Embeddings of cycle-like structures

Dissertation

zur Erlangung des Doktorgrades an der
Fakulät für Mathematik, Informatik und Naturwissenschaften

Fachbereich Mathematik
der Universität Hamburg

vorgelegt von
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Hamburg
2023

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Datum der Disputation:	07. September 2023

urn:nbn:de:gbv:18-ediss-111894

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1. Introduction

This thesis concerns problems in Extremal and Probabilistic Combinatorics. In the core of this work, we will use absorption techniques for embedding spanning subgraphs, which entails applying regularity and probabilistic methods. The thesis is organized in three chapters, each related to a different problem.

The work in Chapter 2 aims at a common generalization of classical embedding results for spanning subgraphs, which rely on a minimum degree condition, and more recent work based on the edge distribution of the host graph. The classical results are first a conjecture by Pósa and Seymour, proven by Hajnal and Szemerédi, that establishes conditions on the minimum degree of graphs that guarantee K_{k+1}-factors and k-th powers of Hamiltonian cycles and second, the Bandwidth Theorem of Böttcher, Taraz and Schacht, that gives a condition on the minimum degree that suffices for the embedding of spanning graphs with bounded chromatic number and sublinear bandwidth. Our aim was to instead of considering the minimum degree, to focus on the distribution of edges. Therefore we worked on the setting where the host graph is dense and inseparable. The minimum degree condition and the dense and inseparable condition are not comparable to each other and, in the continuation presented in this thesis, we address the question of finding a common generalization. We present a setting for the host graph that ensures any tripartite spanning subgraph with bounded degree and small bandwidth. This proof relies on the regularity and absorption methods. This work is the main part of this thesis and we discuss the main result in Section 1.1.

In Chapter 3, we consider the Size-Ramsey number of powers of bounded degree trees and show that it is linear on the number of vertices of the tree. Our proof uses expansion properties of random graphs, algorithmic embedding methods of Friedman and Pippenger, the Kővari-Sós-Turán Theorem and the Local Lemma, moreover we devise an inductive scheme to obtain the result for any (bounded) number of colour. This is joint work with Sören Berger, Yoshiharu Kohayakawa, Taísa Martins, Walner Mendonça, Guilherme Oliveira Mota and Olaf Parczyk. The results on this problem are presented in Section 1.2.

In Chapter 4, we study the model of randomly perturbed graphs, in which we consider the union of a deterministic graph G_α with minimum degree αn and the binomial random graph $G(n,p)$. We want to give a different view to the previous result of Bohman, Frieze, and Martin that set the threshold for a Hamiltonian cycle in the randomly perturbed model with $\alpha > 0$. We are interested in determining the threshold for the sparse case, when $\alpha = o(1)$. Here we use thresholds results on $G(n,p)$ for almost spanning structures combined with the absorption method. This is joint work with Max Hahn-Klimroth, Yannick Mogge, Samuel Mohr and Olaf Parczyk. The results on this problem are presented in Section 1.3.

1.1 Spanning tripartite subgraphs

We study sufficient conditions for the existence of spanning subgraphs in large finite graphs. Our aim is to find a common generalization of previous results for embedding 3-chromatic graphs with small bandwidth and bounded maximum degree. An n-vertex graph $H = (V,E)$ has *bandwidth* at most $b \in \mathbb{R}$ if there is some bijection $\sigma \colon V \longrightarrow [n]$ such that for every edge $xy \in E$, we have $|\sigma(x) - \sigma(y)| \leqslant b$. We denote by $\mathrm{bw}(H)$ the smallest such b.

We start by approaching the problem of embedding the 2nd power of a Hamiltonian cycle. For $k \in \mathbb{N}$ the *k-th power* of a given graph H is the graph H^k on the same vertex set with xy being an edge in H^k if x and y are distinct vertices of H that are connected in H by a path of at most k edges. We refer to a k-th power of a path as a *k-path*. Note that every $k+1$ consecutive vertices of a k-path span a clique and if a graph $G = (V,E)$ contains the k-th power of a Hamiltonian cycle, it also contains $\lfloor \frac{|V|}{k+1} \rfloor$ pairwise vertex disjoint copies of K_{k+1} and G contains a K_{k+1}-factor if $|V|$ is divisible by $k+1$.

Dirac's well known theorem [28] yields a best possible minimum degree condition for embedding a Hamiltonian cycle. The minimum degree of a graph turned out to be an interesting parameter for enforcing a given spanning subgraph and establishing optimal minimum degree conditions for those problems became a fruitful research direction in extremal graph theory (see, e.g., [20] and the references therein). Already about 50 years ago, the minimum degree problem for K_{k+1}-factors was resolved by Corrádi and Hajnal [23] for $k = 2$ and by Hajnal and Szemerédi [45] for every $k \geqslant 3$. Pósa (see [38]) and Seymour [82] asked for a common generalisation of those results on factors and Dirac's theorem and conjectured that the best possible minimum degree conditions for K_{k+1}-factors and k-th powers of Hamiltonian cycles are the same (given that the number of vertices is divisible by $k+1$). The general conjecture was affirmatively resolved for sufficiently large graphs by Komlós, Sárközy, and Szemerédi [59] by establishing the following result.

Theorem 1.1.1 (Komlós, Sarközy & Szemerédi 1998)**.** *For every positive integer k there exists n_0 such that if G is a graph on $n \geqslant n_0$ vertices with minimum degree $\delta(G) \geqslant \frac{k}{k+1}n$, then G contains the k-th power of a Hamiltonian cycle.* □

For spanning graphs with fixed chromatic number and maximum degree and with small bandwidth, we have the following result [20].

Theorem 1.1.2 (Böttcher, Schacht & Taraz 2009)**.** *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds.*

If H is a graph on n vertices with chromatic number $\chi(H) \leqslant r$, with maximum degree $\Delta(H) \leqslant \Delta$, and with bandwidth $\mathrm{bw}(H) \leqslant \beta n$ and if G is a graph on n vertices with minimum degree $\delta(G) \geqslant (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H. □

Strictly speaking, the way we state Theorem 1.1.2, besides the $\gamma > 0$, is not a generalisation of Dirac's theorem, since a Hamiltonian cycle might be 3-chromatic. However, the proof in [20] is robust enough to cover this case and, in fact, to cover any $(r+1)$-chromatic graph H which is "essentially" r-chromatic (see [20, Theorem 2] for details).

Moreover, observe that the condition on the minimum degree is asymptotically optimal, since complete almost balanced r-partite graphs do not contain almost perfect K_r-factors. Therefore, any attempt to weaken the minimum degree condition must introduce some new requirements for G, that prevent those partite lower bound constructions. One possible way to achieve this, relies in restricting the independence number of the large graph G (see, e.g., [6]). Staden and Treglown [84] considered the following robust restriction that uniformly imposes a positive edge density for induced subgraphs on linear sized subsets of vertices.

Definition 1.1.3. We say that a graph $G = (V, E)$ is (ϱ, d)-*dense* for $\varrho > 0$ and $d \in [0, 1]$ if $e(X) \geqslant d\frac{|X|^2}{2} - \varrho|V|^2$ for every subset $X \subseteq V$, where $e(X)$ denotes the number of edges of G that are contained in X.

Staden and Treglown showed that for any d, $\mu > 0$ and sufficiently small $\varrho > 0$ and sufficiently large n, every n-vertex graph G that is (ϱ, d)-dense and has $\delta(G) \geqslant (\frac{1}{2} + \mu)n$ contains any spanning subgraph H with constant bounded maximum degree and sublinear bandwidth (see also [6, Appendix by Reiher and Schacht] for K_{k+1}-factors). Note that the minimum degree requirement for G becomes independent of the chromatic number of H (still the chromatic number of H is implicitly bounded by its maximum degree). This points that the density condition is too strong to distinguish on the chromatic number of H.

The degree condition $\delta(G) \geqslant (\frac{1}{2} + \mu)n$ is essentially optimal, since the graph G consisting of two disjoint cliques on close to $\frac{n}{2}$ vertices (one of them with the number of vertices not divisible by $k+1$) has no clique factor nor the power of a Hamiltonian cycle. However, a bipartite version of Definition 1.1.3, which requires

$$e(X, Y) = \big|\{(x, y) \in X \times Y : xy \in E(G)\}\big| \geqslant \mu\,|X||Y| - \varrho|V|^2\,, \tag{1.1.1}$$

for all subsets X, $Y \subseteq V$, rules out this example. It was observed by Glock and Joos (see [84, Concluding Remarks]) that imposing property (1.1.1) on G allows a further relaxation on the minimum degree condition for G to $\delta(G) \geqslant \mu|V|$ for arbitrary $\mu > 0$. In [EMR+20, 33] it is shown that property (1.1.1) is not needed for arbitrary subsets X and Y; it suffices to assume it only for vertex bipartitions of G as follows.

Definition 1.1.4. A graph $G = (V, E)$ is called μ-*inseparable* for some $\mu > 0$ if for every subset $X \subseteq V$ we have $e(X, V \setminus X) \geqslant \mu\,|X||V \setminus X|$.

Invoking this assumption to subsets X consisting of one vertex only, yields a linear minimum degree condition for μ-inseparable graphs G. Equipped with Definitions 1.1.3 and 1.1.4, the following versions of the theorem for powers of Hamiltonian cycles and the bandwidth theorem are obtained.

Theorem 1.1.5 (Ebsen et al. 2020)**.** *For every d, $\mu \in (0,1]$, and $k \in \mathbb{N}$ there exist $\varrho > 0$ and n_0 such that every (ϱ, d)-dense and μ-inseparable graph G on $n \geqslant n_0$ vertices contains the k-th power of a Hamiltonian cycle.*

Theorem 1.1.6 (Ebsen et al. 2020)**.** *For all $d > 0$, $\mu > 0$ and $\Delta \in \mathbb{N}$, there exist $\varrho, \beta > 0$, and n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices is (ϱ, d)-dense and μ-inseparable, then G contains every n-vertex graph H with $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta n$.

This generalises the result of Staden and Treglown, since $\delta(G) \geqslant (\frac{1}{2} + \mu)|V|$ implies that G is μ-inseparable. In view of Theorems 1.1.1 and 1.1.2, even though Theorems 1.1.5 and 1.1.6 can be applied for sparser graphs, the condition $\delta(G) \geqslant (\frac{r-1}{r} + \gamma)n$ does not imply that G is (ϱ, d)-dense, as an example consider the complete balanced $(r+1)$-partite graph on n vertices.

In a work of Knox and Treglown [55], a new condition on G is introduced, that yields, for the special case of H being bipartite, to a common generalisation for Theorems 1.1.2 and 1.1.6.

Observe that a necessary condition for a Hamiltonian cycle is that G contains a perfect matching (plus a vertex if n is odd) and that G being μ-inseparable alone is not enough to assure a perfect matching, as for example unbalanced complete bipartite graphs show. Moreover, we want a condition which is robust enough to be transferred to the reduced graph corresponding to G and a regular partition of its vertices. The previous bandwidth results were also based on applying a simpler result to the reduced graph, in order to prepare for an application of the blow-up lemma in the original graph G.

In view of Tutte's theorem [85], consider a graph G, a set $S \in V(G)$ and an independent set I which has a vertex in each odd component of $G[V(G) \smallsetminus S]$. Observe that only vertices of S can have more than one neighbour in I. Therefore we turn our attention to

$$N_2(I) = \{v \in V(G) \colon |N(v) \cap I| \geqslant 2\}. \tag{1.1.2}$$

If we ask that for every independent I with $|I| \geqslant cn$ we have $|N_2(I)| \geqslant |I|$, then by Tutte's theorem we get that a maximum matching of G has size at least $\frac{(1-c)n}{2}$. This motivates the following robust version of (1.1.2).

Definition 1.1.7. Let $d \geqslant 0$, $\varrho > 0$ be given. A graph $G = (V, E)$ is (ϱ, d)*-robust matchable* if for every $U \subseteq V$, either

(*i*) $e(U) \geqslant d\frac{|U|^2}{2} - \varrho n^2$ or

(*ii*) $|N_d(U)| = \Big|\{v \in V \smallsetminus U : |N(v) \cap U| \geqslant d|U|\}\Big| \geqslant |U| + dn.$

Obviously (ϱ, d)-dense graphs are (ϱ, d)-robust matchable. The latter does not necessarily require that every subset of vertices U is dense (only large ones), but if a subset is sparse, then the set U must have a large *robust neighbourhood* $N_d(U)$, meaning many vertices outside U having many neighbours inside U.

It is not directly clear that $\delta(G) \geqslant (\frac{1}{2} + \gamma)n$ implies that G is $(\varrho, \frac{\gamma}{2})$-robust matchable for any $\varrho > 0$; we briefly verify it. For $U \subseteq V(G)$ such that $|U| > (\frac{1}{2} - \frac{\gamma}{2})n$, the minimum degree condition implies that each $v \in U$ has at least $\frac{\gamma n}{2}$ neighbours in U, therefore property (*i*) is satisfied for large sets. For $|U| \leqslant (\frac{1}{2} - \frac{\gamma}{2})n$, assume $e(U) < \frac{\gamma}{2} \cdot \frac{|U|^2}{2}$. Then consider the edges between U and the vertices which are neither in U nor in $N_{\frac{\gamma}{2}}(U)$, these edges are at most $\frac{\gamma}{2}|U|n$. We have that

$$|N_{\frac{\gamma}{2}}(U)||U| + \frac{\gamma}{2}|U|n \geqslant e(U, V \smallsetminus U) \geqslant |U|\left(\frac{n}{2} + \gamma n\right) - 2 \cdot \frac{\gamma}{2} \cdot \frac{|U|^2}{2}.$$

From which property (*ii*) follows

$$|N_{\frac{\gamma}{2}}(U)| \geqslant \frac{n}{2} + \gamma n - \frac{\gamma n}{2} - \frac{\gamma n}{2} \geqslant |U| + \frac{\gamma}{2}n.$$

Consequently, the following theorem generalises Theorems 1.1.2 and 1.1.6 for the case of bipartite H.

Theorem 1.1.8 (Knox & Treglown 2012)**.** *For every d, $\mu > 0$, and $\Delta \in \mathbb{N}$, there exist β, $\varrho > 0$, and n_0 such that the following holds.*

If G is a μ-inseparable and (ϱ, d)-robust matchable graph on $n \geqslant n_0$ vertices and H is a bipartite graph on n vertices with $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta n$, then $H \subseteq G$.

Our goal is to find a condition to G that generalises Theorems 1.1.2 and 1.1.6 for the case of 3-chromatic H, and that hopefully can be generalized for all fixed chromatic number. Our approach follows the main ideas from [EMR$^+$20, 33] and makes use of the absorbtion method and the regularity method for graphs.

The first step of the proof is using the absorption method for finding a 2nd power of a Hamiltonian cycle in the reduced graph corresponding to G and a regular partition; that will allow us to prepare G for an application of the blow-up lemma.

For the describing that G is well connected, we use the following notions. We are given a graph $G = (V, E)$ and its robust neighbourhoods, meaning

$$\mathcal{R}_G = \{U_v \subseteq N(v) \colon v \in V(G)\} .$$

Definition 1.1.9. Given $\zeta > 0$ and graph G with $\mathcal{R}_G$, we define the *auxiliary graph* $\mathfrak{A} = A(\mathcal{R}_G, \zeta)$ by setting

$$V(\mathfrak{A}) = V(G) \quad \text{and} \quad E(\mathfrak{A}) = \{uv \colon |E(G[U_u]) \cap E(G[U_v])| \geqslant \zeta |V(G)|^2\} .$$

Definition 1.1.10. Given $\xi > 0$, a graph G and $\mathcal{R}_G$, we say an edge $uv \in E(G)$ is *ξ-connectable* in G, if

$$|\{w \in V(G) \colon uv \in E(G[U_w])\}| \geqslant \xi |V(G)| .$$

Moreover if G contains a 2nd power of a Hamiltonian cycle, then it contains a triangle-factor (if $|V(G)|$ is multiple of 3); we need the following notion.

Definition 1.1.11. Given G, let

$$T_G = \{xyz \colon xyz \text{ is a triangle in } G\} .$$

A *fractional triangle factor of G* is a function $f \colon T_G \to [0, 1]$ such that for every $v \in V(G)$, the *weight of v* satisfies $\Sigma_{xyv \in T_G} f(xyv) \leqslant 1$. We define the *weight of f* by $W(f) = \Sigma_{xyz \in T_G} f(xyz)$.

Given robust neighbourhoods $\mathcal{R}_G$, we say f is a *ξ-connectable triangle factor* if $f(xyz) > 0$ implies that xyz is ξ-connectable in G, meaning that either xy, yz or xz is ξ-connectable in G.

We are ready to state the desired property.

Definition 1.1.12. For $\mu, \varrho, \delta, \zeta, \xi, \eta, \nu \in (0, 1]$ and $\eta \geqslant \max\{2\nu, 4\sqrt{\varrho}\}$, $\xi \geqslant 4\sqrt{\varrho}$. A graph G on n vertices with robust neighbourhoods $\mathcal{R}_G$ is *$(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good* if the following holds.

i. Each $G[U_v]$ is μ-inseparable and contains at least δn^3 different ξ-connectable triangles.

ii. The auxiliary graph $A(\mathcal{R}_G, \zeta)$ is μ-inseparable.

iii. For any $A \subseteq V(G)$ with $|A| \leqslant \nu n$ and $F \subseteq E(G)$ with $|F| \leqslant \varrho n^2$, let $X_F \subseteq V(G)$ be those vertices incident to at least $\sqrt{\varrho} n$ edges of F and $G_{A,F}$ be the graph with

$$V(G_{A,F}) = V(G) \setminus (A \cup X_F) \quad \text{and} \quad E(G_{A,F}) = E(G[V(G_{A,F})]) \setminus F\,.$$

The graph $G_{A,F}$ contains a ξ-connectable (in G) fractional triangle factor $f_{G_{A,F}}$ with

$$W(f_{G_{A,F}}) \geqslant \frac{n}{3} - \eta(n - |A|)\,.$$

Using the absorption method, we are able to proof the following result for the 2nd power of a Hamiltonian cycle.

