Relative Serre functors and pivotality in categorical Morita theory

Dissertation

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare upon oath that I have written the present dissertation independently and have not used further resources and aids than those stated in the dissertation.

David Jaklitsch

David Jaklitsch
Hamburg, den 11.04.2023

Summary

In this thesis we study the interaction of pivotal structures on tensor categories with categorical Morita theory and establish foundational results in the theory from a bicategorical perspective. The dissertation starts with an introductory section presenting the motivation behind the problem of study, its connections with topological field theories and the results achieved in this work. The structure of the document is as follows:

- In Chapter 1 the pertinent definitions and conventions about the theory of finite tensor categories are summarized. Further preliminary material regarding relative Serre functors and Nakayama functors is also included.

- In Chapter 2 we discuss the notion of a categorical Morita context. We describe the two-object bicategory associated to every Morita context. Furthermore, we show that every exact module category induces a *strong* Morita context, and conversely that any such strong Morita context stems from an exact module category.

- In Chapter 3 we investigate the nature of *dualities* for Morita contexts, by showing that its associated bicategory admits duals (adjoints). Moreover, we describe double-duals in terms of relative Serre functors, which leads to Radford theorems for module categories and for Morita context bicategories.

- In Chapter 4 we introduce the notion of *pivotal Morita equivalence*. The first main result of this chapter is a characterization of this notion of equivalence in terms of the pivotal bicategories of pivotal modules. Secondly, we show that pivotal tensor categories that are pivotal Morita equivalent have (pivotal braided) equivalent Drinfeld centers, leading to immediate applications to oriented topological field theories. We further discuss a notion of *sphericality* for module categories.

- In Chapter 5 we make a detour to discuss categorical Morita theory in the *equivariant* setting of tensor categories graded over a finite group G.

- In Chapter 6 we study the properties of traces arising from pivotal structures on bimodule categories with a particular emphasis on the spherical semisimple case.

- In Chapter 7 we reformulate the Turaev-Viro *state sum* construction in terms of structures developed in the thesis, namely the bicategory of spherical modules over a spherical fusion category. The main result of the chapter shows that the constructed topological invariant of oriented 3d-manifolds is independent of the choice of skeleton used to define it.

Zusammenfassung

In dieser Doktorarbeit untersuchen wir die Interaktion zwischen Pivotalstrukturen auf Tensorkategorien und Morita-Theorie, und zeigen wir grundlegende Ergebnisse der Theorie aus einer bikategorialen Perspektive. Die Dissertation beginnt mit einem einleitenden Abschnitt, in dem die Motivation für das zu untersuchende Problem, seine Verbindungen zu topologischen Feldtheorien und die in dieser Arbeit bewiesenen Ergebnisse vorgestellt werden. Die Struktur des Dokuments ist wie folgt:

- In Kapitel 1 werden die notwendigen Definitionen und Konventionen über die Theorie der Tensorkategorien zusammengefasst. Weitere vorbereitende Konzepte und Ergebnisse zu relativen Serre-Funktoren und Nakayama-Funktoren sind ebenfalls enthalten.

- In Kapitel 2 diskutieren wir kategoriale Morita-Kontexte. Wir beschreiben, wie ein Morita-Kontext eine Zwei-Objekt-Bikategorie definiert. Außerdem zeigen wir, dass jede exakte Modulkategorie einen starken Morita-Kontext erzeugt, und dass jeder solche starke Morita-Kontext aus einer exakten Modulkategorie kommt.

- In Kapitel 3 untersuchen wir duale Objekte in Morita-Kontexten. Wir zeigen, dass die Morita-Kontext-Bikategorie Duale (adjungierte 1-Morphismen) hat.
  Außerdem, beschreiben wir die Doppelduale als relative Serre-Funktoren, was zu Radford-Theoremen für Modulkategorien und für Morita-Kontext-Bikategorien führt.

- In Kapitel 4 definieren wir *pivotale Morita-Äquivalenz*. Das erste Hauptergebnis dieses Kapitels zeigt: die pivotale Bikategorie der pivotalen Modulkategorien charakterisiert pivotale Morita-Äquivalenz. Zweitens zeigen wir, dass pivotale Tensorkategorien, die pivotal Morita-äquivalent sind, (pivotal verzopfte) äquivalente Drinfeld-Zentren haben. Dieses Ergebnis hat direkte Anwendungen auf orientierte topologische Feldtheorien. Wir diskutieren außerdem sphärische Modulkategorien.

- In Kapitel 5 diskutieren wir die äquivariante Morita-Theorie für graduierte Tensorkategorien über einer endlichen Gruppe G.

- In Kapitel 6 untersuchen wir die Eigenschaften von Spuren, die von Pivotalstrukturen auf Bimodulkategorien stammen, mit besonderem Interesse am sphärischen und halbeinfachen Fall.

- In Kapitel 7 formulieren wir eine Turaev-Viro-Konstruktion aus der pivotale Bikategorie der sphärischen Modulkategorien über einer sphärischen Fusionskategorie, die in der Dissertation entwickelt wurde. Das Hauptergebnis des Kapitels zeigt: die konstruierte topologische Invariante von orientierten 3D-Mannigfaltigkeiten ist *skeleton*unabhängig.

Contents

Introduction

Associative algebras over a field $\Bbbk$ are ubiquitous fundamental structures in mathematics and physics. A common approach to study an algebraic structure is by means of its representations. Many properties of a $\Bbbk$-algebra A (e.g. semisimple, Noetherian, Artinian, self-injective, etc) are captured by its linear category of modules A-mod, establishing it as a fundamental mathematical object. This leads to the notion of *Morita equivalence*: two algebras are Morita equivalent iff their categories of modules are equivalent, or equivalently, iff there exists an invertible bimodule between them. For example, the algebra $\Bbbk$ is Morita equivalent to the algebra of n-by-n square matrices $M_n(\Bbbk)$ with coefficients in $\Bbbk$ for any order $n \geq 1$. Under certain *finiteness* conditions every $\Bbbk$-linear category is of the form $\mathcal{C} \simeq A$-mod for a finite dimensional algebra A determined only up to Morita equivalence. Frequently, one encounters situations where these categories are given in more abstract terms without making a choice of an algebra realizing it, and therefore from a modern perspective it is desirable to aim for Morita invariant formulations which are purely categorical.

An additional datum to consider on an algebra is that of a symmetric Frobenius form (which can be expressed as a homotopy fixed point [HSV]). These type of structures termed *symmetric Frobenius* $\Bbbk$-*algebras* emerge naturally in countless areas of mathematics, as well: in representation theory of groups in finite characteristic as rings of characters, in differential geometry as cohomology rings of compact oriented manifolds or in commutative algebra as artinian Gorenstein rings.

Topological field theories (TFT's) provide a mechanism to systematically organize algebraic structures. Broadly speaking, these theories are representations of certain geometric categories of manifolds and manifolds with boundary and corners in an algebraic category. Indeed, a topological field theory is defined as a (symmetric monoidal) functor from some category of cobordisms into some target category of algebraic nature. Algebraic objects are then realized as the image of standard manifolds under the TFT functor. The prototypical example is the fact that an oriented two-dimensional TFT with values in the category of vector spaces maps an oriented circle to a vector space with the structure of a commutative (and thus symmetric) Frobenius algebra [Ko].

There are diverse flavours of geometric categories of cobordisms to consider as a source of a topological field theory. These different types of geometry arise, for instance, as tangential structures on manifolds such as G-bundles or framings, i.e. trivializations of the tangent bundle. Framings are additional structure on a manifold, TFT's compensate this by requiring fewer structures on the algebraic countepart: a framed two-dimensional theory evaluated on a suitably framed circle yields a vector space with the structure of a $\Bbbk$-algebra, as opposed to the Frobenius algebras appearing in the oriented case. Consequently, the two structures of an algebra and a symmetric Frobenius algebra have clear distinct geometric counterparts, and thus should be carefully regarded as two different algebraic worlds.

Categories of cobordisms, and hence topological field theories, are defined in any dimension. It is in higher dimensions that TFT's can show their true power and give insights into more intricate objects, more precisely into higher algebraic structures. For example, in three dimensions one comes across *tensor categories*, which serve as categorical analogues for $\Bbbk$-algebras [EGNO]. Instead of vector spaces, $\Bbbk$-linear categories are in play. The product is replaced by a tensor product functor and associativity, a property for algebras, is now promoted to an associator which is an additional structure. In three dimensions, one can again consider different tangential structures on manifolds, which gives rise to distinct higher algebraic objects: tensor categories for framed theories and pivotal tensor categories for oriented theories, as depicted in Table 1. These two worlds of algebraic structures on categories should be carefully told apart, as well.

Dimension	**Framed manifolds**	**Oriented manifolds**
2d	Algebras	Symmetric Frobenius algebras
3d	Tensor categories	Pivotal tensor categories

Table 1: Algebraic structures related to manifold invariants.

Some extent of imbalance is present in the literature: from a geometric and physical point of view, the study of oriented topological field theories is a clear target in current research. On the algebraic side, the theory of categories with pivotal structures is much less developed. One of the main purposes of this thesis is to contribute precisely at this point and thereby provide tools for the study and construction of three-dimensional TFT's with defects and boundaries.

Categorical Morita theory and pivotality

Now, let us be somewhat more specific. As mentioned before, the structure of a finite-dimensional algebra finds its categorical analogue in the notion of a *finite tensor category*, e.g. the category of finite-dimensional representations of a finite-dimensional Hopf algebra [EGNO]. These are finite $\Bbbk$-linear categories together with a linear monoidal structure and a simple monoidal unit. Additionally, the property that every object admits *dual objects* is required. This is a phenomenon appearing at the categorical level that reproduces the notion of dual of a finite-dimensional vector space and that can be interpreted as a relaxed version of invertibility.

Finite-dimensional vector spaces have further the special feature that they are canonically isomorphic to their double-dual in a way that is compatible with the tensor product of vector spaces. In other finite tensor categories, such as the examples coming from Hopf algebras, the situation can be increasingly more complex: duals are obtained using the antipode of the Hopf algebra, the famous Radford S^4 theorem from the 70's only makes a general statement about the fourth power of the antipode, but the square of the antipode may not have additional properties.

The notion of a *pivotal structure* on a tensor category consists of a monoidal trivialization of the double-dual functor (which again could be described as a homotopy fixed point [DSS2]).

Pivotal finite tensor categories are then finite tensor categories together with the choice of a pivotal structure. Such data control the dualities in the tensor category, allow to define traces for endomorphisms and are themselves a prolific source of Frobenius algebras [FuS, Sh2] internal to these categories.

In the study of finite tensor categories there is a well-understood *categorical Morita theory* [EGNO]. Module categories over finite tensor categories play the role of modules over algebras. In the case of the finite tensor category coming from a Hopf algebra H, its module categories are realized by comodule algebras over H. The Morita equivalence relation on tensor categories can be defined essentially in the same way as for algebras: two tensor categories are Morita equivalent iff there is an invertible bimodule category connecting them.

One of the main purposes of this thesis is to develop a comprehensive *pivotal Morita theory* for pivotal tensor categories. In such pivotal setting, some first steps in this direction have been made: crucial concepts such as the notion of a relative Serre functor [FSS], which allows to define pivotal module categories [Sh2], have been recently introduced. As we prove in Chapter 3, in a suitable setting, the image of an object in a module category under the relative Serre functor actually *is* its double-dual. This insight is fundamental, since it provides a conceptual explanation on why appropriate trivializations of the relative Serre functor should be regarded as pivotal structures on module categories. These concepts become foundational for the pivotal Morita theory that we establish in this dissertation.

The study of categorical Morita theory is also significant in the context of three-dimensional topological field theories. In the case of framed theories, it has applications to invertible topological defects between framed modular functors of state-sum type [DSS2]. Morita theory is relevant for oriented TFT's as well: Two fundamental sources of such topological invariants are the state sum construction by Turaev-Viro [TV] and the surgery-based construction due to Reshetikhin-Turaev [RT]. These two are related as follows: the Turaev-Viro theory $\mathrm{TV}_{\mathcal{A}}$ obtained from a spherical fusion category $\mathcal{A}$ is isomorphic to the RT theory arising from its Drinfeld center $\mathcal{Z}(\mathcal{A})$ [TV, Thm. 17.1]. The notion of categorical Morita equivalence is completely captured by the Drinfeld center [EGNO, Thm. 8.12.3]. Hence, the state sum construction applied to Morita equivalent spherical fusion categories produce isomorphic TFT's since these can be obtained as the RT theory associated to their (braided-equivalent) Drinfeld centers. These constructions for oriented manifolds strongly rely on a specific instance of pivotal structure on the input data, namely on a choice of *spherical structure*. The statement only holds true if such structure is preserved under the transit to the centers induced by Morita equivalence, as we prove in Theorem 4.15. From this result we achieve the concrete statement: the oriented three-dimensional Turaev-Viro topological field theories $\mathrm{TV}_{\mathcal{A}}$ and $\mathrm{TV}_{\mathcal{B}}$ arising from *pivotal Morita equivalent* spherical fusion categories $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Organization of the dissertation

The material presented in this thesis is structured as follows: In Chapter 1 the pertinent definitions and conventions about the theory of finite tensor categories are summarized. Further preliminary material regarding relative Serre functors and Nakayama functors is also included.

The Morita equivalence relation on algebras is defined as a property by means of the existence of an invertible bimodule between the corresponding algebras. A choice of such invertible bimodule leads to the concept of a Morita context. This notion has an analog in the categorical Morita theory of finite tensor categories, which we discuss in Chapter 2. Such structure called (*strong*) *categorical Morita context* consists of bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$ between two finite tensor categories together with *mixed tensor products*, i.e. bimodule equivalences

$$\odot: \quad \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\simeq} \mathcal{A} \qquad \text{and} \qquad \boxdot: \quad \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{M} \xrightarrow{\simeq} \mathcal{B}\,.$$

and additional coherence data fulfilling appropriate compatibility conditions. Dropping the requirement that $\odot$ and $\boxdot$ are equivalences gives a weaker notion of a categorical Morita context. In the same spirit as [Mü, Rem. 3.18] the data of such weak Morita context form a bicategory $\mathbb{M}$ with two objects. We show that from every exact module category stems a strong Morita context. In Theorem 2.11 we provide a converse: any strong Morita context comes from an exact module category ${}_{\mathcal{A}}\mathcal{M}$.

In Chapter 3 we investigate the nature of dualities for Morita contexts which is a crucial step in our journey towards the study of pivotal structures. For this goal, we make use of the notion of duals (adjoints) for 1-morphisms in a bicategory, generalizing dual objects in a finite tensor category. We show that the Morita context bicategory $\mathbb{M}$ of an exact module category ${}_{\mathcal{A}}\mathcal{M}$ has adjoints. The duals of objects in $\mathcal{M}$ are module functors from $\mathcal{M}$ to $\mathcal{A}$ given by internal Homs and coHoms. Moreover, all internal Homs and coHoms of the categories in $\mathbb{M}$ can be expressed in terms of actions, mixed tensor products $\odot$ and $\boxdot$ and dual objects in complete analogy to the familiar case of a single tensor category. Furthermore, double duals for objects in ${}_{\mathcal{A}}\mathcal{M}$ correspond to the value of the relative Serre functor on them, and thus the relative Serre functors of the categories in a Morita context combine into a pseudo-functor $\mathbb{S}$ on its associated bicategory $\mathbb{M}$. As a consequence of these results we arrive at a Radford theorem for module categories:

Theorem 3.16. *Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an exact $\mathcal{A}$-module. There is a natural isomorphism*

$$\mathbb{D}_{\mathcal{A}} \triangleright - \triangleleft \mathbb{D}^{-1}_{\mathcal{A}^*_{\mathcal{M}}} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$$

of (*twisted*) *bimodule functors, where $\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ is the relative Serre functor of $\mathcal{M}$ and $\mathbb{D}_{\mathcal{A}}$ and $\mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}$ are the distinguished invertible objects of $\mathcal{A}$ and $\mathcal{A}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$, respectively.*

Theorem 3.16 has the following bicategorical interpretation: there are Radford isomorphisms for every category in the Morita context of $\mathcal{M}$ and these assemble into a trivialization of the square of the relative Serre pseudo-functor $\mathbb{S}$ of the bicategory $\mathbb{M}$. Altogether these results establish natural rigid duality structures for Morita context bicategories.

After setting up the study of dualities, in Chapter 4 we are ready to investigate the interaction of pivotal structures and Morita theory. In view of the bicategorical nature emerging in Chapter 3, we continue regarding it as a guiding principle. There is a notion of pivotal structure for a bicategory with dualities for 1-morphisms. The Morita context bicategory $\mathbb{M}$ arising from a pivotal module turns out to be pivotal as a bicategory. The pivotality of a module category $\mathcal{M}$ just regards its structure as one-sided module; in contrast, the pivotality of the Morita context implies that $\mathcal{M}$ is endowed with a pivotal structure as a bimodule category, and so are all categories in the associated Morita context.

Theorem 4.8. *The Morita context associated to a pivotal module category $\mathcal{M}$ over a pivotal tensor category $\mathcal{A}$ is a pivotal Morita context.*

We define *pivotal Morita equivalence* by means of a connecting invertible pivotal bimodule. The 2-category of pivotal module categories over a pivotal tensor category is naturally endowed with a bicategorical pivotal structure. We prove in Theorem 4.11 that this pivotal bicategory characterizes the notion of pivotal Morita equivalence.

Theorem 4.11. *Two pivotal tensor categories are pivotal Morita equivalent if and only if their associated* 2*-categories of pivotal module categories, module functors and module natural transformations are equivalent as pivotal bicategories.*

We further prove that if two pivotal categories are pivotal Morita equivalent, then their Drinfeld centers are equivalent as pivotal braided tensor categories. The converse of this statement holds in the (non-pivotal) world of finite tensor categories, but for pivotal tensor categories it is an interesting question that still remains open.

Theorem 4.15. *If two pivotal categories are pivotal Morita equivalent, then their Drinfeld centers are equivalent as pivotal braided tensor categories.*

In Section 4.5 the property of sphericality for a pivotal tensor category is studied. The perspective on this condition has recently undergone a change: while trace-sphericality emphasized that left and right quantum traces for endomorphisms in a pivotal category agree, a more recent notion of sphericality [DSS2] requires that the underlying tensor category is unimodular and that the pivotal structure squares to Radford's trivialization of the quadruple dual. The two notions coincide in the semisimple case. We prove that (beyond semisimplicity) the later notion of sphericality is invariant under pivotal Morita equivalence.

Corollary 4.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pivotal tensor categories that are pivotal Morita equivalent. $\mathcal{A}$ is* (*unimodular*) *spherical if and only if $\mathcal{B}$ is* (*unimodular*) *spherical.*

In the same spirit as for tensor categories [DSS2], we propose in Definition 4.21 a notion of sphericality for a pivotal module category ${}_{\mathcal{A}}\mathcal{M}$ by means of the Radford isomorphism from Theorem 3.16. However, for module categories there is a subtlety: this definition is relative to the choice of trivializations of both the distinguished invertible objects of $\mathcal{A}$ and $\mathcal{A}^*_{\mathcal{M}}$. Fixing such trivializations, all monoidal and module categories in the Morita context associated to a spherical module category are spherical. As we point out in Remark 4.20, there are distinguished choices of such trivializations given by Definition 6.12 under the assumption of semisimplicity, which we explore more closely in Chapter 6. These replicate the classical quantum traces for the regular bimodule category, and consequently the notion of trace-sphericality.

The contents of Chapter 6 concern trace-like structures on categories in further detail. These algebraic gadgets are frequently used for assigning scalars to closed graphs on spheres [TV2, GP]. With this goal in mind, we study traces on bimodule categories. For this purpose, we make use of the canonical $\mathbb{N}^r_{\mathcal{X}}$*-twisted trace* [SchW, Def. 2.4] [ShSh, Def. 4.4] associated to any $\Bbbk$-linear category $\mathcal{X}$. Such twisted trace allows to assign scalars, not to endomorphisms, but to a different class of morphisms in the subcategory of projective objects of $\mathcal{X}$. Since, the category underlying a bimodule category is $\Bbbk$-linear, it comes naturally equipped with such twisted trace. In Section 6.3, we explore the interaction of the twisted trace with pivotal structures on bimodule categories. In Proposition 6.10, we obtain partial-trace properties for bimodule categories.

The relative notion of bimodule sphericality from Definition 4.19 finds a distinguished normalization datum in the semisimple setting, as we point out in Remark 6.13. This will lead to traces on spherical bimodule categories. In Proposition 6.14, we show a trace-sphericality property on spherical bimodules that resembles the trace-sphericality property of spherical fusion categories. This will be the basis for the bimodule graphical calculus on spheres developed in Chapter 7.

State sum procedures are fundamental sources of topological invariants. In three dimensions the Turaev-Viro invariant [TV] takes as input the structure of a spherical fusion category. In four dimensions, a construction of topological invariants based on the structure of a *fusion* 2-*category* [DR] has been recently formulated. A bicategorical notion of idempotent completion predominantly appears in this work. In analogy to the fact that the idempotent completion of a semisimple algebra is its category of modules, the 2-category underlying any fusion 2-category can be obtained as some 2-idempotent completion of a multifusion category.

In the case of a spherical fusion category $\mathcal{A}$, we would also like to consider a suitable 2-category of $\mathcal{A}$-modules as an adequate "spherical" 2-idempotent completion, yet to be defined. Based on the theory developed in this thesis, we tentatively consider the structure of the pivotal bicategory $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ of spherical $\mathcal{A}$-module categories as such completion. We test this idea in Chapter 7 by showing that a three-dimensional state sum construction can be formulated from these data. More concretely, to any closed oriented three-dimensional M we assign a scalar $\mathrm{St}(M)$ given by Definition 7.5. This is done in analogy to the classical Turaev-Viro construction by means of the auxiliary datum of a skeleton on M. Since our input is a higher algebraic structure this will require additional considerations. An important step in the construction is the evaluation of $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$-labeled graphs on spheres. This is based on the properties of the traces for 2-morphisms in $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ studied in Section 6.4. The main result of Chapter 7 is Theorem 7.8 which states that the value of $\mathrm{St}(M)$ is independent of the choice of skeleton used to construct it. Moreover, as we point out in Remark 7.7, the input of the construction here presented can be exchanged by a suitable locally Calabi-Yau pivotal bicategory (for instance a spherical *multi*fusion category), thereby generalizing the Turaev-Viro construction.

In Chapter 5 we make a detour to discuss categorical Morita theory in the equivariant setting of tensor categories faithfully graded over a finite group G. The Turaev-Viro and Reshetikhin-Turaev constructions have a generalization for homotopy quantum field theories for G [Tu]. These equivariant constructions require a choice of additional structure on the input data, namely a G-grading and a G-crossing, on spherical categories and modular categories, respectively. The relative center of a graded category with respect to its trivial component, called the *equivariant center*, has the structure of a braided G-crossed tensor category, and plays the role of the Drinfeld center by relating these two constructions [TV3]. In Chapter 5 we extend Morita equivalence to G-graded tensor categories. Interestingly, G-graded module categories are realized by algebras in the trivial component [Ga2], which leads to the definition of *graded Morita equivalence*. We define in Section 5.3 a natural G-action on the bicategory of G-graded module categories over a G-graded tensor category, although trivial to define, is crucial for our results. This bicategory with a G-action provides a characterization of graded Morita equivalence in Theorem 5.18.

Theorem 5.18. *Two G-graded tensor categories are graded Morita equivalent if and only if their associated 2-categories of graded module categories, graded module functors and module natural transformations are equivalent as bicategories with group action.*

In Theorem 5.29 we prove that graded Morita equivalence is characterized by the equivariant center, as well. In Section 5.5 we also verify that dualities and pivotal structures are well-behaved in the equivariant picture.

Relevant contribution

This thesis is partially based on the results in the published article

- C. Galindo, D. Jaklitsch, and C. Schweigert, *Equivariant Morita theory for graded tensor categories*, Bull. Belg. Math. Soc. Simon Stevin 29(2) (2022) 145–171 `[math.QA/2106.07440]`.

that have been obtained after submission of the author's master thesis and the results of the following preprint

- J. Fuchs, C. Galindo, D. Jaklitsch, and C. Schweigert, *Spherical Morita contexts and relative Serre functors*, preprint `[math.QA/2207.07031]`.

that is currently under review. The publication and preprint are also listed in the bibliography as [GJS] and [FGJS]. The results therein were developed together with the coauthors, whose contributions are fully acknowledged by the author of this thesis.

The motivation to study dualities on Morita contexts arose in discussions of the paper [Bh] with Jürgen Fuchs, César Galindo and Christoph Schweigert. In these discussions, the idea on how to describe the second bimodule category in a Morita context came to fruition. Based on this, the author of this thesis had the insight that double duals are related to relative Serre functors. From this observation, most results in Chapters 3 and 4 were developed by the author and found their final form in common meetings.

Most of the results presented in Chapter 5 were obtained after submission of the author's master thesis. From the main results, only Theorem 5.25 is shown in the master thesis [Ja], while Theorem 5.29 also comprises a converse. Furthermore, in Section 5.3 a G-action on the 2-category of graded module categories is introduced and in Theorem 5.18 is shown that this object characterizes the notion of graded Morita equivalence which is a main result not appearing in the master thesis either. These results were obtained by the author after regular interchange of ideas in sessions with César Galindo and Christoph Schweigert.

While the results in Chapter 6 and 7 originated in general discussions about state-sum models, they were obtained largely independently by the author and then only discussed with the supervisors.

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Chapter 1

Preliminaries

In this section we fix notation and conventions and summarize some pertinent concepts and structures. Throughout the entirety of the document, all categories we consider are supposed to be linear abelian over an algebraically closed field $\Bbbk$ of characteristic zero. A linear abelian category where every object is of finite length and all morphism spaces are finite-dimensional is said to be *locally finite*. A *finite* $\Bbbk$-linear abelian category is a linear abelian category that is equivalent to the category of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra. Following [Gu] strict bicategories, i.e. bicategories where the horizontal composition is associative on the nose are termed *2-categories.*

1.1 Tensor categories and module categories

We first recall a few standard definitions in the theory of tensor categories (see [EGNO]). A *multi-tensor category* is a locally finite rigid monoidal category $\mathcal{A}$ whose tensor product functor $\otimes$ is bilinear. $\mathcal{A}$ is said to be a *tensor category* iff in addition its monoidal unit $\mathbf{1}_{\mathcal{A}}$ is a simple object. We take without loss of generality a monoidal category to be strict to simplify the exposition. The *monoidal opposite* $\overline{\mathcal{A}}$ of a monoidal category $\mathcal{A}$ is the monoidal category with the same underlying category as $\mathcal{A}$, but with reversed tensor product, i.e. $a \otimes_{\overline{\mathcal{A}}} b = b \otimes a$ and with the accordingly adjusted associators. When convenient we denote the object in $\overline{\mathcal{A}}$ that corresponds to an object $a \in \mathcal{A}$ by $\overline{a}$.

Concerning dualities on a monoidal category $\mathcal{A}$ our conventions are as follows. A right dual $a^\vee$ of an object $a \in \mathcal{A}$ comes equipped with evaluation and coevaluation morphisms

$$\mathrm{ev}_a : \quad a^\vee \otimes a \to \mathbf{1}_{\mathcal{A}} \qquad \text{and} \qquad \mathrm{coev}_a : \quad \mathbf{1}_{\mathcal{A}} \to a \otimes a^\vee . \tag{1.1}$$

Similarly, a left dual ${}^\vee a$ and of $a \in \mathcal{A}$ comes with evaluation and coevaluation morphism

$$\widetilde{\mathrm{ev}}_a : \quad a \otimes {}^\vee a \to \mathbf{1}_{\mathcal{A}} \qquad \text{and} \qquad \widetilde{\mathrm{coev}}_a : \quad \mathbf{1}_{\mathcal{A}} \to {}^\vee a \otimes a \, . \tag{1.2}$$

A (*left*) *module category* over a tensor category $\mathcal{A}$, or (*left*) $\mathcal{A}$-*module*, for short, is a category $\mathcal{M}$ together with an exact *module action* functor

$$\triangleright : \quad \mathcal{A} \times \mathcal{M} \to \mathcal{M} \tag{1.3}$$

and a mixed associator obeying a pentagon axiom. In order to indicate the tensor category over which $\mathcal{M}$ is a module, we also denote it by ${}_{\mathcal{A}}\mathcal{M}$. Invoking an analogue for module categories of Mac Lane's strictification theorem (see [EGNO, Rem. 7.2.4]), we assume strictness also for module categories. In the case that $\mathcal{A}$ is finite, we require that $\mathcal{M}$ is finite as a linear category as well.

A *right* $\mathcal{B}$-module $\mathcal{N}$ is defined as a left $\overline{\mathcal{B}}$-module; its module action functor is denoted by

$$\triangleleft : \quad \mathcal{N} \times \mathcal{B} \to \mathcal{N}, \tag{1.4}$$

and we also denote it by $\mathcal{N}_{\mathcal{B}}$. Similarly, for finite tensor categories $\mathcal{A}$ and $\mathcal{B}$, an $(\mathcal{A},\mathcal{B})$-*bimodule category* is defined as a (left) module category over the Deligne product $\mathcal{A} \boxtimes \overline{\mathcal{B}}$.

The opposite category $\mathcal{M}^{\text{opp}}$ of the linear category $\mathcal{M}$ that underlies a bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ can be endowed in many different ways with the structure of a $(\mathcal{B},\mathcal{A})$-bimodule category, by twisting the actions with odd powers of duals.

Definition 1.1. Let $\mathcal{M}$ be an $(\mathcal{A},\mathcal{B})$-bimodule category over tensor categories $\mathcal{A}$ and $\mathcal{B}$. We define ${}^{\#}\mathcal{M}$ as the $(\mathcal{B},\mathcal{A})$-bimodule category with underlying category $\mathcal{M}^{\text{opp}}$ and actions given by

$$b \triangleright \overline{m} \triangleleft a := \overline{{}^{\vee}a \triangleright m \triangleleft {}^{\vee}b} \tag{1.5}$$

for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\overline{m} \in \mathcal{M}^{\text{opp}}$. Similarly, $\mathcal{M}^{\#}$ is defined to be the $(\mathcal{B},\mathcal{A})$-bimodule with actions twisted by right duals, i.e.

$$b \triangleright \overline{m} \triangleleft a := \overline{a^{\vee} \triangleright m \triangleleft b^{\vee}} \tag{1.6}$$

for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\overline{m} \in \mathcal{M}^{\text{opp}}$.

An $\mathcal{A}$-module category $\mathcal{M}$ is called *exact* iff $p \triangleright m$ is projective in $\mathcal{M}$ for any projective object $p \in \mathcal{A}$ and any object $m \in \mathcal{M}$. A module category which is not equivalent to a direct sum of two non-zero module categories is said to be *indecomposable*.

Given associative algebras A and B in a tensor category $\mathcal{A}$, the category of right A-modules in $\mathcal{A}$ will be denoted by $\text{Mod}_A(\mathcal{A})$. Similarly, ${}_A\text{Mod}(\mathcal{A})$ denotes the category of left A-modules in $\mathcal{A}$ and ${}_A\text{Bimod}_B(\mathcal{A})$ the category of (A,B)-bimodules in $\mathcal{A}$. An algebra A is said to be *exact* (*indecomposable*) if the $\mathcal{A}$-module category $\text{Mod}_A(\mathcal{A})$ is exact (indecomposable). Moreover, as stated in Theorem 1.8, every indecomposable exact $\mathcal{A}$-module category can be realized in terms of an algebra internal to $\mathcal{A}$.

Module functors

A *module functor* between $\mathcal{A}$-module categories $\mathcal{M}$ and $\mathcal{N}$ is a functor $H\colon \mathcal{M} \to \mathcal{N}$ together with a *module constraint*, i.e. a collection of natural isomorphisms $H(a \triangleright m) \xrightarrow{\cong} a \triangleright H(m)$ for $a \in \mathcal{A}$ and $m \in \mathcal{M}$ obeying appropriate pentagon axioms. A *module natural transformation* between module functors is a natural transformation between the underlying functors that commutes with the respective module structures. We denote by $\text{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ the category which has module functors between two $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$ as objects and module natural transformations as morphisms. We denote by $\text{Nat}_{\text{mod}}(H_1,H_2)$ the space of module natural transformations between given module functors H_1 and H_2. Similarly, $\text{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ and $\text{Lex}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ denote the categories of right exact and left exact module functors, respectively. In the case that $\mathcal{M}$ is exact, every $H \in \text{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is an exact module functor [EGNO, Proposition 7.6.9].

Lemma 1.2. [DSS1, Cor. 2.13] *Let $H\colon \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-module functor. If its underlying functor admits a right (left) adjoint functor, then H admits a right (left) adjoint $\mathcal{A}$-module functor such that the unit and counit of the adjunction are module natural transformations.*

The *dual category* $\mathcal{A}^*_{\mathcal{M}}$ of a tensor category $\mathcal{A}$ with respect to an $\mathcal{A}$-module $\mathcal{M}$ is the category of right exact module endofunctors, $\mathcal{A}^*_{\mathcal{M}} = \mathrm{Rex}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$, with tensor product given by composition of functors. If $\mathcal{M}$ is exact $\mathcal{A}^*_{\mathcal{M}}$ is rigid and in case $\mathcal{M}$ indecomposable, then the identity functor $\mathrm{id}_{\mathcal{M}}$ is simple, making $\mathcal{A}^*_{\mathcal{M}}$ a tensor category. The evaluation of a functor on an object turns $\mathcal{M}$ into an $\mathcal{A}^*_{\mathcal{M}}$-module category.

The category of module functors has the structure of a bimodule category. More specifically, for given bimodule categories ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$, ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{C}}$ and ${}_{\mathcal{D}}\mathcal{L}_{\mathcal{B}}$, $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ becomes a $(\mathcal{B},\mathcal{C})$-bimodule category via the actions

$$b \succ H := H \circ (- \triangleleft b) \qquad \text{and} \qquad H \prec c := (- \triangleleft c) \circ H \tag{1.7}$$

for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$, while $\mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{L})$ inherits a $(\mathcal{D}, \mathcal{A})$-bimodule category structure given by

$$d \succ H := (d \triangleright -) \circ H \qquad \text{and} \qquad H \prec a := H \circ (a \triangleright -)\,. \tag{1.8}$$

Analogously, the categories of right exact and left exact module functors are endowed with the structure of a bimodule category as well.

By realizing module categories in terms of algebras we have from [EGNO, Proposition 7.11.1] that there is an equivalence of bimodule categories

$$\begin{aligned} {}_A\mathrm{Bimod}_B(\mathcal{A}) &\xrightarrow{\simeq} \mathrm{Rex}_{\mathcal{A}}(\mathrm{Mod}_A(\mathcal{A}),\, \mathrm{Mod}_B(\mathcal{A}))\,, \\ M &\longmapsto - \otimes_A M \end{aligned} \tag{1.9}$$

called *the Eilenberg-Watts equivalence.* Furthermore, for $A = B$ and $\mathcal{M} := \mathrm{Mod}_A(\mathcal{A})$ this is a tensor equivalence ${}_A\mathrm{Bimod}_A(\mathcal{A}) \simeq \overline{\mathrm{Rex}_{\mathcal{A}}(\mathcal{M})} = \overline{\mathcal{A}^*_{\mathcal{M}}}$, i.e., it is compatible with composition of functors and the tensor product relative to A.

1.2 Relative Deligne product of module categories

Let $\mathcal{M}$ be a right module and $\mathcal{N}$ a left module over a finite tensor category $\mathcal{B}$.

(i) A $\mathcal{B}$-*balancing* on a bilinear functor $F\colon \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ into a linear category $\mathcal{L}$ is a natural family of isomorphisms

$$F(m \triangleleft b, n) \xrightarrow{\cong} F(m, b \triangleright n) \tag{1.10}$$

for $b \in \mathcal{B}$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$, obeying an obvious pentagon coherence condition (for details, see for instance [Sc2, Def. 2.12]). A bilinear functor endowed with a balancing is called a *balanced functor.*

(ii) A *balanced natural transformation* between balanced functors is a natural transformation between the underlying functors that commutes with the respective balancings.

(iii) Balanced functors $F\colon \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ together with balanced natural transformations form a category, denoted by $\mathrm{Bal}(\mathcal{M} \times \mathcal{N}, \mathcal{L})$. Its full subcategory of right exact balanced functors is denoted by $\mathrm{Bal}^{\mathrm{re}}(\mathcal{M} \times \mathcal{N}, \mathcal{L})$.

(iv) The *relative Deligne product* of $\mathcal{M}_\mathcal{B}$ and ${}_\mathcal{B}\mathcal{N}$ is a linear category $\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}$ equipped with a right exact $\mathcal{B}$-balanced functor $\boxtimes_\mathcal{B}\colon \mathcal{M}\times\mathcal{N}\to\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}$ such that for every linear category $\mathcal{L}$ the functor

$$\begin{aligned} \mathrm{Rex}(\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N},\mathcal{L}) &\xrightarrow{\simeq} \mathrm{Bal}^{\mathrm{re}}(\mathcal{M}\times\mathcal{N},\mathcal{L})\,, \\ F &\longmapsto F\circ\boxtimes_\mathcal{B} \end{aligned} \tag{1.11}$$

is an equivalence of categories. The relative Deligne product exists, different realizations are known, e.g. [DSS1, FSS] and Corollary 2.4 below.

(v) The relative Deligne product of bimodule categories is naturally endowed with the structure of a bimodule category. Accordingly, the equivalence (1.11) descends to an equivalence between the categories of bimodule right exact functors and balanced right exact bimodule functors (see [Sc2, Prop. 3.7]).

We call an object of the form $m\boxtimes n\in\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}$ a *$\boxtimes$-factorized object* of $\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}$.

Lemma 1.3. *Every object in the relative Deligne product $\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}$ is isomorphic to a finite colimit of $\boxtimes$-factorized objects.*

Proof. As shown in [DSS1, Thm. 3.3], the relative Deligne product can be realized as a category of bimodules internal to $\mathcal{B}$. Moreover, as pointed out there in the proof, any such bimodule can be written as a coequalizer of $\boxtimes$-factorized objects. ☐

Proposition 1.4. [DSS2, Prop. 2.4.10] *Let ${}_\mathcal{A}\mathcal{M}_\mathcal{B}$, ${}_\mathcal{A}\mathcal{N}_\mathcal{C}$ and ${}_\mathcal{D}\mathcal{L}_\mathcal{B}$ be bimodule categories over finite tensor categories. The balanced functor*

$$\begin{aligned} \mathrm{Rex}_\mathcal{A}(\mathcal{M},\mathcal{A})\times\mathcal{N} &\longrightarrow \mathrm{Rex}_\mathcal{A}(\mathcal{M},\mathcal{N})\,, \\ (H,n) &\longmapsto H(-)\triangleright n \end{aligned} \tag{1.12}$$

induces an adjoint equivalence

$$\mathrm{Rex}_\mathcal{A}(\mathcal{M},\mathcal{A})\boxtimes_\mathcal{A}\mathcal{N} \xrightarrow{\simeq} \mathrm{Rex}_\mathcal{A}(\mathcal{M},\mathcal{N}) \tag{1.13}$$

of $(\mathcal{B},\mathcal{C})$-bimodule categories. Similarly, there is an adjoint equivalence

$$\mathcal{L}\boxtimes_\mathcal{B}\mathrm{Rex}_\mathcal{B}(\mathcal{M},\mathcal{B}) \xrightarrow{\simeq} \mathrm{Rex}_\mathcal{B}(\mathcal{M},\mathcal{L}) \tag{1.14}$$

of $(\mathcal{D},\mathcal{A})$-bimodule categories.

Corollary 1.5. [DSS2, Cor. 2.4.11] *Let ${}_\mathcal{A}\mathcal{M}_\mathcal{B}$ and ${}_\mathcal{B}\mathcal{N}_\mathcal{C}$ be bimodule categories over finite tensor categories. There are equivalences*

$$\mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}\simeq\mathrm{Rex}_\mathcal{B}({}^{\#}\mathcal{M},\mathcal{N}) \qquad \text{and} \qquad \mathcal{M}\boxtimes_\mathcal{B}\mathcal{N}\simeq\mathrm{Rex}_\mathcal{B}(\mathcal{N}^{\#},\mathcal{M}) \tag{1.15}$$

of $(\mathcal{A},\mathcal{C})$-bimodule categories between relative Deligne products and categories of right exact module functors.

1.3 Internal Hom and coHom

Given a module category ${}_{\mathcal{A}}\mathcal{M}$ over a finite tensor category, for every object $m \in \mathcal{M}$ the action functor $-\triangleright m\colon \mathcal{A} \to \mathcal{M}$ is exact and therefore comes with a right adjoint $\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,-)\colon \mathcal{M} \to \mathcal{A}$, i.e. there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(a \triangleright m, n) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(a, \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n)) \tag{1.16}$$

for $a \in \mathcal{A}$ and $m, n \in \mathcal{M}$. This extends to a left exact functor

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(-,-): \quad \mathcal{M}^{\mathrm{opp}} \times \mathcal{M} \longrightarrow \mathcal{A}\,, \tag{1.17}$$

which is called the *internal Hom* functor of ${}_{\mathcal{A}}\mathcal{M}$. We denote the internal Hom also by $\underline{\mathrm{Hom}}_{\mathcal{M}}$ or just by $\underline{\mathrm{Hom}}$ when it is clear from the context which module category is meant. Additionally, there are canonical natural isomorphisms

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, a \triangleright n) \xrightarrow{\cong} a \otimes \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) \tag{1.18}$$

which turn $- \triangleright m \dashv \underline{\mathrm{Hom}}_{\mathcal{M}}(m,-)$ into an adjunction of $\mathcal{A}$-module functors, as well as

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(a \triangleright m, n) \xrightarrow{\cong} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) \otimes a^{\vee}. \tag{1.19}$$

Together these naturally endow the internal Hom with a bimodule functor structure

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(-,-): \quad {}^{\#}\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}\,. \tag{1.20}$$

The internal Hom is well-behaved with respect to adjunctions of module functors:

Lemma 1.6. [FuS, Lemma 3] *Let $\mathcal{A}$ be a finite tensor category, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-module categories, and $L\colon \mathcal{M} \to \mathcal{N}$ and $R\colon \mathcal{N} \to \mathcal{M}$ be $\mathcal{A}$-module functors. Then L is left adjoint to R if and only if there are natural isomorphisms*

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{N}}(L(m), n) \xrightarrow{\cong} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, R(n)) \tag{1.21}$$

for $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

The monoidal category $\overline{\mathcal{A}^*_{\mathcal{M}}}$ acts on a module category $\mathcal{M}$ by evaluation of a functor on an object. If ${}_{\mathcal{A}}\mathcal{M}$ is exact, then the dual tensor category is rigid, and a left dual for $F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$ is given by its left adjoint F^{la}. In this situation the isomorphisms from Lemma 1.6 turn the internal Hom (1.20) into an $\overline{\mathcal{A}^*_{\mathcal{M}}}$-balanced bimodule functor.

Module categories over a tensor category $\mathcal{A}$ can be realized as categories of modules over an algebra internal to $\mathcal{A}$ by means of internal Hom's: Let $\mathcal{M}$ be an $\mathcal{A}$-module category and denote for $m, n \in \mathcal{M}$ by

$$\mathrm{ev}_{m,n}: \quad \underline{\mathrm{Hom}}_{\mathcal{M}}(m,n) \triangleright m \to n \tag{1.22}$$

the counit of the adjunction (1.16), and for $a \in \mathcal{A}$ by $\eta_{a,m} : a \to \underline{\mathrm{Hom}}_{\mathcal{M}}(m, a \triangleright m)$ the unit of (1.16). Now for objects $m, n, l \in \mathcal{M}$ define the multiplication morphism

$$\circ_{m,n,l}: \quad \underline{\mathrm{Hom}}(n,l) \otimes \underline{\mathrm{Hom}}(m,n) \to \underline{\mathrm{Hom}}(m,l) \tag{1.23}$$

as the image of the following composition

$$\underline{\mathrm{Hom}}(n,l) \otimes \underline{\mathrm{Hom}}(m,n) \triangleright m \xrightarrow{\mathrm{id} \triangleright \mathrm{ev}_{m,n}} \underline{\mathrm{Hom}}(n,l) \triangleright n \xrightarrow{\mathrm{ev}_{n,l}} l \tag{1.24}$$

under the adjunction (1.16).

Remark 1.7. (i) For any object $m \in \mathcal{M}$, the multiplication $\mu := \circ_{m,m,m}$ and unit $u_m := \eta_{\mathbf{1},m}$ furnishes on $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,m)$ the structure of an associative algebra in $\mathcal{A}$.

(ii) Moreover, for every pair $m, n \in \mathcal{M}$, the object $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,n) \in \mathcal{A}$ is endowed with a right $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,m)$-module structure via $\sigma_{m,n} := \circ_{m,m,n}$.

(iii) The functor (1.17) can be seen as a $\mathcal{A}$-module functor

$$\mathcal{M} \longrightarrow \mathrm{Mod}_A(\mathcal{A}), \quad n \mapsto (\,\underline{\mathrm{Hom}}_{\mathcal{M}}(m,n),\, \sigma_{m,n}\,) \tag{1.25}$$

where $A := \underline{\mathrm{Hom}}_{\mathcal{M}}(m,m)$ is the algebra described in (i).

Theorem 1.8. [EO, Theorem 3.17] *Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an indecomposable exact $\mathcal{A}$-module category, then* (1.25) *is an equivalence of $\mathcal{A}$-module categories $\mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{A})$.*

Proposition 1.9. *Let $\mathcal{M}$ be an $\mathcal{A}$-module category decomposed into $\mathcal{A}$-submodule categories as $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$. Then for $m, n \in \mathcal{M}$*

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = \bigoplus_{i \in I} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}_i}(m_i, n_i) \tag{1.26}$$

as objects in $\mathcal{A}$, where m_i and n_i are the components of m and n under the decomposition of $\mathcal{M}$. The decomposition (1.26) *yields to a decomposition of the algebra $\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,m)$ from Remark 1.7,*

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,m) = \prod_{i \in I} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}_i}(m_i, m_i) \tag{1.27}$$

as algebras in $\mathcal{A}$.

