. Consider the quadratic vector space $(\mathbb{R}^{r+s}, q)$ and fix an orientation. Let $e_1, \dots, e_{r+s}$ be any positively oriented q-orthonormal basis. Then the associated **volume element** is

$$\omega \coloneqq e_1 \cdots e_{r+s} \ \in \mathcal{C}\ell(r, s) \tag{2.8}$$

Note that the volume element is independent of the choice of a positively oriented q-orthonormal basis.

Proposition 2.5. *[LM] The volume element (2.8) in $\mathcal{C}\ell(r, s)$ has the following basic properties. Let $n = r + s$. Then:*

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}, \tag{2.9}$$

$$v\omega = (-1)^{n-1}\omega v \quad \textit{for all} \quad v \in \mathbb{R}^n. \tag{2.10}$$

In particular, if n is odd, then the element ω is central in $C\ell(r,s)$. If n is even, then

$$\phi\omega = \omega\alpha(\phi)$$

where α is the automorphism 2.4 for all $\phi \in C\ell(r,s)$.

The property 2.9 can be rewritten as

$$\omega^2 = \begin{cases} (-1)^s & \text{if } \ n \equiv 3 \ \text{ or } \ 4 \ (\mathrm{mod}\, 4), \\ (-1)^{s+1} & \text{if } \ n \equiv 1 \ \text{ or } \ 2 \ (\mathrm{mod}\, 4). \end{cases} \tag{2.11}$$

Proposition 2.6. *[LM] Suppose the volume element ω in $C\ell(r,s)$ satisfies $\omega^2 = 1$ and set*

$$\pi^+ = \frac{1}{2}(1+\omega) \qquad \textit{and} \qquad \pi^- = \frac{1}{2}(1-\omega) \tag{2.12}$$

Then π^+ and π^- satisfy the relations

$$\pi^+ + \pi^- = 1, \tag{2.13}$$

$$(\pi^+)^2 = \pi^+, \qquad \textit{and} \qquad (\pi^-)^2 = \pi^-, \tag{2.14}$$

$$\pi^+\pi^- = \pi^-\pi^+ = 0 \tag{2.15}$$

The above proposition leads to two important facts. The first one describes the conditions under which the volume element of a Clifford algebra induces a grading on it, the so-called **volume grading**, via the idempotents $\pi^{\pm}$:

Proposition 2.7. *[LM] Suppose that the volume element ω in $C\ell(r,s)$ satisfies $\omega^2 = 1$, and that $r+s$ is odd. Then $C\ell(r,s)$ can be decomposed as a direct sum*

$$C\ell(r,s) = C\ell(r,s)^+ \oplus C\ell(r,s)^- \tag{2.16}$$

of isomorphic subalgebras, where $C\ell(r,s)^{\pm} = \pi^{\pm} \cdot C\ell(r,s) = C\ell(r,s) \cdot \pi^{\pm}$ and where $\alpha(C\ell(r,s)^{\pm}) = C\ell(r,s)^{\mp}$.

The second statement describes the conditions under which one obtains a grading on a Clifford module:

Proposition 2.8. *[LM] Suppose that the volume element ω in $\mathcal{C}\ell(r,s)$ satisfies $\omega^2 = 1$ and that $r+s$ is even. Let V be any $\mathcal{C}\ell(r,s)$-module. Then there is a decomposition*

$$V = V^+ \oplus V^-$$

into the $+1$ and -1 eigenspaces for multiplication by ω. In fact,

$$V^+ = \pi^+ \cdot V \qquad \text{and} \qquad V^- = \pi^- \cdot V \tag{2.17}$$

and for any $e \in \mathbb{R}^{r+s}$ with $q(e) \neq 0$, module multiplication by e gives isomorphisms

$$e : V^+ \longrightarrow V^- \qquad \text{and} \qquad e : V^- \longrightarrow V^+. \tag{2.18}$$

Returning to the canonical grading of the Clifford algebra $\mathcal{C}\ell(r,s) = \mathcal{C}\ell^0(r,s) \oplus \mathcal{C}\ell^1(r,s)$ given in (2.5) we have the following relation between a Clifford algebra and the even part of another Clifford algebra.

Proposition 2.9. *[LM] For all r, s there is an algebra isomorphism*

$$\mathcal{C}\ell(r,s) \cong \mathcal{C}\ell^0(r+1,s)$$

2.1.4 Classification of Clifford algebras

The following proposition states how standard Clifford algebras can be generated from an orthonormal basis of $\mathbb{R}^{r+s}$.

Proposition 2.10. *[LM] Let $e_1, \dots, e_{r+s}$ be any q-orthonormal basis of $\mathbb{R}^{r+s} \subset \mathcal{C}\ell(r,s)$. Then $\mathcal{C}\ell(r,s)$ is generated as an algebra by $e_1, \dots, e_{r+s}$ subject to the relation*

$$e_i e_j + e_j e_i = -2\epsilon_i \delta_{ij},$$

where $\epsilon_i := q(e_i, e_i) \in \{\pm 1\}$.

Example 2.11 (Low-dimensional Clifford algebras).

- **the 'trivial' case:** Consider the Clifford algebra $\mathcal{C}\ell(0,0)$ which is associated to the quadratic vector space $(\mathbb{R}, q = 0)$. Then $\mathcal{C}\ell(0,0) \cong \mathbb{R}$ with isomorphism given by $x1 \longleftrightarrow x$.

- $\mathcal{C}\ell(1,0)$**:** This Clifford algebra is generated by one element say $e \in \mathbb{R}$ subject to the relation $e^2 = -1$ and it is isomorphic to $\mathbb{C}$ as a real associative algebra via the isomorphism $x1 + ye \longleftrightarrow x + \iota y$.

- $\mathcal{C}\ell(0,2)$**:** This Clifford algebra is generated by two elements say e_1 and e_2 obeying the relations $e_1^2 = 1 = e_2^2$ and $e_1e_2 = -e_2e_1$. The resulting algebra is isomorphic to the algebra of real 2×2 matrices via the isomorphism:

$$x1 + ye_1 + ze_2 + we_1e_2 \longleftrightarrow \begin{pmatrix} x+y & z+w \\ z-w & x-y \end{pmatrix}$$

- $\mathcal{C}\ell(2,0)$**:** The Clifford algebra $\mathcal{C}\ell(2,0)$ is also generated by two elements e_1 and e_2 satisfying the relations $e_1^2 = -1 = e_2^2$ and $e_1e_2 = -e_2e_1$. Then $\mathcal{C}\ell(0,2) \cong \mathbb{H}$ via the isomorphism:

$$x_01 + x_1e_1 + x_2e_2 + x_3e_1e_2 \longleftrightarrow x_0 + x_1i + x_2j + x_3k$$

- $\mathcal{C}\ell(0,1)$**:** The Clifford algebra $\mathcal{C}\ell(0,1)$ is generated by one element say e subject to the relation $e^2 = 1$. We can define complementary idempotents $\pi_\pm = \frac{1}{2}(1 \pm e)$ which obey the relations $\pi_+ + \pi_- = 1$, $\pi_+\pi_- = 0$ and $\pi_\pm^2 = \pi_\pm$. Therefore $\mathcal{C}\ell(0,1)$ can be decomposed into complementary subspaces. $\mathcal{C}\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$ with explicit isomorphism $x\pi_+ + y\pi_- \longleftrightarrow (x,y)$.

- $\mathcal{C}\ell(1,1)$**:** This Clifford algebra is generated by two elements which we can call e_1 and e_2 satisfying the relations $e_1^2 = -1$ and $e_2^2 = +1$ with $e_1e_2 = -e_2e_1$. It can be identified with $\mathbb{R}[2]$, the algebra of a real 2×2 matrices via the isomorphism

$$x1 + ye_1 + ze_2 + we_1e_2 \longleftrightarrow \begin{pmatrix} x+z & y-w \\ -y-w & x-z \end{pmatrix}.$$

We can obtain an explicit description of the algebras $\mathcal{C}\ell(r,s)$ as real matrix algebras with entries from $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ by using the isomorphisms identifying the lower-dimensional Clifford algebras $\mathcal{C}\ell(1,0)$, $\mathcal{C}\ell(0,1)$, $\mathcal{C}\ell(1,1)$, $\mathcal{C}\ell(2,0)$ and $\mathcal{C}\ell(0,2)$ with matrix algebras and the theorem below.

Theorem 2.12. *[LM] There are isomorphims*

$$\mathcal{C}\ell(0,n+2) \cong \mathcal{C}\ell(n,0) \otimes \mathcal{C}\ell(0,2) \tag{2.19}$$

$$\mathcal{C}\ell(n+2,0) \cong \mathcal{C}\ell(0,n) \otimes \mathcal{C}\ell(2,0) \tag{2.20}$$

$$\mathcal{C}\ell(r+1,s+1) \cong \mathcal{C}\ell(r,s) \otimes \mathcal{C}\ell(1,1) \tag{2.21}$$

for all $n, r, s \geq 0$.

The complete classification is summarised below in Table 2.1.4.

Table 2.1: Classification of $\mathcal{C}\ell(r,s)$

s ↑									
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
	0	1	2	3	4	5	6	7	8
r →									

2.1.5 Spin Groups and their Lie algebras

Let $\mathcal{C}\ell(V,q)^\times$ denote the group of invertible elements in the Clifford algebra $\mathcal{C}\ell(V,q)$. Then we can define two subgroups of $\mathcal{C}\ell(V,q)^\times$ called the Pin-group and the Spin-group as follows,

$$\mathrm{Pin}(V,q) = \{v_1 \dots v_r \in \mathcal{C}\ell(V,q)^\times \;\mid\; q(v_j) = \pm 1 \;\forall\; j\} \tag{2.22}$$

and

$$\mathrm{Spin}(V,q) = \{v_1 \dots v_r \in \mathrm{Pin}(V,q) \;\mid\; r \text{ is even }\}. \tag{2.23}$$

Alternatively, the Spin-group can also be seen as follows,

$$\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathcal{C}\ell^0(V,q).$$

When $V = \mathbb{R}^{r+s}$ and q is the quadratic form (2.7), we denote the Spin and Pin groups by $\mathrm{Spin}(r,s) := \mathrm{Spin}(V,q)$ and $\mathrm{Pin}(r,s) := \mathrm{Pin}(V,q)$ and we have the following short exact sequences.

Theorem 2.13. *[LM] Let V be a finite-dimensional vector space over $\mathbb{R}$ and let q be a non-degenerate quadratic form on V. Then there are the following short exact sequences.*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,q) \longrightarrow 1$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q) \longrightarrow 1.$$

Here the group homomorphism $\widetilde{\mathrm{Ad}}$ is defined by

$$\widetilde{\mathrm{Ad}}_\phi(y) := \alpha(\phi) y \phi^{-1},$$

where α is the automorphism (2.4) and is called the **twisted adjoint representation**.

2.1.5.1 Lie algebra structures

In this section we will state some properties of the Lie algebra associated to the group $\mathrm{Spin}(r,s)$. First note that $\mathfrak{cl}^\times(r,s) := (\mathcal{C}\ell(r,s), [\cdot,\cdot])$ where $[\phi,\psi] = \phi\cdot\psi - \psi\cdot\phi$ is the Lie algebra of the group of units $\mathcal{C}\ell^\times(r,s)$.

Proposition 2.14. *[LM] The Lie subalgebra of $\mathfrak{cl}^\times(r,s)$ corresponding to the subgroup* $\mathrm{Spin}(r,s) \subset \mathcal{C}\ell^\times(r,s)$ *is*

$$\mathfrak{spin}(r,s) = \Lambda^2 \mathbb{R}^{r+s},$$

considered as a subspace of $\mathcal{C}\ell(r,s)$ according to (2.3). In particular, $\Lambda^2\mathbb{R}^{r+s}$ is closed under the bracket operation.

Proof. After delineating cases based on the signature of the metric, the proof to this statement follows analogous to the proof in [LM]. □

The Lie algebra of the orthogonal group $SO(r,s)$ is the vector space below together with the commutator:

$$\mathfrak{so}(r,s) = \left\{ A : \mathbb{R}^{r+s} \longrightarrow \mathbb{R}^{r+s} \;\middle|\; \begin{array}{l} A \text{ is linear and skew-symmetric, i.e.} \\ \langle Ax, y\rangle = \langle x, Ay\rangle \text{ for all } x, y \in \mathbb{R}^{r+s} \end{array} \right\}.$$

There is a natural isomorphism $\phi : \Lambda^2\mathbb{R}^{r+s} \longrightarrow \mathfrak{so}(r,s)$ mapping $v \wedge w$ to $\phi(v \wedge w)$ defined by linearly extending $\phi(v \wedge w)(x) := \langle v, x\rangle w - \langle w, x\rangle v$ for $x \in \mathbb{R}^{r,s}$.

Note that if $e_1, \dots, e_{r+s}$ is a basis of $\mathbb{R}^{r+s}$, then $\phi(e_i \wedge e_j)$ with $1 \le i < j \le r+s$ is a basis of $\mathfrak{so}(r,s)$. From now on, we will suppress ϕ in the notation and write $v \wedge w$ for $\phi(v \wedge w)$, too.

Since the twisted adjoint representation $\widetilde{\mathrm{Ad}}$ gives a surjective homomorphism

$$\mathrm{Spin}(r,s) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(r,s),$$

the differential of this gives a Lie algebra isomorphism

$$\mathfrak{spin}(r,s) \xrightarrow{\widetilde{\mathfrak{ad}}} \mathfrak{so}(r,s).$$

The following two propositions will play an important role in the construction of Dirac generating operators later on, particularly in the proof of Lemma 4.5, where it is shown that a generalised connection on a Courant algebroid E induces an E-connection on any spinor bundle.

Proposition 2.15. *[LM] The Lie algebra isomorphism $\mathfrak{spin}(r,s) \xrightarrow{\widetilde{\mathfrak{ad}}} \mathfrak{so}(r,s)$ induced by the adjoint representation is given explicitly on the basis elements $\{e_ie_j\}_{i<j}$ by*

$$\widetilde{\mathfrak{ad}}(e_ie_j) = 2e_i \wedge e_j. \tag{2.24}$$

Consequently for $v, w \in \mathbb{R}^{r+s}$,

$$\widetilde{\mathfrak{ad}}^{-1}(v \wedge w) = \frac{1}{4}[v, w] \tag{2.25}$$

Proposition 2.16. *[LM] Let $\Delta : \mathrm{Spin}(r, s) \longrightarrow \mathrm{SO}(W)$ be a representation on some vector space W obtained by restriction of a representation $\mathcal{C}\ell(r, s) \longrightarrow \mathrm{End}(W)$ of the Clifford algebra. Let $\Delta_* : \mathfrak{so}(r, s) \longrightarrow \mathfrak{so}(W)$ be the associated representation of the Lie algebra obtained by first pulling back $\mathfrak{so}(r, s)$ to the double covering via $\widetilde{\mathfrak{ad}}^{-1}$. Then on the elementary transformations $v \wedge w \in \mathfrak{so}(r, s)$,*

$$\Delta_*(v \wedge w) = \frac{1}{4}[v, w] \cdot - \tag{2.26}$$

where the dot indicates Clifford module multiplication on W.

In terms of the standard basis $\{e_i \wedge e_j\}_{i<j}$ we have

$$\Delta_*(e_i \wedge e_j) = \frac{1}{2} e_i e_j. \tag{2.27}$$

2.1.6 Representation Theory

We recall here some basic results from Algebra for completeness and to set notations. More details can be found in standard references for the topic such as [L, DK, Ch]. Every vector space is considered over $\mathbb{R}$ unless explicitly specified otherwise. Every algebra is assumed to be a real associative algebra with unit. A real representation V of an algebra A is an $\mathbb{R}$-linear algebra homomorphism between A and the endomorphism algebra $\mathrm{End}_{\mathbb{R}}(V)$. An anti-representation of an algebra $T : A \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ is an $\mathbb{R}$-linear map such that $T(ab) = T(b)T(a)$. A module V over an associative algebra A is a vector space V together with a representation of the algebra. We recall below the notion of equivalence of two representations of algebras.

Definition 2.17. Let A and A' be real algebras. Let V and V' be real vector spaces such that $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ and $\gamma' : A' \longrightarrow \mathrm{End}_{\mathbb{R}}(V')$ are real representations. γ and γ' are said to be equivalent if there exists an isomorphism of algebras $f : A \longrightarrow A'$ and an isomorphism of vectors spaces $g : V \longrightarrow V'$ such that the below diagram commutes,

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "f"] & \mathrm{End}_{\mathbb{R}}(V) \arrow[d, "\mathrm{Ad}(g)"] \\ A' \arrow[r, "\gamma'"] & \mathrm{End}_{\mathbb{R}}(V') \end{tikzcd}$$

where $\mathrm{Ad}(g) : \mathrm{End}_{\mathbb{R}}(V) \longrightarrow \mathrm{End}_{\mathbb{R}}(V')$ is such that $f \mapsto g \circ f \circ g^{-1}$.

Next we define the Schur algebra of a representation, also called the centraliser of a representation. This object will play an important role in our work.

Definition 2.18. Let A be an algebra and $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ a real representation. The Schur algebra of the representation γ is defined as

$$\mathcal{C}(V, \gamma) := \{a \in \mathrm{End}_{\mathbb{R}}(V) \mid [a, \gamma(A)] = 0\}$$

In the proposition below we show that equivalent representations have isomorphic Schur algebras.

Proposition 2.19. *Let A and A' be real algebras. If V and V' are real vector spaces such that $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ and $\gamma' : A' \longrightarrow \mathrm{End}_{\mathbb{R}}(V')$ are equivalent as real representations, then their Schur algebras $\mathcal{C}(V, \gamma)$ and $\mathcal{C}(V', \gamma')$ are isomorphic.*

Proof. If γ and γ' are equivalent, then by definition there exists an isomorphism of algebras $f : A \longrightarrow A'$ and an isomorphism of vectors spaces $g : V \longrightarrow V'$ such that the diagram in Definition 2.17 commutes. Then $\mathrm{Ad}(g)$ defines an isomorphism of the algebras $\mathrm{End}_{\mathbb{R}}(V)$ and $\mathrm{End}_{\mathbb{R}}(V')$ which restricts to an isomorphism of the Schur algebras $\mathcal{C}(V, \gamma)$ and $\mathcal{C}(V', \gamma')$. □

Let A be an algebra over a field $\mathbb{K}$. Recall that a non-zero module is said to be **irreducible** if it has no proper submodules, i.e. if the only submodules are the whole module and the zero module.

Theorem 2.20 (**Schur's Lemma**). *If U and V are irreducible A-modules, then every non-zero homomorphism $f : U \to V$ is an isomorphism.*

Proof. $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are submodules of U and V respectively. Since $f \neq 0$ we can be sure that $\mathrm{Ker}(f) \neq U$ and $\mathrm{Im}(f) \neq 0$ and consequently, we can be sure that, $\mathrm{Ker}(f) = 0$ and $\mathrm{Im}(f) = U$. Therefore, f is both a monomorphism and an epimorphism. □

Recall that a division algebra is an algebra in which every non-zero element is invertible. It is also known as a skew-field. The following corollary is a consequence of Schur's lemma.

Corollary 2.21. *The endomorphism algebra of an irreducible module over algebra A is a real associative division algebra.*

Note the following classification theorem by Frobenius and Peirce for all finite-dimensional real division algebras.

Theorem 2.22 (Frobenius). *Every finite-dimensional real associative division algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Furthermore, every subalgebra of a finite-dimensional division algebra is again a division algebra [DK]. Therefore, in particular, the Schur algebra of a real representation of an algebra is a real associative division algebra.

Definition 2.23. If R and S are non-commutative rings with unit and M is an abelian group, we call M an (S, R)-bimodule if it is a right R-module and a left S-module, and if $s(mr) = (sm)r$ for all $s \in S$, $r \in R$ and $m \in M$.

Example 2.24. $\mathrm{GL}(\mathbb{H})$ is an $(\mathbb{H}, \mathbb{H})$-bimodule with the standard action because $\mathbb{H}$ is associative.

Remark 2.25. If N is a left-R module and M is an (S, R)-bimodule, then the (S, R)-bimodule structure on M implies that for all $s \in S$, $m_i \in M$ and $n_i \in N$,

$$s\Big(\sum_{\text{finite}} m_i \otimes n_i\Big) = \sum_{\text{finite}} (sm_i) \otimes n_i \tag{2.28}$$

and this gives a well-defined action of S under which $M \otimes_R N$ is a left S-module.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, let $\mathbb{K}[n]$ denote the real associative algebra of $n \times n$-matrices with entries from $\mathbb{K}$. The following theorem summarises the representation theory of such an algebra.

Theorem 2.26. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and consider the algebra $\mathbb{K}[n]$ of $n \times n$ matrices with entries from $\mathbb{K}$ as an algebra over $\mathbb{R}$. Then the natural representation ρ of $\mathbb{K}[n]$ on the vector space $\mathbb{K}^n$ is, up to equivalence, the only irreducible real representation of $\mathbb{K}[n]$. The algebra $\mathbb{K}[n] \oplus \mathbb{K}[n]$ has exactly two equivalence classes of irreducible real representations. They are given by*

$$\rho_1(\varphi_1, \varphi_2) = \rho(\varphi_1) \qquad \text{and} \qquad \rho_2(\varphi_1, \varphi_2) = \rho(\varphi_2)$$

acting on $\mathbb{K}^n$.

Proof. The ring $\mathbb{K}[n]$ is the direct sum of n simple left ideals I_i for $i = 1, \ldots, n$, where I_i is the set of matrices with all entries zero except in the i-th column and is thus isomorphic to $\mathbb{K}^n$ as a left module. Since all the left ideals I_i are isomorphic to each other, the ring $\mathbb{K}[n]$ is simple. This means that $\mathbb{K}[n]$ has exactly one simple module due to Corollary 4.6 on page 653 in [L]. Similarly, it can be shown that the ring $\mathbb{K}[n] \oplus \mathbb{K}[n]$ has two isomorphism classes of left ideals into which it decomposes, corresponding to the two equivalence classes of irreducible representations given in the statement of the claim. $\square$

Note that since every Clifford algebra $\mathcal{C}\ell(r,s)$ is of the form $\mathbb{K}[2^m]$ or $\mathbb{K}[2^m] \oplus \mathbb{K}[2^m]$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, see Theorem 2.12, and every representation of a Clifford algebra can be decomposed into irreducible representations, the theorem above also describes their representation theory.

2.1.6.1 Clifford Representations

In the proposition below we state important properties of irreducible real representations of Clifford algebras.

Proposition 2.27. *Let $\gamma : \mathcal{C}\ell(r,s) \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ be an irreducible real representation. Then:*

If $(r-s) \equiv_8 0, 6$ then γ is an isomorphism.

If $(r-s) \equiv_8 1, 5$ then γ is injective but not surjective.

If $(r-s) \equiv_8 2, 4$ then γ is injective but not surjective.

If $(r-s) \equiv_8 7$ then γ is surjective but not injective.

If $(r-s) \equiv_8 3$ then γ is neither surjective nor injective.

Proof. From Theorem 2.26 we know all irreducible real representations of (direct sums of) real, complex or quaternionic matrix algebras.

- $(r-s) \equiv_8 0, 6$:
  In this case the Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathbb{R}[2^m]$ is a real matrix algebra for some $m \in \mathbb{N}$. Its unique irreducible representation is the natural representation on $\mathbb{R}^{2^m}$ and hence γ is clearly an isomorphism.

- $(r-s) \equiv_8 1, 5$:
  In this case the Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathbb{C}[2^m]$ is a complex matrix algebra for some $m \in \mathbb{N}$. Its unique irreducible representation is the natural representation on $\mathbb{C}^{2^m}$ and hence $\gamma : \mathbb{C}[2^m] \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2^m}) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{C}^{2^m})$, if considered as a real representation, is injective and not surjective.

- $(r-s) \equiv_8 2, 4$:
  In this case the Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathbb{H}[2^m]$ is a quaternionic matrix algebra for some $m \in \mathbb{N}$. Its unique irreducible representation is the natural representation on $\mathbb{H}^{2^m}$ and hence $\gamma : \mathbb{H}[2^m] \to \mathrm{End}_{\mathbb{H}}(\mathbb{H}^{2^m}) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{H}^{2^m})$, if considered as a real representation, is injective and not surjective.

- $(r-s) \equiv_8 7$:
  In this case the Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathbb{R}[2^m] \oplus \mathbb{R}[2^m]$ is the direct sum of two real matrix algebras for some $m \in \mathbb{N}$. Its two irreducible representations are the projections of $\mathbb{R}[2^m] \oplus \mathbb{R}[2^m]$ onto a summand and hence they are surjective and not injective.

- $(r-s) \equiv_8 3$:
  In this case the Clifford algebra $\mathcal{C}\ell(r,s) \cong \mathbb{H}[2^m] \oplus \mathbb{H}[2^m]$ is the direct sum of two quaternionic matrix algebras for some $m \in \mathbb{N}$. Its two irreducible representations are given by the embedding $\mathbb{H}[2^m] \oplus \mathbb{H}[2^m] = \mathrm{End}_{\mathbb{H}}(\mathbb{H}^{2^m}) \oplus \mathrm{End}_{\mathbb{H}}(\mathbb{H}^{2^m}) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{H}^{2^m}) \oplus \mathrm{End}_{\mathbb{R}}(\mathbb{H}^{2^m})$ followed by the projections onto a summand and hence they are neither surjective nor injective. $\square$

The isomorphism type of the Schur algebra of irreducible representations of $\mathcal{C}\ell(r,s)$ is a feature on the basis of which we can distinguish the Clifford algebras. For an involved discussion on the Schur algebra of irreducible representations of $\mathcal{C}\ell(r,s)$ see [LS1]. In particular, Proposition 5.3 in [LS1] gives a characterisation of the Schur algebras of irreducible representations of $\mathcal{C}\ell(r,s)$. Clifford algebras can be distinguished by the signature of the quadratic form, the simplicity or isomorphism type of their matrix algebra, and the isomorphism type of the Schur algebra of their irreducible representations. These facts are summarised for convenience in the table below:

$r-s \mod 8$	simplicity	matrix algebra type	Schur algebra of irrep
0,6	simple	real	$\mathbb{R}$
1,5	simple	complex	$\mathbb{C}$
2,4	simple	quaternionic	$\mathbb{H}$
7	non-simple	real	$\mathbb{R}$
3	non-simple	quaternionic	$\mathbb{H}$

So far we have considered only real representations of Clifford algebras. We can also consider $\mathbb{K}$-representations of $\mathcal{C}\ell(r,s)$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Since a complex vector space is just a real vector space W with a real linear map $J : W \longrightarrow W$ such that $J^2 = -\operatorname{Id}$, a complex representation of $\mathcal{C}\ell(r,s)$ is just a real representation $\rho : \mathcal{C}\ell(r,s) \longrightarrow \operatorname{End}_{\mathbb{R}}(W)$ such that $\rho(\phi) \circ J = J \circ \rho(\phi)$ for all $\phi \in \mathcal{C}\ell(r,s)$. Therefore the image of ρ commutes with the subalgebra $\operatorname{span}_{\mathbb{R}}\{\operatorname{Id}, J\} \cong \mathbb{C}$ which is precisely the Schur algebra of the representation ρ. Analogous arguments apply to quaternionic representations of $\mathcal{C}\ell(r,s)$.

2.1.7 Fibre bundles

We collect some standard results from the theory of fibre bundles that we use in our work here for completeness. For details and proofs look at standard texts on the subject like [B]. All manifolds and maps between manifolds in this thesis are assumed to be smooth.

Definition 2.28. Let $\pi : E \longrightarrow M$ be a surjective submersion and let F be another manifold. The tuple (E, π, M, F) is called **fibre bundle with typical fibre** F if around every point $x \in M$ there exists a neighbourhood $U \subseteq M$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \longrightarrow U \times F$ such that $\operatorname{pr}_1 \circ \phi_U = \pi|_{\pi^{-1}(U)}$ where $\operatorname{pr}_1 : U \times F \longrightarrow U$ is the projection onto the first component. The diffeomorphism ϕ_U is called a local trivialisation.

If V is a vector space, then a fibre bundle (E, π, M, V) is called a **vector bundle** if its fibres $\pi^{-1}(x),\ x \in M$, are vector spaces such that there are local trivialisations whose restrictions $\phi_x : \pi^{-1}(x) \longrightarrow \{x\} \times V = V$ to the fibres are linear isomorphisms.