Theorem 1.1.13. *Given $\mu, \delta, \zeta, \xi > 0$ and c_ϱ, there exist n_0 and $\eta, \varrho > 0$ with $\eta, \xi \geqslant c_\varrho \sqrt[8]{\varrho}$ such that the following holds.*

For every $\nu > 0$ such that $\eta, \xi \geqslant 2\nu$, if G is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good, then G contains the second power of a Hamiltonian cycle.

With the stronger assumption that the auxiliary graph $\mathfrak{A}_G$ is complete, we are able to show that there is a regular partition of $V(G)$ for which the reduced graph inherits the good property of G. Consequently, a slightly stronger version of Theorem 1.1.13 (see Theorem 2.2.1) prepares the reduced graph for an application of the blow-up lemma. This way, we obtain a bandwidth theorem for good graphs.

Theorem 1.1.14. *For every $\mu, \delta, \zeta, \xi > 0$ and positive integer Δ, there exist $\beta, \varrho, \eta, \nu > 0$ and n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good and $\mathfrak{A}_G$ is complete and if H on n vertices is such that $\chi(H) \leqslant 3$, $bw(H) \leqslant \beta n$ and $\Delta(H) \leqslant \Delta$, then $H \subseteq G$.

In Section 2.6.4, we show that graphs with minimum degree $\delta(G) \geqslant (\frac{2}{3} + \gamma)n$ are sufficiently good, so that Theorems 1.1.13 and 1.1.14 apply to them. We believe that the same holds for dense and inseparable graphs. Similar and more general results were obtained by Lang and Sanhueza-Matamala in [70].

1.2 The size-Ramsey number of powers of bounded degree trees

Given graphs G and H and a positive integer s, we denote by $G \longrightarrow (H)_s$ the property that any s-colouring of the edges of G contains a monochromatic copy of H. We are interested in the problem proposed by Erdős, Faudree, Rousseau and Schelp [35] of determining the minimum integer m for which there is a graph G with m edges such that property $G \longrightarrow (H)_2$ holds. Formally, the *s-colour size-Ramsey number* $\hat{r}_s(H)$ of a graph H is defined as follows:

$$\hat{r}_s(H) = \min\{e(G) \colon G \to (H)_s\}.$$

Answering a question posed by Erdős [34], Beck [8] showed that $\hat{r}_2(P_n) = O(n)$ by means of a probabilistic proof. Alon and Chung [1] proved the same fact by explicitly constructing a graph G with $O(n)$ edges such that $G \longrightarrow (P_n)_2$. In the last decades many successive improvements were obtained in order to determine the size-Ramsey number of paths (see, e.g., [8, 15, 30] for lower bounds, and [8, 29, 30, 71] for upper bounds). The best known bounds for paths are $\frac{5}{2}n - \frac{15}{2} \leqslant \hat{r}_2(P_n) \leqslant 74n$ from [30]. For any $s \geqslant 2$ colours, Dudek and Prałat [30] and Krivelevich [65] proved that there are positive constants c and C such that $cs^2 n \leqslant \hat{r}_s(P_n) \leqslant Cs^2(\log s)n$.

Moving away from paths, Beck [8] asked whether $\hat{r}_2(H)$ is linear for any bounded degree graph. This question was later answered negatively by Rödl and Szemerédi [81], who constructed a family $\{H_n\}_{n\in\mathbb{N}}$ of n-vertex graphs of maximum degree $\Delta(H_n) \leqslant 3$ such that $\hat{r}_2(H_n) = \Omega(n \log^{1/60} n)$. The current best upper bound for the size-Ramsey number of graphs with bounded degree was obtained in [57] by Kohayakawa, Rödl, Schacht and Szemerédi, who proved that for any positive integer Δ there is a constant c such that, for any graph H with n vertices and maximum degree Δ, we have

$$\hat{r}_2(H) \leqslant cn^{2-1/\Delta} \log^{1/\Delta} n.$$

For more results on the size-Ramsey number of bounded degree graphs see [26, 41, 48, 49, 54, 56].

Let us turn our attention to powers of bounded degree graphs. Let H be a graph with n vertices and let k be a positive integer. The *kth power* H^k of H is the graph with vertex set $V(H)$ in which there is an edge between distinct vertices u and v if and only if u and v are at distance at most k in H. Recently it was proved that the 2-colour size-Ramsey number of powers of paths and cycles is linear [21]. This result was extended to any fixed number s of colours in [46], i.e.,

$$\hat{r}_s(P_n^k) = O_{k,s}(n) \quad \text{and} \quad \hat{r}_s(C_n^k) = O_{k,s}(n). \tag{1.2.1}$$

In our main result (Theorem 1.2.1) we extend (1.2.1) to bounded powers of bounded degree trees. We prove that for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n.

Theorem 1.2.1. *For any positive integers k, Δ and s and any n-vertex tree T with $\Delta(T) \leqslant \Delta$, we have*

$$\hat{r}_s(T^k) = O_{k,\Delta,s}(n).$$

We remark that Theorem 1.2.1 is equivalent to the following result for the 'general' or 'off-diagonal' size-Ramsey number $\hat{r}(H_1, \ldots, H_s) = \min\{e(G) \colon G \longrightarrow (H_1, \ldots, H_s)\}$. If $H_i = T_i^k$ for $i = 1, \ldots, s$ where $T_1, \ldots, T_s$ are bounded degree trees, then $\hat{r}(H_1, \ldots, H_s)$ is linear in $\max_{1 \leqslant i \leqslant s} v(H_i)$. To see this, it is sufficient to apply Theorem 1.2.1 to a tree containing the disjoint union of $T_1, \ldots, T_s$.

The graph that we present to prove Theorem 1.2.1 does not depend on T, but only on Δ, k and n. Moreover, our proof not only gives a monochromatic copy of T^k for a given T, but a monochromatic subgraph that contains a copy of the kth power of every n-vertex tree with maximum degree at most Δ. That is, we prove the existence of so called 'partition universal graphs' with $O_{k,\Delta,s}(n)$ edges for the family of powers T^k of n-vertex trees with $\Delta(T) \leqslant \Delta$.

Theorem 1.2.1 was announced in the extended abstract [12]. While finalizing this paper, we learned that Kamčev, Liebenau, Wood, and Yepremyan [53] proved, among other things, that the 2-colour size-Ramsey number of an n-vertex graph with bounded degree and bounded treewidth is $O(n)$[1]. This is equivalent to our result for $s = 2$. Indeed, any graph with bounded treewidth and bounded maximum

[1]They in fact formulate this for the general 2-colour size-Ramsey number $\hat{r}(H_1, H_2)$.

degree is contained in a suitable blow-up of some bounded degree tree [27, 86] and a blow-up of a bounded degree tree is contained in the power of another bounded degree tree. Conversely, bounded powers of bounded degree trees have bounded treewidth and bounded degree. Therefore, we obtain the following equivalent version of Theorem 1.2.1, which generalises the result from [53] and answers one of their main open questions (Question 5.2 in [53]).

Corollary 1.2.2. *For any positive integers k, Δ and s and any n-vertex graph H with treewidth k and $\Delta(H) \leqslant \Delta$, we have*

$$\hat{r}_s(H) = O_{k,\Delta,s}(n).$$

The proof of Theorem 1.2.1 follows the strategy developed in [46], proving the result by induction on the number of colours s. Very roughly speaking, we start with a graph G with suitable properties and, given any s-colouring of the edges of G ($s \geqslant 2$), either we obtain a monochromatic copy of the power of the desired tree in G, or we obtain a large subgraph H of G that is coloured with at most $s-1$ colours; moreover, the graph H that we obtain is such that we can apply the induction hypothesis on it. Naturally, we design the requirements on our graphs in such a way that this induction goes through. As it turns out, the graph G will be a certain blow-up of a random-like graph. While this approach seems uncomplicated upon first glance, the proof requires a variety of additional ideas and technical details.

To implement the above strategy, we need, among other results, two new and key ingredients which are interesting on their own: (i) a result that states that for any sufficiently large graph G, either G contains a large expanding subgraph or there is a given number of reasonably large disjoint subsets of $V(G)$ without any edge between any two of them (see Lemma 3.2.4[2]); (ii) an embedding result that states that in order to embed a power T^k of a tree T in a certain blow-up of a graph G it is enough to find an embedding of an auxiliary tree T' in G (see Lemma 3.2.6).

[2]We are grateful to the authors of [53], who pointed out to us that similar lemmas have been proved in [77, 78].

1.3 Random perturbation of sparse graphs

For $\alpha \in (0,1)$ we let G_α be an n-vertex graph with minimum degree $\delta(G_\alpha) \geqslant \alpha n$. A famous result by Dirac [28] says that if $\alpha \geqslant \frac{1}{2}$ and $n \geqslant 3$, then G_α contains a Hamiltonian cycle, i.e. a spanning cycle through all vertices of G_α. This motivated the more general questions of determining the smallest α such that G_α contains a given spanning structure. For example, there are results for trees [62], factors [45], powers of Hamiltonian cycles [59, 61], and general bounded degree graphs [20]. This is a problem for deterministic graphs that belongs to the area of extremal graph theory.

We can consider similar questions for random graphs, in particular, for the binomial random graph model $G(n,p)$, which is the probability space over n-vertex graphs with each edge being present with probability p independent of all the others. Analogous to the smallest α we are looking for a function $\hat{p} = \hat{p}(n) \colon \mathbb{N} \to (0,1)$ such that if $p = \omega(\hat{p})$ the probability that $G(n,p)$ contains some spanning subgraph tends to 1 as n tends to infinity and for $p = o(\hat{p})$ it tends to 0. We call this $\hat{p}$ the threshold function for the respective property (an easy sufficient criteria for its existence can be found in [14]) and if the first/second statement holds we say that $G(n,p)$ has/does not have the property asymptotically almost surely (a.a.s.). One often says that $G(n,p)$ undergoes a *phase transition* at $\hat{p}$. For the Hamiltonian cycle problem Posá [79] and Koršunov [63] proved independently that $\hat{p} = \frac{\log n}{n}$ gives the threshold. Thresholds for various other spanning structures were also determined, e.g. for matchings [36], trees [66, 73], factors [50], powers of Hamiltonian cycles [69, 74], and general bounded degree graphs [2, 39, 40, 80]. An extensive survey by Böttcher can be found in [16].

Motivated by the smoothed analysis of algorithms [83], both these worlds were combined by Bohman, Frieze, and Martin [13]. For any fixed $\alpha > 0$, they defined the model of randomly perturbed graphs as the union $G_\alpha \cup G(n,p)$. They showed that $\frac{1}{n}$ is the threshold for a Hamiltonian cycle, meaning that there is a graph G_α such that with $p = o(\frac{1}{n})$ there is a.a.s. no Hamiltonian cycle in $G_\alpha \cup G(n,p)$ and for any G_α and $p = \omega(\frac{1}{n})$ there is a.a.s. a Hamiltonian cycle in $G_\alpha \cup G(n,p)$. It is important to note that in $G(n,p)$, $p = \frac{1}{n}$ is also the threshold for an almost spanning cycle, that is, for any $\varepsilon > 0$ a cycle on at least $(1-\varepsilon)n$ vertices. Another remark is that, if $p = o(\frac{\log n}{n})$ there are a.a.s. isolated vertices in $G(n, p)$ and the purpose of G_α is to compensate this and help turning the almost spanning cycle into a Hamiltonian cycle.

This first result on randomly perturbed graphs [13] motivated subsequent research on thresholds of spanning structures in the randomly perturbed graph model, e.g. trees [17, 51, 68], factors [7], powers of Hamiltonian cycles [9, 19], and general bounded degree graphs [19]. The thresholds for spanning structures in the randomly perturbed model often differ from the thresholds in $G(n, p)$ by a log-factor, like in the case of the Hamiltonian cycle. This difference is due to local restrictions similar to the isolated vertices in the Hamiltonian cycle case. In most cases a G_α that gives the lower bound is the complete imbalanced bipartite graph $K_{\alpha n,(1-\alpha)n}$. In this model there are also results with lower bounds on α [10, 31, 47, 75] and for Ramsey-type problems [24, 25].

1.3.1 Hamiltonicity in randomly perturbed sparse graphs

The aim of this note is to investigate a new direction. Instead of fixing an $\alpha \in (0, 1)$ in advance, we allow α to tend to zero with n. This extends the range of G_α to sparse graphs and we want to determine threshold probabilities in $G_\alpha \cup G(n, p)$. For example, with $\alpha = \frac{1}{\log n}$ we have a sparse deterministic graph G_α with minimum degree $\frac{n}{\log n}$. Then $p = \omega(\frac{1}{n})$ does not suffice in general, but it is sufficient to take $G_\alpha \cup G(n, \frac{\Theta(\log \log n)}{n})$ to guarantee a Hamiltonian cycle with high probability. More generally, we prove the following.

Theorem 1.3.1. *Let $\alpha = \alpha(n) : \mathbb{N} \to (0, 1)$ and $\beta = \beta(\alpha) = -(6 + o(1)) \log(\alpha)$. Then a.a.s. $G_\alpha \cup G(n, \frac{\beta}{n})$ is Hamiltonian.*

This extends the result of Bohman, Frieze, and Martin [13] for constant $\alpha > 0$. For even n a direct consequence of this theorem is the existence of a perfect matching in the same graph. To prove Theorem 1.3.1 we use a result by Frieze [43] to find a very long path in $G(n, p)$ alone and then use the switching technique developed in [19] to turn this into a Hamilton cycle. As it turns out, our method allows to prove the existence of a perfect matching with a slightly lower edge probability.

Theorem 1.3.2. *Let $\alpha = \alpha(n) : \mathbb{N} \to (0, 1)$ and $\beta = \beta(\alpha) = -(4 + o(1)) \log(\alpha)$. Then a.a.s. $G_\alpha \cup G(n, \frac{\beta}{n})$ contains a perfect matching.*

To see that in both theorems β is optimal up to the constant factor, consider $G_\alpha = K_{\alpha n,(1-\alpha)n}$ and note that there cannot be a perfect matching in $G_\alpha \cup G(n, \frac{\beta}{n})$, if in $G(n, \frac{\beta}{n})$ we have more than αn isolated vertices on the $(1-\alpha)n$ side. The number of isolated vertices in $G(n, \frac{\beta}{n})$ is roughly $n(1-\frac{\beta}{n})^{n-1} \cong n\exp(-\beta)$, which is larger than αn if $\beta = o(-\log(\alpha))$.

For proving results in the model of randomly perturbed graphs, we need good almost spanning results. Typically, almost spanning means that for any $\varepsilon > 0$ we can embed the respective structure on at least $(1-\varepsilon)n$ vertices. For paths and cycles in $G(n, \frac{C}{n})$ this can be done using expansion properties and the DFS-algorithm [67]. These almost spanning results are much easier than the spanning counterpart, because there is always a linear sized set of available vertices. For the proof of Theorem 1.3.1 this is not sufficient, because if $\alpha = o(1)$ we will not be able to take care of a linear sized leftover. Thus we have to exploit that we have $G(n, \frac{\beta}{n})$ and use the following result showing that we can find a long cycle consisting of all but sublinearly many vertices.

Lemma 1.3.3 (Frieze [43]). *Let $0 < \beta = \beta(n) \leqslant \log n$. Then $G(n, \frac{\beta}{n})$ a.a.s. contains a cycle of length at least*

$$(1 - (1 - o(1))\beta\exp(-\beta))n.$$

This is optimal, because it is asymptotically the size of the 2-core (maximal subgraph with minimum degree 2) of $G(n,p)$ [42, Lemma 2.16]. A similar result holds for large matchings.

Lemma 1.3.4 (Frieze [43]). *Let $0 < \beta = \beta(n) \leqslant \log n$. Then $G(n, \frac{\beta}{n})$ a.a.s. contains a matching consisting of at least $(1-(1-o(1))\exp(-\beta))n$ vertices.*

Again this is optimal, because the number of isolated vertices is a.a.s. $(1 + o(1))\exp(-\beta)n$ [42, Theorem 3.1]. Observe, that also a bipartite variant of this lemma holds, which can be proved by removing small degree vertices and employing Hall's theorem.

Lemma 1.3.5. *Let $0 < \beta = \beta(n) \leqslant \log n$. Then the bipartite binomial random graph $G(n, n, \frac{\beta}{n})$ a.a.s. contains a matching consisting of at least $(1-(1-o(1))\exp(-\beta))n$ edges.*

1.3.2 Bounded degree trees in randomly perturbed sparse graphs

After Hamilton cycles and perfect matchings, the next natural candidates are n-vertex trees with maximum degree bounded by a constant Δ. In $G(n, p)$ the threshold $\frac{\log n}{n}$ was determined in a breakthrough result by Montgomery [73], in G_α it is enough to have a fixed $\alpha > \frac{1}{2}$ [60], and in $G_\alpha \cup G(n, p)$ with constant $\alpha > 0$ the threshold is $\frac{1}{n}$ [68]. To obtain a result similar to Theorem 1.3.1 for bounded degree trees using our approach, we need an almost spanning result similar to Lemma 1.3.3. With a similar approach as for Theorem 1.3.1 and 1.3.2 we obtain the following modular statement.

