Perverse schobers and cluster categories

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Wissenschaftliche Veröffentlichungen

Folgende drei Veröffentlichungen und Preprints sind aus den Inhalten dieser Dissertation hervorgegangen.

- Ginzburg algebras of triangulated surfaces and perverse schobers [Chr22b],
- Geometric models for the derived categories of Ginzburg algebras of n-angulated surfaces via local-to-global principles [Chr21],
- Cluster theory of topological Fukaya categories [Chr22a].

Teile des Vorlesungsskripts des Autors [Chr22c] basieren auf Teilen des Kapitels 3.1. Kapitel 4 enthält bisher unveröffentlichten Resultate zu relativen Calabi–Yau Strukturen.

Zusammenarbeit mit anderen Wissenschaftlern

Diese Dissertation bedient sich keiner Inhalte des Preprints

- Complexes of stable ∞-categories [CDW23].

Einige wenige Inhalte des Preprints

- Perverse schobers, stability conditions and quadratic differentials [CHQ23]

erscheinen in dieser Dissertation. Dies betrifft Definition 3.18, Remark 3.20, Proposition 3.21, Proposition 3.25, Proposition 3.26, Lemma 3.27, Lemma 5.23 und Definition 5.26. Bis auf Definition 5.26 verfeinern diese Inhalte Resultate aus [Chr22b] und wurden überwiegend vom Autor selbst geschrieben.

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Summary

We introduce a framework for the description of categorified perverse sheaves, called perverse schobers, on surfaces with boundary in terms of constructible sheaves of stable ∞-categories on ribbon graphs. We show that the global sections of some of these sheaves describe the derived categories of a class of relative Calabi–Yau dg-algebras, called relative Ginzburg algebras, associated with n-angulated surfaces. We use local-to-global principles to study the representation theory of these Ginzburg algebras, relating it with the geometry of the underlying surface. Using the derived categories of these relative Ginzburg algebras, we also construct a novel class of additive categorifications of cluster algebras associated with marked surfaces without punctures with coefficients in the boundary arcs. We show that these cluster categories coincide with the topological Fukaya categories of the surfaces with values in the derived category of 1-periodic chain complexes. We further study the relation between perverse schobers, relative Calabi–Yau structures and exact ∞-structures on ∞-categories.

Zusammenfassung

Wir führen ein Modell für die Beschreibung von kategorifizierten perversen Garben, genannt perverse Schober, auf Flächen mit Rand ein. Dieses Modell beschreibt perverse Schober als konstruierbare Garben auf Bandgraphen, die Werte in stabilen ∞-Kategorien annehmen. Wir zeigen, dass die globalen Schnitte mancher dieser Garben die derivierten Kategorien einer Klasse von relativen Calabi–Yau dg-Algebren, genannt relative Ginzburg-Algebren, beschreiben. Diese dg-Algebren werden zu n-angulierten Flächen zugeordnet. Wir wenden lokal-global-Prinzipien an, um die Darstellungstheorie dieser Ginzburg-Algebren zu studieren und mit der Geometrie der zugrunde liegenden Flächen in Verbindung zu setzen. Aus den derivierten Kategorien dieser relativen Ginzburg Algebren konstruieren wir auch eine neue Klasse von additiven Kategorifizierungen von Cluster Algebren, assoziiert zu Flächen mit markierten Punkten und mit Koeffizienten in den Randbögen. Wir zeigen, dass diese Cluster Kategorien mit den topologischen Fukaya-Kategorien, mit Werten in der derivierten Kategorie von 1-periodischen Kettenkomplexes, übereinstimmen. Schließlich studieren wir die Beziehung zwischen perversen Schobern, relative Calabi–Yau Strukturen und exakten ∞-Strukturen auf ∞-Kategorien.

Contents

1 Introduction

Given a stratified topological space, there is an abelian category of perverse sheaves on this space, arising as the heart of the perverse t-structure on the bounded derived category of constructible sheaves. Perverse schobers are a, in general conjectural, notion of categorified perverse sheaves. The first aim of this thesis is to develop a framework for the treatment of perverse schobers on surfaces with boundary, building on prior work of Kapranov-Schechtman on perverse sheaves on surfaces and perverse schobers on the complex line [KS14, KS16]. We realize perverse schobers on a surface with boundary as constructible sheaves on a spanning ribbon graph embedded in the surface, valued in stable ∞-categories and satisfying certain local conditions. Such perverse schobers are referred to as perverse schobers parametrized by the ribbon graph. This framework is explicit, allowing for the study of concrete examples. It also realizes many features of the conjectural full theory of perverse schobers, such as a well-behaved notion of ∞-category of global sections of a perverse schober with support on the ribbon graph or a notion of monodromy.

The second aim of this thesis is to study categorifications of cluster algebras via perverse schobers. Cluster algebras are a class of combinatorially defined commutative algebras introduced by Fomin-Zelevinski [FZ02]. The main feature of cluster algebras is that they come with a notion of mutation, which relates certain subsets of the cluster algebras called clusters. We consider a class of cluster algebras associated with marked surfaces (without punctures). These cluster algebras can be seen as arising from quivers with potentials associated with triangulations of the surfaces [LF09], but also admit a beautiful alternative description as coordinates on the Teichmüller spaces of the surfaces, see [GSV05, FG06, FG09, FT18]. From this perspective, the cluster variables composing the clusters appear as lambda lengths, which describe the lengths of certain geodesic curves for a given hyperbolic metric.

Cluster algebras admit a rich theory of categorification, and their study has lead to many fruitful interactions between cluster algebras and representation theory. A particular class of (additive) categorifications of cluster algebras are formed by so called cluster categories, which are triangulated 2-Calabi-Yau categories, or possibly generalization thereof such as 2-Calabi–Yau Frobenius extriangulated categories in the sense of Nakaoka-Palu [NP19]. The collection of rigid objects in the cluster categories are typically in bijection with the cluster variables (i.e. elements of the clusters) of the corresponding cluster algebras. In the case of a cluster category associated with a marked surface, the rigid objects are further in bijection with the set of arcs, meaning certain embedded curves in the surface. Notice that each such curve defines a Lagrangian submanifold of the surface (equipped with the symplectic volume 2-form). There are different flavors of Fukaya categories associated with surfaces, in which these curves define objects. We focus on the topological Fukaya category of the marked surface, which can be defined purely in terms of higher category and the topology of the surface, see [DK18, HKK17, DK15]. There are further relations between extensions groups in the cluster categories and topological Fukaya categories, which we survey further below. It is thus natural to ask:

Question. *What is the relation between topological Fukaya categories and cluster categories of marked surfaces?*

As it turns out, this question has a surprisingly simple answer, in terms of the topological Fukaya category of the surface valued in the derived category of 1-periodic chain complexes of vector spaces, also called the 1-periodic topological Fukaya category.

Theorem 1. *The* 1*-periodic topological Fukaya category of an unpuctured marked surface is a cluster category, categorifying the corresponding cluster algebra with coefficients in the boundary arcs. This* ∞*-category arises as the global sections of a parametrized perverse schober.*

For the categorification, we equip the 1-periodic topological Fukaya category with a Frobenius exact ∞-structure, which gives rise to an extriangulated structure on the homotopy 1-category. The corresponding stable triangulated category, obtained by the localization at the collection of morphisms factoring through an injective projective object, conjecturally describes the previously known cluster category of the marked surface, which categorifies the cluster algebra without coefficients. This conjecture is verified for certain classes of surfaces.

Theorem 1 is the final result of this thesis, building on the following independent result, which are further explained below.

- We introduce a new class of dg-algebras associated with n-angulated surfaces (allowing punctures), referred to as relative Ginzburg algebras. These dg-algebras are relative weakly left n-Calabi–Yau in the sense of Brav–Dyckerhoff [BD19] and generalize the 3-Calabi–Yau Ginzburg dg-algebras associated with triangulated marked surfaces. We show that their derived categories arise as the global sections of perverse schobers.

- Using local-to-global arguments, we relate the geometry of the surface with the representation theory of the relative Ginzburg algebra. This includes associating objects in these categories with curves and describing their derived Hom's in terms of intersections of curves. These results form what is called a partial geometric model of these categories.

- We describe the 1-periodic topological Fukaya category as a quotient of the derived category of the relative Ginzburg algebra associated with a triangulated surface. We also show that the passage to this quotient can be performed on the level of perverse schobers. From this, it will follow that the partial geometric model for the derived category of the relative Ginzburg algebra gives rise to a geometric model for the 1-periodic topological Fukaya category.

- We describe ways to construct relative Calabi–Yau structures on the global sections of parametrized perverse schobers, which apply to the relative Ginzburg algebras and to the 1-periodic topological Fukaya category. We show that relative weak right 2-Calabi–Yau structures give rise to Frobenius exact ∞-structures, satisfying that the corresponding homotopy categories are Calabi–Yau Frobenius extriangulated.

We proceed with a more detailed review of the contents of this thesis. We begin in Section 1.1, corresponding to Section 3 in the main text, with describing the framework of parametrized perverse schobers on surfaces. Section 1.2, corresponding to Section 4, first recalls the different flavors of (relative) Calabi–Yau structures and then describes their gluing properties and their relation with perverse schobers. We proceed in Section 1.3, corresponding to Section 5, by explaining the relation between derived categories of relative Ginzburg algebras and parametrized perverse schobers, and how this leads to a partial geometric model of their derived categories. In the final Section 1.4, corresponding to Section 6, we introduce cluster categories in more detail and further elaborate on Theorem 1. This thesis is written in the language of ∞-categories, we refer to Section 2 in the main text for background.

1.1 Perverse schobers on surfaces

Perverse schobers are a conjectural notion of categorified perverse sheaves, proposed by Kapranov-Schechtman [KS14]. A direct categorification of perverse sheaves, as objects in the heart of the perverse t-structure, is currently not possible, since it is unclear what the derived category of constructible sheaves valued in stable ∞-categories should be (the problem lies in the deriving).

Perverse schobers as an ad-hoc categorification

The remarkable idea of Kapranov-Schechtman is to use linear algebraic descriptions of the abelian category of perverse sheaves, when available, to find what is called an 'ad-hoc' categorification. For instance, consider the complex line $\mathbb{C}$ with strata $\{0\}$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Then:

Theorem 2 ([GGM85]). *The abelian category of perverse sheaves on $\mathbb{C}$ is equivalent to the category of diagrams of vector spaces*

$$f\colon V \longleftrightarrow N : g$$

satisfying that $fg - \mathrm{id}_V$ and $gf - \mathrm{id}_N$ are invertible.

The vector space V is called the vector space of vanishing cycles and the vector space N is called the vector space of nearby cycles. Theorem 2 is categorified by the following definition:

Definition ([KS14]). A perverse schober on $\mathbb{C}$ is an adjunction of stable ∞-categories

$$F\colon \mathcal{V} \longleftrightarrow \mathcal{N} : G$$

satisfying that

- the twist functor $T_{\mathcal{V}} \coloneqq \mathrm{cof}(\mathrm{id}_{\mathcal{V}} \xrightarrow{\mathrm{u}} GF)\colon \mathcal{V} \to \mathcal{V}$, defined as the cofiber of the unit u of $F \dashv G$ in the stable ∞-category of endofunctors $\mathrm{Fun}(\mathcal{V}, \mathcal{V})$, is an equivalence.

- the cotwist functor $T_{\mathcal{N}} \coloneqq \mathrm{fib}(FG \xrightarrow{\mathrm{cu}} \mathrm{id}_{\mathcal{N}})\colon \mathcal{N} \to \mathcal{N}$, defined as the fiber of the counit cu of $F \dashv G$, is an equivalence.

Such adjunctions were already studied before their relation with perverse sheaves was noticed, they are called spherical adjunctions, and the functors F and G are called spherical functors, see [AL17].

A similar description exists for the abelian category of perverse sheaves on a oriented surfaces with boundary, see [KS16] or Theorem 3.17, of which Theorem 2 is a special case. For this, one chooses a so called spanning graph of the surface, which is a ribbon graph embedded into the surface, onto which the surface retracts. A perverse sheaf can be encoded in terms of a suitable pair of a constructible sheaf and a constructible cosheaf on the graph. The sheaf describes the derived sections of the perverse sheaf with support on the ribbon graph. A remarkable feature is that the derived sections with support on the graph are concentrated in a single degree, thus describing a constructible sheaf of vector spaces on the graph. This admits a categorification, in terms of a constructible sheaf on the graph valued in stable ∞-categories, satisfying certain local properties encoding the perverseness. We refer to such constructible sheaves as perverse schobers parametrized by the ribbon graph. We remark that in the categorification, the constructible cosheaf is not lost, it arises by passage to the adjoint cosheaf of the sheaf (see just below).

As constructible sheaves

Constructible sheaves on a graph Γ can be encoded as functors out of the exit path category $\mathrm{Exit}(\Gamma)$, whose

- objects are the vertices and edges of Γ,

- morphisms are of the form $v \to e$, with v a vertex and e an incident edge,

see also Theorem 3.9 for background. A Γ-parametrized perverse schober $\mathcal{F}$ is thus a functor $\mathcal{F}\colon \mathrm{Exit}(\Gamma) \to \mathrm{St}$ with values in the ∞-category of stable ∞-categories. Passage to the left adjoint diagram of $\mathcal{F}$, meaning the diagram which maps each morphism α in $\mathrm{Exit}(\Gamma)$ to the left adjoint functor of $\mathcal{F}(\alpha)$, defines a diagram $\mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{St}$, which can be regarded as a constructible cosheaf on Γ.

Given any n-valent vertex v of Γ, we consider the subcategory $\mathrm{Exit}(\Gamma)_{v/} \subset \mathrm{Exit}(\Gamma)$ consisting of the vertex v and the n incident halfedges. We show that the restriction of $\mathcal{F}$ to $\mathrm{Exit}(\Gamma)_{v/}$ is a particular form, obtained from applying Dyckerhoff's categorified Dold-Kan correspondence [Dyc21] to a spherical functor $F\colon \mathcal{V} \to \mathcal{N}$, considered as a 2-term complex in degrees $1, 0$. A precise description of this relation is given in Section 3.2. If $\mathcal{V} \neq 0$, we call the vertex v a singularity of $\mathcal{F}$. The perverse schober $\mathcal{F}$ assigns to each edge of Γ an ∞-category which is equivalent to the ∞-category $\mathcal{N}$. We call $\mathcal{N}$ the generic stalk of $\mathcal{F}$. This captures the fact that perverse sheaves on surfaces describe a local system away from their singularities, so that all non-singular stalks are equivalent.

Global sections

The ∞-category of global sections $\mathcal{H}(\Gamma, \mathcal{F})$ of a Γ-parametrized perverse schober $\mathcal{F}$ is defined as the limit of $\mathcal{F}$ in St. Under mild technical assumptions, the global sections of $\mathcal{F}$ are equivalent to a suitable colimit of the left adjoint diagram of F. The ∞-category of global section is a full subcategory of the $(\infty, 2)$-categorical lax limit of $\mathcal{F}$, which describes the ∞-category of local sections. Further, global sections can be glued together from compatible local sections. Formally, this is realized by describing $\mathcal{H}(\Gamma, \mathcal{F})$ as the ∞-category of coCartesian sections of the Grothendieck construction of $\mathcal{F}$, which is a full subcategory of the ∞-category of all sections of the Grothendieck construction, see Definition 3.36.

Note that for a perverse sheaf F, the global sections of the corresponding constructible sheaf on Γ describe the global sections of F with support on Γ. The ∞-category $\mathcal{H}(\Gamma, \mathcal{F})$ of global sections thus categorifies the derived sections with support on Γ.

Given a contraction of ribbon graphs $c\colon \Gamma \to \Gamma'$ which contracts edges, but collides no singularities of $\mathcal{F}$, we show in Proposition 3.47 that there is an induced Γ'-parametrized perverse schober $c_*(\mathcal{F})$ with an equivalent ∞-category of global sections. This allows us to compare perverse schobers parametrized by different spanning graphs of a surface.

Monodromy

Consider a marked surface $\mathbf{S}$ with spanning graph Γ and a Γ-parametrized perverse schober $\mathcal{F}$ with set of singularities P. In Section 3.3.4, we describe how to associate to each loop in $\mathbf{S}\backslash P$ an autoequivalence of the generic stalk of $\mathcal{F}$, called the monodromy. This monodromy possesses all expected properties, such as only depending on the homotopy class of the loop, composing when composing loops with identical basepoints, and not changing under contractions of the ribbon graph Γ. We also show that two parametrized perverse schobers without singularities and with an equivalent generic stalk are equivalent if and only if their monodromies are equivalent, see Proposition 3.63.

1.2 Relative Calabi–Yau structures

Let k be a field. A k-linear triangulated category C with finite dimensional Hom's is called n-Calabi–Yau if there exists an isomorphism of vector spaces

$$\mathrm{Hom}_C(X, Y) \simeq \mathrm{Hom}_C(Y, X[n])^*\,, \tag{1}$$

bifunctorial in $X, Y \in C$. A better behaved version of Calabi–Yau structure for proper k-linear stable ∞-categories (which we assume to be presentable) is the following:

Definition 1.1. Let $\mathcal{C}$ be a proper k-linear stable ∞-category and S the Serre functor, meaning that

$$\mathrm{RHom}_{\mathcal{C}}(X,Y) \simeq \mathrm{RHom}_{\mathcal{C}}(Y,S(X))^* \in \mathcal{D}(k)\,,$$

bifunctorial in compact objects $X,Y \in \mathcal{C}^{\mathrm{c}}$. A weak right n-Calabi–Yau structure on $\mathcal{C}$ consists of an equivalence of functors

$$S \simeq [n]\,.$$

Note that $\mathrm{RHom}(\mathrm{id}_{\mathcal{C}},S) \simeq \mathrm{HH}(\mathcal{C})^*$ describes the k-linear dual Hochschild homology of $\mathcal{C}$. The identification $S \simeq [n]$ may additionally be required to arise from a dual cyclic homology class under the morphism $\mathrm{HH}_{S^1}(\mathcal{C})^* \to \mathrm{HH}(\mathcal{C})^*$ and this leads to the notion of a right n-Calabi–Yau structure.

The Serre functor S is given by the bimodule right dual $\mathrm{id}_{\mathcal{C}}^*$ of the evaluation bimodule. If $\mathcal{C}$ is smooth, but not necessarily proper, the evaluation bimodule admits a left dual, denoted $\mathrm{id}_{\mathcal{C}}^!$, which describes in the smooth and proper case the inverse of the Serre functor.

Definition 1.2 ([Gin06])**.** Let $\mathcal{C}$ be a smooth k-linear stable ∞-category. A weak left n-Calabi–Yau structure on $\mathcal{C}$ consists of an equivalence

$$\mathrm{id}_{\mathcal{C}}^! \simeq \mathrm{id}_{\mathcal{C}}[-n]\,.$$

Note that $\mathrm{RHom}(\mathrm{id}_{\mathcal{C}}^!,\mathrm{id}_{\mathcal{C}}) \simeq \mathrm{HH}(\mathcal{C})$ describes the Hochschild homology of $\mathcal{C}$. Similarly, a left n-Calabi–Yau structure on $\mathcal{C}$ consists of an S^1-equivariant weak left n-Calabi–Yau structure, meaning that the corresponding Hochschild class arises from a negative cyclic homology class.

As emphasized by Brav-Dyckerhoff [BD19], left Calabi–Yau structures should be considered as non-commutative versions of orientations of topological manifolds. Indeed, if X is a closed oriented manifold of dimension n, then the ∞-category $\mathrm{Fun}(X,\mathcal{D}(k))$ of $\mathcal{D}(k)$-valued local systems on X inherits a canonical left n-Calabi–Yau structure. This perspective is extended by Brav-Dyckerhoff by introducing a non-commutative analog of a compact oriented manifold with boundary, referred to as a Calabi–Yau structure on a functor, or as a *relative Calabi–Yau structure*. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a k-linear functor between smooth k-linear stable ∞-categories with right adjoint G. A weak left n-Calabi–Yau structure on F consists of a class in the relative Hochschild homology $\mathrm{HH}(\mathcal{D},\mathcal{C}) \coloneqq \mathrm{cof}(\mathrm{HH}(\mathcal{C}) \xrightarrow{\mathrm{HH}(F)} \mathrm{HH}(\mathcal{D}))$, which gives rise to a specific fiber and cofiber sequence of endofunctors of $\mathcal{D}$ of the form

$$FG[-n] \xrightarrow{\mathrm{counit}} [-n] \longrightarrow \mathrm{id}_{\mathcal{D}}^!\,.$$

This means that the bimodule left dual and the shift functor do not necessarily agree, but there is a canonical map whose fiber is described by the endofunctor $FG[-n]$. There is a similar notion of (weak) relative right Calabi–Yau structure.

Brav-Dyckerhoff formulate their notion of relative Calabi–Yau structure in terms of k-linear dg-categories. We give in Section 4 a purely ∞-categorical formulation of the notion of relative Calabi–Yau structure and this allows us to consider any $\mathbb{E}_\infty$-ring spectrum R as a base. We however do not discuss the S^1-equivariant versions of R-linear Calabi–Yau-structures.

Gluing Calabi–Yau strutures

Given two compact oriented manifolds with two identical boundary components, we can glue the two manifolds together along these boundary components and this yields a new oriented manifold with boundary. Remarkably, there is a non-commutative version of this gluing construction for relative Calabi–Yau structures. For this, consider a pushout diagram of R-linear smooth ∞-categories:

$$\begin{tikzcd} & & \mathcal{B}_3 \arrow[d] \\ & \mathcal{B}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C}_2 \arrow[d] \\ \mathcal{B}_1 \arrow[r] & \mathcal{C}_1 \arrow[r] & \mathcal{D} \end{tikzcd}$$

We assume that each of the above functors admits both a left and a right adjoint functor.

Theorem 3 (Theorem 4.33, [BD19] for $R = k$ a field)**.** *If the functors $\mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{C}_1$ and $\mathcal{B}_2 \times \mathcal{B}_3 \to \mathcal{C}_2$ carry R-linear weak relative left n-Calabi–Yau structures, which are compatible at $\mathcal{B}_2$, then the functor $\mathcal{B}_1 \times \mathcal{B}_3 \to \mathcal{D}$ inherits an R-linear weak relative left n-Calabi–Yau structure.*

There is also an analog of Theorem 3 for weak relative right Calabi–Yau structures, see Theorem 4.35.

Calabi–Yau structures and perverse schobers

We apply the gluing properties of relative Calabi–Yau structures to construct relative Calabi–Yau structures on the global sections of parametrized perverse schobers. One of the main results is the following:

Theorem 4 (Theorem 4.44)**.** *Let $\mathcal{F}$ be a Γ-parametrized perverse schober without singularities, whose generic stalk $\mathcal{N}$ is smooth and admits a weak left $(n-1)$-Calabi–Yau structure. If the monodromy of $\mathcal{F}$ acts trivially on the corresponding Hochschild homology class of $\mathcal{N}$, then $\mathcal{H}(\Gamma, \mathcal{F})$ admits a relative weak left n-Calabi–Yau structure.*

This generalizes [BD19, Thm. 7.2], constructing relative Calabi–Yau structures on k-linear topological Fukaya-categories of framed marked surfaces.

We also describe ways to construct relative Calabi–Yau structures on the global sections of parametrized perverse schobers with non-trivial singularities, which are locally described by spherical functors which carry Calabi–Yau structures. This is applied to construct relative Calabi–Yau structures on the derived categories of relative Ginzburg algebras associated with n-angulated surfaces.

Relation with Frobenius exact ∞-structures

An exact ∞-category, as defined by Barwick [Bar15], consists of an additive ∞-category $\mathcal{C}$, together with chosen subsets of morphisms called inflations and deflations, satisfying ∞-categorical analogs of the axioms of exact 1-categories. Examples of exact ∞-categories are the nerves of exact 1-categories, by keeping the inflations and deflations of the exact 1-category. Other examples are stable ∞-categories, where all morphisms are considered as both inflations and deflations. The homotopy 1-category of an exact ∞-category inherits the structure of an extriangulated 1-category, see [NP20]. The extensions in an extriangulated 1-category C are organized into an additive bifunctor $\mathbb{E}\colon C^{\mathrm{op}} \times C \to \mathrm{Ab}$, valued in abelian groups.

Given an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories, we can pull back the split exact structure on $\mathcal{D}$ to an exact structure on $\mathcal{C}$, such that a fiber and cofiber sequence is exact if and only if its image under F is split exact. Stated differently,

$$\mathbb{E}(X,Y) \subset \mathrm{Ext}^1_{\mathcal{C}}(X,Y)$$

describes the kernel of the map $F\colon \mathrm{Ext}^1_{\mathcal{C}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{D}}(G(X),G(Y))$. If F is a spherical functor, then this exact ∞-structure is Frobenius, meaning that injective and projective objects coincide, see Proposition 4.54 or [BS21, Thm. 4.23]. We show that an additional weak right n-Calabi–Yau structure on the right adjoint G of F gives rise to an isomorphism

$$\mathbb{E}(X,Y) \simeq \mathbb{E}(Y,X[n-2])^{*}\,,$$

bifunctorial in $X,Y \in \mathcal{C}^{\mathrm{c}}$. In particular, if $n=2$, the extriangulated category $\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}$ is 2-Calabi–Yau. This clarifies the relation between weak relative right Calabi–Yau structures and the triangulated Calabi–Yau property (1).

We will apply these results to construct such 2-Calabi–Yau Frobenius extriangulated structures on the homotopy categories of 1-periodic topological Fukaya categories, which will be essential for the categorification of the cluster algebras.

1.3 Ginzburg algebras of n-angulated surfaces

In Section 5, we describe and study the derived categories of relative higher Ginzburg algebras arising from n-angulated surfaces using parametrized perverse schobers.

Ginzburg algebras are a class of weakly left 3-Calabi-Yau dg-algebras, associated to quivers with potential, first considered by Ginzburg in [Gin06]. Their derived categories have been used for, among other things, the categorification of cluster algebras, see [Kel12] for a survey, and the algebraic description of Fukaya categories [Smi15, Smi21, SW23]. Particularly relevant for this work are a class of Ginzburg algebras obtained from a quiver with potential constructed from a marked surface with an ideal triangulation, introduced by Labardini-Fragoso [LF09]. By a marked surface, we mean a compact, oriented topological surface with possibly empty boundary and a non-empty set of marked points on the boundary and in the interior (also called punctures), satisfying that each boundary component contains a marked point.

We generalize the construction of these Ginzburg algebras in two directions: firstly, we consider relative Ginzburg algebras, which are not weakly left 3-Calabi–Yau, but instead weakly relative left 3-Calabi–Yau. Relative Ginzburg algebras were also independently introduced by Yilin Wu in his thesis, see [Wu23b, Wu23a], associated with more general ice quivers with potential. Secondly, we also associate (higher) relative Ginzburg algebras with n-angulated surfaces, meaning surfaces equipped with a decomposition into n-gons. These dg-algebras are weakly relative left n-Calabi–Yau, see Theorem 5.27. We defer the detailed introduction of these relative Ginzburg algebras to Section 5.1. Their connection with perverse schobers is as follows:

Theorem 5 (Theorem 5.15)**.** *Let* $\mathbf{S}$ *be a marked surface equipped with an ideal n-angulation with dual n-valent ribbon graph $\mathcal{T}$. Denote by $\mathscr{G}_{\mathcal{T}}$ the associated relative Ginzburg algebra, see Definition 5.4. There exists a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}(k)$, satisfying that*

$$\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}(k))\,.$$

Theorem 5.15 expresses that the derived categories of relative Ginzburg algebras of n-angulated surfaces are glued together from the derived categories of the relative Ginzburg algebras of the n-gons of the n-angulations, which describe the local categories encoded in the perverse schober.

The perverse schober $\mathcal{F}_{\mathcal{T}}(k)$ is locally described by the spherical adjunction

$$f^*\colon \mathcal{D}(k)) \longleftrightarrow \mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) : f_*\,, \tag{2}$$

where $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))$ denotes the ∞-category of $\mathcal{D}(k)$-valued local systems, and f^* is the pullback functor along the map $f\colon S^n \to *$. Given an $\mathbb{E}_\infty$-ring spectrum R, we can thus obtain an R-linear version of this adjunction, by replacing $\mathcal{D}(k)$ with the stable ∞-category of right R-modules RMod_R. This gives rise to an R-linear version of the perverse schober $\mathcal{F}_{\mathcal{T}}(k)$, which we denote by $\mathcal{F}_{\mathcal{T}}(R)$. We study not only the global sections of $\mathcal{F}_{\mathcal{T}}(k)$, but by the same means also the global sections of $\mathcal{F}_{\mathcal{T}}(R)$.

The geometric model

Fix a marked surface $\mathbf{S}$ with an ideal n-angulation, with dual n-valent ribbon graph $\mathcal{T}$. We develop a geometric model for a full subcategory of the ∞-category of global sections of the perverse schober $\mathcal{F}_{\mathcal{T}}(R)$, meaning that we describe a subset of the objects in terms of curves in $\mathbf{S}$ and their Homs in terms of intersections of the curves. The case $R = k$ thus gives a geometric model for a full subcategory of the derived ∞-category of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$.

The base of the geometric model is formed by so-called matching curves in $\mathbf{S}\backslash M$, which are a class of immersed curves in $\mathbf{S}\backslash M$, where M is the set of marked points.

These curves may either be open, with endpoints at the vertices of $\mathcal{T}$ or on the boundary of $\mathbf{S}\backslash M$, or closed. Matching curves contain the class of curves referred to in cluster theory as arcs, which describe embedded matching curves. We associate a global section M_γ^L of $\mathcal{F}_\mathcal{T}(R)$ to each matching datum (γ, L) in the surface, which consists of a matching curve γ and a 'local value' L, which describes an object in the generic stalk of $\mathcal{F}_\mathcal{T}(R)$. By choosing different values of L, we can realize in the geometric model different classes of objects. In the case $R = k$, these include finite, perfect and also non-perfect $\mathscr{G}_\mathcal{T}$-modules. The class of perfect modules arising from matching data includes the direct summands of the relative Ginzburg algebra $\mathscr{G}_\mathcal{T}$, also known as the projective modules associated with the vertices of the underlying quiver.

For a certain subclass of the matching curves, called pure matching curves, we describe the R-linear morphism objects (i.e. derived Homs) between the M_γ^L's in terms of multiple types of intersections of the curves, see Theorems 5.53 and 5.54. For non-pure matching curves and L arbitrary, the morphism objects do not, in general, simply count intersection.

We also express equivalences between the global sections of the $\mathcal{F}_\mathcal{T}(R)$'s, for different choices of n-valent graph $\mathcal{T}$, whose dual n-angulations differ by a flip of an edge, in the geometric model via rotations of parts of the surface, see Theorem 5.68.

The derived ∞-category of the non-relative Ginzburg algebra associated with the n-angulation can be realized as a full subcategory of $\mathcal{D}(\mathscr{G}_\mathcal{T})$ consisting of those global sections of $\mathcal{F}_\mathcal{T}(k)$ whose evaluation at the external edges of $\mathcal{T}$ vanish. Our results for $\mathcal{D}(\mathscr{G}_\mathcal{T})$ thus restrict to a geometric model for (a full subcategory of) the derived category of the non-relative Ginzburg algebra, and this overlaps considerably with results from the series of papers [Qiu16, Qiu18, QZ19, IQZ20]. The setup based on perverse schober and ∞-categories allows us to arrive at these geometric models via efficient local-to-global arguments. This enables us to substantially extend these previous results in generality. For example, the previous geometric models were restricted to (a subset of) perfect modules, and for $n \geq 4$ to (a subset of) the finite modules.

Section 5 contains two applications of the geometric model. The first is the description of the extended mutation matrices of a class of cluster algebras with coefficients associated to marked surfaces in terms of Ext-groups in $\mathcal{D}(\mathscr{G}_\mathcal{T})$, see Section 5.6.3. The second is the description of the graded homology algebra $H_*(\mathscr{G}_\mathcal{T})$ in the case that the surface has no punctures. We show in Proposition 5.66, that $H_*(\mathscr{G}_\mathcal{T})$ is equivalent to the tensor algebra $\mathscr{J}_\mathcal{T} \otimes_k k[t_{n-2}]$, where $\mathscr{J}_\mathcal{T} = H_0(\mathscr{G}_\mathcal{T})$ is the Jacobian algebra and $k[t_{n-2}]$ the graded polynomial algebra. We will also see that $\mathscr{J}_\mathcal{T}$ is always a gentle algebra, generalizing an observation from [ABCJP10]. However, if $\mathbf{S}$ contains punctures, the Jacobian algebra may be infinite dimensional.

In Section 6, we furthermore show that the partial geometric model for $\mathcal{D}(\mathscr{G}_\mathcal{T})$ gives rise to a full geometric model for the associated cluster category, which arises as a full subcategory of $\mathcal{D}(\mathscr{G}_\mathcal{T})$ (consisting of non-compact objects).

Remarks on the relation to Fukaya categories of Lefschetz fibrations

The handling of Fukaya categories comes with many analytical and geometric difficulties. For an efficient application of the rich intuition which Fukaya categories provide to typical representation theoretic questions, such as classifications problems, an algebraic approach to the construction of Fukaya categories may be desirable. The emerging theory of perverse schobers seeks to give a higher categorical and algebraic approach to the construction of (some classes of) Fukaya categories and other Fukaya-type categories.

For a typical instance where perverse schobers might be applied, consider a Lefschetz fibration $\pi : X \to \mathbf{S}$ from a suitable exact symplectic manifold X to a surface equipped with a set of marked points $M \subset \partial\mathbf{S}$. There should exist a perverse schober on $\mathbf{S}$, parametrized by a spanning graph of $\mathbf{S}$, which can be described in terms of the typical fiber of the fibration and the vanishing cycles at the singular fibers. The category of global sections of the parametrized perverse schober should describe the partially wrapped Fukaya category of X with stops which lie above M. In case that $\mathbf{S}$ is a disc, the resulting theory is supposed be similar to the approach to Fukaya categories of Lefschetz fibrations by Fukaya-Seidel categories, see [Sei08].

These expectations formed the main motivation for the perverse schober description of the derived category of a relative Ginzburg algebra of a triangulated surface in terms of a perverse schober in [Chr22b]. As shown by Ivan Smith [Smi15], the finite derived category of the corresponding non-relative Ginzburg algebra admits an embedding into the (non-wrapped) derived Fukaya category of a Calabi-Yau 3-fold Y, equipped with a Lefschetz fibration to the surface. The typical fiber of the fibration is the cotangent bundle T^*S^2 of the 2-sphere; the generic stalk of the perverse schober is hence given by its Ind-complete wrapped Fukaya category $\operatorname{Ind}\mathcal{W}(T^*S^2) \simeq \mathcal{D}(k[t_1])$. Even though the derived categories of higher Ginzburg algebras of n-angulated surfaces have for $n > 3$ so far not been related to Fukaya categories, we can nevertheless exhibit algebraic versions of many of the usual features possessed by partially wrapped Fukaya categories of Calabi-Yau n-folds equipped with Lefschetz fibrations to surfaces with typical fiber T^*S^{n-1}.

Before we return to relative Ginzburg algebras, we consider the class of derived categories of (graded) gentle algebras, which admit their own geometric model as shown (in the ungraded, respectively underived setting) in [OPS18, BS19]. These derived categories are by [HKK17, LP20] equivalent to the partially wrapped Fukaya categories of surfaces. These categories are further known as topological Fukaya categories and can be described as the global sections of constructible (co)sheaves on ribbon graphs embedded in the surfaces, see [DK18, DK15], which fit into the framework of parametrized perverse schobers. For this class of categories, it is clear how the geometric model, describing objects in terms of decorated curves, relates to the symplectic geometry. These curves themselves simply describe Lagrangians (half-dimensional submanifolds on which the symplectic form vanishes) inside the surface and thus objects in the Fukaya category.

The geometric model for the derived category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ of a relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ does not seek to describe the objects in terms of some half-dimensional subspaces of some speculative Calabi-Yau n-fold Y. Instead, given a Lagrangian $U \subset Y$ whose image under the Lefschetz fibration is a curve γ in the surface, the corresponding object should be given by $M_\gamma^L \in \mathcal{D}(\mathscr{G}_\mathcal{T})$, where $L \subset T^*S^{n-1}$ is the Lagrangian given by the typical fiber of the map $U \to \gamma$. Indeed, in our geometric model, L is an object of $\mathcal{D}(k[t_{n-2}])$, which is by [Abo11] equivalent to the Ind-complete wrapped Fukaya category of T^*S^{n-1}. Two particularly interesting choices of L are the trivial $k[t_{n-2}]$-modules k, which corresponds to the Lagrangian zero-section of T^*S^{n-1}, and also $k[t_{n-2}]$, which corresponds to the Lagrangian fiber of the projection $T^*S^{n-1} \to S^{n-1}$. The other interesting choice $L = k[t_{n-2}^\pm]$, the module of Laurent polynomials, is a non-compact object and thus does not lie in the wrapped Fukaya category, only in its Ind-completion.

Let us further highlight the special case that a Lagrangian U in Y intersects the singular fibers of the Lefschetz fibration. In that case, if U maps to a curve γ, this curve will end in a vertex of the parametrizing ribbon graph; which describe in the case $n = 3$ the singular values of the Lefschetz fibration. If both endpoints of γ are at singular values, then U is a compact Lagrangian sphere and the corresponding object $M_\gamma^k \in \mathcal{D}(\mathscr{G}_\mathcal{T})$ is a spherical object, see Example 5.55. In this case, we thus algebraically recover the well-known construction of Lagrangian matching cycles, see [Sei08].

1.4 Cluster categories and topological Fukaya categories

Background on cluster categories

Cluster categories of acyclic quivers were introduced in [BMR+06] as certain orbit categories of the bounded derived categories of the path algebras of the quivers. Amiot [Ami09] gave a more broadly applicable construction of cluster categories. This construction takes as input a smooth dg-algebra B which is concentrated in positive degrees (in the homological grading convention) and satisfying that the Jacobian algebra $\mathrm{H}_0(B)$ is finite dimensional. Further, we ask that $\mathcal{D}(B)$ is weakly left 3-Calabi-Yau. A typical choice for B would be the Ginzburg algebra of a Jacobi-finite quiver with potential. The smoothness of B implies that the derived category of finite dimensional (over k) modules $\mathcal{D}^{\mathrm{fin}}(B)$ is contained in the perfect derived category $\mathcal{D}^{\mathrm{perf}}(B)$. We may thus form the Verdier quotient

$$\mathcal{C}_B = \mathcal{D}^{\mathrm{perf}}(B)/\mathcal{D}^{\mathrm{fin}}(B) \tag{3}$$

and obtain a triangulated category. This category is called the generalized cluster category of B, but also often simply referred to as the cluster category. The first remarkable property of $\mathcal{C}_B$ is that it is triangulated 2-Calabi-Yau, i.e.

$$\mathrm{Ext}^i_{\mathcal{C}_B}(X,Y) \simeq \mathrm{Ext}^{2-i}_{\mathcal{C}_B}(Y,X)^*\,,$$

functorial in $X, Y \in \mathcal{C}_B$. The second remarkable property is that the image $\pi(B)$ of B in $\mathcal{C}_B$ is a cluster-tilting object, meaning that

- $\pi(B)$ is rigid, i.e. $\mathrm{Ext}^1(\pi(B),\pi(B)) \simeq 0$, and

- $\mathrm{Ext}^1(\pi(B), X) \simeq 0$ implies that X is a finite direct sum of direct summands of $\pi(B)$, for any $X \in \mathcal{C}_B$.

Cluster-tilting objects in triangulated categories admit a well behaved theory of mutations, see [IY08], which categorifies the mutations of clusters in the cluster algebra. For an additive categorification of a cluster algebra, one typically asks that the cluster-tilting objects in $\mathcal{C}_B$ are in bijection with the clusters of the cluster algebra.

The degree 0 homology of the endomorphism dg-algebra of $\pi(B)$ is equivalent to the Jacobian algebra $\mathrm{H}_0(B)$. In the cases where $B = \mathscr{G}$ is the non-relative Ginzburg algebra associated with a triangulated marked surface, the corresponding Jacobian algebra is a gentle algebra. The rigid objects in the corresponding cluster category $\mathcal{C}_{\mathscr{G}}$ are in bijection with the set of arcs in the surface and the cluster titling objects with the set of ideal triangulations of the surface, see [BZ11]. As shown in [ZZZ13], the dimension of the first extension group between two rigid objects (or more generally objects corresponding to open curves) in $\mathcal{C}_{\mathscr{G}}$ is given by the number of crossing intersections of the corresponding arcs (or open curves). These extensions can thus be seen as a subset of the homology of the derived Homs in topological Fukaya categories, whose dimensions are described by the numbers of crossings and boundary intersections.

Given a (suitable) 2-Calabi-Yau triangulated category $\mathcal{C}$ with a cluster-tilting object, there is an associated cluster character, see [Pal08]. A cluster character on $\mathcal{C}$ with values in a commutative ring A is a function $\chi\colon \mathrm{ob}(\mathcal{C}) \to A$, on the set of isomorphism classes of objects in $\mathcal{C}$. A typical choice of A would be a cluster algebra. The cluster character is a kind of exponential map, in the sense that $\chi(X \oplus Y) \simeq X \cdot Y$ for any two $X, Y \in \mathcal{C}$. We note that this is the main difference between an additive categorification and another approach called monoidal categorification of cluster algebras; in the latter the multiplication of the cluster algebra is realized in terms of a symmetric monoidal product (instead of the direct sum). To relate with the cluster algebra, the cluster character has to satisfy a further relation, called the cluster multiplication formula. It states that if $\dim_k \mathrm{Ext}^1(Y, X) = \dim_k \mathrm{Ext}^1(X, Y) = 1$, and $X \to B \to Y$ and $Y \to B' \to X$ are the corresponding non-split extensions, then

$$\chi(X \oplus Y) \simeq \chi(B) + \chi(B')\,.$$

This allows to recover the cluster exchange relations in the cluster algebra in terms of the mutation of cluster-tilting objects in $\mathcal{C}$ and the cluster character.

To summarize, an additive categorification of a cluster algebra A consists of a 2-Calabi-Yau triangulated category whose cluster-tilting objects are in bijection with the clusters of A and which is further equipped with a cluster character to A.

Cluster categories from relative Ginzburg algebras

We consider a marked surface $\mathbf{S}$, which is assumed to have no punctures. We choose an ideal triangulation of $\mathbf{S}$ with dual trivalent ribbon graph $\mathcal{T}$. We have an associated relative 3-Calabi–Yau Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$, whose derived ∞-category is equivalent to the global sections of a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$. At each vertex of $\mathcal{T}$, the perverse schober $\mathcal{F}_{\mathcal{T}}$ is described by the spherical adjunction

$$\phi^*\colon \mathcal{D}(k) \leftrightarrow \mathcal{D}(k[t_1]) :\phi_*\,, \tag{4}$$

where $k[t_1]$ denotes the graded polynomial algebra with generator in degree 1, with k a commutative ring, and ϕ the morphism of dg-algebras $k[t_1] \xrightarrow{t_1\mapsto 0} k$. Note that the adjunctions (2) and (4) are equivalent, see Proposition 5.11.

Since $\mathscr{G}_{\mathcal{T}}$ is smooth, we may formally apply to it Amiot's quotient construction and define the Ind-complete version of the generalized cluster category as

$$\mathcal{C}_{\mathbf{S}} \coloneqq \mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}}) \simeq \operatorname{Ind}\left(\mathcal{D}^{\mathrm{perf}}(\mathscr{G}_{\mathcal{T}})/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\right)\,.$$

We consider the Ind-complete version of the generalized cluster category because of the superior formal properties of presentable ∞-categories. The surprising observation is that $\mathcal{C}_{\mathbf{S}}$ and $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$ arise as the global sections of two perverse subschobers $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ and $\mathcal{F}^{\mathrm{mnd}}_{\mathcal{T}}$ of $\mathcal{F}_{\mathcal{T}}$ and which fit into a cofiber sequence of perverse schobers:

$$\begin{tikzcd} \mathcal{F}^{\mathrm{mnd}}_{\mathcal{T}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{F}_{\mathcal{T}} \arrow[d, two heads] \\ 0 \arrow[r] & \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}} \end{tikzcd} \tag{5}$$

At the vertices of $\mathcal{T}$, the perverse schober $\mathcal{F}^{\mathrm{mnd}}_{\mathcal{T}}$ is described by the spherical adjunction

$$\mathcal{D}(k) \longleftrightarrow \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1])\,, \tag{6}$$

where $\mathcal{D}^{\mathrm{fin}}(k[t_1])$ denotes the derived ∞-category of finite dimensional modules, arising from restricting $\phi^* \dashv \phi_*$. The perverse schober $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ is instead described by the 'quotient' spherical adjunction

$$0 \longleftrightarrow \mathcal{D}(k[t_1])/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1])\,,$$

where $\mathcal{D}(k[t_1])/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \simeq \mathcal{D}(k[t_1^{\pm}])$ describes the derived ∞-category of 1-periodic chain complexes. The perverse schober $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ thus has no singularities and its global sections, which are equivalent to $\mathcal{C}_{\mathbf{S}}$, define a version of topological Fukaya category of $\mathbf{S}$ valued in $\mathcal{D}(k[t_1^{\pm}])$, see also below. Further, we show that $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ has no monodromy, see Proposition 6.10, so that we may call the $\mathcal{C}_{\mathbf{S}}$ *the* 1-periodic topological Fukaya category of $\mathbf{S}$.

We warn the reader that contrary to the name, neither the 1-periodic topological Fukaya categories, nor $\mathcal{D}(k[t_1^{\pm}])$, are 1-periodic ∞-categories in the sense of admitting a trivialization of the suspension functor $[1] \simeq \mathrm{id}$, unless the field has characteristic 2. Nevertheless, the suspension functor acts on objects as the identity. Furthermore, these ∞-categories are 2-periodic, meaning that $[2] \simeq \mathrm{id}$. In Section 6.4.2, we also consider $(n-2)$-periodic topological Fukaya categories, for $n > 3$, which describe the $(n-1)$-cluster categories of marked surfaces.

We formulate the relation between the perverse schobers $\mathcal{F}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ in terms of a notion of semiorthogonal decomposition of perverse schobers. We will see that the spherical adjunction (6) is the monadic adjunction associated to the adjunction monad $\phi_*\phi^*$ of the adjunction $\phi^* \dashv \phi^*$. We expect the semiorthogonal decomposition of $\mathcal{F}_{\mathcal{T}}$ to be a special case of a general principle, by which, under some assumptions such as commutating monodromy equivalences, a perverse schober decomposes into subschobers with monadic or trivial spherical adjunctions.

Properties of the 1-periodic topological Fukaya category

The $\mathbb{Z}$-graded topological Fukaya categories of a framed or graded surface $\mathbf{S}$ with spanning graph Γ arises as the global sections of a Γ-parametrized perverse schober with generic stalk $\mathcal{D}(k)$ and no singularities, see [DK15, HKK17]. The framing or grading data describes the monodromy of the perverse schober, which along any loop is the suspension functor $[i]$, where $i \in \mathbb{Z}$ is the winding number. Given any stable ∞-category $\mathcal{D}$, we can thus consider the global sections of a Γ-perverse schober without singularities as a $\mathcal{D}$-valued version of the topological Fukaya category. If $\mathcal{D}$ is k-linear and 2-periodic, e.g. $\mathcal{D} = \mathcal{D}(k[t_1^{\pm}])$, it turns out that one may further canonically associate a topological Fukaya category to a choice of orientation of $\mathbf{S}$, see [DK18], meaning that no grading or framing datum is required.

The representation type of a $\mathcal{D}(k)$-valued topological Fukaya category is tame, its indecomposable compact object have been classified in [HKK17] in terms of suitable curves (i.e. Lagragians) in the corresponding surface. The dimensions of the Hom's between two objects corresponding to two curves can be described in terms of counts of intersections of the curves. We prove an analogue of these results for the 1-periodic topological Fukaya category. For this, we use that the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}}$ embeds fully faithfully into the derived ∞-category of a relative Ginzburg algebra $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$. As explained in Section 1.3 above, for $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$, we have a partial geometric model in terms of matching data. Each matching datum $(\gamma, k[t_1^{\pm}])$ gives rise to an object $M_\gamma^{k[t_1^{\pm}]} \in \mathcal{C}_{\mathbf{S}} \subset \mathcal{D}(\mathscr{G}_{\mathcal{T}})$. We prove the following geometrization Theorem.

Theorem 6 (Theorem 6.35)**.** *Let $\mathbf{S}$ be a marked surface and $\mathcal{C}_{\mathbf{S}}$ the associated 1-periodic topological Fukaya category. Every compact object $X \in \mathcal{C}_{\mathbf{S}}$ splits uniquely into the direct sum of indecomposable objects associated to matching data with local value $L = k[t_1^{\pm}]$.*

Note that the restriction to compact objects in Theorem 6 arises from the fact that $\mathcal{C}_{\mathbf{S}}$ is an Ind-complete category, the 'perfect' version is the full subcategory of compact objects.

A similar classification result for the objects in the generalized cluster categories of non-relative Ginzburg algebras arising from triangulated surfaces without punctures appears in [BZ11]. The case with punctures was treated in [AP21, QZ17]. We take an original approach to the proof of Theorem 6.35, in that we employ general ∞-categorical techniques on the description of objects in limits of diagrams of ∞-categories in terms of sections of the Grothendieck construction and use the gluing properties of the objects arising from matching curves.

Brav and Dyckerhoff use the gluing properties of relative Calabi-Yau structures to construct relative 1-Calabi-Yau structures on $\mathcal{D}(k)$-valued topological Fukaya categories of framed surfaces. The derived ∞-category $\mathcal{D}(k[t_1^{\pm}])$ of 1-periodic chain complexes is 2-periodic and thus linear over the commutative dg-algebra $k[t_2^{\pm}]$ of graded Laurent polynomials with generator in degree $|t_2| = 2$. As a $k[t_2^{\pm}]$-linear ∞-category, $\mathcal{D}(k[t_1^{\pm}])$ is smooth and proper and admits a weak right 1-Calabi-Yau structure. Applying the gluing results yields a relative weak right 2-Calabi-Yau structure on the $\mathcal{D}(k[t_1^{\pm}])$-valued topological Fukaya category, considered as a $k[t_2^{\pm}]$-linear smooth and proper ∞-category.

Finally, we note that the results of [DK15] show that the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}}$ admits an action of the mapping class group of the surface, see Theorem 6.11. Mapping class group actions on cluster categories of marked surfaces seem to have not been previously constructed.

Additive categorification of cluster algebras with coefficients

Let again $\mathbf{S}$ be a marked surface without punctures with an auxiliary ideal triangulation with dual trivalent ribbon graph $\mathcal{T}$. As explained in Section 1.2, the relative 2-Calabi–Yau structure of the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}}$ gives rise to a Frobenius exact ∞-structure on $\mathcal{C}_{\mathbf{S}}$, whose homotopy 1-category is 2-Calabi–Yau Frobenius extriangulated. We denote the functor of exact extensions in $\mathcal{C}_{\mathbf{S}}$ by $\mathbb{E}$. In terms of the geometric model for $\mathcal{C}_{\mathbf{S}}$ based on matching curves in $\mathbf{S}\backslash M$, these extensions have a very simple interpretation. Given two objects arising from curves in $\mathbf{S}$, a basis of the extensions arises from crossings and directed boundary intersections. The extensions arising from crossings are exact extensions, whereas the extensions arising from directed boundary intersections are not exact.

In an extriangulated category, the notions of rigid object (meaning $\mathrm{Ext}^1(X,X) = 0$) and cluster tilting object need to be adapted, by replacing all Ext^1's with the functor of exact sequences $\mathbb{E}$. The following Theorem is the first part of the categorification of a cluster algebra in terms of $\mathcal{C}_{\mathbf{S}}$.

Theorem 1.3 (Corollary 6.53)**.** *There are canonical bijections between the sets of the following objects.*

- *Clusters of the cluster algebra with coefficients in the boundary arcs associated to* $\mathbf{S}$*.*
- *Ideal triangulations of* $\mathbf{S}$*.*
- *Cluster-tilting objects in the extriangulated category* $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}$*.*

In Section 6.3.3, we also describe the (discrete) endomorphism algebras of the cluster-tilting objects. They are given by finite-dimensional non-smooth gentle algebras.

The notion of cluster character naturally extends to extriangulated categories. The second part of the additive categorification of a cluster algebra in terms of $\mathcal{C}_{\mathbf{S}}$

is the description of a cluster character on $\mathcal{C}_{\mathbf{S}}$ with values in a the commutative Kauffman Skein algebra $\mathrm{Sk}^1_{\mathbf{S}}$ of the surface $\mathbf{S}$ (with parameter $q=1$). The elements of this algebra are links in $\mathbf{S}$, meaning superpositions of curves, modulo relations such as the Kauffman Skein relation, see Definition 6.48. This algebra embeds into the upper cluster algebra of the $\mathbf{S}$ with coefficients in the boundary arcs, see [Mul16].

Theorem 1.4 (Theorem 6.59). *There is a cluster character $\chi\colon \mathrm{obj}(\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}) \to \mathrm{Sk}^1_{\mathbf{S}}$ with values in the commutative Skein algebra of links in $\mathbf{S}$. The character maps an object arising from a matching curve in $\mathbf{S}\backslash M$ to the matching curve considered as a link with a single component. Furthermore, composing χ with the inclusion of $\mathrm{Sk}^1_{\mathbf{S}}$ into the upper cluster algebra yields a cluster character to the upper cluster algebra.*

The stable category of the Frobenius exact ∞-category $\mathcal{C}_{\mathbf{S}}$, obtained as the localization at the collection of morphisms factoring through an injective projective object, is a stable ∞-category, as is shown in [JKPW22], see also Proposition 4.49. We conjecture that this stable ∞-category is equivalent to an enhancement of the standard 2-Calabi-Yau triangulated cluster category arising from $\mathbf{S}$, see Conjecture 6.63. In the case that the non-frozen quiver arising from the triangulation is acyclic, we apply a result of Keller-Reiten [KR08] to show that the two triangulated homotopy categories are equivalent, see Theorem 6.64.

Finally, we wish to mention Yilin Wu's recent approach to the categorification of cluster algebras with coefficients called relative cluster categories and Higgs categories, see [Wu23b]. In loc. cit., a generalization of Amiot's construction is applied to relative Ginzburg algebras, which thus differs from the approach taken here. However, the outcome is also a Frobenius 2-Calabi-Yau extriangulated category with cluster tilting objects, called the Higgs category. The extriangulated structure of the Higgs category arises from the ambient triangulated category, called the generalized cluster category, in which the Higgs category lies as an extension closed subcategory.

2 Background from higher category theory

This thesis is formulated in the language of ∞-categories, as developed by Joyal and Lurie. We make particular use of stable ∞-categories, which provide an enhancement of triangulated categories. It would in principle be possible to formulate many of the results in the framework of dg-categories. One reason to use stable ∞-categories is to gain access to the powerful framework developed in [Lur09, Lur17, Lur18, Lur23]. A further advantage of stable ∞-categories is that they are more general than dg-categories or A_∞-categories. For some explicit computations in the k-linear setting, we will nevertheless make use of dg-categorical constructions.

While we will assume basic familiarity with the theory of ∞-categories, we use this section to fix our notation and review some often used aspects of the theory of ∞-categories. We further discuss in detail how to pass from a dg-category to its stable derived ∞-category. Parts of this discussion were not previously documented in the literature, though certainly known to experts. For extensive treatments of the theory of ∞-categories and stable ∞-categories, we refer to [Lur09, Lur17, Lur23].

Section 2.1 concerns generalities on ∞-category theory. We begin by introducing the different flavors of ∞-categories of ∞-categories in Section 2.1.1. We take a closer look at R-linear ∞-categories, where R is an $\mathbb{E}_\infty$-ring spectrum, in 2.1.2. In Section 2.1.3, we discuss the computation of limits and colimits in ∞-categories of ∞-categories. In Section 2.1.4, we discuss stable ∞-categories of modules over ring spectra and describe the relation between colimits of ring spectra and colimits of the corresponding ∞-categories of right module spectra. Section 2.1.5 reviews the notion of a monadic adjunctions of ∞-categories.

Section 2.2 discusses the passage from dg-categories to k-linear ∞-categories. Section 2.2.1 begins with some generalities on dg-categories and dg-categories of modules. Section 2.2.2 discusses different models for the derived ∞-category of a dg-category. We discuss in the final Section 2.2.3 how the passage to the derived ∞-category of a dg-category forms a functor.

In Section 2.3, we introduce semiorthogonal decompositions of stable ∞-categories (of arbitrary length $n \geq 2$). After discussing some generalities in Section 2.3.1, we treat two sources of semiorthogonal decompositions, as well as their relation, in Sections 2.3.2 and 2.3.3. The two sources are sequences of functors between stable ∞-categories and upper triangular dg-algebras.

2.1 ∞-category theory

2.1.1 ∞-categories of ∞-categories

We use the following notation for different ∞-categories of ∞-categories. We denote by

- $\mathcal{S}$ the ∞-category of spaces.
- Cat_∞ the ∞-category of (small) ∞-categories.

- $\mathrm{St} \subset \mathrm{Cat}_\infty$ the subcategory of stable ∞-categories and exact functors.
- $\mathrm{St}^{\mathrm{idem}} \subset \mathrm{St}$ the full subcategory spanned by idempotent complete stable ∞-categories.
- $\mathcal{P}r^L \subset \mathrm{Cat}_\infty$ the subcategory of presentable ∞-categories and left adjoint functors.
- $\mathcal{P}r^R \subset \mathrm{Cat}_\infty$ the subcategory of presentable ∞-categories and right adjoint functors.
- by $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$ and $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R$ the full subcategories consisting of stable ∞-categories.

A remarkable property of presentable ∞-categories is the adjoint functor theorem, see [Lur09, 5.5.2.9]. Namely, a functor $F\colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories admits

- a right adjoint if and only if F preserves colimits.
- a left adjoint if and only if F preserves limits and κ-filtered colimits for some regular cardinal κ (this condition is also called being an accessible functor).

We further have the following.

Theorem 2.1 ([Lur09, 5.5.3.4])**.** *There exists a canonical equivalence of ∞-categories $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\mathrm{op}}$, restricting to the identity on objects and mapping each functor in $\mathcal{P}r^L$ to its right adjoint.*

For background on adjunctions of ∞-categories, we recommend [Lur23, Cis19].

Let $\mathcal{C}$ be an ∞-category. An object x in an ∞-category $\mathcal{C}$ is called compact if $\mathrm{Map}_{\mathcal{C}}(x,\text{-})\colon \mathcal{C} \to \mathcal{S}$ preserves filtered colimits. We denote by $\mathcal{C}^{\mathrm{c}} \subset \mathcal{C}$ the full subcategory of compact objects. We denote by $\mathrm{Ind}(\mathcal{C})$ the Ind-completion of $\mathcal{C}$. If $\mathcal{C}$ is stable and idempotent complete, then $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^{\mathrm{c}}$.

2.1.2 Linear ∞-categories and morphism objects

For this section, we mostly follow [Lur17, HSS17].

The ∞-category $\mathcal{P}r^L$ admits a symmetric monoidal structure, such that a commutative algebra object in $\mathcal{P}r^L$ amounts to a symmetric monoidal presentable ∞-category $\mathcal{C}$, satisfying that its tensor product $\text{-} \otimes \text{-}\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in both entries, see [Lur17, Section 4.8]. An example of a commutative algebra object in $\mathcal{P}r^L$ is the ∞-category RMod_R of right module spectra over an $\mathbb{E}_\infty$-ring spectrum R. Note that if $R = k$ is a commutative ring, then RMod_k is equivalent as a symmetric monoidal ∞-category to the (unbounded) derived ∞-category $\mathcal{D}(k)$. If $R = \mathbb{S}$ is the sphere spectrum, then $\mathrm{RMod}_{\mathbb{S}} \simeq \mathrm{Sp}$ is equivalent to the symmetric monoidal ∞-category of spectra.

Definition 2.2. Let R be an $\mathbb{E}_\infty$-ring spectrum. The ∞-category $\mathrm{LinCat}_R = \mathrm{RMod}_{\mathrm{LMod}_R}(\mathcal{P}r^L)$ of left modules in $\mathcal{P}r^L$ over RMod_R, is called the ∞-category of R-linear ∞-categories.

As noted in [Lur18, Section D.1.5], R-linear ∞-categories in the above sense are automatically stable.

Definition 2.3 ([Lur17, 4.2.1.28])**.** Let R be an $\mathbb{E}_\infty$-ring spectrum. Let $\mathcal{C}$ be an R-linear ∞-category and let $X, Y \in \mathcal{C}$. A morphism object is an R-module $\mathrm{Mor}_{\mathcal{C}}(X,Y) \in \mathrm{RMod}_R$ equipped with a map $\alpha\colon \mathrm{Mor}_{\mathcal{C}}(X,Y) \otimes X \to Y$ in $\mathcal{C}$ such that for every object $C \in \mathrm{RMod}_R$ composition with α gives rise to an equivalence of spaces

$$\mathrm{Map}_{\mathrm{RMod}_R}(C, \mathrm{Mor}_{\mathcal{C}}(X,Y)) \to \mathrm{Map}_{\mathrm{RMod}_R}(C\otimes X, \mathrm{Mor}_{\mathcal{C}}(X,Y)\otimes X) \to \mathrm{Map}_{\mathcal{C}}(C\otimes X, Y)\,.$$

We thus have $\pi_i \mathrm{Mor}_{\mathcal{C}}(X,Y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(X[i], Y)$ for all $i \in \mathbb{Z}$.
We will also denote $\mathrm{Mor}_{\mathcal{C}}(X,X)$ by $\mathrm{End}(X)$.

Remark 2.4. Morphism objects always exist and the formation of morphism objects forms a functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{RMod}_R$$

which preserves limits in both entries, see [Lur17, 4.2.1.31].

Given a stable ∞-category $\mathcal{C}$ and two objects $A, B \in \mathcal{C}$, the n-th Ext-group is defined as

$$\mathrm{Ext}^n_{\mathcal{C}}(A,B) \coloneqq \pi_0 \mathrm{Map}_{\mathcal{C}}(A, B[n]) \simeq \pi_{-n} \mathrm{Mor}_{\mathcal{C}}(A,B)\,.$$

If $\mathcal{C}$ is k-linear, with k a commutative ring, $\mathrm{Mor}_{\mathcal{C}}(A,B) \in \mathrm{RMod}_k \simeq \mathcal{D}(k)$ describes a chain complex and $\mathrm{Ext}^n_{\mathcal{C}}(A,B)$ is its $(-n)$-th homology group. The abelian group $\mathrm{Ext}^n_{\mathcal{C}}(A,B)$ thus inherits the structure of a (discrete) k-module. If all involved Ext-groups are free k-modules, we denote by

$$\chi \,\mathrm{Ext}^*_{\mathcal{C}}(A,B) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{rk}_k \,\mathrm{Ext}^i_{\mathcal{C}}(A,B)$$

the Euler characteristic.

The ∞-category LinCat_R inherits a symmetric monoidal structure, as the module category over a commutative algebra object. We denote this monoidal product also by $\otimes$. The tensor product of $\mathcal{C}, \mathcal{D} \in \mathrm{LinCat}_R$ arises as the geometric realization of the two-sided bar construction $\mathrm{Bar}(\mathcal{C},\mathcal{D})_*\colon \Delta^{\mathrm{op}} \to \mathcal{P}r^L$, given informally by the formula $\mathrm{Bar}(\mathcal{C},\mathcal{D})_n = \mathcal{C} \otimes_{\mathcal{P}r^L} \mathrm{RMod}_R^{\otimes_{\mathcal{P}r^L} n} \otimes_{\mathcal{P}r^L} \mathcal{D}$, where $\otimes_{\mathcal{P}r^L}$ denotes the symmetric monoidal product of $\mathcal{P}r^L$. The ∞-category LinCat_R admits an internal morphism object:

Lemma 2.5. *There is a functor*

$$\mathrm{Lin}_R(\text{-},\text{-})\colon\ \mathrm{LinCat}_R^{\mathrm{op}} \times \mathrm{LinCat}_R \longrightarrow \mathrm{LinCat}_R$$

satisfying

$$\mathrm{Lin}_R(\mathcal{B}, \mathrm{Lin}_R(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Lin}_R(\mathcal{B} \otimes \mathcal{C}, \mathcal{D}) \,, \tag{7}$$

functorial in $\mathcal{B}, \mathcal{C} \in \mathrm{LinCat}_R^{\mathrm{op}}$ *and* $\mathcal{D} \in \mathrm{LinCat}_R$. *We call* $\mathrm{Lin}_R(\mathcal{C}, \mathcal{D})$ *the* R*-linear* ∞*-category of* R*-linear functors from* $\mathcal{C}$ *to* $\mathcal{D}$.

Proof. To construct the functor $\mathrm{Lin}_R(\text{-},\text{-})$, we follow [Lur17, 4.2.1.31]. Consider the functor

$$\mathrm{Map}_{\mathrm{LinCat}_R}(\text{-} \otimes \text{-}, \text{-})\colon\ \mathrm{LinCat}_R^{\mathrm{op}} \times \mathrm{LinCat}_R^{\mathrm{op}} \times \mathrm{LinCat}_R \longrightarrow \mathcal{S} \,.$$

It is adjoint to a functor $\mathrm{LinCat}_R^{\mathrm{op}} \times \mathrm{LinCat}_R \longrightarrow \mathrm{Fun}(\mathrm{LinCat}_R^{\mathrm{op}}, \mathcal{S})$, and one can show that its image lies in the full subcategory of representable presheaves. Composing with the inverse of the Yoneda embedding $\mathrm{LinCat}_R \to \mathrm{Fun}(\mathrm{LinCat}_R^{\mathrm{op}}, \mathcal{S})$ yields the functor $\mathrm{Lin}_R(\text{-},\text{-})$. By construction, we have

$$\mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{B} \otimes \mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{B}, \mathrm{Lin}_R(\mathcal{C}, \mathcal{D})) \,,$$

functorial in $\mathcal{B}, \mathcal{C}, \mathcal{D}$. It follows that

$$\begin{aligned}\mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{A}, \mathrm{Lin}_R(\mathcal{B}, \mathrm{Lin}_R(\mathcal{C}, \mathcal{D}))) &\simeq \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}, \mathcal{D}) \\ &\simeq \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{A}, \mathrm{LinCat}_R(\mathcal{B} \otimes \mathcal{C}, \mathcal{D})) \,,\end{aligned}$$

functorial in $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{LinCat}_R^{\mathrm{op}}$ and $\mathcal{D} \in \mathrm{LinCat}_R$. Composing again with the inverse of the Yoneda embedding shows (7), concluding the proof. □

If $\mathcal{C} \in \mathrm{LinCat}_R$ is compactly assembled (i.e. a retract of a compactly generated presentable stable ∞-category), then $\mathcal{C}$ is dualizable in the symmetric monoidal ∞-category LinCat_R, see [Lur18, D.7.0.7]. If $\mathcal{C}$ is compactly generated, the dual is given by $\mathcal{C}^\vee := \mathrm{Ind}(\mathcal{C}^{\mathrm{c},\mathrm{op}})$. The duality datum involves the evaluation and coevaluation functors

$$\mathrm{ev}_{\mathcal{C}}\colon\ \mathcal{C}^\vee \otimes \mathcal{C} \longrightarrow \mathrm{RMod}_R$$

and

$$\mathrm{coev}_{\mathcal{C}}\colon\ \mathrm{RMod}_R \longrightarrow \mathcal{C} \otimes \mathcal{C}^\vee \,,$$

satisfying the triangle identity. The evaluation functor $\mathrm{ev}_{\mathcal{C}}$ restricts along

$$\mathcal{C}^{\mathrm{c},\mathrm{c}} \times \mathcal{C}^{\mathrm{c}} \subset \mathcal{C}^\vee \times \mathcal{C} \longrightarrow \mathcal{C}^\vee \otimes \mathcal{C}$$

to the morphism object functor $\mathrm{Mor}_{\mathcal{C}^{\mathrm{c}}}(\text{-},\text{-})$, see [Lur18, D.7.2.3, D.7.7.6].

Definition 2.6. Given an R-linear, compactly assembled ∞-category $\mathcal{C}$, we denote by

$$\mathcal{Y}\colon\ \mathcal{C} \simeq \mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}^\vee} \otimes (\text{-}))} \mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R)$$

the R-linear Yoneda embedding, with inverse

$$\mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R) \xrightarrow{((\text{-}) \otimes \mathrm{id}_{\mathcal{C}}) \circ \mathrm{coev}_{\mathcal{C}}} \mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C}) \simeq \mathcal{C} \,.$$

Lemma 2.7. *Let R be an $\mathbb{E}_\infty$-ring spectrum and $\mathcal{C}$ an R-linear ∞-category. There exist an equivalence of functors* $\mathrm{RMod}_R^\vee \otimes \mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{RMod}_R$

$$\mathrm{ev}_{\mathcal{C}}(C \otimes Y, Z) \simeq \mathrm{ev}_{\mathrm{RMod}_R}(C, \mathrm{ev}_{\mathcal{C}}(Y, Z))\,. \tag{8}$$

Proof. We note that $\mathrm{RMod}_R^\vee \simeq \mathrm{RMod}_R$ and thus $\mathcal{C}^\vee \otimes \mathcal{C} \simeq \mathrm{RMod}_R^\vee \otimes \mathcal{C}^\vee \otimes \mathcal{C}$. Composing with this equivalence, both functors in (8) yield $\mathrm{ev}_{\mathcal{C}}(\text{-},\text{-})$, showing their equivalence. □

2.1.3 Limits and colimit in ∞-categories of ∞-categories

We now recall results on how to compute

i) limits in Cat_∞,

ii) limits and colimits in $\mathcal{P}r^L$, $\mathcal{P}r^L_{\mathrm{St}}$ and $\mathcal{P}r^R$, $\mathcal{P}r^R_{\mathrm{St}}$,

iii) limits and colimits in LinCat_R and

iv) limits and colimits in $\mathrm{St}^{\mathrm{idem}}$.

We begin with ii)-iv) and discuss i) at the end.

ii) Theorem 2.1 implies that we can exchange the computation of limits and colimits of diagrams of presentable ∞-categories. For the computation of limits, we can use i) (see below) and the fact that that the inclusions $\mathcal{P}^L_{\mathrm{St}} \subset \mathcal{P}r^L \subset \mathrm{Cat}_\infty$ and $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R \subset \mathrm{Cat}_\infty$ preserve all limits. Since the equivalence of Theorem 2.1 restricts to the full subcategories of stable ∞-categories, the inclusions $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$, $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R$ also preserve colimits.

iii) The computation of limits and colimits of R-linear ∞-categories reduces to the computation of limits and colimits in $\mathcal{P}r^L$, because the forgetful functor $\mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}r^L) \to \mathcal{P}r^L$ preserves all limits and colimits, see [Lur17, 4.2.3.1,4.2.3.5].

iv) The inclusion functor $\mathrm{St}^{\mathrm{idem}} \subset \mathrm{Cat}_\infty$ preserves all limits. The computation of colimits of idempotent stable ∞-categories can be related to the computation of colimits of presentable stable ∞-categories via the colimit preserving Ind-completion functor $\mathrm{Ind}\colon \mathrm{St}^{\mathrm{idem}} \to \mathcal{P}r^L_{\mathrm{St}}$.

We may thus focus on computing limits in the ∞-category Cat_∞.

i) Given a diagram $F\colon Z \to \mathrm{Cat}_\infty$, with Z any (small) simplicial set, its limit is given by the ∞-category of coCartesian sections of the coCartesian fibration which is classified by F, see [Lur23, 7.4.1.9]. In the case of limits of strictly commuting diagrams indexed by a 1-category C, there is a particularly explicit description of the coCartesian fibration: consider a diagram $D\colon C \to \mathrm{Set}_\Delta$ taking values in ∞-categories. Let $p\colon \Gamma(D) \to \mathrm{N}(C)$ be the relative nerve construction of [Lur09, 3.2.5.2], where $\mathrm{N}(C)$ denotes the nerve of C. We call p or $\Gamma(D)$ the (covariant) *Grothendieck construction* of D. p is a coCartesian fibration classified by D.

The objects and morphisms in the Grothendieck construction $\Gamma(D)$ can be described as follows. The fiber of p over $x \in N(C)$, i.e. the pullback ∞-category $\Gamma(D) \times_{N(C)} \{x\}$, is given by $D(x)$. The set of objects of $\Gamma(D)$ is thus the disjoint union of the sets of objects of the ∞-categories $D(x)$ with $x \in C$. Given $x, y \in C$ and two objects $X \in D(x)$ and $Y \in D(y)$, a morphism $\alpha\colon X \to Y$ in $\Gamma(D)$ consists of

- a morphism $f\colon x \to y$ in C and
- a morphism $D(f)(X) \to Y$ in $D(y)$.

If $D(f)(X) \to Y$ is an equivalence, we call the morphism α a *p-coCartesian* morphism and write $\alpha\colon X \xrightarrow{!} Y$. If $D(f)$ admits a right adjoint, we call the morphism α a *p-Cartesian* morphism if $D(f)(X) \to Y$ is a counit morphism of the adjunction $D(f) \dashv \mathrm{radj}(D(f))$ and write $\alpha\colon X \xrightarrow{*} Y$. For characterizations of p-Cartesian and p-coCartesian morphisms in terms of lifting properties, see [Lur09, 2.4.1.1].

As mentioned above, a choice of limit of the functor of ∞-categories $D'\colon \mathrm{N}(C) \to \mathrm{Cat}_\infty$ (obtained by composing D with the localization $\mathrm{Set}_\Delta \to \mathrm{Cat}_\infty$) is given by the ∞-category of coCartesian sections of $p\colon \Gamma(D) \to \mathrm{N}(C)$. This means the full subcategory of the ∞-category

$$\mathrm{Fun}_{\mathrm{N}(C)}(\mathrm{N}(C), \Gamma(D)) \coloneqq \mathrm{Fun}(\mathrm{N}(C), \Gamma(D)) \times_{\mathrm{Fun}(\mathrm{N}(C),\mathrm{N}(C))} \{\mathrm{id}_{\mathrm{N}(C)}\},$$

spanned by coCartesian sections, i.e. those sections $s\colon N(C) \to \Gamma(D)$ satisfying that $s(e)$ is a p-coCartesian morphism for each morphism e in $N(C)$. Morphisms in the limit of D' are natural transformations between coCartesian sections.

In the cases of interest for us, the diagram D takes values in stable and presentable ∞-categories and limit and colimit preserving functors, which we thus assume in the following. Limits and colimits in the limit of D' can be computed in the respective fibers of p. This can be seen as follows: the computation of limits and colimits in the ∞-category of sections of $p\colon \Gamma(D') \to N(C)$ is equivalent to the computation of p-relative limits and colimits in the ∞-category $\mathrm{Fun}(N(C), \Gamma(D'))$. By [Lur09, 5.1.2.3, 4.3.1.10] limits and colimits are computed pointwise in the respective fibers of p. It thus suffices to note that the property of a section of p being coCartesian is preserved under limits and colimits in the ∞-category of sections of p, which follows from $D'(\alpha)$ being a limit and colimit preserving functor for each 1-simplex α of $\mathrm{N}(C)$.

Finally, we also introduce the following terminology used later on.

Definition 2.8. Let $p\colon \Gamma(D') \to N(C)$ be as above. We define the support of a morphism $\beta\colon s \to s'$ between (not necessarily coCartesian) sections of p to be the subset of objects x of $N(C)$ such that $\beta(x)\colon s(x) \to s'(x)$ is not zero.

Notation 2.9. Let $p\colon \Gamma \to \Delta^n$ be an inner fibration. Given an edge $e\colon a \to a'$ in Γ we write $e\colon a \xrightarrow{!} a'$ if e is a p-coCartesian edge and $e\colon a \xrightarrow{*} a'$ if e is a p-Cartesian edge.

2.1.4 Modules over ring spectra

Consider the symmetric monoidal ∞-category Sp of spectra. Sp is a stable and presentable ∞-category. An $\mathbb{E}_1$-ring spectrum is an object of Alg(Sp), the ∞-category of (coherently associative) algebra objects in Sp. For every such $\mathbb{E}_1$-ring spectrum R, there is a stable and presentable ∞-category RMod_R of right R-modules in Sp. As already noted above, if R can be enhanced to a commutative algebra object of Sp, i.e. an $\mathbb{E}_\infty$-ring spectrum, then RMod_R inherits the structure of a symmetric monoidal ∞-category. In this case, we can form the ∞-category $\mathrm{Alg}(\mathrm{RMod}_R)$ of algebra objects in RMod_R. Given $A \in \mathrm{Alg}(\mathrm{RMod}_R)$, we can again form the ∞-category $\mathrm{RMod}_A(\mathrm{RMod}_R)$ of right A-modules in RMod_R. Alternatively, we can also consider the $\mathbb{E}_1$-ring spectrum $\xi(A) \in \mathrm{Alg}(\mathrm{Sp})$ underlying A obtained as follows. We consider the forgetful functor $\mathrm{RMod}_R \to \mathrm{Sp}$, mapping a right R-module to the underlying spectrum. This functor extends to a functor $\xi\colon \mathrm{Alg}(\mathrm{RMod}_R) \to \mathrm{Alg}(\mathrm{Sp})$, which we apply to A. We can form the ∞-category of right modules $\mathrm{RMod}_{\xi(A)}$ over $\xi(A)$. We will show in Corollary 2.12 that this does not yield a further ∞-category, meaning there exists an equivalence of ∞-categories

$$\mathrm{RMod}_A(\mathrm{RMod}_R) \simeq \mathrm{RMod}_{\xi(A)} \ .$$

Let $\mathcal{D}$ be a stable ∞-category and consider any object $X \in \mathcal{D}$. We can find an $\mathbb{E}_1$-ring spectrum $\mathrm{End}(X) \in \mathrm{Alg}(\mathrm{Sp})$, called the endomorphism algebra, with the following properties, see [Lur17, 7.1.2.2].

- $\pi_n \operatorname{End}(X) \simeq \pi_0 \operatorname{Map}_{\mathcal{D}}(X[n], X)$ for all $n \in \mathbb{Z}$.
- The induced ring structure of $\pi_* \operatorname{End}(X)$ is determined by the composition of endomorphisms in the homotopy category $\mathrm{Ho}(\mathcal{D})$.

The algebra object $\mathrm{End}(X)$ is an endomorphism object of X in the sense of [Lur17, Section 4.7.1] and its existence expresses the enrichment of the stable ∞-category $\mathcal{D}$ in spectra.

Assuming that the stable ∞-category $\mathcal{D}$ is also presentable,, we call an object $X \in \mathcal{D}$ a compact generator if

- X is compact, i.e. $\mathrm{Map}_{\mathcal{D}}(X, \text{-})$ commutes with filtered colimits and
- an object $Y \in \mathcal{D}$ is zero if and only if $\mathrm{Map}_{\mathcal{D}}(X, Y[i]) \simeq *$ for all $i \in \mathbb{Z}$.

The importance of this notion is that if X is a compact generator, there exists an equivalence of ∞-categories $\mathcal{D} \simeq \mathrm{RMod}_{\mathrm{End}(X)}$, as guaranteed by the Schwede-Shipley recognition theorem, see [Lur17, 7.1.2.1].

We now restrict to R-linear ∞-categories where R is an $\mathbb{E}_\infty$-ring spectrum. Suppose that $\mathcal{D}$ is an R-linear ∞-category and $X \in \mathcal{D}$ a compact generator. Lemma 2.10 shows we can lift $\mathrm{End}(X)$ along the forgetful functor $\xi\colon \mathrm{Alg}(\mathrm{RMod}_R) \to \mathrm{Alg}(\mathrm{Sp})$ to an algebra object in RMod_R.

Lemma 2.10. *Let R be an $\mathbb{E}_\infty$-ring spectrum. Let $\mathcal{C}$ be a presentable R-linear ∞-category with a compact generator X. Then there exists an algebra object* $\mathrm{End}_R(X) \in \mathrm{Alg}(\mathrm{RMod}_R)$ *and an equivalence of R-linear ∞-categories*

$$\mathcal{C} \simeq \mathrm{RMod}_{\mathrm{End}_R(X)}(\mathrm{RMod}_R)\,. \tag{9}$$

The algebra object $\mathrm{End}_R(X)$ *is mapped under the functor* $\xi\colon \mathrm{Alg}(\mathrm{RMod}_R) \to \mathrm{Alg}(\mathrm{Sp})$ *to the endomorphism algebra* $\mathrm{End}(X) \in \mathrm{Alg}(\mathrm{Sp})$.

Proof. Using the left tensoring of $\mathcal{C}$ over RMod_R, we can define the R-linear functor $\text{-} \otimes_R X\colon \mathrm{RMod}_R \to \mathcal{C}$. By the adjoint functor theorem, this functor admits a right adjoint G. We denote $\mathrm{End}_R(X) \coloneqq G(X) \in \mathrm{RMod}_R$. The existence of lift of $\mathrm{End}_R(X)$ to $\mathrm{Alg}(\mathrm{RMod}_R)$ and the existence of the equivalence (9) follow from [Lur17, 4.8.5.8], compare also to the proof of [Lur17, 7.1.2.1]. The right adjoint of the composite functor

$$\mathrm{Sp} \xrightarrow{\text{-}\otimes R} \mathrm{RMod}_R \xrightarrow{\text{-}\otimes_R X} \mathcal{C}$$

maps X to the endomorphism object $\mathrm{End}(X)$. By the universal property of $\mathrm{End}(X)$ and the fact that $X \in \mathcal{C} \simeq \mathrm{RMod}_{\xi(\mathrm{End}_R(X))}(\mathrm{Sp})$ is a module, there exists a morphism $\xi(\mathrm{End}_R(X)) \to \mathrm{End}(X)$ in $\mathrm{Alg}(\mathrm{Sp})$, which is an equivalence on underlying spectra and thus an equivalence of $\mathbb{E}_1$-ring spectra. □

Remark 2.11. In the setting of Lemma 2.10, the algebra object $\mathrm{End}_R(X)$ is an endomorphism object of X in the ∞-category $\mathcal{C}$ considered as left tensored over RMod_R. We call $\mathrm{End}_R(X)$ the R-linear endomorphism algebra of X.

Corollary 2.12. *Let R be an $\mathbb{E}_\infty$-ring spectrum and $A \in \mathrm{Alg}(\mathrm{RMod}_R)$. Then there exists an equivalence of ∞-categories*

$$\mathrm{RMod}_A(\mathrm{RMod}_R) \simeq \mathrm{RMod}_{\xi(A)}\,, \tag{10}$$

where $\xi\colon \mathrm{Alg}(\mathrm{RMod}_R) \to \mathrm{Alg}(Sp)$ denotes the forgetful functor.

Proof. The ∞-category $\mathrm{RMod}_A(\mathrm{RMod}_R)$ is presentable by [Lur17, 4.2.3.7], stable by [Lur17, 7.1.1.4] and left-tensored over RMod_R by [Lur17, Section 4.3.2]. Consider the monadic adjunction $\text{-}\otimes A\colon \mathrm{RMod}_R \leftrightarrow \mathrm{RMod}_A(\mathrm{RMod}_R)\colon G$. The adjunction and that G is conservative and accessible imply that A is a compact generator. The R-linear endomorphism algebra of $A \in \mathrm{RMod}_A(\mathrm{RMod}_R)$ is given by $A \in \mathrm{Alg}(\mathrm{RMod}_R)$. The statement thus follows from the second part of Lemma 2.10 and [Lur17, 7.1.2.1]. □

We conclude this section with describing the relation between colimits of algebra objects in RMod_R and the colimits of the corresponding ∞-categories of right modules in LinCat_R. There is a functor $\theta\colon \mathrm{Alg}(\mathrm{RMod}_R) \to \mathrm{LinCat}_R$ that assigns to an algebra object $A \in \mathrm{Alg}(\mathrm{RMod}_R)$ the ∞-category $\mathrm{RMod}_A(\mathrm{RMod}_R)$, see [Lur17, Section 4.8.3]. The functor θ assigns to an edge $\phi\colon A \to B$ in $\mathrm{Alg}(\mathrm{RMod}_R)$ the relative tensor product

$$\theta(\phi) = \text{-} \otimes_A B\colon \mathrm{RMod}_A(\mathrm{RMod}_R) \longrightarrow \mathrm{RMod}_B(\mathrm{RMod}_R)$$

using the right A-module structure on B provided by ϕ. For all $\phi\colon A \to B$, the functor $\theta(\phi)$ admits a right adjoint, given by pullback $\phi^*\colon \mathrm{RMod}_B(\mathrm{RMod}_R) \to \mathrm{RMod}_A(\mathrm{RMod}_R)$ along ϕ, see [Lur17, 4.6.2.17]. The functor θ preserves colimits indexed by contractible simplicial sets (i.e. simplicial sets whose geometric realization is a contractible space), most notably pushouts. Corollary 2.12 shows that $\mathrm{RMod}_A(\mathrm{RMod}_R) \simeq \mathrm{RMod}_A$ and we can thus employ the functor θ to describe RMod_A as a colimit in LinCat_R, provided a description of A as a contractible colimit in $\mathrm{Alg}(\mathrm{RMod}_R)$.

2.1.5 Monadic adjunctions

Let $\mathcal{C}$ be an ∞-category. A monad M on $\mathcal{C}$ is an associative algebra object in the monoidal ∞-category $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C},\mathcal{C})$ of endofunctors. In other words, a monad is a functor $M\colon \mathcal{C} \to \mathcal{C}$, together with a unit $u\colon \mathrm{id}_{\mathcal{C}} \to M$ and a multiplication map $M \circ M \to M$ equipped with data exhibiting coherent associativity and unitality. A main source of monads are adjunctions: given an adjunction $F\colon \mathcal{C} \leftrightarrow \mathcal{D} :G$ of ∞-categories, there is an associated monad $M = GF$, see [Lur17, Section 4.7.3], which we call the adjunction monad. Associated to a monad M on $\mathcal{C}$ is its ∞-category $\mathrm{LMod}_M(\mathcal{C})$ of left modules in $\mathcal{C}$ and the free-forget adjunction

$$\mathrm{Free}\colon \mathcal{C} \longleftrightarrow \mathrm{LMod}_M(\mathcal{C}) :\mathrm{Forget}\,,$$

whose adjunction monad is equivalent to M. The ∞-category $\mathrm{LMod}_M(\mathcal{C})$ is also called the Eilenberg-Moore ∞-category of M. For a typical example, take $\mathcal{C} = \mathcal{D}(k)$ to be the derived ∞-category of a field. If $M = A \otimes_k$ - is the monad arising from tensoring with a dg-algebra A, then $\mathrm{LMod}_M(\mathcal{D}(k)) \simeq \mathcal{D}(A)$. The associated adjunction is equivalent to the usual free-forget adjunction $\mathcal{D}(k) \leftrightarrow \mathcal{D}(A)$.

Given an adjunction $F\colon \mathcal{C} \leftrightarrow \mathcal{D} :G$ with monad $M = GF$, there is an associated functor $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$, making the following diagram commute:

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr, "G"'] & & \mathrm{LMod}_M(\mathcal{C}) \arrow[dl, "\mathrm{Forget}"] \\ & \mathcal{C} & \end{tikzcd}$$

Definition 2.13. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories.

1. The functor G is called monadic if it admits a left adjoint F with adjunction monad $M = GF$ and the associated functor $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$ is an equivalence of ∞-categories.

2. The functor G is called comonadic if the opposite functor $G^{\mathrm{op}}\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is monadic.

If $\mathcal{C}, \mathcal{D}$ are presentable ∞-categories and G preserves (sufficient) colimits, then $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$ admits a fully faithful left adjoint, see [Lur17, Lemma 4.7.3.13].

In this case, $\mathrm{LMod}_M(\mathcal{C})$ is presentable and if $\mathcal{C}$ and $\mathcal{D}$ are furthermore stable, then $\mathrm{LMod}_M(\mathcal{C})$ is also stable, see [Lur17, Prop. 4.2.3.4].

To determine whether a functor is monadic, one can use the ∞-categorical Barr-Beck monadicity theorem, see [Lur17, Thm. 4.7.3.5]. Assuming that all involved ∞-categories are presentable and that the functor preserves all colimits, the theorem reduces to the statement that a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ is monadic if and only if it is conservative, i.e. reflects isomorphisms. Similarly, in this setting a left adjoint F which preserves sufficient limits is comonadic if and only if it is conservative.

2.2 From dg-categories to stable ∞-categories

2.2.1 Differential graded categories and their modules

Let k be a commutative ring. A k-linear dg-category is a 1-category enriched in the 1-category $\mathrm{Ch}(k)$ of chain complexes of k-modules. Given a dg-category C and two objects $x, y \in C$, we write $\mathrm{Hom}_C(x, y)$ or $\mathrm{Hom}(x, y)$ for the mapping complex. We consider dg-algebras as dg-categories with a single object.

Definition 2.14. Let A and B be k-linear dg-algebras.

- A left A-module M is a graded left module over the graded algebra underlying A equipped with a differential d_M, satisfying that

$$d_M(a.m) = d_A(a).m + (-1)^{\deg(a)} a.d_M(m)$$

  for all $a \in A$ and $m \in M$.

- A right A-module M is a graded right module over the graded algebra underlying A equipped with a differential d_M, satisfying that

$$d_M(m.a) = d_M(m).a + (-1)^{\deg(m)} m.d_A(a)$$

  for all $a \in A$ and $m \in M$. We also refer to right A-modules simply as A-modules.

- An A-B-bimodule M is a graded bimodule over the graded algebras underlying A and B equipped endowed with a differential d_M, which exhibits M as a left A-module and a right B-module. If $A = B$, we call M an A-bimodule.

Remark 2.15. Let M be an A-B-bimodule with differential d_M. The shifted A-B-bimodule $M[1]$ can be described as follows.

- The differential is $-d_M$.

- The left action $._{[1]}$ of $a \in A$ on $m \in M[1]$ is given by $a._{[1]}m = (-1)^{\deg(a)} a.m$, where $a.m$ denotes the left action of $a \in A$ on $m \in M$.

- The right action $._{[1]}$ of $b \in B$ on $m \in M[1]$ is given by $m._{[1]}b = m.b$, where $m.b$ denotes the right action of $b \in B$ on $m \in M$.

We can identify left A-modules with dg-functors $A \to \mathrm{Ch}(k)$, right A-modules with dg-functors $A^{op} \to \mathrm{Ch}(k)$ and A-B-bimodules with dg-functors $A \otimes B^{\mathrm{op}} \to \mathrm{Ch}(k)$. The following definition is thus consistent with Definition 2.14.

Definition 2.16. Let C be a dg-category. We call a dg-functor $C^{op} \to \mathrm{Ch}(k)$ a right C-module. We denote by $\mathrm{dgMod}(C)$ the dg-category of right C-modules.

Given a dg-category C and an object $x \in C$, we denote by $\mathrm{End}^{\mathrm{dg}}(x)$ the endomorphism dg-algebra with underlying chain complex given by $\mathrm{Hom}_C(x, x)$ and algebra structure determined by the composition of morphisms in C. We denote the mapping complex between two objects $x, y \in C$ by $\mathrm{Hom}_C(x, y)$ or $\mathrm{Hom}(x, y)$. Given two dg-modules $x, y \in \mathrm{dgMod}(C)$ and a morphism $\alpha\colon x \to y$, we denote by $\mathrm{cone}(\alpha) = x[1] \oplus y$ the cone with differential $d(x, y) = (-d(x), d(y) - \alpha(x))$. Given two dg-modules x, y, the morphism complex $\mathrm{Hom}(x, y)$ has differential $d(f) = d_y \circ f - (-1)^{\deg(f)} f \circ d_x$.

Lemma 2.17. *Let C be a dg-category with finitely many objects $x_1, \dots, x_n$. Then there exists an equivalence of dg-categories $\mathrm{dgMod}(C) \simeq \mathrm{dgMod}(\mathrm{End}^{\mathrm{dg}}(\bigoplus_{i=1}^n x_i))$, where $\mathrm{End}^{\mathrm{dg}}(\bigoplus_{i=1}^n x_i)$ is the endomorphism dg-algebra of $\bigoplus_{i=1}^n x_i$ in $\mathrm{dgMod}(C)$.*

Proof. This follows directly from spelling out the datum of a right module over C and over $\mathrm{End}^{\mathrm{dg}}(\bigoplus_{i=1}^n x_i)$. □

Remark 2.18. A quiver is a directed graph with finitely many directed edges, called arrows, and finitely many vertices. A graded quiver is a quiver, where each arrow carries a $\mathbb{Z}$-label. We will say that a dg-categroy C with finitely many objects arises from a graded quiver Q, if the set of objects of C is given by the set of vertices of Q and the morphism complexes in C are freely generated over k by the (allowed) composites of the graded arrows of Q. In this case, the dg-category C is fully determined by Q and the differentials of the generators, given by the arrows of Q. Lemma 2.17 reduces in this setting to the statement, that the dg-category C is Morita equivalent to the path algebra of the quiver Q with differential determined on generators as in C.

2.2.2 A model for the derived ∞-category of a dg-algebra

Let A be a k-linear dg-algebra. Starting with the dg-category $\mathrm{dgMod}(A)$, we can form the 1-category $\mathrm{dgMod}(A)_0$, with the same objects as $\mathrm{dgMod}(A)$ and with mapping sets given by the 0-cycles. This 1-category admits the projective model structure, where the weak equivalences are given by quasi-isomorphisms and the fibrations are given by degree-wise surjections. All objects of $\mathrm{dgMod}(A)_0$ are fibrant. A description of the cofibrant objects in $\mathrm{dgMod}(A)_0$ can be found for example in [BMR14], where they are called q-semi-projective objects. A right A-module M is cofibrant if and only if

- the ungraded module $\bigoplus_{i \in \mathbb{Z}} M_i$ is a projective right module over the ungraded algebra $\bigoplus_{i \in \mathbb{Z}} A_i$ and

- for all acyclic right A-modules N, the mapping complex $\mathrm{Hom}_A(M, N)$ is acyclic.