Analogously, one defines **algebra bundles**.

Remark 2.29. Let F and F' be smooth manifolds and $\Phi : F \longrightarrow F'$ a diffeomorphism. Then $\pi : E \longrightarrow M$ is a fibre bundle with typical fibre F if and only if $\pi : E \longrightarrow M$ is a fibre bundle with typical fibre F'.

Definition 2.30. Let M and F be manifolds and E a set with a surjection $\pi : E \longrightarrow M$. If $U \subseteq M$ is an open set and $\phi_U : \pi^{-1}(U) \longrightarrow U \times F$ a bijection such that $\mathrm{pr}_1 \circ \phi_U = \pi|_{E_U}$, then we call (U, ϕ_U) a **formal bundle chart** of E. A family $\{(U_i, \phi_{U_i})\}_{i\in\Lambda}$ of formal bundle charts of E is called a **formal bundle atlas** of E if $\{U_i\}_{i\in\Lambda}$ is an open cover of M.

Recall the following characterization of fibre bundles, the statement and proof of which can be found in various references such as for example [B].

Theorem 2.31. *Let M and F be manifolds and E a set with a surjection $\pi : E \longrightarrow M$. Further let $\{(U_i, \phi_{U_i})\}_{i\in\Lambda}$ be a formal bundle atlas of E (with respect to π) such that all transition functions*

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \longrightarrow (U_i \cap U_k) \times F \qquad \forall i, k \in \Lambda$$

are smooth. Then, there exists a unique topology and manifold structure on E such that (E, π, M, F) is a (smooth) fibre bundle with the (smooth) bundle chart $\{(U_i, \phi_i)\}_{i\in\Lambda}$.

Proposition 2.32. *Let $V \longrightarrow M$ be a fibre bundle with typical fibre F, then $\pi : \mathrm{End}(V) := \sqcup_{p\in M} \mathrm{End}(V_p) \longrightarrow M$ is a fibre bundle with typical fibre* $\mathrm{End}(F)$.

Proof. If $\{U_\alpha \mid \alpha \in \Lambda\}$ is an open cover of M and $\{\eta_\alpha \mid \alpha \in \Lambda\}$ is a family of local trivializations of V, then $(\mathrm{End}(V), \pi, M, \mathrm{End}(F))$ is a fibre bundle with the smooth bundle chart $\{\mathrm{Ad}(\eta_\alpha) \mid \alpha \in \Lambda\}$. □

Proposition 2.33. *If $(E, h) \longrightarrow M$ is a pseudo-Riemannian vector bundle, then it has an associated fibre bundle of Clifford algebras $\mathcal{C}\ell(E) \longrightarrow M$.*

Proof. Given a vector bundle with a bilinear form $(E, h) \longrightarrow M$ the construction of a Clifford algebra over a vector space with a bilinear form is performed fibrewise to obtain the fibres of $\mathcal{C}\ell(E) \longrightarrow M$. The topology of $\mathcal{C}\ell(E)$ is inherited from E via an associated bundle construction. □

Next we will prove a general fact about covariant derivatives or connections on vector bundles.

Proposition 2.34. *Every smooth vector bundle admits a covariant derivative.*

Proof. Let V be a vector bundle over a manifold M. We show that there exists a covariant derivative $\nabla : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$.

Let $(U_i)_i$ be an open cover of M such that $V|_{U_i}$ is trivializable and there exists a subordinate partition of unity $(\psi_i)_i$. Let $(x_i^j : U_i \to \mathbb{R})_j$ be coordinates on $U_i \subseteq M$ and let $(s_i : U_i \to V)_i$ be a local frame. Then $\nabla^{(i)} : \Gamma(TU_i) \times \Gamma(V|_{U_i}) \to \Gamma(V|_{U_i}), (\frac{\partial}{\partial x_i^j}, s = \sum_{k=1}^j f_k s_k) \mapsto \sum_k \frac{\partial f_k}{\partial x_i^j} s_k$ defines a covariant derivative on $V|_{U_i}$ for every i, which can be glued together to a covariant derivative on V using a partition of unity $(\psi_i)_i$, as follows. For $X \in \Gamma(TM)$ and $s \in \Gamma(V)$, set $\nabla_X(s) := \sum_i \nabla_X^{(i)}(\psi_i s)$. It is $C^\infty(M)$-linear. Let $f \in C^\infty(M)$. Then,

$$\nabla_{fX}(s) = \sum_i \nabla_{fX}^{(i)}(\psi_i s) = f \sum_i \nabla_X^{(i)}(\psi_i s) = f\nabla_X(s).$$

Furthermore, we have

$$\begin{aligned}\nabla_X(fs) &= \sum_i \nabla_X^{(i)}(\psi_i f s) \\ &= \sum_i X(f)\psi_i s + f\nabla_X^{(i)}(\psi_i s) \\ &= X(f)s + \sum_i f\nabla_X^{(i)}(\psi_i s) \\ &= X(f)s + f\nabla_X(s).\end{aligned}$$

□

2.1.8 Spin Structure

Definition 2.35. Let $E \to M$ be an oriented pseudo-Riemannian vector bundle of rank $n > 2$ and signature (r, s) with oriented orthonormal frame bundle $P_{\mathrm{SO}}(E)$. A **spin structure** on E is a $\mathrm{Spin}(r, s)$-principal bundle $P_{\mathrm{Spin}}(E)$ over M together with a 2-sheeted covering $P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$ which is a $\mathrm{Spin}(r, s)$-equivariant bundle map. Here the $\mathrm{Spin}(r, s)$ acts on $P_{\mathrm{SO}}(E)$ via the double covering $\mathrm{Spin}(r, s) \to \mathrm{SO}(r, s)$.

Proposition 2.36. *A trivializable real vector bundle with an inner product admits a spin structure.*

Proof. Clearly it suffices to show that any trivial pseudo-Riemannian vector bundle $E = M \times \mathbb{R}^{r,s} \to M$ admits a spin structure. The $\mathrm{SO}(r, s)$-principal bundle $P_{\mathrm{SO}}(E)$ of oriented orthonormal frames is the trivial one: $M \times \mathrm{SO}(r, s)$. The trivial $\mathrm{Spin}(r, s)$-principal bundle over M with the obvious bundle map $M \times \mathrm{Spin}(r, s) \to M \times \mathrm{SO}(r, s)$ then gives a spin structure. □

Now we define bundles of irreducible modules over the bundle of Clifford algebras $\mathcal{C}\ell(E)$.

Definition 2.37. Let E be an oriented pseudo-Riemannian vector bundle with a spin structure $\xi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$. A **real spinor bundle** of E is a bundle of the form

$$S(E) = P_{\mathrm{Spin}}(E) \times_\mu V,$$

where V is a left $\mathcal{C}\ell(r,s)$-module, $\mu : \mathrm{Spin}(r,s) \longrightarrow \mathrm{SO}(V)$ is the representation given by left multiplication by elements of $\mathrm{Spin}(r,s) \subset \mathcal{C}\ell^0(r,s)$ and $P_{\mathrm{Spin}}(E) \times_\mu V$ is the associated vector bundle.

Example 2.38. Consider $\mathcal{C}\ell(r,s)$ as a module over itself by left multiplication ℓ. Then the corresponding real spinor bundle is

$$\mathcal{C}\ell_{\mathrm{Spin}(E)} = P_{\mathrm{Spin}(E)} \times_\ell \mathcal{C}\ell(r,s).$$

Every real spinor bundle is also a bundle of modules over the Clifford algebra bundle as can be seen from the proposition below.

Proposition 2.39. *[LM] Let $S(E)$ be a real spinor bundle of E. Then $S(E)$ is a bundle of modules over the bundle of algebras $\mathcal{C}\ell(E)$. In particular then sections of the spinor bundle form a module over the sections of the Clifford bundle.*

2.2 Generalised Geometry

In this section we will define the primary object of study in generalised geometry namely, Courant algebroids. We will look at some examples and then define the notion of connection and torsion in the context of generalised geometry. Standard references for generalised geometry are [GF, GFS, G].

2.2.1 Courant Algebroids

Definition 2.40. A **Courant algebroid** over a manifold M is a vector bundle $E \longrightarrow M$ endowed with the following data:

- a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \in \Gamma(\mathrm{Sym}^2 E^*)$ which we call the scalar product of E,

- a bilinear map $[\cdot,\cdot] : \Gamma(E)\times\Gamma(E) \longrightarrow \Gamma(E)$ which we call the Dorfman bracket,
- a homomorphism $\pi : E \longrightarrow TM$ of vector bundles, called the anchor map,

such that the following axioms are satisfied for all $u, v, w \in \Gamma(E)$:

1. $[u,[v,w]] = [[u,v],w] + [v,[u,w]]$,
2. $\pi(u)\langle v,w\rangle = \langle [u,v],w\rangle + \langle v,[u,w]\rangle$,
3. $\langle [u,v] + [v,u], w\rangle = \pi(w)\langle u,v\rangle$.

Note that the Dorfman bracket is not necessarily skew-symmetric.

Proposition 2.41. *The bracket and the anchor of a Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ satisfy the following equations for $u, v \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$:*

1. $[u, fv] = \pi(u)(f)\cdot v + f[u,v]$
2. $\pi[u,v] = [\pi u, \pi v]$, *where on the right hand side $[\cdot,\cdot]$ stands for the commutator of vector fields.*

Proof. These equalities follow from a straightforward application of the axioms of a Courant algebroid, see [U]. □

If $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ is a Courant algebroid, then we denote by $\pi^* : T^*M \longrightarrow E$ the map obtained by dualising π to obtain $\pi^*\colon T^*M \longrightarrow E^*$ and then identifying $E^* \cong E$ via the scalar product $\langle\cdot,\cdot\rangle$. Explicitly it is given by $\langle \pi^*(\alpha), v\rangle = \alpha(\pi(v))$ for $\alpha \in T^*_pM$, $v \in E_p$, $p \in M$.

Proposition 2.42. *Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be a Courant algebroid. Then the bracket, the anchor π and its dual satisfy the following equations for $\alpha, \beta \in \Gamma(T^*M)$:*

1. $\pi \circ \pi^* = 0$
2. $\langle \pi^*\alpha, \pi^*\beta\rangle = 0$
3. $[\pi^*\alpha, \pi^*\beta] = 0$

Proof. Observe that $T_p^*M = \{d_p f \mid f = \langle v, v\rangle, v \in \Gamma(E)\}$ for any $p \in M$. Therefore we can consider a one-form of the form $\xi = df$ for $f = \langle v, v\rangle$ with $v \in \Gamma(E)$. By Axiom 3 for a Courant algebroid, we have $\pi^*(\xi) = 2[v, v]$. To show that the vector field $\pi(\pi^*(\xi))$ is zero, we evaluate any one-form $\eta \in \Gamma(T^*M)$ on it and obtain

$$\begin{aligned}\eta\left(\pi\left(\pi^*(\xi)\right)\right) &= \langle \pi^*(\eta), \pi^*(\xi)\rangle = 2\left\langle \pi^*(\eta), [v,v]\right\rangle \\ &= 2\eta(\pi([v,v])) = 2\eta([\pi(v), \pi(v)]) = 0.\end{aligned}$$

This shows parts 1 and 2 of the lemma. To prove part 3 we compute for any $e \in \Gamma(E)$

$$\begin{aligned}\left\langle \left[\pi^*(\xi), \pi^*(\eta)\right], e\right\rangle &= \pi\left(\pi^*(\xi)\right)\left\langle \pi^*(\eta), e\right\rangle - \left\langle \pi^*(\eta), \left[\pi^*(\xi), e\right]\right\rangle \\ &= -\eta\left(\pi\left[\pi^*(\xi), e\right]\right) \\ &= -\eta\left(\left[\pi(\pi^*(\xi)), \pi(e)\right]\right) \\ &= 0.\end{aligned}$$

□

Definition 2.43. A Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ is called

1. **regular** if π has constant rank,
2. **transitive** if π is a submersion,
3. **exact** if the sequence below is exact,

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0.$$

Observe that exactness implies transitivity which in turn implies regularity of a Courant algebroid. Now we will look at some examples of Courant algebroids.

Example 2.44. If M is a manifold and TM and T^*M denote its tangent and cotangent bundles respectively, then the vector bundle $\mathbb{T}M = TM \oplus T^*M$, called the **generalised tangent bundle**, may be endowed with the structure of a Courant algebroid by defining the scalar product $\langle\cdot,\cdot\rangle$, the bracket $[\cdot,\cdot]$, the anchor π as follows:

- $\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X))$ for all $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$.
- $[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + d(\xi(Y))$ for all $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$.

- $\pi : \mathbb{T}M \longrightarrow TM$ is the natural projection $X + \xi \mapsto X$.

The generalised tangent bundle is an example of an exact Courant algebroid.

Example 2.45. Consider the generalised tangent bundle $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ with its canonical Courant algebroid structure. If $H \in \Gamma(\Lambda^3 T^*M)$ is a closed 3-form, i.e., such that $dH = 0$, then the bracket defined by

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + H(X, Y, \cdot)$$

is also a Dorfman bracket and defines on $\mathbb{T}M$ another Courant algebroid structure $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$. The Courant algebroid is also called the H-twisted Courant algebroid.

This is another example of an exact Courant algebroid. In fact it can be shown that every exact Courant algebroid is isomorphic to a twisted Courant algebroid [Š].

Example 2.46. Let $\mathfrak{g}$ be a Lie algebra which admits a scalar product that is invariant under the adjoint action, i.e. such that $\langle [u, v], w\rangle + \langle v, [u, w]\rangle = 0$ for all $u, v, w \in \mathfrak{g}$. Then $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called a **quadratic Lie algebra**. A quadratic Lie algebra can be considered as a Courant algebroid over a point $M = \{\mathrm{pt}\}$ together with the trivial anchor map $\pi = 0 : \mathfrak{g} \longrightarrow \{\mathrm{pt}\}$.

This is one of the simplest examples of a transitive Courant algebroid. Another example of a transitive Courant algebroid is the odd exact Courant algebroid on the vector bundle $\mathbb{T}M \oplus \mathbb{R}$ described in [Ru], where $\mathbb{R}$ is the trivial bundle of rank 1 on M. Transitive Courant algebroids have been classified by Vaisman in [V]. Now we will look at a simple example of a regular Courant algebroid.

Example 2.47. A bundle $E \longrightarrow M$ of quadratic Lie algebras can be endowed with a Courant algebroid structure with the trivial anchor map $\pi = 0 : E \longrightarrow TM$, scalar product $\langle\cdot,\cdot\rangle \in \Gamma(\mathrm{Sym}^2 E^*)$ and bracket $[\cdot,\cdot] \in \Gamma(\Lambda^2 E^* \otimes E)$ such that $(E_x, \langle\cdot,\cdot\rangle_x, [\cdot,\cdot]_x)$ is a quadratic Lie algebra for all $x \in M$.

Since $\mathrm{Rk}(\pi) = 0$, the example above describes trivially a regular Courant algebroid which is transitive if and only if M is zero-dimensional. For a complete classification of regular Courant algebroids, see [CSX].

2.2.1.1 Structure of a regular Courant algebroid

In the following lemma we describe the structure of a regular Courant algebroid. This plays a critical role in our proof for the existence of Dirac generating operators in arbitrary signatures. In particular, we will use this lemma to construct a local frame for the Courant algebroid, computing with which then leads us to our desired result, Lemma 4.9, which we need in order to prove Theorem 4.10.

Lemma 2.48. *[CSX] Let E be a regular Courant algebroid with scalar product $\langle\cdot,\cdot\rangle$ of signature (r,s) and anchor $\pi : E \to TM$.*

1. *The bundle $\ker\pi \subset E$ is a co-isotropic subbundle of E, that is $(\ker\pi)^\perp \subset \ker\pi$.*
2. *The bundle E decomposes as $E = \ker\pi \oplus \mathcal{F}$ where $\mathcal{F}$ is isotropic, that is $\mathcal{F} \subset \mathcal{F}^\perp$.*
3. *The bundle $\ker\pi$ decomposes as $\ker\pi = (\ker\pi)^\perp \oplus \mathcal{G}$ where $\mathcal{G}$ is orthogonal to $\mathcal{F}$.*
4. *The decomposition of $E = \big((\ker\pi)^\perp \oplus \mathcal{F}\big) \oplus \mathcal{G}$ into $(\ker\pi)^\perp \oplus \mathcal{F}$ and $\mathcal{G}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$. The restrictions of $\langle\cdot,\cdot\rangle$ to the two factors $(\ker\pi)^\perp \oplus \mathcal{F}$ and $\mathcal{G}$ have signatures (l,l) and $(r-l, s-l)$, respectively.*

Proof. 1. First observe that $\ker\pi \subset E$ is a sub-bundle because the anchor of a regular Courant algebroid has constant rank. Since for any Courant algebroid, $\pi \circ \pi^* = 0$ as seen from Proposition 2.42, that is $\operatorname{Im}\pi^* \subseteq \ker\pi$, it is enough to show that $(\ker\pi)^\perp = \operatorname{Im}\pi^*$. Let $x \in \operatorname{Im}\pi^*$. Then there exists $\xi \in T^*M$ such that $x = \pi^*(\xi)$. If $y \in \ker\pi$, then $\langle\pi^*(\xi), y\rangle = \xi(\pi(y)) = 0$ implying $x \in (\ker\pi)^\perp$. Therefore $\operatorname{Im}\pi^* \subseteq (\ker\pi)^\perp$. Now $(\ker\pi)^\perp \subseteq \operatorname{Im}\pi^*$ follows from comparing dimensions.

2. Denote by $F := \pi(E) \subseteq TM$ and $s := \operatorname{rank} F$. Let $\lambda_0 : F \to E$ be a section of π, that is $\pi \circ \lambda_0 = \mathrm{id}_F$. Define $\rho : F \to F^*$ by $\rho(X)(Y) := \langle\lambda_0(X), \lambda_0(Y)\rangle$ and let $\lambda := \lambda_0 - \frac{1}{2}\pi^* \circ \rho : F \to E$. Since
$$\pi \circ \lambda = \pi \circ \lambda_0 - \frac{1}{2}\pi \circ \pi^* \circ \rho = \mathrm{Id}_F,$$
λ is also a section of π. Now define $\mathcal{F} := \lambda(F)$. For $p \in M$ and $e \in E_p$, we can find $f := \lambda(\pi(e)) \in \mathcal{F}_p$. Furthermore for such an f we have $\pi(e - f) =$

$\pi(e)-\pi(\lambda(\pi(e)))=0$. Thus E_p decomposes as $\mathcal{F}_p+\ker\pi_p$, i.e., $\mathcal{F}$ is transverse to $\ker\pi$. If now $e\in\mathcal{F}\cap\ker\pi$, then $e=\lambda(X)$, for some $X\in F$ and $X=\pi(u)$, for some $u\in E$. Therefore

$$0=\pi(e)=\pi(\lambda(\pi(u)))=\pi(u)=X,$$

and hence $e=\lambda(X)=\lambda(0)=0$. It is left to show that $\mathcal{F}$ is indeed isotropic, i.e., $\langle u,v\rangle=0$ for all $u,v\in\mathcal{F}$. For that, let $u,v\in\mathcal{F}$. Then there exist $X,Y\in F$ such that $u=\lambda(X)$ and $v=\lambda(Y)$. Then

$$\begin{aligned}
\langle u,v\rangle &= \langle\lambda(X),\lambda(Y)\rangle\\
&= \langle\lambda_0(X)-\frac{1}{2}\pi^*(\rho(X)),\lambda_0(Y)-\frac{1}{2}\pi^*(\rho(Y))\rangle\\
&= \langle\lambda_0(X),\lambda_0(Y)\rangle-\frac{1}{2}\langle\lambda_0(X),\pi^*(\rho(Y))\rangle-\frac{1}{2}\langle\pi^*(\rho(X)),\lambda_0(Y)\rangle\\
&\quad+\frac{1}{4}\langle\pi^*(\rho(X)),\pi^*(\rho(Y))\rangle\\
&= \langle\lambda_0(X),\lambda_0(Y)\rangle-\frac{1}{2}\rho(Y)(X))-\frac{1}{2}\rho(X)(Y))\\
&\quad+\frac{1}{4}\rho(X)(\pi(\pi^*(\rho(Y))))\\
&= \langle\lambda_0(X),\lambda_0(Y)\rangle-\frac{1}{2}\langle\lambda_0(Y),\lambda_0(X))\rangle-\frac{1}{2}\langle\lambda_0(X),\lambda_0(Y))\rangle\\
&= 0.
\end{aligned}$$

3. Define $\mathcal{G}:=((\ker\pi)^\perp\oplus\mathcal{F})^\perp=\ker\pi\cap\mathcal{F}^\perp$. Then $\mathcal{G}$ is orthogonal to $\mathcal{F}$. Furthermore

$$\begin{aligned}
(\ker\pi)^\perp\cap\mathcal{G} &= (\ker\pi)^\perp\cap\ker\pi\cap\mathcal{F}^\perp\\
&= (ker\pi)^\perp\cap\mathcal{F}^\perp\\
&= (\ker\pi\oplus\mathcal{F})^\perp\\
&= E^\perp=0.
\end{aligned}$$

Also

$$\begin{aligned}
(\ker\pi)^\perp+\mathcal{G} &= (\ker\pi)^\perp+((\ker\pi)^\perp\oplus\mathcal{F})^\perp\\
&= (\ker\pi\cap((\ker\pi)^\perp\oplus\mathcal{F}))^\perp\\
&\supseteq ((\ker\pi)^\perp)^\perp\\
&= \ker\pi.
\end{aligned}$$

4. It is clear from the arguments so far that the decomposition $\Big((\ker\pi)^{\perp}\oplus\mathcal{F}\Big)\oplus\mathcal{G}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$. In order to determine the signature of the restriction of the scalar product to the factors $\Big((\ker\pi)^{\perp}\oplus\mathcal{F}\Big)$ and $\mathcal{G}$ we first observe that $\operatorname{rank}\ker\pi^{\perp} = \operatorname{rank}\mathcal{F}$ and then we verify that $\langle\cdot,\cdot\rangle$ defines a non-degenerate pairing between $\mathcal{F}$ and $(\ker\pi)^{\perp}$.

   Next due to the non-degenerate pairing between $\mathcal{F}$ and $(\ker\pi)^{\perp}$, for every basis $(u_1,\dots,u_l)$ of $(\ker\pi)^{\perp}$ there exist elements $v_1,\dots,v_l\in\mathcal{F}$ such that $\langle u_i,v_j\rangle = \delta_{ij}$. Now consider the following basis of $\Big((\ker\pi)^{\perp}\oplus\mathcal{F}\Big)$, up to normalization, $(u_1+v_1,\dots,u_l+v_l,u_1-v_1,\dots,u_l-v_l)$. A simple computation with this basis now shows that the restriction of the scalar product $\langle\cdot,\cdot\rangle$ to $\Big((\ker\pi)^{\perp}\oplus\mathcal{F}\Big)$ is of neutral signature (l,l) and therefore its restriction to $\mathcal{G}$ will be of signature $(r-l,s-l)$ and signature of a scalar product being basis independent proves our claim. $\square$

2.2.2 Connections, E-connections and generalised connections

Let $(E,\langle\cdot,\cdot\rangle,[\cdot,\cdot],\pi)$ be a Courant algebroid and let $V\longrightarrow M$ be a vector bundle.

Definition 2.49 (E-Connection)**.** An E-connection is an $\mathbb{R}$-bilinear map

$$D:\Gamma(E)\times\Gamma(V)\longrightarrow\Gamma(V),\quad (e,v)\mapsto D_e(v),$$

such that

1. $D_{(fe)}v = fD_ev$,
2. $D_e(fv) = \pi(e)(f)v + fD_ev$,

for all $f\in\mathcal{C}^\infty(M)$, $e\in\Gamma(E)$ and $v\in\Gamma(V)$.

Definition 2.50 (Generalised Connection)**.** A generalised connection is an E-connection on E that is compatible with the scalar product in the sense that

$$\pi(u)\langle v,w\rangle = \langle D_uv,w\rangle + \langle v,D_uw\rangle$$

for all $u,v,w\in\Gamma(E)$.

Example 2.51. If ∇ is a connection on a vector bundle $V \longrightarrow M$, then we have an induced E-connection on V given by

$$D_e(v) := \nabla_{\pi(e)}(v).$$

The following proposition shows how a covariant derivative on a vector bundle induces a covariant derivative on the associated bundle of Clifford algebras.

Proposition 2.52. *Let $(E, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian vector bundle over the manifold M and ∇^E a covariant derivative on E. Then ∇^E induces a covariant derivative $\nabla^{\mathcal{C}\ell(E)}$ on the Clifford algebra bundle $\mathcal{C}\ell(E) \longrightarrow M$ such that $\nabla^{\mathcal{C}\ell(E)}e = \nabla^E e$ for $e \in \Gamma(E)$ if and only if $\nabla^E\langle\cdot,\cdot\rangle = 0$.*

Proof. The covariant derivative ∇^E on E induces a covariant derivative $\nabla^{\mathcal{T}E}$ on the tensor algebra bundle $\mathcal{T}E = \bigoplus_{k=0}^{\infty} T^k E$ in the standard way. In order to show that this leads to a well-defined covariant connection on $\mathcal{C}\ell(E)$ it is only left to show that $\nabla_X^{\mathcal{T}E}$ for any $X \in \Gamma(TM)$ preserves the ideal generated by the Clifford relation $e \otimes e + \langle e, e\rangle \in \Gamma(\mathcal{T}E)$. Now we can calculate that $\nabla^E\langle\cdot,\cdot\rangle = 0$ is equivalent to

$$\begin{aligned}
\nabla_X(e \otimes e + \langle e, e\rangle) &= \nabla_X(e) \otimes e + e \otimes \nabla_X(e) + \nabla_X(\langle e, e\rangle) \\
&= \nabla_X(e) \otimes e + e \otimes \nabla_X(e) + \langle \nabla_X(e), e\rangle + \langle e, \nabla_X(e)\rangle \\
&= (\nabla_X(e) + e) \otimes (\nabla_X(e) + e) - e \otimes e - \nabla_X(e) \otimes \nabla_X(e) \\
&\quad + \langle \nabla_X(e) + e, \nabla_X(e) + e\rangle - \langle e, e\rangle - \langle \nabla_X(e), \nabla_X(e)\rangle
\end{aligned}$$ □

Lemma 2.53. *If D is a generalised connection on a Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$, then D induces an E-connection $D^{\mathcal{C}\ell(E)}$ on the Clifford algebra bundle $\mathcal{C}\ell(E) \longrightarrow M$ associated to E.*

Proof. The proof follows analogously to the proof of Proposition 2.52. □

2.2.3 Torsion

In classical pseudo-Riemannian geometry, an affine connection on TM has a fundamental tensor field called **torsion**. We also have an analogue of that in generalised geometry which plays an important role in our construction of Dirac generating operators.

Definition 2.54. [G] Let E be a Courant algebroid and D a generalised connection on E. The torsion $T^D \in \Gamma(\Lambda^2 E^* \otimes E)$ of the generalised connection D is defined to be

$$T^D(u,v) = D_u v - D_v u - [u,v] + (Du)^* v$$

for all $u, v \in \Gamma(E)$. Here $(Du)^*$ denotes the adjoint of Du with respect to $\langle \cdot, \cdot \rangle$, that is $\langle (Du)^* v, w\rangle = \langle v, D_w u\rangle$.

Torsion can also be defined as follows for all $u, v, w \in \Gamma(E)$,

$$T^D(u,v,w) \coloneqq \langle T^D(u,v), w\rangle = \langle D_u v - D_v u - [u,v], w\rangle + \langle D_w u, v\rangle. \tag{2.29}$$

In the following lemma we show that torsion is a totally anti-symmetric tensor.

Lemma 2.55. *The torsion of a generalised connection in the sense of* (2.29) *is totally anti-symmetric, i.e.,* $T^D \in \Gamma(\Lambda^3 E^*)$.