Theorem 1.3.6. *Let $\Delta \geqslant 2$ be an integer and suppose that $\alpha, \beta, \varepsilon \colon \mathbb{N} \to [0, 1]$ are such that $4(\Delta + 1)\varepsilon < \alpha^{\Delta+1}$ and a.a.s. $G(n, \frac{\beta}{n})$ contains a given tree with maximum degree Δ on $(1 - \varepsilon)n$ vertices. Then any tree with maximum degree Δ on n vertices is a.a.s. contained in $G_\alpha \cup G(n, \frac{\beta}{n})$.*

Next we discuss the almost spanning results that we can obtain in the relevant regime. Improving on a result of Alon, Krivelevich, and Sudakov [3], Balogh, Csaba, Pei, and Samotij [5] proved that for $\Delta \geqslant 2$ there exists a $C > 0$ such that for $\varepsilon > 0$ a.a.s. $G(n, \frac{\beta}{n})$ contains any tree with maximum degree Δ on at most $(1 - \varepsilon)n$ vertices, provided that $\beta \geqslant \frac{C}{\varepsilon} \log \frac{1}{\varepsilon}$. For the proof they only require that the graph satisfies certain expander properties. This can be extended to the range where $\varepsilon \to 0$ and $\omega(1) = \beta \leqslant \log n$ and following along the lines of their argument we get the following.

Lemma 1.3.7. *For $\Delta \geqslant 2$ there exists a $C > 0$ such that for any $0 < \beta = \beta(n) \leqslant \log n$ and $\varepsilon = \varepsilon(n) > 0$ with $\beta \geqslant \frac{C}{\varepsilon} \log \frac{1}{\varepsilon}$ the following holds. $G(n, \frac{\beta}{n})$ a.a.s. contains any bounded degree tree on at most $(1 - \varepsilon)n$ vertices.*

Then together with Theorem 1.3.6 we obtain the following.

Corollary 1.3.8. *For $\Delta \geqslant 2$ there exists a $C > 0$ such that for $\alpha = \alpha(n) : \mathbb{N} \to (0, 1)$ and $\beta = \beta(\alpha) = C\alpha^{-(\Delta+1)} \log \frac{1}{\alpha}$ the following holds. Any n-vertex tree T with maximum degree Δ is a.a.s. contained in $G_\alpha \cup G(n, \frac{\beta}{n})$.*

The proof for the dense case in [68] uses regularity and it is unlikely to give anything better in the sparse regime. As remarked in [3] the condition on the almost spanning embedding in $G(n, \frac{\beta}{n})$ could possibly be improved to $\beta > \log \frac{C}{\varepsilon}$, then covering almost all non-isolated vertices. More precisely this asks for the following.

Question 1.3.9. For every integer Δ there exists $C > 0$ such that if $0 < \beta = \beta(n) \leqslant \log n$ the following holds. Is any given tree with maximum degree Δ on

$$(1 - C\exp(-\beta))n$$

vertices a.a.s. contained in $G(n, \frac{\beta}{n})$?

With Theorem 1.3.6 this would then give that already $\beta = -(\Delta + 1)\log(C\alpha)$ suffices, which would be optimal up to the constant factors. We want to briefly argue why it is possible to answer this question for large families of trees and what the difficulties are. For simplicity we only discuss the case $\beta = \log\log n$ and note that by Lemma 1.3.7 above we can embed trees on roughly $(1 - \frac{1}{\log\log n})n$ vertices. A very helpful result for handling trees by Krivelevich [66] states that for integers $n, k > 2$, a tree on n vertices either has at least $\frac{n}{4}k$ leaves or a collection of at least $\frac{n}{4}k$ bare paths (internal vertices of the path have degree 2 in the tree) of length k. If there are at least $\frac{n}{4\log\log n}$ leaves, we can embed the tree obtained after removing the leaves. Then we can use a fresh random graph and Lemma 1.3.5 to find a matching for all the leaves, completing the embedding of the tree.

On the other hand, if there are at least $\frac{n\log\log n}{4\log n}$ bare paths of length $\frac{\log n}{\log\log n}$, it is possible to embed all but $\frac{n}{\log n}$ of these paths, which are all but $\frac{n}{\log\log n}$ vertices. Then one has to connect the remaining paths, again using ideas from [73]. In between both cases it is not clear what should be done, because we might have $\frac{n}{\log n}$ leaves and $\frac{n}{4\log\log n}$ bare paths of length $\log\log n$. The length of the paths are too short to connect them and the leaves are too few for the above argument. Answering this questions and thereby improving the result of Alon, Krivelevich, and Sudakov [3] is a challenging open problem.

1.3.3 Other spanning structures

As mentioned above, embeddings of spanning structures in G_α, $G(n,p)$, and $G_\alpha \cup G(n,p)$ for fixed $\alpha > 0$ have also been studied for other graphs such as powers of Hamilton cycles, factors, and general bounded degree graphs. In most of these cases almost spanning embeddings (e.g. Ferber, Luh, and Nguyen [39]) can be generalised such that previous proofs can be extended to the regime $\alpha = o(1)$ with $\beta = \alpha^{-1/C}$, similar to what we do in Corollary 1.3.8. Further improvements seem to be hard, because better almost spanning results are similar in difficulty to spanning results in $G(n,p)$ alone. We want to discuss this on one basic example, the triangle factor, which is the disjoint union of $\frac{n}{3}$ triangles.

In G_α we need $\alpha \geqslant \frac{2}{3}$, in $G(n,p)$ the threshold is $n^{-\frac{2}{3}} \log^{\frac{1}{3}} n$, and in $G_\alpha \cup G(n,p)$ with a fixed $\alpha > 0$ it is $n^{-\frac{2}{3}}$. Note that the log-term in $G(n,p)$ is needed to ensure that every vertex is contained in a triangle, which is essential for a triangle factor. Using Janson's inequality [42, Theorem 21.12] it is not hard to prove the almost spanning result for a triangle factor on at least $(1-\varepsilon)n$ vertices with $p = \omega(n^{-\frac{2}{3}})$. This can be generalised to $G(n, \beta n^{-\frac{2}{3}})$ giving a.a.s. a triangle factor on at least $(1 - \frac{C}{\beta})n$ vertices. Again, this can only give something with $\beta = \alpha^{-\frac{1}{C}}$ in $G_\alpha \cup G(n, \beta n^{-\frac{2}{3}})$ and to improve this we ask the following.

Question 1.3.10. Let $0 < \beta = \beta(n) \leqslant \log^{\frac{1}{3}} n$. Does $G(n, \beta n^{-\frac{2}{3}})$ a.a.s. contain a triangle factor on at least

$$\big(1 - (1-o(1)) \exp(-\beta^3)\big)\, n$$

vertices?

Observe, that this is a.a.s. the number of vertices of $G(n, \beta n^{-\frac{2}{3}})$ that are not contained in a triangle. Similar questions for other factors or more general structures would be of interest. It took a long time until Johannson, Kahn, and Vu [50] determined the threshold for the triangle factor. This conjecture seems to be of similar difficulty, whereas for our purposes it would already be great to obtain a triangle factor on at least $(1 - C\exp(-\beta^3))n$ vertices for some $C > 1$.

2. Spanning tripartite subgraphs

We aim for a good condition on graphs that generalizes Theorems 1.1.2 and 1.1.6. This condition has to first, ensure the existence of a 2nd power of a Hamiltonian cycle. This is done by an application of the absorption method. Second, we show that the reduced graph inherits the good properties of the original graph and prepare for an application of the blow-up lemma.

2.1 Absorption method

We are given a $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good graph G on n vertices. As a consequence of G being good, we have a bound on the minimum degree. Since the auxiliary graph $\mathfrak{A}$ is μ-inseparable, $\delta(\mathfrak{A}) \geqslant \mu(n-1)$. This implies for every $v \in V(G)$, we have $|E(G[U_v])| \geqslant \zeta n^2$. Also, $G[U_v]$ contains at least δn^3 triangles. Therefore,

$$|N(v)| \geqslant |U_v| \geqslant \max\{2\sqrt{\zeta}n, \sqrt[3]{6\delta}n\}\,. \tag{2.1.1}$$

We also observe that there are many $\zeta\mu$-connectable edges contained in $G[U_v]$, for every $v \in V(G)$. Indeed, since $\mathfrak{A} = A(\mathcal{R}, \zeta)$ is μ-inseparable, $N_{\mathfrak{A}}(v) \geqslant \mu(n-1)$ and for every $u \in N_{\mathfrak{A}}(v)$, we have $|E(G[U_v]) \cap E(G[U_u])| \geqslant \zeta n^2$. A standard averaging argument gives us $E'_v \subseteq E(G[U_v])$ such that

$$|E'_v| \geqslant \zeta |E(G)| \geqslant \frac{\zeta n^2}{4}\,, \tag{2.1.2}$$

and for each $ab \in E'_v$ there is $N_{\mathfrak{A}}(v, ab) \subseteq N_{\mathfrak{A}}(v)$ with $|N_{\mathfrak{A}}(v, ab)| \geqslant \zeta |N_{\mathfrak{A}}(v)|$ and $u \in N_{\mathfrak{A}}(v, ab)$ implies $ab \in E(G[U_v]) \cap E(G[U_u])$. Therefore ab is $\zeta\mu$-connectable.

Throughout our application of the absorption method, we refer to the 2nd power of a path/walk as a *triangle path/walk*. We take roughly the following steps:

1. we ensure the abundant existence of so-called absorbers,
2. find an almost perfect cover of only “few” triangle paths, and
3. connect those absorbers and triangle paths to an almost spanning 2nd power of cycle.

Here we state the lemmas that allows us to take these steps. The following lemma will allow us not only to take step 3, but also to connect the absorbers into an absorbing path.

Lemma 2.1.1 (Connecting Lemma). *For every $\mu \in (0,1]$, $\delta, \zeta, \xi > 0$, there exist $c, \xi^* > 0$ and integers L, n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices with robust neighbourhoods $\mathcal{R}_G$ satisfies properties i. and ii. of good graphs, then for every two distinct ξ-connectable pairs $xy, x'y'$, there is some integer $\ell(xy, x'y') = \ell$ with $\ell \leqslant L$ and $\ell \equiv 1 \pmod 3$ such that the number of $(xy, x'y')$-triangle walks with ℓ inner vertices in G is at least cn^ℓ. Moreover if $xyx_1 \ldots x_{3k+1}x'y'$ is such a walk, the edges $x_{3i-1}x_{3i}$, with $i \in [k]$, are ξ^*-connectable.

After establishing that G is well connected, we take step 1 to set aside an absorbing path. A triangle path P in a graph $G = (V, E)$ is α-*absorbing* when given any set $X \subseteq V \setminus V(P)$ of size $|X| \leqslant \alpha|V|$ divisible by 3, there is a triangle path P' with the same ending pairs as P and $V(P') = V(P) \cup X$.

Lemma 2.1.2 (Absorbing Path Lemma). *For every $\mu, \delta, \zeta, \xi > 0$ and $c = c(\delta) > 0$, there exist $\kappa, \alpha_0, \alpha, \xi' > 0$ and n_0, such that the following holds.*

Given $\nu > 0$, a graph G on $n \geqslant n_0$ vertices with $\mathcal{R}_G$ satisfying properties i. and ii. of good graphs and sets $I_1, \ldots, I_m \subseteq V(G)$, with $m \leqslant 2^{\kappa n}$, such that the number of triangles with a ξ-connectable edge in $G[I_i]$ is at least $c(\delta)n^3$, there exist two vertex disjoint triangle paths $P_A, P_I \subseteq G$, such that

(i) $|V(P_A)|, |V(P_I)| \leqslant \min\{\frac{\mu}{8}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\mu\sqrt{\zeta}}{5}, \frac{\nu}{4}\}n - 3$,

(ii) *P_A is α_0-absorbing,*

(iii) *P_A begins with a triangle path $x_1x_2x_3x_4$, where x_1x_2, x_2x_3, x_3x_4 are ξ'-connectable and P_A ends in a ξ'-connectable edge,*

(*iv*) *for all* $i \in [m]$*, the number of triangle paths* $P_5 = y_1y_2y_3y_4y_5$ *contained in* P_I *such that* $y_1, y_2, y_3, y_4, y_5 \in I_i$ *is at least* αn*.*

(*v*) P_I *begins and ends in* ξ'*-connectable edges.*

The path P_I is an addition towards the embedding of graphs with small bandwidth, for the result concerning a 2nd power of a Hamiltonian cycle it is not needed.

Properties i. and ii. of good graphs ensured us steps 1 and 3. For step 2 we use property iii. of good graphs.

Lemma 2.1.3 (Covering Lemma)**.** *Given* $\xi, \alpha > 0$*,* $c_\varrho \geqslant 1$ *there exist* $\eta, \varrho, c > 0$ *with* $\varrho \leqslant \frac{1}{16}$*,* $\xi \geqslant c_\varrho \sqrt[8]{\varrho}$*,* $\eta \geqslant c_\varrho \sqrt[8]{\varrho}$ *and* n_0 *such that the following holds. For every* $\frac{\eta}{2} \geqslant \nu > 0$*, if* G *on* $n \geqslant n_0$ *vertices with* $\mathcal{R}_G$ *satisfies property iii. of good graphs, then there is a collection* $\mathcal{P}$ *of* $|\mathcal{P}| \leqslant c$ *triangle paths with* ξ*-connectable ends covering all but at most* αn *vertices of* G*.*

In the next section we use these lemmas to show that good graphs contain the second power of a Hamiltonian cycle.

2.2 Second Power of a Hamiltonian Cycle

In view of Theorem 1.1.14, we show the following version of Theorem 1.1.13 that includes special segments needed for preparing G for an application of the blow-up lemma.

Theorem 2.2.1. *Given* $\mu, \delta, \zeta, \xi > 0$*,* $c(\delta) > 0$ *and* $c_\varrho \geqslant 1$*, there exist* n_0 *and* $\kappa, \alpha, \eta, \varrho > 0$ *with* $\eta, \xi \geqslant c_\varrho \sqrt[4]{\varrho}$ *such that the following holds.*

For every $\frac{1}{4} \geqslant \nu > 0$ *such that* $\eta \geqslant 2\nu$*, if* G *is* $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$*-good and for some* $m \leqslant 2^{\kappa n}$ *we are given* $I_1, \ldots, I_m \subseteq V(G)$ *such that the number of triangles with a* ξ*-connectable edge contained in each* I_i *is at least* $c(\delta)n^3$*, then* G *contains the second power of a Hamiltonian cycle with segments*

- $P_6 = x_1 \ldots x_6$ *which is the 3rd power of a path,*

- P_I *such that for all* $i \in [m]$*, the number of triangle paths* $y_1y_2y_3y_4y_5$ *contained in* P_I *such that* $y_1, y_2, y_3, y_4, y_5 \in I_i$ *is at least* αn*.*

For embedding graphs with small bandwidth, we apply Theorem 2.2.1 to the reduced graph of G and a certain regular partition $V_0, \ldots, V_t$. The triangle path P_I is needed for redistributing V_0 into the partition classes without losing the regularity of the involved pairs and the 3-path P_6, for the balancing of the sizes of the classes.

Proof of Theorem 2.2.1. Apply the Absorbing Path Lemma (Lemma 2.1.2) with the given $\mu, \delta, \zeta, \xi, c(\delta) > 0$. Get $\kappa, \alpha_0, \alpha, \xi_0 > 0$ and n_0', set $\xi' = \min\{\frac{\xi_0}{2}, \frac{\xi}{8}, \frac{\delta}{8}\}$. Apply the Covering Lemma (Lemma 2.1.3) with $\frac{\xi}{4}, \frac{\alpha}{2}, c_\varrho > 0$ to get n_0'' and $\varrho', \eta, c_0 > 0$ with $\xi, \eta \geqslant c_\varrho \sqrt[8]{\varrho'}$. Apply the Connecting Lemma (Lemma 2.1.1) with $\frac{\mu}{2}, \frac{\delta}{2}, \xi'$ and get $c_1 > 0$ and integers n_0''', L (we also get $\xi^* > 0$, but here we do not need the "moreover" part of the lemma). Set

$$\varrho = \sqrt[4]{4^3 \varrho'}, \quad p = \frac{1}{3} \min\left\{\frac{\mu}{8}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{2\mu\sqrt{\zeta}}{10\sqrt{2}}, \frac{\nu}{4}, \frac{\alpha}{2}\right\},$$

and let n_0 be large enough. Let c_ν and $\nu > 0$ be given such that $\eta \geqslant c_\nu \nu$. Let G be a $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good graph with robust neighbourhoods $\mathcal{R}_G$.

Lemma 2.1.2 gives us disjoint triangle paths P_A and P_I in G. Take

$$G' = G[V(G) \smallsetminus (V(P_A) \dot\cup V(P_I))] \text{ with } \mathcal{R}_{G'} = \{U_v' = U_v \cap N_{G'}(v) \colon v \in V(G')\}.$$

Since $|V(P_A) \dot\cup V(P_I)| \leqslant \min\{\frac{\mu}{4}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{2\mu\sqrt{\zeta}}{5}\}n - 6$, Lemma 2.4.6 gives us that G' satisfies properties i. and ii. of good graphs with $\frac{\mu}{2}, \frac{\delta}{2}, \frac{\zeta}{2}$. Lemma 2.4.7 gives a third power of path $P_6 = x_1'x_2'x_3'x_4'x_5'x_6'$ with $\frac{\delta}{4}$-connectable ends $x_1'x_2'$, $x_5'x_6'$ in G'.