If $A = k$ is a commutative ring, the cofibrant objects are the complexes of projective k-modules. We denote by $\mathrm{dgMod}(A)^\circ \subset \mathrm{dgMod}(A)$ the full dg-subcategory spanned by fibrant-cofibrant objects. We call the dg-nerve $\mathcal{D}(A) := N_{\mathrm{dg}}(\mathrm{dgMod}(A)^\circ)$ the (unbounded) derived ∞-category of A.

Before we can further discuss the properties of $\mathcal{D}(A)$, we need to briefly discuss localizations of ∞-categories.

Definition 2.19. A functor $f\colon \mathcal{C} \to \mathcal{C}'$ between ∞-categories is a reflective localization if f has a fully faithful right adjoint.

In [Lur09], localizations in the sense of Definition 2.19 are simply called localizations. We are however interested in a more general class of localizations, which can be characterized by the following universal property.

Definition 2.20. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms in $\mathcal{C}$. We call an ∞-category $\mathcal{C}'$ the ∞-categorical localization of $\mathcal{C}$ at W if there exists a functor $f\colon \mathcal{C} \to \mathcal{C}'$, such that, for every ∞-category $\mathcal{D}$, composition with f induces a fully faithful functor

$$\chi\colon\ \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})\,,$$

whose essential image consists of those functors $F\colon \mathcal{C} \to \mathcal{D}$ for which $F(\alpha)$ is an equivalence in $\mathcal{D}$ for all $\alpha \in W$. In that case, we also write $\mathcal{C}' = \mathcal{C}[W^{-1}]$.

It is shown in [Lur09, 5.2.7.12], that reflective localizations are localizations in the sense of Definition 2.20. If the collection of morphisms W is closed under homotopy and composition and contains all equivalences in $\mathcal{C}$, we can regard $\mathcal{C}[W^{-1}]$ as a fibrant replacement of $(\mathcal{C}, W)$ in the model category of marked simplicial sets, see also the discussion in the beginning of [Lur17, Section 4.1.7].

Our first goal in this section is to prove the following analogue of [Lur17, 1.3.5.15], which relates the derived ∞-category of A with the ∞-categorical localization of $\mathrm{dgMod}(A)_0$ at the collection of quasi-isomorphisms.

Proposition 2.21. *Let A be a dg-algebra and let W denote the collection of quasi-isomorphisms. There exists an equivalence of ∞-categories*

$$\mathcal{D}(A) \simeq N(\mathrm{dgMod}(A)_0)[W^{-1}]\,.$$

Given a model category C, the ∞-categorical localization of $N(C)$ at the collection of weak equivalences is called the ∞-category underlying C. We refer to [Hin16] for general background. Proposition 2.21 thus shows that the derived ∞-category of A is the ∞-category underlying the model category $\mathrm{dgMod}(A)_0$.

For the proof of Proposition 2.21 we need the following two lemmas.

Lemma 2.22. *Let A be a dg-algebra. The inclusion functor $N(\mathrm{dgMod}(A)_0) \to N_{\mathrm{dg}}(\mathrm{dgMod}(A))$ induces an equivalence of ∞-categories*

$$N(\mathrm{dgMod}(A)_0)[H^{-1}] \to N_{\mathrm{dg}}(\mathrm{dgMod}(A)),$$

where H is the collection of chain homotopy equivalences.

Proof. The proof of [Lur17, 1.3.4.5] applies verbatim. □

Lemma 2.23. *Let A be a dg-algebra. There exists an equivalence of ∞-categories*

$$N_{\mathrm{dg}}(\mathrm{dgMod}(A)^\circ) \simeq N_{\mathrm{dg}}(\mathrm{dgMod}(A))[W^{-1}].$$

Proof. We adapt the proofs of [Lur17, 1.3.4.6, 1.3.5.12]. We show that the inclusion functor

$$i^{\mathrm{op}}\colon\ \mathrm{N_{dg}}(\mathrm{dgMod}(A)^\circ)^{\mathrm{op}} \to \mathrm{N_{dg}}(\mathrm{dgMod}(A))^{\mathrm{op}}$$

admits a left adjoint which exhibits $\mathrm{N_{dg}}(\mathrm{dgMod}(A)^\circ)^{\mathrm{op}}$ as a reflective localization at the collection of quasi-isomorphisms. Note that any functor is a localization if and only if the opposite functor is a localization. We thus conclude that $\mathrm{N_{dg}}(\mathrm{dgMod}(A)^\circ)$ is equivalent as an ∞-category to the localization of $\mathrm{N_{dg}}(\mathrm{dgMod}(A))$ at the collection of quasi-isomorphisms.

We have to show that i^{op} admits a left adjoint G. Further, to show that G is a localization along the collection of quasi-isomorphisms, we need to show by [Lur09, 5.2.7.12] that any edge $e\colon M \to N$ in $\mathrm{N_{dg}}(\mathrm{dgMod}(A))^{op}$ is a quasi-isomorphism if and only if $G(e)$ is an equivalence. Consider a trivial fibration $f\colon Q' \to Q$ in $\mathrm{dgMod}(A)$ given by a cofibrant replacement and any $P \in \mathrm{dgMod}(A)^\circ$. [Lur09, 5.2.7.8] shows the existence of G, provided that the composition with f induces an equivalence of spaces

$$\mathrm{Map}_{\mathrm{N_{dg}}(\mathrm{dgMod}(A)^\circ)}(P, Q') \to \mathrm{Map}_{\mathrm{N_{dg}}(\mathrm{dgMod}(A))}(P, Q).$$

We deduce this from the assertion that composition with f gives a quasi-isomorphism

$$\alpha\colon\ \mathrm{Hom}_{\mathrm{dgMod}(A)}(P, Q') \to \mathrm{Hom}_{\mathrm{dgMod}(A)}(P, Q). \tag{11}$$

The surjectivity of α follows from the lifting property of the cofibration $0 \to P$ with respect to trivial fibrations. The kernel of α is given by $\mathrm{Hom}_{\mathrm{dgMod}(A)}(P, \ker(f))$. Using that f is a quasi-isomorphism, we deduce that $\ker(f)$ is acyclic. The contractibility of the kernel of α thus follows from property of P being cofibrant. We can thus deduce the existence of G. We note that G is pointwise given by choosing a cofibrant replacement. Consider an edge $e\colon M \to N$ in $\mathrm{N_{dg}}(\mathrm{dgMod}(A))^{op}$. If e is a quasi-isomorphism, it follows from Whitehead's theorem for model categories that $G(e)$ is an equivalence. If $G(e)$ is an equivalence, we have the following commutative diagram in $\mathrm{N_{dg}}(\mathrm{dgMod}(A))$.

$$\begin{tikzcd} G(M) \arrow[r, "G(e)"] \arrow[d] & G(N) \arrow[d] \\ M \arrow[r, "e"] & N \end{tikzcd}$$

The vertical edges and the upper horizontal edge are quasi-isomorphisms. It follows that e is also a quasi-isomorphism. □

Proof of Proposition 2.21. By Lemmas 2.22 and 2.23, there exists an equivalence of ∞-categories

$$\left(N(\mathrm{dgMod}(A))[H^{-1}]\right)[W^{-1}] \simeq \mathcal{D}(A)\,.$$

Using that $H \subset W$, the statement follows. □

Let k be a commutative ring. The symmetric monoidal structure of the 1-category $\mathrm{Ch}(k)$ can be used to also endow the ∞-category $\mathcal{D}(k)$ with a symmetric monoidal structure. As shown in [Lur17, 7.1.4.6] there exists an equivalence of ∞-categories

$$\mathrm{N}(\mathrm{Alg}(\mathrm{Ch}^{\otimes}(k)))[W^{-1}] \simeq \mathrm{Alg}(\mathcal{D}(k))\,. \tag{12}$$

The left side of (12) is the ∞-categorical localization of the nerve of the 1-category of dg-algebras at the collection of quasi-isomorphisms. The right side of (12) is the ∞-category of algebra objects in $\mathcal{D}(k)$. The equivalence (12) expresses that every dg-algebra can be considered as an algebra object in $\mathcal{D}(k)$ and that every algebra object in $\mathcal{D}(k)$ can be obtained this way (meaning it can be rectified). Unless stated otherwise, we will omit the identification (12) and consider dg-algebras as algebra objects in the symmetric monoidal ∞-category $\mathcal{D}(k)$.

We can consider k also as an $\mathbb{E}_\infty$-ring spectrum. The ∞-category RMod_k of right modules over k thus inherits a symmetric monoidal structure. The ∞-categories $\mathcal{D}(k)$ and RMod_k are equivalent as symmetric monoidal ∞-categories, see [Lur17, 7.1.2.13].

Let A be a k-linear dg-algebra and X a cofibrant A-module. Consider the Quillen adjunction

$$\text{-} \otimes_k^{\mathrm{dg}} X \colon \mathrm{dgMod}(k) \leftrightarrow \mathrm{dgMod}(A) : \mathrm{Hom}_A(X, \text{-})\,, \tag{13}$$

between the tensor functor on the level of chain complexes and the internal Hom functor composed with the forgetful functor $\mathrm{dgMod}(A) \to \mathrm{dgMod}(k)$. Given a Quillen-adjunction between model categories, there is an associated adjunction between the underlying ∞-categories, see [MG16]. We denote the adjunction of ∞-categories underlying the Quillen adjunction (13) by

$$\text{-} \otimes_k^{\mathrm{dg}} X \colon \mathcal{D}(k) \leftrightarrow \mathcal{D}(A) : \mathrm{RHom}_A(X, \text{-})\,. \tag{14}$$

Lemma 2.24. *Let A be a k-linear dg-algebra. The ∞-category $\mathcal{D}(A)$ admits a structure of a k-linear ∞-category such that for any $X \in \mathcal{D}(A)$ the functor $\text{-} \otimes_k^{dg} X$ is k-linear.*

Proof. The ∞-category $\mathcal{D}(A)$ is stable and presentable by [Lur17, 1.3.5.9, 1.3.5.21]. We now show that $\mathcal{D}(A)$ is left tensored over $\mathcal{D}(k)$. Note that $\mathrm{dgMod}(k)_0 \simeq \mathrm{Ch}(k)$ is a symmetric monoidal model category with respect to the tensor product, which we denote in the following by $\otimes$, see [Lur17, 7.1.2.11]. We further denote the Quillen bifunctor $\mathrm{dgMod}(k) \times \mathrm{dgMod}(A) \to \mathrm{dgMod}(A)$ given by the relative tensor product by $\text{-} \otimes_k^{\mathrm{dg}} \text{-}$. Recall that $\mathcal{LM}^{\otimes}$ denotes the left-module ∞-operad, see [Lur17, 4.2.1.7]. We define a 1-category $O_A^{\otimes}$ as follows.

- An object of $O_A^\otimes$ consists of an object $(\underbrace{a,\dots,a}_{i\text{-many}},\underbrace{m,\dots,m}_{j\text{-many}}) \in \mathcal{LM}^\otimes$ and objects

$$(x_1,\dots,x_i) \in (\mathrm{dgMod}(k)^\circ)^{\times i},\ (m_1,\dots,m_j) \in (\mathrm{dgMod}(A)^\circ)^{\times j}\,.$$

- For $n = 1,2$, consider the object X_n of $O_A^\otimes$ given by $l_n = (\underbrace{a,\dots,a}_{i_n\text{-many}},\underbrace{m,\dots,m}_{j_n\text{-many}}) \in \mathcal{LM}^\otimes$ and

$$(x_1^n,\dots,x_{i_n}^n) \in (\mathrm{dgMod}(k)^\circ)^{\times i_n},\ (m_1^n,\dots,m_{j_n}^n) \in (\mathrm{dgMod}(A)^\circ)^{\times j_n}\,.$$

  A morphism $X_1 \to X_2$ consists of a morphism $\alpha\colon l_1 \to l_2$ in $\mathcal{LM}^\otimes$, which we also consider as a morphism of sets $\tilde{\alpha}\colon \{1,\dots,i_1+j_1\} \to \{1,\dots,i_2+j_2\}$, morphisms

$$\bigotimes_{e\in\tilde{\alpha}^{-1}(i)} a_e^1 \to a_i^2$$

  in $\mathrm{dgMod}(k)^\circ$ for $1 \le i \le i_2$ and morphisms

$$\left(\bigotimes_{e\in\tilde{\alpha}^{-1}(j)\setminus\max(\tilde{\alpha}^{-1}(j))} a_e^1\right) \otimes_k m^1_{\max(\tilde{\alpha}^{-1}(j))-i_1} \to m^2_{j-i_2}$$

  in $\mathrm{dgMod}(A)^\circ$ for $i_1+1 \le j \le i_2+j_2$.

The forgetful functor $N(O_A^\otimes) \to \mathcal{LM}^\otimes$ is a coCartesian fibration of ∞-operads, exhibiting $N((\mathrm{dgMod}(A)^\circ)_0)$ as left-tensored over the symmetric monoidal ∞-category $N((\mathrm{dgMod}(k)^\circ)_0)$. By the discussion following [Lur17, 4.1.7.3] and using that -$\otimes_k^{\mathrm{dg}}$- preserves weak equivalences in both entries, it follows that the left-tensoring passes to the ∞-categorical localizations at the chain homotopy equivalences, meaning that we obtain that $\mathcal{D}(A)$ is left-tensored over $\mathcal{D}(k)$. The action of $\mathcal{D}(k)$ on $\mathcal{D}(A)$ preserves colimits in both variables, as follows from the monoidal product - $\otimes_k$ - being a Quillen-bifunctor. To see that - $\otimes_k^{\mathrm{dg}} X$ is a k-linear functor, we need to describe an extension of - $\otimes_k^{\mathrm{dg}} X$ to a map $\alpha\colon N(O_k^\otimes) \to N(O_A^\otimes)$ of ∞-operads over $\mathcal{LM}^\otimes$. We leave the details of the description of a functor of 1-categories $\alpha'\colon O_k^\otimes \to O_A^\otimes$ whose nerve $N(\alpha')$ defines the desired functor α to the reader. □

Proposition 2.25. *Let A be a k-linear dg-algebra. Using the symmetric monoidal equivalence $\mathcal{D}(k) \simeq \mathrm{RMod}_k$, we can consider $\mathrm{RMod}_A \overset{(10)}{\simeq} \mathrm{RMod}_A(\mathrm{RMod}_k)$ as left-tensored over $\mathcal{D}(k)$. There exists an equivalence*

$$\mathcal{D}(A) \simeq \mathrm{RMod}_A \tag{15}$$

of ∞-categories left-tensored over $\mathcal{D}(k)$.

Proof. Consider the adjunction of ∞-categories -$\otimes_k^{\mathrm{dg}} A\colon \mathcal{D}(k) \leftrightarrow \mathcal{D}(A) :\mathrm{RHom}_A(A,\text{-})$ underlying the Quillen adjunction - $\otimes_k^{\mathrm{dg}} A\colon\ \mathrm{dgMod}(k) \to \mathrm{dgMod}(A) :\mathrm{Hom}_A(A,\text{-})$.

Using the adjunction it can be directly checked that A is a compact generator of $\mathcal{D}(A)$. It follows from [Lur17, 4.8.5.8] that there exists an equivalence

$$\mathcal{D}(A) \simeq \mathrm{RMod}_{\mathrm{End}_k(A)}(\mathcal{D}(k)) \tag{16}$$

of ∞-categories left-tensored over $\mathcal{D}(k)$, where $\mathrm{End}_k(A) \in \mathrm{Alg}(\mathcal{D}(k))$ is the k-linear endomorphism algebra of A, see Remark 2.11. We note that the underlying chain complex satisfies $\mathrm{End}_k(A) \simeq \mathrm{RHom}_A(A, A) \simeq A$. By the universal property of $\mathrm{End}_k(A)$, there exists a morphism of dg-algebras $\chi\colon A \to \mathrm{End}_k(A)$, the underlying morphism of chain complexes of which is induced by the actions $A \otimes_k A \to A$ and $A \otimes_k \mathrm{End}_k(A) \to A$. The latter is induced by the counit of the adjunction $\text{-} \otimes_k^{\mathrm{dg}} A \dashv \mathrm{RHom}_A(A, \text{-})$ and thus equivalent to the former. It follows that χ induces a quasi-isomorphism $\mathrm{End}_k(A) = \mathrm{RHom}_A(A, A) \simeq A$ on underlying chain complexes and is hence a quasi-isomorphism of dg-algebras. In total we obtain, that there also exists an equivalence of k-linear ∞-categories $\mathrm{RMod}_{\mathrm{End}_k(A)}(\mathcal{D}(k)) \simeq \mathrm{RMod}_A(\mathcal{D}(k)) \simeq \mathrm{RMod}_A(\mathrm{RMod}_k)$, which combined with (16) shows the statement. □

Let $A, B \in \mathrm{Alg}(\mathcal{D}(k))$ be dg-algebras and $F\colon \mathrm{RMod}_A \to \mathrm{RMod}_B$ a k-linear functor. Clearly $F(A) \in \mathrm{RMod}_B$ carries the structure of a right B-module. Let $m\colon A \otimes_k A \to A$ be the multiplication map of A. Using the k-linearity of F, we find an action map

$$A \otimes_k F(A) \simeq F(A \otimes_k A) \xrightarrow{F(m)} F(A)\,,$$

which is part of the datum of a left A-module structure on $F(A)$. It turns out that both module structures are compatible, so that we can endow $F(A)$ with the structure of an A-B-bimodule. Since observation can be turned into a functor

$$\phi\colon \mathrm{Lin}_k(\mathrm{RMod}_A, \mathrm{RMod}_B) \to {}_A\mathrm{BMod}_B(\mathcal{D}(k))\,,$$

which is shown in [Lur17, Section 4.8.4] to be an equivalence of ∞-categories. Given a bimodule $M \in {}_A\mathrm{BMod}_B(\mathcal{D}(k))$, we denote by $\text{-} \otimes_A M$ a choice of k-linear functor satisfying that $\phi(\text{-} \otimes_A M) \simeq M$.

Proposition 2.26. *Let A, B be dg-algebras and $M \in {}_A\mathrm{BMod}_B(\mathcal{D}(k)) \simeq \mathcal{D}(A^{op} \otimes_k B)$ and consider the functor of ∞-categories $\text{-}\otimes_A^{\mathrm{dg}} M$ underlying the right Quillen functor $\text{-}\otimes_A^{\mathrm{dg}} M\colon$ $\mathrm{dgMod}(A) \to \mathrm{dgMod}(B)$. There exists a commutative diagram in* LinCat_k *as follows.*

$$\begin{tikzcd} \mathrm{RMod}_A \arrow[r, "\text{-}\otimes_A M"] \arrow[d, "(15)"', "\simeq"] & \mathrm{RMod}_B \arrow[d, "(15)"', "\simeq"] \\ \mathcal{D}(A) \arrow[r, "\text{-}\otimes_A^{\mathrm{dg}} M"] & \mathcal{D}(B) \end{tikzcd} \tag{17}$$

Remark 2.27. As justified by Proposition 2.26, we will not distinguish in notation between the functors $\text{-} \otimes_A M$ and $\text{-} \otimes_A^{\mathrm{dg}} M$ in the remainder of the thesis.

Proof of Proposition 2.26. The k-linear functor

$$\chi\colon \mathrm{RMod}_A \simeq \mathcal{D}(A) \xrightarrow{\text{-}\otimes_A^{\mathrm{dg}} M} \mathcal{D}(B) \simeq \mathrm{RMod}_B$$

is of the form $- \otimes_A N$ for $N \in {}_A\mathrm{BMod}_B(\mathcal{D}(k))$, see [Lur17, Section 4.8.4]. We note that N can be rectified to a strict dg-bimodule and is thus determined by its right B-module structure and its left A-module structure. The right B-module structures of N and M are clearly equivalent. In particular, there exists an equivalence $N \simeq M$ of underlying chain complexes. The left action of A on N is determined by $A \otimes_k N \simeq \chi(A \otimes_k A) \xrightarrow{\chi(m)} \chi(A) \simeq N$, where m denotes the multiplication of A and is thus equivalent to the given left action of A on the A-B-bimodule M. This shows that $N \simeq M$ as bimodules. □

Proposition 2.28. *Let A be a k-linear dg-algebra and $X \in \mathrm{dgMod}(A)$ a cofibrant A-module. The k-linear endomorphism algebra $\mathrm{End}_k(X) \in \mathrm{Alg}(\mathcal{D}(k))$ of X is quasi-isomorphic to the endomorphism dg-algebra $\mathrm{End}^{\mathrm{dg}}(X)$ of X.*

Proof. Proposition 2.26 shows that the functor

$$F\colon \mathcal{D}(k) \simeq \mathrm{RMod}_k \xrightarrow{-\otimes_k X} \mathrm{RMod}_A(\mathrm{RMod}_k) \simeq \mathcal{D}(A)$$

is equivalent to $- \otimes_k^{\mathrm{dg}} X$. The right adjoint G of F is given by $\mathrm{RHom}_A(X, -)$. It follows that $\mathrm{RHom}_A(X, X) \simeq G(X) = \mathrm{End}_k(X)$ in $\mathcal{D}(k)$, see also the definition of $\mathrm{End}_k(X)$ in the proof of Lemma 2.10. Using that $\mathrm{RHom}_A(X, X) = \mathrm{Hom}_A(X, X) = \mathrm{End}^{\mathrm{dg}}(X)$ and the explicit $\mathrm{Hom}_A(X, X)$-module structure on X, it follows from the universal property of the endomorphism object that there exists a morphism of dg-algebras $\alpha\colon \mathrm{RHom}_A(X, X) \to \mathrm{End}_k(X)$, which restricts to the quasi-isomorphism on underlying chain complexes and is hence an quasi-isomorphism of dg-algebras. □

2.2.3 Morita theory

Let k be a commutative ring. We denote by dgCat_k the category of k-linear dg-categories. Given a dg-category $C \in \mathrm{dgCat}_k$, the dg-category $\mathrm{dgMod}(C)$ admits a model structure called the projective model structure. We have already encountered this model structure in Section 2.2.2 in the case where C is a dg-algebra. We define C^{perf} as the full dg-subcategory of $\mathrm{dgMod}(C)$ spanned by fibrant-cofibrant objects x which are compact in the homotopy category $H^0(\mathrm{dgMod}(C))$, i.e. $\mathrm{Hom}(x, -)$ preserves coproducts. This assignment forms a functor

$$(-)^{\mathrm{perf}}\colon \mathrm{dgCat}_k \to \mathrm{dgCat}_k\,.$$

As shown by Tabuada [Tab05], the category dgCat_k admits a model structure where

- the weak equivalences are the quasi-equivalences, that is dg-functors $F\colon A \to B$ satisfying that for all $a, a' \in A$, the morphism between morphism complexes $F(a, a')\colon \mathrm{Hom}_A(a, a') \to \mathrm{Hom}_B(F(a), F(a'))$ is a quasi-isomorphism and such that the induced functor on homotopy categories is an equivalence.

- the fibrations are the dg-functors F such that for all $a, a' \in A$, the morphism between mapping complexes $F(a,a')\colon \operatorname{Hom}_A(a,a') \to \operatorname{Hom}_B(F(a),F(a'))$ is degreewise surjective and satisfies that for any isomorphism $b \to F(a')$ in the homotopy category of B there exists a lift along F to an isomorphism $a \to a'$ in the homotopy category of A.

The ∞-category underlying this model category is given by the ∞-categorical localization $\mathrm{dgCat}_k[W^{-1}]$ of the nerve of dgCat_k at the collection W of weak equivalences. This model structure can be further localized at the collection M of Morita-equivalences, that is dg-functors F such that $(F)^{\mathrm{perf}}$ is a quasi-equivalence. The resulting model structure is called the Morita model structure. The corresponding localization functor

$$L\colon \mathrm{dgCat}_k[W^{-1}] \longrightarrow \mathrm{dgCat}_k[M^{-1}]$$

exhibits $\mathrm{dgCat}_k[M^{-1}]$ as a reflective localization of $\mathrm{dgCat}_k[W^{-1}]$ and thus preserves colimits. Given $C \in \mathrm{dgCat}_k[W^{-1}]$, its image $L(C)$ is quasi-equivalent to C^{perf}. The Morita model structure models the ∞-category of k-linear, stable and idempotent complete ∞-categories, meaning that there exists an equivalence of ∞-categories

$$\mathrm{dgCat}_k[M^{-1}] \simeq \mathrm{Mod}_{N_{\mathrm{dg}}(k^{\mathrm{perf}})}(\mathrm{St}^{\mathrm{idem}})\,, \tag{18}$$

see [Coh13]. The right side of (18) describes the ∞-category of modules in the symmetric monoidal category $\mathrm{St}^{\mathrm{idem}}$ over the algebra object $N_{\mathrm{dg}}(k^{\mathrm{perf}})$. The equivalence (18) maps a dg-category C to the dg-nerve of the dg-category C^{perf}. Ind-completion provides a further colimit preserving functor $\mathrm{Ind}\colon \mathrm{Mod}_{\mathcal{D}(k)^{\mathrm{perf}}}(\mathrm{St}^{\mathrm{idem}}) \to \mathrm{LinCat}_k$. In total we obtain the colimit preserving functor

$$\mathrm{dgCat}_k[W^{-1}] \xrightarrow{L} \mathrm{dgCat}_k[M^{-1}] \simeq \mathrm{Mod}_{\mathcal{D}(k)^{\mathrm{perf}}}(\mathrm{St}^{\mathrm{idem}}) \xrightarrow{\mathrm{Ind}} \mathrm{LinCat}_k \xrightarrow{\mathrm{forget}} \mathcal{P}r^L\,, \tag{19}$$

denoted $\mathcal{D}(\text{-})$. Note that given a dg-algebra A, the derived ∞-category $\mathcal{D}(A)$ is equivalent to the image of A under (19), so that the notation $\mathcal{D}(\text{-})$ for the functor (19) is justified. Furthermore, we can compute colimits in $\mathrm{dgCat}_k[W^{-1}]$ as homotopy colimits in dgCat_k with respect to the quasi-equivalence model structure.

2.3 Semiorthogonal decompositions

In this section we discuss semiorthogonal decompositions of stable ∞-categories of length $n \geq 2$. Some of the treatment is based on the discussion of semiorthogonal decompositions of length $n = 2$ in [DKSS21].

2.3.1 Generalities

Definition 2.29. Let $\mathcal{V}$ and $\mathcal{A}$ be stable ∞-categories. We call $\mathcal{A} \subset \mathcal{V}$ a stable subcategory if the inclusion functor is fully faithful, exact and its image is closed under equivalences.

Notation 2.30. Let $\mathcal{V}$ be a stable ∞-category and $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{V}$ stable subcategories. We denote by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ the smallest stable subcategory of $\mathcal{V}$ containing $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

Definition 2.31. Let $\mathcal{V}$ be a stable ∞-category and let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be stable subcategories of $\mathcal{V}$. Consider the full subcategory $\mathcal{D}$ of $\mathrm{Fun}(\Delta^{n-1}, \mathcal{V})$ spanned by diagrams $D : \Delta^{n-1} \to \mathcal{V}$ satisfying the following two conditions.

- $D(i)$ lies in $\langle \mathcal{A}_{n-i}, \ldots, \mathcal{A}_n \rangle$ for $0 \leq i \leq n-1$.
- The cofiber of $D(i) \to D(i+1)$ in $\mathcal{V}$ lies in $\mathcal{A}_{n-i-1}$ for all $0 \leq i \leq n-2$.

We call the ordered n-tuple $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ a semiorthogonal decomposition of $\mathcal{V}$ of length n if the restriction functor $\mathcal{D} \to \mathcal{V}$ to the vertex $n-1$ is a trivial fibration.

Definition 2.32. Let $\mathcal{V}$ be a stable ∞-category and $\mathcal{A} \subset \mathcal{V}$ a stable subcategory. We define

- the right orthogonal $\mathcal{A}^{\perp}$ to be the full subcategory of $\mathcal{V}$ spanned by those vertices $x \in \mathcal{V}$ such that for all $a \in \mathcal{A}$ the mapping space $\mathrm{Map}_{\mathcal{V}}(a, x)$ is contractible.
- the left orthogonal ${}^{\perp}\mathcal{A}$ to be the full subcategory of $\mathcal{V}$ spanned by those vertices $x \in \mathcal{V}$ such that for all $a \in \mathcal{A}$ the mapping space $\mathrm{Map}_{\mathcal{V}}(x, a)$ is contractible.

The next lemma shows that semiorthogonal decompositions of length n are simply repeated semiorthogonal decompositions of length 2.

Lemma 2.33. *Let $\mathcal{V}$ be a stable ∞-category and $\mathcal{A}_i \subset \mathcal{V}$, for $1 \leq i \leq n$, a stable subcategory. $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a semiorthogonal decomposition of $\mathcal{V}$ if and only if*

i) $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle = \mathcal{V}$ *and*

ii) $(\mathcal{A}_i, {}^{\perp}\mathcal{A}_i)$ *forms a semiorthogonal decomposition of* $\langle \mathcal{A}_i, \ldots, \mathcal{A}_n \rangle$ *for all* $1 \leq i \leq n-1$.

Proof. For $1 \leq i \leq n-1$ and $0 \leq j \leq n-i-1$, denote by

$$\mathcal{D}^i_j \subset \mathrm{Fun}(\Delta^{\{j, \ldots, n-i\}}, \langle \mathcal{A}_i, \ldots, \mathcal{A}_n \rangle)$$

the full subcategory spanned by diagrams D^i_j satisfying that

- $D^i_j(l)$ lies in $\langle \mathcal{A}_{n-l}, \ldots, \mathcal{A}_n \rangle$ for $j \leq l \leq n-i$,
- the cofiber of $D^i_j(l) \to D^i_j(l+1)$ in $\mathcal{V}$ lies in $\mathcal{A}_{n-l+1}$ for all $j \leq l \leq n-i-1$.

We denote by $r_{i,j} \colon \mathcal{D}^i_j \to \langle \mathcal{A}_i, \ldots, \mathcal{A}_n \rangle$ the functor given by the restriction to the vertex $n-i$. Note that for $i < k \leq n-j-1$, there is a trivial fibration $\mathcal{D}^i_j \to \mathcal{D}^i_{n-k} \times_{\langle \mathcal{A}_k, \ldots, \mathcal{A}_n \rangle} \mathcal{D}^k_j$.

Now suppose that $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a semiorthogonal decomposition and let $\mathcal{D} \to \mathcal{V}$ be the corresponding trivial fibration. Condition $i)$ is immediate. For condition $ii)$, we need to show that $r_{i,n-i-1}$ is a trivial fibration for all $1 \leq i \leq n-1$. Using that pullbacks preserve trivial fibrations, it follows that

$$\mathcal{D}' = \mathcal{D} \times_{\mathcal{V}} \langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle \to \langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle \tag{20}$$

is a trivial fibration. We can describe the elements of $\mathcal{D}'$ as the left Kan extensions along the inclusion $\Delta^{\{0,\dots,n-i\}} \to \Delta^n$ of elements of $\mathcal{D}^i_0$. It thus follows from [Lur09, 4.3.2.15] that the restriction functor $\mathcal{D}' \to \mathcal{D}^i_0$ to $\Delta^{\{0,\dots,i\}}$ is a trivial fibration. Using that the functor (20) factors through $r_{i,0} \colon \mathcal{D}^i_0 \to \langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle$, it follows from the 2/3 property of equivalences that also $r_{i,0}$ is a trivial fibration. The following commutative diagram thus shows that $r_{i,n-i-1}$ is a trivial fibration. We have thus shown statement $ii)$.

$$\begin{tikzcd}
D^i_0 \arrow[r, "\text{triv fib}"'] \arrow[rrr, bend left, "r_{i,0}"] & D^i_{n-i-1} \times_{\langle \mathcal{A}_{i+1},\dots,\mathcal{A}_n \rangle} D^{i+1}_0 \arrow[r, "\text{triv fib}"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^i_{n-i-1} \arrow[r, "r_{i,n-i-1}"'] \arrow[d] & \langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle \\
& D^{i+1}_0 \arrow[r, "r_{i+1,0}"] & \langle \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle &
\end{tikzcd} \tag{21}$$

We now show that conditions $i)$ and $ii)$ imply that $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a semiorthogonal decomposition of $\mathcal{V}$. If $n = 2$, the assertion is obvious. We proceed by induction over n. Assume that $(\mathcal{A}_2, \dots, \mathcal{A}_n)$ is a semiorthogonal decomposition of $\langle \mathcal{A}_2, \dots \mathcal{A}_n \rangle$, meaning that $r_{2,0}$ is a trivial fibration. To show that $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a semiorthogonal decomposition of $\mathcal{V} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$, we need to show that $r_{1,0}$ is also a trivial fibration. Condition $ii)$ implies that $r_{1,n-2}$ is a trivial fibration. The diagram (21) for $i = 1$ thus shows that $r_{1,0}$ is also a trivial fibration. □

As it turns out, the functoriality data involved in the definition of semiorthogonal decompositions of length 2 is redundant.

Lemma 2.34. *Let $\mathcal{V}$ be a stable ∞-category and let $\mathcal{A}, \mathcal{B}$ be stable subcategories of $\mathcal{V}$. The pair $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition of length 2 of $\mathcal{V}$ if and only if*

1. *for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{V}}(b, a)$ is contractible and*

2. *for every $x \in \mathcal{V}$, there exists a fiber and cofiber sequence $b \to x \to a$ in $\mathcal{V}$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Proof. A proof is explained in [Lur18, 7.2.0.2]: conditions 1 and 2 above imply that $\mathcal{A}$ and $\mathcal{B}$ form the isles of a t-structure on $\mathcal{V}$, see Definition 1.2.1. in [Lur17]. Then [Lur17, 1.2.1.5] shows that the inclusions $\mathcal{A} \subset \mathcal{V}$ and $\mathcal{B} \subset \mathcal{V}$ admit adjoints. With this, one can prove that the functor $\mathcal{D} \to \mathcal{V}$ from Definition 2.31 is a trivial fibration. □

We proceed with describing the relation between semiorthgonal decompositions and Verdier quotients.

Definition 2.35. An exact sequence of stable, presentable ∞-categories consists of a cofiber sequence in $\mathcal{P}r^L_{\mathrm{St}}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{B} \end{tikzcd}$$

such that i is a fully faithful functor.

Remark 2.36. In the setting of Definition 2.35, the triangulated homotopy category of $\mathcal{B}$ is equivalent to the triangulated Verdier quotient of $\mathcal{C}$ by $\mathcal{A}$, see [BGT13, Prop. 5.9]. We thus call $\mathcal{B}$ the Verdier quotient of $\mathcal{C}$ by $\mathcal{A}$.

The following Lemma shows that the datum of a semiorthogonal decomposition of a stable, presentable ∞-category is equivalent to the datum of an exact sequence in $\mathcal{P}r^L_{\mathrm{St}}$.

Lemma 2.37. *Consider a diagram in* $\mathcal{P}r^L_{\mathrm{St}}$

$$\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{B}\,. \tag{22}$$

Suppose that i and $\mathrm{radj}(\pi)$ *are fully faithful. Then the diagram can be extended to an exact sequence if and only if* $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ *forms a semiorthogonal decomposition of* $\mathcal{C}$.

Proof. Suppose that (22) is part of an exact sequence. We find that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(\iota(a), \mathrm{radj}(\pi)(b)) \simeq \mathrm{Map}_{\mathcal{B}}(\pi\iota(a), b) \simeq \mathrm{Map}_{\mathcal{B}}(0, b)$ is contractible. Let $c \in \mathcal{C}$ and $\mathrm{cu}_c\colon c \to \mathrm{radj}(\pi)(\pi(c))$ be the counit. The map $\pi(\mathrm{cu}_c)$ is an equivalence by the 2/3-property and the fact that $\mathrm{radj}(\pi)$ is fully faithful. We hence have $\pi(\mathrm{fib}(\mathrm{cu}_c)) \simeq 0$. It follows that $\mathrm{fib}(\mathrm{cu}_c) \in \mathrm{Im}(i)$. There thus exists an object $a \in \mathcal{A}$ and an exact sequence $a \to c \to \mathrm{radj}(\pi) \circ \pi(c)$ in $\mathcal{C}$. We have shown that $(i(\mathcal{A}), \mathrm{radj}(\pi)(\mathcal{B}))$ forms a semiorthogonal decomposition of $\mathcal{C}$.

For the converse implication, assume that $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ forms a semiorthogonal decomposition of $\mathcal{C}$. The assertion that (22) can be extended to an exact sequence is via the equivalence $\mathrm{radj}\colon \mathcal{P}r^L_{\mathrm{St}} \simeq (\mathcal{P}r^R_{\mathrm{St}})^{\mathrm{op}}$ equivalent to the assertion that there exists a pullback diagram in Cat_∞ as follows:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathrm{radj}(\pi)"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d, "\mathrm{radj}(i)"] \\ 0 \arrow[r] & \mathcal{A} \end{tikzcd}$$

This is in turn equivalent to the assertion that $\mathrm{radj}(\pi)$ defines an equivalence between $\mathcal{B}$ and the full subcategory of $\mathcal{C}$ spanned by objects $c \in \mathcal{C}$, satisfying that $\mathrm{Map}_{\mathcal{C}}(i(a), c)$ is contractible for all $a \in \mathcal{A}$. Proposition 2.2.4 from [DKSS21] shows that this is a property of the semiorthogonal decomposition, concluding the proof of the converse implication. $\square$

2.3.2 Semiorthogonal decompositions from sequences of functors

A simple source of semiorthogonal decompositions are sequences of functors between stable ∞-categories.

Lemma 2.38. *Let $D\colon \Delta^{n-1} \to \mathrm{Cat}_\infty$ be a diagram taking values in stable ∞-categories, corresponding to n composable functors*

$$\mathcal{A}_1 \xrightarrow{F_1} \mathcal{A}_2 \xrightarrow{F_2} \dots \xrightarrow{F_{n-1}} \mathcal{A}_n\,.$$

(1) The stable ∞-category

$$\{\mathcal{A}_1, \dots, \mathcal{A}_n\} \coloneqq \mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(D))$$

of sections of the Grothendieck construction $p\colon \Gamma(D) \to \Delta^{n-1}$, see Section 2.1.3, admits a semiorthogonal decomposition $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ of length n.

(2) Let R be an $\mathbb{E}_\infty$-ring spectrum. If each functor F_i, $1 \leq i \leq n-1$, is an R-linear functor between R-linear ∞-categories, then the ∞-category $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ further inherits the structure of an R-linear ∞-category such that each inclusion functor $\iota_i\colon \mathcal{A}_i \to \{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ is R-linear.

Proof. We begin by showing part (1). Consider the simplicial set

$$Z = \left(\Delta^0 \times \Delta^{n-1}\right) \amalg_{\Delta^{\{1\}} \times \Delta^{\{1,\dots,n-1\}}} \left(\Delta^1 \times \Delta^{n-2}\right) \amalg \dots \amalg_{\Delta^{\{1,\dots,n-1\}} \times \Delta^{\{1\}}} \left(\Delta^{n-1} \times \Delta^0\right)\,.$$

Let $\mathcal{D}'$ be the full subcategory of $\mathrm{Fun}(Z, \Gamma(D))$ spanned by diagrams given by right Kan extensions along the inclusion $\Delta^{n-1} \times \Delta^0 \to Z$ of a diagram in $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$. By [Lur09, 4.3.2.15], the restriction functor $\mathcal{D}' \to \{\mathcal{A}_1, \dots \mathcal{A}_n\}$ to $\Delta^0 \times \Delta^{n-1}$ is a trivial fibration. We can describe the elements of $\mathcal{D}'$ up to equivalence as diagrams in $\Gamma(D)$ of the form

$$\begin{tikzcd}
 & & & a_1 \arrow[d] \\
 & & a_2 \arrow[r, "\mathrm{id}"] \arrow[d] & a_2 \arrow[d] \\
 & \dots \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \dots \arrow[d] \\
a_n \arrow[r, "\mathrm{id}"] & \dots \arrow[r, "\mathrm{id}"] & a_n \arrow[r, "\mathrm{id}"] & a_n
\end{tikzcd}$$

satisfying that $a_i \in \mathcal{A}_i$. The restriction functor $\mathcal{D}' \to \{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ corresponds in the above description to the restriction to the rightmost column. The ∞-category $\mathcal{D}$ of Definition 2.31 can be identified with the full subcategory of $\mathrm{Fun}(\Delta^{n-1} \times \Delta^{n-1}, \Gamma(\alpha))$ spanned by left Kan extensions along $Z \to \Delta^{n-1} \times \Delta^{n-1}$ of diagrams lying in $\mathcal{D}'$. It follows that the restriction functor $\mathcal{D} \to \mathcal{D}'$ is a trivial fibration and thus that the restriction functor $\mathcal{D} \to \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is a trivial fibration. This shows part (1).

We proceed with showing part (2). Consider the diagram of ∞-operads over $\mathcal{LM}^{\otimes}$

$$D^{\otimes}\colon \mathcal{O}_1^{\otimes} \xrightarrow{F_1^{\otimes}} \mathcal{O}_2^{\otimes} \xrightarrow{F_2^{\otimes}} \ldots \xrightarrow{F_{n-1}^{\otimes}} \mathcal{O}_n^{\otimes}$$

exhibiting the functors F_i as R-linear. The morphism of ∞-operad

$$\mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(D^{\otimes})) \times_{\mathrm{Fun}(\Delta^{n-1},\mathcal{LM}^{\otimes})} \mathcal{LM}^{\otimes} \to \mathcal{LM}^{\otimes}$$

exhibits $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ as left tensored over

$$\mathcal{M} \coloneqq \mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(D^{\otimes})) \times_{\mathrm{Fun}(\Delta^{n-1},\mathcal{LM}^{\otimes})} \mathcal{LM}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathrm{Assoc}^{\otimes}\ .$$

Let $\tilde{D}^{\otimes}\colon \Delta^{n-1} \to \mathrm{Cat}_{\infty}$ be the constant diagram with value $\mathrm{LMod}_R^{\otimes}$. We find $\mathcal{M}$ to be equivalent as a monoidal ∞-category to

$$\mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(\tilde{D}^{\otimes})) \times_{\mathrm{Fun}(\Delta^{n-1},\mathrm{Assoc}^{\otimes})} \mathrm{Assoc}^{\otimes}\ . \tag{23}$$

Pulling back along the monoidal functor $\mathrm{LMod}_R^{\otimes} \to \mathcal{M}$, assigning to $x \in \mathrm{LMod}_R^{\otimes}$ the constant section in (23), we obtain a left-tensoring of $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ over LMod_R. To show that the left-tensoring provides the structure of an R-linear ∞-category, it suffices to show that the monoidal product preserves colimits in the second entry. This follows from the observation that colimits in $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ are computed pointwise, i.e. the n restriction functors $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\} \to \mathcal{A}_i$ preserve colimits. □

We next discuss the notion of gluing functors of semiorthogonal decompositions of length 2, see also [DKSS21].

Definition 2.39. Let $\mathcal{V}$ be a stable ∞-category with a semiorthogonal decomposition $(\mathcal{A}, \mathcal{B})$. We define a simplicial set $\chi(\mathcal{A}, \mathcal{B})$ by defining an n-simplex of $\chi(\mathcal{A}, \mathcal{B})$ to correspond to the following data.

- An n-simplex $j\colon \Delta^n \to \Delta^1$ of Δ^1.
- An n-simplex $\sigma\colon \Delta^n \to \mathcal{V}$ such that $\sigma(\Delta^{j^{-1}(0)}) \subset \mathcal{A}$ and $\sigma(\Delta^{j^{-1}(1)}) \subset \mathcal{B}$.

We define the face and degeneracy maps to act on an n-simplex $(j, \sigma) \in \chi(\mathcal{A}, \mathcal{B})_n$ componentwise.

We denote by $p\colon \chi(\mathcal{A}, \mathcal{B}) \to \Delta^1$ the apparent forgetful functor.

Definition 2.40. Let $\mathcal{V}$ be a stable ∞-category with a semiorthogonal decomposition $(\mathcal{A}, \mathcal{B})$. We call

- $(\mathcal{A}, \mathcal{B})$ Cartesian if the functor $p\colon \chi(\mathcal{A}, \mathcal{B}) \to \Delta^1$ is a Cartesian fibration. In that case, we call the functor classifying the Cartesian fibration p the right gluing functor associated to $(\mathcal{A}, \mathcal{B})$.

- $(\mathcal{A}, \mathcal{B})$ coCartesian if the functor $p\colon \chi(\mathcal{A}, \mathcal{B}) \to \Delta^1$ is a coCartesian fibration. In that case, we call the functor classifying the coCartesian fibration p the left gluing functor associated to $(\mathcal{A}, \mathcal{B})$.

Lemma 2.41 ([DKSS21])**.** *Let $\mathcal{V}$ be a stable ∞-category with a semiorthogonal decomposition $(\mathcal{A}, \mathcal{B})$.*

1. *If $(\mathcal{A}, \mathcal{B})$ is Cartesian, the inclusion functor $\mathcal{A} \to \mathcal{V}$ admits a right adjoint, the restriction of which to $\mathcal{B}$ is the right gluing functor of $(\mathcal{A}, \mathcal{B})$.*

2. *If $(\mathcal{A}, \mathcal{B})$ is coCartesian, the inclusion functor $\mathcal{B} \to \mathcal{V}$ admits a left adjoint, the restriction of which to $\mathcal{A}$ is the left gluing functor of $(\mathcal{A}, \mathcal{B})$.*

The next proposition can be summarized as showing that Cartesian semiorthogonal decompositions of length 2 are fully determined by their left gluing functor and dually that coCartesian semiorthogonal decomposition of length 2 are fully determined by their right gluing functor.

Proposition 2.42 ([DKSS21])**.** *Let $\mathcal{V}$ be a stable ∞-category with a semiorthogonal decomposition $(\mathcal{A}, \mathcal{B})$.*

1. *If $(\mathcal{A}, \mathcal{B})$ is Cartesian with right gluing functor G, there exists an equivalence of ∞-categories $\mathcal{V} \simeq \mathrm{Fun}_{\Delta^1}(\Delta^1, \chi(G))$, where $\chi(G) \to \Delta^1$ is the Cartesian fibration classified by G considered as a functor $\Delta^1 \to \mathrm{Cat}_\infty$.*

2. *If $(\mathcal{A}, \mathcal{B})$ is coCartesian with left gluing functor F, there exists an equivalence of ∞-categories $\mathcal{V} \simeq \mathrm{Fun}_{\Delta^1}(\Delta^1, \Gamma(F))$.*

Definition 2.43. Let $\mathcal{V}$ be a stable ∞-category and let $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a semiorthogonal decomposition of $\mathcal{V}$. We call

- $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ a Cartesian semiorthogonal decomposition if each semiorthogonal decomposition $(\mathcal{A}_i, {}^\perp\mathcal{A}_i)$ is Cartesian. In that case, we call the right gluing functor of $(\mathcal{A}_i, {}^\perp\mathcal{A}_i)$ the i-th right gluing functor of $(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

- $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ a coCartesian semiorthogonal decomposition if each semiorthogonal decomposition $(\mathcal{A}_i, {}^\perp\mathcal{A}_i)$ is coCartesian. If $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is coCartesian, we call the left gluing functor of $(\mathcal{A}_i, {}^\perp\mathcal{A}_i)$ the i-th left gluing functor of $(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

2.3.3 Semiorthogonal decompositions from upper triangular dg-algebras

We now introduce a dg-analogue of the constructions from Lemma 2.38: semiorthogonal decompositions arising from upper triangular dg-algebras concentrated on the diagonal and upper minor diagonal.

Definition 2.44. For $1 \leq i \leq n$, let A_i be a dg-algebra and for $1 \leq i \leq n-1$ let M_i be an A_i-A_{i+1}-bimodule. We denote by

$$\mathbf{A} = \begin{pmatrix} A_1 & M_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & M_2 & \dots & 0 & 0 \\ 0 & 0 & A_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{n-1} & M_{n-1} \\ 0 & 0 & 0 & \dots & 0 & A_n \end{pmatrix}$$

be the upper triangular dg-algebra, i.e. the dg-algebra with underlying chain complex

$$\bigoplus_{1 \leq i \leq n} A_i \oplus \bigoplus_{1 \leq i \leq n-1} M_i$$

and multiplication $\cdot$ given by

$$a_i \cdot a'_j = \delta_{i,j} a_i a'_j\,, \qquad\qquad m_i \cdot m'_j = 0\,,$$
$$a_i \cdot m_j = \delta_{i,j} a_i . m_j\,, \qquad\qquad m_j \cdot a_i = \delta_{j+1,i} m_j . a_i\,,$$

where $a_i \in A_i, a'_j \in A_j$ and $m_i \in M_i, m'_j \in M_j$ and $\delta_{i,j}$ denotes the Kronecker delta.

Proposition 2.45. *Let $\mathbf{A}$ be an upper triangular dg-algebra as in Definition 2.44. Then the stable ∞-category $\mathcal{D}(A)$ admits a semiorthogonal decomposition*

$$(\mathcal{D}(A_1), \dots, \mathcal{D}(A_n))$$

of length n with i-th left gluing functor $- \otimes_{A_i} M_i$.

Proof. The upper triangular dg-algebra $\mathbf{A}$ is quasi-isomorphic to the upper triangular dg-algebra obtained from cofibrantly replacing each M_i. We thus assume without loss of generality that the M_i are cofibrant bimodules. Consider the morphisms of dg-algebras $v^i\colon A_i \to \mathbf{A}$ and $w^i\colon \mathbf{A} \to A_i$, given on the underlying chain complexes by the inclusion of the direct summand A_i and the projection to the summand A_i, respectively. The dg-functor $v^i_! = - \otimes^{\mathrm{dg}}_{A_i} A_i \oplus M_i\colon \mathrm{dgMod}(A_i) \to \mathrm{dgMod}(A)$ and the pullback $(w^i)^*$ determine right A-modules $v^i_!(A_i)$ and $(w^i)^*(A_i)$ with underlying chain complexes $A_i \oplus M_i$, where we set $M_n = 0$, and A_i, respectively. The functors $\mathcal{D}(v^i_!)$ and $\mathcal{D}((w^i)^*)$ both exhibit $\mathcal{D}(A_i) \subset \mathcal{D}(\mathbf{A})$ as a stable subcategory. For concreteness, we denote the stable subcategories obtained from $\mathcal{D}(v^i_!)$ by $\mathcal{D}(A_i)_v$ and the stable subcategories obtained from $\mathcal{D}((w^i)^*)$ by $\mathcal{D}(A_i)_w$. We wish to show that $(\mathcal{D}(A_1)_v, \dots, \mathcal{D}(A_n)_v)$ is a semiorthogonal decomposition of $\mathcal{D}(\mathbf{A})$. For that it suffices to show statements *i*) and *ii*) of Lemma 2.33. To show statement *ii*), it suffices to show conditions 1 and 2 of Lemma 2.34 for the pairs of stable subcategories $\mathcal{D}(A_i)_v, \langle \mathcal{D}(A_{i+1})_v, \dots, \mathcal{D}(A_n)_v \rangle \subset \langle \mathcal{D}(A_i)_v, \dots, \mathcal{D}(A_n)_v \rangle$ for all $1 \leq i \leq n$.

We compute for an A_i-module N_i and an A_j-modules N_j the mapping complex to be

$$\mathrm{Hom}_{\mathrm{dgMod}(\mathbf{A})}(v^i_!(N_i), v^j_!(N_j)) \simeq \begin{cases} \mathrm{Hom}_{\mathrm{dgMod}(A_i)}(N_i, N_j) & \text{if } i = j, \\ \mathrm{Hom}_{\mathrm{dgMod}(A_j)}(N_i \otimes_{A_i} M_i, N_j) & \text{if } i+1 = j, \\ 0 & \text{else.} \end{cases} \tag{24}$$

This shows condition 1 of Lemma 2.34.

We observe that the datum of a right dg-module N over $\mathbf{A}$ is equivalent to the datum of a sequence

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} N_n$$

where N_i is a right $A_i \simeq \mathrm{End}^{\mathrm{dg}}((w^i)^*(A_i))$-module and $f_i \in M_i(N_i, N_{i+1})$. Denote by $N_{\geq i}$ the submodule $N_i \xrightarrow{f_i} \dots \xrightarrow{f_{n-1}} N_n$ of N. We thus find distinguished triangles $N_{\geq i+1} \to N_{\geq i} \to N_i$ in $\mathrm{dgMod}(\mathbf{A})$. As shown in [Fao17, Theorem 4.3.1], the image under the dg-nerve of a distinguished triangle in a dg-category can be extended to a fiber and cofiber sequence. We can thus express $N \in \mathcal{D}(\mathbf{A})$ as repeated cofibers of modules $N_i \in \mathcal{D}(A_i)_w \subset \mathcal{D}(\mathbf{A})$ with $1 \leq i \leq n$. A simple induction, using that there exist distinguished triangles in $\mathrm{dgMod}(\mathbf{A})$ of the form $N_i \to N_i \otimes_{A_i} v_!^i(A_i) \to N_i \otimes_{A_i} M_i$ for $1 \leq i \leq n-1$ and $N_n \in \mathcal{D}(A_n)_w = \mathcal{D}(A_n)_v$, thus shows that $N \in \langle \mathcal{D}(A_1)_v, \dots, \mathcal{D}(A_n)_v \rangle$. It follows that statement $i)$ of Lemma 2.33 is fulfilled.

Consider the subalgebra $\mathbf{A}_{\geq i}$ of $\mathbf{A}$ with underlying chain complex

$$\bigoplus_{i \leq k \leq n} A_k \oplus \bigoplus_{i \leq k \leq n-1} M_k \,.$$

The fully faithful dg-functor $\mathrm{dgMod}(\mathbf{A}_{\geq i}) \to \mathrm{dgMod}(\mathbf{A})$ induces a fully faithful functor of ∞-categories $\iota\colon \mathcal{D}(\mathbf{A}_{\geq \mathbf{i}}) \to \mathcal{D}(\mathbf{A})$. The above arguments show that the essential image of ι is $\langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle$ and can easily be adapted to also show condition 2 of Lemma 2.34. We have thus proven the existence of the desired semiorthogonal decomposition of $\mathcal{D}(\mathbf{A})$.

It remains to determine the i-th left gluing functor of $(\mathcal{D}(A_1)_v, \dots, \mathcal{D}(A_n)_v)$. Consider the fully-faithful left Quillen functor

$$\text{-} \otimes^{\mathrm{dg}}_{A_i} A_i\colon \mathrm{dgMod}(A_i)_0 \to \mathrm{dgMod}(\mathbf{A}_{\geq i})_0 \,.$$

The right adjoint is given by the Quillen functor $\mathrm{Hom}_{\mathrm{dgMod}\,(\mathbf{A}_{\geq i})}(A_i, \text{-})$, the restriction of which to $\mathrm{dgMod}(\mathbf{A}_{\geq i+1})$ is given by $\mathrm{Hom}_{\mathrm{dgMod}(\mathbf{A}_{\geq i+1})}(M_i, \text{-})$, which in turn is left adjoint to $\text{-} \otimes^{\mathrm{dg}}_{A_i} M_i$. Passing to the underlying adjunctions of ∞-categories of the above Quillen adjunctions shows that the i-th left gluing functor of $(\mathcal{D}(A_1), \dots, \mathcal{D}(A_n))$ is given by $\text{-} \otimes_{A_i} M_i$. $\square$

Proposition 2.46. *For $1 \leq i \leq n$, let A_i be a dg-algebra and for $1 \leq i \leq n-1$, let M_i be an A_i-A_{i+1}-bimodule. Denote by $\mathbf{A}$ the upper triangular dg-algebra of Definition 2.44. Consider the diagram $\alpha\colon \Delta^{n-1} \to \mathrm{LinCat}_k$ corresponding to*

$$\mathcal{D}(A_1) \xrightarrow{\text{-}\otimes_{A_1} M_1} \mathcal{D}(A_2) \xrightarrow{\text{-}\otimes_{A_2} M_2} \dots \xrightarrow{\text{-}\otimes_{A_{n-1}} M_{n-1}} \mathcal{D}(A_n) \,.$$

Then there exists an equivalence of ∞-categories

$$\mathcal{D}(\mathbf{A}) \simeq \{\mathcal{D}(A_1), \dots, \mathcal{D}(A_n)\}$$

such that for all $1 \leq i \leq n$, the following diagram commutes.

$$\begin{tikzcd} & \mathcal{D}(A_i) \arrow[dl, "\mathcal{D}(v_!^i)[n-i]"'] \arrow[dr, "\iota_i"] & \\ \mathcal{D}(\mathbf{A}) \arrow[rr, "\simeq"] & & \{\mathcal{D}(A_1), \dots, \mathcal{D}(A_n)\} \end{tikzcd} \tag{25}$$

Proof. The stable ∞-categories $\{\mathcal{D}(A_1), \ldots, \mathcal{D}(A_n)\}$ and $\mathcal{D}(\mathbf{A})$ admit semiorthogonal decompositions $(\mathcal{D}(A_1), \ldots, \mathcal{D}(A_n))$ with equivalent left gluing functors. It thus follows from a repeated application of Proposition 2.42 that there exists an equivalence of ∞-categories $\mathcal{D}(\mathbf{A}) \simeq \{\mathcal{D}(A_1), \ldots, \mathcal{D}(A_n)\}$. The observation that (25) commutes, reduces to the fact that the equivalences of Proposition 2.42 commute with the inclusion functors of the components of the semiorthogonal decomposition, up to delooping. $\square$

3 Perverse schobers on surfaces

We begin in Section 3.1 with an introductory treatment of perverse sheaves on surfaces. The main insight will be that perverse sheaves can be described in terms of constructible (co)sheaves of sections with support on a spanning ribbon graph of the surface. This perspective on perverse sheaves will be the starting point for their categorification, which is split into two parts in Sections 3.2 and 3.3. In Section 3.2, we discuss the local aspects of the categorification. Section 3.3 concerns the global aspects of the categorification and various features of the arising notion of perverse schobers, including their behavior under contractions of the graph and how to define the monodromy of perverse schobers.

3.1 Background on perverse sheaves

The goal of this section is to give an introductory treatment of the notion of a constructible sheaf on a graph and the relation with perverse sheaves on surfaces. We begin in Section 3.1.1 by discussing the general setup, involving marked surfaces with spanning ribbon graphs. In Section 3.1.2, we explain how constructible sheaves on graphs can be encoded via the exit path category and how they allow to facilitate the computation of relative cohomology groups of surfaces. In the final Section 3.1.3, we explain the relation between perverse sheaves on surfaces with boundary and constructible (co)sheaves on ribbon graphs.

3.1.1 Marked surfaces and spanning graphs

Definition 3.1. By a surface $\mathbf{S}$, we mean a smooth, oriented and connected 2-dimensional manifold with possibly empty boundary $\partial\mathbf{S}$. The interior of $\mathbf{S}$ is denoted by $\mathbf{S}^\circ$.

A marked surface is a compact surface $\mathbf{S}$ together with a finite collection of marked points $M \subset \mathbf{S}$. We further require that each boundary component of $\mathbf{S}$ contains at least one marked point and if $\partial\mathbf{S} = \emptyset$, that $M \neq \emptyset$. Note that all boundary components of marked surfaces are circles.

Marked points split into two sets: boundary marked points are elements of $M \cap \partial\mathbf{S}$ and interior marked points are elements of $M \cap \mathbf{S}^\circ$. Interior marked points are also called punctures. We denote the set of punctures by P.

All marked surfaces surfaces are obtained from starting with a closed oriented surface of some genus by removing $k \geq 1$ open discs and adding marked points to the k boundary circles and possibly adding punctures.

Definition 3.2.

- A graph Γ consists of two finite sets Γ_0 of vertices and H_Γ of halfedges (often simply denoted H) together with an involution $\tau\colon \mathrm{H} \to \mathrm{H}$ and a map $\sigma\colon \mathrm{H} \to \Gamma_0$.

- Let Γ be a graph. We denote by Γ_1 the set of orbits of τ. The elements of Γ_1 are called the edges of Γ. An edge is called internal if the orbit contains two elements and called external if the orbit contains a single element. An internal edge is called a loop at $v \in \Gamma_0$ if it consists of two halfedges both being mapped under σ to v. We denote the set of internal edges of Γ by Γ_1° and the set of external edges by Γ_1^∂.
- A ribbon graph consists of a graph Γ together with a choice of a cyclic order on the set $\mathrm{H}(v)$ of halfedges incident to v for each $v \in \Gamma_0$.

Definition 3.3. Let Γ be a graph. We denote by $\mathrm{Exit}(\Gamma)$ the 1-category with

- the set of elements $\Gamma_0 \amalg \Gamma_1$ and
- all non-identity morphisms of the form $v \to e$ with $v \in \Gamma_0$ a vertex and $e \in \Gamma_1$ an edge incident to v. If e is a loop at v, then there are two morphisms $v \to e$.

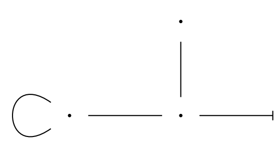
We call $\mathrm{Exit}(\Gamma)$ the exit path category of Γ. We will not distinguish in notation between $\mathrm{Exit}(\Gamma)$ from its nerve $N(\mathrm{Exit}(\Gamma)) \in \mathrm{Set}_\Delta$.

Given a graph Γ, we obtain its geometric realization $|\Gamma|$ by taking the geometric realization of the simplicial set $\mathrm{Exit}(\Gamma)$. We will only consider connected graphs, i.e. graphs whose geometric realization is connected.

Remark 3.4. Let Γ be a graph and $\mathbf{S}$ an oriented surface. Any embedding of $|\Gamma|$ into $\mathbf{S}$ determines a ribbon graph structure on Γ, where the cyclic order of the halfedges at any vertex is so that the cyclic order in the geometric realization is counter-clockwise with respect to the orientation of $\mathbf{S}$.

Notation 3.5. We use a graphical notation for ribbon graphs. We denote the vertices by $\times$, or sometimes by $\cdot$ if we think of them as a non-singular point, and edges by a straight line. We sometimes denote external edges as follows.

Example 3.6. The following diagram



denotes a ribbon graph Γ with three vertices, four edges in total, one external edge and one loop and the cyclic order of the halfedges at each vertex going in the counter-clockwise direction. The exit path category of Γ can be depicted as follows, with

v, v', v'' denoting the vertices of Γ and e, e', e'', e''' denoting the edges of Γ.

$$\mathrm{Exit}(\Gamma) = \begin{tikzcd} & & & v \arrow[d] & \\ & & & e & \\ e''' & v'' \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & e'' & v' \arrow[l] \arrow[u] \arrow[r] & e' \end{tikzcd}$$

Definition 3.7. Let $\mathbf{S}$ be a marked surface and Γ a graph together with an embedding $i\colon |\Gamma| \subset \mathbf{S}\backslash M$. Then Γ is called a spanning graph of $\mathbf{S}$ if

- i is a homotopy equivalence,
- $i\colon i^{-1}(\partial\mathbf{S}\backslash M) \to \partial\mathbf{S}\backslash M$ is a homotopy equivalence and
- each puncture is a vertex of Γ.

We consider spanning graphs as ribbon graphs with the induced ribbon graph structure, see Remark 3.4.

Remark 3.8. Every marked surface admits a spanning graph.

3.1.2 Constructible sheaves on graphs and cohomology

Recall the following statement, of which there exist multiple variations, see [Lur17, A.9.3], [Tan19, Rem 8.6.4] and [PT22].

Theorem 3.9. *Let X be a sufficiently nice conically stratified space and $\mathcal{D}$ a compactly generated ∞-category. Denote by $\mathrm{Shv}_c(X, \mathcal{D})$ the ∞-category of $\mathcal{D}$-valued constructible sheaves on X. There exists a simplicial set $\mathrm{Exit}(X)$, called the ∞-category of exit paths on X, together with an equivalence of ∞-categories*

$$\mathrm{Shv}_c(X, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{D})\,.$$

In the special case that $X = \Gamma$ describes a graph, the exit path category $\mathrm{Exit}(\Gamma)$ is equivalent, as a simplicial set, to the ∞-category of exit paths. This justifies (for compactly generated $\mathcal{D}$) the following definition:

Definition 3.10. Let Γ be a graph. A constructible sheaf on Γ with values in an ∞-category $\mathcal{D}$ is a functor $\mathrm{Exit}(\Gamma) \to \mathcal{D}$.

Definition 3.11. Let $F\colon \Gamma \to \mathcal{D}$ be a constructible sheaf with values in $\mathcal{D}$ and assume that $\mathcal{D}$ admits finite limits. The global sections $H(\Gamma; F) \in \mathcal{D}$ of F are defined as the limit of F.

We proceed with explaining the relation between global sections of constructible sheaves on graphs and relative cohomology groups of surfaces. Consider for this a marked surface $\mathbf{S}$ without punctures, together with a choice of spanning graph Γ. Let L be a local system of finite dimensional vector spaces on $\mathbf{S}$, i.e. a locally constant sheaf. We observe that $M \subset \mathbf{S}\backslash\Gamma$ is a homotopy equivalence. We thus have

$$H^*(\mathbf{S}, M; L) \simeq H^*(\mathbf{S}, \mathbf{S}\backslash\Gamma; L)\,, \tag{26}$$

where we denote by $H^*(\text{-},\text{-};L)$ the relative sheaf cohomology of L.

For all $i \in \mathbb{N}$, we denote by $\underline{H}^i_\Gamma(L) \in \mathrm{Shv}_c(\mathrm{Vect}_k)$ the constructible sheaf of k-vector spaces on Γ, whose value on an open set $W \subset \Gamma$ is given by the following colimit over all open $U \subset \mathbf{S}$ with $W \subset U$.

$$\underline{H}^i_\Gamma(L)(W) \coloneqq \mathrm{colim}_U\, H^i(U, U\backslash\Gamma; L)$$

The sheaves $\underline{H}^*_\Gamma(L)$ compute the (derived) sections of L with support on Γ.

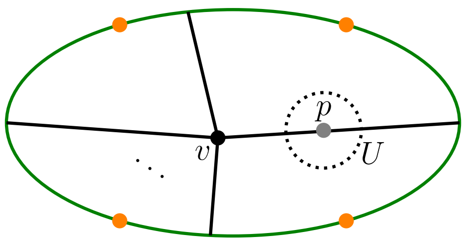
Proposition 3.12. *For all $i \in \mathbb{Z}$, there exists an isomorphism*

$$H(\Gamma; \underline{H}^i_\Gamma(L)) \simeq H^i(\mathbf{S}, M; L)\,.$$

Furthermore, these cohomology groups vanish for $i \neq 1$.

Proposition 3.12 expresses, that the relative cohomology with coefficients in L arise from gluing the local sections of L with support on Γ, as encoded by the constructible sheaf $\underline{H}^*_\Gamma(L)$. The fact that relative cohomology $H^*(\mathbf{S}, M; L)$ is concentrated in the single degree $i = 1$ is an instance of so-called *purity*. The proposition can be proven by using (26) and gluing, i.e. by using Mayer-Vietoris sequences for the involved cohomology groups.

Example 3.13. Let $\mathbf{S}$ be the n-gon and consider the spanning graph Γ depicted below, consisting of a single vertex with n incident external edges.



Let $L = \underline{k}$ be the constant local system with value k. We compute the stalks of $\underline{H}^i_\Gamma(L)$:

Let p be a point on an edge of Γ and U a sufficiently small neighborhood. Then there is an exact sequence:

$$0 \to \underline{H}^0_\Gamma(L)(U\cap\Gamma) \hookrightarrow \underbrace{H^0(U;L)}_{\simeq k} \to \underbrace{H^0(\underbrace{U\backslash\Gamma}_{\simeq *\amalg *};L)}_{\simeq k^{\oplus 2}} \to \underline{H}^1_\Gamma(L)(U\cap\Gamma) \to 0 \to \dots$$

The stalk at p is thus given by $\underline{H}^1_\Gamma(L)_p \simeq k$ and $\underline{H}^i_\Gamma(L)_p \simeq 0$ for $i \neq 1$.

Let v be the n-valent vertex of Γ. Again applying the long exact sequence, a very similar computation shows that the stalk at v is given by $\underline{H}^1_\Gamma(L)_v \simeq k^{\oplus n-1}$ and $\underline{H}^i_\Gamma(L)_v \simeq 0$ for $i \neq 1$.

The reader will readily verify that the stalk at v

$$\underline{H}^1_\Gamma(L)_v \simeq \underline{H}^1_\Gamma(L)(\Gamma) \simeq H^1(\mathbf{S}, M; \underline{k}) \simeq k^{\oplus n-1}$$

computes the relative first cohomology of the pair $(\mathbf{S}, M)$.

3.1.3 Perverse sheaves on surfaces

In this section, we explain how the discussion from Section 3.1.2 generalized from coefficients in local systems to coefficients in a perverse sheaf. We will further see how much information about the perverse sheaf is captured by the associated constructible sheaf on the graph Γ. We fix a marked surface $\mathbf{S}$, allowing punctures, together with a spanning graph Γ.

Definition 3.14. Denote by $\mathrm{Shv}_k(\mathbf{S})$ the abelian category of $\mathrm{Vect}_k^{\mathrm{fin}}$-valued sheaves on $\mathbf{S}$. Let $F \in \mathcal{D}^b(\mathrm{Shv}_k(\mathbf{S}))$. F is called a perverse sheaf with singularities at most at P if it satisfies the following.

- Let $i\colon \mathbf{S}\backslash P \hookrightarrow \mathbf{S}$. We have $H^i(i^*F) = H^i(F|_{\mathbf{S}\backslash P}) = 0$ for $i \neq 0$ and $H^0(F)|_{\mathbf{S}\backslash P}$ is a local system. Note that this implies $H^i(i^!F) \simeq 0$ for $i \neq 2$.
- Let $p \in P$ be a puncture and $j_p\colon \{p\} \hookrightarrow \mathbf{S}$ the inclusion. Then $H^i(j_p^!(F)) = 0$ for $i > 2$ and $H^i(j_p^*(F)) = H^i(F|_p) = 0$ for $i < 0$.

One can show that $H^i(F) = 0$ for all $i \neq 0, 1$.

Perverse sheaves on $\mathbf{S}$ with singularities at most at P form an abelian category denoted $\mathrm{Perv}(\mathbf{S})$, containing the abelian category of local systems as a full subcategory. Clearly $\mathrm{Perv}(\mathbf{S})$ does not depend on the boundary marked points of $\mathbf{S}$, but only on the punctures.

Let $j\colon |\Gamma| \hookrightarrow \mathbf{S}$ be the inclusion of the spanning graph Γ. Given a perverse sheaf F on $\mathbf{S}$, the complex of sheaves $j^!(F) \in \mathcal{D}^b(\mathrm{Shv}_k(\Gamma))$ describes derived sections of F with support on Γ. Remarkably, this complex is again pure, meaning that $H^i(j^!F) = 0$ for $i \neq 1$ and $H^1(j^!F)$ is a constructible sheaf of vector spaces on Γ. For a proof, see [KS16, Prop. 3.2]. This generalizes our previous construction for a local system L, since $H^1(j^!(F)) \simeq \underline{H}^1_\Gamma(L)$. We encode $H^1(j^!F)$ as a functor, denoted $\underline{H}^1_\Gamma(F)\colon \mathrm{Exit}(\Gamma) \to \mathrm{Vect}_k^{\mathrm{fin}}$. We call the global sections of $\underline{H}^1_\Gamma(F)$ the cohomology of $\mathbf{S}$ with support on Γ with coefficients in the perverse sheaf F.

The following theorems explain how much information about F is encoded by the constructible sheaf $\underline{H}^1_\Gamma(F)$.

Theorem 3.15 ([GGM85]). *Let* $\mathbf{D}$ *be the once-punctured* 1*-gon. Let* $\mathcal{A}_1$ *be the abelian category of diagrams of finite dimensional vector spaces*

$$V^1 \underset{s}{\overset{r}{\rightleftarrows}} N$$

satisfying that $sr - \mathrm{id}_N$ *and* $rs - \mathrm{id}_{V^1}$ *are invertible.*

Let Γ_1 *be the graph depicted on the left in Figure 1, also called the* 1*-spider. There is an equivalence of categories:*

$$\begin{aligned} \mathrm{Perv}(\mathbf{D}) &\longrightarrow \mathcal{A}_1 \\ F &\longmapsto \underline{H}^1_{\Gamma_1}(F)(v) \underset{\delta}{\overset{\mathrm{res}}{\rightleftarrows}} \underline{H}^1_{\Gamma_1}(F)(e) \end{aligned}$$

Here $\mathrm{res} = \underline{H}^1_\Gamma(F)(v \to e)$ *is the restriction map. Let* $i\colon \{x\} \hookrightarrow \mathbf{D}$ *be the inclusion of a point* $x \in \partial D \backslash \Gamma$*. Then*

$$\delta\colon \underline{H}^1_\Gamma(F)(e) \simeq H^1(i_*i^*F)(\mathbf{D}) \to H^0(j_!j^!F)(\mathbf{D}) \simeq \underline{H}^1_\Gamma(F)(v)$$

is the connecting homomorphism arising from the distinguished triangle $j_!j^!(F) \to F \to i_*i^*(F)$*.*

V^1 is called the vector space of vanishing cycles and N is called the vector space of nearby cycles.

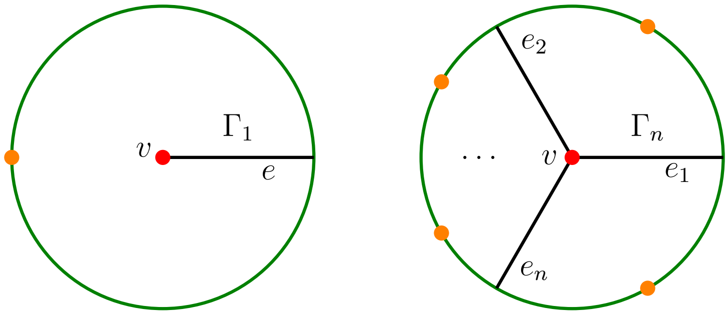


Figure 1: On the left: the once-punctured 1-gon with a spanning graph. On the right: the once-punctured n-gon with a spanning graph with $n \geq 2$.

Theorem 3.16 ([KS16])**.** *Let* $\mathbf{D}$ *be the once-punctured* n*-gon with* $n \geq 2$*. Let* $\mathcal{A}_n$ *be the abelian category whose objects correspond to diagrams of finite dimensional vector spaces*

$$\left(V^n \underset{s_i}{\overset{r_i}{\rightleftarrows}} N_i \right)_{1 \leq i \leq n}$$

satisfying that $r_i \circ s_i = \mathrm{id}_{N_i}$*,* $r_i \circ s_{i+1}$ *is invertible (with* i *modulo* n*) and* $r_i \circ s_j = 0$ *else.*

Let Γ_n *be the graph on the right in Figure 1, also called the* n*-spider. There is an equivalence of categories:*

$$\begin{aligned} \mathrm{Perv}(\mathbf{D}) &\longrightarrow \mathcal{A}_n \\ F &\longmapsto \left(\underline{H}^1_{\Gamma_n}(F)(v) \underset{\delta}{\overset{\mathrm{res}}{\rightleftarrows}} \underline{H}^1_{\Gamma_n}(F)(e_i) \right)_{1 \leq i \leq n} \end{aligned}$$

Using that being perverse is a local condition, Theorems 3.15 and 3.16 combine to the following:

Theorem 3.17 ([KS16, Theorem 3.6]). *Let* $\mathbf{S}$ *be a marked surface with spanning graph* Γ. *Then* $\mathrm{Perv}(\mathbf{S})$ *is equivalent to the abelian category of diagrams*

$$\mathrm{Exit}(\Gamma) \amalg_{\mathrm{ob}(\mathrm{Exit}(\Gamma))} \mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$$

which restrict at each vertex of Γ *and its incident halfedges to a diagram of the form described in Theorems 3.15 and 3.16.*

Given a perverse sheaf F, the functor $X\colon \mathrm{Exit}(\Gamma) \amalg_{\mathrm{ob}(\mathrm{Exit}(\Gamma))} \mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$ can be described as follows: the restriction $X|_{\mathrm{Exit}(\Gamma)}$ is given by the constructible sheaf $\underline{H}^1_\Gamma(F)$ of sections of F with support on Γ. The other restriction $X|_{\mathrm{Exit}(\Gamma)^{\mathrm{op}}}$ is given by the constructible cosheaf $\underline{H}^{-1}_\Gamma(\mathbb{D}(F))^*$, where $\mathbb{D}(F)$ denotes the Verdier dual of F and $(\text{-})^*$ denotes the passage to dual vector spaces, which turns constructible sheaves into constructible cosheaves. A perverse sheaf on $\mathbf{S}$ may thus be encoded in terms of a constructible sheaf on Γ and a constructible cosheaf on Γ, whose (co)stalks are pointwise identified.

3.2 Parametrized perverse schobers locally

We have seen in Section 3.1.3 a description of the category of perverse sheaves on a marked surface in terms of certain diagrams of finite dimensional vector spaces, i.e. linear algebra data. While it is currently unclear how to categorify constructible sheaves and thus perverse sheaves directly, the remarkable idea of [KS14] is to categorify perverse sheaves using such linear algebra descriptions, when available. The goal of this section is to describe such an ad-hoc categorification of perverse sheaves on the disc in terms of categorifications of the linear algebra data from Theorem 3.16. This will give us the local understanding needed in Section 3.3 for the global definition of categorified perverse sheaves on arbitrary marked surfaces.

In Section 3.2.1, we discuss the ad-hoc categorification of the data from Theorem 3.16. This categorification can be realized using Dyckerhoff's categorified Dold-Kan nerve [Dyc21] applied to a spherical functor. As noted in *loc. cit.*, the categorification of the local description of perverse sheaves was one of the motivations for the categorified Dold-Kan correspondence. In Section 3.2.2, we consider the action of the local rotational symmetry of n-gons on the categorified perverse sheaves and relate this with the paracyclic structure on the categorified Dold-Kan nerve.

3.2.1 An ad-hoc categorification

We begin with briefly recalling the concept of a spherical adjunction. Consider an adjunction of stable ∞-categories $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$. We associate with this adjunction the following endofunctors.

- The twist functor $T_{\mathcal{V}}$ is defined as the cofiber in the stable ∞-category $\mathrm{Fun}(\mathcal{V},\mathcal{V})$ of the unit map $\mathrm{id}_{\mathcal{V}} \to GF$ of the adjunction $F \dashv G$.

- The cotwist functor $T_{\mathcal{N}}$ is defined as the fiber in the stable ∞-category $\mathrm{Fun}(\mathcal{N},\mathcal{N})$ of the counit map $FG \to \mathrm{id}_{\mathcal{N}}$ of the adjunction $F \dashv G$.

The adjunction $F \dashv G$ is called spherical if the functors $T_{\mathcal{V}}$ and $T_{\mathcal{N}}$ are equivalences. In this case, the functor F is also called a spherical functor. A spherical functor F admits repeated left and right adjoints, each given by the composite of F or G with a power of the twist or cotwist functor. The notion of a spherical functor was introduced in the setting of dg-categories in [AL17]. For a treatment of spherical adjunctions in the setting of stable ∞-categories, we refer to [DKSS21] and [Chr22d].

Spherical adjunction provide an ad-hoc categorification of the description of perverse sheaves from Theorem 3.15 in terms of sections with support on the 1-spider Γ_1. We categorify Theorem 3.16 as follows:

Definition 3.18.

(1) A perverse schober parametrized by the 1-spider, or on the 1-spider for short, consists of a spherical adjunction

$$F\colon \mathcal{V} \longleftrightarrow \mathcal{N} :G\,.$$

Note that $\mathcal{V}$ categorifies the vector space of vanishing cycles and $\mathcal{N}$ the vector space of nearby cycles. Accordingly, we will call $\mathcal{V}$ the ∞-category of vanishing cycles and $\mathcal{N}$ the ∞-category of nearby cycles.

(2) Let $n \geq 2$. A collection of adjunctions

$$(F_i\colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i :G_i)_{i\in \mathbb{Z}/n}$$

between stable ∞-categories is called a perverse schober parametrized by the n-spider, or a perverse schober on the n-spider for short, if

(a) G_i is fully faithful, i.e. $F_iG_i \simeq \mathrm{id}_{\mathcal{N}_i}$ via the counit,

(b) $F_i \circ G_{i+1}$ is an equivalence of ∞-categories,

(c) $F_i \circ G_j \simeq 0$ if $j \neq i, i+1$,

(d) G_i admits a right adjoint $\mathrm{radj}(G_i)$ and F_i admits a left adjoint $\mathrm{ladj}(F_i)$ and

(e) $\mathrm{fib}(\mathrm{radj}(G_{i+1})) = \mathrm{fib}(F_i)$ as full subcategories of $\mathcal{V}^n$.

We say that a collection of functors $(F_i\colon \mathcal{V}^n \to \mathcal{N}_i)_{i\in\mathbb{Z}/n}$ determines a perverse schober on the n-spider if there exist adjunctions $(F_i \dashv \mathrm{radj}(F_i))_{i\in\mathbb{Z}/n}$ which define a perverse schober on the n-spider. Note that the collection of functors F_i, $i \in \mathbb{Z}/n$, can also be expressed as a functor $\mathrm{Exit}(\Gamma_n) \to \mathrm{St}$.

Remark 3.19. Given a perverse schober on the n-spider, we can pass to the Grothendieck group K_0 to obtain data as in Theorem 3.15 or Theorem 3.16, determining a perverse sheaf of $\mathbb{C}$. The fact that upon categorification, the sheaf and cosheaf parts become adjoint to each other leads to an identification of this perverse sheaf with its Verdier dual, see also the discussion in [KS16]. Such perverse sheaves are also called *polarized*. One might thus more accurately regard perverse schobers as categorified polarized perverse sheaves.

Remark 3.20. We note that condition 5 of Definition 3.18 is equivalent to the condition that $\mathrm{Im}(G_{i+1}) = \mathrm{Im}(\mathrm{ladj}(F_i))$. Since G_{i+1} and $\mathrm{ladj}(F_i)$ are fully faithful, it follows that $\mathrm{ladj}(F_i)$ differs from G_{i+1} by composition with the equivalence $(G_{i+1})^{-1} \circ \mathrm{ladj}(F_i) \colon \mathcal{N}_i \simeq \mathcal{N}_{i+1}$.

Parts 1.-3. of part (2) of Definition 3.18 should be considered as categorifications of the conditions of Theorem 3.16. We proceed by justifying parts 4. and 5., by showing that the datum of a perverse schober on the n-spider is equivalent to the datum of a perverse schober on the 1-spider.

Before that, we illustrate the special case $n = 2$. In this case, the ∞-category $\mathcal{V}^2$ admits a 4-periodic semiorthogonal decomposition $(\mathcal{V}^1, \mathcal{N})$, which implies that the gluing functor $\mathcal{N} \to \mathcal{V}^1$ is spherical, giving rise to the corresponding perverse schober on the 1-spider. We refer to [HLS16,DKSS21] for background on the relation between 4-periodic semiorthogonal decompositions and spherical functors. The two fully faithful functors $\mathcal{N} \simeq \mathcal{N}_i \xrightarrow{G_i} \mathcal{V}^2$, $i = 1, 2$, describe the inclusion of the component $\mathcal{N}$ of the semiorthogonal decomposition and the inclusion of a component of a mutated semiorthogonal decomposition.

Proposition 3.21. *Let $n \geq 2$. Given a perverse schober on the n-spider*

$$(F_i \colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i : G_i)_{i \in \mathbb{Z}/n}$$

and an integer $1 \leq j \leq n$, the collection of functors

$$(F_i|_{\mathrm{fib}(F_j)} \colon \mathrm{fib}(F_j) \longrightarrow \mathcal{N}_i)_{j \neq i \in \mathbb{Z}/n} \tag{27}$$

determines a perverse schober on the $(n-1)$-spider.

Proof. We begin with the case $n \geq 3$. We describe the right adjoints of the functors $F_i|_{\mathrm{fib}(F_j)}$. For $j \neq i, i+1$, we have $F_i \circ G_j \simeq 0$. For $j \neq i, i+1$, the functor G_j, factors through $\mathrm{fib}(F_i) \subset \mathcal{V}^n$ and remains right adjoint to $F_j|_{\mathrm{fib}(F_i)}$.

The adjunction counit defines a natural transformation $\eta \colon G_j F_j G_{j+1} \to G_{j+1}$, which becomes an equivalence after composition with F_j. The image of the cofiber $\mathrm{cof}(\eta)$ of η is hence contained in $\mathrm{fib}(F_j)$. The left adjoint of $\mathrm{cof}(\eta)$ is given by the fiber of a natural transformation $F_{j+1} \to F_{j+1} \,\mathrm{ladj}(F_j) F_j$, which restricts on $\mathrm{fib}(F_j)$ to F_{j+1}. We have thus shown that

$$F_{j+1}|_{\mathrm{fib}(F_j)} \colon \mathrm{fib}(F_j) \longleftrightarrow \mathcal{N}_{j+1} : \mathrm{cof}(\eta)$$

forms an adjunction.

With the above, one readily verifies that the collection of functors (27) and their right adjoints satisfy the conditions of Definition 3.18.

We proceed with the case $n = 2$. Using the adjunctions $F_1 \dashv G_1$ and $F_2 \dashv G_2$, it is easy to see that there are semiorthogonal decompositions

$$(\mathrm{Im}(G_i), \mathrm{fib}(F_i)), (\mathrm{fib}(\mathrm{radj}(G_i)), \mathrm{Im}(G_i))$$

for $i = 1, 2$. The condition $\mathrm{fib}(\mathrm{radj}(G_{i+1}) = \mathrm{fib}(F_i)$ is equivalent to the 4-periodicity of these semiorthogonal decompositions, in the sense of [HLS16, DKSS21]. It follows from [DKSS21, Prop. 2.5.12] that the gluing functor of $(\mathrm{fib}(\mathrm{radj}(G_i)), \mathrm{Im}(G_i))$, given by the restriction of F_i to $\mathrm{fib}(\mathrm{radj}(G_i)) = \mathrm{fib}(F_{i-1})$, is a spherical functor. □

Proposition 3.21 describes how one can pass from perverse schobers on the n-spider to perverse schobers on the 1-spider, i.e. spherical adjunctions. The converse construction assigns a perverse schober on the n-spider to a spherical adjunction, plus a choice of total order of the a priori cyclically ordered edges of the n-spider. This constructed is based on Dyckerhoff's categorified Dold-Kan correspondence [Dyc21]. We begin with briefly summarizing the statement of the categorified Dold-Kan correspondence.

A 2-simplicial stable ∞-category is an $(\infty, 2)$-functor $\Delta^{(\mathrm{op},\text{-})} \to \mathcal{S}t$, from the 2-categorical version of the simplex category to the $(\infty, 2)$-version $\mathcal{S}t$ of the ∞-category St of stable ∞-categories. The categorified Dold-Kan correspondence of [Dyc21] is an adjoint equivalence between the ∞-category of bounded below complexes of stable ∞-categories and the ∞-category of 2-simplicial stable ∞-categories. The right adjoint is called the categorified Dold-Kan nerve $\boldsymbol{N}$. The categorified Dold-Kan nerve $\boldsymbol{N}$ generalizes the well known construction from algebraic K-theory called the Waldhausen $S_\bullet$-construction. More precisely, given a complex of stable ∞-categories concentrated in degrees $0, 1$, the categorified Dold-Kan nerve recovers Waldhausen's relative $S_\bullet$-construction. We refer to [Dyc21] for further details.

Let now $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. We consider the spherical functor $G\colon \mathcal{V} \to \mathcal{N}$ as a complex of stable ∞-categories concentrated in degrees $0, 1$, denoted $G[0]$. We further denote by $\mathcal{N}[1]$ the complex concentrated in degree 1 with value $\mathcal{N}$. Consider the morphism between bounded below complexes of stable ∞-categories $G[0] \to \mathcal{N}[1]$ depicted as follows.

$$\begin{tikzcd}
\mathcal{V} \arrow[d] & \mathcal{N} \arrow[l, "G"'] \arrow[d, "\mathrm{id}_{\mathcal{B}}"'] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
0 & \mathcal{N} \arrow[l] & 0 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Applying the categorified Dold-Kan nerve $\boldsymbol{N}$, we obtain a morphism $\phi_*\colon \boldsymbol{N}(G[0])_* \to \boldsymbol{N}(\mathcal{N}[1])_*$ between the simplicial objects in St underlying the 2-simplicial objects in $\mathcal{S}t$. Spelling out the definition of the categorified Dold-Kan nerve and the properties of Kan extensions, see for instance [Lur09, 4.3.2.15], we obtain the following.

Lemma 3.22. *Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction and $\boldsymbol{N}(G[0])_*$ and $\boldsymbol{N}(\mathcal{N}[1])_*$ as above. There exist the following equivalences between ∞-categories.*

1. $\boldsymbol{N}(G[0])_0 \simeq \mathcal{V}$.

2. $\boldsymbol{N}(G[0])_n \simeq \{\mathcal{V}, \mathcal{N}, \dots, \mathcal{N}\}$ *is, for $n \geq 1$, equivalent to the ∞-category of sections of the diagram $\Delta^{n-1} \to \mathrm{St}$ corresponding to the following sequence of n functors*
$$\mathcal{V} \xrightarrow{F} \mathcal{N} \xrightarrow{\mathrm{id}} \mathcal{N} \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \mathcal{N},$$

see also Lemma 2.38 for the notation.

3. $\boldsymbol{N}(\mathcal{N}[1])_1 \simeq \mathcal{N}$.

Notation 3.23. Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. We denote

- $\mathcal{V}_F^1 = \mathcal{V}$.
- $\mathcal{V}_F^n = \{\mathcal{V}, \underbrace{\mathcal{N}, \dots, \mathcal{N}}_{n-1\text{-many}}\}$ for $n \geq 2$.

Assume that $n \geq 2$. A choice of categorification of the first restriction map $r_1\colon V^n \to N_1$ in Theorem 3.16 is the functor

$$\varrho_1\colon \mathcal{V}_F^n \simeq \boldsymbol{N}(G[0])_{n-1} \xrightarrow{d_0} \boldsymbol{N}(G[0])_{n-2} \xrightarrow{d_0} \dots \xrightarrow{d_0} \boldsymbol{N}(G[0])_1 \xrightarrow{\phi_1} \boldsymbol{N}(\mathcal{N}[1])_1 \simeq \mathcal{N}$$

obtained from composing ϕ_1 with repeated 0th face maps of the simplicial structure of $\boldsymbol{N}(G[0])_*$. The functor ϱ_1 can equivalently be described as the projection functor π_n to the n-th component of the semiorthogonal decomposition $(\mathcal{V}_F^1, \mathcal{N}, \dots, \mathcal{N})$ of length n of $\mathcal{V}_F^n$. If $n = 1$, we categorify the restriction map r_1 simply by $F\colon \mathcal{V}_F^1 \to \mathcal{N}$. To obtain categorification of the further restriction maps, we consider the functors contained in the sequence of adjunctions

$$\varsigma_1 T_{\mathcal{N}}^{-1}[1-n] \dashv \varrho_n \dashv \varsigma_n \dashv \varrho_{n-1} \dashv \dots \dashv \varsigma_2 \dashv \varrho_1 \dashv \varsigma_1\,, \tag{28}$$

where ϱ_1 is as above, ς_i categorifies s_i and ϱ_i categorifies r_i, and $T_{\mathcal{N}}$ denotes the cotwist functor of $F \dashv G$. Explicitly, the functors can be described as follows:

Lemma 3.24. *Let $F \dashv G$ be as above and $n \geq 1$. Consider the following functors $\varrho_i\colon \mathcal{V}_F^n \to \mathcal{N}$ and $\varsigma_i\colon \mathcal{N} \to \mathcal{V}_F^n$ for $1 \leq i \leq n$.*

1. *If $n = 1$, we set $\varrho_1 = F$ and $\varsigma_1 = G$.*
2. *If $n \geq 2$, we set*

$$\varrho_i = \begin{cases} \pi_n & \text{for } i = 1, \\ \mathrm{fib}_{n-i,n-i+1}[i-1] & \text{for } 2 \leq i \leq n-1, \\ \mathrm{rfib}_{1,2}[n-1] & \text{for } i = n. \end{cases}$$

   The functor $\mathrm{rfib}_{1,2}$ denotes the composition of the projection functor to the first two components of the semiorthogonal decomposition with the relative fiber functor that assigns to a vertex $a \to b \in \{\mathcal{V}, \mathcal{N}\}$ the vertex $\mathrm{fib}(F(a) \to b) \in \mathcal{N}$. The functor $\mathrm{fib}_{i-1,i}[n-i]$ denotes the composition of the projection functor to the $(i-1)$-th and i-th component with the fiber functor.