Proof. To show anti-symmetry in the first two components we use the compatibility of D with the scalar product and axiom 3 for a Courant algebroid and we see for all $u, v, w \in \Gamma(E)$ that

$$\begin{aligned} T^D(u,v,w) &= \langle D_u v - D_v u - [u,v], w\rangle + \langle D_w u, v\rangle \\ &= \langle D_u v - D_v u, w\rangle + \langle [v,u], w\rangle - \pi(w)\langle u, v\rangle + \langle D_w u, v\rangle \\ &= \langle -D_v u + D_u v + [v,u], w\rangle - \langle D_w v, u\rangle \\ &= -T^D(v,u,w). \end{aligned}$$

With the help of axiom 2 of a Courant algebroid and compatibility of D with the scalar product, we show anti-symmetry in the last two components:

$$\begin{aligned} T^D(u,v,w) &= \langle D_u v - D_v u - [u,v], w\rangle + \langle D_w u, v\rangle \\ &= \langle D_u v - D_v u, w\rangle + \langle v, [u,w]\rangle - \pi(u)\langle v, w\rangle + \langle D_w u, v\rangle \\ &= -\langle v, D_u w\rangle - \langle D_v u, w\rangle + \langle v, [u,w]\rangle + \langle D_w u, v\rangle \\ &= \langle -D_u w + D_w u + [u,w], v\rangle - \langle D_v u, w\rangle \\ &= -T^D(u,w,v). \end{aligned}$$

□

3

Part I: Clifford Module Bundles

I saw all the mirrors on earth and
none of them reflected me..

Jorge Luis Borges, The Aleph and Other Stories

3.1 Module bundles

All algebras we consider in this thesis are finite-dimensional semi-simple real unital associative algebras.
Let M be a smooth manifold. Without loss of generality we will take $\{U_\alpha \mid \alpha \in \Lambda\}$ to be a fixed open cover of M such that all bundles defined over M appearing in this thesis, be they vector bundles, algebra bundles or any other fibre bundle, admit a family of local trivialisations over it. Let us now fix some notation that will be used throughout this part. $U_{\alpha\beta} := U_\alpha \cap U_\beta$ will denote the intersection of the members of the cover, the restriction of a map f to the subset U_α will be denoted by f_α and furthermore, if $\pi : V \longrightarrow M$ is any fibre bundle, then we use the notation $V_\alpha := \pi^{-1}(U_\alpha)$.

Definition 3.1. Let $\mathbb{A} \longrightarrow M$ be a bundle of algebras whose typical fibre is the algebra A, let $V \longrightarrow M$ be a real vector bundle of rank N, $\Gamma : \mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ a morphism of algebra bundles and let $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ be a fixed representation of the typical fibre of $\mathbb{A}$ over the typical fibre of V. We say that (V, Γ) **is a (left) module bundle of type** $[\gamma]$ **over the algebra bundle** $\mathbb{A}$ if and only if there exists a family of local trivialisations for the algebra bundle $\mathbb{A}$, $\{\xi_\alpha \mid \alpha \in \Lambda\}$, and the vector bundle V, $\{\eta_\alpha \mid \alpha \in \Lambda\}$, such that for every $\alpha \in \Lambda$ the following diagram commutes.

$$\begin{tikzcd}
\mathbb{A}_\alpha \arrow[r, "\Gamma_\alpha"] \arrow[d, "\xi_\alpha"] & \mathrm{End}_{\mathbb{R}}(V_\alpha) \arrow[d, "\mathrm{Ad}(\eta_\alpha)"] \\
U_\alpha \times A \arrow[r, "\mathrm{Id}_\alpha \times \gamma"] & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)
\end{tikzcd}$$

Observe that the representations $\Gamma_p : \mathbb{A}_p \longrightarrow \mathrm{End}_{\mathbb{R}}(V_p)$ are by definition equivalent to the representation $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ for every $p \in M$.

Remark 3.2. Analogously we define 'right' module bundles by considering anti-representations.

In fact, for every algebra bundle $\mathbb{A}$ and vector bundle V that have an algebra bundle homomorphism $\Gamma : \mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ defined, we can always find a family of local trivialisations of the algebra bundle and the vector bundle, (ξ_α) and (η_α) respectively, such that the above diagram commutes. To show this, in Proposition 3.3 below, we verify that for any local frame $\{f_i\}$ for $\mathbb{A}$ there exists a local frame $\{s_i\}$ for V such that the matrix representation of $\Gamma(f_i)$ is equal to the corresponding matrix representation of $\gamma_i := \gamma(\xi_\alpha(f_i))$.

Proposition 3.3. *Let $\mathbb{A} \longrightarrow M$ be an algebra bundle whose typical fibre is an algebra A and let $(e_i)_{i=1}^n$ be a basis of A. Let $\Gamma : \mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ be an algebra bundle morphism over a real vector bundle $V \longrightarrow M$ of rank N and let $\gamma : A \longrightarrow \mathrm{Mat}_{\mathbb{R}}(\mathbb{R}^N)$ be a fixed real representation of A that is equivalent to Γ_x for all $x \in M$. Let ξ be a family of local trivialisations of $\mathbb{A}$ and let η be a family of local trivialisations of V over a fixed open cover. Then $A_i := \gamma(e_i) \in \gamma(A)$ is a system of real $N \times N$-matrices which satisfy certain relations for $i = 1, \ldots, n$. For an open set $U \subset \mathbb{R}^k$, then $B_i(x) := \mathrm{Ad}(\eta_\alpha)(\Gamma_x({\xi_\alpha}^{-1}(x, e_i)))$ is another system of $N \times N$-matrices which depend smoothly on $x \in U$ and satisfy the same relations. Assume that for every $x \in U$ there exists an invertible $N \times N$-matrix $C(x)$ such that for every i, $A_i C(x) = C(x) B_i(x)$ but $x \mapsto C(x)$ is not necessarily smooth. Then there exist invertible $N \times N$-matrices $D(x)$ defined in a neighbourhood of some $x_0 \in U$ which depend smoothly on x such that $A_i D(x) = D(x) B_i(x)$ for all i.*

Proof. For all $x \in U \subset \mathbb{R}^k$ consider the equation $A_i Y = Y B_i(x)$ for $Y \in \mathbb{R}[N]$. The vector space of solutions to this equation denoted by $\mathrm{Sol}_x \subset \mathbb{R}[N]$ is clearly non-trivial because $C(x) \in \mathrm{Sol}_x$. Furthermore, $\dim(\mathrm{Sol}_x) = \dim \mathcal{C}(\{A_i\})$, where $\mathcal{C}(\{A_i\}) := \{Y \in \mathbb{R}[N] \mid A_i Y = Y A_i \text{ for all } i = 1, \dots, n\}$ is the Schur algebra of the system $\{A_i\}$. This is because for every $T \in \mathcal{C}(\{A_i\})$ the map $T \mapsto TC(x)$ maps T into Sol_x and for every $Y \in \mathrm{Sol}_x$ the map $Y \mapsto YC(x)^{-1}$ maps Y into $\mathcal{C}(\{A_i\})$. Thus the dimension of Sol_x is constant in x. Therefore $\mathrm{Sol} := \sqcup_{x \in U} \mathrm{Sol}_x \longrightarrow U$ can be given a smooth vector bundle structure. The smoothness follows from the smooth dependence of the Gauss algorithm on the parameters of the linear system which in turn are smooth functions of x. We are assured then that there exists a smooth section $D \in \Gamma(U', \mathrm{Sol})$ such that $D(x_0) = C(x_0)$ where $x_0 \in U' \subseteq U$. Furthermore, since D is invertible at x_0, it is invertible in some neighbourhood of x_0. $\square$

Now it is easily seen that Proposition 3.3 proves the equivalence stated below.

Proposition 3.4. *Let $\mathbb{A} \longrightarrow M$ be a bundle of algebras whose typical fibre is a algebra A, let $V \longrightarrow M$ be a real vector bundle of rank N and $\Gamma : \mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ a morphism of algebra bundles with $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ being a fixed representation of the typical fibre of $\mathbb{A}$ over the typical fibre of V. Then (V, Γ) is a module bundle of type $[\gamma]$ over the algebra bundle $\mathbb{A}$ if and only if for every $p \in M$, $\Gamma_p : \mathbb{A}_p \longrightarrow \mathrm{End}_{\mathbb{R}}(V_p)$ is equivalent to $\gamma : A \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ as a representation of algebras.*

We obtain Corollary 3.5 to Proposition 3.3 which shows that we can always choose frames for the algebra bundle $\mathbb{A}$ and the vector bundle V in such a way that the action of the frame of $\mathbb{A}$ on the frame of V has constant coefficients.

Corollary 3.5. *Let (V, Γ) be a module bundle over the algebra bundle $\mathbb{A}$ of type $[\gamma]$. Then there exist local frames (e_i) of $\mathbb{A}$ and (σ_k) of V such that $e_i \sigma_k = \sum_\beta C_{ik}^\beta \sigma_\beta$ where C_{ik}^β are constants.*

Proof. Recall that due to Proposition 3.3 we have families of local trivialisations ξ_α and η_α such that the below diagram commutes.

$$\begin{array}{ccc} \mathbb{A}_\alpha & \xrightarrow{\Gamma_\alpha} & \mathrm{End}_{\mathbb{R}}(V_\alpha) \\ \Big\downarrow{\scriptstyle \xi_\alpha} & & \Big\downarrow{\scriptstyle \mathrm{Ad}(\eta_\alpha)} \\ U_\alpha \times A & \xrightarrow{\mathrm{Id}_\alpha \times \gamma} & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N). \end{array}$$

Let (e_i) be a local frame of $\mathbb{A}$ that corresponds to the local trivialisation ξ and let (σ_k) be a local frame of V that corresponds to the local trivialisation η. More specifically, if (x_i) is a basis of the algebra A, then set $e_i(p) := \xi_\alpha^{-1}(p, x_i)$. Likewise, if $(\hat{x}_j)$ is a basis of $\mathbb{R}^N$, then set $\sigma_i^\alpha(p) := \eta_\alpha^{-1}(p, \hat{x}_i)$. Then,

$$\begin{aligned}\Gamma_\alpha(e_i(p))(\sigma_j^\alpha) &= \Gamma_\alpha(\xi_\alpha^{-1}(p, x_i))(\eta_\alpha^{-1}(p, \hat{x}_j)) \\ &= \mathrm{Ad}(\eta_\alpha)^{-1}(\gamma(p, x_i))(\eta_\alpha^{-1}(p, \hat{x}_j)) \\ &= \eta_\alpha^{-1}\gamma(p, x_i)\eta_\alpha(\eta_\alpha^{-1}(p, \hat{x}_j)) \\ &= \eta_\alpha^{-1}(p, \sum_k C_{ij}^k \hat{x}_k) = \sum_k C_{ij}^k \sigma_k(p)\end{aligned}$$

where C_{ij}^k are constants such that $\gamma(x_i)(\hat{x}_j) = \sum_k C_{ij}^k \hat{x}_k$. $\square$

3.1.1 Irreducible Clifford Module Bundles

Let $\mathcal{C}\ell(E) \longrightarrow M$ be the bundle of real Clifford algebras associated to the quadratic vector bundle $(E, q) \longrightarrow M$ where q is of signature (r, s). We now consider module bundles over $\mathcal{C}\ell(E) \longrightarrow M$, a bundle of real Clifford algebras with typical fibre $\mathcal{C}\ell(r, s)$.

If $V \longrightarrow M$ is a real vector bundle with typical fibre $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and $\Gamma : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_\mathbb{R}(V)$ is a morphism of algebra bundles with $\gamma : \mathcal{C}\ell(r, s) \longrightarrow \mathrm{End}_\mathbb{R}(\mathbb{R}^N)$ being a fixed irreducible representation of the typical fibre of $\mathcal{C}\ell(E)$ over the typical fibre of V, then applying Definition 3.1 to this setting, we say that (V, Γ) is an **irreducible Clifford module bundle over the Clifford algebra bundle** $\mathcal{C}\ell(E)$ **of type** $[\gamma]$ if there exists a family of local trivialisations for the Clifford algebra bundle $\mathcal{C}\ell(E)$, $\{\xi_\alpha \mid \alpha \in \Lambda\}$, and the vector bundle V, $\{\eta_\alpha^V \mid \alpha \in \Lambda\}$, such that for every $\alpha \in \Lambda$ the following diagram commutes,

$$\begin{tikzcd} \mathcal{C}\ell(E)_\alpha \arrow[r, "\Gamma_\alpha"] \arrow[d, "\xi_\alpha"] & \mathrm{End}_\mathbb{R}(V_\alpha) \arrow[d, "\mathrm{Ad}(\eta_\alpha^V)"] \\ U_\alpha \times \mathcal{C}\ell(r, s) \arrow[r, "\mathrm{Id}_\alpha \times \gamma"] & U_\alpha \times \mathrm{End}_\mathbb{R}(\mathbb{R}^N). \end{tikzcd}$$

Applying Proposition 3.3 to the specific case of Clifford module bundles, we show next that given any family of local trivialisations of $\mathcal{C}\ell(E)$ and any family of local trivialisations of V we can obtain trivialisations of V such that the above diagram commutes. Let $V \longrightarrow M$ be a vector bundle, $\Gamma : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_\mathbb{R}(V)$ an algebra bundle morphism and $\gamma : \mathcal{C}\ell(r, s) \longrightarrow \mathrm{End}_\mathbb{R}(\mathbb{R}^N)$ a fixed representation of the typical fibre. For a fixed open cover $\{U_\alpha\}$ of M, if ξ_α is a family of local trivialisations of the module bundle $\mathcal{C}\ell(E)$, and ${\eta'}^V_\alpha$ is a family of local trivialisations of the vector bundle $V \longrightarrow M$, not necessarily such that it makes (V, Γ) a $\mathcal{C}\ell(E)$-module bundle, then we can apply Proposition 3.3 by taking $A_i = \gamma(e_i)$ and $B_i(x) = \mathrm{Ad}({\eta'}^V_\alpha)(\Gamma_\alpha(\xi_\alpha^{-1}(x, e_i))) = \mathrm{Ad}(C(x))A_i = \mathrm{Ad}(D(x))A_i$ and $\eta^V_\alpha = D \circ {\eta'}^V$ to show that we can find local trivialisations η^V of V such that (V, Γ) becomes a $\mathcal{C}\ell(E)$-module bundle.

The corollary below summarises the fact that if we have two irreducible $\mathcal{C}\ell(E)$-module bundles S and Σ and fix a trivialisation for E, then we can find trivialisations of S and Σ such that they are simultaneously compatible with the algebra bundle homomorphisms Γ^S and Γ^Σ.

Corollary 3.6. *Let (S, Γ^S) and (Σ, Γ^Σ) be $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$. Then there exists an open cover of M, a family of local trivialisations $\{\xi_\alpha \mid \alpha \in \Lambda\}$ of $\mathcal{C}\ell(E)$, a family of local trivialisations $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ of the vector bundle $S \longrightarrow M$ and a family of local trivialisations $\{\eta^\Sigma_\alpha \mid \alpha \in \Lambda\}$ of the vector bundle $\Sigma \longrightarrow M$, such that*

$$\begin{array}{ccc}
\mathcal{C}\ell(E)_\alpha & \xrightarrow{\quad\Gamma^S_\alpha\quad} & \mathrm{End}_{\mathbb{R}}(S_\alpha) \\
\downarrow{\scriptstyle \xi_\alpha} & & \downarrow{\scriptstyle \mathrm{Ad}(\eta^S_\alpha)} \\
U_\alpha \times \mathcal{C}\ell(r,s) & \xrightarrow{\mathrm{Id}\times\gamma} & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N) \\
\downarrow{\scriptstyle \xi_\alpha^{-1}} & & \downarrow{\scriptstyle \mathrm{Ad}((\eta^\Sigma_\alpha)^{-1})} \\
\mathcal{C}\ell(E)_\alpha & \xrightarrow{\quad\Gamma^\Sigma_\alpha\quad} & \mathrm{End}_{\mathbb{R}}(\Sigma_\alpha)
\end{array}$$

i.e., such that $\mathrm{Ad}((\eta^\Sigma_\alpha)^{-1} \circ \eta^S_\alpha) \circ \Gamma^S_\alpha = \Gamma^\Sigma_\alpha$.

Proof. That η^S_α and η^Σ_α can be determined once we fix ξ_α, follows from Proposition 3.3. $\square$

3.2 Intertwiners of $\mathcal{C}\ell(E)$-module bundles

In this section, we will construct the vector bundle of $\mathcal{C}\ell(E)$-equivariant homomorphisms between two Clifford module bundles and show that it is a module bundle over the Schur algebra bundle. To begin, we define the Schur algebra bundle of a Clifford module bundle.

Definition 3.7. For a module bundle (S, Γ) over a Clifford algebra bundle $\mathcal{C}\ell(E)$, the sub-bundle of algebras $\mathcal{C}(S, \Gamma) \hookrightarrow \mathrm{End}_{\mathbb{R}}(S)$ of the endomorphism bundle defined by the subspace $\mathcal{C}(S, \Gamma) := \sqcup_{p \in M} \mathcal{C}(S_p, \Gamma_p) \subset \mathrm{End}_{\mathbb{R}}(S)$ where, $\mathcal{C}(S_p, \Gamma_p) := \{a \in \mathrm{End}_{\mathbb{R}}(S_p) \mid [a, \Gamma_p(\mathcal{C}\ell(E_p))] = 0\}$ is called the **Schur algebra bundle of the Clifford module bundle** (S, Γ).

In the lemma below we show the compatibility between the transition functions of a Clifford module bundle with the action of the Clifford algebra.

Lemma 3.8. *Let $S \longrightarrow M$ be a $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$ where $\gamma : \mathcal{C}\ell(r, s) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ is an irreducible representation of $\mathcal{C}\ell(r, s)$ and $\mathcal{C}\ell(E) \longrightarrow M$ is a bundle of Clifford algebras. If $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ is a family of local trivialisations of S as a $\mathcal{C}\ell(E)$-module bundle and $\{g^S_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(\mathbb{R}^N)\}$ denotes the corresponding family of transition functions, then*

$$g^S_{\alpha\beta}(p)(\gamma_p(e)(v)) = \gamma_p(e)(g^S_{\alpha\beta}(p)(v)) \tag{3.1}$$

for all $e \in \mathcal{C}\ell(r, s)$, $\quad p \in M \quad$ and $\quad v \in \mathbb{R}^N$

Proof. Let $\{\xi_\alpha \mid \alpha \in \Lambda\}$ be a family of local trivialisations of the algebra bundle $\mathcal{C}\ell(E) \longrightarrow M$ such that $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ is a family of local trivialisations of $S \longrightarrow M$ as a $\mathcal{C}\ell(E)$-module bundle and $\{g^S_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{GL}(\mathbb{R}^N)\}$ is the family of transition functions corresponding to $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ i.e.,

$$\begin{aligned} \eta^S_\beta \circ (\eta^S_\alpha)^{-1} : U_{\alpha\beta} \times \mathbb{R}^N &\longrightarrow U_{\alpha\beta} \times \mathbb{R}^N \\ (p, v) &\longmapsto (p, g^S_{\alpha\beta}(p)(v)), \end{aligned}$$

then $g^S_{\alpha\beta}(p)(v) := \eta^S_\beta \circ (\eta^S_\alpha)^{-1}(v)$ for all $p \in U_{\alpha\beta}$ and $v \in \mathbb{R}^N$, and

$$\begin{aligned} g^S_{\alpha\beta}(p)(\gamma_p(e)(v)) &= \eta^S_\beta \mid_p \circ (\eta^S_\alpha)^{-1} \mid_p (\gamma_p(e)(v)) \\ &= \eta^S_\beta \mid_p \circ \Gamma^S_\alpha \mid_p (\xi^{-1}_\alpha \mid_p (e)) \circ (\eta^S_\alpha)^{-1} \mid_p (v) \\ &= \eta^S_\beta \mid_p \circ \Gamma^S_\beta \mid_p (\xi^{-1}_\beta \mid_p (e)) \circ (\eta^S_\alpha)^{-1} \mid_p (v) \\ &= \gamma_p(e)(g^S_{\alpha\beta}(v)), \end{aligned}$$

because $\eta^S_\beta \mid_p \circ \Gamma^S_\beta \mid_p (\xi^{-1}_\beta \mid_p (e))(v) = \gamma_p(\xi_\beta \mid_p (\xi^{-1}_\beta \mid_p (e)))(\eta^S_\beta \mid_p (v))$. □

In the lemma below we construct a vector bundle of $\mathcal{C}\ell(E)$-equivariant homomorphisms between any two Clifford module bundles.

Lemma 3.9. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ with the family of local trivialisations $\{\eta^S_\alpha : \alpha \in \Lambda\}$ and $\{\eta^\Sigma_\alpha \mid \alpha \in \Lambda\}$ respectively and where $\gamma : \mathcal{C}\ell(r,s) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ is an irreducible representation of $\mathcal{C}\ell(r,s)$. Then there exists a real vector bundle $L := \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \longrightarrow M$ with the typical fibre $\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N)$.*

Proof. Let $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) := \sqcup_{p\in M} \mathrm{Hom}_{\mathcal{C}\ell(E_p)}(S_p,\Sigma_p)$ and $\pi : \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \longrightarrow M$ be a surjection such that $\pi^{-1}(U) = \mathrm{Hom}_{\mathcal{C}\ell(E_U)}(S_U,\Sigma_U)$ for any open set $U \subset M$.

Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be a fixed open cover of M over which we have the local trivialisations η^S_α and η^Σ_α. We define a formal bundle chart as follows,

$$\begin{aligned} \Phi_\alpha : \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \\ \psi_p &\longmapsto (p,\ \eta^\Sigma_p \circ \psi_p \circ (\eta^S_p)^{-1}) \end{aligned}$$

where $p \in U_\alpha$. With a simple calculation below we verify that Φ_α is well-defined. Indeed,

$$\begin{aligned} \eta^\Sigma_p \circ \psi_p \circ (\eta^S_p)^{-1}(\gamma_p(e)(v)) &= \eta^\Sigma_p(\psi_p(\eta^{S\,-1}_p(\eta^S_p(\Gamma^S_p(\xi^{-1}_p(e))((\eta^S)^{-1}(v)))))) \\ &= \eta^\Sigma_p(\psi_p(\Gamma^S_p(\xi^{-1}_p(e))((\eta^S)^{-1}(v)))) \\ &= \eta^\Sigma_p(\Gamma^\Sigma_p(\xi^{-1}_p(e))(\psi_p((\eta^S)^{-1}(v)))) \\ &= \eta^\Sigma_p(\Gamma^\Sigma_p(\xi^{-1}_p(e))(\eta^{\Sigma\,-1}_p(\eta^\Sigma_p(\psi_p((\eta^S)^{-1}(v)))))) \\ &= \gamma_p(e)(\eta^\Sigma_p(\psi_p((\eta^S)^{-1}(v)))) \end{aligned}$$

for every $e \in \mathcal{C}\ell(E_p)$, $v \in \mathbb{R}^N$ and $p \in M$. It is clear now that $\{\Phi_\alpha\}$ defines a formal bundle atlas. Furthermore the below map,

$$\begin{aligned} \Phi_\alpha \circ (\Phi_\beta)^{-1} : U_{\alpha\beta} \times \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) &\longrightarrow U_{\alpha\beta} \times \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \\ (p,f) &\longmapsto (p,\ \eta^\Sigma_\alpha|_p\ (\eta^\Sigma_\beta)^{-1}|_p\ f\ \eta^S_\beta|_p\ (\eta^S_\alpha)^{-1}|_p) \end{aligned}$$

is smooth because η^Σ_α and η^S_β are smooth. Therefore, from Theorem 2.31 we can assert that there exists a real vector bundle $L := \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \longrightarrow M$ with the typical fibre $\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N)$ whose bundle charts are given by Φ_α. □

In the lemma below we show that the vector bundle of $\mathcal{C}\ell(E)$-equivariant homomorphisms between two Clifford module bundles, as constructed above, is a module bundle over the Schur algebra bundle.

Lemma 3.10. *Let S and Σ be $\mathcal{C}\ell(E)$-module bundles. Consider the real vector bundle $L := \mathrm{Hom}_{\mathcal{C}\ell(E)}(S, \Sigma) \longrightarrow M$ and let $\theta : \mathcal{C}(\mathbb{R}^N, \gamma) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N))$ be the anti-representation given by $(f \longmapsto (g \longmapsto g \circ f))$. Then $L \longrightarrow M$ is a right $\mathcal{C}(S, \Gamma)$-module bundle of type $[\theta]$.*

Proof. In order to show that L is indeed a $\mathcal{C}(S, \Gamma)$-module bundle of type $[\theta]$, we will need to verify that upon fixing an open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of M and local trivialisations of $\mathcal{C}\ell(E)$-module bundles S and Σ, η^S_α and η^Σ_α respectively, for any α the below diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(S,\Gamma)_\alpha \arrow[r, "\Theta_\alpha"] \arrow[d, "\mathrm{Ad}(\eta^S_\alpha)"] & \mathrm{End}_{\mathbb{R}}(L_\alpha) \arrow[d, "\mathrm{Ad}(\Phi_\alpha)"] \\
U_\alpha \times \mathcal{C}(\mathbb{R}^N, \gamma) \arrow[r, "\mathrm{Id}_\alpha \times \theta"] & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N))
\end{tikzcd}$$

Here Φ_α are the charts defined in Lemma 3.9 and Θ_α are restrictions of the bundle morphism Θ which are defined fibrewise as $\Theta_p : f \longmapsto (g \longmapsto g \circ f)$. For $f \in \mathcal{C}(S, \Gamma)_p$, $v \in \mathbb{R}^N$ and $\tau \in \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N)$. Then,

$$\begin{aligned}
\mathrm{Ad}(\Phi_p) \circ \Theta_p(f)(\tau(v)) &= \Phi_p(\Theta_p(f)(\Phi_p^{-1}(\tau(v)))) \\
&= \eta^\Sigma_p({\eta^\Sigma}^{-1}_p(\tau(\eta^S_p(f((\eta^S)^{-1}_p(v)))))) \\
&= \tau(\eta^S_p(f((\eta^S)^{-1}_p(v))),
\end{aligned}$$

and

$$\theta \circ \mathrm{Ad}(\eta^S_p)(f)(\tau(v)) = \tau(\eta^S_p(f((\eta^S_p)^{-1}(v))).$$

With the above calculation, we have shown that the diagram commutes and therefore L is indeed a right $\mathcal{C}(S, \Gamma)$-module bundle of type $[\theta]$. □

In the next lemma we show that a $\mathcal{C}\ell(E)$-module bundle is naturally a left module bundle over its Schur algebra bundle.

Lemma 3.11. *Let $S \longrightarrow M$ be a $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$ and let $\mathcal{C}(S, \Gamma) \longrightarrow M$ its Schur algebra bundle. Then S inherits the structure of a $\mathcal{C}(S, \Gamma)$-module bundle with $\mathcal{C}(S, \Gamma)$ acting on S fibrewise from the left.*

Proof. Let us fix an open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of M and let $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ be a family of local trivialisations of the $\mathcal{C}\ell(E)$-module bundle S over that cover. Since

$\mathcal{C}(S,\Gamma) \longrightarrow M$ is a subalgebra bundle of the endomorphism bundle $\mathrm{End}_{\mathbb{R}}(S) \longrightarrow M$, $\{\mathrm{Ad}(\eta^S_\alpha) \mid \alpha \in \Lambda\}$ will also be a family of local trivialisations of the Schur algebra bundle $\mathcal{C}(S,\Gamma)$ as seen in Proposition 2.32. If $\Psi : \mathcal{C}(S,\Gamma) \longrightarrow \mathrm{End}_{\mathbb{R}}(S)$ is an algebra bundle morphism which is defined fibre-wise as $\Psi_p : f \longmapsto (g \longmapsto f \circ g)$ and $\psi : \mathcal{C}(\mathbb{R}^N, \gamma) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N)$ is the representation given by $f \longmapsto (g \longmapsto f \circ g)$. It is easy now to see that the below diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(S,\Gamma)_\alpha \arrow[r, "\Psi_\alpha"] \arrow[d, "\mathrm{Ad}(\eta^S_\alpha)"] & \mathrm{End}_{\mathbb{R}}(S_\alpha) \arrow[d, "\mathrm{Ad}(\eta^S_\alpha)"] \\
U_\alpha \times \mathcal{C}(\mathbb{R}^N,\gamma) \arrow[r, "\mathrm{Id}_\alpha \times \psi"] & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathbb{R}^N).
\end{tikzcd}$$

□

3.3 Classification of Irreducible Clifford Module Bundles

It is clear that given $L \longrightarrow M$, the bundle of all equivariant homomorphisms between two $\mathcal{C}\ell(E)$-module bundles S and Σ as defined above, we can construct the tensor product bundle $L \otimes_{\mathbb{R}} S \longrightarrow M$ over $\mathbb{R}$. We will now show that this tensor product bundle is in turn a $\mathcal{C}\ell(E)$-module bundle in a rather natural way.