Proposition 1.10. *Let L be an exact algebra in a finite tensor category $\mathcal{A}$.*

(i) *There is an algebra decomposition $L = \prod_{i \in I} L_i$, where L_i are exact indecomposable algebras in $\mathcal{A}$.*

(ii) *The $\mathcal{A}$-module category of L-modules in $\mathcal{A}$ can be decomposed as*

$$\mathrm{Mod}_L(\mathcal{A}) \simeq \bigoplus_{i \in I} \mathrm{Mod}_{L_i}(\mathcal{A})\,. \tag{1.28}$$

(iii) *The multi-tensor category of L-bimodules in $\mathcal{A}$ decomposes as*

$${}_L\mathrm{Bimod}_L(\mathcal{A}) \simeq \bigoplus_{i,j \in I} {}_{L_i}\mathrm{Bimod}_{L_j}(\mathcal{A})\,. \tag{1.29}$$

Proof. Since L is exact, according to [EGNO, Proposition 7.6.7] there is a decomposition of the form

$$\mathcal{M} := \mathrm{Mod}_L(\mathcal{A}) = \bigoplus_{i \in I} \mathcal{M}_i \tag{1.30}$$

where $\mathcal{M}_i$ is an exact indecomposable $\mathcal{A}$-submodule category of $\mathrm{Mod}_L(\mathcal{A})$ for each $i \in I$. In particular the regular module can be decomposed as $L = \oplus_{i \in I}\, m_i$. It follows from Remark 1.7

and Theorem 1.8 that $L_i := \underline{\mathrm{Hom}}_{\mathcal{M}_i}(m_i, m_i)$ is an algebra in $\mathcal{A}$ and there is an equivalence of $\mathcal{A}$-module categories

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}_i}(m_i, -): \quad \mathcal{M}_i \xrightarrow{\simeq} \mathrm{Mod}_{L_i}(\mathcal{A}) \tag{1.31}$$

for each $i \in I$, and thus the algebras L_i are exact and indecomposable. These equivalences lead to an equivalence of $\mathcal{A}$-module categories

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(L, -) = \bigoplus_{i \in I} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}_i}(m_i, -): \quad \mathrm{Mod}_L(\mathcal{A}) \xrightarrow{\simeq} \bigoplus_{i \in I} \mathrm{Mod}_{L_i}(\mathcal{A}) \tag{1.32}$$

which shows (ii). Now recall from [EGNO, Example 7.9.8] that the internal Hom is given by $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, n) = (m \otimes_L {}^*n)^*$ for $m, n \in \mathcal{M}$. In particular one can verify that $\underline{\mathrm{Hom}}_{\mathcal{M}}(L, L) = L$ as algebras in $\mathcal{A}$. It follows from Proposition 1.9 that

$$L = \underline{\mathrm{Hom}}_{\mathcal{M}}(L, L) = \prod_{i \in I} L_i \tag{1.33}$$

as algebras in $\mathcal{A}$, and thus statement (i) holds. Lastly, from the Eilenberg-Watts equivalence (1.9) follows

$${}_L\mathrm{Bimod}_L(\mathcal{A}) \simeq \bigoplus_{i,j \in I} \mathrm{Fun}_{\mathcal{A}}\Big(\mathrm{Mod}_{L_i}(\mathcal{A}), \mathrm{Mod}_{L_j}(\mathcal{A})\Big) \simeq \bigoplus_{i,j \in I} {}_{L_i}\mathrm{Bimod}_{L_j}(\mathcal{A}) \tag{1.34}$$

providing the desired decomposition in (iii). □

Internal coHom

Dual to the notion of internal Hom, for $m \in \mathcal{M}$ the action functor $- \triangleright m\colon \mathcal{A} \to \mathcal{M}$ has a left adjoint

$$\mathrm{Hom}_{\mathcal{M}}(n, a \triangleright m) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, n), a)\,, \tag{1.35}$$

called the *internal coHom*, where $a \in \mathcal{A}$ and $m, n \in \mathcal{M}$. This extends to a right exact functor $\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(-,-)\colon \mathcal{M}^{\mathrm{opp}} \times \mathcal{M} \to \mathcal{A}$. The internal Hom and coHom are related by

$$\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, n) \cong {}^\vee\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(n, m) \qquad \text{for} \;\; m, n \in \mathcal{M}\,. \tag{1.36}$$

Hence analogously to (1.18) and (1.19), there are coherent natural isomorphisms

$$\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, a \triangleright n) \xrightarrow{\cong} a \otimes \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, n) \tag{1.37}$$

and

$$\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(a \triangleright m, n) \xrightarrow{\cong} \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, n) \otimes {}^\vee a \tag{1.38}$$

turning the internal coHom into an $\mathcal{A}$-bimodule functor

$$\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(-,-): \quad \mathcal{M}^{\#} \times \mathcal{M} \longrightarrow \mathcal{A}\,. \tag{1.39}$$

In view of the relation (1.36), Lemma 1.6 takes the following form:

Lemma 1.11. *Let $\mathcal{M}$ and $\mathcal{N}$ be module categories over a finite tensor category $\mathcal{A}$, and $L\colon \mathcal{M} \to \mathcal{N}$ and $R\colon \mathcal{N} \to \mathcal{M}$ be $\mathcal{A}$-module functors. Then L is left adjoint to R if and only if there are natural isomorphisms*

$$\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(R(n), m) \xrightarrow{\cong} \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{N}}(n, L(m)) \tag{1.40}$$

for $n \in \mathcal{N}$ and $m \in \mathcal{M}$.

Again, if $\mathcal{M}$ is exact, then the isomorphisms (1.40) provide an $\overline{\mathcal{A}^*_{\mathcal{M}}}$-balancing to the internal coHom bimodule functor (1.39).

1.4 Relative Serre functors and Nakayama functors

A Serre functor $\mathbb{S}$ of a linear additive category $\mathcal{X}$ with finite dimensional morphism spaces furnishes natural isomorphisms between the vector spaces $\mathrm{Hom}_{\mathcal{X}}(x,y)$ and $\mathrm{Hom}_{\mathcal{X}}(y,\mathbb{S}(x))^*$. In case $\mathcal{X}$ is finite abelian, a Serre functor only exists if $\mathcal{X}$ is semisimple. But in the case that $\mathcal{X}=\mathcal{M}$ is a finite module category over a finite tensor category $\mathcal{A}$, then there is internalized version of the Serre functor which exists beyond the semisimple case:

Definition 1.12. [FSS, Def. 4.22] Let $\mathcal{M}$ be a left $\mathcal{A}$-module category. A (*right*) *relative Serre functor* on $\mathcal{M}$ is an endofunctor $\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\colon \mathcal{M}\to\mathcal{M}$ together with a family

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n)^{\vee} \xrightarrow{\cong} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}\big(n,\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\big) \tag{1.41}$$

of natural isomorphisms for $m,n\in\mathcal{M}$. Similarly, a (*left*) *relative Serre functor* $\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}$ comes with a family

$${}^{\vee}\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) \xrightarrow{\cong} \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}\big(\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(n),m\big) \tag{1.42}$$

of natural isomorphisms.

According to [FSS, Prop. 4.24] a module category admits relative Serre functors if and only if it is exact. In that case the left and right relative Serre functors are quasi-inverses of each other and can be uniquely identified (see also [Sh2, Lemmas 3.3-3.5]) by the formulas

$$\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\cong\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,-)^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) \qquad\text{and}\qquad \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m)\cong\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,-)^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}})\,. \tag{1.43}$$

Proposition 1.13. *For $\mathcal{M}$ an exact $(\mathcal{A},\mathcal{B})$-bimodule category, there are coherent natural isomorphisms*

$$\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(a\triangleright m)\cong a^{\vee\vee}\triangleright\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \qquad\textit{and}\qquad \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m\triangleleft b)\cong\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\triangleleft b^{\vee\vee} \tag{1.44}$$

which turn the relative Serre functor $\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ into a twisted bimodule equivalence. In a similar manner, its quasi-inverse comes with coherent natural isomorphisms

$$\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(a\triangleright m)\cong {}^{\vee\vee}a\triangleright\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m) \qquad\textit{and}\qquad \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m\triangleleft b)\cong\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m)\triangleleft {}^{\vee\vee}b \tag{1.45}$$

making of $\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}$ a twisted bimodule equivalence.

Proof. For left module categories this was already shown in [FSS, Lemma 4.23]. Now for $b\in\mathcal{B}$ we have an adjunction $(-\triangleleft b)\dashv(-\triangleleft b^{\vee})$ of $\mathcal{A}$-module functors. It follows that there is a chain

$$\begin{aligned}\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(n,\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m\triangleleft b))&\cong\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m\triangleleft b,n)^{\vee}\cong\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n\triangleleft b^{\vee})^{\vee}\\&\cong\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(n\triangleleft b^{\vee},\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m))\cong\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(n,\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\triangleleft b^{\vee\vee})\end{aligned} \tag{1.46}$$

of natural isomorphisms, where $m,n\in\mathcal{M}$ and $b\in\mathcal{B}$. Hence the desired family of isomorphisms is granted by the Yoneda Lemma for internal Hom's, and by construction these are coherent with respect to the tensor product. □

Remark 1.14. A right module category $\mathcal{N}_{\mathcal{B}}$ can be seen as a left module category ${}_{\overline{\mathcal{B}}}\mathcal{N}$ over the monoidal opposite of $\mathcal{B}$. Since right duals in $\mathcal{B}$ correspond to left duals in its monoidal opposite and vice versa, the module structure on the relative Serre functors is twisted according to

$$\mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n\triangleleft b)\cong\mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n)\triangleleft {}^{\vee\vee}b \qquad\text{and}\qquad \overline{\mathbb{S}}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n\triangleleft b)\cong\overline{\mathbb{S}}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n)\triangleleft b^{\vee\vee} \tag{1.47}$$

for $n\in\mathcal{N}$ and $b\in\mathcal{B}$. In the case of a bimodule category ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$, the actions of $\mathcal{A}$ are twisted by

$$\mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{N}}(a\triangleright n)\cong {}^{\vee\vee}a\triangleright\mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n) \qquad\text{and}\qquad \overline{\mathbb{S}}^{\overline{\mathcal{B}}}_{\mathcal{N}}(a\triangleright n)\cong a^{\vee\vee}\triangleright\overline{\mathbb{S}}^{\overline{\mathcal{B}}}_{\mathcal{N}}(n)\,. \tag{1.48}$$

Proposition 1.15. *Given exact $\mathcal{A}$-module categories $\mathcal{M}$ and $\mathcal{N}$, for every module functor $F\colon \mathcal{M} \to \mathcal{N}$ there is a natural isomorphism*

$$\Lambda_F : \quad \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ F \xRightarrow{\cong} F^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \tag{1.49}$$

of twisted module functors, where F^{rra} is the double right adjoint of F. Λ_F is compatible with composition of module functors, i.e. the diagram

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{A}}_{\mathcal{L}} \circ H \circ F \arrow[r, Rightarrow, "\Lambda_{H\circ F}"] \arrow[d, Rightarrow, swap, "\Lambda_H \circ \mathrm{id}"] & (H \circ F)^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[d, Rightarrow, "\cong"] \\
H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ F \arrow[r, Rightarrow, swap, "\mathrm{id}\circ\Lambda_F"] & H^{\mathrm{rra}} \circ F^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}
\end{tikzcd} \tag{1.50}$$

commutes for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})$ and $F \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. Analogously, in the case of bimodule categories and a bimodule functor F, the natural isomorphism (1.49) *is an isomorphism of twisted bimodule functors.*

Proof. The first part of the statement referring to left modules is shown in [Sh2, Thm. 3.10]. For bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$ and $F\colon \mathcal{M} \to \mathcal{N}$ a bimodule functor, we now check that the isomorphism Λ_F in (1.49) is compatible with the $\mathcal{B}$-module functor structures: Consider the diagram

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ F(m \triangleleft b) \arrow[r, "\psi"] \arrow[d, swap, "\Lambda_F \circ \mathrm{id}"] \arrow[dr, "\Lambda_{F(-)\triangleleft b}"] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}}\,(F(m) \triangleleft b) \arrow[r, "\Lambda_{-\triangleleft b}"] \arrow[dr, "\Lambda_{F(-\triangleleft b)}"] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ F(m) \triangleleft b^{\vee\vee} \arrow[d, "\Lambda_F \triangleleft \mathrm{id}"] \\
F^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m \triangleleft b) \arrow[r, swap, "\mathrm{id}\circ\Lambda_{-\triangleleft b}"] & F^{\mathrm{rra}}\Big(\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \triangleleft b^{\vee\vee}\Big) \arrow[r, swap, "\psi^{\mathrm{rra}}"] & F^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \triangleleft b^{\vee\vee}
\end{tikzcd} \tag{1.51}$$

where $\psi_{m,b}\colon F(m \triangleleft b) \xrightarrow{\cong} F(m) \triangleleft b$ denotes the $\mathcal{B}$-module functor structure on F and $\Lambda_{-\triangleleft b}$ is the twisted $\mathcal{B}$-module structure of the relative Serre functor. Both of the triangles in (1.51) are realizations of diagram (1.50) and are thus commutative. The middle square in (1.51) commutes owing to naturality of (1.49) with respect to F. ☐

Remark 1.16. An analogous statement as in Proposition 1.15 holds for left relative Serre functors: Given $F \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, there is a natural isomorphism

$$\overline{\Lambda}_F : \quad \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ F \xRightarrow{\cong} F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \tag{1.52}$$

of twisted module functors, compatible with composition of module functors in a similar manner as in (1.50), i.e. the diagram

$$\begin{tikzcd}
\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{L}} \circ H \circ F \arrow[r, Rightarrow, "\overline{\Lambda}_{H\circ F}"] \arrow[d, Rightarrow, swap, "\overline{\Lambda}_H \circ \mathrm{id}"] & (H \circ F)^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \arrow[d, Rightarrow, "\cong"] \\
H^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ F \arrow[r, Rightarrow, swap, "\mathrm{id}\circ\overline{\Lambda}_F"] & H^{\mathrm{lla}} \circ F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}
\end{tikzcd} \tag{1.53}$$

commutes for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})$ and $F \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. The isomorphisms (1.52) are related with (1.49) by the commutative diagram

$$\begin{tikzcd}
\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ F \arrow[rr, Rightarrow, "\overline{\Lambda}_F"] \arrow[d, Rightarrow, "\cong"'] & & F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \arrow[d, Rightarrow, "\cong"] \\
\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ F \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} & & \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \arrow[ll, Rightarrow, "\mathrm{id}\circ\Lambda_{F^{\mathrm{lla}}}\circ\mathrm{id}"]
\end{tikzcd} \tag{1.54}$$

considering that the left and right relative Serre functors are mutual quasi-inverses. Likewise also the statements for bimodule functors hold.

Nakayama functors

Given a finite linear category $\mathcal{X}$, its right exact *Nakayama functor* is the image of the identity functor under the Eilenberg-Watts correspondence [FSS, Def. 3.14], i.e. explicitly

$$\mathbb{N}^r_{\mathcal{X}} := \int^{x\in\mathcal{X}} \mathrm{Hom}_{\mathcal{X}}(-,x)^* \otimes x\,. \tag{1.55}$$

It comes equipped with a family of natural isomorphisms

$$\mathbb{N}^r_{\mathcal{X}} \circ F \xRightarrow{\cong} F^{\mathrm{rra}} \circ \mathbb{N}^r_{\mathcal{X}} \tag{1.56}$$

for every $F \in \mathrm{Rex}(\mathcal{X},\mathcal{X})$ having a right adjoint that is right exact as well. The left exact analogue of the Nakayama functor is a left adjoint to $\mathbb{N}^r_{\mathcal{X}}$ which is explicitly given by

$$\mathbb{N}^l_{\mathcal{X}} := \int_{x\in\mathcal{X}} \mathrm{Hom}_{\mathcal{X}}(x,-) \otimes x\,. \tag{1.57}$$

In a similar manner as $\mathbb{N}^r_{\mathcal{X}}$, the functor $\mathbb{N}^l_{\mathcal{X}}$ is endowed with a family of natural isomorphisms

$$\mathbb{N}^l_{\mathcal{X}} \circ F \xRightarrow{\cong} F^{\mathrm{lla}} \circ \mathbb{N}^l_{\mathcal{X}} \tag{1.58}$$

for every $F \in \mathrm{Lex}(\mathcal{X},\mathcal{X})$ having a left adjoint that is left exact as well.

The Nakayama functors of the $\Bbbk$-linear category underlying a finite multitensor category $\mathcal{A}$ [FSS, Lemma 4.10] can be described, with the help of (1.56) and (1.58), as

$$\mathbb{N}^r_{\mathcal{A}} \cong \mathbb{D}^{-1}_{\mathcal{A}} \otimes (-)^{\vee\vee} \quad \text{and} \quad \mathbb{N}^l_{\mathcal{A}} \cong \mathbb{D}_{\mathcal{A}} \otimes {}^{\vee\vee}(-) \tag{1.59}$$

where $\mathbb{D}_{\mathcal{A}} := \mathbb{N}^l_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}})$ is a *distinguished invertible object* and $\mathbb{D}^{-1}_{\mathcal{A}} = \mathbb{N}^r_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}})$ [FSS, Lemma 4.11]. Additionally, for every such finite multitensor category $\mathcal{A}$ the invertible object $\mathbb{D}_{\mathcal{A}}$ comes with a monoidal natural isomorphism

$$r_{\mathcal{A}}: \quad \mathbb{D}_{\mathcal{A}} \otimes - \otimes \mathbb{D}^{-1}_{\mathcal{A}} \xRightarrow{\cong} (-)^{\vee\vee\vee\vee} \tag{1.60}$$

known as the *Radford isomorphism*. Composing the isomorphisms (1.59) with the Radford isomorphism we obtain

$$\mathbb{N}^r_{\mathcal{A}} \cong {}^{\vee\vee}(-) \otimes \mathbb{D}^{-1}_{\mathcal{A}} \quad \text{and} \quad \mathbb{N}^l_{\mathcal{A}} \cong (-)^{\vee\vee} \otimes \mathbb{D}_{\mathcal{A}}\,. \tag{1.61}$$

A finite tensor category $\mathcal{A}$ is said to be *unimodular* iff its distinguished invertible object $\mathbb{D}_{\mathcal{A}}$ is isomorphic to the monoidal unit $\mathbf{1}_{\mathcal{A}}$.

More generally, given an $(\mathcal{A}, \mathcal{B})$-bimodule category $\mathcal{M}$, by means of the isomorphisms (1.56) and (1.58), the Nakayama functors are endowed with a twisted module functor structure [FSS, Thm. 4.4-4.5]:

$$\mathbb{N}^l_{\mathcal{M}}(a \triangleright m \triangleleft b) \cong a^{\vee\vee} \triangleright \mathbb{N}^l_{\mathcal{M}}(m) \triangleleft {}^{\vee\vee}b \quad \text{and} \quad \mathbb{N}^r_{\mathcal{M}}(a \triangleright m \triangleleft b) \cong {}^{\vee\vee}a \triangleright \mathbb{N}^r_{\mathcal{M}}(m) \triangleleft b^{\vee\vee} . \tag{1.62}$$

The relative Serre functors of ${}_{\mathcal{A}}\mathcal{M}$ are related to the Nakayama functors by

$$\mathbb{D}_{\mathcal{A}} \triangleright \mathbb{N}^r_{\mathcal{M}} \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \qquad \text{and} \qquad \mathbb{D}^{-1}_{\mathcal{A}} \triangleright \mathbb{N}^l_{\mathcal{M}} \cong \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \tag{1.63}$$

as twisted module functors [FSS, Thm. 4.26]. It can be checked by direct computation that for a bimodule category these are indeed isomorphisms of twisted bimodule functors.

1.5 The relative Drinfeld center

Given a tensor category $\mathcal{A}$ and a full tensor subcategory $\mathcal{D} \subset \mathcal{A}$, the *relative center* of $\mathcal{A}$ with respect to $\mathcal{D}$ [GNN, Section 2B] is a tensor category $\mathcal{Z}_{\mathcal{D}}(\mathcal{A})$ whose objects are pairs (a, σ) where $a \in \mathcal{A}$ and σ is a half-braiding relative to $\mathcal{D}$, i.e., a natural isomorphism $\sigma_{d,a} : d \otimes a \xrightarrow{\cong} a \otimes d$ for $d \in \mathcal{D}$, obeying the corresponding hexagon axiom. The morphisms are morphisms in $\mathcal{A}$ commuting with the associated half-braidings. Notice that the Drinfeld center $\mathcal{Z}(\mathcal{D})$ is a tensor subcategory of $\mathcal{Z}_{\mathcal{D}}(\mathcal{A})$. The relative center $\mathcal{Z}_{\mathcal{D}}(\mathcal{A})$ comes equipped with a *relative braiding* with respect to $\mathcal{Z}(\mathcal{D})$, this is a natural isomorphism given for $(a, \sigma) \in \mathcal{Z}(\mathcal{D})$ and $(d, \delta) \in \mathcal{Z}_{\mathcal{D}}(\mathcal{A})$ by

$$\omega_{(a,\sigma),(d,\delta)} := \delta_{a,d} : \quad a \otimes d \xrightarrow{\cong} d \otimes a \tag{1.64}$$

which obeys the corresponding hexagon axioms. In the case of $\mathcal{D} = \mathcal{A}$ we have that $\mathcal{Z}_{\mathcal{A}}(\mathcal{A}) = \mathcal{Z}(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$ and the relative braiding corresponds to its braided structure.

Remark 1.17. Let $\mathcal{A}$ be a finite tensor category, then the relative center with respect to a tensor subcategory $\mathcal{D}$ has the following properties:

(i) There is a commutative diagram of forgetful functors.

$$\begin{tikzcd}
\mathcal{Z}(\mathcal{A}) \arrow[rr, "(a,\sigma)\mapsto(a,\sigma|_{\mathcal{D}})"] \arrow[dr, "(a,\sigma)\mapsto a"'] & & \mathcal{Z}_{\mathcal{D}}(\mathcal{A}) \arrow[dl, "(a,\sigma)\mapsto a"] \\
& \mathcal{A} &
\end{tikzcd} \tag{1.65}$$

According to [EO, Proposition 3.39] the functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is surjective, thus the forgetful functor $F : \mathcal{Z}_{\mathcal{D}}(\mathcal{A}) \to \mathcal{A}$ is surjective as well; this means that every $a \in \mathcal{A}$ is a subquotient of an object of the form $F(Z)$ with $Z \in \mathcal{Z}_{\mathcal{D}}(\mathcal{A})$.

(ii) The category $\mathcal{A}$ is naturally endowed with the structure of a $\mathcal{D} \boxtimes \overline{\mathcal{A}}$-module category via

$$(d \boxtimes \overline{a}) \triangleright b := d \otimes b \otimes a . \tag{1.66}$$

There is a distinguished tensor equivalence $\mathcal{Z}_{\mathcal{D}}(\mathcal{A}) \simeq (\mathcal{D} \boxtimes \overline{\mathcal{A}}\,)^*_{\mathcal{A}}$ between the dual tensor category of this module and the relative center [GNN, Rem. 2.4 (i)].

(iii) The Frobenius-Perron dimension of the relative center [GNN, Rem. 2.4 (ii)] is given by

$$\mathrm{FPdim}(\mathcal{Z}_{\mathcal{D}}(\mathcal{A})) = \mathrm{FPdim}(\mathcal{D})\,\mathrm{FPdim}(\mathcal{A})\ .$$

(iv) The category $\mathcal{A}$ is an exact module category over $\mathcal{Z}_{\mathcal{D}}(\mathcal{A})$, with module structure induced by the forgetful functor $F : \mathcal{Z}_{\mathcal{D}}(\mathcal{A}) \to \mathcal{A}$: exactness follows from (ii) and [EGNO, Lemma 7.12.7].

The following result shows a relationship between module categories and the Drinfeld center:

Theorem 1.18. [Sch, Theorem 3.3] *Let A be an algebra in a finite tensor category $\mathcal{A}$. There is a braided tensor equivalence*

$$\begin{aligned} \mathrm{S}: \quad \mathcal{Z}(\mathcal{A}) &\xrightarrow{\simeq} \mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A})), \\ (X,\sigma) &\longmapsto (A\otimes X, \delta)\,, \end{aligned} \tag{1.67}$$

where the object $A\otimes X$ becomes an A-bimodule by means of the regular action and the isomorphism $\sigma_{X,A}$ and for $M \in {}_A\mathrm{Bimod}_A(\mathcal{A})$, the half-braiding δ in $\mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A}))$ is defined by the composition

$$M \otimes_A A \otimes X \cong M \otimes X \xrightarrow{\sigma_{M,X}} X \otimes M \cong X \otimes A \otimes_A M \xrightarrow{\sigma^{-1}_{A,X}\otimes_A M} A \otimes X \otimes_A M\,. \tag{1.68}$$

According to Theorem 1.8, every $\mathcal{A}$-module category is of the form $\mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{A})$ and thus composing (1.67) with the Eilenberg-Watts equivalence (1.9) leads to a braided equivalence $\mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\overline{\mathcal{A}^*_{\mathcal{M}}})$. Explicitly it can be given, without making a choice of an algebra in $\mathcal{A}$, as [Sh3, Thm. 3.13]

$$\begin{aligned} \Sigma: \quad \mathcal{Z}(\mathcal{A}) &\xrightarrow{\simeq} \mathcal{Z}(\,\overline{\mathcal{A}^*_{\mathcal{M}}}\,), \\ (a,\sigma) &\longmapsto (a \triangleright -, \gamma)\,, \end{aligned} \tag{1.69}$$

where σ endows $a\triangleright -$ with an $\mathcal{A}$-module functor structure, and for $F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$ the half-braiding $\gamma_{F,a\triangleright -} : a \triangleright F \xrightarrow{\cong} F(a\triangleright -)$ is given by the module functor structure of F.

1.6 Categorical Morita equivalence

In classical algebra two rings are Morita equivalent iff their categories of modules are equivalent as abelian categories. This notion finds a categorical analogue in the setting of finite tensor categories. The following more technical definition is in practice easier to check.

Definition 1.19. [EGNO, Def. 7.12.17] Two tensor categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *categorically Morita equivalent* iff there exists an $\mathcal{A}$-module category $\mathcal{M}$ together with a tensor equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$.

Example 1.20.

- Let G be a finite group, then the tensor category of G-graded vector spaces Vec_G and the tensor category of group representations $\mathrm{Rep}(G)$ are categorically Morita equivalent.
- More generally, given a finite dimensional Hopf algebra H, the tensor categories comod-H and H-mod of comodules and modules over H are categorically Morita equivalent.

Remark 1.21. In the situation of Definition 1.19, the module category $\mathcal{M}$ is necessarily:

(i) Indecomposable: Follows from the fact that the identity functor of $\mathcal{M}$ has to be simple in $\mathcal{A}^*_{\mathcal{M}}$ [EO, Lemma 3.24].

(ii) Exact: Rigidity of $\mathcal{A}^*_{\mathcal{M}}$ implies that every endofunctor of $\mathcal{M}$ is exact, which means that the $\mathcal{A}$-module category $\mathcal{M}$ is exact [EO, Proposition 3.16].

Definition 1.19 indeed yields an equivalence relation on finite tensor categories:

(i) Reflexivity: Let $\mathcal{A}$ be a finite tensor category and consider the regular module category ${}_{\mathcal{A}}\mathcal{A}$. Then the tensor functor

$$\begin{aligned} \mathcal{A} &\xrightarrow{\simeq} \overline{\mathrm{End}_{\mathcal{A}}(\mathcal{A})} = \overline{\mathcal{A}^*_{\mathcal{A}}}\,, \\ a &\longmapsto - \otimes a \end{aligned} \tag{1.70}$$

is a tensor equivalence [EGNO, Ex. 7.12.3].

(ii) Symmetry: Let $\mathcal{M}$ be an exact $\mathcal{A}$-module category. Regarding $\mathcal{M}$ as a left module category over $\mathcal{A}^*_{\mathcal{M}}$, one can consider the double dual tensor category $(\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}$. There is a canonical tensor equivalence [EGNO, Thm. 7.12.11]

$$\begin{aligned} \mathrm{can}: \quad \mathcal{A} &\xrightarrow{\simeq} \mathrm{End}_{\mathcal{A}^*_{\mathcal{M}}}(\mathcal{M}) = (\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}\ , \\ a &\longmapsto a \triangleright - \,, \end{aligned} \tag{1.71}$$

where the $\mathcal{A}^*_{\mathcal{M}}$-module functor structure on the functor $a \triangleright -$ is given for $F \in \mathcal{A}^*_{\mathcal{M}}$ and $m \in \mathcal{M}$ by the $\mathcal{A}$-module functor structure of F: $a \triangleright F(m) \xrightarrow{\cong} F(a \triangleright m)$.

(iii) Transitivity is proven in [EGNO, Prop. 7.12.18].

The modules of a finite tensor category form a 2-category $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ with objects exact left $\mathcal{A}$-module categories, 1-morphisms module functors and 2-morphisms module natural transformations. As expected Morita equivalent finite tensor categories have equivalent 2-categories of exact modules: Given an exact $\mathcal{A}$-module category $\mathcal{M}$, the assignment

$$\begin{aligned} \mathbf{Mod}^{\mathrm{ex}}(\mathcal{A}) &\xrightarrow{\simeq} \mathbf{Mod}^{\mathrm{ex}}\left(\overline{\mathcal{A}^*_{\mathcal{M}}}\right), \\ \mathcal{N} &\longmapsto \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \end{aligned} \tag{1.72}$$

is an equivalence of 2-categories [EGNO, Thm. 7.12.16]. The converse also holds and seems to be well-known to experts; but it is not easy to find in the literature and thus a proof is here included.

Theorem 1.22. *Two finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ and $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{B})$ are equivalent as 2-categories.*

Proof. One implication follows from [EGNO, Thm. 7.12.16] as previously explained. To prove the converse, consider an equivalence of 2-categories

$$\Phi: \quad \mathbf{Mod}^{\mathrm{ex}}(\mathcal{B}) \xrightarrow{\simeq} \mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})\,. \tag{1.73}$$

Let $\mathcal{M}$ be the image of the regular module category ${}_{\mathcal{B}}\mathcal{B}$ under Φ. Since Φ is a 2-equivalence, at the level of Hom-categories, it comes with an equivalence

$$\Phi_{\mathcal{B},\mathcal{B}}: \quad \mathcal{B}^*_{\mathcal{B}} = \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M}) = \mathcal{A}^*_{\mathcal{M}} \tag{1.74}$$

of categories. Additionally, amongst the data of the 2-functor Φ there is a natural isomorphism

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \times \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \arrow[r, "\circ"] \arrow[d, "\Phi\times\Phi"'] & \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\gamma"'] \\
\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M}) \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M}) \arrow[r, "\circ"'] & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M})
\end{tikzcd} \tag{1.75}$$

which endows $\Phi_{\mathcal{B},\mathcal{B}}$ with a tensor structure. Pre-composing (1.74) with (1.70) we obtain the desired tensor equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$. □

The characterization of Morita equivalence given in Theorem 1.22 is more symmetric than Definition 1.19 and thus clearly shows that, indeed, we are dealing with an equivalence relation on finite tensor categories. There is a further characterization: The Drinfeld center completely captures the notion of Morita equivalence.

Theorem 1.23. [EGNO, Thm. 8.12.3] *Two finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if their Drinfeld centers $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are equivalent as braided tensor categories.*

Chapter 2

Categorical Morita contexts

In the theory of rings or $\Bbbk$-algebras the notion of Morita equivalence finds a generalization in the structure of a Morita context or pre-equivalence data [Ba, Ch.2 §3]. These data involving two (not necessarily invertible) bimodules between two rings form a category with two objects. We now study the analogue of this notion for categorical Morita equivalence of finite tensor categories.

Definition 2.1.
(i) A (categorical) *Morita context* consists of the following data:

1. Two finite (multi-)tensor categories $\mathcal{A}$ and $\mathcal{B}$.
2. Two bimodule categories ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$.
3. Two bimodule functors

$$\underline{\ominus}: \quad \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \longrightarrow \mathcal{A} \qquad \text{and} \qquad \underline{\boxminus}: \quad \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{M} \longrightarrow \mathcal{B}\,. \tag{2.1}$$

4. Two bimodule natural isomorphisms α and β of the form

$$\begin{tikzcd}
(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}) \boxtimes_{\mathcal{A}} \mathcal{M} \arrow[rr, "\simeq"] \arrow[d, swap, "\underline{\ominus}\boxtimes\mathrm{id}"] & & \mathcal{M} \boxtimes_{\mathcal{B}} (\mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}) \arrow[d, "\mathrm{id}\boxtimes\underline{\boxminus}"] \arrow[dll, Rightarrow, swap, "\alpha"] \\
\mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{M} \arrow[dr, swap, "\triangleright"] & & \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{B} \arrow[dl, "\triangleleft"] \\
& \mathcal{M} &
\end{tikzcd} \tag{2.2}$$

and

$$\begin{tikzcd}
(\mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}) \boxtimes_{\mathcal{B}} \mathcal{N} \arrow[rr, "\simeq"] \arrow[d, swap, "\underline{\boxminus}\boxtimes\mathrm{id}"] & & \mathcal{N} \boxtimes_{\mathcal{A}} (\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}) \arrow[d, "\mathrm{id}\boxtimes\underline{\ominus}"] \arrow[dll, Rightarrow, swap, "\beta"] \\
\mathcal{B} \boxtimes_{\mathcal{B}} \mathcal{N} \arrow[dr, swap, "\triangleright"] & & \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{A} \arrow[dl, "\triangleleft"] \\
& \mathcal{N} &
\end{tikzcd} \tag{2.3}$$

These data are required to fulfill the condition that the pentagon diagrams

$$\begin{tikzcd}
& (m_1 \underline{\ominus} n_1) \otimes_{\mathcal{A}} (m_2 \underline{\ominus} n_2) \arrow[dl, "\phi_{m_1 \underline{\ominus} n_1, m_2 \boxtimes n_2}"'] \arrow[dr, "\phi'_{m_1 \boxtimes n_1, m_2 \underline{\ominus} n_2}"] & \\
((m_1 \underline{\ominus} n_1) \triangleright m_2) \underline{\ominus} n_2 & & m_1 \underline{\ominus} (n_1 \triangleleft (m_2 \underline{\ominus} n_2)) \arrow[d, "\mathrm{id} \underline{\ominus} \beta_{n_1, m_2, n_2}"] \\
(m_1 \triangleleft (n_1 \underline{\boxminus} m_2)) \underline{\ominus} n_2 \arrow[u, "\alpha_{m_1, n_1, m_2} \underline{\ominus} \mathrm{id}"] \arrow[rr, "\cong"] & & m_1 \underline{\ominus} ((n_1 \underline{\boxminus} m_2) \triangleright n_2)
\end{tikzcd} \tag{2.4}$$

and

$$\begin{tikzcd}
& (n_1 \underline{\boxminus} m_1) \otimes_{\mathcal{B}} (n_2 \underline{\boxminus} m_2) \arrow[dl, "\psi_{n_1 \underline{\boxminus} m_1, n_2 \boxtimes m_2}"'] \arrow[dr, "\psi'_{n_1 \boxtimes m_1, n_2 \underline{\boxminus} m_2}"] & \\
((n_1 \underline{\boxminus} m_1) \triangleright n_2) \underline{\boxminus} m_2 & & n_1 \underline{\boxminus} (m_1 \triangleleft (n_2 \underline{\boxminus} m_2)) \arrow[d, "\mathrm{id} \underline{\boxminus} \alpha_{m_1, n_2, m_2}"] \\
(n_1 \triangleleft (m_1 \underline{\ominus} n_2)) \underline{\boxminus} m_2 \arrow[u, "\beta_{n_1, m_1, n_2} \underline{\boxminus} \mathrm{id}"] \arrow[rr, "\cong"] & & n_1 \underline{\boxminus} ((m_1 \underline{\ominus} n_2) \triangleright m_2)
\end{tikzcd} \tag{2.5}$$

commute for all $m_1, m_2 \in \mathcal{M}$ and $n_1, n_2 \in \mathcal{N}$, where ϕ and ϕ' are the bimodule structures of the functor $\underline{\ominus}$ and ψ and ψ' are the bimodule structures of $\underline{\boxminus}$.

(ii) We say that a Morita context is *strict* iff $\mathcal{A}$ and $\mathcal{B}$ are strict tensor categories, $\mathcal{M}$ and $\mathcal{N}$ are strict bimodules, $\underline{\ominus}$ and $\underline{\boxminus}$ have strict bimodule functor structures and α and β are identities.

(iii) A Morita context is said to be *strong* iff $\underline{\ominus}$ and $\underline{\boxminus}$ are equivalences. In that case, the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is *invertible* and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$ is *its inverse.*

Remark 2.2.

(i) Via the equivalence (1.11) coming from the universal property of the relative Deligne product, the bimodule functors (2.1) in a Morita context correspond to bimodule balanced functors

$$\ominus : \quad \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{A} \qquad \text{and} \qquad \boxminus : \quad \mathcal{N} \times \mathcal{M} \longrightarrow \mathcal{B} \,. \tag{2.6}$$

As a consequence, a Morita context can be equivalently defined by replacing (2.2) and (2.3) with bimodule balanced natural isomorphisms

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{N} \times \mathcal{M} \arrow[r, "\mathrm{id} \times \boxminus"] \arrow[d, "\ominus \times \mathrm{id}"'] & \mathcal{M} \times \mathcal{B} \arrow[d, "\triangleleft"] \arrow[dl, Rightarrow, "\alpha"] & \text{and} & \mathcal{N} \times \mathcal{M} \times \mathcal{N} \arrow[r, "\mathrm{id} \times \ominus"] \arrow[d, "\boxminus \times \mathrm{id}"'] & \mathcal{N} \times \mathcal{A} \arrow[d, "\triangleleft"] \arrow[dl, Rightarrow, "\beta"] \\
\mathcal{A} \times \mathcal{M} \arrow[r, "\triangleright"'] & \mathcal{M} & & \mathcal{B} \times \mathcal{N} \arrow[r, "\triangleright"'] & \mathcal{N}
\end{tikzcd} \tag{2.7}$$

obeying relations analogous to (2.4) and (2.5).

(ii) The four categories in a Morita context interact with each other via the tensor products of $\mathcal{A}$ and $\mathcal{B}$ and their actions on the bimodule categories $\mathcal{M}$ and $\mathcal{N}$. Accordingly the functors $\ominus$ and $\boxminus$ play the role of mixed products between the categories $\mathcal{M}$ and $\mathcal{N}$.

2.1 The bicategory of a Morita context

In the spirit of [Mü, Rem. 3.18] the data of a Morita context form a bicategory. The construction is as follows. Given a Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \ominus, \boxminus, \alpha, \beta)$, define a bicategory $\mathbb{M}$ consisting of two objects $\{+, -\}$ and Hom-categories

$$\begin{aligned} \mathbb{M}(+,+) &:= \mathcal{A}\,, \qquad & \mathbb{M}(-,+) &:= \mathcal{M}\,, \\ \mathbb{M}(-,-) &:= \mathcal{B}\,, \qquad & \mathbb{M}(+,-) &:= \mathcal{N}\,. \end{aligned} \tag{2.8}$$

Pictorially the bicategory $\mathbb{M}$ can be portrayed by

$$\begin{tikzcd} + \arrow[loop left, "\mathcal{A}"] \arrow[r, bend right, "\mathcal{N}"'] & - \arrow[l, bend right, "\mathcal{M}"'] \arrow[loop right, "\mathcal{B}"] \end{tikzcd} \quad . \tag{2.9}$$

The horizontal composition

$$\circ_{i,j,k} : \quad \mathbb{M}(\mathbf{j},\mathbf{k}) \times \mathbb{M}(\mathbf{i},\mathbf{j}) \longrightarrow \mathbb{M}(\mathbf{i},\mathbf{k}) \qquad \text{for} \quad \mathbf{i},\mathbf{j},\mathbf{k} \in \{+,-\} \tag{2.10}$$

in the bicategory $\mathbb{M}$ is given by the eight functors

$$\begin{aligned} \otimes_{\mathcal{A}} &: \quad \mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, & \otimes_{\mathcal{B}} &: \quad \mathcal{B} \times \mathcal{B} \to \mathcal{B}\,, \\ \triangleright_{\mathcal{M}} &: \quad \mathcal{A} \times \mathcal{M} \to \mathcal{M}\,, & \triangleright_{\mathcal{N}} &: \quad \mathcal{B} \times \mathcal{N} \to \mathcal{N}\,, \\ \triangleleft_{\mathcal{N}} &: \quad \mathcal{N} \times \mathcal{A} \to \mathcal{N}\,, & \triangleleft_{\mathcal{M}} &: \quad \mathcal{M} \times \mathcal{B} \to \mathcal{M}\,, \\ \ominus &: \quad \mathcal{M} \times \mathcal{N} \to \mathcal{A}\,, & \boxminus &: \quad \mathcal{N} \times \mathcal{M} \to \mathcal{B}\,, \end{aligned} \tag{2.11}$$

i.e. by the tensor products, module actions and mixed products in the Morita context.
The associativity constraints in $\mathbb{M}$ are natural isomorphisms

$$\begin{tikzcd} \mathbb{M}(\mathbf{j},\mathbf{k}) \times \mathbb{M}(\mathbf{i},\mathbf{j}) \times \mathbb{M}(\mathbf{h},\mathbf{i}) \arrow[r, "\mathrm{id}\times\circ_{h,i,j}"] \arrow[d, "\circ_{i,j,k}\times\mathrm{id}"'] & \mathbb{M}(\mathbf{j},\mathbf{k}) \times \mathbb{M}(\mathbf{h},\mathbf{j}) \arrow[d, "\circ_{h,j,k}"] \arrow[dl, Rightarrow, "\gamma_{h,i,j,k}"] \\ \mathbb{M}(\mathbf{i},\mathbf{k}) \times \mathbb{M}(\mathbf{h},\mathbf{i}) \arrow[r, "\circ_{h,i,k}"'] & \mathbb{M}(\mathbf{h},\mathbf{k}) \end{tikzcd} \tag{2.12}$$

for each quadruple of objects $\mathbf{h},\mathbf{i},\mathbf{j},\mathbf{k} \in \{+,-\}$. We require these sixteen constraints to be the following:

- The two associativity constraints from the tensor products of $\mathcal{A}$ and $\mathcal{B}$.
- The six module associativity constraints from the actions on the bimodules $\mathcal{M}$ and $\mathcal{N}$.
- Four constraints coming from the bimodule functor structures of $\ominus$ and $\boxminus$.
- The two balancings of $\ominus$ and $\boxminus$.
- The coherence data of the Morita context, i.e. the two isomorphisms α and β.

These data provide indeed a bicategory:

Theorem 2.3. *The data of a Morita context form a bicategory* $\mathbb{M}$ *with two objects.*

Proof. Clearly, the data given above are those needed for a bicategory. Thus it remains to check that the associators defined for $\mathbb{M}$ obey the relevant axiom in a bicategory, i.e. that the diagram

$$\begin{tikzcd}
 & (f_{k,l} \circ f_{j,k}) \circ (f_{i,j} \circ f_{h,i}) \arrow[dl, "\gamma_{h,i,j,l}"'] \arrow[dr, "\gamma_{h,k,j,l}"] & \\
((f_{k,l} \circ f_{j,k}) \circ f_{i,j}) \circ f_{h,i} \arrow[d, "\gamma_{i,j,k,l} \circ \mathrm{id}"'] & & f_{k,l} \circ (f_{j,k} \circ (f_{i,j} \circ f_{h,i})) \arrow[d, "\mathrm{id} \circ \gamma_{h,i,j,k}"] \\
(f_{k,l} \circ (f_{j,k} \circ f_{i,j})) \circ f_{h,i} \arrow[rr, "\gamma_{h,i,k,l}"'] & & f_{k,l} \circ ((f_{j,k} \circ f_{i,j}) \circ f_{h,i})
\end{tikzcd} \tag{2.13}$$

commutes for all $\mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \{+, -\}$ and every quadruple of 1-morphisms $f_{k,l} \in \mathbb{M}(\mathbf{k}, \mathbf{l})$, $f_{j,k} \in \mathbb{M}(\mathbf{j}, \mathbf{k})$, $f_{i,j} \in \mathbb{M}(\mathbf{i}, \mathbf{j})$ and $f_{h,i} \in \mathbb{M}(\mathbf{h}, \mathbf{i})$. It can be verified that these conditions correspond to the following thirty-two diagrams:

1. Two pentagons obeyed by the associativity constraints of $\mathcal{A}$ and $\mathcal{B}$.
2. Four pentagons fulfilled by the left and right module constraints of $\mathcal{M}$ and $\mathcal{N}$.
3. Four pentagons fulfilled by the middle associativity constraints of the bimodules $\mathcal{M}$ and $\mathcal{N}$.
4. Six compatibility conditions on the bimodule functor structures of $\ominus$ and $\boxminus$.
5. The two diagrams that describe the conditions on the balancings of $\ominus$ and $\boxminus$.
6. Four conditions corresponding to the compatibility between the bimodule structures and the balancings of $\ominus$ and $\boxminus$.
7. Four compatibility conditions characterizing α and β as bimodule natural transformations.
8. Four conditions defining α and β as balanced natural transformations.
9. The diagrams (2.4) and (2.5), which exhibit the compatibility between the coherence data α and β of the Morita context.

Thus all the conditions are satisfied by construction. □

Remark 2.4.

(i) The notion of a Morita context as a bicategory studied in [Mü] is an instance of Theorem 2.3 where a (Frobenius) algebra is chosen to realize the bimodules in the Morita context.

(ii) Theorem 2.3 justifies Definition 2.1. Coherence results for bicategories, such as Corollary 2.7 of [Gu], imply that all paths between any two possible bracketings of a product of multiple objects in the Morita context through associativity constraints are the same isomorphism.

As Theorem 2.3 indicates, the bicategorical setting emerges as the natural framework for studying Morita contexts. Let us recall a few pertinent notions from this setting. A *pseudo-functor* $U : \mathcal{F} \to \mathcal{G}$ between bicategories consists of assignments at the level of objects and at the level of Hom-categories together with invertible 2-morphisms that witness their compatibility with horizontal composition and which obey a pentagon axiom. A *pseudo-natural equivalence*

$\eta\colon U \overset{\cong}{\Longrightarrow} V$ between pseudo-functors amounts to an invertible 1-morphism $\eta_x\colon U(x) \xrightarrow{\cong} V(x)$ for every object $x \in \mathcal{F}$ and an invertible 2-morphism $\eta_f\colon \eta_y \circ U(f) \overset{\cong}{\Longrightarrow} V(f) \circ \eta_x$ for every 1-morphism $f\colon x \to y$, respecting horizontal composition. Complete definitions can for instance be found in [Sc2, App. A.2].

Definition 2.5. An *equivalence of Morita contexts* is a pseudo-equivalence between their bicategories.

Corollary 2.6 (Coherence for Morita contexts)**.** *Every Morita context is equivalent to a strict Morita context.*

Proof. Using that every bicategory is equivalent to a strict 2-category [Gu, Cor. 2.7], this follows directly from Theorem 2.3. □

2.2 The Morita context derived from an exact module category

Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an exact $\mathcal{A}$-module. There is a strong Morita context associated to ${}_{\mathcal{A}}\mathcal{M}$. To see this, first recall that the category $\mathcal{A}^*_{\mathcal{M}}$ of module endofunctors has the structure of a tensor category and that the evaluation of a functor on an object turns $\mathcal{M}$ into an exact $\mathcal{A}^*_{\mathcal{M}}$-module category. More specifically, hereby $\mathcal{M}$ becomes an $(\mathcal{A}, \overline{\mathcal{A}^*_{\mathcal{M}}})$-bimodule category.

As described in (1.7), the category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ of module functors is naturally endowed with the structure of an $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \mathcal{A})$-bimodule category, via the actions

$$F \succ H := H \circ F \qquad \text{and} \qquad H \prec a := H(-) \otimes a = (- \otimes a) \circ H \tag{2.14}$$

for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$, $F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$ and $a \in \mathcal{A}$.

Having obtained two tensor categories and two bimodules categories from the exact $\mathcal{A}$-module $\mathcal{M}$, we can proceed to define the mixed products in the Morita context:

Definition 2.7. The $\overline{\mathcal{A}^*_{\mathcal{M}}}$*-valued mixed product* of $\mathcal{M}$ is the functor

$$\begin{aligned} \boxdot :\quad \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \times \mathcal{M} &\longrightarrow \overline{\mathcal{A}^*_{\mathcal{M}}}\,, \\ (H, m) &\longmapsto H(-) \triangleright m = (- \triangleright m) \circ H\,, \end{aligned} \tag{2.15}$$

which is a special case of (1.13). The $\mathcal{A}$*-valued mixed product* of $\mathcal{M}$ is the functor given by evaluation

$$\begin{aligned} \odot :\quad \mathcal{M} \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) &\longrightarrow \mathcal{A}\,, \\ (m, H) &\longmapsto H(m)\,. \end{aligned} \tag{2.16}$$

Lemma 2.8. *The mixed product* $\boxdot$ *is an* $\mathcal{A}$*-balanced* $\overline{\mathcal{A}^*_{\mathcal{M}}}$*-bimodule functor, and the mixed product* $\odot$ *is an* $\overline{\mathcal{A}^*_{\mathcal{M}}}$*-balanced* $\mathcal{A}$*-bimodule functor.*

Proof. First notice that the functor $\boxdot$ comes with an $\mathcal{A}$-balancing given by the module associativity constraints of ${}_{\mathcal{A}}\mathcal{M}$:

$$(H \prec a) \boxdot m = (H(-) \otimes a) \triangleright m \xrightarrow{\cong} H(-) \triangleright (a \triangleright m) = H \boxdot (a \triangleright m)\,. \tag{2.17}$$

Furthermore, the identities

$$F \triangleright (H \boxdot m) = (- \triangleright m) \circ H \circ F = (F \succ H) \boxdot m \tag{2.18}$$

and the isomorphisms

$$(H \boxdot m) \triangleleft F = F(H(-) \triangleright m) \xrightarrow{\cong} H(-) \triangleright F(m) = H \boxdot F(m) \tag{2.19}$$

that come from the module functor structure of F endow $\boxdot$ with the structure of an $\overline{\mathcal{A}^*_{\mathcal{M}}}$-bimodule functor. These are compatible with the balancing because the module functor structure of F is compatible with the associativity constraints of ${}_{\mathcal{A}}\mathcal{M}$.