3. *If $n \geq 2$, we set ς_1 to be the functor that assigns to $b \in \mathcal{N}$ the object*

$$G(b) \xrightarrow{*} b \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} b \in \mathcal{V}_F^n\,,$$

see also Notation 2.9, and set for $2 \le i \le n$

$$\varsigma_i = (\iota_{\mathcal{N}})_{n-i+2}[-i+2]\,,$$

where $(\iota_{\mathcal{N}})_j$ *is the inclusion of the* j*-th component of the semiorthogonal decomposition.*

These functors form the sequence of adjunctions (28)*.*

Proof. The adjunctions $\varrho_i \dashv \varsigma_i$ for $1 \le i \le n$ and $\varsigma_j \dashv \varrho_{j-1}$ for $2 \le j \le n$ are straightforward to show. The final adjunction $\varsigma_1 T_{\mathcal{N}}^{-1}[1-n] \dashv \varrho_n$ is shown in the proof of Lemma 3.31 below. □

We have indeed constructed a perverse schober on the n-spider starting with a spherical adjunction $F \dashv G$:

Proposition 3.25. *The collection of adjunctions*

$$(\varrho_i \colon \mathcal{V}^n_F \longleftrightarrow \mathcal{N} : \varsigma_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$$

defines a perverse schober on the n*-spider. We denote by* $\mathcal{F}_n(F)\colon \mathrm{Exit}(\Gamma_n) \to \mathrm{St}$ *the corresponding functor.*

Proof. Inspecting Lemma 3.24, parts 1. to 3. of Definition 3.18 are immediate. Parts 4. and 5. follow from the sequence of adjunctions (28). □

The next proposition shows that every perverse schober on the n-spider is equivalent, in the appropriate sense, to a perverse schober of the type described in Proposition 3.25.

Proposition 3.26. *Let*

$$(F_i \colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i : G_i)_{i\in\mathbb{Z}/n}$$

be a perverse schober on the n*-spider* Γ_n *and consider the corresponding functor* $\mathcal{F}\colon \mathrm{Exit}(\Gamma_n) \to \mathrm{St}$ *(describing the functors* F_i*,* $i \in \mathbb{Z}/n$*). Let* $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ *be the spherical adjunction obtained by* $n-1$ *times applying Proposition 3.21 to the perverse schober on* Γ_n*, removing in each step any choice of edge from the* n*-spider. Then there exists an equivalence*

$$\mathcal{F} \simeq \mathcal{F}_n(F)$$

in $\mathrm{Fun}(\mathrm{Exit}(\Gamma_n), \mathrm{St})$.

Proof. An ∞-category with a semiorthogonal decomposition can be recovered up to equivalence from its gluing functor, see for instance [DKSS21, Chr22d]. We can thus apply the Lemma 3.27 below $(n-1)$-times to obtain an equivalence $\mathcal{F}(v) \simeq \mathcal{F}_n(F)(v)$, where v denotes the vertex of Γ_n. Let j be the position of the first edge removed from Γ_n upon application of Proposition 3.21. The functors $\mathcal{F}(v \to j)$ and $\varrho_j = \mathcal{F}_n(F)(v \to j)$ both describe the projections to the final component of the semiorthogonal decomposition of $\mathcal{F}(v) \simeq \mathcal{F}_n(F)(v)$. By Remark 3.20 and Equation (28), the remaining functors $\mathcal{F}(v \to i)$ and $\mathcal{F}_n(F)(v \to i)$ are obtained, up to postcomposition with equivalences, as repeated adjoints of $\mathcal{F}(v \to j)$ and $\varrho_j = \mathcal{F}_n(F)(v \to j)$, respectively. These identifications assemble into the desired equivalence $\mathcal{F} \simeq \mathcal{F}_n(F)$ in $\mathrm{Fun}(\mathrm{Exit}(\Gamma_n), \mathrm{St})$. $\square$

Lemma 3.27. *Let $(F_i \colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i : G_i)_{i \in \mathbb{Z}/n}$ be a perverse schober on the n-spider. Then there is a semiorthogonal decomposition $(\mathrm{fib}(F_i), \mathrm{Im}(G_{i+1}))$ for any $i \in \mathbb{Z}/n$. If $n = 2$, the gluing functor is spherical and if $n \geq 3$, the gluing functor from $\mathrm{Im}(G_{i+1})$ to $\mathrm{fib}(F_i)$ is fully faithful.*

Proof. The semiorthogonal decomposition follows from $\mathrm{fib}(F_i) = \mathrm{fib}(\mathrm{radj}(G_{i+1}))$. In the case $n = 2$, the sphericalness of the gluing functor follows from the 4-periodicity of the semiorthogonal decomposition, shown in the proof of Proposition 3.21. We next consider the case $n \geq 3$. Denote by ι the fully faithful functor $\mathrm{fib}(F_i) \to \mathcal{V}^n$. The gluing functor $\mathcal{N}_{i+1} \xrightarrow{G_{i+1}} \mathcal{V}^n \xrightarrow{\mathrm{radj}(\iota)} \mathrm{fib}(F_i)$ is the right adjoint of $F_{i+1}|_{\mathrm{fib}(F_i)}$, and thus fully faithful by Proposition 3.21. $\square$

3.2.2 The paracyclic structure

We begin by recalling the definition of the paracyclic 1-category Λ_∞.

Definition 3.28. For $n \geq 0$, let $[n]$ denote the set $\{0, \ldots, n\}$. The objects of Λ_∞ are the sets $[n]$. The morphism in Λ_∞ are generated by morphisms

- $\delta^0, \ldots, \delta^n \colon [n-1] \to [n]$,
- $\sigma^0, \ldots, \sigma^{n-1} \colon [n] \to [n-1]$,
- $\tau^{n,i} \colon [n] \to [n]$ with $i \in \mathbb{Z}$

subject to the simplicial relations and the further relations

$$\begin{aligned} \tau^{n,i} \circ \tau^{n,j} &= \tau^{n,i+j}, & \tau^{n,0} &= \mathrm{id}_{[n]}, \\ \tau^{n,1}\delta^i &= \delta^{i-1}\tau^{n-1,1} \text{ for } i > 0, & \tau^{n,1}\delta^0 &= \delta^n, \\ \tau^{n,1}\sigma^i &= \tau^{n+1,1}\sigma^{i-1} \text{ for } i > 0, & \tau^{n,1}\sigma^0 &= \sigma^n\tau^{n+1,2}\,. \end{aligned}$$

The simplex category Δ is a subcategory of Λ_∞. A paracyclic object in an ∞-category $\mathcal{C}$ is a functor $\Lambda_\infty^{op} \to \mathcal{C}$, where we identify the 1-category Λ_∞^{op} with its nerve. A paracyclic object in $\mathcal{C}$ is thus a simplicial object $X_* \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ with face maps d_i and degeneracy maps s_i together with a sequence of paracyclic isomorphisms $t_n \colon X_n \to X_n$ satisfying

$$d_i t_n = t_{n-1} d_{i-1} \text{ for } i > 0, \quad d_0 t_n = d_n \quad \text{and} \tag{29}$$

$$s_i t_n = t_{n+1} s_{i-1} \text{ for } i > 0, \quad s_0 t_n = t_{n+1}^2 s_n\,. \tag{30}$$

Let $F \dashv G$ be a spherical adjunction. As shown in [DKSS21], the simplicial object $\mathcal{N}(G[0])_*$ can be lifted to a paracyclic object. The action of t_{n-1} on the $(n-1)$-cells corresponds to the rotational symmetry of the n-spider. In the following, we explicitly describe the paracyclic automorphism t_{n-1} of the n-cells $\mathcal{V}_F^n \simeq \boldsymbol{N}(G[0])_{n-1}$. We realize t_{n-1} as the twist functor $T_{\mathcal{V}_F^n}$ of a spherical adjunction $F' \dashv G'$ described below in Lemma 3.31. We call $T_{\mathcal{V}_F^n}$ the paracyclic twist functor. We then proceed to show that this isomorphisms indeed realizes the rotational symmetry (up to monodromy) of the functors ϱ_i and ς_i, corresponding geometrically to a rotation of the n-spider.

Construction 3.29. Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. Consider the full subcategory $\mathcal{M}$ of the ∞-category of diagrams $\operatorname{Fun}(\Delta^1 \times \Delta^1, \Gamma(F))$ of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d, "!"'] \arrow[dr] & a' \arrow[d, "*"] \\ b' \arrow[r] & b \end{tikzcd}$$

with $a, a' \in \mathcal{V}$ and $b, b' \in \mathcal{N}$. The restriction functor $\operatorname{res}\colon \mathcal{M} \to \{\mathcal{V}, \mathcal{N}\}$, given by the projection to the edge $a \to b$ is a trivial fibration. As shown in [DKSS21], it follows from the sphericalness of the adjunction $F \dashv G$, that the fiber functor in the horizontal direction $\mathcal{M} \to \{\mathcal{V}, \mathcal{N}\}$ is also an equivalence. By choosing a section of the trivial fibration res and composing with the fiber functor we obtain an autoequivalence $\tau\colon \{\mathcal{V}, \mathcal{N}\} \to \{\mathcal{V}, \mathcal{N}\}$, called the *relative suspension functor* in *loc. cit.*

Lemma 3.30. *Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction with cotwist functor $T_{\mathcal{N}}$. Denote the left adjoint of F by E. The left adjoint of the functor*

$$\mathcal{V}_F^2 = \{\mathcal{V}, \mathcal{N}\} \xrightarrow{\text{rfib}} \mathcal{N} \tag{31}$$

is given by the functor that assigns $E(b) \xrightarrow{*} T_{\mathcal{N}}^{-1}(b)$ to $b \in \mathcal{N}$.

Proof. As shown in [Chr22d, Lemma 1.30], the stable subcategories

$$\mathcal{V}^{\perp}, \mathcal{V}, \mathcal{N}, {}^{\perp}\mathcal{N} \subset \mathcal{V}_F^2$$

form semiorthogonal decompositions $(\mathcal{V}^{\perp}, \mathcal{V}), (\mathcal{V}, \mathcal{N}), (\mathcal{N}, {}^{\perp}\mathcal{N})$ of $\mathcal{V}_F^2$. We denote by $i_{\mathcal{N}}, i_{\mathcal{V}^{\perp}}$ the inclusion functors of $\mathcal{N}$ and $\mathcal{N} \simeq \mathcal{V}^{\perp}$ into $\mathcal{V}_F^2$, respectively. The functor $i_{\mathcal{V}^{\perp}}$ assigns to $b \in \mathcal{N} \simeq \mathcal{V}^{\perp}$ the object $G(b) \xrightarrow{!} b \in \mathcal{V}_F^2$. It is easily checked that there is a sequence of adjunctions

$$\text{rfib}[1] \dashv i_{\mathcal{N}} \dashv \pi_0 \dashv i_{\mathcal{V}^{\perp}}\,. \tag{32}$$

Composing with the adjunction $\tau^{-1} \dashv \tau$, where τ is the relative suspension functor from Construction 3.29, with the sequence of adjunction (32) yields the sequence of adjunctions

$$\pi_0[1] \dashv i_{\mathcal{V}^{\perp}}[-1] \dashv T_{\mathcal{N}}^{-1}\,\text{rfib}[1] \dashv i_{\mathcal{N}} T_{\mathcal{N}}\,.$$

We have thus established the desired adjunction $i_{\mathcal{V}^{\perp}} T_{\mathcal{N}}^{-1} \dashv \text{rfib}$. $\square$

Lemma 3.31. *Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction with cotwist functor $T_{\mathcal{N}}$. For $n \geq 2$, consider the functor*

$$F'\colon \mathcal{V}_F^n \longrightarrow \mathcal{N}^{\times n}$$

with components $F' = (\varrho_1, \ldots, \varrho_n)$.

(1) The functor F' admits left and right adjoints E', respectively, G', given by

$$\begin{aligned} E' &= (\varsigma_2, \ldots, \varsigma_n, \varsigma_1 T_{\mathcal{N}}^{-1}[n-1])\,, \\ G' &= (\varsigma_1, \ldots, \varsigma_n)\,. \end{aligned}$$

(2) The adjunction $F' \dashv G'$ is spherical. We denote the twist functor by $F' \dashv G'$ by $T_{\mathcal{V}_F^n}$, and call it the paracyclic twist functor.

Proof. Denote the left adjoint of F by E.

We begin with showing part (1). The adjunction $F' \dashv G'$ follows from composing the adjunctions $\varrho_i \dashv \varsigma_i$ with the adjunction $\Delta \dashv \oplus$ between the constant diagram functor $\Delta\colon \mathcal{N} \to \mathcal{N}^{\times n}$ and its right adjoint given by the direct sum functor. Again by composing adjunctions, we obtain that to show that E' is left adjoint to F' it suffices to show that $\varsigma_1 T_{\mathcal{N}}^{-1}[n-1]$ is left adjoint to ϱ_n. This follows directly from the following observations.

- The functor ϱ_n factors as
$$\mathcal{V}_F^n \xrightarrow{\pi_{1,2}} \mathcal{V}_F^2 \xrightarrow{\mathrm{rfib}[n-1]} \mathcal{N}\,.$$
- The left adjoint of $\mathrm{rfib}\colon \mathcal{V}_F^2 \to \mathcal{N}$ was determined in Lemma 3.30 and is given by the functor that maps $b \in \mathcal{N}$ to $E(b) \xrightarrow{*} T_{\mathcal{N}}^{-1}(b)$.
- The left adjoint of $\pi_{1,2}$ is given by the functor that maps $E(b) \xrightarrow{*} T_{\mathcal{N}}^{-1}(b) \in \mathcal{V}_F^2$ to $E(b) \xrightarrow{*} T_{\mathcal{N}}^{-1}(b) \xrightarrow{\mathrm{id}} \ldots \xrightarrow{\mathrm{id}} T_{\mathcal{N}}^{-1}(b) \in \mathcal{V}_F^n$.

For part (2), consider the endofunctor $M = F'G'\colon \mathcal{N}^{\times n} \to \mathcal{N}^{\times n}$ of the adjunction $F' \dashv G'$ with cotwist functor $T_{\mathcal{N}^{\times n}}$. We can depict M as the following matrix.

$$\begin{pmatrix} \mathrm{id}_{\mathcal{N}} & \mathrm{id}_{\mathcal{N}} & 0 & \ldots & 0 & 0 \\ 0 & \mathrm{id}_{\mathcal{N}} & \mathrm{id}_{\mathcal{N}} & \ldots & 0 & 0 \\ 0 & 0 & \mathrm{id}_{\mathcal{N}} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathrm{id}_{\mathcal{N}} & \mathrm{id}_{\mathcal{N}} \\ T_{\mathcal{N}}[n-1] & 0 & 0 & \ldots & 0 & \mathrm{id}_{\mathcal{N}} \end{pmatrix}$$

The counit $\mathrm{cu}\colon M \to \mathrm{id}_{\mathcal{N}^{\times n}}$ is the projection to the diagonal, so that we deduce that the cotwist $T_{\mathcal{N}^{\times n}}$ is an equivalence. We further observe that there exists an equivalence $\mathrm{cu} \circ T_{\mathcal{N}^{\times n}} \simeq T_{\mathcal{N}^{\times n}} \circ \mathrm{cu}$. The left adjoint $E'\colon \mathcal{N}^{\times n} \to \mathcal{V}^n$ clearly satisfies $G' \circ T_{\mathcal{N}^{\times n}}^{-1}$. We have shown that all conditions of [Chr22d, Proposition 4.5] are fulfilled and it follows that the adjunction $F' \dashv G'$ spherical. $\square$

Proposition 3.32. *Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction and consider the paracyclic twist functor $T_{\mathcal{V}^n_F}$ from Lemma 3.31. Then there exist equivalences of functors*

$$\varrho_i \circ T_{\mathcal{V}^n_F} = \begin{cases} \varrho_{i+1} & \text{for } 1 \leq i \leq n-1 \\ T_{\mathcal{N}}[n-1] \circ \varrho_1 & \text{for } i = n \end{cases} \tag{33}$$

and

$$T_{\mathcal{V}^n_F}^{-1} \circ \varsigma_i = \begin{cases} \varsigma_{i+1} & \text{for } 1 \leq i \leq n-1 \\ \varsigma_1 \circ T_{\mathcal{N}}^{-1}[1-n] & \text{for } i = n \end{cases} \tag{34}$$

Proof. By the 2/4 property of spherical adjunctions there exists an equivalence $T_{\mathcal{V}^n_F}^{-1} G' \simeq E'$, showing the identities (34). The identities (33) follow from passing to left adjoints. ☐

Proposition 3.32 implies that $\varrho_i \circ T^n_{\mathcal{V}^n_F} \simeq T_{\mathcal{N}}[n-1]\varrho_i$. We should consider the change in a perverse schober on the n-spider obtained by composition with $T_{\mathcal{V}^n_F}$ as the effect of an $\frac{1}{n}$-th of a full rotation. The effect of a full rotation on a perverse schober on the n-spider is composition by the shifted twist functor $T_{\mathcal{N}}[n-1]$, which describes up to shift the monodromy of the perverse schober around its singularity at the vertex of the n-spider.

3.3 Parametrized perverse schobers globally

In Section 3.3.1 we define the notion of a perverse schober parametrized by a ribbon graph and its ∞-category of global sections. In Section 3.3.2, we discuss how perverse schobers parametrized by different ribbon graphs can be related. In Section 3.3.3, we introduce a notion of semiorthogonal decomposition of parametrized perverse schobers, which give rise to semiorthogonal decompositions of the ∞-categories of global sections. In Section 3.3.4, we discuss how to define the monodromy along a loop of a perverse schober.

3.3.1 Parametrized perverse schobers

Given a ribbon graph Γ and a vertex $v \in \Gamma_0$ of valency n, the under-category $\mathrm{Exit}(\Gamma)_{v/}$ has $n+1$ objects, which can be identified with v and its n incident halfedges and non-identify morphisms going from v to these halfedges. The datum of a diagram $\mathcal{F}\colon \mathrm{Exit}(\Gamma)_{v/} \to \mathrm{St}$ thus amounts to n functors $\mathcal{F}(v) \to \mathcal{F}(a)$, with a some halfedge, together with a cyclic order of these functors. There is a functor $\mathrm{Exit}(\Gamma)_{v/} \to \mathrm{Exit}(\Gamma)$, which is fully faithful if Γ has no loops incident to v.

Definition 3.33. Let Γ be a ribbon graph. A functor $\mathcal{F}\colon \mathrm{Exit}(\Gamma) \to \mathrm{St}$ is called a Γ-parametrized perverse schober if for each vertex v of Γ, the restriction of $\mathcal{F}$ to $\mathrm{Exit}(\Gamma)_{v/}$ determines a perverse schober parametrized by the n-spider in the sense of Definition 3.18.

Γ-parametrized perverse schobers form an ∞-category, given by the full subcategory of the functor category $\mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathrm{St})$ spanned by perverse schobers.

We will also call a functor $\mathcal{F}\colon \operatorname{Exit}(\Gamma) \to \mathcal{C}$, with $\mathcal{C} = \mathcal{P}r^L_{\mathrm{St}}, \mathcal{P}r^R_{\mathrm{St}}, \mathrm{LinCat}_R$, with R an $\mathbb{E}_\infty$-ring spectrum, a Γ-parametrized perverse schober if its composite with the forgetful functor to St defines a Γ-parametrized perverse schober in the sense of Definition 3.33.

Remark 3.34. If $\mathcal{F}$ is a Γ-parametrized perverse schober and $v \in \Gamma_0$, we can repeatedly apply Proposition 3.21 to the perverse schober on the n-spider $\mathcal{F}|_{\operatorname{Exit}(\Gamma)_{v/}}$ to obtain a spherical functor $F\colon \mathcal{V}_v \to \mathcal{N}$. This resulting spherical functor does not depend on the chosen integers $1 \leq j \leq n$ up to composition with equivalences of ∞-categories, as follows from Proposition 3.26. We say that $\mathcal{F}$ is described or encoded at v by F. We call v a singularity of $\mathcal{F}$ if $\mathcal{V}_v \neq 0$.

Since we assumed ribbon graphs to be connected, a Γ-parametrized perverse schober assigns to each edge of Γ an equivalent ∞-category $\mathcal{N}$, which we call the generic stalk of the perverse schober.

Remark 3.35. Given a Γ-parametrized perverse schober $\mathcal{F}$ and a morphism α in $\operatorname{Exit}(\Gamma)$, the functor $\mathcal{F}(\alpha)$ always admits both left and right adjoints, as follows from the sequence of adjunctions (28). If $\mathcal{F}$ takes values in presentable ∞-categories, it thus automatically factors through the forgetful functors $\mathcal{P}r^R_{\mathrm{St}}, \mathcal{P}r^L_{\mathrm{St}} \to \mathrm{St}$. Most perverse schobers we encounter will be of this form, and we refer to these as presentable parametrized perverse schobers.

Definition 3.36. Let Γ be a ribbon graph and $\mathcal{F}$ a Γ-parametrized perverse schober.

- The ∞-category of global sections $\mathcal{H}(\Gamma, \mathcal{F}) \in \mathrm{St}$ of $\mathcal{F}$ is defined as the limit of $\mathcal{F}$. We usually identify $\mathcal{H}(\Gamma, \mathcal{F})$ with the ∞-category of coCartesian sections of the Grothendieck construction $p\colon \Gamma(\mathcal{F}) \to \operatorname{Exit}(\Gamma)$ of $\mathcal{F}$, see Section 2.1.3.
- The ∞-category of local sections $\mathcal{L}(\Gamma, \mathcal{F}) = \operatorname{Fun}_{\operatorname{Exit}(\Gamma)}(\operatorname{Exit}(\Gamma), \Gamma(\mathcal{F})) \in \mathrm{St}$ of $\mathcal{F}$ is defined as the ∞-category of all sections of the Grothendieck construction p.

Let $\mathcal{F}$ be a Γ-parametrized perverse schober. Given an edge e of Γ, we denote by $\mathrm{ev}_e\colon \mathcal{H}(\Gamma, \mathcal{F}) \to \mathcal{F}(e)$ the evaluation functor, which maps a coCartesian section s of p to its value $s(e)$ at e. The functors ev_e preserve all finite limits and colimits by [Lur09, 5.1.2.3, 4.3.1.10, 4.3.1.16]. If we assume that $\mathcal{F}$ is presentable, the functors ev_e further preserve all (small) limits and colimits. In this case, by the ∞-categorical adjoint functor theorem, ev_e admits a left and right adjoints.

Definition 3.37. Let $\mathcal{F}$ be a presentable Γ-parametrized perverse schober. The functor

$$\partial\mathcal{F}\colon \prod_{e \in \Gamma_1^\partial} \mathcal{F}(e) \longrightarrow \mathcal{H}(\Gamma, \mathcal{F})$$

is defined as the left adjoint of

$$\prod_{e \in \Gamma_1^\partial} \mathrm{ev}_e\colon \mathcal{H}(\Gamma, \mathcal{F}) \to \prod_{e \in \Gamma_1^\partial} \mathcal{F}(e)\,. \tag{35}$$

Remark 3.38. In the study of partially wrapped Fukaya categories, particularly Fukaya-Seidel categories, the functor $\partial\mathcal{F}$ is often called the cap functor and $\prod_{e\in\Gamma_1^\partial}\mathrm{ev}_e$ is called the Orlov functor or cup functor, see for instance [Syl19] for background.

Proposition 3.39. *Let $\mathcal{F}$ be a Γ-parametrized perverse schober taking values in presentable ∞-categories. Assume that the generic stalk $\mathcal{N}$ of $\mathcal{F}$ admits a compact generator $X\in\mathcal{N}$ and that at each vertex of Γ the perverse schober $\mathcal{F}$ is encoded by a conservative spherical functor. For $e\in\Gamma_1$, let $\mathrm{ev}_e^*\colon\mathcal{F}(e)\to\mathcal{H}(\Gamma,\mathcal{F})$ be the left adjoint of ev_e. Then the image of $(X,\dots,X)$ under*

$$\prod_{e\in\Gamma_1}\mathcal{N}\simeq\prod_{e\in\Gamma_1}\mathcal{F}(e)\xrightarrow{\prod_{e\in\Gamma_1}\mathrm{ev}_e^*}\mathcal{H}(\Gamma,\mathcal{F})$$

is a compact generator of $\mathcal{H}(\Gamma,\mathcal{F})$.

Proof. Let v be a m-valent vertex of Γ with incident halfedges $a_1,\dots,a_m$, corresponding to edges $e_1,\dots,e_m$. Let $F\colon\mathcal{V}\to\mathcal{N}$ be the spherical functor encoding $\mathcal{F}$ at v. Consider an object $Y\in\mathcal{V}_F^m\simeq\mathcal{F}(v)$, which corresponds to a diagram of the form

$$a\to b_1\to\cdots\to b_{m-1}\,,$$

with $a\in\mathcal{V}$ and $b_i\in\mathcal{N}$, such that $\mathcal{F}(v\xrightarrow{a_i}e_i)(Y)\simeq 0$ for all $1\leq i\leq m$. Since $\mathcal{F}|_{\mathrm{Exit}(\Gamma)_{v/}}\simeq\mathcal{F}_m(F)$, this implies that $b_{m-1}\simeq\varrho_1(Y)\simeq 0$. Proceeding with an iterative argument, we find that $b_i[m-1-i]\simeq\varrho_{m-i}(Y)\simeq 0$ for all $1\leq i\leq m-1$. Finally, since F is conservative, it follows from $F(a)[m-1]\simeq\varrho_m(Y)\simeq 0$ that $a\simeq 0$ and thus also $Y\simeq 0$.

A global section of $\mathcal{F}$ vanishes if and only if its value at each object $x\in\mathrm{Exit}(\Gamma)$ vanishes. The above computation shows that the values of a global section at all $x\in\mathrm{Exit}(\Gamma)$ vanish if and only if the values at all edges $e\in\Gamma_1$ vanish. This is equivalent to the assertion that $(X,\dots,X)$ is a compact generator. □

Definition 3.40. The ∞-categories of left/right presentable Γ-parametrized perverse schobers

$$\mathfrak{P}^L(\Gamma)\subset\mathrm{Fun}(\mathrm{Exit}(\Gamma),\mathcal{P}r^L_{\mathrm{St}})$$

and

$$\mathfrak{P}^R(\Gamma)\subset\mathrm{Fun}(\mathrm{Exit}(\Gamma),\mathcal{P}r^R_{\mathrm{St}})$$

are defined as the full subcategories spanned by Γ-parametrized perverse schobers.

Remark 3.41. Consider a ribbon graph Γ. Given a Γ-parametrized perverse schober $\mathcal{F}$, we can pass to the right adjoint or left adjoint diagrams $\mathcal{F}^R,\mathcal{F}^L\colon\ \mathrm{Exit}(\Gamma)^{\mathrm{op}}\longrightarrow\mathrm{St}$. Lemma 3.31 implies that there exists an equivalence $\mathcal{F}^R\simeq\mathcal{F}^L$ in $\mathrm{Fun}(\mathrm{Exit}(\Gamma)^{\mathrm{op}},\mathrm{St})$, which restricts on each vertex v with corresponding spherical adjunction $F_v\dashv G_v$ to the twist functor of the spherical adjunction $F'_v\dashv G'_v$.

If $\mathcal{F}$ takes values in presentable ∞-categories, we have factorizations

$$\mathcal{F}^R,\mathcal{F}^L\colon\ \mathrm{Exit}(\Gamma)^{\mathrm{op}}\to\mathcal{P}r^L_{\mathrm{St}}\to\mathrm{St}$$

and there exist equivalences of ∞-categories

$$\mathcal{H}(\Gamma,\mathcal{F}) \simeq \operatorname*{colim}_{\mathcal{P}r^L} \mathcal{F}^L \simeq \operatorname*{colim}_{\mathcal{P}r^L} \mathcal{F}^R \,.$$

We can thus, assuming presentability, equivalently express parametrized perverse schobers and their ∞-categories of global sections in terms of their adjoint constructible cosheaves.

Notation 3.42. We will use a graphical notation for perverse schobers parametrized by ribbon graphs similar to the graphical notation for ribbon graphs introduced in Notation 3.5. We denote a parametrized perverse schober by specifying the spherical functor at each vertex of the corresponding ribbon graph and specifying the functor associated to each non-identity morphism in the exit path category.

Example 3.43. Let $F\colon \mathcal{V} \to \mathcal{N}$ be a spherical functor and $T\colon \mathcal{N} \to \mathcal{N}$ some autoequivalence. The diagram

$$\begin{tikzcd}
 & F \arrow[d, dash, "(T\circ\varrho_1,\varrho_1)"] & \\
0_{\mathcal{N}} \arrow[loop left, dash, "(\varrho_3,\varrho_2)"] \arrow[r, dash, "(\varrho_1,\varrho_2)"] & 0_{\mathcal{N}} \arrow[r, "\varrho_3", -|] & {}
\end{tikzcd} \tag{36}$$

corresponds to the parametrized perverse schober given by the following $\operatorname{Exit}(\Gamma)$-indexed diagram in St,

$$\begin{tikzcd}
 & & & \mathcal{V} \arrow[d, "T\circ F"] & \\
 & & & \mathcal{N} & \\
\mathcal{N} & \mathcal{V}^3_{0_{\mathcal{N}}} \arrow[l, bend right, "\varrho_2"'] \arrow[l, bend left, "\varrho_3"] \arrow[r, "\varrho_1"] & \mathcal{N} & \mathcal{V}^3_{0_{\mathcal{N}}} \arrow[l, "\varrho_2"'] \arrow[u, "\varrho_1"'] \arrow[r, "\varrho_3"] & \mathcal{N}
\end{tikzcd}$$

where $0_{\mathcal{N}}\colon 0 \to \mathcal{N}$ denotes the spherical zero functor.

3.3.2 Contractions of ribbon graphs

The goal of this section is to show that that parametrized perverse schobers can be transported along contractions of ribbon graphs which do not contract any edges connecting two singularities, such that the ∞-categories of global sections are preserved up to equivalence.

Definition 3.44.

- Let Γ be a ribbon graph and $e \in \Gamma_1$ an edge connecting two distinct vertices v_1, v_2. Let $\{e_1, e_2\}$ be the orbit representing the edge e. We define a ribbon graph Γ' with

– $\Gamma_0' = \Gamma_0/(v_1 \sim v_2)$ is the set obtained from Γ_0 obtained by identifying v_1 and v_2,

– $\mathrm{H}_{\Gamma'} = \mathrm{H}_\Gamma \setminus \{e_1, e_2\}$,

– $\tau\colon \mathrm{H}_{\Gamma'} \to \mathrm{H}_{\Gamma'}$ is the restriction of $\tau\colon \mathrm{H}_\Gamma \to \mathrm{H}_\Gamma$.

– $\sigma\colon \mathrm{H}_{\Gamma'} \to \Gamma_0'$ is the composite map : $\mathrm{H}_{\Gamma'} \subset \mathrm{H}_\Gamma \xrightarrow{\sigma} \Gamma_0 \to \Gamma_0'$.

– the cyclic order on $\mathrm{H}_{\Gamma'}(v)$ with $v \in \Gamma_0' \setminus [v_1]$ is identical to the cyclic order on $\mathrm{H}_\Gamma(v)$. Choose any two linear orders of the elements of $\mathrm{H}_\Gamma(v_1)\setminus\{e_1\}$ and $\mathrm{H}_\Gamma(v_2)\setminus\{e_2\}$ compatible with the given cyclic ordering. Consider the total order on

$$\mathrm{H}_{\Gamma'}([v_1]) = \Big(\mathrm{H}_\Gamma(v_1)\setminus\{e_1\}\Big) \cup \Big(\mathrm{H}_\Gamma(v_2)\setminus\{e_2\}\Big)$$

which restricts to the given total orders on $\mathrm{H}_\Gamma(v_1)\setminus\{e_1\}, \mathrm{H}_\Gamma(v_2)\setminus\{e_2\}$ and such that all elements of $\mathrm{H}_\Gamma(v_2)\setminus\{e_2\}$ follow the elements in $\mathrm{H}_\Gamma(v_1)\setminus\{e_1\}$). We let the cyclic order on $\mathrm{H}_{\Gamma'}([v_1])$ to be the cyclic order induced by the above total order.

We call Γ' the edge contraction of Γ at e.

- Let Γ and Γ' be ribbon graphs. We say that there exists a contraction from Γ to Γ' if Γ' is obtained as a (finitely many times) repeated edge contraction of Γ. We write $c\colon \Gamma \to \Gamma'$.

Lemma 3.45. *Let $F\colon \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction and denote by $0_\mathcal{N}\colon 0 \to \mathcal{N}$ the spherical zero functor. Let $m, n \geq 1$ and consider the stable ∞-categories $\mathcal{V}^m_{0_\mathcal{N}}$ and $\mathcal{V}^n_F$, with corresponding functors from Lemma 3.24 denoted $\varrho^1_i\colon \mathcal{V}^m_{0_\mathcal{N}} \to \mathcal{N}$, for $i = 1, \dots, m$, and $\varrho^2_j\colon \mathcal{V}^n_F \to \mathcal{N}$, for $j = 1, \dots, n$.*

1. *There exists a pullback diagram in Cat_∞ as follows.*

$$\begin{tikzcd} \mathcal{V}^{n+m-2}_F \arrow[r, "\alpha"] \arrow[d, "\beta"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{V}^n_F \arrow[d, "\varrho^2_1"] \\ \mathcal{V}^m_{0_\mathcal{N}} \arrow[r, "\varrho^1_m"] & \mathcal{N} \end{tikzcd} \tag{37}$$

2. *Consider the functors $\varrho_1, \dots, \varrho_{n+m-2}\colon \mathcal{V}^{n+m-2}_F \to \mathcal{N}$. There exist equivalences of functors $\varrho_j \simeq \varrho^1_j \circ \beta$ and $\varrho_{i+m-2} \simeq \varrho^2_i \circ \alpha$ for $j = 1, \dots, m-1$ and $i = 2, \dots n$.*

Proof. Let $D_1\colon \Delta^{m-2} \to \mathrm{St}$ be the constant diagram with value $\mathcal{N}$ and $D_2\colon \Delta^{n-1} \to \mathrm{St}$, $D\colon \Delta^{n+m-3} \to \mathrm{St}$ be the diagrams obtained from the sequences of composable functors

$$\mathcal{V} \xrightarrow{G} \mathcal{N} \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \mathcal{N}\,.$$

The diagram D restricts to the diagrams D_1 and D_2 on $\Delta^{\{0,\dots,n-1\}}$ and $\Delta^{\{n-1,\dots,n+m-3\}}$, respectively. The inclusion functor $\Delta^{\{0,\dots,n-1\}} \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,\dots,n+m-3\}} \to \Delta^{n+m-3}$ is inner anodyne. It follows that the restriction functor

$$\mathrm{res}\colon \mathrm{Fun}(\Delta^{n+m-3}, \Gamma(D)) \to \mathrm{Fun}(\Delta^{n-1}, \Gamma(D_1)) \times_\mathcal{N} \mathrm{Fun}(\Delta^{m-2}, \Gamma(D_2))$$

is a trivial fibration, from which we obtain a further trivial fibration

$$\mathrm{Fun}_{\Delta^{n+m-3}}(\Delta^{n+m-3},\Gamma(D)) \to \mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1},\Gamma(D_1)) \times_{\mathcal{N}} \mathrm{Fun}_{\Delta^{m-2}}(\Delta^{m-2},\Gamma(D_2))\,.$$

Using the equivalences of ∞-categories

$$\begin{aligned} \mathcal{V}^n_F &\simeq \mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1},\Gamma(D_2))\,,\\ \mathcal{V}^m_{0_{\mathcal{N}}} &\simeq \mathrm{Fun}_{\Delta^{m-2}}(\Delta^{m-2},\Gamma(D_1))\,,\\ \mathcal{V}^{n+m-3}_F &\simeq \mathrm{Fun}_{\Delta^{n+m-3}}(\Delta^{n+m-3},\Gamma(D))\,, \end{aligned}$$

it follows that there exists a pullback diagram of the form (37). The functors $\alpha[2-m]$ and β in this pullback diagram are given by the restriction functors to the first $m-1$ and last n components, respectively. The description of the categorified restriction maps can thus be checked directly. □

Construction 3.46. Consider the setup of Lemma 3.45 and the following diagram,

$$\begin{tikzcd} & \mathcal{V}^n_F \arrow[d, "\varrho^2_j"] \\ \mathcal{V}^m_{0_{\mathcal{N}}} \arrow[r, "\varrho^1_i"] & \mathcal{N} \end{tikzcd} \tag{38}$$

where $1 \leq i \leq m$ and $1 \leq j \leq n$ are arbitrary. We can use the paracyclic twist functors $\left(T_{\mathcal{V}^n_F}\right)^{1-j}$ and $\left(T_{\mathcal{V}^m_{0_{\mathcal{N}}}}\right)^{m-i}$, see Section 3.2.2 and Proposition 3.32 in particular, to find a natural equivalence between the diagram (38) and the following diagram.

$$\begin{tikzcd} & \mathcal{V}^n_F \arrow[d, "\varrho^2_1"] \\ \mathcal{V}^m_{0_{\mathcal{N}}} \arrow[r, "\varrho^1_m"] & \mathcal{N} \end{tikzcd} \tag{39}$$

The limits of the diagrams (38) and (39) are therefore both equivalent to $\mathcal{V}^{n+m-2}_F$. Proposition 3.32 further shows that under this equivalence the resulting functors $\varrho_i\colon \mathcal{V}^{n+m-2}_F \to \mathcal{N}$ are cyclically permuted and may each further change by post-composition with an autoequivalence of the form $(T_{\mathcal{N}}[n-1])^l$ for some $l \in \mathbb{Z}$.

Proposition 3.47. *Let $c\colon \Gamma \to \Gamma'$ be a contraction of ribbon graphs.*

(1) Let $\mathcal{F}$ be a Γ-parametrized perverse schober and assume that c contracts no edges incident to two singularities of $\mathcal{F}$. Then there exists a canonical Γ'-parametrized perverse schober $c_*(\mathcal{F})$ and an equivalence of global sections

$$c_*\colon \mathcal{H}(\Gamma,\mathcal{F}) \simeq \mathcal{H}(\Gamma', c_*(\mathcal{F}))\,.$$

(2) Let $\mathcal{F}'$ be a Γ'-parametrized perverse schober. For each choice of subset $S \subset \Gamma_0$, such that $c|_S\colon S \to \Gamma'_0$ defines a bijection between S and the singularities of $\mathcal{F}'$, there exists a Γ-parametrized perverse schober $c^*(\mathcal{F}')$, satisfying $c_*c^*(\mathcal{F}') \simeq \mathcal{F}'$. There thus exists an equivalence of global sections

$$c^*\colon \mathcal{H}(\Gamma',\mathcal{F}') \simeq \mathcal{H}(\Gamma, c^*(\mathcal{F}'))\,.$$

Proof. We begin by proving part (1). It suffices to show the statement in the case that c is the contraction of an edge $e \in \Gamma_1$ connecting two vertices v_1, v_2 such that v_1 is not a singularity. The edge contraction c induces a functor $\mathrm{Exit}(c): \mathrm{Exit}(\Gamma) \to \mathrm{Exit}(\Gamma')$ determined by mapping

- $x \in \Gamma_0 \backslash \{v_1, v_2\} \subset \mathrm{Exit}(\Gamma)$ to $x \in \Gamma_0 \backslash \{v_1, v_2\} \subset \mathrm{Exit}(\Gamma')$,
- $v_1, v_2 \in \Gamma_0 \subset \mathrm{Exit}(\Gamma)$ to $[v_1]$,
- $f \in \Gamma_1 \backslash \{e\} \subset \mathrm{Exit}(\Gamma)$ to $f \in \Gamma_1 \backslash \{e\} \subset \mathrm{Exit}(\Gamma')$ and
- $e \in \Gamma_1 \subset \mathrm{Exit}(\Gamma)$ to $[v_1] \in \Gamma'_0 \subset \mathrm{Exit}(\Gamma')$.

We define $\mathcal{E}$ to be the poset determined by the following properties.

- There exist fully faithful functors $\mathrm{Exit}(\Gamma'), \mathrm{Exit}(\Gamma) \to \mathcal{E}$.
- The induced functor $\mathrm{Exit}(\Gamma') \amalg \mathrm{Exit}(\Gamma) \to \mathcal{E}$ is bijective on objects
- For $x' \in \mathrm{Exit}(\Gamma')$ and $x \in \mathrm{Exit}(\Gamma)$, there exists a unique morphism from x' to x in $\mathcal{E}$ if and only if there exists a morphism $x' \to \mathrm{Exit}(c)(x)$. There are no morphisms from x to x'.

Note that the poset $\mathcal{E}$ can be equivalently described as the total space of a Cartesian fibration classified by the functor $\mathrm{Exit}(c)\colon \Delta^1 \to \mathrm{Cat}_\infty$.

We define a functor $c_*\colon \mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathrm{St}) \to \mathrm{Fun}(\mathrm{Exit}(\Gamma'), \mathrm{St})$ as the composition of the right Kan extension functor along the inclusion $\mathrm{Exit}(\Gamma) \to \mathcal{E}$ with the restriction functor to $\mathrm{Exit}(\Gamma')$. and define $c_*(\mathcal{F})$ as the image of $\mathcal{F}$ under c_*. It follows from Lemma 3.45 and Construction 3.46 that $c_*(\mathcal{F})$ is a Γ'-parametrized perverse schobers. The fact that c_* preserves global sections (i.e. limits) follows from the fact that iterated right Kan extensions along inclusions are again right Kan extensions, or alternatively a decomposition result for limits [Lur09, 4.2.3.10].

For part (2), it suffices to treat the case that c contracts a single edge e of Γ incident to vertices v_1, v_2, whose image in Γ' is denoted v. Swapping the labels of v_1 and v_2 if necessary, we may assume that $S = \{v_2\}$. Let m_i be the valency of v_i, $i = 1, 2$, and $n = m_1 + m_2 - 2$ the valency of v. Replacing $\mathcal{F}'$ by an equivalent Γ'-parametrized perverse schober, we may assume that near v, $\mathcal{F}'$ is given by the following diagram:

$$\begin{tikzcd}
\mathcal{N} & & \mathcal{N} \\
\cdots & \mathcal{V}^n_{F_v} \arrow[ul, "\varrho_1"'] \arrow[ur, "\varrho_n"] \arrow[dl, "\varrho_{m_1-1}"] \arrow[dr, "\varrho_{m_1}"'] & \cdots \\
\mathcal{N} & & \mathcal{N}
\end{tikzcd}$$

We define $c^*(\mathcal{F}')$ to be identical to $\mathcal{F}'$ away from v_1, v_2 and near v_1, v_2 as the following diagram,

$$\begin{tikzcd}
\mathcal{N} & & & & \mathcal{N} \\
\dots & \mathcal{V}^{m_1}_{0_{\mathcal{N}}} \arrow[ul, "\varrho_1"'] \arrow[dl, "\varrho_{m_1-1}"] \arrow[r, "\varrho_{m_1}"] & \mathcal{N} & \mathcal{V}^{m_2}_{F_v} \arrow[l, "\varrho_1"] \arrow[ur, "\varrho_{m_2}"] \arrow[dr, "\varrho_2"] & \dots \\
\mathcal{N} & & & & \mathcal{N}
\end{tikzcd}$$

where we denote by $0_{\mathcal{N}}$ the spherical functor $0\colon 0 \to \mathcal{N}$. By [Chr22b, Lemma 4.26], we then get $c_*c^*(\mathcal{F}') \simeq \mathcal{F}'$. The second statement of part 2) now follows from part 1). $\square$

3.3.3 Semiorthogonal decompositions of perverse schobers

Let Γ be a ribbon graph. We note that a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of Γ-parametrized perverse schobers in one of the ∞-categories $\mathfrak{P}^L(\Gamma)$ or $\mathfrak{P}^R(\Gamma)$, see Definition 3.40, is simply a natural transformation $\Delta^1 \times \operatorname{Exit}(\Gamma) \to \mathcal{P}r^L_{\mathrm{St}}$ or $\Delta^1 \times \operatorname{Exit}(\Gamma) \to \mathcal{P}r^R_{\mathrm{St}}$ between the diagrams defining $\mathcal{F}$ and $\mathcal{G}$.

Definition 3.48. Let Γ be a ribbon graph and let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of Γ-parametrized perverse schobers in $\mathfrak{P}^L(\Gamma)$ or $\mathfrak{P}^R(\Gamma)$.

1. We call α an inclusion of perverse schobers in $\mathfrak{P}^L(\Gamma)$ or $\mathfrak{P}^R(\Gamma)$ if $\alpha(x)\colon \mathcal{F}(x) \to \mathcal{G}(x)$ is the inclusion of a stable subcategory (in particular fully faithful) for all $x \in \operatorname{Exit}(\Gamma)$.

2. Suppose that α is an inclusion in $\mathfrak{P}^L(\Gamma)$. We call α right admissible if there exists a morphism $\beta\colon \mathcal{G} \to \mathcal{F}$ in $\mathfrak{P}^R(\Gamma)$ such that $\beta(x)$ is right adjoint to $\alpha(x)$ for all $x \in \operatorname{Exit}(\Gamma)$.

3. Suppose that α is an inclusion in $\mathfrak{P}^R(\Gamma)$. We call α left admissible if there exists a morphism $\beta\colon \mathcal{G} \to \mathcal{F}$ in $\mathfrak{P}^L(\Gamma)$ such that $\beta(x)$ is left adjoint to $\alpha(x)$ for all $x \in \operatorname{Exit}(\Gamma)$.

Remark 3.49. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be an inclusion of Γ-parametrized perverse schobers. Spelling out the condition that α is left admissible yields the requirement that for each morphism $v \to e$ in $\operatorname{Exit}(\Gamma)$ the diagram

$$\begin{tikzcd}
\mathcal{F}(v) \arrow[r, "\alpha(v)"] \arrow[d, "\mathcal{F}(v\to e)"'] & \mathcal{G}(v) \arrow[d, "\mathcal{G}(v\to e)"] \\
\mathcal{F}(e) \arrow[r, "\alpha(e)"] & \mathcal{G}(e)
\end{tikzcd} \tag{40}$$

is left adjointable, meaning that the diagram

$$\begin{tikzcd} \mathcal{F}(v) \arrow[d, "\mathcal{F}(v\to e)"'] & \mathcal{G}(v) \arrow[l, "\beta(v)"'] \arrow[d, "\mathcal{G}(v\to e)"] \\ \mathcal{F}(e) & \mathcal{G}(e) \arrow[l, "\beta(e)"'] \end{tikzcd}$$

commutes, with $\beta(x)$ left adjoint to $\alpha(x)$ for $x = v, e$. Analogously, the condition that α is right admissible is equivalent to the right adjointability of the diagram (40). Adjointability of a commutative square is also called the *Beck-Chevalley* property.

Definition 3.50. Let Γ be a ribbon graph. A semiorthogonal decomposition $\{\mathcal{F}_1, \mathcal{F}_2\}$ of a Γ-parametrized perverse schober $\mathcal{G}$ consists of

- an inclusion $\alpha_1 \colon \mathcal{F}_1 \to \mathcal{G}$ in $\mathfrak{P}^R(\Gamma)$ and
- an inclusion $\alpha_2 \colon \mathcal{F}_2 \to \mathcal{G}$ in $\mathfrak{P}^L(\Gamma)$, satisfying that
- α_1 is left admissible and that α_2 is right admissible and that
- the cofiber of α_2 in $\mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathcal{P}r^L_{\mathrm{St}})$ is given by $\mathcal{F}_1$.

Remark 3.51. Consider a Γ-parametrized perverse schober $\mathcal{G}$ with a semiorthogonal decomposition $\{\mathcal{F}_1, \mathcal{F}_2\}$.

1. Then for each $x \in \mathrm{Exit}(\Gamma)$, the ∞-category $\mathcal{G}(x)$ admits a semiorthogonal decomposition $(\mathcal{F}_1(x), \mathcal{F}_2(x))$.
2. The condition that $\mathcal{F}_1$ is the cofiber of α_2 in $\mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathcal{P}r^L_{\mathrm{St}})$ is equivalent to the condition that $\mathcal{F}_2$ is the fiber of the pointwise right adjoint $\beta_1 \colon \mathcal{G} \to \mathcal{F}_1$ of α_1 in $\mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathcal{P}r^R_{\mathrm{St}})$. This follows from the fact that limits and colimits in functor categories are determined pointwise, see [Lur09, 5.1.2.3]. Since limits commute with limits, we find that there exists a fiber sequence in $\mathcal{P}r^L_{\mathrm{St}}$

   $$\mathcal{H}(\Gamma, \mathcal{F}_2) \xrightarrow{i} \mathcal{H}(\Gamma, \mathcal{G}) \xrightarrow{\pi} \mathcal{H}(\Gamma, \mathcal{F}_1)\,.$$

   Passing to right adjoints yields a cofiber sequence in $\mathcal{P}^R_{\mathrm{St}}$, showing by Lemma 2.37 that $\mathcal{H}(\Gamma, \mathcal{G})$ admits a semiorthogonal decomposition

   $$\Big(\mathrm{radj}(\pi)(\mathcal{H}(\Gamma, \mathcal{F}_1)), i(\mathcal{H}(\Gamma, \mathcal{F}_2))\Big)\,.$$

Definition 3.52. Let $\mathcal{F}$ be a Γ-parametrized perverse schober. Recall that given a vertex v of Γ, $\mathcal{F}$ is described near v by a spherical functor $F \colon \mathcal{V}_v \to \mathcal{N}$, where $\mathcal{V}_v$ is called the ∞-category of vanishing cycles and $\mathcal{N}$ the ∞-category of nearby cycles. We call

- $\mathcal{F}$ vanishing-monadic, if at each vertex v of Γ the spherical functor $F_v \colon \mathcal{V}_v \to \mathcal{N}$ describing $\mathcal{F}$ at v is a monadic functor.

- $\mathcal{F}$ nearby-monadic, if at each vertex v of Γ the right adjoint of the spherical functor F_v describing $\mathcal{F}$ at v is a monadic functor.

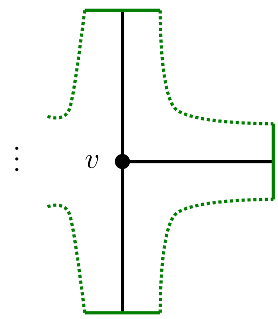
- $\mathcal{F}$ locally constant if $\mathcal{F}$ has no singularities.

Remark 3.53. A spherical functor between presentable ∞-categories is monadic if and only if it is comonadic if and only if it is conservative, as follows from the ∞-categorical Barr-Beck theorem [Lur17, 4.7.3.5].

3.3.4 Monodromy of perverse schobers

Let Γ be a spanning graph of a marked surface $\mathbf{S}$ and fix a Γ-parametrized perverse schober $\mathcal{F}$. We denote by P the set of vertices of Γ which are singularities of $\mathcal{F}$, see Remark 3.34. In analogy with perverse sheaves on $\mathbf{S}$, which restrict to a local system on $\mathbf{S}\backslash P$, we wish to associate a monodromy equivalence of $\mathcal{F}$ along any loop $S^1 \subset \mathbf{S}\backslash P$. We construct these by composing local identifications, which we refer to as transports. We begin in Construction 3.55 by considering a perverse schober on the n-spider Γ_n and describe how to obtain the local transport along a path in the once-punctured n-gon. For technical convenience, we however replace the n-gon by the homotopic surface Σ_n described in the following remark:

Remark 3.54. Let Γ be a ribbon graph. To each vertex v of Γ of valency n we associate a (non-compact) surface, denoted Σ_v or also Σ_n, with an embedding of v and its n incident halfedges. We depict Σ_v as follows (in green). The dotted lines correspond to open ends, whereas the solid lines indicate the boundary.

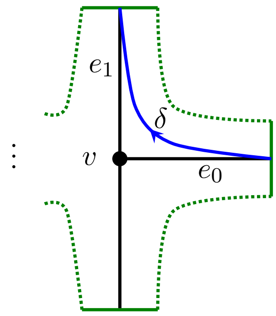


We define the thickening of Γ to be the surface Σ_Γ, obtained from gluing the surfaces Σ_v, whenever two vertices are incident to the same edge, at their boundary components corresponding to the edge. The surface Σ_Γ comes with an embedding of $|\Gamma|$, which is also a homotopy equivalence.

If Γ is a spanning graph of a marked surface $\mathbf{S}$, then we further choose a homotopy equivalence $\Sigma_\Gamma \to \mathbf{S}$, which maps boundary to boundary, the open boundary of Σ_Γ in the limit to $M \cap \partial\mathbf{S}$, and satisfying that the composite $|\Gamma| \to \Sigma_\Gamma \to \mathbf{S}\backslash M$ is the given embedding of the spanning ribbon graph Γ.

Construction 3.55. Let $n \geq 2$, the case $n = 1$ is addressed at the end. Consider the n-spider Γ_n with central vertex v embedded in Σ_n. Consider a curve $\delta\colon [0,1] \to \Sigma_n\backslash\{v\}$ satisfying that $\delta(0), \delta(1) \in \partial\Sigma_n \cap |\Gamma_n|$ and that the edge e_1 of Γ_n containing

$\delta(1)$ lies one step in the counterclockwise direction of the edge e_0 containing $\delta(0)$. We can depict this setup as follows:



Given a Γ_n-parametrized perverse schober $\mathcal{F}$, we define the transport $\mathcal{F}^{\to}(\delta)$ of $\mathcal{F}$ along δ as the equivalence

$$\mathcal{F}(e_0) \xrightarrow{\operatorname{ladj}(\mathcal{F}(v\to e_0))} \mathcal{F}(v) \xrightarrow{\mathcal{F}(v\to e_1)} \mathcal{F}(e_1)\,.$$

We note that the transport of $\mathcal{F}$ along a path γ only depends on the homotopy class of γ under homotopies preserving endpoints.

Reversing the orientation of δ yields a path δ^{rev} going one step in the clockwise direction along Γ_n and the transport of $\mathcal{F}$ along δ^{rev} is defined as

$$\mathcal{F}^{\to}(\delta^{\mathrm{rev}}) \coloneqq \mathcal{F}(v\to e_0)\circ \operatorname{radj}(\mathcal{F}(v\to e_1))\colon \mathcal{F}(e_1) \longrightarrow \mathcal{F}(e_0)\,.$$

We thus have $\mathcal{F}^{\to}(\delta^{\mathrm{rev}}) \simeq (\mathcal{F}^{\to}(\delta))^{-1}$.

Consider a now an arbitrary path $\delta\colon [0,1] \to \Sigma_n\backslash\{v\}$, such that $\delta(0),\delta(1) \in \partial\Sigma_v \cap |\Gamma_n|$. This path starts at some edge e_0 of Γ_n and ends at some edge e_1 and goes $i \in \mathbb{Z}$ steps counterclockwise. Replacing δ by a homotopic path, with the homotopy fixing endpoints and not crossing v, we can assume that $\delta = \delta_{|i|} * \cdots * \delta_1$ is the composite of $|i| \in \mathbb{N}$ paths $\delta_1,\dots,\delta_{|i|}$, as above, each wrapping one step counterclockwise if $i > 0$ and one step clockwise if $i < 0$. If $i = 0$, then after the homotopy, δ is the constant path and we define the transport $\mathcal{F}^{\to}(\delta)$ as $\mathrm{id}_{\mathcal{F}(e_0)}$. For $i > 0$ (i.e. a counterclockwise path), we define the transport of $\mathcal{F}$ along δ as the equivalence

$$\mathcal{F}^{\to}(\delta) \coloneqq \mathcal{F}^{\to}(\delta_{|i|})\circ\cdots\circ\mathcal{F}^{\to}(\delta_1)[i-1]\,, \tag{41}$$

and for $i < 0$ (i.e. a clockwise path) as

$$\mathcal{F}^{\to}(\delta) \coloneqq \mathcal{F}^{\to}(\delta_{|i|})\circ\cdots\circ\mathcal{F}^{\to}(\delta_1)[1-i]\,. \tag{42}$$

The suspension is chosen, so that the transport along a counterclockwise loop of a perverse schober on the n-gon with no singularity at v is given by [1]. Note that the monodromy defined below will thus be $\mathrm{id}_{\mathcal{F}(e_0)}$. If the perverse schober instead has a singularity at v, the transport along a counterclockwise loop can be identified with the inverse cotwist autoequivalence of the spherical adjunction describing the perverse schober near v.

We conclude with the case $n = 1$. Given a perverse schober $\mathcal{F}$ parametrized by the 1-spider, with vertex v and edge e, i.e. a spherical functor $F\colon \mathcal{V} = \mathcal{F}(v) \to \mathcal{N} = \mathcal{F}(e)$, and a loop $\delta\colon [0,1] \to \Sigma_1 \backslash \{v\}$ with the endpoint the unique point in $\partial\Sigma_1 \cap |\Gamma_1|$, wrapping m times around v in the counterclockwise direction, we define the transport as $\mathcal{F}^{\to}(\delta) := (T_{\mathcal{N}})^{-m}[1-m]$, where $T_{\mathcal{N}}$ denotes the cotwist functor of the adjunction $F \dashv \mathrm{radj}(F)$. If δ instead wraps clockwise around v, we set $\mathcal{F}^{\to}(\delta) := (T_{\mathcal{N}})^{m}[m-1]$.

Construction 3.56. Let Γ be a ribbon graph and consider a path $\eta\colon [0,1] \to \Sigma_\gamma \backslash \Gamma_0$, mapping 0 and 1 to edges e_0, e_1 of Γ. We can write η as the composite of a minimal number of paths $\delta_1, \dots, \delta_m$, with δ_i contained in $\Sigma_{v_i} \subset \Sigma_\Gamma$ for some vertex $v_i \in \Gamma_0$ and $\delta_1(0) = x$. Concretely, the paths δ_i near a given vertex v are obtained as the components of the intersection of γ with $\Sigma_v \subset \Sigma_\Gamma$.

Let $\mathcal{F}$ be an Γ-parametrized perverse schober. We define the transport $\mathcal{F}^{\to}(\eta)$ of $\mathcal{F}$ along η as the equivalence $\mathcal{F}(e_0) \simeq \mathcal{F}(e_1)$ given by the composite

$$\mathcal{F}^{\to}(\delta_m) \circ \cdots \circ \mathcal{F}^{\to}(\delta_1)\,.$$

Lemma 3.57. *Let $\mathcal{F}$ be a Γ-parametrized perverse schober with singularities P. Let $\eta, \eta'\colon [0,1] \to \Sigma_\Gamma \backslash \Gamma_0$ be two paths with $\eta_1(0) = \eta_2(0), \eta_1(1) = \eta_2(1)$ lying on edges of Γ. If γ_1, γ_2 are homotopic in $\Sigma_\Gamma \backslash P$ relative $\partial\Sigma_\Gamma$ and their endpoints, then*

$$\mathcal{F}^{\to}(\eta_1) \simeq \mathcal{F}^{\to}(\eta_2)\,.$$

Proof. It is clear by the definition of the transport, that homotopies in $\Sigma_\Gamma \backslash \Gamma_0$ relative the boundary do not affect the transport. It thus suffices to show that for a non-singular vertex $v \in \Gamma_0 \backslash P$, the transports of two paths δ, δ' in Σ_v going from an edge e to an edge f, where δ goes clockwise and δ' goes counterclockwise, are equivalent. Using the local model for perverse schobers from Proposition 3.25, this is readily verified. $\square$

The transport of a composite of paths is not necessarily the composite of the transports of the paths, as there may arise suspenions or deloopings from (41). This is fixed as follows:

Construction 3.58. Let $\eta = \delta_m * \cdots * \delta_1, \eta' = \delta'_{m'} * \cdots * \delta'_1\colon [0,1] \to \Sigma_\gamma$ be paths as in Construction 3.56. We note that $\eta = \eta'$ being a loop is allowed. We associate to the pair (η, η') the following integer $N(\eta, \eta') \in \{-1, 0, 1\}$.

If δ_m, δ'_1 lie at distinct vertices, we set $N(\eta, \eta') = 0$. Assume that δ_m, δ'_1 lie at the same vertex v. If the end point of δ_m, denoted $\delta_m(1)$, and the starting point of δ'_1, denoted $\delta'_1(0)$, lie on distinct boundary components of $\partial\Sigma_{v_m}$ (whose images in Σ_Γ are however be same), we also set $N(\eta, \eta') = 0$. Assume that $\delta_m(1), \delta'_1(0)$ lie on the same boundary component of Σ_{v_m}, denoted B. Replacing $\delta_m, \delta'_1 \subset \Sigma_{v_m}$ by homotopic paths, we may assume that the two do not intersect. We set $N(\eta, \eta') = 1$, if the intersection point $\delta'_1(0) \in B$ follows the intersection point $\delta_m(1) \in B$ in the counterclockwise direction. Otherwise, we set $N(\eta, \eta') = -1$.

Lemma 3.59. *Let $\eta, \eta' \colon [0,1] \to \Sigma_\Gamma \backslash \Gamma_0$ be two paths with $\eta_1(1) = \eta_2(0)$ and composite $\eta' * \eta$. Then*

$$\mathcal{F}^{\rightarrow}(\eta' * \eta) \simeq \mathcal{F}^{\rightarrow}(\eta') \circ \mathcal{F}^{\rightarrow}(\eta)[N(\eta, \eta')] \,. \tag{43}$$

Proof. The minimal decompositions $\eta = \delta_m * \cdots * \delta_1, \eta' = \delta'_{m'} * \cdots * \delta'_1$ give rise to a minimal decomposition of $\eta' * \eta$. If $N(\eta, \eta') = 0$, then this decomposition is

$$\delta'_{m'} * \cdots * \delta'_1 * \delta_m * \cdots * \delta_1 \,,$$

and (43) follows from the definition of transport. If $N(\eta, \eta') = \pm 1$, then the decomposition is

$$\delta'_{m'} * \cdots * (\delta'_1 * \delta_m) * \cdots * \delta_1 \,.$$

with $(\delta'_1 * \delta_m)$ describing a curve in Σ_v for some vertex v. Inspecting (41) and (42), one observes that

$$\mathcal{F}^{\rightarrow}(\delta'_1 * \delta_m) \simeq \mathcal{F}^{\rightarrow}(\delta'_1) \circ \mathcal{F}^{\rightarrow}(\delta_m)[N(\eta, \eta')]$$

and (43) follows. □

Definition 3.60. Let Γ be a ribbon graph and $\mathcal{F}$ a Γ-parametrized perverse schober. Consider a loop $\gamma \colon S^1 \simeq [0,1]/(0 \sim 1) \to \Sigma_\Gamma \backslash \Gamma_0$, mapping the chosen basepoint $x = 0 \in [0,1]/(0 \sim 1) \simeq S^1$ to an edge $e \in \Gamma_1$. Let $\eta \colon [0,1] \to [0,1]/(0 \sim 1) \xrightarrow{\gamma} \Sigma_\Gamma \backslash \Gamma_0$. The monodromy of $\mathcal{F}$ along γ is defined as the autoequivalence of $\mathcal{F}(e)$

$$\mathcal{F}^{\rightarrow}(\gamma, e) = \mathcal{F}^{\rightarrow}(\eta)[N(\eta, \eta)] \,.$$

More generally, given a loop $\gamma \colon S^1 \simeq [0,1]/(0 \sim 1) \to \Sigma_\Gamma \backslash P$, mapping 0 to $e \in \Gamma_1$, we can define the monodromy of $\mathcal{F}$ along γ by choosing a homotopic path γ' in $\Sigma_\Gamma \backslash \Gamma_0$ and defining $\mathcal{F}^{\rightarrow}(\gamma, e) := \mathcal{F}^{\rightarrow}(\gamma', e)$. This is well-defined by Lemma 3.57.

Example 3.61. Consider a perverse schober $\mathcal{F}$ on the once-punctured disc $\mathbb{C}$, parametrized by the 1-spider Γ_1, and described by a spherical functor $F \colon \mathcal{V} \to \mathcal{N}$. Let G be the right adjoint of F and $T_\mathcal{N}$ the cotwist functor of $F \dashv G$. The monodromy of $\mathcal{F}$ along a loop wrapping once clockwise around the singularity is given by $T_\mathcal{N}[1]$. Passing to Grothiendieck groups, i.e. applying $K_0(\text{-})$, we obtain a perverse sheaf on $\mathbb{C}$ with a single singularity. We have

$$K_0(T_\mathcal{N}) = K_0(F)K_0(G) - K_0(\mathrm{id}_\mathcal{N}) \,.$$

The isomorphism

$$K_0(T_\mathcal{N}[1]) = -K_0(T_\mathcal{N}) = K_0(\mathrm{id}_\mathcal{N}) - K_0(F)K_0(G)$$

describes the (usual) monodromy of this perverse sheaf.

Similarly, given a perverse schober on the n-times punctures disc, the monodromy along a clockwise loop enclosing all singularities is given by the composite of the suspensions of the cotwist functors of the corresponding spherical adjunctions.

Lemma 3.62. *Let $\mathcal{F}$ be a Γ-parametrized perverse schober and $\gamma\colon S^1 \to \mathbf{S}\backslash P$ a loop, mapping the basepoint to an edge e of Γ. Let $c\colon \Gamma \to \Gamma'$ be a contraction of ribbon graphs contracting no edges connecting two singularities of $\mathcal{F}$ and not contracting e. Then*

$$\mathcal{F}^{\to}(\gamma, e) \simeq c_*(\mathcal{F})^{\to}(\gamma, e)\,.$$

Proof. It suffices to consider the case that c contracts a single edge e'. Since $\mathcal{F}$ and $c_*(\mathcal{F})$ are identical away from a neighborhood of e', it suffices to show that the transports in the neighborhood of e' of $\mathcal{F}$ and $c_*(\mathcal{F})$ coincide. Using the explicit local models for $\mathcal{F}$ and $c_*(\mathcal{F})$ near e' from Lemma 3.45, this is straightforward to verify. □

Proposition 3.63. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two Γ-parametrized perverse schobers without singularities. Assume that there exists an edge e of Γ, such that $\mathcal{F}_1(e) = \mathcal{F}_2(e)$. The following two are equivalent:*

1) There exists an equivalence of Γ-parametrized perverse schobers $\mathcal{F}_1 \simeq \mathcal{F}_2$.

2) Given any loop $\gamma\colon S^1 \to \mathbf{S}\backslash P$, mapping the basepoint to a point in $e \subset \mathbf{S}$, the monodromy of $\mathcal{F}_1$ and $\mathcal{F}_2$ along γ are equivalent.

Proof. It is clear that 1) implies 2). We proceed by proving the converse. Denote by $\mathcal{N} = \mathcal{F}_1(e)$ the generic stalk. Choose a contraction $c\colon \Gamma \to \Gamma'$, such that e is not contracted and Γ' has only a single vertex. Lemma 3.62 implies that the monodromies of $c_*(\mathcal{F}_1)$ and $c_*(\mathcal{F}_2)$ along any loop γ mapping the basepoint to e are equivalent. We choose a total order of the m halfedges incident to the vertex v of Γ', compatible with their given cyclic order. We denote the i-th halfedge by a_i, and its corresponding edge by e_i. We can replace $c_*(\mathcal{F}_1)$ and $c_*(\mathcal{F}_2)$ by equivalent Γ'-parametrized perverse schobers, denoted $\mathcal{G}_1$ and $\mathcal{G}_2$, such that for $j = 1, 2$ and $1 \leq i \leq m$

$$\mathcal{G}_j(v) \simeq \mathcal{V}^m_{0_{\mathcal{N}}}\,,$$
$$\mathcal{G}_j(e_i) = \mathcal{N}\,,$$

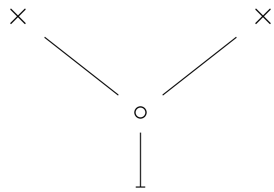
and

$$\mathcal{G}_j(v \xrightarrow{a_i} e_i) = S_{i,j} \circ \varrho_i\,,$$

where $S_{i,j}$ is some autoequivalence of $\mathcal{N}$. The monodromy along a path γ starting at e and going around a given loop of Γ', composed of halfedges $a_i, a_{i'}$ of $\mathcal{G}_j$, is given by $S_{i',j} \circ S_{i,j}^{-1}[i' - i + 1]$. We thus find $S_{i',1}S_{i,1}^{-1} \simeq S_{i',2}S_{i,2}^{-1}$. We can additionally assume that $S_{i',2} = S_{i',1} = \mathrm{id}_{\mathcal{G}(e)}$, by replacing $\mathcal{G}_1, \mathcal{G}_2$ by equivalent perverse schobers once more. We thus conclude $S_{i,1} \simeq S_{i,2}$ as well. Performing this argument for all loops of Γ' shows that $\mathcal{G}_1 \simeq \mathcal{G}_2$, concluding the proof. □

The assumption in Proposition 3.63, that the perverse schobers have no singularities, is necessary. The following example shows that there exist non-equivalent perverse schobers with equivalent singularity data and equivalent monodromy.

Example 3.64. Consider the following spanning graph Γ of the twice-punctured 1-gon:



A Γ-parametrized perverse schober $\mathcal{F}$ with two singularities at the vertices labeled $\times$ and no singularity at the vertices labeled $\circ$ corresponds, up to natural equivalence and using Notation 3.42, to a diagram

$$\begin{tikzcd}
F_1 & & F_2 \\
& 0_{\mathcal{N}} \arrow[ul, dash, "(S_1\circ\varrho_1,\varrho_3)"] \arrow[ur, dash, "(\varrho_2,S_2\circ\varrho_1)"'] \arrow[d, dash, "\varrho_1"'] & \\
& {} &
\end{tikzcd}$$

with $F_1 \colon \mathcal{V}_1 \to \mathcal{N}, F_2 \colon \mathcal{V}_2 \to \mathcal{N}$ spherical functors and S_1, S_2 autoequivalences of $\mathcal{N}$. Composing with the autoequivalence $\mathrm{Fun}(\Delta^1, S_2^{-1})$ of $\mathcal{V}^3_{0_{\mathcal{N}}}$ replaces S_1 by $S_2^{-1}S_1$ and S_2 by $\mathrm{id}_{\mathcal{N}}$. Up to equivalence of perverse schobers, we may thus assume that $S_2 = \mathrm{id}_{\mathcal{N}}$. The equivalence class of the perverse schober $\mathcal{F}$ is hence determined by the functors F_1, F_2, and an autoequivalence S_1 of $\mathcal{N}$. The choice of S_1 affects the monodromy of $\mathcal{F}$ along a clockwise loop wrapping one around the first singularity by conjugation with S_1, since the cotwist functor of $S_1F_1 \dashv \mathrm{radj}(F_1)S_1^{-1}$ is given by $S_1T_{\mathcal{N}}S_1^{-1}$, with $T_{\mathcal{N}}$ the cotwist of $F_1 \dashv \mathrm{radj}(F_1)$.

Let k be a field with $\mathrm{char}(k) = 0$. We choose $F_1 = \phi^* \colon \mathcal{D}(k) \to \mathcal{D}(k[t_2])$ with ϕ the morphism of dg-algebras $k[t_2] \xrightarrow{t_2 \mapsto 0} k$, see Section 5.2 for more details. The cotwist is then equivalent to $T_{\mathcal{N}} \simeq [3]$, see Proposition 5.12. For this particular choice of F_1, we may thus choose S_1 arbitrarily without affecting the monodromy of $\mathcal{F}$. The choices $S_1 = \mathrm{id}_{\mathcal{N}}$ and S_1 the pullback along $k[t_2] \xrightarrow{t_2 \mapsto -t_2} k[t_2]$ clearly give rise to two non-equivalent perverse schobers.

4 Relative Calabi–Yau structures

In this section, we extend the notion of a relative Calabi–Yau structure defined for dg-categories in [BD19] to R-linear ∞-categories, where R is any $\mathbb{E}_\infty$-ring spectrum. A motivating example is the 1-periodic topological Fukaya category of a marked surface, which carries a relative weak right 2-Calabi–Yau structure as a smooth and proper $k[t_2^{\pm}]$-linear ∞-category. Here, $k[t_2^{\pm}]$ denotes the graded Laurent algebra, so that $\mathcal{D}(k[t_2^{\pm}])$ is the derived ∞-category of 2-periodic chain complexes. We further discuss the construction of relative Calabi–Yau structures on the ∞-categories of global sections of perverse schobers.

We begin in Section 4.1 by recalling the notions of duality for bimodules, basics about smooth and proper R-linear ∞-categories, and R-linear Hochschild homology. Section 4.2 proceeds with defining weak left and right R-linear relative Calabi–Yau structures and an extension of the gluing results for relative Calabi–Yau structure of [BD19] to the R-linear setting.

In Section 4.3, we give construct relative Calabi–Yau structures on the sections of perverse schobers, both locally and globally. A main result is a construction of a relative Calabi–Yau structure on the ∞-category of global sections of a locally constant perverse schober with trivial monodromy and a Calabi–Yau generic stalk.

In Section 4.4, we show that relative Calabi–Yau structures give rise to Frobenius exact ∞-categories, whose extriangulated homotopy 1-categories are 2-Calabi–Yau.

4.1 Bimodules and Hochschild homology

The goal of this section, is to review background material on the duality of bimodules, smooth and proper ∞-categories and Hochschild homology over an arbitrary base $\mathbb{E}_\infty$-ring spectrum R. Most of this material appears in a similar form in [Lur17, HSS17, Lur18, BD19, BD21].

4.1.1 Bimodules

We fix a base $\mathbb{E}_\infty$-ring spectrum R. Suppose we are given two R-linear ring spectra A, A'. The ∞-category of A-A'-bimodules ${}_A\mathrm{BMod}_{A'}$ is equivalent to the ∞-category $\mathrm{Lin}_R(\mathrm{RMod}_A, \mathrm{RMod}_{A'})$ of R-linear functors between the respective module ∞-categories, see [Lur17, 4.8.4.1, 4.3.2.7]. In terms of functors, left and right duals of bimodules (if they exist) correspond to left and right adjoints of the corresponding functors.

Let $\mathcal{C}$ be a compactly generated R-linear ∞-category. Recall that we denote by $\mathcal{C}^\vee = \mathrm{Ind}(\mathcal{C}^{c,\mathrm{op}})$ its dual in the symmetric monoidal ∞-category LinCat_R, see Section 2.1.2. We have evaluation and coevaluation functors $\mathrm{ev}_\mathcal{C}\colon \mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{RMod}_R$, $\mathrm{coev}_\mathcal{C}\colon \mathrm{RMod}_R \to \mathcal{C} \otimes \mathcal{C}^\vee$, see again Section 2.1.2.

Remark 4.1. Given a compact objects preserving R-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ between compactly generated R-linear ∞-categories, we denote by $F^\vee \coloneqq \mathrm{Ind}(f^{\mathrm{op}})$ the functor on Ind-completions arising from the opposite functor of its restriction to compact objects $f\colon \mathcal{C}^{c} \to \mathcal{D}^{c}$. We note that if F admits a left adjoint $\mathrm{ladj}(F)$, then the right adjoint of $F^\vee$ is given by $\mathrm{radj}(F^\vee) \simeq \mathrm{ladj}(F)^\vee$.

We are especially interested in the adjoints of functors $\mathcal{C} \otimes \mathcal{C}^\vee \to \mathrm{RMod}_R$ or $\mathrm{RMod}_R \to \mathcal{C} \otimes \mathcal{C}^\vee$. This corresponds as a special case to studying modules over the enveloping algebra $A^e = A \otimes_R A^{\mathrm{rev}}$ of some R-linear ring spectrum A. We have the following equivalences.

Lemma 4.2.

(1) The R-linear functor $\Phi_{\mathcal{C}}$, defined as

$$\mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R) \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes (\text{-})} \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C}, \mathcal{C}) \xrightarrow{(\text{-})\circ(\mathrm{coev}_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}})} \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})\,,$$

is an equivalence with inverse $\Phi_{\mathcal{C}}^{-1}$ given by

$$\mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \xrightarrow{\mathrm{id}_{\mathcal{C}^\vee} \otimes (\text{-})} \mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathcal{C}^\vee \otimes \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathcal{C}} \circ (\text{-})} \mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R)\,.$$

(2) The R-linear functor $\Psi_{\mathcal{C}}$, defined as

$$\mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C} \otimes \mathcal{C}^\vee) \xrightarrow{(\text{-})\otimes \mathrm{id}_{\mathcal{C}}} \mathrm{Lin}_R(\mathcal{C}, \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C}) \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathcal{C}} \circ (\text{-})} \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})\,,$$

is an equivalence with inverse $\Psi_{\mathcal{C}}^{-1}$ given by

$$\mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \xrightarrow{(\text{-})\otimes \mathrm{id}_{\mathcal{C}^\vee}} \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathcal{C} \otimes \mathcal{C}^\vee) \xrightarrow{(\text{-})\circ \mathrm{coev}_{\mathcal{C}}} \mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C} \otimes \mathcal{C}^\vee)\,.$$

Proof. We begin by proving part (1). The equivalence of ∞-categories

$$\mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \xrightarrow{\mathrm{Lin}_R(\mathcal{C}^\vee, \mathcal{Y})} \mathrm{Lin}_R(\mathcal{C}, \mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R)) \simeq \mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R)\,,$$

where $\mathcal{Y}$ denotes the R-linear Yoneda embedding, maps an endofunctor $X\colon \mathcal{C} \to \mathcal{C}$ to $\mathrm{ev}_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}^\vee} \otimes X)$. This shows that $\Phi_{\mathcal{C}}^{-1}$ is essentially surjective. The triangle identity for $\mathrm{ev}_{\mathcal{C}}$ and $\mathrm{coev}_{\mathcal{C}}$ implies that $\Phi_{\mathcal{C}} \circ \Phi_{\mathcal{C}}^{-1} \simeq \mathrm{id}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}$. It follows that $\Phi_{\mathcal{C}}^{-1}$ is faithful, and in fact a split inclusion on Hom spaces. Using that all objects $Y \in \mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R)$ are of the form $Y \simeq \mathrm{ev}_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}^\vee} \otimes X)$, we find $\Phi_{\mathcal{C}}^{-1} \circ \Phi_{\mathcal{C}}(Y) \simeq Y$. Using that $\Phi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}^{-1}$ are exact, we find that $\Phi_{\mathcal{C}}^{-1}$ is also full, showing that $\Phi_{\mathcal{C}}^{-1}$ is an equivalence. Since $\Phi_{\mathcal{C}} \circ \Phi_{\mathcal{C}}^{-1} \simeq \mathrm{id}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}$, the inverse of $\Phi_{\mathcal{C}}^{-1}$ is given by $\Phi_{\mathcal{C}}$.

For part (2), a similar argument as above applies, using that the equivalence of ∞-categories

$$\mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \simeq \mathrm{Lin}_R(\mathrm{RMod}_R, \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})) \simeq \mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C} \otimes \mathcal{C}^\vee)$$

maps a functor $X\colon \mathcal{C} \to \mathcal{C}$ to $(X \otimes \mathrm{id}_{\mathcal{C}^\vee}) \circ \mathrm{coev}_{\mathcal{C}}$. ☐

Notation 4.3. We denote by τ the R-linear equivalence $\mathcal{C} \otimes \mathcal{C}^\vee \simeq \mathcal{C}^\vee \otimes \mathcal{C}$ which permutes the factors.

We can use the equivalences $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ to define the dual of an R-linear endofunctor $\mathcal{C} \to \mathcal{C}$, considered as a functor $\mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{RMod}_R$ or $\mathrm{RMod}_R \to \mathcal{C} \otimes \mathcal{C}^\vee$.

Definition 4.4. Let $X \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ be an R-linear endofunctor.

1. We call X left dualizable if $\Phi_{\mathcal{C}}^{-1}(X)$ admits a left adjoint. In this case, we call
$$X^! := \Psi_{\mathcal{C}}(\tau \circ \mathrm{ladj}(\Phi_{\mathcal{C}}^{-1}(X))) \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$$
the left dual of X.

2. We call X right dualizable if $\Phi_{\mathcal{C}}^{-1}(X)$ admits a right adjoint. In this case, we call
$$X^* := \Psi_{\mathcal{C}}(\tau \circ \mathrm{radj}(\Phi_{\mathcal{C}}^{-1}(X))) \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$$
the right dual of X.

Lemma 4.5. *Let $X \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$.*

(1) If X is left dualizable, then
$$X^! \simeq \Phi_{\mathcal{C}}(\tau \circ \mathrm{radj}(\Psi_{\mathcal{C}}^{-1}(X)))\,.$$

(2) If X is right dualizable, then
$$X^* \simeq \Phi_{\mathcal{C}}(\tau \circ \mathrm{ladj}(\Psi_{\mathcal{C}}^{-1}(X)))\,.$$

Proof. Part (1) follows from Lemma 4.6 below and the observation that
$$\mathrm{Lin}_R(\Phi_{\mathcal{C}}^{-1}(X), \mathrm{RMod}_R) \dashv \mathrm{Lin}_R(\mathrm{ladj}(\Phi_{\mathcal{C}}^{-1}(X)), \mathrm{RMod}_R)\,.$$
Part (2) follows from a similar argument. □

Lemma 4.6. *Let $X \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$. There is a commutative diagram,*

$$\begin{tikzcd}
\mathrm{RMod}_R \arrow[r, "\Psi_{\mathcal{C}}^{-1}(X)"] \arrow[d, "よ"'] & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[r, "\Phi_{\mathcal{C}^\vee}^{-1}(X)"] \arrow[d, "よ"'] & \mathrm{RMod}_R \arrow[d, "よ"] \\
\mathrm{Lin}_R(\mathrm{RMod}_R, \mathrm{RMod}_R) \arrow[r, "\mathrm{Lin}_R(\Phi_{\mathcal{C}}^{-1}(X), \mathrm{RMod}_R)"'] & \mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R) \arrow[r, "\mathrm{Lin}_R(\Psi_{\mathcal{C}}^{-1}(X), \mathrm{RMod}_R)"'] & \mathrm{Lin}_R(\mathrm{RMod}_R, \mathrm{RMod}_R)
\end{tikzcd}$$

with よ the R-linear Yoneda embedding, see Definition 2.6.

Proof. The evaluation bimodule
$$\mathrm{ev}_{\mathcal{C} \otimes \mathcal{C}^\vee} \colon \mathcal{C}^\vee \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}^\vee \simeq (\mathcal{C} \otimes \mathcal{C}^\vee)^\vee \otimes \mathcal{C} \otimes \mathcal{C}^\vee \longrightarrow \mathrm{RMod}_R$$
is, after reordering the factors of the tensor product, given by the tensor product of the evaluation bimodules of $\mathcal{C}$ and $\mathcal{C}^\vee$. Note that these two evaluation bimodules

are themselves equivalent, up to composition with $\tau\colon \mathcal{C}\otimes\mathcal{C}^\vee \simeq \mathcal{C}^\vee\otimes\mathcal{C}$. The Yoneda embedding

$$\mathcal{Y}\colon \operatorname{Lin}_R(\operatorname{RMod}_R, \mathcal{C}\otimes\mathcal{C}^\vee) \simeq \mathcal{C}\otimes\mathcal{C}^\vee \to \operatorname{Lin}_R(\mathcal{C}^\vee\otimes\mathcal{C}, \operatorname{RMod}_R)$$

is thus given by the functor

$$(\mathrm{ev}_\mathcal{C}\otimes\mathrm{ev}_\mathcal{C})\circ(\mathrm{id}_{\mathcal{C}^\vee}\otimes(\text{-})\otimes\mathrm{id}_\mathcal{C})\ . \tag{44}$$

Using this, the commutativity follows from the triangle identities for the evaluation and coevaluation bimodules, as well as the equivalences

$$\begin{aligned}\Psi_\mathcal{C}^{-1}(X) &\simeq (X\otimes\mathrm{id}_{\mathcal{C}^\vee})\circ\mathrm{coev}_\mathcal{C}\,,\\ \Phi_{\mathcal{C}^\vee}^{-1}(X) &\simeq \mathrm{ev}_{\mathcal{C}^\vee}\circ(X\otimes\mathrm{id}_{\mathcal{C}^\vee})\,,\\ \operatorname{Lin}_R(\Phi_\mathcal{C}^{-1}(X),\operatorname{RMod}_R) &\simeq (\text{-})\circ\mathrm{ev}_{\mathcal{C}^\vee}\circ(X\otimes\mathrm{id}_{\mathcal{C}^\vee})\,,\\ \operatorname{Lin}_R(\Psi_\mathcal{C}^{-1}(X),\operatorname{RMod}_R) &\simeq (\text{-})\circ(X\otimes\mathrm{id}_{\mathcal{C}^\vee})\circ\mathrm{coev}_\mathcal{C}\ .\end{aligned}$$

□

Remark 4.7. We denote by $\operatorname{Lin}_R^{\mathrm{ld}}(\mathcal{C},\mathcal{C})\subset\operatorname{Lin}_R(\mathcal{C},\mathcal{C})$ the stable subcategories of left dualizable functors. We similarly denote by $\operatorname{Lin}_R^{\mathrm{rd}}((\mathcal{C},\mathcal{C})\subset\operatorname{Lin}_R(\mathcal{C},\mathcal{C})$ the stable subcategory of right dualizable functors. Since passing to adjoints can be made functorial, see [Lur09, 5.2.6.2], we have exact functors

$$(\text{-})^!\colon \operatorname{Lin}_R^{\mathrm{ld}}(\mathcal{C},\mathcal{C})\to\operatorname{Lin}_R(\mathcal{C},\mathcal{C})^{\mathrm{op}}$$

and

$$(\text{-})^*\colon \operatorname{Lin}_R^{\mathrm{rd}}(\mathcal{C},\mathcal{C})\to\operatorname{Lin}_R(\mathcal{C},\mathcal{C})^{\mathrm{op}}\,.$$

Concretely, we find that $\operatorname{Lin}_R^{\mathrm{ld}}(\mathcal{C},\mathcal{C})=\operatorname{Lin}_R(\mathcal{C},\mathcal{C})^{\mathrm{c}}$ is given by the subcategory of compact objects. This follows from the adjoint functor theorem and the observation that a compact object in

$$\mathcal{C}\otimes\mathcal{C}^\vee \overset{\mathcal{Y}}{\simeq} \operatorname{Lin}_R(\mathcal{C}^\vee\otimes\mathcal{C},\operatorname{RMod}_R)\simeq\operatorname{Lin}_R(\mathcal{C},\mathcal{C})$$

gives via the Yoneda embedding rise to an exact functor $\mathcal{C}^\vee\otimes\mathcal{C}\to\operatorname{RMod}_R$ which also preserves filtered limits, and hence all limits. An endofunctor is right dualizable if and only if its image under $\Phi_\mathcal{C}^{-1}$ in $\operatorname{Lin}_R(\mathcal{C}^\vee\otimes\mathcal{C},\operatorname{RMod}_R)$ preserves compact objects, since in this case the right adjoint preserves colimits and is thus R-linear.

For the following, we fix a compact objects preserving R-linear functor $F\colon\mathcal{C}\to\mathcal{D}$ between compactly generated, R-linear ∞-categories. We denote the R-linear right adjoint of F by $G\colon\mathcal{D}\to\mathcal{C}$.

Lemma 4.8. *(1) There exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C} \otimes \mathcal{C}^\vee) \arrow[r, "\Psi_{\mathcal{C}}"] \arrow[d, "(F\otimes F^\vee)\circ(\text{-})"'] & \mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \arrow[d, "F\circ(\text{-})\circ G"] \\
\mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{D} \otimes \mathcal{D}^\vee) \arrow[r, "\Psi_{\mathcal{D}}"] & \mathrm{Lin}_R(\mathcal{D}, \mathcal{D})
\end{tikzcd}$$

In particular, it follows that the R-linear functor

$$F_! \coloneqq F \circ (\text{-}) \circ G\colon \ \mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \to \mathrm{Lin}_R(\mathcal{D}, \mathcal{D})$$

preserves compact objects.

(2) There exists a commutative diagram

$$\begin{tikzcd}
\mathrm{Lin}_R(\mathcal{D}^\vee \otimes \mathcal{D}, \mathrm{RMod}_R) \arrow[r, "\Phi_{\mathcal{D}}"] \arrow[d, "(\text{-})\circ(F^\vee\otimes F)"'] & \mathrm{Lin}_R(\mathcal{D}, \mathcal{D}) \arrow[d, "G\circ(\text{-})\circ F"] \\
\mathrm{Lin}_R(\mathcal{C}^\vee \otimes \mathcal{C}, \mathrm{RMod}_R) \arrow[r, "\Phi_{\mathcal{C}}"] & \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})
\end{tikzcd}$$

In particular, it follows that the R-linear functor

$$F^* \coloneqq G \circ (\text{-}) \circ F\colon \ \mathrm{Lin}_R(\mathcal{D}, \mathcal{D}) \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$$

preserves compact objects.

Proof. We only prove part (1), part (2) is analogous. The adjunction $F \dashv G$ implies that

$$\mathrm{ev}_{\mathcal{D}} \circ (F^\vee \otimes \mathrm{id}_{\mathcal{D}}) \simeq \mathrm{ev}_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}^\vee} \otimes G)\,.$$

It follows that

$$\begin{aligned}
\Psi_{\mathcal{D}}((F \otimes F^\vee) \circ \alpha) &\simeq (\mathrm{id}_{\mathcal{D}} \otimes \mathrm{ev}_{\mathcal{D}}) \circ (F \otimes F^\vee \otimes \mathrm{id}_{\mathcal{D}}) \circ (\alpha \otimes \mathrm{id}_{\mathcal{D}}) \\
&\simeq (\mathrm{id}_{\mathcal{D}} \otimes \mathrm{ev}_{\mathcal{C}}) \circ (F \otimes \mathrm{id}_{\mathcal{D}^\vee} \otimes G) \circ (\alpha \otimes \mathrm{id}_{\mathcal{D}}) \\
&\simeq F \circ \Psi_{\mathcal{C}}(\alpha) \circ G\,,
\end{aligned}$$

functorial in $\alpha\colon \ \mathrm{RMod}_R \to \mathcal{C} \otimes \mathcal{C}^\vee$. □

4.1.2 Smooth and proper linear ∞-categories

We fix an $\mathbb{E}_\infty$-ring spectrum R and an R-linear ∞-category $\mathcal{C}$.

Definition 4.9.

(1) The ∞-category $\mathcal{C}$ is called smooth if it is compactly generated and $\mathrm{id}_{\mathcal{C}} \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ is left dualizable. In this case, the bimodule left dual $\mathrm{id}_{\mathcal{C}}^!$ is also called the inverse dualizing bimodule.

(2) The ∞-category $\mathcal{C}$ is called proper if it is compactly generated and for any two compact objects $x, y \in \mathcal{C}^c$ the R-linear morphism object $\mathrm{Mor}_{\mathcal{C}}(x, y) \in \mathrm{RMod}_R$ is compact. Since $\mathrm{Mor}_{\mathcal{C}}(x, y) \simeq \Phi_{\mathcal{C}}^{-1}(\mathrm{id}_{\mathcal{C}})(x \otimes y)$, $\mathcal{C}$ being proper equivalent to the functor $\mathrm{id}_{\mathcal{C}}$ being right dualizable.

In the following, we suppose that $\mathcal{C}$ is compactly generated. We denote $(\text{-})^* = \mathrm{Mor}_{\mathrm{RMod}_R}(\text{-}, R)\colon (\mathrm{RMod}_R^{\mathrm{c}})^{\mathrm{op}} \to \mathrm{RMod}_R^{\mathrm{c}}$

Definition 4.10. Suppose that $\mathcal{C}$ is proper. We call an endofunctor $U \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ a Serre functor of $\mathcal{C}$ if there exists a natural equivalence

$$\mathrm{Mor}_{\mathcal{C}}(\text{-}_1, \text{-}_2) \simeq \mathrm{Mor}_{\mathcal{C}}(\text{-}_2, U(\text{-}_1))^*\colon \mathcal{C}^{\mathrm{c,op}} \times \mathcal{C}^{\mathrm{c}} \to \mathrm{RMod}_R^{\mathrm{c}}\,.$$

Lemma 4.11. *Let U, U' be two Serre functors of $\mathcal{C}$. Then $U \simeq U' \in \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$.*

Proof. Since U and U' are both Serre functors, there exist natural equivalences

$$\mathrm{Mor}_{\mathcal{C}}(\text{-}_1, U(\text{-}_2)) \simeq \mathrm{Mor}_{\mathcal{C}}(\text{-}_2, \text{-}_1)^* \simeq \mathrm{Mor}_{\mathcal{C}}(\text{-}_1, U'(\text{-}_2))\,.$$

Applying $\mathrm{Map}_{\mathrm{RMod}_R}(R, \text{-})$ to this equivalence yields

$$\mathrm{Map}_{\mathcal{C}}(\text{-}_1, U(\text{-}_2)) \simeq \mathrm{Map}_{\mathcal{C}}(\text{-}_1, U'(\text{-}_2))\,.$$

It follows that $U \simeq U'$ on compact objects by (a corollary of) the Yoneda lemma, see for instance [Cis19, Cor. 5.8.14]. Passing to Ind-completions shows $U \simeq U'$. $\square$

Lemma 4.12.

(1) Suppose that $\mathcal{C}$ is proper. The right dual $\mathrm{id}_{\mathcal{C}}^*$ is a Serre functor of $\mathcal{C}$.

(2) Suppose that $\mathcal{C}$ is smooth and proper. The functors $\mathrm{id}_{\mathcal{C}}^*$ and $\mathrm{id}_{\mathcal{C}}^!$ are inverse equivalences.

Remark 4.13. Part (1) of Lemma 4.12 is stated without proof in [Lur18, 11.1.5.2].