Lemma 3.12. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$ and $L \longrightarrow M$, the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the $\mathcal{C}\ell(E)$-module bundles S and Σ. Then the tensor product bundle $L \otimes_{\mathbb{R}} S \longrightarrow M$ over $\mathbb{R}$ is a $\mathcal{C}\ell(E)$-module bundle.*

Proof. Let us fix an open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of M and let $\{\xi_\alpha \mid \alpha \in \lambda\}$ be a family of local trivialisations of the algebra bundle $\mathcal{C}\ell(E) \longrightarrow M$ such that $\{\eta^S_\alpha \mid \alpha \in \Lambda\}$ and $\{\eta^\Sigma_\alpha \mid \alpha \in \Lambda\}$ are families of local trivialisations of the $\mathcal{C}\ell(E)$-module bundle S and Σ, respectively, over that cover. Then $\Gamma^S : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(S)$ denotes, as usual, the algebra bundle morphism. If $\Delta : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(L \otimes_{\mathbb{R}} S)$ denotes the algebra bundle morphism defined fibre-wise as

$$\Delta_p : e \longmapsto (l \otimes s \longmapsto l \otimes \Gamma^S(e)(s))$$

and we consider the algebra morphism

$$\delta : \mathcal{C}\ell(r,s) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N),$$

$$w \longmapsto (h \otimes r \longmapsto h \otimes \gamma(w)(r)),$$

then we can show that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{C}\ell(E)_\alpha & \xrightarrow{\quad\quad\Delta_\alpha\quad\quad} & \mathrm{End}_{\mathbb{R}}\,(L_\alpha \otimes_{\mathbb{R}} S_\alpha) \\ \Big\downarrow{\scriptstyle \xi_\alpha} & & \Big\downarrow{\scriptstyle \mathrm{Ad}(\Phi_\alpha \otimes \eta^S_\alpha)} \\ U_\alpha \times \mathcal{C}\ell(r,s) & \xrightarrow{\mathrm{Id}_\alpha \times \delta} & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N). \end{array}$$

Observe that $\mathrm{Ad}(\Phi_\alpha \otimes \eta^S_\alpha) : \mathrm{End}_{\mathbb{R}}\,(L_\alpha \otimes_{\mathbb{R}} S_\alpha) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N)$ takes $\mathrm{End}_{\mathbb{R}}\,(L_\alpha \otimes_{\mathbb{R}} S_\alpha) \ni \tau_\alpha := \tau^L_\alpha \otimes \tau^S_\alpha$ where $\tau^L_\alpha \in \mathrm{End}_{\mathbb{R}}(L\,|_\alpha)$ and $\tau^S_\alpha \in \mathrm{End}_{\mathbb{R}}(S\,|_\alpha)$ to $\mathrm{Ad}(\Phi_\alpha \otimes \eta^S_\alpha)(\tau_\alpha) = \mathrm{Ad}(\Phi_\alpha)(\tau^L_\alpha) \otimes \mathrm{Ad}(\eta^S_\alpha)(\tau^S_\alpha)$.
For every $p \in M$, $e \in \mathcal{C}\ell(E)_p$ and $h \otimes r \in \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N$ we have,

$$\begin{aligned} \mathrm{Ad}(\Phi_p \otimes \eta^S_p) \circ \Delta_p(e)(h \otimes r) &= (\mathrm{Ad}(\Phi_p \otimes \eta^S_p) \circ \mathrm{Id}_{L_p}(\cdot) \otimes \Gamma^S(e)(\cdot))(h \otimes r) \\ &= (\mathrm{Ad}(\Phi_p)(\mathrm{Id}_{L_p}))(h) \otimes (\mathrm{Ad}(\eta^S_p)(\Gamma^S(e))(r) \\ &= h \otimes \gamma(\xi_p(e))(r) \\ &= (\delta \circ \xi_p(e))(h \otimes r). \end{aligned}$$

Thus we have shown that the real vector bundle $L \otimes_{\mathbb{R}} S \longrightarrow M$ is a $\mathcal{C}\ell(E)$-module bundle of type $[\delta]$. □

In the lemma below we show that there is a surjective and $\mathcal{C}\ell(E)$-equivariant bundle homomorphism between $L \otimes_{\mathbb{R}} S$ and Σ.

Lemma 3.13. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$ and $L \longrightarrow M$, the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the two $\mathcal{C}\ell(E)$-module bundles S and Σ. If $L \otimes_{\mathbb{R}} S \longrightarrow M$ is the $\mathcal{C}\ell(E)$-module bundle of type $[\delta]$ where $\delta : \mathcal{C}\ell(r,s) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N)$ is the algebra morphism $w \longmapsto (h \otimes r \longmapsto h \otimes \gamma(w)(r))$. Then the bundle morphism $\Upsilon : L \otimes_{\mathbb{R}} S \longrightarrow \Sigma$ given by $l \otimes s \longmapsto l(s)$ is $\mathcal{C}\ell(E)$-equivariant and surjective.*

Proof. The bundle morphism $\Upsilon : L \otimes_{\mathbb{R}} S \longrightarrow \Sigma$ given by $l \otimes s \longmapsto l(s)$ is clearly surjective because, given $\sigma \in \Sigma_p$ for any $p \in M$, since S_p and Σ_p are irreducible modules of the Clifford algebra $\mathcal{C}\ell(E)_p$, we are assured a $\mathcal{C}\ell(E)_p$-equivariant isomorphism $f_p : S_p \xrightarrow{\cong} \Sigma_p$ such that $s := f_p^{-1}(\sigma)$ and $\Upsilon(f_p \otimes s) = \sigma$.

To show the $\mathcal{C}\ell(E)$-equivariance of Υ, take any $e \in \mathcal{C}\ell(E)_p$ and any $l \otimes s \in L_p \otimes_{\mathbb{R}} S_p$ and observe

$$\begin{aligned}\Upsilon(\Delta(e)_p(l \otimes s)) &= \Upsilon(l \otimes \Gamma^S(e)(s)) \\ &= l(\Gamma^S(e)(s)) \\ &= \Gamma^\Sigma(e)(l(s)).\end{aligned}$$ □

3.3.1 An algebraic interlude

We recall here the algebraic notion of tensoring modules over an algebra. For reference see any standard text on Algebra such as [DF, JS].

Definition 3.14. Let A be a real associative algebra, V a right A-module, W a left A-module and $\mathfrak{U} \subset V \otimes_{\mathbb{R}} W$ the subspace

$$\mathfrak{U} := \mathrm{Span}_{\mathbb{R}}\{v \cdot a \otimes w - v \otimes a \cdot w \mid v \in V,\ w \in W \ \text{ and } a \in A\}.$$

The **tensor product of V and W over the algebra** A is given by the quotient space $V \otimes_A W := \frac{V \otimes_{\mathbb{R}} W}{\mathfrak{U}}$.

The usual notion of tensor product is recovered if we take A to be the algebra of real numbers. We see in the proposition below that the quotient space $V \otimes_A W$ satisfies a universal property.

Proposition 3.15. *Let $V \otimes_A W$ be the quotient space defined in the definition above and $\pi : V \otimes_{\mathbb{R}} W \longrightarrow V \otimes_A W$ be the corresponding quotient map. Then $(V \otimes_A W, \pi)$ satisfies the following* ***universal property****:*

1. $\pi \circ (\rho_V \otimes \mathrm{Id}_W) = \pi \circ (\mathrm{Id}_V \otimes \rho_W)$, *where $\rho_V : V \otimes A \to V$ and $\rho_W : A \otimes W \to W$ denote the action maps.*

2. *For any other vector space X and linear map $p : V \otimes_{\mathbb{R}} W \to X$ such that $p \circ (\rho_V \otimes \mathrm{Id}_W) = p \circ (\mathrm{Id}_V \otimes \rho_W)$ there exists a unique linear map $\varphi : V \otimes_A W \to X$ such that $\varphi \circ \pi = p$, as in the following diagram:*

$$\begin{tikzcd} V \otimes_{\mathbb{R}} A \otimes_{\mathbb{R}} W \arrow[r, shift left, "\rho_V \otimes \mathrm{Id}_W"] \arrow[r, shift right, "\mathrm{Id}_V \otimes \rho_W"'] & V \otimes_{\mathbb{R}} W \arrow[r, "\pi"] \arrow[dr, "p"'] & V \otimes_A W \arrow[d, dotted, "\exists! \varphi"] \\ & & X \end{tikzcd}$$

Proof.

1. For any $v \otimes a \otimes w \in V \otimes_{\mathbb{R}} A \otimes_{\mathbb{R}} W$

$$\pi \circ (\rho_V \otimes \mathrm{Id}_W)(v \otimes a \otimes w) = \pi(v \cdot a \otimes w) = [v \cdot a \otimes w] \tag{3.2}$$

and

$$\pi \circ (\mathrm{Id}_V \otimes \rho_W)(v \otimes a \otimes w) = \pi(v \otimes a \cdot w) = [v \otimes a \cdot w] \tag{3.3}$$

Clearly the quantities in equations (3.2) and (3.3) are equal if and only if $v \cdot a \otimes w - v \otimes a \cdot w \in \mathfrak{U}$, which is the case. By linearity this extends to prove part 1 of the proposition.

2. Define $\varphi : V \otimes_A W \longrightarrow X$ as $\varphi(v \otimes_A w) := p(v \otimes w)$ for every $v \otimes_A w \in V \otimes_A W$. We show that this is well defined as follows: observe that $\varphi(\mathfrak{U}) = 0$ because $\varphi(v \cdot a \otimes w - v \otimes a \cdot w) = p(v \cdot a \otimes w) - p(v \otimes a \cdot w) = 0$. If $v \otimes_A w = v' \otimes_A w'$, then $\varphi(v \otimes_A w - v' \otimes_A w') = \varphi(v \otimes_A w) - \varphi(v' \otimes_A w') \in \varphi(\mathfrak{U}) = 0$.

   For uniqueness, see that if $\varphi' : V \otimes_A W \longrightarrow X$ is any linear map such that $\varphi' \circ \pi = p$, then $\varphi'(v \otimes_A w) = p(v \otimes_{\mathbb{R}} w) = \varphi(v \otimes_A w)$. □

Thus we could have defined the tensor product over A equivalently via the universal property. We note the following general fact with respect to A-modules.

Proposition 3.16. *Let A be an algebra, V a left A-module and A_A be the algebra A considered as a right A-module. Then there exists a canonical isomorphism $\Phi : A \otimes_A V \xrightarrow{\cong} V$ of vector spaces.*

Proof. Consider the epimorphism of vector spaces $\hat{\Phi} : A \otimes_{\mathbb{R}} V \longrightarrow V$ which takes $(a \otimes v \longmapsto a \cdot v)$. Since

$$\hat{\Phi}(b \cdot a \otimes v - b \otimes a \cdot v) = (b \cdot a) \cdot v - b \cdot (a \cdot v) = 0,$$

by linearity, this implies $\mathfrak{U} \subset \operatorname{Ker} \hat{\Phi}$.

Consider $x \in \operatorname{Ker} \hat{\Phi}$ such that $x = \sum_i t_i \otimes w_i$ for $t_i \in A,\ w_i \in V$. Then $\hat{\Phi}(x) = 0 \Leftrightarrow \sum_i t_i \cdot w_i = 0$. Observe $x = \sum_i t_i \otimes w_i = \sum_i 1 \cdot t_i \otimes w_i - \sum_i 1 \otimes t_i \cdot w_i$ and by linearity this implies that $\operatorname{Ker} \hat{\Phi} \subset \mathfrak{U}$. Thus we have shown that $\Phi := \hat{\Phi} : A \otimes_A V \xrightarrow{\cong} V$ is an isomorphism of vector spaces. □

The above proposition about A-modules inspires the following lemma.

Lemma 3.17. *Let $\mathcal{C}\ell(E,q)$ be the real Clifford algebra generated by the quadratic vector space (E,q), S and Σ be equivalent irreducible $\mathcal{C}\ell(E,q)$-modules defined by representations γ^S and γ^Σ respectively, $\mathcal{C}(S,\gamma^S)$ be the Schur algebra of S, and L the space of all $\mathcal{C}\ell(E,q)$-equivariant homomorphisms from S to Σ. Then there exists a canonical isomorphism of vector spaces $\Phi : L \otimes_{\mathcal{C}(S,\gamma^S)} S \xrightarrow{\cong} \Sigma$.*

Proof. Note that composing from the right makes L a right $\mathcal{C}(S,\gamma^S)$-module and acting from the left makes S a left $\mathcal{C}(S,\gamma^S)$-module. Define

$$\mathfrak{U} := \mathrm{Span}_{\mathbb{R}} \left\{ -(l\cdot\lambda)\otimes s + l \otimes (\lambda\cdot s) \;\middle|\; \begin{array}{l} \lambda \in \mathcal{C}(S,\gamma^S) \\ l \in L \\ s \in S \end{array} \right\}$$

and $L \otimes_{\mathcal{C}(S,\gamma^S)} S := \frac{L\otimes_{\mathbb{R}} S}{\mathfrak{U}}$. Let $\widetilde{\Phi} : L\otimes S \longrightarrow \Sigma$ be the map $l\otimes s \mapsto l(s)$ and observe that it is surjective since there exists a $\mathcal{C}\ell(E)$-equivariant isomorphism from S to Σ. Since

$$\widetilde{\Phi}(-(l\cdot\lambda)\otimes s + l\otimes(\lambda\cdot s)) = -(l\cdot\lambda)(s) + l(\lambda\cdot s) = 0,$$

by linearity we can assert that $\mathfrak{U} \subset \operatorname{Ker}\widetilde{\Phi}$.

Let $f : S \xrightarrow{\cong} \Sigma$ be a fixed isomorphism of $\mathcal{C}\ell(E)$-modules and $x = \sum_i l_i \otimes s_i \in \operatorname{Ker}\widetilde{\Phi}$. Then observe,

$$x = \sum_i l_i \otimes s_i = -\sum_i (f \otimes (f^{-1}\circ l_i)(s_i) - f\circ (f^{-1}\circ l_i)\otimes s_i)$$

and $f^{-1}\circ l_i \in \mathcal{C}(S,\gamma^S)$. By linearity this implies that $\operatorname{Ker}\widetilde{\Phi} \subset \mathfrak{U}$. From the fundamental theorem of homomorphisms we know that there exists a unique isomorphism $\Phi : L\otimes_{\mathcal{C}(S,\gamma^S)} S \xrightarrow{\cong} \Sigma$ through which $\widetilde{\Phi}$ factorises. □

3.3.2 Tensoring over the Schur algebra bundle

Henceforth we will assume that $S \longrightarrow M$ and $\Sigma \longrightarrow M$ are two $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$ and $L \longrightarrow M$ is the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the two $\mathcal{C}\ell(E)$-module bundles S and Σ. From Lemma 3.10 we know that L is a right-$\mathcal{C}(S,\Gamma^S)$-module bundle of type $[\theta]$ and from Lemma 3.11 we know that $S \longrightarrow M$ is a left-$\mathcal{C}(S,\Gamma^S)$-module bundle of type $[\psi]$.

In the lemma below we show that one can quotient the real vector bundle $L\otimes_{\mathbb{R}} S$ by a sub bundle that preserves the action of the Schur algebra on L and S.

Lemma 3.18. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be two $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$ and $L \longrightarrow M$ is the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the $\mathcal{C}\ell(E)$-module bundles S and Σ. Let Θ and Ψ be bundle homomorphisms between $\mathcal{C}(S,\Gamma^S)$ and $\mathrm{End}_{\mathbb{R}}(L)$ and $\mathrm{End}_{\mathbb{R}}(S)$ such that L and S are right and left $\mathcal{C}(S,\Gamma^S)$-module bundles respectively. Then, there exists a real $\mathcal{C}\ell(E)$-module bundle $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ where $(L \otimes_{\mathcal{C}(S,\Gamma^S)} S)_p := \frac{L_p \otimes_{\mathbb{R}} S_p}{\mathfrak{U}_p}$ and*

$$\mathfrak{U}_p := \mathrm{Span}_{\mathbb{R}} \left\{ -\Theta_p(\lambda)(l) \otimes s + l \otimes (\Psi_p(\lambda)(s)) \;\middle|\; \begin{array}{l} \lambda \in \mathcal{C}(S_p, \Gamma^S_p) \\ l \in L_p \\ s \in S_p \end{array} \right\}.$$

Proof. Let $L \otimes_{\mathcal{C}(S,\Gamma^S)} S := \bigsqcup_{p \in M} \frac{L_p \otimes_{\mathbb{R}} S_p}{\mathfrak{U}_p}$ and $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ be the natural projection. Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be a fixed open cover of M and Φ_α a family of local trivialisations of L. A bundle atlas can be defined as

$$\begin{gathered} \Xi_\alpha : \bigsqcup_{p \in U_\alpha} \frac{L_p \otimes_{\mathbb{R}} S_p}{\mathfrak{U}_p} \longrightarrow U_\alpha \times \frac{\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N}{\mathfrak{U}} \\ (l \otimes s \longmapsto (p, \Phi_\alpha(l) \otimes \eta^S_\alpha(s))) \end{gathered}$$

where

$$\mathfrak{U} = \mathrm{Span}_{\mathbb{R}} \left\{ -\theta(\lambda')(l') \otimes s' + l' \otimes (\psi(\lambda')(s')) \;\middle|\; \begin{array}{l} \lambda' \in \mathcal{C}(\mathbb{R}^N, \gamma) \\ l' \in \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \\ s' \in \mathbb{R}^N \end{array} \right\}.$$

To show that Ξ_α defined above is not just a diffeomorphism between $L_\alpha \otimes_{\mathbb{R}} S_\alpha$ and $U_\alpha \times \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes \mathbb{R}^N$ but descends to an isomorphism between the corresponding quotients, we observe that for $\lambda \in \mathcal{C}(S_p, \Gamma^S_p)$, $l \in L_p$, $s \in S_p$ and family of local trivialisations Φ_α of L,

$$\begin{aligned} &\Xi_p(-(\Theta_p(\lambda)(l)) \otimes s + l \otimes (\Psi_p(\lambda)(s))) \\ &\qquad = -\Phi_p(\Theta_p(\lambda)(l)) \otimes \eta^S_p(s) + \Phi_p(l) \otimes \eta^S_p(\Psi_p(\lambda)(s)) \\ &\qquad = -(\eta^\Sigma_p \circ l \circ {\eta^{S}}^{-1}_p \circ \eta^S_p \circ \lambda \circ {\eta^{S}}^{-1}_p) \otimes \eta^S_p(s) \\ &\qquad\qquad + \Phi_p(l) \otimes \eta^S_p(\lambda({\eta^{S}}^{-1}_p(\eta^S_p(s)))) \\ &\qquad = -l' \circ \lambda' \otimes s' + l' \otimes \lambda'(s') \\ &\qquad = -(\theta(\lambda')(l')) \otimes s' + l' \otimes (\psi(\lambda')(s')), \end{aligned}$$

where $l' = \eta^{\Sigma}_p \circ l \circ {\eta^{S}_p}^{-1} = \Phi_p(l) \in \mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N)$, $s' = \eta^S_p(s) \in \mathbb{R}^N$ and $\lambda' = \eta^S_p \circ \lambda \circ {\eta^{S}_p}^{-1} \in \mathcal{C}(\mathbb{R}^N, \gamma)$. Therefore, by linearity this implies $\Xi_p(\mathfrak{U}_p) = \mathfrak{U}$. Theorem 2.31 now guarantees the existence of a real vector bundle structure $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$.

From Lemma 3.12 we know that $L \otimes_{\mathbb{R}} S \longrightarrow M$ is a $\mathcal{C}\ell(E)$-module bundle. To show that $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ inherits the structure of a real $\mathcal{C}\ell(E)$-module bundle, it is enough to show that $\mathfrak{U}_p$ and $\mathfrak{U}$ are in fact $\mathcal{C}\ell(E)_p$- and $\mathcal{C}\ell(r,s)$-submodules respectively. To this end, as in Lemma 3.12, let $\Delta : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(L \otimes_{\mathbb{R}} S)$ denote the algebra bundle morphism defined fibre-wise as $\Delta_p : e \longmapsto (l \otimes s \longmapsto l \otimes \Gamma^S(e)(s))$ and $\delta : \mathcal{C}\ell(r,s) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{R}^N)$, the algebra morphism $w \longmapsto (h \otimes r \longmapsto h \otimes \gamma(w)(r))$. We verify that $\Delta_p(\mathcal{C}\ell(E)_p)(\mathfrak{U}_p) = \mathfrak{U}_p$ with the following calculation.

For any $e \in \mathcal{C}\ell(E)_p$, $\lambda \in \mathcal{C}(S_p, \Gamma^S_p)$, $l \in L_p$ and $s \in S_p$,

$$\begin{aligned}
\Delta_p(e)(-(\Theta_p(\lambda)(l)) \otimes s + l \otimes (\Psi_p(\lambda)(s))) &= (-(\Theta_p(\lambda)(l)) \otimes \Gamma^S(e)(s)) \\
&\quad + (l \otimes \Gamma^S(e)(\Psi_p(\lambda)(s))) \\
&= (-(\Theta_p(\lambda)(l)) \otimes (\Gamma^S(e)(s))) \\
&\quad + (l \otimes (\Psi_p(\lambda)(\Gamma^S(e)(s)))).
\end{aligned}$$

By linearity we establish the claim. With an analogous calculation it can be shown that $\delta(\mathcal{C}\ell(r,s))(\mathfrak{U}) = \mathfrak{U}$. From this it follows that the diagram below commutes and hence $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ is a real $\mathcal{C}\ell(E)$-module bundle.

$$\begin{tikzcd}
\mathcal{C}\ell(E)_\alpha \arrow[r, "\Delta_\alpha"] \arrow[d, "\xi_\alpha"] & \mathrm{End}_{\mathbb{R}}\left(L_\alpha \otimes_{\mathcal{C}(S,\Gamma^S)} S_\alpha\right) \arrow[d, "\mathrm{Ad}(\Phi_\alpha \otimes \eta^S_\alpha)"] \\
U_\alpha \times \mathcal{C}\ell(r,s) \arrow[r, "\mathrm{Id}_\alpha \times \delta"] & U_\alpha \times \mathrm{End}_{\mathbb{R}}(\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N, \mathbb{R}^N) \otimes_{\mathcal{C}(\mathbb{R}^N,\gamma)} \mathbb{R}^N).
\end{tikzcd}$$

□

Proposition 3.19. *The real vector bundle $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ defined as above satisfies the following* ***universal property****:*

1. *$\pi \circ (\rho_L \otimes \mathrm{Id}_S) = \pi \circ (\mathrm{Id}_L \otimes \rho_S)$, where $\rho_L : L \otimes \mathcal{C}(S, \Gamma^S) \to L$ and $\rho_S : \mathcal{C}(S, \Gamma^S) \otimes S \to S$ denote the action maps defined by composition from the right and composition from the left, respectively.*

2. *For any other vector bundle X and bundle morphism $p : L\otimes_{\mathbb{R}} S \to X$ such that $p\circ(\rho_L\otimes \mathrm{Id}_S) = p\circ(\mathrm{Id}_L \otimes\rho_S)$ there exists a unique linear map $\varphi : L\otimes_{\mathcal{C}(S,\Gamma^S)} S \to X$ such that $\varphi\circ\pi = p$, as in the following diagram:*

$$\begin{tikzcd} L \otimes_{\mathbb{R}} \mathcal{C}(S,\Gamma^S) \otimes_{\mathbb{R}} S \arrow[r, shift left, "\rho_L\otimes \mathrm{Id}_S"] \arrow[r, shift right, "\mathrm{Id}_S \otimes\rho_S"'] & L \otimes_{\mathbb{R}} S \arrow[r, "\pi"] \arrow[dr, "p"'] & L \otimes_{\mathcal{C}(S,\Gamma^S)} S \arrow[d, dotted, "\exists!\varphi"] \\ & & X \end{tikzcd}$$

Proof. The proof is the same as the proof of Proposition 3.15, *mutatis mutandis.* The difference is that in this proposition we deal with vector bundles while we deal with modules in Proposition 3.15. □

3.3.3 Classification Theorem

In the theorem below we bring together all the facts that we have established so far to answer the question of how two irreducible Clifford module bundles of a type are related. This is the first main result of this thesis.

Theorem 3.20. *Let $S \longrightarrow M$ and $\Sigma \longrightarrow M$ be two $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$. Let $L = \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \longrightarrow M$ be the bundle of all $\mathcal{C}\ell(E)$-equivariant homomorphisms between the module bundles S and Σ. Then $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ and $\Sigma \longrightarrow M$ are isomorphic as $\mathcal{C}\ell(E)$-module bundles.*

Proof. From Lemma 3.13, we know that the bundle morphism $\Upsilon : L \otimes_{\mathbb{R}} S \longrightarrow \Sigma$ which maps $(l \otimes s \longmapsto l(s))$ is $\mathcal{C}\ell(E)$-equivariant and surjective. Lemma 3.17 shows that fibrewise Υ_p descends to a $\mathcal{C}\ell(E)_p$-equivariant isomorphism $\Upsilon_p : (L \otimes_{\mathcal{C}(S,\Gamma^S)} S)_p \longrightarrow \Sigma_p$. Thus we have established that $\pi : L \otimes_{\mathcal{C}(S,\Gamma^S)} S \longrightarrow M$ and $\Sigma \longrightarrow M$ are isomorphic as real $\mathcal{C}\ell(E)$-module bundles. □

We can now establish the below immediate consequences of the theorem. The corollary below describes how the choice of S and Σ affects the isomorphism $L\otimes_{\mathcal{C}(S,\Gamma^S)} S \cong \Sigma$.

Corollary 3.21. *Suppose $S \longrightarrow M$ and $S' \longrightarrow M$ are $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ which are not necessarily isomorphic. Let $\Sigma \longrightarrow M$ be another $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$. Then*

$$\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \otimes_{\mathcal{C}(S,\Gamma^S)} S \cong \mathrm{Hom}_{\mathcal{C}\ell(E)}(S',\Sigma) \otimes_{\mathcal{C}(S',\Gamma^{S'})} S' \cong \Sigma.$$

The corollary below describes how the bundle $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma)\otimes_{\mathcal{C}(S,\Gamma^S)} S$ depends on the choice of Σ.

Corollary 3.22. *Let $S \longrightarrow M$ be any $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$ and let $\Sigma \longrightarrow M$ and $\Sigma' \longrightarrow M$ be two $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ which are not isomorphic. Then $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma)\otimes_{\mathcal{C}(S,\Gamma^S)} S \ncong \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma')\otimes_{\mathcal{C}(S,\Gamma^S)} S$ and in particular $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) \ncong \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma')$ as bundles of right-$\mathcal{C}(S,\Gamma^S)$-modules.*

We define below a right regular Schur module bundle.

Definition 3.23. Let $S \longrightarrow M$ be a $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$ where γ is an irreducible representation of $\mathcal{C}\ell(r,s)$ on $\mathbb{R}^N$. Then a real algebra bundle $V \longrightarrow M$ with typical fibre $\mathcal{C}(\mathbb{R}^N,\gamma)$ is said to be a **right $\mathcal{C}(S,\Gamma^S)$-module bundle of the 'regular' type** if and only if $V \longrightarrow M$ is a right module bundle over the algebra bundle $\mathcal{C}(S,\Gamma^S) \longrightarrow M$ with the action of $\mathcal{C}(\mathbb{R}^N,\gamma)$ on $\mathcal{C}(\mathbb{R}^N,\gamma)$ given by the right regular representation $\mathrm{reg}:\mathcal{C}(\mathbb{R}^N,\gamma) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathcal{C}(\mathbb{R}^N,\gamma))$ where $\mathrm{reg}:\lambda \longmapsto (f \mapsto f\cdot\lambda)$.

Remark 3.24. Observe that $L \longrightarrow M$ as constructed in Lemma 3.9 is an example of such a right $\mathcal{C}(S,\Gamma^S)$-module bundle because $\mathrm{Hom}_{\mathcal{C}\ell(r,s)}(\mathbb{R}^N,\mathbb{R}^N) \cong \mathcal{C}(\mathbb{R}^N,\gamma)$ as right $\mathcal{C}(\mathbb{R}^N,\gamma)$-module bundles. Furthermore, note that the rank of such right $\mathcal{C}(S,\Gamma^S)$-module bundles of the regular type is the same as the rank of the vector bundle $\mathcal{C}(S,\Gamma^S) \longrightarrow M$.