Similarly, for the functor $\odot$ the identity morphisms $(m \triangleleft F) \odot H = H \circ F(m) = m \odot (F \succ H)$ provide an $\overline{\mathcal{A}^*_{\mathcal{M}}}$-balancing. Moreover, there is a natural $\mathcal{A}$-bimodule functor structure on $\odot$, namely

$$a \otimes (m \odot H) = a \otimes H(m) \cong H(a \triangleright m) = (a \triangleright m) \odot H \tag{2.20}$$

and

$$(m \odot H) \otimes a = H(m) \otimes a = m \odot (H \prec a) \tag{2.21}$$

given by the module functor structure on H and the identity morphisms, respectively. ☐

Theorem 2.9. *Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. Then the data*

$$\Big(\mathcal{A},\, \overline{\mathcal{A}^*_{\mathcal{M}}},\, \mathcal{M},\, \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}),\ \odot\,,\ \boxdot\Big) \tag{2.22}$$

form a Morita context.

Proof. Notice that the diagram

$$\begin{tikzcd}
\mathcal{M} \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \times \mathcal{M} \arrow[r, "\mathrm{id}\times\boxdot"] \arrow[d, "\odot\times\mathrm{id}"'] & \mathcal{M} \times \overline{\mathcal{A}^*_{\mathcal{M}}} \arrow[d, "\triangleleft"] \\
\mathcal{A} \times \mathcal{M} \arrow[r, "\triangleright"'] & \mathcal{M}
\end{tikzcd} \tag{2.23}$$

commutes strictly owing to $(m \odot H) \triangleright n = H(m) \triangleright n = m \triangleleft (H \boxdot n)$, and thus the identities serve as the bimodule natural isomorphism (2.2). On the other hand, the diagram

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \times \mathcal{M} \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \arrow[r, "\mathrm{id}\times\odot"] \arrow[d, "\boxdot\times\mathrm{id}"'] & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \times \mathcal{A} \arrow[d, "\prec"] \\
\overline{\mathcal{A}^*_{\mathcal{M}}} \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \arrow[r, "\succ"'] & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})
\end{tikzcd} \tag{2.24}$$

commutes up to the isomorphism

$$(H_1 \boxdot m) \succ H_2 = H_2(H_1(-) \triangleright m) \cong H_1(-) \otimes H_2(m) = H_1 \prec (m \odot H_2) \tag{2.25}$$

that comes from the module functor structure of H_2. One can directly verify that the associated conditions (2.4) and (2.5) are satisfied. ☐

Remark 2.10. As we will see in Proposition 3.9, the Morita context derived from an exact module category $\mathcal{M}$ is in fact a strong Morita context.

2.3 Characterization of strong Morita contexts

Exact module categories provide examples of strong Morita contexts, as stated in Remark 2.10. It turns out that, conversely, every strong Morita context in the sense of Definition 2.1(iii) is equivalent to the Morita context of an exact module category.

Theorem 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite tensor categories and ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$ be finite bimodule categories. Consider a strong Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \ominus, \boxminus, \alpha, \beta)$. Then the following statements hold:*

(i) *$\mathcal{M}$ and $\mathcal{N}$ are exact indecomposable bimodule categories.*

(ii) *The assignments*

$$\begin{array}{ccc} \mathcal{N} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) & \quad \textit{and} \quad & \mathcal{N} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \\ n \longmapsto - \ominus n & & n \longmapsto n \boxminus - \end{array} \tag{2.26}$$

are equivalences of $(\mathcal{B}, \mathcal{A})$-bimodule categories.

(iii) *The assignments*

$$\begin{array}{ccc} \mathcal{M} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{B}}(\mathcal{N}, \mathcal{B}) & \quad \textit{and} \quad & \mathcal{M} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{A}) \\ m \longmapsto - \boxminus m & & m \longmapsto m \ominus - \end{array} \tag{2.27}$$

are equivalences of $(\mathcal{A}, \mathcal{B})$-bimodule categories.

(iv) *The assignments*

$$\begin{array}{ccc} R_{\mathcal{M}}: \quad \mathcal{B} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) & \quad \textit{and} \quad & L_{\mathcal{N}}: \quad \mathcal{B} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{N}) \\ b \longmapsto - \triangleleft b & & b \longmapsto b \triangleright - \end{array} \tag{2.28}$$

are equivalences of tensor categories and of $\mathcal{B}$-bimodule categories.

(v) *The assignments*

$$\begin{array}{ccc} R_{\mathcal{N}}: \quad \mathcal{A} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{B}}(\mathcal{N}, \mathcal{N}) & \quad \textit{and} \quad & L_{\mathcal{M}}: \quad \mathcal{A} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M}) \\ a \longmapsto - \triangleleft a & & a \longmapsto a \triangleright - \end{array} \tag{2.29}$$

are equivalences of tensor categories and of $\mathcal{A}$-bimodule categories.

(vi) *The commutativity of the diagrams*

$$\begin{tikzcd} \mathcal{M} \times \mathcal{N} \arrow[r, "\ominus"] \arrow[d, "(m,n)\mapsto(m,-\ominus n)"'] & \mathcal{A} \arrow[d, "\mathrm{id}_{\mathcal{A}}"] & \textit{and} & \mathcal{N} \times \mathcal{M} \arrow[r, "\boxminus"] \arrow[d, "(n,m)\mapsto(-\ominus n,m)"'] & \mathcal{B} \arrow[d, "R_{\mathcal{M}}"] \\ \mathcal{M} \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \arrow[r, "\odot"'] & \mathcal{A} & & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \times \mathcal{M} \arrow[r, "\boxdot"'] & \overline{\mathcal{A}^*_{\mathcal{M}}} \end{tikzcd} \tag{2.30}$$

is witnessed by the identity and by $\alpha_{-,n,m}\colon - \triangleleft (n \boxminus m) \xrightarrow{\cong} (-\ominus n) \triangleright m$, respectively.

Proof. First notice that (i) follows from (iv) and (v). For instance, assuming that $R_{\mathcal{M}}$ is an equivalence, rigidity of $\mathcal{B}$ implies that every $\mathcal{A}$-module endofunctor of $\mathcal{M}$ is exact and thus ${}_{\mathcal{A}}\mathcal{M}$ is an exact module [EGNO, Prop. 7.9.7(2)]. Also, since the monoidal unit of $\mathcal{B}$ is simple, then $\mathcal{M}$ is an indecomposable $\mathcal{A}$-module. To prove the remaining statements we consider categories of right exact module functors. Once the statements are checked, exactness of the bimodules $\mathcal{M}$ and $\mathcal{N}$ will follow, and thus that every module functor is exact.

We now prove (ii). That $\ominus\colon \mathcal{M}\times\mathcal{N}\to\mathcal{A}$ is a balanced bimodule functor implies that $-\ominus n$ has an $\mathcal{A}$-module structure and that the functor

$$\begin{aligned} \mathcal{N} &\longrightarrow \mathrm{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \\ n &\longmapsto -\ominus n \end{aligned} \tag{2.31}$$

is endowed with a $(\mathcal{B},\mathcal{A})$-bimodule functor structure. On the other hand, we have an equivalence $\underline{\boxminus}\colon \mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}\simeq\mathcal{B}$, and in view of Lemma 1.3 there is an object $\mathrm{colim}_{i\in I}\, n_i\boxtimes m_i\in\mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}$ such that $\mathbf{1}_{\mathcal{B}}\cong\underline{\boxminus}(\mathrm{colim}_{i\in I}\, n_i\boxtimes m_i)=\mathrm{colim}_{i\in I}\, n_i\boxminus m_i$. Then the functor

$$\begin{aligned} \mathrm{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) &\longrightarrow \mathcal{N} \\ H &\longmapsto \mathrm{colim}_{i\in I}\, n_i\triangleleft H(m_i) \end{aligned} \tag{2.32}$$

defines a quasi-inverse to the bimodule functor (2.31). Indeed, given $n\in\mathcal{N}$ there is a natural isomorphism

$$\mathrm{colim}_{i\in I}\, n_i\triangleleft(m_i\ominus n)\ \xrightarrow[\cong]{\beta}\ \mathrm{colim}_{i\in I}\,(n_i\boxminus m_i)\triangleright n\cong\mathbf{1}_{\mathcal{B}}\triangleright n=n\,. \tag{2.33}$$

Similarly, for a module functor $H\in\mathrm{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ and an object $n\in\mathcal{N}$ we have a natural isomorphism

$$\begin{aligned} \mathrm{colim}_{i\in I}\, n\ominus(n_i\triangleleft H(m_i)) &\cong \mathrm{colim}_{i\in I}\,(n\ominus n_i)\otimes H(m_i)\cong\mathrm{colim}_{i\in I}\, H((n\ominus n_i)\triangleright m_i) \\ &\xrightarrow[\cong]{\alpha}\ \mathrm{colim}_{i\in I}\, H(n\triangleleft(n_i\boxminus m_i))\cong H(n\triangleleft\mathbf{1}_{\mathcal{B}})=H(n)\,, \end{aligned} \tag{2.34}$$

where the first isomorphism is the right $\mathcal{A}$-module structure on $\ominus$, the second isomorphism is the module structure of H, and the last isomorphism after α expresses the fact that the colimit is preserved since H is right exact, the action functor is exact and $\underline{\boxminus}$ is an equivalence. The proof for the functor

$$\begin{aligned} \mathcal{N} &\longrightarrow \mathrm{Rex}_{\mathcal{B}}(\mathcal{M},\mathcal{B}) \\ n &\longmapsto n\boxminus - \end{aligned} \tag{2.35}$$

is obtained by merely exchanging the roles of $\ominus$ and $\boxminus$. Similarly, the assertion (iii) follows by symmetry.

Next we prove (iv). We have a $\mathcal{B}$-bimodule equivalence

$$\phi:\quad \mathcal{B}\xrightarrow{\boxminus^{-1}}\mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}\xrightarrow{(2.31)}\mathrm{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{A})\boxtimes_{\mathcal{A}}\mathcal{M}\xrightarrow{(1.13)}\mathrm{Rex}_{\mathcal{A}}(\mathcal{M},\mathcal{M})\,. \tag{2.36}$$

We write $\phi(\mathbf{1}_{\mathcal{B}})=:I$ and follow the argument given in [ENOM, Prop. 4.2]. The bimodule structure on ϕ provides natural isomorphisms

$$\phi(b)\cong b\succ I=I\circ(-\triangleleft b)\qquad\text{and}\qquad\phi(b)\cong I\prec b=(-\triangleleft b)\circ I \tag{2.37}$$

for $b \in \mathcal{B}$. Since ϕ is an equivalence, there exists an object $B \in \mathcal{B}$ such that $\mathrm{id}_{\mathcal{M}} \cong I \circ (- \triangleleft B) \cong (- \triangleleft B) \circ I$, which means that I is invertible and therefore the endofunctor $(- \circ I)\colon \mathrm{Rex}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) \to \mathrm{Rex}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ is an equivalence. From (2.37) we have in particular $\phi \cong (- \circ I) \circ R_{\mathcal{M}}$, and thus $R_{\mathcal{M}}$ must be an equivalence as well. The same type of argument can be applied to the bimodule equivalence

$$\mathcal{B} \xrightarrow{\boxminus^{-1}} \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{M} \xrightarrow{(2.27)} \mathcal{N} \boxtimes_{\mathcal{A}} \mathrm{Rex}_{\mathcal{A}}(\mathcal{N}, \mathcal{A}) \xrightarrow{(1.14)} \mathrm{Rex}_{\mathcal{A}}(\mathcal{N}, \mathcal{N})\,, \tag{2.38}$$

leading to the second part of the statement. And similarly assertion (v) follows by symmetry. Finally, (vi) holds by direct calculation. □

Remark 2.12. Theorem 2.11 implies in particular that every strong Morita context is equivalent to the Morita context of an exact module category. More explicitly, together with the isomorphisms (2.30) the equivalences $\mathcal{N} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ and $R_{\mathcal{M}}\colon \mathcal{B} \xrightarrow{\simeq} \overline{\mathcal{A}^*_{\mathcal{M}}}$ furnish an equivalence

$$(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \ominus\,, \boxminus\,) \xrightarrow{\simeq} \big(\mathcal{A},\, \overline{\mathcal{A}^*_{\mathcal{M}}},\, \mathcal{M},\, \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}),\, \odot,\, \boxdot\big) \tag{2.39}$$

between Morita contexts, i.e. a pseudo-equivalence between the associated bicategories.

Chapter 3

Dualities in a Morita context

There is a notion of dualities for a 1-morphism in a bicategory $\mathcal{F}$, see for instance [Sc2, App. A.3]: A *right dual (or right adjoint)* to a 1-morphism $a \in \mathcal{F}(x,y)$ consists of a 1-morphism $a^\vee \in \mathcal{F}(y,x)$ and 2-morphisms $\mathbf{1}_y \Rightarrow a \circ a^\vee$ and $a^\vee \circ a \Rightarrow \mathbf{1}_x$ that fulfill the appropriate snake relations; left duals are defined similarly. A bicategory for which every 1-morphism has both duals is said to be a *bicategory with dualities*. It is worth noting that the existence of duals is a property, rather than extra structure, of a bicategory. In the present section we explore this notion of dualities for the bicategory $\mathbb{M}$ described in Theorem 2.3, that is associated to the Morita context given by an exact module category, cf. Theorem 2.9.

3.1 Existence of dualities in the bicategory $\mathbb{M}$

For $\mathcal{M}$ an exact module category over a finite tensor category $\mathcal{A}$ we consider its (strong) Morita context $\big(\mathcal{A},\, \overline{\mathcal{A}^*_\mathcal{M}},\, \mathcal{M},\, \mathrm{Fun}_\mathcal{A}(\mathcal{M},\mathcal{A}),\, \odot,\, \boxdot\big)$. Since the tensor categories $\mathcal{A}$ and $\overline{\mathcal{A}^*_\mathcal{M}}$ are rigid, their objects come with dualities. Also, since $\mathcal{M}$ is exact, an object $H \in \mathrm{Fun}_\mathcal{A}(\mathcal{M},\mathcal{A})$ has left and right adjoints. We now show that in the whole Morita context every object is equipped with dualities, in such a way that the associated bicategory $\mathbb{M}$ is a bicategory with dualities.

Definition 3.1. Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$ and $\big(\mathcal{A},\, \overline{\mathcal{A}^*_\mathcal{M}},\, \mathcal{M},\, \mathrm{Fun}_\mathcal{A}(\mathcal{M},\mathcal{A}),\, \odot,\, \boxdot\big)$ the associated Morita context.

(i) The *right dual* of an object $m \in \mathcal{M}$ is the triple $(m^\vee, \underline{\mathrm{ev}}_m, \underline{\mathrm{coev}}_m)$ consisting of the following data: First, the $\mathcal{A}$-module functor

$$m^\vee := \underline{\mathrm{Hom}}^\mathcal{A}_\mathcal{M}(m,-) \;\in \mathrm{Fun}_\mathcal{A}(\mathcal{M},\mathcal{A})\,. \tag{3.1}$$

Second, the evaluation morphisms with which the internal Hom comes naturally endowed, which form a module natural transformation

$$\underline{\mathrm{ev}}_m : \quad m^\vee \boxdot m = \underline{\mathrm{Hom}}_\mathcal{M}(m,-) \triangleright m \Longrightarrow \mathrm{id}_\mathcal{M}\,, \tag{3.2}$$

defined as the counit of the adjunction $(-\triangleright m) \dashv \underline{\mathrm{Hom}}_\mathcal{M}(m,-)$. And third, the coevaluation morphism

$$\underline{\mathrm{coev}}_m : \quad \mathbf{1}_\mathcal{A} \longrightarrow \underline{\mathrm{Hom}}_\mathcal{M}(m,m) = m \odot m^\vee, \tag{3.3}$$

which is defined as the component of the unit of the adjunction $(-\triangleright m) \dashv \underline{\mathrm{Hom}}_\mathcal{M}(m,-)$ corresponding to $\mathbf{1}_\mathcal{A}$.

(ii) The *left dual* of an object $m \in \mathcal{M}$ is the triple $({}^{\vee}m, \overline{\mathrm{ev}}_m, \overline{\mathrm{coev}}_m)$ consisting of the following data: First, the $\mathcal{A}$-module functor

$$ {}^{\vee}m := \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,-) \ \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})\,. \tag{3.4}$$

Second, the evaluation morphism

$$\overline{\mathrm{ev}}_m : \quad m \odot {}^{\vee}m = \underline{\mathrm{coHom}}_{\mathcal{M}}(m,m) \longrightarrow \mathbf{1}_{\mathcal{A}}\,, \tag{3.5}$$

defined as the component of the counit of the adjunction $\underline{\mathrm{coHom}}_{\mathcal{M}}(m,-) \dashv (-\triangleright m)$ that corresponds to $\mathbf{1}_{\mathcal{A}}$. And third, the coevaluation morphism

$$\overline{\mathrm{coev}}_m : \quad \mathrm{id}_{\mathcal{M}} \Longrightarrow \underline{\mathrm{coHom}}_{\mathcal{M}}(m,-) \triangleright m = {}^{\vee}m \boxdot m\,, \tag{3.6}$$

defined as the unit of the adjunction $\underline{\mathrm{coHom}}_{\mathcal{M}}(m,-) \dashv (-\triangleright m)$.

(iii) The *right dual* of an object $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ is the triple $(H^{\vee}, \mathrm{ev}_m, \mathrm{coev}_m)$ consisting of: The object

$$H^{\vee} := H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) \ \in \mathcal{M}\,, \tag{3.7}$$

together with an evaluation morphism

$$\mathrm{ev}_H : \quad H^{\vee} \odot H = H \circ H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) \longrightarrow \mathbf{1}_{\mathcal{A}} \tag{3.8}$$

given by the component of the counit of the adjunction $H \dashv H^{\mathrm{ra}}$ corresponding to $\mathbf{1}_{\mathcal{A}}$, and with a coevaluation morphism

$$\mathrm{coev}_H : \quad \mathrm{id}_{\mathcal{M}} \Longrightarrow H^{\mathrm{ra}} \circ H \cong H(-) \triangleright H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) = H \boxdot H^{\vee} \tag{3.9}$$

given by the unit of the adjunction $H \dashv H^{\mathrm{ra}}$.

(iv) The *left dual* of an object $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ is the triple $({}^{\vee}H, \widetilde{\mathrm{ev}}_m, \widetilde{\mathrm{coev}}_m)$ consisting of: The object

$${}^{\vee}H := H^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}) \ \in \mathcal{M} \tag{3.10}$$

together with an evaluation morphism

$$\widetilde{\mathrm{ev}}_H : \quad H \boxdot {}^{\vee}H = H(-) \triangleright H^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}) \cong H^{\mathrm{la}} \circ H \Longrightarrow \mathrm{id}_{\mathcal{M}} \tag{3.11}$$

provided by the adjunction $H^{\mathrm{la}} \dashv H$ and with a coevaluation morphism

$$\widetilde{\mathrm{coev}}_H : \quad \mathbf{1}_{\mathcal{A}} \longrightarrow H \circ H^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}) = {}^{\vee}H \boxdot H \tag{3.12}$$

being the component of the unit of the adjunction $H^{\mathrm{la}} \dashv H$ corresponding to $\mathbf{1}_{\mathcal{A}}$.

Since they are the unit and counit of an adjunction, the morphisms (3.2) and (3.3), respectively (3.5) and (3.6), obey the snake relations for a right and left duality. For the same reason, the morphisms (3.8) and (3.9), respectively (3.11) and (3.12), obey the relevant snake relations as well. In summary, the left and right duals introduced in Definition 3.1 indeed provide proper dualities, so that we have determined dualities for all 1-morphisms in the bicategory $\mathbb{M}$.

We have thus established:

Theorem 3.2. *Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ be an exact $\mathcal{A}$-module category. The associated bicategory $\mathbb{M}$ is a bicategory with dualities.*

3.2 Properties of duals in a Morita context

Some further properties familiar from the duals in tensor categories are again fulfilled by duals in a Morita context. For instance, an object serves as left (right) dual of its right (left) dual, and the dual of a product is the product of the duals in the reversed order up to isomorphism.

Proposition 3.3. *There are natural isomorphisms*

$$\begin{aligned} &\text{(i)} \quad {}^{\vee}(m^{\vee}) \cong m \cong ({}^{\vee}m)^{\vee}\,, && \text{(v)} \quad (F \succ H)^{\vee} \cong H^{\vee} \triangleleft F^{\vee}\,, \\ &\text{(ii)} \quad {}^{\vee}(H^{\vee}) \cong H \cong ({}^{\vee}H)^{\vee}\,, && \text{(vi)} \quad (H \prec a)^{\vee} \cong a^{\vee} \triangleright H^{\vee}\,, \\ &\text{(iii)} \quad (a \triangleright m)^{\vee} \cong m^{\vee} \prec a^{\vee}\,, && \text{(vii)} \quad (m \odot H)^{\vee} \cong H^{\vee} \odot m^{\vee}\,, \\ &\text{(iv)} \quad (m \triangleleft F)^{\vee} \cong F^{\vee} \succ m^{\vee}\,, && \text{(viii)} \quad (H \boxdot m)^{\vee} \cong m^{\vee} \boxdot H^{\vee} \end{aligned} \tag{3.13}$$

for $a \in \mathcal{A}$, $m \in \mathcal{M}$, $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ *and* $F \in \overline{\mathcal{A}^{*}_{\mathcal{M}}}$. *Analogous relations are valid for left duals.*

Proof. All these isomorphisms follow immediately from the definitions and Theorem 3.2. We provide nonetheless the proof for some of the statements. For instance, the definition of duals directly implies (i):

$${}^{\vee}(m^{\vee}) = \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, -)^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{A}} \triangleright m \cong m\,. \tag{3.14}$$

Similarly, (ii) follows with the help of Lemma 1.6:

$$({}^{\vee}H)^{\vee} = \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(H^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}), -) \cong \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}}, H(-)) = H(-) \otimes \mathbf{1}_{\mathcal{A}}{}^{\vee} \cong H\,. \tag{3.15}$$

The isomorphism in (iii) corresponds to the module functor structure on the internal Hom given by (1.19). Finally, (viii) comes from the composition

$$\begin{aligned} (H \boxdot m)^{\vee} &= [(- \triangleright m) \circ H]^{\mathrm{ra}} = H^{\mathrm{ra}} \circ \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, -) \\ &\cong H^{\mathrm{ra}}\left(\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, -) \otimes \mathbf{1}_{\mathcal{A}}\right) \\ &\cong \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, -) \triangleright H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) = m^{\vee} \boxdot H^{\vee}, \end{aligned} \tag{3.16}$$

where we first identify the adjoint of a composite with the composition of the adjoints in reversed order, and where the last isomorphism corresponds to the module functor structure of H^{ra}. □

Dualities in a tensor category provide an equivalence between its opposite category and its monoidal opposite. Analogously, the duals we just defined for a module category exhibit how the bimodule category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ plays the role of an opposite to $\mathcal{M}$.

Proposition 3.4. *Let* $\mathcal{M}$ *be an exact* $\mathcal{A}$*-module category. The dualities on the bicategory* $\mathbb{M}$ *induce equivalences*

$$(-)^{\vee} : \quad \mathcal{M} \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})^{\#} \qquad \textit{and} \qquad {}^{\vee}(-) : \quad \mathcal{M} \xrightarrow{\simeq} {}^{\#}\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \tag{3.17}$$

of $(\mathcal{A}, \overline{\mathcal{A}^{*}_{\mathcal{M}}})$*-bimodule categories, and equivalences*

$$(-)^{\vee} : \quad \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \xrightarrow{\simeq} \mathcal{M}^{\#} \qquad \textit{and} \qquad {}^{\vee}(-) : \quad \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \xrightarrow{\simeq} {}^{\#}\mathcal{M} \tag{3.18}$$

of $(\overline{\mathcal{A}^{*}_{\mathcal{M}}}, \mathcal{A})$*-bimodule categories.*

Proof. Proposition 3.3 shows that right and left duals are mutual quasi-inverses, and the isomorphisms from Proposition 3.3(iii) and 3.3(iv) endow the duality functor with a bimodule structure. The statements for the remaining equivalences follow by analogous reasoning. □

There are further relations involving the dualities which constitute a duality calculus in the Morita context, a powerful tool for performing calculations such as the computation of relative Serre functors.

Remark 3.5.

(i) Recall that for the regular module ${}_{\mathcal{A}}\mathcal{A}$ we have

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{A}}(a,b) = b\otimes a^{\vee} \qquad \text{and} \qquad \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{A}}(a,b) = b\otimes {}^{\vee}a\,. \tag{3.19}$$

for $a,b \in \mathcal{A}$. Similarly it follows from the definition of the mixed products and of the dualities that

$$\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = n\odot m^{\vee} \qquad \text{and} \qquad \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = n\odot {}^{\vee}m\,. \tag{3.20}$$

for $m,n \in \mathcal{M}$.

(ii) Acting with dualities on the bimodules in the Morita context leads to the adjunctions

$$(a^{\vee}\triangleright -) \dashv (a\triangleright -) \dashv ({}^{\vee}a\triangleright -) \qquad \text{and} \qquad (-\triangleleft F^{\mathrm{la}}) \dashv (-\triangleleft F) \dashv (-\triangleleft F^{\mathrm{ra}}) \tag{3.21}$$

as well as

$$(F^{\mathrm{ra}}\succ -) \dashv (F\succ -) \dashv (F^{\mathrm{la}}\succ -) \qquad \text{and} \qquad (-\prec {}^{\vee}a) \dashv (-\prec a) \dashv (-\prec a^{\vee}) \tag{3.22}$$

for $a\in\mathcal{A}$ and $F\in\mathcal{A}^*_{\mathcal{M}}$.

It turns out that the relations given in Remark 3.5 can be extended to the entire Morita context:

Proposition 3.6. *There are natural isomorphisms*

$$\begin{aligned}
&\text{(i)} && \mathrm{Hom}_{\mathcal{M}}(a\triangleright m,n) \cong \mathrm{Hom}_{\mathcal{A}}(a,n\odot m^{\vee})\,,\\
&\text{(ii)} && \mathrm{Hom}_{\mathcal{M}}(n,a\triangleright m) \cong \mathrm{Hom}_{\mathcal{A}}(n\odot {}^{\vee}m,a)\,,\\
&\text{(iii)} && \mathrm{Hom}_{\mathcal{M}}(m\triangleleft F,n) \cong \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(F,{}^{\vee}m\boxdot n)\,,\\
&\text{(iv)} && \mathrm{Hom}_{\mathcal{M}}(n,m\triangleleft F) \cong \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(m^{\vee}\boxdot n,F)\,,\\
&\text{(v)} && \mathrm{Hom}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(F\succ H_1,H_2) \cong \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(F,H_2\boxdot H_1^{\vee})\,,\\
&\text{(vi)} && \mathrm{Hom}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_2,F\succ H_1) \cong \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(H_2\boxdot {}^{\vee}H_1,F)\,,\\
&\text{(vii)} && \mathrm{Hom}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_1\prec a,H_2) \cong \mathrm{Hom}_{\mathcal{A}}(a,{}^{\vee}H_1\odot H_2)\,,\\
&\text{(viii)} && \mathrm{Hom}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_2,H_1\prec a) \cong \mathrm{Hom}_{\mathcal{A}}(H_1^{\vee}\odot H_2,a)
\end{aligned} \tag{3.23}$$

for $a\in\mathcal{A}$, $m,n\in\mathcal{M}$, $H_1,H_2\in\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ *and* $F\in\overline{\mathcal{A}^*_{\mathcal{M}}}$.

Proof. (i) and (ii) follow directly from the definition of the duals and the defining properties of the internal Hom and coHom. The bijection

$$\mathrm{Hom}_{\mathcal{M}}(F(m), n) \longrightarrow \mathrm{Nat}_{\mathrm{mod}}(F, \underline{\mathrm{coHom}}(m, -) \triangleright n) \tag{3.24}$$

in (iii) is an arrow assigning to $f\colon F(m) \to n$ a module natural transformation whose component at $l \in \mathcal{M}$ is given by the composition

$$F(l) \xrightarrow{F(\overline{\mathrm{coev}})} F(\underline{\mathrm{coHom}}(m,l) \triangleright m) \cong \underline{\mathrm{coHom}}(m,l) \triangleright F(m) \xrightarrow{\mathrm{id} \triangleright f} \underline{\mathrm{coHom}}(m,l) \triangleright n\,. \tag{3.25}$$

The inverse of (3.24) is given by the assignment

$$\eta \;\longmapsto\; F(m) \xrightarrow{\eta_m} \underline{\mathrm{coHom}}(m,m) \triangleright n \xrightarrow{\overline{\mathrm{ev}}_m \triangleright \mathrm{id}_n} n \tag{3.26}$$

for $\eta \in \mathrm{Nat}_{\mathrm{mod}}(F, \underline{\mathrm{coHom}}(m,-) \triangleright n) = \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(F, {}^{\vee}m \boxdot n)$. The bijection in (iv) is defined in a similar fashion. To prove (v) consider the bijection

$$\mathrm{Nat}_{\mathrm{mod}}(H_1 \circ F, H_2) \longrightarrow \mathrm{Nat}_{\mathrm{mod}}(F, H_1^{\mathrm{ra}} \circ H_2) \tag{3.27}$$

which assigns to $\eta\colon H_1 \circ F \Rightarrow H_2$ the natural transformation

$$F \Longrightarrow H_1^{\mathrm{ra}} \circ H_1 \circ F \xRightarrow{\mathrm{id} \circ \eta} H_1^{\mathrm{ra}} \circ H_2\,. \tag{3.28}$$

Given $\gamma \in \mathrm{Nat}_{\mathrm{mod}}(F, H_1^{\mathrm{ra}} \circ H_2) = \mathrm{Hom}_{\mathcal{A}^*_{\mathcal{M}}}(F, H_2 \boxdot H_1^{\vee})$ the assignment

$$\gamma \;\longmapsto\; H_1 \circ F \xRightarrow{\mathrm{id} \circ \gamma} H_1 \circ H_1^{\mathrm{ra}} \circ H_2 \Longrightarrow H_2 \tag{3.29}$$

serves as inverse of (3.27). The isomorphism (vi) is defined analogously. To obtain (vii), consider the function

$$\begin{aligned} \mathrm{Nat}_{\mathrm{mod}}\left((- \otimes a) \circ H_1, H_2\right) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}\Big(a,\; H_2(H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}))\Big) \\ \eta &\longmapsto a \to (- \otimes a) \circ H_1 \circ H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}) \xrightarrow{\eta_{H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}})}} H_2(H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}))\,, \end{aligned} \tag{3.30}$$

which has as inverse the arrow that assigns to $f\colon a \to H_2(H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}))$ the composition

$$(- \otimes a) \circ H_1 \xRightarrow{(-\otimes f) \circ \mathrm{id}} \Big(- \otimes H_2(H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}}))\Big) \circ H_1 \cong H_2 \circ H_1^{\mathrm{la}} \circ H_1 \Rightarrow H_2\,, \tag{3.31}$$

where the isomorphism is the module structure of $H_2 \circ H_1^{\mathrm{la}}$ and the last arrow is the counit of the adjunction $H_1^{\mathrm{la}} \dashv H_1$. □

Remark 3.7. Notice that the adjunctions in Proposition 3.6 describe the internal Homs and coHoms of the bimodule categories in the Morita context in terms of the products and dualities:

$$\begin{aligned}
&\text{(i)} && \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = n \odot m^{\vee}, \\
&\text{(ii)} && \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = n \odot {}^{\vee}m\,, \\
&\text{(iii)} && \underline{\mathrm{Hom}}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}}(m,n) = {}^{\vee}m \boxdot n = \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,-) \triangleright n\,, \\
&\text{(iv)} && \underline{\mathrm{coHom}}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}}(m,n) = m^{\vee} \boxdot n = \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,-) \triangleright n\,, \\
&\text{(v)} && \underline{\mathrm{Hom}}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_1,H_2) = H_2 \boxdot H_1^{\vee} = H_1^{\mathrm{ra}} \circ H_2\,, \\
&\text{(vi)} && \underline{\mathrm{coHom}}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_1,H_2) = H_2 \boxdot {}^{\vee}H_1 = H_1^{\mathrm{la}} \circ H_2\,, \\
&\text{(vii)} && \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_1,H_2) = {}^{\vee}H_1 \odot H_2 = H_2 \circ H_1^{\mathrm{la}}(\mathbf{1}_{\mathcal{A}})\,, \\
&\text{(viii)} && \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H_1,H_2) = H_1^{\vee} \odot H_2 = H_2 \circ H_1^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}})\,.
\end{aligned} \tag{3.32}$$

In particular the formulas (iii) and (iv) relate the internal Homs and coHoms of the module categories ${}_{\mathcal{A}}\mathcal{M}$ and ${}_{\mathcal{A}^*_{\mathcal{M}}}\mathcal{M}$.

Lemma 3.8. *Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. Then the assignment*

$$\begin{aligned}
\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) &\longrightarrow \mathrm{Fun}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}(\mathcal{M},\, \overline{\mathcal{A}^*_{\mathcal{M}}})\,, \\
H &\longmapsto H \boxdot -
\end{aligned} \tag{3.33}$$

is an equivalence of $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \mathcal{A})$-bimodule categories.

Proof. According to Lemma 2.8 the functor $\boxdot$ is a balanced bimodule functor. It follows that $H \boxdot -$ is an $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module functor and that the functor (3.33) has an $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \mathcal{A})$-bimodule structure. The following functor is a quasi-inverse to (3.33):

$$\begin{aligned}
\mathrm{Fun}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}(\mathcal{M},\, \overline{\mathcal{A}^*_{\mathcal{M}}}) &\longrightarrow \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})\,, \\
K &\longmapsto {}^{\vee}[K^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}})] = \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(K^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}}),-)
\end{aligned} \tag{3.34}$$

Indeed, given $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ we have

$${}^{\vee}[(H \boxdot -)^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}})] \cong {}^{\vee}(H^{\vee} \triangleleft \mathrm{id}_{\mathcal{M}}) \cong H\,, \tag{3.35}$$

where the first isomorphism comes from Proposition 3.6(iii) and the second isomorphism from Proposition 3.3(ii). Conversely, for $K \in \mathrm{Fun}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}(\mathcal{M},\, \overline{\mathcal{A}^*_{\mathcal{M}}})$ and $n \in \mathcal{M}$ we have the chain

$$\begin{aligned}
{}^{\vee}[K^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}})] \boxdot n = \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(K^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}}),-) \triangleright n &\cong \underline{\mathrm{Hom}}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}}(K^{\mathrm{la}}(\mathrm{id}_{\mathcal{M}}),n) \\
&\cong \underline{\mathrm{Hom}}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{A}^*_{\mathcal{M}}}(\mathrm{id}_{\mathcal{M}},K(n)) \cong K(n)
\end{aligned} \tag{3.36}$$

of natural isomorphisms, where the first isomorphism is from Remark 3.7(iii), while the second is Lemma 1.6. □

Proposition 3.9. *Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. Then the Morita context $(\mathcal{A},\, \overline{\mathcal{A}^*_{\mathcal{M}}},\, \mathcal{M},\, \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}),\, \odot,\, \boxdot)$ from Theorem 2.9 is strong.*

Proof. By invoking Proposition 1.4 it follows immediately that the mixed product $\boxdot$ descends to an equivalence $\underline{\boxdot} : \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \boxtimes_{\mathcal{A}} \mathcal{M} \longrightarrow \overline{\mathcal{A}^*_{\mathcal{M}}}$. Thus it remains to verify that $\odot$ descends to an equivalence $\underline{\odot} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \longrightarrow \mathcal{A}$, with $\mathcal{B} := \overline{\mathcal{A}^*_{\mathcal{M}}}$, as well. To see this, consider the canonical equivalence [EGNO, Thm. 7.12.11]

$$\mathrm{can}: \quad \mathcal{A} \xrightarrow{\simeq} (\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}\,, \qquad a \mapsto a \triangleright - \tag{3.37}$$

and the bimodule equivalence from Lemma 3.8. The diagram

$$\begin{tikzcd}
\mathcal{M} \boxtimes_{\mathcal{B}} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \arrow[rr, "\underline{\odot}"] \arrow[d, "(3.33)"'] & & \mathcal{A} \arrow[d, "\mathrm{can}"] \\
\mathcal{M} \boxtimes_{\mathcal{B}} \mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{B}) \arrow[r, "(1.14)"'] & \mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{M}) &
\end{tikzcd} \tag{3.38}$$

commutes strictly: we have $m \triangleleft (H \boxdot n) = H(m) \triangleright n = (m \odot H) \triangleright n$ for $n \in \mathcal{M}$. Since all other functors in the diagram are equivalences, it follows that $\underline{\odot}$ is an equivalence, too. □

Remark 3.10. Above we have focused our attention on Morita contexts associated to exact module categories. The reason for this is the following: According to Proposition 3.9 the Morita context derived from an exact module category, as described in Theorem 2.9, is strong. In view of Theorem 2.11, every strong Morita context is of this type. As a consequence, Theorem 3.2 and related statements, valid for the Morita context of an exact module category, also hold for any arbitrary strong Morita context.

3.3 Double duals and relative Serre functors

Given a tensor category $\mathcal{A}$, the relative Serre functor of the regular module ${}_{\mathcal{A}}\mathcal{A}$ corresponds to the double right dual, $\mathbb{S}^{\mathcal{A}}_{\mathcal{A}}(a) \cong a^{\vee\vee}$. Similarly, The double duals of objects in the Morita context of an exact module ${}_{\mathcal{A}}\mathcal{M}$ admit the following description involving the relative Serre functors:

Proposition 3.11. *For $\mathcal{M}$ an exact module category over a finite tensor category $\mathcal{A}$, let $m \in \mathcal{M}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$. We have isomorphisms*

$$\begin{aligned}
&\text{(i)} \quad m^{\vee\vee} \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\,, &&\text{(iii)} \quad H^{\vee\vee} \cong H^{\mathrm{rra}} \cong \mathbb{S}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H),\\
&\text{(ii)} \quad {}^{\vee\vee}m \cong \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m)\,, &&\text{(iv)} \quad {}^{\vee\vee}H \cong H^{\mathrm{lla}} \cong \mathbb{S}^{\overline{\mathcal{A}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H).
\end{aligned} \tag{3.39}$$

Proof. Combining the realization (1.43) of the relative Serre functor with the description of right duals in Definition 3.1(i) and (iii) we directly get

$$\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \cong \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,-)^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}) = m^{\vee\vee}. \tag{3.40}$$

Similarly for a module functor we have

$$H^{\vee\vee} = \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}),-) \cong (- \triangleright H^{\mathrm{ra}}(\mathbf{1}_{\mathcal{A}}))^{\mathrm{ra}} \cong H^{\mathrm{rra}}. \tag{3.41}$$

The second isomorphism in (iii) follows as

$$\mathbb{S}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H) \cong \underline{\mathrm{Hom}}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H,-)^{\mathrm{ra}}(\mathrm{id}_{\mathcal{M}}) \cong (- \boxdot H^{\vee})^{\mathrm{ra}}(\mathrm{id}_{\mathcal{M}}) \cong \mathrm{id}_{\mathcal{M}} \prec H^{\vee\vee} \tag{3.42}$$

with the help of Remark 3.7(v) and Proposition 3.6(vi). The statements for double left duals follow in a similar manner. □

Corollary 3.12. *The relative Serre functors of* $\mathcal{M}$ *are related by*

$$\mathbb{S}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^*}(m) \cong \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}}(m) \qquad \textit{and} \qquad \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \cong \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^*}(m)\,. \tag{3.43}$$

Proof. The statements follow by considering again the standard realization of the relative Serre functors together with the duality calculus in the Morita context. For instance, combining Remark 3.7(iii), Proposition 3.6(iv) and Proposition 3.11(ii) yields the first isomorphism:

$$\mathbb{S}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^*}(m) \cong \underline{\mathrm{Hom}}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^*}(m,-)^{\mathrm{ra}}(\mathrm{id}_{\mathcal{M}}) \cong ({}^{\vee}m \boxdot -)^{\mathrm{ra}}(\mathrm{id}_{\mathcal{M}}) \cong {}^{\vee\vee}m \triangleleft \mathrm{id}_{\mathcal{M}} \cong \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}}(m)\,. \tag{3.44}$$

The second isomorphism is obtained in a similar manner. □

Double duals in a tensor category are compatible with tensor products: the double dual of a product is isomorphic to the product of the double duals of the factors. This property extends to any bicategory $\mathcal{F}$ with dualities. Moreover, the double duals of 1-morphisms form a pseudo-equivalence $(-)^{\vee\vee}\colon \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$.

In the case of the bicategory $\mathbb{M}$ associated with the Morita context of a module category, Proposition 3.11 implies that the double-dual functors are isomorphic to relative Serre functors. The compatibility between double duals and products ensures that there are coherent natural isomorphisms

$$\begin{array}{llll}
\text{(i)} & \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(a \triangleright m) \cong a^{\vee\vee} \triangleright \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m)\,, & \text{(iv)} & \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m \triangleleft F) \cong \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \triangleleft F^{\mathrm{rra}}\,, \\
\text{(ii)} & (F \succ H)^{\mathrm{rra}} \cong F^{\mathrm{rra}} \succ H^{\mathrm{rra}}\,, & \text{(v)} & (H \prec a)^{\mathrm{rra}} \cong H^{\mathrm{rra}} \prec a^{\vee\vee}\,, \\
\text{(iii)} & (m \odot H)^{\vee\vee} \cong \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \odot H^{\mathrm{rra}}\,, & \text{(vi)} & H^{\mathrm{rra}} \boxdot \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \cong (H \boxdot m)^{\mathrm{rra}}
\end{array} \tag{3.45}$$

for $a \in \mathcal{A}$, $m \in \mathcal{M}$, $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ and $F \in \overline{\mathcal{A}_{\mathcal{M}}^*}$. These isomorphisms can be obtained by iterating the isomorphisms from Proposition 3.3. In particular, (i) and (iv) recover the twisted bimodule functor structure of $\mathbb{S}_{\mathcal{M}}^{\mathcal{A}}$. Put differently, the isomorphisms (3.45) relate the value of the relative Serre functor of a product with the product of the relative Serre functors evaluated in the corresponding factors. For instance, (iii) exhibits the coherence data

$$\mathbb{S}_{\mathcal{A}}^{\mathcal{A}}(m \odot H) \cong (m \odot H)^{\vee\vee} \cong \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \odot H^{\mathrm{rra}} \cong \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(m) \odot \mathbb{S}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}^{\overline{\mathcal{A}_{\mathcal{M}}^*}}(H) \tag{3.46}$$

for the composition of $m \in \mathcal{M}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ in $\mathbb{M}$. In this spirit the relative Serre functors of the categories in $\mathbb{M}$ assemble into a pseudo-equivalence:

Definition 3.13 (Relative Serre pseudo-functor)**.**
Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. The *relative Serre pseudo-functor* on the bicategory $\mathbb{M}$ consists of the assignment

$$\begin{aligned}
\mathbb{S}:\quad \mathbb{M} &\xrightarrow{\simeq} \mathbb{M}, \\
x &\longmapsto x\,, \\
\mathbb{M}(x,y) \ni a &\longmapsto \mathbb{S}_{\mathbb{M}(x,y)}(a) \in \mathbb{M}(x,y)
\end{aligned} \tag{3.47}$$

together with the natural isomorphisms (3.45), which witness the compatibility with the horizontal composition in $\mathbb{M}$.

3.4 The Radford pseudo-equivalence

For a finite tensor category $\mathcal{A}$, the functor taking double left duals is naturally isomorphic to the double right dual functor up to the action of the distinguished invertible object $\mathbb{D}_{\mathcal{A}}$. We explore how this extends to a bimodule category and to the entirety of the Morita context of an exact module category.

Theorem 3.14 (Radford isomorphism of a bimodule category)**.**
Let ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be an exact bimodule category. There exists a natural isomorphism

$$\mathcal{R}_{\mathcal{L}}: \quad \mathbb{D}_{\mathcal{C}}^{-1} \triangleright \mathbb{S}_{\mathcal{L}}^{\mathcal{C}}(-) \xRightarrow{\cong} \mathbb{S}_{\mathcal{L}}^{\overline{\mathcal{D}}}(-) \triangleleft \mathbb{D}_{\mathcal{D}}^{-1} \tag{3.48}$$

of twisted bimodule functors.

Proof. We can regard $\mathcal{L}$ both as a left $\mathcal{C}$-module and as a left $\overline{\mathcal{D}}$-module. Therefore from (1.63) we obtain the desired isomorphism

$$\mathbb{D}_{\mathcal{C}}^{-1} \triangleright \mathbb{S}_{\mathcal{L}}^{\mathcal{C}}(-) \cong \mathbb{N}_{\mathcal{L}}^{r}(-) \cong \mathbb{D}_{\mathcal{D}}^{-1} \mathbin{\overline{\triangleright}} \mathbb{S}_{\mathcal{L}}^{\overline{\mathcal{D}}}(-) \equiv \mathbb{S}_{\mathcal{L}}^{\overline{\mathcal{D}}}(-) \triangleleft \mathbb{D}_{\mathcal{D}}^{-1} \tag{3.49}$$

of twisted bimodule functors. □

As a first consequence of Theorem 3.14 together with the computation of the relative Serre functors of $\mathcal{M}$ we arrive to a description of the distinguished invertible object of the dual tensor category.

Proposition 3.15. *Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. There is an isomorphism*

$$\mathbb{D}_{\mathcal{A}_{\mathcal{M}}^{*}} \cong \mathbb{D}_{\mathcal{A}} \triangleright \left(\overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}}\right)^{2} \cong \mathbb{N}_{\mathcal{M}}^{l} \circ \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}} \tag{3.50}$$

of $\mathcal{A}$-module functors, where $\mathbb{D}_{\mathcal{A}_{\mathcal{M}}^{*}}$ is the distinguished invertible object of the dual tensor category $\mathcal{A}_{\mathcal{M}}^{*}$.

Proof. Applying Theorem 3.14 to the $(\mathcal{A}, \overline{\mathcal{A}_{\mathcal{M}}^{*}})$-bimodule category $\mathcal{M}$ we obtain an isomorphism

$$\mathbb{D}_{\mathcal{A}}^{-1} \triangleright \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \cong \mathbb{S}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^{*}}(-) \triangleleft \mathbb{D}_{\mathcal{A}_{\mathcal{M}}^{*}}^{-1} \cong \mathbb{D}_{\mathcal{A}_{\mathcal{M}}^{*}}^{-1} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^{*}} \tag{3.51}$$

of bimodule functors. The result now follows by taking into account that $\mathbb{S}_{\mathcal{M}}^{\mathcal{A}_{\mathcal{M}}^{*}} \cong \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}}$. The second isomorphism in (3.50) comes from the isomorphism $\mathbb{N}_{\mathcal{M}}^{l} \cong \mathbb{D}_{\mathcal{A}} \triangleright \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}}$ in (1.63). □

We find that Radford's theorem can be extended to exact module categories, with the relative Serre functor playing the role of the double right dual functor.

Corollary 3.16 (Radford isomorphism of a module category)**.**
Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an exact $\mathcal{A}$-module. There is a natural isomorphism

$$r_{\mathcal{M}}: \quad \mathbb{D}_{\mathcal{A}} \triangleright - \triangleleft \mathbb{D}_{\mathcal{A}_{\mathcal{M}}^{*}}^{-1} \xRightarrow{\cong} \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \tag{3.52}$$

of twisted bimodule functors.

Proof. The statement follows from Proposition 3.15 by reformulating the description of $\mathbb{D}_{\mathcal{A}^*_\mathcal{M}}$. From the isomorphism (3.50) we obtain

$$\mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \circ \mathbb{S}^\mathcal{A}_\mathcal{M} \circ \mathbb{S}^\mathcal{A}_\mathcal{M} \cong \mathbb{N}^l_\mathcal{M} \circ \overline{\mathbb{S}}^\mathcal{A}_\mathcal{M} \circ \mathbb{S}^\mathcal{A}_\mathcal{M} \circ \mathbb{S}^\mathcal{A}_\mathcal{M} \cong \mathbb{N}^l_\mathcal{M} \circ \mathbb{S}^\mathcal{A}_\mathcal{M} \cong \mathbb{D}_\mathcal{A} \triangleright - \,, \tag{3.53}$$

where we use the fact that $\mathbb{S}^\mathcal{A}_\mathcal{M}$ and $\overline{\mathbb{S}}^\mathcal{A}_\mathcal{M}$ are quasi-inverses and the isomorphism coming from (1.63). □

Note that Equation (3.52) takes a very symmetric form because we see the $\mathcal{A}$-module category $\mathcal{M}$ as an $(\mathcal{A}, \overline{\mathcal{A}^*_\mathcal{M}})$-bimodule: the distinguished invertible objects of both $\mathcal{A}$ and $\overline{\mathcal{A}^*_\mathcal{M}}$ enter in (3.52) on the same footing.