Proof of Lemma 4.12. We begin by proving part (1). We denote by

$$\widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, \text{-})\colon \mathcal{C}^{\vee} \times \mathcal{C} \to \mathrm{RMod}_R$$

the functor obtained by passing to Ind-completions from the restriction of $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \text{-})$ to $\mathcal{C}^{\mathrm{c,op}} \times \mathcal{C}^{\mathrm{c}}$. Let $X \in \mathcal{C}^{\mathrm{c}}$ and consider the adjunction

$$(\text{-}) \otimes \widehat{\mathrm{Mor}}_{\mathcal{C}}(X, \text{-})\colon\ \mathrm{RMod}_R \longleftrightarrow \mathrm{Lin}_R(\mathcal{C}, \mathrm{RMod}_R)\ \colon \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C}, \mathrm{RMod}_R)}(\widehat{\mathrm{Mor}}_{\mathcal{C}}(X, \text{-}), \text{-})\,.$$

Using that $\widehat{\mathrm{Mor}}_{\mathcal{C}}(X, \text{-}) \simeq \widehat{\mathrm{Mor}}_{\mathcal{C}^{\vee}}(\text{-}, X)$ and the fully faithfulness of the R-linear Yoneda embedding of $\mathcal{C}^{\vee}$, we find that the right adjoint is given by evaluation at X, i.e.

$$\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C}, \mathrm{RMod}_R)}(\widehat{\mathrm{Mor}}_{\mathcal{C}}(X, \text{-}), \text{-}) \simeq \mathrm{ev}_X\,.$$

Using the identification

$$\mathrm{Lin}_R(\mathcal{C}, \mathrm{RMod}_R) \otimes \mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R) \simeq \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R)$$

we define the functor

$$\mathrm{ev}'_X \colon \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R) \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{\mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R)}} \mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R)$$

with left adjoint

$$\mathrm{ladj}(\mathrm{ev}'_X) \colon \mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R) \xrightarrow{\left((\text{-}) \otimes \widehat{\mathrm{Mor}}_{\mathcal{C}}(X, \text{-})\right) \otimes \mathrm{id}} \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R)\,.$$

Informally, the left adjoint $\mathrm{ladj}(\mathrm{ev}'_X)$ is given by

$$(c' \mapsto F(c')) \mapsto \Big((c \otimes c') \mapsto \widehat{\mathrm{Mor}}_{\mathcal{C}}(X, c) \otimes F(c')\Big)\,.$$

The right adjoint of the evaluation bimodule $\mathrm{ev}_{\mathcal{C}} \colon \mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{RMod}_R$ is equivalent to $(\mathrm{id}_{\mathcal{C}^\vee} \otimes \mathrm{id}^*_{\mathcal{C}}) \circ \mathrm{coev}_{\mathcal{C}^\vee}$. Using the description the the Yoneda embedding in (44), it follows that the right adjoint of the functor

$$\widetilde{\mathrm{ev}} \colon \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R) \simeq \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\mathrm{ev}_{\mathcal{C}}} \mathrm{RMod}_R$$

is equivalent to $(\text{-}) \otimes \mathrm{ev}_{\mathcal{C}^\vee} \circ (\mathrm{id}^*_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}^\vee})$.

In total, we obtain that the right adjoint of

$$\mathrm{Lin}_R(\mathcal{C}^\vee, \mathrm{RMod}_R) \xrightarrow{\mathrm{ladj}(\mathrm{ev}'_X)} \mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R) \xrightarrow{\widetilde{\mathrm{ev}}} \mathrm{RMod}_R$$

is given by $(\text{-}) \otimes \widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, \mathrm{id}^*_{\mathcal{C}}(X))$. To make the notation more transparent, we denote $\mathrm{Mor}'_{\mathcal{C}}(\text{-},\text{-}) = \mathrm{ev}_{\mathcal{C}} \colon (\mathcal{C}^\vee \otimes \mathcal{C})^{\mathrm{c}} \to \mathrm{RMod}_R$ and $\mathrm{Mor}'_{\mathcal{C}^\vee}(\text{-},\text{-}) = \mathrm{ev}_{\mathcal{C}^\vee} \colon (\mathcal{C} \otimes \mathcal{C}^\vee)^{\mathrm{c}} \to \mathrm{RMod}_R$ in the following. Using the above adjunctions, the fully faithfulness of the R-linear Yoenda embedding and Lemma 2.7, we find the following equivalences in RMod_R, functorial in $X \otimes Y \in (\mathcal{C} \otimes \mathcal{C}^\vee)^{\mathrm{c}}$,

$$\begin{aligned}
\mathrm{Mor}'_{\mathcal{C}}(X, Y)^* &= \mathrm{Mor}_{\mathrm{RMod}_R}(\mathrm{Mor}'_{\mathcal{C}}(X, Y), R)\\
&\simeq \mathrm{Mor}_{\mathrm{RMod}_R}(\widetilde{\mathrm{ev}} \circ \mathrm{ladj}(\mathrm{ev}'_X)(\widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, Y)), R)\\
&\simeq \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C} \otimes \mathcal{C}^\vee, \mathrm{RMod}_R)}(\mathrm{ladj}(\mathrm{ev}'_X)(\widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, Y)), \mathrm{Mor}'_{\mathcal{C}}(\text{-}, \mathrm{id}^*_{\mathcal{C}}(\text{-})))\\
&\simeq \mathrm{Mor}_{\mathrm{Lin}(\mathcal{C}^\vee, \mathrm{RMod}_R)}(\widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, Y), \widehat{\mathrm{Mor}}_{\mathcal{C}}(\text{-}, \mathrm{id}^*_{\mathcal{C}}(X)))\\
&\simeq \mathrm{Mor}'_{\mathcal{C}}(Y, \mathrm{id}^*_{\mathcal{C}}(X))\,.
\end{aligned}$$

Restricting the above equivalence of functors along $\mathcal{C}^{\mathrm{c}} \times \mathcal{C}^{\mathrm{c,op}} \to (\mathcal{C} \otimes \mathcal{C}^\vee)^{\mathrm{c}}$ shows that that $\mathrm{id}^*_{\mathcal{C}}$ is indeed a Serre functor.

We proceed with proving part (2). We have

$$\mathrm{id}_{\mathcal{C}} \simeq (\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathcal{C}}) \circ (\mathrm{coev}_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}}) \tag{45}$$

and passing to the right adjoint yields

$$\mathrm{id}_{\mathcal{C}} \simeq (\mathrm{radj}(\mathrm{coev}_{\mathcal{C}}) \otimes \mathrm{id}_{\mathcal{C}}) \circ (\mathrm{id}_{\mathcal{C}} \otimes \mathrm{radj}(\mathrm{ev}_{\mathcal{C}}))\,.$$

We have

$$\mathrm{radj}(\mathrm{ev}_{\mathcal{C}}) \simeq (\mathrm{id}_{\mathcal{C}^\vee} \otimes \mathrm{id}_{\mathcal{C}}^*) \circ \mathrm{coev}_{\mathcal{C}^\vee}$$

and by Lemma 4.5 further

$$\mathrm{radj}(\mathrm{coev}_{\mathcal{C}}) \simeq \mathrm{ev}_{\mathcal{C}^\vee} \circ (\mathrm{id}_{\mathcal{C}}^! \otimes \mathrm{id}_{\mathcal{C}^\vee})\,.$$

Combining the above equivalences yields $\mathrm{id}_{\mathcal{C}}^* \circ \mathrm{id}_{\mathcal{C}}^! \simeq \mathrm{id}_{\mathcal{C}}$. The identity $\mathrm{id}_{\mathcal{C}}^! \circ \mathrm{id}_{\mathcal{C}}^* \simeq \mathrm{id}_{\mathcal{C}}$ arises from a similar argument by passing to the left adjoint of (45). □

Definition 4.14. Given a compactly generated R-linear ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{fin}} \subset \mathcal{C}$ the full subcategory of objects Y, satisfying that $\mathrm{Mor}_{\mathcal{C}}(X,Y) \in \mathrm{RMod}_R^{\mathrm{c}}$ is compact for all $X \in \mathcal{C}^{\mathrm{c}}$.

The following lemma provides the analog of part (1) of Lemma 4.12 for smooth, but not necessarily proper R-linear ∞-categories.

Lemma 4.15. *Let $\mathcal{C}$ be a smooth R-linear ∞-category. Then*

$$\mathrm{Mor}_{\mathcal{C}}(X,Y)^* \simeq \mathrm{Mor}_{\mathcal{C}}(\mathrm{id}^!(Y),X)\,,$$

functorial in $X \in \mathcal{C}^{\mathrm{c}}$ and $Y \in (\mathcal{C}^{\mathrm{fin}})^{\mathrm{op}}$.

Proof of Lemma 4.15. The exact inclusion $\mathcal{C}^{\mathrm{fin}} \subset \mathcal{C}$ gives rise to an R-linear functor $\mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \to \mathcal{C}$. The R-linear functor

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{fin}} \colon \mathcal{C}^\vee \otimes \mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \longrightarrow \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\mathrm{ev}_{\mathcal{C}}} \mathrm{RMod}_R$$

preserves compact objects by the definition of $\mathcal{C}^{\mathrm{fin}}$ and thus admits an R-linear right adjoint $\mathrm{radj}(\mathrm{ev}_{\mathcal{C}}^{\mathrm{fin}})$. We define the R-linear functor $U\colon \mathcal{C} \to \mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}}$ as the composite

$$\mathcal{C} \xrightarrow{\mathcal{C}\otimes \mathrm{radj}(\mathrm{ev}_{\mathcal{C}}^{\mathrm{fin}})} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \xrightarrow{\mathrm{ev}_{\mathcal{C}} \otimes \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}}}} \mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}}\,.$$

The functor U admits a left adjoint, given by the composite

$$\mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \xrightarrow{\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})\otimes \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}}}} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathcal{C}}^{\mathrm{fin}}} \mathcal{C}\,,$$

which describes the restriction of $\mathrm{id}_{\mathcal{C}}^!$ along the R-linear functor $\mathrm{Ind}\,\mathcal{C}^{\mathrm{fin}} \to \mathcal{C}$.

A very similar proof as for part (1) of Lemma 4.12 shows that

$$\mathrm{Mor}_{\mathcal{C}}(X,Y)^* \simeq \mathrm{Mor}_{\mathcal{C}}(Y,U(X))\,, \tag{46}$$

functorial in $Y \in \mathcal{C}^{\mathrm{fin,op}}$ and $X \in \mathcal{C}^{\mathrm{c}}$. By the above adjunction, we have

$$\mathrm{Mor}_{\mathcal{C}}(Y,U(X)) \simeq \mathrm{Mor}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}^!(Y),X)),$$

which combined with (46) yields the desired equivalence. □

4.1.3 Hochschild homology

Let R be an $\mathbb{E}_\infty$-ring spectrum and $\mathcal{C}$ a compactly generated R-linear ∞-category. The R-linear Hochschild homology of $\mathcal{C}$ is defined as the trace

$$\mathrm{HH}(\mathcal{C}) := \mathrm{ev}_{\mathcal{C}} \circ \tau \circ \mathrm{coev}_{\mathcal{C}}(R) \in \mathrm{RMod}_R \ .$$

When R is the sphere spectrum, $\mathrm{HH}(\mathcal{C})$ describes topological Hochschild homology. When $R = k$ is a field, $\mathrm{HH}(\mathcal{C})$ describes the usual k-linear Hochschild homology.

Lemma 4.16. *Let $\mathcal{C}$ be a compactly generated R-linear ∞-category.*

(1) If $\mathcal{C}$ is smooth, then $\mathrm{HH}(\mathcal{C})$ is canonically equivalent to

$$\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}^!_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \,.$$

(2) If $\mathcal{C}$ is proper, then $\mathrm{HH}(\mathcal{C})^* = \mathrm{Mor}_{\mathrm{RMod}_R}(\mathrm{HH}(\mathcal{C}), R)$ is canonically equivalent to

$$\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}^*_{\mathcal{C}}) \,.$$

Proof. Suppose that $\mathcal{C}$ is smooth. Then we have an adjunction

$$\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}) \circ (\text{-})\colon\ \mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C}^\vee \otimes \mathcal{C}) \longleftrightarrow \mathrm{Lin}_R(\mathrm{RMod}_R, \mathrm{RMod}_R) : \mathrm{ev}_{\mathcal{C}} \circ (\text{-}) \,,$$

whose unit is given by precomposition with the unit of $\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}) \dashv \mathrm{ev}_{\mathcal{C}}$. It follows that

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}^!_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) &\simeq \mathrm{Mor}_{\mathrm{Lin}_R(\mathrm{RMod}_R, \mathcal{C}\otimes\mathcal{C}^\vee)}(\tau \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}), \mathrm{coev}_{\mathcal{C}}) \\ &\simeq \mathrm{Mor}_{\mathrm{RMod}_R}(R, \mathrm{ev}_{\mathcal{C}} \circ \tau \circ \mathrm{coev}_{\mathcal{C}}) \\ &\simeq \mathrm{HH}(\mathcal{C}) \,.\end{aligned}$$

If $\mathcal{C}$ is a proper, a similar argument applies. $\square$

Remark 4.17. Let $\mathcal{C}$ be smooth. If we make two different choices of left duals/adjoints

$$\mathrm{id}^!_{\mathcal{C}} = \Psi_{\mathcal{C}}(\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}))\,, \quad (\mathrm{id}^!_{\mathcal{C}})' = \Psi_{\mathcal{C}}(\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})')$$

and two choices of units, there is a contractible space of equivalences $\alpha\colon (\mathrm{id}^!_{\mathcal{C}})' \simeq \mathrm{id}^!_{\mathcal{C}}$, compatible with the unit, see [Cis19, Prop. 6.1.9]. Any such equivalence α assembles with the equivalences from Lemma 4.16 into to a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}^!_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \arrow[rr, "\mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\alpha{,}\mathrm{id}_{\mathcal{C}})"] \arrow[dr, "\simeq"'] & & \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}((\mathrm{id}^!_{\mathcal{C}})', \mathrm{id}_{\mathcal{C}}) \arrow[dl, "\simeq"] \\
& \mathrm{HH}(\mathcal{C}) &
\end{tikzcd}$$

Stated differently, this means that the equivalence in part (1) of Lemma 4.16 is independent of the choice of left dual. A similar statement holds for the equivalence in part (2).

One can use the formalism of traces to turn R-linear Hochschild homology into a functor $\mathrm{HH}(\text{-})\colon \mathrm{LinCat}_R^{\mathrm{cpt}} \to \mathrm{RMod}_R$ defined on the subcategory $\mathrm{LinCat}_R^{\mathrm{cpt}} \subset \mathrm{LinCat}_R$ of compactly generated ∞-categories and compact objects preserving functors, see [HSS17]. We adopt a more explicit approach to describe a weaker version of this functoriality, which is suited for the purposes of this work. We expect the approach of [HSS17] to restrict to our approach, but do not prove this here. The description of $\mathrm{HH}(\text{-})$ used here also appears for smooth dg-categories in [BD21, Prop. 4.4].

Construction 4.18. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an R-linear compact objects preserving functor between compactly generated R-linear ∞-categories and G the R-linear right adjoint of F.
Case 1: Suppose that $\mathcal{C}, \mathcal{D}$ are smooth.

We denote by

$$F_!(\text{-}) = F \circ (\text{-}) \circ G\colon \mathrm{Lin}_R(\mathcal{C},\mathcal{C}) \longrightarrow \mathrm{Lin}_R(\mathcal{D},\mathcal{D})$$

the functor from Lemma 4.8, and by $\mathrm{cu}\colon F_!(\mathrm{id}_{\mathcal{C}}) \to \mathrm{id}_{\mathcal{D}}$ the counit transformation of $F \dashv G$. We define the *unit* $\mathrm{u}\colon \mathrm{id}_{\mathcal{D}}^! \to F_!(\mathrm{id}_{\mathcal{C}}^!)$ as the image under $\Psi_{\mathcal{D}}$ of the natural transformation

$$\begin{aligned}\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) &\to \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \circ \mathrm{ev}_{\mathcal{C}} \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})\\ &\to \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \circ \mathrm{ev}_{\mathcal{D}} \circ (F \otimes F^{\vee}) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})\\ &\to (F \otimes F^{\vee}) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})\end{aligned}$$

composed with the equivalence

$$\Psi_{\mathcal{D}}((F \otimes F^{\vee}) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})) \simeq F_!(\mathrm{id}_{\mathcal{C}}^!)$$

from Lemma 4.8. The transformation u is indeed a unit if F admits a left adjoint, see Lemma 4.21.

We define the morphism $\mathrm{HH}(F)$ in RMod_R as

$$\mathrm{HH}(F) \coloneqq \mathrm{cu} \circ F_!(\text{-}) \circ \mathrm{u}\colon \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}^!, \mathrm{id}_{\mathcal{C}}) \longrightarrow \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{D},\mathcal{D})}(\mathrm{id}_{\mathcal{D}}^!, \mathrm{id}_{\mathcal{D}})\,.$$

Explicitly, $\mathrm{HH}(F)$ maps a degree m-morphism $\alpha\colon \mathrm{id}_{\mathcal{C}}^! \to \mathrm{id}_{\mathcal{C}}[m]$ to the morphism $\mathrm{id}_{\mathcal{D}}^! \to \mathrm{id}_{\mathcal{D}}[m]$ obtained as the following composite:

$$\begin{tikzcd} \mathrm{id}_{\mathcal{D}}^! \arrow[r, "\mathrm{u}"] & F_!(\mathrm{id}_{\mathcal{C}}^!) \arrow[d, "F_!(\alpha)"] & \\ & F_!(\mathrm{id}_{\mathcal{C}})[m] \arrow[r, "\mathrm{cu}[m]"] & \mathrm{id}_{\mathcal{D}}[m] \end{tikzcd}$$

Case 2: Suppose that $\mathcal{C}, \mathcal{D}$ are proper.

Consider the functor

$$F^*(\text{-}) = G \circ (\text{-}) \circ F\colon \mathrm{Lin}_R(\mathcal{D},\mathcal{D}) \longrightarrow \mathrm{Lin}_R(\mathcal{C},\mathcal{C})$$

from Lemma 4.8 and denote by $\mathrm{u}\colon \mathrm{id}_{\mathcal{C}} \to F^*(\mathrm{id}_{\mathcal{D}})$ the unit of $F \dashv G$. Let $E^\vee$ denote the right adjoint of $F^\vee$. Applying $\Psi_{\mathcal{D}^\vee}^{-1}$ to the counit $F^\vee E^\vee \to \mathrm{id}_{\mathcal{D}}$ defines a natural transformation

$$(F^\vee \otimes F) \circ \mathrm{coev}_{\mathcal{C}^\vee} \longrightarrow \mathrm{coev}_{\mathcal{D}^\vee}\ .$$

We use this to define the *counit* $\mathrm{cu}\colon F^*(\mathrm{id}_{\mathcal{D}}^*) \to \mathrm{id}_{\mathcal{C}}^*$ as the image under $\Phi_{\mathcal{C}}$ of the natural transformation

$$\begin{aligned}\mathrm{ladj}(\mathrm{coev}_{\mathcal{D}^\vee}) \circ (F^\vee \otimes F) &\to \mathrm{ladj}(\mathrm{coev}_{\mathcal{D}^\vee}) \circ (F^\vee \otimes F) \circ \mathrm{coev}_{\mathcal{C}^\vee} \circ \mathrm{ladj}(\mathrm{coev}_{\mathcal{C}^\vee}) \\ &\to \mathrm{ladj}(\mathrm{coev}_{\mathcal{D}^\vee}) \circ \mathrm{coev}_{\mathcal{D}^\vee} \circ \mathrm{ladj}(\mathrm{coev}_{\mathcal{C}^\vee}) \\ &\to \mathrm{ladj}(\mathrm{coev}_{\mathcal{C}^\vee})\end{aligned}$$

composed with the identification $\Phi_{\mathcal{C}}(\mathrm{ladj}(\mathrm{coev}_{\mathcal{D}^\vee}) \circ (F^\vee \otimes F)) \simeq F^*(\mathrm{id}_{\mathcal{D}}^*)$ from Lemma 4.8. We define the morphism $\mathrm{HH}^*(F)$ in RMod_R as

$$\mathrm{HH}^*(F) \coloneqq \mathrm{cu} \circ F^*(\text{-}) \circ \mathrm{u}\colon\ \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{D},\mathcal{D})}(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}^*) \longrightarrow \mathrm{Mor}_{\mathrm{Lin}_R(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}^*)\,.$$

The notation $\mathrm{HH}^*(F)$ emphasizes that $\mathrm{HH}^*(F)$ is not manifestly the dual of $\mathrm{HH}(F)$.

Lemma 4.19. *The assignments from Construction 4.18 define functors between homotopy* 1*-categories*

$$\mathrm{HH}(\text{-})\colon\ \mathrm{ho}\,\mathrm{LinCat}_R^{\mathrm{cpt,sm}} \longrightarrow \mathrm{ho}\,\mathrm{RMod}_R$$

and

$$\mathrm{HH}^*(\text{-})\colon\ \mathrm{ho}\left(\mathrm{LinCat}_R^{\mathrm{cpt,prop}}\right)^{\mathrm{op}} \longrightarrow \mathrm{ho}\,\mathrm{RMod}_R\ .$$

Here, $\mathrm{LinCat}_R^{\mathrm{cpt,sm}}, \mathrm{LinCat}_R^{\mathrm{cpt,prop}} \subset \mathrm{LinCat}_R^{\mathrm{cpt}}$ *denote the full subcategories consisting of smooth, respectively, proper R-linear ∞-categories.*

Proof. We only prove part (1), a similar proof applies to part (2). Consider compact objects preserving, R-linear functors $F_1\colon \mathcal{C} \to \mathcal{D}$ and $F_2\colon \mathcal{D} \to \mathcal{E}$ between smooth R-linear ∞-categories. By construction of $\mathrm{HH}(\text{-})$, it suffices to show that composing the units $\mathrm{u}_2\colon \mathrm{id}_{\mathcal{E}}^! \to (F_2)_! \mathrm{id}_{\mathcal{D}}^!$ and $(F_2)_! \circ \mathrm{u}_1\colon (F_2)_! \mathrm{id}_{\mathcal{D}}^! \to (F_2)_!(F_1)_! \mathrm{id}_{\mathcal{C}}^!$ yields the unit $\mathrm{u}\colon \mathrm{id}_{\mathcal{E}}^! \to (F_2F_1)_! \mathrm{id}_{\mathcal{C}}^! \simeq (F_2)_!(F_1)_! \mathrm{id}_{\mathcal{C}}^!$. This is straightforward to verify using the commutativity of the following diagram,

$$\begin{tikzcd}
\mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{C}} \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[rr] \arrow[d] & & \mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{D}} \circ (F_1^\vee \otimes F_1) \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[dd] \\
\mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{D}} \circ \mathrm{ev}_{\mathcal{D}}^L \circ \mathrm{ev}_{\mathcal{C}} \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[r] \arrow[d] & \mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{D}} \circ \mathrm{ev}_{\mathcal{D}}^L \circ \mathrm{ev}_{\mathcal{D}} \circ (F_1^\vee \otimes F_1) \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[ur] \arrow[d] & \\
\mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{E}} \circ (F_2^\vee \otimes F_2) \circ \mathrm{ev}_{\mathcal{D}}^L \circ \mathrm{ev}_{\mathcal{C}} \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[r] & \mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{E}} \circ (F_2^\vee \otimes F_2) \circ \mathrm{ev}_{\mathcal{D}}^L \circ \mathrm{ev}_{\mathcal{D}} \circ (F_1^\vee \otimes F_1) \circ \mathrm{ev}_{\mathcal{C}}^L \arrow[r] & \mathrm{ev}_{\mathcal{E}}^L \circ \mathrm{ev}_{\mathcal{C}} \circ (F_2^\vee F_1^\vee \otimes F_2 F_1) \circ \mathrm{ev}_{\mathcal{C}}^L
\end{tikzcd}$$

where we denote for brevity $\mathrm{ev}_{\mathcal{E}}^L = \mathrm{ladj}(\mathrm{ev}_{\mathcal{E}}), \mathrm{ev}_{\mathcal{D}}^L = \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}), \mathrm{ev}_{C}^L = \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})$. $\square$

Notation 4.20. Consider an R-linear, compact objects preserving functor $F\colon \mathcal{C} \to \mathcal{D}$.

(1) Suppose that $\mathcal{C}, \mathcal{D}$ are smooth. We denote $\mathrm{HH}(\mathcal{D}, \mathcal{C}) = \operatorname{cof} \mathrm{HH}(F)$.

(2) Suppose that $\mathcal{C}, \mathcal{D}$ are proper. We denote $\mathrm{HH}^*(\mathcal{D}, \mathcal{C}) = \operatorname{cof} \mathrm{HH}^*(F)$.

Lemma 4.21. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a compact objects preserving R-linear functor which admits a left adjoint E.*

(1) Suppose that $\mathcal{C}, \mathcal{D}$ are smooth. Then there exists a natural equivalence $F_!(\mathrm{id}^!_{\mathcal{C}}) \simeq FE\,\mathrm{id}^!_{\mathcal{D}}$, such that the composite of

$$\mathrm{u}\colon\ \mathrm{id}^!_{\mathcal{D}} \to F_!(\mathrm{id}^!_{\mathcal{C}})$$

with this equivalence describes a unit of $E \dashv F$ composed with $\mathrm{id}^!_{\mathcal{D}}$.

(2) Suppose that $\mathcal{C}, \mathcal{D}$ are proper. Then there exists a natural equivalence $F^*(\mathrm{id}^*_{\mathcal{D}}) \simeq \mathrm{id}^*_{\mathcal{C}}\,EF$, such that the composite of

$$\mathrm{cu}\colon\ F^*(\mathrm{id}^*_{\mathcal{D}}) \to \mathrm{id}^*_{C}$$

with this equivalence describes a counit of $E \dashv F$ composed with $\mathrm{id}^*_{\mathcal{C}}$.

Proof. We only prove part (1), part (2) is similar. The adjunction $E \dashv F$ implies that

$$\mathrm{ev}_{\mathcal{C}} \circ (E^\vee \otimes \mathrm{id}_{\mathcal{C}}) \simeq \mathrm{ev}_{\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}^\vee} \otimes F)\,.$$

The units of the adjunctions

$$(F^\vee \otimes \mathrm{id}_{\mathcal{C}}) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}) \dashv \mathrm{ev}_{\mathcal{C}} \circ (E^\vee \otimes \mathrm{id}_{\mathcal{C}})$$

and

$$(\mathrm{id}_{\mathcal{D}^\vee} \otimes E) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \dashv \mathrm{ev}_{\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}^\vee} \otimes F)$$

are hence equivalent. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{id}_{\mathrm{RMod}_R} \arrow[r] \arrow[d] & \mathrm{ev}_{\mathcal{C}} \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}) \arrow[r] & \mathrm{ev}_{\mathcal{C}} \circ (E^\vee F^\vee \otimes \mathrm{id}_{\mathcal{C}}) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}}) \arrow[d, "\simeq"] \\
\mathrm{ev}_{\mathcal{D}} \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \arrow[r] & \mathrm{ev}_{\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}^\vee} \otimes FE) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \arrow[r, "\simeq"] & \mathrm{ev}_{\mathcal{D}} \circ (F^\vee \otimes F) \circ \mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})
\end{tikzcd}$$

The naturality of the unit $\mathrm{id}_{\mathcal{D}^\vee} \otimes \mathrm{id}_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}^\vee} \otimes FE$ and the counit $\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\,\mathrm{ev}_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}^\vee} \otimes \mathrm{id}_{\mathcal{D}}$ gives rise to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\,\mathrm{ev}_{\mathcal{D}}\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \arrow[r] \arrow[d] & \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \arrow[d] \\
\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\,\mathrm{ev}_{\mathcal{D}}(\mathrm{id}_{\mathcal{D}^\vee} \otimes FE)\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \arrow[r] & (\mathrm{id}_{\mathcal{D}^\vee} \otimes FE)\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})
\end{tikzcd}$$

Combining these two diagrams, we find that the natural transformation $u\colon \mathrm{id}^!_{\mathcal{D}} \to F_!(\mathrm{id}^!_{\mathcal{C}})$ is equivalent to the image under $\Psi_{\mathcal{D}}$ of the natural transformation

$$\begin{aligned}\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) &\to \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\,\mathrm{ev}_{\mathcal{D}}\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\\ &\to \mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\\ &\to (\mathrm{id}_{\mathcal{D}^\vee}\otimes FE)\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}})\\ &\simeq (F^\vee\otimes F)\,\mathrm{ladj}(\mathrm{ev}_{\mathcal{C}})\,.\end{aligned}$$

The desired description of u follows via the triangle identity for the adjunction $\mathrm{ladj}(\mathrm{ev}_{\mathcal{D}}) \dashv \mathrm{ev}_{\mathcal{D}}$. □

4.2 Relative Calabi–Yau structures

The goal of this section is to introduce and study R-linear weak relative Calabi–Yau structures. We begin in Sections 4.2.1 and 4.2.2 with their definition. We then describe the behavior of Calabi–Yau structures under tensor products in Section 4.2.3 and generalize the gluing properties of relative Calabi–Yau structures of [BD19] to the R-linear setting in Section 4.2.4. Finally, we describe two basic examples of ∞-categories with relative Calabi–Yau structures in Section 4.2.5, which will later be used to construct relative Calabi–Yau structures on relative Ginzburg algebras and 1-periodic topological Fukaya categories.

4.2.1 Left Calabi–Yau structures

Let R be an $\mathbb{E}_\infty$-ring spectrum and let $F\colon \mathcal{C} \to \mathcal{D}$ be a compact objects preserving, R-linear functor between smooth R-linear ∞-categories. By Construction 4.18, an R-linear relative Hochschild class $\sigma\colon R[n] \to \mathrm{HH}(\mathcal{D},\mathcal{C})$ amounts to a diagram

$$\begin{tikzcd} \mathrm{id}^!_{\mathcal{D}} \arrow[r, "\mathrm{u}"] & F_!(\mathrm{id}^!_{\mathcal{C}}) \arrow[d] & \\ & F_!(\mathrm{id}_{\mathcal{C}})[1-n] \arrow[r, "\mathrm{cu}[1-n]"] & \mathrm{id}_{\mathcal{D}}[1-n] \end{tikzcd}$$

together with a choice of null-homotopy of the composite $\mathrm{id}^!_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}[1-n]$. It hence gives rise to a diagram with horizontal fiber and cofiber sequences as follows:

$$\begin{tikzcd} \mathrm{id}^!_{\mathcal{D}} \arrow[r, "\mathrm{u}"] \arrow[d] & F_!(\mathrm{id}^!_{\mathcal{C}}) \arrow[r] \arrow[d] & \mathrm{cof} \arrow[d] \\ \mathrm{fib} \arrow[r] & F_!(\mathrm{id}_{\mathcal{C}})[1-n] \arrow[r, "\mathrm{cu}[1-n]"] & \mathrm{id}_{\mathcal{D}}[1-n] \end{tikzcd} \tag{47}$$

We call the Hochschild class σ *non-degenerate* if all vertical maps in the diagram (47) are equivalences.

Definition 4.22. An (R-linear) weak left n-Calabi–Yau structure on the functor F consists of a non-degenerate Hochschild class $\sigma\colon R[n] \to \mathrm{HH}(\mathcal{D},\mathcal{C})$. If $F = 0$, we also say that $\mathcal{D}$ carries a weak left n-Calabi–Yau structure.

Weak left n-Calabi–Yau structures are also sometimes called bimodule Calabi–Yau structures.

Remark 4.23. The notion of weak left Calabi–Yau structure on a functor F only depends on the functor and the relative Hochschild class and not on any further choices made in its definition. These include choices of adjoints and (co)unit, for instance the choice of right adjoint of F together with the counit, of which there each exists a contractible space of choices. Inspecting the definition one finds that making a different choice yields an equivalent diagram in (47) and thus the same condition of the Hochschild class being non-degenerate.

4.2.2 Right Calabi–Yau structures

Let R be an $\mathbb{E}_\infty$-ring spectrum and let $F\colon \mathcal{C} \to \mathcal{D}$ be a compact objects preserving, R-linear functor between proper R-linear ∞-categories. By Construction 4.18, an R-linear dual relative Hochschild class $\sigma\colon R[n] \to \mathrm{HH}^*(\mathcal{D}, \mathcal{C})$ amounts to a diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{D}} \arrow[r, "\mathrm{u}"] & F^*(\mathrm{id}_{\mathcal{D}}) \arrow[d] & \\
& F^*(\mathrm{id}_{\mathcal{D}}^*)[1-n] \arrow[r, "{\mathrm{cu}[1-n]}"] & \mathrm{id}_{\mathcal{C}}^*[1-n]
\end{tikzcd}$$

together with a choice of null-homotopy of the composite $\mathrm{id}_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}^*[1-n]$. It hence gives rise to a diagram with horizontal fiber and cofiber sequences as follows:

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{D}} \arrow[r, "\mathrm{u}"] \arrow[d] & F^*(\mathrm{id}_{\mathcal{D}}) \arrow[r] \arrow[d] & \mathrm{cof} \arrow[d] \\
\mathrm{fib} \arrow[r] & F^*(\mathrm{id}_{\mathcal{D}}^*)[1-n] \arrow[r, "{\mathrm{cu}[1-n]}"] & \mathrm{id}_{\mathcal{C}}^*[1-n]
\end{tikzcd}$$

We call the dual Hochschild class σ *non-degenerate* if all vertical maps in the above diagram are equivalences.

Definition 4.24. Am (R-linear) weak right n-Calabi–Yau structure on the functor F consists of a non-degenerate dual Hochschild class $\sigma\colon R[n] \to \mathrm{HH}(\mathcal{D}, \mathcal{C})^*$. If $\mathcal{D} = 0$, we also say that $\mathcal{C}$ carries a weak right n-Calabi–Yau structure.

Remark 4.25. A weak right n-Calabi–Yau structure on $\mathcal{C}$ equivalently consists of an equivalence in RMod_R

$$\mathrm{Mor}_{\mathcal{C}}(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(Y, X[n])^*\,,$$

bifunctorial in $X, Y \in \mathcal{C}^{\mathrm{c}}$.

4.2.3 Behavior under tensor products

Lemma 4.26. *Let $\mathcal{C}, \mathcal{D}$ be R-linear ∞-categories. There is a canonical equivalence*

$$\mathrm{HH}(\mathcal{C}) \otimes \mathrm{HH}(\mathcal{D}) \simeq \mathrm{HH}(\mathcal{C} \otimes \mathcal{D})\,.$$

Proof. Using that $\mathrm{ev}_{\mathcal{C}\otimes\mathcal{D}} \simeq \mathrm{ev}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathcal{D}}$ and $\mathrm{coev}_{\mathcal{C}\otimes\mathcal{D}} \simeq \mathrm{coev}_{\mathcal{C}} \otimes \mathrm{coev}_{\mathcal{D}}$, we find

$$\begin{aligned}
\mathrm{HH}(\mathcal{C} \otimes \mathcal{D}) &= \mathrm{ev}_{\mathcal{C}\otimes\mathcal{D}} \circ \tau \circ \mathrm{coev}_{\mathcal{C}\otimes\mathcal{D}}(R) \\
&\simeq (\mathrm{ev}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathcal{D}}) \circ \tau \circ (\mathrm{coev}_{\mathcal{C}} \otimes \mathrm{coev}_{\mathcal{D}})(R) \\
&\simeq (\mathrm{ev}_{\mathcal{C}} \circ \tau \circ (\mathrm{coev}_{\mathcal{C}})(R) \otimes (\mathrm{ev}_{\mathcal{D}} \circ \tau \circ \mathrm{coev}_{\mathcal{D}})(R) \\
&= \mathrm{HH}(\mathcal{C}) \otimes \mathrm{HH}(D)\,.
\end{aligned}$$

□

Remark 4.27. If $\mathcal{C}, \mathcal{D}$ are smooth, we have $\mathrm{id}^!_{\mathcal{C}\otimes\mathcal{D}} \simeq \mathrm{id}^!_{\mathcal{C}} \otimes \mathrm{id}^!_{\mathcal{D}}$. A pair of morphisms $\alpha\colon \mathrm{id}^!_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}[-n]$, $\beta\colon \mathrm{id}^!_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}[-m]$ gives under the identifications from Lemmas 4.16 and 4.26 rise to the morphism

$$\mathrm{id}^!_{\mathcal{C}\otimes\mathcal{D}}[-n-m] \simeq \mathrm{id}^!_{\mathcal{C}}[-n] \otimes \mathrm{id}^!_{\mathcal{D}}[-m] \xrightarrow{\alpha\otimes\beta} \mathrm{id}_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{D}} \simeq \mathrm{id}_{\mathcal{C}\otimes\mathcal{D}}\,.$$

The tensor product of a Calabi–Yau functor with a Calabi–Yau category is thus again Calabi–Yau:

Proposition 4.28. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be R-linear smooth ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a compact objects preserving R-linear functor*

(1) Let $\sigma\colon R[n] \to \mathrm{HH}(\mathcal{D}, \mathcal{C})$ and $\sigma'\colon R[m] \to \mathrm{HH}(\mathcal{E})$ be weak left Calabi–Yau-structures on F and $\mathcal{E}$, respectively. The class

$$R[n+m] \simeq R[n] \otimes R[m] \xrightarrow{\sigma\otimes\sigma'} \mathrm{HH}(\mathcal{D}, \mathcal{C}) \otimes \mathrm{HH}(\mathcal{E}) \to \mathrm{HH}(\mathcal{D} \otimes \mathcal{E}, \mathcal{C} \otimes \mathcal{E})$$

defines a weak left $n+m$-Calabi–Yau structure on

$$F \otimes \mathcal{E}\colon \mathcal{D} \otimes \mathcal{E} \to \mathcal{C} \otimes \mathcal{E}\,.$$

(2) Let $\sigma\colon R[n] \to \mathrm{HH}^*(\mathcal{D}, \mathcal{C})$ and $\sigma'\colon R[m] \to \mathrm{HH}(\mathcal{E})^*$ be weak right Calabi–Yau-structures on F and $\mathcal{E}$, respectively. The class

$$R[n+m] \simeq R[n] \otimes R[m] \xrightarrow{\sigma\otimes\sigma'} \mathrm{HH}^*(\mathcal{D}, \mathcal{C}) \otimes \mathrm{HH}(\mathcal{E})^* \to \mathrm{HH}^*(\mathcal{D} \otimes \mathcal{E}, \mathcal{C} \otimes \mathcal{E})$$

defines a weak right $n+m$-Calabi–Yau structure on

$$F \otimes \mathcal{E}\colon \mathcal{D} \otimes \mathcal{E} \to \mathcal{C} \otimes \mathcal{E}\,.$$

Proof. We only prove part (1), part (2) is analogous. The class $R[n+m] \to \mathrm{HH}(\mathcal{D}\otimes\mathcal{E}, \mathcal{C}\otimes\mathcal{E})$ gives rise to the following diagram in $\mathrm{Lin}_R(\mathcal{D}\otimes\mathcal{E}, \mathcal{C}\otimes\mathcal{E})$, up to equivalence.

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{D}}^! \otimes \mathrm{id}_{\mathcal{E}}^! \arrow[r] \arrow[d] & F_!(\mathrm{id}_{\mathcal{C}}^!) \otimes \mathrm{id}_{\mathcal{E}}^! \arrow[r] \arrow[d] & \mathrm{cof} \otimes \mathrm{id}_{\mathcal{E}}^! \arrow[d] \\
\mathrm{fib} \otimes \mathrm{id}_{\mathcal{E}}[m] \arrow[r] & F_!(\mathrm{id}_{\mathcal{C}})[1-n] \otimes \mathrm{id}_{\mathcal{E}}[-m] \arrow[r] & \mathrm{id}_{\mathcal{D}}[1-n] \otimes \mathrm{id}_{\mathcal{E}}[-m]
\end{tikzcd}$$

The horizontal sequences in the above diagram are fiber and cofiber sequences as tensor products of such with $\mathrm{id}_{\mathcal{E}}^!$ or $\mathrm{id}_{\mathcal{E}[m]}$. The vertical maps are equivalences as tensor products of equivalences, showing the non-degeneracy of the class. $\square$

4.2.4 Gluing Calabi–Yau structures

In this section, we discuss a generalization of the gluing theorem for left Calabi–Yau structures on k-linear dg-categories of [BD19, Theorem 6.1] to weak left and right Calabi–Yau structures on R-linear ∞-categories. The gluing theorem boils down to a simple description of objects in pullbacks/pushouts of stable, presentable ∞-categories in terms of their restrictions, see Lemma 4.30.

Consider the cospan simplicial set $Z = \Delta^{\{0,1\}} \amalg_{\Delta^{\{0\}}} \Delta^{\{0,1'\}}$ with objects $0, 1, 1'$ and two non-degenerate 1-simplicies $0 \to 1, 1'$. We fix a diagram $D\colon Z \to \mathrm{LinCat}_R$ with colimit $\mathcal{C}$, satisfying that D maps each morphism to a compact objects preserving functor. For $z \in Z$, we denote $\mathcal{C}_z = D(z)$. Let further $i_z\colon \mathcal{C}_z \to \mathcal{C}$ denote the functor from the colimit cone, $j_z = \mathrm{radj}(i_z)$ its right adjoint, and cu_z the counit of $i_z \dashv j_z$.

The fact that counits compose to counits provides us with a commutative square

$$\phi_D\colon Z^{\triangleright} \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$$

which can be depicted as follows:

$$\begin{tikzcd}
i_0 j_0 \arrow[r] \arrow[d] \arrow[dr, "\mathrm{cu}_0"] & i_1 j_1 \arrow[d, "\mathrm{cu}_1"] \\
i_{1'} j_{1'} \arrow[r, "\mathrm{cu}_{1'}"] & \mathrm{id}_{\mathcal{C}}
\end{tikzcd}$$

Proposition 4.29. *The square $\phi_D\colon Z^{\triangleright} \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ is biCartesian.*

Proof. Using that the forgetful functor $\mathrm{Lin}_R(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is exact and that colimits in functor categories are computed pointwise, see [Lur09, 5.1.2.3], the statement reduces to Lemma 4.30. $\square$

Lemma 4.30. *Let $c \in \mathcal{C}$ and consider the diagram*

$$\phi_c\colon Z^{\triangleright} \to \mathcal{C}\,,$$

given by evaluating ϕ_D at c. Then ϕ_c describes a biCartesian square in $\mathcal{C}$.

Proof. We can identify the colimit $\mathcal{C}$ of D with the ∞-category of coCartesian sections of the Grothendieck construction $\pi\colon \Gamma(\mathrm{radj}(D)) \to Z^{\mathrm{op}}$ of the diagram obtained from D by passing to right adjoint functors. Denote by $\mathcal{L}$ the ∞-category of all sections of π and by $\kappa\colon \mathcal{C} \hookrightarrow \mathcal{L}$ the fully faithful inclusion with left adjoint ζ. For $z \in Z$, we denote by $\tilde{j}_z\colon \mathcal{L} \to \mathcal{C}_z$ the evaluation functor at z, satisfying that $\tilde{j}_z \circ \kappa \simeq j_z$. The left adjoint of $\tilde{j}_z$ is denoted $\tilde{i}_z$, it satisfies $\zeta \circ \tilde{i}_z = i_z$. An object $c \in \mathcal{C}$ corresponding to a coCartesian section of π is of the form

$$\begin{tikzcd} c_1 \arrow[dr] & & c_{1'} \arrow[dl] \\ & c_0 & \end{tikzcd}$$

with $c_i \in \mathcal{C}_i$. By [Lur09, 4.3.2.17], $\tilde{i}_z$ is given by the π-relative left Kan extension functor and the objects $\tilde{i}_z j_z(c)$, $z \in Z$ are hence given as follows.

$$\tilde{i}_0 j_0(c) \simeq \begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ & c_0 & \end{tikzcd}$$

$$\tilde{i}_1 j_1(c) \simeq \begin{tikzcd} c_1 \arrow[dr] & & 0 \arrow[dl] \\ & c_0 & \end{tikzcd} \qquad \tilde{i}_{1'} j_{1'}(c) \simeq \begin{tikzcd} 0 \arrow[dr] & & c_{1'} \arrow[dl] \\ & c_0 & \end{tikzcd}$$

These assemble into a square $\tilde{\phi}_c$ in $\mathcal{L}$ of the form

$$\begin{tikzcd} \tilde{i}_0 j_0(c) \arrow[r] \arrow[d] & \tilde{i}_1 j_1(c) \arrow[d] \\ \tilde{i}_{1'} j_{1'}(c) \arrow[r] & c \end{tikzcd}$$

which restricts at $0 \in Z$ to the constant diagram with value c_0, up to equivalence, and at $i = 1, 1' \in Z$ to the fiber sequence of the map $c_i \to 0$ in $\mathcal{C}_i$. Using that limits in the ∞-category $\mathcal{L}$ of sections of the Grothendieck construction are computed componentwise in Z, it follows that $\tilde{\phi}_c$ is a limit diagram in $\mathcal{L}$. Using that $\zeta\colon \mathcal{L} \to \mathcal{C}$ is exact, we conclude that $\phi_c \simeq \zeta \circ \tilde{\phi}_c$ is a limit diagram as well. □

Proposition 4.29 implies that R-linear smoothness is preserved under finite colimits along compact objects preserving R-linear functors. A variant of this observation appears for k-linear ∞-categories in [ST16, Lemma 8.21].

Corollary 4.31. *Let W be a finite simplicial set and $D\colon W \to \mathrm{LinCat}_R$ a functor taking values in smooth R-linear ∞-categories and compact objects preserving functors. The colimit of D is again smooth.*

Proof. Any finite colimit can be computed in terms of a pushout and finite coproducts. Smoothness is clearly preserved under finite coproducts of R-linear ∞-categories. It remains to check that the pushout of a span of smooth R-linear ∞-categories along compact objects preserving functors is again smooth. This follows from Lemma 4.30, part (1) of Lemma 4.8 and the fact that pushouts of compact objects are again compact. $\square$

Proposition 4.29 admits a 'dual' version as follows. Consider a diagram $D\colon Z^{\mathrm{op}} \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ taking values in limit preserving functors. Set $\mathcal{C} \coloneqq \lim D$. For $z \in Z$, consider the adjunction $h_z \dashv i_z$ with $i_z\colon \mathcal{C} \to \mathcal{C}_z$ the functor from the limit cone. Denote by cu_z the counit of $h_z \dashv i_z$. We have a diagram $\phi^D\colon Z^{\triangleright} \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$, which can depicted as follows:

$$\begin{tikzcd} h_0 i_0 \arrow[r] \arrow[d] \arrow[dr, "\mathrm{cu}_0"] & h_1 i_1 \arrow[d, "\mathrm{cu}_1"] \\ h_{1'} i_{1'} \arrow[r, "\mathrm{cu}_{1'}"] & \mathrm{id}_{\mathcal{C}} \end{tikzcd}$$

An analogue of the proof of Proposition 4.29 shows the following.

Proposition 4.32. *The diagram $\phi^D\colon Z^{\triangleright} \to \mathrm{Lin}_R(\mathcal{C}, \mathcal{C})$ is a biCartesian square.*

The above discussion provides us with the tools needed for proving the gluing results for Calabi–Yau structures. We turn to the gluing of weak left Calabi–Yau structures. For this, fix a colimit diagram in LinCat_R, valued in smooth ∞-categories and functors which preserve compact objects and limits, of the following form:

$$\begin{tikzcd} & & \mathcal{B}_3 \arrow[d, "F_{3,2}"] \\ & \mathcal{B}_2 \arrow[r, "F_{2,2}"] \arrow[d, "F_{2,1}"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C}_2 \arrow[d, "F_2"] \\ \mathcal{B}_1 \arrow[r, "F_{1,1}"] & \mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{D} \end{tikzcd} \tag{48}$$

We form the following diagram in RMod_R:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{HH}(\mathcal{B}_2) \arrow[r] \arrow[d, "(\mathrm{id},-\mathrm{id})"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ \bigoplus_{i=1,2} \mathrm{HH}(\mathcal{C}_i, \mathcal{B}_i \times \mathcal{B}_{i+1})[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{HH}(B_2)^{\oplus 2} \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "\mathrm{HH}(F_{2,1})\oplus\mathrm{HH}(F_{2,2})"] & \mathrm{HH}(\mathcal{B}_2) \arrow[d] \\ \mathrm{HH}(\mathcal{B}_1) \oplus \mathrm{HH}(\mathcal{B}_3) \arrow[r, "\mathrm{HH}(F_{1,1})\oplus\mathrm{HH}(F_{3,2})"] & \mathrm{HH}(\mathcal{C}_1) \oplus \mathrm{HH}(\mathcal{C}_2) \arrow[r, "(\mathrm{HH}(F_1),\mathrm{HH}(F_2))"'] & \mathrm{HH}(\mathcal{D}) \end{tikzcd}$$

The outer square of the above diagram, though not necessarily biCartesian, induces a morphism $X \to \mathrm{HH}(\mathcal{D}, \mathcal{B}_1 \oplus \mathcal{B}_3)[-1]$. A class $R[n] \to X[1]$ corresponds to two classes $R[n] \to \mathrm{HH}(\mathcal{C}_1, \mathcal{B}_1 \times \mathcal{B}_2), \mathrm{HH}(\mathcal{C}_2, \mathcal{B}_2 \times \mathcal{B}_3)$, whose restrictions to $\mathrm{HH}(B_2)[1]$ are *not* identical, but differ exactly by a reversal of the sign, i.e. composition with $-\mathrm{HH}(\mathrm{id}_{\mathcal{B}_2})$. In this case, we say that the restrictions of the classes to $\mathrm{HH}(B_2)[1]$ are compatible.

Theorem 4.33. *Consider two classes $\sigma_i \colon R[n] \to \mathrm{HH}(\mathcal{C}_i, \mathcal{B}_i \times \mathcal{B}_{i+1})$, with $i = 1, 2$, whose restrictions to $\mathrm{HH}(\mathcal{B}_2)[1]$ are compatible and let $\sigma \colon R[n] \to \mathrm{HH}(\mathcal{D}, \mathcal{B}_1 \times \mathcal{B}_3)$ be the arising class. If σ_1 and σ_2 are non-degenerate, i.e. define weak left n-Calabi–Yau structures on the functors*

$$\mathcal{B}_i \times \mathcal{B}_{i+1} \longrightarrow \mathcal{C}_i\,,$$

then σ is non-degenerate as well and thus defines a weak left n-Calabi–Yau structure on the functor

$$\mathcal{B}_1 \times \mathcal{B}_3 \longrightarrow \mathcal{D}\,.$$

Proof. For $\mathcal{X} = \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_1, \mathcal{C}_2$, denote by $i_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{D}$ the functor from (48). Let $j_{\mathcal{X}}$ be the right adjoint of $i_{\mathcal{X}}$. Since the restriction of (48) to $\mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$ is a pushout diagram, we find by Proposition 4.29 a biCartesian square in $\mathrm{Lin}_R(\mathcal{D}, \mathcal{D})$, which is depicted as follows.

$$\begin{tikzcd}
i_{\mathcal{B}_2} j_{\mathcal{B}_2} \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] \arrow[dr, phantom, "\square"] & i_{\mathcal{C}_1} j_{\mathcal{C}_1} \arrow[d, "\beta_1"] \\
i_{\mathcal{C}_2} j_{\mathcal{C}_2} \arrow[r, "\beta_2"] & \mathrm{id}_{\mathcal{D}}
\end{tikzcd}$$

The sequence

$$i_{\mathcal{B}_2} j_{\mathcal{B}_2} \xrightarrow{(\alpha_1, -\alpha_2)} i_{\mathcal{C}_1} j_{\mathcal{C}_1} \oplus i_{\mathcal{C}_2} j_{\mathcal{C}_2} \xrightarrow{(\beta_1, \beta_2)} \mathrm{id}_{\mathcal{D}}$$

is hence a fiber and cofiber sequence.

Using the pasting law for biCartesian squares, this gives rise to the following commutative diagram in $\mathrm{Lin}_R(\mathcal{D}, \mathcal{D})$, where all squares are biCartesian and all objects are compact.

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2} j_{\mathcal{B}_2} \arrow[r] \arrow[d, "{(\mathrm{id}, -\mathrm{id})}"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2} j_{\mathcal{B}_2} \oplus i_{\mathcal{B}_2} j_{\mathcal{B}_2} \arrow[r, "{(\mathrm{id}, \mathrm{id})}"] \arrow[d, "\alpha_1 \oplus \alpha_2"] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2} j_{\mathcal{B}_2} \arrow[d] \\
i_{\mathcal{B}_1} j_{\mathcal{B}_1} \oplus i_{\mathcal{B}_3} j_{\mathcal{B}_3} \arrow[r] & i_{\mathcal{C}_1} j_{\mathcal{C}_1} \oplus i_{\mathcal{C}_2} j_{\mathcal{C}_2} \arrow[r, "\beta_1 \oplus \beta_2"] & \mathrm{id}_{\mathcal{D}}
\end{tikzcd} \tag{49}$$

By assumption the functor $i_{\mathcal{X}}$ preserves limits and thus admits a left adjoint $\mathrm{ladj}(i_{\mathcal{X}})$, for $\mathcal{X} = \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_1, \mathcal{C}_2$. By passing to left adjoints, the counit $\mathrm{cu}_{\mathcal{X}} \colon i_{\mathcal{X}} j_{\mathcal{X}} \to \mathrm{id}_{\mathcal{C}}$ induces the unit $\mathrm{id}_{\mathcal{C}} \to i_{\mathcal{X}} \,\mathrm{ladj}(i_{\mathcal{X}})$ of the adjunction $\mathrm{ladj}(i_{\mathcal{X}}) \dashv i_{\mathcal{X}}$. Lemma 4.21 thus shows that the image under $(\text{-})^!$ of the counit $\mathrm{cu}_{\mathcal{X}}$ is given, after composition with an equivalence, by the unit $\mathrm{id}_{\mathcal{C}}^! \to i_{\mathcal{X}} \,\mathrm{id}_{\mathcal{X}}^!\, j_{\mathcal{X}}$ from Construction 4.18. Applying the exact contravariant functor (-)$^!$ to (49) thus yields the following diagram, up to equivalence.

$$\begin{tikzcd}
\mathrm{id}^!_{\mathcal{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2}\, \mathrm{id}^!_{\mathcal{B}_2}\, j_{\mathcal{B}_2} \arrow[r] \arrow[d, "{(\mathrm{id},-\mathrm{id})}"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
i_{\mathcal{C}_1}\, \mathrm{id}^!_{\mathcal{C}_1}\, j_{\mathcal{C}_1} \oplus i_{\mathcal{C}_2}\, \mathrm{id}^!_{\mathcal{C}_2}\, j_{\mathcal{C}_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2}\, \mathrm{id}^!_{\mathcal{B}_2}\, j_{\mathcal{B}_2} \oplus i_{\mathcal{B}_2}\, \mathrm{id}^!_{\mathcal{B}_2}\, j_{\mathcal{B}_2} \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d] \arrow[dr, phantom, "\square"] & i_{\mathcal{B}_2}\, \mathrm{id}^!_{\mathcal{B}_2}\, j_{\mathcal{B}_2} \arrow[d] \\
i_{\mathcal{B}_1}\, \mathrm{id}^!_{\mathcal{B}_1}\, j_{\mathcal{B}_1} \oplus i_{\mathcal{B}_3}\, \mathrm{id}^!_{\mathcal{B}_3}\, j_{\mathcal{B}_3} \arrow[r] & U^! \arrow[r] & V^!
\end{tikzcd}$$

The classes σ_1, σ_2 define an equivalence between the lower left squares and upper right squares of the lower diagram and the $(1-n)$-th suspension of the upper diagram. These equivalences extend to an equivalence of the entire diagrams by using that the lower right and upper left squares are biCartesian. Restricting the equivalence to the outer biCartesian squares provides us with a diagram in $\mathrm{Lin}_R(\mathcal{D}, \mathcal{D})$

$$\begin{tikzcd}
\mathrm{id}^!_{\mathcal{D}} \arrow[r, "\mathrm{u}"] \arrow[d, "\simeq"] & (i_{\mathcal{B}_1} j_{\mathcal{B}_1})^! \oplus (i_{\mathcal{B}_3} j_{\mathcal{B}_3})^! \arrow[r] \arrow[d, "\simeq"] & V^! \arrow[d, "\simeq"] \\
V[n-1] \arrow[r] & i_{\mathcal{B}_1} j_{\mathcal{B}_1}[1-n] \oplus i_{\mathcal{B}_3} j_{\mathcal{B}_3}[1-n] \arrow[r, "\mathrm{cu}"] & \mathrm{id}_{\mathcal{D}}[1-n]
\end{tikzcd}$$

with horizontal fiber and cofiber sequences. Using Lemma 4.19, we find that this diagram arises from the class σ, hence showing its non-degeneracy. □

Remark 4.34. In the analogue of Theorem 4.33 in [BD19], it is not assumed that the functors in (48) preserve limits. This restriction is needed in the argument for showing that the image under (-)$^!$ of the counit $i_{\mathcal{X}} j_{\mathcal{X}} \to \mathrm{id}_{\mathcal{C}}$ describes the unit $\mathrm{id}^!_{\mathcal{C}} \to i_{\mathcal{X}}\, \mathrm{id}^!_{\mathcal{X}}\, j_{\mathcal{X}}$, up to composition with an equivalence $i_{\mathcal{X}}\, \mathrm{id}^!_{\mathcal{X}}\, j_{\mathcal{X}} \simeq (i_{\mathcal{X}} j_{\mathcal{X}})^!$. This latter statement is likely true without this restriction.

We conclude this section with describing the gluing property of right Calabi–Yau structures, dual to Theorem 4.35. Consider a limit diagram in LinCat_R valued in compact objects and limit preserving functors and proper R-linear ∞-categories of the following form:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "F_2"] \arrow[d, "F_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_2 \arrow[r, "F_{2,3}"] \arrow[d, "F_{2,2}"] & \mathcal{B}_3 \\
\mathcal{C}_1 \arrow[r, "F_{1,2}"] \arrow[d, "F_{1,1}"] & \mathcal{B}_2 & \\
\mathcal{B}_1 & &
\end{tikzcd} \tag{50}$$

We form the following diagram in RMod_R:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{HH}(\mathcal{B}_2)^* \arrow[r] \arrow[d, "{(\mathrm{id},-\mathrm{id})}"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\bigoplus_{i=1,2} \mathrm{HH}^*(\mathcal{B}_i \times \mathcal{B}_{i+1}, \mathcal{C}_i)[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathrm{HH}(\mathcal{B}_2)^*)^{\oplus 2} \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "\mathrm{HH}^*(F_{2,1}) \oplus \mathrm{HH}^*(F_{2,2})"] & \mathrm{HH}(\mathcal{B}_2)^* \arrow[d] \\
\mathrm{HH}(\mathcal{B}_1)^* \oplus \mathrm{HH}(\mathcal{B}_3)^* \arrow[r, "\mathrm{HH}^*(F_{1,1}) \oplus \mathrm{HH}^*(F_{3,2})"] & \mathrm{HH}(\mathcal{C}_1)^* \oplus \mathrm{HH}(\mathcal{C}_2)^* \arrow[r, "{(\mathrm{HH}^*(F_1),\mathrm{HH}^*(F_2))}"'] & \mathrm{HH}(\mathcal{D})^*
\end{tikzcd}$$

Similar to the smooth case, a class in X consists of classes in $\mathrm{HH}^*(\mathcal{B}_i \times \mathcal{B}_{i+1}, \mathcal{C}_i)$, with $i = 1, 2$, whose restrictions to $\mathrm{HH}(\mathcal{B}_2)^*[1]$ differ by sign, and we again call such classes compatible. The above diagram induces a morphism $\omega\colon X \to \mathrm{HH}^*(\mathcal{B}_1 \times \mathcal{B}_3, \mathcal{D})$.

Theorem 4.35. *Consider two classes $\sigma_i\colon R[n] \to \mathrm{HH}^*(\mathcal{B}_i \times \mathcal{B}_{i+1}, \mathcal{C}_i)$, with $i = 1, 2$, whose restrictions to $\mathrm{HH}(\mathcal{B}_2)^*[1]$ are compatible and let $\sigma\colon R[n] \to \mathrm{HH}^*(\mathcal{B}_1 \times \mathcal{B}_3, \mathcal{D})$ be the arising class. If σ_1 and σ_2 are non-degenerate, i.e. define weak right n-Calabi–Yau structures on the functors*

$$\mathcal{B}_i \times \mathcal{B}_{i+1} \longrightarrow \mathcal{C}_i\,,$$

then σ is non-degenerate as well and thus defines a weak right n-Calabi–Yau structure on the functor

$$\mathcal{B}_1 \times \mathcal{B}_3 \longrightarrow \mathcal{D}\,.$$

Proof. Analogous to the proof of Theorem 4.33. □

4.2.5 Examples

We collect some examples of functors which admit a relative Calabi–Yau structure which we will need later on.

Theorem 4.36 ([BD19])**.** *Let M be a compact oriented manifold of dimension n with boundary $i\colon \partial M \subset M$ and k a field. The left adjoint $i_!$ of the pullback functor*

$$i^*\colon \mathrm{Fun}(M, \mathcal{D}(k)) \longrightarrow \mathrm{Fun}(\partial M, \mathcal{D}(k))$$

admits a left n-Calabi–Yau structure.

Given a commutative ring k and $m \geq 0$, we denote by $k[t_m^{\pm}]$ the algebra of graded Laurent polynomials, with generator t_m in degree m.

Lemma 4.37. *Assume that $\mathrm{char}(k) \neq 2$ and let $m \geq 1$ be odd. Then $\mathcal{D}(k[t_m^{\pm}])$ is smooth and proper as a $k[t_{2m}^{\pm}]$-linear ∞-category and further admits weak left and right m-Calabi–Yau structures.*

Proof. The $k[t_{2m}^{\pm}]$-linear enveloping algebra A^e of $A := k[t_m^{\pm}]$ is given by the graded commutative dg-algebra $k[t^{\pm}, s^{\pm}]/(s^2 - t^2)$, with generators t, s in degrees m satisfying $st = (-1)^{m^2} ts = -ts$ (graded commutativity) and $s^2 = t^2$. As a right A^e-module, A is equipped with the action $1.t = t$ and $1.s = -t$. As a left $(A^e)^{\mathrm{op}}$-module, A is equipped with the action $t.1 = -t$ and $s.1 = t$. We denote by $\hat{A}$ the the right A^e-module A with the action $1.t = t$ and $1.s = t$.

We consider A^e as a right module over itself. There is a retract of right A^e-modules

$$A \xrightarrow{1 \mapsto 1 - s^{-1}t} A^e \xrightarrow{1 \mapsto 1} A$$

since the composite is given by multiplication by $1 - 1.s^{-1}t = 1 + t^{-1}t = 2 \neq 0$ and thus invertible. There is a similar retract

$$\hat{A} \xrightarrow{1 \mapsto 1 + s^{-1}t} A^e \xrightarrow{1 \mapsto 1} \hat{A}$$

and $A^e \simeq A \oplus \hat{A}$.

It follows that A is compact as a right A^e-module. The inverse dualizing bimodule $\mathrm{id}^!_{\mathcal{D}(A)}$ is given by the tensor product with the left A^e-module $A^! = \mathrm{RHom}_{A^e}(A, A^e) = \mathrm{Hom}_{A^e}(A, A^e)$. One finds $\mathrm{Hom}_{A^e}(A, \hat{A}) \simeq 0$. We thus have $\mathrm{RHom}_{A^e}(A, A^e) \simeq \mathrm{Hom}_{A^e}(A, A) \simeq A$ on k-linear homology, with generator as a graded algebra the map $\phi\colon A \to A^e$, $1 \mapsto 1 - s^{-1}t$. The element $t \in A$ corresponds to $\phi'\colon 1 \mapsto t - s$. The left action of A^e on $\mathrm{Hom}_{A^e}(A, A^e) \simeq A$ is determined by $t.\phi = \phi'$ and $s.\phi = -\phi'$. It follows that $A^! = \mathrm{RHom}_{A^e}(A, A^e) \simeq A[m]$ as left R^e-modules, since the shift by m preserves the homology of A over k but flips the signs of the actions of t and s (since m is odd). This shows that $\mathrm{id}^!_{\mathcal{D}(A)} \simeq \mathrm{id}_{\mathcal{D}(A)}[m]$, as desired. Composing with $\mathrm{id}^*_{\mathcal{D}(A)}$ also yields $\mathrm{id}_{\mathcal{D}(A)} \simeq \mathrm{id}^*_{\mathcal{D}(A)}[m]$. □

4.3 Calabi–Yau structures for perverse schobers

In Section 4.3.1, we describe how relative Calabi–Yau structures arise from perverse schobers on discs. We then discuss relative Calabi–Yau structures on global sections of perverse schobers in Section 4.3.2. One of the main results is that the global sections of locally constant perverse schober with trivial monodromy, whose generic stalk is Calabi–Yau, is relative Calabi–Yau.

4.3.1 Calabi–Yau structures locally

The goal of this section is to prove the following Proposition.

Proposition 4.38. *Let $F\colon \mathcal{C} \leftrightarrow \mathcal{D} :G$ be a spherical adjunction of R-linear ∞-categories. Consider the adjoint functors, see Lemma 3.24,*

$$\mathrm{R}^m_F := (\varrho_1[-1], \varrho_2[-2], \dots, \varrho_m[-m])\colon \mathcal{V}^m_F \longleftrightarrow \mathcal{D}^{\times m} :\mathrm{S}^m_F := (\varsigma_1[1], \varsigma_2[2], \dots, \varsigma_m[m])\,. \tag{51}$$

(1) Suppose that $\mathcal{C}, \mathcal{D}$ are smooth and that the class $\sigma_G\colon R[n] \to \mathrm{HH}(\mathcal{C}, \mathcal{D})$ defines a weak left n-Calabi–Yau structure on G, which restricts to a weak left $(n-1)$–Calabi–Yau structure $\sigma_{\mathcal{D}}\colon R[n-1] \to \mathrm{HH}(\mathcal{D})$ on $\mathcal{D}$. Then the functor S_F^m admits a canonical weak left n-Calabi–Yau structure, which restricts on $\mathcal{D}^{\times m}$ to

$$\sigma_{\mathcal{D}}^{\times m}\colon R[n-1] \longrightarrow \mathrm{HH}(\mathcal{D}^{\times n}) \simeq \mathrm{HH}(\mathcal{D})^{\oplus n}\,.$$

(2) Suppose that $\mathcal{C}, \mathcal{D}$ are proper and that the class $\sigma_F\colon R[n] \to \mathrm{HH}^*(\mathcal{D}, \mathcal{C})$ defines a weak right n-Calabi–Yau structure on F, which restricts to a weak right $(n-1)$–Calabi–Yau structure $\sigma_{\mathcal{D}}\colon R[n-1] \to \mathrm{HH}(\mathcal{D})^*$ on $\mathcal{D}$. Then the functor R_F^m admits a canonical weak right n-Calabi–Yau structure, which restricts on $\mathcal{D}^{\times m}$ to

$$\sigma_{\mathcal{D}}^{\times m}\colon R[n-1] \longrightarrow \mathrm{HH}(\mathcal{D}^{\times n})^* \simeq (\mathrm{HH}(\mathcal{D})^*)^{\oplus n}\,.$$

Using the gluing properties of Calabi–Yau structures, we will reduce the proof of Proposition 4.38 to the case $m = 3$ and $F = 0$. This case is then first solved for $\mathcal{D} = \mathrm{RMod}_R$, and then for arbitrary $\mathcal{D}$ by tensoring, see Proposition 4.28.

Construction 4.39. Let $\mathcal{D} = \mathrm{RMod}_R$, considered as an R-linear smooth and proper ∞-category. We construct two inverse equivalences $U, T\colon \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ via Kan extension.

Consider the ∞-category $\mathcal{X}$ of diagrams in $\mathcal{D}$ of the follow form.

$$\begin{tikzcd}
a' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] & \\
0 \arrow[r] & a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
& 0 \arrow[r] & a'' \arrow[r] & b''
\end{tikzcd}$$

Formally, the ∞-category $\mathcal{X}$ can be characterized as consisting of diagrams which are repeated Kan extensions of their restriction to $a \to b$. The restriction functor to $a \to b$ thus defines by [Lur09, 4.3.2.15] a trivial fibration $\phi\colon \mathcal{X} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$. It hence admits an inverse, unique up to contractible space of choices. We denote by $\psi'\colon \mathcal{X} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ the restriction functor to $a' \to b'$ and by $\psi''\colon \mathcal{X} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ the restriction functor to $a'' \to b''$.

The functor U is defined as the suspension of the composite functor

$$\mathrm{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\phi^{-1}} \mathcal{X} \xrightarrow{\psi'} \mathrm{Fun}(\Delta^1, \mathcal{D})\,,$$

and the functor T is defined as the delooping of the composite functor

$$\mathrm{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\phi^{-1}} \mathcal{X} \xrightarrow{\psi''} \mathrm{Fun}(\Delta^1, \mathcal{D})\,.$$

The functor U from Construction 4.39 is informally given by the assignment

$$U\colon \big(a \longrightarrow b\big) \quad \mapsto \quad \big(b \longrightarrow \mathrm{cof}(a \to b)\big)\,.$$

Lemma 4.40. *Let $\mathcal{D} = \mathrm{RMod}_R$. The R-linear ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{D})$ is smooth and proper and the functor U from Construction 4.39 is a Serre functor of $\mathrm{Fun}(\Delta^1, \mathcal{D})$.*

Proof. Consider the fully faithful, R-linear functors

$$\iota_0 \colon \mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D}), d \mapsto (d \to 0)\,,$$

$$\iota_1 \colon \mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D}), d \mapsto (0 \to d)\,.$$

The functors ι_0, ι_1 admit repeated left and right adjoints, and in particular preserve compact objects. Observing that the evaluation functors $\mathrm{ev}_0 = \mathrm{ladj}(\iota_0), \mathrm{ev}_1 = \mathrm{radj}(\iota_1) \colon \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ arise as adjoints of ι_0, ι_1, we find that if an object $a \to b \in \mathrm{Fun}(\Delta^1, \mathcal{D})$ is compact, then $a, b \in \mathcal{D} = \mathrm{RMod}_R$ are compact too.

It follows from Lemma 4.8, that $\mathrm{ladj}(\iota_0)^*(\mathrm{id}_\mathcal{D}) \simeq \iota_0 \circ \mathrm{ladj}(\iota_0)$ and $(\iota_1)_!(\mathrm{id}_\mathcal{D}) \simeq \iota_1 \circ \mathrm{radj}(\iota_1)$ are compact in $\mathrm{Lin}_R(\mathrm{Fun}(\Delta^1, \mathcal{D}), \mathrm{Fun}(\Delta^1, \mathcal{D}))$. We observe that $\mathrm{id}_\mathcal{D}$ is given by the cofiber of a natural transformation $\iota_0 \circ \mathrm{ladj}(\iota_0)[-1] \to \iota_1 \circ \mathrm{radj}(\iota_1)$. In particular, we find that $\mathrm{id}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}$ is also compact, showing that $\mathrm{Fun}(\Delta^1, \mathcal{D})$ is smooth.

Any object $a \to b$ in $\mathrm{Fun}(\Delta^1, \mathcal{D})$ is given as the cofiber of a map $(a[-1] \to 0) \to (0 \to b)$. We thus find for any $a, a', b, b' \in \mathrm{RMod}_R$ an equivalence

$$\mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}(a \to b, a' \to b') \simeq \mathrm{fib}\left(\mathrm{Mor}_\mathcal{D}(b, b') \to \mathrm{Mor}_\mathcal{D}(a, \mathrm{cof}(a' \to b'))\right), \tag{52}$$

implying that $\mathrm{Fun}(\Delta^1, \mathcal{D})$ is proper.

Let $a \to b, a' \to b' \in \mathrm{Fun}(\Delta^1, \mathcal{D})$ be compact objects. Using that $\mathrm{id}_\mathcal{D}$ is an R-linear Serre functor, we find equivalences in RMod_R

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}(a \to b, a' \to b') &\simeq \mathrm{fib}\left(\mathrm{Mor}_\mathcal{D}(b, b') \to \mathrm{Mor}_\mathcal{D}(a, \mathrm{cof}(a' \to b'))\right) \\
&\simeq \mathrm{cof}\left(\mathrm{Mor}_\mathcal{D}(a, \mathrm{cof}(a' \to b'))^* \to \mathrm{Mor}_\mathcal{D}(b, b')^*\right)^* \\
&\simeq \mathrm{cof}\left(\mathrm{Mor}_\mathcal{D}(\mathrm{cof}(a' \to b'), a) \to \mathrm{Mor}_\mathcal{D}(b', b)\right)^* \\
&\simeq \mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}(b' \to \mathrm{cof}(a' \to b'), \mathrm{cof}(a \to b) \to a[1])^* \\
&\simeq \mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}(a' \to b', U(a \to b))^*\,,
\end{aligned}$$

functorial in $a \to b \in \mathrm{Fun}(\Delta^1, \mathcal{D})^{\mathrm{c,op}}$, $a' \to b' \in \mathrm{Fun}(\Delta^1, \mathcal{D})^{\mathrm{c}}$. The second to last equivalence arises in the same way as the equivalence (52). The last equivalence uses that the sequences $a' \to b' \to \mathrm{cof}(a' \to b')$ and $b \to \mathrm{cof}(a \to b) \to a[1]$ are fiber and cofiber sequences. This shows that U is a Serre functor and concluding the proof. □

Lemma 4.41. *Let $\mathcal{D} = \mathrm{RMod}_R$. We let $\sigma_\mathcal{D} \colon R \xrightarrow{\mathrm{id}} \mathrm{HH}(\mathcal{D}) \simeq R$ denote the apparent weak left 0-Calabi–Yau structure on $\mathcal{D}$ and $\sigma_\mathcal{D}^* \colon R \xrightarrow{\mathrm{id}} \mathrm{HH}(\mathcal{D})^* \simeq R$ denote the apparent weak right 0-Calabi–Yau structure on $\mathcal{D}$.*

(1) The R-linear functor

$$S \coloneqq (\varsigma_1[1], \varsigma_2[2], \varsigma_3[3]) \colon \mathcal{D}^{\times 3} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{D})$$

admits a unique weak left 1-Calabi–Yau structure which restricts to the weak left 0-Calabi–Yau structure $\sigma_\mathcal{D}^{\times 3}$ on $\mathcal{D}^{\times 3}$.

(2) The R-linear functor

$$(\varrho_1[-1], \varrho_2[-2], \varrho_3[-3])\colon \operatorname{Fun}(\Delta^1, \mathcal{D}) \longrightarrow \mathcal{D}^{\times 3}$$

admits a unique weak right 1*-Calabi–Yau structure which restricts to the weak right* 0*-Calabi–Yau structure* $(\sigma_{\mathcal{D}}^*)^{\times 3}$ *on* $\mathcal{D}^{\times 3}$*.*

Proof. We only prove part (1), part (2) is similar. The split localization sequence

$$\mathcal{D} \xrightarrow{\varsigma_2[2]} \operatorname{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\varrho_3[-3]} \mathcal{D}$$

provides us with a splitting $\operatorname{HH}(\operatorname{Fun}(\Delta^1, \mathcal{D})) \simeq \operatorname{HH}(\mathcal{D}) \oplus \operatorname{HH}(\mathcal{D}) \simeq R^{\oplus 2}$. Using the adjunctions $\varrho_3[-3] \dashv \varsigma_3[3]$ and $\varsigma_2[2] \dashv \varrho_1[-2]$, we find that $\operatorname{HH}(\varrho_3[-3])$ and $\operatorname{HH}(\varrho_1[-2])$ are the two projection maps $R^{\oplus 2} \to R$ and $\operatorname{HH}(\varsigma_3[3]), \operatorname{HH}(\varsigma_2[2])$ are the two inclusion maps $R \to R^{\oplus 2}$ of the direct summands.

With the above, we have

$$\operatorname{HH}\left(\operatorname{Fun}(\Delta^1, \mathcal{D}), \mathcal{D}^{\oplus 3}\right) \simeq \operatorname{cof}(R^{\oplus 3} \xrightarrow{\mathscr{M}} R^{\oplus 2}) \simeq R[1]\,,$$

where $\mathscr{M} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$. The formula for $\mathscr{M}$ follows from

$$\varrho_1[-2] \circ \varsigma_1[1] \simeq [-1]$$

and

$$\varrho_3[-3] \circ \varsigma_1[1] \simeq [-1]\,,$$

see also Remark 4.42.

We let σ be the the class $R[1] \xrightarrow{\mathrm{id}} R[1] \simeq \operatorname{HH}(\operatorname{Fun}(\Delta^1, \mathcal{D}), \mathcal{D}^{\oplus 3})$. The observation that $\mathscr{M}(1,1,1) = 0$ implies that σ indeed restricts to $\sigma_{\mathcal{D}}^{\times 3}$ on $\mathcal{D}^{\times 3}$. Furthermore, σ is clearly unique with this property. To complete the proof, it remains to show that σ is non-degenerate.

The class σ determines a diagram

$$\begin{tikzcd}
\mathrm{id}^!_{\operatorname{Fun}(\Delta^1,\mathcal{D})} \arrow[r, "\mathrm{u}"] & S_!(\mathrm{id}^!_{\mathcal{D}^{\times 3}}) \arrow[d, "S_!(\sigma_{\mathcal{D}}^{\times 3})"] & \\
& S_!(\mathrm{id}_{\mathcal{D}^{\times 3}}) \arrow[r, "\mathrm{cu}"] & \mathrm{id}_{\operatorname{Fun}(\Delta^1,\mathcal{D})}
\end{tikzcd}$$

together with a null-homotopy of the composite functor $\mathrm{id}^!_{\operatorname{Fun}(\Delta^1,\mathcal{D})} \to \mathrm{id}_{\operatorname{Fun}(\Delta^1,\mathcal{D})}$. Composing the first two morphisms in the above diagram, we obtain the sequence

$$\mathrm{id}^!_{\operatorname{Fun}(\Delta^1,\mathcal{D})} \longrightarrow S_!(\mathrm{id}_{\mathcal{D}^{\times 3}}) \xrightarrow{\mathrm{cu}} \mathrm{id}_{\operatorname{Fun}(\Delta^1,\mathcal{D})}\,. \tag{53}$$

The morphism u is by Lemma 4.21 equivalent to the composite of the unit of the adjunction $S^L \dashv S$ with $\mathrm{id}^!_{\operatorname{Fun}(\Delta^1,\mathcal{D})}$. By Lemmas 4.11, 4.12 and 4.40, there exists an equivalence between $\mathrm{id}^!_{\operatorname{Fun}(\Delta^1,\mathcal{D})}$ and the functor T from Construction 4.39. It is straightforward to see, that the functor T is furthermore equivalent to the cotwist functor of the adjunction $S \dashv S^R$. There thus exists a fiber and cofiber sequence

$$T \xrightarrow{\mathrm{u}'} SS^R \xrightarrow{\mathrm{cu}} \mathrm{id}_{\mathrm{Fun}(\Delta^1,\mathcal{D})}\,, \tag{54}$$

where u' is up to equivalence a unit of the adjunction $S^L \dashv S$ composed with T, see [DKSS21] or [Chr22d, Remark 2.10]. The respective counit maps in (53) and (54) describe the same counit map. The respective unit maps are also equivalent, up to composition with an autoequivalence of SS^R. To show that this autoequivalence may be chosen trivially, we inspect the R-module of all possible autoequivalences of $SS^R = S_!(\mathrm{id}_{\mathcal{D}^{\times 3}})$. We have

$$\mathrm{Map}(S_!(\mathrm{id}_{\mathcal{D}^{\times 3}}), S_!(\mathrm{id}_{\mathcal{D}^{\times 3}})) \simeq \mathrm{Map}(\mathrm{ladj}(S_!)S_!(\mathrm{id}_{\mathcal{D}^{\times 3}}), \mathrm{id}_{\mathcal{D}^{\times 3}})\,,$$

where $\mathrm{ladj}(S_!) = S^L(\text{-})S^{RR}$ is the left adjoint of $S_!$, with S^{RR} the right adjoint of S^R. Since $S^R \simeq S^L \circ T$, we have $S^{RR} \simeq T^{-1} \circ S$ and thus

$$\mathrm{ladj}(S_!)S_!(\mathrm{id}_{\mathcal{D}^{\times 3}}) \simeq S^L SS^R S^{RR} \simeq S^L SS^L S\,.$$

The functor $S^L S$ splits as

$$S^L S \simeq \mathrm{id}_{\mathcal{D}^{\times 3}} \oplus P\,,$$

where P is the twist functor of the adjunction $S^L \dashv S$. It acts via 'rotation', meaning that P sends the i-th component of the direct sum to the $(i-1)$-th component of the direct sum for all $i \in \mathbb{Z}/3\mathbb{Z}$ and then acts with some suspensions on the three components. It is straightforward to see, that there are no non-zero natural transformations between $\mathrm{id}_{\mathcal{D}^{\times 3}}$ and P or P^2. It follows that the morphism

$$\mathrm{RMod}_R^{\oplus 3} \simeq \mathrm{Map}_{\mathrm{Lin}_R(\mathcal{D}^{\times 3},\mathcal{D}^{\times 3})}(\mathrm{id}_{\mathcal{D}^{\times 3}}, \mathrm{id}_{\mathcal{D}^{\times 3}}) \xrightarrow{S_!} \mathrm{Map}_{\mathrm{Lin}_R(\mathrm{Fun}(\Delta^1,\mathcal{D}),\mathrm{Fun}(\Delta^1,\mathcal{D}))}(SS^R, SS^R)$$

is an equivalence. This shows that every possible autoequivalence SS^R can be accommodated by choosing a different Hochschild-class in $\mathcal{D}^{\times 3}$. We may thus conclude from the existence of the cofiber sequence (54) that there exists some choice of Hochschild class $(\sigma')^{\times 3} \in R^{\oplus 3}$, which turns (53) into a cofiber sequence. Since $\mathrm{ladj}(S_!)S_!$ contains the identity as a direct summand, we find that $S_!$ is a conservative functor. The fact that $\sigma'_{\mathcal{D}^{\times 3}}$ induces an equivalence $S_!(\mathrm{id}^!_{\mathcal{D}^{\times 3}}) \simeq S_!(\mathrm{id}_{\mathcal{D}^{\times 3}})$ thus implies that $\sigma'\colon \mathrm{id}^!_{\mathcal{D}^{\times 3}} \to \mathrm{id}_{\mathcal{D}^{\times 3}}$ is already an equivalence. It follows that $\sigma' \in \pi_0(R)$ must be an invertible element. Upon composing σ' with its inverse in the ring $\pi_0(R)$ the cofiber sequence (53) clearly remains a cofiber sequence. We may thus choose $\sigma' = \sigma_{\mathcal{D}}$ as desired, concluding the proof. □

Remark 4.42. Let again $\mathcal{D} = \mathrm{RMod}_R$. We show below that a variation of the first part of the proof of Lemma 4.41 can be used to prove the equivalence

$$\mathrm{HH}([1]) \simeq -\,\mathrm{HH}(\mathrm{id}_{\mathcal{D}}) \simeq -\,\mathrm{id}_R\,.$$

We can further relax the assumptions of this argument as follows: let $E\colon \mathrm{LinCat}_R^{\mathrm{cpt}} \to \mathrm{RMod}_R$ be an additive invariant, on the ∞-category of compactly generated R-linear ∞-categories and compact objects preserving R-linear functors, meaning that E sends split-exact sequences in $\mathrm{LinCat}_R^{\mathrm{cpt}}$ to split cofiber sequences. Then E satisfies $E([1]) \simeq -\mathrm{id}_{E(\mathcal{C})}$, for any $\mathcal{C} \in \mathrm{LinCat}_R^{\mathrm{cpt}}$ and $[1]\colon \mathcal{C} \to \mathcal{C}$ the suspension functor.

Applying HH(-) to the split-exact sequence

$$\mathcal{D} \xrightarrow{\varsigma_1} \mathrm{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\varrho_2} \mathcal{D}$$

yields the split cofiber sequence

$$R \xrightarrow{(\mathrm{HH}([-1]),\mathrm{HH}([-1]))} R^{\oplus 2} \xrightarrow{\mathrm{HH}(\varrho_2)} R\,,$$

showing that $\mathrm{HH}(\varrho_2) \simeq (-\alpha, \alpha)$ for some autoequivalence $\alpha\colon R \to R$.

We have $\varrho_2 \circ \varsigma_2[2] \simeq [2]$ and $\varrho_2 \circ \varsigma_3[3] \simeq [3]$. Applying HH(-) to the functors

$$([2],[3])\colon \mathcal{D}^{\oplus 2} \xrightarrow{(\varsigma_2[2],\varsigma_3[3])} \mathrm{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\varrho_2} \mathcal{D}$$

yields

$$R^{\oplus 2} \xrightarrow{(\mathrm{id}_R,\mathrm{id}_R)} R^{\oplus 2} \xrightarrow{(-\alpha,\alpha)} R\,.$$

We obtain that $\mathrm{HH}([2]) \simeq -\alpha \simeq -\mathrm{HH}([3])$, as desired.

Proof of Proposition 4.38. We only prove part (2), the proof of part (1) is analogous.

We first prove the proposition in the case that $F = 0_{\mathcal{D}}\colon 0 \to \mathcal{D}$. By Lemma 3.45, there is a pullback diagram in LinCat_R

$$\begin{tikzcd} \mathcal{V}^m_{0_{\mathcal{D}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{V}^{m-1}_{0_{\mathcal{D}}} \arrow[d, "\varrho_1[-1]"] \\ \mathcal{V}^3_{0_{\mathcal{D}}} \arrow[r, "\varrho_3[-3]"] & \mathcal{D} \end{tikzcd}$$

such that the functor $\varrho_i[-i]\colon \mathcal{V}^m_{0_{\mathcal{D}}} \to \mathcal{D}$ factors for $i = 1, 2$ as

$$\mathcal{V}^m_{0_{\mathcal{D}}} \to \mathcal{V}^3_{0_{\mathcal{D}}} \xrightarrow{\varrho_i[-i]} \mathcal{D}$$

and for $i = 3, \dots, m$ as

$$\mathcal{V}^m_{0_{\mathcal{D}}} \to \mathcal{V}^{m-1}_{0_{\mathcal{D}}} \xrightarrow{\varrho_{i-2}[2-i]} \mathcal{D}\,.$$

To show that $\mathrm{R}^m_{\mathcal{D}}$ admits the desired weak right Calabi–Yau structure, it thus suffices by Theorem 4.35 to show this in the case $m = 3$. This case follows from combining Lemma 4.41 and Proposition 4.28.

Suppose now that $F\colon \mathcal{C} \to \mathcal{D}$ is any spherical functor with a weak right Calabi–Yau structure. Again, by Lemma 3.45, there exists a pullback diagram in LinCat_R.

$$\begin{tikzcd} \mathcal{V}^m_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{V}^{m+1}_{0_{\mathcal{D}}} \arrow[r, "\varrho_{m+1}[-m-1]"] & \mathcal{D} \end{tikzcd}$$

By Theorem 4.35, the above constructed weak right Calabi–Yau structure on $\mathrm{R}^{m+1}_{0_{\mathcal{D}}}$ glues with the Calabi–Yau structure of F to the desired weak right Calabi–Yau structure on R^m_F. $\square$

4.3.2 Calabi–Yau structures globally

We begin by recording a direct consequence of the gluing property of Calabi–Yau structures.

Theorem 4.43. *Let $\mathcal{F}\colon \operatorname{Exit}(\Gamma) \to \operatorname{LinCat}_R$ be a Γ-parametrized perverse schober.*

(i) Suppose that $\mathcal{F}$ takes values in smooth R-linear ∞-categories. Suppose further that

- *for each vertex v of Γ with incident halfedges $a_1, \dots, a_m$ and corresponding edges $e_1, \dots, e_m$, the functor*
$$\prod_{i=1}^m \operatorname{ladj}(\mathcal{F}(v \xrightarrow{a_i} e_i))\colon \prod_{i=1}^m \mathcal{F}(e_i) \longrightarrow \mathcal{F}(v)$$
carries a weak left n-Calabi–Yau structure
$$\sigma_v\colon R[n] \to \operatorname{HH}(\mathcal{F}(v), \prod_{i=1}^m \mathcal{F}(e_i))\,.$$
- *for each vertex v and $1 \leq i \leq m$, the restriction of σ_v along the functor $\operatorname{ladj}(\mathcal{F}(v \xrightarrow{a_i} e_i))$ defines a weak left $(n-1)$-Calabi–Yau structure on $\mathcal{F}(e_i)$, denoted*
$$\sigma_{e,a_i}\colon R[n-1] \to \operatorname{HH}(\mathcal{F}(e_i))\,.$$
- *for each internal edge $e \in \Gamma_1^\circ$ with incident halfedges $a \neq b$, we have $\sigma_{e,a} \simeq -\sigma_{e,b}$.*

Then the R-linear ∞-category of global sections $\mathcal{H}(\Gamma, \mathcal{F})$ is smooth and the functor from Definition 3.37
$$\partial\mathcal{F}\colon \prod_{e \in \Gamma_1^\partial} \mathcal{F}(e) \longrightarrow \mathcal{H}(\Gamma, \mathcal{F})$$
admits a weak left n-Calabi–Yau structure.

(ii) Suppose that $\mathcal{F}$ takes values in proper R-linear ∞-categories and limit preserving functors. Suppose further that

- *for each vertex v of Γ with incident halfedges $a_1, \dots, a_m$ and corresponding edges $e_1, \dots, e_m$, the functor*
$$\prod_{i=1}^m \mathcal{F}(v \xrightarrow{a_i} e_i)\colon \mathcal{F}(v) \longrightarrow \prod_{i=1}^m \mathcal{F}(e_i)$$

carries a weak right n-Calabi–Yau structure

$$\sigma_v\colon R[n] \to \mathrm{HH}^*(\prod_{i=1}^{m} \mathcal{F}(e_i), \mathcal{F}(v))\,.$$

- *for each vertex v and $1 \leq i \leq m$, the restriction of σ_v along the functor $\mathcal{F}(v \xrightarrow{a_i} e_i)$ defines a weak right $(n-1)$-Calabi–Yau structure on $\mathcal{F}(e_i)$, denoted*
$$\sigma_{e,a_i}\colon R[n-1] \to \mathrm{HH}^*(\mathcal{F}(e_i))\,.$$
- *for each internal edge $e \in \Gamma_1^\circ$ with incident halfedges $a \neq b$, we have $\sigma_{e,a} \simeq -\sigma_{e,b}$.*
- *the R-linear ∞-category of global sections $\mathcal{H}(\Gamma, \mathcal{F})$ is proper*[1].

Then the evaluation functor at the external edges

$$\prod_{e\in\Gamma_1^\partial} \mathrm{ev}_e\colon \mathcal{H}(\Gamma,\mathcal{F}) \longrightarrow \prod_{e\in\Gamma_1^\partial} \mathcal{F}(e)$$

admits a weak right n-Calabi–Yau structure.

Proof. Part ii) follows from repeated application of Theorem 4.35, by using that we can compute the limit over $\mathrm{Exit}(\Gamma)$ via repeated pullbacks. Part i) follows by a similar argument from Theorem 4.33, when using that $\mathcal{H}(\Gamma, \mathcal{F})$ is equivalent to the colimit in LinCat_R of the left adjoint diagram of $\mathcal{F}$. □

Given a parametrized perverse schober without singularities, also called a locally constant perverse schober, whose generic stalk admits a Calabi–Yau structure, we show in Theorem 4.44 that its global sections admit a Calabi–Yau structure if its monodromy, see Section 3.3.4, acts trivially on the corresponding (dual) Hochschild class. Note that a direct variation on this result for arbitrary perverse schobers does not hold, as follows from a variant of Example 3.64.

Theorem 4.44. *Let $\mathcal{F}\colon \mathrm{Exit}(\Gamma) \to \mathrm{LinCat}_R$ be a Γ-parametrized perverse schober without singularities. Fix an edge e of Γ and let $\mathcal{N} = \mathcal{F}(e)$ be the generic stalk of $\mathcal{F}$.*

(i) Suppose that $\mathcal{N}$ is smooth and admits a weak left $(n-1)$-Calabi–Yau structure

$$\sigma\colon R[n-1] \to \mathrm{HH}(\mathcal{N})\,.$$

Suppose further that for each loop $\gamma\colon S^1 \to \mathbf{S}\backslash\Gamma_0$, mapping the chosen basepoint to e, composition with the monodromy equivalence in RMod_R

$$\mathrm{HH}(\mathcal{F}^{\to}(\gamma, e))\colon\ \mathrm{HH}(\mathcal{N}) \longrightarrow \mathrm{HH}(\mathcal{N})$$

preserves σ. Then the functor

$$\partial\mathcal{F}\colon \prod_{e\in\Gamma_1^\partial} \mathcal{F}(e) \longrightarrow \mathcal{H}(\Gamma,\mathcal{F}) \tag{55}$$

admits a weak left n-Calabi–Yau structure.

[1] If Γ is a spanning graph of a marked surface, one can show that this is the case if and only if each boundary component of the marked surface has at least one marked point.

(ii) Suppose that $\mathcal{N}$ *and* $\mathcal{H}(\Gamma,\mathcal{F})$ *are proper and that* $\mathcal{N}$ *admits a weak right* $(n-1)$*-Calabi–Yau structure*

$$\sigma\colon R[n-1]\to \mathrm{HH}(\mathcal{N})^*\,.$$

Suppose further that for each loop $\gamma\colon S^1\to \mathbf{S}\backslash\Gamma_0$*, mapping the chosen basepoint to* e*, composition with the dual of the monodromy equivalence in* RMod_R

$$\mathrm{HH}(\mathcal{F}^{\to}(\gamma,e))^*\colon\ \mathrm{HH}(\mathcal{N})^*\longrightarrow \mathrm{HH}(\mathcal{N})^*$$

preserves σ*. Then the functor*

$$\prod_{e\in\Gamma_1^{\partial}}\mathrm{ev}_e\colon \mathcal{H}(\Gamma,\mathcal{F})\longrightarrow \prod_{e\in\Gamma_1^{\partial}}\mathcal{F}(e)$$

admits a weak right n*-Calabi–Yau structure.*

Proof. We only prove part (1), part (2) is analogous. Using Proposition 3.47 and Lemma 3.62, we may assume that Γ has a single vertex v. Let m be the valency of v. We choose a total order of the halfedges incident to v. Applying Proposition 4.38 to the spherical adjunction $F=0_{\mathcal{N}}\colon 0\leftrightarrow\mathcal{N}:G$, with weak left n-Calabi–Yau structure on G arising from σ, yields a weak left n-Calabi–Yau structure on $R^m_{0_{\mathcal{N}}}$, which restricts on $\mathcal{N}^{\times m}$ to $\sigma^{\times m}$. The diagram $R^m_{0_{\mathcal{N}}}$ gives rise to a perverse schober $\mathcal{G}'_v$ on the m-spider Γ_m, assigning to the incidence of the i-th halfedge with v the functor $\varrho_i[-i]$.