Lemma 3.25. *Let $S \longrightarrow M$ be a fixed $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$. Then, $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,-)$ defines an injective mapping from the set of isomorphism classes of $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ to the set of isomorphism classes of right $\mathcal{C}(S,\Gamma^S)$-module bundles of the regular type.*

Proof. If Σ and Σ' are two isomorphic $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$ and $\phi:\Sigma \longrightarrow \Sigma'$ is an isomorphism between them, then

$$\begin{aligned}\widehat{\phi}:\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,\Sigma')\\ f &\longmapsto \phi\circ f.\end{aligned}$$

Injectivity follows from Corollary 3.22. $\square$

Lemma 3.26. *Let $S \longrightarrow M$ be a fixed $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$. Then, $-\otimes_{\mathcal{C}(S,\Gamma^S)} S$ defines a mapping from the set of isomorphism classes of right $\mathcal{C}(S,\Gamma^S)$-module bundles of the regular type to the set of isomorphism classes of irreducible $\mathcal{C}\ell(E)$-module bundles of type $[\gamma]$.*

Proof. If V and V' are two isomorphic right $\mathcal{C}(S,\Gamma^S)$-module bundles of the regular type and $\psi : V \longrightarrow V'$ is an isomorphism between them, then

$$\begin{aligned} \widehat{\psi} : V \otimes_{\mathcal{C}(S,\Gamma^S)} S &\xrightarrow{\cong} V' \otimes_{\mathcal{C}(S,\Gamma^S)} S \\ v \otimes s &\longmapsto \psi(v) \otimes s. \end{aligned}$$

□

In the theorem below we establish a classification of irreducible Clifford module bundles. This is the second main result of this thesis.

Theorem 3.27. *Let $S \longrightarrow M$ be a fixed irreducible $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$. Then there exists a bijection between the following sets,*

$$A := \{\textit{isomorphism classes of irreducible } \mathcal{C}\ell(E)\textit{-module bundles of type } [\gamma]\}$$

$$\updownarrow$$

$$B := \{\textit{isomorphism classes of right } \mathcal{C}(S,\Gamma^S)\textit{-module bundles of regular type }\}.$$

Proof. From Theorem 3.20, it follows that $(-\otimes_{\mathcal{C}(S,\Gamma^S)} S) \circ (\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,-)) = \mathrm{Id}_A$. To show that $(\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,-)) \circ (-\otimes_{\mathcal{C}(S,\Gamma^S)} S) = \mathrm{Id}_B$, take a right $\mathcal{C}(S,\Gamma^S)$-module bundle of the regular type $V \longrightarrow M$. First we observe that

$$\begin{aligned} V \otimes_{\mathcal{C}(S,\Gamma^S)} \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,S) &\cong \mathrm{Hom}_{\mathcal{C}\ell(E)}(S, V \otimes_{\mathcal{C}(S,\Gamma^S)} S) \\ \sum_{i,j} e_i \otimes f_j &\longmapsto (s \longmapsto \sum_{i,j} e_i \otimes f_j(s)) \end{aligned}$$

for every $e_i \in V$, $f_i \in \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,S)$ and $s \in S$, is an isomorphism of right $\mathcal{C}(S,\Gamma^S)$-module bundles where the action of $\mathcal{C}(S,\Gamma^S)$ on $V \otimes_{\mathcal{C}(S,\Gamma^S)} \mathrm{Hom}_{\mathcal{C}\ell(E)}(S,S)$ is given by $(\lambda, e \otimes f) \longmapsto e \otimes f \circ \lambda$. Since by definition $\mathrm{Hom}_{\mathcal{C}\ell(E)}(S,S) = \mathcal{C}(S,\Gamma^S)$, we have

$$\mathrm{Hom}_{\mathcal{C}\ell(E)}(S, V \otimes_{\mathcal{C}(S,\Gamma^S)} S) \cong V \otimes_{\mathcal{C}(S,\Gamma^S)} \mathcal{C}(S,\Gamma^S).$$

Furthermore note that

$$V \otimes_{\mathcal{C}(S,\Gamma^S)} \mathcal{C}(S,\Gamma) \cong V$$

$$v \otimes f \mapsto v \cdot f$$

is again an isomorphism of right $\mathcal{C}(S, \Gamma^S)$-module bundles of the regular type. With this we have shown that $(\mathrm{Hom}_{\mathcal{C}\ell(E)}(S, -)) \circ (- \otimes_{\mathcal{C}(S,\Gamma^S)} S) = \mathrm{Id}_B$ and that there is a bijection between the sets A and B as stated in the theorem. $\square$

4

Part II: Dirac Generating Operators

Sometimes I am convinced that
triangle is another name for
stupidity, that eight times eight is
madness or a dog.

Julio Cortázar, Hopscotch

4.1 Dirac generating operators of Courant algebroids

In this part, we will apply the classification of Clifford module bundles from Part I to extend the result of Alekseev and Xu [AX] on the existence of local Dirac generating operators of Courant algebroids in neutral signature (r, r), to arbitrary signatures. We begin with the definition of Dirac generating operators of Courant algebroids (DGOs) where we allow the Courant algebroids to have arbitrary signatures. Recall from Definition 2.40 that a Courant algebroid consists of a vector bundle $E \to M$ with a scalar product $\langle \cdot, \cdot \rangle$, a Dorfman bracket $[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ and an anchor map $\pi : E \to TM$.

To simplify our notation henceforth, whenever we talk of a $\mathcal{C}\ell(E)$-module bundle (S, Γ^S) of type $[\gamma]$, we will abuse the notation and use γ to indicate both γ and Γ^S.

Definition 4.1. Given a Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ and an irreducible $\mathcal{C}\ell(E)$-module bundle (S, γ) a **Dirac generating operator (DGO) of a Courant algebroid** E is a first-order differential operator $\not{d} : \Gamma(S) \longrightarrow \Gamma(S)$ such that for all $e, e_1, e_2 \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$

1. $[\{\not{d}, \gamma(e_1)\}, \gamma(e_2)] = \gamma([e_1, e_2])$,
2. $\{[\not{d}, f], \gamma(e)\} = \pi(e)(f)$,
3. $\not{d}^2 \in \mathcal{C}^\infty(M)$,

where $[\cdot,\cdot]$ denotes the commutator on $\mathrm{End}(\Gamma(S))$ or the Dorfman bracket on E and $\{\cdot,\cdot\}$ denotes the anti-commutator on $\mathrm{End}(\Gamma(S))$.

Remark 4.2. We would now like to highlight some subtleties regarding the definitions of Dirac generating operators that exist in literature. In the unpublished manuscript of Alekseev and Xu [AX], they define a so-called 'generating operator' for a pseudo-Riemannian vector bundle $(E, \langle\cdot,\cdot\rangle)$ as a first-order odd operator D on the sections of an irreducible module bundle (S, γ) over the Clifford algebra bundle $\mathcal{C}\ell(E)$ associated to $(E, \langle\cdot,\cdot\rangle)$ satisfying the following properties:

1. For any function $f \in \mathcal{C}^\infty(M)$, $[D, f] \in \mathrm{im}(\gamma)$.
2. For any two sections $e_1, e_2 \in \Gamma(E)$, $[[D, e_1], e_2] \in \mathrm{im}(\gamma)$.
3. $D^2 \in \mathcal{C}^\infty(M)$.

They also show that every generating operator on a pseudo-Riemannian vector bundle $(E, \langle\cdot,\cdot\rangle)$ of neutral signature induces a Dorfman bracket and an anchor map on E and thus induces a Courant algebroid structure on E. Observe that generating operators only induce a Courant algebroid structure on E when γ is injective, i.e., only when $\mathcal{C}\ell(E)$ is a bundle of simple algebras.

Assume now that a Courant algebroid E is already given. Consider a generating operator associated to the pseudo-Riemannian vector bundle underlying the Courant algebroid E such that the Dorfman bracket and anchor map induced by the operator coincides with the Dorfman bracket and anchor map of E. Such a generating operator is called (Dirac) generating operator of a Courant algebroid. In a large majority of literature starting from Alekseev and Xu in [AX] followed by [LWX, CS, CD1, GF, CS, CCLX], Dirac generating operators are defined on Courant algebroids of neutral signature. This is either explicitly stated or implicitly assumed. In some places, such as in [CS, LWX] Courant algebroids are not assumed to be of neutral signature but the DGOs are defined over sections of only those $\mathcal{C}\ell(E)$-module bundles that admit a $\mathbb{Z}_2$-grading that is compatible with the canonical $\mathbb{Z}_2$-grading of the Clifford algebra bundle. Furthermore, it is also always assumed to be an odd operator and the definition is stated in terms of the super-commutator on $\mathrm{End}(\Gamma(S))$. From Proposition 2.8 from the theory of Clifford modules we know that a natural $\mathbb{Z}_2$-grading on Clifford modules compatible with the canonical $\mathbb{Z}_2$-grading of the Clifford algebra exists when the volume element of $\mathcal{C}\ell(E)$ satisfies $\omega^2 = 1$, which, while true for some signatures of E, does not hold in all signatures, see Equation (2.11). Since in this thesis we consider Courant algebroids of arbitrary signature, we drop the assumption that Dirac generating operators are odd operators and state its properties in terms of the anti-commutator and commutator instead of a super-commutator.

Now we look at the classical examples of DGOs.

Example 4.3. Consider the generalized tangent bundle $\mathbb{T}M$ from Example 2.44 on a smooth manifold M. The sections $\Gamma(\mathbb{T}M)$ of $\mathbb{T}M$ act on the space of differential forms $\Gamma(S) = \Omega^\bullet(M) = \Gamma(\bigwedge^\bullet T^*M)$ as follows:

$$(X + \xi) \cdot \mu = \iota_X \mu + \xi \wedge \mu$$

where $X \in \Gamma(TM)$, $\xi \in \Gamma(T^*M)$ and $\mu \in \Omega^\bullet(M)$. A straightforward computation shows us that this action is also a Clifford action and therefore it turns $\bigwedge^\bullet T^*M$ into a module bundle over the Clifford algebra bundle $\mathcal{C}\ell(\mathbb{T}M)$. The de Rham differential d on the co-chain complex of differential forms on M defines a DGO over $\mathbb{T}M$ as can be verified from a straightforward computation.

Similarly, on the H-twisted generalized tangent bundle $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$ from Example 2.45, the following operator defines a DGO:

$$d_H = d + H \wedge \cdot$$

In both of the above cases the DGOs square to zero [AX].

4.2 The Existence of Local DGOs

The main theorem of this section is stated below and we will prove it in several steps.

Theorem 4.4. *Let E be a regular Courant algebroid with scalar product of arbitrary signature. Every $\mathcal{C}\ell(E)$-module bundle S of type γ, where γ is an irreducible representation, admits locally a Dirac generating operator.*

Henceforth DGO will always mean Dirac generating operator of a Courant algebroid E, whether stated explicitly or not. The signature of the scalar product on E will be denoted by (r, s). Recall that a regular Courant algebroid is a Courant algebroid whose anchor map has constant rank.

While the basic strategy for the proof in arbitrary signatures is similar to the proof in the neutral signature case as in [CD1], we need to generalise this proof in certain key places and use results about Clifford module bundles that we have obtained in Part I. To prove the existence of local Dirac generating operators we first show that, for each pair of generalised connection D on a regular Courant algebroid E and an irreducible module bundle over (S, γ) over $\mathcal{C}\ell(E)$, there exists an E-connection on (S, γ) that is compatible with the generalised connection D. By compatibility we mean that the E-connection on (S, γ) respects the Clifford action. In proving the existence of such an E-connection on (S, γ) we use in critical ways the results that we have obtained in Part I. Specifically, we use our result on the relation between two irreducible $\mathcal{C}\ell(E)$-module bundles of the same type to induce an E-connection on S from the given generalised connection D on E.

We next use this E-connection on (S, γ) to define a Dirac type operator $\not{D}$. We then propose an ansatz for a DGO, namely, $\not{d} = \not{D} + \gamma_T$, where T denotes the torsion of the generalised connection D, and show that it satisfies the first two properties of a Dirac generating operator of the Courant algebroid E. At this point the generalised connection D is arbitrary and, in particular, globally defined, in virtue of which $\not{d}$ is globally defined. To show that the ansatz for $\not{d}$ also satisfies the third property, however, we must choose a specific local generalised connection, namely, the generalised connection D induced from a parallel connection with respect to a certain structure-preserving local frame for E. This means that $\not{d}$ can then be defined only locally. By structure-preserving we mean that this local frame is constructed so that it preserves the structure of the regular Courant algebroid E. Here the signature of the Courant algebroid plays an important role. We next compute $\not{d}^2$ with respect to this local frame and verify that it satisfies the third property of a DGO, namely $\not{d}^2 \in \mathcal{C}^\infty(M)$.

In the lemma below we show that upon fixing a generalised connection there exists a D-compatible E-connection D^S on S.

Lemma 4.5. *Let E be a regular Courant algebroid with arbitrary signature and D a generalised connection on E. If (S,γ) is an irreducible module bundle over $\mathcal{C}\ell(E)$, then there exists an E-connection D^S on S which is compatible with D, i.e.,*

$$D^S_e(\gamma_a(s)) = \gamma_{D_e(a)}(s) + \gamma_a(D^S_e(s)) \tag{4.1}$$

for all $e \in \Gamma(E)$, $a \in \Gamma(\mathcal{C}\ell(E))$, $s \in \Gamma(S)$.

Proof. Without loss of generality, we fix $\{U_\alpha \mid \alpha \in \Lambda\}$ to be an open cover of M such that all bundles defined over M appearing in this proof, be they vector bundles, algebra bundles or module bundles, admit a family of local trivializations over it.

Since the bundle $(E, \langle\cdot,\cdot\rangle)$ locally always admits a spin structure, we can associate a spinor bundle Σ over every open set U_α. For example, Σ can be chosen to be the trivial $\mathcal{C}\ell(E)$-module bundle of type $[\gamma]$ over U_α. We show below how the E-connection D on E induces an E-connection D^Σ on Σ. As we will show, the connection form of this D^Σ with respect to the local trivialisation of the spin structure is one half of the connection form of D with respect to the corresponding local orthonormal frame of E. Both connection forms can be considered as sections of $E^* \otimes \mathfrak{so}(E)$ after the identification $\mathfrak{spin}(E) \cong \mathfrak{so}(E)$ via the adjoint representation $\mathrm{ad} : \mathfrak{spin}(E) \longrightarrow \mathfrak{so}(E)$, $\mathrm{ad}_u(v) = uv - vu$. For the following calculations, let (e_i) be an orthonormal frame of $E|_U$ and (σ_α) a frame of Σ such that $e_i\sigma_\alpha = \sum_\beta C^\beta_{i\alpha}\sigma_\beta$ where $C^\beta_{i\alpha}$ are constants. We know that such a frame exists because of Corollary 3.5.

Since D is a generalised connection on E, due to the compatibility of D with $\langle\cdot,\cdot\rangle$, we have the following expression:

$$D_v(e_k) = -\epsilon_k \sum_j \omega_{jk}(v) e_j$$

for $v \in \Gamma(E|_U)$ and $\epsilon_j = \langle e_j, e_j\rangle \in \{\pm 1\}$ and (ω_{jk}) is the skew symmetric matrix of 1-forms. When $e^*_j(e_i) = \delta_{ij} = \epsilon_j \langle e_i, e_j\rangle$ we have

$$\sum_{(j,p)} \omega_{pj}(v)(\epsilon_p e^*_p \otimes e_j - \epsilon_j e^*_j \otimes e_p)(e_k) = \sum_{(j,p)} \omega_{pj}(v)(\epsilon_p \delta_{pk} e_j - \epsilon_j \delta_{jk} e_p)$$

$$
\begin{aligned}
&= \sum_j \omega_{kj}(v)(\epsilon_k e_j) + \sum_p \omega_{kp}(v)(\epsilon_k e_p) \\
&= -2\epsilon_k \sum_j \omega_{jk}(v) e_j.
\end{aligned}
$$

Furthermore, since

$$
\sum_{(j,p)} \omega_{pj}(v)(\epsilon_p e_p^* \otimes e_j - \epsilon_j e_j^* \otimes e_p)(e_k) = 2\sum_{j<p} \omega_{pj}(v)(\epsilon_p e_p^* \otimes e_j - \epsilon_j e_j^* \otimes e_p)(e_k),
$$

we have

$$
\begin{aligned}
D_v(e_k) &= -\epsilon_k \sum_j \omega_{jk}(v) e_j \\
&= \sum_{j<p} \omega_{pj}(v)(\epsilon_p e_p^* \otimes e_j - \epsilon_j e_j^* \otimes e_p)(e_k) \\
&= \sum_{j<p} \omega_{pj}(v)(e_p \wedge e_j)(e_k).
\end{aligned}
$$

The element $\frac{1}{2} e_p e_j \in \mathfrak{spin}(E) \subset \mathcal{C}\ell(E)$ when $i \neq j$ acts under the adjoint representation as $\epsilon_p e_p^* \otimes e_j - \epsilon_j e_j^* \otimes e_p \in \mathfrak{so}(E)$ which corresponds to $e_p \wedge e_j \in \wedge^2 E$, see (2.27) for more details. The following computations show that $D_e^\Sigma(\sigma_\alpha) := \frac{1}{2}\sum_{j<p} \omega_{jp}(e) e_j e_p \sigma_\alpha$ is compatible with $D|_U$ for basis elements.

$$
\begin{aligned}
D_v(e_k) \cdot \sigma_m + e_k \cdot D_v^\Sigma(\sigma_m) &= \sum_{i<j} \omega_{ji}(v)(\epsilon_j e_j^* \otimes e_i - \epsilon_i e_i^* \otimes e_j)(e_k)\sigma_m \\
&\qquad + \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_k e_i e_j \sigma_m \\
&= \sum_{i<j} \omega_{ij}(v)(\epsilon_i e_i^* \otimes e_j - \epsilon_j e_j^* \otimes e_i)(e_k)\sigma_m \\
&\qquad + \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_k e_i e_j \sigma_m \\
&= \sum_{i<j} \omega_{ij}(v)(\epsilon_i \delta_{ik} e_j - \epsilon_j \delta_{jk} e_i)\sigma_m + \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_k e_i e_j \sigma_m \\
&= \frac{1}{2}\sum_{i<j} \omega_{ij}(v)(e_i(-2\epsilon_j \delta_{jk} - e_k e_j) - (-2\epsilon_i \delta_{ik} - e_i e_k) e_j)\sigma_m \\
&\qquad + \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_k e_i e_j \sigma_m \\
&= \frac{1}{2}\sum_{i<j} \omega_{ij}(v)(e_i e_j e_k - e_k e_i e_j)\sigma_m + \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_k e_i e_j \sigma_m \\
&= \frac{1}{2}\sum_{i<j} \omega_{ij}(v) e_i e_j e_k \sigma_m
\end{aligned}
$$

and

$$\begin{aligned} D_v^{\Sigma}(e_k\sigma_m) &= D_v^{\Sigma}(\sum_{\beta} C_{km}^{\beta}\sigma_{\beta}) \\ &= \sum_{\beta} C_{km}^{\beta} D_v^{\Sigma}(\sigma_{\beta}) \\ &= \sum_{\beta} C_{km}^{\beta}\left(\frac{1}{2}\sum_{i<j}\omega_{ij}(v)e_ie_j\sigma_{\beta}\right) \\ &= \frac{1}{2}\sum_{i<j}\omega_{ij}(v)e_ie_je_k\sigma_m. \end{aligned}$$

For any general local section, we can express them in terms of the local frame and apply Leibniz rule to show the same compatibility.

Now consider the bundle $L := \mathrm{Hom}_{\mathcal{C}\ell(E)}(\Sigma, S) \longrightarrow M$ that we know to exist from Lemma 3.9 and consider the family of local trivialisations Φ_α of L constructed there. Let (l_i) be the local frame of L that corresponds to the local trivialisation Φ_α. Let (λ_j) be the local frame of the Schur algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$ that corresponds to the local trivialization $\mathrm{Ad}(\eta_\alpha^{\Sigma})$ restricted to $\mathcal{C}(\Sigma_\alpha, \Gamma^{\Sigma}) \subset \mathrm{Hom}_{\mathcal{C}\ell(E)}(\Sigma_\alpha, \Sigma_\alpha)$. Here η_α^{Σ} is the $\mathcal{C}\ell(E)$-equivariant local trivialization of Σ underlying the local frame (σ_k).

From Lemma 3.10 we know that L is a right module bundle over the Schur algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$. Therefore, from Corollary 3.5 it follows that $l_i \cdot \lambda_j = \sum_k N_{ij}^k l_k$ where N_{ij}^k are constants. Likewise, from Lemma 3.11 we know that Σ is a left module bundle over the Schur algebra bundle $\mathcal{C}(\Sigma, \Gamma^{\Sigma})$ so again from Corollary 3.5 we know that $\lambda_r \cdot \sigma_k = \sum_q B_{rk}^q \sigma_q$ for some constants B_{rk}^q.

Consider the trivial connection on L with respect to the local trivialisation Φ_α and let D^L denote the induced E-connection. Then by construction we have $D_v^L(l_i) = 0$ for $v \in \Gamma(E)$ and for all i. Let $D^{L\otimes_{\mathbb{R}}\Sigma}$ denote the local connection over the tensor bundle $L \otimes_{\mathbb{R}} \Sigma \longrightarrow M$ induced by the connections D^L and D^{Σ}. With the following computation we show that this local connection descends to a local connection on the Clifford module bundle $S \cong L \otimes_{\mathcal{C}(\Sigma,\Gamma)} \Sigma \longrightarrow M$, i.e. $D_v^{L\otimes_{\mathbb{R}}\Sigma}$ for all $v \in \Gamma(E)$ leaves invariant the space of sections of the sub-bundle $\widetilde{\mathfrak{U}} := \cup_{p\in M}\mathfrak{U}_p \longrightarrow M$, where $\mathfrak{U}_p := \mathrm{Span}_{\mathbb{R}}\{-(l\cdot\lambda)\otimes\sigma + l\otimes(\lambda\cdot\sigma) \mid \ l \in L_p, \lambda \in \mathcal{C}(\Sigma,\Gamma)_p \text{ and } \sigma \in \Sigma_p\}$ is the subspace defined in Lemma 3.18. For $v \in \Gamma(E)$,

$$\begin{aligned} D_v^{L\otimes_{\mathbb{R}}\Sigma}(-(l_i\cdot\lambda_k)\otimes\sigma_j + l_i\otimes(\lambda_k\cdot\sigma_j)) = D_v^{L\otimes_{\mathbb{R}}\Sigma}&(-(l_i\cdot\lambda_k)\otimes\sigma_j) \\ &+ D_v^{L\otimes_{\mathbb{R}}\Sigma}(l_i\otimes(\lambda_k\cdot\sigma_j)) \end{aligned}$$

$$\begin{aligned}
&= D_v^L(-(l_i \cdot \lambda_k)) \otimes \sigma_j + (-l_i \cdot \lambda_k) \otimes D_v^\Sigma(\sigma_j) \\
&\quad + D_v^L(l_i) \otimes \lambda_k \cdot \sigma_j + l_i \otimes D_v^\Sigma(\lambda_k \cdot \sigma_j) \\
&= -(l_i \cdot \lambda_k) \otimes D_v^\Sigma(\sigma_j) + l_i \otimes D_v^\Sigma(\lambda_k \cdot \sigma_j) \\
&= -(l_i \cdot \lambda_k) \otimes D_v^\Sigma(\sigma_j) + l_i \otimes \lambda_k D_v^\Sigma(\sigma_j) \\
&\in \Gamma(\mathfrak{U}),
\end{aligned}$$

where in the third step we use $l_i \cdot \lambda_j = \sum_k N_{ij}^k l_k$ and $D_v^L(l_i) = 0$. The last step follows due to the fact that $\lambda_r \cdot \sigma_k = \sum_q B_{rk}^q \sigma_q$ for some constants B_{rk}^q and because $\lambda_k \in \Gamma(\mathcal{C}(\Sigma, \Gamma^\Sigma))$ they commute with $e_i \in \Gamma(E)$ as can be seen from the calculation below.

$$\begin{aligned}
D_v^\Sigma(\lambda_k \cdot \sigma_j) &= D_v^\Sigma(\sum_q B_{kj}^q \sigma_q) \\
&= \sum_q B_{kj}^q \left(\frac{1}{2} \sum_{l<m} \omega_{lm}(v) e_l e_m \sigma_q \right) \\
&= \frac{1}{2} \sum_{l<m} \omega_{lm}(v) e_l e_m \left(\sum_q B_{kj}^q \sigma_q \right) \\
&= \frac{1}{2} \sum_{l<m} \omega_{lm}(v) e_l e_m \lambda_k \cdot \sigma_j \\
&= \lambda_k \frac{1}{2} \sum_{l<m} \omega_{lm}(v) e_l e_m \sigma_j \\
&= \lambda_k D_v^\Sigma(\sigma_j).
\end{aligned}$$

Furthermore, we can express any local section of $\mathfrak{U}$ over an open set U_α as a $\mathcal{C}^\infty(U)$-linear combination of elements $-(l_i \cdot \lambda_k) \otimes \sigma_j + l_i \otimes (\lambda_k \cdot \sigma_j)$ and apply the Leibniz rule to show that $D_v^{L \otimes_\mathbb{R} \Sigma}(s)$ is again in $\Gamma(\mathfrak{U})$ for any $v \in \Gamma(E)$. Therefore locally $D^{L \otimes_\mathbb{R} \Sigma}$ descends to an E-connection $D^{L \otimes_{\mathcal{C}(\Sigma,\Gamma)} \Sigma}$ on $L \otimes_{\mathcal{C}(\Sigma,\Gamma)} \Sigma$ and therefore an E-connection $D^{S|_{U_\alpha}}$ on $S|_{U_\alpha} \cong L \otimes_{\mathcal{C}(\Sigma,\Gamma)} \Sigma|_{U_\alpha}$. Now by gluing the E-connections $D^{S|_{U_\alpha}}$ with a partition of unity subordinate to the cover (U_α) we get an E-connection D^S. □

We recall the notion of 'Dirac operator' in the context of generalised geometry given an E-connection D^S on a $\mathcal{C}\ell(E)$ module bundle (S, Γ^S).

Definition 4.6. A **Dirac operator** is a first-order differential operator on S such that

$$\not{D} = -\frac{1}{2} \sum_{i=1}^{r+s} \tilde{e}_i D_{e_i}^S,$$

where (e_i) is any local frame of E and $\tilde{e}_i$ is the metrically dual frame, $\langle e_i, \tilde{e}_j\rangle = \delta_{ij}$.

Note that the Dirac operator is independent of the chosen basis. Next in the lemma below, we propose an ansatz for a Dirac generating operator and show that it satisfies the first two properties of a DGO:

Lemma 4.7. *Let $T \in \Gamma(\wedge^3 E^*) \cong \Gamma(\wedge^3 E) \subset \Gamma(\mathcal{C}\ell(E))$ denote the torsion of D where D is any generalised connection and let D^S be a D compatible E-connection. Then the operator*

$$\not{d} = \not{D} + \frac{1}{4}\gamma_T \tag{4.2}$$

satisfies conditions (1) *and* (2) *from Definition 4.1.*

Proof. We will first show that condition (2) in Definition 4.1 is satisfied. Since $[\gamma_T, f] = 0$ for $f \in \mathcal{C}^\infty(M)$, we have $\{[\not{d}, f], \gamma_v\} = \{[\not{D}, f], \gamma_v\}$ for $v \in \Gamma(E)$. With $[\not{D}, f] = \frac{1}{2}\sum_{i=1}^{r+s} \pi(e_i)(f)\gamma_{\tilde{e}_i}$ it follows that $\{[\not{D}, f], \gamma_v\} = \pi(v)(f)$ and therefore condition (2) is satisfied.