Consider $X \subseteq V(G)$ containing the vertices in the ending pairs $x_I y_I, x_I' y_I'$ of P_I, the vertices x_1, x_2, x_3, x_4 in the initial segment of P_A, the ending pair $x_A y_A$ of P_A and the vertices in the ending pairs $x_1'x_2', x_5'x_6'$ of P_6, note that $|X| = 14$. Take

$$G^* = G\big[V(G) \smallsetminus \big((V(P_A) \dot\cup V(P_I) \dot\cup V(P_6)) \smallsetminus X\big)\big] \quad \text{and}$$

$$\mathcal{R}_{G^*} = \{U_v^* = U_v \cap N_{G^*}(v) \colon v \in V(G^*)\}.$$

Since $|V(P_A) \dot\cup V(P_I) \dot\cup V(P_6)| \leqslant \min\{\frac{\mu}{4}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{2\mu\sqrt{\zeta}}{5}, \frac{\nu}{2}\}n$ and $\nu \leqslant \frac{1}{4}$, $\varrho \leqslant \frac{1}{16}$, we have that G^* is $(\frac{\mu}{2}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{\varrho^2}{4}, \frac{\xi}{2}, \eta, \frac{\nu}{2})$-good.

Choose a reservoir set $S \subseteq V(G) \smallsetminus (V(P_A) \dot\cup V(P_I) \dot\cup V(P_6))$ by including vertices independently at random with probability $p > 0$. We show that with positive probability there is a choice of S such that the following holds:

(a) $|S| \leqslant \min\left\{\frac{\mu}{8}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{2\mu\sqrt{\zeta}}{10\sqrt{2}}, \frac{\nu}{4}, \frac{\alpha}{2}\right\}|V(G^*)| - |X|$;

(b) for two distinct ξ'-connectable (in G^*) pairs $xy, x'y' \in E(G^*)$ there is $\ell \leqslant L$, with $\ell \equiv 1 \pmod 3$, such that there are at least $\frac{c_1 p^L}{4}|V(G^*)|^\ell$ distinct $(xy, x'y')$-triangle paths in G^* with ℓ inner vertices, all in S.

For property (a), observe that $p|V(G^*)| \geqslant \frac{p|V(G)|}{2} \geqslant 28p + 14 = 2p|X| + |X|$ and use Markov's inequality to get

$$\mathbb{P}\left(|S| \geqslant 3p|V(G^*)| - |X|\right) \leqslant \mathbb{P}\left(|S| \geqslant 2p(|V(G^*)| + |X|)\right) \leqslant \frac{1}{2}. \tag{2.2.1}$$

For property (b), the Connecting Lemma (Lemma 2.1.1) guarantees for every two ξ'-connectable pairs $xy, x'y'$, at least $c_1|V(G^*)|^{\ell(xy,x'y')}$ different triangle walks between xy and $x'y'$ with $\ell(xy, x'y') \leqslant L$ inner vertices (note that if G^* is good with $\frac{\xi}{2}$, it is also good with ξ'). At most half of these walks have vertex repetition or an inner vertex in X, thus we have at least $\frac{c_1}{2}|V(G^*)|^{\ell(xy,x'y')}$ triangle paths between xy and $x'y'$ with $\ell(xy, x'y')$ inner vertices in $V(G) \smallsetminus (V(P_A) \dot\cup V(P_I) \dot\cup V(P_6))$.

Let $X(\ell(xy, x'y'), xy, x'y')$ be the number of triangle paths between xy and $x'y'$ with all its $\ell = \ell(xy, x'y')$ inner vertices in S. For an application of the Azuma-Hoeffding inequality, note that the inclusion or exclusion of a vertex in S changes $X(\ell, xy, x'y')$ by at most $\ell|V(G^*)|^{\ell-1}$. Thus,

$$\begin{aligned}\mathbb{P}\Big(X(\ell, xy, x'y') \leqslant \frac{c_1(|V(G^*)|p)^\ell}{4}\Big) &\leqslant \exp\Big(-\frac{c_1^2(|V(G^*)|p)^{2\ell}}{16 \cdot 2\ell^2|V(G^*)|^{2\ell-1}}\Big) \\ &\leqslant \exp\Big(-\frac{c_1^2 p^{2L}}{32L^2}|V(G^*)|\Big).\end{aligned} \tag{2.2.2}$$

There are up to $L \cdot |V(G^*)|^4$ triples $(\ell, xy, x'y')$, thus by the union bound and for n_0 large enough, the probability that there is a triple such that (2.2.2) holds is smaller than $\frac{1}{2}$. Considering this and (2.2.1), we may fix an instance S satisfying properties (a) and (b).

Consider

$$G'' = G[V(G) \smallsetminus (V(P_A) \dot\cup V(P_I) \dot\cup V(P_6) \dot\cup S)] = G^*[V(G^*) \smallsetminus (S \dot\cup X)] \quad \text{and}$$

$$\mathcal{R}_{G''} = \{U''_v = U_v \cap N_{G''}(v) \colon v \in V(G'')\}.$$

Since $|X \dot\cup S| \leqslant \min\left\{\frac{\mu}{8}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{2\mu\sqrt{\zeta}}{10\sqrt{2}}, \frac{\nu}{4}, \frac{\alpha}{2}\right\}|V(G^*)|$, we have G'' is $(\frac{\mu}{4}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\varrho^4}{4^3}, \frac{\xi}{4}, \eta, \frac{\nu}{4})$-good.

Lemma 2.1.3 gives us a collection $\mathcal{Q}_0$ of at most c_0 triangle paths with $\frac{\xi}{4}$-connectable (in G'') ends covering all vertices but at most $\frac{\alpha}{2}|V(G'')|$ vertices of G''. Let $\mathcal{Q}$ be the collection of triangle paths containing $\mathcal{Q}_0$ and in addition P_A, P_I and P_6.

We note that all paths in $\mathcal{Q}$ have ξ'-connectable ends in G^*. First, the paths in $\mathcal{Q}_0$ have $\frac{\xi}{4}$-connectable ends xy in G''. If $xy \in E(G''[U''_v])$, since $G''[U''_v] \subseteq G^*[U^*_v]$ and $|V(G'')| \geqslant \frac{|V(G^*)|}{2}$, we have that xy is $\frac{\xi}{8}$-connectable in G^*. Second, if xy is an end pair of P_A or P_I, then $x, y \in X \subseteq V(G^*)$ and $x, y \in U_v$ for $\xi_0 n$ different $v \in V(G)$. If $v \in V(G^*)$, then $x, y \in U^*_v$. Since $|V(G^*)| \geqslant \frac{n}{2}$, we have that x, y are in at least $\frac{\xi_0}{2}n$ different U^*_v. Third, if xy is an end pair of $P_6 = x'_1x'_2x'_3x'_4x'_5x'_6$, then $x, y \in X$ and $x, y \in U'_v$ for $\frac{\delta}{4}|V(G')|$ different $v \in V(G')$. Since $V(G^*) \supseteq V(G') \setminus \{x'_3, x'_4\}$, if $v \notin \{x'_3, x'_4\}$, then $x, y \in U^*_v$. Moreover $|V(G')| = |V(G^*)| - 8$, thus xy is $\frac{\delta}{8}$-connectable in G^*.

Using the reservoir S, we connect the triangle paths in this collection into a single triangle path in G. Consider a maximal subset $\mathcal{Q}' \subseteq \mathcal{Q}$ such that there is a triangle path $P_{\mathcal{Q}'}$ in G with ξ'-connectable (in G^*) ends containing all triangle paths from $\mathcal{Q}'$, intersecting no other element of $\mathcal{Q} \setminus \mathcal{Q}'$ and containing at most $L(|\mathcal{Q}'| - 1)$ vertices from S. The set $\mathcal{Q}'$ is non-empty and we show that $\mathcal{Q}' = \mathcal{Q}$.

Otherwise, let $Q \in \mathcal{Q} \setminus \mathcal{Q}'$ be a triangle path with an end $x_Q y_Q$ and let $x_{Q'} y_{Q'}$ be an end of $P_{\mathcal{Q}'}$, both are ξ'-connectable pairs in G^*. Property (b) of S gives $\frac{c_1 p^L}{4}|V(G^*)|^\ell$ different $(x_Q y_Q, x_{Q'} y_{Q'})$-triangle paths in G^* with all its $\ell \leqslant L$ inner vertices in S. The path $P_{\mathcal{Q}'}$ intersects at most $L(|\mathcal{Q}'| - 1)\ell|V(G^*)|^{\ell-1}$ of these paths and for n_0 large, it assures at least one triangle path between $x_Q y_Q$ and $x_{Q'} y_{Q'}$ with inner vertices in S that is disjoint from $P_{\mathcal{Q}'}$. Thus we may extent $P_{\mathcal{Q}'}$ to $P_{\mathcal{Q}' \cup \{Q\}}$ intersecting S in at most $L(|\mathcal{Q}' \cup \{Q\}| - 1)$ vertices.

We find the second power of a cycle C containing roughly all $P_{\mathcal{Q}}$. At this point we shall address the parity issue. Since the absorbing path P_A absorbs triples of vertices, we have to ensure that $|V(G)| - |V(C)|$ is a multiple of 3. For that, we take P_A to be the first triangle path in $P_{\mathcal{Q}}$, and therefore $P_{\mathcal{Q}}$ begins with $x_1x_2x_3x_4$ such that all edges are ξ'-connectable in G^*; say $x_Q y_Q$ is the other end of $P_{\mathcal{Q}}$. For $x_i x_j \in E(x_1x_2x_3x_4)$, property (b) of S gives many $(x_i x_j, x_Q y_Q)$-triangle paths with $3k_{ij}+1$ inner vertices in S. If $|V(G)|-|V(P_{\mathcal{Q}})|$ is of the form $3k'$, we will connect the pairs x_2x_3 and $x_{\mathcal{Q}}y_{\mathcal{Q}}$ and $|V(G)|-|V(C)|$ is of the form $3(k'-k_{23})-1+1$, since x_1 is a left over. If $|V(G)|-|V(P_{\mathcal{Q}})|$ is of the form $3k'+1$, we connect the pairs x_1x_2 and $x_{\mathcal{Q}}y_{\mathcal{Q}}$. If $|V(G)|-|V(P_{\mathcal{Q}})|$ is of the form $3k'+2$, we connect the pairs x_3x_4 and $x_{\mathcal{Q}}y_{\mathcal{Q}}$.

Property (b) of S gives at least $\frac{c_1p^L}{4}|V(G^*)|^\ell$ triangle paths between the selected pairs, having its inner vertices in S. By the same argument as above, there is a path with all its ℓ inner vertices in $S\smallsetminus V(P_{\mathcal{Q}})$. Consequently there exists the second power of a cycle C with $|V(G)|-|V(C)|$ a multiple of 3 and containing $P_{\mathcal{Q}}$ except by possibly its first two vertices x_1, x_2.

The paths in $\mathcal{Q}_0$ leave at most $\frac{\alpha}{2}|V(G'')|$ uncovered vertices in G'' and since $V(P_A)\smallsetminus\{x_1,x_2\}, V(P_I), V(P_6)\subseteq V(P_{\mathcal{Q}})$, the triangle path $P_{\mathcal{Q}}$ leaves at most $|S|+\frac{\alpha}{2}|V(G'')|+2$ uncovered vertices in G. Since $|S|\leqslant\frac{\alpha}{2}|V(G^*)|-14$ and P_A is α-absorbing in G, we may absorb all uncovered vertices, obtaining the second power of a Hamiltonian cycle containing the special segments P_I and P_6. $\square$

In the following sections we prove Lemmas 2.1.1, 2.1.2 and 2.1.3.

2.3 Connecting

We use the fact that between any two vertices in a μ-inseparable graph there are many paths with a certain number of inner vertices. This is shown in [EMR$^+$20] and we also include the proof here.

Lemma 2.3.1 (Many paths)**.** *For every $\mu\in(0,1]$, there exist $c>0$ and integers L, n_0 such that every μ-inseparable graph $G=(V,E)$ on $|V|=n\geqslant n_0$ vertices satisfies the following. For every two distinct vertices x, $y\in V$, there is some integer ℓ with $1\leqslant\ell\leqslant L$ such that the number of (x,y)-walks with ℓ inner vertices in G is at least cn^ℓ.*

Proof. Given μ we define

$$L=\left\lceil\frac{8}{\mu}\right\rceil+1,\qquad \delta_i=\left(\frac{\mu^2}{3}\right)^i\left(\frac{1}{2}\right)^{\binom{i+1}{2}},\qquad\text{and}\qquad c=\frac{\mu^2}{48}\delta^2_{\lceil 4/\mu\rceil}\,. \tag{2.3.1}$$

Let G be a sufficiently large μ-inseparable graph on n vertices and x, y be two distinct vertices of G. Consider for each $i\geqslant 0$ the set of vertices v that can be reached from x by “many” walks in G with i inner vertices. For that we define

$$X_i = \big\{v \in V : \text{there are } \delta_i n^i\ (x,v)\text{-walks with } i \text{ inner vertices}\big\} \ \text{ and } X^i = \bigcup_{0\leqslant j\leqslant i} X_j\,.$$

Analogously, consider the vertices v that can be reached from y by $\delta_i n^i$ walks in G with i inner vertices and define the sets Y_i and Y^i in the same way.

Observe that $X_0 = X^0 = N(x)$ and since G is μ-inseparable, $|N(x)| \geqslant \mu(n-1)$. Moreover, $X^i \subseteq X^{i+1}$ and we shall show that as long as $|X^i|$ is not too large, then $|X^{i+1}|$ is substantially larger than $|X^i|$. More precisely, we show for every $i \geqslant 0$ that

$$|X^i| \leqslant \frac{2}{3}n \quad \Longrightarrow \quad |X^{i+1} \smallsetminus X^i| \geqslant \frac{\mu}{6}n\,. \tag{2.3.2}$$

Before verifying (2.3.2), we conclude the proof of Lemma 2.3.1. In fact, (2.3.2) implies that there is some $i_0 < \lfloor \frac{4}{\mu} \rfloor$ such that $|X^{i_0}| > \frac{2n}{3}$. Applying the same argument for Y^i, we get some $j_0 < \lfloor \frac{4}{\mu} \rfloor$ such that $|Y^{j_0}| > \frac{2n}{3}$ and, hence, $|X^{i_0} \cap Y^{j_0}| \geqslant \frac{n}{3}$.

Each vertex $v \in X^{i_0} \cap Y^{j_0}$ can be used to create many (y,x)-walks with possibly different number of inner vertices. However, by the pigeonhole principle there are integers a, b with $0 \leqslant a \leqslant i_0$ and $0 \leqslant b \leqslant j_0$ such that

$$|X_a \cap Y_b| \geqslant \frac{|X^{i_0} \cap Y^{j_0}|}{(i_0+1)(j_0+1)} \geqslant \frac{\mu^2 n}{48}\,. \tag{2.3.3}$$

For each $v \in X_a \cap Y_b$ there exist $\delta_a n^a$ (x,v)-walks and $\delta_b n^b$ (v,y)-walks with a and b inner vertices, respectively. Concatenating these walks leads to at least $\delta_a \delta_b n^{a+b} \cdot |X_a \cap Y_b|$ different (x,y)-walks, with $\ell = a+b+1$ inner vertices. Owing to the choice of constants in (2.3.1) we conclude the proof.

It is left to verify (2.3.2). Suppose $|X^i| \leqslant \frac{2n}{3}$ and consider the complement $Z = V \smallsetminus X^i$. Owing to the μ-inseparability of G we have

$$e(X^i, Z) \geqslant \mu |X^i||Z|\,. \tag{2.3.4}$$

Note that each vertex v with at least $\frac{\delta_{j+1}}{\delta_j}n$ neighbours in X_j belongs to X_{j+1}. Since Z is disjoint from X^i, we have

$$e(X^{i-1}, Z) < |Z| \cdot \sum_{j=0}^{i-1} \frac{\delta_{j+1}}{\delta_j} n\,. \tag{2.3.5}$$

Moreover, supposing by contradiction that (2.3.2) fails, we also have

$$e(X_i, Z) < |Z| \cdot \frac{\delta_{i+1}}{\delta_i} n + \frac{\mu}{6}|X_i|n\,. \tag{2.3.6}$$

Combining (2.3.5) and (2.3.6) we arrive at

$$e(X^i, Z) < |Z| \cdot \sum_{j=0}^{i-1} \frac{\delta_{j+1}}{\delta_j} n + |Z| \cdot \frac{\delta_{i+1}}{\delta_i} n + \frac{\mu}{6} n|X_i| = |Z| \cdot \sum_{j=0}^{i} \frac{\delta_{j+1}}{\delta_j} n + \frac{\mu}{6} n|X_i|\,. \tag{2.3.7}$$

Owing to the choice of δ_j in (2.3.1) we have

$$\sum_{j=0}^{i} \frac{\delta_{j+1}}{\delta_j} = \frac{\mu^2}{3} \sum_{j=0}^{i} \left(\frac{1}{2}\right)^{j+1} \leqslant \frac{\mu^2}{3}\,.$$

Furthermore, since $|X^i| \geqslant |X^0| = |N(x)| \geqslant \mu(n-1)$ and $|Z| = |V \setminus X^i| \geqslant n/3$, we derive for sufficiently large n from (2.3.7) that

$$e(X^i, Z) < \frac{\mu^2}{3}|Z|n + \frac{\mu}{6}|X_i|n \leqslant \frac{\mu}{2}|Z||X^i| + \frac{\mu}{2}|X_i||Z| \leqslant \mu|X^i||Z|\,,$$

which contradicts (2.3.4). □

Our triangle paths between connectable ordered pairs xy and $x'y'$ will be obtained by including a vertex between the vertices in every second edge of a (y, x')-path (Figure 2.3.1). Thus we need that the number of inner vertices in these paths is even. We can obtain this, whenever we have μ-inseparability and many triangles.