Consider an internal edge h of Γ, which is by assumption a loop. The loop consists of two halfedges which lie in positions $1\le i<j\le m$. We modify $\mathcal{G}'_v$ by composing the functor $\mathcal{G}'_v(v\xrightarrow{j}h)$ with the suspension $[1]\colon\mathcal{N}\to\mathcal{N}$. We do this for each such internal edge h and denote the arising perverse schober on the m-spider by $\mathcal{G}_v$. We let $\mathcal{G}$ be the Γ-parametrized perverse schober, which restricts along $\mathrm{Exit}(\Gamma_m)\to\mathrm{Exit}(\Gamma)$ to $\mathcal{G}_v$. Since $\mathrm{HH}([1])=-\mathrm{HH}(\mathrm{id}_{\mathcal{N}})$, we find that $\mathcal{G}$ satisfies the assumptions of Theorem 4.43.

The monodromy of $\mathcal{G}$ is fully determined by the monodromy along a generating set of $\pi_1(\Sigma_\Gamma)$. As the generating set, we choose the loops homotopic to the loops of Γ. Using Lemma 3.59, one can show that the monodromy is independent of the chosen basepoint of these loops. We may thus let h be such a loop, with halfedge $i<j$, and $\delta\colon[0,1]\to\Sigma_v$ a curve satisfying $\delta(0),\delta(1)\in\partial\Sigma_v\cap h$, wrapping from i to j in the counterclockwise direction. Using that $\varrho_l[-l]\circ\varsigma_{l-1}[l-1]\simeq[-1]$ for $l\ge 2$, we find that the monodromy $\mathcal{G}^{\to}(\delta)\colon\mathcal{N}\to\mathcal{N}$ is equivalent to the identity.

By Proposition 3.63, the perverse schober $\mathcal{F}$ differs from $\mathcal{G}$, up to equivalence of perverse schobers, by changing the monodromy along each loop of Γ by the monodromy of $\mathcal{G}$ along that loop. Explicitly, this change of monodromy can be achieved by composing $\mathcal{G}(v\xrightarrow{j}e)$ with the monodromy equivalence $\mathcal{F}^{\to}(\delta)$, with e a loop with halfedges $i<j$ and δ the loop going from i to j. Since by assumption $\mathrm{HH}(\mathcal{F}^{\to}(\delta))\circ(\text{-})$ acts trivially on σ and $\mathcal{G}$ satisfies the assumptions of Theorem 4.43, we find that $\mathcal{F}$ does so as well. Theorem 4.43 thus shows that the functor (55) admits the desired weak left n-Calabi–Yau structure. □

4.4 Exact structures from relative Calabi–Yau structures

Consider an exact functor $G\colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories, meaning a functor which preserves finite limits and finite colimits. We can pull back the split-exact structure on $\mathcal{D}$ to an exact structure on the ∞-category $\mathcal{C}$, meaning that a sequence in $\mathcal{C}$ is exact if and only if its image under G is a split-exact sequence in $\mathcal{D}$, see Example 4.53. The goal of this section is to show that if G carries a right 2-Calabi–Yau structure and is spherical, then the exact structure on $\mathcal{C}$ is Frobenius and its extriangulated homotopy category is 2-Calabi–Yau.

In Section 4.4.1, we recall the notion of an exact ∞-category and describe pullbacks of exact structures. In Section 4.4.2, we recall what is an extriangulated category and why it is natural to consider the extriangulated homotopy categories of exact ∞-categories. In Section 4.4.3, we describe the Frobenius exact structure arising from a spherical functor carrying a 2-Calabi–Yau structure.

4.4.1 Exact ∞-categories

Definition 4.45 ([Bar15]). An exact ∞-category is a triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, where $\mathcal{C}$ is an additive ∞-category and $\mathcal{C}_\dagger, \mathcal{C}^\dagger \subset \mathcal{C}$ are subcategories (called subcategories of inflations and deflations), satisfying that

(1) every morphism $0 \to X$ in $\mathcal{C}$ lies in $\mathcal{C}_\dagger$ and every morphism $X \to 0$ in $\mathcal{C}$ lies in $\mathcal{C}^\dagger$.

(2) pushouts in $\mathcal{C}$ along morphisms in $\mathcal{C}_\dagger$ exist and lie in $\mathcal{C}_\dagger$. Dually, pullbacks in $\mathcal{C}$ along morphisms in $\mathcal{C}^\dagger$ exist and lie in $\mathcal{C}^\dagger$.

(3) Given a commutative square in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & Y \arrow[d, "c"] \\ X' \arrow[r, "d"] & Y' \end{tikzcd}$$

the following are equivalent.

- The square is pullback, $c \in \mathcal{C}_\dagger$ and $d \in \mathcal{C}^\dagger$.
- The square is pushout, $b \in \mathcal{C}_\dagger$ and $a \in \mathcal{C}^\dagger$.

When the exact structure is clear from the context, we also simply refer to $\mathcal{C}$ as an exact ∞-category.

Definition 4.46. An exact sequence $X \to Y \to Z$ in an exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ consists of a fiber and cofiber sequence in $\mathcal{C}$ as follows, with $a \in \mathcal{C}_\dagger$ and $b \in \mathcal{C}^\dagger$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "b"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

Example 4.47.

- Let $\mathcal{C}$ be an additive ∞-category. Then there is an exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ with inflations consisting of inclusions of direct summands and deflations consisting of projections onto direct summands, called the split-exact structure. The exact sequences are the split-exact sequences $X \hookrightarrow X \oplus Y \twoheadrightarrow Y$.

- Let $\mathcal{C}$ be a stable ∞-category. Then $(\mathcal{C}, \mathcal{C}, \mathcal{C})$ is an exact ∞-category.

Definition 4.48. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ be an exact ∞-category.

1) An object $P \in \mathcal{C}$ is called projective if every exact sequence $X \to Y \to P$ is split-exact. An object $I \in \mathcal{C}$ is called injective if every exact sequence $I \to Y \to Z$ is split-exact.

2) We say that $\mathcal{C}$ has enough projectives if for each object $X \in \mathcal{C}$ there exists an exact sequence $X \to P \to Y$ with P projective. Similarly, we say that $\mathcal{C}$ has enough injectives if for each object $Y \in \mathcal{C}$ there exists an exact sequence $Y \to I \to X$ with I injective.

3) We say that $\mathcal{C}$ is Frobenius if $\mathcal{C}$ has enough projectives and injectives and the classes of projective and injective objects coincide.

The ∞-categorical version of the stable 1-category of a Frobenius exact 1-category is the following.

Proposition 4.49 ([JKPW22])**.** *Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ be a Frobenius exact ∞-category and W the class of morphisms $f\colon X \to Y$ which fit into an exact sequence*

$$X \xrightarrow{(f,g)} Y \oplus I \longrightarrow J$$

with I and J injective and g arbitrary. Then, the ∞-categorical localisation $\bar{\mathcal{C}} := \mathcal{C}[W^{-1}]$ of $\mathcal{C}$ at W is a stable ∞-category.

Proof idea. Applying [Cis19, Prop. 7.5.6] and its dual version, one finds that $\bar{\mathcal{C}}$ admits finite limits and colimits and pushouts and pullbacks coincide. □

Lemma 4.50. *Suppose that $\mathcal{C}$ is a k-linear ∞-category and $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ a Frobenius exact ∞-category. Then the stable ∞-category $\bar{\mathcal{C}}$ from Proposition 4.49 inherits the structure of a k-linear ∞-category in the sense of [Lur18, D.1.1.1].*

Proof. As argued in [Lur18, Section D.1.2], the datum of a k-linear structure on a presentable ∞-category $\mathcal{D}$ is equivalent to the datum of a monoidal functor $\mathrm{LMod}_k^{\mathrm{ff}} \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ with domain the symmetric monoidal ∞-category of (automatically free) k-modules of finite rank. In turn, we can identify this datum with the datum of an object of $\mathrm{LMod}_k^{\mathrm{ff}} \times_{\mathrm{Alg}(\mathrm{Cat}_\infty)} \mathrm{LMod}(\mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \mathcal{D}$, see [Lur17, Section 4.7.1], where Cat_∞ is considered as a monoidal ∞-category with the Cartesian monoidal structure.

Let $Z = \coprod_W \Delta^1$ and $\bar{Z} = \coprod_W \Delta^0$. We have apparent functors $\pi\colon Z \to \bar{Z}$ and $Z \to \mathcal{C}$. Using the module structure of $\mathcal{C}$, the latter functor gives rise to a functor $f\colon Z \times \mathrm{LMod}_k^{\mathrm{ff}} \to \mathcal{C} \times \mathrm{LMod}_k^{\mathrm{ff}} \to \mathcal{C}$. We define the k-linear ∞-category $\mathcal{C}'$ as the pushout

$$\begin{tikzcd} Z \times \mathrm{LMod}_k^{\mathrm{ff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C} \arrow[d] \\ \bar{Z} \times \mathrm{LMod}_k^{\mathrm{ff}} \arrow[r] & \mathcal{C}' \end{tikzcd}$$

in $\mathrm{LMod}_k^{\mathrm{ff}} \times_{\mathrm{Alg}(\mathrm{Cat}_\infty)} \mathrm{LMod}(\mathrm{Cat}_\infty^\times)$. By [Lur17, 4.2.3.5], the underlying diagram in Cat_∞ is also pushout. For any ∞-category $\mathcal{A}$, we obtain a pullback square in Cat_∞ as follows.

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}', \mathcal{A}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\mathcal{C}, \mathcal{A}) \arrow[d] \\ \mathrm{Fun}(\bar{Z} \times \mathrm{LMod}_k^{\mathrm{ff}}, \mathcal{A}) \arrow[r, hook] & \mathrm{Fun}(Z \times \mathrm{LMod}_k^{\mathrm{ff}}, \mathcal{A}) \end{tikzcd}$$

Since W is closed under the action by $\mathrm{LMod}_k^{\mathrm{ff}}$, we find that $\mathcal{C}'$ is the ∞-categorical localization of $\mathcal{C}$ at W, see [Cis19, Def. 7.1.2]. This concludes the argument, showing that $\bar{\mathcal{C}}$ inherits a k-linear structure. $\square$

Definition 4.51.
1. An exact functor $G\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ between exact ∞-categories consists of a functor $G\colon \mathcal{C} \to \mathcal{D}$ which preserves zero objects, inflations, deflations, pushouts along inflations and pullbacks along deflations.

2. A sub-exact structure of an exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ consists of an exact ∞-category $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$, satisfying that $\mathcal{C}_\ddagger \subset \mathcal{C}_\dagger$ and $\mathcal{C}^\ddagger \subset \mathcal{C}^\dagger$.

Sub-exact structures can be pulled back along exact functors as follows.

Lemma 4.52. *Let $G\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be an exact functor between exact ∞-categories and $(\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$ a sub-exact structure of $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$. Then there exists a sub-exact structure $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$ of $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, such that G defines an exact functor $G\colon (\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger) \to (\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$.*

Proof. We set $\mathcal{C}_\ddagger \subset \mathcal{C}_\dagger$ to be the subcategory of morphisms whose image under G lies in $\mathcal{D}_\ddagger$. We define $\mathcal{C}^\ddagger$ similarly. It is straightforward to verify that $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$ is an exact ∞-category and that $G\colon (\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger) \to (\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$ is an exact functor. $\square$

Example 4.53. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories and $G\colon \mathcal{C} \to \mathcal{D}$ is an exact functor in the usual sense, i.e. a functor that preserves finite limits and colimits. Then G defines an exact functor $G\colon (\mathcal{C}, \mathcal{C}, \mathcal{C}) \to (\mathcal{D}, \mathcal{D}, \mathcal{D})$ between exact ∞-categories. Applying Lemma 4.52, we obtain a non-trivial additional exact structure $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ on $\mathcal{C}$ by pulling back the split-exact structure $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ on $\mathcal{D}$. A fiber and cofiber sequence in $\mathcal{C}$ is exact in $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ if and only if its image under G is split-exact.

Proposition 4.54. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. If G is a spherical functor, then the exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ from Example 4.53 is Frobenius. The subcategory of injective and projective objects of $\mathcal{C}$ consists of the additive closure of the essential image of the right adjoint H of G.*

Remark 4.55. A result similar to Proposition 4.54 appears for triangulated categories in [BS21], see Theorem 4.23 and Remark 4.24 in loc. cit.

Proof of Proposition 4.54. Let F be the left adjoint of G and H the right adjoint of G. By the sphericalness of G, we have that $H \simeq F \circ T_{\mathcal{D}}$, with $T_{\mathcal{D}}$ the cotwist functor of the adjunction $G \dashv H$, see [DKSS21, Cor. 2.5.16]. In particular, it follows that the essential images of F and H, and hence also their additive closures in $\mathcal{C}$, agree.

We begin by showing that every object of the form $F(d) \in \mathcal{C}$ with $d \in \mathcal{D}$ is injective. A dual argument shows that every object of the form $H(d) \in \mathcal{C}$ is projective. Let $c \in \mathcal{C}$ and consider an extension $\alpha \in \operatorname{Ext}^1_{\mathcal{C}}(F(d), c)$. The extension α corresponds to a fiber and cofiber sequence $c \to z \to F(d)$ in $\mathcal{C}$. The image of this fiber and cofiber sequence under G splits (i.e. defines an exact sequence) if and only if $G(\alpha) \simeq 0$. The commutative diagram,

$$\begin{tikzcd} & \operatorname{Ext}^1_{\mathcal{D}}(GF(d), G(c)) \arrow[dr, "\mathrm{u}"] & \\ \operatorname{Ext}^1_{\mathcal{C}}(F(d), c) \arrow[ur, "G"] \arrow[rr, "\simeq"'] & & \operatorname{Ext}^1_{\mathcal{D}}(d, G(c)) \end{tikzcd}$$

with u the unit of $F \dashv G$, shows that $G(\alpha) \simeq 0$ if and only if $\alpha \simeq 0$. This is the case, if and only if the fiber and cofiber sequence $c \to z \to F(d)$ already splits. Since c is chosen arbitrarily, it follows that $F(d)$ is injective.

Next, we show that $\mathcal{C}$ has enough projective objects, a similar argument shows that $\mathcal{C}$ has enough injective objects. Let $c \in \mathcal{C}$. We have a fiber and cofiber sequence $c \xrightarrow{\mathrm{u}_c} HG(x) \to T_{\mathcal{C}}(c)$, where u_c is a unit map and $T_{\mathcal{C}}$ the twist functor of $G \dashv H$. We apply G and extend to the diagram

$$\begin{tikzcd} & T_{\mathcal{D}}G(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ G(c) \arrow[r, "G\,\mathrm{u}_c"'] \arrow[d] \arrow[dr, phantom, "\square"] & GHG(c) \arrow[r, "\mathrm{cu}_{G(c)}"'] \arrow[d] \arrow[dr, phantom, "\square"] & G(c) \arrow[d] \\ 0 \arrow[r] & GT_{\mathcal{C}}(c) \arrow[r] & 0 \end{tikzcd}$$

with $\mathrm{cu}_{G(c)}$ a counit map and all squares biCartesian. This shows that the sequence $G(c) \xrightarrow{G\,\mathrm{u}_c} GHG(x) \to GT_{\mathcal{C}}(c)$ in $\mathcal{D}$ splits and hence that that $c \xrightarrow{\mathrm{u}_c} HG(x) \to T_{\mathcal{C}}(c)$ is exact, as desired

Finally, we show that any injective or projective object lies in the additive closure (closure under direct summands) of the essential image of H. Let I be injective.

Then we have an exact sequence $I \to HG(I) \to T_{\mathcal{C}}(I)$ in $\mathcal{C}$. Since I is injective, this sequence splits, showing that I is a direct summand of $HG(I)$, as desired. If P is projective, we similarly find an exact sequence $T_{\mathcal{C}}^{-1} \to FG(P) \to P$, showing that P is a direct summand of $FG(P) \simeq HT_{\mathcal{D}}^{-1}G(P)$, as desired, concluding the proof. $\square$

4.4.2 Extriangulated categories

Extriangulated categories were introduced by Nakaoka-Palu in [NP19] as a simultanious generalization of triangulated and exact 1-categories. An extriangulated category $(C, \mathbb{E}, \mathfrak{s})$ consists of

- an additive 1-category C,
- an additive bifunctor $\mathbb{E}\colon C^{\mathrm{op}} \times C \to \mathrm{Ab}$ to the additive 1-category of abelian groups, of which we think as describing an interesting class of extensions in C and
- a sequence $(Y \to Z \to X) = \mathfrak{s}(\alpha)$, called realization, associated to each $\alpha \in \mathbb{E}(X, Y)$,

subject to number of conditions, see [NP19].

Examples of extriangulated categories are, besides triangulated and exact categories, extension closed subcategories of triangulated categories. A further natural source of extriangulated categories are the homotopy 1-categories of exact ∞-categories. To see this, note that by [Kle20] any exact ∞-category admits a stable hull, meaning that it can be embedded as an extension closed subcategory in a stable ∞-category. Passing to homotopy categories, one obtains an extriangulated structure on the homotopy category, as it is an extension closed subcategory of the triangulated homotopy category of the stable hull. An independent and more direct proof that the homotopy category is extriangulated also appears in [NP20].

Given an exact ∞-category $\mathcal{C}$, the additive functor of extensions $\mathbb{E}\colon \mathrm{ho}\,\mathcal{C}^{\mathrm{op}} \times \mathrm{ho}\,\mathcal{C} \to \mathrm{Ab}$ in the extriangulated structure of $\mathrm{ho}\,\mathcal{C}$ describes isomorphism classes of exact sequences in $\mathcal{C}$. The extriangulated structure on $\mathrm{ho}\,\mathcal{C}$ can be useful when studying the exact ∞-category $\mathcal{C}$. For instance, one can use it to formulate a notion of cluster-tilting object in the exact ∞-category $\mathcal{C}$, which is simply an object which is cluster-tilting when considered as an object in the extriangulated homotopy category $\mathrm{ho}\,\mathcal{C}$ in the sense of Definition 4.61 below.

The definition of a Frobenius extriangulated category is analogous to the definition of a Frobenius exact ∞-category, see Definition 4.48.

Definition 4.56. Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangualated category.

1) An object $P \in C$ is called projective if $\mathbb{E}(P, X) \simeq 0$ for all $X \in C$ and injective if $\mathbb{E}(X, P) \simeq 0$ for all $X \in C$.

2) We say that C has enough projectives if for each object $X \in C$ there exists an exact sequence $X \to P \to Y$ with P projective. Similarly, we say that C

has enough injectives if for each object $Y \in C$ there exists an exact sequence $Y \to I \to X$ with I injective.

3) We say that C is Frobenius if C has enough projectives and injectives and the classes of projective and injective objects coincide.

The condition of being a projective or injective object in an exact ∞-category can be tested in the extriangulated homotopy category. We further have the following.

Lemma 4.57. *An exact ∞-category is Frobenius if and only if its extriangulated homotopy category is Frobenius.*

Remark 4.58. Given a Frobenius extriangulated category C, its stable category $\bar{C}$ is defined as the quotient category by the ideal of morphisms which factor through an injective object. The stable category $\bar{C}$ inherits the structure of a triangulated category, see Corollary 7.4 and Remark 7.5 in [NP19].

If $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a Frobenius exact ∞-category, the triangulated homotopy category of its associated stable ∞-category $\bar{\mathcal{C}}$ can be identified with the triangulated stable category $\overline{\mathrm{ho}\,\mathcal{C}}$ of the Frobenius extriangulated homotopy category $\mathrm{ho}\,\mathcal{C}$.

Definition 4.59. An extriangulated category $(C, \mathbb{E}, \mathfrak{s})$ is called extrianguled 2-Calabi–Yau if there exists an isomorphism $\mathbb{E}(X, Y) \simeq \mathbb{E}(Y, X)^*$, bifunctorial in X and Y.

Lemma 4.60. *Let $(\mathcal{C}, \mathcal{C}^\dagger, \mathcal{C}_\dagger)$ be an exact ∞-category, whose extriangulated homotopy category is extriangulated 2-Calabi–Yau. Then the homotopy category $\mathrm{ho}\,\bar{C}$ of its stable category $\bar{C}$ is triangulated 2-Calabi–Yau.*

Proof. We denote by $\mathbb{E}\colon \mathrm{ho}\,\mathcal{C}^{\mathrm{op}} \times \mathrm{ho}\,\mathcal{C} \to \mathrm{Ab}$ the additive bifunctor parametrizing extensions in the extriangulated homotopy category of $\mathcal{C}$. For any $X, Y \in \bar{\mathcal{C}}$ arising from $X', Y' \in \mathcal{C}$, the abelian group $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(X, Y)$ can be seen as describing equivalence classes of fiber and cofiber sequences $Y \to Z \to X$, which arise from an exact sequence $Y' \to Z' \to X'$ in $\mathcal{C}$. We thus find that $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(X, Y) \simeq \mathbb{E}(X', Y')$, and this description is bifunctorial in X and Y on the level of the homotopy category of $\bar{\mathcal{C}}$.

Fix $X, Y \in \bar{C}$. We denote by $Y[-1]$ and $Y[2]$ the shifts of Y with respect to the triangulated structure of $\bar{C}$. We have

$$\mathrm{Hom}_{\bar{C}}(X, Y) \simeq \mathbb{E}(X', Y[-1]') \simeq \mathbb{E}(Y[-1]', X')^* \simeq \mathrm{Hom}_{\bar{C}}(Y, X[2])^* \,.$$

These equivalences are bifunctorial in X and Y, showing the claim. □

We conclude this section with stating the definition of a cluster-tilting object in an extriangulated category.

Definition 4.61. Let X be an object in an extriangulated category $(C, \mathbb{E}, \mathfrak{s})$.

- X is called basic if it can be decomposed into finitely many indecomposable direct summands which are pairwise non-isomorphic.

- X is called rigid if
$$\mathbb{E}(X,X) \simeq 0\,.$$
Further, a rigid object X is called maximal rigid, if there exist no objects $Y \in C$, satisfying that $X \oplus Y$ is rigid and Y is not a direct sum of summands of C.

- X is called cluster-tilting if it is basic, rigid and if for $Y \in C$ the conditions $\mathbb{E}(X,Y) \simeq 0$ and $\mathbb{E}(Y,X) \simeq 0$ each imply that Y is a finite direct sum of direct summands of X.

4.4.3 Exact structures via relative Calabi–Yau structures

Let $\kappa = k$ be a field or $\kappa = k[t_{2n}^{\pm}]$, for $n \geq 1$, the commutative dg-algebra of graded Laurent polynomials over a field with generator in degree $|t_{2n}| = 2n$. We fix a functor $G\colon \mathcal{B} \to \mathcal{A}$ between κ-linear smooth and proper ∞-categories, which is spherical and carries a weak right n-Calabi–Yau structure.

We begin this section by exhibiting a relative version of the triangulated n-Calabi–Yau condition $\operatorname{Ext}^i(X,Y) \simeq \operatorname{Ext}^{n-i}(Y,X)^*$ for $\mathcal{B}$, using the relative Calabi–Yau structure of $\mathcal{B}$. We then describe a Frobenius exact ∞-structure on $\mathcal{B}$ arising from G.

Definition 4.62. Let $X, Y \in \mathcal{B}^{\mathrm{c}}$ be compact objects.

1. We denote by $\operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y) \subset \operatorname{Mor}_{\mathcal{B}}(X,Y) \in \mathcal{D}(\kappa)$ the maximal direct summand satisfying that the composite with
$$G\colon\ \operatorname{Mor}_{\mathcal{B}}(X,Y) \longrightarrow \operatorname{Mor}_{\mathcal{A}}(G(X),G(Y))$$
yields the zero morphism in $\mathcal{D}(\kappa)$. We call $\operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y)$ the *Calabi–Yau morphism object.*

2. We denote by $\operatorname{Ext}_{\mathcal{B}}^{i,\mathrm{CY}}(X,Y) \coloneqq \mathrm{H}_0 \operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y[i]) \in \mathrm{N}(\mathrm{Vect}_k)$ the k-vector space of *Calabi–Yau extensions.*

Lemma 4.63. *(1) The Calabi–Yau morphism objects assemble into a functor*
$$\operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathcal{D}(\kappa)\,.$$
There further exists a natural transformation $\operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-}) \to \operatorname{Mor}_{\mathcal{B}}(\text{-},\text{-})$ *which at a point* $(X,Y) \in \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}}$ *is given by the inclusion of the direct summand* $\operatorname{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y) \subset \operatorname{Mor}_{\mathcal{B}}(X,Y)$*.*

(2) The Calabi–Yau extensions form the maximal subfunctor
$$\operatorname{Ext}_{\mathcal{B}}^{i,\mathrm{CY}}(\text{-},\text{-}) \subset \operatorname{Ext}_{\mathcal{B}}^{i}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathrm{N}(\mathrm{Vect}_k)$$
satisfying that $\operatorname{Ext}_{\mathcal{B}}^{i,\mathrm{CY}}(G(\text{-}),G(\text{-})) \simeq 0$*.*

Proof. Consider the mapping space functor $\mathrm{Map}_{\mathcal{B}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathcal{S}$ and the functor $\pi_0 \mathrm{Map}_{\mathcal{B}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathcal{S}$, obtained from composing $\mathrm{Map}_{\mathcal{B}}(\text{-},\text{-})$ with the functor $\mathcal{S} \xrightarrow{\pi_0} N(\mathrm{Set}) \hookrightarrow \mathcal{S}$ taking connected components. Given $X, Y \in \mathcal{B}$, we denote by $(\pi_0 \mathrm{Map}_{\mathcal{B}})^{\mathrm{CY}}(X,Y) \subset \pi_0 \mathrm{Map}_{\mathcal{B}}(X,Y)$ the subset of homotopy classes of morphisms, satisfying that their image under G is a zero morphism in $\mathcal{A}$. Since zero morphisms are closed under composition, we find that $(\pi_0 \mathrm{Map}_{\mathcal{B}})^{\mathrm{CY}}(\text{-},\text{-})$ defines a subfunctor of $\pi_0 \mathrm{Map}_{\mathcal{B}}(X,Y)$. We define the Calabi–Yau mapping space functor $\mathrm{Map}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathcal{S}$ as the pullback of the diagram

$$\begin{tikzcd} & \mathrm{Map}_{\mathcal{B}}(\text{-},\text{-}) \arrow[d] \\ (\pi_0 \mathrm{Map}_{\mathcal{B}})^{\mathrm{CY}}(\text{-},\text{-}) \arrow[r] & \pi_0 \mathrm{Map}_{\mathcal{B}}(\text{-},\text{-}) \end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}}, \mathcal{S})$. Using the fully-faithfulness of the Yoneda embedding, we can define a functor $\mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathcal{D}(\kappa)$ via

$$\mathrm{Map}_{\mathcal{D}(\kappa)}(C, \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-})) \simeq \mathrm{Map}_{\mathcal{B}}^{\mathrm{CY}}(\text{-} \otimes C, \text{-})\,.$$

The natural transformation $\mathrm{Map}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-}) \subset \mathrm{Map}_{\mathcal{B}}(\text{-},\text{-})$ induces a natural transformation $\eta\colon \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-}) \to \mathrm{Mor}_{\mathcal{B}}(\text{-},\text{-})$. Note that for any pair $(X,Y) \in \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}}$, the natural transformation η evaluates on the i-th homology group to the inclusion

$$(\pi_0 \mathrm{Map}_{\mathcal{B}})^{\mathrm{CY}}(X[i],Y) \subset \pi_0 \mathrm{Map}_{\mathcal{B}}(X[i],Y)\,. \tag{56}$$

Using that any object in $\mathcal{D}(\kappa)$ is equivalent to its homology, which is the direct sum of suspensions of copies of κ, we find that the inclusion $\mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y) \subset \mathrm{Mor}_{\mathcal{B}}(X,Y)$ of the Calabi–Yau morphism object evaluates on the i-th homology group to the inclusion (56). This implies that the functor $\mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\text{-},\text{-})$ indeed describes the Calabi–Yau morphism objects and η evaluates pointwise to their inclusion. This shows part (1).

Part (2) follows from the fact that on i-th homology, η exhibits $\mathrm{Ext}_{\mathcal{B}}^{i,\mathrm{CY}}(\text{-},\text{-})$ as the desired maximal subfunctor of $\mathrm{Ext}_{\mathcal{B}}^{i}(\text{-},\text{-})$, by the description in (56). □

Recall that given $A \in \mathcal{D}(\kappa)$, we denote $A^* = \mathrm{Mor}_{\mathcal{D}(\kappa)}(A, \kappa)$. Given a k-vector space B, we denote $B^* = \mathrm{Hom}_{\mathrm{Vect}_k}(B, k)$.

Proposition 4.64. *Let $X, Y \in \mathcal{B}^c$ be compact objects.*

(1) There exists an equivalence in $\mathcal{D}(\kappa)$

$$\mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X,Y) \simeq \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(Y[-n],X)^*\,,$$

bifunctorial in $X \in \mathcal{B}^{\mathrm{c,op}}$ and $Y \in \mathcal{B}^{\mathrm{c}}$.

(2) There exists an equivalence in $N(\mathrm{Vect}_k)$

$$\mathrm{Ext}_{\mathcal{B}}^{i,\mathrm{CY}}(X,Y) \simeq \mathrm{Ext}_{\mathcal{B}}^{n-i,\mathrm{CY}}(Y,X)^*\,,$$

bifunctorial in $X \in \mathcal{B}^{\mathrm{c,op}}$ and $Y \in \mathcal{B}^{\mathrm{c}}$.

Part (2) of Proposition 4.64 should be seen as a relative version of the triangulated n-Calabi–Yau property. We postpone the proof of Proposition 4.64 to the end of this subsection.

Let $(\mathcal{B}, \mathcal{B}_\dagger, \mathcal{B}^\dagger)$ be the exact ∞-category obtained by pulling back the split-exact structure on $\mathcal{A}$, see Example 4.53. We denote by $(\mathrm{ho}\,\mathcal{B}^{\mathrm{c}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}, \mathfrak{s})$ the arising extriangulated homotopy category. Here, we use that $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}$ describes the extensions in this extriangulated category, as follows from Lemma 4.65. To be precise, we also abuse notation by labeling by $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}$ the second functor in the factorization

$$\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}} \colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \longrightarrow \mathrm{ho}\,\mathcal{B}^{\mathrm{c,op}} \times \mathrm{ho}\,\mathcal{B}^{\mathrm{c}} \longrightarrow N(\mathrm{Vect}_k)\,.$$

Lemma 4.65. *Let $X, Y \in \mathcal{B}^{\mathrm{c}}$. Consider a fiber and cofiber sequence $X \to Z \xrightarrow{\alpha} Y$ in $\mathcal{B}$ and let $\beta \colon X \to Y[1]$ be the cofiber morphism of α. Then β lies in $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}(X,Y) \subset \mathrm{Ext}^{1}_{\mathcal{B}}(X,Y)$ if and only if the image of the fiber and cofiber sequence under G splits.*

Proof. We show that the fiber and cofiber sequence $G(X) \to G(Z) \xrightarrow{G(\alpha)} G(Y)$ splits if and only if the cofiber morphism $G(\beta)$ vanishes. By definition, the latter is equivalent to β being a Calabi–Yau extension. The forward implication is clear. For the converse, suppose that $G(\beta)$ is zero. Then its fiber morphism, given by $G(\alpha)$, is equivalent to $G(X) \oplus G(Y) \xrightarrow{(0,\mathrm{id})} G(Y)$. This shows that the fiber and cofiber sequence splits. □

Proposition 4.66. *The extriangulated category $(\mathrm{ho}\,\mathcal{B}^{\mathrm{c}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}, \mathfrak{s})$ is Frobenius and extriangled 2-Calabi–Yau. The exact ∞-category $(\mathcal{B}^{\mathrm{c}}, \mathcal{B}^{\mathrm{c}}_\dagger, \mathcal{B}^{\mathrm{c},\dagger})$ is hence also Frobenius.*

Proof. The Frobenius property is shown in Proposition 4.54. The statement about the 2-Calabi–Yau property follows directly from Proposition 4.64. □

Remark 4.67. The proof of Proposition 4.54 shows that the suspension functor of the stable ∞-category $\bar{\mathcal{B}}^{\mathrm{c}}$ arising from $(\mathcal{B}^{\mathrm{c}}, \mathcal{B}^{\mathrm{c}}_\dagger, \mathcal{B}^{\mathrm{c},\dagger})$, see Proposition 4.49, acts on objects as the twist functor of the spherical adjunction $G \dashv H$, or by the relative 2-Calabi–Yau structure equivalently as the delooping (or negative shift) of the Serre functor of $\mathcal{B}$.

It is an interesting problem to determine whether $\bar{\mathcal{B}}^{\mathrm{c}}$ inherits a weak right 2-Calabi–Yau structure from the relative weak right 2-Calabi–Yau structure of $\mathcal{B}$.

We conclude this section with the proof of Proposition 4.64.

Proof of Proposition 4.64. Part (2) follows from part (1). We proceed with part (1). Let H be the right adjoint of G and $\mathrm{u} \colon \mathrm{id}_{\mathcal{B}} \to HG$ be the unit. Note that there is a commutative triangle as follows.

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \arrow[r, "G"] \arrow[dr, "\mathrm{u}\circ -"'] & \mathrm{Map}_{\mathcal{A}}(G(\text{-}_1), G(\text{-}_2)) \arrow[d, "\simeq"] \\ & \mathrm{Map}_{\mathcal{B}}(\text{-}_1, HG(\text{-}_2)) \end{tikzcd} \tag{57}$$

We thus have a commutative diagram in $\mathrm{Fun}(\mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}}, \mathcal{D}(\kappa))$

$$\begin{tikzcd}
\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \arrow[r] \arrow[d] & \mathrm{Mor}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \arrow[d, "\mathrm{u} \circ \text{-}"] \\
0 \arrow[r] & \mathrm{Mor}_{\mathcal{B}}(\text{-}_1, HG(\text{-}_2)))
\end{tikzcd}$$

which induces by the relative Calabi–Yau structure a natural transformation

$$\xi \colon \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \to \mathrm{fib}(\mathrm{u} \circ \text{-}) \simeq \mathrm{Mor}_{\mathcal{B}}(\text{-}_1, \mathrm{id}^*_{\mathcal{B}}(\text{-}_2)[-n])\,.$$

Consider the natural transformation $\nu \colon \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \to \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_2[-n], \text{-}_1)^*$ appearing in the following diagram.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(\text{-}_1, \mathrm{id}^*_{\mathcal{B}}(\text{-}_2)[-n]) \arrow[r, "\simeq"] & \mathrm{Mor}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(\text{-}_2)[-n], \mathrm{id}^*_{\mathcal{B}}(\text{-}_1))^* \arrow[d] & \\
\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \arrow[u, "\xi"] \arrow[r] \arrow[rr, bend right, "\nu"'] & \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(\text{-}_2)[-n], \mathrm{id}^*_{\mathcal{B}}(\text{-}_1))^* \arrow[r, "\simeq"] & \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(\text{-}_2[-n], \text{-}_1)^*
\end{tikzcd}$$

To prove part (1), we show that ν is a natural equivalence, meaning that it evaluates at any pair $X, Y \in \mathcal{B}^c$ to an equivalence. Consider the sequence $HG(Y)[-1] \to \mathrm{id}^*_{\mathcal{B}}(Y)[-n] \to Y \xrightarrow{\mathrm{u}} HG(Y)$ in $\mathcal{B}^{\mathrm{c}}$, where any three consecutive terms form a fiber and cofiber sequence. Applying $\mathrm{Mor}_{\mathcal{B}}(X, \text{-})$, we obtain the upper sequence in $\mathcal{D}(\kappa)$ in the following diagram.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(X, HG(Y)[-1]) \arrow[r] & \mathrm{Mor}_{\mathcal{B}}(X, \mathrm{id}^*_{\mathcal{B}}(Y)[-n]) \arrow[r, "\alpha"] & \mathrm{Mor}_{\mathcal{B}}(X, Y) \arrow[r] & \mathrm{Mor}_{\mathcal{B}}(X, HG(Y)) \\
0 \arrow[u] \arrow[r] & \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(X, Y) \arrow[u, hook] \arrow[r, "\mathrm{id}"] & \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(X, Y) \arrow[u, hook] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Note that any three consecutive horizontal terms in the above diagram form a fiber and cofiber sequence. Note also that the inclusions

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{B}}(X, Y) \subset \mathrm{Mor}_{\mathcal{B}}(X, \mathrm{id}^*_{\mathcal{B}}(Y)[-n]), \mathrm{Mor}_{\mathcal{B}}(X, Y)$$

split, as any morphism in $\mathcal{D}(\kappa)$ splits (into a direct sum of equivalences and zero morphisms). It follows that the Calabi–Yau morphism object is the maximal simultaneous direct summand of both $\mathrm{Mor}_{\mathcal{B}}(X, Y)$ and $\mathrm{Mor}_{\mathcal{B}}(X, \mathrm{id}^*_{\mathcal{B}}(Y)[-n])$, being preserved by α.

Using the natural equivalence $\mathrm{Mor}_{\mathcal{B}}(\text{-}_1, \text{-}_2) \simeq \mathrm{Mor}_{\mathcal{B}}(\text{-}_2, \mathrm{id}^*_{\mathcal{B}}(\text{-}_1))^*$, we see that the upper part of the above diagram is equivalent to the upper parts of the following diagram.

$$
\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(HG(Y)[-1], \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[r, "\delta"] \arrow[d] & \mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[r, "\beta"] \arrow[d, two heads] & {} \\
0 \arrow[r] & \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[r, "\mathrm{id}"] & {}
\end{tikzcd}
\tag{58}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\beta"] & \mathrm{Mor}_{\mathcal{B}}(Y, \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[r] \arrow[d, two heads] & \mathrm{Mor}_{\mathcal{B}}(HG(Y), \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[d] \\
{} \arrow[r, "\mathrm{id}"] & \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))^* \arrow[r] & 0
\end{tikzcd}
\tag{59}
$$

The left adjoint F of G is given by $(\mathrm{id}_{\mathcal{B}}^*[1-n])^{-1} \circ H$, where $\mathrm{id}_{\mathcal{B}}^*[1-n]$ is equivalent to the twist functor of the adjunction $G \dashv H$, see [DKSS21, Corollary 2.5.16]. Furthermore by [Chr22d, Lemma 2.11], the morphism (from the above sequence in $\mathcal{B}^c$)

$$FG \circ \mathrm{id}_{\mathcal{B}}^*(Y)[-n] \simeq HG(Y)[-1] \to \mathrm{id}_{\mathcal{B}}^*(Y)[-n]$$

is a counit morphism of the adjunction $F \dashv G$, which is adjoint under $FG \dashv HG$ to the unit of $G \dashv H$. The morphism δ in the above sequence is thus equivalent to the dual of the morphism

$$
\begin{aligned}
\mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X)) \xrightarrow{\mathrm{u}\circ -} & \mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], HG \circ \mathrm{id}_{\mathcal{B}}^*(X)) \,. \\
& \simeq \mathrm{Mor}_{\mathcal{B}}(FG \circ \mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X)) \\
& \simeq \mathrm{Mor}_{\mathcal{B}}(HG(Y)[-1], \mathrm{id}_{\mathcal{B}}^*(X))
\end{aligned}
$$

arising from postcomposition with the unit u of $G \dashv H$. Using the diagram (57), this shows that the Calabi–Yau morphism object $\mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))$ fits into the diagrams (58) and (59) as indicated, such that the entire diagram formed by (58) and (59) commutes. Again, the vertical morphisms split and it follows that $\mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))$ forms the maximal direct summand which is preserved by β. This shows that the composite

$$
\begin{aligned}
\mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(X, Y) \hookrightarrow & \mathrm{Mor}_{\mathcal{B}}(X, \mathrm{id}_{\mathcal{B}}^*(Y)[-n]) \\
& \simeq \mathrm{Mor}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))^* \\
& \twoheadrightarrow \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(\mathrm{id}_{\mathcal{B}}^*(Y)[-n], \mathrm{id}_{\mathcal{B}}^*(X))^* \\
& \simeq \mathrm{Mor}_{\mathcal{B}}^{\mathrm{CY}}(Y, X[n])^*
\end{aligned}
$$

and hence also ν evaluated at (X, Y) are equivalences. This concludes the proof of part (1) and the proof. $\square$

5 Ginzburg algebras of n-angulated surfaces

We begin in Section 5.1 with introducing Ginzburg algebras of quivers with potentials and their relative higher variants associated with n-angulated surfaces. We then discuss in Section 5.2 spherical adjunctions arising from ∞-categories of local systems on the n-sphere S^n, and how the arising ∞-categories can be described algebraically. This provides preparatory computations for Section 5.3, where we show that the derived ∞-categories of relative Ginzburg algebras of n-angulated surfaces arise as the global sections of perverse schobers build from these spherical adjunctions. We also construct relative Calabi–Yau structures on the derived categories of the relative Ginzburg algebras in Section 5.3.3.

In Section 5.4, we describe how one can associate to each suitable curve in the surface a global section of these perverse schobers, i.e. a module over the Ginzburg algebras. In Section 5.5, we then describe the morphism objects between the global sections associated with such curves in terms of the intersections of the curves. Sections 5.4 and 5.5 together build the foundation of a so called partial geometric model for the derived ∞-category of the relative Ginzburg algebra of an n-angulated surface.

First applications of this geometric model are given in Section 5.6. We describe in Section 5.6.1 the homology of the relative Ginzburg algebras in terms of the Jacobian gentle algebras. In Section 5.6.2, we associate to each flip of the n-angulation a derived equivalence of the global sections of the corresponding perverse schobers and describe the action of this equivalence in terms of the partial geometric model. In the final Section 5.6.3, we show that the extended mutation matrices of cluster algebras with coefficients associated with multi-laminated surface can be recovered in terms of the Euler-characteristics of Ext-groups in the derived category of the relative Ginzburg algebra.

5.1 Introduction to Ginzburg algebras

5.1.1 The Ginzburg algebra of a quiver with potential

We begin by recalling the definition of the Ginzburg algebra associated with a quiver with potential. References include [Gin06, KY11, Kel11]. Let Q be a finite quiver and k a commutative ring. We denote by kQ the k-linear path algebra of Q (without taking any completions).

Definition 5.1. A potential W of Q is an element of kQ consisting of a k-linear sum of cycles, meaning paths which begin and end at the same vertex.

Let $c \in kQ$ be a cycle and a an arrow of Q. We define the cyclic derivative of c at a as

$$\partial_a c \coloneqq \sum_{c=uav} vu\,,$$

where the sum runs over all paths u, v (possibly lazy paths), such that $c = uav$. The cyclic derivative acts on potentials by extending the action on cycles k-linearly.

Quivers with potential may be considered up to cyclic equivalence, as the cyclic derivative does not change under cyclic equivalence, see for instance [KY11] for background.

Definition 5.2. Let (Q, W) be a quiver with potential. Let $\bar{Q}$ be the graded quiver with the same vertices as Q and

- an arrow $a\colon i \to j$ in degree 0 for each arrow $a\colon i \to j$ in Q,
- an arrow $a^*\colon j \to i$ in degree 1 for each arrow $a\colon i \to j$ in Q,
- a loop $l_i\colon i \to i$ in degree 2 for each vertex i of Q.

We define the Ginzburg dg-algebra $\mathscr{G}_{(Q,W)}$ as the graded path algebra $k\bar{Q}$ with differential determined on generators by

- $d(a) = 0$ for all arrows a in Q,
- $d(a^*) = \partial_a W$ for all arrows a in Q,
- $d(l_i) = \sum_a e_i(aa^* - a^*a)e_i$, where sum runs over all arrows in Q, and e_i denotes the lazy path at the vertex i.

The zero'th homology $\mathscr{J}_{(Q,W)} \coloneqq \mathrm{H}_0\, \mathscr{G}_{(Q,W)}$ is called the Jacobian algebra of (Q, W).

Many sources also consider the complete version of the dg-algebra $\mathscr{G}_{(Q,W)}$, obtained by replacing $k\bar{Q}$ with the path algebra completed at the ideal generated by the arrows. Under some mild assumptions on the potential W, the dg-algebra $\mathscr{G}_{(Q,W)}$ is left 3-Calabi–Yau, see [Kel11, Yeu16].

Given a quiver with potential (Q, W), such that Q has no loops or 2-cycles, one can mutate it at any vertex v of Q, yielding a new quiver with potential, see [DWZ08]. Such a quiver encodes a skew-symmetric mutation matrix, and the process of mutation corresponds to matrix mutation in the sense of Definition 6.40. A quiver with potential (Q, W), where Q has no loops or 2-cycles, is called non-degenerate if under repeated mutations no quivers with 2-cycles appear. Quiver mutation induces a derived equivalence between the corresponding Ginzburg algebras, see [KY11].

An interesting class of quivers with potentials, and thus of Ginzburg algebras, arises from marked surfaces equipped with an ideal triangulation (see Definition 6.31), cf. [LF09, GLFS16]. We fix such a surface $\mathbf{S}$. The vertices of the quiver Q are given by the internal edges of the ideal triangulation, also referred to as arcs. The arrows in Q are obtained by inscribing a clockwise triangle into each triangle. The potential $W = W' + W''$ consists of two terms. The first term is $W' = \sum_f T(f)$, where the sum runs over the interior faces of the triangulation and $T(f)$ denotes the clockwise 3-cycle inscribed into the face. The second term $W'' = \sum_{p \in P} c_p$ is the sum of the counterclockwise n-cycles obtained by going around the punctures in $\mathbf{S}$. One may also consider the quiver with potential (Q, W'), with the caveat that this quiver with potential is degenerate in most cases. The operation of mutation of quiver with potential corresponds geometrically to flipping the ideal triangulation, see [LF09].

5.1.2 Relative Ginzburg algebras from n-angulated surfaces

Before discussing relative Ginzburg algebras, we explain what we mean by n-angulated marked surfaces. Given a marked surface $\mathbf{S}$ with a spanning graph Γ, we can pass to its dual graph Γ^{dual} with vertices and edges as follows:

- The set of vertices of Γ^{dual} is the set of marked points $M \subset \mathbf{S}$. There is hence a vertex for each face of Γ, i.e. connected component of $\mathbf{S}\backslash\Gamma$.
- The edges of Γ^{dual} are in bijection with the edges of Γ, they are obtained by connecting two vertices of Γ^{dual} for each edge lying in the intersection of the corresponding faces. External edges of Γ give rise to edges of Γ^{dual} lying on the boundary of $\mathbf{S}$.

There is a canonical embedding of the realization of the dual graph $|\Gamma^{\mathrm{dual}}| \subset \mathbf{S}$.

Definition 5.3. A graph Γ is called n-valent if each vertex of Γ has valency n, i.e. n incident halfedges. We mostly denote such graphs by $\mathcal{T}$.

Given spanning graph Γ of a marked surface $\mathbf{S}$, the embedding of the realization of its dual graph $|\Gamma^{\mathrm{dual}}| \subset \mathbf{S}$ decomposes $\mathbf{S}$ into polygons with vertices the marked points and possibly self-folded edges. If $\Gamma = \mathcal{T}$ is furthermore n-valent, for some $n \geq 3$, these polygons are n-gons and this decomposition of $\mathbf{S}$ may be called an ideal n-angulation. These generalize ideal triangulations of marked surfaces, corresponding to the case $n = 3$, see also Definition 6.31. Examples of 4-angulated surfaces are depicted in Figure 3.

Note that not all marked surfaces admit an ideal triangulation, and much less an ideal n-angulation. For example, spheres with less than 3 punctures, the unpunctured 1-gon and the unpunctured 2-gon do not admit an ideal triangulation.

The dual edges of loops of spanning graphs (i.e. internal edges whose two endpoints coincide) are precisely the self-folded edges of the dual decomposition into polygons.

We fix a marked surface $\mathbf{S}$, a commutative ring k and $n \geq 3$. Definition 5.4 defines the relative higher Ginzburg algebra associated with an n-valent spanning graph $\mathcal{T}$ of $\mathbf{S}$.

Definition 5.4. Let $\mathcal{T}$ be an n-valent spanning graph of $\mathbf{S}$. We define a graded quiver $\tilde{Q}_{\mathcal{T}}$ with vertices the edges of $\mathcal{T}$ and the following graded arrows.

- An arrow $a_{v,i,j}\colon i \to j$ of degree $l-1$ for each vertex $v \in \mathcal{T}_0$ at which a halfedge of i follows a halfedge of j in the (counterclockwise) cyclic order after $1 \leq l \leq n-1$ steps. The arrows thus go in the clockwise direction and the loops of $\mathcal{T}$ give rise to loops.
- A loop $L_i : i \to i$ of degree $n-1$ for each internal edge i.

Given two edges $i, j \in \mathcal{T}_1$ incident to $v \in \mathcal{T}_0$, we denote by $j - i \in \{0, \dots, n-1\}$ the number of steps after which i follows j in the cyclic order at v.

The relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ is defined as the dg-algebra with underlying graded algebra $k\tilde{Q}_{\mathcal{T}}$ and with differential d determined on the generators as follows.

For the generators $a_{v,i,j}$, we set

$$d(a_{v,i,j}) = \sum_{j<k<i} (-1)^{j-k-1} a_{v,k,j} a_{v,i,k}\,,$$

where the sum runs over all edges k appearing between j and i in the cyclic order. For an internal edge i incident to two (possibly identical) vertices $v_1, v_2 \in \mathcal{T}_0$, we set

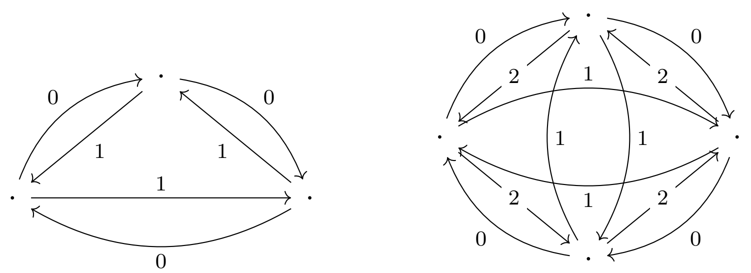
$$d(L_i) = \sum_{j\neq i} (-1)^{j-i} a_{v_1,j,i} a_{v_1,i,j} + (-1)^{n-1} \sum_{j\neq i} (-1)^{j-i} a_{v_2,j,i} a_{v_2,i,j}\,. \tag{60}$$

Note that we implicitly chose and order of v_1, v_2, but making a different choice changes $\mathscr{G}_{\mathcal{T}}$ only up to isomorphism of dg-algebras (by mapping L_i by $-L_i$).

Remark 5.5. The formula for the differential of $a_{v,i,j}$ in Definition 5.4 should be regarded as some version of graded cyclic derivative of a potential

$$W'_{\mathcal{T}} = \sum_{v\in\mathcal{T}_0} \sum_{j<k<i} \pm a_{v,k,j} a_{v,i,k} a_{v,j,i}\,.$$

Example 5.6. For $n = 3, 4$, let $\mathbf{S}$ be the n-gon and $\mathcal{T}$ the unique n-valent spanning graph. The graded algebras underlying the relative Ginzburg algebras $\mathscr{G}_{\mathcal{T}}$ are given by the path algebras of the following graded quivers. The labels indicate degrees.



The differentials of the arrows in degree 0 vanish and the differential of an arrow $a\colon x \to y$ of degree m consists, modulo signs, of the sum of all paths composed of two arrows of degrees less than m which compose to a path $x \to y$.

An ice quiver with potential (Q, F, W) consists of a quiver with potential (Q, W) together with a subquiver F of Q, whose vertices and edges are referred to as frozen. Yilin Wu has recently extended the definition of Ginzburg algebra to relative Ginzburg algebras associated with ice quivers with potential, see [Wu23b, Wu23a]. Wu also considers 'higher' versions of relative Ginzburg algebras associated with quivers with potential. Wu shows in [Wu23b] that these are equivalent to the relative n-Calabi–Yau completions of [Yeu16], which implies that these are relative (weakly) left n-Calabi–Yau. The relative Ginzburg algebras associated with triangulated surfaces defined above are special cases of Wu's more general setup. Here, the underlying ice quiver with potential arises from a variant of the quiver with potential of Section 5.1.1: the difference is that the boundary edges of the triangulation are included as frozen vertices in the quiver. The relative Ginzburg algebras associated with n-angulated surfaces are of a more general kind (they are for instance not always Calabi–Yau).

5.2 Spherical adjunctions from local systems on S^n

By a spherical fibration $f\colon X \to Y$, we mean a Kan fibration between Kan complexes whose fiber is homotopy equivalent to the singular set of the topological m-sphere, for some $m \geq 0$, denoted S^m. We also refer to $S^m \in \mathcal{S}$ as the m-sphere. Given a stable ∞-category $\mathcal{D}$, we call the ∞-category of functors $\mathrm{Fun}(S^m, \mathcal{D})$ the ∞-category of $\mathcal{D}$-valued local systems on S^m. The fibration f induces a pullback functor

$$f^*\colon \mathrm{Fun}(Y, \mathcal{D}) \longrightarrow \mathrm{Fun}(X, \mathcal{D}),$$

which admits left and right adjoints $f_!, f_*$, see [Chr22d, Section 3.2]. The functor $f_!$ is given by left Kan extension and the functor f_* is given by right Kan extension. The sphericalness of the fibration f implies that $f^* \dashv f_*$ is spherical:

Theorem 5.7 ([KS14, Chr22d])**.** *Let $f\colon X \to Y$ be a spherical fibration and $\mathcal{D}$ a stable ∞-category. Then the adjunction*

$$f^*\colon \mathrm{Fun}(Y, \mathcal{D}) \longleftrightarrow \mathrm{Fun}(X, \mathcal{D}) : f_*$$

is spherical.

In the following, we consider the apparent spherical fibration $f\colon S^{n-1} \to *$ with $n \geq 3$. We further specialize to the case that $\mathcal{D} = \mathrm{RMod}_R$ is the symmetric monoidal ∞-category of right module spectra over an $\mathbb{E}_\infty$-ring spectrum R. The arising spherical functor f^* will be used to construct the perverse schobers studied in the following sections.

Remark 5.8. Let Z be a simplicial set and R an $\mathbb{E}_\infty$-ring spectrum. The ∞-category $\mathrm{Fun}(Z, \mathrm{RMod}_R)$ admits a symmetric monoidal structure, such that the pullback functor h^* along $h\colon Z \to *$ is a symmetric monoidal functor, see for example [Chr22d, Section 3.3]. We can thus consider $\mathrm{Fun}(Z, \mathrm{RMod}_R)$ as a left module in $\mathcal{P}r^L_{\mathrm{St}}$ over itself and the functor h^* as a morphism of algebra objects in $\mathcal{P}r^L$. Pulling back along h^* provides $\mathrm{Fun}(Z, \mathrm{RMod}_R)$ with the structure of a left module over RMod_R and thus with the structure of a left-tensoring over RMod_R. This shows that $\mathrm{Fun}(Z, \mathrm{RMod}_R)$ is an R-linear ∞-category such that the functor h^* is R-linear.

Our goal is to find an algebraic description of the ∞-category of local systems $\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)$. We denote by $R[t_{n-1}]$ the free algebra object in RMod_R generated by $R[n-1]$. Note that if $R = k$ is a commutative ring, there exists an equivalence $R[t_{n-1}] \simeq k[t_{n-1}]$, where $k[t_{n-1}]$ denotes the graded polynomial algebra, with generator in degree $|t_{n-1}| = n-1$.

We let L denote the simplicial set consisting of a single vertex and a single non-degenerate 1-simplex. We use Remark 5.8 to lift $\mathrm{Fun}(L, \mathcal{D}(k))$ to a k-linear ∞-category.

Lemma 5.9. *Consider the morphism of simplicial sets $g\colon L \to *$ and the associated pullback functor $g^*\colon \mathrm{RMod}_R \to \mathrm{Fun}(L, \mathrm{RMod}_R)$. There exists an equivalence of R-linear ∞-categories*

$$\mathrm{Fun}(L, \mathrm{RMod}_R) \simeq \mathrm{RMod}_{R[t_0]}$$

such that the following diagram commutes.

$$\begin{tikzcd} & \mathrm{RMod}_R \arrow[dl, "g^*"'] \arrow[dr, "\phi^*"] & \\ \mathrm{Fun}(L, \mathrm{RMod}_R) \arrow[rr, "\simeq"] & & \mathrm{RMod}_{R[t_0]} \end{tikzcd} \tag{61}$$

Here ϕ^* denotes the pullback functor along the morphism of R-algebras $R[t_0] \to R$, determined on the generator by the morphism $R \xrightarrow{\mathrm{id}} R$ in RMod_R.

Proof. Consider the object $X \in \mathrm{Fun}(L, \mathrm{RMod}_R)$ given by the diagram $R[t_0] \xrightarrow{\cdot t_0} R[t_0]$ in RMod_R. Let $h\colon * \to L$ be the morphism of simplicial sets given by inclusion of the unique vertex and consider the associated pullback functor $h^*\colon \mathrm{RMod}_R \to \mathrm{Fun}(L, \mathrm{RMod}_R)$ with right adjoint h_* given by evaluation at $* \in L$. We prove that $X \simeq h^*(R)$ by showing that $\mathrm{Mor}_{\mathrm{Fun}(L,\mathrm{RMod}_R)}(X, \text{-}) \simeq h_*$.

Let $Y \in \mathrm{Fun}(L, \mathrm{RMod}_R)$. The morphism object $\mathrm{Mor}_{\mathrm{Fun}(L,\mathrm{RMod}_R)}(X, Y)$ is equivalent to the equalizer

$$\textstyle\prod_{i\in\mathbb{N}} h_*(Y) \xrightarrow{\simeq} \mathrm{Mor}_R(R[t_0], Y) \underset{Y(l)\circ(\text{-})}{\overset{(\text{-})\circ t_0}{\rightrightarrows}} \mathrm{Mor}_R(R[t_0], Y) \xrightarrow{\simeq} \prod_{i\in\mathbb{N}} h_*(Y) \quad ,$$

where l is the unique non-degenerate 1-simplex of L. This can be seen as follows. Consider the simplicial set L' consisting of four vertices x_1, x_2, x_3, x_4 and four non-degenerate 1-simplicies l_1, l_2, l_3, l_4 arranged as follows.

$$\begin{tikzcd} x_1 \arrow[r, "l_1"'] \arrow[d, "l_4"] & x_3 \\ x_4 & x_2 \arrow[u, "l_2"] \arrow[l, "l_3"'] \end{tikzcd}$$

The morphism of simplicial sets $p\colon L' \to L$, mapping all vertices to $* \in L$, l_1 to l and l_2, l_3, l_4 to the degenerate 1-simplex, induces a fully faithful R-linear functor $p^*\colon \mathrm{Fun}(L, \mathrm{RMod}_R) \to \mathrm{Fun}(L', \mathrm{RMod}_R)$. The description of

$$\mathrm{Mor}_{\mathrm{Fun}(L,\mathrm{RMod}_R)}(X, Y) \simeq \mathrm{Mor}_{\mathrm{Fun}(L',\mathrm{RMod}_R)}(p^*(X), p^*(Y))$$

as an equalizer can now be obtained by using a pushout description of $p^*(X) \simeq X_1 \amalg_{X_3} X_2$, with

$$X_1 = \begin{matrix} R[t_0] & \xrightarrow{\simeq} & R[t_0] \\ \downarrow\simeq & & \uparrow \\ R[t_0] & \longleftarrow & 0 \end{matrix} \quad , \quad X_2 = \begin{matrix} 0 & \longrightarrow & R[t_0] \\ \downarrow & & \simeq\uparrow \\ R[t_0] & \xleftarrow{\simeq} & R[t_0] \end{matrix} \quad , \quad X_3 = \begin{matrix} 0 & \longrightarrow & R[t_0] \\ \downarrow & & \uparrow \\ R[t_0] & \longleftarrow & 0 \end{matrix}$$

and that $\mathrm{Mor}_{\mathrm{Fun}(L',\mathrm{RMod}_R)}(\text{-}, p^*(Y))$ is an exact functor. The equalizer is given by $h_*(Y)$, the morphism of R-modules $h_*(Y) \to \prod_{i\in\mathbb{N}} h_*(Y)$ is informally given by mapping $z \in h_*(Y)$ to $(Y(l)^i(z))_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} h_*(Y)$. We note that the equivalence

$\mathrm{Mor}_{\mathrm{Fun}(L,\mathrm{RMod}_R)}(X,Y) \simeq h_*(Y)$ is functorial in Y, so that we find the desired equivalence $\mathrm{Mor}_{\mathrm{Fun}(L,\mathrm{RMod}_R)}(X,\text{-}) \simeq h_*$.

It follows that X is a compact generator of $\mathrm{Fun}(L,\mathrm{RMod}_R)$. Applying [Lur17, 4.1.1.18], we further obtain an equivalence of R-linear ring spectra $\mathrm{End}_R(X) \simeq R[t_0]$, showing the existence of an equivalence of R-linear ∞-categories $\mathrm{Fun}(L,\mathrm{RMod}_R) \simeq \mathrm{RMod}_{R[t_0]}$.

The commutativity of the diagram (61) can be checked using the fact that the R-linear functors $\phi^*, g^*\colon \mathrm{RMod}_R \to \mathrm{RMod}_{R[t_0]}$ are fully determined by $\phi^*(R)$, respectively $g^*(R)$, see [Lur17, Section 4.8.4]. □

We denote by $i\colon * \to S^{n-1}$ the inclusion of any point and the corresponding pullback functor by $i^*\colon \mathrm{Fun}(S^{n-1},\mathrm{RMod}_R) \to \mathrm{RMod}_R$. The functor i^* admits a left adjoint $i_!$. Similarly, $g_!$ denotes the left adjoint of the pullback functor g^*.

Lemma 5.10.

1. *There exists a pushout diagram in* LinCat_R *as follows.*

$$\begin{tikzcd} \mathrm{Fun}(L,\mathrm{RMod}_R) \arrow[r, "g_!"] \arrow[d, "g_!"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{RMod}_R \arrow[d, "i_!"] \\ \mathrm{RMod}_R \arrow[r, "i_!"'] & \mathrm{Fun}(S^2,\mathrm{RMod}_R) \end{tikzcd} \tag{62}$$

2. *Let $m \geq 2$. There exists a pushout diagram in* LinCat_R *as follows.*

$$\begin{tikzcd} \mathrm{Fun}(S^{m-1},\mathrm{RMod}_R) \arrow[r, "f_!"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{RMod}_R \arrow[d, "i_!"] \\ \mathrm{RMod}_R \arrow[r, "i_!"'] & \mathrm{Fun}(S^m,\mathrm{RMod}_R) \end{tikzcd} \tag{63}$$

Proof. We begin by showing statement 2. Consider the following pushout diagram of spaces.

$$\begin{tikzcd} S^{m-1} \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d, "i"] \\ * \arrow[r, "i"'] & S^m \end{tikzcd}$$

The above diagram is also pushout in Cat_∞. Applying the limit preserving functor $\mathrm{Fun}(\text{-},\mathrm{RMod}_R)\colon \mathrm{Cat}_\infty^{op} \to \mathrm{Cat}_\infty$, we obtain from this pushout diagram the following pullback diagram in $\mathcal{P}r^R$.

$$\begin{tikzcd} \mathrm{Fun}(S^m,\mathrm{RMod}_R) \arrow[r, "i^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{RMod}_R \arrow[d, "f^*"] \\ \mathrm{RMod}_R \arrow[r, "f^*"'] & \mathrm{Fun}(S^{m-1},\mathrm{RMod}_R) \end{tikzcd}$$

The left adjoint diagram is the diagram (63) and thus pushout in LinCat_R.

We proceed with showing statement 1. The geometric realization of L is equivalent to the topological 1-sphere. There thus exists a morphism of simplicial sets $\iota\colon L \to S^1$ such that the limit functor $g_* = \lim\colon \mathrm{Fun}(L, \mathrm{RMod}_R) \to \mathrm{RMod}_R$ restricts via the fully faithful pullback functor $\iota^*\colon \mathrm{Fun}(S^1, \mathrm{RMod}_R) \to \mathrm{Fun}(L, \mathrm{RMod}_R)$ to the limit functor f_*. The left adjoint $g^*\colon \mathrm{RMod}_R \to \mathrm{Fun}(L, \mathrm{RMod}_R)$ thus factors through $\mathrm{Fun}(S^1, \mathrm{RMod}_R)$. It thus follows from the explicit model for limits in Cat_∞ that the right adjoint diagram of diagram (62) is pullback in $\mathcal{P}r^R$. It follows that the diagram (62) is pushout in LinCat_R. $\square$

Proposition 5.11. *Let $n \geq 3$. There exists an equivalence of R-linear ∞-categories*

$$\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R) \simeq \mathrm{RMod}_{R[t_{n-2}]}\,, \tag{64}$$

such that the following diagram in LinCat_R commutes.

$$\begin{tikzcd}
\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R) \arrow[rr, "\simeq"] \arrow[dr, "i^*"] & & \mathrm{RMod}_{R[t_{n-2}]} \arrow[dl, "G"] \\
& \mathrm{RMod}_R \arrow[dl, "f^*"'] \arrow[dr, "\phi^*"] & \\
\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R) \arrow[rr, "\simeq"] & & \mathrm{RMod}_{R[t_{n-2}]}
\end{tikzcd} \tag{65}$$

Here G denoted the monadic functor and ϕ^* the pullback functor along the morphism of R-algebras $\phi\colon R[t_{n-2}] \to R$ determined on the generator by the morphism $R[n-2] \xrightarrow{0} R$ in RMod_R.

Proof. Let $m \geq 1$ and consider the following biCartesian square in RMod_R.

$$\begin{tikzcd}
R[m-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & R[m]
\end{tikzcd}$$

Applying the colimit preserving free R-algebra functor $\mathrm{RMod}_R \to \mathrm{Alg}(\mathrm{RMod}_R)$ yields the following pushout diagram of R-algebras.

$$\begin{tikzcd}
R[t_{m-1}] \arrow[r, "t_{m-1}\mapsto 0"] \arrow[d, "t_{m-1}\mapsto 0"'] \arrow[dr, phantom, "\ulcorner"] & R \arrow[d] \\
R \arrow[r] & R[t_m]
\end{tikzcd} \tag{66}$$

Consider the morphism of ring spectra $R[t_0] \xrightarrow{t_0 \mapsto t_0+1} R[t_0]$, determined via the universal property by the morphism $R \xrightarrow{1 \mapsto 1+t_0} R[t_0]$ in RMod_R. Using the commutativity of the diagram

$$\begin{tikzcd} R[t_0] \arrow[r, "t_0\mapsto 1"] \arrow[d, "t_0\mapsto t_0+1"'] & R \\ R[t_0] \arrow[ur, "t_0\mapsto 0"'] & \end{tikzcd}$$

and Lemma 5.9, it follows that for $m = 1$ the image of the diagram (66) under the functor $\theta\colon \mathrm{Alg}(\mathrm{RMod}) \to \mathrm{LinCat}_R$, see Section 2.1.4, is equivalent to the pushout diagram in (62). It follows that there exists an equivalence of R-linear ∞-categories $\mathrm{Fun}(S^2, \mathrm{RMod}_R) \simeq \mathrm{RMod}_{R[t_1]}$. Using that the monadic functor G is equivalent to the pullback along $R \to R[t_1]$, we obtain that the upper half of the diagram (65) commutes.

Since $f \circ i = \mathrm{id}\colon * \to *$, we have that $f_! \circ i_! \simeq \mathrm{id}_{\mathrm{RMod}_R}$, and thus obtain the following commutative diagram.

$$\begin{tikzcd} \mathrm{Fun}(L, \mathrm{RMod}_R) \arrow[r, "g_!"] \arrow[d, "g_!"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{RMod}_R \arrow[d, "i_!"] \arrow[ddr, bend left, "\mathrm{id}"] & \\ \mathrm{RMod}_R \arrow[r, "i_!"] \arrow[drr, bend right, "\mathrm{id}"] & \mathrm{Fun}(S^2, \mathrm{RMod}_R) \arrow[dr, "f_!"] & \\ & & \mathrm{RMod}_R \end{tikzcd} \tag{67}$$

The diagram (67) is equivalent to the image under θ of the following diagram in $\mathrm{Alg}(\mathrm{RMod}_R)$.

$$\begin{tikzcd} R[t_0] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & R \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\ R \arrow[r] \arrow[drr, bend right, "\mathrm{id}"] & R[t_1] \arrow[dr] & \\ & & R \end{tikzcd} \tag{68}$$

By the universal property of the pushout in $\mathrm{Alg}(\mathrm{RMod}_R)$ there exists a unique morphism of ring spectra $R[t_1] \to R$ such that (68) commutes. Such a map is given by ϕ. It follows that the functor $f_!$ is equivalent to $\theta(\phi)$ and, using the adjunction between $\theta(\phi)$ and the pullback along ϕ, see [Lur17, 4.6.2.17], also that the functor f^* is equivalent to the pullback functor along ϕ.

For $m \geq 3$, we can continue by induction and as before. The image of (66) under the functor θ is the pushout diagram in (63). We thus find the desired equivalence $\mathrm{Fun}(S^m, \mathrm{RMod}_R) \simeq \mathrm{RMod}_{R[t_{m-1}]}$ so that the upper half of diagram (65) commutes. Analogous to the case $m = 1$, it can be checked that the lower half of the diagram (65) commutes. □

We next describe the twist functor $T_{\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)}$ of the adjunction $\phi_! \dashv \phi^*$. For the full description, we restrict to the case that $R = k$ is a commutative ring.

Proposition 5.12. *Let $n \geq 3$.*

(1) Let $X \in \mathrm{RMod}_R$. There exist equivalences in $\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)$

$$T_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(f^*(X)) \simeq f^*(X)[1-n]\,,$$
$$T_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(i_*(X)) \simeq i_*(X)[1-n]\,.$$

(2) Assume that $R = k$ is a commutative ring and consider the morphism of dg-algebras

$$\varphi\colon k[t_{n-2}] \xrightarrow{t_{n-2}\mapsto(-1)^n t_{n-2}} k[t_{n-2}]\,.$$

There exists a commutative diagram in LinCat_k *as follows:*

$$\begin{tikzcd}
\mathrm{Fun}(S^{n-1},\mathrm{RMod}_k) \arrow[r, "T_{\mathrm{Fun}(S^n,\mathrm{RMod}_k)}"] \arrow[d, "(64)"', "\simeq"] & \mathrm{Fun}(S^{n-1},\mathrm{RMod}_k) \arrow[d, "(64)"', "\simeq"] \\
\mathrm{RMod}_{k[t_{n-2}]} \arrow[r, "\varphi^*[1-n]"] & \mathrm{RMod}_{k[t_{n-2}]}
\end{tikzcd}$$

Proof. We begin with the proof of part (1). Let T_{RMod_R} be the twist functor of the adjunction $f^* \dashv f_*$. The description of image under the cotwist $T_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(f^*(X))$ directly follows from the equivalences of functors $f^*T_{\mathrm{RMod}_R} \simeq T_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}f^*$, see [Chr22d, Lemma 2.2], and $T_{\mathrm{RMod}_R} \simeq [1-n]$, see [Chr22d, Section 3.1].

For the description of $T_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(i_*(X))$, we begin by recalling some notation from [Chr22d, Section 3.1], where the sphericalness of the adjunction $f^* \dashv f_*$ is proven. Given a simplicial set Z, we denote by $Z^{\triangleright} = Z * \Delta^0$ the simplicial join. Consider the recursively defined simplicial sets $P_0 = S^0 = \Delta^0 \amalg \Delta^0$ and

$$P_n \coloneqq P_{n-1}^{\triangleright} \coprod_{P_{n-1}} P_{n-1}^{\triangleright}\,.$$

We denote any vertex in $P_n \backslash P_{n-1}$ by n. Let $g\colon P_n \to *$ and $g^*\colon \mathcal{D} \to \mathrm{Fun}(P_n, \mathcal{D})$ be the pullback functor with right adjoint g_* given by the limit functor. The ∞-category $\mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)$ embeds fully faithfully into $\mathrm{Fun}(P_n, \mathrm{RMod}_R)$, with image the functors mapping all edges in P_n to equivalences in RMod_R, and the adjunction $g^* \dashv g_*$ restricts to $f^* \dashv f_*$ along this inclusion. Consider the simplicial sets

$$Z_1 = \{n\} \times \Delta^1 \amalg_{\{n\}\times\Delta^{\{1\}}} P_n \times \Delta^{\{1\}}$$
$$Z_2 = (P_n\backslash\{n\}) \times \Delta^1 \amalg_{(P_n\backslash\{n\})\times\Delta^{\{0\}}} P_n \times \Delta^{\{0\}}$$
$$Z_3 = P_n \times \Delta^1$$

We define the following ∞-categories via Kan extensions.

- Let $\mathcal{D}_1 \subset \mathrm{Fun}(Z_1, \mathrm{RMod}_R)$ be the full subcategory spanned by right Kan extensions along $P_n \times \Delta^{\{1\}} \hookrightarrow Z_1$.
- Let $\mathcal{D}_2 \subset \mathrm{Fun}(Z_2, \mathrm{RMod}_R)$ be the full subcategory spanned by left Kan extensions along $P_n \times \Delta^{\{0\}} \hookrightarrow Z_2$.

- Let $\mathcal{D}_1' \subset \operatorname{Fun}(Z_3, \operatorname{RMod}_R)$ be the full subcategory spanned by left Kan extensions along $Z_1 \to Z_3$ of functors in $\mathcal{D}_1$.
- Let $\mathcal{D}_2' \subset \operatorname{Fun}(Z_3, \operatorname{RMod}_R)$ be the full subcategory spanned by right Kan extensions along $Z_2 \to Z_3$ of functors in $\mathcal{D}_2$.

The diagrams in $\mathcal{D}_1'$ have the property that their restriction to $P_n \times \Delta^{\{0\}}$ is everywhere zero, except at $(n, 0)$, where their value is identical to their value at $(n, 1)$. Similarly, the diagrams in $\mathcal{D}_2'$ have the property that their restriction to $P_n \times \Delta^{\{1\}}$ is equivalent to the their restriction to $P_n \times \Delta^{\{0\}}$, expect for having value 0 at the vertex $(n, 1)$. For $i = 1, 2$, the functors

$$R_i \colon \mathcal{D}_i' \to \mathcal{D}_i \to \operatorname{Fun}(P_n \times \Delta^{\{1\}}, \operatorname{RMod}_R)$$

are trivial fibrations by [Lur09, 4.3.2.15]. We can thus choose an essentially unique section of R_i, denoted R_i^{-1}. Consider the functor

$$\operatorname{colim}_{\Delta^1} \colon \operatorname{Fun}(Z_3, \operatorname{RMod}_R) \simeq \operatorname{Fun}(\Delta^1, \operatorname{Fun}(P_n, \operatorname{RMod}_R)) \xrightarrow{\operatorname{Fun}(\Delta^1, \operatorname{colim})} \operatorname{Fun}(\Delta^1, \operatorname{RMod}_R)$$

We find that the composite of R_1^{-1} and the restriction of $\operatorname{colim}_{\Delta^1}$ to $\mathcal{D}_1'$ describes a natural transformation $\eta \colon i^* \to f_!$. The cofiber of this natural transformation in the ∞-category $\operatorname{Fun}(\operatorname{Fun}(S^{n-1}, \operatorname{RMod}_R), \operatorname{RMod}_R)$ can be described as the composite ν of R_2^{-1}, $\lim_{\Delta^1}$ restricted to $\mathcal{D}_2'$ and the evaluation functor at $1 \in \Delta^1$. A small computation reveals that ν is equivalent to $i^*[n-1]$.

Passing to right adjoints, we obtain a fiber and cofiber sequence $i_*[1-n] \to f^* \to i_*$ in $\operatorname{Fun}(\operatorname{RMod}_R, \operatorname{Fun}(S^{n-1}, \operatorname{RMod}_R))$, which evaluated at $X \in \operatorname{RMod}_R$ yields a fiber and cofiber sequence $i_*(X)[1-n] \to f^*(X) \xrightarrow{\alpha} i_*(X)$. Composing with the equivalence $f^*(X) \simeq f^* f_* i_*(X)$, one sees that the morphism α is a counit map of the adjunction $f^* \dashv f_*$. It thus follows $T_{\operatorname{Fun}(S^{n-1}, \operatorname{RMod}_R)}(i_*(X)) \simeq i^*(X)[1-n]$, concluding the proof of part (1).

We proceed with the proof of part (2). The dg-category of $k[t_{n-2}]$-bimodules is equivalent to the dg-category $\operatorname{dgMod}(k[t_{n-2}] \otimes_k k[t_{n-2}]^{\mathrm{op}})$. The former dg-category thus inherits a model structure from the projective model structure of the latter, whose underlying ∞-category is equivalent to the ∞-category of k-linear endofunctors of $\operatorname{RMod}_{k[t_{n-2}]}$. Let $\odot$ denote the multiplication in $k[t_{n-2}]$. We denote by $\widehat{k[t_{n-2}]}$ the $k[t_{n-2}]$-bimodule $\widehat{k[t_{n-2}]}$ with

- underlying chain complex $k[t_{n-2}]$,
- left action on $a \in \widehat{k[t_{n-2}]}$ determined by $t_{n-2}^i.a = (-1)^{in} t_{n-2}^i \odot a$ and
- right action on $a \in \widehat{k[t_{n-2}]}$ determined by $a.t_{n-2}^i = a \odot t_{n-2}^i$.

Note that $\varphi^* \simeq - \otimes_{k[t_{n-2}]} \widehat{k[t_{n-2}]}$. We can thus prove part (2) by showing that the composite of the twist functor $T'_{\operatorname{Fun}(S^{n-1}, \operatorname{RMod}_k)}$ of the spherical adjunction $f_! \dashv f^*$

with the equivalence (64) is equivalent to $-\otimes \widehat{k[t_{n-2}]}[n-1]$. Using the commutativity of the lower part of diagram (65), it suffices to show that the twist functor T of the spherical adjunction $\phi_! \dashv \phi^*$ is equivalent to $-\otimes \widehat{k[t_{n-2}]}[n-1]$.

We find $\phi^*\phi_!(k[t_{n-2}]) \simeq k \in \mathrm{RMod}_{k[t_{n-1}]}$, with k the trivial $k[t_{n-2}]$-module. The k-linear functor $\phi^*\phi_!$ is thus equivalent to the functor $-\otimes_{k[t_{n-2}]} k$, for a $k[t_{n-2}]$-bimodule k. There is but a unique such bimodule, which carries the action $t_{n-2}.1 = 0 = 1.t_{n-2} \in k$. A cofibrant replacement of the $k[t_{n-2}]$-bimodule k is given the cone of the morphism of bimodules

$$\alpha\colon \widehat{k[t_{n-2}]}[n-2] \to k[t_{n-2}]\,.$$

$$t_{n-2}^i \mapsto t_{n-2}^{i+1}$$

To see that α indeed defines a morphism of bimodules, note that by the definition of $\widehat{k[t_{n-2}]}$ and the sign rule for the shift of left modules, see Remark 2.15, the left action of $k[t_{n-2}]$ on $\widehat{k[t_{n-2}]}[n-2]$ is determined by $t_{n-2}.1 = (-1)^{(n-2)+(n-2)}t_{n-2} = t_{n-2}$. We deduce that the twist functor T is equivalent to the functor given by tensoring with the homotopy pushout in the following diagram of cofibrant $k[t_{n-2}]$-bimodules.

$$\begin{tikzcd}
k[t_{n-2}] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{cone}(\alpha) \arrow[d] \\
0 \arrow[r] & \widehat{k[t_{n-2}]}[n-1]
\end{tikzcd}$$

We have shown $T \simeq -\otimes_{k[t_{n-2}]} \widehat{k[t_{n-2}]}[n-1]$, as desired. □

Lemma 5.13. *There exists an equivalence of R-modules*

$$\mathrm{End}(f^*(R)) = \mathrm{Mor}_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(f^*(R), f^*(R)) \simeq R \oplus R[1-n]\,.$$

Proof. The R-linear functor $f^*\colon \mathrm{RMod}_R \to \mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)$ is fully determined by the image of R, see [Lur17, Section 4.8.4], and thus equivalent to the R-linear functor

$$-\otimes_R f^*(R)\colon \mathrm{RMod}_R \to \mathrm{Fun}(S^{n-1}, \mathrm{RMod}_R)\,.$$

Its right adjoint f_* is equivalent to $\mathrm{Mor}_{\mathrm{Fun}(S^{n-1},\mathrm{RMod}_R)}(f^*(R), -)$. The equivalence $f_*f^*(R) \simeq R \oplus R[1-n]$ is shown in [Chr22d, Section 3.1]. □

5.3 Ginzburg algebras and perverse schobers

Fix a marked surface $\mathbf{S}$ with n-valent spanning graph $\mathcal{T}$ and a choice of $\mathbb{E}_\infty$-ring spectrum R. In the following construction, we describe a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_\mathcal{T}(R)$, with the property that its ∞-category of global sections $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$ is equivalent to $\mathcal{D}(\mathscr{G}_\mathcal{T})$, if $R = k$ is a commutative ring, see Theorem 5.15.

Construction 5.14. Locally at each vertex of $\mathcal{T}$, the perverse schober $\mathcal{F}_{\mathcal{T}}(R)$ is described by the spherical adjunction

$$\phi^*\colon \operatorname{RMod}_R \longleftrightarrow \operatorname{RMod}_{R[t_{n-2}]} : \phi_*$$

where $R[t_{n-2}]$ denotes the free R-linear ring spectrum generated by $R[n-2]$ and ϕ^* is the pullback along the morphism of ring spectra $\phi\colon R[t_{n-2}] \xrightarrow{t_{n-2}\mapsto 0} R$. Note that we can by Proposition 5.11 equivalently locally describe the perverse schober in terms of the spherical adjunction $f^*\colon \operatorname{RMod}_R \longleftrightarrow \operatorname{Fun}(S^{n-1}, \operatorname{RMod}_R) : f_*$, arising form the pullback functor along $f\colon S^n \to *$. We will freely pass between these two perspectives, as they are each particularly convenient in different situations.

We thus want to define $\mathcal{F}_{\mathcal{T}}(R)$ as the gluing of the local perverse schobers $\mathcal{F}_v(\phi^*)\colon C_v \to \operatorname{St}$, i.e. as the diagram $\operatorname{Exit}(\mathcal{T}) \to \operatorname{St}$ which restricts at C_v to $\mathcal{F}_v(\phi^*)$, see also Proposition 3.25. Note that this definition involves choosing for each vertex of $\mathcal{T}$ a total order of its incident halfedges, compatible with their given cyclic order. These choices change the monodromy of $\mathcal{F}_{\mathcal{T}}(R)$, at least if n is odd, and we need to slightly modify the perverse schobers $\mathcal{F}_v(\phi^*)$, to remove this monodromy. This also ensures that we match the signs in the differentials of the Ginzburg algebras later on.

For each edge e of $\mathcal{T}$, we consider its two incident (possibly identical) vertices v_1, v_2. We denote by $i_1 \in \{1, \dots, n\}$ the position of the halfedge of e lying at v_1 in the chosen total order of the n halfedges incident to v_1. We similarly denote by $i_2 \in \{1, \dots, n\}$ the position of the halfedge of e at v_2 in the chosen total order of halfedges at v_2. If $i_1 - i_2$ is even, we change $\mathcal{F}_{v_1}(\phi^*)$ by composing $\mathcal{F}_{v_1}(\phi^*)(v_1 \to e)$ with the autoequivalence T of $\operatorname{RMod}_{R[t_{n-2}]}$, given by the the pullback functor along the morphism of ring spectra $R[t_{n-2}] \xrightarrow{t_{n-2}\mapsto(-1)^n t_{n-2}} R[t_{n-2}]$. Note that for $R = k$ a commutative ring, the functor $T[n-1]$ is equivalent to the cotwist functor of $\phi^* \dashv \phi_*$, see [Chr22b, Prop. 5.7], and further $T \simeq \operatorname{id}_{\operatorname{RMod}_{k[t_{n-2}]}}$ if n is even. If $i_1 - i_2$ is odd, we do nothing. The perverse schober $\mathcal{F}_{\mathcal{T}}(R)$ is now defined as the gluing of the above modifications of the $\mathcal{F}_v(\phi^*)$'s. If $R = k$ is a commutative ring, we also sometimes write $\mathcal{F}_{\mathcal{T}}$ for $\mathcal{F}_{\mathcal{T}}(k)$.

Theorem 5.15. *There exists an equivalence of ∞-categories*

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}(k)) \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}}) \tag{69}$$

between the ∞-category of global sections of $\mathcal{F}_{\mathcal{T}}(k)$ and the derived ∞-category of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$.

Sections 5.3.1 and 5.3.2 are dedicated to the proof of Theorem 5.15. Section 5.3.3 discusses relative Calabi–Yau structures on $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}(k))$.

5.3.1 Local computations

We again fix a commutative ring k and $n \geq 3$. We refer to Section 2.2.1 for the sign conventions on dg-modules, cones and shifts. Consider the morphism $\phi\colon k[t_{n-2}] \xrightarrow{t_{n-2}\mapsto 0} k$ from Proposition 5.11.

Lemma 5.16. *Let*

$$\mathbf{A_m} = \begin{pmatrix} k & k[1-n] & 0 & \dots & 0 & 0 \\ 0 & k[t_{n-2}] & k[t_{n-2}] & 0 & \dots & 0 \\ 0 & 0 & k[t_{n-2}] & k[t_{n-2}] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k[t_{n-2}] & k[t_{n-2}] \\ 0 & 0 & 0 & \dots & 0 & k[t_{n-2}] \end{pmatrix}$$

be the upper triangular dg-algebra. There exists an equivalence of ∞-categories

$$\mathcal{V}^m_{\phi^*} \simeq \mathcal{D}(\mathbf{A_m}) \tag{70}$$

Proof. Spelling out the definition of $\mathcal{V}^m_{\phi^*}$, this immediately follows from Proposition 2.46 by using that $\phi_* \simeq \phi_![1-n] = \mathrm{ladj}(\phi^*)[1-n] \simeq \left(\text{-} \otimes_{k[t_{n-2}]} k\right)[1-n]$. □

Remark 5.17. Consider the graded quiver Q_m

$$\begin{tikzcd} x_0 \arrow[r, "a_{0,1}"] & x_1 \arrow[loop above, "l_1"] \arrow[r, "a_{1,2}"] & \dots \arrow[r, "a_{m-2,m-1}"] & x_{m-1} \arrow[loop above, "l_{m-1}"] \end{tikzcd} \tag{71}$$

with $|a_{i,i+1}| = 0$ and $|l_i| = n-2$. The dg-algebra $\mathbf{A_m}$ is Morita-equivalent to the dg-category B_m with objects the vertices of Q_m and morphisms freely generated by the arrows of Q_m subject to the relations $l_1 \circ a_{0,1} = 0$, $a_{i+1,i+2} \circ a_{i,i+1} = 0$ for $i \geq 0$ and $a_{i,i+1}l_i = l_{i+1}a_{i,i+1}$ for $i \geq 1$.

For $m \geq 3$, we define D_m to be the dg-category with objects $z_1, \dots, z_m$ and morphisms freely generated by $b_{i,j}\colon z_i \to z_j$ for all $i \neq j$ in degree $j-i-1$ if $j > i$ and $n+j-i-1$ if $j < i$ and with differentials determined by

$$d(b_{i,j}) = \begin{cases} \sum_{i<k<j} (-1)^{j-k+1} b_{k,j} b_{i,k} & \text{if } j > i\,, \\ \sum_{i<k\leq m} (-1)^{j-k+n+1} b_{k,j} b_{i,k} + \sum_{1\leq k<j} (-1)^{j-k+1} b_{k,j} \circ b_{i,k} & \text{if } j < i\,. \end{cases}$$

Note that if $m = n$, the dg-category D_m is Morita equivalent to the relative Ginzburg algebra of the n-gon, and is depicted in the cases $m = n = 3$ and $m = n = 4$ in Example 5.6.

Lemma 5.18. *The homology of the mapping complexes in D_m is given by*

$$\mathrm{H}_* \operatorname{Hom}_{D_m}(z_i, z_j) \simeq \begin{cases} 0 & j \neq i, i+1 \\ k[t_{n-2}] & j = i \\ k[t_{n-2}] & j = i+1 \\ k[t_{n-2}][n-m] & j = 1, i = m \end{cases}$$

Proof. Let c be a cycle in a morphism complex in D_m. We can decompose c into the k-linear sum of morphisms composed of the generating morphisms of D_m. We call the length of c the maximal number of generating morphisms appearing in a summand of c. We show the following statements via an induction on the length of c.

1) If $c\colon z_i \to z_i$, then c is homologous to

$$\lambda \left(\sum_{j<i} (-1)^{n+j-i} b_{j,i} b_{i,j} + \sum_{i<j} (-1)^{j-i} b_{j,i} b_{i,j} \right)^l \tag{72}$$

with $\lambda \in k$ and $l \in \mathbb{N}$.

2) If $c\colon z_i \to z_{i+1}$, with $i+1$ considered modulo m, then c is homologous to

$$\lambda b_{i,i+1} \left(\sum_{j<i} (-1)^{n+j-i} b_{j,i} b_{i,j} + \sum_{i<j} (-1)^{j-i} b_{j,i} b_{i,j} \right)^l \tag{73}$$

with $\lambda \in k$ and $l \in \mathbb{N}$.

3) Otherwise, c is nullhomologous.

We note that (72) and (73) define nonzero homology classes. One simple way to see this is to observe that their image in C_m, see the proof of Proposition 5.19 below, define nonzero homology classes. The assertion follows.

We continue with showing 1),2) and 3). We denote the cycles of the form (72) by l_i and the cycles of the form (73) by $b_{i,i+1} l_i$. We consider all indices i, j of the $b_{i,j}$ modulo m.

We begin with 2), as it is the easiest case. We consider a cycle $c\colon z_i \to z_{i+1}$. Since the morphisms $b_{j,l}$ freely generate D_m, we can write c as $c = \sum_{k \neq i+1} b_{k,i+1} u_{i,k}$ for some chains $u_{i,k}$. The condition $d(c) = 0$ implies $d(u_{i,i+2}) = 0$. By the induction assumption there exists a chain $v_{i,i+2}$ with $d(v_{i,i+2}) = u_{i,i+2}$. We thus find

$$c + (-1)^{n-1} d(b_{i+2,i+1} v_{i,i+2}) = \sum_{k \neq i+1, i+2} b_{k,i+1} \left(u_{i,k} + (-1)^{s_{k,i}+i-k} b_{i+2,k} v_{i,i+2} \right)$$

with $s_{k,i} = 0$ if $i+2 < k \leq m$ and $s_{k,i} = n$ if $1 \leq k \leq i$. This shows that c is homologous to a cycle $c_2 = \sum_{k \neq i+1,i+2} b_{k,i+1} u_{i,k}$ for some other chains also denoted $u_{i,k}$. Repeating this argument $m-2$ times, we see that c is homologous to a cycle of the form $b_{i,i+1} u_{i,i}$ and by the induction hypothesis we find $u_{i,i} = l_i + d(v_{i,i})$ for some chain $v_{i,i}$. It follows that c is homologous to $b_{i,i+1} l_i$, showing 2).

For 3), we consider a cycle $c\colon z_i \to z_j$ with $j \neq i, i+1$ and assume without loss of generality that $i < j$. We write $c = \sum_{k \neq j} b_{k,j} u_{i,k}$ for some chains $u_{i,k}$ with $d(u_{i,j+1}) = 0$. If $j+1 \neq i$, then by the induction assumption $u_{i,j+1} = d(v_{i,j+1})$. As in the case ii), we thus find that c is homologous to a cycle of the form $\sum_{k \neq j,j+1} b_{k,j} u_{i,k}$ for some chains also labeled $u_{i,k}$. Repeating this process a few times, we find that

c is homologous to a cycle of the form $\sum_{k,\neq j,\dots,i-1} b_{k,j}u_{i,k}$ with $d(u_{i,i})=0$. Applying the induction assumption, we find that $u_{i,i}=l_i+d(v_{i,i})$ for some chain $v_{i,i}$. The condition $d(c)=0$ then implies that $(-1)^{j-i}b_{i+1,j}b_{i,i+1}l_i = b_{i+1,j}d(u_{i,i+1})$. Since $b_{i,i+1}l_i$ is not a boundary unless $l_i=0$, it follows that $l_i=0$. We obtain that c is homologous to $\sum_{k\neq j,\dots,i} b_{k,j}u_{i,k}$, for some chains also labeled $u_{i,k}$ with $d(u_{i,i+1})=0$. If $j=i+2$, the assertion now follows and otherwise we argue as before to obtain that c is homologous to $\sum_{k,\neq j,\dots,i,i+1} b_{k,j}u_{i,k}$ with $d(u_{i,i+2})=0$. The induction assumption implies that $u_{i,i+2}$ is a boundary, from which we obtain that c is homologous to $\sum_{k\neq j,\dots,i+2} b_{k,j}u_{i,k}$ for some chains also labeled $u_{i,k}$. Repeating this argument a few times, we can finally conclude that c is a boundary.