There are similar Radford isomorphisms for the categories $\mathrm{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ and $\overline{\mathcal{A}^*_\mathcal{M}}$ in the Morita context $\mathbb{M}$, where again the corresponding relative Serre functors play the role of the double right dual. These Radford isomorphisms assemble into a trivialization of the square of the relative Serre pseudo-functor (3.47), i.e. a trivialization of the fourth power of the pseudo-functor of dualities of $\mathbb{M}$.

Theorem 3.17 (Radford pseudo-equivalence of a Morita context)**.**
Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$ and $\mathbb{M}$ the bicategory associated to its Morita context. There is a pseudo-natural equivalence

$$\mathcal{R}: \quad \mathrm{id}_\mathbb{M} \xRightarrow{\cong} \mathbb{S}^2, \tag{3.54}$$

where $\mathbb{S}$ is the relative Serre pseudo-functor (3.47)*.*

Proof. To construct the pseudo-natural equivalence, consider the following data:

(i) For the objects 0 and 1 of $\mathbb{M}$ the distinguished invertible 1-morphisms

$$\mathcal{R}_0 := \mathbb{D}_\mathcal{A} \qquad \text{and} \qquad \mathcal{R}_1 := \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \,. \tag{3.55}$$

(ii) For 1-morphisms in $\mathbb{M}$, the following invertible 2-morphisms:

- For $a \in \mathcal{A}$ and $F \in \overline{\mathcal{A}^*_\mathcal{M}}$, the natural isomorphisms

$$\mathcal{R}_a: \quad \mathbb{D}_\mathcal{A} \otimes a \xrightarrow{\cong} a^{\vee\vee\vee\vee} \otimes \mathbb{D}_\mathcal{A} \tag{3.56}$$

  and

$$\mathcal{R}_m: \quad \mathbb{D}_\mathcal{A} \triangleright m \xrightarrow{\cong} \mathbb{S}^\mathcal{A}_\mathcal{M} \circ \mathbb{S}^\mathcal{A}_\mathcal{M}(m) \triangleleft \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \tag{3.57}$$

  coming from (1.60) and from (3.52), respectively.

- For $F \in \overline{\mathcal{A}^*_\mathcal{M}}$, the natural isomorphism

$$\mathcal{R}_F: \quad \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \circ^{\mathrm{opp}} F \xRightarrow{\cong} F^{\mathrm{rrrra}} \circ^{\mathrm{opp}} \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \tag{3.58}$$

  given by the composite

$$F \circ \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \cong F \circ \mathbb{D}_\mathcal{A} \triangleright (\overline{\mathbb{S}}^\mathcal{A}_\mathcal{M})^2 \cong \mathbb{D}_\mathcal{A} \triangleright (\overline{\mathbb{S}}^\mathcal{A}_\mathcal{M})^2 \circ F^{\mathrm{rrrra}} \cong \mathbb{D}_{\mathcal{A}^*_\mathcal{M}} \circ F^{\mathrm{rrrra}}, \tag{3.59}$$

  where the first and last isomorphisms come from (3.50) and the middle isomorphism uses the module structure of F and the twisted structure (1.52) of the relative Serre functor twice.

– Analogously, for any $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ a natural isomorphism

$$\mathcal{R}_H : \quad \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \succ H \xRightarrow{\cong} H^{\mathrm{rrrra}} \prec \mathbb{D}_{\mathcal{A}} \tag{3.60}$$

given by the composite

$$H \circ \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \cong H \circ \mathbb{D}_{\mathcal{A}} \triangleright \left(\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}\right)^2 \cong \mathbb{D}_{\mathcal{A}} \otimes {}^{\vee\vee\vee\vee}(-) \circ H^{\mathrm{rrrra}} \cong (- \otimes \mathbb{D}_{\mathcal{A}}) \circ H^{\mathrm{rrrra}}, \tag{3.61}$$

where again we first use (3.50) and then the twisted structure (1.52) of the relative Serre functor (also taking into account that $\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{A}} \cong {}^{\vee\vee}(-)$), and where the last isomorphism is the Radford isomorphism of $\mathcal{A}$.

The claim now reduces to making the routine check of the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{D} \circ s \circ t \arrow[r, "\mathcal{R}_{s \circ t}"] \arrow[d, "\mathcal{R}_s \circ \mathrm{id}"'] & (s \circ t)^{\vee\vee\vee\vee} \circ \mathbb{D} \arrow[d, "(3.45)"] \\
s^{\vee\vee\vee\vee} \circ \mathbb{D} \circ t \arrow[r, "\mathrm{id} \circ \mathcal{R}_t"'] & s^{\vee\vee\vee\vee} \circ t^{\vee\vee\vee\vee} \circ \mathbb{D}
\end{tikzcd} \tag{3.62}$$

where the symbol $\circ$ denotes the horizontal composition in the bicategory $\mathbb{M}$ and s and t are composable 1-morphisms. ☐

The bicategorical formulation in Theorem 3.17 unifies Radford's theorems for tensor and module categories. It also offers a natural home to the invertible objects in Radford-type theorems, since they become part of the data of a pseudo-natural equivalence.

3.5 Relative Serre functors of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ and Radford theorems

Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ exact $\mathcal{A}$-module categories. Composition of module functors turns the category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ of module functors into a $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category, or equivalently a $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \overline{\mathcal{A}^*_{\mathcal{N}}})$-bimodule category with action given by

$$F_1 \succ H \prec F_2 := F_2 \circ H \circ F_1 \qquad \text{for} \quad F_1 \in \overline{\mathcal{A}^*_{\mathcal{M}}}\,, \quad F_2 \in \overline{\mathcal{A}^*_{\mathcal{N}}} \quad \text{and} \quad H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})\,. \tag{3.63}$$

More generally, if $\mathcal{M}$ and $\mathcal{N}$ are bimodules, then $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ becomes a bimodule category with actions given by (1.7).

Lemma 3.18. *Given exact and indecomposable module categories $\mathcal{M}$ and $\mathcal{N}$ over a finite tensor category $\mathcal{A}$, the $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *is invertible.*

Proof. Applying Proposition 3.9 to the $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, we deduce that the associated Morita context is strong. On the other hand, the right action on $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ in the Morita context coincides under

$$\mathrm{Fun}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}\Big(\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}), \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})\Big) \xrightarrow{(1.72)} \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{N}) = \mathcal{A}^*_{\mathcal{N}} \tag{3.64}$$

with the action defined by (3.63). We conclude that $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is invertible as an $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \overline{\mathcal{A}^*_{\mathcal{N}}})$-bimodule category. ☐

Lemma 3.19. *Let $\mathcal{M}$ and $\mathcal{N}$ be exact module categories over a finite tensor category $\mathcal{A}$.*

(i) *The relative Serre functors of the $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ are given by*

$$\mathbb{S}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong H^{\mathrm{rra}}\,, \qquad \overline{\mathbb{S}}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong H^{\mathrm{lla}}\,, \tag{3.65}$$

$$\mathbb{S}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong H^{\mathrm{lla}} \qquad \textit{and} \qquad \overline{\mathbb{S}}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong H^{\mathrm{rra}} \tag{3.66}$$

for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$.

(ii) *The Nakayama functors of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ are given by*

$$\mathbb{N}^r_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{N}^r_{\mathcal{N}} \circ H \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \mathbb{N}^r_{\mathcal{M}} \tag{3.67}$$

and

$$\mathbb{N}^l_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{N}^l_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \cong \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \mathbb{N}^l_{\mathcal{M}} \tag{3.68}$$

for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$.

(iii) *For exact bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{C}}$, the relative Serre functors of the bimodule category ${}_{\mathcal{B}}\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_{\mathcal{C}}$ are given by*

$$\begin{aligned} &\mathbb{S}^{\overline{\mathcal{C}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}} \circ H \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\,, &&\overline{\mathbb{S}}^{\overline{\mathcal{C}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \overline{\mathbb{S}}^{\overline{\mathcal{C}}}_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}\,, \\ &\overline{\mathbb{S}}^{\mathcal{B}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\overline{\mathcal{B}}}_{\mathcal{M}}\,, &&\mathbb{S}^{\mathcal{B}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}} \end{aligned} \tag{3.69}$$

for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$.

Proof.

(i) There is a bijection analogous to (3.27) for module functors in $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. The statement thus follows in complete analogy to the computation for $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ in Proposition 3.11, once one takes into account that $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is exact over $\mathcal{A}^*_{\mathcal{M}}$ [EGNO, Prop. 7.12.14].

(ii) We show (3.68); the isomorphisms (3.67) follow in the same manner. According to Proposition 3.15 the distinguished object of the dual tensor category is $\mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \cong \mathbb{N}^l_{\mathcal{M}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}$. Thus

$$\mathbb{N}^l_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \succ \overline{\mathbb{S}}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong H^{\mathrm{lla}} \circ \mathbb{N}^l_{\mathcal{M}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \cong \mathbb{N}^l_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}\,, \tag{3.70}$$

where the second isomorphism uses the second isomorphism in (3.65) and the last isomorphism is the twisted module structure of the Nakayama functor. Moreover, a factor of $\mathbb{D}_{\mathcal{A}}$ can be juggled between the Nakayama functor of $\mathcal{N}$ and the relative Serre functor of $\mathcal{M}$ using the module structure of H, leading to the second isomorphism in (3.68).

(iii) According to [Sc2, Prop. 4.15], the inner-homs of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ with respect to $\mathcal{B}$ and $\mathcal{C}$ are described in terms of those of ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{C}}$ and their respective module actions; this ensures exactness of ${}_{\mathcal{B}}\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_{\mathcal{C}}$. All four isomorphisms in (3.69) follow then from the relation (1.63) between relative Serre and Nakayama functors. For instance, the last isomorphism in (3.69) is given by the composite

$$\mathbb{S}^{\mathcal{B}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{D}_{\mathcal{B}} \succ \mathbb{N}^r_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ H \circ \mathbb{N}^r_{\mathcal{M}}(-) \triangleleft \mathbb{D}_{\mathcal{B}} \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ H \circ \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}}\,, \tag{3.71}$$

where the second isomorphism comes from (3.67) and the last one is an instance of (1.63).

□

Theorem 3.17 can be extended to the bicategory $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ of exact module categories over a finite tensor category $\mathcal{A}$. $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ is a bicategory with dualities for 1-morphisms, given by $H^* := H^{\mathrm{la}}$ and ${}^*H := H^{\mathrm{ra}}$ for every module functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. The existence of dualities for 1-morphisms extends to a pseudo-functor of *left adjoints*

$$\begin{aligned} (-)^* : \quad \mathbf{Mod}^{\mathrm{ex}}(\mathcal{A}) &\xrightarrow{\simeq} \mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})^{\,\mathrm{op,op}}, \\ \mathcal{M} &\longmapsto \mathcal{M}\,, \\ \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \ni H &\longmapsto H^{\mathrm{la}} \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{M})\,. \end{aligned} \tag{3.72}$$

In complete analogy to Definition 3.13 relative Serre functors define a pseudo-functor on $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ which, by Lemma 3.19 (i), is equivalent to the double left adjoint pseudo-functor, i.e. the square of (3.72).

Theorem 3.20 (Radford pseudo-equivalence of the 2-category of module categories)**.**
Let $\mathcal{A}$ be a finite tensor category and $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})$ *its* 2*-category of exact module categories. There is a pseudo-natural equivalence*

$$\mathcal{R} : \quad \mathrm{id}_{\mathbf{Mod}^{\mathrm{ex}}(\mathcal{A})} \overset{\cong}{\Longrightarrow} (-)^{****} \tag{3.73}$$

where $(-)^{****}$ is the fourth power of the adjoint pseudo-functor (3.72)*.*

Proof. The following data assemble into the desired pseudo-natural equivalence:

(i) For any exact $\mathcal{A}$-module $\mathcal{M}$ consider the distinguished invertible 1-morphism $\mathcal{R}_{\mathcal{M}} := \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}$.

(ii) The isomorphisms coming from (3.50) and (1.52) induce a module natural isomorphism

$$\mathcal{R}_H : \quad \mathbb{D}_{\mathcal{A}^*_{\mathcal{N}}} \circ H \overset{\cong}{\Longrightarrow} H^{\mathrm{llla}} \circ \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}\,, \tag{3.74}$$

for every module functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$.

It only remains to be checked the commutativity of the diagram

$$\begin{tikzcd} \mathbb{D}_{\mathcal{A}^*_{\mathcal{L}}} \circ H_2 \circ H_1 \arrow[r, Rightarrow, "\mathcal{R}_{H_2 \circ H_1}"] \arrow[d, Rightarrow, "\mathcal{R}_{H_2} \circ \mathrm{id}"'] & (H_2 \circ H_1)^{\mathrm{llla}} \circ \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \arrow[d, Rightarrow, "\cong"] \\ H_2^{\mathrm{llla}} \circ \mathbb{D}_{\mathcal{A}^*_{\mathcal{N}}} \circ H_1 \arrow[r, Rightarrow, "\mathrm{id} \circ \mathcal{R}_{H_1}"'] & H_2^{\mathrm{llla}} \circ H_1^{\mathrm{llla}} \circ \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \end{tikzcd} \tag{3.75}$$

for all module functors $H_1 : \mathcal{M} \to \mathcal{N}$ and $H_2 : \mathcal{N} \to \mathcal{L}$. This follows from the compatibility condition (1.53) once one takes into account that $\mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}} \cong \mathbb{D}_{\mathcal{A}} \triangleright \left(\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}\right)^2$ according to (3.50).

□

Chapter 4

On pivotality and Morita theory

4.1 Pivotal module categories

An additional structure that a tensor category $\mathcal{A}$ can carry is a *pivotal structure*, that is, a monoidal natural isomorphism $p\colon \mathrm{id}_{\mathcal{A}} \overset{\cong}{\Longrightarrow} (-)^{\vee\vee}$ between the identity functor and the double-dual functor. The monoidal opposite $\overline{\mathcal{A}}$ of a pivotal tensor category is endowed with a canonical pivotal structure, given by

$$\overline{p}_{\overline{a}} := p^{-1}_{{}^{\vee\vee}a} : \quad \overline{a} \xrightarrow{\cong} \overline{{}^{\vee\vee}a} = \overline{a}^{\vee\vee}. \tag{4.1}$$

A pivotal structure on a module category over a pivotal tensor category can be defined as follows:

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be pivotal finite tensor categories.

(i) ([Sc2, Def. 5.2] and [Sh2, Def. 3.11]) A *pivotal structure* on an exact left $\mathcal{A}$-module category $\mathcal{M}$ is a natural isomorphism $\widetilde{p}\colon \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ such that the diagram

$$\begin{tikzcd}
a \triangleright m \arrow[rr, "\widetilde{p}_{a\triangleright m}"] \arrow[dr, "p_a \triangleright \widetilde{p}_m"'] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(a \triangleright m) \arrow[dl, "(1.44)"] \\
& a^{\vee\vee} \triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) &
\end{tikzcd} \tag{4.2}$$

commutes for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. A module category together with a module structure is said to be a *pivotal module category.*

(ii) An exact right $\mathcal{B}$-module category $\mathcal{N}$ is said to be *pivotal* if the left module category ${}_{\overline{\mathcal{B}}}\mathcal{N}$ has a pivotal structure.

(iii) A *pivotal bimodule category* is an exact bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ together with the structure $\widetilde{p}\colon \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ of a pivotal $\mathcal{A}$-module and the structure $\widetilde{q}\colon \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}}$ of a pivotal $\mathcal{B}$-module, such that the diagrams

$$\begin{tikzcd}
m \triangleleft b \arrow[rr, "\widetilde{p}_{m\triangleleft b}"] \arrow[dr, "\widetilde{p}_m \triangleleft q_b"'] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m \triangleleft b) \arrow[dl, "(1.44)"] \\
& \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \triangleleft b^{\vee\vee} &
\end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd}
a\triangleright m \arrow[rr, "\widetilde{q}_{a\triangleright m}"] \arrow[dr, "p^{-1}_{\vee\vee a}\triangleright\widetilde{q}_m"'] & & \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}}(a\triangleright m) \arrow[dl, "(1.48)"] \\
& {}^{\vee\vee}a\triangleright\mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}}(m) &
\end{tikzcd} \tag{4.4}$$

commute for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $m\in\mathcal{M}$.

The dual tensor category of a pivotal module inherits the structure of a pivotal finite multitensor category:

Proposition 4.2. *Let $\mathcal{A}$ be a pivotal tensor category and $\mathcal{M}$ a pivotal $\mathcal{A}$-module category. Then*

(i) [Sh2, Thm. 3.13] *The dual tensor category $\mathcal{A}^*_{\mathcal{M}}$ has a pivotal structure given by the composite*

$$q_F:\quad F\xRightarrow{\mathrm{id}\circ\widetilde{p}} F\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\xRightarrow{(1.49)}\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\circ F^{\mathrm{lla}}\xRightarrow{\widetilde{p}^{-1}\circ\mathrm{id}} F^{\mathrm{lla}} \tag{4.5}$$

for a module endofunctor $F\in\mathcal{A}^*_{\mathcal{M}}$, with $\widetilde{p}$ the pivotal structure of $\mathcal{M}$.

(ii) *Given a pivotal bimodule category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$, the assignment*

$$\mathcal{B}\longrightarrow\overline{\mathcal{A}^*_{\mathcal{M}}},\qquad b\longmapsto -\triangleleft b \tag{4.6}$$

is a pivotal tensor functor.

Proof. The statement in [Sh2, Thm. 3.13] concerns $\overline{\mathcal{A}^*_{\mathcal{M}}}$, the monoidal opposite of the dual tensor category, with its pivotal structure described by

$$\overline{q}_F:\quad F\xRightarrow{\widetilde{p}\circ\mathrm{id}}\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\circ F\xRightarrow{(1.49)} F^{\mathrm{rra}}\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\xRightarrow{\mathrm{id}\circ\widetilde{p}^{-1}} F^{\mathrm{rra}}. \tag{4.7}$$

Considering the opposite pivotal structure (4.1) on $\mathcal{A}^*_{\mathcal{M}}$ we obtain (4.5). Now, assertion (ii) states that the diagram

$$\begin{tikzcd}
m\triangleleft b \arrow[r, "\widetilde{p}_{m\triangleleft b}"] \arrow[d, "\mathrm{id}\triangleleft q_b"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m\triangleleft b) \arrow[d, "(1.49)"] \\
m\triangleleft b^{\vee\vee} \arrow[r, "\widetilde{p}_m\triangleleft\mathrm{id}"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)\triangleleft b^{\vee\vee}
\end{tikzcd} \tag{4.8}$$

commutes for every $m\in\mathcal{M}$ and $b\in\mathcal{B}$. This diagram is precisely the fulfilled condition (4.3). □

For bimodule categories $\mathcal{M}$ and $\mathcal{N}$, the category of module functors $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ becomes a bimodule category with actions given by (1.7). Pivotal structures are transferred to the category of module functors as we show next.

Proposition 4.3. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be pivotal finite tensor categories, and consider exact bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{C}}$.*

(i) *Suppose that ${}_{\mathcal{A}}\mathcal{M}$ and $\mathcal{N}_{\mathcal{C}}$ are pivotal modules. Then $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ has the structure of a pivotal $\mathcal{C}$-module category.*

(ii) *Suppose that ${}_{\mathcal{A}}\mathcal{N}$ and $\mathcal{M}_{\mathcal{B}}$ are pivotal modules. Then $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ has the structure of a pivotal $\mathcal{B}$-module category.*

(iii) *If* ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ *and* ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{C}}$ *are pivotal bimodules, then* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *has the structure of a pivotal* $(\mathcal{B},\mathcal{C})$*-bimodule category.*

Proof. Lemma 3.19 provides a description of the relative Serre functors of the category of module functors. We construct the corresponding trivialization in each situation. To show (i) denote by $\widehat{q}\colon \mathrm{id}_{\mathcal{N}} \xRightarrow{\cong} \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}$ and by $\widetilde{p}\colon \mathrm{id}_{\mathcal{M}} \xRightarrow{\sim} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ the pivotal structures of the pivotal modules ${}_{\mathcal{A}}\mathcal{M}$ and $\mathcal{N}_{\mathcal{C}}$. Define for $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ a natural isomorphism

$$\widetilde{Q}_H : \quad H \xRightarrow{\widehat{q}\,\circ\,\mathrm{id}_H\,\circ\,\widetilde{p}} \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}} \circ H \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \cong \mathbb{S}^{\overline{\mathcal{C}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H)\,; \tag{4.9}$$

this serves as a $\mathcal{C}$-pivotal structure for $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. It remains to check that the diagram

$$\begin{tikzcd}
H \prec c \arrow[rr, Rightarrow, "\widetilde{Q}_{H\prec c}"] \arrow[dr, Rightarrow, "\widetilde{Q}_H \prec\, \overline{q}_{\overline{c}}"'] & & \mathbb{S}^{\overline{\mathcal{C}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H\prec c) \arrow[dl, Rightarrow, "(1.47)"] \\
& \mathbb{S}^{\overline{\mathcal{C}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \prec {}^{\vee\vee}c &
\end{tikzcd} \tag{4.10}$$

commutes for every $c \in \mathcal{C}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, where $\overline{q}$ is the pivotal structure of $\overline{\mathcal{C}}$. Now indeed, for every $m \in \mathcal{M}$ the diagram

$$\begin{tikzcd}
H(m) \triangleleft c \arrow[r, "\widehat{q}_{H(m)\triangleleft c}"] \arrow[d, "\mathrm{id}\,\triangleleft\, \overline{q}_{\overline{c}}"'] & \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}(H(m) \triangleleft c) \arrow[r, "\mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}\circ(-\triangleleft c)\circ H(\widetilde{p}_m)"] \arrow[d, "(1.47)"] & \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}(H(\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)) \triangleleft c) \arrow[d, "(1.47)"] \\
H(m) \triangleleft {}^{\vee\vee}c \arrow[r, "\widehat{q}_{H(m)}\triangleleft\mathrm{id}"'] & \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}(H(m)) \triangleleft {}^{\vee\vee}c \arrow[r, "\mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}\circ H(\widetilde{p}_m)\triangleleft\mathrm{id}"'] & \mathbb{S}^{\overline{\mathcal{C}}}_{\mathcal{N}}(H(\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m))) \triangleleft {}^{\vee\vee}c
\end{tikzcd} \tag{4.11}$$

commutes: the square on the left corresponds to the condition fulfilled by $\widehat{q}$ of being a $\overline{\mathcal{C}}$-pivotal structure for $\mathcal{N}$, while the square on the right commutes owing to naturality of $\widetilde{p}$.

The claim (ii) follows analogously by considering as $\mathcal{B}$-pivotal structure for $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ the natural isomorphism

$$\widetilde{P}_H : \quad H \xRightarrow{\widehat{p}\,\circ\,\mathrm{id}_H\,\circ\,\widetilde{q}} \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}} \cong \mathbb{S}^{\mathcal{B}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H)\,, \tag{4.12}$$

where $\widehat{p}\colon \mathrm{id}_{\mathcal{N}} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{N}}$ and $\widetilde{q}\colon \mathrm{id}_{\mathcal{M}} \xRightarrow{\cong} \mathbb{S}^{\overline{\mathcal{B}}}_{\mathcal{M}}$ are the corresponding pivotal structures of ${}_{\mathcal{A}}\mathcal{N}$ and $\mathcal{M}_{\mathcal{B}}$.

Assertion (iii) is verified by making the routine check of the diagrams (4.3) and (4.4). ☐

Remark 4.4. According to Corollary 1.5 the category of module functors is a model for the relative Deligne product. Therefore Proposition 4.3 implies that the product of two pivotal bimodules inherits a pivotal structure.

4.2 Pivotal Morita theory

Given a bicategory $\mathcal{F}$, the existence of dualities for 1-morphisms extends to a pseudo-functor

$$\begin{aligned} (-)^{\vee} : \quad \mathcal{F} &\longrightarrow \mathcal{F}^{\mathrm{op,op}}, \\ x &\longmapsto x\,, \\ (a\colon x \to y) &\longmapsto (a^{\vee}\colon y \to x)\,. \end{aligned} \tag{4.13}$$

A *pivotal structure* on a bicategory $\mathcal{F}$ with dualities is a pseudo-natural equivalence

$$\mathbf{P}: \quad \mathrm{id}_{\mathcal{F}} \overset{\cong}{\Longrightarrow} (-)^{\vee\vee} \tag{4.14}$$

obeying $\mathbf{P}_x = \mathrm{id}_x$ for every object $x \in \mathcal{F}$. A bicategory together with a pivotal structure is called a *pivotal bicategory.*

Remark 4.5. A tensor category $\mathcal{A}$ can be seen as a bicategory with a single object $\mathbb{A}$. The requirement that $\mathbf{P}_x = \mathrm{id}_x$ imposed on the pseudo-natural equivalence (4.14) ensures that a pivotal structure on the bicategory $\mathbb{A}$ recovers a pivotal structure on the tensor category $\mathcal{A}$. A pseudo-natural equivalence (4.14) without this requirement corresponds to the notion of a *quasi-pivotal* structure on $\mathcal{A}$ [Sh1, Sec. 4], that is, a pair (d,γ) where $d \in \mathcal{A}$ is an invertible object and $\gamma = \{\gamma_a : d \otimes a \xrightarrow{\cong} a^{\vee\vee} \otimes d\}$ is a twisted half-braiding.

Definition 4.6. A Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \odot, \boxdot)$ is said to be *pivotal* iff its associated bicategory $\mathbb{M}$ is pivotal.

Next we establish that the Morita context of a pivotal module is indeed pivotal, thus justifying the terminology.

Lemma 4.7. *Let $\mathcal{M}$ be a pivotal module category over a pivotal tensor category $\mathcal{A}$. Then we have:*

(i) *$\mathcal{M}$ has the structure of a pivotal $(\mathcal{A}, \overline{\mathcal{A}^*_{\mathcal{M}}})$-bimodule category.*

(ii) *For every $\mathcal{A}$-module category $\mathcal{N}$, the $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module category* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *inherits a pivotal structure.*

(iii) *The functor category* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ *has the structure of a pivotal $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \mathcal{A})$-bimodule category.*

Proof. Denote by $\widetilde{p}\colon \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ the pivotal structure of ${}_{\mathcal{A}}\mathcal{M}$. According to Corollary 3.12 we have $\mathbb{S}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}} \cong \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}$, so that for any $m \in \mathcal{M}$ we can define a natural isomorphism

$$\widetilde{q}_m := \widetilde{p}^{\,-1}_{\overline{\mathbb{S}}_{\mathcal{M}}(m)} : \quad m \xrightarrow{\cong} \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m) \cong \mathbb{S}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}}(m)\,. \tag{4.15}$$

Recall from Proposition 4.2 that $\mathcal{A}^*_{\mathcal{M}}$ is endowed with the pivotal structure (4.5). In view of Remark 1.16 this pivotal structure coincides with the composition

$$q_F: \quad F \xRightarrow{\widetilde{q}\,\circ\,\mathrm{id}} \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \circ F \xRightarrow{(1.52)} F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \xRightarrow{\mathrm{id}\,\circ\,\widetilde{q}^{-1}} F^{\mathrm{lla}}. \tag{4.16}$$

Now we verify that $\widetilde{q}$ is compatible with this pivotal structure, i.e. that the diagram

$$\begin{tikzcd}
F(m) \arrow[rr, "\widetilde{q}_{F(m)}"] \arrow[dr, "q_F \triangleright \widetilde{q}_m"'] & & \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \circ F(m) \arrow[dl, "(1.52)"] \\
& F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m) &
\end{tikzcd} \tag{4.17}$$

commutes for every $F \in \mathcal{A}^*_{\mathcal{M}}$ and $m \in \mathcal{M}$. In fact, by invoking the relevant definitions, the diagram translates to

$$\begin{tikzcd}
F(m) \arrow[rrr, "\widetilde{q}_{F(m)}"] \arrow[d, "\widetilde{q}_{F(m)}"'] \arrow[rrd, dotted, "q_F"] & & & \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \circ F(m) \arrow[d, "(1.52)"] \\
\overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \circ F(m) \arrow[r, "(1.52)"'] & F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m) \arrow[r, "\mathrm{id} \circ \widetilde{q}_m^{-1}"'] & F^{\mathrm{lla}}(m) \arrow[r, "\mathrm{id} \circ \widetilde{q}_m"'] & F^{\mathrm{lla}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}}(m)
\end{tikzcd} \tag{4.18}$$

which commutes trivially. Thus it is established that ${}_{\mathcal{A}}\mathcal{M}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}$ is a pivotal bimodule category.
Statement (ii) follows from (i) and Proposition 4.3(ii). Explicitly, the pivotal structure is the composite

$$\widehat{p}_H : \quad H \xRightarrow{\widetilde{p}^{\mathcal{N}} \circ \mathrm{id}} \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \xRightarrow{(1.49)} H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \xRightarrow{\mathrm{id} \circ \left(\widetilde{p}^{\mathcal{M}}\right)^{-1}} H^{\mathrm{rra}} \cong \mathbb{S}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(H) \tag{4.19}$$

for a module functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$.
Similarly, claim (iii) follows from (i) and Proposition 4.3(iii) by considering ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{A}} = \mathcal{A}$ as the regular bimodule category. In this situation the pivotal structures are explicitly given by

$$\widehat{p}_H : \quad H \xRightarrow{p \circ \mathrm{id}} (-)^{\vee\vee} \circ H \xRightarrow{(1.49)} H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \xRightarrow{\mathrm{id} \circ \widetilde{p}^{-1}} H^{\mathrm{rra}} \cong \mathbb{S}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H) \tag{4.20}$$

and

$$\widehat{q}_H : \quad H \xRightarrow{\mathrm{id} \circ \widetilde{p}} H \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \xRightarrow{(1.49)} (-)^{\vee\vee} \circ H^{\mathrm{lla}} \xRightarrow{p^{-1} \circ \mathrm{id}} H^{\mathrm{lla}} \cong \mathbb{S}^{\overline{\mathcal{A}}}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})}(H) \tag{4.21}$$

for a module functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$. ☐

Theorem 4.8. *For $\mathcal{M}$ a pivotal module category over a pivotal tensor category $\mathcal{A}$, its Morita context $(\mathcal{A}$, $\overline{\mathcal{A}^*_{\mathcal{M}}}$, $\mathcal{M}$, $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$, $\odot$, $\boxdot)$ is a pivotal Morita context.*

Proof. Let $\mathbb{M}$ be the bicategory associated to the Morita context of $\mathcal{M}$. We need to construct a pseudo-natural equivalence $\mathbf{P} \colon \mathrm{id}_{\mathbb{M}} \xRightarrow{\cong} (-)^{\vee\vee}$, subject to the condition that the components of $\mathbf{P}$ on any object is the identity. For any 1-morphism $a \in \mathcal{A}$ and any $F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$, $m \in \mathcal{M}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ we define

(i) $\mathbf{P}_a \colon a \xrightarrow{\cong} a^{\vee\vee}$ as the pivotal structure p_a of $\mathcal{A}$;

(ii) $\mathbf{P}_F \colon F \xRightarrow{\cong} F^{\mathrm{rra}}$ as the isomorphism $\overline{q}_F$ in (4.7) which serves as a pivotal structure of $\mathcal{A}^*_{\mathcal{M}}$;

(iii) $\mathbf{P}_m \colon m \xrightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)$ as the pivotal structure $\widetilde{p}_m$ of the module ${}_{\mathcal{A}}\mathcal{M}$;

(iv) $\mathbf{P}_H \colon H \xRightarrow{\cong} H^{\mathrm{rra}}$ as the isomorphism $\widehat{p}_H$ in (4.20) which serves as a pivotal structure of the module ${}_{\overline{\mathcal{A}^*_{\mathcal{M}}}}\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$.

These assignments are natural for 2-morphisms in $\mathbb{M}$. The compatibility with composition of 1-morphisms reduces to the commutativity of the diagram

$$\begin{tikzcd}
s \circ t \arrow[rr, Rightarrow, "\mathbf{P}_{s \circ t}"] \arrow[dr, Rightarrow, "\mathbf{P}_s \circ \mathbf{P}_t"'] & & (s \circ t)^{\vee\vee} \arrow[dl, Rightarrow, "\cong"] \\
& s^{\vee\vee} \circ t^{\vee\vee} &
\end{tikzcd} \tag{4.22}$$

for s and t composable 1-morphisms in $\mathbb{M}$. These translate into the following eight conditions, which are indeed all satisfied:
The tensor categories $\mathcal{A}$ and $\overline{\mathcal{A}^*_{\mathcal{M}}}$ are pivotal, i.e.:

(i) the monoidality condition of $p\colon \mathrm{id}_{\mathcal{A}} \xRightarrow{\cong} (-)^{\vee\vee}$ in the case $s = a$ and $t = b$ in $\mathcal{A}$;

(ii) the monoidality condition of $\overline{q}\colon \mathrm{id}_{\overline{\mathcal{A}^*_{\mathcal{M}}}} \xRightarrow{\cong} (-)^{\mathrm{rra}}$ in the case $s = F_1$ and $t = F_2$ in $\overline{\mathcal{A}^*_{\mathcal{M}}}$.

Further, the bimodules $\mathcal{M}$ and $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ are pivotal ($\widetilde{p}$ and $\widehat{p}$ are bimodule natural transformations), i.e.:

(iii) in case $s = a \in \mathcal{A}$ and $t = m \in \mathcal{M}$, the condition (4.2) fulfilled by $\widetilde{p}$;

(iv) for $s = m \in \mathcal{M}$ and $t = F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$, the condition (4.3) fulfilled by $\widetilde{p}$;

(v) in case $s = F \in \overline{\mathcal{A}^*_{\mathcal{M}}}$ and $t = H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$, the condition (4.2) fulfilled by $\widehat{p}$;

(vi) for $s = H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ and $t = a \in \mathcal{A}$, the condition (4.3) fulfilled by $\widehat{p}$.

Finally, two additional conditions involving the mixed products:

(vii) for $m \in \mathcal{M}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ the commutativity of the diagram

$$\begin{tikzcd}
m \odot H \arrow[rr, "p_{m \odot H}"] \arrow[dr, "\widetilde{p}_m \odot \widehat{p}_H"'] & & (m \odot H)^{\vee\vee} \arrow[dl, "(1.49)"] \\
& \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \odot H^{\mathrm{rra}} &
\end{tikzcd} \tag{4.23}$$

which can be rewritten in the more explicit form

$$\begin{tikzcd}
H(m) \arrow[rr, "p_{H(m)}"] \arrow[d, "p \circ \mathrm{id}"'] \arrow[dd, dotted, "\widehat{p}_H", to=3-2] & & H(m)^{\vee\vee} \arrow[dd, "(1.49)"] \\
(-)^{\vee\vee} \circ H(m) \arrow[d, "(1.49)"'] & & \\
H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) \arrow[r, "\mathrm{id} \circ \widetilde{p}_m^{-1}"'] & H^{\mathrm{rra}}(m) \arrow[r, "\mathrm{id} \circ \widetilde{p}_m"'] & H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)
\end{tikzcd} \tag{4.24}$$

which trivially commutes;

(viii) similarly, for $m \in \mathcal{M}$ and $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ the commutativity of the diagram

$$\begin{tikzcd}
H \boxdot m \arrow[rr, Rightarrow, "\overline{q}_{H \boxdot m}"] \arrow[dr, Rightarrow, swap, "\widehat{p}_H \boxdot \widetilde{p}_m"] & & (H \boxdot m)^{\mathrm{rra}} \arrow[dl, Rightarrow, "\cong"] \\
& H^{\mathrm{rra}} \boxdot \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m) &
\end{tikzcd} \tag{4.25}$$

which is the same as the diagram

$$\begin{tikzcd}
(-\triangleright m) \circ H \arrow[r, Rightarrow, "\widetilde{p} \circ \mathrm{id}"] \arrow[d, Rightarrow, swap, "(- \triangleright \widetilde{p}_m) \circ p \circ \mathrm{id}"] & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ (-\triangleright m) \circ H \arrow[r, Rightarrow, "(1.49)"] \arrow[dl, Rightarrow, "(1.49)"] & {[(-\triangleright m) \circ H]^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}} \arrow[r, Rightarrow, "\mathrm{id} \circ \widetilde{p}^{-1}"] \arrow[d, Rightarrow, "\cong"] & {[(-\triangleright m) \circ H]^{\mathrm{rra}}} \arrow[d, Rightarrow, "\cong"] \\
(-\triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)) \circ (-)^{\vee\vee} \circ H \arrow[rr, Rightarrow, swap, "(1.49)"] & & (-\triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)) \circ H^{\mathrm{rra}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[r, Rightarrow, swap, "\mathrm{id} \circ \widetilde{p}^{-1}"] & (-\triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(m)) \circ H^{\mathrm{rra}}
\end{tikzcd} \tag{4.26}$$

which is commutative: the left triangle corresponds to the condition of $\widetilde{p}$ being a pivotal structure for $\mathcal{M}$, the rightmost square commutes due to naturality, and the square in the middle is the compatibility (1.50).

This shows that $\mathbf{P}\colon \mathrm{id}_{\mathbb{M}} \overset{\cong}{\Longrightarrow} (-)^{\vee\vee}$ is a pivotal structure on the bicategory $\mathbb{M}$, and thus the claim is proven. □

Definition 4.9. Two pivotal tensor categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *pivotal Morita equivalent* iff there exists a pivotal $\mathcal{A}$-module category $\mathcal{M}$ together with a pivotal equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$.

Proposition 4.10. *Let $\mathcal{A}$ be a pivotal category and $\mathcal{M}$ a pivotal $\mathcal{A}$-module.*

(i) *The tensor equivalence*

$$\mathcal{A} \xrightarrow{\simeq} \overline{\mathcal{A}^*_{\mathcal{A}}}, \qquad a \longmapsto - \otimes a \tag{4.27}$$

from (1.70) *is pivotal.*

(ii) *The canonical tensor equivalence*

$$\mathrm{can}: \quad \mathcal{A} \xrightarrow{\simeq} (\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}, \qquad a \longmapsto a \triangleright - \tag{4.28}$$

from (1.71) *is pivotal.*

Proof.

(i) The pivotal structure for the functor $- \otimes a$ in $\overline{\mathcal{A}^*_{\mathcal{A}}}$ is given by

$$q_{-\otimes a}: \quad (- \otimes a) \xRightarrow{p_{-\otimes a}} (- \otimes a)^{\vee\vee} \xRightarrow{(1.49)} (-)^{\vee\vee} \otimes a^{\vee\vee} \xRightarrow{p_-^{-1} \otimes \mathrm{id}} (- \otimes a^{\vee\vee}). \tag{4.29}$$

But since $p\colon \mathrm{id}_{\mathcal{A}} \overset{\simeq}{\Longrightarrow} (-)^{\vee\vee}$ is monoidal, we have $q_{-\otimes a} = \mathrm{id}_- \otimes p_a$.

(ii) According to Corollary 3.12 there is an isomorphism $\overline{\mathbb{S}}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}} \cong \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$. Taking this into consideration the pivotal structure on $(\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}$ is given by

$$\overline{\overline{q}}_{a \triangleright -}: \quad (a \triangleright -) \xRightarrow{\widetilde{p} \circ \mathrm{id}} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\, (a \triangleright -) \xRightarrow{(1.44)} a^{\vee\vee} \triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \xRightarrow{\mathrm{id} \circ \widetilde{p}^{-1}} (a^{\vee\vee} \triangleright -) = (a \triangleright -)^{\mathrm{lla}}. \tag{4.30}$$

But in a similar manner the defining condition (4.2) of $\widetilde{p}$ being a pivotal structure for $\mathcal{M}$ implies that $\mathrm{can}(p_a) = \overline{\overline{q}}_{a \triangleright -}$. □

4.3 The 2-category of pivotal modules

Pivotal modules over a pivotal tensor category form a pivotal bicategory, as presented in [Sc2, Def. 5.2] in terms of inner-product module categories. Here we express this fact in the language of relative Serre functors. Let $\mathcal{A}$ be a pivotal tensor category, and denote by $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ the 2-category that has pivotal $\mathcal{A}$-module categories as objects, $\mathcal{A}$-module functors as 1-morphisms and module natural transformations as 2-morphisms. Since pivotal modules are exact [FSS, Prop. 4.24], every module functor $H\colon {}_{\mathcal{A}}\mathcal{N}_1 \to {}_{\mathcal{A}}\mathcal{N}_2$ comes with adjoints

$$H^* := H^{\mathrm{la}} : \; {}_{\mathcal{A}}\mathcal{N}_2 \longrightarrow {}_{\mathcal{A}}\mathcal{N}_1 \qquad \text{and} \qquad {}^*H := H^{\mathrm{ra}} : \; {}_{\mathcal{A}}\mathcal{N}_2 \longrightarrow {}_{\mathcal{A}}\mathcal{N}_1 \,. \tag{4.31}$$

These turn $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ into a bicategory with dualities for 1-morphisms. Moreover, $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ is endowed with a pivotal structure (4.14). Indeed, given any 1-morphism $H\colon {}_{\mathcal{A}}\mathcal{N}_1 \to {}_{\mathcal{A}}\mathcal{N}_2$ in $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$, define

$$\mathbf{P}_H : \quad H \xRightarrow{\mathrm{id}\circ\widetilde{p}_1} H \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}_1} \xRightarrow{(1.49)} \mathbb{S}^{\mathcal{A}}_{\mathcal{N}_2} \circ H^{\mathrm{lla}} \xRightarrow{(\widetilde{p}_2)^{-1}\circ\mathrm{id}} H^{\mathrm{lla}}, \tag{4.32}$$

where $\widetilde{p}_i$ are the pivotal structures of the module categories $\mathcal{N}_i$. The 2-morphisms $\mathbf{P}_H$ are invertible and natural in H. Moreover, (1.50) implies that they are compatible with the composition of module functors. Therefore $\mathbf{P}$ constitutes a pivotal structure on the 2-category $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$.

Theorem 4.11. *Two pivotal tensor categories $\mathcal{A}$ and $\mathcal{B}$ are pivotal Morita equivalent if and only if $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ and $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{B})$ are equivalent as pivotal bicategories.*

Proof. Given a pivotal $\mathcal{A}$-module $\mathcal{M}$, according to Lemma 4.7(ii) for every $\mathcal{N} \in \mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ the $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is endowed with a pivotal structure. By Theorem 7.12.16 of [EGNO] this assignment extends to a 2-equivalence

$$\begin{aligned} \Psi : \quad \mathbf{Mod}^{\mathrm{piv}}(\mathcal{A}) &\longrightarrow \mathbf{Mod}^{\mathrm{piv}}(\overline{\mathcal{A}^*_{\mathcal{M}}}) \,, \\ \mathcal{N} &\longmapsto \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \,. \end{aligned} \tag{4.33}$$

Moreover, Ψ preserves the pivotal structure: To a 1-morphism $H\colon {}_{\mathcal{A}}\mathcal{N}_1 \to {}_{\mathcal{A}}\mathcal{N}_2$ in $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ it assigns

$$\Psi(H) : \quad \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}_1) \xrightarrow{H\circ -} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}_2) \,. \tag{4.34}$$

The component at $\Psi(H)$ of the pivotal structure of $\mathbf{Mod}^{\mathrm{piv}}(\overline{\mathcal{A}^*_{\mathcal{M}}})$ is the composite

$$\mathbf{P}_{\Psi(H)} : \quad (H\circ -) \xRightarrow{\mathrm{id}\circ\widehat{p}_1} H\circ(-)^{\mathrm{rra}} \xRightarrow{\cong} (H^{\mathrm{lla}}\circ -)^{\mathrm{rra}} \xRightarrow{(\widehat{p}_2)^{-1}\circ\mathrm{id}} (H^{\mathrm{lla}}\circ -) \,, \tag{4.35}$$

where, for $i=1,2$, $\widehat{p}_i$ is the pivotal structure of the module category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}_i)$ given by (4.19). The task at hand is to verify that the diagram

$$\begin{tikzcd} \Psi(H) \arrow[rr, Rightarrow, "\mathbf{P}_{\Psi(H)}"] \arrow[dr, Rightarrow, "\Psi(\mathbf{P}_H)"'] & & \Psi(H)^{\mathrm{lla}} \arrow[dl, Rightarrow, "\cong"] \\ & \Psi(H^{\mathrm{lla}}) & \end{tikzcd} \tag{4.36}$$

commutes for every 1-morphism H in $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$. By inserting the definitions, this diagram translates to

$$\begin{tikzcd}
(H\circ -) \arrow[r, Rightarrow, "\mathrm{id}\circ\widetilde{p}_{\mathcal{N}_1}\circ\mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id}\circ\widetilde{p}_{\mathcal{N}_1}\circ\mathrm{id}"'] & (H\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{N}_1}\circ -) \arrow[r, Rightarrow, "(1.49)"] \arrow[dl, Rightarrow, "\mathrm{id}"] & H\circ(-)^{\mathrm{rra}}\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[r, Rightarrow, "\mathrm{id}\circ(\widetilde{p}_{\mathcal{M}})^{-1}"] \arrow[ddr, Rightarrow, "\cong"'] & H\circ(-)^{\mathrm{rra}} \arrow[d, Rightarrow, "\cong"] \\
(H\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{N}_1}\circ -) & & & (H^{\mathrm{lla}}\circ -)^{\mathrm{rra}} \arrow[d, Rightarrow, "\mathrm{id}\circ\widetilde{p}_{\mathcal{M}}"] \\
(\mathbb{S}^{\mathcal{A}}_{\mathcal{N}_2}\circ H^{\mathrm{lla}}\circ -) \arrow[u, Rightarrow, "(1.49)"] & (H^{\mathrm{lla}}\circ -) \arrow[l, Rightarrow, "\widetilde{p}_{\mathcal{N}_2}\circ\mathrm{id}"] \arrow[r, Rightarrow, "\widetilde{p}_{\mathcal{N}_2}\circ\mathrm{id}"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}_2}\circ(H^{\mathrm{lla}}\circ -) \arrow[r, Rightarrow, "(1.49)"'] \arrow[ull, Rightarrow, "(1.49)"'] & (H^{\mathrm{lla}}\circ -)^{\mathrm{rra}}\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}
\end{tikzcd} \tag{4.37}$$

This is indeed a commutative diagram: The pentagon in the middle commutes owing to the compatibility (1.50), and the triangle and squares in the periphery commute trivially.
To see the converse implication, consider a pivotal 2-equivalence

$$\Phi : \quad \mathbf{Mod}^{\mathrm{piv}}(\mathcal{B}) \xrightarrow{\simeq} \mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})\,. \tag{4.38}$$

Define $\mathcal{M}$ as the image of the regular pivotal module ${}_{\mathcal{B}}\mathcal{B}$ under Φ. Since Φ is a 2-equivalence, we obtain an equivalence

$$\Phi_{\mathcal{B},\mathcal{B}} : \quad \mathcal{B}^*_{\mathcal{B}} = \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M}) = \mathcal{A}^*_{\mathcal{M}} \tag{4.39}$$

of categories. In addition, amongst the data of the 2-functor Φ there is a natural isomorphism

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B})\times\mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \arrow[r, "\circ"] \arrow[d, "\Phi\times\Phi"'] & \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\gamma"] \\
\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M})\times\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M}) \arrow[r, "\circ"'] & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M})
\end{tikzcd} \tag{4.40}$$

whereby $\Phi_{\mathcal{B},\mathcal{B}}$ is endowed with a tensor structure. Furthermore, $\Phi_{\mathcal{B},\mathcal{B}}$ is pivotal. To see this, notice that an object $F\in\mathcal{A}^*_{\mathcal{M}}$ is a 1-morphism in $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$. Now on the one hand the pivotal structure (4.5) of the tensor category $\mathcal{A}^*_{\mathcal{M}}$ provides an isomorphism $q_F\colon F \overset{\cong}{\Longrightarrow} F^{\mathrm{lla}}$, while on the other hand the pivotal structure (4.32) in the bicategory $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ for F is an isomorphism $\mathbf{P}_F\colon F \overset{\cong}{\Longrightarrow} F^{\mathrm{lla}}$, and in fact it coincides with q_F. The same argument holds for $\mathcal{B}^*_{\mathcal{B}}$; since Φ preserves $\mathbf{P}$, it follows that $\Phi_{\mathcal{B},\mathcal{B}}$ is a pivotal equivalence. We have thus obtained an equivalence

$$\mathcal{B} \xrightarrow{(4.27)} \overline{\mathcal{B}^*_{\mathcal{B}}} \xrightarrow{\Phi_{\mathcal{B},\mathcal{B}}} \overline{\mathcal{A}^*_{\mathcal{M}}} \tag{4.41}$$

of pivotal tensor categories; hence $\mathcal{A}$ and $\mathcal{B}$ are pivotal Morita equivalent. □

Remark 4.12. Theorem 4.11 implies that pivotal Morita equivalence is indeed an equivalence relation on pivotal tensor categories. Proposition 4.10 already shows reflexivity and symmetry.