Corollary 2.3.2. *Given $\mu \in (0, 1]$ and $\delta > 0$, there exist $c > 0$ and L such that the following holds. If G is μ-inseparable and contains δn^3 triangles, then for any different $x, y \in V$, there is an odd (even) integer $\ell_o(x, y) \leqslant L$ ($\ell_e(x, y) \leqslant L$) such that there are $cn^{\ell_o(x,y)}$ ($cn^{\ell_e(x,y)}$) walks with $\ell_o(x, y)$($\ell_e(x, y)$) inner vertices between x and y.*

Proof. Let μ, δ be given. Apply Lemma 2.3.1 with μ and get $c' > 0$, L'. Set

$$c = \frac{\delta c'^2}{L'^2} \qquad \text{and} \qquad L = 2L' + 3.$$

Let T_G be the triangles in G. Consider different vertices $x, y \in V$. For each $abc = T \in T_G$, Lemma 2.3.1 gives $c'n^{\ell_{xa}}$ walks with $\ell_{xa} \leqslant L'$ inner vertices between x

and a and $c'n^{\ell_{yc}}$ walks with $\ell_{yc} \leqslant L'$ inner vertices between y and c. Each triangle is associated to a pair (ℓ_{xa}, ℓ_{yc}) and there is a pair (ℓ_x, ℓ_y) common to at least $\frac{\delta}{L'^2}n^3$ triangles, let $T_G(x,y) \subseteq T_G$ be such triangles.

If $abc \in T_G(x,y)$ and $\ell_x + \ell_y$ is odd, then we use the edge ac to complete and odd walk between x and y. In total we have

$$c'^2 n^{\ell_x+\ell_y} \frac{\delta n^3}{L'^2} \cdot \frac{1}{n}$$

walks between x and y with $\ell_x + \ell_y + 2 = \ell_o(x,y)$ inner vertices (the factor $1/n$ excludes the possible n triangles on the same edge ac). If $\ell_x + \ell_y$ is even, we complete walks between x and y by taking the segment abc of T and get an odd number $\ell_x + \ell_y + 3 = \ell_o(x,y)$ of inner vertices. The total number of walks in this case is

$$c'^2 n^{\ell_x+\ell_y} \frac{\delta n^3}{L'^2} \,.$$

Completing the walks with abc when $\ell_x + \ell_y$ is odd and with ac when it is even, we get many paths between x and y with an even number of inner vertices. $\square$

We are ready to prove the Connecting Lemma.

Proof of Lemma 2.1.1. We are given μ, δ, ζ, ξ, apply Lemma 2.3.1 with μ and get c_0 and L_0 apply Corollary 2.3.2 with μ, δ to get c_1 and L_1. Take $c' = \min\{c_0, c_1\}$, $L' = \max\{L_0, L_1\}$,

$$L = (L'+2)\Big(\frac{3L'}{2}+1\Big) + 2(L'+1), \qquad \xi^* \leqslant \frac{c'^2\xi^2\zeta^{L'+1}}{8L'^2} \quad \text{and}$$

$$c = \Big(\frac{\zeta^2}{L^2}\Big)^{L'+1}\Big(\frac{c'}{2}\Big)^{L'+2}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{(L'+2)L} .$$

Let G with robust neighbourhoods $\mathcal{R}_G$ be given. For any edge $ab \in E(G)$, set

$$U(ab) = \{v \in V(G) \colon ab \in E(G[U_v])\} \,.$$

Let $xy, x'y'$ be ξ-connectable edges, thus we have $|U(xy)|, |U(x'y')| \geqslant \xi n$.

Consider the auxiliary graph $\mathfrak{A} = A(\mathcal{R}, \zeta)$. For any $U, W \subseteq V(G)$ and $\gamma \in (0,1]$, define

$$d_\gamma(U, W) = \min\{i \colon \exists \gamma n^{i+1} \text{ walks in } \mathfrak{A} \text{ between } U \text{ and } W \text{ of length } i\}\,.$$

Note that $d_\gamma(U, W) = 0$, iff $|U \cap W| \geqslant \gamma n$.

Observe that $d_\gamma(U(xy), U(x'y')) \leqslant L'+1$, with $\gamma = \frac{\xi^2 c'}{4L'}$. Indeed, assume $|U(xy) \cap U(x'y')| \leqslant \frac{\xi}{2} n$. Since the $\mathfrak{A}$ is μ-inseparable, Lemma 2.3.1 gives for each distinct $u \in U(xy)$, $v \in U(x'y')$, a value $1 \leqslant \ell(u, v) \leqslant L'$. There exists an ℓ which is the same for at least $\frac{\xi^2}{4L'} n^2$ pairs. Thus, we get at least $\frac{\xi^2 c'}{4L'} n^{\ell+2} = \gamma n^{\ell+2}$ walks between $U(xy)$ and $U(x'y')$ of length $\ell + 1$.

We now prove the lemma by induction on $d_\gamma(U(xy), U(x'y'))$. For the base case, assume $d_{\gamma'}(U(xy), U(x'y')) = 0$ for $\gamma' = \zeta^{L'+1}\gamma$. For $u \in U(xy) \cap U(x'y')$, we have that $xy, x'y' \in E(G[U_u])$. Since $G[U_u]$ is μ-inseparable and contains δn^3 triangles, Corollary 2.3.2 gives us $\ell_e(u) \leqslant L'$ and $c' n^{\ell_e(u)}$ different (y, x')-walks with $\ell_e(u)$ inner vertices in $G[U_u]$. By the pigeonhole principle there is ℓ_e common to at least $\frac{\gamma' n}{L'}$ vertices in $U(xy) \cap U(x'y')$.

By a standard averaging argument we have a set $\mathcal{P}$ of (y, x')-paths with ℓ_e inner vertices with $|\mathcal{P}| \geqslant \frac{c'}{2} n^{\ell_e}$ such that for each $P \in \mathcal{P}$, there is $U(P) \subseteq U(xy) \cap U(x'y')$ with $|U(P)| \geqslant \frac{c'\gamma'}{2L'} n$ and $u \in U(P)$ implies $P \subseteq G[U_u]$. Observe that all edges in P are $\frac{c'\gamma'}{2L'}$-connectable.

We note that including some vertices of $U(P)$ into P gives a triangle walk between xy and $x'y'$. Say $P = y v_1 \ldots v_{\ell_e} x'$; for any choice,

$$\left\{ u_i \in U(P) \colon i \in \left[\frac{\ell_e}{2} + 1\right] \right\},$$

we have that $x y u_1 v_1 v_2 u_2 \cdots u_{\ell_e/2+1} x' y'$ is a triangle walk.

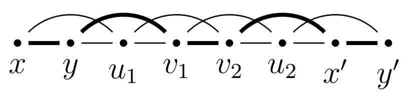


Figure 2.3.1: Including some vertices of $U(P)$ into P.

The number of triangle walks between xy and $x'y'$ with $\frac{3\ell_e}{2} + 1 \leqslant \frac{3L'}{2} + 1 \leqslant L$ inner vertices is at least

$$|\mathcal{P}||U(P)|^{\ell_e/2+1} \geqslant \frac{c'}{2}\Big(\frac{c'\gamma'}{2L'}\Big)^{\ell_e/2+1} n^{3\ell_e/2+1} \geqslant \frac{c'}{2}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{L} n^{3\ell_e/2+1}\,.$$

For the induction hypothesis, let $1 \leqslant \ell' \leqslant L'+1$ and assume that if $ab, a'b' \in E(G)$ are such that $d_{\gamma'}(U(ab), U(a'b')) \leqslant \ell' - 1$ for $\gamma' = \zeta^{L'-(\ell'-1)+1}\gamma$, then there is $\ell = 1 \pmod 3$,

$$\ell \leqslant \ell'\Big(\frac{3L'}{2} + 1\Big) + 2(\ell' - 1) \leqslant L\,,$$

such that the number of triangle walks with ℓ inner vertices between ab and $a'b'$ is at least

$$\Big(\frac{\zeta^2}{L^2}\Big)^{\ell'-1}\Big(\frac{c'}{2}\Big)^{\ell'}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{\ell' L} n^{\ell}\,.$$

Moreover if $abx_1 \ldots x_\ell a'b'$ is such a walk, the edges $x_{3i-1}x_{3i}$, with $i \in [\frac{\ell-1}{3}]$, are ξ^*-connectable.

Now consider that $d_{\gamma'}(U(xy), U(x'y')) = \ell'$ for $\gamma' = \zeta^{L'-\ell'+1}\gamma$. Set

$$U_1 = \{u_1 \colon \exists \text{ walk } u_0u_1 \ldots u_{\ell'-1}u_{\ell'} \text{ in } \mathfrak{A} \text{ with } u_0 \in U(xy), u_{\ell'} \in U(x'y')\}\,.$$

Note that the set $E_{\mathfrak{A}}(U(xy), U_1)$ of edges in $E(\mathfrak{A})$ between $U(xy)$ and U_1 is such that $|E_{\mathfrak{A}}(U(xy), U_1)| \geqslant \gamma' n^2$.

We want a large set $E' \subseteq E(G)$ such that for any $ab \in E'$, we have $d_{\gamma''}(U(xy), U(ab)) = 0$ and $d_{\gamma''}(U(ab), U(x'y')) \leqslant \ell' - 1$ for $\gamma'' = \zeta\gamma'$. Indeed, considering each $uv \in E_{\mathfrak{A}}(U(xy), U_1)$, since $uv \in E(\mathfrak{A})$ we have $|E(G[U_u]) \cap E(G[U_v])| \geqslant \zeta n^2$. By a standard averaging argument, there is $E' \in E(G)$ with $|E'| \geqslant \zeta|E(G)|$ such that $ab \in E'$ implies that there is $E_{\mathfrak{A}}(ab) \subseteq E_{\mathfrak{A}}(U(xy), U_1)$ with $|E_{\mathfrak{A}}(ab)| \geqslant \zeta|E_{\mathfrak{A}}(U(xy), U_1)| \geqslant \zeta\gamma' n^2$ and for every $uv \in E_{\mathfrak{A}}(ab)$, we have $ab \in E(G[U_u]) \cap E(G[U_v])$. Thus $|U(ab) \cap U(xy)| \geqslant \zeta\gamma' n$ and $|U(ab) \cap U_1| \geqslant \zeta\gamma' n = \zeta^{L'-(\ell'-1)+1}\gamma$, which for $\gamma'' = \zeta\gamma'$, gives us that ab is γ''-connectable, that $d_{\gamma''}(U(xy), U(ab)) = 0$ and $d_{\gamma''}(U(ab), U(x'y')) \leqslant \ell' - 1$.

According to our induction hypothesis, there are $\ell_1', \ell_2' \leqslant L$, with $\ell_1', \ell_2' = 1 \pmod 3$, such that the number of triangle walks between xy and ab with ℓ_1' inner vertices and between ab and $x'y'$ with ℓ_2' inner vertices are at least

$$\frac{c'}{2}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{L} n^{\ell_1'} \qquad \text{and} \qquad \Big(\frac{\zeta^2}{L^2}\Big)^{\ell'-1}\Big(\frac{c'}{2}\Big)^{\ell'}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{\ell' L} n^{\ell_2'}\,.$$

We have values ℓ_1 and ℓ_2 common to at least $\frac{\zeta|E(G)|}{L^2} \geqslant \frac{\zeta^2 n^2}{L^2}$ edges in E'. For each such edge ab, we concatenate the walks between xy and ab and between ab and $x'y'$. The number of $(xy, x'y')$-triangle walks with $\ell_1 + \ell_2 + 2 = 1 \pmod 3$ inner vertices that we obtain is at least

$$\Big(\frac{\zeta^2}{L^2}\Big)^{\ell'}\Big(\frac{c'}{2}\Big)^{\ell'+1}\Big(\frac{c'\zeta^{L'+1}\gamma}{2L'}\Big)^{(\ell'+1)L}n^{\ell_1+\ell_2+2}\,.$$

Moreover,

$$\begin{aligned}\ell_1+\ell_2+2 &\leqslant \frac{3L'}{2}+1+\ell'\Big(\frac{3L'}{2}+1\Big)+2(\ell'-1)+2\\ &\leqslant (\ell'+1)\Big(\frac{3L'}{2}+1\Big)+2\ell' \leqslant L\,.\end{aligned}$$ □

2.4 Absorbing

We start by introducing the desired "absorbing" structures. Roughly, the structure consists of disjoint triangle paths with the property that they can be modified to include other vertices, without changing the ending (ordered) edges of each triangle path.

To describe the absorbers precisely, we use the following structure, see Figure 2.4.1 for guidance. Given a positive integer k, an *open C_4-path* L_k is obtained from disjoint $C_1,\dots,C_k$ and $u_1,\dots,u_{k-1}$, where C_i ($i\in[k]$) are C_4's and u_i ($i\in[k-1]$) are vertices such that $C_i, C_{i+1}\subseteq N(u_i)$. For vertices x,y disjoint from L_k, such that $C_1\subseteq N(x)$ and $C_k\subseteq N(y)$, we say xL_ky is a *C_4-path* connecting x and y and we refer to k as its length.

The following definition describes the absorbers.

Definition 2.4.1. Given $\xi>0$, a graph G, vertices $x,y,z\in V(G)$ and positive integers k_1,k_2,k_3, we say a tuple $A=(S,L_{k_1},L_{k_2},L_{k_3})$ is a $(k_1,k_2,k_3)_\xi$*-absorber* for x,y and z with *switch* S, when

1. S is a $K_{3,3,2}$ with parts $\{w_1,s_4,w_7\}$, $\{w_2,s_5,w_8\}$, $\{w_3,s_6\}$ and such that the edges w_1w_2 and w_7w_8 are ξ-connectable.

2. $xL_{k_1}s_4$, $yL_{k_2}s_5$ and $zL_{k_3}s_6$ are C_4-paths of lengths k_1,k_2,k_3 respectively and for each C_i in these C_4-paths, we have that all edges in $E(C_i)$ are ξ-connectable.

3. The vertices x,y,z, the open C_4-paths and S are disjoint.

We also say A is a $(k_1,k_2,k_3)_\xi$*-absorber* if it is a $(k_1,k_2,k_3)_\xi$-absorber for some triple of vertices x, y and z.

In the following observation we describe how the vertices in a $(k_1, k_2, k_3)_\xi$-absorber for the vertices x, y, z can be covered by disjoint triangle paths and how they can be modified to include x, y, z.

Observation 2.4.2. *The vertices in a $(k_1, k_2, k_3)_\xi$-absorber $A = (S, L_{k_1}, L_{k_2}, L_{k_3})$ can be covered by disjoint triangle paths with ξ-connectable ending pairs. Let the C_4-path $xL_{k_1}s_4$ be such that $C_i = v_{2i-1}v_{2i}v'_{2i-1}v'_{2i}$, then take the triangle paths $v_{2i-1}v_{2i}u_iv'_{2i-1}v'_{2i}$ for $i \in [k_1 - 1]$ and $v_{2k_1-1}v_{2k_1}s_4v'_{2k_1-1}v'_{2k_1}$, do it similarly for $yL_{k_2}s_5$ and $zL_{k_3}s_6$, and for the switch S take $w_1w_2w_3w_7w_8$.*

Moreover if A is a $(k_1, k_2, k_3)_\xi$-absorber for $x, y, z \in V(G)$, we say A absorbed x, y, z when we consider the following disjoint triangle paths covering all vertices of A plus x, y, z. For the C_4-path $xL_{k_1}s_4$ take $v_1v_2xv'_1v'_2$ and $v_{2i-1}v_{2i}u_{i-1}v'_{2i-1}v'_{2i}$ for $2 \leqslant i \leqslant [k_1]$, proceed similarly for $yL_{k_2}s_5$ and $zL_{k_3}s_6$, for the switch S take $w_1w_2w_3s_4s_5s_6w_7w_8$. All the ending pairs of these triangle paths are the same as when covering A without absorbing x, y, z.

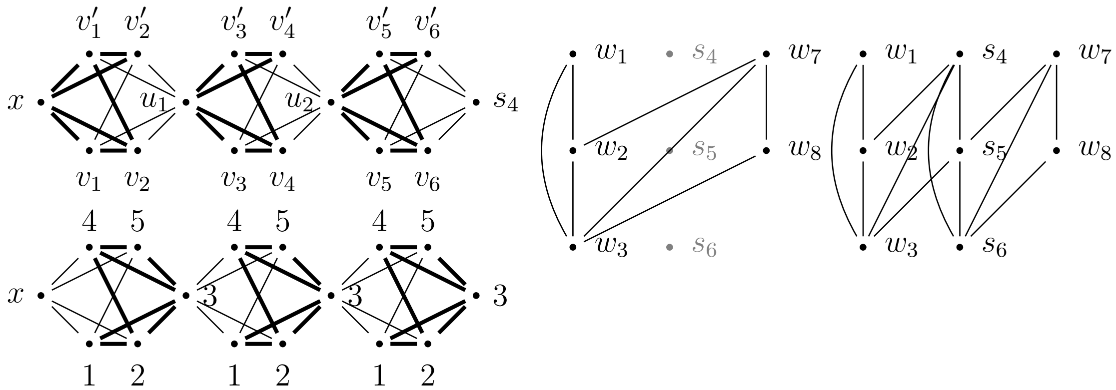


Figure 2.4.1: A C_4-path xL_2s_4 and switch both absorbing x and not.

We take step 1 in our application of the absorption method and show the existence of many absorbers in a good graph. For that we use the following result of Erdős [37], that ensures many r-partite complete hypergraphs $K^{(r)}(\ell_1, \ldots, \ell_r)$ in a dense r-uniform hypergraph $G^{(r)}$.

Theorem 2.4.3 (Erdős 1964)**.** *Given integers $r, \ell_1, \ldots, \ell_r$ and $c > 0$, there is $c' > 0$ and n_0 such that, if $G^{(r)}$ is an r-uniform hypergraph on $n \geqslant n_0$ vertices and at least cn^r edges, then there are $c'n^{\ell_1+\cdots+\ell_r}$ copies of $K^{(r)}(\ell_1, \ldots, \ell_r)$ in $G^{(r)}$.*

The following lemma guarantees that good graphs have many absorbers.