For 1), we consider a cycle $c\colon z_i\to z_i$, which we can write as $c=\sum_{k\neq i} b_{k,i}u_{i,k}$ for some chains $u_{i,k}$ with $d(u_{i,i+1})=0$. Using the induction assumption, we find a chain $v_{i,i+1}$ with $u_{i,i+1}=d(v_{i,i+1})-b_{i,i+1}l_i$. It follows that c is homologous to a cycle of the form $c_1=\sum_{k\neq i,i+1} b_{k,i}u_{i,k}-b_{i+1,i}b_{i,i+1}l_i$ for some other chains also labeled $u_{i,k}$. This constitutes the base case for an induction on j of the following assertion.

For all $1\leq j\leq m-1$, the cycle c is homologous to

$$c_j=\sum_{k\in I} b_{k,i}u_{i,k}+\left(\sum_{1\leq k\leq i+j-m<i}(-1)^{n+k-i}b_{k,i}b_{i,k}+\sum_{i<k\leq i+j,m}(-1)^{k-i}b_{k,i}b_{i,k}\right)l_i\,,$$

where I is the set of $1\leq k\leq m$ such that $k>i+j$ or $k<i+j-m$ and the $u_{i,k}$ are some chains.

For the induction step, we consider the case $i+j\leq m$. The case $i+j>m$ is dealt with analogously. Suppose that c is homologous to c_j. Evaluating the condition $d(c_j)=0$ at the summands beginning with $b_{i+j+1,i}$ yields

$$0=(-1)^{n-j}b_{i+j+1,i}d(u_{i,i+j+1})+\left(\sum_{i<k\leq i+j}(-1)^{n-j+k-i}b_{i+j+1,i}b_{k,i+j+1}b_{i,k}\right)l_i\,,$$

so that by the induction hypothesis (of the induction over the length of c) we have $u_{i,i+j+1}=d(v_{i,i+j+1})+(-1)^{j+1}b_{i,i+j+1}l_i$ for some chain $v_{i,i+j+1}$. It follows that c is homologous to c_{j+1}. This completes the induction step. Setting $j=m-1$, we obtain that c is homologous to

$$\left(\sum_{k<i}(-1)^{n+k-i}b_{k,i}b_{i,k}+\sum_{i<k}(-1)^{k-i}b_{k,i}b_{i,k}\right)l_i$$

and thus of the form (72). This concludes the proof. □

Proposition 5.19.

(1) There exists an equivalence of ∞-categories

$$\mathcal{V}^m_{\phi^*}\simeq\mathcal{D}(D_m)\,. \tag{74}$$

(2) For $1 \leq i \leq m-1$, the composite of the equivalence (74) with the functor $\mathcal{D}(k[t_{n-2}]) \simeq \mathrm{RMod}_{k[t_{n-2}]} \xrightarrow{\varsigma_{m-i+1}} \mathcal{V}^m_{\phi^*}$ is equivalent to the image under $\mathcal{D}(\text{-})$ of the dg-functor $\iota_i\colon k[t_{n-2}] \to D_m$ determined by

$$\iota_i(t_{n-2}) = (-1)^{m+in}\left(\sum_{j<i}(-1)^{n+j-i}b_{j,i}b_{i,j} + \sum_{i<j}(-1)^{j-i}b_{j,i}b_{i,j}\right).$$

Furthermore, if $n = m$, the composite of the equivalence (74) with the functor $\mathcal{D}(k[t_{n-2}]) \simeq \mathrm{RMod}_{k[t_{n-2}]} \xrightarrow{\varsigma_1} \mathcal{V}^m_{\phi^*}$ is equivalent to the image under $\mathcal{D}(\text{-})$ of the dg-functor $\iota_0\colon k[t_{n-2}] \to D_m$ determined by

$$\iota_m(t_{n-2}) = (-1)^m\left(\sum_{j<m}(-1)^{n+j-m}b_{j,m}b_{m,j}\right).$$

Proof. We recursively define objects $y_{i+1} = \mathrm{cone}(y_i \xrightarrow{\alpha_i} x_i)$ for $i \geq 0$ in $\mathrm{dgMod}(B_m)$, where $y_1 = x_0$ and for $i \geq 1$

$$\alpha_i = (0,\dots,0,a_{i-1,i}) \in \bigoplus_{j=0}^{i-1}\mathrm{Hom}_{B_m}\left(x_j[i-j-1], x_i\right) \simeq \mathrm{Hom}_{\mathrm{dgMod}(B_m)}(y_i, x_i)\,,$$

where the splitting holds only on the level of graded k-modules. We denote by $\langle x_1,\dots,x_{m-1},y_m\rangle \subset \mathrm{dgMod}(B_m)$ the full dg-subcategory spanned by $x_1,\dots,x_{m-1}$ and y_m. Note that $x_1,\dots,x_{m-1},y_m$ compactly generate $\mathcal{D}(B_m)$ so that there exists an equivalence of ∞-categories

$$\mathcal{D}(\langle x_1,\dots,x_{m-1},y_m\rangle) \simeq \mathcal{D}(B_m)\,.$$

A direct computation shows that $\langle x_1,\dots,x_{m-1},y_m\rangle$ is quasi-equivalent to the dg-category C_m with objects $x_1,\dots,x_{m-1},y_m$, generated by the morphisms

- $a_{i,i+1}\colon x_i \to x_{i+1}$ in degree 0,
- $a_{i,m}\colon x_i \to y_m$ in degree $m-i-1$ and
- $a_{m,i}\colon y_m \to x_i$ in degree $n-m+i-1$

subject to the relations $a_{i,i+1}a_{i-1,i} = 0$ for $2 \leq i \leq m-2$ and $a_{m,i}a_{j,m} = 0$ for $i \neq j$ and with differentials determined on generators by

- $d(a_{i,i+1}) = 0$ for $1 \leq i \leq m-1$ and $d(a_{m,1}) = 0$,
- $d(a_{i,m}) = (-1)^{m-i}a_{i+1,m}a_{i,i+1}$ for $i \neq m-1$,
- $d(a_{m,i}) = a_{i-1,i}a_{m,i-1}$ for $i \neq 1$.

The morphisms $a_{i,m}$ and $a_{m,i}$ are given by the images under the quasi-equivalence $\langle x_1, \ldots, x_{m-1}, y_m \rangle \to C_m$ of

$$(0, \ldots, \mathrm{id}_{x_i}, \ldots, 0) \in \bigoplus_{j=0}^{m-1} \mathrm{Hom}_{B_m}(x_i, x_j[m-j-1]) \simeq \mathrm{Hom}_{\mathrm{dgMod}(B_m)}(x_i, y_m)$$

and

$$(0, \ldots, (-1)^{i(n-1)} l_i, \ldots, 0) \in \bigoplus_{j=0}^{m-1} \mathrm{Hom}_{B_m}(x_j[m-j-1], x_i) \simeq \mathrm{Hom}_{\mathrm{dgMod}(B_m)}(y_m, x_i)\,, \tag{75}$$

respectively. For example, for $m = 4$, we can depict the generating morphisms of C_m as follows.

$$\begin{tikzcd}
 & x_2 \arrow[dr, bend left, "a_{2,3}"] \arrow[dd, bend right, "a_{2,4}"'] & \\
x_1 \arrow[ur, bend left, "a_{1,2}"] \arrow[dr, "a_{1,4}"] & & x_3 \arrow[dl, bend left, "a_{3,4}"] \\
 & y_4 \arrow[ul, bend left, "a_{4,1}"] \arrow[uu, bend right, "a_{4,2}"'] \arrow[ur, "a_{4,3}"] &
\end{tikzcd}$$

Using Lemma 5.18, we find that the dg-functor $\mu_m \colon D_m \to C_m$ determined by

- $\mu_m(z_i) = x_i$ for $i \neq m$ and $\mu_m(z_m) = y_m$,
- $\mu_m(b_{i,j}) = \begin{cases} a_{i,j} & \text{if } j = i+1 \text{ or } i = m \text{ or } j = m \\ 0 & \text{else} \end{cases}$

is a quasi-equivalence. We thus find equivalences of ∞-categories

$$\mathcal{D}(D_m) \xrightarrow{\mathcal{D}(\mu_m)} \mathcal{D}(C_m) \simeq \mathcal{D}(B_m) \simeq \mathcal{V}^m_{f^*}\,,$$

showing part (1).

We proceed with part (2). By inspecting the construction of the equivalence $\mathcal{D}(B_m) \simeq \mathcal{V}^m_{f^*}$, one finds that for $1 \leq i \leq m-1$ the functor ς_{m-i+1} is modeled by the dg-functor $k[t_{n-2}] \to B_m$, determined by mapping t_{n-2} to l_i, note for this also the commutative diagram in Proposition 2.46. The commutative diagram of dg-categories

$$\begin{tikzcd}
 & k[t_{n-2}] \arrow[dl, "\iota_i"'] \arrow[dr, "t_{n-2} \mapsto l_i"] & \\
D_m \arrow[r] & C_m \arrow[r] & \mathrm{dgMod}(B_m)
\end{tikzcd}$$

whose horizontal morphisms are Morita equivalences hence shows that ς_{m-i+1} is modeled by ι_i; note that the sign in ι_i follows from the sign $(-1)^{m-i}$ of the summand $b_{m,i}b_{i,m}$ in (72) and the sign $(-1)^{i(n-1)}$ in (75). In the case $n = m$, the remaining assertion that ς_1 is modeled by ι_1 follows from the cyclic symmetry of D_m and the sequence of adjunctions (28). $\square$

5.3.2 Gluing

Let k be a commutative ring and $n \geq 3$. When gluing dg-algebras arising from graded quivers, it is convenient to instead consider corresponding Morita-equivalent dg-categories with finitely many objects, see Remark 2.18. For instance, the dg-category corresponding to the relative Ginzburg algebra of an n-gon is described in Section 5.3.1, where it is denoted D_n. We denote by $D_{\mathfrak{T}}$ the dg-category with finitely many objects arising from the graded quiver $\tilde{Q}_{\mathfrak{T}}$ from Definition 5.4, and with differentials of the generators given as in the relative Ginzburg algebra $\mathscr{G}_{\mathfrak{T}}$.

Remark 5.20. The dg-category D_n is quasi-equivalent to the dg-category denoted D_n^{cfbr} with

- objects $1, \dots, n$,
- free generating morphisms $b_{i,j} \colon i \to j$ for all $1 \leq i, j \leq n$, $i \neq j$, as well as $l_i, L_i \colon i \to i$ for all $1 \leq i \leq n$. The degree of $b_{i,j}$ is given by

$$\deg(b_{i,j}) = \begin{cases} j-i-1 & \text{if } j > i\,, \\ j-i+n-1 & \text{if } j < i\,. \end{cases}$$

  The degrees of L_i and l_i are given by

$$\deg(L_i) = n-1\,, \quad \deg(l_i) = n-2\,.$$

- The differentials are determined on the generators by

$$d(b_{i,j}) = \begin{cases} \sum_{i \leq k \leq j} (-1)^{j-k-1} b_{k,j} b_{i,k} & \text{if } j > i \\ \sum_{i \leq k \leq n} (-1)^{j-k+n-1} b_{k,j} b_{i,k} + \sum_{1 \leq k \leq j} (-1)^{j-k-1} b_{k,j} b_{i,k} & \text{if } j < i\,, \end{cases}$$

$$d(l_i) = 0\,,$$

  and

$$d(L_i) = -l_i + \sum_{j<i} (-1)^{n+j-i} b_{j,i} b_{i,j} + \sum_{i<j} (-1)^{j-i} b_{j,i} b_{i,j}\,. \tag{76}$$

The advantage of considering D_n^{cfbr} is that the dg-functor

$$k[t_{2-n}]^{\amalg n} \longrightarrow D_n^{\mathrm{cfbr}}\,,$$

determined by mapping the generator t_{2-n} in the i-th component $k[t_{2-n}]$ to l_i, defines a cofibration between cofibrant dg-categories, with respect to the quasi-equivalence model structure on dgCat. To see this, one may directly verify the left lifting property with respect to acyclic fibrations.

Construction 5.21. For each vertex v of $\mathcal{T}$, we choose a total order of its incident halfedges, compatible with the given cyclic order. We choose for each edge e of $\mathcal{T}$ a halfedge of e. We emphasize that the outcome of this construction does not depend on these choices up to dg-isomorphism.

We define a functor

$$\underline{D}_{\mathcal{T}}\colon \operatorname{Exit}(\mathcal{T})^{\mathrm{op}} \to \mathrm{dgCat}$$

by mapping

- each vertex v of $\mathcal{T}$ to the dg-category D_n^{cfbr},
- each edge e of $\mathcal{T}$ to the dg-algebra $k[t_{n-2}]$ and
- each incidence $\alpha\colon v \xrightarrow{i} e$ of a vertex with an edge, given by the i-th halfedge in the chosen total order, to the dg-functor $k[t_{n-2}] \to D_n$ determined by
$$t_{n-2} \mapsto (-1)^{\operatorname{sgn}(\alpha)} l_{n-i}\,. \tag{77}$$
Here $\operatorname{sgn}(\alpha) = n$ if the halfedge i was chosen in the beginning, and $\operatorname{sgn}(\alpha) = 0$ if it was not. Note that (77) reverses the total order from counterclockwise (orientation of the ribbon graph), to clockwise (orientation of the quiver underlying D_n^{cfbr}).

Lemma 5.22. *The* 1*-categorical colimit of the diagram* $\underline{D}_{\mathcal{T}}\colon \operatorname{Exit}(\mathcal{T})^{\mathrm{op}} \to \mathrm{dgCat}$ *is quasi-equivalent to* $D_{\mathcal{T}}$ *and hence Morita-equivalent to the relative Ginzburg algebra* $\mathscr{G}_{\mathcal{T}}$*.*

Proof. The 1-categorical colimit of $\underline{D}_{\mathcal{T}}$ is described by the dg-category arising from a graded quiver obtained by adding to $\tilde{Q}_\Gamma$ at each vertex v an additional loop in degree $n-2$, denoted here l_v, whose differential in $\underline{D}_{\mathcal{T}}$ vanishes, as well as an additional loop in degree $n-1$ (whose differential we specify below).

Given a vertex v of $\tilde{Q}_\Gamma$ corresponding to an external edge of $\mathcal{T}$, we thus have a single loop L_v in degree $n-1$ at v. The differential is as in (76). It is straightforward to see that discarding the loops L_v, l_v does not affect the homology of $\operatorname{colim} \underline{D}_{\mathcal{T}}$.

Similarly, given a vertex v of $\tilde{Q}_\Gamma$ corresponding to an internal edge of $\mathcal{T}$, we have two loops L_v, L'_v in degree $n-1$ at v and a single loop l_v in degree $n-2$ at v. The differential of $\pm(L_v + (-1)^{n-1} L'_v)$ is given as in (60) (replacing a's by b's). Note that the sign $(-1)^{n-1} = -(-1)^n$ arises from the sign in (77). We thus have a well-defined dg-functor $D_{\mathcal{T}} \to \operatorname{colim} \underline{D}_{\mathcal{T}}$, which maps all generators $a_{v,i,j}$ to the corresponding generators in $\operatorname{colim} \underline{D}_{\mathcal{T}}$, and each loop L_v to $\pm(L_v + (-1)^{n-1} L'_v)$. This dg-functor describes the desired quasi-equvialence. □

Proof of Theorem 5.15. We consider $\mathcal{F}_{\mathcal{T}}$ as a diagram $\operatorname{Exit}(\mathcal{T}) \to \mathcal{P}r^R_{\mathrm{St}}$ and let $\mathcal{F}^L_{\mathcal{T}}$ be the left adjoint diagram of $\mathcal{F}_{\mathcal{T}}$, obtained by composing $\mathcal{F}_{\mathcal{T}}$ with the equivalence of ∞-categories $\operatorname{ladj}(\text{-})\colon \mathcal{P}r^R_{\mathrm{St}} \simeq (\mathcal{P}r^L_{\mathrm{St}})^{\mathrm{op}}$. We hence have an equivalence of ∞-categories

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}) = \lim_{\operatorname{Exit}(\mathcal{T})} \mathcal{F}_{\mathcal{T}} \simeq \operatorname{colim}_{\operatorname{Exit}(\mathcal{T})^{\mathrm{op}}} \mathcal{F}^L_{\mathcal{T}}\,.$$

To prove the Theorem, we may thus proceed with describing $\operatorname{colim}_{\mathrm{Exit}(\mathcal{T})^{\mathrm{op}}} \mathcal{F}^L_{\mathcal{T}}$.

The functor $\mathcal{F}^L_{\mathcal{T}}$ factors by Proposition 5.19, up to equivalence, through the colimit preserving functor $\mathcal{D}(\text{-})\colon \mathrm{dgCat}[W^{-1}] \to \mathcal{P}r^L_{\mathrm{St}}$ via the composite of the localization functor $\mathrm{dgCat} \to \mathrm{dgCat}[W^{-1}]$ with the functor $\underline{D}_{\mathcal{T}}\colon \mathrm{Exit}(\mathcal{T})^{\mathrm{op}} \to \mathrm{dgCat}$ from Construction 5.21. By Lemma 5.22, $\mathscr{G}_{\mathcal{T}}$ is Morita-equivalent to the colimit of $\underline{D}_{\mathcal{T}}$.

It follows from Remark 5.20 and Lemma 5.23 below that $\underline{D}_{\mathcal{T}}$ defines a cofibrant object with respect to the projective model strutures on $\mathrm{Fun}(\mathrm{Exit}(\mathcal{T})^{\mathrm{op}}, \mathrm{dgCat})$. Hence, the colimit of $\underline{D}_{\mathcal{T}}$ coincides with its homotopy colimit. The localization functor $\mathrm{dgCat} \to \mathrm{dgCat}[W^{-1}]$ turns homotopy colimits into ∞-categorical colimits and we thus find the desired equivalence of ∞-categories

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}) \simeq \operatorname{colim}_{\mathrm{Exit}(\mathcal{T})^{\mathrm{op}}} \mathcal{D}(\underline{D}_{\mathcal{T}}) \simeq \mathcal{D}(\operatorname{colim}_{\mathrm{Exit}(\mathcal{T})^{\mathrm{op}}} \underline{D}_{\mathcal{T}}) \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})\,.$$

$\square$

Lemma 5.23. *Let P be a finite, bipartite poset with partition sets $X, Y \subset P$ and morphisms going from X to Y. Let C be a model category with finite coproducts and $F\colon P \to C$ a diagram valued in cofibrant objects. Assume further that, for all $y \in P$, the morphism*

$$\coprod_{\alpha: x\to y \in X/y} F(\alpha)\colon \coprod_{\alpha: x\to y \in X/y} F(x) \longrightarrow F(y)$$

is a cofibration in C, where $X/y = X \times_P P/y$ is the relative over-category. Then F defines a cofibrant object in the category $\mathrm{Fun}(P, C)$ with the projective model structure. In particular, the colimit of F coincides with the homotopy colimit.

Proof. We need to check the right lifting property of F with respect to acyclic fibrations in $\mathrm{Fun}(P, C)$, meaning we need to solve the lifting problem

$$\begin{tikzcd} & G \arrow[d, "\eta"] \\ F \arrow[ur, dashed, "\mu"] \arrow[r, "\nu"] & H \end{tikzcd}$$

where $G, H\colon P \to C$ and $\eta\colon G \to H$ is a acyclic fibration, meaning that $\eta(p)$ is an acyclic fibration in C for all $p \in P$. For each $x \in X$, we can use that $F(x) \in C$ is cofibrant to lift $\nu(x)$ along $\eta(x)$, defining the morphism $\mu(x)\colon F(x) \to G(x)$. Let $y \in Y$ and consider the composite morphism in C

$$\xi_y\colon \coprod_{\alpha: x\to y \in X/y} F(x) \xrightarrow{\coprod_{\alpha: x\to y \in X/y} \mu(x)} \coprod_{\alpha: x\to y \in X/y} G(x) \xrightarrow{\coprod_{\alpha: x\to y \in X/y} G(\alpha)} G(y)\,.$$

Using that $\coprod_{\alpha: x\to y \in X/y} F(\alpha)$ is a cofibrantion and $\eta(y)$ a trivial cofibration, we can solve the lifting problem

$$\begin{tikzcd} \coprod_{\alpha: x\to y \in X/y} F(x) \arrow[r, "\xi_y"] \arrow[d, "\coprod_{\alpha: x\to y \in X/y} F(\alpha)"'] & G(y) \arrow[d, "\eta(y)"] \\ F(y) \arrow[ur, dashed, "\mu(y)"] \arrow[r, "\nu(y)"] & H(y) \end{tikzcd}$$

defininig $\mu(y)$. Inspecting the construction, it is immediate that these choices of $\mu(x)$ and $\mu(y)$ for $x \in X, y \in Y$ assemble into a natural transformation μ. Further, by construction, $\eta(x) \circ \mu(x) = \nu(x)$ and $\eta(y) \circ \mu(y) = \nu(y)$ for all $x \in X, y \in Y$ and thus also $\eta \circ \mu = \nu$. This shows that μ is the desired lift, concluding the proof. $\square$

Let $e \in \mathcal{T}_1$ be an edge and $R = k$ a commutative ring. Recall that $\mathrm{ev}_e^*\colon \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}) \to \mathcal{F}_{\mathcal{T}}(e) = \mathrm{RMod}_{k[t_{n-2}]}$ denotes be the left adjoint of the evaluation functor ev_e of co-Cartesian sections at e.

Proposition 5.24. *If $R = k$ is a commutative ring, the projective $\mathscr{G}_{\mathcal{T}}$-module $p_e\mathscr{G}_{\mathcal{T}}$, where $p_e \in k\tilde{Q}_{\mathcal{T}}$ is the lazy path at e, is identified under the equivalence (69) with $\mathrm{ev}_e^*(k[t_{n-2}])$.*

Proof. Let v be a vertex incident to e. The functor ev_e factors through the restriction functors
$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}) \to \mathcal{H}(\mathcal{T}_{v/}, \mathcal{F}_{\mathcal{T}}|_{\mathrm{Exit}(\mathcal{T}_{v/})}) \to \mathcal{F}(e)\,.$$
The left adjoint thus factors as
$$\mathcal{F}(e) \to \mathcal{H}(\mathcal{T}_{v/}, \mathcal{F}_{\mathcal{T}}|_{\mathrm{Exit}(\mathcal{T}_{v/})}) \to \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})\,.$$
The first functor in this factorization is modeled by Proposition 5.19 by a dg-functor $k[t_{n-2}] \to D_n \simeq D_n^{\mathrm{cfbr}}$. The latter functor in this factorization is modeled by the proof of Theorem 5.15 by the functor $D_n^{\mathrm{cfbr}} \to \operatorname{colim} \underline{D}_{\mathcal{T}}$. Composing with the Morita equivalence $\underline{D}_{\mathcal{T}} \to (\mathscr{G}_{\mathcal{T}})^{\mathrm{perf}}$, the arising dg-functor $k[t_{n-2}] \to (\mathscr{G}_{\mathcal{T}})^{\mathrm{perf}}$ maps the unique object of $k[t_{n-2}]$ to $p_e\mathscr{G}_{\mathcal{T}}$. This shows that $\mathrm{ev}_e^*(k[t_{n-2}]) \simeq p_e\mathscr{G}_{\mathcal{T}}$. $\square$

5.3.3 The relative Calabi–Yau structure

For this section, we fix a field k and an integer $n \geq 3$.

Lemma 5.25.

(1) There exists an equivalence
$$\mathrm{HH}(\mathcal{D}(k[t_{n-2}])) \simeq k[t_{n-2}] \oplus k[t_{n-2}][n-1]\,.$$

(2) Let $\varphi\colon k[t_{n-2}] \xrightarrow{t_{n-2} \mapsto (-1)^n t_{n-2}} k[t_{n-2}]$. There exists a commutative diagram
$$\begin{tikzcd} \mathrm{HH}(k[t_{n-2}]) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}(\varphi^*)"'] & k[t_{n-2}] \oplus k[t_{n-2}][n-1] \arrow[d, "\varphi\oplus\varphi[n-1]"] \\ \mathrm{HH}(k[t_{n-2}]) \arrow[r, "\simeq"] & k[t_{n-2}] \oplus k[t_{n-2}][n-1] \end{tikzcd}$$

Proof. Note that $k[t_{n-2}]^e \simeq k[s_{n-2}, t_{n-2}]$ is the graded commutative polynomial dg-algebra in two variables in degree $n-2$. A free $k[s_{n-2}, t_{n-2}]$-resolution of the right $k[s_{n-2}, t_{n-2}]$-module $k[t_{n-2}]$ is given by
$$k[s_{n-2}, t_{n-2}][n-2] \xrightarrow{(s_{n-2}+(-1)^n t_{n-2})\cdot(\text{-})} k[s_{n-2}, t_{n-2}]\,.$$

It follows that

$$\mathrm{HH}(k[t_{n-2}]) \simeq k[t_{n-2}] \otimes_{k[t_{n-2}]^e} k[t_{n-2}] \simeq k[t_{n-2}] \oplus k[t_{n-2}][n-1]\,,$$

showing part (1). For part (2), we note that the action of $\mathrm{HH}(\varphi)$ is equivalent to the apparent action of φ on the tensor product $k[t_{n-2}] \otimes_{k[t_{n-2}]^e} k[t_{n-2}]$, as follows for instance by combining [BD19, Prop. 4.3] and [BD21, Prop. 4.4], implying the commutativity of the diagram. □

Definition 5.26. Let $\mathbf{S}$ be a marked surface with an ideal n-angulation $\mathcal{T}$ with n even. We call $\mathcal{T}$ orientable if there exist choices of orientations of the edges of $\mathcal{T}$, such that the directions of the halfedges at any vertex of $\mathcal{T}$ alternative in their cyclic order.

Theorem 5.27. *Let $\mathbf{S}$ be a marked surface with an ideal n-angulation $\mathcal{T}$. The k-linear ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ is a smooth. If n is odd, or $\mathcal{T}$ is orientable in the sense of Definition 5.26, then the k-linear functor*

$$\partial\mathcal{F}_{\mathcal{T}} \colon \prod_{e\in\mathcal{T}_1^\partial} \mathcal{F}(e) \longrightarrow \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$$

admits a weak left n-Calabi–Yau structure.

Remark 5.28. If $\mathbf{S}$ has empty boundary, then Theorem 5.27 shows that $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ is weakly left n-Calabi–Yau (without the adjective relative). A related result is shown in [CHQ23, Prop. 7.6], namely that the derived endomorphism algebra of the simple $\mathscr{G}_{\mathcal{T}}$-modules associated with the vertices of the underlying quiver is a version of right n-Calabi–Yau. The condition of orientability of Definition 5.26 also appears there.

Proof of Theorem 5.27. The smoothness follows from Corollary 4.31. By Theorem 4.36, $\mathcal{D}(k[t_{n-2}]) \simeq \mathrm{Fun}(S^{n-1}, \mathcal{D}(k))$ admits a weak left $(n-1)$-Calabi–Yau structure. We can take the corresponding Hochschild class $\sigma_{k[t_{n-2}]} \colon k[n-1] \to \mathrm{HH}(\mathcal{D}(k[t_{n-2}])$ to be given by $1 \in k \simeq \mathrm{H}_{n-1}\,\mathrm{HH}(\mathcal{D}(k[t_{n-2}])$. Composition with the functor $\mathrm{HH}(\varphi^*)$ maps $\sigma_{k[t_{n-2}]}$ by Lemma 5.25 to $(-1)^n\sigma_{k[t_{n-2}]}$.

Theorem 4.36 further shows that the functor $f_! \colon \mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) \to \mathcal{D}(k)$ admits a weak left $(n-1)$-Calabi–Yau structure. Since $f_* \simeq f_![-n]$ is modeled by ϕ_*, it follows that ϕ_* also admits a weak left $(n-1)$-Calabi–Yau structure $\sigma \colon k[n] \to \mathrm{HH}(\mathcal{D}(k[t_{n-2}]), \mathcal{D}(k))$, which restricts to $\sigma_{k[t_{n-2}]} \colon k[n-1] \to \mathrm{HH}(\mathcal{D}(k[t_{n-2}])$. The functor

$$\mathcal{D}(k[t_{n-2}])^{\times n} \xrightarrow{\prod \varsigma_i} \mathcal{V}^n_{\phi^*}$$

inherits by Proposition 4.38 a weak left n-Calabi–Yau structure which restricts on the i-th copy of $\mathcal{D}(k[t_{n-2}])$ to $(-1)^i\sigma_{k[t_{n-2}]}$. Inspecting Construction 5.14 and, if n is odd using that φ^* acts on $\sigma_{k[t_{n-2}]}$ by reversing the sign, we find that the conditions of Theorem 4.43 are satisfied for $\mathcal{F}_{\mathcal{T}}$, yielding the desired relative weak left n-Calabi–Yau structure. □

Remark 5.29. In the case $n = 3$, the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ arises from an ice quiver with potential in the sense of [Wu23b] and is thus an instance of Yeung's deformed relative Calabi-Yau completion [Yeu16]. This gives an alternative proof of Theorem 5.27 in the case $n = 3$, which additionally shows that $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ is relative left n-Calabi–Yau (without the adjective weak).

5.4 Objects from curves

In Section 5.4.1, we introduce the class of curves forming the base for the geometric model, referred to as matching curves. We proceed in Section 5.4.2 with the construction of the global sections associated with matching data, consisting of matching curves together with further data. In Section 5.4.3, we show that the projective $\mathscr{G}_{\mathcal{T}}$-modules associated with the vertices of the underlying quiver can be realized in terms of global sections associated with matching data with underlying pure matching curves.

5.4.1 Matching curves

We fix a marked surface $\mathbf{S}$ with an n-valent spanning graph $\mathcal{T}$. We also fix a base $\mathbb{E}_\infty$-ring spectrum R. For each vertex v of $\mathcal{T}$, we have an immersion $\Sigma_v \to \Sigma_{\mathcal{T}}$, see Remark 3.54. This immersion is an embedding if no edge of $\mathcal{T}$ incident to v is a loop.

Definition 5.30. Let $v \in \mathcal{T}_0$. A segment at v is an embedded curve $\delta\colon [0,1] \to \Sigma_v$, which does not hit v away from the endpoints and which is of one of the following two types.

(1) One end lies at v, the other on the boundary of Σ_v.

(2) Both ends lie on the boundary of Σ_v. In this case, since the segment δ is embedded, it wraps $1 \leq a \leq n$ steps around the vertex v.

We consider segments at v as equivalence classes under homotopies relative $\partial\Sigma_v \cup \{v\}$.

A segment in $\Sigma_{\mathcal{T}}$ is a segment at any $v \in \mathcal{T}_0$. We do not distinguish in notation its representatives from the curves in $\Sigma_{\mathcal{T}}$ obtained from composing with the immersion $\Sigma_v \to \Sigma_{\mathcal{T}}$.

The two types of segments are depicted in Figure 2.

We can always assume that a given end of a segment which does not lie at a vertex of $\mathcal{T}_0$ ends on an edge e of $\mathcal{T}$. This will be useful for specifying at which boundary component of Σ_v the segments begins or ends. If the segment is of the first type and ends at e, we also say that the segment exits the vertex through e.

Definition 5.31. Let δ be a segment in $\Sigma_{\mathcal{T}}$. If δ is of the first type, we define its degree as $d(\delta) = 0$. If δ is of the second type, we define its degree as $d(\delta) = a-1$ if δ goes in the counterclockwise direction and as $d(\delta) = 1-a$ if δ goes in the clockwise direction. We call a segment pure if $d(\delta) = 0$.

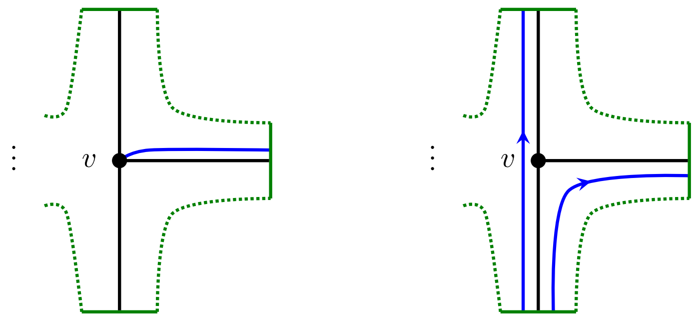


Figure 2: A segment of the first type at v (in blue, on the left) with unspecified direction and two segments of the second type at v (in blue, on the right), wrapping around v in the counterclockwise and clockwise direction by $a = 1$ and $a = n - 2$ steps, respectively.

For $N \in \mathbb{N} = \{1, 2, 3, \dots\}$, we consider the sets $[N] \coloneqq \{1, \dots, N\}$ and $\mathbb{Z}/N\mathbb{Z} \coloneqq \{1, \dots, N+1\}/N+1 \sim 1$. The set $[N]$ is ordered linearly, the set $\mathbb{Z}/N\mathbb{Z}$ has a cyclic order.

Definition 5.32. Let $I = \mathbb{Z}/N\mathbb{Z}, [N], \mathbb{N}, \mathbb{Z}$ for $N \in \mathbb{N}$. Consider a collection of segments $\{\delta_i\}_{i \in I}$ in $\Sigma_{\mathcal{T}}$ satisfying for all $i \in I$ and $i \neq N$ if $I = [N]$ that

- $\delta_{i+1}(0) = \delta_i(1)$ in $\Sigma_{\mathcal{T}}$.
- if the segments δ_i and δ_{i+1} both lie at the same vertex v of $\mathcal{T}$, then $\delta_i(1)$ lies on a loop of $\mathcal{T}$ and the two points $\delta_i(1), \delta_{i+1}(0)$ lie on different boundary components of Σ_v, (this means that that their composite wraps around the loop).

Consider the curve $\gamma \colon U \to \Sigma_{\mathcal{T}}$ arising from composing (representatives of) the curves $\{\delta_i\}_{i \in I}$ (in their given order). We further suppose that

- The curve γ does not cut out (by tracing along a connected part of the curve) any unmarked discs in $\Sigma_{\mathcal{T}}$.
- If $I = \mathbb{Z}/N\mathbb{Z}$, that $d(\gamma) = 0$, see Definition 5.33, and that γ is not homotopic relative $\partial\Sigma_{\mathcal{T}} \cup \mathcal{T}_0$ to the composite of multiple identical closed curves in $\Sigma_{\mathcal{T}}$.

We call the curve $\gamma \colon U \to \Sigma_{\mathcal{T}}$ a curve in $\Sigma_{\mathcal{T}}$ composed of segments. Being composed of segments is a property, the segments can be recovered from γ by intersecting with the Σ_v's.

Reparametrizing γ if necessary, we can assume that $U = S^1$ if $I = \mathbb{Z}/N\mathbb{Z}$, $U = [0, 1]$ if $I = [N]$, $U = [0, \infty)$ if $I = \mathbb{N}$ and $U = (-\infty, \infty)$ if $I = \mathbb{Z}$. If $U = S^1$, we call γ closed. Otherwise, we call γ open.

Definition 5.33. Let γ be a curve in $\Sigma_{\mathcal{T}}$ composed of segments.

(1) We call γ regular, if it is only composed of segments of the second type which do not wrap around the vertices by n steps. We call γ singular, if it is not regular.

(2) Suppose that γ is composed of finitely many segments $\delta_1, \ldots, \delta_m$. We define the degree of γ as

$$d(\gamma) = \sum_{i=1}^{m} d(\delta_i)\,.$$

Definition 5.34.

- Let $U = S^1, [0,1], [0,\infty), (-\infty,\infty)$. Consider a curve $\gamma\colon U \to \Sigma_{\mathcal{T}}$ composed of segments, see Definition 5.32. The curve γ is called a matching curve in $\Sigma_{\mathcal{T}}$ if for all $x \in \partial U$, $\gamma(x)$ lies in $\mathcal{T}_0$ or in $\partial\Sigma_{\mathcal{T}}$. We consider matching curves as equivalence classes under homotopies relative $\partial\Sigma_{\mathcal{T}} \cup \mathcal{T}_0$.
- A matching curve in $\mathbf{S}\backslash M$ is defined to be a homotopy class relative $(\partial\mathbf{S}\backslash M) \cup \mathcal{T}_0$ of curves $U \to \mathbf{S}\backslash M$ which contains a representative given by the composite of a matching curve in $\Sigma_{\mathcal{T}}$ with the homotopy equivalence $\Sigma_{\mathcal{T}} \to \mathbf{S}\backslash M$ of Remark 3.54.
- A matching curve in $\Sigma_{\mathcal{T}}$ or $\mathbf{S}\backslash M$ is called finite if it is composed of finitely many segments.
- A matching curve in $\Sigma_{\mathcal{T}}$ or $\mathbf{S}\backslash M$ is called pure if it is only composed of pure segments.

Note that matching curves do not intersect $\mathcal{T}_0$ nor the boundary of the surface except at the endpoints.

Remark 5.35. The notion of matching curve in $\mathbf{S}\backslash M$ does not depend on the choice of n-valent spanning graph $\mathcal{T}$. A homotopy class of open curves in $\mathbf{S}\backslash M$ with endpoints in $(\partial\mathbf{S}\backslash M) \cup \mathcal{T}_0$, but which is away from its endpoints disjoint from $(\partial\mathbf{S}\backslash M) \cup \mathcal{T}_0$, arises as a matching curve if and only if it has a representative which cuts out no discs and which is not contractible to a point in $(\partial\mathbf{S}\backslash M) \cup \mathcal{T}_0$. For closed curves, additionally the degree needs to vanish.

Notation 5.36. Let γ be an open curve in $\mathbf{S}\backslash M$ composed of segments with index set I. Let $i \in I$ and δ_i be the corresponding segment of γ. We denote by $\gamma < \delta_i$ the curve obtained by the composite of the segments $\{\delta_j\}_{j\in I, j<i}$ and by $\gamma \leq \delta_i$ the curve obtained by the composite the segments $\{\delta_j\}_{j\in I, j\leq i}$. We similarly define the curves $\delta_i < \gamma$ and $\delta_i \leq \gamma$, and given two segments $\delta_i, \delta_{i'}$ of γ, the curves $\delta_i < \gamma < \delta_{i'}$, $\delta_i \leq \gamma < \delta_{i'}$, $\delta_i < \gamma \leq \delta_{i'}$ and $\delta_i \leq \gamma \leq \delta_{i'}$.

Lemma 5.37. *Let $n = 3$. There exists a bijection between*

1) pure matching curves in $\mathbf{S}\backslash M$ and

2) curves in $\mathbf{S}\backslash M$ which do not cut out any discs in $\mathbf{S}\backslash M$ and whose endpoints lie in $\mathcal{T}_0$ or $\partial\mathbf{S}\backslash M$, considered modulo homotopies relative $\partial\mathbf{S}\backslash M$ which fix the endpoints in $\mathcal{T}_0$.

Proof. Let v be a vertex of $\mathcal{T}$. In Σ_v, each segment of the second type wrapping around v by two steps is homotopic relative $\partial\Sigma_v$ to a segment of the second type wrapping only a single step around v, which is thus pure. Note that this uses, that the homotopy is allowed to cross the vertex v. Similarly, each segment of the second type wrapping around v by 3 steps is homotopic relative $\partial\Sigma_v$ to a point.

Given a curve in $\mathbf{S}\backslash M$ which cuts out no discs, it arises as a matching curve. It follows by the above that this curve is homotopy equivalent, relative $\partial\mathbf{S}\backslash M$ and fixing endpoints in $\mathcal{T}_0$, to a pure matching curve. We can thus produce from each curve as in 2) a pure matching curve in $\mathbf{S}\backslash M$. Conversely, any pure matching curve clearly defines a curve as in 2). These assignments are inverse bijections. □

5.4.2 Objects from matching curves

We fix an n-valent spanning graph $\mathcal{T}$ of a marked surface $\mathbf{S}$ and an $\mathbb{E}_\infty$-ring spectrum R.

Definition 5.38. A matching datum (γ, L) in $\mathbf{S}\backslash M$ consists of

- a matching curve γ in $\mathbf{S}\backslash M$,
- an object $L \in \mathrm{RMod}_{R[t_{n-2}]}$ called the local value,
- if γ is singular an object $Q \in \mathrm{RMod}_R$ and an equivalence $\phi^*(Q) \simeq L$,
- if γ is closed, an integer $a \geq 1$ and a monodromy equivalence

$$\mu \in \mathrm{Map}_{\mathrm{RMod}_{R[t_{n-2}]}}(L^{\oplus a}, L^{\oplus a})\,.$$

We further assume that L satisfies the technical condition explained in Remark 5.39, which is always fulfilled if $R = k$ is a commutative ring.

We call (γ, L) open if γ is open and closed if γ is closed. We define the rank of (γ, L) as a if γ is closed and as 1 if γ is open.

Given a segment δ and an object $L \in \mathrm{RMod}_{R[t_{n-2}]}$ (also subject to the conditions in Remark 5.39), we associate below a section M_δ^L of the $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}(R)$ from Construction 5.14. By gluing sections, we then produce for each matching datum (γ, L) a global section M_γ^L of $\mathcal{F}_{\mathcal{T}}(R)$, see Proposition 5.44.

Remark 5.39. For computational reasons, we always assume that $L \in \mathrm{RMod}_{R[t_{n-2}]}$ satisfies $T_{\mathrm{RMod}_{R[t_{n-2}]}}(L) \simeq L[1-n]$, where $T_{\mathrm{RMod}_{R[t_{n-2}]}}$ is the twist functor of the spherical adjunction $\phi^* \dashv \phi_*$. If $R = k$ is a commutative ring, then this is satisfied for any L, as follows from [Chr22b, Prop. 5.7]. If R is arbitrary, then $L = \phi^*(R)$ and $L = R[t_{n-2}]$ also satisfy the requirement, see part (2) of Proposition 5.12.

Construction 5.40. Let $p\colon \Gamma(\mathcal{F}_{\mathcal{T}}(R)) \to \mathrm{Exit}(\Gamma)$ be the Grothendieck construction of $\mathcal{F}_{\mathcal{T}}(R)$ and $\mathcal{L}$ the ∞-category of sections of $\mathcal{F}_{\mathcal{T}}(R)$, i.e. sections of p, see Definition 3.36.

We consider a vertex $v \in \mathcal{T}_0$, with cyclically ordered incident halfedges $a_1, \dots, a_n$, which are part of the edges $e_1, \dots, e_n$.

1) For $1 \le i \le n$, we denote by δ^i the segment of the first type at v ending at $e_i \cap \partial\Sigma_v$. Consider the paracyclic twist functor $T_{\mathcal{V}^n_{\phi^*}}$ from Lemma 3.31. Let $L \simeq \phi^*(Q)$. We define the section $M^L_{\delta^i}$ of $\mathcal{F}_{\mathcal{T}}(R)$ as the p-relative left Kan extension along the inclusion $\Delta^0 \xrightarrow{v} \mathrm{Exit}(\mathcal{T})$ of the functor $\Delta^0 \to \mathcal{F}_{\mathcal{T}}(R)(v) \subset \mathcal{L}$ with value

$$M_{\delta^i}(v) = T^{i-n}_{\mathcal{V}^n_{\phi^*}}(Q \to 0 \to \dots)\,[1-n] \in \mathcal{V}^n_{f^*} = \mathcal{F}_{\mathcal{T}}(R)(v)\,.$$

Spelling out the definition, one sees that the section M_{δ^i} is concentrated at the elements $v, e_i \in \mathrm{Exit}(\mathcal{T})$ and takes up to equivalence the values

$$M^L_{\delta^i}(v) \simeq \left(Q \xrightarrow{!} L \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \underbrace{L}_{(n-i+1)\text{-th}} \to 0 \to \dots \to 0\right)[-i+1] \in \mathcal{V}^n_{\phi^*} = \mathcal{F}_{\mathcal{T}}(R)(v) \tag{78}$$

$$M^L_{\delta^i}(e_i) \simeq L \in \mathrm{RMod}_{R[t_{n-2}]} = \mathcal{F}_{\mathcal{T}}(R)(e_i) \tag{79}$$

and assigns to the edge $v \to e_i$ a coCartesian morphism, describing an apparent equivalence $\varrho_i(M_{\delta^i}(v)) \simeq L \simeq M_\delta(e_i)$. The notation $\xrightarrow{*}$ and $\xrightarrow{!}$ in (78) and below refer to p-Cartesian and p-coCartesian morphisms, respectively, see also Section 2.1.3.

2) For $1 \le i,j \le n$ and $i \ne j$, we denote by $\delta^{i,j}$ the segment at v which starts at $e_i \cap \partial\Sigma_v$ and ends at $e_j \cap \partial\Sigma_v$ going in the counterclockwise direction. For $L \in \mathrm{RMod}_{R[t_{n-2}]}$, we define $M^L_{\delta^{i,j}}$ as the p-relative left Kan extension along $\Delta^0 \xrightarrow{v} \mathrm{Exit}(\mathcal{T})$ of the functor $\Delta^0 \to \mathcal{F}_{\mathcal{T}}(R)(v) \subset \mathcal{L}$ with value

$$M^L_{\delta^{i,j}}(v) = T^{i-1}_{\mathcal{V}^n_{\phi^*}}\left(0 \to \dots \to 0 \to \underbrace{L}_{(n-j+i+1)\text{-th}} \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} L\right) \in \mathcal{V}^n_{\phi^*} = \mathcal{F}_{\mathcal{T}}(R)(v)\,.$$

Spelling out the definition, one sees that the section $M^L_{\delta^{i,j}}$ is concentrated at e_i, e_j and v. To the edges it assigns

$$M^L_{\delta^{i,j}}(e_l) \simeq \begin{cases} L & l = i\,, \\ L[j-i-1] & k = j > i\,, \\ L[n+j-i-1] & k = j < i\,, \\ 0 & \text{else.} \end{cases} \tag{80}$$

This uses that $T_{\mathrm{RMod}_{R[t_{n-2}]}}(L) \simeq L[1-n]$, see Remark 5.39.

The value of $M_{\delta^{i,j}}$ at v is given as follows. If $i < j$, we have

$$M_{\delta^{i,j}}(v) \simeq \left(0 \to \dots \to 0 \to \underbrace{L}_{(n-j+2)\text{-th}} \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \underbrace{L}_{(n-i+1)\text{-th}} \to 0 \to \dots \to 0\right)[-i+1]\,,$$

if $j < i < n$

$$M_{\delta^{i,j}}(v) \simeq \Big(\phi_*(L) \xrightarrow{!} \phi^*\phi_*(L) \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \underbrace{\phi^*\phi_*(L)}_{(n-i+1)\text{-th}} \xrightarrow{\mathrm{cu}_L} L \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \underbrace{L}_{(n-j+1)\text{-th}} \to 0 \to \dots\Big)[n-i]\,,$$

where cu_L denotes the counit map of the adjunction $\phi^* \dashv \phi_*$ at L, and in the case $j < i = n$ we have

$$M_{\delta^{n,j}}(v) \simeq \Big(\phi_*(L) \xrightarrow{*} L \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \underbrace{L}_{(n-j+1)\text{-th}} \to 0 \to \dots \to 0\Big)\,.$$

3) Let $1 \leq i \leq n$ and $L \simeq \phi^*(Q)$. Denote by $\delta^{i,i}$ the segment of the second type starting and ending at $e_i \cap \partial\Sigma_v$, wrapping around v by n steps in the counterclockwise direction. We define $M^L_{\delta^{i,i}}$ as the p-relative left Kan extension along $\Delta^0 \xrightarrow{v} \mathrm{Exit}(\mathcal{T})$ of the functor $\Delta^0 \to \mathcal{F}_{\mathcal{T}}(R)(v) \subset \mathcal{L}$ with value

$$M^L_{\delta^{i,i}}(v) \simeq \Big(\phi_*(L) \xrightarrow{!} \phi^*\phi_*(L) \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \underbrace{\phi^*\phi_*(L)}_{(n-i+1)\text{-th}} \to 0 \to \dots \to 0\Big)[n-i]$$

if $i \neq n$ and value

$$M^L_{\delta^{n,n}}(v) \simeq \Big(\phi_*(L) \to 0 \to \dots \to 0\Big)$$

if $i = n$. One finds apparent equivalences

$$M_{\delta^{i,i}}(e_l) \simeq \begin{cases} \phi^*\phi_*(L)[n-1] \simeq \phi^*\phi_!(L) & l = i \\ 0 & l \neq i \end{cases}$$

We have a splitting $\phi^*\phi_!(L) \simeq \phi^*\phi_!\phi^*(Q) \simeq \phi^*(Q) \oplus \phi^*(Q)[n-1] \simeq L \oplus L[n-1]$ arising from the following diagram:

$$\begin{tikzcd} \phi^*(Q) \arrow[r, "\mathrm{u}_{\phi^*(Q)}"] \arrow[d] \arrow[dr, phantom, "\square"] & \phi^*\phi_!\phi^*(Q) \arrow[r, "\phi^* o \mathrm{cu}_Q"] \arrow[d] \arrow[dr, phantom, "\square"] & \phi^*(Q) \arrow[d] \\ 0 \arrow[r] & \phi^*(Q)[n-1] \arrow[r] & 0 \end{tikzcd}$$

Above u and cu denote the unit and counit of $\phi_! \dashv \phi^*$, respectively.

Remark 5.41. Consider a segment $\delta^{i,j}$ of the second type with $i \neq j$. The suspensions arising in the definition of the $M^L_{\delta^{i,j}}$ in (80) correspond to the degree of $\delta^{i,j}$. Namely, one finds $M^L_{\delta^{i,j}}(e_j) \simeq L[d(\delta^{i,j})]$ if $i \neq j$ and $M^L_{\delta^{i,i}}(e_j) = L \oplus L[d(\delta^{i,j})]$ if $i = j$.

Construction 5.42. Let γ be an open curve composed of segments in $\mathbf{S}\backslash M$. Let $L \in \mathrm{RMod}_{R[t_{n-2}]}$. In the following we give the construction of a section $M_\gamma^L \in \mathcal{L}$ of $\mathcal{F}_{\mathcal{T}}(R)$. If $L = \phi^*(R)$, we will also use the notation $M_\gamma = M_\gamma^{\phi^*(R)}$. If $R = k$ is a commutative ring and $L = k[t_{n-2}^\pm]$, we will also write $N_\gamma = M_\gamma^{k[t_{n-2}^\pm]}$ (this will only be used in Section 6).

Let I be the index set of the segments of γ. We denote (if they exist) the minimal and maximal elements of I by $\min(I), \max(I)$. We set $I' = I\backslash\{\max(I)\}$, and if I has no maximal element, we agree that $\{\max(I)\} = \emptyset$. We set $I'' = I\backslash\{\min(I), \max(I)\}$, and if I has no minimal element, we again set $\{\min(I), \max(I)\} = \emptyset$. Recall that the segments of γ are denoted by δ_i with $i \in I$ (ordered compatibly with their appearance in γ). We denote the edge of $\mathcal{T}$ where δ_i begins by e^i and the edge where δ_i ends by e^{i+1}. For later use, we also denote by $v^i \in \mathcal{T}_0$ the vertex at which δ_i lies. We define M_γ^L as the colimit of a diagram D_γ in the ∞-category $\mathcal{L}$ of sections of $\mathcal{F}_{\mathcal{T}}(R)$, which is given as follows.

The domain of D_γ is the coequalizer E_γ in the 1-category of simplicial sets of the diagram

$$\coprod_{i\in I''} \Delta^0 \underset{\coprod_{i\in I''} \Delta^{\{2\}}\times\{i-1\}}{\overset{\coprod_{i\in I''} \Delta^{\{1\}}\times\{i\}}{\rightrightarrows}} \coprod_{i\in I'} \Lambda_0^2 \times \{i\}$$

where the horn $\Lambda_0^2 \times \{i\} \simeq \Lambda_0^2$ is the poset with objects $(0,i),(1,i),(2,i)$ and morphisms $(0,i) \to (1,i),(2,i)$, i.e. a span. For $l = 1,2$ and $j \in I'$, the morphism $\Delta^{\{l\}} \times \{j\}$ is the inclusion $\Delta^0 \to \amalg_{i\in I'}\Lambda_0^2 \times \{i\}$ determined by mapping $0 \in (\Delta^0)_0$ to $(l,j) \in \Lambda_0^2 \times \{j\} \subset \amalg_{i\in I'}\Lambda_0^2 \times \{i\}$. Recall that for each segment δ_i, we defined an associated section $M_{\delta_i}^L \in \mathcal{L}$ in Construction 5.40. We further denote by $Z_{e^j}^L \in \mathcal{L}$ the section concentrated at e^j with value L. The diagram D_γ is determined via the its restrictions to $\Lambda_0^2 \times \{i\}$ with $i \in I'$, which is for $i \geq 1$ given by

$$\begin{tikzcd} & Z_{e^i}^L[d(\delta_1 \leq \gamma < \delta_{i+1})] \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M_{\delta_i}^L[d(\delta_1 \leq \gamma < \delta_i)] & & M_{\delta_{i+1}}^L[d(\delta_1 \leq \gamma < \delta_{i+1})] \end{tikzcd}$$

and for $i \leq 0$ given by

$$\begin{tikzcd} & Z_{e^i}^L[-d(\delta_{i+1} \leq \gamma < \delta_1)] \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M_{\delta_i}^L[-d(\delta_i \leq \gamma < \delta_1)] & & M_{\delta_{i+1}}^L[-d(\delta_{i+1} \leq \gamma < \delta_1)] \end{tikzcd}$$

where α and β are the apparent (pointwise in $\mathrm{Exit}(\mathcal{T})$) inclusions.

Construction 5.43. Let (γ, L) be a closed matching datum in $\mathbf{S}\backslash M$ of rank a with monodromy equivalence μ. Let η be the open curve composed of the segments $\delta_, 1\dots,\delta_N$ of γ, where δ_N and δ_1 have not been composed. Let e be the edge of

$\mathcal{T}$ where the curve η starts and ends. We consider the section M_η^L from Construction 5.42. We define M_γ^L as the coequalizer in $\mathcal{L}$ of

$$(Z_e^L)^{\oplus a} \underset{\iota'\circ\mu^{-1}}{\overset{\iota}{\rightrightarrows}} (M_\eta^L)^{\oplus a} \tag{81}$$

where ι and ι' are the morphisms with support at e determined by the inclusions of $L^{\oplus a}$ into $M_\eta^L(e)^{\oplus a}$ arising from the two ends of η at e and $\mu\colon L^{\oplus a} \to L^{\oplus a}$ is the monodromy equivalence. We note that upon changing the basepoint $1 \in I = \mathbb{Z}/N\mathbb{Z}$, i.e. relabeling the elements of I but keeping their cyclic order, the curve γ does not change. One can show, that the section M_γ^L also only changes up to equivalence by such a relabeling.

The section M_η^L is obtained by gluing together local sections defined via Kan extensions. Making different choices of these Kan extensions further yields a different section M_γ^L, which however only differs by a change in the monodromy equivalence μ given by composition with an invertible diagonal $a \times a$-matrix with entries in $\mathrm{Map}_{k[t_{n-2}]}(L, L)$. By varying over all μ, we thus always construct the same class of objects, independent of the choices of Kan extensions. One can remove this ambiguity by fixing choices of these Kan extensions. Alternatively, in the special case $n = 3$ and $R = k$ a field and $L = k[t_1^\pm]$, the proof of Theorem 6.35 also describes a way to associate a unique monodromy equivalence with a global section $M_\gamma^{k[t_1^\pm]}$.

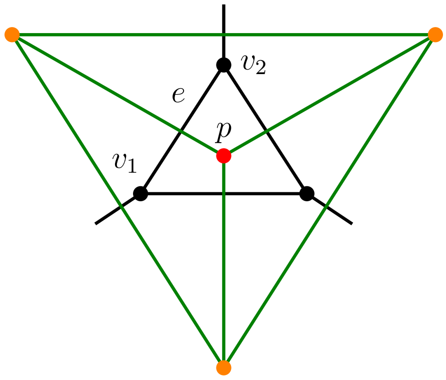
Proposition 5.44. *Let (γ, L) be matching datum. The section M_γ^L of $\mathcal{F}_\mathcal{T}(R)$ defined in Construction 5.42 or Construction 5.43 is a global section.*

Proof. One needs to show that M_γ^L is a coCartesian section. This directly follows from unraveling the construction of M_γ^L and is left to the reader. □

Remark 5.45. Given a matching datum (γ, L), there is a canonical matching datum with underlying matching curve γ^{rev} obtained by reversing the orientation of γ. If γ is an open matching curve, it is easy to see that $M_{\gamma^{\mathrm{rev}}}^L \simeq M_\gamma[-d(\gamma)]$. If γ is a closed matching curve, reversing the orientation of γ in the construction of M_γ^L interchanges the two maps in (81). Hence, to obtain the same global section, we need to replace the monodromy equivalence in the matching datum by its inverse.

Remark 5.46. It is apparent that Construction 5.40 can be translated to a construction, which associates for an arbitrary perverse schober $\mathcal{F}$ with generic stalk $\mathcal{N} \in \mathrm{St}$ to segments of the second type and choice of object $L \in \mathcal{N}$ a local section of $\mathcal{F}$. Additional data is required to associate local sections to segments of the first type. One can then proceed to glue these local sections to associate global sections to certain curves, as in the above constructions.

Example 5.47. We illustrate Construction 5.42 and Construction 5.43 in an example with $R = k$ and $n = 3$. Consider the once-punctured 3-gon (in green), with the puncture called p and the ideal triangulation depicted in black.



The perverse schober $\mathcal{F}_{\mathcal{T}}(k)$ is up to natural equivalence given by the following diagram,

$$\begin{tikzcd}
 & & \mathcal{N}_{\phi^*} & & \\
 & \mathcal{N}_{\phi^*} & \mathcal{V}^3_{\phi^*} \arrow[l, "\varrho_1"'] \arrow[r, "\varrho_2"] \arrow[u, "\varrho_3"] & \mathcal{N}_{\phi^*} & \\
\mathcal{N}_{\phi^*} & \mathcal{V}^3_{\phi^*} \arrow[l, "\varrho_3"'] \arrow[r, "\varrho_1"] \arrow[u, "\varrho_2"] & \mathcal{N}_{\phi^*} & \mathcal{V}^3_{\phi^*} \arrow[l, "\varrho_2"'] \arrow[r, "\varrho_3"] \arrow[u, "\varrho_1"] & \mathcal{N}_{\phi^*}
\end{tikzcd}$$

where $\mathcal{V}^3_{\phi^*}$ denotes the value of $\mathcal{F}_{\mathcal{T}}(k)$ at the vertices and $\mathcal{N}_{\phi^*} = \mathrm{RMod}_{k[t_1]} \simeq \mathcal{D}(k[t_1])$ denotes the value of $\mathcal{F}_{\mathcal{T}}(k)$ at the edges of $\mathcal{T}$. Algebraically, one can describe $\mathcal{V}^3_{f*}$ as the derived ∞-category of the relative Ginzburg algebra of the 3-gon, which we denote in the following by $\mathscr{G}_\Delta$. For a depiction of $\mathscr{G}_\Delta$ see Example 5.6.

The singular matching curve γ_e given by the edge e connecting v_1 and v_2 gives rise to the following global section M_{γ_e}, which describes a 3-spherical object.

$$\begin{tikzcd}
 & & 0 & & \\
 & \phi^*(k) & s_1 \arrow[l] \arrow[r] \arrow[u] & 0 & \\
0 & s_2 \arrow[l] \arrow[r] \arrow[u] & 0 & 0 \arrow[l] \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Above, we denote $s_1 = (k \xrightarrow{*} \phi^*(k) \xrightarrow{\mathrm{id}} \phi^*(k))$, $s_2 = (k \xrightarrow{*} \phi^*(k) \to 0) \in \mathcal{V}^3_{\phi^*}$. Algebraically, s_1 and s_2 describe simple $\mathscr{G}_\Delta$-modules(in the abelian 1-category of modules), each associated with a vertex of the underlying quiver, and $\phi^*(k)$ is the unique $k[t_1]$-module with value k.

Consider any module $L \in \mathrm{RMod}_{k[t_1]}$, and a matching datum (γ, L) of rank 1 with γ the closed pure matching curve wrapping around m. The associated global section M_γ^L is of the form

$$\begin{tikzcd}
 & & 0 & & \\
 & L & \iota_3(L) \arrow[l] \arrow[r] \arrow[u] & L & \\
0 & \iota_3(L) \arrow[l] \arrow[r] \arrow[u] & L & \iota_3(L) \arrow[l] \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

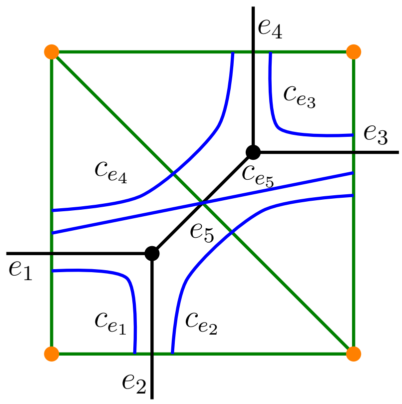
where $\iota_3(L) = (0 \to 0 \to L) \in \mathcal{V}^3_{f^*}$ is the object concentrated in the third component of the semiorthogonal decomposition of $\mathcal{V}^3_{f^*}$ with value L. Algebraically, one can describe $\iota_3(L)$ as the derived tensor product $L \otimes_{k[t_1]} p_e \mathscr{G}_\Delta$ with the projective $\mathscr{G}_\Delta$-module associated to some vertex of the quiver underlying $\mathscr{G}_\Delta$.

5.4.3 Projective modules via pure matching curves

We fix a marked surface $\mathbf{S}$ with an n-valent spanning graph $\mathcal{T}$ and an $\mathbb{E}_\infty$-ring spectrum R.

Let e be an edge of $\mathcal{T}$ and v_1, v_2 the vertices incident to e. Consider the curve composed of segments c_e^i with $i = 1, 2$ whose first segment lies at v_i, which begins at e and whose segments are all pure of the second type, wrapping exactly one step in the counterclockwise direction around a vertex of $\mathcal{T}$. We define the curve c_e as the composite of c_e^1 with the curve obtained by reversing the orientation of c_e^2. Note that c_e is a pure, regular matching curve.

Example 5.48. Below, we depict the 4-gon with an ideal triangulation $\mathcal{T}$ with edges $e_1, \dots, e_5$ and associated matching curves $c_{e_1}, \dots, c_{e_5}$.



Proposition 5.49. *For each $L \in \mathrm{RMod}_{R[t_{n-2}]}$ and edge $e \in \mathcal{T}_1$, there exists an equivalence in $\mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R))$*

$$M_{c_e}^L \simeq \mathrm{ev}_e^*(L) \tag{82}$$

with $\mathrm{ev}_e^* = \mathrm{ladj}(\mathrm{ev}_e)$ the left adjoint of the evaluation functor $\mathrm{ev}_e \colon \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T}(R)) \to \mathcal{F}_\mathcal{T}(R)(e) = \mathrm{RMod}_{R[t_{n-2}]}$.

Thus, by Proposition 3.39, the global section $M_{c_e}^{R[t_{n-2}]}$ is the direct summand of a compact generator of the ∞-category $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}(R))$ associated with the edge e. Proposition 5.24 further shows that in the case that $R = k$ is a commutative ring, the global section $M_{c_e}^{k[t_{n-2}]} \simeq p_e \mathscr{G}_{\mathcal{T}}$ describes the projective $\mathscr{G}_{\mathcal{T}}$ module associated with the edge e.

Proof. Consider the curve c_e as above and let I be the index set of its segments. The two first segments of the curves c_e^1 and c_e^2, lying at v_1, respectively, v_2, yield segments δ_x and δ_{x+1} of c_e with $x \in I$.

Consider the R-linear ∞-categories $\mathcal{D} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}(R))$ and $\mathcal{L}$ of global sections, respectively, all sections of $\mathcal{F}_{\mathcal{T}}$. Let $v \in \mathcal{T}_0$ be a vertex with incident edges labeled $e_1, \dots, e_n$ and let $\delta^{i,i+1}$ be the pure segment of the second type lying at v passing from e_i to e_{i+1}. The functor $\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta^{i,i+1}}, \text{-})\colon \mathcal{L} \to \mathrm{RMod}_R$ is equivalent to the functor

$$\widetilde{\mathrm{ev}}^L_{v,i}\colon \mathcal{L} \xrightarrow{\mathrm{ev}_v} \mathcal{V}^n_{\phi^*} \xrightarrow{\varrho_i} \mathrm{RMod}_{R[t_{n-2}]} \xrightarrow{\mathrm{Mor}(L,\text{-})} \mathrm{RMod}_R\,,$$

as can be seen using that $M^L_{\delta^{i,i+1}}$ is a p-relative left Kan extension of its restriction to v. Similarly, for each edge e of $\mathcal{T}$, we find the functor $\mathrm{Mor}_{\mathcal{L}}(Z^L_e, \text{-})\colon \mathcal{L} \to \mathrm{RMod}_R$ to be equivalent to

$$\widetilde{\mathrm{ev}}^L_e\colon \mathcal{L} \xrightarrow{\mathrm{ev}_e} \mathrm{RMod}_{R[t_{n-2}]} \xrightarrow{\mathrm{Mor}(L,\text{-})} \mathrm{RMod}_R\ .$$

Note that the composites of $\widetilde{\mathrm{ev}}_{v,i}$ and $\widetilde{\mathrm{ev}}_{e_i}$ with the inclusion $\mathcal{D} \to \mathcal{L}$ are equivalent, we denote this functor by $\mathrm{ev}^L_{e_i}$.

Using the definition of $M^L_{c_e}$ and that $\mathrm{Mor}_{\mathcal{L}}(\text{-},\text{-})$ preserves limits in the first entry, we obtain that $\mathrm{Mor}_{\mathcal{L}}(M^L_{c_e}, \text{-})\colon \mathcal{L} \to \mathrm{RMod}_R$ is given by the limit of the diagram $\mathrm{Mor}_{\mathcal{L}}(D_{c_e}, \text{-})\colon E^{\mathrm{op}}_{c_e} \to \mathrm{Fun}(\mathcal{L}, \mathrm{RMod}_R)$. Composing with the limit preserving pullback functor $\mathrm{Fun}(\mathcal{L}, \mathrm{RMod}_R) \to \mathrm{Fun}(\mathcal{D}, \mathrm{RMod}_R)$ along the inclusion $\mathcal{D} \to \mathcal{L}$, one obtains the diagram in $\mathrm{Fun}(\mathcal{D}, \mathrm{RMod}_R)$, which assigns, up to equivalence, to $\Lambda^2_0 \times \{x\}$ the constant diagram with value ev^L_e, to $(0,j) \to (2,j)$ for $j > x$ and to $(0,j) \to (1,j)$ for $j < x$ the identity on $\mathrm{ev}^L_{e^{j-1}}$, where e^{j-1} is the edge where δ^{j-1} ends and δ^j begins. The limit $\mathrm{Mor}_{\mathcal{D}}(M^L_{c_e}, \text{-})$ is thus equivalent to the functor ev^L_e. Evaluating the left adjoints at R shows the desired equivalence (82). $\square$

5.5 Morphisms from intersections

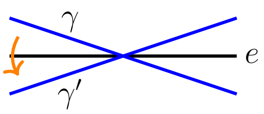
For the entirety of Section 5.5, we fix an $\mathbb{E}_\infty$-ring spectrum R and a marked surface $\mathbf{S}$ with an n-valent spanning graph $\mathcal{T}$. In this section, we describe the morphism objects between the global sections of $\mathscr{F}_{\mathcal{T}}(R)$ arising from matching data with pure underlying matching curves. We begin by describing the different types of intersections between matching curves.

Definition 5.50. Let γ, γ' be two matching curves in $\mathbf{S}$. We choose representatives of γ and γ' with the minimal number of intersections.

- We define the number of singular intersections $i^{\mathrm{sg}}(\gamma, \gamma')$ as the number of intersections of γ and γ' at their endpoints in $\mathcal{T}_0$. If $\gamma = \gamma'$ with distinct endpoints, we define $i^{\mathrm{sg}}(\gamma,\gamma)$ as the number of endpoints of γ in $\mathcal{T}_0$. If $\gamma=\gamma'$ with two identical endpoints, we set $i^{\mathrm{sg}}(\gamma,\gamma)=4$.

- We define the number of directed boundary intersections $i^{\mathrm{bdry}}(\gamma,\gamma')$ as the number of intersections of γ and γ' with the same connected component of $\partial\mathbf{S}\backslash M$ such that the intersection of γ and $\partial\mathbf{S}\backslash M$ precedes the intersection of γ' and $\partial\mathbf{S}\backslash M$ in the orientation of $\partial\mathbf{S}\backslash M$ induced by the clockwise orientation of $\mathbf{S}$. If $\gamma=\gamma'$, we only count directed boundary intersections of distinct endpoints.
- We denote by $i^{\mathrm{cr}}(\gamma,\gamma')$ the number of crossings from γ to γ', see Definition 5.51. If $\gamma=\gamma'$, then $i^{\mathrm{cr}}(\gamma,\gamma)$ counts each self-crossing only once.

Definition 5.51. Consider an intersection of two matching curves γ,γ' in $\mathbf{S}\backslash M$ away from their endpoints. This intersection can be chosen to lie in a small neighborhood of an edge e of $\mathcal{T}$. We say that the intersection is a crossing from γ to γ' if in this neighborhood, the curves are arranged as follows.



The orange arrow goes in the counterclockwise direction. If the curves are arranged in the opposite way, we say that the intersection is a crossing from γ' to γ.

Notation 5.52.

- Consider a singular intersection of γ and γ' at a vertex v, with segments $\delta=\delta^i$ and $\delta'=\delta^j$ at v and $1\leq i,j\leq n$ in the notation of part **1)** of Construction 5.40. We denote by $\deg=i-j$, if $i<j$, and $\deg=i-j-n$, if $j<i$, the number of steps after which the segment δ' follows the segment δ at v in the counterclockwise direction. If $i=j$, then both δ and δ' exit Σ_v through the same boundary component $f\subset\partial\Sigma_v$. Choosing γ,γ' with the minimal number of intersections, we set $\deg=0$ if the intersection of δ with f precedes the intersection of δ' with f; otherwise we set $\deg=-n$.
- Given $L,L'\in\mathrm{RMod}_{R[t_{n-2}]}$, we denote by $\mathrm{Mor}(L,L')=\mathrm{Mor}_{\mathrm{RMod}_{R[t_{n-2}]}}(L,L')$ the morphism object.
- Given $Q,Q'\in\mathrm{RMod}_R$, we denote by $\mathrm{Mor}_R(Q,Q')=\mathrm{Mor}_{\mathrm{RMod}_R}(Q,Q')$ the morphism object.

Theorem 5.53. *Let $(\gamma,L),(\gamma',L')$ be two matching data in $\mathbf{S}\backslash M$, such that $\gamma\neq\gamma'$, γ,γ' are pure and have no common infinite ends, see Remark 5.57. Let a be the rank of (γ,L) and a' the rank of (γ',L'). Consider the associated global sections $M_\gamma^L,M_{\gamma'}^{L'}\in\mathcal{D}:=\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}(R))$.*

The morphism object $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L,M_{\gamma'}^{L'})\in\mathrm{RMod}_R$ is equivalent to

$$\mathrm{Mor}(L,L')^{\oplus i^{\mathrm{bdry}}(\gamma,\gamma'))+aa'(i^{\mathrm{cr}}(\gamma,\gamma')}\oplus\mathrm{Mor}(L,L')[-1]^{\oplus aa' i^{\mathrm{cr}}(\gamma',\gamma)}\oplus\bigoplus_{i^{\mathrm{sg}}(\gamma,\gamma')}\mathrm{Mor}_R(Q,Q')[\deg]\,.$$

Theorem 5.54. *Let (γ, L) and (γ, L') be two matching data in $\mathbf{S}\backslash M$, whose underlying matching curves are identical. Suppose that γ is pure.*

i) Suppose that γ is open and regular. The morphism object $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_\gamma^{L'}) \in \mathrm{RMod}_R$ is equivalent to

$$\mathrm{Mor}(L,L')^{\oplus 1+i^{\mathrm{cr}}(\gamma,\gamma)+i^{\mathrm{bdry}}(\gamma,\gamma)} \oplus \mathrm{Mor}(L,L')[-1]^{\oplus i^{\mathrm{cr}}(\gamma,\gamma)}$$

ii) Suppose that γ is open and singular. The morphism object $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_\gamma^{L'}) \in \mathrm{RMod}_R$ is equivalent to

$$\left(\mathrm{Mor}(L,L') \oplus \mathrm{Mor}(L,L')[-1]\right)^{\oplus i^{\mathrm{cr}}(\gamma,\gamma)} \oplus \bigoplus_{i^{\mathrm{sg}}(\gamma,\gamma)} \mathrm{Mor}_R(Q,Q')[\deg]\,.$$

iii) Let $R = k$ be a field and assume that γ is a closed matching curve (thus automatically regular) and that $\mathrm{Map}(L,L) \simeq k$. Assume further that the monodromy equivalence of (γ, L) is given by a single $a \times a$-Jordan block. The morphism object $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_\gamma^{L}) \in \mathrm{RMod}_R$ is equivalent to

$$\left(\mathrm{Mor}(L,L) \oplus \mathrm{Mor}(L,L)[-1]\right)^{\oplus a+a^2 i^{\mathrm{cr}}(\gamma,\gamma)}\,.$$

Example 5.55. Let $(\gamma, \phi^*(R))$ be a matching datum in $\mathbf{S}\backslash M$, such that γ is finite, pure and has no self-intersections.

1. If both ends of γ lie at vertices of $\mathcal{T}_0$, then

$$\mathrm{Mor}_{\mathcal{D}}(M_\gamma, M_\gamma) \simeq R \oplus R[-n]\,,$$

   meaning that M_γ is an n-spherical object.

2. If γ begins at a vertex of $\mathcal{T}$ and ends on the boundary of $\mathbf{S}$, then M_γ is an exceptional object, i.e. $\mathrm{Mor}_{\mathcal{D}}(M_\gamma, M_\gamma) \simeq R$.

3. If γ begins and ends on the boundary of $\mathbf{S}$, then

$$\mathrm{Mor}_{\mathcal{D}}(M_\gamma, M_\gamma) \simeq R \oplus R[1-n]\,,$$

   meaning that M_γ is an $(n-1)$-spherical object.

4. If $(\gamma, \phi^*(R))$ is closed and of rank 1 and $R = k$ a field, then

$$\mathrm{Mor}_{\mathcal{D}}(M_\gamma, M_\gamma) \simeq k \oplus k[-1] \oplus k[1-n] \oplus k[-n]\,.$$

Remark 5.56. Let (γ, L) and (γ', L') be two matching data, such that γ and γ' are not necessarily pure. The method of proof of Theorem 5.53 also applies to compute the morphism object $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$. In this more general setting, there are however exceptions to the simple rule that $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$ counts intersections, unless $L, L' \in \mathrm{Im}(\phi^*)$. We discuss this in Section 5.5.1 and Example 5.61. In the cases that there are segments of the second type wrapping around a vertex by n steps, the proofs of Theorems 5.53 and 5.54 do not directly apply and would need some minor adaptions. Giving a systematic description of the morphism objects in the non-pure setting would also require introducing gradings of the matching curves and the surface, as for example done in [IQZ20]. For the applications we have in mind, the pure matching curves are the most important ones.

Remark 5.57. Consider two distinct matching curves $\gamma\colon U \to \Sigma$ and $\gamma'\colon U' \to \Sigma$. We say that γ and γ' have a common infinite end if there exist immersions $I_1\colon \mathbb{R}_{\geq 0} \to U$, $I_2\colon \mathbb{R}_{\geq 0} \to U'$ such that the curves $\gamma|_{I_1}$ and $\gamma|_{I_2}$ are composed of infinitely many identical segments. Note that this definition allows that one of the two curves γ or γ' is closed. If in Theorem 5.53, γ and γ' are open with a common infinite end, then the R-module $\mathrm{Mor}_{\mathcal{D}}(M^L_\gamma, M^{L'}_{\gamma'})$ consists of infinitely many copies of $\mathrm{Mor}(L, L')$.

The proofs of Theorems 5.53 and 5.54 consist of gluing arguments. We decompose γ and γ' into segments and begin in Section 5.5.1 by describing all morphisms between the associated local sections. In Section 5.5.1, we also allow non-pure segments. In Section 5.5.2, we then describe $\mathrm{Mor}_{\mathcal{D}}(M^L_\gamma, M^{L'}_{\gamma'})$ via the colimit of a diagram of morphism objects between the local sections associated to the segments. In Section 5.5.3 we combine the findings of Section 5.5.1 and Section 5.5.2 to prove Theorems 5.53 and 5.54.

5.5.1 Intersections locally

In this section, we exclude all segments of the second type wrapping around a vertex by n steps and all curves composed of segments in which such a segment appear.

Let $\mathcal{L}$ denote the R-linear ∞-category of (all) sections of $\mathcal{F}_{\mathcal{T}}(R)$, see Definition 3.36. In the following, we describe the morphism objects $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_\eta)$ and $\mathrm{Mor}_{\mathcal{L}}(Z^L_e, M^{L'}_\eta)$ where $L, L' \in \mathrm{RMod}_{R[t_{n-2}]}$, δ is a segment in $\Sigma_{\mathcal{T}}$, e is an edge of $\mathcal{T}$ and η is any matching curve in $\mathbf{S}\backslash M$ or an open curve composed of segments, see also Construction 5.42 for the notation. If δ (or η) is singular, we require as always that $L \simeq \phi^*(Q)$ (or $L' \simeq \phi^*(Q')$).

We begin by determining the morphism objects between sections M^L_δ and $M^{L'}_{\delta'}$ associated to segments δ, δ'.

If the segments δ and δ' are not located at the same vertex of $\mathcal{T}$, one finds $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'}) \simeq 0$, see also Lemma 5.58. We thus assume that δ, δ' are located at the same vertex $v \in \mathcal{T}_0$. We choose representatives of δ and δ' with the minimal number of intersections. With the exception of some cases described further below, we find that $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'})$ is the direct sum of R-modules given as follows.

- Each directed boundary intersection from δ to δ' in Σ_v contributes a copy of $\mathrm{Mor}(L, L')$, up to suspensions, to $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'})$. The corresponding morphisms have support at v, e_i, where e_i is the edge of $\mathcal{T}$ intersecting the same component of $\partial\Sigma_v$ as δ, δ'. If δ, δ' are pure, no suspensions appear.

- If $L \simeq \phi^*(Q)$ and $L' \simeq \phi^*(Q')$, then each singular intersection of $\delta = \delta^i$ and $\delta' = \delta^j$ contributes a single copy of $\mathrm{Mor}_R(Q,Q')[b]$ to $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'})$, where $b = i-j$ if $j > i$, $b = i-j-n$ if $j < i$, and $b = 0$ for $i = j$. The support of the corresponding morphisms is given by v if $\delta \neq \delta'$ and by v, e_i if $\delta = \delta' = \delta^i$.

- Each crossing of δ and δ' contributes a copy of $\mathrm{Mor}(L, L')$, up to suspensions, to $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'})$, corresponding to morphisms with support at v. There are no crossings if both δ and δ' are pure.

- If $\delta = \delta' = \delta^{i,j}$ is of the second type with $i \neq j$, then $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^L_{\delta'}) \simeq \mathrm{Mor}(L, L')$.

There are three possible exceptions to the above description, which appear if L and L' do not lie in the image of ϕ^* and at least one of the segments δ, δ' is not pure. In these exceptional cases, we have that δ and δ' are of the second type, $\delta \neq \delta'$ and the two segments have either two crossings, two boundary intersections or a crossing and a boundary intersection. Further below, we describe the outcome in these cases in more detail.

The above descriptions of $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'})$ follow from the universal properties of the involved Kan extensions and some basic computations, as we now explain. We denote the Grothendieck construction of $\mathcal{F}_{\mathcal{T}}(R)$ by $p : \Gamma(\mathcal{F}_{\mathcal{T}}(R)) \to \mathrm{Exit}(\mathcal{T})$. The sections M^L_δ and $M^{L'}_{\delta'}$ were defined as the p-relative left Kan extensions of their restrictions to $\{v\} \subset \mathrm{Exit}(\mathcal{T})$. Using the universal property of Kan extensions and arguing as in the proof of Lemma 5.58 below, we find that the restriction functor induces an equivalence of R-modules

$$\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'}) \simeq \mathrm{Mor}_{\mathcal{V}^n_{\phi^*}}(M^L_\delta(v), M^{L'}_{\delta'}(v))\,,$$
$$\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta'}, M^{L'}_{\delta}) \simeq \mathrm{Mor}_{\mathcal{V}^n_{\phi^*}}(M^L_{\delta'}(v), M^{L'}_{\delta}(v))\,.$$

The resulting morphism objects in $\mathcal{V}^n_{\phi^*}$ can be directly determined by a case by case analysis, using again universal properties of Kan extensions and making use of the paracyclic twist $T_{\mathcal{V}^n_{\phi^*}}$ from Lemma 3.31. We collect the resulting morphisms objects for all possible pairs δ, δ' in Tables 1 to 3. For the description of δ, δ', we use the notation of Construction 5.40. In Table 1, we consider the cases $\delta = \delta^1$ and $\delta' = \delta^i$ with $1 \leq i \leq n$ and $L \simeq \phi^*(Q)$, $L' \simeq \phi^*(Q')$. In Table 2, we consider the cases $\delta = \delta^i$ and $\delta' = \delta^{j,n}$ with $1 \leq i, j \leq n$ and $j \neq n$ and $L \simeq \phi^*(Q)$. In Table 3, we consider (a subset of) the cases $\delta = \delta^{1,j}$ and $\delta' = \delta^{i',j'}$ with $1 \leq i', j, j' \leq n$ and $1 \neq j, i' \neq j'$. Any unordered pair δ, δ' is described by one of the pairs considered below, up to the rational symmetry of Σ_v, on the categorical level realized by the action of the paracyclic twist functor $T_{\mathcal{V}^n_{\psi^*}}$ on $\mathcal{F}_{\mathcal{T}}(R)(v)$.

In Table 3, the three cases where the simple description of the morphisms objects in terms of intersections can fail are separated from the other cases. In the case of $\delta \neq \delta'$ with two boundary intersections, the morphism objects match the number of intersections, but the support of the morphisms does not behave as expected (unless $L, L' \in \mathrm{Im}(\phi^*)$). See also Example 5.61 for the consequences of this phenomenon. To

Intersections	δ^1, δ^i	$\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta^1}, M^{L'}_{\delta^i})$	$\mathrm{Mor}_{\mathcal{L}}(M^{L'}_{\delta^i}, M^L_{\delta^1})$	Support
1x singular	$i > 1$	$\mathrm{Mor}_R(Q, Q')[1-i]$	$\mathrm{Mor}_R(Q', Q)[i-1-n]$	at v
1x singular	$i = 1$	$\mathrm{Mor}_R(Q, Q')$	$\mathrm{Mor}_R(Q', Q)$	at v, e_1

Table 1: All possible pairs of two segments δ^1, δ^i of the first type up to rotational symmetry.

Intersections	$\delta^i, \delta^{j,n}$	$\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta^i}, M^{L'}_{\delta^{j,n}})$	$\mathrm{Mor}_{\mathcal{L}}(M^{L'}_{\delta^{j,n}}, M^L_{\delta^i})$	Support
none	$i < j < n$	0	0	/
1x crossing	$j < i < n$	$\mathrm{Mor}(L, L')[i-j-1]$	$\mathrm{Mor}(L', L)[j-i]$	at v
1x boundary	$j = i < n$	0	$\mathrm{Mor}(L', L)[j-i]$	at v, e_i
1x boundary	$j < i = n$	$\mathrm{Mor}(L, L')[n-j-1]$	0	at v, e_n

Table 2: All possible pairs of one segment δ^i of the first type and one segments $\delta^{j,n}$ of the second type up to rotational symmetry.

Intersections	$\delta^{1,j}, \delta^{i',j'}$	$\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta^{1,i}}, M^{L'}_{\delta^{i',j'}})$	$\mathrm{Mor}_{\mathcal{L}}(M^{L'}_{\delta^{i',j'}}, M^L_{\delta^{1,i}})$	Support
none	$1 < j < i' < j'$	0	0	/
none	$1 < i' < j' < j$	0	0	/
1x crossing	$1 < i' < j < j'$	$\mathrm{Mor}(L, L')[1-i']$	$\mathrm{Mor}(L', L)[i'-2]$	at v
1x boundary	$1 < i' < j = j'$	$\mathrm{Mor}(L, L')[1-i']$	0	at v, e_j
1x boundary	$1 < i' = j < j'$	0	$\mathrm{Mor}(L', L)[i'-2]$	at v, e_j
1x boundary	$1 = i' < j < j'$	$\mathrm{Mor}(L, L')$	0	at v, e_1
2x boundary	$1 = i' < j = j'$	$\mathrm{Mor}(L, L')$	$\mathrm{Mor}(L', L)$	at v, e_1, e_j
2x crossing	$1 < j' < i' < j$	$\mathrm{Mor}(L, \phi^*\phi_*(L'))[n-i']$	$\mathrm{Mor}(L', \phi^*\phi_*(L))[i'-2]$	at v
2x boundary	$1 = j' < j = i'$	$\mathrm{Mor}(L, L')[n-i']$	$\mathrm{Mor}(L', L)[i'-2]$	at v, e_1, e_j
1x crossing + 1x boundary	$1 < j' < j = i'$	$\mathrm{Mor}(L, L')[n-i']$	$\mathrm{Mor}(L', \phi^*\phi_*(L))[i'-2]$	at v, e_j

Table 3: All possible pairs of two segments $\delta^{1,j}$ and $\delta^{i',j'}$ of the second type up to a swap and rotational symmetry.

summarize our computations, the exceptions to the simple rule "#intersections=dim of Homs" arise if δ and δ' cut out a vertex of $\mathcal{T}_0$.

Lemma 5.58. *Let η be a curve in $\mathbf{S}\backslash M$ composed of segments $\{\delta_i\}_{i\in I}$. Given a subset $\tilde{I}\subset I$, we denote $\tilde{I}^{\geq 1}=\tilde{I}\cap\mathbb{N}$ if $I\neq\mathbb{Z}/N\mathbb{Z}$ and $\tilde{I}^{\geq 1}=\tilde{I}$ if $I=\mathbb{Z}/N\mathbb{Z}$. We denote $\tilde{I}^{\leq 0}=\tilde{I}\backslash\tilde{I}^{\geq 1}$.*

(1) Let δ be a segment lying at a vertex $v\in\mathcal{T}_0$ and let $I_v\subset I$ be the set of segments of η lying at v. Then there exists an equivalence of R-modules

$$\begin{aligned}\operatorname{Mor}_{\mathcal{L}}(M_\delta^L,M_\eta^{L'})\simeq\bigoplus_{\delta'\in I_v^{\geq 1}}&\operatorname{Mor}_{\mathcal{L}}(M_\delta^L,M_{\delta'}^{L'}[d(\delta^1\leq\eta<\delta')])\oplus\\ \bigoplus_{\delta'\in I_v^{\leq 0}}&\operatorname{Mor}_{\mathcal{L}}(M_\delta^L,M_{\delta'}^{L'}[d(\delta'\leq\eta<\delta^1)])\,.\end{aligned}$$

(2) Let e be an edge of $\mathcal{T}$. Then there exists an equivalence of R-modules

$$\operatorname{Mor}_{\mathcal{L}}(Z_e^L,M_\eta^{L'})\simeq\operatorname{Mor}(L,M_\eta^{L'}(e))\,.$$

Denote

$$N=\bigoplus_{\delta'\in I_e^{\geq 1}}\operatorname{Mor}(L,L')[d(\delta^1\leq\eta<\delta')]\oplus\bigoplus_{\delta'\in I_e^{\leq 0}}\operatorname{Mor}(L,L')[d(\delta'\leq\eta<\delta^1)]\,,$$

where $I_e\subset I$ is the set of segments of η which begin at e.

If η does not end at e, there exists an equivalence of R-modules

$$\operatorname{Mor}_{\mathcal{L}}(Z_e^L,M_\eta^{L'})\simeq N \tag{83}$$

If η ends at e, then there exists an equivalence of R-modules

$$\operatorname{Mor}_{\mathcal{L}}(Z_e^L,M_\eta^{L'})\simeq N\oplus\operatorname{Mor}(L,L')[d(\delta^1\leq\eta)]\,. \tag{84}$$

Proof. Note that M_δ^L and Z_e^L are left Kan extensions relative the Grothendieck construction of $\mathcal{F}_{\mathcal{T}}(R)$ of their restrictions to v, respectively, e. Using the universal property of Kan extensions, see [Lur09, 4.3.2.17], it follows that for any section $X\in\mathcal{L}$ the restriction morphisms of R-modules

$$\operatorname{Mor}_{\mathcal{L}}(M_\delta^L,X)\longrightarrow\operatorname{Mor}_{\mathcal{V}^n_{\phi^*}}(M_\delta^L(v),X(v))$$
$$\operatorname{Mor}_{\mathcal{L}}(Z_e^L,X)\longrightarrow\operatorname{Mor}(L,X(e))$$

restrict to equivalences on all homotopy groups so that they are equivalences of R-modules.

By construction of $M_\eta^{L'}$, we find an equivalence in $\mathcal{V}^n_{\phi^*}$

$$M_\eta^{L'}(v)\simeq\bigoplus_{\delta'\in I_v^{\geq 1}}M_{\delta'}^{L'}(v)[d(\delta^1\leq\eta<\delta')]\oplus\bigoplus_{\delta'\in I_v^{\leq 0}}M_{\delta'}^{L'}(v)[d(\delta'\leq\eta<\delta^1)]$$

showing statement (1). Similarly, there exists an equivalence in $\mathrm{RMod}_{R[t_{n-2}]}$

$$M_\eta^{L'}(e) \simeq \bigoplus_{\delta' \in I_e^{\geq 1}} L'[d(\delta^1 \leq \eta < \delta')] \oplus \bigoplus_{\delta' \in I_e^{\leq 0}} L'[d(\delta' \leq \eta < \delta^1)]$$

if η does not end at e and

$$M_\eta^{L'}(e) \simeq \bigoplus_{\delta' \in I_e^{\geq 1}} L'[d(\delta^1 \leq \eta < \delta')] \oplus \bigoplus_{\delta' \in I_e^{\leq 0}} L'[d(\delta' \leq \eta < \delta^1)] \oplus L'[d(\delta^1 \leq \eta)]$$

if η ends at e, see also Remark 5.41 for the shifts. This shows statement (2). □

Remark 5.59. The support of the morphisms objects in Tables 1 to 3 refers to the support of the corresponding morphisms between sections in the sense of Definition 2.8. It has the following further interpretation: let δ, δ' be two segment in $\mathbf{S}$ both lying at a vertex $v \in \mathcal{T}_0$. Suppose that δ starts at an edge e and ends at an edge f. Let $\mathrm{Mor}(L, L')[l]$, with $l \in \mathbb{Z}$, be a summand of $\mathrm{Mor}_{\mathcal{L}}(M_\delta^L, M_{\delta'}^{L'})$ identified in Tables 1 to 3, corresponding to morphisms with support at v and the edges $J \subset \{e, f\}$ ($J = \emptyset$ is possible). The composite c of the inclusion morphism of R-modules

$$\mathrm{Mor}(L, L')[l] \longrightarrow \mathrm{Mor}_{\mathcal{L}}(M_\delta^L, M_{\delta'}^{L'})$$

with the morphism

$$\mathrm{Mor}_{\mathcal{L}}(M_\delta^L, M_{\delta'}^{L'}) \longrightarrow \mathrm{Mor}_{\mathcal{L}}(Z_e^L, M_{\delta'}^{L'})$$

obtained from precomposing with the pointwise inclusion of the section Z_e^L into M_δ^L can be described as follows.

- If $e \notin J$, then c is zero.
- If $e \in J$, then c is the inclusion of a direct summand under the equivalence (83) or (84).

An analogous description holds for the morphism c arising by replacing the edge e with f.

The three cases at the end of Table 3 are exceptions to the above descriptions (unless $L, L' \in \mathrm{Im}(\phi^*)$).

5.5.2 Intersections globally

We fix two matching data (γ, L) and (γ', L') in $\mathbf{S}\backslash M$, such that γ, γ' are composed of pure segments. We also assume that γ, γ' do not have a common infinite end. We choose representatives of γ and γ' with the minimal number of intersections. We also assume in this section that γ is open (and in particular has rank 1). To ease notation, we further assume that the rank a' of γ' is also 1, the general case is entirely analogous.

The segments of γ are denoted $\{\delta_i\}_{i\in I}$ and the segments of γ' are denoted $\{\delta'_j\}_{j\in I'}$. Recall that M_γ^L is defined in Construction 5.42 as the colimit of the diagram $D_\gamma\colon E_\gamma \to \mathcal{L}$. Using that the functor

$$\mathrm{Mor}_{\mathcal{L}}(\text{-}, M_{\gamma'}^{L'})\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathrm{RMod}_R$$

preserves limits, it follows that $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_{\gamma'}^{L'})$ is the limit of the diagram

$$\mathrm{Mor}_{\mathcal{L}}(\text{-}, M_{\gamma'}^{L'}) \circ D_\gamma^{\mathrm{op}}\colon E_\gamma^{\mathrm{op}} \longrightarrow \mathrm{RMod}_R\,. \tag{85}$$

In the following we fully describe the diagram (85). We will see that the diagram (85) is equivalent to the direct sum in the stable ∞-category $\mathrm{Fun}(E_\gamma^{\mathrm{op}}, \mathrm{RMod}_R)$ of a collection of very manageable diagrams. A subset of these diagrams correspond to the intersections of γ and γ' which we will show in the case that γ, γ' are matching curves to be the only summands with nonzero limits in RMod_R.