To show that condition (1) in Definition 4.1 is satisfied, we need to compute $[\{\not{D}, \gamma_v\}, \gamma_w]$ for $v, w \in \Gamma(E)$. To do this, we first compute

$$\begin{aligned}
\{\not{D}, \gamma_v\} &= -\frac{1}{2}\sum_i (\gamma_{\tilde{e}_i} D^S_{e_i}\gamma_v + \gamma_v\gamma_{\tilde{e}_i} D^S_{e_i}) \\
&= -\frac{1}{2}\sum_i (\gamma_{\tilde{e}_i} D^S_{e_i}\gamma_v - \gamma_{\tilde{e}_i}\gamma_v D^S_{e_i} - 2\gamma_{\langle \tilde{e}_i, v\rangle} D^S_{e_i}) \\
&= -\frac{1}{2}\sum_i \gamma_{\tilde{e}_i}(D^S_{e_i}\gamma_v - \gamma_v D^S_{e_i}) + D^S_v \\
&= -\frac{1}{2}\sum_i \gamma_{\tilde{e}_i}(\gamma_{D_{e_i}v} + \gamma_v D^S_{e_i} - \gamma_v D^S_{e_i}) + D^S_v \\
&= -\frac{1}{2}\sum_i \gamma_{\tilde{e}_i}\gamma_{D_{e_i}v} + D^S_v.
\end{aligned}$$

Now,

$$\begin{aligned}
[\{\not{D}, \gamma_v\}, \gamma_w] &= -\frac{1}{2}\sum_i [\gamma_{\tilde{e}_i}\gamma_{D_{e_i}v}, \gamma_w] + \gamma_{D_v w} \\
&= -\sum_i (\gamma_{\langle \tilde{e}_i, w\rangle}\gamma_{D_{e_i}v} - \gamma_{\langle D_{e_i}v, w\rangle}\gamma_{\tilde{e}_i}) + \gamma_{D_v w} \\
&= \sum_i \gamma_{\langle D_{e_i}v, w\rangle}\gamma_{\tilde{e}_i} + \gamma_{D_v w - D_w v}.
\end{aligned}$$

Using the definition of torsion of D we get

$$[\{\not{D}, \gamma_v\}, \gamma_w] = \gamma_{T(v,w)} + \gamma_{[v,w]}.$$

For any 3-form we can further show that

$$[\{\gamma_T, \gamma_v\}, \gamma_w] = -4\gamma_{T(v,w)}.$$

With this we have shown that

$$[\{\not{d}, \gamma_v\}, \gamma_w] = [\{\not{D} + \frac{1}{4}\gamma_T, \gamma_v\}, \gamma_w] = \gamma_{[v,w]}. \qquad \square$$

If we can show that $\not{d}$ as defined in the previous lemma also satisfies condition (3) of Definition 4.1, then we have shown the existence of a local Dirac generating operator on regular Courant algebroids. To analyse condition (3) of Definition 4.1, we use the Lemma 2.48 on the structure of a regular Courant algebroid. Observe that Lemma 2.48 implies that for a regular Courant algebroid E of signature (r, s) over any $U \subset M$ sufficiently small, the bundle $E|_U$ admits a frame

$$(p_1, \dots, p_{\min(r,s)}, q_1, \dots, q_{\min(r,s)}, t_1, \dots, t_{|r-s|}) \tag{4.3}$$

such that there exists an l, with $1 \le l \le \min(r, s)$, such that

$$\begin{aligned} \ker \pi^\perp &= \operatorname{span}_{\mathbb{R}}\{p_a | a \le l\} \\ \mathcal{F} &= \operatorname{span}_{\mathbb{R}}\{q_a | a \le l\} \\ \ker \pi &\supset \operatorname{span}_{\mathbb{R}}\{q_a | a \ge l+1\} \end{aligned}$$

and

$$\begin{aligned} \langle p_a, q_b \rangle &= \delta_{ab} \\ [\pi(q_a), \pi(q_b)] &= 0 \end{aligned}$$

for any $1 \le a, b \le \min(r, s)$,

$$\begin{aligned} \langle p_a, t_m \rangle &= 0 \\ \langle q_a, t_m \rangle &= 0 \end{aligned}$$

for all $1 \le m \le |r - s|$ and

$$\langle t_i, t_j \rangle = \epsilon \delta_{ij}$$

for $1 \leq i, j \leq |r-s|$, where $\epsilon = -1$ if $r < s$ and $\epsilon = +1$ if $s < r$.

We can obtain such a frame as follows: Property 2 of Proposition 2.41 states that the image of π is an involutive distribution and thus by the Frobenius theorem we can choose a basis q_a for $a = 1, \dots, l$ of $\mathcal{F}$ such that $[\pi(q_a), \pi(q_b)] = 0$ for any $1 \leq a, b \leq l$. Next, choose a basis of $(\ker \pi)^\perp$, p_a where $a = 1, \dots, l$ such that $\langle p_a, q_b \rangle = \delta_{ab}$ for any $1 \leq a, b \leq l$. Finally we choose a basis

$$\{p_{l+1}, \dots, p_{\min(r,s)}, q_{l+1}, \dots, q_{\min(r,s)}, t_1, \dots, t_{|r-s|}\} \subset \mathcal{G}$$

in such a way that the relations $\langle p_a, p_b \rangle = \langle q_a, q_b \rangle = 0$, $\langle p_a, q_b \rangle = \delta_{ab}$ hold for all $a, b \geq l+1$, $\langle t_i, t_j \rangle = \epsilon \delta_{ij}$, $\langle p_a, t_i \rangle = 0$ and $\langle q_a, t_i \rangle = 0$ for all a, b, i and where $\epsilon = -1$ if $r < s$ and $\epsilon = +1$ if $s < r$.

The following lemma will help us in proving that the operator in (4.2) will indeed satisfy the third property of Definition 4.1.

Lemma 4.8. *For any $\sigma \in \Gamma(\ker \pi)$, $\sum_a \pi(q_a)\langle \sigma, p_a \rangle + \sum_{i=1}^k \pi(t_i)\langle \sigma, t_i \rangle = 0$.*

Proof. Each term in $\sum_a \pi(q_a)\langle \sigma, p_a \rangle + \sum_{i=1}^k \pi(t_i)\langle \sigma, t_i \rangle$ vanishes: if $a \leq s$, then $p_a \in \Gamma(\ker \pi^\perp)$ and $\langle \sigma, p_a \rangle = 0$. If $a \geq s+1$, then $q_a \in \Gamma(\ker \pi)$ and $\pi(q_a) = 0$. Since $t_i \in \Gamma(\ker \pi)$, we have $\pi(t_i)\langle \sigma, t_i \rangle = 0$ for all i. $\square$

To show the existence of a DGO, we will choose a connection ∇ on $E|_U$ with respect to which the frame $((p_a)_a, (q_a)_a, (t_m)_m)$, as constructed above, is parallel. Such a ∇ is flat, preserves the scalar product $\langle \cdot, \cdot \rangle$ of E and $S|_U$ admits a flat connection ∇^S compatible with ∇ due to Lemma 4.5. Then ∇ induces a generalised connection D on E and ∇^S an E-connection D^S on S which is compatible with D. The next lemma will show us that the operator 4.2 satisfies Property 3 of a Dirac generating operator.

Lemma 4.9. *The operator $\not{d} = \not{D} + \frac{1}{4}\gamma_T$ constructed using D and D^S satisfies $\not{d}^2 \in \mathcal{C}^\infty(U)$.*

Proof. The Dirac operator $\not{D}$ has the expression:

$$\not{D} = -\frac{1}{2}\left(\sum_a p_a D^S_{q_a} + \sum_a q_a D^S_{p_a} + \epsilon \sum_{i=1}^k t_i D^S_{t_i} \right) = -\frac{1}{2}\left(\sum_a p_a D^S_{q_a} \right) \tag{4.4}$$

because $p_a, t_i \in \Gamma(\ker \pi)$. Observe that $((q_a)_a, (p_a)_a, \epsilon(t_m)_m)$ is the frame that is metrically dual to the frame $((p_a)_a, (q_a)_a, (t_m)_m)$, where $\epsilon = -1$ if $r < s$ and $\epsilon = +1$ if $s < r$.

Since $\not{d}^2 = (\not{D})^2 + \frac{1}{4}\{\not{D}, \gamma_T\} + \frac{1}{16}\gamma_T^2$, we compute each of the terms on the right hand side starting with the square of $\not{D}$ below:

$$\begin{aligned}
(\not{D})^2 &= \frac{1}{4}\left(\sum_a p_a D^S_{q_a}\right)^2 \\
&= \frac{1}{4}\sum_{a,b} p_a p_b D^S_{q_a} D^S_{q_b} \\
&= -\frac{1}{2}\sum_{a,b} \langle p_a, p_b\rangle D^S_{q_a} D^S_{q_b} \\
&= 0.
\end{aligned}$$

This follows from the flatness of ∇^S, that is, $\nabla^S p_a = 0$ and $[\pi(q_a), \pi(q_b)] = 0$ for any a, b. The last step follows from the fact that $(p_a)_a$ span an isotropic subspace.

Next we compute $\{\not{D}, \gamma_T\}$. The torsion T of D is written as $T = \frac{1}{6}\sum T^{ijk} e_{ijk} \in \mathcal{C}\ell(E)$ where (e_i) is a D-parallel orthonormal frame and $e_{ijk} := e_i e_j e_k$. The coefficients T^{ijk} are given by

$$T^{ijk} = T(\tilde{e_i}, \tilde{e_j}, \tilde{e_k}) = \epsilon_i\epsilon_j\epsilon_k T(e_i, e_j, e_k) = -\epsilon_i\epsilon_j\epsilon_k\langle [e_i, e_j], e_k\rangle, \tag{4.5}$$

where $(\tilde{e_i})$ is the frame of $E|_U$ that is metrically dual to (e_i), i.e., $\tilde{e_i} = \epsilon_i e_i$ with $\epsilon_i = \langle e_i, e_i\rangle$. We use the abbreviation $\gamma_{ijk} = \gamma_{e_i e_j e_k}$ and we write

$$\begin{aligned}
\{\not{D}, \gamma_T\} &= -\frac{1}{12}\left\{\sum_l \gamma_{\tilde{e}_l} D^S_{e_l}, \sum_{ijk} T^{ijk}\gamma_{ijk}\right\} \\
&= -\frac{1}{12}\left(\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk} + \gamma_l T^{ijk} D^S_{e_l}(\gamma_{ijk}) + T^{ijk}\gamma_{ijk}\gamma_{e_l} D^S_{e_l})\right) \\
&= -\frac{1}{12}\left(\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk} + \gamma_l T^{ijk}\gamma_{ijk} D^S_{e_l} + T^{ijk}\gamma_{ijk}\gamma_{e_l} D^S_{e_l})\right) \\
&= -\frac{1}{12}\left(\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk} + T^{ijk}\{\gamma_l, \gamma_{ijk}\} D^S_{e_l})\right).
\end{aligned}$$

For a fixed l we have

$$\begin{aligned}
\sum_{ijk} T^{ijk}\{\gamma_{\tilde{e}_l}, \gamma_{ijk}\} &= \epsilon_l \sum_{ijk} T^{ijk}(\gamma_l\gamma_{ijk} + \gamma_{ijk}\gamma_l) \\
&= \epsilon_l \sum_{ijk} T^{ijk}(-2\langle e_l, e_i\rangle\gamma_{jk} - \gamma_{iljk} + \gamma_{ijk}\gamma_l)
\end{aligned}$$

$$\begin{aligned}
&= \epsilon_l \sum_{ijk} T^{ijk}(-2\langle e_l, e_i\rangle \gamma_{jk} + 2\langle e_l, e_j\rangle \gamma_{ik} - 2\langle e_l, e_k\rangle \gamma_{ij}) \\
&= -\sum_{jk} 2T^{ljk}\delta_{ll}\gamma_{jk} + \sum_{ik} 2T^{ilk}\delta_{ll}\gamma_{ik} - \sum_{ij} 2T^{ijl}\delta_{ll}\gamma_{ij} \\
&= -6\sum_{jk} T^{ljk}\gamma_{jk}.
\end{aligned}$$

Further,

$$\begin{aligned}
\sum_{ijkl} T^{ijk}\{\gamma_{\tilde{e_l}}, \gamma_{ijk}\} D^S_{e_l} &= -6\sum_{jk} \gamma_{jk} D^S_{\sum_l \epsilon_l T^{ljk} e_l} \\
&= 6\sum_{jk} \gamma_{jk} D^S_{[\tilde{e_j}, \tilde{e_k}]} \\
&= 6\sum_{jk} \epsilon_j \epsilon_k \ \gamma_{jk} D^S_{([e_j, e_k])}.
\end{aligned}$$

Therefore,

$$\{\not{D}, \gamma_T\} = -\frac{1}{12}\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk}) - \frac{1}{2}\sum_{jk} \epsilon_j\epsilon_k \ \gamma_{jk} D^S_{([e_j, e_k])}.$$

To compute the last term we choose the orthonormal frame (e_i) to be

$$(e_i)_{i=1,\ldots,r+s} = \left(\left(\frac{1}{\sqrt{2}}(p_a + q_a)\right)_a, \left(\frac{1}{\sqrt{2}}(p_a - q_a)\right)_a, \ (t_m)_m\right),$$

where $((p_a)_a, (q_a)_a, (t_m)_m)$ is the frame (4.3) constructed before. Since $[\pi(q_a), \pi(q_b)] = 0$, we have $\pi([e_j, e_k]) = 0$. Therefore,

$$\{\not{D}, \gamma_T\} = -\frac{1}{12}\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk}). \tag{4.6}$$

So we have

$$\not{d}^2 = -\frac{1}{16}\left(\frac{1}{3}\sum_{ijkl} \epsilon_l(\gamma_l \pi(e_l)(T^{ijk})\gamma_{ijk}) - \gamma_T^2\right).$$

First observe

$$\begin{aligned}
\sum_{i,j} e_i e_j &= \sum_{i,j}(-2\langle e_i, e_j\rangle - e_j e_i) \\
&= -2\sum_{i,j}\langle e_i, e_j\rangle - \sum_{i,j} e_i e_j = -\sum_{i,j}\langle e_i, e_j\rangle = -\sum_i \epsilon_i.
\end{aligned}$$

Next we compute

$$(\gamma_T)^2 = \left(\gamma_{\frac{1}{6}\sum_{ijk} T^{ijk} e_{ijk}}\right)\left(\gamma_{\frac{1}{6}\sum_{lmn} T^{lmn} e_{lmn}}\right)$$

$$
\begin{aligned}
&= \frac{1}{36} \sum_{ijklmn} T^{ijk} T^{lmn} \gamma_{ijklmn} \\
&= \frac{1}{36} {\sum_{ijklmn}}' T^{ijk} T^{lmn} \gamma_{ijk} \gamma_{lmn} - \frac{9}{36} \sum_{ijlmn} \epsilon_l T^{ijl} T^{lmn} \gamma_{ijmn} \\
&= -\frac{1}{4} \sum_{ijlmn} \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn},
\end{aligned}
$$

where the primed summation sign here, and in the following, runs only over pairwise distinct indices.

Note that in the above calculation, if we let I denote the index set of ordered tuples $(ijklmn)$ over which our summation runs, and consider the involution $\beta : I \to I$ where $(ijklmn) \mapsto (lmnijk)$, then

$$
\frac{1}{36} {\sum_{ijklmn}}' \frac{T^{ijk} T^{lmn} \gamma_{ijk} \gamma_{lmn}}{2} + \frac{1}{36} {\sum_{\beta(ijklmn)}}' \frac{T^{ijk} T^{lmn} \gamma_{ijk} \gamma_{lmn}}{2} = 0.
$$

Observe that

$$
\begin{aligned}
\sum_{ijlmn} \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} &= {\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} + \sum_{ijln} \epsilon_l T^{lij} T^{lin} \gamma_{ijin} + \sum_{ijlm} \epsilon_l T^{lij} T^{lmi} \gamma_{ijmi} \\
&\quad + \sum_{ijln} \epsilon_l T^{lij} T^{ljn} \gamma_{ijjn} + \sum_{ijlm} \epsilon_l T^{lij} T^{lmj} \gamma_{ijmj} \\
&= {\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} + 4 \sum_{ijln} \epsilon_l T^{lij} T^{lin} \gamma_{ijin} \\
&= {\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} - 4 \sum_{ijln} \epsilon_l T^{lij} T^{lin} \gamma_{iijn} \\
&= {\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} + 4 \sum_{ijl} \epsilon_l \epsilon_j T^{lij} T^{lij} \gamma_{ii} \\
&= {\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} - 4 \sum_{ijr} \epsilon_r \epsilon_j \epsilon_i (T^{rij})^2.
\end{aligned}
$$

Therefore,

$$
(\gamma_T)^2 = -\frac{1}{4} \left({\sum_{ijlmn}}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} - 4 \sum_{ijr} \epsilon_r \epsilon_j \epsilon_i (T^{ijr})^2 \right).
$$

Similarly,

$$
\begin{aligned}
\sum_{ijkl} \pi(e_l)(T^{ijk}) \gamma_{\tilde{e}_l} \gamma_{ijk} &= {\sum_{ijkl}}' \pi(e_l)(T^{ijk}) \gamma_{\tilde{e}_l} \gamma_{ijk} + \sum_{jkl} \pi(e_l)(T^{ljk}) \gamma_{\tilde{e}_l} \gamma_{ljk} \\
&\quad + \sum_{ikl} \pi(e_l)(T^{ilk}) \gamma_{\tilde{e}_l} \gamma_{ilk} + \sum_{ijl} \pi(e_l)(T^{ijl}) \gamma_{\tilde{e}_l} \gamma_{ijl}
\end{aligned}
$$

$$= \sum_{ijkl}{}' \pi(e_l)(T^{ijk})\gamma_{\tilde{e}_l}\gamma_{ijk} + 3\sum_{jkl}\pi(e_l)(T^{ljk})\gamma_{\tilde{e}_l}\gamma_{ljk}$$
$$= \sum_{ijkl}{}' \pi(e_l)(T^{ijk})\gamma_{\tilde{e}_l}\gamma_{ijk} - 3\sum_{jkl}\pi(e_l)(T^{ljk})\gamma_{jk}.$$

Combining the above two expressions and taking into account that for any fixed j and k we have

$$\sum_l \pi(e_l)(T^{ljk}) = -\epsilon_j\epsilon_k \sum_l \pi(e_l)\langle [e_j, e_k], e_l\rangle \epsilon_l$$
$$= -\epsilon_j\epsilon_k \sum_a \pi(q_a)\langle [e_j, e_k], p_a\rangle - \sum_i \pi(t_i)\langle [e_j, e_k], t_i\rangle = 0,$$

Above we have used Lemma 4.8. Next we obtain the expression

$$\not{d}^2 = \frac{1}{16}\left(\frac{1}{3}\sum_{ijkl}{}' \pi(e_l)(T^{ijk})\gamma_{\tilde{e}_l}\gamma_{ijk} + \frac{1}{4}\sum_{ijlmn}{}' \epsilon_l T^{lij}T^{lmn}\gamma_{ijmn}\right) - \frac{1}{16}\sum_{ijr}\epsilon_r\epsilon_j\epsilon_i (T^{rij})^2.$$

We aim to show that

$$\not{d}^2 = -\frac{1}{16}\left(\sum_{ijr}\epsilon_i\epsilon_j\epsilon_r (T^{ijr})^2\right). \tag{4.7}$$

For this we need to show that

$$\frac{1}{3}\sum_{ijkl}{}' \pi(e_l)(T^{ijk})\gamma_{\tilde{e}_l}\gamma_{ijk} + \frac{1}{4}\sum_{ijlmn}{}' \epsilon_l T^{lij}T^{lmn}\gamma_{ijmn} = 0. \tag{4.8}$$

To show that the equation (4.7) is true we use the property that $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$ for all $u, v, w \in \Gamma(E)$ and we metrically raise and lower indices as needed:

$$\begin{aligned}
0 &= [e_i, [e_j, e_k]] - [[e_i, e_j], e_k] - [e_j, [e_i, e_k]] \\
&= \sum_l \Big(- [e_i, T_{jk}{}^l e_l] + [T_{ij}{}^l e_l, e_k] + [e_j, T_{ik}{}^l e_l]\Big) \\
&= \sum_l (-\pi(e_i)(T_{jk}{}^l e_l) + \pi(e_j)(T_{ik}{}^l)e_l) + \sum_{l,m}(T_{jk}{}^l T_{il}{}^m - T_{ik}{}^l T_{jl}{}^m)e_m \\
&\quad + \sum_l (-[e_k, T_{ij}{}^l e_l] + \pi^* d\langle T_{ij}{}^l e_l, e_k\rangle) \\
&= \sum_l (-\pi(e_i)(T_{jk}{}^l e_l) + \pi(e_j)(T_{ik}{}^l)e_l) + \sum_{l,m}(T_{jk}{}^l T_{il}{}^m - T_{ik}{}^l T_{jl}{}^m)e_m + \pi^* dT_{ijk} \\
&\quad - \sum_l \pi(e_k)(T_{ij}{}^l)e_l + \sum_{l,m} T_{ij}{}^l T_{kl}{}^m e_m \\
&= \sum_l \pi(e_l)(T_{ijk})\tilde{e}_l - \sum_{(i,j,k)\text{ cyclic}}\sum_l \left(\pi(e_i)(T_{jk}{}^l)e_l - \sum_m T_{ij}{}^l T_{kl}{}^m e_m\right).
\end{aligned}$$

In the above we have used that $\pi^* df = \sum_l \pi(e_l)(f)\tilde{e}_l$ for all $f \in \mathcal{C}^\infty(M)$. For any fixed i, j, k, l we therefore find

$$\pi(e_l)(T_{ijk})\tilde{e}_l - \sum_{(i,j,k)\text{ cyclic}} \sum_l \left(\pi(e_i)(T_{jk}{}^l)e_l - \sum_m T_{ij}{}^l T_{kl}{}^m e_m \right) = 0.$$

Let i, j, k, l all be pairwise distinct. Now we multiply the above equation with γ^{ijk} and sum over i, j, k, l which are pairwise distinct

$$\begin{aligned}
0 &= \sum_{ijkl}{}' \left(\pi(e_l)(T_{ijk})\tilde{e}_l - \sum_{(i,j,k)\text{ cyclic}} \sum_l \left(\pi(e_i)(T_{jk}{}^l)e_l - \sum_m T_{ij}{}^l T^m_{kl} e_m \right) \gamma^{ijk} \right) \\
&= \sum_{ijkl}{}' \pi(e_l)(T_{ijk})\gamma^{lijk} - \sum_{ijkl}{}' \sum_{(i,j,k)\text{ cyclic}} \pi(e_i)(T_{jkl})\gamma^{lijk} \\
&\quad + \sum_{ijkl}{}' \left(\sum_{(i,j,k)\text{ cyclic}} \sum_m T_{ij}{}^m T_{kml} \right) \gamma^{lijk}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{ijkl}{}' \sum_{(i,j,k)\text{ cyclic}} \pi(e_i)(T_{jkl})\gamma^{lijk} &= 3 \sum_{ijkl}{}' \pi(e_i)(T_{jkl})\gamma^{lijk} = 3 \sum_{ijkl}{}' \pi(e_l)(T_{jki})\gamma^{iljk} \\
&= -3 \sum_{ijkl}{}' \pi(e_l)(T_{ijk})\gamma^{lijk}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{ijkl}{}' \left(\sum_{(i,j,k)\text{ cyclic}} \sum_m T_{ij}{}^m T_{kml} \right) \gamma^{lijk} &= \sum_{ijklm}{}' \left(\sum_{(i,j,k)\text{ cyclic}} T_{ij}{}^m T_{kml} \right) \gamma^{lijk} \\
&= 3 \sum_{ijklm}{}' T_{ij}{}^m T_{kml} \gamma^{lijk} \\
&= 3 \sum_{ijklm}{}' T_{ij}{}^l T_{klm} \gamma^{mijk}.
\end{aligned}$$

Combining all the above we obtain

$$4 \sum_{i,j,k,l}{}' \pi(e_l)(T^{ijk})\gamma_{\tilde{e}_l}\gamma_{ijk} + 3 \sum_{i,j,l,m,n}{}' \epsilon_l T^{lij} T^{lmn} \gamma_{ijmn} = 0$$

which is essentially Equation (4.8) subject to re-organisation of indices. With this we can conclude that $\not{d}^2 \in \mathcal{C}^\infty(U)$. $\square$

Lemma 4.7 shows that $\not{d} = \not{D} + \frac{1}{4}\gamma_T$ satisfies the first two properties of a DGO and Lemma 4.9 shows that it satisfies the third, so we conclude that $\not{d} = \not{D} + \frac{1}{4}\gamma_T$ is a Dirac generating operator – proving Theorem 4.4.

4.3 The space of local DGOs in signature $(p, p+1)$:

In this section, we aim to describe the space of local Dirac generating operators in signature $(p, p+1)$. Having a description of this space is not only important to gain a better understanding of DGOs of Courant algebroids of the said signature type but it is also a first step towards constructing the canonical DGO, which would then be a globally defined operator.

Theorem 4.10. *Suppose there exists a DGO $\not{d}$ for a Courant algebroid $E \longrightarrow M$ of signature $(p, p+1)$ on an irreducible $\mathcal{C}\ell(E)$-module bundle (S, γ). Then, the set of DGOs for E on (S, γ) has the structure of an affine space modelled on*

$$V_{\not{d}} := \{e \in \Gamma(E) \mid \{\not{d}, \gamma_e\} \in \mathcal{C}^\infty(M)\}. \tag{4.9}$$

In particular, $V_{\not{d}}$ is independent of the choice of the DGO $\not{d}$.

The proof of Theorem 4.10 is in Section 4.3.2.

In all that follows we assume that $E \longrightarrow M$ is a Courant algebroid and (S, γ) an irreducible $\mathcal{C}\ell(E)$-module bundle. In the lemma below, we show that if $\not{d}$ is a DGO, then $\not{d} + \gamma_e$ is also a DGO for any $e \in V_{\not{d}}$, where $V_{\not{d}}$ is as described in Theorem 4.10 above. Note that the lemma below holds in all signatures even though we apply it only in the $(p, p+1)$ case.

Lemma 4.11. *Suppose there exists a DGO $\not{d}$ for a Courant algebroid $E \longrightarrow M$ of signature (r, s) on an irreducible $\mathcal{C}\ell(E)$-module bundle (S, γ). Then for every $e \in V_{\not{d}}$, where $V_{\not{d}}$ is as in Equation (4.9), $\not{d} + \gamma_e$ is a DGO for E.*

Proof. For any $e \in V_{\not{d}}$, we verify below that $\not{d}' := \not{d} + \gamma_e$ is a DGO.

1. Let $e_1, e_2 \in \Gamma(E)$. Then

$$\begin{aligned} [\{\not{d} + \gamma_e, \gamma_{e_1}\}, \gamma_{e_2}] &= [\{\not{d}, \gamma_{e_1}\}, \gamma_{e_2}] + [\{\gamma_e, \gamma_{e_1}\}, \gamma_{e_2}] \\ &= \gamma_{[e_1, e_2]} + [\gamma_e\gamma_{e_1}, \gamma_{e_2}] + [\gamma_{e_1}\gamma_e, \gamma_{e_2}] \\ &= \gamma_{[e_1, e_2]}. \end{aligned}$$

   In the last step $[\gamma_e\gamma_{e_1}, \gamma_{e_2}] + [\gamma_{e_1}\gamma_e, \gamma_{e_2}] = 0$ due to the Clifford relation.

2. Let $e_1 \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$. Since $[\gamma_e, f] = 0$,

$$\{[\not{d} + \gamma_e, f], \gamma_{e_1}\} = \{[\not{d}, f], \gamma_{e_1}\} + \{[\gamma_e, f], \gamma_{e_1}\} = \pi(e)(f).$$

   With this we have shown that $\not{d}'$ satisfies condition 2.

3. If $e \in \Gamma(E)$ such that $\{\not{d}, \gamma_e\} \in \mathcal{C}^\infty(M)$, then

$$(\not{d} + \gamma_e)(\not{d} + \gamma_e) = \not{d}^2 + \{\not{d}, \gamma_e\} + \gamma_e\gamma_e \in \mathcal{C}^\infty(M).$$

Therefore, $\not{d}'$ satisfies condition 3. □

Now we will prove the converse, that is, in signature $(p, p+1)$ we show that $\not{d} - \not{d}' = \gamma_e$ for some $e \in V_{\not{d}}$ for any two DGOs $\not{d}$ and $\not{d}'$. To do this we proceed as follows. We first observe that $[L, f]$ is a 0th order operator for $L := \not{d} - \not{d}'$ and any smooth function f. Then we show that $[L, f] = 0$ in signature $(p, p+1)$ and from this we conclude that L is a 0-th order operator. Next, we show that $\{L, f\}$ belongs to the Schur algebra bundle and is therefore a scalar operator. For this the signature $(p, p+1)$ is important. In the final step, due to surjectivity of γ and since $\{L, f\}$ is a scalar operator in signature $(p, p+1)$, we conclude that $L = \gamma_e$ for some $e \in \Gamma(E)$. To this end, we begin with a lemma.