4.4 Pivotality of the center and pivotal Morita equivalence

Recall that the *Drinfeld center* of a tensor category $\mathcal{A}$ is the braided tensor category $\mathcal{Z}(\mathcal{A})$ whose objects are pairs (a,σ) consisting of an object $a\in\mathcal{A}$ and a *half-braiding* σ, i.e. a natural isomorphism $\sigma_{b,a}\colon b\otimes a \xrightarrow{\cong} a\otimes b$ for $b\in\mathcal{A}$ obeying the appropriate hexagon axiom. According to [EGNO, Prop. 8.10.10] the Drinfeld center $\mathcal{Z}(\mathcal{A})$ is unimodular. A pivotal structure $p_a\colon a \xrightarrow{\cong} a^{\vee\vee}$ on $\mathcal{A}$ induces a pivotal structure on $\mathcal{Z}(\mathcal{A})$ via $p_{(a,\sigma)} := p_a$ [EGNO, Ex. 7.13.6].

Lemma 4.13. *Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an $\mathcal{A}$-module, and let $(a,\sigma)\in\mathcal{Z}(\mathcal{A})$.*

(i) *There is a natural isomorphism $\iota_a\colon a^{\vee\vee} \xrightarrow{\cong} {}^{\vee\vee}a$ in $\mathcal{Z}(\mathcal{A})$.*

(ii) *The diagram*

$$\begin{tikzcd}
a^{\vee\vee} \triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[rr, Rightarrow, "\iota_a \triangleright \mathrm{id}"] \arrow[dr, Rightarrow, "(1.44)"'] & & {}^{\vee\vee}a \triangleright \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[dl, Rightarrow, "(1.49)"] \\
& \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(a \triangleright -) &
\end{tikzcd} \tag{4.42}$$

commutes, where the natural isomorphism (1.49) *is applied to the module functor $a\triangleright -$.*

Proof. The desired isomorphism in (i) comes from the composition

$$a^{\vee\vee}\otimes\mathbb{D}_{\mathcal{A}} \xrightarrow{(1.60)} \mathbb{D}_{\mathcal{A}}\otimes {}^{\vee\vee}a \xrightarrow{{}^{\vee\vee}\sigma} {}^{\vee\vee}a\otimes\mathbb{D}_{\mathcal{A}}\,. \tag{4.43}$$

Now denote by $F_a := a\triangleright -$ the module functor induced by (a,σ). Its double adjoints are $F_a^{\mathrm{lla}} = a^{\vee\vee}\triangleright -$ and $F_a^{\mathrm{rra}} = {}^{\vee\vee}a\triangleright -$. Assertion (ii) is thus implied by the commutativity of the diagram

$$\begin{tikzcd}
a^{\vee\vee}\triangleright\mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[rrrr, Rightarrow, "\iota_a\triangleright\mathrm{id}"] \arrow[dr, Rightarrow, "(1.63)"'] \arrow[dd, Rightarrow, "(1.44)"'] & & & & {}^{\vee\vee}a\triangleright\mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[dd, Rightarrow, "(1.49)"] \\
& a^{\vee\vee}\otimes\mathbb{D}_{\mathcal{A}}\triangleright\mathbb{N}^r_{\mathcal{M}} \arrow[r, Rightarrow, "(1.60)"] & \mathbb{D}_{\mathcal{A}}\otimes{}^{\vee\vee}a\triangleright\mathbb{N}^r_{\mathcal{M}} \arrow[r, Rightarrow, "{}^{\vee\vee}\sigma"] \arrow[d, Rightarrow, "(1.56)"] & {}^{\vee\vee}a\otimes\mathbb{D}_{\mathcal{A}}\triangleright\mathbb{N}^r_{\mathcal{M}} \arrow[ur, Rightarrow, "(1.63)"'] & \\
\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(a\triangleright -) \arrow[rr, Rightarrow, "(1.63)"'] & & \mathbb{D}_{\mathcal{A}}\triangleright\mathbb{N}^r_{\mathcal{M}}(a\triangleright -) \arrow[rr, Rightarrow, "(1.63)"'] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}(a\triangleright -)
\end{tikzcd} \tag{4.44}$$

where (1.56) and (1.49) are applied to the functor F_a. The pentagon at the top of this diagram commutes owing to naturality and the definition of ι_a. Commutativity of the pentagon on the left is the condition that (1.63) is an isomorphism of twisted module functors. Similarly, the pentagon on the right is secretly the diagram

$$\begin{tikzcd}
F_a^{\mathrm{rra}}\circ\mathbb{D}_{\mathcal{A}}\triangleright\mathbb{N}^r_{\mathcal{M}} \arrow[r, Rightarrow, "(1.63)"] \arrow[d, Rightarrow, "(1.56)\circ\sigma"'] & F_a^{\mathrm{rra}}\circ\mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[d, Rightarrow, "(1.49)"] \\
\mathbb{D}_{\mathcal{A}}\triangleright\mathbb{N}^r_{\mathcal{M}}\circ F_a \arrow[r, Rightarrow, "(1.63)"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}\circ F_a
\end{tikzcd} \tag{4.45}$$

which again commutes because (1.63) is an isomorphism of twisted bimodule functors. □

Proposition 4.14. *For any pivotal $\mathcal{A}$-module category $\mathcal{M}$ the braided equivalence Σ between $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\overline{\mathcal{A}_{\mathcal{M}}^{*}})$ defined in* (1.69) *is pivotal.*

Proof. From Lemma 4.13 it follows that the pivotal structure in $\mathcal{Z}(\overline{\mathcal{A}_{\mathcal{M}}^{*}})$ for the object $\Sigma(a,\sigma)$ is given by the composite

$$a \triangleright - \xRightarrow{\widetilde{p}\circ \mathrm{id}} \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}(a \triangleright -) \xRightarrow{(1.44)} a^{\vee\vee} \triangleright \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \xRightarrow{\mathrm{id}\circ \widetilde{p}^{-1}} (a^{\vee\vee} \triangleright -)\,. \tag{4.46}$$

The condition (4.2) on $\widetilde{p}$ means that $\Sigma(p_a)$ coincides with the morphism (4.46), and thus the assertion holds. □

Proposition 4.14 immediately implies

Theorem 4.15. *If two pivotal categories $\mathcal{A}$ and $\mathcal{B}$ are pivotal Morita equivalent, then their Drinfeld centers $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are equivalent as pivotal braided tensor categories.*

4.5 Sphericality for bimodule categories

A notion of sphericality for pivotal tensor categories that is defined through the Radford isomorphism (1.60) has been studied, under the assumption of unimodularity, in [DSS2]. In the semisimple case this notion is equivalent to trace-sphericality [DSS2, Prop. 3.5.4], i.e. to the property that right and left traces coincide. Let $\mathcal{C}$ be a unimodular finite tensor category. The monoidal functor given by conjugation with $\mathbb{D}_{\mathcal{C}}$ can be canonically identified with the identity functor of $\mathcal{C}$. Consider any trivialization $\mathbf{u}_{\mathcal{C}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_{\mathcal{C}}^{-1}$ of the distinguished invertible object, then

$$\mathrm{id}_{\mathcal{C}} \xRightarrow{\mathbf{u}^{*}\otimes \mathrm{id}\otimes \mathbf{u}} \mathbb{D}_{\mathcal{C}} \otimes - \otimes \mathbb{D}_{\mathcal{C}}^{-1} \tag{4.47}$$

does not depend of the choice of $\mathbf{u}_{\mathcal{C}}$ since $\mathbf{1}$ is simple and thus $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbb{D}_{\mathcal{C}})$ is one-dimensional. By precomposing the Radford isomorphism (1.60) with (4.47), we obtain a canonical monoidal isomorphism

$$\overline{r}_{\mathcal{C}} : \quad \mathrm{id}_{\mathcal{C}} \xRightarrow{\cong} (-)^{\vee\vee\vee\vee} \tag{4.48}$$

trivializing the fourth power of the right dual functor.

Definition 4.16. [DSS2, Def. 3.5.2] A unimodular pivotal tensor category $\mathcal{C}$ is called *spherical* iff the diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[rr, Rightarrow, "\overline{r}_{\mathcal{C}}"] \arrow[dr, Rightarrow, "p"'] & & (-)^{\vee\vee\vee\vee} \\
& (-)^{\vee\vee} \arrow[ur, Rightarrow, "p^{\vee\vee}"'] &
\end{tikzcd} \tag{4.49}$$

commutes, where p is the pivotal structure of $\mathcal{C}$.

It turns out that sphericality is a property on unimodular pivotal tensor categories that is invariant under pivotal Morita equivalence, as we show next.

Theorem 4.17. *Let $\mathcal{M}$ be a pivotal module category over a* (*unimodular*) *spherical tensor category $\mathcal{A}$. Then the following statements hold:*

(i) *The dual tensor category* $\mathcal{A}^*_{\mathcal{M}}$ *is unimodular.*

(ii) *The pivotal structure* (4.5) *induced on the dual tensor category* $\mathcal{A}^*_{\mathcal{M}}$ *is spherical.*

Proof. According to Proposition 3.15, the distinguished invertible object of $\mathcal{A}^*_{\mathcal{M}}$ is given by $\mathbb{D}^{-1}_{\mathcal{A}^*_{\mathcal{M}}} \cong \mathbb{D}^{-1}_{\mathcal{A}} \triangleright (\mathbb{S}_{\mathcal{M}})^2$. Any trivialization $\mathbf{u}_{\mathcal{A}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}^{-1}_{\mathcal{A}}$ of the distinguished invertible object of $\mathcal{A}$ furnishes an $\mathcal{A}$-module natural isomorphism $\mathbb{D}^{-1}_{\mathcal{A}^*_{\mathcal{M}}} \cong (\mathbb{S}_{\mathcal{M}})^2$ where $(\mathbb{S}_{\mathcal{M}})^2$ is endowed with the module structure

$$(\mathbb{S}_{\mathcal{M}})^2\,(a \triangleright m) \xrightarrow{(1.44)^2} a^{\vee\vee\vee\vee} \triangleright (\mathbb{S}_{\mathcal{M}})^2\,(m) \xleftarrow{\overline{r}_{\mathcal{A}}} a \triangleright (\mathbb{S}_{\mathcal{M}})^2\,(m)\ . \tag{4.50}$$

Now the square of the pivotal structure $\tilde{p}$ of $\mathcal{M}$ trivializes $(\mathbb{S}_{\mathcal{M}})^2$ with module structure

$$(\mathbb{S}_{\mathcal{M}})^2\,(a \triangleright m) \xrightarrow{(1.44)^2} a^{\vee\vee\vee\vee} \triangleright (\mathbb{S}_{\mathcal{M}})^2\,(m) \xleftarrow{p^{\vee\vee} \circ p} a \triangleright (\mathbb{S}_{\mathcal{M}})^2\,(m)\ , \tag{4.51}$$

by the defining condition (4.2) of $\tilde{p}$. Sphericality of $\mathcal{A}$ ensures that the two resulting module structures on $(\mathbb{S}_{\mathcal{M}})^2$ coincide. And thus, the composition

$$\mathbf{u}_{\mathcal{A}^*_{\mathcal{M}}} :\quad \mathrm{id}_{\mathcal{M}} \xRightarrow{\tilde{p} \circ \tilde{p}} (\mathbb{S}_{\mathcal{M}})^2 \xRightarrow{\mathbf{u}_{\mathcal{A}} \triangleright \mathrm{id}} \mathbb{D}^{-1}_{\mathcal{A}} \triangleright (\mathbb{S}_{\mathcal{M}})^2 \xRightarrow{(3.50)} \mathbb{D}^{-1}_{\mathcal{A}^*_{\mathcal{M}}} \tag{4.52}$$

provides an isomorphism of $\mathcal{A}$-module functors, which proves (i). In order to prove (ii) we need to check the commutativity of the triangle

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{A}^*_{\mathcal{M}}} \arrow[rr, Rightarrow, "\overline{r}_{\mathcal{A}^*_{\mathcal{M}}}"] \arrow[dr, Rightarrow, "q"'] & & (-)^{\mathrm{llla}} \\
& (-)^{\mathrm{lla}} \arrow[ur, Rightarrow, "q^{\mathrm{lla}}"'] &
\end{tikzcd} \tag{4.53}$$

where q is the pivotal structure of the dual tensor category given by the isomorphism (4.5), and $\overline{r}_{\mathcal{A}^*_{\mathcal{M}}}$ is the canonical isomorphism (4.48) of the dual tensor category $\mathcal{A}^*_{\mathcal{M}}$. Since $\overline{r}_{\mathcal{A}^*_{\mathcal{M}}}$ does not depend on the trivialization of $\mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}$ chosen, we can very well use (4.52). By inserting the definitions, it is not hard to see that this diagram reads, for $F \in \mathcal{A}^*_{\mathcal{M}}$,

$$\begin{tikzcd}
F \arrow[rr, Rightarrow, "\mathrm{id} \circ \tilde{p} \circ \tilde{p}"] \arrow[d, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] & & F \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[rr, Rightarrow, "(1.49)"] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ F^{\mathrm{lla}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[rr, Rightarrow, "(1.49)"] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ F^{\mathrm{llla}} \arrow[rr, Rightarrow, "(\tilde{p} \circ \tilde{p})^{-1} \circ \mathrm{id}"] & & F^{\mathrm{llla}} \\
F \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[urr, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] \arrow[drr, Rightarrow, "(1.49)"'] & & & & & & & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ F^{\mathrm{llla}} \arrow[ull, Rightarrow, "\tilde{p} \circ \mathrm{id}"] \arrow[u, Rightarrow, "\tilde{p}^{-1} \circ \mathrm{id}"'] \\
& & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ F^{\mathrm{lla}} \arrow[uurr, Rightarrow, "\mathrm{id} \circ \tilde{p}"] \arrow[rr, Rightarrow, "\tilde{p}^{-1} \circ \mathrm{id}"'] & & F^{\mathrm{lla}} \arrow[rr, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] & & F^{\mathrm{lla}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[uull, Rightarrow, "\tilde{p} \circ \mathrm{id}"'] \arrow[urr, Rightarrow, "(1.49)"'] & &
\end{tikzcd} \tag{4.54}$$

This diagram indeed commutes: the upper-left and upper-right triangles commute trivially; the remaining squares commute due to functoriality of functor composition. ☐

Corollary 4.18. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two pivotal Morita equivalent pivotal tensor categories.* $\mathcal{A}$ *is (unimodular) spherical if and only if* $\mathcal{B}$ *is (unimodular) spherical.*

We will now explore sphericality for module categories in a similar vein. Let ${}_\mathcal{C}\mathcal{L}_\mathcal{D}$ be an exact bimodule category over unimodular pivotal tensor categories. Associated to $\mathcal{L}$, there is a Radford isomorphism (3.48) of twisted bimodule functors

$$\mathcal{R}_\mathcal{L} : \quad \mathbb{D}_\mathcal{C}^{-1} \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(-) \xRightarrow{\cong} \mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L}(-) \triangleleft \mathbb{D}_\mathcal{D}^{-1} \quad : \quad \mathcal{L} \longrightarrow {}_{{}^{\vee\vee}(-)}\mathcal{L}_{(-)^{\vee\vee}}$$

involving the distinguished invertible objects of both $\mathcal{C}$ and $\mathcal{D}$. The pivotal structures of $\mathcal{C}$ and $\mathcal{D}$ turn the functors $\mathbb{D}_\mathcal{C}^{-1} \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}$ and $\mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L}(-) \triangleleft \mathbb{D}_\mathcal{D}^{-1}$ into bimodule functors and $\mathcal{R}_\mathcal{L}$ becomes an isomorphism of bimodule functors. Now we pick trivializations

$$\mathbf{u}_\mathcal{C} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_\mathcal{C}^{-1} \quad \text{and} \quad \mathbf{u}_\mathcal{D} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_\mathcal{D}^{-1} \tag{4.55}$$

which together with $\mathcal{R}_\mathcal{L}$ furnish an isomorphism

$$\overline{\mathcal{R}}_\mathcal{L} : \quad \mathbb{S}^\mathcal{C}_\mathcal{L} \xRightarrow{\cong} \mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L} \tag{4.56}$$

of bimodule functors. There are some subtleties in the definition of (4.56). First of all, unlike $\overline{r}_\mathcal{C}$, the isomorphism $\overline{\mathcal{R}}_\mathcal{L}$ is not canonical and depends on the choice of $\mathbf{u}_\mathcal{C}$ and $\mathbf{u}_\mathcal{D}$. Moreover, this is an isomorphism of bimodule functors by considering on $\mathbb{S}^\mathcal{C}_\mathcal{L}$ the left module structure

$$\mathbb{S}^\mathcal{C}_\mathcal{L}(c \triangleright x) \xrightarrow{(1.44)} c^{\vee\vee} \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(x) \xleftarrow{\overline{r}_\mathcal{C}} {}^{\vee\vee}c \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(x) \xrightarrow{p_{{}^{\vee\vee}c}} c \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(x) \ , \tag{4.57}$$

and analogous module structure morphisms on $\mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L}$. Now a pivotal structure on ${}_\mathcal{C}\mathcal{L}$ is a trivialization of $\mathbb{S}^\mathcal{C}_\mathcal{L}$ with module functor structure given by

$$\mathbb{S}^\mathcal{C}_\mathcal{L}(c \triangleright x) \xrightarrow{(1.44)} c^{\vee\vee} \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(x) \xleftarrow{p_c} c \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L}(x) \ . \tag{4.58}$$

In the case $\mathcal{C}$ is spherical, the module structures (4.57) and (4.58) on $\mathbb{S}^\mathcal{C}_\mathcal{L}$ coincide by definition. A similar consideration holds for $\mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L}$ under sphericality of $\mathcal{D}$. We are now ready define sphericality for bimodule categories by means of the Radford isomorphism $\overline{\mathcal{R}}_\mathcal{L}$:

Definition 4.19 (Spherical bimodule category)**.**
Let $\mathcal{C}$ and $\mathcal{D}$ be (unimodular) spherical tensor categories and $\mathbf{u}_\mathcal{C} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_\mathcal{C}^{-1}$ and $\mathbf{u}_\mathcal{D} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_\mathcal{D}^{-1}$ trivializations of the corresponding distinguished invertible objects.
A pivotal bimodule category ${}_\mathcal{C}\mathcal{L}_\mathcal{D}$ is called $(\mathbf{u}_\mathcal{C}, \mathbf{u}_\mathcal{D})$-*spherical* or simply *spherical* iff the diagram

$$\begin{tikzcd}
\mathbb{S}^\mathcal{C}_\mathcal{L} \arrow[rr, Rightarrow, "\overline{\mathcal{R}}_\mathcal{L}"] & & \mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L} \\
& \mathrm{id}_\mathcal{L} \arrow[ul, Rightarrow, "\tilde{p}"] \arrow[ur, Rightarrow, "\tilde{q}"'] &
\end{tikzcd} \tag{4.59}$$

commutes, where $\overline{\mathcal{R}}_\mathcal{L}$ is the composition

$$\mathbb{S}^\mathcal{C}_\mathcal{L} \xRightarrow{\mathbf{u}_\mathcal{C} \triangleright \mathrm{id}} \mathbb{D}_\mathcal{C}^{-1} \triangleright \mathbb{S}^\mathcal{C}_\mathcal{L} \xRightarrow{\mathcal{R}_\mathcal{L}} \mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L}(-) \triangleleft \mathbb{D}_\mathcal{D}^{-1} \xRightarrow{\mathrm{id} \triangleleft \mathbf{u}_\mathcal{D}^{-1}} \mathbb{S}^{\overline{\mathcal{D}}}_\mathcal{L} \tag{4.60}$$

and $\tilde{p}$ and $\tilde{q}$ denote the pivotal structures of ${}_\mathcal{C}\mathcal{L}$ and $\mathcal{L}_\mathcal{D}$ respectively.

Remark 4.20. The notion of sphericality for bimodule categories given in Definition 4.19 is relative to the choice of isomorphisms $\mathbf{u}_{\mathcal{C}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_{\mathcal{C}}^{-1}$ and $\mathbf{u}_{\mathcal{D}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_{\mathcal{D}}^{-1}$. As we will see in Definition 6.12, there is a special choice of such trivializations in the semisimple setting.

We define sphericality of left module categories in terms of its associated invertible bimodule. Under the hypothesis of Theorem 4.17, a choice of isomorphisms

$$\mathbf{u}_{\mathcal{A}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_{\mathcal{A}}^{-1} \qquad \text{and} \qquad \mathbf{u}_{\mathcal{A}^*_{\mathcal{M}}} : \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}^{-1} \tag{4.61}$$

turns (3.50) into an $\mathcal{A}$-module isomorphism

$$\overline{r}_{\mathcal{M}} : \quad \mathrm{id}_{\mathcal{M}} \xRightarrow{\mathbf{u}_{\mathcal{A}^*_{\mathcal{M}}}} \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}^{-1} \xRightarrow{(3.50)} \mathbb{D}_{\mathcal{A}}^{-1} \triangleright \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \xRightarrow{\mathbf{u}_{\mathcal{A}}^{-1} \triangleright \mathrm{id}} \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \,, \tag{4.62}$$

where $\mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}}$ is the functor with module structure coming from the isomorphism $\overline{r}_{\mathcal{A}} = p^{\vee\vee} \circ p$.

Definition 4.21 (Spherical module category)**.**
Let $\mathcal{M}$ be a pivotal module category over a (unimodular) spherical tensor category $\mathcal{A}$ and $\mathbf{u}_{\mathcal{A}} : \mathbf{1} \xrightarrow{\cong} \mathbb{D}_{\mathcal{A}}^{-1}$ and $\mathbf{u}_{\mathcal{A}^*_{\mathcal{M}}} : \mathrm{id}_{\mathcal{M}} \overset{\cong}{\Longrightarrow} \mathbb{D}_{\mathcal{A}^*_{\mathcal{M}}}^{-1}$ be trivializations of the distinguished invertible objects. The pivotal module ${}_{\mathcal{A}}\mathcal{M}$ is called *spherical* iff the pivotal $(\mathcal{A}, \overline{\mathcal{A}^*_{\mathcal{M}}})$-bimodule category $\mathcal{M}$ is spherical, i.e. the diagram

$$\begin{tikzcd}
\mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \arrow[rr, Rightarrow, "\overline{\mathcal{R}}_{\mathcal{M}}"] & & \overline{\mathbb{S}}_{\mathcal{M}}^{\mathcal{A}} \arrow[dl, Rightarrow, "\tilde{p}"] & \text{or equivalently} & \mathrm{id}_{\mathcal{M}} \arrow[rr, Rightarrow, "\overline{r}_{\mathcal{M}}"] \arrow[dr, Rightarrow, "\tilde{p}"'] & & \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \circ \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \\
 & \mathrm{id}_{\mathcal{M}} \arrow[ul, Rightarrow, "\tilde{p}"] & & & & \mathbb{S}_{\mathcal{M}}^{\mathcal{A}} \arrow[ur, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] &
\end{tikzcd} \tag{4.63}$$

commutes, where $\widetilde{p}$ is the pivotal structure of $\mathcal{M}$.

Remark 4.22. In [Ya] the choice of trivialization $\mathbf{u}_{\mathcal{A}^*_{\mathcal{M}}}$ is termed *unimodular structure* on $\mathcal{M}$.

Proposition 4.23. *Let $\mathcal{M}$ and $\mathcal{N}$ be spherical $\mathcal{A}$-module categories, then the $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ *is spherical.*

Proof. According to Proposition 4.3 (iii) $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ inherits a pivotal structure as a $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category. Sphericality of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is proved by checking the commutativity of the following diagram

$$\begin{tikzcd}
H \arrow[rr, Rightarrow, "\overline{r}_{\,\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}"] \arrow[dr, Rightarrow, "(4.12)"'] & & \left(\mathbb{S}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}\right)^2 (H) \\
 & \mathbb{S}_{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}^{\overline{\mathcal{A}^*_{\mathcal{M}}}}(H) \arrow[ur, Rightarrow, "(4.12)^{\mathrm{lla}}"'] &
\end{tikzcd} \tag{4.64}$$

for every module functor $H : \mathcal{M} \to \mathcal{N}$. Considering (3.69), this diagram turns explicitly into

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\overline{r}_{\mathcal{N}} \circ \mathrm{id}"] \arrow[d, Rightarrow, "\hat{p} \circ \mathrm{id}"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \arrow[r, Rightarrow, "\mathrm{id} \circ \overline{r}^{-1}_{\mathcal{M}}"] \arrow[dr, Rightarrow, "\mathrm{id} \circ \tilde{q}"] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \\
\mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \arrow[r, Rightarrow, "\mathrm{id} \circ \tilde{q}"'] \arrow[ur, Rightarrow, "\hat{p} \circ \mathrm{id}"] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \arrow[r, Rightarrow, "\hat{p} \circ \mathrm{id}"'] & \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{N}} \circ H \circ \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \arrow[u, Rightarrow, "\mathrm{id} \circ \tilde{q}"']
\end{tikzcd} \tag{4.65}$$

where $\widehat{p}\colon \mathrm{id}_{\mathcal{N}} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{N}}$ is the pivotal structure of ${}_{\mathcal{A}}\mathcal{N}$ and $\widetilde{q}\colon \mathrm{id}_{\mathcal{M}} \xRightarrow{\cong} \overline{\mathbb{S}}^{\mathcal{A}}_{\mathcal{M}} \cong \mathbb{S}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathcal{M}}$ is the pivotal structure coming from (4.15). The commutativity of the left triangle is precisely the sphericality condition of the pivotal module ${}_{\mathcal{A}}\mathcal{N}$ and the sphericality of ${}_{\mathcal{A}}\mathcal{M}$ implies the commutativity of the right triangle. The diagram in the middle commutes due to functoriality of the composition functor. □

Remark 4.24. Given a (unimodular) spherical tensor category $\mathcal{A}$, and a spherical module category ${}_{\mathcal{A}}\mathcal{M}$, Proposition 4.23 implies that the $(\overline{\mathcal{A}^*_{\mathcal{M}}}, \mathcal{A})$-bimodule category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is spherical. The dual tensor category $\mathcal{A}^*_{\mathcal{M}}$ is also spherical by Theorem 4.17, and so is every Hom-category of the pivotal bicategory $\mathbb{M}$ associated to ${}_{\mathcal{A}}\mathcal{M}$.

The sphericality condition on a pivotal module category can be then interpreted in the bicategorical setting as the statement that the pivotal structure of $\mathbb{M}$ squares to the Radford pseudo-natural equivalence (3.54). Beyond unimodularity, the non-triviality of the distinguished invertible objects in (3.54) suggests that such a bicategorical interpretation could be studied in the framework of quasi-pivotal structures, as considered in Remark 4.5.

Definition 4.25. A pivotal Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \odot, \boxdot)$ is said to be *spherical* iff the Radford pseudo-equivalence (3.54) of its associated bicategory $\mathbb{M}$ is equivalent to the square of the pivotal structure of $\mathbb{M}$.

Chapter 5

On G-equivariant Morita theory

5.1 Equivariant setting background

This section revisits some notions and G-structures on categories for a finite group G. Most of the content seem to be well-known to experts, but we would like to highlight Remark 5.3 and Theorem 5.8.

Group actions on tensor categories and equivariantization

Let G be a finite group and denote by $\underline{G}$ the strict monoidal category whose objects are elements in G, all morphisms are identities and the tensor product is given by the group law. Given a monoidal category $\mathcal{C}$ denote by $\mathrm{Aut}_{\otimes}(\mathcal{C})$ the strict monoidal category of tensor autoequivalences of $\mathcal{C}$ and morphisms monoidal natural isomorphisms. A monoidal *G-action* on $\mathcal{C}$ is the datum of a monoidal functor $T : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$. We refer to the pair $(\mathcal{C}, T)$ as a *monoidal G-category.*

Let $(\mathcal{C}, T)$ be a monoidal G-category. A *G-equivariant object* is an object $X \in \mathcal{C}$ together with a choice of isomorphisms $\{u_g : T_g(X) \xrightarrow{\sim} X\}_{g \in G}$, fulfilling a compatibility condition with the tensor structure of the G-action functor T [GNN, Def, 2.6]. The category of equivariant objects is denoted by $\mathcal{C}^G$ and is called the *equivariantization of* $(\mathcal{C}, T)$. The equivariantization $\mathcal{C}^G$ inherits a monoidal structure from $\mathcal{C}$.

Graded tensor categories

Let G be a finite group and $\mathcal{A}$ a tensor category. A *G-grading* on $\mathcal{A}$ consists of a decomposition

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

into a direct sum of full abelian subcategories, such that for $g, h \in G$ the tensor product restricts to $\otimes : \mathcal{A}_g \times \mathcal{A}_h \to \mathcal{A}_{gh}$. If $\mathcal{A}_g \neq 0$ for all $g \in G$ the G-grading is called *faithful.* In this case we also say that $\mathcal{A}$ is a *G-extension* of the trivial component $\mathcal{A}_e$ and it holds that $\mathrm{FPdim}(\mathcal{A}) = |G| \, \mathrm{FPdim}(\mathcal{A}_e)$. We will only consider faithful gradings.

Braided G-crossed tensor categories and central G-extensions

The following notion is an analog to the notion of braided tensor category in the equivariant setting and was introduced in [Tu].

Definition 5.1. (Braided G-crossed tensor category)
A *braided G-crossed tensor category* is a tensor category $\mathcal{C}$ equipped with,

- A G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$.
- A compatible monoidal G-action $g \mapsto T_g$, i.e., $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ for all $g, h \in G$.
- A *G-braiding* which consists of a natural collection of isomorphisms,

$$\omega_{c,x} : \quad x \otimes c \xrightarrow{\cong} T_g(c) \otimes x, \qquad x \in \mathcal{C}_g,\ c \in \mathcal{C}$$

which satisfy certain compatibility conditions with the tensor structure $\eta_{g,h} : T_g \circ T_h \overset{\cong}{\Longrightarrow} T_{gh}$ of the G-action T and the tensor structure of T_g. A complete definition can be found in [GNN, Definition 2.10].

The equivariantization $\mathcal{C}^G$ of any braided G-crossed tensor category $\mathcal{C}$ inherits the structure of a braided tensor category, as explained in [Mü]: Consider $(x, \{u_g\}_{g \in G})$ and $(c, \{v_g\}_{g \in G})$ objects in $\mathcal{C}^G$ with $x \in \mathcal{C}_h$, then the braiding is given by extending additively the following composition

$$\widetilde{\omega}_{x,c} : \quad x \otimes c \xrightarrow{\omega_{x,c}} T_h(c) \otimes x \xrightarrow{v_h \otimes \mathrm{id}_x} c \otimes x \tag{5.1}$$

where $\omega_{x,c}$ is the G-braiding of $\mathcal{C}$. As an additional structure $\mathcal{C}^G$ contains a Tannakian subcategory $\mathrm{Rep}(G)$, i.e., there is a braided fully faithful functor

$$\mathrm{Rep}(G) = \mathrm{Vec}^G \longrightarrow \mathcal{C}^G \tag{5.2}$$

whose essential image is given by the possible equivariant structures on objects which are multiples of the monoidal unit of $\mathcal{C}$.

Definition 5.2. Let $\mathcal{D}$ be a braided tensor category. A *central G-extension of $\mathcal{D}$* is a G-extension $\mathcal{C}$ of $\mathcal{D}$ together with a braided tensor functor $\iota : \mathcal{D} \to \mathcal{Z}(\mathcal{C})$ such that its composition with the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ coincides with the inclusion $\mathcal{D} \to \mathcal{C}$. We also just say that $\mathcal{C}$ is a *central G-extension.*

Notice that given a central G-extension $\mathcal{C}$ of $\mathcal{D}$, the datum of the braided tensor functor $\iota : \mathcal{D} \to \mathcal{Z}(\mathcal{C})$ is the same as a *relative braiding* on $\mathcal{C}$ with respect to $\mathcal{D}$, i.e., a natural isomorphism

$$\gamma_{d,c} : d \otimes c \xrightarrow{\cong} c \otimes d, \qquad \text{for} \quad d \in \mathcal{D},\ c \in \mathcal{C}$$

fulfilling the hexagon axioms of a braiding.

Remark 5.3. To any braided G-crossed tensor category $\mathcal{C}$ a central G-extension of $\mathcal{C}_e$ is associated up to equivalence, and vice versa. This is described in [DN, Proposition 8.11] as a 2-equivalence between the 2-groupoids of central G-extensions and braided G-crossed tensor categories. We will therefore use both notions indistinctly.

De-equivariantization

The inverse construction to the equivariantization of a braided G-crossed tensor category is known as de-equivariantization, as formulated in detail in [DGNO, Section 4.4].

Let $\mathcal{D}$ be a braided tensor category together with the additional datum of a braided fully faithful functor $\mathrm{Rep}(G) \to \mathcal{D}$. The group G acts by left translations on the set $\mathrm{Fun}(G, \Bbbk)$ of functions, thereby turning it into an object in $\mathrm{Rep}(G)$. Moreover, this object has a canonical structure of a commutative special Frobenius algebra in $\mathrm{Rep}(G)$. This algebra is called *the regular algebra of functions*; denote by Υ_G the image in $\mathcal{D}$ of $\mathrm{Fun}(G, \Bbbk)$ under the functor $\mathrm{Rep}(G) \to \mathcal{D}$. The *de-equivariantization of* $\mathcal{D}$ is a braided G-crossed tensor category $\mathcal{D}_G$, whose underlying category is the category of modules ${}_{\Upsilon_G}\mathrm{Mod}(\mathcal{D})$ with tensor product $\otimes_{\Upsilon_G}$.

The processes of equivariantization and de-equivariantization are mutual inverses providing a 2-equivalence between the 2-categories of braided G-crossed fusion categories and of braided fusion categories containing $\mathrm{Rep}(G)$. As it is mentioned at the start of [DGNO, Section 4.4] these constructions and results hold in the non-semisimple case as well, leading in particular to the following statements (for a proof without the semisimplicity assumption we also refer to [Ja, Section 3.4]):

(i) For every braided G-crossed tensor category $\mathcal{C}$, there is an equivalence $\mathcal{C} \xrightarrow{\simeq} (\mathcal{C}^G)_G$ of braided G-crossed tensor categories.

(ii) Given a braided tensor category $\mathcal{D}$ together with a braided fully faithful functor $\Gamma : \mathrm{Rep}(G) \to \mathcal{D}$, there exists a braided equivalence $\mathcal{D} \xrightarrow{\simeq} (\mathcal{D}_G)^G$, which commutes with Γ and the braided functor (5.2) coming from the equivariantization process.

Graded module categories

Definition 5.4. Let $\mathcal{A}$ be a G-graded tensor category.

(i) A *G-graded module category over $\mathcal{A}$* is an $\mathcal{A}$-module category with a decomposition

$$\mathcal{M} = \bigoplus_{g \in G} \mathcal{M}_g \tag{5.3}$$

into a direct sum of full abelian subcategories, with $\mathcal{M}_g \neq 0$ for every $g \in G$, and such that for $g, h \in G$ the $\mathcal{A}$-action restricts to $\triangleright : \mathcal{A}_g \times \mathcal{M}_h \to \mathcal{M}_{gh}$.

(ii) A G-graded $\mathcal{A}$-module category $\mathcal{M}$ is called *indecomposable* if it is not equivalent to a non-trivial direct sum of G-graded module categories, and is called *exact* if it is exact as a $\mathcal{A}$-module category.

The following propositions regarding graded module categories seem to be well-known but are difficult to find in the literature; hence, they are here included.

Proposition 5.5. *Let $\mathcal{A}$ be a G-graded finite tensor category and $\mathcal{M}$ a G-graded $\mathcal{A}$-module category. For $m \in \mathcal{M}_g$ and $n \in \mathcal{M}_h$, then $\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, n) \in \mathcal{A}_{hg^{-1}}$.*

Proof. It follows from the definition of the internal Hom considering that $\mathcal{A}$ is graded. □

Proposition 5.6. *Let $\mathcal{A}$ be a G-graded tensor category and let A be an algebra in the trivial component $\mathcal{A}_e$.*

(i) $\mathrm{Mod}_A(\mathcal{A})$ *is a G-graded $\mathcal{A}$-module category with decomposition*

$$\mathrm{Mod}_A(\mathcal{A}) = \bigoplus_{g\in G} \mathrm{Mod}_A(\mathcal{A}_g)$$

where $\mathrm{Mod}_A(\mathcal{A}_g)$ is the subcategory of A-modules with underlying object in $\mathcal{A}_g$.

(ii) *The category ${}_A\mathrm{Bimod}_A(\mathcal{A})$ is G-graded with decomposition*

$$_A\mathrm{Bimod}_A(\mathcal{A}) = \bigoplus_{g\in G} {}_A\mathrm{Bimod}_A(\mathcal{A}_g)$$

where ${}_A\mathrm{Bimod}_A(\mathcal{A}_g)$ is the subcategory of bimodules with underlying object in $\mathcal{A}_g$, and this decomposition is compatible with $\otimes_A$.

(iii) *Let $\mathcal{C}$ be a central G-extension and A be an exact commutative algebra in the trivial component $\mathcal{C}_e$. Then $\mathrm{Mod}_A(\mathcal{C})$ has the structure of a G-graded tensor category.*

Proof.

(i) Consider a collection of right A-modules $\{(M_g, r_g : M_g \otimes A \to M_g)\}_{g\in G}$ with $M_g \in \mathcal{A}_g$, and let $M := \oplus_{g\in G} M_g$, then the morphism

$$M \otimes A = \bigoplus_{g\in G} M_g \otimes A \xrightarrow{\oplus_{g\in G} r_g} \bigoplus_{g\in G} M_g = M$$

provides a right A-module structure on M.

Conversely given a module $(M, r : M \otimes A \to M) \in \mathrm{Mod}_A(\mathcal{A})$, since $\mathcal{A}$ is graded there is a decomposition of the underlying object $M = \bigoplus_{g\in G} M_g$ in $\mathcal{A}$. For each $g \in G$ one can check that the object M_g acquires a right A-module structure via the morphism

$$r_g : M_g \otimes A \xrightarrow{\iota_g \otimes \mathrm{id}_A} M \otimes A \xrightarrow{r} M \xrightarrow{p_g} M_g$$

this determines a decomposition of $(M, r) = \bigoplus_{g\in G} (M_g, r_g)$ in $\mathrm{Mod}_A(\mathcal{A})$.

The decomposition above is compatible with the module category structure of $\mathrm{Mod}_A(\mathcal{A})$: Given an object $X \in \mathcal{A}_g$ and a module $(M, r) \in \mathrm{Mod}_A(\mathcal{A}_h)$, then $X \otimes M \in \mathcal{A}_{gh}$ meaning that $(X \otimes M, \mathrm{id}_X \otimes r) \in \mathrm{Mod}_A(\mathcal{A}_{gh})$.

(ii) The decomposition of bimodules follows in complete analogy to (i). It remains to verify that this G-decomposition is compatible with $\otimes_A$:
Let $(M, r_M, l_M) \in {}_A\mathrm{Bimod}_A(\mathcal{A}_g)$ and $(N, r_N, l_N) \in {}_A\mathrm{Bimod}_A(\mathcal{A}_h)$, and consider the coequalizer diagram

$$\begin{tikzcd} M \otimes A \otimes N \arrow[r, shift left, "r_M \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes l_N"'] & M \otimes N \arrow[r] & M \otimes_A N . \end{tikzcd}$$

If $M \otimes_A N$ were not of degree gh, then the coequalizer morphism would be a zero morphism, but since it is epic, then $M \otimes_A N$ would be a zero object.

(iii) Given a module $(M, r) \in \mathrm{Mod}_A(\mathcal{C})$ the following composition defines a compatible left A-action on M

$$A \otimes M \xrightarrow{\omega_{A,M}} M \otimes A \xrightarrow{r} M$$

where $\omega_{A,M}$ is the relative braiding of $\mathcal{C}$. This construction induces a fully faithful functor $\mathrm{Mod}_A(\mathcal{C}) \to {}_A\mathrm{Bimod}_A(\mathcal{C})$ and $\mathrm{Mod}_A(\mathcal{C})$ is closed under the tensor product $\otimes_A$ in ${}_A\mathrm{Bimod}_A(\mathcal{C})$ providing the tensor structure on $\mathrm{Mod}_A(\mathcal{C})$.

□

The notions of induction and restriction of modules categories play an important role in the context of graded categories by relating them with their trivial component. Let $\mathcal{A}$ be a G-graded finite tensor category and $\mathcal{N} \simeq \mathrm{Mod}_A(\mathcal{A}_e)$ an $\mathcal{A}_e$-module category, where A is an algebra in $\mathcal{A}_e$. The *induced module category* is defined as the $\mathcal{A}$-module category $\mathrm{Ind}^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{N}) := \mathrm{Mod}_A(\mathcal{A})$. Given a G-graded $\mathcal{A}$-module category $\mathcal{M}$, the *restricted* $\mathcal{A}_e$*-module category* is denoted by $\mathrm{Res}^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{M}) := \mathcal{M}_e$.

Lemma 5.7. [MM, Lemma 4.5] *Let $\mathcal{A}$ be a G-graded tensor category.*

(i) *$\mathcal{M}$ is an exact $\mathcal{A}$-module category if and only if $Res^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{M})$ is an exact $\mathcal{A}_e$-module category.*

(ii) *$\mathcal{N}$ is an exact $\mathcal{A}_e$-module category if and only if $Ind^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{N})$ is an exact $\mathcal{A}$-module category.*

There is a correspondence between exact graded module categories over a graded tensor category and exact module categories over its trivial component.

Theorem 5.8. [Ga2, Theorem 3.3]
Let $\mathcal{A}$ be a G-graded finite tensor category. Induction and restriction of module categories determine a 2-equivalence between exact G-graded $\mathcal{A}$-module categories and exact $\mathcal{A}_e$-module categories.

Proof. The statement in [Ga2, Theorem 3.3] is for the fusion case, but the proof does not require semisimplicity, and in view of Lemma 5.7 the result holds restricting to the class of exact module categories. □

The following corollary is an immediate consequence from Theorem 5.8.

Corollary 5.9. *For any exact G-graded $\mathcal{A}$-module category $\mathcal{M}$, there exists an algebra A in the trivial component $\mathcal{A}_e$ such that $\mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{A})$ as G-graded $\mathcal{A}$-module categories.*

Graded category of module functors

Definition 5.10. Let $\mathcal{A}$ be a G-graded tensor category and let $\mathcal{M}$ and $\mathcal{N}$ be G-graded $\mathcal{A}$-module categories. An $\mathcal{A}$-module functor $H\colon \mathcal{M} \to \mathcal{N}$ satisfying $H(\mathcal{M}_x) \subseteq \mathcal{N}_{xg}$ for every $x \in G$ is called *homogeneous of degree* $g \in G$. A *grading preserving* module functor is a homogeneous module functor of trivial degree.

The full subcategory of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ whose objects are homogeneous module functors of degree $g \in G$ is denoted by $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_g$.

Proposition 5.11. [Ja, Prop. 4.9] *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ be G-graded $\mathcal{A}$-module categories, then the composition of module functors restricts to*

$$\circ \ : \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})_g \times \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_h \to \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{L})_{hg} \tag{5.4}$$

and the category of module functors decomposes as

$$\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = \bigoplus_{g\in G} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_g \,.$$

In particular, the dual category $\mathcal{A}^*_{\mathcal{M}}$ is G^{opp}-graded and $\mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{M})$ is a G^{opp}-graded module category over $\mathcal{A}^*_{\mathcal{M}}$. Similarly, the opposite dual category $\overline{\mathcal{A}^*_{\mathcal{M}}}$ is a G-graded tensor category and $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is a G-graded module category over $\overline{\mathcal{A}^*_{\mathcal{M}}}$.

Proof. It is straightforward to check the compatibility of composition with the grading as expressed in (5.4).

Given a family of functors $\{(H_g,\phi_g)\}_{g\in G}$ with $(H_g,\phi_g)\in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_g$, their sum is defined as the functor

$$\bigoplus_{g\in G} H_g : \mathcal{M} \longrightarrow \mathcal{N}, \quad m \longmapsto \bigoplus_{g\in G} H_g(m) \tag{5.5}$$

and the isomorphisms defined for $a\in\mathcal{A}$ and $m\in\mathcal{N}$ by

$$\bigoplus_{g\in G} H_g(a\triangleright m) \xrightarrow{\oplus_{g\in G}(\phi_g)_{a,m}} \bigoplus_{g\in G} a\triangleright H_g(m) = a\triangleright \bigoplus_{g\in G} H_g(m) \tag{5.6}$$

provide a $\mathcal{A}$-module functor structure on $\bigoplus\limits_{g\in G} H_g$.

Conversely for $(H,\phi)\in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ and $g,y\in G$ consider the composition of functors

$$\mathcal{M}_y \xrightarrow{\iota_y} \mathcal{M} \xrightarrow{H} \mathcal{N} \xrightarrow{p_{yg}} \mathcal{N}_{yg}$$

and define a homogeneous functor $H_g := \bigoplus\limits_{y\in G} p_{yg}\circ H\circ \iota_y$ with $\mathcal{A}$-module structure $(\phi_g)_{a,m}$ given for homogeneous objects $a\in\mathcal{A}_x$ and $m\in\mathcal{M}_y$ by

$$H_g(a\triangleright m) = H(a\triangleright m)_{xyg} \xrightarrow{p_{xyg}(\phi_{a,m})} (a\triangleright H(m))_{xyg} = a\triangleright\, (H(m))_{yg} = a\triangleright H_g(m)$$

then $(H_g,\phi_g)\in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_g$. Moreover, their sum correspond to a decomposition of H since for $m\in\mathcal{M}$

$$\bigoplus_{g\in G} H_g(m) = \bigoplus_{y,g\in G} H(m_y)_{yg} = \bigoplus_{y\in G} H(m_y) = H(m)$$

where the last line follows since H preserves finite sums. □

A direct computation shows that in the equivariant setting the Eilenberg-Watts equivalence is compatible with the grading:

Proposition 5.12. [Ja, Prop. 4.10] *Given algebras A and B in the trivial component of a G-graded tensor category $\mathcal{A}$, the Eilenberg-Watts equivalence*

$${}_A\mathrm{Bimod}_B(\mathcal{A}) \xrightarrow{\sim} \mathrm{Rex}_{\mathcal{A}}(\mathrm{Mod}_A(\mathcal{A}),\mathrm{Mod}_B(\mathcal{A}))$$

given by equation (1.9) *is grading preserving.*

5.2 Graded Morita equivalence

Definition 5.13. [Ja, Def. 4.14] Two G-graded tensor categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *graded Morita equivalent* if there exists a G-graded $\mathcal{A}$-module category $\mathcal{M}$ together with a G-graded tensor equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$.

Remark 5.14. For a G-graded finite tensor category $\mathcal{A}$, similarly to the non-graded case, we have the following remarks:

(i) Consider the regular graded module category $\mathcal{A}$, then the tensor equivalence (1.70)

$$\mathcal{A} \xrightarrow{\simeq} \overline{\mathcal{A}^*_{\mathcal{A}}}, \quad a \mapsto - \otimes a\,, \tag{5.7}$$

is grading preserving.