We proceed with the constructions of the summands of (85) associated to the different types of intersections.

Singular intersections

For this case, we assume that $\gamma \neq \gamma'$. Assume that the endpoints of γ, γ' intersect in a vertex $v \in \mathcal{T}_0$. Note that in this case γ and γ' are singular and thus $L \simeq \phi^*(Q), L' \simeq \phi^*(Q')$. Since γ and γ' are pure, reversing their orientations does not change M_γ^L or $M_{\gamma'}^L$. We may thus assume that γ and γ' both start at v.

Using the rotational symmetry at v, we may assume that the first segment δ_1 is given by the segment δ^1 of the first type at v. We distinguish two cases. Either the first segment $\delta'_1 = \delta^i$ of γ' is identical to δ_1, i.e. $i = 1$, or it is not. We begin with the case $i \neq 1$. By Table 1, we have $\mathrm{Mor}_{\mathcal{L}}(M_{\delta^1}^L, M_{\delta^i}^{L'}) \simeq \mathrm{Mor}_R(Q, Q')[1-i]$ and the corresponding morphisms have support at v. Using Remark 5.59, it follows that there is a direct summand of the diagram (85) which restricts at $(\Lambda_0^2 \times \{1\})^{\mathrm{op}}$ to the diagram

$$\begin{tikzcd} \mathrm{Mor}_R(Q,Q')[1-i] \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$$

and vanishes on $(\Lambda_0^2 \times \{i\})^{\mathrm{op}}$ for $1 < i \in I'$. Passing to limits, we get a direct summand $\mathrm{Mor}_R(Q, Q')[1-i]$ of $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_{\gamma'}^{L'})$.

We now consider the case $i = 1$. In that case, the matching curves γ and γ' are composed of m identical segments $\delta_1 = \delta'_1, \ldots, \delta_m = \delta'_m$, starting at v, such that $\delta_{m+1} \neq \delta'_{m+1}$ (this uses that $\gamma \neq \gamma'$). Let $v' \in \mathcal{T}_0$ be the vertex where $\delta_{m+1}, \delta'_{m+1}$ lie. We choose $\delta_{m+1}, \delta'_{m+1}$ such that they have a minimal number of intersections.

We distinguish the following two cases.

1) $\delta_{m+1}(0) \in \partial\Sigma_{v'}$ precedes $\delta'_{m+1}(0) \in \partial\Sigma_{v'}$ in the clockwise orientation of $\partial\Sigma_{v'}$.

2) $\delta_{m+1}(0) \in \partial\Sigma_{v'}$ follows $\delta'_{m+1}(0) \in \partial\Sigma_{v'}$ in the clockwise orientation of $\partial\Sigma_{v'}$

We find in either case for $2 \leq i \leq m$

$$\begin{aligned} \mathrm{Mor}_{\mathcal{L}}(M^L_{\delta_1}, M^{L'}_{\delta'_1}) &\simeq \mathrm{Mor}_R(Q, Q') \\ \mathrm{Mor}_{\mathcal{L}}(M^L_{\delta_i}, M^{L'}_{\delta'_i}) &\simeq \mathrm{Mor}(L, L') \end{aligned}$$

In case 1), we find

$$\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta_{m+1}}, M^{L'}_{\delta'_{m+1}}) \simeq \mathrm{Mor}(L, L')\,,$$

whereas in case 2), we find

$$\mathrm{Mor}_{\mathcal{L}}(M_{\delta_{m+1}}, M_{\delta'_{m+1}}) \simeq 0\,,$$

see Section 5.5.1. By part (1) of Lemma 5.58, each of the above R-modules also gives rise to a direct summand of the morphism object $\mathrm{Mor}_{\mathcal{L}}(M^L_{\delta}, M^{L'}_{\gamma'})$. In the case 1), using again Remark 5.59, we thus find a summand of (85) which restricts on $(\Lambda^2_0 \times \{1\})^{\mathrm{op}}$ to the diagram

$$\begin{tikzcd} \mathrm{Mor}_R(Q, Q') \arrow[dr] & & \mathrm{Mor}(L, L') \arrow[dl, "\simeq"'] \\ & \mathrm{Mor}(L, L') & \end{tikzcd} \tag{86}$$

on $(\Lambda^2_0 \times \{i\})^{\mathrm{op}}$ for $2 \leq i \leq m$ to the constant diagram with value $\mathrm{Mor}(L, L')$ and vanishes on the remaining parts of E^{op}_{γ}. The limit of this summand gives us a direct summand $\mathrm{Mor}_R(Q, Q') \subset \mathrm{Mor}_{\mathcal{L}}(M_{\gamma}, M_{\gamma'})$, as desired. In the case 2), we analogously find a summand of (85) which restricts on $(\Lambda^2_0 \times \{1\})^{\mathrm{op}}$ to the diagram (86), on $(\Lambda^2_0 \times \{i\})^{\mathrm{op}}$ for $2 \leq i < m$ to the constant diagram with value $\mathrm{Mor}(L, L')$, on $(\Lambda^2_0 \times \{m\})^{\mathrm{op}}$ to the diagram

$$\begin{tikzcd} \mathrm{Mor}(L, L') \arrow[dr, "\simeq"] & & 0 \arrow[dl, "\simeq"'] \\ & \mathrm{Mor}(L, L') & \end{tikzcd}$$

and vanishes on the remaining parts of E^{op}_{γ}. Using the equivalences $\mathrm{Mor}(L, L') \simeq \mathrm{Mor}_R(Q, \phi_*\phi^*(Q'))$ and $\phi_*\phi^*(Q') \simeq Q' \oplus Q'[1-n]$, one finds the limit to be given by the direct summand $\mathrm{Mor}_R(Q, Q')[-n] \subset \mathrm{Mor}_{\mathcal{D}}(M_{\gamma'}, M_{\gamma})$.

Crossings

Assume that γ and γ' have a crossing and consider the segments δ and δ' of γ and γ', respectively, describing the curves at the crossing. The segments δ and δ' are located at a vertex $v \in \mathcal{T}_0$. Since pure segments cannot have crossings between themselves, the crossing between γ and γ' does not arise as a crossing between segments in Σ_v.

Before and after the crossing, the two curves are composed of $m \geq 0$ identical segments. If m were infinite, we could find different representatives for γ, γ' with one intersection less which would contradict our assumptions. We can thus assume m to be finite.

We can choose representatives of γ and γ' such that the crossing lies on an edge connecting two vertices $v, v' \in \mathcal{T}_0$. We assume that γ and γ' are oriented such that locally around the crossing, they both first pass through Σ_v and then $\Sigma_{v'}$. We consider all segments δ_{x+i} and δ'_{y+i} with $x \in I$, $y \in I'$ and $0 \leq i \leq m+1$ for which there exist representatives of γ and γ', such that the induced representatives of the segments δ_{x+i} and δ'_{y+i} share the crossing in question. Note also that by assumption $\delta_{x+i} = \delta_{y+i}$ for $1 \leq i \leq m$.

The segments δ_x and δ'_y both lie at a vertex $v_1 \in \mathcal{T}_0$ and the segments δ_{x+m+1} and δ'_{y+m+1} also both lie at a vertex $v_2 \in \mathcal{T}_0$. We distinguish the following two cases.

1) The point $\delta'_y(1) \in \partial\Sigma_{v_1}$ follows the point $\delta_x(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\delta'_{y+m+1}(0) \in \partial\Sigma_{v_2}$ follows the point $\delta_{x+m+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$. This means that the crossing goes from γ to γ'

2) The point $\delta'_y(1) \in \partial\Sigma_{v_1}$ precedes the point $\delta_x(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\delta'_{y+m+1}(0) \in \partial\Sigma_{v_2}$ precedes the point $\delta_{x+m+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$. This means that the crossing goes form γ' to γ.

It follows from Section 5.5.1 that there exist direct summands

$$\mathrm{Mor}(L, L') \subset \mathrm{Mor}_{\mathcal{L}}(M^L_{\delta_{x+i}}, M^{L'}_{\delta'_{y+i}})$$

for $0 \leq i \leq m+1$ in the case 1) and $1 \leq i \leq m$ in the case 2).

In the case 1), we thus find a direct summand of the diagram (85) which restricts on $(\Lambda^2_0 \times \{x+i\})^{\mathrm{op}}$ for $0 \leq i \leq m$ up to equivalence to the constant diagram with value $\mathrm{Mor}(L, L')$ and vanishes on the remaining parts of E^{op}_γ. Passing to limits, we obtain the desired summand $\mathrm{Mor}(L, L') \subset \mathrm{Mor}_{\mathcal{L}}(M^L_\gamma, M^{L'}_{\gamma'})$. In the case 2), we similarly find a direct summand of the diagram (85) which restricts on $(\Lambda^2_0 \times \{x\})^{\mathrm{op}}$ to the diagram

$$\begin{tikzcd} 0 \arrow[dr] & & \mathrm{Mor}(L,L') \arrow[dl, "\simeq"'] \\ & \mathrm{Mor}(L,L') & \end{tikzcd}$$

on $(\Lambda^2_0 \times \{x+i\})^{\mathrm{op}}$ for $1 \leq i \leq m-1$ to the constant diagram with value $\mathrm{Mor}(L, L')$, on $(\Lambda^2_0 \times \{x+m\})^{\mathrm{op}}$ to the diagram

$$\begin{tikzcd} \mathrm{Mor}(L,L') \arrow[dr, "\simeq"] & & 0 \arrow[dl] \\ & \mathrm{Mor}(L,L') & \end{tikzcd}$$

and vanishes on the remaining parts of E^{op}_γ. Passing to limits, we find the direct summand $\mathrm{Mor}(L, L')[-1] \subset \mathrm{Mor}_{\mathcal{L}}(M^L_\gamma, M^{L'}_{\gamma'})$.

Boundary intersections

We assume that γ and γ' both intersect a boundary component B of $\mathbf{S}\backslash M$ and distinguish two cases.

1) The intersection of γ' and B follows the intersection of γ and B in the orientation of B induced by the clockwise orientation of $\mathbf{S}$.

2) The intersection of γ' and B precedes the intersection of γ and B in the orientation of B induced by the clockwise orientation of $\mathbf{S}$.

We can assume that both γ and γ' start at B and have m identical segments $\delta_1 = \delta_1', \dots, \delta_m = \delta_m'$, and the segments $\delta_{m+1} \neq \delta'_{m+1}$ both lie at a vertex $v \in \mathcal{T}_0$. In the case 1), we find that $\delta'_{m+1}(0) \in \partial\Sigma_v$ follows $\delta_{m+1}(0) \in \partial\Sigma_v$ on a boundary component of $\partial\Sigma_v$ in the clockwise direction. In the case 2), we find that $\delta'_{m+1}(0) \in \partial\Sigma_v$ precedes $\delta_{m+1}(0) \in \partial\Sigma_v$ in the clockwise orientation of $\partial\Sigma_v$.

We thus find direct summands

$$\operatorname{Mor}(L, L') \subset \operatorname{Mor}_{\mathcal{L}}(M^L_{\delta_i}, M^{L'}_{\delta_i'})\,, \tag{87}$$

with $1 \leq i \leq m+1$ in the case 1) and $1 \leq i \leq m$ in the case 2).

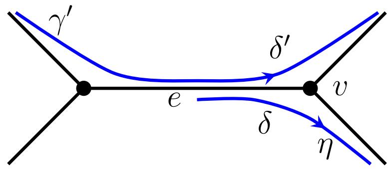
In the case 1), the summands (87) assemble using Lemma 5.58 and Remark 5.59 into a direct summand of (85) which has constant value $\operatorname{Mor}(L, L')$ on $(\Lambda^2_0 \times \{i\})^{\mathrm{op}}$ for $1 \leq i \leq m$ and vanishes on the remainder of E^{op}_γ. Passing to limits, we obtain the desired summand $\operatorname{Mor}(L, L') \subset \operatorname{Mor}_{\mathcal{L}}(M^L_\gamma, M^{L'}_{\gamma'})$. In the case 2), the summands (87) assemble using Lemma 5.58 into a direct summand of (85) which has constant value $\operatorname{Mor}(L, L')$ on $(\Lambda^2_0 \times \{i\})^{\mathrm{op}}$ for $1 \leq i \leq m-1$, takes the value

$$\begin{tikzcd} \operatorname{Mor}(L,L') \arrow[dr, "\simeq"] & & 0 \arrow[dl] \\ & \operatorname{Mor}(L,L') & \end{tikzcd}$$

on $(\Lambda^2_0 \times \{m\})^{\mathrm{op}}$ and vanishes on the remainder of E^{op}_γ. The limit of this summand vanishes.

Remark 5.60. Let η be a curve composed of segments, which is not necessarily a matching curve. The above arguments generalize to describe direct summands of $\operatorname{Mor}_{\mathcal{L}}(M^L_\eta, M^{L'}_{\gamma'})$ associated to singular intersections, crossings and directed boundary intersections (defined as for matching curves) of η and γ'.

In the case that η begins or ends at an internal edge e, and γ' has a segment δ' which begins or ends at e, we have the following further direct summands. Reorienting η if necessary, we may assume that η starts at e. We denote by δ the first segment of η and by $v \in \mathcal{T}_0$ the vertex at which η is located. We assume that δ' also lies at v and, reorienting γ' if necessary, that δ' also begins at e. We choose δ' in such a way that it has the minimal number of intersections with δ. This arrangement roughly looks as follows (for $n = 3$).



If $\delta'(0) \in \partial\Sigma_v$ follows $\delta(0) \in \partial\Sigma_v$ in the clockwise direction in the boundary component of $\partial\Sigma_v$, we find $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'}) \simeq \mathrm{Mor}(L, L')$, if not then $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'}) \simeq 0$. Assuming that we are in the former case, the above construction for directed boundary intersections generalizes to this situation and provides us with a direct summand $\mathrm{Mor}(L, L') \subset \mathrm{Mor}_{\mathcal{L}}(M^L_\eta, M^{L'}_{\gamma'})$.

Non-intersections

A relevant non-intersection appears every time both curves γ, γ' pass through $\Sigma_v \subset \Sigma_{\mathcal{T}}$ for $v \in \mathcal{T}_0$, so that the corresponding sections δ, δ' at v satisfy that $\mathrm{Mor}_{\mathcal{L}}(M^L_\delta, M^{L'}_{\delta'}) \neq 0$ or $\mathrm{Mor}_{\mathcal{L}}(M^{L'}_{\delta'}, M^L_\delta) \neq 0$, even though δ and δ' do not have a singular intersection and are neither part of a crossing or a boundary intersection.

Since δ and δ' do not have a crossing, we find by the computations of Section 5.5.1 that there must exist a boundary component B of Σ_v which intersects both γ and γ'. We choose to orient γ and γ' so that both δ and δ' start at B. Before and after δ and δ', the curves γ and γ' are composed of m identical segments. If m is infinite, γ and γ' have a common infinite end. We may thus assume that m is finite. We find $x \in I$ and $y \in I'$, such that the m common segments of γ and γ' are $\delta_{x+i} = \delta'_{y+i}$ with $1 \leq i \leq m$. We assume without loss of generality that $x, y \geq 1$. The segments δ_x and δ'_y both lie at a vertex $v_1 \in \mathcal{T}_0$ and the segments δ_{x+m+1} and δ'_{y+m+1} both lie at a vertex $v_2 \in \mathcal{T}_0$. The discussion now resembles the discussion in the case of a crossing above. However in contrast to the situation there, we find the following two possibilities.

1) The point $\delta'_y(1) \in \partial\Sigma_{v_1}$ follows the point $\delta_x(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\delta'_{y+m+1}(0) \in \partial\Sigma_{v_2}$ precedes the point $\delta_{x+m+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$.

2) The point $\delta_x(1) \in \partial\Sigma_{v_1}$ precedes the point $\delta'_y(1) \in \partial\Sigma_{v_1}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_1}$ and the point $\delta_{x+m+1}(0) \in \partial\Sigma_{v_2}$ follows the point $\delta'_{y+m+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise direction on the intersected boundary component of $\partial\Sigma_{v_2}$.

We continue with the case 1), the case 2) is analogous. The direct summand of (85) corresponding to the non-intersection is given by the diagram which restricts on each $\Lambda^2_0 \times \{x+i\}$ for $0 \leq i \leq m$ to the diagram

$$\begin{tikzcd} \mathrm{Mor}(L, L') \arrow[dr, "\simeq"] & & \mathrm{Mor}(L, L') \arrow[dl, "\simeq"'] \\ & \mathrm{Mor}(L, L') & \end{tikzcd}$$

restricts on $\Lambda_0^2 \times \{x+m+1\}$ to the following diagram

$$\begin{tikzcd}
\mathrm{Mor}(L,L') \arrow[dr, "\simeq"] & & 0 \arrow[dl] \\
& \mathrm{Mor}(L,L') &
\end{tikzcd}$$

and vanishes on the remainder of E_γ^{op}. The limit of this summand thus vanishes, as desired.

5.5.3 The proofs of Theorems 5.53 and 5.54

Proof of Theorem 5.53. We distinguish the following two cases.

Case 1: γ is open.
In Section 5.5.2 we have associated to each intersection (or relevant non-intersection) of γ, γ' a direct summand of the diagram (85), which passing to limits gave direct summands of $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$, matching exactly the desired description of the morphism object in Theorem 5.53. Note that if γ' is closed of rank a', then it is locally equivalent to an a'-fold direct sum. In this case, each direct summand of $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$ thus appears with multiplicity a'.

By Lemma 5.58, the diagram (85) fully arises from morphisms between the segments of γ and γ'. These morphisms are all each accounted for in exactly one of the direct summands of the diagram (85) described above. These direct summands thus describe the entirety of the diagram (85) and we may conclude that Theorem 5.53 holds for γ not closed.

Case 2: γ is closed.
The global section M_γ^L is given by the coequalizer of the diagram (81), so that the R-module $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_{\gamma'}^{L'})$ is equivalent to the equalizer of the following diagram in RMod_R.

$$\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'}) \rightrightarrows \mathrm{Mor}_{\mathcal{L}}((Z_e^L)^{\oplus a}, M_{\gamma'}^{L'}) \tag{88}$$

The R-module $\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'}) \simeq \mathrm{Mor}_{\mathcal{L}}(M_\eta^L, M_{\gamma'}^{L'})^{\oplus a}$ can be determined using its description as the limit of the a-fold direct sum of the diagram (85). All direct summands of (85) associated to intersections between η and γ' yield direct summands of the diagram (88) of the form

$$N \rightrightarrows 0$$

so that passing to limits yields the direct summands $N \subset \mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$. However, not all summands of $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$ have to be of this form, because there can be crossings between γ and γ' which do not restrict to a crossing between η and γ'. These remainder of this proof consists of an account of these summands.

Since η is not a matching curve, the direct summands associated to the intersections of η and γ' in general do not describe the entirety of $\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'})$. We additionally have to include the direct summands described in Remark 5.60 to obtain the entire morphism object $\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'})$. The closed curve γ was opened at e to the curve η. We denote the first segment of η by δ_2 and the last segment of η by δ_1. We denote the compose of δ^1 and δ^2 at e by μ. We now show the following.

a) Every crossing from γ to γ' (or from γ' to γ), which does not give rise to a crossing of μ and γ', leads to a direct summands $\mathrm{Mor}(L, L')^{\oplus a}$ (or $\mathrm{Mor}(L, L')^{\oplus a}[-1]$) in the equalizer of (88).

b) Direct summands of $\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'})$ as described in Remark 5.60 and direct summands of $\mathrm{Mor}_{\mathcal{L}}((Z_e^L)^{\oplus a}, M_{\gamma'}^{L'})$ do not persist in the equalizer of (88) if they cannot be accounted for by a crossing as above.

If γ' is closed, statement a) needs to be modified, as in the previous case where γ is not closed, to include the a'-fold multiplicity. Together with the previous discussion, the statements a) and b) then imply that $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_{\gamma'}^{L'})$ is the direct sum of the desired number of suspensions or deloopings of $\mathrm{Mor}(L, L')$, concluding this proof.

We begin by showing part a). The curves γ and γ' can be chosen so that their crossing restricts to an intersection between γ and the composite of two segments δ'_y and δ'_{y+1}, with $y \in I'$, of γ' which end, respectively, begin at e. We denote the two vertices incident to e by v_1 and v_2 and, reorienting η, γ' if necessary, can assume that both δ'_y and δ_1 lie at v_1 and both δ'_{y+1} and δ_2 lie at v_2.

We distinguish the following two cases.

1) $\delta_1(1) \in \partial\Sigma_{v_1}$ precedes $\delta'_y(1) \in \partial\Sigma_{v_1}$ in the clockwise orientation on the intersected boundary component of $\partial\Sigma_{v_1}$ and $\delta_2(0) \in \partial\Sigma_{v_2}$ precedes $\delta'_{y+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise orientation on the intersected boundary component of $\partial\Sigma_{v_2}$. This means that the crossing goes from γ to γ'

2) $\delta_1(1) \in \partial\Sigma_{v_1}$ follows $\delta_y(1) \in \partial\Sigma_{v_1}$ in the clockwise orientation on the intersected boundary component of $\partial\Sigma_{v_1}$ and $\delta_2(0) \in \partial\Sigma_{v_2}$ follows $\delta'_{y+1}(0) \in \partial\Sigma_{v_2}$ in the clockwise orientation on the intersected boundary component of $\partial\Sigma_{v_2}$. This means that the crossing goes from γ' to γ.

In the case 1), we find by Remark 5.60 a direct summand $\mathrm{Mor}(L, L')^{\oplus a} \oplus \mathrm{Mor}(L, L')^{\oplus a} \subset \mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'})$, where the first copy of $\mathrm{Mor}(L, L')^{\oplus a}$ arises from the boundary intersection of δ_2 and δ'_{y+1} and the second copy of $\mathrm{Mor}(L, L')^{\oplus a}$ arises from the boundary intersection of δ_1 and δ'_y. In terms of the diagram (88), the crossing corresponds to a direct summand of (88) of the form

$$\mathrm{Mor}(L, L')^{\oplus a} \oplus \mathrm{Mor}(L, L')^{\oplus a} \underset{(0,\mathrm{id})}{\overset{(\mathrm{id},0)}{\rightrightarrows}} \mathrm{Mor}(L, L')^{\oplus a}$$

whose equalizer gives a direct summand $\mathrm{Mor}(L, L')^{\oplus a} \subset \mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_{\gamma'}^{L'})$.

In the case 2), there are no morphisms in $\mathrm{Mor}_{\mathcal{L}}((M_\eta^L)^{\oplus a}, M_{\gamma'}^{L'})$ associated to the crossing. Using part (2) of Lemma 5.58, we thus find a direct summand of (88) corresponding to the crossing of the following form.

$$0 \rightrightarrows \mathrm{Mor}(L,L')^{\oplus a}$$

Passing to equalizers, we thus obtain the direct summand $\mathrm{Mor}(L,L')^{\oplus a}[-1]$ of $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_{\gamma'}^{L'})$. This concludes the proof of a).

For part b), we assume that part of γ' passes along e but does partake in a crossing of γ' and γ. We employ the same notation for the segments of γ' at e as above. In this case, either $\delta_1(1) \in \partial\Sigma_{v_1}$ and $\delta'_{y+1}(0) \in \partial\Sigma_{v_2}$ both follow or both precede $\delta_y(1) \in \partial\Sigma_{v_1}$ and $\delta_2(0) \in \partial\Sigma_{v_2}$, respectively. The corresponding direct summand of (88) is thus of the following form.

$$\mathrm{Mor}(L,L')^{\oplus a} \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Mor}(L,L')^{\oplus a} \tag{89}$$

The equalizer of (89) vanishes, showing b). $\square$

Proof of Theorem 5.54. The proof of Theorem 5.54 goes along the same lines as the proof of Theorem 5.53.

Case 1: γ is open.
As shown in Section 5.5.2, each self-crossing or directed boundary self-intersection gives rise to direct summands of $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_\gamma^{L'})$ given by suspensions or deloopings of $\mathrm{Mor}(L,L')$ or $\mathrm{Mor}_R(Q,Q')$ of the desired form.

Assume that all segments of γ are of the second type. Given a segment δ of the second type, we have $\mathrm{Mor}_{\mathcal{L}}(M_\delta^L, M_\delta^{L'}) \simeq \mathrm{Mor}(L,L')$, see Table 3. Similarly, we have $\mathrm{Mor}_{\mathcal{L}}(Z_e^L, Z_e^{L'}) \simeq \mathrm{Mor}(L,L')$. The constant diagram with value $\mathrm{Mor}(L,L')$ thus defines a direct summand of (85). Passing to limits, we obtain the direct summand $\mathrm{Mor}(L,L') \subset \mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_\gamma^{L'})$.

Assume that exactly one segment of γ is of the first type. The curve γ is thus singular and exactly one end lies at a vertex of $\mathcal{T}$. Reorienting γ if necessary, we can assume that γ begins at the vertex. The endomorphisms of the segments of γ thus yield a direct summand of (85), which assigns to $(\Lambda_0^2)^{\mathrm{op}} \times \{1\}$ the diagram (86) and is constant on the remainder of E_γ^{op} with value $\mathrm{Mor}(L,L')$. The limit of this direct summand is given by $\mathrm{Mor}_R(Q,Q') \subset \mathrm{Mor}_{\mathcal{L}}(M_\gamma, M_\gamma)$.

If exactly two segments of γ are of the first type, then γ is singular, and begins and ends at vertices of $\mathcal{T}$. Let N be the number of segments of γ. The endomorphisms of the segments of γ yield a direct summand of (85), which assigns to $(\Lambda_0^2)^{\mathrm{op}} \times \{1\}$ the diagram (86), to $(\Lambda_0^2)^{\mathrm{op}} \times \{N-1\}$ the diagram

$$\begin{tikzcd} \mathrm{Mor}(L,L') \arrow[dr, "\simeq"] & & \mathrm{Mor}_R(Q,Q') \arrow[dl] \\ & \mathrm{Mor}(L,L') & \end{tikzcd}$$

and to the remainder of E_γ^{op} the constant diagram with value $\mathrm{Mor}(L,L')$. To compute the limit of this diagram, one uses $\mathrm{Mor}(L,L')\simeq \mathrm{Mor}_R(Q,\phi_*\phi^*(Q'))$ and $\phi_*\phi^*(Q')\simeq Q'\oplus Q'[1-n]$. The resulting direct summand is given by $\mathrm{Mor}_R(Q,Q')\oplus \mathrm{Mor}_R(Q,Q')[-n]\subset \mathrm{Mor}_{\mathcal{L}}(M_\gamma,M_\gamma)$. If the endpoints of γ furthermore coincide, then the singular intersections of $\mathrm{Mor}_{\mathcal{L}}(M_\gamma,M_\gamma)$ produce two further direct summands given by suspensions or deloopings of $\mathrm{Mor}_R(Q,Q')$.

The above identified direct summand of $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L,M_\gamma^{L'})$ account for the entire morphism object and match the count given in Theorem 5.54. This thus concludes the proof in the case that γ is not closed.

Case 2: γ is closed.
We need to compute the equalizer of (88) with $R=k$ a field. Showing that each self-crossing of γ contributes a direct summand given by $(\mathrm{Mor}(L,L)\oplus\mathrm{Mor}(L,L)[-1])^{\oplus a^2}$ to the equalizer of (88) is analogous to the discussion in the proof of Theorem 5.53 in the case that γ is closed. A novel argument is required to determine the endomorphisms not corresponding to self-crossings. The morphisms from M_η^L to M_γ^L arising from the morphisms between the sections associated to the common segments of η,γ (all of the second type) contribute a direct summand $\mathrm{Mor}(L,L)^{\oplus a^2}\subset \mathrm{Mor}(M_\eta^L,M_\gamma^L)$. Each of the two composites with the pointwise inclusion $Z_e^L\to M_\eta^L$ in $\mathcal{L}$ arising from an end of η at e yields an equivalence between the direct summand $\mathrm{Mor}(L,L)^{\oplus a^2}$ of both $\mathrm{Mor}_{\mathcal{L}}(M_\eta^L,M_\gamma^L)$ and $\mathrm{Mor}_{\mathcal{L}}(Z_e^L,M_\gamma^L)$. As we explain below, these equivalences give rise to the following direct summand of (88).

$$\mathrm{Mor}(L,L)^{\oplus a^2} \underset{\mathscr{J}\circ(\text{-})\circ\mathscr{J}^{-1}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Mor}(L,L)^{\oplus a^2} \tag{90}$$

Above $\mathscr{J}$ denotes the monodromy equivalence, which was assumed to be a single Jordan block with eigenvalue $\lambda\in k\backslash\{0\}$. The matrix $\mathscr{J}^{-1}$ is the inverse matrix. Using the equivalence $\mathrm{Mor}(L,L)^{\oplus a^2}\simeq \mathrm{Mor}(L^{\oplus a},L^{\oplus a})$, the morphism $\mathscr{J}\circ(\text{-})\circ\mathscr{J}^{-1}$ takes a map $L^{\oplus a}\to L^{\oplus a}$, precomposes it with the endomorphism of $L^{\oplus a}$ given by $\mathscr{J}^{-1}$ and postcomposes it with the endomorphism of $L^{\oplus a}$ given by $\mathscr{J}$. The equalizer of (90) is equivalent to the fiber of the morphism

$$\mathrm{Mor}(L,L)^{\oplus a^2}\xrightarrow{\mathrm{id}-\mathscr{J}\circ(\text{-})\circ\mathscr{J}^{-1}}\mathrm{Mor}(L,L)^{\oplus a^2}\,. \tag{91}$$

A direct computation shows, that the morphism from (91) maps an $a\times a$-matrix $(m_{i,j})_{1\leq i,j\leq a}$ with entries in $\mathrm{Mor}(L,L)$ to the $a\times a$-matrix $(m'_{i,j})_{1\leq i,j\leq a}$ with

$$-m'_{i,j}=\frac{m_{i+1,j}}{\lambda}+\sum_{l>0}(-1)^l\frac{\lambda m_{i,j-l}+m_{i+1,j-l}}{\lambda^{l+1}}$$

where we set $m_{i,j}=0$ for $j\leq 0$ or $i>a$. The kernel of (91) thus consists of upper triangular matrices $(m_{i,j})_{1\leq i,j\leq a}$, satisfying that $m_{i,j}=m_{i+1,j+1}$ for all $1\leq i,j\leq a-1$. The fiber of (91) splits as its kernel, which is equivalent to $\mathrm{Mor}(L,L)^{\oplus a}$ and the delooping of its cokernel, which is given by $\mathrm{Mor}(L,L)^{\oplus a}[-1]$. This shows that we obtain the desired direct summand of $\mathrm{Mor}_{\mathcal{D}}(M_\gamma^L, M_\gamma^L)$. We have again determined the entire morphism object $\mathrm{Mor}_{\mathcal{L}}(M_\gamma^L, M_\gamma^L)$, showing Theorem 5.54.

To arrive at the direct summand (90), we need to describe the equivalence of the direct summand $\mathrm{Mor}(L,L)^{\oplus a^2}$ obtained from composing the two equivalences contained in the following diagram in RMod_k, arising from restricting morphisms to the two endpoints of η.

$$\begin{tikzcd}
 & \mathrm{Mor}(L,L)^{\oplus a^2} \arrow[d, hook] \arrow[dr, "\simeq"] & \\
\mathrm{Mor}(L,L)^{\oplus a^2} \arrow[ur, "\simeq"] \arrow[d, hook] & \mathrm{Mor}_{\mathcal{L}}\left(\left(M_\eta^L\right)^{\oplus a}, M_\gamma^L\right) \arrow[dl] \arrow[dr] & \mathrm{Mor}(L,L)^{\oplus a^2} \arrow[d, hook] \\
\mathrm{Mor}_{\mathcal{L}}\left(\left(Z_e^L\right)^{\oplus a}, M_\gamma^L\right) & & \mathrm{Mor}_{\mathcal{L}}\left(\left(Z_e^L\right)^{\oplus a}, M_\gamma^L\right)
\end{tikzcd} \tag{92}$$

This equivalence is affected by two kinds of monodromy. Firstly, the monodromy of the perverse schober $\mathcal{F}_{\mathcal{T}}(k)$ along γ, see also Section 3.3.4, which we show to be trivial. Secondly, the monodromy equivalence of (γ, L), given by the Jordan-block $\mathscr{J}$.

Consider a segment δ_i of γ lying at v^i and connecting e^i and e^{i+1}. We assume that δ_i turns counterclockwise, the clockwise case is analogous. Up to the action of the paracyclic twist functor $T_{\mathcal{F}(v^i)}$, see Lemma 3.31, we can assume that $\mathcal{F}_{\mathcal{T}}(k)(v^i \to e^i)$ and $\mathcal{F}_{\mathcal{T}}(k)(v^i \to e^{i+1})$ are given by $T_1 \circ \varrho_1$, respectively, $T_2 \circ \varrho_2$, with T_1, T_2 each given by one of the two autoequivalences id, T, using the notation from Construction 5.14. The left adjoint ς_2 of ϱ_1 is right adjoint to ϱ_2, see Lemma 3.24. Since ς_2 is a fully faithful functor, it thus follows that $\varrho_2 \circ \varsigma_2 \simeq \mathrm{id}_{\mathrm{RMod}_{R[t_{n-2}]}}$. The transport $\mathcal{F}_{\mathcal{T}}(k)^{\to}(\delta_i)$ along δ_i is thus a power of the involution T. The total monodromy $\mathcal{F}_{\mathcal{T}}(k)^{\to}(\gamma, e)$ of $\mathcal{F}_{\mathcal{T}}(k)$ along γ is hence, for any choice edge e along which γ passes, given by i-th power of the involution T for some $i \in \mathbb{Z}$. The integer i is even, as follows from inspecting the construction of $\mathcal{F}_{\mathcal{T}}(k)$ and from the observation that there are an equal number of halfedges being transversed by γ which carry an even or odd labeling in the chosen total orders. It follows that the monodromy of $\mathcal{F}_{\mathcal{T}}(k)$ along γ is trivial.

We proceed by spelling out in detail how the equivalence (92) arises from the composition of the two kinds of monodromy.

The morphism object $\mathrm{Mor}(M_\eta^L, M_\gamma^L)$ is given by the limit of (85). Instead of directly computing the equivalence in (92), we can thus equivalently compute the equivalences obtained by tracing along the segments of η, i.e. compose the endomorphisms of $\mathrm{Mor}(L,L)^{\oplus a^2}$ contained in the commutative diagrams

$$\begin{tikzcd}
\operatorname{Mor}(L,L)^{\oplus a^2} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \operatorname{Mor}(L,L)^{\oplus a^2} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \operatorname{Mor}(L,L)^{\oplus a^2} \arrow[d, "\simeq"] \\
\operatorname{Mor}\left(Z^L_{e^i}(e^i), M^L_{\delta_i}(e^i)\right)^{\oplus a^2} \arrow[d, "\simeq"] & \operatorname{Mor}\left(M^L_{\delta_i}(e^i), M^L_{\delta_i}(e^i)\right)^{\oplus a^2} \arrow[l, "\simeq"'] & \operatorname{Mor}_{\mathcal{V}^n_{\phi^*}}\left(M^L_{\delta_i}(v^i), M^L_{\delta_i}(v^i)\right)^{\oplus a^2} \arrow[l, "\mathcal{F}_{\mathcal{T}}(v^i\to e^i)"'] \arrow[d, hook] \\
\operatorname{Mor}_{\mathcal{L}}\left(\left(Z^L_{e^i}\right)^{\oplus a}, M^L_\gamma\right) & & \operatorname{Mor}_{\mathcal{L}}\left(\left(M^L_{\delta_i}\right)^{\oplus a}, M^L_\gamma\right) \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Mor}(L,L)^{\oplus a^2} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \operatorname{Mor}(L,L)^{\oplus a^2} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \operatorname{Mor}(L,L)^{\oplus a^2} \arrow[d, "\simeq"] \\
\operatorname{Mor}_{\mathcal{V}^n_{\phi^*}}\left(M^L_{\delta_i}(v^i), M^L_{\delta_i}(v^i)\right)^{\oplus a^2} \arrow[r, "\mathcal{F}_{\mathcal{T}}(v^i\to e^{i+1})"'] \arrow[d, hook] & \operatorname{Mor}\left(M^L_{\delta_i}(e^{i+1}), M^L_{\delta_i}(e^{i+1})\right)^{\oplus a^2} \arrow[r, "\simeq"] & \operatorname{Mor}\left(Z^L_{e^{i+1}}(e^{i+1}), M^L_{\delta_i}(e^{i+1})\right)^{\oplus a^2} \\
\operatorname{Mor}_{\mathcal{L}}\left(\left(M^L_{\delta_i}\right)^{\oplus a}, M^L_\gamma\right) \arrow[rr] & & \operatorname{Mor}_{\mathcal{L}}\left(\left(Z^L_{e^{i+1}}\right)^{\oplus a}, M^L_\gamma\right) \arrow[u, hook']
\end{tikzcd}$$

with $1 \le i \le N$, where γ has N segments δ_i, $i \in I = \mathbb{Z}/N\mathbb{Z}$, lying at $v^i \in \mathcal{T}_0$ and beginning and ending at the edge e^i, respectively, e^{i+1}. For $i = N$, the morphism $\mathcal{F}_{\mathcal{T}}(v^n \to e^1)$ in the above diagram needs to additionally be composed with the endomorphism $\mathscr{J} \circ (\text{-})$. To justify this, we make a choice of coequalizer M^L_γ of (81), which assigns to each edge $v \to e$ the morphism $M^L_\eta(v \to e)^{\oplus a}$ (strictly and not just up to equivalence), except for the morphism $v^n \to e^1$, where $M^L_\eta(v^n \to e^1)^{\oplus a}$ is composed with the equivalence $\mathscr{J}$. The precomposition with $\mathscr{J}^{-1}$ arises from the appearance of $\mathscr{J}^{-1}$ in (81).

Before explaining why the equivalences in the above diagram also describe the monodromy of $\mathcal{F}_{\mathcal{T}}(k)$, we have to take care of some further contributions. These are the equivalences

$$E_i\colon \operatorname{Mor}(M^L_{\delta_{i-1}}(e_i), M^L_{\delta_{i-1}}(e_i))^{\oplus a^2} \simeq \operatorname{Mor}(Z^L_{e_i}(e_i), Z^L_{e_i}(e_i))^{\oplus a^2} \simeq \operatorname{Mor}(M^L_{\delta_i}(e_i), M^L_{\delta_i}(e_i))^{\oplus a^2}$$

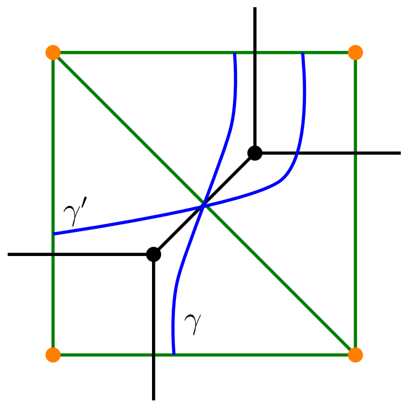
which arise from the fact that the inclusions $Z^L_{e_i} \to M^L_{\delta_i}, M^L_{\delta_{i-1}}$ were only specified up to k-linear equivalence (since the local sections were defined as Kan extensions). Under the equivalences with $\operatorname{Mor}(L,L)^{\oplus a^2}$, the equivalence E_i corresponds to an endomorphism $\mathscr{D}^{-1}(\text{-})\mathscr{D}$ of $\operatorname{Mor}(L,L)^{\oplus a^2}$, where $\mathscr{D}$ is some invertible diagonal $a\times a$-matrix with entries in $\pi_0 \operatorname{Map}(L,L)$, all of whose diagonal entries are identical (since we took the direct sum of local sections in the gluing). It follows that $\mathscr{D}^{-1}(\text{-})\mathscr{D}$ is the identity on $\operatorname{Mor}(L,L)^{\oplus a^2}$. Thus, the equivalences E_i do not contribute to the diagram (90).

The left and right adjoints of the functors $\mathcal{F}_{\mathcal{T}}(v^i \to e^i)$ contained in the middle parts of the above diagrams are fully faithful and hence define right inverses of $\mathcal{F}_{\mathcal{T}}(v^i \to e^i)$. If δ_i wraps clockwise, then $M^L_{\delta_i}(v^i) \simeq \operatorname{radj}(\mathcal{F}_{\mathcal{T}}(v^i \to e^i))(M^L_{\delta_i}(e^i))$.

Similarly, if δ_i wraps counterclockwise, then $M^L_{\delta_i}(v^i) \simeq \mathrm{ladj}(\mathcal{F}_{\mathcal{T}}(v^i \to e^i))(M^L_{\delta_i}(e^i))$. Tracing along the middle part of the above diagram thus yields a contribution of the monodromy of $\mathcal{F}_{\mathcal{T}}(k)$. This concludes the argument, why the direct summand (90) of the diagram (88) appears. □

We describe in Example 5.61 global sections arising from matching data, whose matching curves are non-pure, and for which an arising morphism object does not simply count intersections.

Example 5.61. Consider the 4-gon with an ideal triangulation with dual trivalent spanning $\mathcal{T}$ and two matching curves γ, γ', which can be depicted as follows.



The matching curve γ is pure, whereas γ' is not pure. There is a directed boundary intersection from γ to γ' and a crossing from γ' to γ. For any $L, L' \in \mathrm{RMod}_{R[t_1]}$, there are apparent matching data (γ, L) and (γ', L').

The computation of $\mathrm{Mor}_{\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}(R))}(M^L_\gamma, M^{L'}_{\gamma'})$ boils down to the computation of the limit of the E^{op}_γ-indexed diagram:

$$
\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dr] & & \mathrm{Mor}(L,L')[1] \arrow[dl, "\alpha"'] \arrow[dr, "\simeq"] & \\
0 & & \mathrm{Mor}(L,L') & & \mathrm{Mor}(L,L')[1]
\end{tikzcd} \tag{93}
$$

Spelling out its construction, we see that the morphism α is given by precomposition with the morphism $s_L \colon L \to T_{\mathrm{RMod}_{R[t_1]}}(L)[1] \simeq L[-1]$, arising from the fiber and cofiber sequence of endofunctors of $\mathrm{RMod}_{R[t_1]}$

$$\phi^*\phi_* \to \mathrm{id}_{\mathrm{RMod}_{R[t_1]}} \xrightarrow{s} T_{\mathrm{RMod}_{R[t_1]}}[1]$$

describing the twist functor of the adjunction $\phi^* \dashv \phi_*$. The natural transformation s is also called the section of the twist functor. It evaluates to a zero morphism at L if $L \in \mathrm{Im}(\phi^*)$, but evaluates non-trivially for other L (such as $L = R[t_1]$). The limit of the diagram (93) is given by the fiber of the morphism α. If $\alpha = 0$, the limit thus consists of the sum of two copies of suspensions or deloopings of $\mathrm{Mor}(L, L)$, matching the number of intersections of γ and γ'. If $\alpha \neq 0$, this is not true.

5.5.4 Indecomposability of objects

Corollary 5.62. *Let $R = k$ be a field and $L \in \mathrm{RMod}_{k[t_{n-2}]}$, such that $H_0 \operatorname{End}(L) \simeq k$.*

(1) Let (γ, L) be an open matching datum in $\mathbf{S}\backslash M$, such that γ is finite and pure. The discrete endomorphism ring $H_0 \operatorname{End}(M_\gamma^L)$ is local and M_γ^L thus indecomposable.

(2) Let (γ, L) be a closed matching datum, whose monodromy matrix is a single Jordan block and such that γ is pure. The discrete endomorphism ring $H_0 \operatorname{End}(M_\gamma^L)$ is local and M_γ^L thus indecomposable.

Remark 5.63. For $L = \phi^*(k), k[t_{n-1}], k[t_{n-1}^{\pm}] \in \mathrm{RMmod}_{k[t_{n-1}]}$, we have an equivalence $H_0(\operatorname{End}(L)) \simeq k$. We can thus apply Corollary 5.62 to matching data with local values of different sizes.

Proof of Corollary 5.62. We begin with proving part (1). By Theorem 5.54, we have a complete description of the discrete endomorphism algebra $H_0 \operatorname{End}(M_\gamma^L)$. The degree 0 morphisms arising from crossings, directed boundary intersections and singular intersections of distinct endpoints generate an ideal J. Every endomorphism of M_γ^L is the sum of an endomorphism in J and a k-linear multiple of the identity. We argue below, that an endomorphism $\alpha = \beta + \lambda \operatorname{id}_{M_\gamma^L}$ of M_γ^L with $\beta \in J$ is an equivalence if and only if $\lambda \in k$ is nonzero. It thus follows, that J is the unique maximal ideal, showing that $H_0 \operatorname{End}(M_\gamma^L)$ is a local ring. This shows part (1).

The argument generalizes to the setting in part (2). We can again form a maximal ideal out of the contributions from crossings, out of the copy of $H_0 \operatorname{Mor}(L, L)^{\oplus a-1}$ consisting of strictly upper triangular matrices in the kernel of (91) and the copy of $H_0(\operatorname{Mor}(L, L)^{\oplus a}[-1])$.

Let $\beta \in J$ and $\lambda \in k$. We conclude this proof by showing that the endomorphism $\alpha = \beta + \lambda \operatorname{id}_{M_\gamma^L}$ of M_γ^L is invertible if and only if $\lambda \neq 0$. The morphism α is a natural transformation between sections and thus invertible if and only if the morphism $\alpha(x) : M_\gamma^L(x) \to M_\gamma^L(x)$ in $\mathcal{F}_\mathcal{T}(x)$ is an equivalence for all $x \in \operatorname{Exit}(\mathcal{T})$. Locally, near a vertex v of $\mathcal{T}$ with incident edges $e_1, \dots, e_n$, $\mathcal{F}_\mathcal{T}$ is described by the conservative functor

$$(\varrho_1, \dots, \varrho_n) : \mathcal{F}_\mathcal{T}(v) \simeq \mathcal{V}_{f^*}^n \longrightarrow \mathrm{RMod}_{R[t_{n-2}]}^{\times n} \simeq \prod_{i=1}^n \mathcal{F}_\mathcal{T}(e_i)\,.$$

It thus suffices to consider the case that $x = e \in \operatorname{Exit}(\mathcal{T})$ is a vertex of $\mathcal{T}$.

Before we proceed, let us note that the endomorphisms of M_γ^L appearing in the direct summand $H_0\left(\operatorname{Mor}(L, L)[-1]^{\oplus i^{\mathrm{cr}}(\gamma,\gamma)}\right)$ evaluate to zero at all objects $x \in \operatorname{Exit}(\mathcal{T})$. This is easily seen from tracing through their construction. We thus do not need to consider these endomorphisms in the following to determine whether $\alpha(x)$ is invertible.

Let now $x = e$ be an edge of $\mathcal{T}$ and choose a halfedge with incident vertex v. The object $M_\gamma^L(e) \in \mathcal{V}_{\phi^*}^n$ decomposes into the sum of the evaluations at e of the local sections associated with the different segments of γ at v. There are three pure segments at v with an intersection with e, two of the second type and one of the first type. The corresponding local sections are pairwise semiorthogonal, since the morphisms between these sections arise from directed boundary intersections in Σ_v at the boundary component intersecting e. From this directedness, it follows that $\alpha(e)$ is of a block upper triangular shape and hence an equivalence if and only if for any such segment δ at v, the restriction of $\alpha(e)$ to an endomorphism of $\bigoplus_{\delta_i=\delta} M_\delta^L(e)$ is an equivalence, where the sum runs over the segments δ_i with $i \in I$ of γ at v which are identical to δ.

If δ is of the first type at v ending at e, it is obvious that the restriction of $\alpha(e)$ to $\bigoplus_{\delta_i=\delta} M_\delta^L(e)$ is an equivalence if and only if $\lambda \neq 0$.

Suppose that δ is a segment of the second type at v ending at e. Let B be the boundary component of Σ_v corresponding to the chosen halfedge of e (this essentially means B intersects e, but remains correct if e is a loop). We consider each segment $\delta_i = \delta$ as oriented, so that it ends at B. An orientation of a segment δ_i, with $i \in I$, determines an orientation of γ. We can choose the map γ into $\mathbf{S}\backslash M$ such that all crossings of γ involving two segments $\delta_i = \delta_j = \delta$ with $i \neq j$ are such that they appear after δ_i in the induced orientation of γ by δ_i (and thus also after δ_j in the induced orientation of γ by δ_j). The curve γ induces an embedding of the segments $\{\delta_i = \delta\}_{i\in I}$ into $\Sigma_v \subset \mathbf{S}\backslash M$, which endows these segments with a total order, given by the clockwise order of the intersections with B. This total order has the following good property: an endomorphism of γ arising from a self-crossing which evaluates non-trivially at e always arises via evaluating at e a morphism from $M_{\delta_i}^L$ to $M_{\delta_j}^L$ with $\delta_i = \delta_j = \delta$ and $i < j$.

If there is a directed boundary self-intersection of γ so that it goes from the segment $\delta_i = \delta$ to the segment $\delta_j = \delta$, we further distinguish two cases. In the first case, the directed boundary intersection follows after δ_i in the induced oriented of γ (by δ_i). It is immediate that $i < j$. In the second case, the directed boundary is before δ_i in the induced orientation of γ. In this case, we have $j = i+1$, as otherwise there would have to be crossings appearing before δ_i in the orientation of γ. We swap the order of i and $j = i+1$.

In the constructed order of the segments $\delta_i = \delta$, we find that if $\alpha(e)$ induces a nonzero morphism from $M_{\delta_i}^L$ to $M_{\delta_j}^L$ with $\delta^i = \delta^j = \delta$, then $i < j$. This means that $\alpha(e)$ restricts to an endomorphism of $\bigoplus_{\delta_i=\delta} M_\delta^L(v)$ given by an upper triangular matrix, with diagonal entries λ. It follows that any edge e, $\alpha(e)$ is an equivalence if and only if $\lambda \neq 0$, concluding the proof. □

5.6 Further topics

5.6.1 The Jacobian gentle algebra

We fix a marked surface $\mathbf{S}$ with an n-valent spanning graph $\mathcal{T}$.

Definition 5.64. Let k be a commutative ring. The Jacobian algebra $\mathscr{J}_\mathcal{T}$ is defined as the 0-th homology k-algebra $\mathscr{J}_\mathcal{T} = H_0(\mathscr{G}_\mathcal{T})$ of the relative Ginzburg algebra $\mathscr{G}_\mathcal{T}$ (defined over k).

Consider the sub-quiver $P_{\mathcal{T}}$ of the quiver $\tilde{Q}_{\mathcal{T}}$ of Definition 5.4 consisting of all vertices and the arrows $a_{v,i,i+1}$, where $i+1$ denotes the counterclockwise halfedge predecessor of the halfedge i, meaning that $P_{\mathcal{T}}$ consists of all arrows of $\tilde{Q}_{\mathcal{T}}$ lying in degree 0. It is immediate from the definition of $\mathscr{G}_{\mathcal{T}}$ that $\mathscr{J}_{\mathcal{T}} \simeq kP_{\mathcal{T}}/I$, where the ideal $I = \{a_{v,i,i+1}a_{v,i-1,i}\}$ consists of certain paths of length. With this, it is straightforward to see that the Jacobian algebra $\mathscr{J}_{\mathcal{T}}$ is a gentle algebra, in the sense recalled in Definition 5.65. While the gentle algebra $\mathscr{J}_{\mathcal{T}}$ is finite dimensional if $\mathbf{S}$ has no punctures, it is infinite dimensional if there are punctures, as the cycles wrapping around the puncture do not lie in the ideal I.

Definition 5.65. A k-algebra is called a gentle algebra if it is isomorphic to the path algebra kQ/I of a finite quiver Q modulo an ideal I generated by paths of length 2, such that

- every vertex of Q has at most two incoming and two outgoing arrows.
- For each arrow a, there is at most one arrow b such that ab lies in I and there is at most one arrow b such that $ba \in I$.
- For each arrow a, there is at most one arrow b such that $ab \notin I$ and there is at most one arrow b such that $ba \notin I$.

The main result of this section is the following description of the homology algebra $H_*(\mathscr{G}_{\mathcal{T}})$ in the case that $\mathbf{S}$ has no punctures.

Proposition 5.66. *Let $R = k$ be a commutative ring and suppose that $\mathbf{S}$ has no punctures. There exists an isomorphism of dg-algebras with vanishing differentials between $H_*(\mathscr{G}_{\mathcal{T}})$ and the tensor algebra $\mathscr{J}_{\mathcal{T}} \otimes_k k[t_{n-2}]$.*

Other classes of Ginzburg algebras whose homology has been computed include non-relative Ginzburg algebras of acyclic quivers, see [Her16], and some classes of relative Ginzburg algebras whose homology is concentrated in degree 0, see [Wu23b, Section 8.2].

Remark 5.67. If $\mathbf{S}$ has punctures, the curves c_e associated to the edges of $\mathcal{T}$ have common infinite ends. We expect that a generalization of Theorem 5.53 which allows common infinite ends would allow to extend Proposition 5.66 to arbitrary surfaces.

Proof of Proposition 5.66. Let $L = k[t_{n-2}] \in \mathrm{RMod}_{k[t_{n-2}]}$. By Proposition 5.49, there exists an isomorphism of dg-algebras

$$\mathscr{G}_{\mathcal{T}} \simeq \mathrm{End}\left(\bigoplus_e M_{c_e}^L\right), \tag{94}$$

so that it suffices for part (1) to construct an isomorphism between $H_* \mathrm{End}\left(\bigoplus_e M_{c_e}^L\right)$ and $\mathscr{J}_{\mathcal{T}} \otimes_k k[t_{n-2}]$.

Given two edges e_1, e_2 of $\mathcal{T}$, the associated pure matching curves c_{e_1}, c_{e_2} do not intersect, except for directed boundary intersections. Applying Theorem 5.53, we obtain for each directed boundary intersection a direct summand

$$k[t_{n-2}] \simeq \mathrm{Mor}(k[t_{n-2}], k[t_{n-2}]) \subset \mathrm{End}\left(\bigoplus_e M^L_{c_e}\right).$$

This shows that there exists an equivalence in RMod_k

$$H_* \,\mathrm{End}\left(\bigoplus_e M^L_{c_e}\right) \simeq H_0 \,\mathrm{End}\left(\bigoplus_e M^L_{c_e}\right) \otimes_k k[t_{n-2}]\,. \tag{95}$$

Since there exists an isomorphism of k-algebras $H_0 \,\mathrm{End}\left(\bigoplus_e M^L_{c_e}\right) \simeq \mathscr{J}_{\mathcal{T}}$ by (94), to conclude this proof, it suffices to show that (95) is also an isomorphism of dg-algebras (with vanishing differentials). For that, we need to compare the composition in the two dg-algebras.

We call two directed boundary intersections from c_{e_1} to c_{e_2} and c_{e_2} to c_{e_3} composable if they lie at the same boundary component B of $\mathbf{S}\backslash M$. In this case, starting at B, the curves are composed of identical segments such that c_{e_1} shares the same segments with both c_{e_2} and c_{e_3} and the two curves c_{e_2} and c_{e_3} share at least as many segments with each other as with c_{e_1}. Let $a, b \in \{1,2,3\}$ with $a \leq b$. Each generating morphisms given by $t^i_{n-2} \in k[t_{n-2}] \subset \mathrm{End}(\bigoplus_e M^L_{c_e})$ with $i \geq 0$ associated to the boundary intersections of c_{e_a}, c_{e_b} at B, or the endomorphisms of $M^L_{c_{e_a}}$ if $a = b$, corresponds to a morphism between the sections $M^L_{c_{e_a}}, M^L_{c_{e_b}}$ which restricts for each shared segment δ of c_{e_a}, c_{e_b} to the endomorphism $t^i_{n-2} \in k[t_{n-2}] \simeq \mathrm{End}(M^L_\delta)$. We thus see, that the composite of $t^i_{n-2}\colon M^L_{c_{e_a}} \to M^L_{c_{e_b}}$ with $t^j_{n-2}\colon M^L_{c_{e_b}} \to M^L_{c_{e_c}}$ is given by $t^{i+j}_{n-2}\colon M^L_{c_{e_a}} \to M^L_{c_{e_c}}$ for all $a \leq b \leq c \in \{1,2,3\}$.

We also note that if two boundary intersections are not composable, then the corresponding endomorphisms of $\bigoplus_e M^L_{c_e}$ compose to zero.

Comparing the two sides of (94) in degree 0, one obtains that dircted boundary intersection from c_{e_1} to c_{e_2} are in bijection with nonzero paths from e_1 to e_2 in $\mathscr{J}_{\mathcal{T}}$. Using that by construction $H_0(\alpha)$ is an isomorphism of k-algebras, we further obtain that two boundary intersections are composable if and only if the corresponding paths in $\mathscr{J}_{\mathcal{T}}$ are composable with nonzero composite. The description of the product of the generating morphisms of $\mathrm{End}\left(M^L_{c_e}\right)$ given above implies that α commutes with the multiplications and is thus an isomorphism of dg-algebras, concluding this proof. $\square$

5.6.2 Derived equivalences from flips of the n-angulation

In this section, we construct derived equivalences between the global sections of the perverse schobers $\mathcal{F}_{\mathcal{T}}(R)$ arising from changing the n-valent spanning graph $\mathcal{T}$ by a flip of an edge. In particular, in the case $R = k$, we thus obtain derived equivalences between relative Ginzburg algebras. Related results on derived equivalences of non-relative and relative Ginzburg algebras from mutations of quivers with potentials were obtained in [KY11,Wu23a]. We then further describe the action of these derived equivalences in terms of the partial geometric model.

Given a decomposition of a surface into n-gons ($n \geq 3$) and an edge e of one of the n-gons which is not a loop, there are $n-2$ possible *flips* of the decomposition, which are obtained by replacing e by a different diagonal contained in the $(2n-2)$-gon formed by the two adjacent n-gons of e. For example, the local pictures of the two possible flips of a decomposition into 4-gons are depicted in Figure 3.

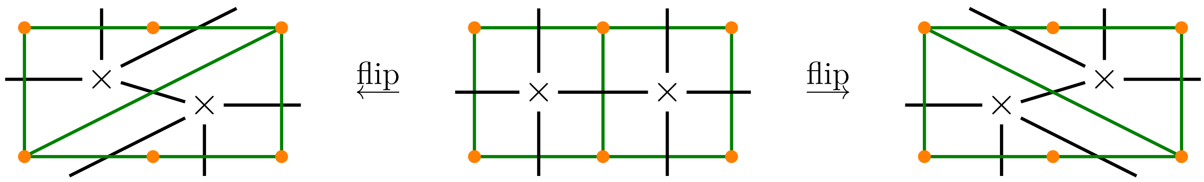
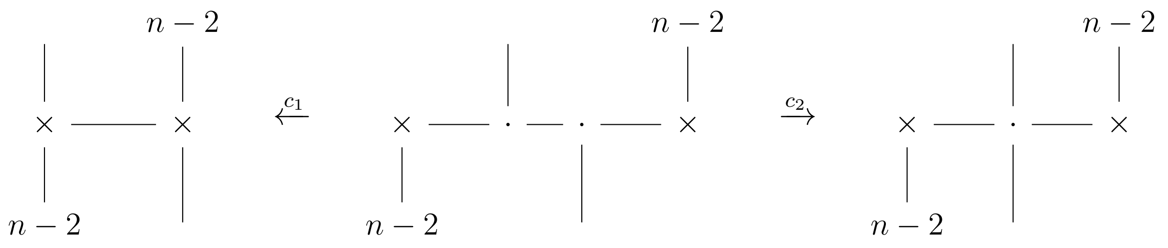


Figure 3: The two possible flips of a decomposition into 4-gons (in green) at an edge and the corresponding change in the dual ribbon graphs (in black).

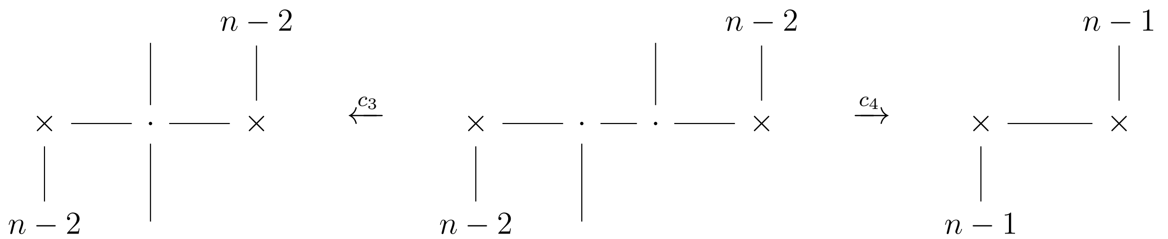
Starting with a flip of a decomposition into n-gons of the $2n-2$-gon and passing to the dual ribbon graphs, we obtain the local description of a flip of a ribbon graph at an edge e. The flip of an n-valent ribbon graph $\mathcal{T}$ at a an edge e which is not an internal loop is defined by locally at e changing $\mathcal{T}$ as above and away from e not changing $\mathcal{T}$.

We proceed with describing flips of n-valent ribbon graphs in terms of contractions of ribbon graphs. It suffices to restrict to a flip at an edge, which moves the edge by one step in the counterclockwise direction. We use the following graphical notation for ribbon graphs from Notation 3.5. If an edge ends in an integer, that means that this edge represents that number of edges.

The flip by one step is realized by two spans of contractions which are everywhere trivial, except near the edge which is being flipped, where they can be depicted as follows.



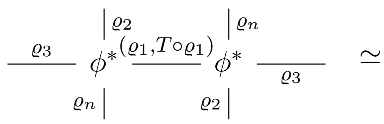
(96)

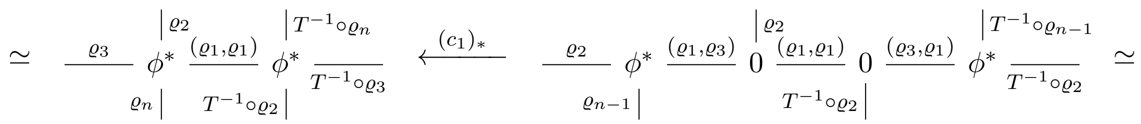


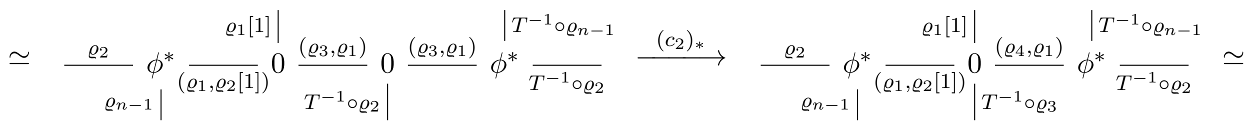
(97)

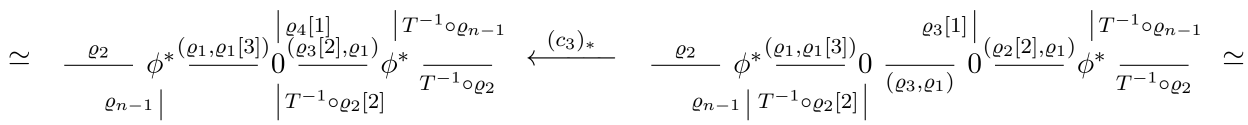
Applying Proposition 3.47, we can use the above contractions of ribbon graphs to produce an equivalence between ∞-categories of global sections of perverse schobers parametrized by the involved ribbon graphs. To this end, we describe below a collection of parametrized perverse schobers, using Notation 3.42. Below, T denotes the autoequivalence of $\mathrm{RMod}_{R[t_{n-2}]}$ from Construction 5.14.

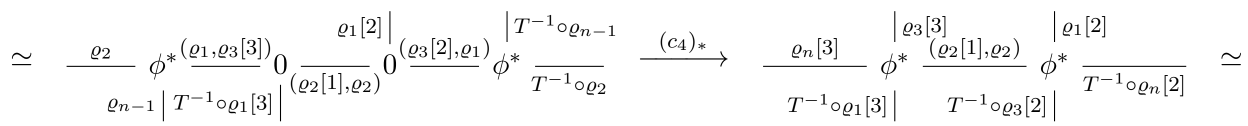
From now on we fix a marked surface $\mathbf{S}$ and an $\mathbb{E}_\infty$-ring spectrum R. Let $\mathcal{T}_1, \mathcal{T}_2$ be two n-valent spanning graphs of $\mathbf{S}$ which differ by a flip at any edge e of $\mathcal{T}_1$ which is not a loop by one step in the counterclockwise direction. We find a collection of parametrized perverse schobers, related by equivalences and contractions, which are everywhere identical except at e and its two incident vertices v, v', where they are given as follows, starting with $\mathcal{F}_{\mathcal{T}_1}(R)$ and ending with $\mathcal{F}_{\mathcal{T}_2}(R)$. For better readability, we do not depict all edges below.

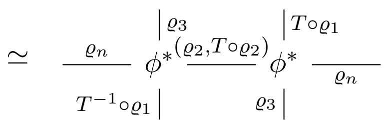

(98)


(99)


(100)


(101)


(102)


(103)

The above equivalences of parametrized perverse schober are each nontrivial only at one or two vertices with label 0, where they are each given by a power of the paracyclic twist functor $T_{\mathcal{V}^i_0}$ with $i = 3, 4$, see Section 2.1.4, except for the equivalence between the parametrized perverse schober in (98) and the left parametrized perverse schober in (99) and the equivalence between the right parametrized perverse schober of (102) and the parametrized perverse schober of (103). The former is nontrivial only at the right vertex labeled ϕ^*, where it is given by the autoequivalence ϵ of $\mathcal{V}^n_{\phi^*}$, defined by restricting on each of the $n-1$ components $\mathrm{RMod}_{R[t_{n-2}]}$ of the semiorthogonal decomposition to T and on the component RMod_R of the semiorthogonal decomposition to the identity functor. The latter equivalence of parametrized perverse schobers is nontrivial at the three objects of $\mathrm{Exit}(\mathcal{T}')$ corresponding to e and the two incident vertices. At the left vertex, the equivalence is given by $[3]$, at the right vertex by $\epsilon^{-1} \circ [2]$ and at e by $[-2]$.

We thus obtain an equivalence

$$\mu^1_e \colon \mathcal{H}(\mathcal{T}_1, \mathcal{F}_{\mathcal{T}_1}(R)) \xrightarrow{\simeq} \mathcal{H}(\mathcal{T}_2, \mathcal{F}_{\mathcal{T}_2}(R))\,, \tag{104}$$

which we call the mutation equivalence. We denote the repeated mutation by $\mu^i_e := (\mu^1_e)^i$ for $i \in \mathbb{Z}$.

In the remainder of this section, we give a geometric description of μ^1_e in terms of a homeomorphism

$$D_e\left(\frac{1}{n-1}\pi\right) : \mathbf{S}\backslash M \to \mathbf{S}\backslash M$$

which we now describe.

Let v, v' be the two vertices of $\mathcal{T}_1$ and $\mathcal{T}_2$ incident to e. Recall from Remark 3.54, that $\Sigma_{\mathcal{T}_1}$ and $\Sigma_{\mathcal{T}_2}$ are embedded in $\mathbf{S}\backslash M$. The two subspaces $\Sigma_v \cup_{\Sigma_{\mathcal{T}_1}} \Sigma_{v'}$ and $\Sigma_v \cup_{\Sigma_{\mathcal{T}_2}} \Sigma_{v'}$ of $\mathbf{S}\backslash M$ are both clearly homeomorphic to the closed unit disc in $\mathbb{R}^2$ with $2n-2$ intervals removed from the boundary. For concreteness, we arrange the homeomorphism so that it maps v to $(-\frac{1}{4}, 0)$ and v' to $(\frac{1}{4}, 0)$.

We set $D_e(\frac{1}{n-1}\pi)$ to be any homeomorphism that

- restricts to a homeomorphisms between $\Sigma_v \cup_{\Sigma_{\mathcal{T}_1}} \Sigma_{v'}$ and $\Sigma_v \cup_{\Sigma_{\mathcal{T}_2}} \Sigma_{v'}$ which under the above homeomorphisms with the unit disc is an automorphism of the disc which keeps the boundary fixed and rotates the convex hull of v and v' by $\frac{1}{n-1}\pi$.
- is constant on the remainder of $\mathbf{S}\backslash M$.

Theorem 5.68. *Let $\mathcal{T}_1$ be an n-valent spanning graph of $\mathbf{S}$ and let $\mathcal{T}_2$ be the n-valent spanning graph of $\mathbf{S}$ obtained by a flip of an edge e of $\mathcal{T}$ which is not a loop by one step in the counterclockwise direction. Let (γ, L) be a matching datum in $\mathbf{S}\backslash M$, such that γ is pure. There exists a matching datum $(D_e(\frac{1}{n-1}\pi) \circ \gamma, L)$ and an equivalence in $\mathcal{H}(\mathcal{T}_2, \mathcal{F}_{\mathcal{T}_2}(R))$*

$$\mu^1_e(M^L_\gamma) \simeq M^L_{D_e(\frac{1}{n-1}\pi)\circ\gamma}[m]\,, \tag{105}$$

where

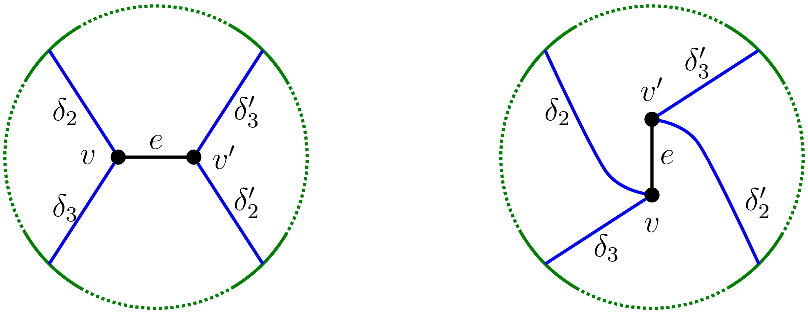


Figure 4: The 4-gon with two vertices v, v', the edge e and some matching curves (in blue) on the left and their image under $D_e(\frac{1}{2}\pi)$ on the right.

- *$m = 1$ if the first or last segment of γ is of the first type and lies at v, exiting v through the edge e.*
- *$m = 0$ if γ is not as above. This includes all cases in which γ is regular, i.e. all its endpoints lie in $\partial\mathbf{S}\backslash M$.*

Remark 5.69. Note that the matching curve $D_e(\frac{1}{n-1}\pi) \circ \gamma$ in Theorem 5.68 is not necessarily pure. Theorem 5.68 can be extended to the global sections associated to arbitrary matching data, but this requires a more systematic discussion of grading structures on curves.

Remark 5.70. Under the flip of an edge e of the n-valent spanning graph $\mathcal{T}$, we rotate the vertices incident to e. This is a matter of convention: we could equally well keep the vertices of $\mathcal{T}$ fixed and considered these as part of the data of $\mathbf{S}$. In the latter convention, the statement of Theorem 5.68 becomes simpler, see also [CHQ23, Lemma 4.20], but that convention obscures the fact that flipping n times reverts back to the original n-valent spanning graph.

Proof of Theorem 5.68. We note that it suffices to show that

$$\mu_e^1(M_\gamma^L) \simeq M^L_{D_e(\frac{1}{n-1}\pi)\circ\gamma}[m] \tag{106}$$

for each pure matching curve γ in the $(2n-2)$-gon. The theorem then follows, using that μ_e^1 and the object M_γ^L associated to the matching curve γ in $\mathbf{S}$ are defined via gluing. This leaves finitely many cases, which are directly verified by tracing through the equivalences between the global sections of the parametrized perverse schobers defining μ_e^1.

To help the reader appreciate the appearance or absence of suspensions in the theorem, without having to trace through the definition of μ_e^1, we offer the following hints.

To see the absence of suspensions for pure, regular segments, one simply observes that the values of the corresponding sections at the external edges of the $(2n-2)$-gon remain unchanged under μ_e^1 and in particular do not acquire any suspensions.

Denote the vertices of $\mathcal{T}_1$ incident to e by v, v'. Let γ be a pure matching curve starting at v (for v' the argument is analogous). One has for $Q \in \mathrm{RMod}_R$ an equivalence

$$M_\gamma^L(v) \simeq \left(Q \xrightarrow{!} \phi^*(Q) \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \phi^*(Q)\right) \in \mathcal{V}_{\phi^*}^n$$

(where M_γ^L is the section of the schober (98)) and to verify the appearance of the suspension, one computes

$$\mu_e^1(M_\gamma^L)(v) \simeq \left(Q \xrightarrow{!} \phi^*(Q) \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \phi^*(Q) \to 0\right) \in \mathcal{V}_{\phi^*}^n\,,$$

(where $\mu_e^1(M_\gamma^L)$ is a section of the schober (103)), which is the suspension of (78) (for $i = 2$). $\square$

Remark 5.71. Similar descriptions of derived equivalences in terms of rotations of a disc by fractions of π also appear for instance in [Qiu16], [OPS18, Thm. 5.1] (apply the Thm. 5.1 to the case of unpunctured n-gons), and [DJL21, Prop. 3.5.1]. The automorphism μ_e^{n-1} acts objectwise as the cotwist functor of the n-spherical object M_{γ_e}.

5.6.3 Categorification of the extended mutation matrix

While the derived categories of Ginzburg algebras can be used to construct generalized cluster categories, there are also further and more direct links between Ginzburg algebras and the combinatorics of cluster algebras. For example, the mutation matrix of a cluster algebra can be recovered via the Euler-characteristics of the Ext-complexes of the simple 3-spherical modules over the Ginzburg algebra associated to the vertices of the underlying quiver. This observation is made, formulated in the more general setting of cluster collections, in [KS08, Section 8.1]. As an application of the partial geometric model for relative Ginzburg algebras, we extend in this section the relation between the mutation matrices and Ginzburg algebras to extended mutation matrices and relative Ginzburg algebras of triangulated surfaces. The extended mutation matrix consists of the mutation matrix and the c-matrix, the latter encodes the coefficients of the cluster algebra.

We begin by recalling the definition of the class of cluster algebras with coefficients introduced in [FT18], associated to a fixed marked surface $\mathbf{S}$ (possibly with punctures) equipped with an ideal triangulation and a multi-lamination (see below). We denote the dual trivalent spanning graph of the ideal triangulation by $\mathcal{T}$.

Definition 5.72 ([FST08, Definition 4.1])**.** We arbitrarily label the internal edges of $\mathcal{T}$ by $e_1, \dots, e_m$ and define the quiver $Q_\mathcal{T}$ as follows.

- The vertices are the internal edges of $\mathcal{T}$.
- Let $e_i \neq e_j$ be two edges which are not loops. We add an arrow $a\colon e_i \to e_j$ for each vertex v of $\mathcal{T}$ incident to halfedges of e_i, e_j at which the halfedge of e_j precedes the halfedge of e_i in the cyclic (counterclockwise) order. The arrows of $Q_\mathcal{T}$ thus go in the clockwise direction.

- For each loop e_i, we add further arrows obtained as follows. Consider the unique edge e_j that such e_i and e_j are incident to the same vertex of $\mathcal{T}$, meaning that e_j is dual to the outer edge of the self-folded ideal triangle containing the dual of e_i. For $l \neq i, j$, we add an arrow $e_l \to e_i$ for each arrow $e_l \to e_j$ and an arrow $e_i \to e_l$ for each arrow $e_j \to e_l$.

The signed adjacency matrix of $\mathcal{T}$ is the skew-symmetric $m \times m$-matrix $B(\mathcal{T}) = (b_{i,j})$ where $b_{i,j}$ is the number of arrows from e_i to e_j minus the number of arrows from e_j to e_i in $Q_{\mathcal{T}}$.

Definition 5.73 ([FT18, Definition 12.1]). A lamination curve in $\mathbf{S}$ is a curve $\gamma\colon U \to \mathbf{S}\backslash M$ with $U = S^1, [0,1], [0,\infty), (-\infty,\infty)$ such that

- γ does not self-intersect.
- all endpoints of γ lie in $\partial\mathbf{S}\backslash M$.
- the curve does not bound any unpunctured disc, once-punctured disc or unpunctured 1-gon in $\mathbf{S}$.
- if U is not compact, then at the infinite ends the curve spirals around a puncture.
- if $U = (-\infty, \infty)$, then γ is not homotopic to a curve both of whose ends spiral around the same puncture p and which lies in a contractible neighborhood of p containing no further punctures.

Laminations curves are considered as equivalence classes under homotopies fixing endpoints. A lamination λ on $\mathbf{S}$ is a collection of pairwise non-intersecting lamination curves in $\mathbf{S}$. A multi-lamination $\Lambda = (\lambda_1, \ldots, \lambda_l)$ on $\mathbf{S}$ is a collection of $l \geq 1$ laminations on $\mathbf{S}$.

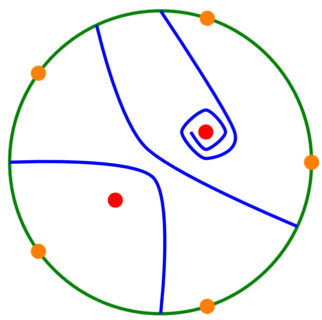


Figure 5: A lamination (in blue) with one spiraling curve in a surface with boundary (in green) with 7 marked points. Boundary marked points are in orange, punctures in red.

For more examples and counterexamples of laminations, see [FT18, Figures 32 and 33].

Definition 5.74. Denote the internal edges of $\mathcal{T}$ by $e_1, \dots, e_m$.

- Let γ_i be a lamination curve and e_j not a loop. We call a crossing of γ_i with e_j positive (or negative), if their local arrangement is as depicted on the left (or right) in Figure 6. We denote the signed count of such crossings of γ_i and e_j by (e_j, γ_i).

- Let γ_i be a lamination curve and e_j a loop. Let e_l be unique other edge incident to the same vertex of $\mathcal{T}$ as e_j. We define $(e_j, \gamma_i) \coloneqq (e_l, \tilde{\gamma}_i)$, where $\tilde{\gamma}_i$ is the lamination curve obtained by replacing each infinite end of γ_i spiraling around a puncture p by the infinite end spiraling around p in the opposite direction.

- The shear coordinates of a lamination λ with respect to $\mathcal{T}$ are given by the m-tuple $v_{\lambda,\mathcal{T}} \in \mathbb{Z}^n$ whose j-th entry is given by

$$(v_{\lambda,\mathcal{T}})_j = \sum_{\gamma \in \lambda} (e_j, \gamma)\,.$$

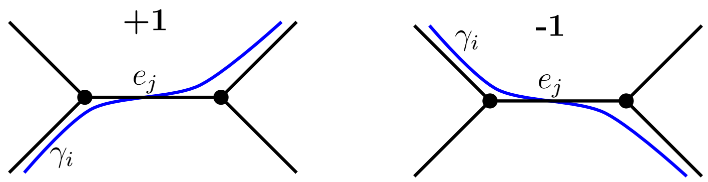


Figure 6: A crossing of a lamination curve γ_i (in blue) with an edge e_j of the triangulation contributes to the shear coordinates $+1$ if the crossing is as on the left and -1 if the crossing is as on the right.

Definition 5.75. Denote the internal edges of $\mathcal{T}$ by $e_1, \dots, e_m$. Let $\Lambda = (\lambda_1, \dots, \lambda_l)$ be a multi-lamination on $\mathbf{S}$. The extended mutation matrix $B(\mathcal{T}, \Lambda)$ is defined as the $m \times (m + l)$-matrix with

- the upper $m \times m$-submatrix is given by the signed adjacency matrix $B(\mathcal{T})$,

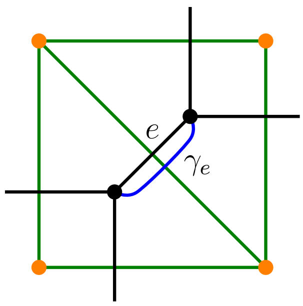
- the $(m+l)$-th row of $B(\mathcal{T}, \Lambda)$ is given by the shear coordiantes $v_{\lambda_i,\mathcal{T}}$ of λ_i with respect to $\mathcal{T}$, for $1 \leq i \leq l$.

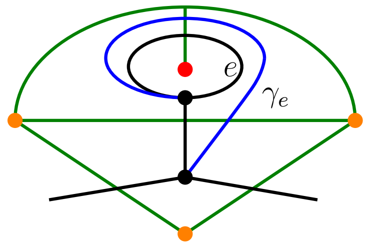
The c-matrix $C(\mathcal{T}, \Lambda)$ is the $m \times l$ submatrix of $B(\mathcal{T}, \Lambda)$ consisting of the columns $m+1, \dots, m+l$.

Remark 5.76. In the case Λ consists of the boundary arcs of $\mathbf{S}$, each considered as a lamination, then the extended mutation matrix $B(\mathcal{T}, \Lambda)$ coincides with the extended mutation matrix from Definition 6.43 (defined under the assumption that $\mathbf{S}$ has no punctures).

To describe the extended mutation matrix $B(\mathcal{T}, \Lambda)$ categorically, we regard the lamination curves as pure matching curve and consider the associated finite $\mathscr{G}_{\mathcal{T}}$-modules.

Notation 5.77. Let e be an internal edge of the ribbon graph $\mathcal{T}$. If e is not a loop, we let γ_e denote the pure matching curve which traces along e. If e is a loop, we let γ_e denote the pure matching curve which traces along e (in any direction, e.g. clockwise) and then traces along the other edge incident to the same vertex as e. We depict γ_e in the these two cases as follows.





Notation 5.78. Let λ be a lamination of $\mathbf{S}$. Using Lemma 5.37, we can consider each lamination curve as a pure matching curve. We denote by $M_\lambda = \bigoplus_{\gamma \in \lambda} M_\gamma \in \mathcal{D}(\mathscr{G}_{\mathcal{T}})$ the direct sum of the objects associated to the lamination curves (recall that $M_\gamma := M_\gamma^{\phi^*(k)}$). For e an internal edge of $\mathcal{T}$, we denote by $M_{\gamma_e} \in \mathcal{D}(\mathscr{G}_{\mathcal{T}})$ the object associated to the pure matching curve γ_e.

Definition 5.79. Denote the internal edges of $\mathcal{T}$ by $e_1, \ldots, e_m$. Let $\Lambda = (\lambda_1, \ldots, \lambda_l)$ be a multi-lamination on $\mathbf{S}$. The categorical extended mutation matrix $\hat{B}(\mathcal{T}, \Lambda) = (\hat{b}_{i,j})$ is defined as the $m \times (m+l)$-matrix with

- $\hat{b}_{i,j} = \chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_{\gamma_{e_j}})$ for $1 \leq i, j \leq m$ and
- $\hat{b}_{i,m+j} = \frac{1}{2} \chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_{\lambda_j})$ for $1 \leq i \leq m$ and $1 \leq j \leq l$,

where χ denotes the Euler-characteristic, see Section 2.1.2. The categorical c-matrix $\hat{C}(\mathcal{T}, \Lambda)$ is the $m \times l$ submatrix of $\hat{B}(\mathcal{T}, \Lambda)$ consisting of the columns $m+1, \ldots, m+l$.

Remark 5.80. We will see below that in the setting of Definition 5.79

$$\chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_{\gamma_{e_j}}) = \dim_k \operatorname{Ext}^2(M_{\gamma_{e_i}}, M_{\gamma_{e_j}}) - \dim_k \operatorname{Ext}^1(M_{\gamma_{e_i}}, M_{\gamma_{e_j}})$$

and

$$\frac{1}{2} \chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_{\lambda_j}) = \dim_k \operatorname{Ext}^2(M_{\gamma_{e_i}}, M_{\lambda_j}) - \dim_k \operatorname{Ext}^1(M_{\gamma_{e_i}}, M_{\lambda_j}) \,.$$

Theorem 5.81. *Let Λ be a multi-lamination of $\mathbf{S}$. The extended mutation matrices $\hat{B}(\mathcal{T}, \Lambda)$ and $B(\mathcal{T}, \Lambda)$ are identical.*

Proof of Theorem 5.81. We begin by showing that the upper $m \times m$-submatrix of $\hat{B}(\mathcal{T}, \Lambda)$ agrees with the signed adjacency matrix $B(\mathcal{T})$. Let e_i, e_j be two edges of $\mathcal{T}$. If $e_i = e_j$, it is obvious that $\hat{b}_{i,i} = 0 = b_{i,i}$. We can thus assume that $e_i \neq e_j$. Assume that e_i, e_j are not a loop and a non-loop incident to the same vertex (i.e. dual to the two edges of a self-folded ideal triangle). Theorem 5.53 implies that $\dim_k \operatorname{Ext}^2(M_{\gamma_{e_i}}, M_{\gamma_{e_j}})$ counts the number of singular intersections where γ_{e_j} follows γ_{e_i} in the clockwise order. This number is equal to the number of arrows from e_i to e_j in $Q_{\mathcal{T}}$. Similarly, $\dim_k \operatorname{Ext}^1(M_{\gamma_{e_i}}, M_{\gamma_{e_j}})$ is equal to the number of arrows from e_j to e_i in $Q_{\mathcal{T}}$. All other Ext-groups vanish and it follows that $\hat{b}_{i,j} = \chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_{\gamma_{e_j}}) = b_{i,j}$. In the case that e_i, e_j are a loop and a non-loop incident to the same vertex, with e_i the loop, we have $b_{i,j} = b_{j,i} = 0$ and $\operatorname{Ext}^*(\gamma_{e_i}, \gamma_{e_j}) = k \oplus k[-1]$ and $\operatorname{Ext}^*(\gamma_{e_j}, \gamma_{e_i}) = k[-2] \oplus k[-3]$ so that also $\hat{b}_{i,j} = \hat{b}_{j,i} = 0$.

We continue by showing that the c-matrices are identical. Using the additivity of Ext, it suffices to verify that for each lamination curve γ and each edge e_i there exists an equality

$$\frac{1}{2}\chi \operatorname{Ext}^*(M_{e_i}, M_\gamma) = (e_i, \gamma)\,. \tag{107}$$

We begin with the case that e_i is not a loop. By Theorem 5.53, we have that $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma)$ is the direct sum of contributions arising from crossings of e_i and γ. If a crossing of γ and e_i is as on the left in Figure 6, then $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) \simeq k \oplus k[-2]$ and $\frac{1}{2}\chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) = 1$ and the intersection thus contributes the same amount to both sides of (107). Similarly, if the crossing of γ and γ_{e_i} is as on the right in Figure 6, then $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) = k[-1] \oplus k[-3]$ and the intersection also contributes with -1 to both sides of (107).

Consider now the case that e_i is a loop and let e_i' be the unique other edge of $\mathcal{T}$ incident to e_i. If γ does not have an infinite end spiraling around the puncture at which e_i lies, then both sides of (107) vanish. We thus assume that such a spiraling infinite end exists. Consider the vertex v incident to e_i' at which e_i does not lie and consider the two edges $e_1 \neq e_2$ incident to v such their cyclic (counterclockwise) order is given by e_i', e_1, e_2, e_i'. There are four possible arrangements: the end of γ_i either arrives at e_i' first passing along e_1 or e_2 and the infinite end either spirals clockwise or counterclockwise. In the clockwise case, one finds $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) \simeq k[-1] \oplus k[-3]$ if γ passes along e_1 and $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) \simeq k \oplus k[-1] \oplus k[-2] \oplus k[-3]$ if γ passes along e_2 (in this case, there are two crossings). In the counterclockwise case, one finds $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) \simeq 0$ if γ passes along e_1 and $\operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) \simeq k \oplus k[-2]$ if γ passes along e_2. In each case, we thus find as desired

$$\frac{1}{2}\chi \operatorname{Ext}^*(M_{\gamma_{e_i}}, M_\gamma) = \frac{1}{2}\chi \operatorname{Ext}^*(M_{\gamma_{e_i'}}, M_{\tilde{\gamma}}) = (e_i', \tilde{\gamma}) = (e_i, \gamma)\,,$$

where $\tilde{\gamma}$ is as in Definition 5.74. We have shown that the c-matrices are also identical, concluding the proof. $\square$

6 Cluster categories of unpunctured surfaces

We fix a field k and an unpunctured marked surface $\mathbf{S}$ with a trivalent spanning graph $\mathcal{T}$. Construction 5.14 yields a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}(k)$, in the following denoted $\mathcal{F}_{\mathcal{T}}$, whose ∞-category of global sections $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$ is equivalent to the unbounded derived ∞-category of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ associated with $\mathcal{T}$. The goal of this section is to describe the generalized cluster category

$$\mathcal{C}_{\mathbf{S}} \coloneqq \operatorname{Ind}\left(\mathcal{D}^{\mathrm{perf}}(\mathscr{G}_{\mathcal{T}})/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\right)$$

associated to $\mathscr{G}_{\mathcal{T}}$ and show that it is an additive categorification of a cluster algebra with coefficients associated with $\mathbf{S}$. This splits as follows. Section 6.1 describes the generalized cluster category $\mathcal{C}_{\mathbf{S}}$ abstractly, as a 1-periodic version of the topological Fukaya category of $\mathbf{S}$. Section 6.2 proceeds with a detailed description of $\mathcal{C}_{\mathbf{S}}$, including a classification of all indecomposable, compact objects in terms of pure matching curves in $\mathbf{S}$. Section 6.3 introduces the relevant cluster and skein algebras and describes their categorification in terms of $\mathcal{C}_{\mathbf{S}}$. The relation of $\mathcal{C}_{\mathbf{S}}$ with the 'usual' triangulated 2-Calabi–Yau cluster category of $\mathbf{S}$, as well as variants of the results of this section for n-angulated surfaces are discussed in Section 6.4.

Since in this section, we are working exclusively linearly over a field k, we will write $\mathcal{D}(A)$ for the derived ∞-category of a k-linear dg-algebra A, instead of RMod_A, see also Proposition 2.25. We write $\mathcal{D}^{\mathrm{perf}}(A) \subset \mathcal{D}(A)$ for the full subcategory consisting of compact objects and $\mathcal{D}^{\mathrm{fin}}(A) \subset \mathcal{D}(A)$ for the full subcategory consisting of modules with finite dimensional total homology over k.

6.1 Generalized cluster categories and perverse schobers

We begin in Section 6.1.1 by showing that the Verdier quotient $\mathcal{D}^{\mathrm{perf}}(k[t_1])/\mathcal{D}^{\mathrm{fin}}(k[t_1])$ is equivalent to the derived ∞-category $\mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])$ of the graded Laurent algebra $k[t_1^{\pm}]$, and relating the non-vanishing of this quotient to the failure of the monadicity of the functor $\phi_*\colon \mathcal{D}(k[t_1]) \to \mathcal{D}(k)$, where $\phi\colon k[t_1] \xrightarrow{t_1 \mapsto 0} k$. This provides a 'local' computation, to be applied in Section 6.1.2, where we show that $\mathcal{F}_{\mathcal{T}}$ splits into a semiorthogonal decomposition of perverse schobers. In Section 6.1.3, we show that the global sections of one component of this semiorthogonal decomposition describes the 1-periodic topological Fukaya category of $\mathbf{S}$. In the final Section 6.1.4, we show that the 1-periodic topological Fukaya category describes the generalized cluster category $\mathcal{C}_{\mathbf{S}}$.

6.1.1 Graded Laurent algebras and monadicity

For background on monadic adjunctions, see Section 2.1.5. We fix an integer $n \geq 3$. We are interested in the monadic adjunction arising from the spherical adjunction

$$f^*\colon \mathcal{D}(k) \longleftrightarrow \operatorname{Fun}(S^{n-1},\mathcal{D}(k)) : f_*$$

from Section 5.2. In the case $n = 3$, this adjunction describes the singularities of the perverse schober $\mathcal{F}_{\mathcal{T}}$ from Construction 5.14. Let $M = f_*f^*$ be the adjunction monad and denote by $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}} := \mathrm{LMod}_M(\mathcal{D}(k))$ the stable, presentable ∞-category of left modules over M, also called the Eilenberg-Moore ∞-category of the monad M. We have an associated fully faithful functor $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}} \to \mathrm{Fun}(S^{n-1}, \mathcal{D}(k))$, which we denote by i_{mnd}. In the following, we identify $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}}$ with its essential image under i_{mnd}.

Lemma 6.1. *The object $f^*(k) \in \mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}}$ is a compact generator.*

Proof. The restriction of f_* to $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}}$ is by definition monadic. Its left adjoint $(f^*)^{\mathrm{mnd}}$ is obtained from f^* by restricting the target. The k-linear functor $(f^*)^{\mathrm{mnd}}$ is fully determined by the image of k, which is $f^*(k)$. It is thus equivalent to the functor $(\text{-}) \otimes f^*(k)$ (defined using the k-linear structure). The adjunction

$$(\text{-}) \otimes f^*(k) \dashv \mathrm{Mor}_{\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}}}(f^*(k), \text{-}) \,,$$

and the facts that the right adjoint $\mathrm{Mor}_{\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}}}(f^*(k), \text{-}) \simeq (f_*)^{\mathrm{mnd}}$ is conservative, because monadic, and that $k \in \mathcal{D}(k)$ is compact generator, now imply that $f^*(k)$ compactly generates $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}}$. □

Recall from Proposition 5.11, that there exists an equivalence of k-linear ∞-categories

$$\mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) \simeq \mathcal{D}(k[t_{n-2}]) \,.$$

Lemma 6.2. *There exist equivalences of k-linear ∞-categories*

$$\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}} \simeq \mathrm{Ind}\,\mathrm{Fun}(S^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}]) \,,$$

compatible with their inclusions into $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))$.