Lemma 4.12. *The commutator of any first-order differential operator L and any smooth function f is a 0-th order operator. Furthermore, if L is a first-order operator such that $[L, f] = 0$ for all f, then L is of 0-th order.*

Proof. Let M be a smooth manifold of dimension n and $S \longrightarrow M$ a real vector bundle of rank m. Recall that a first-order differential operator on M is a linear map $L : \Gamma(S) \longrightarrow \Gamma(S)$ that has the following property. Each point of $x \in M$ has a neighbourhood U with local coordinates $(x_1, \ldots, x_n)$ and local trivialisations $\chi_U^S : S \mid_U \longrightarrow U \times \mathbb{R}^m$ in which L can be written in the form:

$$L = \sum_{i=1}^{m} A^i(x) \frac{\partial^i}{\partial x^i} + A^0(x),$$

where each $A^i(x)$ is an $m \times m$-matrix of smooth real-valued functions such that $A^i \neq 0$ for some i. Let $(\xi_1, \ldots, \xi_m)$ be a local frame for S over the neighbourhood U. Write any $g \in \Gamma(S|_U)$ as $g = \sum_{i=1}^m g_i \xi_i$ where $g_i \in \mathcal{C}^\infty(U)$. Now by the Leibniz rule,

$$\begin{aligned}(Lf - fL)(g_1, \ldots, g_m) &= \left(\sum_{i=1}^{m} A^i(x) \frac{\partial}{\partial x^i} + A^0(x)\right)(f(g_1, \ldots, g_m)) \\ &\quad - f\left(\sum_{i=1}^{m} A^i(x) \frac{\partial}{\partial x^i} + A^0(x)\right)((g_1, \ldots, g_m))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} A^i(x) \left(\frac{\partial (f(g_1, \ldots, g_m))}{\partial x^i} \right) \\
&\quad - f \sum_{i=1}^{m} A^i(x) \left(\frac{\partial ((g_1, \ldots, g_m)}{\partial x^i} \right) \\
&= \sum_{i=1}^{m} A^i(x)(g_1, \ldots, g_m) \left(\frac{\partial f}{\partial x^i} \right).
\end{aligned}
$$

Since each component of the above m-tuple is a $\mathcal{C}^\infty(U)$ function, $[L, f]$ is a 0-th order operator for all $f \in \mathcal{C}^\infty(U)$.

Now to show that L is of 0-th order when $[L, f] = 0$, let $f = x_j$ for any $1 \leq j \leq n$. Given that $[L, f] = 0$ for all $f \in \mathcal{C}^\infty(U)$ we have the following:

$$
\left[\left(\sum_{i=1}^{m} A^i(x) \frac{\partial}{\partial x^i} + A^0(x) \right), x_j \right] ((g_1, \ldots, g_m)) = 0.
$$

$$
\sum_{i=1}^{m} A^i(x)(g_1, \ldots, g_m) \left(dx_j(\partial_{x_i}) \right) = 0.
$$

With this we have shown that $A^j(x)(g_1, \ldots, g_m) = 0$ for all $x \in U$ and all j. Without loss of generality, now let g be such that g_i are nowhere vanishing functions for all i. This would imply that $A^j(x) = 0$ for all j. Thus L is of 0-th order. $\square$

In particular, from Lemma 4.12 it follows that if $L := \not{d} - \not{d}'$, where $\not{d}$ and $\not{d}'$ are DGOs on a Courant algebroid $E \longrightarrow M$, then, $[L, f]$ is a 0-th order operator for all $f \in \mathcal{C}^\infty(M)$.

Remark 4.13. From property 2 of Definition 4.1, it is clear that $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$.

The lemma below shows that $[L, f]$ is $\mathcal{C}\ell^0(E)$-equivariant.

Lemma 4.14. *If $L := \not{d} - \not{d}'$ where $\not{d}$ and $\not{d}'$ are DGOs on a Courant algebroid $E \longrightarrow M$ of signature (r, s), then $[L, f]$, where $f \in \mathcal{C}^\infty(M)$, commutes with $\mathcal{C}\ell^0(E)$ and anti-commutes with $\mathcal{C}\ell^1(E)$.*

Proof. For $L := \not{d} - \not{d}'$ it follows from property 2 of DGOs that $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$. Now from the Clifford relations the proof follows. $\square$

Let (r, s) be the signature of the Courant algebroid $E \longrightarrow M$. We show below that when the Clifford algebras $\mathcal{C}\ell(E)_x$ are isomorphic to $\mathbb{R}[N] \oplus \mathbb{R}[N]$ or $\mathbb{H}[N] \oplus \mathbb{H}[N]$ for

some $N \in \mathbb{N}$, i.e., when $r - s \equiv 7 \bmod 8$ or $r - s \equiv 3 \bmod 8$ respectively, $[L, f]$ is in fact zero. Note that metric of signature $(p, p+1)$ is a special case of $r - s \equiv 7 \bmod 8$. The fact that $[L, f]$ is zero is necessary for our proof of Theorem 4.10.
Note that in both of the above stated cases, there are two inequivalent irreducible representations, $\gamma_\pm$ of $\mathcal{C}\ell(E)$ and a unique irreducible representation $\widetilde{\gamma}$ of $\mathcal{C}\ell^0(E)$ such that $\gamma_\pm |_{\mathcal{C}\ell^0(E)} \cong \widetilde{\gamma}$ as representations. This fact is critical in our proof showing that $[L, f]$ is zero.

Lemma 4.15. *Let $E \longrightarrow M$ be a Courant algebroid whose signature (r, s) is such that $r - s \equiv 7 \bmod 8$ or $r - s \equiv 3 \bmod 8$ and let $L := \not{d} - \not{d}'$ where $\not{d}$ and $\not{d}'$ are DGOs. Then $[L, f] = 0$ for all $f \in \mathcal{C}^\infty(M)$.*

Proof.

- **Case** $r - s \equiv 7 \bmod 8$: Let $\gamma : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(S)$ be an irreducible representation of $\mathcal{C}\ell(E)$ and let $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})$ be the Schur algebra bundle of the restriction of γ to the even part $\mathcal{C}\ell^0(E)$. Clearly $\Gamma(\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})) \subset \Gamma(\mathrm{End}_{\mathbb{R}}(S))$. Since $[L, f]$ is a 0-th order operator, $[L, f] \in \Gamma(\mathrm{End}_{\mathbb{R}}(S))$. Furthermore, $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$ implies $[[L, f], \gamma(\Gamma(\mathcal{C}\ell^0(E)))] = 0$. Therefore $[L, f] \in \Gamma(\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)}))$. Since $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})_p \cong \mathbb{R}$, $[L, f]_p = \lambda_p \mathrm{Id}_{S_p}$ for some $\lambda_p \in \mathbb{R}$ for all $p \in M$. Observe that $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$ also implies that $\{[L, f], \gamma(\Gamma(\mathcal{C}\ell^1(E)))\} = 0$, that is for all $\eta \in \gamma(\Gamma(\mathcal{C}\ell^1(E)))$:

$$\begin{aligned} \{[L, f], \eta\}_p &= \lambda_p \mathrm{Id}_{S_p} \eta_p + \eta_p \lambda_p \mathrm{Id}_{S_p} \\ &= 2\lambda_p \eta_p \\ &= 0. \end{aligned}$$

  This can only hold for all $\eta_p \in \gamma_p(\mathcal{C}\ell^1(E))$ and for all $p \in M$, if $\lambda_p = 0$. Thus we have shown that $[L, f] = 0$.

- **Case** $r - s \equiv 3 \bmod 8$: Let $\gamma : \mathcal{C}\ell(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(S)$ be an irreducible representation on $\mathcal{C}\ell(E)$ and let $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})$ be the Schur algebra bundle of the restriction of γ to the even part $\mathcal{C}\ell^0(E)$. Clearly $\Gamma(\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})) \subset \Gamma(\mathrm{End}_{\mathbb{R}}(S))$. Since $[L, f]$ is a 0-th order operator, $[L, f] \in \Gamma(\mathrm{End}_{\mathbb{R}}(S))$. Furthermore, $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$ implies $[[L, f], \gamma(\Gamma(\mathcal{C}\ell^0(E)))] = 0$. Therefore $[L, f] \in \Gamma(\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)}))$. Recall that $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})_p \cong \mathbb{H}$. Let $I_p, J_p, K_p \in \mathrm{End}_{\mathbb{R}}(S_p)$ be the endomorphisms that correspond to the quaternionic units $\{i, j, k\}$ via the identification $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})_p \cong \mathbb{H}$. Then

$$[L, f]_p = a \,\mathrm{Id}_{S_p} + bI_p + cJ_p + dK_p$$

for some $a, b, c, d \in \mathbb{R}$. Observe that $\{[L, f], \gamma_e\} = 0$ for all $e \in \Gamma(E)$ also implies that $\{[L, f], \gamma(\Gamma(\mathcal{C}\ell^1(E)))\} = 0$, that is for all $\eta \in \gamma(\Gamma(\mathcal{C}\ell^1(E)))$,

$$\{[L, f], \eta\}_p = 2a(\eta_p \operatorname{Id}_{S_p}) + b(I_p\eta_p + \eta_p I_p) + c(J_p\eta_p + \eta_p J_p) + d(K_p\eta_p + \eta_p K_p).$$

Note that by the natural action of $\mathcal{C}(S, \gamma)_p \cong \mathbb{H}$, S_p is a left $\mathbb{H}$-module. Furthermore, since by definition $\gamma(\mathcal{C}\ell(E)_p)$ commutes with $\mathcal{C}(S, \gamma)_p \cong \mathbb{H}$, it is clear that the action of $\mathcal{C}\ell(E)_p$ on S_p is $\mathbb{H}$-linear. Observe that $\mathcal{C}(S, \gamma|_{\mathcal{C}\ell^0(E)})_p = \mathcal{C}(S, \gamma)_p \cong \mathbb{H}$, therefore, we have:

$$\{[L, f], \eta\}_p = 2a(\eta_p \operatorname{Id}_{S_p}) + 2b(\eta_p I_p) + 2c(\eta_p J_p) + 2d(\eta_p K_p) = 0.$$

Since $\{\operatorname{Id}_{S_p}, I_p, J_p, K_p\}$ are linearly independent, this can only hold for all $\eta_p \in \gamma_p(\mathcal{C}\ell^1(E))$ and for all $p \in M$, if $a = b = c = d = 0$. Thus we have shown that $[L, f] = 0$. □

The next lemma about Clifford modules will supply us with the final part necessary to prove Theorem 4.10.

Lemma 4.16. *Let $\mathcal{C}\ell(E)$ be a Clifford algebra whose underlying pseudo-Euclidean vector space has signature $(p, p+1)$ and let (S, γ) be an irreducible $\mathcal{C}\ell(E)$-module. Let $A \in \operatorname{End}(S)$ and assume that for all $v \in E$ we have $\{A, \gamma_v\} = \lambda \operatorname{Id}_S$ for some $\lambda \in \mathbb{R}$. Then $A = \gamma_u$ for some $u \in E$.*

For the proof of this lemma, which is written in Section 4.3.1 in full detail, we find a basis of $\operatorname{End}(S)$ in terms of the Clifford generators of $\mathcal{C}\ell(E)$. We do this by first determining the kernel of γ and then use the surjectivity of γ when the signature of E is $(p, p+1)$. Note that, to avoid cumbersome notation, unlike in the rest of the text we use the symbol $\mathcal{C}\ell(E)$ here to indicate a Clifford algebra instead of Clifford algebra bundle.

We begin with the observation that if $\mathcal{C}\ell(E)$ is a Clifford algebra whose underlying pseudo-Euclidean vector space has signature $(p, p+1)$, then

$$\mathcal{C}\ell(E) \cong \mathcal{C}\ell(p, p+1) \cong \mathcal{C}\ell(0,1) \otimes \underbrace{\mathcal{C}\ell(1,1) \otimes \cdots \otimes \mathcal{C}\ell(1,1)}_{\text{p times}}.$$

This is a standard fact about Clifford algebras which follows from Theorem 2.12. Now we fix algebra isomorphisms $\mathcal{C}\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$ and $\mathcal{C}\ell(1,1) \cong \mathbb{R}[2]$. We do this by allowing $\{(1,1),(1,-1)\}$ to be the images of Clifford generators $\{\mathrm{Id},\widetilde{\Gamma}\}$ of $\mathcal{C}\ell(0,1)$ and the matrices below to be the images of Clifford generators $\{\mathrm{Id},\Gamma_1,\Gamma_1,\Gamma_3\}$ of $\mathcal{C}\ell(1,1)$:

$$\mathrm{Id}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad I=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad J=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad K=IJ=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

The lemma below gives an explicit form for an isomorphism $\mathcal{C}\ell(p,p+1)\cong(\mathbb{R}\oplus\mathbb{R})\otimes\underbrace{\mathbb{R}[2]\otimes\cdots\otimes\mathbb{R}[2]}_{\text{p times}}$ in terms of the above generators.

Lemma 4.17. *Let $\{\Gamma_1,\Gamma_2,\dots,\Gamma_{2p+1}\}$ be a set of generators of $\mathcal{C}\ell(p,p+1)$ such that $\Gamma_i^2=-1$ for $i\le p$ and $\Gamma_i^2=1$ for $i>p$. Then there exists an isomorphism of Clifford algebras*

$$\Phi:\mathcal{C}\ell(p,p+1)\longrightarrow(\underbrace{\mathbb{R}[2]\otimes\cdots\otimes\mathbb{R}[2]}_{p\ times})\oplus(\underbrace{\mathbb{R}[2]\otimes\cdots\otimes\mathbb{R}[2]}_{p\ times})$$

which maps the generators as follows.

$$\begin{aligned}
\Phi(\Gamma_l):=&(\underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{l-1}\otimes I\otimes\underbrace{K\otimes\cdots\otimes K}_{p-l},\ \underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{l-1}\otimes I\otimes\underbrace{K\otimes\cdots\otimes K}_{p-l})\\
\Phi(\Gamma_{p+1}):=&(\underbrace{K\otimes\cdots\otimes K}_{p},\ \underbrace{-K\otimes\cdots\otimes K}_{p})\\
\Phi(\Gamma_{p+1+l}):=&(\underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{l-1}\otimes J\otimes\underbrace{K\otimes\cdots\otimes K}_{p-l},\ \underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{l-1}\otimes J\otimes\underbrace{K\otimes\cdots\otimes K}_{p-l})
\end{aligned}$$

for all $1\le l\le p$.

Proof. First we verify that the images $\Phi(\Gamma_l)$, for $1\le l\le 2p+1$, of the Clifford generators satisfy the Clifford relations:

$$\begin{aligned}
\Phi(\Gamma_l)^2&=-\mathrm{Id}, && \forall 1\le l\le p,\\
\Phi(\Gamma_{p+1+l})^2&=\mathrm{Id}, && \forall 0\le l\le p,\\
\Phi(\Gamma_m)\Phi(\Gamma_l)&=-\Phi(\Gamma_l)\Phi(\Gamma_m), && \forall 1\le m,l\le 2p+1, m\ne l.
\end{aligned}$$

This is a straightforward verification, using the relations between I, J and K. So we conclude that Φ defines a homomorphism of algebras.

Observe that $\mathcal{C}\ell(p,p+1)$ has dimension 2^{2p+1} and $\mathbb{R}[2]^{\otimes p}\oplus\mathbb{R}[2]^{\otimes p}$ has dimension $2(4^p)=2^{2p+1}$. Therefore, in order to show that Φ is an isomorphism it is sufficient to prove that Φ is surjective.

Surjectivity of Φ follows from the fact that the elements $\Phi(\Gamma_l)$, for $1 \leq l \leq 2p+1$, generate the algebra $\mathbb{R}[2]^{\otimes p} \oplus \mathbb{R}[2]^{\otimes p}$. By explicit computation one can verify that the elements in the set below can all be obtained as products of elements $\Phi(\Gamma_l)$, for $1 \leq l \leq 2p+1$.

$$G := \left\{ A \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{l-1} \otimes T \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{p-l} \;\middle|\; \begin{array}{l} A \in \{(1,1),(1,-1)\} \\ T \in \{I, J, K\} \\ 1 \leq l \leq p \end{array} \right\}$$

Since G generates the algebra $(\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{R}[2]^{\otimes p}$, this concludes our proof. $\square$

Remark 4.18. Denote by $(\Gamma_j^{[p]})$ the Clifford generators of $\mathcal{C}\ell(p, p+1)$. Then note that from Lemma 4.17 it follows that $\Phi(\Gamma_j^{[p]}) = \Phi(\Gamma_j^{[p-1]}) \otimes K$ for all $1 \leq j < p$ and $\Phi(\Gamma_j^{[p]}) = \Phi(\Gamma_{j-1}^{[p-1]}) \otimes K$ for all $p < j < 2p+1$.

Without loss of generality, hence forth we will consider the irreducible representation $\gamma : \mathcal{C}\ell(p, p+1) \longrightarrow \mathrm{End}(\mathbb{R}^{2^p})$ that linearly maps $\Gamma_i \mapsto \pi_1 \circ \Phi(\Gamma_i)$ for all generators Γ_i of $\mathcal{C}\ell(p, p+1)$ where π_1 is the projection of $\mathcal{C}\ell(p, p+1)$ onto the first component of $\mathbb{R}[2]^{\otimes p} \oplus \mathbb{R}[2]^{\otimes p}$.

In what follows we will obtain an explicit description of the kernel of γ in terms of the generators of the Clifford algebra. Before we give the abstract description of the kernel of γ for $\mathcal{C}\ell(p, p+1)$ in full generality, we will take a look at some examples in lower dimensions. We explicitly compute below the kernel of γ for $\mathcal{C}\ell(1,2)$, $\mathcal{C}\ell(2,3)$ and $\mathcal{C}\ell(3,4)$.

Example 4.19. $\boldsymbol{\mathcal{C}\ell(1,2)}$

Let $\{\mathrm{Id}, \Gamma_1, \Gamma_2, \Gamma_3\}$ be a set of generators for $\mathcal{C}\ell(1,2)$. Then

$$\{\mathrm{Id}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_1\Gamma_2, \Gamma_1\Gamma_3, \Gamma_2\Gamma_3, \Gamma_1\Gamma_2\Gamma_3\}$$

forms a basis for $\mathcal{C}\ell(1,2)$. Explicitly, an isomorphism Φ as in Lemma 4.17 maps the above basis of $\mathcal{C}\ell(1,2)$ to the following elements in $\mathbb{R}[2] \oplus \mathbb{R}[2]$:

$$\begin{aligned} \mathrm{Id} &\mapsto (\mathbb{1}, \mathbb{1}) \\ \Gamma_1 &\mapsto (I, I) \\ \Gamma_2 &\mapsto (K, -K) \\ \Gamma_3 &\mapsto (J, J) \end{aligned}$$

$$\begin{aligned}
\Gamma_1\Gamma_2 &\mapsto (-J, J)\\
\Gamma_1\Gamma_3 &\mapsto (K, K)\\
\Gamma_2\Gamma_3 &\mapsto (I, -I)\\
\Gamma_1\Gamma_2\Gamma_3 &\mapsto (-\mathbb{1}, \mathbb{1})
\end{aligned}$$

It is now easy to see that the kernel of γ is as follows:

$$\mathrm{Ker}(\gamma) = \mathrm{Span}_{\mathbb{R}}\{\Gamma_1 - \Gamma_2\Gamma_3, \Gamma_2 - \Gamma_1\Gamma_3, \Gamma_3 + \Gamma_1\Gamma_2, \Gamma_1\Gamma_2\Gamma_3 + \mathrm{Id}\}.$$

Example 4.20. $\mathcal{C}\ell(\mathbf{2}, \mathbf{3})$

Let $\{\mathrm{Id}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ be a set of generators for $\mathcal{C}\ell(2, 3)$. Then the isomorphism Φ as in Lemma 4.17 maps a basis of $\mathcal{C}\ell(2, 3)$ constructed from the generators of $\mathcal{C}\ell(2, 3)$ to elements in $\mathbb{R}[4] \oplus \mathbb{R}[4]$ as follows.

$$\begin{aligned}
\mathrm{Id} &\mapsto (\mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{1}) & \Gamma_1\Gamma_2\Gamma_3 &\mapsto (-J \otimes -I, -J \otimes I)\\
\Gamma_1 &\mapsto (I \otimes K, I \otimes K) & \Gamma_1\Gamma_2\Gamma_4 &\mapsto (K \otimes -I, K \otimes -I)\\
\Gamma_2 &\mapsto (\mathbb{1} \otimes I, \mathbb{1} \otimes I) & \Gamma_1\Gamma_2\Gamma_5 &\mapsto (I \otimes \mathbb{1}, I \otimes \mathbb{1})\\
\Gamma_3 &\mapsto (K \otimes K, -K \otimes K) & \Gamma_1\Gamma_3\Gamma_4 &\mapsto (-\mathbb{1} \otimes K, \mathbb{1} \otimes K)\\
\Gamma_4 &\mapsto (J \otimes K, J \otimes K) & \Gamma_1\Gamma_3\Gamma_5 &\mapsto (-J \otimes J, J \otimes J)\\
\Gamma_5 &\mapsto (\mathbb{1} \otimes J, \mathbb{1} \otimes J) & \Gamma_1\Gamma_4\Gamma_5 &\mapsto (K \otimes J, K \otimes J)\\
\Gamma_1\Gamma_2 &\mapsto (I \otimes J, I \otimes J) & \Gamma_2\Gamma_3\Gamma_4 &\mapsto (I \otimes I, I \otimes -I)\\
\Gamma_1\Gamma_3 &\mapsto (-J \otimes \mathbb{1}, J \otimes \mathbb{1}) & \Gamma_2\Gamma_3\Gamma_5 &\mapsto (K \otimes -\mathbb{1}, K \otimes \mathbb{1})\\
\Gamma_1\Gamma_4 &\mapsto (K \otimes \mathbb{1}, K \otimes \mathbb{1}) & \Gamma_2\Gamma_4\Gamma_5 &\mapsto (J \otimes -\mathbb{1}, J \otimes -\mathbb{1})\\
\Gamma_1\Gamma_5 &\mapsto (I \otimes I, I \otimes I) & \Gamma_3\Gamma_4\Gamma_5 &\mapsto (I \otimes J, -I \otimes J)\\
\Gamma_2\Gamma_3 &\mapsto (K \otimes -J, K \otimes J) & \Gamma_1\Gamma_2\Gamma_3\Gamma_4 &\mapsto (-\mathbb{1} \otimes J, -\mathbb{1} \otimes -J)\\
\Gamma_2\Gamma_4 &\mapsto (J \otimes -J, J \otimes -J) & \Gamma_2\Gamma_3\Gamma_4\Gamma_5 &\mapsto (I \otimes K, I \otimes -K)\\
\Gamma_2\Gamma_5 &\mapsto (\mathbb{1} \otimes K, \mathbb{1} \otimes K) & \Gamma_1\Gamma_3\Gamma_4\Gamma_5 &\mapsto (-\mathbb{1} \otimes I, \mathbb{1} \otimes I)\\
\Gamma_3\Gamma_4 &\mapsto (I \otimes \mathbb{1}, -I \otimes \mathbb{1}) & \Gamma_1\Gamma_2\Gamma_4\Gamma_5 &\mapsto (K \otimes -K, K \otimes -K)\\
\Gamma_3\Gamma_5 &\mapsto (K \otimes I, -K \otimes I) & \Gamma_1\Gamma_2\Gamma_3\Gamma_5 &\mapsto (-J \otimes -K, -J \otimes K)\\
\Gamma_4\Gamma_5 &\mapsto (J \otimes I, J \otimes I) & \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 &\mapsto (-\mathbb{1} \otimes \mathbb{1}, -\mathbb{1} \otimes -\mathbb{1})
\end{aligned}$$

The kernel of γ is as follows:

$$\ker(\gamma) = \mathrm{span}_{\mathbb{R}}\{\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 + \mathrm{Id}, \Gamma_1\Gamma_2\Gamma_3\Gamma_4 + \Gamma_5, \Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \Gamma_1,$$

$$\Gamma_1\Gamma_3\Gamma_4\Gamma_5 + \Gamma_2, \Gamma_1\Gamma_2\Gamma_4\Gamma_5 + \Gamma_3, \Gamma_1\Gamma_2\Gamma_3\Gamma_5 - \Gamma_4,$$
$$\Gamma_1\Gamma_2\Gamma_3 - \Gamma_4\Gamma_5, \Gamma_1\Gamma_2\Gamma_4 + \Gamma_3\Gamma_5, \Gamma_1\Gamma_2\Gamma_5 - \Gamma_3\Gamma_4,$$
$$\Gamma_1\Gamma_3\Gamma_4 + \Gamma_2\Gamma_5, \Gamma_1\Gamma_3\Gamma_5 - \Gamma_2\Gamma_4, \Gamma_1\Gamma_4\Gamma_5 + \Gamma_2\Gamma_3,$$
$$\Gamma_2\Gamma_3\Gamma_4 - \Gamma_1\Gamma_5, \Gamma_2\Gamma_3\Gamma_5 + \Gamma_1\Gamma_4, \Gamma_2\Gamma_4\Gamma_5 - \Gamma_1\Gamma_3,$$
$$\Gamma_3\Gamma_4\Gamma_5 - \Gamma_1\Gamma_2\}.$$

Example 4.21. $\mathcal{C}\ell(\mathbf{3}, \mathbf{4})$

Let $\{\mathrm{Id}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\}$ be a set of generators for $\mathcal{C}\ell(3, 4)$. Then explicitly, an isomorphism Φ as in Lemma 4.17 maps these generators to the following elements in $\mathbb{R}[6] \oplus \mathbb{R}[6]$.

$$\begin{aligned}
\mathrm{Id} &\mapsto (\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1})\\
\Gamma_1 &\mapsto (I \otimes K \otimes K, I \otimes K \otimes K)\\
\Gamma_2 &\mapsto (\mathbb{1} \otimes I \otimes K, \mathbb{1} \otimes I \otimes K)\\
\Gamma_3 &\mapsto (\mathbb{1} \otimes \mathbb{1} \otimes I, \mathbb{1} \otimes \mathbb{1} \otimes I)\\
\Gamma_4 &\mapsto (K \otimes K \otimes K, -K \otimes K \otimes K)\\
\Gamma_5 &\mapsto (J \otimes K \otimes K, J \otimes K \otimes K)\\
\Gamma_6 &\mapsto (\mathbb{1} \otimes J \otimes K, \mathbb{1} \otimes J \otimes K)\\
\Gamma_7 &\mapsto (\mathbb{1} \otimes \mathbb{1} \otimes J, \mathbb{1} \otimes \mathbb{1} \otimes J)
\end{aligned}$$

The kernel of γ is spanned by the elements

$$\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 - \mathrm{Id}$$

and

$$\Gamma_{i_1} \dots \Gamma_{i_k} (\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 - \mathrm{Id}),$$

where $1 \le k \le 3$, $i_k \in \{1, \dots, 7\}$ and $i_1 < \dots < i_k$.

Below we give a general description for kernel of γ.

Lemma 4.22. *The set*

$$\left\{ \Gamma_\mu \left(\Gamma_1\Gamma_2 \dots \Gamma_{2p+1} + (-1)^{\frac{p(p+1)}{2}+1} \mathrm{Id} \right) \middle| \begin{array}{c} \mu = (i_1, i_2, \dots, i_k), \\ 0 \le |\mu| = k \le p, \\ 1 \le i_1 < \dots < i_k \le 2p+1 \end{array} \right\}$$

forms a basis of the kernel of the irreducible representation $\gamma : \mathcal{C}\ell(p, p+1) \longrightarrow \mathrm{End}(\mathbb{R}^{2^p})$ *where* $\gamma_{\Gamma_\mu} = \mathrm{Id}$ *when* $|\mu| = 0$.