Lemma 2.4.4. *For every $\mu, \delta, \zeta > 0$, there exist $c, \xi^* > 0$ and integers L, n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices with robust neighbourhoods $\mathcal{R}_G$ satisfies properties i. and ii. of good graphs, then for every distinct $x, y, z \in V(G)$ there are integers $k_1, k_2, k_3 \leqslant L$ such that the number of $(k_1, k_2, k_3)_{\xi^*}$-absorbers for x, y, z is at least $cn^{5k_1+5k_2+5k_3+5}$.

Proof. Given $\mu, \delta, \zeta > 0$, apply Lemma 2.1.1 with $\mu, \delta, \zeta, \xi = \zeta\mu > 0$ and get $c', \xi^* > 0$ and L, assume wlog $\xi^* \leqslant \zeta\mu$. Constants c_0, c_1, c_2, c_3 will be given by applications of Theorem 2.4.3. Take

$$c = \frac{c_0 c_1 c_2 c_3}{2L^3}.$$

We want to apply Theorem 2.4.3 to find many switches in G, for that we need many triangles with a connectable edge. Equation (2.1.2) gives us that for any $v \in V(G)$, if E'_v are the $\zeta\mu$-connectable edges in $G[U_v]$, then $|E'_v| \geqslant \frac{\zeta n^2}{4}$. For each $ab \in E'_v$ we get a triangle abv, thus there are at least $\frac{\zeta n^3}{12}$ triangles with at least one $\zeta\mu$-connectable edge in G.

Take a random partition of $V(G)$ into $\{V_1, V_2, V_3\}$, where $\mathbb{P}(v \in V_i) = \frac{1}{3}$. Let X be the number of triangles $v_1v_2v_3$ with a connectable edge and $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$. The expected value is

$$\mathbb{E}X \geqslant \frac{\zeta n^3}{12} \cdot \frac{6}{27} = \frac{\zeta n^3}{54}.$$

Thus there is a partition $\{V_1, V_2, V_3\}$ with at least $\frac{\zeta n^3}{54}$ such triangles and by averaging, we may assume without loss of generality that at least $\frac{\zeta n^3}{162}$ of these triangles have its connectable edge between V_1, V_2.

Consider the hypergraph $G^{(3)}$ on $V(G)$ and $v_1v_2v_3 \in E(G^{(3)})$ when $v_1v_2v_3$ is a triangle in G with $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$ and v_1v_2 connectable. Theorem 2.4.3, gives us $c_0 n^8$ copies of $K^{(3)}(3,3,2)$. Since all hyperedges had a connectable pair between V_1 and V_2, each of these $K^{(3)}(3,3,2)$ corresponds to a possible switch in G, let $\mathcal{S}$ be the set containing these corresponding $K_{3,3,2}$'s.

Consider vertices x, y, z and $S \in \mathcal{S}$ (with vertices labelled as in Figure 2.4.1). We want a value k_1 and many C_4-paths $xL_{k_1}s_4$. Take $\zeta\mu$-connectable edges $v_1v_2 \in E'_x$ and $v'_1v'_2 \in E'_{s_4}$. Lemma 2.1.1 gives us $\ell(v_1v_2, v'_1v'_2) = \ell \leqslant L$ and $\ell = 1 \pmod 3$. Excluding pairs of connectable edges that intersect, there are at least $\frac{\zeta^2 n^4}{17}$ different choices of $(v_1v_2, v'_1v'_2)$ and thus there is a value $\ell_1 = 3(k_1 - 2) + 1$ common to at least $\frac{\zeta^2 n^4}{17L}$ such choices. The number of $(v_1v_2, v_1'v_2')$-triangle walks of form

$$v_1v_2\ x_1x_2x_3x_4 \dots x_{\ell_1}\ v_1'v_2'\,,$$

is at least $c'n^{\ell_1}$ and the edges $x_{3i-1}x_{3i}$ $(i \in [k_1-2])$ are ξ^*-connectable. For all choices of $(v_1v_2, v_1'v_2')$, let $\mathcal{W}$ be the set of all such walks.

Consider the partite hypergraph $G^{(\ell_1+4)}$ with

$$V(G^{(\ell_1+4)}) = V_1 \dot\cup \dots \dot\cup V_{\ell_1+4}, \quad V_i = V(G) \quad \text{and} \quad E(G^{(\ell_1+4)}) = \mathcal{W}\,.$$

We have that $|E(G^{(\ell_1+4)})| \geqslant \frac{\zeta^2 c'}{17L} n^{\ell_1+4}$. Theorem 2.4.3 gives us that the number of copies of partite cliques $K^{(\ell_1+4)}(2,2,1,2,2,1\dots,1,2,2)$ is at least $c_1 n^{5(k_1-2)+9}$ and at least half of them have no vertex repetition. In G, each such partite clique forms an open C_4-path

$$L_{k_1} = C_1u_1 \dots u_{k_1-1}C_{k_1}\,.$$

We have that $xL_{k_1}s_4$ is a C_4-path where the edges in $E(C_i)$ are ξ^*-connectable.

Repeating the argument for the pairs (y, s_5) and (z, s_6), we get $c_2n^{5k_2-1}$ different $yL_{k_2}s_5$ of length k_2 and $c_3n^{5k_3-1}$ different $zL_{k_3}s_6$ of length k_3. For x, y, x, there are at least $\frac{c_0 n^8}{L^3}$ switches associated to the same values k_1, k_2, k_3. Picking such a switch S and C_4-paths $xL_{k_1}s_4$, $yL_{k_2}s_5$, $zL_{k_3}s_6$, then excluding choices with vertex repetition, the number of $(k_1,k_2,k_3)_{\xi^*}$-absorbers for x, y, z is at least

$$\frac{1}{2L^3} c_0 n^8 c_1 n^{5k_1-1} c_2 n^{5k_2-1} c_3 n^{5k_3-1} = c n^{5k_1+5k_2+5k_3+5}\,. \qquad \square$$

To get an absorbing path, we will start with a collection of disjoint absorbers. Then one by one we connect the triangle paths that cover the first absorber, then the second and so on, always making sure that the triangle path we constructed so far and the following one to be incorporated do not intersect. For ensuring these connections, we show that the good property of G holds for subgraphs obtained after the removal of few vertices.

We need the following property of μ-inseparable graphs.

Property 2.4.5. *If G is a μ-inseparable graph on n vertices, $\beta \in (0, \frac{1}{2})$ and $U \subseteq V(G)$ has $|U| \leqslant \beta\mu n$, then $G' = G[V(G) \setminus U]$ is $(1-2\beta)\mu$-inseparable.*

Indeed, assume for a contradiction that this is not valid, then there exists $X \subseteq V' = V(G) \setminus U$ with $|X| \leqslant \frac{|V'|}{2} \leqslant \frac{n}{2}$ such that

$$e_{G'}(X, V' \setminus X) < (1-2\beta)\mu|X||V' \setminus X|\,.$$

Consider the partition of $V(G)$ into the sets X and $(V' \setminus X) \cup U = V(G) \setminus X$. We have that

$$\begin{aligned} e_G(X, V(G) \setminus X) &< (1-2\beta)\mu|X||V' \setminus X| + |U||X| \\ &= (1-2\beta)\mu|X|(|V(G)| - |U| - |X|) + |U||X| \\ &= \mu|X||V(G) \setminus X| - 2\beta\mu|X||V(G) \setminus X| + (1-(1-2\beta)\mu)|U||X|\,. \end{aligned}$$

Since $|V(G) \setminus X| \geqslant \frac{n}{2}$, $\beta\mu n \geqslant |U|$, and $\beta < \frac{1}{2}$ we have

$$2\beta\mu|X||V(G) \setminus X| \geqslant \beta\mu n|X| \geqslant |U||X| \geqslant (1-(1-2\beta)\mu)|U||X|\,.$$

We derive that $e_G(X, V(G) \setminus X) < \mu|X||V(G) \setminus X|$, which is a contradiction.

In the following lemma we show that good graphs have a similar property.

Lemma 2.4.6. *Given $\mu, \delta, \zeta > 0$, if G satisfies properties i. and ii. of good graphs and $X \subseteq V(G)$ is such that $|X| \leqslant \min\{\frac{\mu}{4}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{2\mu\sqrt{\zeta}}{5}\}n$, then $G' = G[V(G) \setminus X]$ with $\mathcal{R}_{G'} = \{U'_v = U_v \cap N_{G'}(v) \colon v \in V(G')\}$ satisfies properties i. and ii. of good graphs with $\frac{\mu}{2}, \frac{\delta}{2}, \frac{\zeta}{2}$.*

Moreover, given $\varrho, \xi, \eta, \nu > 0$, $\varrho \leqslant \frac{1}{16}$, $\nu \leqslant \frac{1}{4}$, if G satisfies property iii. of good graphs and $X \subseteq V$ is such that $|X| \leqslant \frac{\nu n}{2}$, then G' satisfies property iii. with $\frac{\varrho^2}{4}, \frac{\xi}{2}, \eta, \frac{\nu}{2}$.

Proof of Lemma 2.4.6. Given G satisfying properties i. and ii. and given X such that $|X| \leqslant \min\{\frac{\mu}{4}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{2\mu\sqrt{\zeta}}{5}\}n$, we check that G' with $\mathcal{R}_{G'}$ has property i. of good graphs. We have for any $v \in V(G) \setminus X$, that

$$|X| \leqslant \frac{2\mu}{5}\sqrt{\zeta}n = \frac{\mu}{4}\Big(2\sqrt{\zeta} - \frac{2\sqrt{\zeta}}{5}\Big)n \leqslant \frac{\mu}{4}(|U_v| - |X|) \leqslant \frac{\mu}{4}|U'_v|\,.$$

Property 2.4.5 gives us that each $G'[U'_v]$ is $\mu/2$-inseparable. The number of triangles in $G[U_v]$ containing a vertex in X is at most $|X||U_v|^2 \leqslant \frac{\delta}{2}n^3$, thus each $G'[U'_v]$ contains at least $\frac{\delta}{2}n^3 \geqslant \frac{\delta}{2}|V(G')|^3$ triangles.

For property ii., observe that $|E(G) \smallsetminus E(G')| \leqslant |X|n \leqslant \frac{\zeta}{2}n^2$ and if $u, v \in V(G')$ are such that $uv \in E(A(\mathcal{R}, \zeta))$, then $|E(G'[U'_u]) \cap E(G'[U'_v])| \geqslant \frac{\zeta}{2}n^2$. Thus for $\mathfrak{A}_{G'} = A(\mathcal{R}_{G'}, \zeta/2)$ and $\mathfrak{A}_G = A(\mathcal{R}, \zeta)$, we have $\mathfrak{A}_{G'} = \mathfrak{A}_G[V(G) \smallsetminus X]$ and $|X| \leqslant \frac{\mu}{4}n$ implies that $\mathfrak{A}_{G'}$ is $\mu/2$-inseparable.

Next consider G satisfying property iii. and $|X| \leqslant \frac{\nu}{2}n$ be given. We check that G' satisfies property iii. with $\frac{\varrho^2}{4}, \frac{\xi}{2}, \eta, \frac{\nu}{2}$, thus take $A' \subseteq V(G')$ with $|A'| \leqslant \frac{\nu}{2}|V(G')|$ and $F' \subseteq E(G')$ with $|F'| \leqslant \frac{\varrho^2}{4}|V(G')|^2$. Consider $X_{F'} \subseteq V(G')$ and $G'_{A',F'}$ as in property iii. and take

$$A = A' \cup X \quad \text{and} \quad F = \{uv \in E(G) \colon u \in X_{F'}\} \cup F',$$

since $|X| \leqslant \frac{\nu}{2}n$ and $|X_{F'}| \leqslant \frac{\varrho}{2}n$, we have that $|A| \leqslant \nu n$ and $|F| \leqslant \varrho n^2$. Thus $G_{A,F}$ contains a ξ-connectable fractional triangle factor with

$$W(f_{G_{A,F}}) \geqslant \frac{n}{3} - \eta(n - |A|) = \frac{n}{3} - \eta(|V(G')| - |A'|).$$

We check that $xyz \in T_{G_{A,F}}$ implies $xyz \in T_{G'_{A',F'}}$. Indeed, if $xy \in E(G_{A,F})$, then by the choice of A and F, we have $x, y \notin A' \cup X \cup X_{F'}$ and $xy \notin F'$, thus $xy \in E(G'_{A',F'})$. We set $f_{G'_{A',F'}}(xyz) = f_{G_{A,F}}(xyz)$ for $xyz \in T_{G'_{A',F'}} \cap T_{G_{A,F}}$ and $f_{G_{A',F'}}(xyz) = 0$ otherwise, then we have that $W(f_{G'_{A',F'}}) = W(f_{G_{A,F}})$.

We check that this is a $\frac{\xi}{2}$-connectable fractional triangle factor. We have the robust neighbourhoods

$$\begin{aligned}
\mathcal{R}_{G_{A,F}} &= \{U_{A,F}(v) = U_v \cap N_{G_{A,F}}(v) \colon v \in V(G_{A,F})\} \quad \text{and} \\
\mathcal{R}_{G'_{A',F'}} &= \{U'_{A',F'}(v) = U_v \cap N_{G'_{A',F'}}(v) \colon v \in V(G'_{A',F'})\}.
\end{aligned}$$

Take $xy \in E(G_{A,F})$ and $x, y \in U_{A,F}(v)$ for $\xi|V(G_{A,F})|$ different $v \in V(G_{A,F})$. We already observed that $E(G_{A,F}) \subseteq E(G'_{A',F'})$, thus $x, y \in N_{G'_{A',F'}}(v)$ and $x, y \in U'_{A',F'}(v)$. Considering $\nu \leqslant \frac{1}{4}$ and $\varrho \leqslant \frac{1}{16}$, the number of different $v \in V(G'_{A',F'})$ for which $x, y \in U_{A',F'}(v)$ is at least

$$\xi|V(G_{A,F})| \geqslant \xi(n - |X| - |A'| - |X_F|) \geqslant \xi\Big(1 - \frac{\nu}{2} - \frac{\nu}{2} - \sqrt{\varrho}\Big)n \geqslant \frac{\xi}{2}|V(G'_{A',F'})|. \ \square$$

In Lemma 2.1.2, we require that the absorbing path P_A starts with a triangle path $x_1x_2x_3x_4$ and in Theorem 2.2.1 we require P_6, which is the 3rd power of a path on 6 vertices. Here we show that a good graph has many of these structures.

Lemma 2.4.7. *For every $\mu, \delta > 0$ there are $c_1, c_2 > 0$ such that if $G = (V, E)$ satisfies property i. of good graphs, then G contains c_1n^4 triangle paths $x_1x_2x_3x_4$, where all 5 edges are $\frac{\delta}{2}$-connectable and c_2n^6 third power of paths $x_1x_2x_3x_4x_5x_6$, where x_1x_2 and x_5x_6 are $\frac{\delta}{2}$-connectable.*

Proof. Take $c_1 = 3\delta^2$ and c_2 will be given by an application of Theorem 2.4.3.

Since the number of triangles in $G[U_v]$ is at least δn^3, by a standard averaging argument we have T' a set of triangles abc such that $|T'| \geqslant \frac{\delta}{2}n^3$ and abc is in at least $\frac{\delta}{2}n$ different $G[U_v]$. Therefore, all edges in abc are $\frac{\delta}{2}$-connectable.

For finding triangle paths $x_1x_2x_3x_4$, denote by $d_{T'}(ab)$ the number of triangles in T' containing ab. A standard application of Cauchy-Schwartz gives us

$$\sum_{ab \in E(G)} d_{T'}^2(ab) \geqslant \frac{2(3|T'|)^2}{n^2} \geqslant \frac{9\delta^2}{2}n^4\,,$$

and excluding vertex repetitions, there are at least $\frac{9\delta^2}{3}n^4 = c_1n^4$ triangle paths $x_1x_2x_3x_4$ with all 5 edges being $\frac{\delta}{2}$-connectable.

For finding the third power of paths, consider the set $\mathcal{K}_4$ of $K_4 = abcv$, where $abc \in T'$ and $abc \subseteq G[U_v]$, then $|\mathcal{K}_4| \geqslant \left(\frac{\delta}{2}\right)^2 n^4$ and all edges in abc are connectable. Take a random partition $V(G) = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$, where $v \in V_i$ with $p = 1/4$. Let X be the number of $v_1v_2v_3v_4 \in \mathcal{K}_4$ with $v_i \in V_i$. We have that

$$\mathbb{E}X \geqslant \frac{4!}{4^4}\left(\frac{\delta}{2}\right)^2 n^4 = \frac{3\delta^2}{128}n^4\,.$$

Fix a partititon with at least $\frac{3\delta^2}{128}n^4$ such K_4's. By the pigeonhole principle there is $\mathcal{K}_4' \subseteq \mathcal{K}_4$ with $|\mathcal{K}_4'| \geqslant \frac{3\delta^2}{256}n^4$, such that $v_1v_2v_3v_4 \in \mathcal{K}_4'$ iff $v_i \in V_i$ and v_1v_2 is $\frac{\delta}{2}$-connectable.

Define the partite hypergraph $G^{(4)}$ on $V(G^{(4)}) = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$ and $E(G^{(4)}) = \mathcal{K}_4'$. Theorem 2.4.3 gives us c_2n^6 copies of $K^{(4)}(2, 2, 1, 1)$. Let $x_j, x_j' \in V_j$ for $j \in [2]$, $x_3 \in V_3$ and $x_4 \in V_4$ be the vertices of such a $K^{(4)}(2, 2, 1, 1)$, then $x_1x_2x_3x_4x_1'x_2' \subseteq G$ is such that every 4 consecutive vertices form a K_4 and the first and last edges are connectable, as desired. □

We prove the Absorbing Path Lemma.