Proof. Under the equivalence $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) \simeq \mathcal{D}(k[t_{n-2}])$, the forgetful functor

$$\mathrm{Mor}_{\mathcal{D}(k[t_{n-2}])}(k[t_{n-2}], \text{-})\colon \mathcal{D}(k[t_{n-2}]) \to \mathcal{D}(k)$$

corresponds to the evaluation functor $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) \to \mathcal{D}(k)$ at any point $x \in S^{n-1}$. This implies that $\mathrm{Fun}(S^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$, which yields

$$\mathrm{Ind}\,\mathrm{Fun}(S^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$$

by passing to Ind-completions.

We proceed by showing the equivalence $\mathrm{Fun}(S^{n-1}, \mathcal{D}(k))^{\mathrm{mnd}} \simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. Consider the trivial $k[t_{n-2}]$-module with homology k (corresponding to $f^*(k)$ under the equivalence $\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}]) \simeq \mathrm{Fun}(S^{n-1}, \mathcal{D}(k)^{\mathrm{perf}})$). Let $X \in \mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. We show by induction on the dimension of the homology of X, that X lies in the stable closure $\langle k \rangle \subset \mathcal{D}(k[t_{n-2}])$ of k, i.e. the smallest stable subcategory containing k. The induction beginning is the case $X \simeq 0$ and thus clear. For the induction step, suppose that $m \in \mathbb{Z}$ is the maximal degree in which the homology $\mathrm{H}_* \mathrm{Mor}_{\mathcal{D}(k[t_{n-2}])}(k[t_{n-2}], X)$ of X is nontrivial. We find a corresponding non-zero morphism $\alpha\colon k[t_{n-2}][m] \to X$, such that the composite with $k[t_{n-2}][m+n-2] \to$

$k[t_{n-2}][m]$ is zero. The morphism α thus induces a non-zero morphism $\alpha'\colon k \to X$, which is injective on homology. Taking the fiber of α', we obtain a module $\mathrm{fib}(\alpha')$ whose homology has one dimension less. We have by the induction assumption, that $\mathrm{fib}(\alpha') \in \langle k \rangle$. From $X \simeq \mathrm{cof}(\mathrm{fib}(\alpha') \to k)$, it follows that X also lies in $\langle k \rangle$. Since any object in $\langle k \rangle$ also lies in $\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$, we obtain $\langle k \rangle = \mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. This ∞-category is stable and also idempotent-complete, as having finite dimensional homology is a condition preserved under retracts. Consider the k-linear endomorphism dg-algebras $\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k) \simeq \mathrm{End}_{\mathrm{Fun}(S^{n-1},\mathcal{D}(k))}(f^*(k)) \simeq k \oplus k[1-n]$. Since $f^*(k)$ is by Lemma 6.1 a compact generator of $\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}}$, we have $\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}} \simeq \mathcal{D}(\mathrm{End}_{\mathrm{Fun}(S^{n-1},\mathcal{D}(k))}(f^*(k))) \simeq \mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k))$, see [Lur17, 7.1.2.1]. The perfect derived ∞-category $\mathcal{D}^{\mathrm{perf}}(\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k))$ (consisting of compact objects) is by [Lur17, 7.2.4.1, 7.2.4.4] the smallest stable subcategory of $\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}} \simeq \mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k))$ containing k which is closed under retracts and thus equivalent to $\langle k \rangle = \mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. It follows that

$$\begin{aligned}\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}} &\simeq \mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k))\\ &\simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{perf}}(\mathrm{End}_{\mathcal{D}(k[t_{n-2}])}(k))\\ &\simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])\,,\end{aligned}$$

concluding the proof. □

We denote by $k[t_{n-2}^{\pm}]$ the dg-algebra of graded Laurent polynomials with generator in degree $|t_{n-2}| = n-2$. Note that if n is even, then $k[t_{n-2}]$ and $k[t_{n-2}^{\pm}]$ are graded commutative dg-algebras, whereas if n is odd, then $k[t_{n-2}]$ and $k[t_{n-2}^{\pm}]$ are not graded commutative, because $t_{n-2}^2 \neq 0$. By Lemma 2.37, the following Lemma shows that $\mathcal{D}(k[t_{n-2}^{\pm}])$ arises as the Verdier quotient of $\mathcal{D}(k[t_{n-2}])$ by $\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. By Lemma 6.2, this Verdier quotient is also equivalent to the Verdier quotient of $\mathrm{Fun}(S^{n-1},\mathcal{D}(k))$ by $\mathrm{Fun}(S^{n-1},\mathcal{D}(k))^{\mathrm{mnd}}$.

Lemma 6.3. *Let $n \geq 1$. The ∞-category $\mathcal{D}(k[t_{n-2}])$ admits a semiorthogonal decomposition $(\mathcal{D}(k[t_{n-2}^{\pm}]), \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}]))$.*

Proof. For the proof, we show that $\mathcal{D}(k[t_{n-2}^{\pm}]) = \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])^{\perp}$ is the stable subcategory of $\mathcal{D}(k[t_{n-2}])$ consisting of objects b, such that $\mathrm{Mor}_{\mathcal{D}(k[t_{n-2}])}(a,b) \simeq 0$ for all $a \in \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])$. Using that the inclusion $\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}]) \subset \mathcal{D}(k[t_{n-2}])$ admits a right adjoint, the Lemma then follows from [DKSS21, Prop. 2.2.4, Prop. 2.3.2].

The objects of $\mathcal{D}(k[t_{n-2}])$ can be identified with dg-modules over the dg-algebra $k[t_{n-2}]$. Under this identification, we observe that $\mathcal{D}(k[t_{n-2}^{\pm}]) \subset \mathcal{D}(k[t_{n-2}])$ is the full subcategory consisting of $k[t_{n-2}]$-modules $M_\bullet$, satisfying that $t_{n-2}\colon M_i \to M_{i+n-2}$ is an isomorphism of k-vector spaces for all $i \in \mathbb{Z}$. We have

$$\mathrm{Mor}_{\mathcal{D}(k[t_{n-2}])}(k[t_{n-2}][i], M_\bullet) \simeq M_{\bullet+i}\,.$$

Using that $k \simeq \mathrm{cof}(k[t_{n-2}][n-2] \to k[t_{n-2}])$, we thus have

$$\mathrm{Mor}_{\mathcal{D}(k[t_{n-2}])}(k[i], M_\bullet) \simeq \mathrm{cof}(t_{n-2}\colon M_{\bullet+i} \to M_{\bullet+i+n-2}) \in \mathcal{D}(k)\,.$$

We thus find that $t_{n-2}\colon M_i \to M_{i+n-2}$ is an isomorphism for all $i \in \mathbb{Z}$ if and only if $M_\bullet \in \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])^\perp$. This shows the desired equality

$$\mathcal{D}(k[t_{n-2}^{\pm}]) = \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-2}])^\perp\,,$$

concluding the proof. □

Lemma 6.4. *There exists an equivalence of* 1*-categories* $\mathrm{Vect}_k \simeq \operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$*, where* Vect_k *denotes the* 1*-category of* k*-vector spaces and* $\operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$ *denotes the homotopy* 1*-category of* $\mathcal{D}(k[t_1^{\pm}])$*.*

Proof. Since any complex of k-vector spaces is quasi-isomorphic to its homology, any 1-periodic complex with values in k is equivalent to an object in the image of $N(\mathrm{Vect}_k) \hookrightarrow \mathcal{D}(k) \xrightarrow{\text{-}\otimes k[t_1^{\pm}]} \mathcal{D}(k[t_1^{\pm}])$. This functor is thus essentially surjective, and using that $\pi_0 \operatorname{Map}_{\mathcal{D}(k[t_1^{\pm}])}(k[t_1^{\pm}], k[t_1^{\pm}]) \simeq \mathrm{H}_0(k[t_1^{\pm}]) \simeq k$, one sees that this functor is also fully faithful on the level of homotopy 1-categories. □

In the triangulated homotopy 1-category $\operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$, distinguished triangles take the form

$$\ker(\alpha) \oplus \operatorname{coker}(\alpha) \xrightarrow{(\iota,0)} k[t_1^{\pm}]^{\oplus I} \xrightarrow{\alpha} k[t_1^{\pm}]^{\oplus J} \xrightarrow{(0,\pi)} \ker(\alpha) \oplus \operatorname{coker}(\alpha)$$

with ι being the kernel map and π being the cokernel map and I, J two sets.

6.1.2 A semiorthogonal decomposition of perverse schobers

The $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$ is locally at each vertex of $\mathcal{T}$ encoded by the spherical adjunction

$$\phi^*\colon \mathcal{D}(k) \longleftrightarrow \mathcal{D}(k[t_1]) : \phi^*\,,$$

with $\phi\colon k[t_1] \xrightarrow{t_1 \mapsto 0} k$. The functor ϕ^* is conservative and the adjunction $\phi^* \dashv \phi_*$ thus comonadic. The adjunction $\phi^* \dashv \phi_*$ is however not monadic. The Eilenberg-Moore ∞-category of the monad $\phi_*\phi^*$ can be identified with the full subcategory $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \subset \mathcal{D}(k[t_1]))$, see Lemma 6.2. Using this observation, we can obtain a vanishing-monadic and nearby-monadic $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$, see Definition 3.52, by restricting in the construction of $\mathcal{F}_{\mathcal{T}}$ in Construction 5.14 the spherical adjunction $\phi^* \dashv \phi^*$ to the spherical monadic and comonadic adjunction

$$(\phi^*)^{\text{Ind-fin}}\colon \mathcal{D}(k) \longleftrightarrow \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) : \phi_*^{\text{Ind-fin}}\,.$$

We remark that to give this definition of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$, we also use that in the construction of $\mathcal{F}_{\mathcal{T}}$ all appearing autoequivalences of $\mathcal{D}(k[t_1])$ preserve the full subcategory $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \subset \mathcal{D}(k[t_1])$.

By construction, there is an inclusion of perverse schobers $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$.

Lemma 6.5. *The inclusion* $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ *is right admissible.*

Proof. By Remark 3.49, we need to show that for $1 \leq i \leq 3$, the diagram

$$\begin{tikzcd} \mathcal{V}^3_{(\phi^*)^{\text{Ind-fin}}} \arrow[r, hook] \arrow[d, "\varrho_i"'] & \mathcal{V}^3_{\phi^*} \arrow[d, "\varrho_i"] \\ \operatorname{Ind}\mathcal{D}^{\text{fin}}(k[t_1]) \arrow[r, hook] & \mathcal{D}(k[t_1]) \end{tikzcd} \tag{108}$$

is right adjointable. We only consider the case $i = 1$, the other cases can be treated analogously or also follow by Proposition 3.32 from the $i = 1$ case.

The right adjoint of the inclusion functor $i : \mathcal{V}^3_{(\phi^*)^{\text{Ind-fin}}} \hookrightarrow \mathcal{V}^3_{\phi^*}$ can be determined from the following diagram in $\mathcal{P}r^R_{\text{St}}$ containing two pullback squares:

$$\begin{tikzcd} \mathcal{V}^3_{\phi^*} \arrow[rr, two heads, "\text{radj}(i)"] \arrow[dr] \arrow[dd] & & \mathcal{V}^3_{(\phi^*)^{\text{Ind-fin}}} \arrow[dr] \arrow[dd] & \\ & \operatorname{Fun}(\Delta^2, \mathcal{D}(k[t_1])) \arrow[rr, two heads, "\operatorname{Fun}(\Delta^2,\pi_{\text{mnd}})"] \arrow[dd, "\text{ev}_0"] \arrow[dl, phantom, "\lrcorner"] & & \operatorname{Fun}(\Delta^2, \operatorname{Ind}\mathcal{D}^{\text{fin}}(k[t_1])) \arrow[dd, "\text{ev}_0"] \arrow[dl, phantom, "\lrcorner"] \\ \mathcal{D}(k) \arrow[rr, "\text{id}"] \arrow[dr, "\phi^*"'] & & \mathcal{D}(k) \arrow[dr, "\phi^*"] & \\ & \mathcal{D}(k[t_1]) \arrow[rr, two heads, "\pi_{\text{mnd}}"] & & \operatorname{Ind}\mathcal{D}^{\text{fin}}(k[t_1]) \end{tikzcd}$$

The functor π_{mnd} is the right adjoint of the inclusion. Recall that the functor ϱ_1 is defined in Section 3.2.1 by evaluating at the vertex $2 \in \Delta^2$. The above diagram shows that evaluating $\text{radj}(i)$ at 2 gives π_{mnd}. It follows that the diagram (108) is right adjointable. □

Definition 6.6. We define the functor $\mathcal{F}^{\text{clst}}_{\mathcal{T}} \colon \operatorname{Exit}(\mathcal{T}) \to \text{St}$ as the cofiber of the inclusion $\mathcal{F}^{\text{mnd}}_{\mathcal{T}} \hookrightarrow \mathcal{F}_{\mathcal{T}}$ in $\operatorname{Fun}(\operatorname{Exit}(\mathcal{T}), \text{St})$.

Lemma 6.7. *The functor $\mathcal{F}^{\text{clst}}_{\mathcal{T}}$ is a $\mathcal{T}$-parametrized perverse schober. At each vertex of $\mathcal{T}$, it is described by the trivial spherical adjunction*

$$0 \leftrightarrow \mathcal{D}(k[t_1^{\pm}])\,.$$

Proof. Using Lemma 6.3, this is straightforward to check by using that pushouts in the functor category $\operatorname{Fun}(\operatorname{Exit}(\mathcal{T}), \text{St})$ are computed pointwise in $\operatorname{Exit}(\mathcal{T})$. □

Remark 6.8. The abbreviation clst stands for 'cluster'. It also forms an anagram with lcst, which stands for 'locally constant'. Note that the perverse schober $\mathcal{F}^{\text{clst}}_{\mathcal{T}}$ is locally constant, see Definition 3.52, i.e. has no singularities.

Proposition 6.9. *The pair $\{\mathcal{F}^{\text{clst}}_{\mathcal{T}}, \mathcal{F}^{\text{mnd}}_{\mathcal{T}}\}$ forms a semiorthogonal decomposition of the $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$.*

Proof. The inclusion $\alpha \colon \mathcal{F}^{\text{mnd}}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}$ is by Lemma 6.5 right admissible. A similar argument as in the proof Lemma 6.5 further shows that the cofiber morphism $\mathcal{F}_{\mathcal{T}} \to \mathcal{F}^{\text{clst}}_{\mathcal{T}}$ arising from the definition of $\mathcal{F}^{\text{clst}}_{\mathcal{T}}$ is the pointwise left adjoint of a left admissible inclusion $\mathcal{F}^{\text{clst}}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}$. This shows that $\{\mathcal{F}^{\text{clst}}_{\mathcal{T}}, \mathcal{F}^{\text{mnd}}_{\mathcal{T}}\}$ indeed forms a semiorthogonal decomposition of $\mathcal{F}_{\mathcal{T}}$. □

6.1.3 The 1-periodic topological Fukaya category

The perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is locally constant, i.e. has no singularities, and its generic stalk is the derived ∞-category $\mathcal{D}(k[t_1^{\pm}])$ of 1-periodic chain complexes. It is thus locally at its vertices described by the 1-periodic derived category of the A_2-quiver. We hence refer to its ∞-category of global sections $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ as the topological Fukaya category of $\mathbf{S}$ with values in the derived ∞-category of 1-periodic chain complexes, or the 1-periodic topological Fukaya category for short.

Proposition 6.10. *The monodromy of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ along any loop in $\mathbf{S}\backslash M$ is trivial.*

Proof. Any loop γ in $\mathbf{S}\backslash M$ is by Lemma 5.37 homotopic to a pure matching curve in $\mathbf{S}\backslash M$ (where the homotopies are allowed to cross the vertices of $\mathcal{T}$). Since $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ has no singularities, we may assume by Lemma 3.57 that γ is a closed pure matching curve, without changing the monodromy.

The monodromy of $\mathcal{F}_{\mathcal{T}}$ along γ is computed in the proof Theorem 5.54 and shown to be trivial. The transport equivalences, and hence also the monodromy equivalences, fix the subcategories $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \subset \mathcal{F}_{\mathcal{T}}(e)$ for all edges e. It follows that the monodromy of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ along γ is also trivial. $\square$

Proposition 6.10 justifies calling $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ *the* 1-periodic topological Fukaya category of $\mathbf{S}$. Indeed, combining Proposition 6.10 with Proposition 3.63 as well as Lemma 3.62, we can further deduce that $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ does not depend on the choice of trivalent spanning graph $\mathcal{T}$, up to equivalence. We thus write

$$\mathcal{C}_{\mathbf{S}} \coloneqq \mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})\,.$$

This ∞-category can further be considered an ∞-categorical (and Ind-complete) avatar of the topological Fukaya (dg-)category in the sense of [DK18] with coefficients in the cyclic 2-Segal object arising from the Waldhausen $S_\bullet$-construction of the 2-periodic dg-category $\mathrm{dgMod}_{k[t_1^{\pm}]}$ of dg $k[t_1^{\pm}]$-modules. In particular, Corollary 3.4.7 of [DK18] shows the following.

Theorem 6.11. *The k-linear ∞-category $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ is acted upon by automorphisms in $\mathrm{ho}\,\mathrm{LinCat}_k$ by the mapping class group of the marked surface $(\mathbf{S},M)$ of isotopy classes of orientation preserving diffeomorphisms $\mathbf{S}\to\mathbf{S}$ restricting to the identity on $\partial\mathbf{S}$.*

Finally, we also record a description of the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}}$ in terms of a gentle algebra.

Proposition 6.12. *There exists a(n ungraded) gentle algebra gtl and an equivalence of ∞-categories $\mathcal{C}_{\mathbf{S}} \simeq \mathcal{D}(\mathrm{gtl})\otimes_k \mathcal{D}(k[t_1^{\pm}])$.*

Proof. Choose a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. As observed in [HKK17], the ∞-category of global sections of a locally constant perverse schober with generic stalk $\mathcal{D}(k)$ on $\mathbf{S}$ parametrized by $\mathcal{T}$ admits a formal generator whose endomorphism algebra is a finite dimensional, and in general graded, gentle algebra, denoted gtl'. Using that tensoring with $\mathcal{D}(k[t_1^{\pm}])$ (with respect to the symmetric monoidal structure of LinCat_k) preserves colimits in LinCat_k, we find an equivalence $\mathcal{C}_{\mathbf{S}} \simeq \mathcal{D}(\mathrm{gtl}') \otimes_k \mathcal{D}(k[t_1^{\pm}])$ in LinCat_k. Let gtl be the ungraded gentle algebra obtained from gtl' by discarding the grading. It is easy to see that $\mathcal{D}(\mathrm{gtl}')\otimes_k \mathcal{D}(k[t_1^{\pm}]) \simeq \mathcal{D}(\mathrm{gtl}) \otimes_k \mathcal{D}(k[t_1^{\pm}])$ in LinCat_k, showing the claim. □

6.1.4 The generalized cluster category of a triangulated surface

Let $\mathcal{D}$ be a smooth k-linear ∞-category with a compact generator $X \in \mathcal{D}$. Then $\mathcal{D} \simeq \mathcal{D}(\mathrm{End}(X))$ is equivalent to the derived ∞-category of the derived endomorphism dg-algebra $\mathrm{End}(X)$. It follows from [Kel08, Lemma 4.1] that the derived ∞-category $\mathcal{D}^{\mathrm{fin}}(\mathrm{End}(X))$ of modules with finite dimensional total homology over k is a subcategory of the perfect derived ∞-category $\mathcal{D}^{\mathrm{perf}}(\mathrm{End}(X))$. Note that $\mathcal{D}^{\mathrm{fin}}(\mathrm{End}(X)) \simeq \mathcal{D}^{\mathrm{fin}}$, using the notation of Definition 4.14, and $\mathcal{D}^{\mathrm{perf}}(\mathrm{End}(X)) \simeq \mathcal{D}^{c}$. Passing to Ind-completions yields $\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}} \subset \mathcal{D}$.

Definition 6.13. The generalized cluster category of $\mathcal{D}$ is defined as the Ind-complete Verdier quotient

$$\mathcal{D}/\,\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}\,,$$

i.e. as the cofiber in $\mathcal{P}r^L_{\mathrm{St}}$.

Remark 6.14. Let $\mathcal{D}$ be as in Definition 6.13. Then $\mathcal{D}$ admits a semiorthogonal decomposition $(\mathcal{D}/\,\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}, \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}})$ into its generalized cluster category and its Ind-finite part, see Lemma 2.37.

Theorem 6.15. *The generalized cluster category of $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ is equivalent to the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$.*

To prove Theorem 6.15, we show that the ∞-category of global sections of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ is equivalent to $\mathrm{Ind}\,\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})^{\mathrm{fin}}$ and then make use of the semiorthogonal decomposition $\{\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}\}$ of $\mathcal{F}_{\mathcal{T}}$.

Definition 6.16. We denote by $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \subset \mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$ the full subcategory of global sections $X \in \mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$ satisfying that $\mathrm{ev}_e(X) \in \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_1])$ for all edges e of $\mathcal{T}$.

Lemma 6.17. *The ∞-category $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$ is compactly generated by the object $\bigoplus_{e\in\mathcal{T}_1} M_{c_e}^{\phi^*(k)}$ associated to finite matching data $(c_e,\phi^*(k))$, with c_e as defined in Section 5.4.3.*

Proof. Using that $\phi^*(k)$ is a compact generator of $\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_1])$, it follows from Proposition 5.49 that the objects $M_{c_e}^{\phi^*(k)}$ associated with the collection of matching data $(c_e,\phi^*(k))_{e\in\mathcal{T}_1}$ compactly generate $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. We remark that for the fact that the matching curves giving rise to this compact generator are finite, it is crucial that the marked surface has a marked point on each boundary component. □

Proposition 6.18. *There exist equivalences of ∞-categories*

$$\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \simeq \mathrm{Ind}\,\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}\,.$$

Proof. It follows from the definition of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$, that the image on global sections of the inclusion $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ consists of global sections which evaluate on each edge of $\mathcal{T}$ to an object in $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \subset \mathcal{D}(k[t_1])$. We thus see that $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$.

We proceed with showing the equivalence $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \simeq \operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}$. We can realize both of these ∞-categories as presentable, stable subcategories of $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$. An object of $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ is finite if and only if its values at the edges of $\mathcal{T}$ lie in $\mathcal{D}^{\mathrm{fin}}(k[t_1]) \subset \mathcal{D}(k[t_1])$. It follows that $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. Using that the objects in $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\mathrm{fin}}$ are compact in $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$, we further get $\operatorname{Ind}\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. Lemma 6.17, in combination with the fact that the global sections associated to finite, pure matching data with finite local value lie by construction in $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$, now implies that both $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$ and $\operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}$ are compactly generated by the same set of objects and thus equivalent. □

Proof of Theorem 6.15. Combine Proposition 6.9, part 2 of Remark 3.51 and Proposition 6.18. □

The dg-algebra $k[t_2^{\pm}]$ is commutative and the ∞-category $\mathcal{D}(k[t_1^{\pm}])$ clearly comes with a $k[t_2^{\pm}]$-linear structure. This can be used to construct a factorization of the perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ through $\mathrm{LinCat}_{k[t_2^{\pm}]} \to \mathrm{St}$. In particular, its limit $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ inherits a $k[t_2^{\pm}]$-linear structure

Proposition 6.19. *The $k[t_2^{\pm}]$-linear ∞-category $\mathcal{C}_{\mathbf{S}}$ is smooth and proper.*

Proof. Using the duality $\mathcal{P}r^L_{\mathrm{St}} \simeq (\mathcal{P}r^R_{\mathrm{St}})^{\mathrm{op}}$, we find that $\mathcal{C}_{\mathbf{S}}$ is equivalent as an ∞-category to the colimit of the right adjoint diagram of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ in $\mathcal{P}r^L_{\mathrm{St}}$. Since the forgetful functor $\mathrm{LinCat}_{k[t_2^{\pm}]} \to \mathcal{P}r^L_{\mathrm{St}}$ preserves colimits, we see that that $\mathcal{C}_{\mathbf{S}}$ is furthermore equivalent as a $k[t_2^{\pm}]$-linear ∞-category to the colimit in LinCat_k The smoothness thus follows from the fact that finite colimits of smooth ∞-categories along compact objects preserving functors are again smooth, see Corollary 4.31.

To see that $\mathcal{C}_{\mathbf{S}}$ is proper, consider its compact generator $\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}^*(k[t_1^{\pm}])$, see Proposition 3.39. Proposition 5.49 shows that $\mathrm{ev}_e^*(k[t_1^{\pm}]) \simeq M_{c_e}^{k[t_1^{\pm}]}$. Theorems 6.27 and 6.28 imply see that $\mathrm{End}(\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}^*(k[t_1^{\pm}]))$ is perfect in $\mathcal{D}(k[t_2^{\pm}])$, showing that $\mathcal{C}_{\mathbf{S}}$ is proper. □

Theorem 6.20. *Assume that* $\mathrm{char}(k) \neq 2$. *The $k[t_2^{\pm}]$-linear functor*

$$\prod_{e\in\mathcal{T}_1^{\partial}} \mathrm{ev}_e \colon \mathcal{C}_{\mathbf{S}} \longrightarrow \prod_{e\in\mathcal{T}_1^{\partial}} \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e)$$

admits a weak right 2-Calabi–Yau structure.

Proof. Combine Lemma 4.37, Theorem 4.44 and Proposition 6.10. □

Corollary 6.21. *Assume that* $\mathrm{char}(k) \neq 2$. *The adjunction*

$$\partial\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}} \colon \prod_{e\in\mathcal{T}_1^{\partial}} \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \longleftrightarrow \mathcal{C}_{\mathbf{S}} : \prod_{e\in\mathcal{T}_1^{\partial}} \mathrm{ev}_e$$

is spherical.

Proof. The twist functor of the adjunction $\partial\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}} \dashv \prod_{e\in\mathcal{T}_1^\partial}\mathrm{ev}_e$ is by Theorem 6.20 a suspension of the inverse Serre functor $\mathrm{id}^!_{\mathcal{C}_{\mathbf{S}}}$ and thus invertible by Lemma 4.12. The cotwist functor of the adjunction $\partial\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}} \dashv \prod_{e\in\mathcal{T}_1^\partial}\mathrm{ev}_e$ can be readily determined using the equivalence $\mathrm{ev}_e^*(k[t_1^\pm]) \simeq M_{c_e}^{k[t_1^\pm]}$ from Proposition 5.49, and shown to be an equivalence. It acts by permuting cyclically the copies of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \simeq \mathcal{D}(k[t_1^\pm])$ corresponding to the external edges of each boundary circle. □

6.2 The geometric model

In this section, we describe a full geometric model for the generalized cluster category $\mathcal{C}_{\mathbf{S}}$, including a classification of all indecomposable objects in terms of pure matching curves. We begin in Section 6.2.1 by specializing the partial geometric model for $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ to its parts applicable to $\mathcal{C}_{\mathbf{S}} \subset \mathcal{D}(\mathscr{G}_{\mathcal{T}})$. This includes an association of objects to matching data with local value $k[t_1^\pm]$, and a description of the derived Hom's in terms of intersections. In Section 6.2.2, we compare the computation of cones in $\mathcal{C}_{\mathbf{S}}$ of morphisms arising from intersections with the Kauffman Skein relation. In the final Section 6.2.3, we prove the geometrization Theorem 6.35, which states that every compact object in $\mathcal{C}_{\mathbf{S}}$ decomposes uniquely into the direct sum of objects associated with pure matching data and local value $k[t_1^\pm]$.

In the remainder of Section 6, we always assume that k is an algebraically closed field. For the entirety of this section, we also fix a marked surface $\mathbf{S}$ and an auxiliary trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$.

6.2.1 Objects and Hom's

We begin by characterizing the global sections arising from matching data which lie in the generalized cluster category $\mathcal{C}_{\mathbf{S}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$.

Lemma 6.22. *Let (γ, L) be a matching datum in $\mathbf{S}\backslash M$. The following two are equivalent:*

i) The global section $M_\gamma^L \in \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ lies in the generalized cluster category $\mathcal{C}_{\mathbf{S}}$.

ii) The object L lies in the full subcategory

$$\mathcal{D}(k[t_1^\pm]) \subset \mathcal{D}(k[t_1]) \simeq \mathrm{RMod}_{k[t_1]}\ .$$

Proof. We note that the full subcategory $\mathcal{C}_{\mathbf{S}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ consists of those global sections which evaluate at each edge e of $\mathcal{T}$ to an object in $\mathcal{D}(k[t_1^\pm]) \subset \mathrm{RMod}_{k[t_1]} = \mathcal{F}(e)$. This follows for instance by observing that these objects are exactly the right orthogonal objects to $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}})$, as follows from Lemma 6.17. This lemma thus follows from the further observation that an object M_γ^L evaluates at all edges of $\mathcal{T}$ to $\mathcal{D}(k[t_1^\pm]) \subset \mathcal{D}(k[t_1])$ if and only if $L \in \mathcal{D}(k[t_1^\pm]) \subset \mathcal{D}(k[t_1])$. □

Notation 6.23. Given a matching datum $(\gamma, k[t_1^\pm])$ in $\mathbf{S}\backslash M$, we denote $N_\gamma := M_\gamma^{k[t_1^\pm]}$.

We note that given a matching datum $(\gamma, k[t_1^\pm])$ with local value $k[t_1^\pm]$, all endpoints of γ lie in $\partial\mathbf{S}\backslash M$. Specializing the definition of a matching datum (γ, L) to the case $L = k[t_1^\pm]$, we thus find that $(\gamma, k[t_1^\pm])$ is fully determined by γ, if γ is open, or by γ, the rank a and the monodromy equivalence $\mu\colon k[t_1^\pm]^{\oplus a} \simeq k[t_1^\pm]^{\oplus a}$, if γ is closed.

Remark 6.24. Given a matching datum $(\gamma, k[t_1^\pm])$ with γ closed and of rank a, the monodromy equivalence $\mu\colon k[t_1^\pm]^{\oplus a} \simeq k[t_1^\pm]^{\oplus a}$ can be considered via Lemma 6.4 as an $a \times a$-matrix with entries in k. Since k is algebraically closed, μ is similar to a Jordan block matrix. Replacing μ by a similar matrix has no effect on N_γ. We will thus always assume that μ is a Jordan matrix.

If there is more than one Jordan block in the Jordan normal form of μ, N_γ splits into multiple direct summands. We will thus always assume that μ consists of a single Jordan block. Its eigenvalue $\lambda \in k^\times$, together with the rank a of $(\gamma, k[t_1^\pm])$ fully determine μ. We may thus equivalently specify μ or λ. We call λ the monodromy datum of $(\gamma, k[t_1^\pm])$.

For the geometric model for $\mathcal{C}_\mathbf{S}$ it suffices to consider pure matching curves:

Lemma 6.25. *Consider a matching datum $(\gamma, k[t_1^\pm])$. Then there exists a canonical matching datum $(\tilde{\gamma}, k[t_1^\pm])$, such that $\tilde{\gamma}$ is pure and homotopic to γ relative $\partial\mathbf{S}\backslash M$. The pure matching curve $\tilde{\gamma}$ is furthermore uniquely determined by this property. Furthermore, there exists an equivalence of global sections*

$$N_\gamma^{k[t_1^\pm]} \simeq N_{\tilde{\gamma}}^{k[t_1^\pm]} \in \mathcal{C}_\mathbf{S}\,.$$

Proof. Using that $\phi_*(k[t_1^\pm]) \simeq 0$, it is easy to see that for any $1 \leq i, j \leq 3$, $i \neq j$, that $M_{\delta^{i,j}}^{k[t_1^\pm]} \simeq M_{\delta^{j,i}}^{k[t_1^\pm]}$ as sections of $\mathcal{F}_\mathcal{T}$, using the notation from Construction 5.40. In other words, on the level of segments, the associated local sections do not distinguish between segments which are homotopic by a homotopy crossing vertices of $\mathcal{T}$. Replacing segments of γ of the form $\delta^{i,j}$ with $2 = j - i \in \mathbb{Z}/3\mathbb{Z}$ by segments $\delta^{j,i}$, we thus obtain a pure matching curve $\tilde{\gamma}$, which is homotopic to γ relative $\partial\mathbf{S}\backslash M$, and unique with this property. The curve $\tilde{\gamma}$ gives rise to a matching datum $(\tilde{\gamma}, k[t_1^\pm])$, and since the change in the segments did not change the associated local sections, up to equivalence, we find that the glued global sections are also equivalent. □

Remark 6.26. Given a matching curve γ, we call the unique pure matching curve $\tilde{\gamma}$ which is homotopic to γ relative $\partial\mathbf{S}\backslash M$ the purification of γ.

Specializing Theorems 5.53 and 5.54 to $L = k[t_1^\pm]$, we obtain the following.

Theorem 6.27. *Let γ, γ' be two distinct pure matching curves in $\mathbf{S}\backslash M$. Let a be the rank of γ and a' the rank of γ'. There exists an equivalence in $\mathcal{D}(k[t_2^\pm])$*

$$\mathrm{Mor}_{\mathcal{C}_\mathbf{S}}(N_\gamma, N_{\gamma'}) \simeq k[t_1^\pm]^{\oplus aa' i^{\mathrm{cr}}(\gamma,\gamma') \oplus i^{\mathrm{bdry}}(\gamma,\gamma')}\,.$$

Theorem 6.28. *(i) Let $\gamma\colon [0,1] \to \mathbf{S}$ be an open matching curve in $\mathbf{S}\backslash M$. There exists an equivalence in $\mathcal{D}(k[t_2^{\pm}])$*

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) \simeq k[t_1^{\pm}]^{\oplus 1+2i^{\mathrm{cr}}(\gamma,\gamma)+i^{\mathrm{bdry}}(\gamma,\gamma)}\,.$$

(ii) Let $\gamma\colon S^1 \to \mathbf{S}$ be a closed matching curve in $\mathbf{S}\backslash M$ of rank a. There exists an equivalence in $\mathcal{D}(k[t_2^{\pm}])$

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) \simeq k[t_1^{\pm}]^{\oplus 2a+2a^2 i^{\mathrm{cr}}(\gamma,\gamma)}\,.$$

We end this section with the definitions of arcs and ideal triangulations.

Definition 6.29. An open, pure matching curve $\gamma\colon [0,1] \to \mathbf{S}\backslash M$ is called an arc if it has no self-crossings. An arc is called a boundary arc, if it cuts out a monogon. An arc is called an internal arc, if it is not a boundary arc.

Remark 6.30. The notion of an internal arc in the sense of Definition 6.29 coincides with the notion of an arc from [FT18, Def. 5.2], except that we use a different convention regarding endpoints. For us, endpoints lie in $\partial\mathbf{S}\backslash M$, whereas in loc. cit. endpoints lie in M. These two perspectives are however equivalent, as can be seen by expanding the marked points to intervals, contracting their complements to points and using that arcs are considered up to homotopy.

For more background and examples of ideal triangulations, we refer to [FST08].

Definition 6.31. Two arcs in $\mathbf{S}\backslash M$ are called compatible, if they do not have any crossings. An ideal triangulation of $\mathbf{S}$ consists of a maximal (by inclusion) collection I of pairwise compatible arcs in $\mathbf{S}\backslash M$.

We note that any ideal triangulation of $\mathbf{S}$ contains all boundary arcs of $\mathbf{S}$ and that any two ideal triangulations of $\mathbf{S}$ have the same cardinality.

6.2.2 Skein relations from mapping cones

In this section, we collect geometric descriptions of the cones of the morphisms between the objects of $\mathcal{C}_{\mathbf{S}}$ associated to the matching curves. Similar descriptions of cones in the derived categories of gentle algebras appear in [OPS18]. We fix two open, pure matching curves γ, γ' in $\mathbf{S}\backslash M$ and distinguish two cases.

Case 1: γ and γ' have a crossing.

There are two possible *smoothings* of this crossing, each consisting of two matching curves in $\mathbf{S}\backslash M$, denoted γ_1, γ_2 and γ_3, γ_4. The first curve in each of the two smoothings is obtained by starting at an endpoint of γ and tracing along γ up to that crossing, and then tracing along γ' in one of two possible directions. Similarly, the second curve in the two smoothings is obtained by starting at the other endpoint of γ, tracing along γ up to the crossing, and then tracing along γ' to the other end. This process is locally at the crossing illustrated on the left in Figure 7.

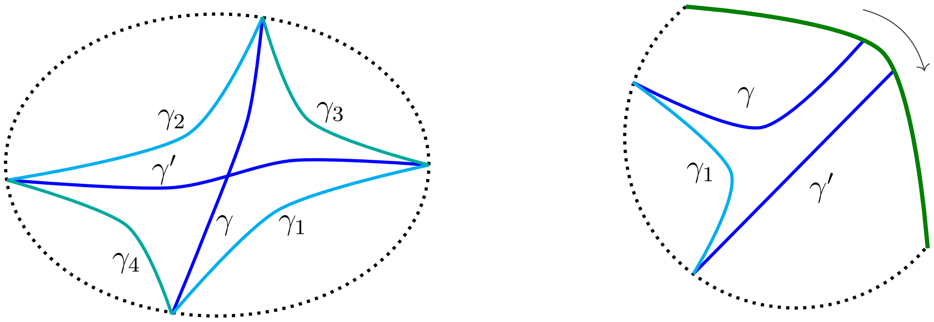


Figure 7: On the left: a crossing of two matching curves γ, γ (in blue) and the two possible smoothings γ_1, γ_2 and γ_3, γ_4 (in different shades of blue/green). On the right: a directed boundary intersection of two matching curves γ, γ' and the corresponding smoothed composite γ_1.
The boundary of $\mathbf{S}$ is depicted in green. Outside of the depicted parts of $\mathbf{S}$, the matching curves continue identically.

Case 2: there is a directed boundary intersection from γ to γ'.

We can compose γ with part of a boundary component of $\mathbf{S}\backslash M$ and γ' to a curve, which we then smooth to a matching curve γ_1. This process is illustrated on the right in Figure 7.

To both types of intersection, Theorems 6.27 and 6.28 associate a direct summand of the morphism object $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ and also $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ in case of a crossing. Proposition 6.32 describes the cones of these morphisms in terms of the above smoothings.

Proposition 6.32. *Let γ, γ' be two matching curves in $\mathbf{S}\backslash M$.*

(1) Suppose that γ and γ' have a crossing. There exist fiber and cofiber sequences in $\mathcal{C}_{\mathbf{S}}$

$$N_{\gamma_1} \oplus N_{\gamma_2} \to N_\gamma \xrightarrow{\alpha} N_{\gamma'}\,, \qquad N_{\gamma_3} \oplus N_{\gamma_4} \to N_{\gamma'} \xrightarrow{\beta} N_\gamma\,,$$

with γ_1, γ_2 and γ_3, γ_4 being the two possible smoothings of the crossing. The morphisms α and β describe any non-zero degree 0 elements of the direct summands $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$, $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_{\gamma'}, N_\gamma)$ associated to the crossing in Theorem 6.27.

(2) Suppose that there is a directed boundary intersection from γ to γ'. There exist a fiber and cofiber sequence in $\mathcal{C}_{\mathbf{S}}$

$$N_{\gamma_1} \to N_\gamma \xrightarrow{\alpha} N_{\gamma'}\,,$$

with γ_1 the smoothed composite of γ, γ'. The morphism α describes any non-zero degree 0 element of the direct summand $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ associated to the directed boundary intersection in Theorem 6.27.

Proof. The Proposition follows from a direct computation, using the descriptions of N_γ and $N_{\gamma'}$ as coCartesian sections of the Grothendieck construction of $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$. $\square$

Remark 6.33. The matching curves γ_i appearing as the smoothings of intersections in Proposition 6.32 are not necessarily pure. We may however replace γ_i by its purification $\tilde{\gamma}_i$, since $N_{\gamma_i} \simeq N_{\tilde{\gamma}_i}$, see Lemma 6.25.

Remark 6.34. Part (1) of Proposition 6.32 matches the $q = 1$ Kauffman Skein relations, see Definition 6.48. After purifying as indicated in Remark 6.33 Proposition 6.32 also matches for arc the Ptolemy cluster exchange relations, see Definition 6.45.

6.2.3 The geometrization Theorem

We introduce an equivalence relation $\sim$ on pure matching data $(\gamma, k[t_1^\pm])$ generated by the following relation.. Let γ^{rev} be the matching curve obtained by reversing the orientation of γ. If γ is open, we set $(\gamma, k[t_1^\pm]) \sim (\gamma^{\mathrm{rev}}, k[t_1^\pm])$. If γ is closed of rank a with monodromy datum $\lambda \in k^\times$, we set $(\gamma, k[t_1^\pm]) \sim (\gamma^{\mathrm{rev}}, k[t_1^\pm])$, where $(\gamma^{\mathrm{rev}}, k[t_1^\pm])$ is the matching datum with rank a and monodromy datum λ^{-1}. Note that $N_\gamma \simeq N_{\gamma^{\mathrm{rev}}}$ by Remark 5.45.

The main result of this section is the following Theorem, proving that all compact objects in $\mathcal{C}_{\mathbf{S}}$ are geometric, meaning that they arise as direct sums of object associated to pure matching curves.

Theorem 6.35 (The geometrization Theorem)**.** *Let $X \in \mathcal{C}_{\mathbf{S}}$ be a compact object. Then there exists a unique and finite set J of equivalence classes of pure matching data with local value $k[t_1^\pm]$, under the relation defined above, and an equivalence in $\mathcal{C}_{\mathbf{S}}$*

$$X \simeq \bigoplus_{(\gamma, k[t_1^\pm]) \in J} N_\gamma \,.$$

Corollary 6.36. *The full subcategory $\mathcal{C}^c_{\mathbf{S}} \subset \mathcal{C}_{\mathbf{S}}$ of compact objects is Krull-Schmidt, meaning that every objects splits into a direct sum of objects with local endomorphism rings.*

Proof of Corollary 6.36. As shown in the proof of Theorem 6.35, every compact object X of $\mathcal{C}_{\mathbf{S}}$ splits as the direct sum $X \simeq \bigoplus_{\gamma \in J} N_\gamma$, with J a set of pure matching curves. The endomorphism rings of these objects are local by Corollary 5.62. $\square$

The proof of the geometrization Theorem makes use of the gluing description of global sections arising from matching curves and the description of global sections in terms of coCartesian sections of the Grothendieck construction. Given a compact object X of $\mathcal{C}_{\mathbf{S}}$, corresponding to a global section of $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$, we can evaluate it at any edge e of the ribbon graph $\mathcal{T}$. If the value is non-zero we can choose a direct sum decomposition of this value with a non-zero direct summand. Let v be a vertex incident to e. As we show in Construction 6.37 that we can find a lift along the functor $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(v \to e)$ of the direct sum decomposition at e to a direct sum decomposition of the value of X to v. This then give rise to direct sum decompositions at further edges incident to v. Proceeding this way, we choose repeated direct sum decompositions of the values of X at all vertices and edges of $\mathcal{T}$. The main difficulty with this argument is, that the direct sum decompositions which we find are not necessarily preserved under $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(v \to e)$. A fiddly argument shows that we can further tweak the chosen direct sum decompositions, so that they are preserved, and thus glue to a global section, which is a non-zero direct summands of X. The summand of X so constructed is obtained from gluing local sections associated to segments and hence geometric, i.e. arises from a pure matching curve. Repeating this argument, we find a splitting of X into geometric direct summands. The uniqueness of the set C in the geometrization Theorem is proven separately.

Construction 6.37. Let $\mathbf{S}$ be the 3-gon and e an edge of a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. Let $X \in \mathcal{C}_{\mathbf{S}}^{c}$ and let $\mathrm{ev}_e(X) \simeq B \oplus A \oplus B'$ with $A \neq 0$. We construct two direct sum decompositions $X \simeq Z \oplus Y \oplus Z'$ and $X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'$, called the counterclockwise and clockwise splittings. These further satisfy that the arising equivalences

$$\mathrm{ev}_e(Z) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(Z') \simeq B \oplus A \oplus B'$$
$$\mathrm{ev}_e(\tilde{Z}) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(\tilde{Z}') \simeq B \oplus A \oplus B'$$

are lower triangular with invertible diagonals.

The generalized cluster category $\mathcal{C}_{\mathbf{S}}$ of $\mathbf{S}$ is equivalent to the functor category $\mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$, whose objects are diagrams $x \to y$ with $x, y \in \mathcal{D}(k[t_1^{\pm}])$. In the following, we set $\mathcal{C}_{\mathbf{S}} = \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$. The functor $\mathrm{ev}_e\colon \mathcal{C}_{\mathbf{S}} \to \mathcal{D}(k[t_1^{\pm}])$ can be chosen to evaluate a functor $\Delta^1 \to \mathcal{D}(k[t_1^{\pm}])$ at its value at $1 \in \Delta^1$ (recall that Δ^1 has the vertices $0, 1$). There are three indomposable objects in $\mathcal{C}_{\mathbf{S}}$, corresponding to the three boundary arcs in $\mathbf{S}\backslash M$ (there are no further pure matching curves in $\mathbf{S}\backslash M$). Explicitly, these objects can be described as follows.

$$N_1 = \left(0 \to k[t_1^{\pm}]\right)$$
$$N_2 = \left(k[t_1^{\pm}] \xrightarrow{\simeq} k[t_1^{\pm}]\right)$$
$$N_3 = \left(k[t_1^{\pm}] \to 0\right)$$

The homotopy category of $\mathcal{D}(k[t_1^{\pm}])$ is equivalent to the abelian 1-category of k-vector spaces, see Lemma 6.4. We thus treat objects in $\mathcal{D}(k[t_1^{\pm}])$ as vector spaces in the following.

All compact objects in $\mathcal{D}(k[t_1^{\pm}])$ are a finite direct sums of copies of $k[t_1^{\pm}]$. Using that every morphism in $\mathcal{D}(k[t_1^{\pm}])$ splits, we may assume that the object $X \in \mathcal{C}_{\mathbf{S}}$ is equivalent to an object of the form

$$k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus l} \xrightarrow{\mathscr{M}} k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i},$$

with $\mathscr{M} = \begin{pmatrix} \mathrm{id}_{k[t_1^{\pm}]^{\oplus j}} & 0 \\ 0 & 0 \end{pmatrix}$ and $i, j, l \geq 0$. We thus have $\mathrm{ev}_e(X) = k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i}$ and a splitting

$$X \simeq \left(k[t_1^{\pm}]^{\oplus l} \to 0\right) \oplus \left(k[t_1^{\pm}]^{\oplus j} \hookrightarrow k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i}\right) .$$

We denote the first summand by O and the second by U. Note that $\mathrm{ev}_e(O) = 0$.

Let $C = (B \oplus A) \cap k[t_1^{\pm}]^{\oplus j}$. We define V as the diagram $C \to B \oplus A$ in $\mathcal{D}(k[t_1^{\pm}])$. It is easy to see that the apparent morphism $V \to U$ in $\mathcal{C}_{\mathbf{S}}$ admits a retraction. We can thus choose a splitting $U \simeq V \oplus Q'$. By construction, we have $\mathrm{ev}_e(V) = B \oplus A$. Next, we define $D = B \cap k[t_1^{\pm}]^{\oplus j}$. Then we again obtain a direct summand $Q = (D \to B) \hookrightarrow V$. We choose a direct sum complement $V \simeq Q \oplus Y$. We have $\mathrm{ev}_e(Q) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(Q') \simeq B \oplus A \oplus B'$ and this equivalence is lower triangular with invertible diagonals.

We set $Z = Q$, $Z' = Q' \oplus O$ and $\tilde{Z} = Q \oplus O$, $\tilde{Z}' = Q'$. This gives the desired splittings $X \simeq Z \oplus Y \oplus Z'$ and $X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'$.

Lemma 6.38. *Consider the setup of Construction 6.37 and the counterclockwise and clockwise splittings*

$$X \simeq Z \oplus Y \oplus Z' \quad \text{and} \quad X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'\,.$$

Denote by e_1 the edge of $\mathcal{T}$ following the edge e in the counterclockwise direction and by e_2 the edge of $\mathcal{T}$ following the edge e in the clockwise direction.

(1) Suppose that $Y \simeq N_1$. Then any autoequivalence ϕ of $\mathrm{ev}_{e_1}(Z) \oplus \mathrm{ev}_{e_1}(Y) \oplus \mathrm{ev}_{e_1}(Z')$ given by a lower triangular matrix lifts to an autoequivalence of $Z \oplus Y \oplus Z'$ given by a lower triangular matrix.

(2) Suppose that $Y \simeq N_2$. Any autoequivalence ϕ of $\mathrm{ev}_{e_2}(\tilde{Z}) \oplus \mathrm{ev}_{e_2}(Y) \oplus \mathrm{ev}_{e_2}(\tilde{Z}')$ given by a lower triangular matrix lifts to an autoequivalence of $\tilde{Z} \oplus Y \oplus \tilde{Z}'$ given by a lower triangular matrix.

Proof. We only prove part (1), part (2) is analogous. By construction, we can find splittings $Z \simeq \bigoplus_{i_1} N_1 \oplus \bigoplus_{j_1} N_2$ and $Z' \simeq \bigoplus_{i_2} N_1 \oplus \bigoplus_{j_2} N_2 \oplus \bigoplus_l N_3$. We have $\mathrm{ev}_{e_1}(N_2) \simeq 0$ and $\mathrm{ev}_{e_1}(N_3) \simeq \mathrm{ev}_{e_1}(N_1) \simeq k[t_1^{\pm}]$. We further find $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_x, N_y) \xrightarrow{\mathrm{ev}_{e_1}} \mathrm{Mor}_{\mathcal{D}(k[t_1^{\pm}])}(k[t_1^{\pm}, k[t_1^{\pm}])$ to be an equivalence for $x = y = 1$, $x = 3, y = 1$ or $x = y = 3$. It is thus clear that we can find a unique lift of ϕ, which restricts to the identity on $\bigoplus_{j_1} N_2 \oplus \bigoplus_{j_2} N_2$. □

As shown in Lemma 6.38, the clockwise and counterclockwise splitting have the advantage that we can lift certain autoequivalences of their values at an edge to autoequivalences of the splitting. We will need these autoequivalences to tweak the choices of local splittings in the proof of the geometrization Theorem 6.35, by replacing some splittings with their image under such an autoequivalence. After the necessary tweaks, we will glue the arising two-term splittings $Y \oplus (Z \oplus Z') \simeq X$ or $Y \oplus (\tilde{Z} \oplus \tilde{Z}') \simeq X$.

Proof of Theorem 6.35, part 1: existence of the decomposition. Given an edge e of $\mathcal{T}$ and $X \in \mathcal{C}_{\mathbf{S}}$, we denote by $\mathrm{ev}_e(X) \in \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(e) \simeq \mathcal{D}(k[t_1^{\pm}])$ the value of the coCartesian section X at e. Similarly, given a vertex v of $\mathcal{T}$, we denote by $\mathrm{res}_v(X)$ the restriction of the coCartesian section X to v and its three incident edges. The object $\mathrm{res}_v(X)$ can be considered as an object in $\mathcal{C}_{\mathbf{S}_v}$, with $\mathbf{S}_v$ a 3-gon. Note also that $\mathcal{C}_{\mathbf{S}_v} \simeq \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(v)$, with the equivalence given by evaluation at v. We show that if $X \neq 0$, then there is a splitting $X \simeq N_\gamma \oplus T$ for a matching datum $(\gamma, k[t_1^{\pm}])$. By a descending induction on the total dimension over k of $\bigoplus_{e \in \mathcal{T}_1} \mathrm{ev}_e(X)$, the existence of the desired decomposition then follows.

Let $0 \neq X \in \mathcal{C}_{\mathbf{S}}$ be compact. Let e_0 be an external edge of $\mathcal{T}$ with $\mathrm{ev}_{e_0}(X) \neq 0$. If such an external edge does not exist, choose e_0 instead to be an internal edge with $\mathrm{ev}_{e_0}(X) \neq 0$. Choose any direct sum decomposition $B_0 \oplus A_0 \oplus B_0' \simeq \mathrm{ev}_{e_0}(X)$ with $A_0 \simeq k[t_1^{\pm}]$. Let v_1 be a trivalent vertex incident to e_0.

With this starting data, we iteratively make choices of edges e_i with incident vertices v_i and v_{i+1} and find splittings $B_i \oplus A_i \oplus B_i' \simeq \mathrm{ev}_{e_i}(X)$ and $T_i \oplus S_i \oplus T_i' \simeq \mathrm{res}_{v_i}(X)$ as follows.

Suppose the data for i has been chosen. Let $v_{i+1} \neq v_i$ be the other vertex incident to e_i. The summand $S_i \coloneqq Y$ appearing in both the clockwise and counterclockwise splittings of $\mathrm{res}_{v_{i+1}}(X)$ is of the form $Y \simeq N_\delta$, with δ a segment starting at the edge e_i and ending at another edge, called e_{i+1}. If e_{i+1} follows e_i in the clockwise direction, we use the clockwise splitting from Construction 6.37 arising from $B_i \oplus A_i \oplus B_i'$ to obtain a splitting $T_{i+1} \oplus S_{i+1} \oplus T_{i+1}'$ of $\mathrm{res}_{v_{i+1}}(X)$. If e_{i+1} instead follows e_i in the counterclockwise direction, we instead use the counterclockwise splitting. By Construction 6.37, we have an equivalence $\phi\colon \mathrm{ev}_{e_i}(T_{i+1}) \oplus \mathrm{ev}_{e_i}(S_{i+1}) \oplus \mathrm{ev}_{e_i}(T_{i+1}') \simeq \mathrm{ev}_{e_i}(X) \simeq B_i \oplus A_i \oplus B_i'$ which is given by a lower triangular matrix with invertible diagonal. By Lemma 6.38, we can find compatible autoequivalences of $T_j \oplus S_j \oplus T_j'$ for all $1 \leq j \leq i-1$ and of $B_j \oplus A_j \oplus B_j'$ for all $0 \leq j \leq i-1$, such that the composite at $B_i \oplus A_i \oplus B_i'$ with ϕ is a diagonal matrix. We redefine all $S_j, T_j, T_j', A_j, B_j, B_j'$ by their images under these autoequivalences. We then set $A_{i+1} = \mathrm{ev}_{e_{i+1}}(S_{i+1})$ and $B_{i+1} = \mathrm{ev}_{e_{i+1}}(T_{i+1})$, $B_{i+1}' = \mathrm{ev}_{e_{i+1}}(T_{i+1}')$.

We proceed with making these choices, until we come to a stop in one the following cases below. If e_0 is external, we always come to a stop in the following case, as otherwise one can find a contradiction to X being a coCartesian section.

Case 1) We have started at an external edge e_0 and arrive at an external edge e_N with $N \geq 1$.

In this case, the S_i's, with $1 \leq i \leq N$, glue to a global section $S \subset X$, satisfying $\mathrm{res}_v(S) = \bigoplus_{v_j = v} S_j$. Similarly, there is a global section $T \subset X$ satisfying $\mathrm{res}_v(T) = \bigcap_{v_j = v} T_j \oplus T_j'$ if there is at least one $1 \leq j \leq N$ with $v_j = v$ and $\mathrm{res}_v(T) = \mathrm{res}_v(X)$ if there are no j with $v_j = v$. The arising map $S \oplus T \to X$ restricts pointwise to an equivalence, and is hence also an equivalence of global sections. At each vertex v_i, we have by construction $S_i \simeq N_{\delta_i}$ for a segment δ_i at v_i. We compose these segments to an open pure matching curve γ. We find that $S \simeq N_\gamma$ is geometric. This gives the desired splitting $X \simeq N_\gamma \oplus T$.

Suppose now that we did not stop in case 1). We may then assume that $\mathrm{ev}_e(X) \simeq 0$ for all external edges e.

Case 2) There is a non-empty set $J \subset \{0, \ldots, N-1\}$, such that $e_N = e_j$ for all $j \in J$ and $A_N \subset \langle A_j \rangle_{j \in J}$ lies in the submodule of $\mathrm{ev}_{e_N}(X)$ generated by the A_j's. Note that in this case J contains the element 0, as otherwise we again get a contradiction to X being coCartesian. We obtain a direct summand $S \subset X$ with complement T, satisfying $\mathrm{res}_v(S) = \bigoplus_{v_j=v,\, j<N} S_j$ and $\mathrm{res}_v(T) = \bigcap_{v_j=v,\, j<N} T_j \oplus T'_j$ or $\mathrm{res}_v(T) = \mathrm{res}_v(X)$ if there are no $j < N$, such that $v_j = v$.

Again, at each vertex v_i we have by construction $S_i \simeq N_{\delta_i}$ for a segment δ_i at v_i. We compose these segments to a closed curve γ. If γ is given by the composite of a identical closed curves γ', we have that the number N of segments of γ is a multiple of a and $e_i = e_{i+jN/a}$ for $1 \leq j \leq a-1$ and $0 \leq i \leq N/a$. We relabel $A_i = \bigoplus_{0\leq j\leq a-1} A_{i+jN/a}$ and $S_i = \bigoplus_{0\leq j\leq a-1} S_{i+jN/a}$ for $0 \leq i < N/a$. We have chosen γ' so that it is not again given by the composite of multiple identical closed curves. To extend γ' to a matching datum, we need to specify a rank and a monodromy datum.

We choose an ordered basis U_0 of $A_0 = \mathrm{ev}_{e_0}(S_1) \in \mathcal{D}(k[t_1^{\pm}])$ with cardinality a. Consider the basis U'_1 of $\mathrm{res}_{v_1}(S_1)$ which is mapped under $S_1(v_1 \to e_0)$ to the chosen basis U_0. The image of U'_1 under $S_1(v_1 \to e_1)$ defines an ordered basis U_1 of A_1. From this, we again find a basis U'_2 of $\mathrm{res}_{v_2}(S_2)$. Proceeding in this way, performing these steps N/a times, we obtain a second ordered basis $U_{N/a}$ of $A_{N/a} = A_0$. Consider the Jordan normal form of the linear map, which maps $U_{N/a}$ to U_0. For each Jordan $b\times b$-block with eigenvalue $\lambda \neq 0$, we can equip γ with rank b and monodromy datum λ. With these choices of monodromy data, we finally have $S \simeq \bigoplus_{\text{blocks}} N_\gamma$, where the sum runs over the Jordan blocks. We thus have the desired splitting $X \simeq \bigoplus_{\text{blocks}} N_\gamma \oplus T$, concluding the proof. □

Proof of Theorem 6.35, part 2: uniqueness of decomposition. Part 1 of the proof of Theorem 6.35 and the proof of Corollary 6.36 show that $\mathcal{C}_{\mathbf{S}}$ is Krull-Schmidt, or equivalently the homotopy 1-category $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}$ is Krull-Schmidt. The essential uniqueness of a decomposition into indecomposables in a Krull-Schmidt category is shown in [Kra15, Theorem 4.2]. It thus suffices to show that for two pure matching data $(\gamma, k[t_1^{\pm}]), (\gamma', k[t_1^{\pm}])$, we have $N_\gamma \simeq N_{\gamma'}$ if and only if $(\gamma, k[t_1^{\pm}]) \sim (\gamma', k[t_1^{\pm}])$. It is now easy to see that $N_\gamma \simeq N_{\gamma'}$ implies that γ and γ' are either both open or both closed. In the former case, we can argue by using Theorem 6.27, by testing $N_\gamma, N_{\gamma'}$ against objects arising from different matching curves to conclude $(\gamma, k[t_1^{\pm}]) \sim (\gamma', k[t_1^{\pm}])$.

If γ and γ' are closed, we distinguish the cases that the matching curves underlying γ and γ' are identical (up to reversal of orientation) or not. In the first case, one can show that $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) \simeq 0$ if the monodromy data of $(\gamma, k[t_1^{\pm}]) \sim (\gamma', k[t_1^{\pm}])$ are distinct; it follows that $N_\gamma, N_{\gamma'}$ are also distinct. In the second case, a similar argument as in the case of open matching curves applies to show that the open curves composed of segments η and η', obtained from cutting γ and γ' open at an edge of $\mathcal{T}$, are identical. This shows that γ and γ' must be identical, concluding the case distinction and the proof. □

6.3 Categorification of the cluster algebra with coefficients

For the entire section, we fix an unpunctured marked surface $\mathbf{S}$. The field k is still assumed to be algebraically closed. We further assume that $\mathrm{char}(k) \neq 2$.

We begin in the Sections 6.3.1 and 6.3.2 by recalling the definitions of the cluster algebra and of the commutative Skein algebra associated with $\mathbf{S}$. In Section 6.3.3, we show that the cluster-tilting objects in the exact ∞-/extriangulated generalized cluster category $\mathcal{C}_{\mathbf{S}}$ are in bijection with the clusters of this cluster algebra. We also describe the endomorphisms algebras of the relative cluster-tilting objects in $\mathcal{C}_{\mathbf{S}}$ in terms of gentle algebras and discuss two simple examples. In Section 6.3.4, we describe a cluster character on $\mathcal{C}_{\mathbf{S}}$ with values in the commutative Skein algebra.

6.3.1 Cluster algebras of marked surfaces

The goal of this section is to define the cluster algebra associated to $\mathbf{S}$ with coefficients in the boundary arcs. For the definition of cluster algebra, we follow [FWZ16, Chapter 3].

Definition 6.39. Let $\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_{m_1+m_2})$ be the field of rational functions in $m_1 + m_2$ variables. A (labeled) seed $\left(\mathbf{x}, \tilde{M}\right)$ in $\mathscr{F}$ consists of

- an $m_1 + m_2$-tuple $\mathbf{x} = (x_1, \ldots, x_{m_1+m_2})$ in $\mathscr{F}$ forming a free generating set of $\mathscr{F}$ and
- an $(m_1 + m_2) \times m_1$-matrix $\tilde{M}$, such that the upper $m_1 \times m_1$-matrix is skew-symmetric[2].

The tuple $\mathbf{x}$ is called a cluster and the elements $x_1, \ldots, x_{m_1+m_2}$ are called cluster variables. The elements $x_{m_1+1}, \ldots, x_{m_1+m_2}$ are also called the frozen cluster variables. The matrix $\tilde{M}$ is called the extended mutation matrix.

Definition 6.40. Let $\left(\mathbf{x}, \tilde{M}\right)$ be a seed and $l \in \{1, \ldots, m_1\}$. The seed mutation at l is given by the seed $\left(\mathbf{x}', \mu_l(\tilde{M})\right)$ with

- $\mu_l(\tilde{M})_{i,j} = \begin{cases} -\tilde{M}_{i,j} & \text{if } i = l \text{ or } j = l \\ \tilde{M}_{i,j} + \tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l} > 0 \text{ and } \tilde{M}_{l,j} > 0 \\ \tilde{M}_{i,j} - \tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l} < 0 \text{ and } \tilde{M}_{l,j} < 0 \\ \tilde{M}_{i,j} & \text{else.} \end{cases}$
- $\mathbf{x}' = (x_1, \ldots, x_{l-1}, x_l', x_{l+1}, \ldots, x_{m_1+m_2})$, where x_l' is determined by the cluster exchange relation
$$x_l' x_l = \prod_{j \text{ with } \tilde{M}_{j,l} > 0} x_j^{\tilde{M}_{j,l}} + \prod_{j \text{ with } \tilde{M}_{j,l} < 0} x_j^{-\tilde{M}_{j,l}}.$$

Definition 6.41. Let $(\mathbf{x}, \tilde{M})$ be a seed.

- The associated cluster algebra CA $\subset \mathscr{F}$ is the $\mathbb{Q}$-subalgebra of $\mathscr{F}$ generated by all cluster variables in all seeds obtained from $(\mathbf{x}, \tilde{M})$ via iterated seed mutation.

[2] A possible generalization considers skew-symmetrizable matrices, see [FWZ16, Definition 3.1.1].

- The associated upper cluster algebra UCA $\subset \mathscr{F}$ is the $\mathbb{Q}$-subalgebra consisting of those elements which, for every cluster of CA, are Laurent polynomials in the cluster variables of that cluster.

Remark 6.42. The cluster algebra or upper cluster algebra associated to a seed only depends on the extended mutation matrix, up to isomorphism of $\mathbb{Q}$-algebras. We can thus speak of the cluster algebra or upper cluster algebra associated to an extended mutation matrix.

We proceed in Definition 6.43 with associating an extended mutation matrix to a choice of ideal triangulation I of $\mathbf{S}$ in the sense of Definition 6.31. Choosing a different ideal triangulation changes the extended mutation matrix by matrix mutations.

Definition 6.43 ([FST08, Definition 4.1], [FT18])**.** Let I be an ideal triangulation of $\mathbf{S}$ with m_1 interior arcs, labeled arbitrarily as $1, \ldots, m_1$, and m_2 boundary arcs, labeled arbitrarily as $m_1+1, \ldots, m_1+m_2$.

The extended signed adjacency matrix $\tilde{M}_I = \sum_\Delta M_\Delta$ of I is the $(m_1+m_2) \times m_1$-matrix given by the sum over all vertices ideal triangles Δ of I of the $(m_1+m_2) \times m_1$-matrices defined by

$$(M_\Delta)_{i,j} = \begin{cases} 1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the counterclockwise direction,} \\ -1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the clockwise direction,} \\ 0 & \text{else.} \end{cases}$$

The upper $m_1 \times m_1$-matrix of $\tilde{M}_I$ is skew-symmetric and called the signed adjacency matrix.

Definition 6.44.

- Let $\mathbf{S}$ be an oriented marked surface with an ideal triangulation I. We define $\mathrm{CA}_\mathbf{S}$ to be the cluster algebra associated to the extended mutation matrix given by the extended signed adjacency matrix $\tilde{M}_I$ of I. Similarly, $\mathrm{UCA}_\mathbf{S}$ is defined as the associated upper cluster algebra.

- We define $\mathrm{CA}_\mathbf{S}^{\mathrm{loc}}$ as the localization of $\mathrm{CA}_\mathbf{S}$ at the frozen cluster variables $x_{m_1+1}, \ldots, x_{m_1+m_2}$, where m_1 is the number of interior arc in I and m_2 is the number of boundary arcs in I.

Definition 6.45. Let γ, γ' be two arcs in $\mathbf{S}\backslash M$. Suppose that γ, γ' have a crossing as in Figure 7. Then the Ptolemy relation is defined as $\gamma \cdot \gamma' = \gamma_1 \cdot \gamma_2 + \gamma_3 \cdot \gamma_4$.

Theorem 6.46. *The cluster variables of* $\mathrm{CA}_\mathbf{S}$ *are canonically in bijection with the arcs in* $\mathbf{S}\backslash M$. *A set of cluster variables of* $\mathrm{CA}_\mathbf{S}$ *forms a cluster if and only if the corresponding arcs form an ideal triangulation of* $\mathbf{S}$. *The cluster exchange relations are the Ptolemy relations.*

Proof. This is [FT18, Theorem 8.6] specialized to the case that $\mathbf{S}$ has no punctures. □

6.3.2 Commutative Skein algebras

We proceed with defining the commutative $q = 1$ Skein algebra $\mathrm{Sk}^1(\mathbf{S})$ of links in $\mathbf{S} \backslash M$. As shown in [Mul16], this algebra embeds into the upper algebra $\mathrm{UCA}_{\mathbf{S}}$.

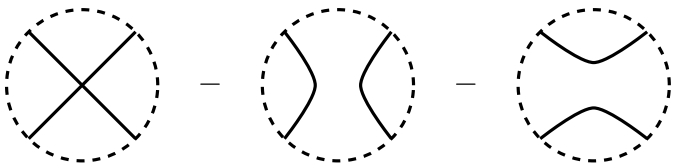
Definition 6.47. A link is a homotopy class relative $\partial\mathbf{S}\backslash M$ of continuous maps $\gamma \colon U \to \mathbf{S}\backslash M$, with U a finite disjoint union of $[0,1]$'s and S^1's, satisfying that all existent endpoints of γ (possibly none) lie in $\partial\mathbf{S}\backslash M$ and that away from the endpoints, γ is disjoint from $\partial\mathbf{S}\backslash M$.

We consider links up to reversal of orientation. We refer to the curves with domain $U = [0,1], S^1$ constituting a link as its components. There is an empty link with $U = \emptyset$. We denote by $\mathscr{L}(\mathbf{S})$ the set of all links in $\mathbf{S}\backslash M$.

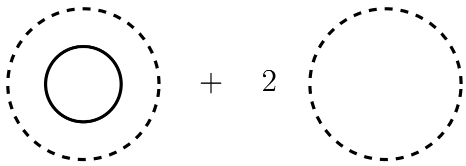
The $\mathbb{Q}$-vector space $\mathbb{Q}^{\mathcal{L}(\mathbf{S})}$ inherits the structure of a commutative $\mathbb{Q}$-algebra, by defining the product of two links to be the union of the two links and extending this product $\mathbb{Q}$-bilinearly.

Definition 6.48. We define the $q = 1$ Skein algebra $\mathrm{Sk}^1(\mathbf{S})$ as the quotient of the $\mathbb{Q}$-algebra $\mathbb{Q}^{\mathcal{L}(\mathbf{S})}$ by the ideal generated by the following elements.

1) The $q = 1$ Kauffman Skein relation.



2) The value of the unknot.



3) Any component with domain $[0,1]$ which is homotopic relative endpoints to a subset of $\partial\mathbf{S}\backslash M$.

The relations in 1) and 2) are understood to describe local relations inside a small disc (indicated by the dotted circle) in $\mathbf{S}$, applicable to any subset of components of a link. The depicted curves are identical outside the small disc.

Remark 6.49. The $q = 1$ Kauffman Skein relation recover the cluster exchange relations given by the Ptolemy relation, see Definition 6.45.

We remark that Definition 6.48 is equivalent to the definition of the q-Skein algebra given by Muller in [Mul16], with q set to 1, as can be seen by using Remarks 3.4 and 3.6 in [Mul16].

Definition 6.50. We define the localized $q = 1$ Skein algebra $\mathrm{Sk}^{1,\mathrm{loc}}(\mathbf{S})$ as the localization of $\mathrm{Sk}^1(\mathbf{S})$ at the set of boundary arcs.

Theorem 6.51 ([Mul16])**.** *There exist injective morphisms of $\mathbb{Q}$-algebras*

$$\mathrm{CA}^{\mathrm{loc}}_{\mathbf{S}} \hookrightarrow \mathrm{Sk}^{1,\mathrm{loc}}(\mathbf{S}) \hookrightarrow \mathrm{UCA}_{\mathbf{S}}\,. \tag{109}$$

If $\mathbf{S}$ additionally has at least two marked points, then the maps in (109) are equivalences of $\mathbb{Q}$-algebras.

6.3.3 Classification of cluster-tilting objects

We choose a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. Consider the perverse schober $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ from Definition 6.6 with the $k[t_2^{\pm}]$-linear smooth and proper ∞-category of global sections given by the generalized cluster category $\mathcal{C}_{\mathbf{S}}$. We let G be the right adjoint of the adjunction

$$F \coloneqq \partial\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}} \colon \prod_{e\in\mathcal{T}_1^\partial} \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(e) \longleftrightarrow \mathcal{C}_{\mathbf{S}} : G \coloneqq \prod_{e\in\mathcal{T}_1^\partial} \mathrm{ev}_e$$

defined in Definition 3.37. The functor G admits a $k[t_2^{\pm}]$-linear 2-Calabi–Yau structure, see Theorem 6.20 and the adjunction $F \dashv G$ is spherical by Corollary 6.21. By the results of Section 4.4.3, we obtain a Frobenius, 2-Calabi–Yau extrianguled category $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ arising from of a Frobenius exact ∞-structure on $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$.

Theorem 6.52.

i) Consider a pure matching datum $(\gamma, k[t_1^{\pm}])$ in $\mathbf{S}\backslash M$. Then N_γ is rigid in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ if and only if γ is an arc.

ii) Consider a finite collection I of distinct arcs in $\mathbf{S}\backslash M$. Then $\bigoplus_{\gamma\in I} N_\gamma$ is a cluster-tilting object in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ if and only if I is an ideal triangulation of $\mathbf{S}$.

We prove Theorem 6.52 further below.

Corollary 6.53. *There are canonical bijections between the sets of the following objects.*

- *Clusters of the cluster algebra with coefficients $\mathrm{CA}_{\mathbf{S}}$ of $\mathbf{S}$.*
- *Ideal triangulations of $\mathbf{S}$.*
- *Cluster-tilting objects in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ up to equivalence.*

Proof. Combining the geometrization Theorem 6.35 and Theorem 6.52, we obtain that there is a bijection between the sets of equivalence classes of cluster-tilting objects and ideal triangulations of $\mathbf{S}$. The bijection between clusters and ideal triangulation is Theorem 6.46. □

Lemma 6.54.

(1) Let γ, γ' be two distinct pure matching curves in $\mathbf{S}\backslash M$. Then

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$$

consists of those direct summands identified in Theorem 6.27 which correspond to crossings of γ and γ'.

(2) Let γ be an open matching curve in $\mathbf{S}\backslash M$. Then

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)$$

consists of those direct summands identified in Theorem 6.28 which corresponding to self-crossings.

(3) Let γ be a closed matching curve in $\mathbf{S}\backslash M$. Then

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)\,.$$

Proof. Inspecting the construction of the direct summands of $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ corresponding to the different types of intersections, one finds the following.

- Boundary intersections give rise to morphisms which evaluate non-trivially at an external edge of $\mathcal{T}$, i.e. do not get mapped by G to zero.
- Crossings give rise to morphisms which restrict to zero on all external edges of $\mathcal{T}$ and thus define direct summands lying in $\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$.
- If γ is open, then the direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)$ does not evaluate to zero under G.
- If γ is closed, then $G(N_\gamma) \simeq 0$ and hence $\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)$.

This shows the Lemma. □

Proof of Theorem 6.52. We begin with part i). If γ is a closed matching curve, then N_γ is not rigid because there is always a direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)$. We thus assume that γ is open. Theorem 6.27 and Lemma 6.54 imply that N_γ is rigid if and only if γ has no self crossings, meaning that γ is an arc, showing part i).

Using Theorem 6.27 and Lemma 6.54, we find that given two arcs γ, γ', we have an isomorphism $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) \simeq 0$ if and only if γ and γ' are compatible in the sense of Definition 6.31. By part i) and the maximality of an ideal triangulation, we find that any basic, maximal rigid object is of the form $\bigoplus_{\gamma\in I} N_\gamma$ for an ideal triangulation I and conversely any ideal triangulation I gives rise to such a basic, maximal rigid object. Since any cluster-tilting object is basic and maximal rigid, it remains to verify that

$$\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}\Big(\bigoplus_{\gamma\in I} N_\gamma, N_{\gamma'}\Big) \not\simeq 0 \tag{110}$$

for I an ideal triangulation and $(\gamma', k[t_1^{\pm}])$ a pure matching datum with $\gamma' \notin I$.

Let thus I be an ideal triangulation. Then I decomposes $\mathbf{S}$ into triangles with edges the arcs in I. If a matching curve γ' crosses an edge of one of these triangles, we find a direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$, showing (110). If γ' does not cross any arcs in I, then γ' is contained in an ideal triangle and hence already in I. This shows (110) and that $\bigoplus_{\gamma \in I} N_\gamma$ is cluster-tilting. $\square$

Recall that given a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$, we denote the associated relative Ginzburg algebra by $\mathscr{G}_{\mathcal{T}}$.

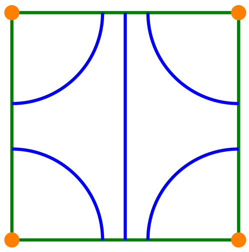
Proposition 6.55. *Consider an ideal triangulation I of $\mathbf{S}$ with dual trivalent spanning graph $\mathcal{T}$ and let $X = \bigoplus_{\gamma \in I} N_\gamma$ be the associated cluster-tilting object. There exists an isomorphism of dg-algebras (with vanishing differentials)*

$$\mathrm{H}_* \,\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(X, X) \simeq \mathrm{H}_0(\mathscr{G}_{\mathcal{T}}) \otimes_k k[t_1^{\pm}]\,.$$

In particular, the discrete endomorphism algebra $\mathrm{Ext}^0_{\mathcal{C}_{\mathbf{S}}}(X, X)$ of X is a gentle algebra.

Proof. The object N_γ associated with an edge γ of I, dual to an edge e of $\mathcal{T}$, is given by N_{c_e}. The proposition thus follows from a minor variation of the proof of Proposition 5.66. $\square$

Example 6.56. Consider the 4-gon $\mathbf{S}$, depicted as follows, with an ideal triangulation (in blue).

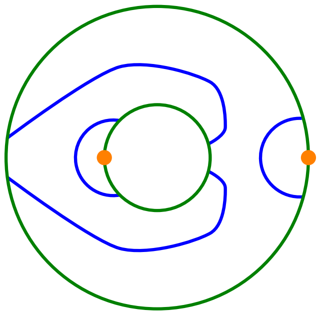


The generalized cluster category $\mathcal{C}_{\mathbf{S}}$ is equivalent to the 1-periodic derived ∞-category of the A_3-quiver, which can be defines as $\mathcal{D}(kA_3 \otimes_k k[t_1^{\pm}])$. The arising Jacobian gentle algebra is given by quotient of the path algebra of the quiver

$$\begin{tikzcd}
1 \arrow[dd, "a"'] & & 2' \arrow[dl, "b'"'] \\
& 3 \arrow[ul, "c"'] \arrow[dr, "c'"'] & \\
2 \arrow[ur, "b"'] & & 1' \arrow[uu, "a'"']
\end{tikzcd}$$

by the ideal $(ba, cb, ac, b'a', c'b', a'c')$.

Example 6.57. Consider the annulus $\mathbf{S}$ with a marked point on each boundary component and an ideal triangulation depicted as follows.



The Jacobian gentle algebra is given by the quotient of the path algebra of the quiver

$$
\begin{tikzcd}
& 3 \arrow[dl, "b"'] & \\
1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "a'"'] & & 2 \arrow[ul, "b"'] \arrow[dl, "b'"] \\
& 3' \arrow[ul, "c'"] &
\end{tikzcd}
$$

by the ideal $(ba, cb, ac, b'a', c'b', a'c')$. This is a relative version of the Kronecker quiver. While the mapping class group of the 4-gon in Example 6.56 is trivial, the mapping class group of the annulus $\mathbf{S}$ is given by $\mathbb{Z}$. The generator $1 \in \mathbb{Z}$ corresponds to a diffeomorphism rotating one boundary circle by one full rotation and fixing the other boundary circle. Under the action of an element α of the mapping class group, an object $N_\gamma \in \mathcal{C}_{\mathbf{S}}$ is mapped to $N_{\alpha\circ\gamma}$.

6.3.4 A cluster character on $\mathcal{C}_{\mathbf{S}}$

We begin with describing the notion of a cluster character on an extriangulated category, generalizing the notion of a cluster character of [Pal08].

Definition 6.58. Let $(C, \mathbb{E}, \mathfrak{s})$ be a k-linear extriangulated category. We denote by $\operatorname{obj}(C)$ the set of equivalence classes of objects in C. A cluster character χ on C with values in a commutative ring R is a map

$$\chi\colon \operatorname{obj}(C) \to R$$

such that for all $X, Y \in \mathcal{C}$ the following holds.

- $\chi(X) = \chi(Y)$ if $X \simeq Y$.
- $\chi(X \oplus Y) = \chi(X)\chi(Y)$.
- $\chi(X)\chi(Y) = \chi(B) + \chi(B')$ if $\dim_k \mathbb{E}(X, Y) = \dim_k \mathbb{E}(Y, X) = 1$ and $B, B' \in C$ are the middle terms of the corresponding non-split extensions of X by Y and Y by X.

The last property is called the cluster multiplication formula.

Given a pure matching curve γ in $\mathbf{S}\backslash M$, we consider the underlying curve as a link in $\mathbf{S}\backslash M$ with a single component, denoted $l(\gamma)$.

Theorem 6.59. *Let k be an algebraically closed field. Consider the map*

$$\chi\colon \operatorname{obj}(\operatorname{ho}\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}) \to \mathrm{Sk}^1_{\mathbf{S}}$$

determined by

- $\chi(A) = \chi(B)$ *if* $A \simeq B$
- $\chi(A \oplus B) = \chi(A)\chi(B)$
- $\chi(N_\gamma) = l(\gamma)$ *for any pure matching datum* $(\gamma, k[t_1^{\pm}])$ *in* $\mathbf{S}\backslash M$*.*

Then χ is a cluster character on $(\operatorname{ho}\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$.

Remark 6.60. Composing χ with $\mathrm{Sk}^1_{\mathbf{S}} \hookrightarrow \mathrm{Sk}^{1,\mathrm{loc}}_{\mathbf{S}} \hookrightarrow \mathrm{UCA}_{\mathbf{S}}$ defines a cluster character to the upper cluster algebra of $\mathbf{S}$. If $\mathbb{S}$ has at least two marked points, the upper cluster algebra is equivalent to the cluster algebra with coefficients $\mathrm{CA}^{\mathrm{loc}}_{\mathbf{S}}$ localized at the boundary arcs, see Theorem 6.51.

Proof of Theorem 6.59. The geometrization Theorem 6.35 shows that χ is well-defined. It is also clear that χ satisfies all parts of Definition 6.58, except for the cluster multiplication formula. We thus consider two pure matching data $(\gamma, k[t_1^{\pm}])$ and $(\gamma', k[t_1^{\pm}])$ in $\mathbf{S}\backslash M$, which satisfy $\dim_k \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) = 1$ and let $B, B' \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ be the corresponding extensions. Suppose that γ and γ' are distinct. By Theorem 6.27, $\dim_k \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) = 1$ implies that γ, γ' have a single crossing. The cluster multiplication formula $\chi(N_\gamma)\chi(N_{\gamma'}) = \chi(B) + \chi(B')$ thus follows from Proposition 6.32, see also Remark 6.34.

If $\gamma = \gamma'$, we find that γ is closed and that $(\gamma, k[t_1^{\pm}]), (\gamma', k[t_1^{\pm}])$ have the same monodromy datum. Denote by a and a' the ranks of γ and γ'. We show that $\dim_k \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ is always an even number and hence not equal to 1. Self-crossings of γ give rise to two crossings of γ, γ' and thus contribute the number $2aa'$ to $\dim_k \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$. The global section N_γ arises as a coequalizer, see (81). It follows that $\operatorname{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ is equivalent to the fiber of some morphism

$$k[t_1^{\pm}]^{\oplus aa'} \longrightarrow k[t_1^{\pm}]^{\oplus aa' + 2aa' i^{\mathrm{cr}}(\gamma,\gamma')}\,. \tag{111}$$

Inspecting the construction of the morphism (111), one finds that it lies in the image of the functor $\zeta^*\colon \mathcal{D}(k[t_1^{\pm}]) \to \mathcal{D}(k[t_2^{\pm}])$. Using that every morphism in $\mathcal{D}(k[t_1^{\pm}])$ splits into a direct sum of an equivalence and a zero morphism, it follows that $\operatorname{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'}) = \operatorname{Mor}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_{\gamma'})$ is free of even rank in $\mathcal{D}(k[t_1^{\pm}])$, which implies that $\dim_k \operatorname{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(N_\gamma, N_\gamma)$ is indeed an even number. This concludes the case distinction and the proof. □

Remark 6.61. The proof of Theorem 6.59 shows that there is some flexibility in the formula for the cluster character. In fact, it appears that one can assign arbitrary values to the closed matching curves of rank $a \geq 2$.

6.4 Further topics

In Section 6.4.1, we discuss the relation between the generalized cluster category $\mathcal{C}_{\mathbf{S}}$ and the previously known 2-Calabi–Yau triangulated cluster category associated with $\mathbf{S}$. In Section 6.4.2, we describe the generalized n-cluster categories arising from relative higher Ginzburg algebras associated to n-angulated marked surfaces.

6.4.1 The stable ∞-category of $\mathcal{C}_{\mathbf{S}}$

Let $\mathbf{S}$ be a marked surface and $\mathcal{C}_{\mathbf{S}}$ the associated generalized cluster category, considered as a Frobenius exact ∞-category. We begin by noting that the mapping class group action on $\mathcal{C}_{\mathbf{S}}$ induces an action on the stable ∞-category $\bar{\mathcal{C}}_{\mathbf{S}}$.

Proposition 6.62. *The mapping class group action on $\mathcal{C}_{\mathbf{S}}$ induces an action on $\bar{\mathcal{C}}_{\mathbf{S}}$ by automorphisms in* $\mathrm{ho}\,\mathrm{St}$*.*

Proof. The action of the mapping class group does not affect the values of the global sections at the external edges of $\mathcal{T}$. It follows that the action preserves the set W from Proposition 4.49 and hence induces an action on the localization. □

The homotopy 1-category $\mathrm{ho}\,\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ of the stable ∞-category $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ is a triangulated category and triangulated 2-Calabi–Yau, see Lemma 4.60. There is a second 2-Calabi–Yau triangulated category associated to $\mathbf{S}$, the cluster category denoted $\mathrm{ho}\,\mathcal{C}'_{\mathbf{S}}$. The stable ∞-category $\mathcal{C}'_{\mathbf{S}}$ arises as the generalized cluster category of the non-relative Ginzburg algebra associated to any choice of ideal triangulation I of $\mathbf{S}$, see also Section 5.1.1, which we denote by $\mathscr{G}_I$. The triangulated homotopy category of $\mathcal{C}'_{\mathbf{S}}$ was considered for instance in [BZ11]. One can observe a number of similarities between $\mathcal{C}'_{\mathbf{S}}$ and $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$.

- The classifications of indecomposables in $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ and $\mathcal{C}'_{\mathbf{S}}$, see [BZ11], match.
- It is shown in [ZZZ13] that the dimensions of the Ext^1's in $\mathcal{C}'_{\mathbf{S}}$ match the dimensions of the Ext^1's in $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$, which are obtained by counting crossings of the pure matching curves.
- Remark 4.67 shows that $S^{-1}[2] \simeq \mathrm{id}$ on objects in $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$, where S is the Serre functor of $\mathcal{C}_{\mathbf{S}}$. The action of the Serre functor can be described in terms of the matching curves in terms of a partial rotation of the boundary circles of $\mathbf{S}$. A similar description of the shift functor in $\mathcal{C}'_{\mathbf{S}}$ is given in [QZ17, Thm. 5.2]. One can also compare this description in the acyclic case to the orbit category construction of the cluster category.

We hence expect the following.

Conjecture 6.63. *There exist an equivalence of stable ∞-categories*

$$\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}} \simeq \mathcal{C}'_{\mathbf{S}}\,.$$

We prove a triangulated version of the conjecture in the acyclic case using the results of [KR08].

Theorem 6.64. *Suppose that there exist an ideal triangulation I of* $\mathbf{S}$*, such that the quiver underlying $\mathscr{G}_I$ is acyclic. Then there exists an equivalence of triangulated categories* $\operatorname{ho}\bar{\mathcal{C}}^{c}_{\mathbf{S}} \simeq \operatorname{ho}\mathcal{C}'_{\mathbf{S}}$.

Proof. The stable ∞-category $\bar{\mathcal{C}}^{c}_{\mathbf{S}}$ admits the structure of a k-linear ∞-category, see Lemma 4.50. Using that $\mathcal{C}^{c}_{\mathbf{S}}$ is Krull-Schmidt and the classification of indecomposable objects, it is straightforward to check that $\bar{\mathcal{C}}^{c}_{\mathbf{S}}$ is also idempotent complete. Hence, it arises as the perfect derived ∞-category of a dg-category and the triangulated homotopy category $\operatorname{ho}\bar{\mathcal{C}}^{c}_{\mathbf{S}}$ is an algebraic triangulated category. The triangulated category $\operatorname{ho}\bar{\mathcal{C}}^{c}_{\mathbf{S}}$ is triangulated 2-Calabi–Yau by Lemma 4.60. It also has a cluster-tilting object X with discrete endomorphism algebra $\mathrm{H}_0(\mathscr{G}_I)$ as follows from Proposition 6.55 and the fact that

$$\operatorname{Hom}_{\operatorname{ho}\bar{\mathcal{C}}^{c}_{\mathbf{S}}}(X,X) \simeq \operatorname{Hom}_{\operatorname{ho}\mathcal{C}^{c}_{\mathbf{S}}}(X,X)/\mathcal{P}\,,$$

with $\mathcal{P}$ the ideal of morphisms factoring through an injective projective object. The Theorem now directly follows from the main result of [KR08]. $\square$

Remark 6.65. The surfaces which admit an ideal triangulation I, such that the quiver underlying $\mathscr{G}_I$ is acyclic include the disc and the annulus, each with any collection of boundary marked points.

6.4.2 Generalized n-cluster categories of marked surfaces

The study of 2-Calabi–Yau cluster categories admits a generalization to triangulated $(n-1)$-Calabi–Yau categories with $n \geq 3$, called $(n-1)$-cluster categories. Their representation theoretic behavior is very similar to the 2-cluster categories. For example, there is a notion of $(n-1)$-cluster-tilting objects which admit mutations similar to the 2-Calabi–Yau case. $(n-1)$-Cluster categories for $n \geq 3$ however do not categorify cluster algebras. They can be constructed in the same ways as the 2-cluster categories, for example as orbit categories or explicit geometric constructions. There is also a higher Calabi–Yau-dimensional analog of Amiot's construction of generalized cluster categories, see [Guo11]. A possible input for this construction is a higher Ginzburg algebra, meaning an n-Calabi–Yau version of the usual Ginzburg algebra.

We fix a marked surface $\mathbf{S}$ and choose an n-valent spanning graph $\mathcal{T}$ of $\mathbf{S}$. With this setup, we described in Section 5 a relative higher Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ and a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$, with global sections its derived ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$. In this section, we describe the generalized $(n-1)$-cluster category, denoted $\mathcal{C}^{n-1}_{\mathbf{S}}$, arising from $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ and survey how our results generalize to this category.

Locally at each vertex of $\mathcal{T}$, the perverse schober $\mathcal{F}_{\mathcal{T}}$ is encoded by the spherical adjunction

$$\phi^*\colon \mathcal{D}(k) \longleftrightarrow \mathcal{D}(k[t_{n-2}]) :\phi_*\,,$$

where $\phi\colon k[t_{n-2}] \xrightarrow{t_{n-2}\mapsto 0} k$. By Lemma 6.3, the quotient $\mathcal{D}(k[t_{n-2}])/\operatorname{Ind}\mathcal{D}(k[t_{n-2}])^{\mathrm{fin}}$ is equivalent to the derived ∞-category $\mathcal{D}(k[t_{n-2}^{\pm}])$ of $(n-2)$-periodic chain complexes.

Theorem 6.66. *There exist a vanishing-monadic and nearby-monadic $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ and a locally constant $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ with generic stalk $\mathcal{D}(k[t_{n-2}^{\pm}])$ satisfying the following.*

i) There is a semiorthogononal decomposition $\{\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}\}$ of $\mathcal{F}_{\mathcal{T}}$.

ii) There exists an equivalence

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\,.$$

iii) There exists an equivalence

$$\mathcal{C}_{\mathbf{S}}^{n-1} := \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\,.$$

Proof. Analogous to the proofs of Proposition 6.9 and Theorem 6.15. □

We thus find the generalized $(n-1)$-Calabi–Yau cluster category

$$\mathcal{C}_{\mathbf{S}}^{n-1} \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$$

of the n-angulated marked surface $\mathbf{S}$ to be the topological Fukaya category of $\mathbf{S}$ with coefficients in the derived ∞-category of $(n-2)$-periodic chain complexes. In particular, the well-known 2-periodic topological Fukaya category of a surface may thus be seen as the generalized 3-cluster category of the surface.

The ∞-category $\mathcal{C}_{\mathbf{S}}^{n-1}$ is smooth and proper as an $(n-2)$-periodic, or $2(n-2)$-periodic if n is odd, stable ∞-category. The functor

$$\prod_{e\in\mathcal{T}_1^{\partial}} \mathrm{ev}_e\colon \mathcal{C}_{\mathbf{S}}^{n-1} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \longrightarrow \prod_{e\in\mathcal{T}_1^{\partial}} \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \simeq \prod_{e\in\mathcal{T}_1^{\partial}} \mathcal{D}(k[t_{n-2}^{\pm}])$$

further admits a weak right $(n-1)$-Calabi–Yau structure.

The results of Section 5 imply a geometric model for $\mathcal{C}_{\mathbf{S}}^{n-1}$, describing (a priori a subset of) objects in $\mathcal{C}_{\mathbf{S}}^{n-1}$ in terms of matching data with local value $k[t_{n-2}^{\pm}][i]$, $0 \leq i \leq n-3$. The main difference to the geometric model for $\mathcal{C}_{\mathbf{S}}^{n}$ is thus that we may associate objects to matching curves equipped with a grading datum in $\mathbb{Z}/(n-3)\mathbb{Z}$. Since in the n-gon, with $n > 3$, not every arc is homotopic to a pure matching curve, objects associated with non-pure matching curves must be considered for $\mathcal{C}_{\mathbf{S}}^{n-1}$. Theorems 5.53 and 5.54 thus do not describe all morphism objects between the associated global sections. This should be considered as a technical artifact arising from the approach to the geometric model which passes through the derived ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ of the relative higher Ginzburg algebra. A direct approach is applicable to prove analogs of Theorems 5.53 and 5.54 for all objects in $\mathcal{C}_{\mathbf{S}}^{n-1}$ coming from matching curves. Furthermore, the proof of the geometrization Theorem 6.35

applies with minor modifications also to $\mathcal{C}_{\mathbf{S}}^{n-1}$, showing that all compact objects in $\mathcal{C}_{\mathbf{S}}^{n-1}$ arise from collections of graded matching curves in $\mathbf{S}\backslash M$.

As in Section 4.4.2, we can use the functor $\prod_{e\in\mathcal{T}_1^{\partial}}\mathrm{ev}_e$ to define an exact ∞-structure on $\mathcal{C}_{\mathbf{S}}^{n-1,\mathrm{c}}$ and hence also an extriangulated structure on the homotopy category $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}^{n-1,\mathrm{c}}$. This exact ∞-structure is again Frobenius.

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