Proof. Let $(\Gamma_j^{[p]})$ be generators of $\mathcal{C}\ell(p,p+1)$. First we will show by induction on p that

$$\gamma(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{2p+1}^{[p]}) = \otimes_{l=1}^{p} K^{l-1} I K^l J.$$

Observe that for $\mathcal{C}\ell(1,2)$, the volume element gets mapped by γ as follows,

$$\gamma(\Gamma_1^{[1]}\Gamma_2^{[1]}\Gamma_3^{[1]}) = IKJ.$$

Assume that for $p-1$ the volume element gets mapped as

$$\gamma(\omega) = \gamma(\Gamma_1^{[p-1]}\Gamma_2^{[p-1]}\ldots\Gamma_{2p-1}^{[p-1]}) = \otimes_{l=1}^{p-1} K^{l-1} I K^l J.$$

Then

$$\gamma(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{2p+1}^{[p]}) = \gamma((-1)^{p-1}\Gamma_p^{[p]}(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{p-1}^{[p]}\Gamma_{p+1}^{[p]}\ldots\Gamma_{2p}^{[p]})\Gamma_{2p+1}^{[p]}).$$

From Remark 4.18 it follows that

$$\gamma(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{2p+1}^{[p]}) = \gamma((-1)^{p-1}\Gamma_p^{[p]}(\Gamma_1^{[p-1]}\Gamma_2^{[p-1]}\ldots\Gamma_{2p-1}^{[p-1]}\otimes K^{2p-1})\Gamma_{2p+1}^{[p]}).$$

Now due to the induction hypothesis we have

$$\gamma(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{2p+1}^{[p]}) = (-1)^{p-1}\gamma(\Gamma_p^{[p]})(\otimes_{l=1}^{p-1} K^{l-1} I K^l J\otimes K^{2p-1})\gamma(\Gamma_{2p+1}^{[p]}).$$

By substituting $\gamma(\Gamma_p^{[p]}) = \underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{p-1}\otimes I$ and $\gamma(\Gamma_{2p+1}^{[p]}) = \underbrace{\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}_{p-1}\otimes J$ into the previous equation and simplifying we get

$$\begin{aligned}
\gamma(\Gamma_1^{[p]}\Gamma_2^{[p]}\ldots\Gamma_{2p+1}^{[p]}) &= \otimes_{l=1}^{p-1} K^{l-1} I K^l J\otimes(-1)^{p-1} I K^{2p-1} J\\
&= \otimes_{l=1}^{p-1} K^{l-1} I K^l J\otimes(-1)^{p-1} I K^{p-1} K^p J\\
&= \otimes_{l=1}^{p} K^{l-1} I K^l J.
\end{aligned}$$

From now on we will drop the superscript p from the symbol $\Gamma_j^{[p]}$ denoting the generators of $\mathcal{C}\ell(p,p+1)$ and we observe the following:

$$\begin{aligned}
\gamma(\Gamma_1\Gamma_2\ldots\Gamma_{2p+1}) &= \otimes_{l=1}^{p} K^{l-1} I K^l J\\
&= \otimes_{l=1}^{p} (-1)^{l-1} I K^{2l-1} J\\
&= \otimes_{l=1}^{p} (-1)^{l-1} IKJ\\
&= \otimes_{l=1}^{p} (-1)^{l}
\end{aligned}$$

$$= (-1)^{\frac{p(p+1)}{2}} \operatorname{Id}.$$

From this, we can not only conclude that $\Gamma_1\Gamma_2 \dots \Gamma_{2p+1} + (-1)^{\frac{p(p+1)}{2}+1} \operatorname{Id}$ belongs to the kernel of γ but also that the set described in the statement of this lemma does. We can further simplify and show that the elements of this set are either of the form $\Gamma_1\Gamma_2 \dots \Gamma_{2p+1} + (-1)^{\frac{p(p+1)}{2}+1} \operatorname{Id}$ or of the form $\Gamma_{i_1} \dots \Gamma_{i_k} \pm \Gamma_{i_{k+1}} \dots \Gamma_{i_{2p+1}}$ where $1 \leq k \leq p$, $i_1, \dots, i_{2p+1} \in \{1, \dots, 2p+1\}$, $i_1 < \dots < i_k$ and $i_{k+1} < \dots < i_{2p+1}$. It is now easy to see that these elements are linearly independent. A straightforward combinatorial computation shows that the total number of elements of the stated form is $\sum_{j=0}^{p} \binom{2p+1}{j}$ which is equal to $\dim \mathcal{C}\ell(p, p+1) - \dim \operatorname{End} \mathbb{R}^{2^p}$. Thus we can conclude that these elements form a basis of the kernel of γ. $\square$

Using dimensional arguments and the basis for the kernel of γ from Lemma 4.22, we can further conclude that the set described below forms a basis of $\operatorname{End}(\mathbb{R}^{2^p})$.

$$\{\gamma_{\Gamma_\mu} \mid \mu = (i_1, i_2, \dots, i_k),\ 0 \leq \mid \mu \mid \leq p,\ 1 \leq i_1 < \dots < i_k \leq 2p+1 \text{ and } 1 \leq k \leq p\} \tag{4.10}$$

where $\gamma_{\Gamma_\mu} = \operatorname{Id}$ when $|\mu| = 0$.

Before concluding the proof of Lemma 4.16, we will prove it in the special case of $\mathcal{C}\ell(2,3)$ as an illustration.

Lemma 4.23. *Consider the Clifford algebra $\mathcal{C}\ell(2,3)$ and let (S, γ) be an irreducible $\mathcal{C}\ell(2,3)$-module. Let $A \in \operatorname{End}(S)$ and assume that for all $v \in E$ and for some $\lambda \in \mathbb{R}$ we get $\{A, \gamma_v\} = \lambda \operatorname{Id}_S$. Then $A = \gamma_u$ for some $u \in E$.*

Proof. Consider $\mathcal{C}\ell(2,3)$ and the irreducible representation $\gamma : \mathcal{C}\ell(2,3) \longrightarrow \operatorname{End}(\mathbb{R}^{2^2})$ that linearly maps $\Gamma_i \mapsto \pi_1 \circ \Phi(\Gamma_i)$ for all generators Γ_i of $\mathcal{C}\ell(2,3)$ where π_1 is the projection onto the first component of $\mathbb{R}[4] \oplus \mathbb{R}[4]$. Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ be a set of generators for $\mathcal{C}\ell(2,3)$. Then $A \in \operatorname{End}(\mathbb{R}^{2^2})$ has the following expression

$$A = \lambda \operatorname{Id} + \sum_{i=1}^{5} \lambda_i \gamma_i + \sum_{\substack{i<j \\ i,j=1}}^{5} \lambda_{ij} \gamma_{ij},$$

where $\gamma_{ijk} := \gamma_{\Gamma_i}\gamma_{\Gamma_j}\gamma_{\Gamma_k}$. Imposing the condition that $\{A, \gamma_i\} = \eta_i \operatorname{Id}$ for some $\eta_i \in \mathbb{R}$, we obtain the following equations.

$$\eta_1 \operatorname{Id} = 2\lambda\gamma_1 - \lambda_1 \operatorname{Id} + 2\lambda_{23}\gamma_{123} + 2\lambda_{24}\gamma_{124} + 2\lambda_{25}\gamma_{125} + 2\lambda_{34}\gamma_{134} + 2\lambda_{35}\gamma_{135} + 2\lambda_{45}\gamma_{145}$$

$$\eta_2 \operatorname{Id} = 2\lambda\gamma_2 - \lambda_2 \operatorname{Id} + 2\lambda_{13}\gamma_{213} + 2\lambda_{14}\gamma_{214} + 2\lambda_{15}\gamma_{215} + 2\lambda_{34}\gamma_{234} + 2\lambda_{35}\gamma_{235} + 2\lambda_{45}\gamma_{245}$$

$$\eta_3 \operatorname{Id} = 2\lambda\gamma_3 + \lambda_3 \operatorname{Id} + 2\lambda_{12}\gamma_{123} + 2\lambda_{14}\gamma_{314} + 2\lambda_{15}\gamma_{315} + 2\lambda_{24}\gamma_{324} + 2\lambda_{25}\gamma_{325} + 2\lambda_{45}\gamma_{345}$$

$$\eta_4 \operatorname{Id} = 2\lambda\gamma_4 + \lambda_4 \operatorname{Id} + 2\lambda_{12}\gamma_{124} + 2\lambda_{13}\gamma_{134} + 2\lambda_{15}\gamma_{415} + 2\lambda_{23}\gamma_{234} + 2\lambda_{25}\gamma_{254} + 2\lambda_{35}\gamma_{354}$$

$$\eta_5 \operatorname{Id} = 2\lambda\gamma_5 + \lambda_5 \operatorname{Id} + 2\lambda_{12}\gamma_{125} + 2\lambda_{13}\gamma_{135} + 2\lambda_{14}\gamma_{145} + 2\lambda_{23}\gamma_{235} + 2\lambda_{24}\gamma_{245} + 2\lambda_{34}\gamma_{345}$$

Since for any five pairwise distinct i, j, k, l, m we have $\gamma_{ijk} = \pm\gamma_{lm}$, the latter of which belongs to the basis of $\operatorname{End}(\mathbb{R}^{2^2})$, we can conclude that if $\{A, \gamma_v\} = \eta \operatorname{Id}$ for all $v \in \mathbb{R}^{2,3}$, then $A = \gamma_u$ for some $u \in \mathbb{R}^{2,3}$. □

4.3.1 Proof of Lemma 4.16

We are now ready to prove Lemma 4.16 in full generality:

Proof of Lemma 4.16. Consider $\mathcal{C}\ell(p, p+1)$ and let $\{\Gamma_1, \ldots, \Gamma_{2p+1}\}$ be a set of Clifford generators. Consider the irreducible representation $\gamma : \mathcal{C}\ell(p, p+1) \longrightarrow \operatorname{End}(\mathbb{R}^{2^p})$ that linearly maps $\Gamma_i \mapsto \pi_1 \circ \Phi(\Gamma_i)$ for all generators Γ_i of $\mathcal{C}\ell(p, p+1)$ where π_1 is the projection onto the first component of $\mathbb{R}[2]^{\otimes p} \oplus \mathbb{R}[2]^{\otimes p}$. Let $\gamma_\mu := \gamma_{\Gamma_\mu}$ where $\mu = (i_1, i_2, \ldots, i_k)$, $0 \leq |\mu| \leq p$, $1 \leq i_1 < \cdots < i_k \leq 2p+1$ and $1 \leq k \leq p$. Then $A \in \operatorname{End}(\mathbb{R}^{2^p})$ has the following expression in terms of the basis of $\operatorname{End}(\mathbb{R}^{2^p})$ as in (4.10):

$$\begin{aligned} A &= \sum_\mu \lambda_\mu \gamma_\mu \\ &= \lambda_0 \operatorname{Id} + \sum_{t=1}^{2p+1} \left(\sum_{\substack{i_1, \ldots, i_t \\ i_1 < \cdots < i_t \\ i_1, \ldots, i_t \in \{1, \ldots, 2p+1\}}} \lambda_{i_1 \ldots i_t} \gamma_{i_1 \ldots i_t} \right) \end{aligned} \tag{4.11}$$

where $\lambda_\mu \in \mathbb{R}$. Let $\epsilon_l = +1$ for all $1 \leq l \leq p$ and $\epsilon_l = -1$ for all $p+1 \leq l \leq 2p+1$. For each k such that $1 \leq k \leq 2p+1$ by imposing the condition that $\{A, \gamma_k\} = \eta_k \operatorname{Id}$ where $\eta_k \in \mathbb{R}$, we obtain the following equation:

$$\eta_k \operatorname{Id} = 2\lambda_0 \gamma_k + \sum_{t=1}^{2p+1} \left(\sum_{\substack{i_1, \ldots, i_t \\ i_1 < \cdots < i_t \\ i_1, \ldots, i_t \in \{1, \ldots, 2p+1\}}} \lambda_{i_1 \ldots i_t} \{\gamma_{i_1 \ldots i_t}, \gamma_k\} \right) \tag{4.12}$$

Observe that there are four cases to consider in order to determine $\{\gamma_\mu, \gamma_k\}$ when $|\mu| \geq 2$ namely:

Case (1) $|\mu|$ is even and $k \in \mu$:

In this case, the length of the string μ is $|\mu| = 2r$ for some $r \in \mathbb{N}$. Since the index $k \in \mu$ its position in the string μ will divide μ into parts of length x and y

$$(\underbrace{\ldots\ldots}_{x}k\underbrace{\ldots\ldots}_{y})$$

such that $2r = x + 1 + y$. Since this would imply that $x + y = 2r - 1$, this would further imply that only one of x and y is even while the other is odd. From this we can conclude that $\{\gamma_\mu, \gamma_k\} = 0$.

Case (2) $|\mu|$ is even and $k \notin \mu$:

When $k \notin \mu$ and the length of μ is even $\{\gamma_\mu, \gamma_k\} = 2\gamma_{\mu'}$ where μ' is a multi index of length $|\mu| + 1$.

Case (3) $|\mu|$ is odd and $k \in \mu$:

In this case, the length of the string μ is $|\mu| = 2r + 1$ for some $r \in \mathbb{N}$. Since the index k is such that $k \in \mu$, its position in the string μ will divide μ into parts of length x and y

$$(\underbrace{\ldots\ldots}_{x}k\underbrace{\ldots\ldots}_{y})$$

such that $2r+1 = x+1+y$. Since this would imply that $x+y = 2r$, we would have two possibilities, x and y are both even or both odd. When x and y are both even, $\{\gamma_\mu, \gamma_k\} = 2\epsilon_k\gamma_{\mu'}$ where μ' is a multi-index of length $|\mu| - 1$. When x and y are both odd, $\{\gamma_\mu, \gamma_k\} = -2\epsilon_k\gamma_{\mu'}$.

Case (4) $|\mu|$ is odd and $k \notin \mu$:

It is clear that in this case again $\{\gamma_\mu, \gamma_k\} = 0$.

In the simplified expression for $\{A, \gamma_k\}$ after incorporating the results for $\{\gamma_\mu, \gamma_k\}$ coming from the above considerations, no two terms will have the same basis element of $\operatorname{End}\mathbb{R}^{2^p}$. To demonstrate this, consider the expression (4.11) that we have for A in terms of the basis in (4.10) of $\operatorname{End}\mathbb{R}^{2p}$. The indices of the basis elements γ_μ have length $|\mu| \in \{0, \ldots, p\}$. In the anti-commutator $\{A, \gamma_k\}$ the terms $\{\gamma_\mu, \gamma_k\}$ will be equal to $a\gamma_{\mu'}$ for some appropriate scalar $a \in \mathbb{R}$. By carefully observing the way the length of μ changes upon the action of the anti-commutator with γ_k on A according to the above analysis, we find that if μ is of even length, then the term becomes 0 or $|\mu'| = |\mu|+1$ and if μ is of odd length, then the term becomes 0 or $|\mu'| = |\mu|-1$. This means that all the terms whose basis elements have indices of length 1 will either become 0 or some scalar times Id, all the terms whose basis elements have indices of length 2 will either become 0 or some scalar times basis elements whose indices are of length 3 while all the terms whose basis elements have indices of length 3 will either become 0 or some scalar times basis elements whose indices are of length 2 and so on.

Observe that if p is odd, then all the terms whose basis elements have indices of length p will either become 0 or some scalar times basis elements whose indices are of length $p-1$. There will be no basis elements whose indices are of length $p-1$ that come from anywhere other than this reduction. Therefore we can be certain that in the simplified expression for $\{A, \gamma_k\}$ after incorporating the results for $\{\gamma_\mu, \gamma_k\}$, no two terms will have the same basis element. Below is a helpful diagram to illustrate this. Here the numbers denote the length of μ and the arrows denote the length to which μ changes upon the action of the anti-commutator with γ_k on A, when the terms don't vanish:

$$\begin{tikzcd}
0 \arrow[dr, shift left] & & 2 \arrow[dr, shift left] & & 4 \arrow[dr, shift left] & & \cdots & & p-1 \arrow[dr, shift left] & \\
& 1 \arrow[ul, shift left] & & 3 \arrow[ul, shift left] & & 5 \arrow[ul, shift left] & & \cdots & & p \arrow[ul, shift left]
\end{tikzcd}$$

If p is even, then all the terms whose basis elements have indices of length p will either become 0 or some scalar times basis elements whose indices are of length $p+1$. When $|\mu| = p+1$ we know from the description of the kernel of γ (cf. Lemma 4.22) that $\gamma_\mu = \pm\gamma_{\mu^c}$ where $\mu^c := \{1, \ldots, 2p+1\} \setminus \mu$ is the complement of the set underlying μ in the set $\{1, \ldots, 2p+1\}$. Since in this situation there are no basis elements whose indices are of length p that are coming from anywhere other than this reduction, we can be certain that in the simplified expression for $\{A, \gamma_k\}$ after incorporating the results for $\{\gamma_\mu, \gamma_k\}$, no two terms will have the same basis element. Below is a helpful illustrative diagram, analogous to the one above:

$$\begin{tikzcd}
0 \arrow[dr, shift left] & & 2 \arrow[dr, shift left] & & 4 \arrow[dr, shift left] & & \cdots & & p \arrow[dr, shift left] & \\
& 1 \arrow[ul, shift left] & & 3 \arrow[ul, shift left] & & 5 \arrow[ul, shift left] & & \cdots & & p+1 \arrow[ul, shift left]
\end{tikzcd}$$

Computing $\{A, \gamma_k\}$ for every $1 \leq k \leq 2p+1$ in this way, we obtain a system of equations of the below form:

$$\begin{aligned}
\eta_k \,\mathrm{Id} &= \sum_{\substack{\mu \\ |\mu|=2q \\ k \notin \mu}} \lambda_\mu \{\gamma_\mu, \gamma_k\} + \sum_{\substack{\mu \\ |\mu|=2q+1 \\ k \in \mu}} \lambda_\mu \{\gamma_\mu, \gamma_k\} \\
&= \sum_{\substack{\mu \\ |\mu|=2q \\ k \notin \mu \\ |\mu'|=|\mu|+1}} 2\lambda_\mu \gamma_{\mu'} - \sum_{\substack{\mu \\ |\mu|=2q+1 \\ k \in \mu \\ |\mu'|=|\mu|-1}} 2\lambda_\mu \epsilon_k \gamma_{\mu'}
\end{aligned} \tag{4.13}$$

where $q \geq 0$ and in each equation no two terms will have the same basis element. Comparing the two sides of the equation, we can then conclude that the only non-zero coefficients are λ_k where $1 \leq k \leq 2p+1$. From this we can conclude that $A = \gamma_u$ for some $u \in E \cong \mathbb{R}^{2p+1}$. $\square$

We are now ready to prove Theorem 4.10:

4.3.2 Proof of Theorem 4.10

Proof of Theorem 4.10. From Lemma 4.11 it is clear that if there exists a DGO $\not{d}$ for a Courant algebroid $E \longrightarrow M$ on an irreducible $\mathcal{C}\ell(E)$-module bundle (S, γ), then $\not{d} + \gamma_e$ is a DGO for every $e \in V_{\not{d}} = \{e \in \Gamma(E) \mid \{\not{d}, \gamma_e\} \in \mathcal{C}^\infty(M)\}$. Conversely, we have to show that given DGOs $\not{d}$ and $\not{d}'$ there exists an $e \in V_{\not{d}}$ such that $L := \not{d}' - \not{d} = \gamma_e$. First we observe that $[L, f]$ is a 0-th order operator for all $f \in \mathcal{C}^\infty(M)$ due to Lemma 4.12. Then we see from Lemma 4.14 that $[L, f]$ is not only $\mathcal{C}\ell^0(E)$-equivariant but it also anti-commutes with $\Gamma(\mathcal{C}\ell^1(E))$. From this, it follows that $[L, f] = 0$ as seen in Lemma 4.15. Given this, we can now conclude that L is in fact of 0-th order due to Lemma 4.12. Now if L is of 0-th order and γ is surjective, $L = \gamma_a$ for some $a \in \Gamma(\mathcal{C}\ell(E))$. Next we observe that $\{L, \gamma_v\}$ commutes with $\Gamma(\mathcal{C}\ell(E))$ in virtue of the property 1 of DGOs as in Definition 4.1 and therefore it is a section of the Schur bundle of the representation γ. Since the signature of E is $(p, p+1)$ the Schur algebra bundle $\mathcal{C}(S, \gamma)$ is isomorphic to the trivial real line bundle therefore we can conclude that $\{L, \gamma_v\}$ is a scalar operator. From Lemma 4.16, we can conclude that $L = \gamma_e$ where $e \in \Gamma(E)$. Now property 3 of the Definition 4.1 applied to $\not{d}' = \not{d} + \gamma_e$ implies that $e \in V_{\not{d}}$.

The claim that $V_{\not{d}}$ is independent of the choice of $\not{d}$ is now immediate because if $\not{d}$ and $\not{d}'$ are two DGOs, then $\not{d}' = \not{d} + \gamma_e$ for $e \in V_{\not{d}} \subset \Gamma(E)$. Now applying property

3 of Definition 4.1 to $\not{d}'$ and due to the symmetry in exchanging $\not{d}'$ and $\not{d}$, it follows that $V_{\not{d}} = V_{\not{d}'}$. $\square$

Remark 4.24. We have stated and proved Theorem 4.10 for the signature $(p, p+1)$. Since in virtue of the classification of Clifford algebras, any Clifford algebra $\mathcal{C}\ell(r, s)$ for $r - s \equiv 7 \bmod 8$ is isomorphic to $\mathcal{C}\ell(p, p+1)$ for some p, the result is true more generally also in these cases.

Bibliography

[AM] Alekseev, A. and Meinrenken, E. *The non-commutative Weil algebra*. Inventiones mathematicae 139.1 (2000): 135-172. arXiv:math/9903052v1 [math.DG]

[AX] Alekseev, A., Xu, P., *Derived brackets and Courant algebroids*, un-published, available at https://www.personal.psu.edu/pxx2/anton-final.pdf, 2001.

[B] Baum, H. *Eichfeldtheorie*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[CCLX] Cai, L., Chen, Z., Lang, H. and Xiang, M. *Dirac generating operators of split Courant algebroids*. arXiv:2206.10091 [math.DG]

[CaG] Cavalcanti, G. and Gualtieri, M. *Generalized complex geometry and T-duality*. arXiv:1106.1747 [math.DG]

[CS] Chen, Z. and Stiénon, M. *Dirac generating operators and Manin triples*. Journal of the London Mathematical Society 79.2 (2009): 399-421. arxiv:0803.2376v3 [math.DG]

[CSX] Chen, Z., Stiénon, M. and Xu, P. *On regular Courant algebroids*. Journal of Symplectic Geometry 11.1 (2013): 1-24. arXiv:0909.0319 [math.DG]

[Ch] Chevalley, C. *The algebraic theory of spinors*. The Algebraic Theory of Spinors. Columbia University Press, 1954.

[CD1] Cortés, V. and David, L., *Generalised connections, spinors and integrability of generalized Courant algebroids*, Moscow Math. J., Vol 21 Issue 4, pp. 695–736, 2021.

[CD2] Cortés, V. and David, L., *T-duality for transitive Courant algebroids*, arXiv:2101.07184v2 [math.DG], 2021.

[Co] Courant, T. J. *Dirac manifolds.* Transactions of the American Mathematical Society 319.2 (1990): 631-661.

[DK] Drozd, Y.A. and Kirichenko, V.V. *Finite dimensional algebras.* Springer Science & Business Media, 2012.

[DF] Dummit, D. S. and Foote, R. M. *Abstract algebra.* Vol. 3. Hoboken: Wiley, 2004.

[F] Figueroa-O'Farrill, J. *Spin geometry.* Lecture Notes. University of Edinburgh (2010). Version of 18th May 2017 retrieved from https://empg.maths.ed.ac.uk/Activities/Spin/SpinNotes.pdf

[FG] Figueroa-O'Farrill, J. and Gadhia, S. *Supersymmetry and spin structures.* Classical and Quantum Gravity 22.20 (2005): L121. arxiv:hep-th/0506229v1

[FV] Freedman, D. and Van Proeyen, A. *Supergravity.* Cambridge University Press, 2012. Available at http://www.nucleares.unam.mx/~alberto/apuntes/freedman.pdf

[F] Friedrich, T. *Dirac operators in Riemannian geometry.* Vol. 25. American Mathematical Soc., 2000.

[FT] Friedrich, T. and Trautman, A. *Spin spaces, Lipschitz groups, and spinor bundles.* Annals of Global Analysis and Geometry 18.3 (2000): 221-240. arxiv:math/9901137

[GF] Garcia-Fernandez, M. *Ricci flow, Killing spinors, and T-duality in generalized geometry.* Advances in Mathematics 350 (2019): 1059-1108. arxiv:1611.08926v3 [math.DG]

[GFS] Garcia-Fernandez, M. and Streets, J. *Generalized Ricci Flow.* Vol. 76. American Mathematical Soc., 2021. arXiv:2008.07004v2 [math.DG]

[GMX] Grützmann, M., Michel, J.-P. and Xu, P. *Weyl quantization of degree 2 symplectic graded manifolds.* Journal de Mathématiques Pures et Appliquées 154 (2021): 67-107. arXiv:1410.3346v2 [math.DG]

[G] Gualtieri, M. *Generalized complex geometry.* PhD Thesis, University of Oxford, 2003. arXiv:math/0401221

[Hi] Hitchin, N. *Generalized Calabi–Yau manifolds.* Quarterly Journal of Mathematics 54.3 (2003): 281-308. arXiv:math/0209099

[JS] Jantzen, J. C. and Schwermer J. *Algebra.* Springer-Verlag, 2013.

[K] Kosmann-Schwarzbach, Y. *Courant algebroids. A short history.* SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 9 (2013): 014. arXiv:1212.0559v2 [math.HO]

[Ko] Kostant, B. *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups.* Duke Mathematical Journal 100.3 (1999): 447-501.

[L] Lang, S. *Algebra.* Vol. 211. Springer Science & Business Media, 2012.

[LM] Lawson, H. B. and Michelsohn, M.-L. *Spin Geometry (PMS-38), Volume 38.* Vol. 84. Princeton University Press, 2016.

[LS1] Lazaroiu, C.I. and Shahbazi, C.S. *Real pinor bundles and real Lipschitz structures.* Asian Journal of Mathematics, Vol. 23, No. 5 (2019), pp. 749-836. arXiv:1606.07894 [math.DG]

[LS2] Lazaroiu, C. I., and C. S. Shahbazi. *Complex Lipschitz structures and bundles of complex Clifford modules.* Differential Geometry and its Applications 61 (2018): 147-169. arXiv:1711.07765 [math.DG]

[LWX] Liu, Z.-J., Weinstein, A. and Xu, P. *Manin triples for Lie bialgebroids.* Journal of Differential Geometry 45.3 (1997): 547-574. arXiv:dg-ga/9508013v3

[N] Nakamura, N. *$Pin^-(2)$-monopole invariants.* Journal of Differential Geometry 101.3 (2015): 507-549. arXiv:1303.4870v3 [math.GT]

[R1] Roytenberg, D. *Courant algebroids, derived brackets and even symplectic supermanifolds.* University of California, Berkeley, 1999. arXiv:math/9910078

[R2] Roytenberg, D. *On the structure of graded symplectic supermanifolds and Courant algebroids.* Contemporary Mathematics 315 (2002): 169-186. arXiv:math/0203110v1 [math.SG]

[Ru] Rubio, R. *Generalized geometry of type B_n.* PhD thesis, University of Oxford, 2014.

[Sa] Salamon, D. *Spin geometry and Seiberg-Witten invariants.* Lecture Notes, ETH Zurich, (1996). Available at https://people.math.ethz.ch/~salamon/PREPRINTS/witsei.pdf

[Š] Ševera, P. *Letters to Alan Weinstein about Courant algebroids.* arXiv:1707.00265 [math.DG]

[ŠV] Ševera, P. and Valach, F. *Courant algebroids, Poisson–Lie T-duality, and type II supergravities.* Communications in Mathematical Physics 375.1 (2020): 307-344. arXiv:1810.07763v2 [math.DG]

[U] Uchino, K. *Remarks on the definition of a Courant algebroid.* Letters in Mathematical Physics 60.2 (2002): 171-175. arXiv:math/0204010 [math.DG]

[V] Vaisman, I. *Transitive Courant algebroids.* International Journal of Mathematics and Mathematical Sciences 2005.11 (2005): 1737-1758. arXiv:math/0407399v1 [math.DG]

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 16.09.2022

Manasa Manjunatha