Proof of Lemma 2.1.2. We are given $\mu, \delta, \zeta, \xi, c(\delta) > 0$. Apply Lemma 2.4.4 with μ, δ, ζ, get $\xi_1, c_1 > 0$ and L_1. Take

$$\xi^* = \min\left\{\frac{\delta}{2}, \xi_1\right\} \quad \text{and} \quad \xi' = \frac{\xi^*}{2}.$$

Apply Lemma 2.1.1 with $\frac{\mu}{2}, \frac{\delta}{2}, \frac{\zeta}{2}, \frac{\xi^*}{2}$, get $c_2 > 0$ and L_2 (we also get a constant ξ_2, but here we do not require the moreover part of Lemma 2.1.1). Moreover $c > 0$ is given by an application of Theorem 2.4.3. Set

$$p' \leqslant \min\left\{\frac{c_1}{24 \cdot 400 \cdot L_1^8}, \frac{c_2}{27(5+L_2)(3L_1+1)L_1^6}, \frac{\min\left\{\frac{\mu}{4}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\mu\sqrt{\zeta}}{5}, \frac{\nu}{4}, \frac{\xi^*}{2}, \frac{c(\delta)}{2}\right\}}{4(5+L_2)(3L_1+1)L_1^6}\right\},$$

$$p'' \leqslant \min\left\{\frac{c}{4 \cdot 75}, \frac{c_2}{24(5+L_2)}, \frac{\min\left\{\frac{\mu}{4}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\mu\sqrt{\zeta}}{5}, \frac{\nu}{4}\right\}}{4(5+L_2)}\right\},$$

$$\alpha_0 = 3\frac{c_1 p'}{8}, \quad \alpha = \frac{p''c}{4} \quad \text{and} \quad \kappa = \frac{p''c}{16}.$$

Let $\nu > 0$ be given. First we find the path P_A and put it to the side. In the remaining graph, we proceed in a very similar way to find P_I and conclude the proof.

Fix a triangle path $x_1x_2x_3x_4$ given by Lemma 2.4.7. Let

$$\mathcal{A} = \{(k_1, k_2, k_3)_{\xi^*}\text{-absorber} \subseteq G\colon k_1, k_2, k_3 \in [L_1]\}.$$

Consider a random choice A of absorbers in $\mathcal{A}$, where each $(k_1, k_2, k_3)_{\xi^*}$-absorber is included independently with probability

$$p_{k_1k_2k_3} = p' n^{-5(k_1+k_2+k_3)-4}.$$

We prove that with positive probability, A is such that the following holds.

(a) For each $k_1, k_2, k_3 \in [L_1]$, the number of $(k_1, k_2, k_3)_{\xi^*}$-absorbers in A is less than $3L_1^3 p' n$.

(b) For every $x, y, z \in V(G)$, there exist $k_1, k_2, k_3 \in [L_1]$ such that the number of $(k_1, k_2, k_3)_{\xi^*}$-absorbers for x, y, z in A that do not intersect $x_1x_2x_3x_4$ is at least $\frac{c_1 p' n}{4}$.

(c) The number of pairs of absorbers in A that share at least one vertex is less than $\frac{c_1 p' n}{8}$.

For property (a), let $A(k_1,k_2,k_3)$ be the number of $(k_1,k_2,k_3)_{\xi^*}$-absorbers in A. We have that $\mathbb{E}A(k_1,k_2,k_3) \leqslant n^{5(k_1+k_2+k_3)+5} p_{k_1k_2k_3} = p'n$ and by Markov's inequality,

$$\mathbb{P}\left[A(k_1,k_2,k_3) \geqslant 3L_1^3 p'n\right] \leqslant \mathbb{P}\left[A(k_1,k_2,k_3) \geqslant 3L_1^3 \mathbb{E}A(k_1,k_2,k_3)\right] \leqslant \frac{1}{3L_1^3}.$$

Consequently the union bound gives us that property (a) holds with probability at least $\frac{2}{3}$.

For property (b), fix $x,y,z \in V(G)$ and Lemma 2.4.4 assures $k_1,k_2,k_3 \in [L_1]$ such that the number of $(k_1,k_2,k_3)_{\xi^*}$-absorbers for x,y,z is at least $c_1 n^{5(k_1+k_2+k_3)+5}$. At least half of these absorbers do not intersect $x_1x_2x_3x_4$. Let A_{xyz} be the number of $(k_1,k_2,k_3)_{\xi^*}$-absorbers for x,y,z in A not intersecting $x_1x_2x_3x_4$. We have by Chernoff's inequality, that

$$\mathbb{P}\left[A_{xyz} \leqslant \frac{c_1 p' n}{4}\right] \leqslant \mathbb{P}\left[A_{xyz} \leqslant \frac{\mathbb{E}A_{xyz}}{2}\right] \leqslant \exp\left(-\frac{c_1 p' n}{16}\right).$$

In view of the union bound for all triples $x,y,z \in V(G)$, we derive that a.a.s. A enjoys property (b).

For property (c), fix $k_1,k_2,k_3,k_1',k_2',k_3' \in [L_1]$. For a $(k_1,k_2,k_3)_{\xi^*}$-absorber A_1, the number of $(k_1',k_2',k_3')_{\xi^*}$-absorbers A_2 that intersect A_1 is at most

$$(5(k_1+k_2+k_3)+5)(5(k_1'+k_2'+k_3')+5)n^{5(k_1'+k_2'+k_3')+4} \leqslant 400L_1^2 n^{5(k_1'+k_2'+k_3')+4}.$$

The number of intersecting pairs (A_1,A_2) is at most

$$400L_1^2 n^{5(k_1+k_2+k_3+k_1'+k_2'+k_3')+9}.$$

Let $X(k_1,k_2,k_3,k_1',k_2',k_3')$ be the number of such intersecting pairs in A, we have

$$\begin{aligned}\mathbb{E}X(k_1,k_2,k_3,k_1',k_2',k_3') &\leqslant p_{k_1k_2k_3} p_{k_1'k_2'k_3'} \cdot 400L_1^2 n^{5(k_1+k_2+k_3+k_1'+k_2'+k_3')+9} \\ &\leqslant 400L_1^2 p'^2 n \leqslant \frac{c_1 p' n}{24L_1^6}.\end{aligned}$$

Since this upper bound is independent of $k_1, k_2, k_3, k_1', k_2', k_3'$, for the number of pairs of absorbers in A sharing a vertex,

$$X = \sum_{k_1,k_2,k_3,k_1',k_2',k_3' \in [L_1]} X(k_1, k_2, k_3, k_1', k_2', k_3')\,,$$

we have $\mathbb{E}X \leqslant \frac{c_1 p' n}{24}$ and Markov's inequality yields

$$\mathbb{P}\left[X \geqslant \frac{c_1 p' n}{8}\right] \leqslant \frac{1}{3}.$$

From the previous observations we conclude that there exists an instance set of absorbers satisfying properties (a), (b) and (c). Removing one absorber for each intersecting pair in this instance yields a set of disjoint absorbers B satisfying (a) and

(b') for every $x, y, z \in V(G)$, there exist $k_1, k_2, k_3 \in [L_1]$ such that there are at least $\frac{c_1 p' n}{8} = \frac{\alpha_0 n}{3}$ different $(k_1, k_2, k_3)_{\xi^*}$-absorbers for x, y, z in B that are also disjoint from the triangle path $x_1 x_2 x_3 x_4$.

The next step is to obtain P_A by "connecting all absorbers from B". Observation 2.4.2 tells us that a $(k_1, k_2, k_3)_{\xi^*}$-absorber in B consists of $k_1+k_2+k_3+1 \leqslant 3L_1+1$ triangle paths on 5 vertices with ξ^*-connectable ends. Let $\mathcal{Q}(B)$ be the set of all such paths, considering (a) we have that

$$|\mathcal{Q}(B)| \leqslant L_1^3 \cdot 3L_1^3 p' n(3L_1+1) \leqslant \frac{c_2}{5 \cdot 8} n\,. \tag{2.4.1}$$

The paths in $\mathcal{Q}(B)$ will be connected by repeated applications of Lemma 2.1.1. Consider a maximal subset $\mathcal{Q}' \subseteq \mathcal{Q}(B)$ such that there exists a path $P_{\mathcal{Q}'}$ in G on at most $(5+L_2)|\mathcal{Q}'|$ vertices that contains every path Q of $\mathcal{Q}'$ and is disjoint from each path in $\mathcal{Q}(B) \smallsetminus \mathcal{Q}'$. We shall show that $\mathcal{Q}' = \mathcal{Q}(B)$.

Otherwise, let $Q \in \mathcal{Q}(B) \smallsetminus \mathcal{Q}'$ and let xy, uv be ending pairs of Q and $P_{\mathcal{Q}'}$ respectively. We have that

$$|V(P_{\mathcal{Q}'}) \cup V(Q)| \leqslant (5+L_2)|\mathcal{Q}'| + 5 \overset{(2.4.1)}{\leqslant} (5+L_2)(3L_1+1)3L_1^6 p' n + 5 \leqslant \frac{c_2}{8} n\,. \tag{2.4.2}$$

By Lemma 2.1.1, there is $\ell \leqslant L_2$ and at least $c_2 n^\ell$ different (xy, uv)-triangle walks in G with ℓ inner vertices, at least half of these are triangle paths. At most $\frac{c_2}{8}n^\ell$ of these paths intersect $V(P_{\mathcal{Q}'}) \cup V(Q)$ and at most $\frac{c_2}{8}n^\ell$ intersect $V(\mathcal{Q}(B) \smallsetminus (\mathcal{Q}' \cup Q))$. Then $P_{\mathcal{Q}'}$ can be extended to a path $P_{\mathcal{Q}' \cup \{Q\}}$ with number of vertices at most

$$|V(P_{\mathcal{Q}'})| + |Q| + L_2 \leqslant (5 + L_2)(|\mathcal{Q}'| + 1) = (5 + L_2)|\mathcal{Q}' \cup \{Q\}|\,.$$

Take $P_A = P_{\mathcal{Q}(B)}$, equation (2.4.2) gives us that

$$|V(P_A)| \leqslant \min\left\{\frac{\mu}{4}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\mu\sqrt{\zeta}}{5}, \frac{\xi^*}{2}, \frac{c(\delta)}{2}, \frac{\nu}{4}\right\} n - 3\,,$$

and we observe that the path P_A is α_0-absorbing. In fact, given any set $X \subseteq V(G) \smallsetminus V(P_A)$ with $|X| \leqslant \alpha_0 n$ divisible by 3, it can be split in up to $\frac{\alpha_0 n}{3}$ triples (x, y, z) and owing to property (b') and the disjointness of the absorbers in P_A, we may inductively absorb all triples, as in Observation 2.4.2.

Consider $G' = G[V(G) \smallsetminus V(P_A)]$ and locate P_I in G'. Lemma 2.4.6 gives us that G' with $\mathcal{R}_{G'}$ satisfies properties i. and ii. of good graphs with $\frac{\mu}{2}, \frac{\delta}{2}, \frac{\zeta}{2}$. Moreover if $ab \in E(G') \subseteq E(G)$ and ab is ξ^*-connectable in G, then $a, b \notin V(P_A)$ and $ab \in U_w$ for $\xi^* n$ different w. For $w \notin V(P_A)$, we have $a, b \in U_w \cap N_{G'}(w)$. Since $|V(P_A)| \leqslant \frac{\xi^* n}{2}$, if ab is ξ^*-connectable in G, it is $\frac{\xi^*}{2}$-connectable in G'.

For every $i \in [m]$, set $I_i' = I_i \smallsetminus V(P_A)$ and since $|V(P_A)| < \frac{c(\delta)}{2}n$ we have that the number of triangles with a $\frac{\xi^*}{2}$-connectable edge in $G'[I_i']$ is at least $\frac{c(\delta)}{2}n^3$. Theorem 2.4.3 gives us at least cn^5 disjoint triangle paths on 5 vertices with $\frac{\xi^*}{2}$-connectable end pairs in $G'[I_i']$. Thus for each $i \in [m]$, let

$$B(i) = \left\{y_1y_2y_3y_4y_5 \colon \text{disjoint triangle path with } \frac{\xi^*}{2}\text{-connectable ends in } G'[I_i']\right\}$$
$$\text{and} \quad |B(i)| \geqslant cn^5\,.$$

We carry out a similar argument as before; we fix the probability $p = p''n^{-4}$. Consider a random set $\mathcal{B}$ of triangle paths on 5 vertices, P_5, each included independently with probability p. We show that with positive probability $\mathcal{B}$ enjoys the following properties.

(a) The number of P_5 in $\mathcal{B}$ is at most $3p''n$.

(b) For all $i \in [m]$ the number of $P_5 \in \mathcal{B}$ such that $P_5 \in B(i)$ is least $\frac{p''c}{2}n$.

(c) The number of pairs (P_5, P_5') sharing a vertex is at most $75p''^2 n \leqslant \frac{p''c}{4}n$.

Similarly as in the selection of A for P_A, it follows from Markov's inequality that each of properties (a) and (c) holds with probability at least $\frac{2}{3}$.

For property (b), let X_i ($i \in [m]$) be the number of $P_5 \in \mathcal{B}$ such that $P_5 \in B(i)$. We have that $\mathbb{E}X_i \geqslant p''cn$. Chernoff's inequality yields,

$$\mathbb{P}\Big[X_i \leqslant \frac{p''c}{2}n\Big] \leqslant \exp\Big(-\frac{p''c}{8}n\Big).$$

Since $m \leqslant 2^{\kappa n}$, the union bound over all $i \in [m]$ gives that a.a.s. $\mathcal{B}$ enjoys property (b). Hence, there exists a set of triangle paths on 5 vertices satisfying properties (a), (b), and (c). After deleting one such triangle path for each intersecting pair (P_5, P_5'), we arrive at a set B of disjoint triangle paths on 5 vertices satisfying (a) and

(b') For all $i \in [m]$, we have $|B \cap B(i)| \geqslant \frac{p''c}{4}n = \alpha n$.

Let $B' = \bigcup_{i\in[m]} B \cap B(i)$ and $B^* \subseteq B'$ be maximal such that there is a triangle path $P_{B^*} \subseteq G'$ containing all $P_5 \in B^*$, being disjoint from $B' \setminus B^*$, having $\frac{\xi^*}{2}$-connectable end pairs $xy, x'y'$ and $|V(P_{B^*})| \leqslant (5 + L_2)|B^*|$.

We show that $B^* = B'$, otherwise take $y_1y_2y_3y_4y_5 \in B' \setminus B^*$, Lemma 2.1.1 gives us $\ell \leqslant L_2$ and $\frac{c_2}{2}n^\ell$ different $(x'y', y_1y_2)$-triangle paths in G' with ℓ inner vertices. We have that

$$|V(P_{B^*})| \leqslant (5 + L_2)|B'| \overset{(a)}{\leqslant} (5 + L_2)(3p''n) \leqslant \frac{c_2}{8}n.$$

Thus we have at least $\frac{c_2}{4}n^\ell$ different $(x'y', y_1y_2)$-triangle paths that are disjoint from P_{B^*} and each path in $B' \setminus (B^* \cup \{y_1y_2y_3y_4y_5\})$. Thus we may use such a $(x'y', y_1y_2)$-triangle path to get a (xy, y_4y_5)-triangle path $P_{B^*\cup\{y_1y_2y_3y_4y_5\}}$, such that

$$|V(P_{B^*\cup\{y_1y_2y_3y_4y_5\}})| \leqslant |V(P_{B^*})| + L_2 + 5 \leqslant (5 + L_2)|B^* \cup \{y_1y_2y_3y_4y_5\}|,$$

contradicting the maximality of B^*.

Set $P_I = P_{B'}$ and

$$|V(P_I)| \leqslant (5 + L_2)|B'| \overset{(a)}{\leqslant} (5 + L_2)(3p''n) \leqslant \min\Big\{\frac{\mu}{4}, \frac{\delta}{4}, \frac{\zeta}{4}, \frac{\mu\sqrt{\zeta}}{5}, \frac{\nu}{4}\Big\}n - 3.$$

Property (b') gives us that for each I_i, there are at least αn different $y_1y_2y_3y_4y_5 \subseteq P_I$ with $y_1, y_2, y_4, y_5 \in I_i$. $\square$

2.5 Covering

For step 2 in our application of the absorption method, we will use property iii. of good graphs and the Regularity Lemma. Our aim is to obtain a triangle factor in the reduced graph and from each (ε, d)-regular triangle ijk in this factor, extract a triangle path covering of the vertices in V_i, V_j, V_k.

In order to transform a fractional triangle factor in the reduced graph into an (integer) triangle factor, we need to refine the regular partition. We use the following lemma.

Lemma 2.5.1. *For every $\varepsilon > 0$, $\gamma_1, \gamma_2 \geqslant 0$, $\frac{\varepsilon}{56} \geqslant \gamma' > 0$, integers $t > 0$ and $s(i) \geqslant 0$ for $i \in [t]$, and weights $f_i(j) > 0$ for $j \in [s(i)]$ such that $\sum_{j\in[s(i)]} f_i(j) \leqslant 1$, the following is true for sufficiently large n.*

For any graph G on n vertices and partition $\mathcal{P} = \{V_0, V_1, \ldots, V_t\}$ of $V(G)$ with

$$(1-\gamma_1)\frac{n}{t} \leqslant |V_i| \leqslant (1+\gamma_2)\frac{n}{t}\,,$$

there exists a refinement $\mathcal{P}' = \{V_0\} \cup \{V_{i0}, V_{ij} \colon i \in [t], j \in [s(i)]\}$ with $\bigcup_{0\leqslant j\leqslant s(i)} V_{ij} = V_i$ such that,

i. for $i \in [t]$ and $j \in [s(i)]$,

$$(1-\gamma')(1-\gamma_1)\frac{n}{t} \cdot f_i(j) \leqslant |V_{ij}| \leqslant (1+\gamma')(1+\gamma_2)\frac{n}{t} \cdot f_i(j)\,,$$

ii. if $(V_i, V_{i'})$ is $(\varepsilon, d_{ii'})$-regular, then $(V_{ij}, V_{i'j'})$ is $(5\varepsilon^{1/4}, d_{ii'})$-regular for all $j \in [s(i)]$, $j' \in [s(i')]$.

iii. if we are given $c > 0$ and for each $v \in V(G)$, a set $U_v \subseteq V(G)$ such that $|U_v \cap V_i| \geqslant c|V_i|$, then $|U_v \cap V_{ij}| \geqslant \frac{1-\gamma'}{1+\gamma'}c|V_{ij}|$ for all $j \in [s(i)]$.