(ii) Let $\mathcal{M}$ be an exact G-graded $\mathcal{A}$-module category, then $\mathcal{M}$ is naturally a G^{opp}-graded module category over $\mathcal{A}^*_{\mathcal{M}}$, and the double dual $(\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}$ is a G-graded tensor category.

(iii) Notice that the canonical tensor equivalence (1.71) is G-graded

$$\mathrm{can}: \mathcal{A} \xrightarrow{\simeq} (\mathcal{A}^*_{\mathcal{M}})^*_{\mathcal{M}}\,, \quad a \mapsto a \triangleright - \tag{5.8}$$

taking into consideration (ii).

(iv) From Corollary 5.9 and Proposition 5.12, the notion of graded Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$ can be described by the existence of an exact algebra A in $\mathcal{A}_e$ together with a G-graded tensor equivalence $\mathcal{B} \simeq {}_A\mathrm{Bimod}_A(\mathcal{A})$.

Proposition 5.15. *The notion of graded Morita equivalence is an equivalence relation on G-graded finite tensor categories.*

Proof. Remark 5.14 (i) exhibits reflexivity as well as (iii) implies symmetry. Now transitivity follows in the same manner as in the non-graded case shown in [EGNO, Proposition 7.12.18], if one takes into consideration that the algebras involved are always in the trivial component of the corresponding graded tensor category. □

From Definition 5.13 it immediately follows that two graded Morita equivalent graded tensor categories are also Morita equivalent just as tensor categories. As shown next, their trivial components are Morita equivalent as well.

Proposition 5.16. *If two G-graded finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent, then their trivial components $\mathcal{A}_e$ and $\mathcal{B}_e$ are Morita equivalent.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent there is an exact G-graded $\mathcal{A}$-module category $\mathcal{M} = \mathrm{Mod}_A(\mathcal{A})$ and we have that

$$(\mathcal{A}^*_{\mathcal{M}})_e = \mathrm{End}_{\mathcal{A}}(\mathcal{M})_e \simeq \overline{{}_A\mathrm{Bimod}_A(\mathcal{A}_e)} \simeq \mathrm{End}_{\mathcal{A}_e}(\mathcal{M}_e) = (\mathcal{A}_e)^*_{\mathcal{M}_e} \tag{5.9}$$

where the first equivalence takes into account that the Eilenberg-Watts equivalence preserves the grading as shown in Proposition 5.12. Consequently, a graded tensor equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$ induces a tensor equivalence $\mathcal{B}_e \simeq \overline{(\mathcal{A}_e)^*_{\mathcal{M}_e}}$. □

Remark 5.17. The converse of Proposition 5.16 does not hold in general. Morita equivalent tensor categories can have G-extensions which are not graded Morita equivalent:
Consider for instance the finite cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ with p prime and the tensor category $\mathcal{A}_e = \mathcal{B}_e = \mathrm{Vec}$. The G-extensions $\mathcal{A} = \mathrm{Vec}_G$ and $\mathcal{B} = \mathrm{Vec}_G^{\omega}$ with $\omega \in H^3(G;\mathcal{C}^{\times})$ a non-trivial 3-cocycle have different number of indecomposable module categories according to [Os, Example 2.1]. Hence $\mathcal{A}$ and $\mathcal{B}$ are not Morita equivalent and therefore $\mathcal{A}$ and $\mathcal{B}$ cannot be G-graded Morita equivalent.

5.3 The 2-category of graded module categories

For every G-graded tensor category $\mathcal{A}$ there is associated a 2-category $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ whose objects are exact G-graded $\mathcal{A}$-module categories, the 1-morphisms are graded module functors and the 2-morphisms are module natural transformations.

Recall the notion of group actions on 2-categories, as discussed for example in [HSV] and [BGM]. A strict G-action on a 2-category $\mathcal{F}$ is a collection of 2-functors $\{g. : \mathcal{F} \to \mathcal{F}\}_{g\in G}$ such that $g. \circ h. = (gh).$ for every $g, h \in G$. An equivalence of 2-categories with group action is a 2-equivalence Ψ with a G-structure $\gamma_g : \Psi \circ g. \overset{\sim}{\Rightarrow} g. \circ \Psi$ fulfilling certain conditions, see [BGM, Definition 2.3] for a complete definition.

The G-grading of $\mathcal{A}$ induces an additional structure on the 2-category $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$, namely a strict left action of G which is given by shifting the grading: For $\mathcal{N} \in \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ and $g \in G$, define a G-graded $\mathcal{A}$-module category $g.\mathcal{N}$ by $\mathcal{N}$ as a $\mathcal{A}$-module category, but with G-grading described by the following homogeneous components

$$[g.\mathcal{N}]_x := \mathcal{N}_{xg}, \quad \text{for } x \in G\,.$$

Every 1-morphism $H : \mathcal{N} \to \mathcal{L}$ induces a 1-morphism $g.H : g.\mathcal{N} \to g.\mathcal{L}$, $n \mapsto H(n)$, and the assignment on 2-morphisms is similarly defined.

Notice that for $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ and $g \in G$

$$\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, g.\mathcal{N})_e = \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_g = \mathrm{Fun}_{\mathcal{A}}(g.^{-1}\mathcal{M},\mathcal{N})_e \tag{5.10}$$

which are $\mathcal{A}$-module functors $H : \mathcal{M} \to \mathcal{N}$ such that $H(\mathcal{M}_x) \subset \mathcal{N}_{xg}$ for all $x \in G$.

The action of G on the graded module categories of a graded tensor category plays an important role in the notion of graded Morita equivalence.

Theorem 5.18. *Two G-graded finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent if and only if $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ and $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{B})$ are equivalent as 2-categories with G-action.*

Proof. Given an exact graded $\mathcal{A}$-module category $\mathcal{M}$, from [EGNO, Theorem 7.12.16] and considering Proposition 5.11 we have a 2-equivalence

$$\begin{aligned} \Psi :\ \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}) &\longrightarrow \mathbf{Mod}^{\mathrm{Gr}}(\overline{\mathcal{A}^*_{\mathcal{M}}}) \\ \mathcal{N} &\longmapsto \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \end{aligned}$$

which has a strict G-structure, i.e., the diagram of 2-functors

$$\begin{tikzcd} \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}) \arrow[r, "\Psi"] \arrow[d, "g."'] & \mathbf{Mod}^{\mathrm{Gr}}(\overline{\mathcal{A}^*_{\mathcal{M}}}) \arrow[d, "g."] \\ \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}) \arrow[r, "\Psi"'] & \mathbf{Mod}^{\mathrm{Gr}}(\overline{\mathcal{A}^*_{\mathcal{M}}}) \end{tikzcd}$$

strictly commutes for every $g \in G$. Indeed, for $\mathcal{N} \in \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ notice that both $\Psi(g.\mathcal{N}) = \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, g.\mathcal{N})$ and $g.\Psi(\mathcal{N}) = g.\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ are equal to $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ as $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module categories. Moreover, the G-gradings coincide:
For a homogeneous functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, g.\mathcal{N})_h$ it holds that

$$H(\mathcal{M}_x) \subset [g.\mathcal{N}]_{xh} = \mathcal{N}_{xhg}$$

for all $x \in G$, which means that $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})_{hg} = [\, g.\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})\,]_h$.

Conversely, consider a 2-equivalence

$$\Phi : \ \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{B}) \simeq \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$$

which is compatible with the corresponding G-actions. Denote by $\mathcal{M} := \Phi(\mathcal{B})$ the image under Φ of the regular graded $\mathcal{B}$-module category. Then for every $g \in G$ we have an equivalence

$$\mathrm{End}_{\mathcal{B}}(\mathcal{B})_g = \mathrm{Fun}_{\mathcal{B}}(\mathcal{B}, g.\mathcal{B})_e \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \Phi(g.\mathcal{B}))_e \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, g.\mathcal{M})_e = \mathrm{End}_{\mathcal{A}}(\mathcal{M})_g$$

and thus Φ induces a graded equivalence of categories

$$\Omega : \ \mathrm{End}_{\mathcal{B}}(\mathcal{B}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}}(\mathcal{M}) = \mathcal{A}^*_{\mathcal{M}} \,.$$

Moreover, since Φ is a 2-functor there is a natural isomorphism γ

$$\begin{tikzcd}
\mathrm{End}_{\mathcal{B}}(\mathcal{B})_g \times \mathrm{End}_{\mathcal{B}}(\mathcal{B})_h \arrow[r, "\circ"] \arrow[d, "\Phi\times\Phi"'] & \mathrm{End}_{\mathcal{B}}(\mathcal{B})_{hg} \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\gamma"] \\
\mathrm{End}_{\mathcal{A}}(\mathcal{M})_g \times \mathrm{End}_{\mathcal{A}}(\mathcal{M})_h \arrow[r, "\circ"'] & \mathrm{End}_{\mathcal{A}}(\mathcal{M})_{hg}
\end{tikzcd}$$

which endows Ω with a monoidal structure. We therefore obtain a G-graded tensor equivalence

$$\mathcal{B} \simeq \mathrm{End}_{\mathcal{B}}(\mathcal{B})^{\mathrm{opp}} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{M})^{\mathrm{opp}} = \overline{\mathcal{A}^*_{\mathcal{M}}}$$

which means that $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent. □

Remark 5.19. In Theorem 5.18 it is necessary to keep the information encoded in the G-action on $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$. If $\mathcal{A}$ is a G-graded tensor category, any group automorphism $f \in \mathrm{Aut}(G)$, defines a new G-grading by $(\mathcal{A}^f)_g := \mathcal{A}_{f(g)}$, where $g \in G$. Depending of the group automorphism and the graded tensor category, it is possible that the 2-categories $\mathrm{Mod}^{\mathrm{Gr}}(\mathcal{A})$ and $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}^f)$ are equivalent as 2-categories, but not as 2-categories with G-action. For instance, if $\mathcal{A} = \mathrm{Vec}^{\omega}_{G}$, and $f \in \mathrm{Aut}(G)$ is such that $f^*(\omega)$ and ω are not cohomologous, then $\mathcal{A}$ and $\mathcal{A}^f$ are not graded Morita equivalent, but the 2-categories $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ and $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}^f)$ are equivalent.

The equivariantization construction has an analog for 2-categories with group action. Let $\mathcal{A}$ be a G-graded finite tensor category, then for $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ the 2-category of equivariant objects corresponds to $\mathbf{Mod}(\mathcal{A})$ the 2-category of (not necessarily graded) exact module categories over $\mathcal{A}$. In order to see this consider the following:

The 2-equivalence in Theorem 5.8 is given by induction of module categories and can be described in terms of the relative Deligne product [ENOM]

$$\mathbf{Mod}(\mathcal{A}_e) \to \mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}), \qquad \mathcal{N} \mapsto \mathrm{Ind}^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{N}) = \mathcal{A} \boxtimes_{\mathcal{A}_e} \mathcal{N} \tag{5.11}$$

and its inverse is given by restriction

$$\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A}) \to \mathbf{Mod}(\mathcal{A}_e), \qquad \mathcal{M} \mapsto \mathrm{Res}^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{M}) = \mathcal{M}_e \,. \tag{5.12}$$

Since G acts on $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$ we can transport this G-action structure to the 2-category $\mathbf{Mod}(\mathcal{A}_e)$ via the 2-equivalence (5.12). We obtain for each $g \in G$ a 2-functor

$$g_\times \,:\mathbf{Mod}(\mathcal{A}_e) \to \mathbf{Mod}(\mathcal{A}_e), \quad \mathcal{N} \mapsto \mathrm{Res}^{\mathcal{A}}_{\mathcal{A}_e}(g.\,\mathrm{Ind}^{\mathcal{A}}_{\mathcal{A}_e}(\mathcal{N})) = \mathcal{A}_g \boxtimes_{\mathcal{A}_e} \mathcal{N}$$

and the $\mathcal{A}_e$-bimodule equivalences $M_{g,h} : \mathcal{A}_g \boxtimes_{\mathcal{A}_e} \mathcal{A}_h \xrightarrow{\sim} \mathcal{A}_{gh}$ coming from the tensor product of $\mathcal{A}$ provide the corresponding pseudonatural equivalences $g_\times \circ h_\times \overset{\sim}{\Rightarrow} gh_\times$.

We notice that this G-action coincides with the G-action on $\mathbf{Mod}(\mathcal{A}_e)$ presented in [BGM, Theorem 5.4]. Moreover, the authors also show that the 2-category of equivariant objects in $\mathbf{Mod}(\mathcal{A}_e)$ under this action corresponds to $\mathbf{Mod}(\mathcal{A})$. Consequently, by considering equivariant objects in $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})$, one recovers the (not necessarily graded) module categories over $\mathcal{A}$.

Corollary 5.20. *Given a G-graded finite tensor category $\mathcal{A}$. The equivariantization $\mathbf{Mod}^{\mathrm{Gr}}(\mathcal{A})^G$ is 2-equivalent to $\mathbf{Mod}(\mathcal{A})$.*

5.4 The equivariant center and graded Morita equivalence

Given a G-graded finite tensor category $\mathcal{A}$, there is a construction [GNN, TV2] that associates to $\mathcal{A}$ a braided G-crossed tensor category called the *equivariant center* and denoted by $\mathcal{Z}_G(\mathcal{A})$, whose underlying tensor category is the relative center $\mathcal{Z}_{\mathcal{A}_e}(\mathcal{A})$ of $\mathcal{A}$ with respect to the trivial component $\mathcal{A}_e$.

On the other hand, the ordinary Drinfeld center of a G-graded finite tensor category $\mathcal{A}$ is a braided tensor category endowed with a fully faithful braided functor

$$\mathrm{Rep}(G) \longrightarrow \mathcal{Z}(\mathcal{A})\,, \qquad (\Bbbk^n, \rho) \mapsto (\mathbf{1}^n, \gamma_{-,\mathbf{1}^n})\,, \tag{5.13}$$

where for $a \in \mathcal{A}_g$ the half-braiding is defined via $\gamma_{a,\mathbf{1}^n} : a \otimes \mathbf{1}^n \xrightarrow{\mathrm{id}_a \otimes \rho(g)} a \otimes \mathbf{1}^n = \mathbf{1}^n \otimes a$. Therefore $\mathcal{Z}(\mathcal{A})$ is a possible input for the de-equivariantization construction. The following result shown in [GNN] states that the category of equivariant objects in $\mathcal{Z}_G(\mathcal{A})$ is braided equivalent to the Drinfeld center $\mathcal{Z}(\mathcal{A})$.

Theorem 5.21. [GNN, Theorem 3.5] *Let $\mathcal{A}$ be a G-graded finite tensor category, there is an equivalence of braided tensor categories*

$$\mathcal{Z}_G(\mathcal{A})^G \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A}) \tag{5.14}$$

compatible with the canonical inclusions of Rep(G) given by (5.2) *and* (5.13)*.*

Remark 5.22. Since equivariantization and de-equivariantization are inverse procedures, it follows from Theorem 5.21 that the *equivariant center* $\mathcal{Z}_G(\mathcal{A})$ can be characterized as the de-equivariantization of $\mathcal{Z}(\mathcal{A})$, i.e. $\mathcal{Z}_G(\mathcal{A}) \simeq {}_{\Upsilon_G}\mathrm{Mod}(\mathcal{Z}(\mathcal{A}))$, where Υ_G is the image of the algebra of functions under the inclusion (5.13).

Proposition 5.23. [Ja, Thm. 4.13] *Let $\mathcal{A}$ be a G-graded finite tensor category and $\mathcal{M} = \mathrm{Mod}_A(\mathcal{A})$ an exact indecomposable G-graded $\mathcal{A}$-module category, where A is an algebra in $\mathcal{A}_e$. The braided equivalence $\mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\overline{\mathcal{A}^*_{\mathcal{M}}})$ given by* (1.69) *induces, via de-equivariantization, an equivalence*

$$\mathcal{Z}_G(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}_G({}_A\mathrm{Bimod}_A(\mathcal{A})) \xrightarrow{\simeq} \mathcal{Z}_G(\,\overline{\mathcal{A}^*_{\mathcal{M}}}\,) \tag{5.15}$$

of braided G-crossed tensor categories.

Proof. First, notice that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{Z}(\mathcal{A}) \arrow[dd, "S"] \\
\mathrm{Rep}(G) \arrow[ur, "(5.13)"] \arrow[dr, "(5.13)"'] & \\
 & \mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A}))
\end{tikzcd} \tag{5.16}$$

where S refers to equivalence (1.67). Explicitly we have for the G-graded tensor category ${}_A\mathrm{Bimod}_A(\mathcal{A})$ that the functor (5.13) is given by

$$\mathrm{Rep}(G) \to \mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A})), \quad (\Bbbk^n, \rho) \mapsto (A \otimes \mathbf{1}^n, \beta_{\_,A^n})$$

where for $N \in {}_A\mathrm{Bimod}_A(\mathcal{A}_g)$, the half-braiding β_{N,A^n} is given by

$$N \otimes_A A \otimes \mathbf{1}^n \xrightarrow{\mathrm{id}_N \otimes_A \mathrm{id}_A \otimes \rho(g)} N \otimes_A A \otimes \mathbf{1}^n \cong N \otimes \mathbf{1}^n = \mathbf{1}^n \otimes N \cong \mathbf{1}^n \otimes A \otimes_A N = A \otimes \mathbf{1}^n \otimes_A N.$$

On the other hand, composing S with the functor (5.13) corresponding to $\mathcal{A}$ leads to

$$(\mathbf{1}^n, \gamma_{-,\mathbf{1}^n}) \mapsto (A \otimes \mathbf{1}^n, \delta_{\_,A^n})$$

where for $M \in {}_A\mathrm{Bimod}_A(\mathcal{A}_g)$, the half-braiding δ_{M,A^n} is given by the composition

$$M \otimes_A A \otimes \mathbf{1}^n \cong M \otimes \mathbf{1}^n \xrightarrow{\mathrm{id}_M \otimes \rho(g)} M \otimes \mathbf{1}^n = \mathbf{1}^n \otimes M \cong \mathbf{1}^n \otimes A \otimes_A M \xrightarrow{\gamma^{-1}_{A,\mathbf{1}^n} \otimes_A \mathrm{id}_M} A \otimes \mathbf{1}^n \otimes_A M\,.$$

but since $A \in \mathcal{A}_e$, then $\gamma_{A,\mathbf{1}^n} = \mathrm{id}_A \otimes \rho(e) = \mathrm{id}_{A^n}$ and thus the half-braidings β and δ coincide, which implies the commutativity of the diagram (5.16). It follows by applying de-equivariantization, that Schauenburg's equivalence (1.67) induces an equivalence of braided G-crossed tensor categories

$$\mathcal{Z}_G(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}_G({}_A\mathrm{Bimod}_A(\mathcal{A}))\,.$$

Similarly, from Proposition 5.12 the Eilenberg-Watts equivalence is graded, but the construction of the inclusion (5.13) is determined by the G-grading, therefore the diagram

$$\begin{tikzcd}
 & \mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A})) \arrow[dd] \\
\mathrm{Rep}(G) \arrow[ur, "(5.13)"] \arrow[dr, "(5.13)"'] & \\
 & \mathcal{Z}(\overline{\mathcal{A}^*_{\mathcal{M}}})
\end{tikzcd}$$

is commutative and by applying de-equivariantization, we conclude that $\mathcal{Z}_G({}_A\mathrm{Bimod}_A(\mathcal{A}))$ and $\mathcal{Z}_G(\overline{\mathcal{A}^*_{\mathcal{M}}})$ are equivalent as braided G-crossed tensor categories as well. □

Remark 5.24. Given a G-graded finite tensor category $\mathcal{A}$ the equivalence from Proposition 5.23 makes the following diagram commute

$$\begin{tikzcd}
\mathcal{Z}(\mathcal{A}) \arrow[r, "S"] \arrow[d] & \mathcal{Z}({}_A\mathrm{Bimod}_A(\mathcal{A})) \arrow[d] \\
\mathcal{Z}_G(\mathcal{A}) \arrow[r, "S_G"] & \mathcal{Z}_G({}_A\mathrm{Bimod}_A(\mathcal{A}))
\end{tikzcd}$$

where the vertical arrows are the forgetful functors mentioned in Remark 1.17 (i) and S is equivalence (1.67). More explicitly S_G is given by the assignment $(X, \sigma) \longmapsto (A \otimes X, \delta)$, where for $M \in {}_A\mathrm{Bimod}_A(\mathcal{A}_e)$, the half-braiding δ is defined by the composition

$$M \otimes_A A \otimes X \cong M \otimes X \xrightarrow{\sigma_{M,X}} X \otimes M \cong X \otimes A \otimes_A M \xrightarrow{\sigma^{-1}_{A,X} \otimes_A M} A \otimes X \otimes_A M\,.$$

In particular Proposition 5.23 implies the following extension of Theorem 1.18 which corresponds to the case of the trivial group:

Theorem 5.25. [Ja, Rem. 4.15] *If two G-graded finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent, then their equivariant centers $\mathcal{Z}_G(\mathcal{A})$ and $\mathcal{Z}_G(\mathcal{B})$ are equivalent as braided G-crossed tensor categories.*

Converse of Theorem 5.25

Morita equivalence of finite tensor categories can be completely detected by braided equivalence of their Drinfeld centers. In the graded case an analogous result will be proven in Theorem 5.29 for the equivariant center. To this end, we need to prove the converse of Theorem 5.25 and we will closely follow the approach in [EGNO, Section 8.12] for the non-graded case.

Given a G-graded finite tensor category $\mathcal{A}$, the forgetful functor

$$F: \quad \mathcal{Z}_G(\mathcal{A}) \to \mathcal{A}, \quad (X, \gamma) \mapsto X \tag{5.17}$$

endows $\mathcal{A}$ with the structure of an exact G-graded module category over $\mathcal{Z}_G(\mathcal{A})$, according to Remark 1.17 (iv).

The functor $\underline{\mathrm{Hom}}(\mathbf{1}, -) : \mathcal{A} \to \mathcal{Z}_G(\mathcal{A})$ is right adjoint to F: From the definition of the internal Hom, given $Z \in \mathcal{Z}_G(\mathcal{A})$ and $X \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}}(F(Z), X) = \mathrm{Hom}_{\mathcal{A}}(Z \otimes \mathbf{1}, X) \cong \mathrm{Hom}_{\mathcal{Z}_G(\mathcal{A})}(Z, \underline{\mathrm{Hom}}(\mathbf{1}, X)) \tag{5.18}$$

where $Z \otimes \mathbf{1}$ stands for the action of Z on the monoidal unit $\mathbf{1} \in \mathcal{A}$.

Notice that the adjunction (5.18) is a special case of (1.16) for the exact G-graded $\mathcal{Z}_G(\mathcal{A})$-module category $\mathcal{A}$. Denote by $\xi_X := \mathrm{ev}_{\mathbf{1},X} : \underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \mathbf{1} \to X$ the counit morphism (1.22). Now consider for every $X \in \mathcal{A}$ the following morphisms:

$$\sigma_X := \circ_{\mathbf{1},\mathbf{1},X} : \underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \underline{\mathrm{Hom}}(\mathbf{1}, \mathbf{1}) \to \underline{\mathrm{Hom}}(\mathbf{1}, X) \tag{5.19}$$

i.e., the image of the composition

$$\underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \underline{\mathrm{Hom}}(\mathbf{1}, \mathbf{1}) \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes \xi_{\mathbf{1}}} \underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \mathbf{1} \xrightarrow{\xi_X} X \tag{5.20}$$

under the adjunction (5.18), and define

$$\rho_X : \quad \underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \otimes \underline{\mathrm{Hom}}(\mathbf{1},X) \to \underline{\mathrm{Hom}}(\mathbf{1},X) \tag{5.21}$$

as the image of the composition

$$\underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \otimes \mathbf{1} \otimes \underline{\mathrm{Hom}}(\mathbf{1},X) \xrightarrow{\xi_{\mathbf{1}}\otimes\mathrm{id}} \mathbf{1} \otimes \underline{\mathrm{Hom}}(\mathbf{1},X) \cong \underline{\mathrm{Hom}}(\mathbf{1},X) \otimes \mathbf{1} \xrightarrow{\xi_X} X \tag{5.22}$$

under the adjunction (5.18), where monoidal units should be inserted and removed where necessary.

From Remark 1.7 we know that

(i) $A := (\underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}), m := \sigma_{\mathbf{1}}, u_{\mathbf{1}})$ is an algebra in $\mathcal{Z}_G(\mathcal{A})$. Moreover, A is in the trivial component $\mathcal{Z}_G(\mathcal{A})_e$ according to Proposition 5.5.

(ii) $R(X) := (\underline{\mathrm{Hom}}(\mathbf{1},X), \sigma_X)$ is a right A-module in $\mathcal{Z}_G(\mathcal{A})$, for every $X \in \mathcal{A}$.

(iii) From Theorem 1.8 the assignment $R : \mathcal{A} \longrightarrow \mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A})),\ X \mapsto R(X)$ is an equivalence of $\mathcal{Z}_G(\mathcal{A})$-module categories, and from Proposition 5.5 it is G-graded.

Proposition 5.26. (a) *The algebra A is commutative in $\mathcal{Z}_G(\mathcal{A})_e = \mathcal{Z}(\mathcal{A}_e)$.*

(b) *For every $X \in \mathcal{A}$, the morphism $\rho_X : A \otimes R(X) \to R(X)$ from equation* (5.21) *coincides with the composition*

$$A \otimes R(X) \xrightarrow{c_{A,R(X)}} R(X) \otimes A \xrightarrow{\sigma_X} R(X) \tag{5.23}$$

providing a structure of an A-bimodule in $\mathcal{Z}_G(\mathcal{A})$ on $R(X)$.

Proof. Given $X \in \mathcal{A}$, the following diagram commutes due to the naturality of the braiding (see equation (1.64))

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \otimes \underline{\mathrm{Hom}}(\mathbf{1},X) \arrow[r, "\xi_{\mathbf{1}}\otimes id"] \arrow[dd, "c_{A,R(X)}"'] & \mathbf{1} \otimes \underline{\mathrm{Hom}}(\mathbf{1},X) \arrow[dd, "c_{\mathbf{1},R(X)}"] \arrow[dr, "\sim"] & & \\
 & & \underline{\mathrm{Hom}}(\mathbf{1},X) \otimes \mathbf{1} \arrow[r, "\xi_X"] & X \\
\underline{\mathrm{Hom}}(\mathbf{1},X) \otimes \underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \arrow[r, "id\otimes\xi_{\mathbf{1}}"'] & \underline{\mathrm{Hom}}(\mathbf{1},X) \otimes \mathbf{1} \arrow[ur, "="'] & &
\end{tikzcd} \tag{5.24}$$

From the definition of σ_X and ρ_X, the commutativity of the diagram (5.24) translates to $\sigma_X \circ c_{A,R(X)} = \rho_X$ under the isomorphism

$$\mathrm{Hom}_{\mathcal{A}}\left(\underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \otimes \underline{\mathrm{Hom}}(\mathbf{1},X), X\right) \cong \mathrm{Hom}_{\mathcal{Z}_G(\mathcal{A})}(\underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1}) \otimes \underline{\mathrm{Hom}}(\mathbf{1},X), \underline{\mathrm{Hom}}(\mathbf{1},X))$$

coming from the adjunction (5.18). The case $X = \mathbf{1}$ corresponds to commutativity of the algebra $A = \underline{\mathrm{Hom}}(\mathbf{1},\mathbf{1})$. □

In particular since A is a commutative algebra in the trivial component of $\mathcal{Z}_G(\mathcal{A})$, it follows from Proposition 5.6 that $\mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A}))$ is a G-graded tensor category. Now the goal is to define a tensor structure on the graded functor $R : \mathcal{A} \to \mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A}))$. It is important to point out that the construction of such tensor structure does not involve the grading on $\mathcal{A}$, and thus is reduced to the non-graded case. Define for $X, Y \in \mathcal{A}$ a morphism

$$\varphi_{X,Y} : \underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \underline{\mathrm{Hom}}(\mathbf{1}, Y) \to \underline{\mathrm{Hom}}(\mathbf{1}, X \otimes Y) \tag{5.25}$$

as the image of $\xi_X \otimes \xi_Y : \underline{\mathrm{Hom}}(\mathbf{1}, X) \otimes \underline{\mathrm{Hom}}(\mathbf{1}, Y) \to X \otimes Y$ under (5.18). A direct computation shows that $\varphi_{X,Y}$ is a cone under the coequalizer diagram

$$\begin{tikzcd}
R(X) \otimes A \otimes R(Y) \arrow[r, shift left, "\sigma_X \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \rho_Y"'] & R(X) \otimes R(Y) \arrow[r] \arrow[dr, "\varphi_{X,Y}"'] & R(X) \otimes_A R(Y) \arrow[d, dashed, "\widetilde{\varphi}_{X,Y}"] \\
 & & R(X \otimes Y)
\end{tikzcd} \tag{5.26}$$

and moreover one can check that the morphisms $\widetilde{\varphi}_{X,Y} : R(X) \otimes_A R(Y) \to R(X \otimes Y)$, given by the universal property of the coequalizer, fulfill the axioms of a weak tensor structure on R. Furthermore, $\widetilde{\varphi}_{X,Y} : R(X) \otimes_A R(Y) \to R(X \otimes Y)$ is an isomorphism for every $X, Y \in \mathcal{A}$:

(i) For an object $Z \in \mathcal{Z}_G(\mathcal{A})$ the canonical isomorphism (1.18) provides an isomorphism of A-modules $R(F(Z)) = \underline{\mathrm{Hom}}(\mathbf{1}, Z \otimes \mathbf{1}) \cong Z \otimes \underline{\mathrm{Hom}}(\mathbf{1}, \mathbf{1}) = Z \otimes A$, where F is the forgetful functor (5.17) and in this case $\widetilde{\varphi}_{F(Z),Y}$ corresponds to the isomorphism

$$R(F(Z)) \otimes_A R(Y) \cong Z \otimes \underline{\mathrm{Hom}}(\mathbf{1}, Y) \cong \underline{\mathrm{Hom}}(\mathbf{1}, Z \otimes Y) = R(F(Z) \otimes Y) \tag{5.27}$$

where the second isomorphism comes from (1.18) once more.

(ii) Every projective object $P \in \mathcal{A}$ is a direct summand of an object of the form $F(Z)$: Since F is surjective there are $Z \in \mathcal{Z}_G(\mathcal{A})$ and $W \in \mathcal{A}$, such that P is a subobject of W and W is a quotient of $F(Z)$. Now from [EGNO, Proposition 6.1.3] P is injective, then P is a direct summand of W and therefore a quotient of $F(Z)$. But from projectivity of P it follows that P is a direct summand of $F(Z)$.

(iii) For every projective object $P \in \mathcal{A}$ the morphism $\widetilde{\varphi}_{P,Y}$ is an isomorphism: From (ii) there exists $Z \in \mathcal{Z}_G(\mathcal{A})$ with $F(Z) = P \oplus T$ for some $T \in \mathcal{A}$ and thus $\widetilde{\varphi}_{F(Z),Y} = \widetilde{\varphi}_{P,Y} \oplus \widetilde{\varphi}_{T,Y}$. It follows that if $\widetilde{\varphi}_{P,Y}$ is not an isomorphism, then $\widetilde{\varphi}_{F(Z),Y}$ is not an isomorphism, but this is a contradiction with (i).

(iv) For an arbitrary $X \in \mathcal{A}$, the morphism $\widetilde{\varphi}_{X,Y}$ is an isomorphism: Consider a projective cover $p : P \to X$. By naturality of $\widetilde{\varphi}_{X,Y}$ the following diagram commutes

$$\begin{tikzcd}
R(P) \otimes_A R(Y) \arrow[r, "\widetilde{\varphi}_{P,Y}"] \arrow[d, "R(p) \otimes_A \mathrm{id}"'] & R(P \otimes Y) \arrow[d, "R(p \otimes \mathrm{id})"] \\
R(X) \otimes_A R(Y) \arrow[r, "\widetilde{\varphi}_{X,Y}"'] & R(X \otimes Y)
\end{tikzcd} \tag{5.28}$$

Now from (iii) the top arrow $\widetilde{\varphi}_{P,Y}$ is an isomorphism, and since p is epic and R and $\otimes$ are exact, then $R(p \otimes \mathrm{id})$ is epic and thus $\widetilde{\varphi}_{X,Y}$ has to be epic as well. An analogous argument using an injective hull of X, shows that $\widetilde{\varphi}_{X,Y}$ is also mono.

Lemma 5.27. *Let $\mathcal{A}$ be a G-graded finite tensor category, then there exists a commutative algebra A in $\mathcal{Z}_G(\mathcal{A})_e = \mathcal{Z}(\mathcal{A}_e)$ and an equivalence $\mathcal{A} \simeq \mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A}))$ of G-graded tensor categories.*

Proof. Follows considering the construction given above. □

Proposition 5.28. *Let $\mathcal{A}$ and $\mathcal{B}$ be G-graded finite tensor categories. Provided that $\mathcal{Z}_G(\mathcal{A})$ and $\mathcal{Z}_G(\mathcal{B})$ are equivalent as central G-extensions, then $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent.*

Proof. Let B be the commutative algebra in $\mathcal{Z}_G(\mathcal{B})_e$ constructed in Lemma 5.27 and let $\Lambda : \mathcal{Z}_G(\mathcal{B}) \simeq \mathcal{Z}_G(\mathcal{A})$ be an equivalence of central G-extensions, then $L := \Lambda(B)$ is a commutative algebra in $\mathcal{Z}_G(\mathcal{A})_e$ and

$$\mathcal{B} \simeq \mathrm{Mod}_B(\mathcal{Z}_G(\mathcal{B})) \simeq \mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A})) \tag{5.29}$$

as G-graded tensor categories, where the first equivalence comes from Lemma 5.27 and the second equivalence is induced by Λ.

Now notice that $\mathrm{Mod}_{F(L)}(\mathcal{A})$ is an exact G-graded $\mathcal{A}$-module category: let A be the commutative algebra in $\mathcal{Z}_G(\mathcal{A})_e$ from Lemma 5.27, then $\mathcal{A} \simeq \mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A}))$.

(i) Since $\mathcal{B} \simeq \mathrm{Mod}_B(\mathcal{Z}_G(\mathcal{B}))$ is exact over $\mathcal{Z}_G(\mathcal{B})$, then $\mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A}))$ is exact as a $\mathcal{Z}_G(\mathcal{A})$-module category.

(ii) From [EGNO, Proposition 7.12.14] and Proposition 5.11, the category

$$\mathrm{Fun}_{\mathcal{Z}_G(\mathcal{A})}(\mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A})), \mathcal{A}) \simeq {}_L\mathrm{Bimod}_A(\mathcal{Z}_G(\mathcal{A}))$$

is an exact G^{opp}-graded module category over $(\mathcal{Z}_G(\mathcal{A}))^*_{\mathcal{A}} \simeq \overline{{}_A\mathrm{Bimod}_A(\mathcal{Z}_G(\mathcal{A}))}$.

(iii) Since $\mathcal{A} \simeq \mathrm{Mod}_A(\mathcal{Z}_G(\mathcal{A}))$, then $\mathrm{Mod}_{F(L)}(\mathcal{A}) \simeq {}_L\mathrm{Bimod}_A(\mathcal{Z}_G(\mathcal{A}))$.

(iv) From (ii) and (iii) it follows that $\mathrm{Mod}_{F(L)}(\mathcal{A})$ is an exact G^{opp}-graded module category over

$$(\mathcal{Z}_G(\mathcal{A}))^*_{\mathcal{A}} \simeq ((\mathcal{A}_e \boxtimes \overline{\mathcal{A}}\,)^*_{\mathcal{A}})^*_{\mathcal{A}} \simeq \mathcal{A}_e \boxtimes \overline{\mathcal{A}}$$

therefore it is in particular exact over $\mathcal{A}_e$, and thus $\mathcal{M}_e = \mathrm{Mod}_{F(L)}(\mathcal{A}_e)$ is an exact $\mathcal{A}_e$-module category. From Lemma 5.7 it follows that $\mathrm{Mod}_{F(L)}(\mathcal{A})$ is exact over $\mathcal{A}$.

On the other hand, the module category $\mathcal{B} \simeq \mathrm{Mod}_B(\mathcal{Z}_G(\mathcal{B}))$ is indecomposable over the dual category $(\mathcal{B}_e \boxtimes \overline{\mathcal{B}}\,)^*_{\mathcal{B}} \simeq \mathcal{Z}_G(\mathcal{B})$, hence under Λ the module category $\mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A}))$ is indecomposable over $\mathcal{Z}_G(\mathcal{A})$. But the forgetful image $\mathrm{Mod}_{F(L)}(\mathcal{A})$ might be decomposable over $\mathcal{A}$. Since $\mathrm{Mod}_{F(L)}(\mathcal{A})$ is exact over $\mathcal{A}$, it follows from Proposition 1.10 that there is a decomposition of $\mathcal{A}$-module categories

$$\mathrm{Mod}_{F(L)}(\mathcal{A}) \simeq \bigoplus_{i\in I} \mathrm{Mod}_{L_i}(\mathcal{A}) \tag{5.30}$$

where L_i is an exact indecomposable algebra in $\mathcal{A}_e$ for all $i \in I$ and $F(L)$ decomposes as $\prod_{i\in I} L_i$ as an algebra. Furthermore, the category of bimodules decomposes as follows

$${}_{F(L)}\mathrm{Bimod}_{F(L)}(\mathcal{A}) \simeq \bigoplus_{i,j\in I} {}_{L_i}\mathrm{Bimod}_{L_j}(\mathcal{A})\,. \tag{5.31}$$

Now consider the following commutative diagram,

$$\begin{tikzcd}
& \mathcal{Z}_G(\mathcal{A}) \arrow[rr, "S_G"] \arrow[d, "Z\mapsto Z\otimes L"'] & & \mathcal{Z}_G\left({}_{L_i}\mathrm{Bimod}_{L_i}(\mathcal{A})\right) \arrow[d, "F_i"] \\
\mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A})) \subset {}_L\mathrm{Bimod}_L(\mathcal{Z}_G(\mathcal{A})) \arrow[rr, "\overline{F}"'] & & {}_{F(L)}\mathrm{Bimod}_{F(L)}(\mathcal{A}) \arrow[r, "\pi_i"'] & {}_{L_i}\mathrm{Bimod}_{L_i}(\mathcal{A})
\end{tikzcd} \tag{5.32}$$

where the equivalence S_G comes from Proposition 5.23, π_i is the canonical projection of the direct sum (5.31) and $\overline{F}$ is the functor between the categories of bimodules induced by the forgetful functor F. Now since the forgetful functor F_i is surjective (see Remark 1.17 (i)), then we have a surjective graded functor

$$H_i := \pi_i \circ \overline{F} :\ \mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A})) \longrightarrow {}_{L_i}\mathrm{Bimod}_{L_i}(\mathcal{A}) \tag{5.33}$$

between tensor categories of the same Frobenius-Perron dimension, and thus H_i is an equivalence. Indeed,

(i) From [EGNO, Corollary 7.16.7] $\mathrm{FPdim}({}_{L_i}\mathrm{Bimod}_{L_i}(\mathcal{A})) = \mathrm{FPdim}(\mathcal{A})$.

(ii) Since Λ is an equivalence,

$$\frac{1}{|G|}\mathrm{FPdim}(\mathcal{B})^2 = \mathrm{FPdim}(\mathcal{Z}_G(\mathcal{B})) = \mathrm{FPdim}(\mathcal{Z}_G(\mathcal{A})) = \frac{1}{|G|}\mathrm{FPdim}(\mathcal{A})^2 \tag{5.34}$$

and thus $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})$.

(iii) From (ii) and equivalence (5.29) it follows that

$$\mathrm{FPdim}(\mathrm{Mod}_L(\mathcal{Z}_G(\mathcal{A}))) = \mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\,. \tag{5.35}$$

Summarizing, there is an exact G-graded $\mathcal{A}$-module category $\mathcal{M} = \mathrm{Mod}_{L_i}(\mathcal{A})$ and a graded tensor equivalence $\mathcal{B} \simeq {}_{L_i}\mathrm{Bimod}_{L_i}(\mathcal{A}) \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$. ☐

Considering the results of Proposition 5.28 and Theorem 5.25, we obtain the following theorem.

Theorem 5.29. *Two G-graded finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ are graded Morita equivalent if and only if $\mathcal{Z}_G(\mathcal{A})$ and $\mathcal{Z}_G(\mathcal{B})$ are equivalent as braided G-crossed tensor categories.*

In view of [DN, Proposition 8.11], Theorem 5.29 can be described considering the equivariant centers as central G-extensions as well.

5.5 Dualities and pivotality for equivariant Morita Theory

The previous study of graded Morita equivalence has not yet considered pivotal structures. In this section we investigate the interaction of duals and pivotal structures with Morita theory in the equivariant setting. One expects that the equivalence data for the notion of graded Morita equivalence are endowed with a graded structure in a compatible manner. That is, the categories in the Morita context of a G-graded module category should be graded and be related via grading preserving actions. Indeed we have

Proposition 5.30. *Let $\mathcal{A}$ be a G-graded tensor category and $\mathcal{M}$ a G-graded $\mathcal{A}$-module category and consider ($\mathcal{A}$, $\overline{\mathcal{A}^*_{\mathcal{M}}}$, $\mathcal{M}$, $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$, $\odot$, $\boxdot$) the Morita context associated to it.*

(i) *$\overline{\mathcal{A}^*_{\mathcal{M}}}$ is a G-graded tensor category.*

(ii) *$\mathcal{M}$ and $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ are G-graded bimodule categories.*

(iii) *The mixed products $\odot$ and $\boxdot$ are compatible with the group law of G.*

Proof. That $\overline{\mathcal{A}^*_{\mathcal{M}}}$ is a G-graded tensor category and $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ is a G-graded $\overline{\mathcal{A}^*_{\mathcal{M}}}$-module is part of the statement in Proposition 5.11. The remaining assertions follow from the fact that, by the definition of the gradings, we have

$$\begin{aligned} m \triangleleft F = F(m) \,\in \mathcal{M}_{gy}\,, \qquad & H \triangleleft a = H(-) \otimes a \,\in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})_{xh}\,, \\ m \odot H = H(m) \,\in \mathcal{M}_{gx}\,, \qquad & H \boxdot m = H(-) \triangleright m \,\in \left(\overline{\mathcal{A}^*_{\mathcal{M}}}\right)_{xg} \end{aligned} \tag{5.36}$$

for all $m \in \mathcal{M}_g$, $a \in \mathcal{A}_h$, $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})_x$ and $F \in (\overline{\mathcal{A}^*_{\mathcal{M}}})_y$. □

Remark 5.31. Proposition 5.30 can be seen as a statement about the bicategory $\mathbb{M}$ associated to the Morita context of an exact G-graded module category: $\mathbb{M}$ is a bicategory enriched in the (non-symmetric) monoidal 2-category of G-graded linear abelian categories and grading preserving functors.

A feature of a G-graded tensor category is that the duals of a homogeneous object are again homogeneous, with inverse degree. This holds for the Morita context of an exact G-graded module category as well:

Proposition 5.32. *Let $\mathcal{A}$ be a G-graded finite tensor category and $\mathcal{M}$ an exact G-graded $\mathcal{A}$-module category.*

(i) *For a homogeneous object $m \in \mathcal{M}_g$, the duals $m^\vee$ and ${}^\vee m$ are homogeneous of degree g^{-1} in $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$.*

(ii) *For a homogeneous object $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})_g$, the duals $H^\vee$ and ${}^\vee H$ are homogeneous of degree g^{-1} in $\mathcal{M}$.*

(iii) *The relative Serre functors of $\mathcal{M}$ and of $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})$ are grading preserving.*

Proof. According to Proposition 5.5 we have $\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) \in \mathcal{A}_{hg^{-1}}$ for $m \in \mathcal{M}_g$ and $n \in \mathcal{M}_h$. This also implies that $\underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) \cong {}^\vee\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(n,m) \in \mathcal{A}_{hg^{-1}}$, which proves (i).
To show (ii), notice that the adjoints of a module functor $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{A})_g$ are homogeneous module functors in $\mathrm{Fun}_{\mathcal{A}}(\mathcal{A},\mathcal{M})_{g^{-1}}$. Since $\mathbf{1} \in \mathcal{A}_e$, it follows that $H^{\mathrm{ra}}(\mathbf{1}), H^{\mathrm{la}}(\mathbf{1}) \in \mathcal{M}_{g^{-1}}$.
Assertion (iii) follows from (i) and (ii) together with Proposition 3.11. □

Now we will consider a G-graded tensor category $\mathcal{A}$ endowed with the additional datum of a pivotal structure.

Definition 5.33. An exact G-graded module $\mathcal{M}$ over a G-graded pivotal category $\mathcal{A}$ is said to be *pivotal* iff the underlying module category has the structure of a pivotal module.

Proposition 5.34. *Let $\mathcal{M}$ be an exact G-graded module category over a G-graded pivotal tensor category $\mathcal{A}$. A pivotal structure on ${}_{\mathcal{A}}\mathcal{M}$ is the same as a pivotal structure on ${}_{\mathcal{A}_e}\mathcal{M}_e$.*

Proof. First notice that for $m, n \in \mathcal{M}_e$ we have $\underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m,n) = \underline{\mathrm{Hom}}^{\mathcal{A}_e}_{\mathcal{M}_e}(m,n)$, and thus the restriction of the relative Serre functor obeys $\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}|_{\mathcal{M}_e} = \mathbb{S}^{\mathcal{A}_e}_{\mathcal{M}_e}$. The pivotal structure of $\mathcal{A}$ turns both $\mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ and $\mathbb{S}^{\mathcal{A}_e}_{\mathcal{M}_e}$ into module functors. According to Proposition 5.32, relative Serre functors are grading preserving, and thus a pivotal structure on ${}_{\mathcal{A}}\mathcal{M}$ is an isomorphism $\mathrm{id}_{\mathcal{M}} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ in $(\mathcal{A}^*_{\mathcal{M}})_e$. Now Theorem 5.8 implies that restriction induces an equivalence

$$(\mathcal{A}^*_{\mathcal{M}})_e = \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{M})_e \simeq \mathrm{Fun}_{\mathcal{A}_e}(\mathcal{M}_e,\mathcal{M}_e) = (\mathcal{A}_e)^*_{\mathcal{M}_e} \tag{5.37}$$

under which a module natural isomorphism $\mathrm{id}_{\mathcal{M}} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}}_{\mathcal{M}}$ corresponds to a module natural isomorphism $\mathrm{id}_{\mathcal{M}_e} \xRightarrow{\cong} \mathbb{S}^{\mathcal{A}_e}_{\mathcal{M}_e}$. ☐

Definition 5.35. Two G-graded pivotal categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *graded pivotal Morita equivalent* iff there exists a G-graded pivotal $\mathcal{A}$-module category $\mathcal{M}$ together with a G-graded pivotal equivalence $\mathcal{B} \simeq \overline{\mathcal{A}^*_{\mathcal{M}}}$.

The Drinfeld center of a pivotal tensor category inherits a pivotal structure [EGNO, Ex. 7.13.6]. In particular for a G-graded pivotal tensor category $\mathcal{A}$, the Drinfeld center $\mathcal{Z}(\mathcal{A})$ has a canonical pivotal structure $p\colon \mathrm{id}_{\mathcal{Z}(\mathcal{A})} \xRightarrow{\cong} (-)^{\vee\vee}$. According to [KO, Thm. 1.17], p serves as pivotal structure for the equivariant center ${}_{\Upsilon_G}\mathrm{Mod}(\mathcal{Z}(\mathcal{A})) \simeq \mathcal{Z}_G(\mathcal{A})$ (Remark 5.22).

Proposition 5.36. *Let $\mathcal{M}$ be a G-graded pivotal $\mathcal{A}$-module category. The braided G-crossed equivalence* (5.15) *is pivotal.*

Proof. By Proposition 4.14 the equivalence (1.69) is pivotal. Further, (5.15) is induced by (1.69) and thus preserves p as well. ☐

Corollary 5.37. *If two G-graded pivotal categories $\mathcal{A}$ and $\mathcal{B}$ are graded pivotal Morita equivalent, then their equivariant centers $\mathcal{Z}_G(\mathcal{A})$ and $\mathcal{Z}_G(\mathcal{B})$ are equivalent as pivotal braided G-crossed tensor categories.*

Chapter 6

Categories with traces and dimensions

Traces are a crucial ingredient for the construction of topological invariants. These structures on categories allow to assign scalars to closed graphs on spheres, which is an important step in constructions such as the Turaev-Viro state sum from semisimple categories or the invariants discussed in [GP, GPV] from non-semisimple categories.

In this chapter, we study traces arising from pivotal structures on bimodule categories. For this purpose, we make use of the canonical *Nakayama twisted trace* (6.15) associated to any $\Bbbk$-linear category. Such twisted trace allows to assign scalars, not to endomorphisms, but to a different class of morphisms in the subcategory of projective objects. Since, the category underlying a bimodule category is $\Bbbk$-linear, it comes naturally equipped with such twisted trace. In Section 6.3, we explore the interaction of the twisted trace with pivotal structures on bimodule categories. We obtain partial-trace properties for bimodule categories.

In Section 6.4, we further explore the properties of the traces constructed for 2-morphisms the pivotal bicategory $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ of spherical module categories over a spherical fusion category $\mathcal{A}$. In Chapter 7, this will allow to evaluate $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$-labeled graphs on spheres to produce a topological invariant.

In this chapter, we will give use of the graphical calculus of string diagrams in bicategories and pivotal bicategories following the conventions summarized in Appendix A. In particular, the graphical calculus will be used in the context of a tensor category seen as a one-object bicategory.

6.1 Calabi-Yau categories and Nakayama twisted traces

Given a finite $\Bbbk$-linear category $\mathcal{X}$, we denote by $\mathrm{Proj}\,\mathcal{X}$ the full subcategory of projective objects in $\mathcal{X}$ and $\mathrm{Irr}\,\mathcal{X}$ stands for the set of equivalence classes of simple objects.

Definition 6.1. [He, Def. 2.9]

1. A *Calabi-Yau category* $(\mathcal{X}, \mathrm{tr}^{\mathcal{X}})$ consists of a finite $\Bbbk$-linear category $\mathcal{X}$, together with a family of *traces*, i.e. $\Bbbk$-linear maps

$$\mathrm{tr}^{\mathcal{X}}_x : \quad \mathrm{Hom}_{\mathcal{X}}(x,x) \longrightarrow \Bbbk \tag{6.1}$$

for each $x \in \mathcal{X}$, obeying the following two conditions.

   (i) Symmetry: Given $x, y \in \mathcal{X}$, it holds

$$\mathrm{tr}^{\mathcal{X}}_y(f \circ g) = \mathrm{tr}^{\mathcal{X}}_x(g \circ f)\ . \tag{6.2}$$

for every $f \in \mathrm{Hom}_{\mathcal{X}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{X}}(y, x)$

(ii) Non-degeneracy: The induced pairing

$$\langle -, - \rangle_{\mathcal{X}} : \quad \mathrm{Hom}_{\mathcal{X}}(x, y) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{X}}(y, x) \longrightarrow \Bbbk, \quad f \otimes g \mapsto \mathrm{tr}_x^{\mathcal{X}}(g \circ f) \tag{6.3}$$

is non-degenerate, i.e. the $\Bbbk$-linear map

$$\mathrm{Hom}_{\mathcal{X}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(y, x)^*, \quad f \longmapsto \mathrm{tr}_x^{\mathcal{X}}(- \circ f) \tag{6.4}$$

is an isomorphism for every $x, y \in \mathcal{X}$.

2. Given Calabi-Yau categories $(\mathcal{X}, \mathrm{tr}^{\mathcal{X}})$ and $(\mathcal{Y}, \mathrm{tr}^{\mathcal{Y}})$ a functor $F \in \mathrm{Rex}(\mathcal{X}, \mathcal{Y})$ is said to be a *Calabi-Yau* functor, if

$$\mathrm{tr}_x^{\mathcal{X}}(f) = \mathrm{tr}_{F(x)}^{\mathcal{Y}}(F(f)) \tag{6.5}$$

for every $x \in \mathcal{X}$ and $f \in \mathrm{Hom}_{\mathcal{X}}(x, x)$.

3. A *Calabi-Yau* natural transformation between Calabi-Yau functors is just a natural transformation.

Remark 6.2. The datum of a Calabi-Yau structure on a finite semisimple $\Bbbk$-linear category $\mathcal{X}$ is equivalent to the data of a collection of natural isomorphisms

$$\mathrm{Hom}_{\mathcal{X}}(x, y) \cong \mathrm{Hom}_{\mathcal{X}}(y, x)^* \tag{6.6}$$

for objects $x, y \in \mathcal{X}$ [Sc, Prop. 4.1].

In a Calabi-Yau category there is the notion of *dimension of an object* $x \in \mathcal{X}$ which is given by

$$\dim(x) := \mathrm{tr}_x^{\mathcal{X}}(\mathrm{id}_x) \in \Bbbk. \tag{6.7}$$

The *categorical dimension of* $\mathcal{X}$ is defined as the scalar

$$\dim(\mathcal{X}) := \sum_{x \in \mathrm{Irr}\mathcal{X}} \dim(x)^2 \in \Bbbk. \tag{6.8}$$

Proposition 6.3. *Let* $(\mathcal{X}, \mathrm{tr}^{\mathcal{X}})$ *be a semisimple Calabi-Yau category.*

(i) *Given a simple object* $h \in \mathrm{Irr}\mathcal{X}$*, any endomorphism* $f \in \mathrm{Hom}_{\mathcal{X}}(h, h)$ *can be expressed as*

$$f = \frac{\mathrm{tr}_h^{\mathcal{X}}(f)}{\dim(h)} \cdot \mathrm{id}_h \, . \tag{6.9}$$

(ii) *For every object* $x \in \mathcal{X}$ *it holds that*

$$\dim(x) = \sum_{h \in \mathrm{Irr}\mathcal{X}} \dim(h) \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{X}}(h, x) \, . \tag{6.10}$$

Proof. To prove (i) notice that, due to simplicity of h, $f = \lambda . \mathrm{id}_h$ for some $\lambda \in \Bbbk^*$. It follows that $\mathrm{tr}_h^{\mathcal{X}}(f) = \lambda \cdot \dim(h)$, which leads to the result. Now since $\mathcal{X}$ is semisimple hom-spaces have a decomposition of the form

$$\mathrm{Hom}_{\mathcal{X}}(x, x) \cong \bigoplus_{h \in \mathrm{Irr}\mathcal{X}} \mathrm{Hom}_{\mathcal{X}}(x, h) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{X}}(h, x) \tag{6.11}$$

and thus for each $h \in \operatorname{Irr} \mathcal{X}$, there are bases $\{\psi_i^*\}$ and $\{\psi_i\}$ of $\operatorname{Hom}_{\mathcal{X}}(h,x)$ and $\operatorname{Hom}_{\mathcal{X}}(x,h)$ such that

$$\psi_i \circ \psi_j^* = \delta_{i,j} \cdot \mathrm{id}_h, \qquad \text{and} \qquad \sum_{h \in \operatorname{Irr}\mathcal{X}} \sum_i \psi_i^* \circ \psi_i = \mathrm{id}_x \tag{6.12}$$

Summing over i and j gives $\sum_i \psi_i \circ \psi_i^* = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{X}}(h,x) \cdot \mathrm{id}_h$, and thus

$$\dim(x) = \operatorname{tr}_x^{\mathcal{X}}(\mathrm{id}_x) \overset{(6.12)}{=} \sum_{h \in \operatorname{Irr}\mathcal{X}} \sum_i \operatorname{tr}_x^{\mathcal{X}}(\psi_i^* \circ \psi_i) \overset{(6.2)}{=} \sum_{h \in \operatorname{Irr}\mathcal{X}} \sum_i \operatorname{tr}_h^{\mathcal{X}}(\psi_i \circ \psi_i^*) \tag{6.13}$$

$$= \sum_{h \in \operatorname{Irr}\mathcal{X}} \dim(h) \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{X}}(h,x) \tag{6.14}$$

which proves statement (ii). □

The canonical Nakayama twisted trace of a linear category

In general, non-semisimple $\Bbbk$-linear categories may not admit Calabi-Yau structures [He, Example 3.12]. However, there is a canonical trace structure on any such category once we consider a different class of morphisms to be traced: Recall that every finite $\Bbbk$-linear category $\mathcal{X}$ comes with a distinguished right exact endofunctor called the Nakayama functor (1.55). For every projective object $p \in \operatorname{Proj} \mathcal{X}$ there exists a linear map

$$\mathbf{t}_p^{\mathcal{X}} : \quad \operatorname{Hom}_{\mathcal{X}}(p, \mathbb{N}_{\mathcal{X}}^r(p)) \longrightarrow \Bbbk \tag{6.15}$$

named the $\mathbb{N}_{\mathcal{X}}^r$-*twisted trace* or simply the twisted trace of $\mathcal{X}$ [SchW, Def. 2.4] [ShSh, Def. 4.4].

Lemma 6.4. [SchW, Lemma 2.5] *Let $\mathcal{X}$ be a finite $\Bbbk$-linear category. The twisted trace* (6.15) *satisfies the following properties:*

(i) *Cyclicity: Given $p, q \in \operatorname{Proj} \mathcal{X}$*

$$\mathbf{t}_q^{\mathcal{X}}(f \circ g) = \mathbf{t}_p^{\mathcal{X}}\left(\mathbb{N}_{\mathcal{X}}^r(g) \circ f \right) \tag{6.16}$$

for every $f \in \operatorname{Hom}_{\mathcal{X}}(p, \mathbb{N}_{\mathcal{X}}^r(q))$ and $g \in \operatorname{Hom}_{\mathcal{X}}(q,p)$.

(ii) *Non-degeneracy: The induced pairing*

$$\langle -,- \rangle_{\mathcal{X}} : \quad \operatorname{Hom}_{\mathcal{X}}(p,x) \otimes_{\Bbbk} \operatorname{Hom}_{\mathcal{X}}(x, \mathbb{N}_{\mathcal{X}}^r(p)) \longrightarrow \Bbbk, \qquad f \otimes g \mapsto \mathbf{t}_p^{\mathcal{X}}(g \circ f) \tag{6.17}$$

is non-degenerate for $p \in \operatorname{Proj} \mathcal{X}$ and $x \in \mathcal{X}$.

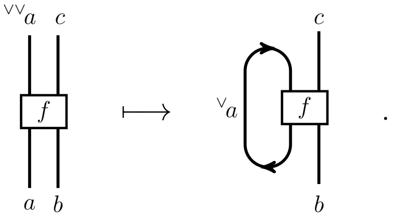
6.2 Traces on tensor categories

Tensor categories are a classical source of categories with traces. The monoidal structure on a category permit the definition of *partial traces*. These lead to the so-called *quantum traces*, quantities defined only for certain class of morphisms in a tensor category, where appropriate instances of double-duals appear in their target. This section discusses their interaction with the Nakayama twisted trace.

Definition 6.5. Let $\mathcal{A}$ be a finite tensor category.

(i) For objects $a, b, c \in \mathcal{A}$, the *left partial trace* with respect to a is the map given by

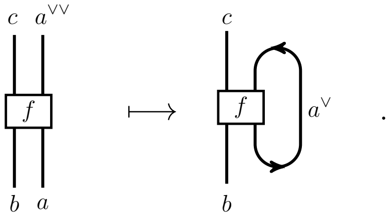
$$\mathrm{tr}_{\mathrm{l}}^{a} : \quad \mathrm{Hom}_{\mathcal{A}}(a \otimes b, {}^{\vee\vee}a \otimes c) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(b, c)$$



(6.18)

(ii) Similarly, the *right partial trace* with respect to a is defined by the map

$$\mathrm{tr}_{\mathrm{r}}^{a} : \quad \mathrm{Hom}_{\mathcal{A}}(b \otimes a, c \otimes a^{\vee\vee}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(b, c)$$



(6.19)

Now, in a finite tensor category, the monoidal unit is a simple object. Thus, it is natural to think of its endomorphisms as scalars in the base field. In the literature [TV, Sec. 4.2.3] [EGNO, Rem. 4.7.2] it is standard to make this identification via the map

$$\begin{aligned} \mathrm{tr}_{\mathbf{1}}^{\mathcal{A}} : \quad \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) &\longrightarrow \Bbbk, \\ \mathrm{id}_{\mathbf{1}} &\longmapsto 1 \end{aligned} \tag{6.20}$$

which is an isomorphism of $\Bbbk$-algebras. Left and right (quantum) traces are then simply defined as the scalars associated to partial traces under this identification.

Definition 6.6. [EGNO, Def. 4.7.1] Let $\mathcal{A}$ be a finite tensor category. Define the *left* (*quantum*) *trace* as the map

$$\mathrm{tr}_{\mathrm{L}}^{\mathcal{A}} : \quad \mathrm{Hom}_{\mathcal{A}}(a, {}^{\vee\vee}a) \xrightarrow{\mathrm{tr}_{\mathrm{l}}^{a}} \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathrm{tr}_{\mathbf{1}}^{\mathcal{A}}} \Bbbk \tag{6.21}$$

for an object $a \in \mathcal{A}$. *Right* (*quantum*) *traces* can be defined in complete analogy as

$$\mathrm{tr}_{\mathrm{R}}^{\mathcal{A}} : \quad \mathrm{Hom}_{\mathcal{A}}(a, a^{\vee\vee}) \xrightarrow{\mathrm{tr}_{\mathrm{r}}^{a}} \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathrm{tr}_{\mathbf{1}}^{\mathcal{A}}} \Bbbk \,. \tag{6.22}$$

The double-duals at the target of the morphisms in Definition 6.6 can be trivialized with the use of a pivotal structure leading to traces on endomorphisms. However, there is another obstacle to obtain a Calabi-Yau category: In general, quantum traces may be degenerate if the underlying category is non-semisimple [EGNO, Rem. 4.8.5].

To avoid this, we now consider the $\mathbb{N}_{\mathcal{A}}^{r}$-twisted trace associated to the $\Bbbk$-linear category underlying a finite tensor category $\mathcal{A}$. We can extend the partial traces to morphisms with target in the image of the Nakayama functor [SchW, Def. 3.3]. This is done by means of its twisted $\mathcal{A}$-bimodule functor structure. Explicitly, the *left partial trace* is given for $a, b \in \mathcal{A}$ by the composition:

$$\mathrm{tr}_{\mathrm{l}}^{a} : \quad \mathrm{Hom}_{\mathcal{A}}(a \otimes b, \mathbb{N}_{\mathcal{A}}^{r}(a \otimes b)) \xrightarrow{(1.62)} \mathrm{Hom}_{\mathcal{A}}(a \otimes b, {}^{\vee\vee}a \otimes \mathbb{N}_{\mathcal{A}}^{r}(b)) \xrightarrow{(6.18)} \mathrm{Hom}_{\mathcal{A}}(b, \mathbb{N}_{\mathcal{A}}^{r}(b)) \tag{6.23}$$

Similarly, the *right partial trace* is defined by the following composition.

$$\mathrm{tr}_{\mathrm{r}}^{a}: \quad \mathrm{Hom}_{\mathcal{A}}(b \otimes a, \mathbb{N}_{\mathcal{A}}^{r}(b \otimes a)) \xrightarrow{(1.62)} \mathrm{Hom}_{\mathcal{A}}(b \otimes a, \mathbb{N}_{\mathcal{A}}^{r}(b) \otimes a^{\vee\vee}) \xrightarrow{(6.19)} \mathrm{Hom}_{\mathcal{A}}(b, \mathbb{N}_{\mathcal{A}}^{r}(b)) \quad (6.24)$$

for all objects $a, b \in \mathcal{A}$. The morphisms obtained by applying partial traces can still be interpreted as scalars under (6.15) at the cost of restricting our attention to projective objects.

Proposition 6.7. [SchW, Prop. 3.5]
Let $\mathcal{A}$ be a finite tensor category, for any projective object $p \in \mathrm{Proj}\,\mathcal{A}$, the twisted trace (6.15)

$$\mathbf{t}_{p}^{\mathcal{A}}: \quad \mathrm{Hom}_{\mathcal{A}}(p, \mathbb{N}_{\mathcal{A}}^{r}(p)) \longrightarrow \Bbbk$$

satisfies the right and left partial trace properties:
Given $a \in \mathcal{A}$, and morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(p \otimes a,\, \mathbb{N}_{\mathcal{A}}^{r}(p \otimes a))$ and $g \in \mathrm{Hom}_{\mathcal{A}}(a \otimes p,\, \mathbb{N}_{\mathcal{A}}^{r}(a \otimes p))$

$$\mathbf{t}_{p\otimes a}^{\mathcal{A}}(f) = \mathbf{t}_{p}^{\mathcal{A}}\, \mathrm{tr}_{\mathrm{r}}^{a}(f), \qquad \mathbf{t}_{a\otimes p}^{\mathcal{A}}(g) = \mathbf{t}_{p}^{\mathcal{A}}\, \mathrm{tr}_{\mathrm{l}}^{a}(g) \quad (6.25)$$

i.e. the following diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(p \otimes a, \mathbb{N}_{\mathcal{A}}^{r}(p \otimes a)) \arrow[rr, "\mathbf{t}_{p\otimes a}^{\mathcal{A}}"] \arrow[d, "\mathrm{tr}_{\mathrm{r}}^{a}"'] & & \Bbbk & \mathrm{Hom}_{\mathcal{A}}(a \otimes p, \mathbb{N}_{\mathcal{A}}^{r}(a \otimes p)) \arrow[rr, "\mathbf{t}_{a\otimes p}^{\mathcal{A}}"] \arrow[d, "\mathrm{tr}_{\mathrm{l}}^{a}"'] & & \Bbbk \\
\mathrm{Hom}_{\mathcal{A}}(p, \mathbb{N}_{\mathcal{A}}^{r}(p)) \arrow[urr, "\mathbf{t}_{p}^{\mathcal{A}}"'] & & & \mathrm{Hom}_{\mathcal{A}}(p, \mathbb{N}_{\mathcal{A}}^{r}(p)) \arrow[urr, "\mathbf{t}_{p}^{\mathcal{A}}"'] & &
\end{tikzcd}$$

(6.26)

Proof. In [SchW, Prop. 3.5] the authors prove the right partial trace property. The argument strongly relies on the fact that the Nakayama functor $\mathbb{N}_{\mathcal{A}}^{r}$ comes with a canonical right twisted structure $\mathbb{N}_{\mathcal{A}}^{r}(- \otimes a) \cong \mathbb{N}_{\mathcal{A}}^{r}(-) \otimes a^{\vee\vee}$. The same argument can be made into a proof of the left partial trace property by mutatis mutandis where now the canonical left twisted structure $\mathbb{N}_{\mathcal{A}}^{r}(a \otimes -) \cong {}^{\vee\vee}a \otimes \mathbb{N}_{\mathcal{A}}^{r}(-)$ coming from (1.62) is in play. □

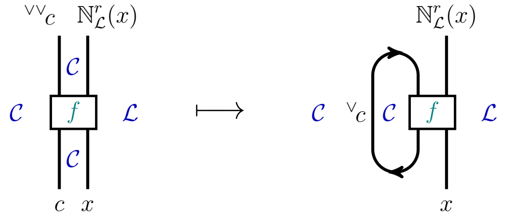
One can further consider a choice of pivotal structure on the finite tensor category $\mathcal{A}$ as pointed out in [SchW, Thm. 3.6] and [ShSh, Thm. 6.8]. We leave this aspect for Section 6.3, where we study a more general situation involving bimodule categories.

6.3 Traces on module categories

In [Sc] the notion of a module trace in the fusion setting is defined. In [ShSh] the authors study twisted traces for right module categories beyond the semisimplicity requirement. We explore in this section the Nakayama twisted trace on bimodule categories under the presence of pivotal structures.

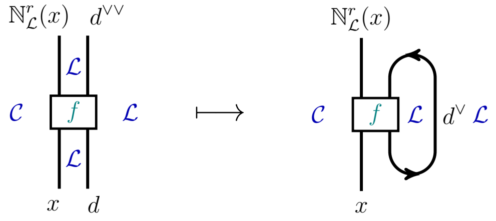
In the situation of a bimodule category ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ over finite tensor categories, the underlying categories are $\Bbbk$-linear and thus each comes naturally equipped with Nakayama twisted traces. The structure of bimodule category on $\mathcal{L}$ together with the twisted bimodule functor structure on $\mathbb{N}^r_{\mathcal{L}}$ allow to define *left and right partial traces* in complete analogy to the case of a tensor category. These are given for objects $x \in \mathcal{L}$, $c \in \mathcal{C}$ and $d \in \mathcal{D}$ by the compositions:

$$\mathrm{tr}^c_{\mathrm{l}} : \quad \mathrm{Hom}_{\mathcal{L}}\,(c \triangleright x, \mathbb{N}^r_{\mathcal{L}}(c \triangleright x)) \xrightarrow{(1.62)} \mathrm{Hom}_{\mathcal{L}}(c \triangleright x, {}^{\vee\vee}c \triangleright \mathbb{N}^r_{\mathcal{L}}(x)) \longrightarrow \mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^r_{\mathcal{L}}(x))$$



(6.27)

$$\mathrm{tr}^d_{\mathrm{r}} : \quad \mathrm{Hom}_{\mathcal{L}}(x \triangleleft d, \mathbb{N}^r_{\mathcal{L}}(x \triangleleft d)) \xrightarrow{(1.62)} \mathrm{Hom}_{\mathcal{L}}(x \triangleleft d, \mathbb{N}^r_{\mathcal{L}}(x) \triangleleft d^{\vee\vee}) \longrightarrow \mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^r_{\mathcal{L}}(x))$$



(6.28)

In Section 3.1 we studied dualities for strong Morita contexts. This notion of duals allows to extend partial traces with respect to objects in invertible bimodule categories. These are set up by means of the following isomorphisms which are analogous to the twisted module structure of the Nakayama functor.

Lemma 6.8. *Let ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be an invertible bimodule category over finite tensor categories. There are natural isomorphisms*

$$\mathbb{N}^r_{\mathcal{L}}(c \triangleright y) \cong \mathbb{N}^r_{\mathcal{C}}(c) \triangleright y^{\vee\vee} \qquad \text{and} \qquad \mathbb{N}^r_{\mathcal{L}}(y \triangleleft d) \cong {}^{\vee\vee}y \triangleleft \mathbb{N}^r_{\mathcal{D}}(d) \tag{6.29}$$

for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $y \in \mathcal{L}$.

Proof. The isomorphisms can be obtained by using that double-duals are isomorphic to relative Serre functors and juggling distinguished invertible objects around. Explicitly, for the first isomorphism in (6.29) consider the composition

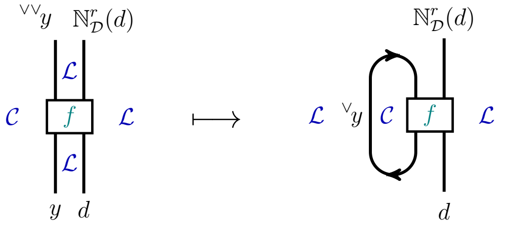
$$\mathbb{N}^r_{\mathcal{L}}(c \triangleright y) \cong {}^{\vee\vee}c \triangleright \mathbb{N}^r_{\mathcal{L}}(y) \cong {}^{\vee\vee}c \otimes \mathbb{D}^{-1}_{\mathcal{C}} \triangleright \mathbb{S}^{\mathcal{C}}_{\mathcal{L}}(y) \cong \mathbb{N}^r_{\mathcal{C}}(c) \triangleright y^{\vee\vee} \tag{6.30}$$

where the first isomorphism comes from the twisted structure (1.62) of the Nakayama functor, the second is (1.63) and the last one uses (1.61) and Proposition 3.11 (i). □

Definition 6.9. Let ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be an invertible bimodule category over finite tensor categories.

(i) Given $y \in \mathcal{L}$ and $d \in \mathcal{D}$, the *left partial trace* with respect to y is defined by the composition

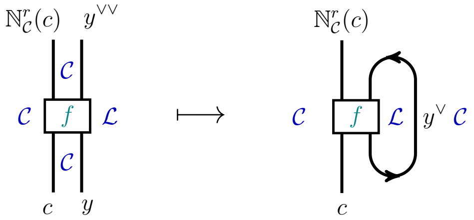
$$\mathrm{tr}^y_{\mathrm{l}} : \quad \mathrm{Hom}_{\mathcal{L}}(y \triangleleft d, \mathbb{N}^r_{\mathcal{L}}(y \triangleleft d)) \xrightarrow{(6.29)} \mathrm{Hom}_{\mathcal{L}}(y \triangleleft d, {}^{\vee\vee}y \triangleleft \mathbb{N}^r_{\mathcal{D}}(d)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(d, \mathbb{N}^r_{\mathcal{D}}(d))$$



(6.31)

(ii) Similarly, for objects $y \in \mathcal{L}$ and $c \in \mathcal{C}$ the *right partial trace* with respect to y is given by the composition

$$\mathrm{tr}^y_{\mathrm{r}} : \quad \mathrm{Hom}_{\mathcal{L}}(c \triangleright y, \mathbb{N}^r_{\mathcal{L}}(c \triangleright y)) \xrightarrow{(6.29)} \mathrm{Hom}_{\mathcal{L}}(c \triangleright y, \mathbb{N}^r_{\mathcal{C}}(c) \triangleright y^{\vee\vee}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(c, \mathbb{N}^r_{\mathcal{C}}(c))$$



(6.32)

Proposition 6.10. *Let ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be a bimodule category over finite tensor categories. For any projective object $x \in \mathrm{Proj}\,\mathcal{L}$, the twisted trace* (6.15)

$$\mathbf{t}^{\mathcal{L}}_x : \quad \mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^r_{\mathcal{L}}(x)) \longrightarrow \Bbbk$$

satisfies right and left partial trace properties: Given $x \in \mathrm{Proj}\,\mathcal{L}$, $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and

$$\mathbf{t}^{\mathcal{L}}_{x \triangleleft d}(f) = \mathbf{t}^{\mathcal{L}}_x\, \mathrm{tr}^d_{\mathrm{r}}(f), \qquad \textit{where} \;\; f \in \mathrm{Hom}_{\mathcal{L}}(x \triangleleft d,\, \mathbb{N}^r_{\mathcal{L}}(x \triangleleft d)) \tag{6.33}$$

$$\mathbf{t}^{\mathcal{L}}_{c \triangleright x}(g) = \mathbf{t}^{\mathcal{L}}_x\, \mathrm{tr}^c_{\mathrm{l}}(g), \qquad \textit{where} \;\; g \in \mathrm{Hom}_{\mathcal{L}}(c \triangleright x,\, \mathbb{N}^r_{\mathcal{L}}(c \triangleright x)) \tag{6.34}$$

i.e. the following diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{L}}(x \triangleleft d, \mathbb{N}^r_{\mathcal{L}}(x \triangleleft d)) \arrow[r, "\mathbf{t}^{\mathcal{L}}_{x \triangleleft d}"] \arrow[d, "\mathrm{tr}^d_{\mathrm{r}}"'] & \Bbbk & \mathrm{Hom}_{\mathcal{L}}(c \triangleright x, \mathbb{N}^r_{\mathcal{L}}(c \triangleright x)) \arrow[r, "\mathbf{t}^{\mathcal{L}}_{c \triangleright x}"] \arrow[d, "\mathrm{tr}^c_{\mathrm{l}}"'] & \Bbbk \\
\mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^r_{\mathcal{L}}(x)) \arrow[ur, "\mathbf{t}^{\mathcal{L}}_x"'] & & \mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^r_{\mathcal{L}}(x)) \arrow[ur, "\mathbf{t}^{\mathcal{L}}_x"'] &
\end{tikzcd} \tag{6.35}$$

Furthermore, if $\mathcal{L}$ is invertible, then for objects $y \in \mathcal{L}$, $p \in \mathrm{Proj}\,\mathcal{C}$ and $q \in \mathrm{Proj}\,\mathcal{D}$

$$\mathbf{t}^{\mathcal{L}}_{y \triangleleft q}(f) = \mathbf{t}^{\mathcal{D}}_q\, \mathrm{tr}^y_{\mathrm{l}}(f), \qquad \textit{where} \;\; f \in \mathrm{Hom}_{\mathcal{L}}(y \triangleleft q,\, \mathbb{N}^r_{\mathcal{L}}(y \triangleleft q)) \tag{6.36}$$

$$\mathbf{t}^{\mathcal{L}}_{p \triangleright y}(g) = \mathbf{t}^{\mathcal{C}}_p\, \mathrm{tr}^y_{\mathrm{r}}(g), \qquad \textit{where} \;\; g \in \mathrm{Hom}_{\mathcal{L}}(p \triangleright y,\, \mathbb{N}^r_{\mathcal{L}}(p \triangleright y)) \tag{6.37}$$

i.e. the following diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{L}}(p \triangleright y, \mathbb{N}^r_{\mathcal{L}}(p \triangleright y)) \arrow[r, "\mathbf{t}^{\mathcal{L}}_{p \triangleright y}"] \arrow[d, "\mathrm{tr}^y_{\mathrm{r}}"'] & \Bbbk & \mathrm{Hom}_{\mathcal{L}}(y \triangleleft q, \mathbb{N}^r_{\mathcal{L}}(y \triangleleft q)) \arrow[r, "\mathbf{t}^{\mathcal{L}}_{y \triangleleft q}"] \arrow[d, "\mathrm{tr}^y_{\mathrm{l}}"'] & \Bbbk \\
\mathrm{Hom}_{\mathcal{C}}(p, \mathbb{N}^r_{\mathcal{C}}(p)) \arrow[ur, "\mathbf{t}^{\mathcal{C}}_p"'] & & \mathrm{Hom}_{\mathcal{D}}(q, \mathbb{N}^r_{\mathcal{D}}(q)) \arrow[ur, "\mathbf{t}^{\mathcal{D}}_q"'] &
\end{tikzcd} \tag{6.38}$$

Proof. The result follows in complete analogy to Proposition 6.7. The commutativity of the diagrams (6.35) is obtained by using the same argument on the twisted structure of $\mathbb{N}^r_{\mathcal{L}}$ given by (1.62) (see also [ShSh, Lemma 5.5]). The second part of the proposition requires invertibility for two reasons: On the one hand, this implies exactness on $\mathcal{L}$ and thus $p \triangleright y$ and $y \triangleleft q$ are projective which ensures that the traces $\mathbf{t}^{\mathcal{L}}_{y \triangleleft q}(f)$ and $\mathbf{t}^{\mathcal{L}}_{p \triangleright y}(g)$ in equations (6.36) and (6.37) are defined. Secondly, invertibility allows to define partial traces for objects in $\mathcal{L}$ as described in Definition 6.9. Again, the statement can be proved by means of the argument in Proposition 6.7 mutatis mutandis where the isomorphisms (6.29) play the role of the twisted structure of the Nakayama functor. $\square$

The role of pivotality

The addition of pivotal structures in the mix permit to trivialize the Nakayama functors up to the action of the distinguished invertible objects. Under the assumption of unimodularity we can apply the Nakayama twisted traces to endomorphisms of projective objects.

Definition 6.11. Let $\mathcal{C}$ and $\mathcal{D}$ be pivotal finite tensor categories and ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be a pivotal bimodule category with pivotal structures $\widetilde{p} : \mathrm{id}_{\mathcal{L}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\mathcal{C}}_{\mathcal{L}}$ and $\widetilde{q} : \mathrm{id}_{\mathcal{L}} \overset{\cong}{\Longrightarrow} \mathbb{S}^{\overline{\mathcal{D}}}_{\mathcal{L}}$.
The *left trace* is the map defined as the composition

$$\mathrm{tr}^{\mathcal{L}}_{\mathrm{L}\ x} : \quad \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{D}^{-1}_{\mathcal{C}} \triangleright x\right) \xrightarrow{\tilde{p}\circ -} \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{D}^{-1}_{\mathcal{C}} \triangleright \mathbb{S}^{\mathcal{C}}_{\mathcal{L}}(x)\right) \xrightarrow{(1.63)} \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{N}^r_{\mathcal{L}}(x)\right) \xrightarrow{(6.15)} \Bbbk \tag{6.39}$$

for any projective object $x \in \mathrm{Proj}\,\mathcal{L}$. Similarly, the *right trace* is given by

$$\mathrm{tr}^{\mathcal{L}}_{\mathrm{R}\ x} : \quad \mathrm{Hom}_{\mathcal{L}}\left(x, x \triangleleft \mathbb{D}^{-1}_{\mathcal{D}}\right) \xrightarrow{\tilde{q}\circ -} \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{S}^{\overline{\mathcal{D}}}_{\mathcal{L}}(x) \triangleleft \mathbb{D}^{-1}_{\mathcal{D}}\right) \xrightarrow{(1.63)} \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{N}^r_{\mathcal{L}}(x)\right) \xrightarrow{(6.15)} \Bbbk\,. \tag{6.40}$$

We obtain traces for endomorphisms of projective objects, if we require unimodularity. However, these traces will depend on the choice of trivializations of the corresponding distinguished invertible objects. One could further define a notion of trace-sphericality subject to such choices, in a similar fashion as bimodule sphericality (see Definition 4.19).

For now, we leave this analysis on the side and we focus on the semisimple setting. In that case, tensor categories are unimodular and every object is projective. Moreover, there is a special trivialization of the distinguished invertible object compatible with the twisted trace associated to the monoidal unit.

Definition 6.12. Let $\mathcal{C}$ be a fusion category. The *standard trivialization of* $\mathbb{D}_{\mathcal{C}}$ is the isomorphism $\mathbf{s}_{\mathcal{C}} : \mathbf{1} \xrightarrow{\cong} \mathbb{N}^r_{\mathcal{C}}(\mathbf{1}) = \mathbb{D}^{-1}_{\mathcal{C}}$ obeying that the composition

$$\mathrm{Hom}_{\mathcal{C}}\left(\mathbf{1}, \mathbf{1}\right) \xrightarrow{\mathbf{s}_{\mathcal{C}}\,\circ\,-} \mathrm{Hom}_{\mathcal{C}}\left(\mathbf{1}, \mathbb{N}^r_{\mathcal{C}}(\mathbf{1})\right) \xrightarrow{(6.15)} \Bbbk \tag{6.41}$$

is equal to the $\Bbbk$-linear map $\mathrm{tr}^{\mathcal{C}}_{\mathbf{1}}$ from equation (6.20), i.e. the assignment $\mathrm{id}_{\mathbf{1}} \mapsto 1$.

Let ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ be a pivotal bimodule category as in Definition 6.11 with the additional assumption of semisimplicity. It holds that every object in $\mathcal{L}$ is projective and that $\mathcal{C}$ and $\mathcal{D}$ are unimodular. Thus, one can trace arbitrary endomorphisms via the standard trivialization of the distinguished invertible objects: Explicitly, for every $x \in \mathcal{L}$ we define the *left trace* for endomorphisms as the composition

$$\mathrm{tr}^{\mathcal{L}}_{\mathrm{L}\ x} : \quad \mathrm{Hom}_{\mathcal{L}}\left(x, x\right) \xrightarrow{\mathbf{s}_{\mathcal{C}} \triangleright \mathrm{id}\,\circ\,-} \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{D}^{-1}_{\mathcal{C}} \triangleright x\right) \xrightarrow{(6.39)} \Bbbk \tag{6.42}$$

and the *right trace* for endomorphisms given by

$$\mathrm{tr}^{\mathcal{L}}_{\mathrm{R}\,x}: \quad \mathrm{Hom}_{\mathcal{L}}(x,x) \xrightarrow{\mathrm{id}\triangleleft \mathbf{s}_{\mathcal{D}}\circ -} \mathrm{Hom}_{\mathcal{L}}\left(x, x\triangleleft \mathbb{D}_{\mathcal{D}}^{-1}\right) \xrightarrow{(6.40)} \Bbbk \tag{6.43}$$

These traces endow the category $\mathcal{L}$ with two Calabi-Yau structures $(\mathcal{L}, \mathrm{tr}^{\mathcal{L}}_{\mathrm{L}})$ and $(\mathcal{L}, \mathrm{tr}^{\mathcal{L}}_{\mathrm{R}})$ in the sense of Definition 6.1. These two agree under sphericality, as we will see in Proposition 6.14. In a purely pivotal (non-spherical) situation, this leads to the notion of left and right dimensions for any object $x \in \mathcal{L}$

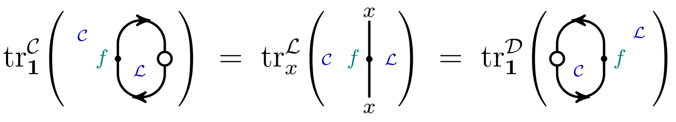
$$\dim_{\mathrm{L}}(x) := \mathrm{tr}^{\mathcal{L}}_{\mathrm{L}\,x}(\mathrm{id}_x), \qquad \dim_{\mathrm{R}}(x) := \mathrm{tr}^{\mathcal{L}}_{\mathrm{R}\,x}(\mathrm{id}_x)\,. \tag{6.44}$$

Remark 6.13. Recall that the notion of sphericality on a pivotal bimodule category ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ from Definition 4.19 depends on the choices of trivialization of the distinguished invertible objects of $\mathcal{C}$ and $\mathcal{D}$. In the semisimple setting, we will always consider sphericality relative to the standard trivializations $\mathbf{s}_{\mathcal{C}}$ and $\mathbf{s}_{\mathcal{D}}$ given in Definition 6.12. The same consideration will be taken for spherical module categories over a spherical fusion category.

Proposition 6.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be spherical fusion categories and ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ a spherical bimodule category. Then*

$$\mathrm{tr}^{\mathcal{L}}_{\mathrm{L}\,x}(f) = \mathrm{tr}^{\mathcal{L}}_{\mathrm{R}\,x}(f) =: \mathrm{tr}^{\mathcal{L}}_{x}(f) \tag{6.45}$$

for all $x \in \mathcal{L}$ and endomorphisms $f \in \mathrm{Hom}_{\mathcal{L}}(x,x)$. The trace $\mathrm{tr}^{\mathcal{L}}$ endows the category $\mathcal{L}$ with the structure of a Calabi-Yau category in the sense of Definition 6.1.
Moreover, in case ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ is invertible, it holds that



(6.46)

for every $f \in \mathrm{Hom}_{\mathcal{L}}(x,x)$, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) \arrow[dr, "\mathrm{tr}^{\mathcal{C}}_{\mathbf{1}}"] & \\
\mathrm{Hom}_{\mathcal{L}}(x,x) \arrow[u, "\mathrm{tr}^{x}_{\mathrm{r}}"] \arrow[d, "\mathrm{tr}^{x}_{\mathrm{l}}"'] \arrow[r, "\mathrm{tr}^{\mathcal{L}}_{x}"] & \Bbbk \\
\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},\mathbf{1}) \arrow[ur, "\mathrm{tr}^{\mathcal{D}}_{\mathbf{1}}"'] &
\end{tikzcd} \tag{6.47}$$

for every object $x \in \mathcal{L}$, where $\mathrm{tr}^{\mathcal{C}}_{\mathbf{1}}$ and $\mathrm{tr}^{\mathcal{D}}_{\mathbf{1}}$ are the $\Bbbk$-algebra isomorphisms given by (6.20).

Proof. Equation (6.45) states that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{D}_{\mathcal{C}}^{-1}\triangleright x\right) \arrow[r, "(\mathrm{id}\triangleright\tilde{p})\circ -"] & \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{D}_{\mathcal{C}}^{-1}\triangleright \mathbb{S}^{\mathcal{C}}_{\mathcal{L}}(x)\right) \arrow[dd, "\mathcal{R}_{\mathcal{L}}\circ -"] \arrow[dr, "(1.63)"] & & \\
\mathrm{Hom}_{\mathcal{L}}(x,x) \arrow[u, "\mathbf{s}_{\mathcal{C}}\triangleright\mathrm{id}\circ -"] \arrow[d, "\mathrm{id}\triangleleft\mathbf{s}_{\mathcal{D}}\circ -"'] & & \mathrm{Hom}_{\mathcal{L}}(x, \mathbb{N}^{r}_{\mathcal{L}}(x)) \arrow[r, "(6.15)"] & \Bbbk \\
\mathrm{Hom}_{\mathcal{L}}\left(x, x\triangleleft \mathbb{D}_{\mathcal{D}}^{-1}\right) \arrow[r, "(\tilde{q}\triangleleft\mathrm{id})\circ -"'] & \mathrm{Hom}_{\mathcal{L}}\left(x, \mathbb{S}^{\overline{\mathcal{D}}}_{\mathcal{L}}(x)\triangleleft \mathbb{D}_{\mathcal{D}}^{-1}\right) \arrow[ur, "(1.63)"'] & &
\end{tikzcd}$$

(6.48)

This is indeed the case, since the diagram in the left is the sphericality condition on $\mathcal{L}$ and the triangle in the right is the definition of the Radford isomorphism. The relation (6.46) follows from (6.37) and (6.36) applied to the objects $\mathbf{1} \in \mathcal{C}$ and $\mathbf{1} \in \mathcal{D}$ which are projective due to semisimplicity. □

Remark 6.15. The traces (6.42) and (6.43) applied to the regular bimodule category ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ give back the classical quantum traces from Definition 6.6 in the semisimple pivotal setting. In this case, Proposition 6.14 is the statement that, under semisimplicity, sphericality implies trace-sphericality.

6.4 Traces in the 2-category of spherical module categories

We will focus our attention on a specific type of 2-category with traces for 2-endomorphims. Let $\mathcal{A}$ be spherical fusion category, and denote by $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ the 2-category that has spherical $\mathcal{A}$-module categories as objects, $\mathcal{A}$-module functors as 1-morphisms and module natural transformations as 2-morphisms. Roughly speaking, this 2-category is, at the level of Hom-categories, a collection of invertible spherical bimodule categories: Given two indecomposable spherical module categories $\mathcal{M}$ and $\mathcal{N}$, the $(\mathcal{A}^*_{\mathcal{N}}, \mathcal{A}^*_{\mathcal{M}})$-bimodule category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is invertible by Lemma 3.18 and naturally endowed with a spherical structure by Proposition 4.23. Hence, $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ is a Calabi-Yau category according to Proposition 6.14, in other words, the 2-category $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ is locally Calabi-Yau.

Moreover, as described in Section 4.3, $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ has the structure of a pivotal bicategory given by (4.32). From now on, the identity module functor $\mathrm{id}_{\mathcal{M}}$ of an $\mathcal{A}$-module $\mathcal{M}$ will be denoted by $\mathbf{1}_{\mathcal{M}}$ to emphasize its role as monoidal unit of $\mathcal{A}^*_{\mathcal{M}}$.

Remark 6.16. Recall that in a spherical fusion category $\mathcal{A}$, the following properties involving traces and dimensions hold:

(i) The dimension of an object $a \in \mathcal{A}$ equals the dimension of its dual: $\dim(a) = \dim(a^\vee)$.

(ii) Traces are multiplicative: given objects $a, b \in \mathcal{A}$,

$$\mathrm{tr}^{\mathcal{A}}_{a\otimes b}(f \otimes g) = \mathrm{tr}^{\mathcal{A}}_{a}(f)\ \mathrm{tr}^{\mathcal{A}}_{b}(g) \tag{6.49}$$

for every $f \in \mathrm{Hom}_{\mathcal{A}}(a,a)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(b,b)$.

(iii) Dimensions are multiplicative: for $a, b \in \mathcal{A}$ it holds that

$$\dim(a \otimes b) = \dim(a)\ \dim(b)\ . \tag{6.50}$$

These familiar properties generalize to $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$, as we show next.

Proposition 6.17. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ be indecomposable spherical module categories over a spherical fusion category $\mathcal{A}$.*

(i) *For every $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, it holds that*

$$\dim(H) = \dim(H^{\mathrm{la}})\ . \tag{6.51}$$

(ii) *Given* $H_1 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *and* $H_2 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})$ *and module natural transformations* $\alpha : H_1 \Rightarrow H_1$ *and* $\beta : H_2 \Rightarrow H_2$

$$\mathrm{tr}_{H_2 \circ H_1}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{L})}(\beta \circ \alpha) = \mathrm{tr}_{H_2}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})}(\beta) \cdot \mathrm{tr}_{H_1}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\alpha) \tag{6.52}$$

i.e. the horizontal composition functor

$$\circ : \quad \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L}) \boxtimes \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{L}) \tag{6.53}$$

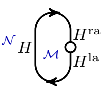
is a Calabi-Yau functor.

(iii) *Given module functors* $H_1 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *and* $H_2 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})$*, it holds that*

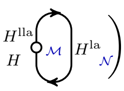
$$\dim(H_2 \circ H_1) = \dim(H_2) \cdot \dim(H_1) \ . \tag{6.54}$$

Proof. Assertion (i) follows from the computation

$$\dim(H) = \mathrm{tr}_H^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\mathrm{id}_H) = \mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}\Bigg($$



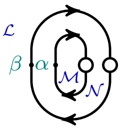
$$\Bigg) = \mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}\Bigg($$



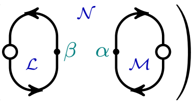
$$\Bigg) = \mathrm{tr}_{H^{\mathrm{la}}}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{M})}(\mathrm{id}_{H^{\mathrm{la}}}) = \dim(H^{\mathrm{la}}) \tag{6.55}$$

where the second and fourth equality make use of (6.46). Similarly, to prove (ii) notice that

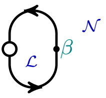
$$\mathrm{tr}_{H_2 \circ H_1}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{L})}(\beta \circ \alpha) = \mathrm{tr}_{\mathbf{1}_\mathcal{L}}^{\mathcal{A}^*_\mathcal{L}}\Bigg($$



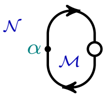
$$\Bigg) = \mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}\Bigg($$



$$\Bigg) = \mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}\Bigg($$



$$\Bigg) \cdot \mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}\Bigg($$



$$\Bigg) = \mathrm{tr}_{H_2}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{L})}(\beta) \cdot \mathrm{tr}_{H_1}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\alpha) \tag{6.56}$$

where the second equality is (6.46) applied to the 2-morphism $\beta \cdot \mathrm{tr}_{\mathrm{r}}^{H_1}(\alpha)$ and the third step follows from the multiplicativity of $\mathrm{tr}_{\mathbf{1}_\mathcal{N}}^{\mathcal{A}^*_\mathcal{N}}$. First and last steps also follow from (6.46). Statement (iii) directly follows by applying (ii) to the respective identity 2-morphisms. □

Further identities proper of the semisimple setting generalize to $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ as well.

Proposition 6.18. *Let* $\mathcal{A}$ *be a spherical fusion category,* $\mathcal{M}, \mathcal{N}$ *indecomposable spherical* $\mathcal{A}$*-module categories, and* $H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ *a simple module functor.*

(i) *For every module natural transformation* $\alpha : H \Rightarrow H$*, it holds that*

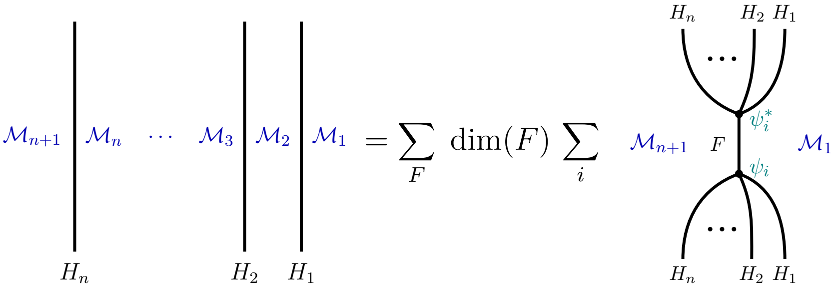
$$\alpha = \frac{\mathrm{tr}_{\mathrm{H}}^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\alpha)}{\dim(H)}\ \mathrm{id}_H \tag{6.57}$$

(ii) *Let* $\beta : H \Rightarrow H$ *and* $\alpha : H \Rightarrow H$ *be module natural transformations, then*

$$\dim(H) \cdot \mathrm{tr}_H^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\beta \cdot \alpha) = \mathrm{tr}_H^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\beta) \cdot \mathrm{tr}_H^{\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})}(\alpha) \ . \tag{6.58}$$

Given indecomposable spherical $\mathcal{A}$*-module categories* $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_{n+1}$*:*

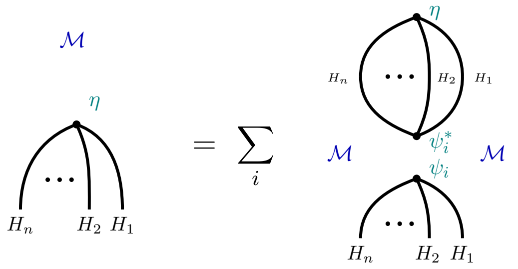
(iii) *For composable simple module functors* $H_1, H_2, \ldots, H_n$*; where* $H_k \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}_k, \mathcal{M}_{k+1})$*, it holds that*



(6.59)

where the sum on F *runs over all simple objects in* $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_{n+1})$ *and* $\{\psi_i^*\}$ *and* $\{\psi_i\}$ *is a pair of dual bases of the vector spaces* $\mathrm{Nat}_{\mathrm{mod}}(H_n \circ \cdots \circ H_1, F)$ *and* $\mathrm{Nat}_{\mathrm{mod}}(F, H_n \circ \cdots \circ H_1)$*.*

(iv) *Assume in the situation of (iii) that* $\mathcal{M} := \mathcal{M}_1 = \mathcal{M}_{n+1}$*. Given a module natural transformation* $\eta \in \mathrm{Nat}_{\mathrm{mod}}(H_n \circ \cdots \circ H_1, \mathrm{id}_{\mathcal{M}})$*, it holds that*



(6.60)

for a pair $\{\psi_i^*\}$ *and* $\{\psi_i\}$ *of dual bases of the vector spaces* $\mathrm{Nat}_{\mathrm{mod}}(H_n \circ \cdots \circ H_1, F)$ *and* $\mathrm{Nat}_{\mathrm{mod}}(F, H_n \circ \cdots \circ H_1)$*.*

Proof.

(i) The statement is simply Proposition 6.3 (i) applied to the category $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ with associated Calabi-Yau structure given by (6.45).

(ii) It directly follows from (i).

(iii) The statement is a slight generalization of [BK, Lemma 1.1] to this 2-categorical situation and follows from semisimplicity. Without loss of generality, it is enough to prove the case $n = 1$ for every simple $K \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_{n+1})$. In that case, $\mathrm{Nat}_{\mathrm{mod}}(K, F)$ is non-zero and one-dimensional only for $F \cong K$, then $\psi \circ \psi^* = \lambda . \mathrm{id}_K$ for some $\lambda \in \Bbbk^*$. Therefore, $\lambda \dim(K) = \mathrm{tr}(\psi \circ \psi^*) = \mathrm{tr}(\psi^* \circ \psi) = 1$, which produces the factor of $\dim(K)$ on (6.59). Since we are working on a semisimple setting, extending the previous argument to direct sums implies the full statement.

(iv) From Schur's lemma, $\mathrm{Nat}_{\mathrm{mod}}(F, \mathbf{1}_{\mathcal{M}}) = 0$ in case F is not isomorphic to the identity functor. Additionally, by the normalization condition $\dim(\mathbf{1}_{\mathcal{M}}) = 1$ and thus the claim follows from (iii).

□

Chapter 7

State sum construction for oriented manifolds

State sum constructions are a fundamental source of topological invariants for oriented manifolds. The classical case in three dimensions is the Turaev-Viro construction [TV] based on the structure of a spherical fusion category. In the last years, the story has been extended to four dimensions: The higher structure of a *fusion 2-category* has been introduced and, together with an adequate 2-spherical structure, it has been used for the construction of topological invariants of four-dimensional oriented manifolds [DR]. An interesting aspect of this work is the role played by the notion of idempotent completion. In the same way that the idempotent completion of a semisimple algebra is its category of modules, the 2-category underlying any fusion 2-category can be obtained as some 2-idempotent completion of a multifusion category.

In the case of a spherical fusion category $\mathcal{A}$, we would like to consider an adequate 2-idempotent completion as well. This should be as suitable 2-category of $\mathcal{A}$-modules. We tentatively consider the pivotal bicategory $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ of spherical $\mathcal{A}$-module categories studied in Section 6.4. We regard it as a spherical 2-idempotent completion of $\mathcal{A}$, though the precise meaning of this is something that remains to be worked out.

In this chapter, we will test this idea by showing that a three-dimensional state-sum construction can be formulated from these data. We expect this construction to be equivalent to the original Turaev-Viro invariant based on $\mathcal{A}$, but the details of this statement also remain to be worked out. The main result of the chapter is Theorem 7.8 where we prove that our construction does not depend on the choice of skeleton used to produce the invariant. This is a non-trivial insight: this displays that the geometric *primary moves* on manifold skeletons interact in synergy with the algebraic structures of the spherical module categories developed in this thesis. Moreover, as we point out in Remark 7.7, the input of the construction here presented can be exchanged by a suitable locally Calabi-Yau pivotal bicategory (for instance a spherical *multi*fusion category), thereby generalizing the Turaev-Viro construction.

7.1 Skeletons on manifolds *à la* Turaev-Virelizier

In this section we summarize the pertinent definitions about an auxiliary datum on a manifold called *skeleton* and introduced in [TV, Sec. 11].

Definition 7.1. [TV, Sec. 11.1]

(i) A *graph* is a topological space obtained from a disjoint union of intervals $[0,1]$ by identifying endpoints. The images of the intervals are called the *edges* of the graph and the endpoints are called *vertices*. Each edge connects two (possibly coinciding) vertices, and each vertex is incident to at least one edge. The images of the half-intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ in the graph are called *half-edges*.

(ii) A 2-*polyhedron* S is a compact topological space with the property that it can be triangulated using a finite number of simplices of dimension 2 or lower.

(iii) A *stratification* on a 2-polyhedron S is a graph $S^{(1)}$ embedded in S such that $S \backslash \mathrm{Int}(S) \subset S^{(1)}$, where the interior $\mathrm{Int}(S)$ of S is the set of points in S that have a neighbourhood homeomorphic to $\mathbb{R}^2$.

(iv) Given a stratified 2-polyhedron S, the edges and vertices of the graph $S^{(1)}$ are called *edges and vertices of* S and the sets containing them are denoted by $v(S)$ and $e(S)$, respectively. The connected components of $S \setminus S^{(1)}$ are called the *regions* of S. The finite set of regions of S is denoted by $\mathrm{Reg}(S)$.

(v) A *branch* of a stratified 2-polyhedron S at an edge $l \in e(S)$ is a germ of a region $r \in \mathrm{Reg}(S)$ adjacent to l, i.e. a homotopy class of embeddings $\sigma : [0,1) \to S$ such that $\sigma(0) \in l$ and $\sigma(0,1) \subset S \backslash l$. The set of branches at an edge $l \in e(S)$ is denoted by S_l.

(vi) An *orientation* of a stratified 2-polyhedron S is an orientation of the surface $S \setminus S^{(1)}$, i.e. a choice of orientation for each region of S.

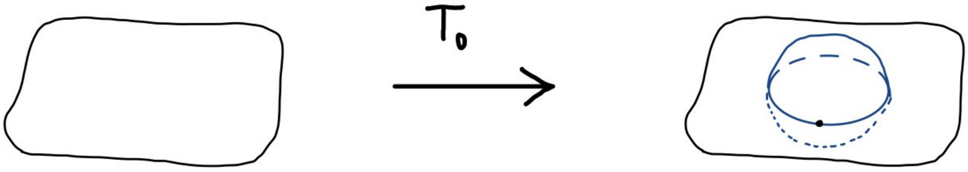
(vii) The *boundary* ∂S of a stratified 2-polyhedron S consists of the edges adjacent to only one region of S.

Definition 7.2. [TV, Sec. 11.2.1] Let M be a closed 3-manifold. A *skeleton* of M is an oriented stratified polyhedron S embedded in M such that $\partial S = \emptyset$ and $M \setminus S$ is a disjoint union of open 3-balls.

The set of open 3-balls determined by an skeleton S of M (also called the 3-*cells* of (M, S)) is denoted by $|M \backslash S|$, and by abuse of notation $|M \backslash S|$ will also denote its cardinality, i.e. the number of such 3-balls.

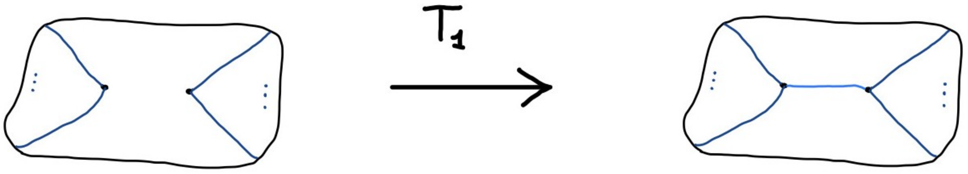
We summarize next the study [TV, Sec. 11.3.1] of local transformations on skeletons known as moves. Two arbitrary skeletons S and S' of a closed 3-manifold are connected by a finite sequence of so-called *primary moves*: These occur locally on a subset of M represented in what follows by a plane consisting of regions adjacent to common connected components of $M \backslash S$.

T_0 : The *bubble move* can be made by removing an open disk from a region in a first step and gluing afterwards a sphere along the equator to the boundary of the removed disk.



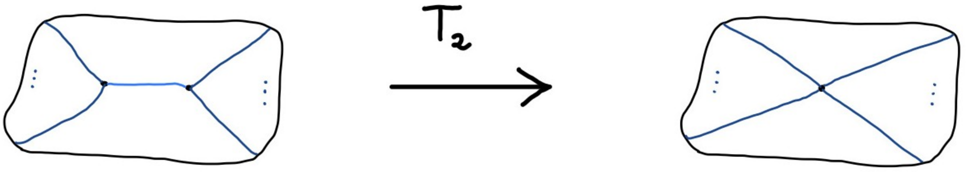
A new vertex and an edge along the equator of the sphere are added to S' and thus $|v(S')| = |v(S)| + 1$ and $|e(S')| = |e(S)| + 1$. The hemispheres are new regions $|\mathrm{Reg}(S')| = |\mathrm{Reg}(S)| + 2$ and the 3-ball inside the sphere is an additional 3-cell $|M \backslash S'| = |M \backslash S| + 1$.

T_1 : The *phantom edge move* adds a new edge by gluing its endpoints to two existing distinct vertices of S. We have that $|e(S')| = |e(S)| + 1$.



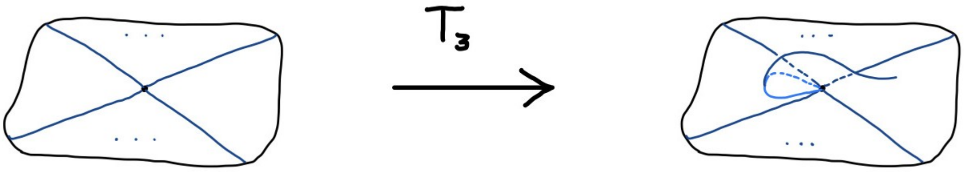
No vertices are added $|v(S')| = |v(S)|$ and depending whether the regions adjacent to the new edge coincide or not we have that $|\text{Reg}(S')| = |\text{Reg}(S)|(\text{or } +1)$. The number of 3-cells is preserved $|M\backslash S'| = |M\backslash S|$.

T_2 : The *contraction move* collapses an edge in S with distinct endpoints into a single point.



Consequently, the number of vertices and edges decrease by one $|v(S')| = |v(S)| - 1$ $|e(S')| = |e(S)| - 1$. The regions and 3-cells are preserved: $|\text{Reg}(S')| = |\text{Reg}(S)|$ and $|M\backslash S'| = |M\backslash S|$.

T_3 : The *percolation move* pushes a branch across a vertex v in S. An open disk whose boundary contains v is removed from a branch. The boundary of the removed disk is glued on another branch at v. This creates a new edge $|e(S')| = |e(S)|+1$ whose endpoints are both v. The disk bounded by this edge is a new region $|\text{Reg}(S')| = |\text{Reg}(S)| + 1$.



There is no change in vertices $|v(S')| = |v(S)|$ or in 3-cells $|M\backslash S'| = |M\backslash S|$.

Theorem 7.3. [TV, Thm. 11.1] *Any two skeletons of a closed* 3*-manifold* M *can be related by primary moves.*

7.2 The state sum invariant

Consider an oriented closed 3-manifold M with an oriented skeleton S. We construct a scalar invariant of M following very closely the procedure in [TV, Sec. 13.1.1]. The main difference is that our input will be a special type of pivotal bicategory, which will require additional considerations.

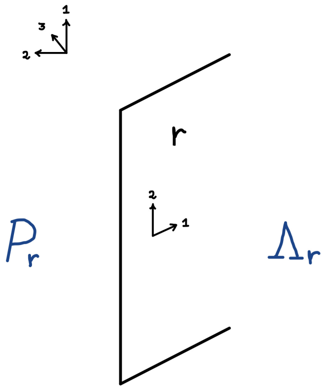
Let $\mathcal{A}$ be a spherical fusion category and consider the pivotal bicategory $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$. We fix a choice of representatives of the equivalence classes of simple objects in the 2-category $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ and denote by $\mathrm{Irr}\,\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ the set of representatives. The construction consists of multiple steps described next.

1. **Labeling:** We define a labeling on (M, S) as a pair $\varphi := (\varphi_3, \varphi_2)$ of functions of the following form.

   (i) Every 3-cell in $|M\backslash S|$ is labeled by a simple object in $\mathrm{Irr}\,\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$, i.e. an indecomposable spherical $\mathcal{A}$-module category. This defines the first function

   $$\varphi_3 : \quad |M\backslash S| \longrightarrow \mathrm{Irr}\,\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A}) \ . \tag{7.1}$$

   (ii) Every region $r \in \mathrm{Reg}(S)$ is adjacent to two 3-cells. The orientation on r and the global orientation of the 3-manifold M determine a direction normal to r, and thus we could say that one 3-cell P_r is to the "right" of the region and the other 3-cell Λ_r is to the "left".

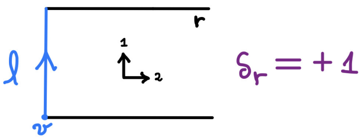
   We label the region r with a simple object $H \in \mathrm{Fun}_{\mathcal{A}}(\varphi_3(\Lambda_r), \varphi_3(P_r))$, i.e. a simple $\mathcal{A}$-module functor between the spherical module categories labeling the 3-cells adjacent to r. This assignment determines the second function

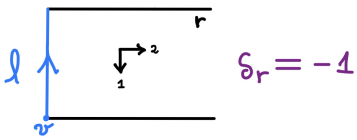
   $$\varphi_2 : \quad \mathrm{Reg}(S) \longrightarrow \mathrm{Irr} \bigoplus_{|M\backslash S|} \mathrm{Fun}_{\mathcal{A}}\big(\varphi_3(\Lambda_r), \varphi_3(P_r)\big) \ . \tag{7.2}$$

   For now let us fix such a labeling $\varphi := (\varphi_3, \varphi_2)$ on (M, S).

2. **Assignment of vector spaces:** We assign to each half-edge $l \equiv (l, v)$ of the skeleton S a vector space V_l. For this purpose we make use of the set S_l of branches at l, see Definition 7.1 (v).

(i) Each branch $r \in S_l$ at the edge l is contained in a region $\tilde{r}$ bounding l. We assign to r a label depending on both the label of the region $\tilde{r}$ and a sign δ_r. The value of the sign is given by $\delta_r = +1$ in case the orientation of r restricts to the orientation of l and $\delta_r = -1$ otherwise.



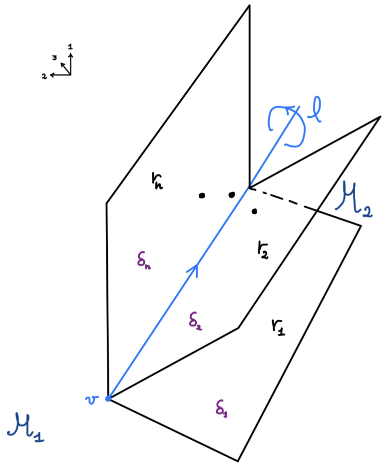


Let $H \in \mathrm{Fun}_{\mathcal{A}}(\varphi_3(\Lambda_{\tilde{r}}), \varphi_3(P_{\tilde{r}}))$ be the label of the region $\tilde{r} \in \mathrm{Reg}(S)$. Then the branch r gets labeled by

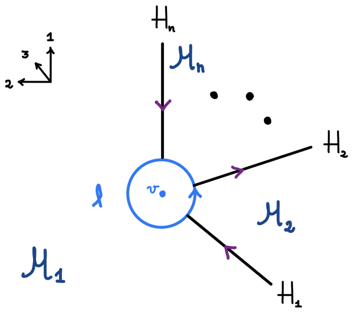
$$H^{\delta_r} = \begin{cases} H\,, & \text{if } \delta_r = +1 \\ H^{\mathrm{ra}}\,, & \text{if } \delta_r = -1 \end{cases} \tag{7.3}$$

(choosing right adjoints instead of left adjoints is simply a mere convention since the pivotal structure on the bicategory identifies these two options.)

(ii) The global orientation of the 3-manifold M together with the orientation of l determine a direction on a circle around the half-edge l. This defines a cyclic order on the set of branches $S_l = \{r_1, ..., r_n\}$ bounding l.



We pick a *starting branch* r_1 and compose the labels $H_i^{\delta_i}$ of the adjacent branches r_i following the cyclic order on S_l. The resulting composition is an endofunctor of the spherical module category labeling the 3-cell to the left of r_1. We assign to l a 2-hom-space in the pivotal bicategory $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$.



$$\longmapsto \mathrm{Nat}_{\mathrm{mod}}\left(1_{\mathcal{M}_1}, H_n^{\delta_n} \circ \cdots \circ H_1^{\delta_1}\right) \tag{7.4}$$

This vector space only depends up to canonical isomorphism on the cyclic order of S_l and not on the linear order determined by the starting branch. More explicitly, this vector space can be defined as the tip of the limit cone of the diagram defined by the choices of linear orders compatible with the cyclic order of S_l [FSY, Def. 2.9]:

$$V_l := \lim_\rho \mathrm{Nat}_{\mathrm{mod}} \left(1_{\mathcal{M}_{\rho(1)}}, H_{\rho(n)}^{\delta_{\rho(n)}} \circ \cdots \circ H_{\rho(1)}^{\delta_{\rho(1)}} \right) \tag{7.5}$$

where ρ is a cyclic permutation of the ordered tuple $(1, ..., n)$.

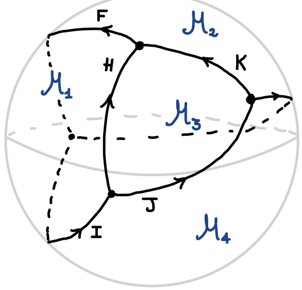
Each edge in S consists of two half-edges l and $\bar{l}$. Tensoring over all edges gives a vector space associated to the labeled skeleton S on M:

$$V(M, S, \varphi) := \bigotimes_{l \in e(S)} V_l \otimes_{\Bbbk} V_{\bar{l}} \; . \tag{7.6}$$

3. **Distinguished vector:** For each edge in the skeleton, we can realize the vector spaces V_l and $V_{\bar{l}}$ associated to its half-edges l and $\bar{l}$ as hom-spaces in a dual category $\mathcal{A}^*_{\mathcal{M}}$. Since $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ is locally Calabi-Yau, in particular $\mathcal{A}^*_{\mathcal{M}}$ is Calabi-Yau (see Section 6.4) and thus the non-degeneracy condition of the traces yields an isomorphism $V_{\bar{l}} \cong V_l^*$. The coevaluation morphism determines a vector $*_l \in V_l \otimes_{\Bbbk} V_{\bar{l}}$.
   We obtain a *distinguished vector* by tensoring over all edges in S:

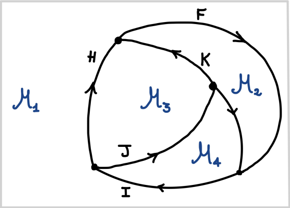
$$*_\varphi := \bigotimes_{l \in e(S)} *_l \; \in V(M, S, \varphi) \; . \tag{7.7}$$

4. **Evaluation at vertices:** For any given vertex $v \in v(S)$, consider a ball B_v in M around v. The intersection $B_v \cap S$ with the skeleton is a closed graph $\mathcal{G}$ on the sphere $\mathtt{S}^2 = \partial B_v$ bounding the ball B_v: Denote by $e_v(S)$ the set of half-edges with endpoint being the vertex v. The sphere intersects the edges in $e_v(S)$ at points and the regions bounding these edges at strands connecting the corresponding points. This intersection generates a graph $\mathcal{G}$ on the sphere. The labels on the regions of the skeleton $\mathrm{Reg}(S)$ and the distinguished vector $*_\varphi$ induce a labeling of $\mathcal{G}$. An intersecting 3-cell of the skeleton meets the sphere at a *patch*, i.e. a connected component of $\mathtt{S}^2 \setminus \mathcal{G}$. Similarly, the labels of the 3-cells induce a label on the patches. We obtain a closed $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$-labeled graph on the sphere.



(7.8)

The structure on the 2-category $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ allows to define a graphical calculus and evaluate closed graphs on the sphere to a scalar. Let $\mathcal{G}$ be such a labeled graph. First, we choose one of the patches on the sphere, i.e. one connected component of $\mathtt{S}^2 \setminus \mathcal{G}$. By puncturing the chosen patch, we obtain a closed graph $\tilde{\mathcal{G}}$ on the plane. The labeling on the graph determine a planar string diagram in $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$.



$$\longmapsto \mathrm{Nat}_{\mathrm{mod}}\left(1_{\mathcal{M}_1}, 1_{\mathcal{M}_1}\right) \xrightarrow{\mathrm{tr}^{\mathcal{A}^*_{\mathcal{M}}}_{\mathbf{1}_{\mathcal{M}}}} \Bbbk \tag{7.9}$$

The string diagram represents a 2-endomorphism which is invariant under isotopies of the graph $\tilde{\mathcal{G}}$ on the plane. Such invariance in general pivotal bicategories is established in [FSY, Prop. 2.2]. Further, we trace the associated 2-endomorphism into an scalar by means of the local Calabi-Yau structure, i.e. of the trace map of the corresponding Hom-category. This value will be independent on the choice of patch used to transition from the graph $\mathcal{G}$ on the sphere to a graph $\tilde{\mathcal{G}}$ on the plane, see Lemma 7.4 below. Altogether, the graphical calculus on the sphere determines a linear map by tensoring over all the vertices of the skeleton

$$\mathrm{tr}_{\varphi}: \quad V(M, S, \varphi) = \bigotimes_{v \in v(S)} \bigotimes_{l \in e_v(S)} V_l \longrightarrow \Bbbk\,. \tag{7.10}$$

Lemma 7.4. *The scalar assigned to a closed* $\mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$*-labeled graph* $\mathcal{G}$ *on the sphere* S^2 *is invariant under isotopies of* $\mathcal{G}$ *and independent of the patch chosen to define it.*

Proof. Without loss of generality, assume that the graph $\mathcal{G}$ determines two patches on the sphere. Then Proposition 6.46 ensures that by puncturing either patch we obtain the same scalar. In case of a larger number of patches, the result follows by transitively applying the previous argument to any pair of patches adjacent to a common strand in the graph. ☐

5. **State sum:** Now we define the state sum invariant as a weighted average of the value of all possible labelings on (M, S):

$$\sum_{\varphi_3} \omega_3 \sum_{\varphi_2} \omega_2 \cdot \mathrm{tr}_{\varphi}(*_{\varphi})\,. \tag{7.11}$$

Given a labeling $\varphi = (\varphi_3, \varphi_2)$ we define the weight ω_2 associated to the function (7.2) by

$$\omega_2 := \prod_{r \in \mathrm{Reg}(S)} \dim\left(\varphi_2(r)\right)^{\chi_r} \tag{7.12}$$

where for a label $\varphi_2(r) = H \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$, $\dim(H)$ is given by (6.7) and χ_r denotes the Euler characteristic of the region $r \in \mathrm{Reg}(S)$. For the function (7.1) we designate a weight ω_3 given by

$$\omega_3 := \prod_{\Gamma \in |M \backslash S|} \left(\dim \mathcal{A}^*_{\varphi_3(\Gamma)} \cdot \#\varphi_3(\Gamma)\right)^{-1} \tag{7.13}$$

where for $\mathcal{M} \in \mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ the value $\dim \mathcal{A}^*_{\mathcal{M}}$ is the categorical dimension of the fusion category $\mathcal{A}^*_{\mathcal{M}}$ and $\#\mathcal{M}$ is the number of equivalence classes of simple objects in the connected component of $\mathcal{M}$, i.e. the number of equivalence classes of simple objects $\mathcal{N} \in \mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ for which there are non-zero 1-morphisms $\mathcal{M} \to \mathcal{N}$.

In our setting, these two values simplify: For any indecomposable $\mathcal{A}$-module $\mathcal{M}$ its categorical dimension satisfies $\dim \mathcal{A}^*_{\mathcal{M}} = \dim \mathcal{A}$ [EGNO, Prop. 9.3.9]. Additionally, since $\mathcal{A}$ is fusion, $\#\mathcal{M} =: \#\mathcal{A}$ is equal to the number of equivalence classes of indecomposable $\mathcal{A}$-module categories which is a finite number according to [EGNO, Cor. 9.1.6 (ii)]. This simplifies the value of ω_3 to the expression

$$\omega_3 = \Big(\dim \mathcal{A} \cdot \#\mathcal{A}\Big)^{-|M \setminus S|} . \tag{7.14}$$

Definition 7.5. Let M be a closed oriented 3-manifold together with a skeleton S and $\mathcal{A}$ be a spherical fusion category. Denote by $\mathfrak{C} = \mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ the 2-category of spherical $\mathcal{A}$-modules. The *state sum* assigned to M is the scalar

$$\mathrm{St}_{\mathfrak{C}}(M, S) := \Big(\dim \mathcal{A} \cdot \#\mathcal{A}\Big)^{-|M \setminus S|} \sum_{\varphi = (\varphi_3, \varphi_2)} \omega_2 \cdot \mathrm{tr}_{\varphi}(*_{\varphi}) \in \Bbbk , \tag{7.15}$$

where the sum runs over all possible labelings (7.1) and (7.2), the value of ω_2 is given by (7.12), $\dim \mathcal{A}$ denotes the categorical dimension of $\mathcal{A}$ and $\#\mathcal{A}$ is the number of equivalence classes of indecomposable $\mathcal{A}$-module categories.

Remark 7.6. The Turaev-Viro construction based on a spherical fusion category $\mathcal{A}$ appears in the state sum (7.15) as the term corresponding to the constant labeling φ_3 equal to the regular spherical module category ${}_{\mathcal{A}}\mathcal{A}$.

Remark 7.7. The state sum invariant of Definition 7.5 can be generalized by changing the input to a certain pivotal bicategory $\mathfrak{C}$ that resemble a Calabi-Yau category in the bicategorical setting. For instance, consider a multifusion category $\mathcal{C} = \oplus_{i,j \in I} \mathcal{C}_{i,j}$ together with a choice of spherical structure for each fusion category $\mathcal{C}_i := \mathcal{C}_{i,i}$ and a spherical structure on the $(\mathcal{C}_i, \mathcal{C}_j)$-bimodule category $\mathcal{C}_{i,j}$ for all $i, j \in I$. There is a bicategory $\mathfrak{C}$ associated to $\mathcal{C}$ [EGNO, Rem. 4.3.7]. The objects of $\mathfrak{C}$ are elements of I, the Hom-categories are given by $\mathfrak{C}(j, i) := \mathcal{C}_{i,j}$ and the tensor product of $\mathcal{C}$ furnishes the horizontal composition of $\mathfrak{C}$. By assumption each $\mathcal{C}_{i,j}$ comes the structure of a spherical bimodule, which together endow the bicategory $\mathfrak{C}$ with a pivotal structure. Furthermore, by [EGNO, Prop. 7.17.5] each $\mathcal{C}_{i,j}$ is invertible and consequently, Proposition 6.14 implies that $\mathfrak{C}$ is locally Calabi-Yau.

Theorem 7.8. *Given a closed oriented* 3*-manifold* M*, the state sum scalar* $\mathrm{St}_{\mathfrak{C}}(M, S)$ *is independent of the choice of skeleton* S*.*

Proof. By means of Theorem 7.3 it is enough to check the invariance under the primary moves. This is verified in further detail in Appendix B. □

Appendix A

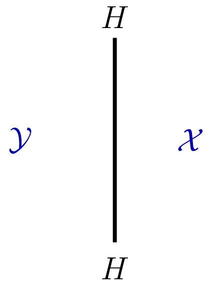
String diagrams in pivotal bicategories

The graphical notation of morphisms in a pivotal 2-category is a useful tool for making computations. A practical survey for such graphical calculus can be found for instance in [FSY, Sec. 2]. In this appendix we settle the conventions to be used in the document.

Given a 2-category $\mathcal{F}$ we have the following graphical representations on the plane:

(i) Objects: $\mathcal{X} \in \mathcal{F}$ is represented by a label on a region of the plane.

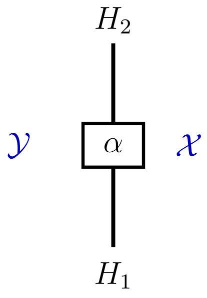
(ii) For a 1-morphism $H \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ we draw a line between appropriately labeled regions. The diagram is meant to be read from right to left, i.e the source of H is on the right of the line and the target on the left.



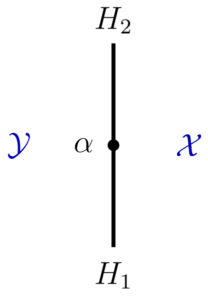
(A.1)

The identity 1-morphism $\mathbf{1}_{\mathcal{X}} \in \mathcal{F}(\mathcal{X}, \mathcal{X})$ is depicted by a transparent line.

(iii) A 2-morphism $\alpha : H_1 \Rightarrow H_2$ is represented by a rectangular box between the corresponding lines where the source of α is the line at the bottom and the target the one at the top. In order to simplify some diagrams, the rectangular box is often replaced by a labeled dot.



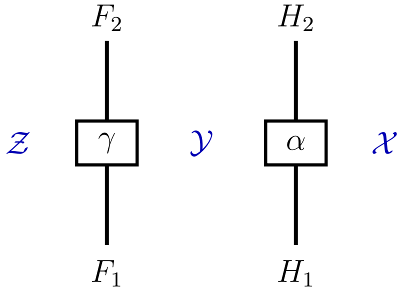
or simply



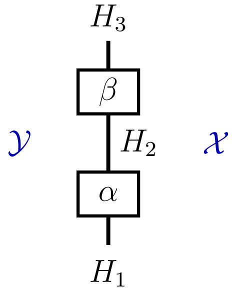
(A.2)

The identity 2-morphism $\mathrm{id}_H : H \Rightarrow H$ is represented by (A.1).

(iv) Horizontal and vertical compositions of 2-morphisms in $\mathcal{F}$ are depicted by



and

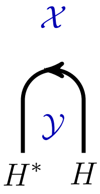


(A.3)

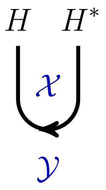
correspondingly, i.e. the first diagram represents the 2-morphism $\gamma \circ \alpha : F_1 \circ H_1 \Rightarrow F_2 \circ H_2$ and the second diagram stands for $\beta \cdot \alpha : H_1 \Rightarrow H_3$.

Recall that there is a notion of duals for a 1-morphism in a 2-category $\mathcal{F}$, see for instance [Sc2, App. A.3], we will use the following graphical description.

(v) A *right dual* to a 1-morphism $H \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ consists of a 1-morphism $H^* \in \mathcal{F}(\mathcal{Y}, \mathcal{X})$ and 2-morphisms $\mathrm{ev}_H : H^* \circ H \Longrightarrow \mathbf{1}_{\mathcal{X}}$ and $\mathrm{coev}_H : \mathbf{1}_{\mathcal{Y}} \Longrightarrow H \circ H^*$ depicted by



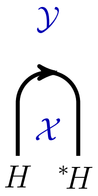
and



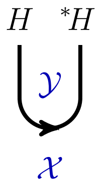
(A.4)

These pair of 2-morphisms obey appropriate snake relations.

(vi) Similarly, a *left dual* to $H \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ is a 1-morphism ${}^*H \in \mathcal{F}(\mathcal{Y}, \mathcal{X})$ together with 2-morphisms $\mathrm{ev}'_H : H \circ {}^*H \Longrightarrow \mathbf{1}_{\mathcal{Y}}$ and $\mathrm{coev}'_H : \mathbf{1}_{\mathcal{X}} \Longrightarrow {}^*H \circ H$ portrayed by



and



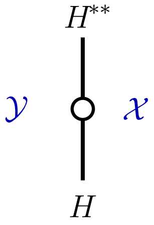
(A.5)

If the bicategory $\mathcal{F}$ admits dualities for every 1-morphism one can further consider a *pivotal structure*, i.e. pseudo-natural equivalence

$$\mathbf{P} : \quad \mathrm{id}_{\mathcal{F}} \xRightarrow{\cong} (-)^{**} \tag{A.6}$$

obeying $\mathbf{P}_{\mathcal{X}} = \mathrm{id}_{\mathcal{X}}$ for every object $\mathcal{X} \in \mathcal{F}$.

(vii) Given a 1-morphism $H \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$, the component $\mathbf{P}_H : H \Longrightarrow H^{**}$ is depicted by

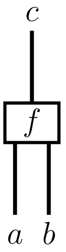


(A.7)

or, in case the labeling of the lines is clear, the circumference is not made explicit.

A.1 The pivotal 2-categories $\mathbb{M}$ and $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$

We look a bit closer into the graphical calculus on the bicategories of interest for this thesis. Any tensor category $\mathcal{A}$ can be seen as a bicategory $\mathbb{A}$ with a single object also known as its *delooping*. The End-category of the unique object in $\mathbb{A}$ is the category underlying $\mathcal{A}$, and the composition between 1-morphisms $a, b \in \mathcal{A}$ is given by the tensor product $a \circ b := a \otimes b$. A morphism $f : a \otimes b \to c$ in $\mathcal{A}$ is graphically represented by

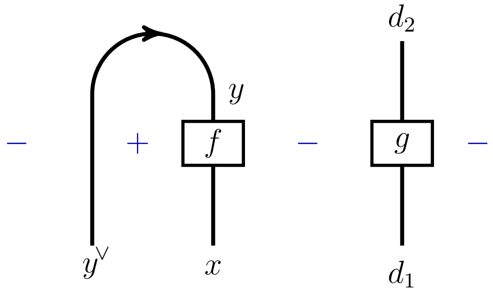


(A.8)

as a string diagram in $\mathbb{A}$. Left and right duals in $\mathcal{A}$ correspond to bicategorical left and right duals in $\mathbb{A}$. A pivotal structure on $\mathcal{A}$ endows the bicategory $\mathbb{A}$ with a pivotal structure in the bicategorical sense, see Remark 4.5.

A second pivotal bicategory of interest is the 2-category $\mathbf{Mod}^{\mathrm{piv}}(\mathcal{A})$ of pivotal modules over a pivotal tensor category discussed in Section 4.3. In this 2-category the right dual of a module functor is its left adjoint and vice versa. The conventions presented in this appendix for drawing string diagrams are set up for this pivotal 2-category.

Lastly, recall that associated to an invertible bimodule ${}_{\mathcal{C}}\mathcal{L}_{\mathcal{D}}$ there is a strong Morita context $(\mathcal{C}, \mathcal{D}, \mathcal{L}, \mathcal{K})$ which defines a two-object bicategory $\mathbb{M}$ as explained in Section 2.1. Moreover, owing to Theorem 3.2, $\mathbb{M}$ is a bicategory with dualities. Consequently, objects in $\mathcal{L}$ and the actions of $\mathcal{C}$ and $\mathcal{D}$ can be graphically represented in terms of string diagrams. For instance given morphisms $f \in \mathrm{Hom}_{\mathcal{L}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(d_1, d_2)$ the diagram



(A.9)

stands for the morphism $\mathrm{ev}_y \triangleright \mathrm{id}_{d_2} \circ \mathrm{id}_{y^\vee} \boxminus f \triangleleft g : \quad y^\vee \boxminus x \triangleleft d_1 \to \mathbf{1} \otimes_{\mathcal{D}} d_2$ in $\mathcal{D}$.

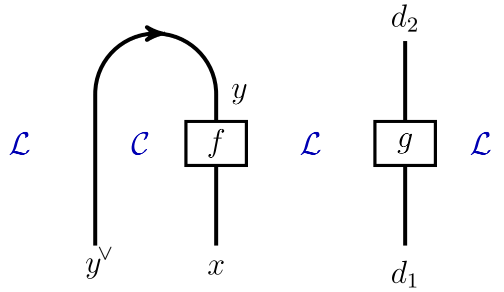
Let $\mathbb{M}^{\mathbf{1(op)}}$ be a bicategory with the same objects than $\mathbb{M}$, but where the Hom-categories are given by $\mathbb{M}^{\mathbf{1(op)}}(x, y) = \mathbb{M}(y, x)$, this means that only the 1-morphisms are reversed. The bicategory $\mathbb{M}^{\mathbf{1(op)}}$ can be embedded into the 2-category $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{C})$ of exact $\mathcal{C}$-module categories, i.e. there is a fully faithful pseudo-functor

$$\mathbb{M}^{\mathbf{1(op)}} \longrightarrow \mathbf{Mod}^{\mathrm{ex}}(\mathcal{C}) \tag{A.10}$$

sending $+ \mapsto \mathcal{C}$ and $- \mapsto \mathcal{L}$ at the level of objects, and equivalences at the level of Hom-categories (Theorem 2.11) given by the assignments

$$\begin{array}{cc} \mathbb{M}^{\mathbf{1(op)}}(+,+) = \mathcal{C} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})\,, & \mathbb{M}^{\mathbf{1(op)}}(+,-) = \mathcal{L} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{L})\,, \\ c \mapsto - \otimes c & y \mapsto - \triangleright y \\ \mathbb{M}^{\mathbf{1(op)}}(-,-) = \mathcal{D} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{L},\mathcal{L})\,, & \mathbb{M}^{\mathbf{1(op)}}(-,+) = \mathcal{K} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{L},\mathcal{C}) \\ d \mapsto - \triangleleft d & k \mapsto - \ominus k \;\; . \end{array} \tag{A.11}$$

Under this identifications one might very well represent the morphism (A.9) by the diagram



(A.12)

as a 2-morphism in the 2-category $\mathbf{Mod}^{\mathrm{ex}}(\mathcal{C})$. Furthermore, in the pivotal setting the bicategory $\mathbb{M}$ inherits a pivotal structure according to Theorem 4.8 and the pseudo-functor factors through a pivotal pseudo-functor

$$\mathbb{M}^{\mathbf{1(op)}} \longrightarrow \mathbf{Mod}^{\mathrm{piv}}(\mathcal{C})\ . \tag{A.13}$$

Appendix B

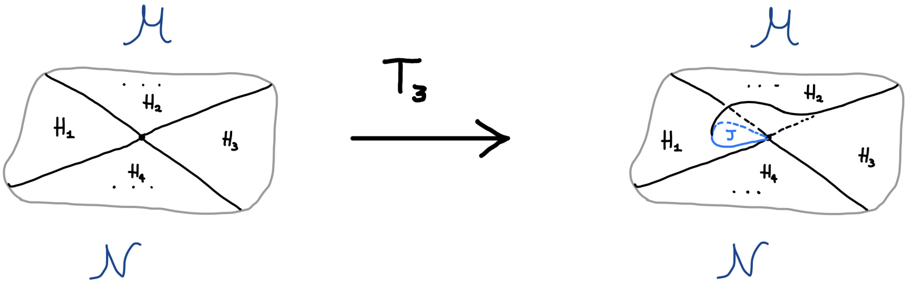
Skeleton independence of the state sum

Proof of Theorem 7.8

Theorem 7.8. *Given a closed oriented* 3*-manifold* M*, the state sum scalar* $\mathrm{St}_{\mathfrak{C}}(M,S)$ *is independent of the choice of skeleton* S*.*

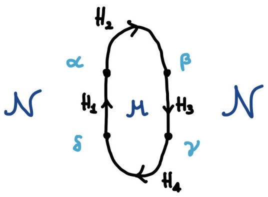
Proof. According to Theorem 7.3, every pair of skeletons on M can be related by a sequence of the primary moves described in Section 7.1. Consequently, the invariance of $\mathrm{St}_{\mathfrak{C}}(M,S)$ under the primary moves implies the independence of skeleton in the construction. We check next the invariance under primary moves.

Let us start with the percolation move $T_3 : S \to S'$. Fix a labeling φ of the skeleton S. The move pushes a branch across a vertex $v \in v(S)$. This creates a new edge whose endpoints are both equal to v. This edge bounds a new region. Each simple object $J \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N},\mathcal{N})$ assigned to the region created by the move extends φ to a labeling of the new skeleton S'.



(B.1)

The contribution of the vertex v to the state sum before the move is the value of the graph on the sphere, namely the trace of the string diagram

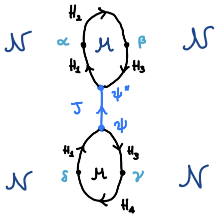


(B.2)

where $\alpha, \beta, \gamma, \delta$ are the module natural transformations coming from the distinguished vector.

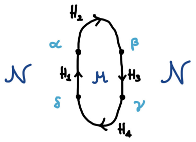
The contribution of v after the move is the trace of the string diagram

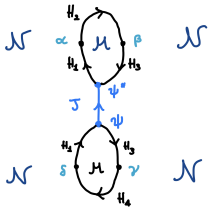
$$\sum_{J\in\mathrm{Irr}\mathcal{A}^*_{\mathcal{N}}} \dim(J) \tag{B.3}$$



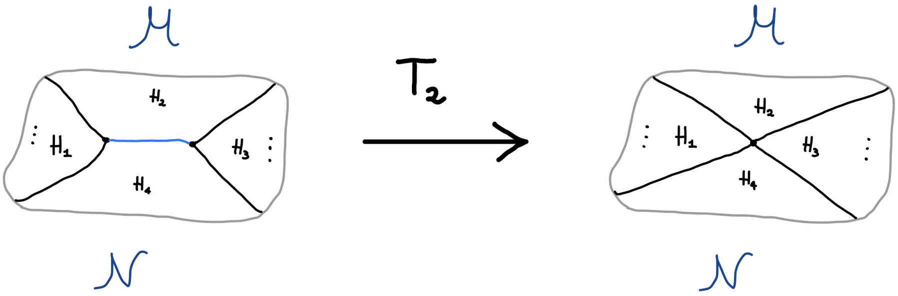
where ψ and ψ^* are dual bases since the two half-edges associated to the new edge in S' share endpoint. The value $\dim(J)$ comes from the factor in ω_2 associated to the new region, and the sum accounts for all the possible labelings of S' extending φ. It follows from Proposition 6.18 (iii) that

$$\mathrm{tr}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) = \sum_{J\in\mathrm{Irr}\mathcal{A}^*_{\mathcal{N}}} \dim(J)\,\mathrm{tr}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) \tag{B.4}$$





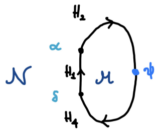
proving the invariance under the percolation move T_3.
Now, for the contraction move $T_2 : S \to S'$, we fix again a labeling φ of S. The move collapses an edge $l \in e(S)$ into a single vertex $v \in v(S')$.

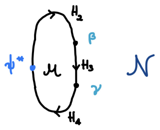


(B.5)

The contribution of the endpoints of the edge l before the move is

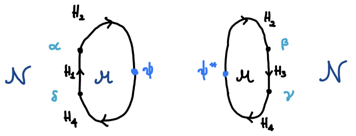
$$\mathrm{tr}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) \cdot \mathrm{tr}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) \tag{B.6}$$

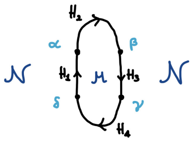




where ψ and ψ^* are again dual bases since they come from the two half-edges associated to l. By the multiplicativity of the trace we have that (B.6) equals to

$$tr^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) = \mathrm{tr}^{\mathcal{A}^*_{\mathcal{N}}}_{\mathbf{1}_{\mathcal{N}}}\Bigg(\quad \Bigg) \tag{B.7}$$

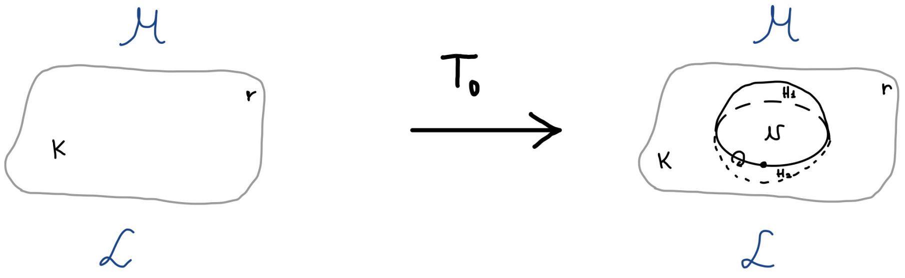




where the last equality follows from Proposition 6.18 (iv). On the other hand, the graph on the sphere around the vertex v after the move is again of the form (B.2). Hence, the contribution of v is precisely the right hand side of (B.7), which proves invariance under T_2.

For the bubble move $T_0 : S \to S'$, we consider a labeling $\varphi = (\varphi_3, \varphi_2)$ of S once more. The interior of the newly created bubble is an additional 3-cell of S'. Each choice of spherical module category $\mathcal{N} \in \operatorname{Irr} \mathbf{Mod}^{\mathrm{sph}}(\mathcal{A})$ extends the function φ_3 to the first component of a labeling $\varphi' = (\varphi'_3, \varphi'_2)$ of S'.

Now, fix such a function φ'_3 or equivalently such a spherical module $\mathcal{N}$. The choice of simple module functors $H_1 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ and $H_2 \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{L})$ extend the function φ_2 to the second component of a labeling of S'.



(B.8)

The new edge $l \in v(S')$ created along the equator has coinciding endpoints, namely the new vertex $v \in v(S')$, and thus dual bases label its half-edges. Consequently, the value of the graph around v equals the dimension of the vector space $\mathrm{Nat}_{\mathrm{mod}}(\mathbf{1}_{\mathcal{N}}, K^{\delta} \circ H_2^{\delta_2} \circ H_1^{\delta_1})$ associated to l. Hence, we have for a fixed φ'_3 that the contribution of v to the state sum is

$$\begin{aligned}
&\sum_{H_1, H_2} \dim(H_1^{\delta_1}) \dim(H_2^{\delta_2}) \dim_{\Bbbk} \mathrm{Nat}_{\mathrm{mod}}(\mathbf{1}_{\mathcal{N}}, K^{\delta} \circ H_2^{\delta_2} \circ H_1^{\delta_1}) \\
=&\sum_{H_1, H_2} \dim(H_1) \dim(H_2) \dim_{\Bbbk} \mathrm{Nat}_{\mathrm{mod}}(\mathbf{1}_{\mathcal{N}}, K \circ H_2 \circ H_1^{\mathrm{ra}}) \\
=&\sum_{H_2} \dim(H_2) \sum_{H_1} \dim(H_1) \dim_{\Bbbk} \mathrm{Nat}_{\mathrm{mod}}(H_1, K \circ H_2) \\
=&\sum_{H_2} \dim(H_2) \dim(K \circ H_2) = \dim(K) \sum_{H_2} \dim(H_2)^2 \\
=&\dim(K) \dim \mathrm{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{L}) = \dim(K) \dim \mathcal{A}^*_{\mathcal{N}}
\end{aligned} \tag{B.9}$$

where the first step makes use of Proposition 6.17 (i) and in the second equality we use that H_1 and H_1^{ra} are duals. The third step follows from (6.10). The fourth equality uses Proposition 6.17 (iii). Finally, we use the definition of the categorical dimension and the fact that $\dim \mathrm{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{L}) = \dim \mathcal{A}^*_{\mathcal{N}}$ according to [ENO, Prop. 2.17].

Now, the sum over all φ'_3 contributes with a factor of $\#\mathcal{A}$ which together with the factor $\dim \mathcal{A}^*_{\mathcal{N}} = \dim \mathcal{A}$ from (B.9) cancel out the prefactor in ω arising from the inclusion of the new 3-cell. The factor of $\dim(K)$ in (B.9) gives account of the change in Euler characteristic of the region r labeled by K (removing a disk from r reduces χ_r by 1), which demonstrates the invariance under T_0. Invariance under the move T_1 is proven in complete analogy to the proof in [TV, Thm. 13.1] by using Proposition 6.17 (ii) and Proposition 6.18 (i) and (iv). □

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