Moreover, if $\sum_{j\in[s(i)]} f_i(j) = 1$, then $V_{i0} = \varnothing$.

For the proof of Lemma 2.5.1, we use an equivalence between a pair being (ε, d)-regular and having small number of homomorphisms of C_4 in the bipartite graph induced by the pair. We use the following definition.

Definition 2.5.2. Let $G = (V, E)$ be a graph and X, $Y \subseteq V$ be non-empty and disjoint subsets of vertices. Given $\varepsilon > 0$ and $d \in [0, 1]$, the pair (X, Y) is (ε, d)-*minimal*, if

$$e(X, Y) \geqslant (d - \varepsilon)|X||Y| \quad \text{and} \quad \hom(C_4, X, Y) \leqslant (d^4 + \varepsilon)|X|^2|Y|^2 \,,$$

where $\hom(C_4, X, Y)$ denotes the number of partite graph homomorphisms from C_4 to $G[X, Y]$ for a fixed ordered bipartition of C_4.

We use the following equivalence.

Theorem 2.5.3. *For every graph $G = (V, E)$ and non-empty disjoint subsets X, $Y \subseteq V$, and $\varepsilon > 0$ and $d \in [0, 1]$ the following holds.*

i. If the pair (X, Y) is (ε, d)-regular, then it is $(4\varepsilon, d)$-minimal.

ii. If the pair (X, Y) is (ε, d)-minimal, then it is $(3\varepsilon^{1/4}, d)$-regular.

We now prove that we can refine the regular partition and keep the regularity in the subpairs.

Proof of Lemma 2.5.1. We are given $\varepsilon > 0$, $\gamma_1, \gamma_2 \geqslant 0$, $\frac{\varepsilon}{56} \geqslant \gamma' > 0$, integers $t > 0$ and $s(i) \geqslant 0$ for $i \in [t]$, and weights $f_i(j) > 0$ for $j \in [s(i)]$ such that $\sum_{j \in [s(i)]} f_i(j) \leqslant 1$. We are also given G on large enough n vertices and a partition $\mathcal{P}$.

Take a random refinement $\mathcal{P}'$ of $\mathcal{P}$, where each $v \in V_i$ is included in V_{ij} independently with probability $f_i(j)$. We shall derive from the sharp concentration of the binomial distribution, that a.a.s. $\mathcal{P}'$ satisfies properties i., ii. and iii. of the lemma.

Chernoff's inequality gives for fixed $i \in [t]$, $j \in [s(i)]$ that

$$\mathbb{P}\Big(\Big||V_{ij}| - |V_i| f_i(j)\Big| \geqslant \gamma' |V_i| f_i(j)\Big) \leqslant 2 \exp\left(-\frac{\gamma'^2 f_i(j)(1 - \gamma_1)n}{3t}\right).$$

Therefore by the union bound, a.a.s. for all $i \in [t]$ and $j \in [s(i)]$ ($s(i) \leqslant (1 + \gamma_2)\frac{n}{t}$), we have that

$$(1 - \gamma')\,|V_i| f_i(j) \leqslant |V_{ij}| \leqslant (1 + \gamma')\,|V_i| f_i(j). \tag{2.5.1}$$

For property ii. of the lemma, we appeal to Theorem 2.5.3. We have that $(V_i, V_{i'})$ is $(\varepsilon, d_{ii'})$-regular, then part i. of Theorem 2.5.3 gives that

$$\mathbb{E}\hom(C_4, V_{ij}, V_{i'j'}) = \hom(C_4, V_i, V_{i'}) f_i(j)^2 f_{i'}(j')^2 \leqslant (d_{ii'}^4 + 4\varepsilon)|V_i|^2|V_{i'}|^2 f_i(j)^2 f_{i'}(j')^2.$$

Next we apply the Azuma-Hoeffding inequality for establishing the concentration of $\hom(C_4, V_{ij}, V_{i'j'})$. Note that the inclusion or exclusion of a vertex in V_{ij} or $V_{i'j'}$ changes $\hom(C_4, V_{ij}, V_{i'j'})$ by at most $\max\{|V_i||V_{i'}|^2, |V_{i'}||V_i|^2\}$, thus we obtain

$$\begin{aligned}&\mathbb{P}\left(\hom(C_4, V_{ij}, V_{i'j'}) \geqslant (d_{ii'}^4 + 5\varepsilon)|V_i|^2|V_{i'}|^2 f_i(j)^2 f_{i'}(j')^2\right)\\ &\leqslant \exp\left(-\frac{\varepsilon^2|V_i|^4|V_{i'}|^4 f_i(j)^4 f_{i'}(j')^4}{4|V_i|^5|V_{i'}|^2}\right) \leqslant \exp\left(-\frac{\varepsilon^2 f_i(j)^4 f_{i'}(j')^4(1-\gamma_1)^2 n}{4(1+\gamma_2)t}\right).\end{aligned}$$

Therefore a.a.s. for all ε-regular $(V_i, V_{i'})$ and $j \in [s(i)]$, $j' \in [s(i')]$, we have that

$$\hom(C_4, V_{ij}, V_{i'j'}) \leqslant (d_{ii'}^4 + 5\varepsilon)|V_i|^2|V_{i'}|^2 f_i(j)^2 f_{i'}(j')^2. \tag{2.5.2}$$

Similarly, another application of the Azuma-Hoeffding inequality gives us that a.a.s. for all ε-regular $(V_i, V_{i'})$ and $j \in [s(i)]$, $j' \in [s(i')]$, we have that

$$e(V_{ij}, V_{i'j'}) \geqslant (d_{ii'} - 2\varepsilon)|V_i||V_{i'}| f_i(j) f_{i'}(j')\,. \tag{2.5.3}$$

For property iii. of the lemma, fix $v \in V(G)$, $i \in [t]$ and $j \in [s(i)]$. An application of the Chernoff's inequality implies

$$\mathbb{P}\left(|U_v \cap V_{ij}| \leqslant (1-\gamma')c|V_i| f_i(j)\right) \leqslant \exp\left(-\frac{\gamma'^2 c(1-\gamma_1) f_i(j) n}{2t}\right).$$

By the union bound, a.a.s. for all $v \in V(G)$, $i \in [t]$, $j \in [s(i)]$, we have that

$$|U_v \cap V_{ij}| \geqslant (1-\gamma')c|V_i| f_i(j)\,. \tag{2.5.4}$$

Consequently there exists a refinement $\mathcal{P}'$ satisfying (2.5.1), (2.5.2), (2.5.3) and (2.5.4). Thus for all ε-regular $(V_i, V_{i'})$ and $j \in [s(i)]$, $j' \in [s(i')]$,

$$\hom(C_4, V_{ij}, V_{i'j'}) \leqslant \frac{(d_{ii'}^4 + 5\varepsilon)}{(1-\gamma')^4}|V_{ij}|^2|V_{i'j'}|^2\,,$$

and

$$e(V_{ij}, V_{i'j'}) \geqslant \frac{(d_{ii'} - 2\varepsilon)}{(1+\gamma')^2} |V_{ij}||V_{i'j'}|\,.$$

Since $\gamma' \leqslant \varepsilon/56$, we have

$$\frac{d_{ii'} - 2\varepsilon}{(1+\gamma')^2} \geqslant \frac{d_{ii'} - 2\varepsilon}{1+3\gamma'} \geqslant d_{ii'} - 3\varepsilon \quad \text{and} \quad \frac{d_{ii'}^4 + 5\varepsilon}{(1-\gamma')^4} \leqslant \frac{d_{ii'}^4 + 5\varepsilon}{1-8\gamma'} \leqslant d_{ii'}^4 + 6\varepsilon\,.$$

Thus $(V_{ij}, V_{i'j'})$ is $(6\varepsilon, d_{ii'})$-minimal. Consequently part ii. of Theorem 2.5.3 gives that $(V_{ij}, V_{i'j'})$ is $(5\varepsilon^{1/4}, d_{ii'})$-regular. □

Next we prove the Covering Lemma.

Proof of Lemma 2.1.3. We are given $\xi, \alpha > 0$ and $c_\varrho \geqslant 1$. Take

$$0 < \eta \leqslant \frac{\alpha}{8}, \quad \varrho \leqslant \min\left\{\left(\frac{\xi}{c_\varrho}\right)^8, \left(\frac{\eta}{c_\varrho}\right)^8, \frac{1}{16}\right\} \quad \text{and}$$

$$0 < d_0 \leqslant \varrho, \quad \varepsilon = \min\left\{\left(\frac{d_0^3}{14}\right)^8, \left(\frac{\alpha}{6}\right)^{16}\right\}, \quad t_0 = \frac{1}{\varepsilon}\,.$$

Apply the Regularity Lemma with $\frac{\varepsilon}{2}, t_0$ and get n_0, T_0, take

$$c = \frac{45 T_0^9}{d_0^3 \varepsilon^3 \eta^3}\,.$$

Let $\frac{\eta}{2} \geqslant \nu > 0$ and G with $\mathcal{R}_G$ satisfying property iii. with ξ, ν, ϱ, η be given. Consider the subgraphs G_C and $\overline{G}_C$ of G on the same vertex set $V(G)$, where G_C contains the ξ-connectable edges and $E(\overline{G}_C) = E(G) \smallsetminus E(G_C)$. By the Regularity Lemma there is a partition $\mathcal{P} = \{V_0, \dots, V_t\}$ of $V(G)$ with $t_0 \leqslant t \leqslant T_0$ which is $\frac{\varepsilon}{2}$-regular for both G_C and $\overline{G}_C$. Thus $\mathcal{P}$ is a ε-regular partition for G.

We remove a set F of edges in $E(G)$, namely those which are incident to V_0, inside partition classes, between pairs which are not ε-regular, pairs with density less than d_0 in G, and for pairs with density less than d_0 in G_C, delete the $\frac{\xi}{2}$-connectable edges. We have,

$$|F| \leqslant \varepsilon n^2 + t\frac{n^2}{2t^2} + \varepsilon t^2 \frac{n^2}{t^2} + d_0 \frac{n^2}{2} \leqslant \frac{5\varepsilon + d_0}{2} n^2 \leqslant \varrho n^2.$$

Take $\varnothing = A \subseteq V(G)$ and consider X_F, $G_{A,F}$ as in property iii. of good graphs and get a ξ-connectable fractional triangle factor $f_{G_{A,F}}$ of weight

$$W(f_{G_{A,F}}) \geqslant \frac{n}{3} - \eta n\,.$$

Take the cleaned graph $G' = G(V(G), E(G) \smallsetminus F)$, since $G_{A,F} \subseteq G'$ we may view $f_{G_{A,F}}$ also as a fractional triangle factor for G'.

Consider the reduced graph $\mathfrak{R} = R(\mathcal{P}, d_0, \varepsilon)$. Since $\varepsilon < \frac{d_0^3}{3}$, it follows from the triangle counting lemma that $T_{G'[V_i,V_j,V_k]} \neq \varnothing$ if and only if $ijk \in T_{\mathfrak{R}}$. Define a fractional triangle factor $f_{\mathfrak{R}}$ of $\mathfrak{R}$ in the following way. For distinct $i, j, k \in [t]$, let $W(ijk)$ be the total weight of the triangles in $G'[V_i, V_j, V_k]$ under $f_{G_{A,F}}$. Set

$$f_{\mathfrak{R}}(ijk) = W(ijk)\frac{t}{n}\,.$$

We check that for every $i_0 \in [t]$, we have that $\sum_{i_0jk\in T_{\mathfrak{R}}} f_{\mathfrak{R}}(i_0jk) \leqslant 1$. Since the sum of weights of $f_{G_{A,F}}$ in a vertex is at most 1 and $|V_{i_0}| \leqslant \frac{n}{t}$,

$$\sum_{i_0jk\in T_{\mathfrak{R}}} f_{\mathfrak{R}}(i_0jk) = \sum_{i_0jk\in T_{\mathfrak{R}}} W(i_0jk)\frac{t}{n} = \frac{t}{n} \cdot \sum_{v\in V_{i_0}} \sum_{vxy\in T_{G'}} f_{G_{A,F}}(vxy) \leqslant \frac{t}{n} \cdot \frac{n}{t} = 1.$$

Moreover,

$$W(f_{\mathfrak{R}}) = W(f_{G_{A,F}})\frac{t}{n} \geqslant \Big(\frac{1}{3} - \eta\Big)t\,.$$

Consider a refined partition $\mathcal{P}'$ in the following way. Let $T_{\eta,\mathfrak{R}}$ be all triangles $T \in T_{\mathfrak{R}}$ with $f_{\mathfrak{R}}(T) \geqslant \frac{\eta}{t^2}$ and for $i \in [t]$, let $T_{\eta,\mathfrak{R}}(i) = \{T_1^i, \ldots, T_{s(i)}^i\}$ be all triangles in $T_{\eta,\mathfrak{R}}$ containing i. Apply Lemma 2.5.1 with $\varepsilon, \gamma_1 = \gamma_2 = 0, \gamma' = \frac{\varepsilon}{56}, t, s(i)$ and for $j \in [s(i)]$, the weight $f_i(j) = f_{\mathfrak{R}}(T_j^i)$ (here we set $n' = |V(G) \smallsetminus V_0|$). We get $\mathcal{P}^* = \{V_0\} \cup \{V_{i0}, V_{iT_j^i} \colon i \in [t], j \in [s(i)]\}$ such that

i. $(1-\gamma')\,\frac{n'}{t} f_{\mathfrak{R}}(T_j^i) \leqslant |V_{iT_j^i}| \leqslant (1+\gamma')\,\frac{n'}{t} f_{\mathfrak{R}}(T_j^i)$;

ii. If $(V_i, V_{i'})$ is (ε, d)-regular, then for any $j \in [s(i)], j' \in [s(i')]$, the pair $(V_{iT_j^i}, V_{i'T_{j'}^{i'}})$ is $(5\varepsilon^{1/4}, d)$-regular.

For the number of vertices in $\bigcup_{i\in[t]} V_{i0}$, we observe that

$$
\begin{aligned}
\sum_{i\in[t]} |V_{i0}| &\leqslant \sum_{i\in[t]} \Big(\frac{n'}{t} - \sum_{T\in T_{\eta,\mathfrak{R}}(i)} (1-\gamma') f_{\mathfrak{R}}(T)\frac{n'}{t}\Big) \\
&\leqslant \frac{n'}{t}\Big(t - \sum_{i\in[t]}\sum_{T\in T_{\eta,\mathfrak{R}}(i)} f_{\mathfrak{R}}(T) + \gamma' \sum_{i\in[t]}\sum_{T\in T_{\eta,\mathfrak{R}}(i)} f_{\mathfrak{R}}(T)\Big) \\
&\leqslant \frac{n'}{t}\Big(t - 3\sum_{T\in T_{\mathfrak{R}}} f_{\mathfrak{R}}(T) + 3\sum_{T\in T_{\mathfrak{R}}\setminus T_{\eta,\mathfrak{R}}} f_{\mathfrak{R}}(T) + 3\gamma' \sum_{T\in T_{\mathfrak{R}}} f_{\mathfrak{R}}(T)\Big) \\
&\leqslant \frac{n'}{t}\Big(t - (1-3\eta)t + t^3\frac{\eta}{t^2} + \gamma' t\Big) = (4\eta+\gamma')n'\,.
\end{aligned}
$$

Take $V_0' = V_0 \cup \bigcup_{i\in[t]} V_{i0}$ and set $\mathcal{P}' = \{V_0'\} \cup \{V_{iT_j^i} \colon i\in[t], j\in[s(i)]\}$.

If $ijk = T \in T_{\eta,\mathfrak{R}}$, then there is $u_iu_ju_k \in T_{G'[V_i,V_j,V_k]}$ with $f_{G_{A,F}}(u_iu_ju_k) > 0$. Since $f_{G_{A,F}}$ is a ξ-connectable fractional triangle factor in G, we have that one of its edges is connectable, say $u_iu_j \in E(G_C)$.

We have $G_{A,F} \subseteq G'$ and G' only contains ξ-connectable edges between dense pairs in G_C. Thus (V_{iT}, V_{jT}) is $(5\varepsilon^{1/4}, d)$-regular in G_C for some $d \geqslant d_0$. The number of triangles in $T_{G'[V_{iT},V_{jT},V_{kT}]}$ with a ξ-connectable edge between V_{iT}, V_{jT} is at least

$$
(d_0^3 - 15\sqrt[4]{\varepsilon})|V_{iT}||V_{jT}||V_{kT}| \geqslant (d_0^3 - 15\sqrt[4]{\varepsilon})(1-\gamma')^3(1-\varepsilon)^3\Big(\frac{\eta}{T_0^2}\Big)^3\Big(\frac{n}{t}\Big)^3 \geqslant c_0\Big(\frac{n}{t}\Big)^3\,.
$$

Consider the 3-uniform partite hypergraph $G_1^{(3)}$ with $V(G_1^{(3)}) = V_{iT}\dot\cup V_{jT}\dot\cup V_{kT}$ and $v_iv_jv_k \in E(G_1^{(3)})$ iff $v_iv_jv_k \in T_{G'[V_{iT},V_{jT},V_{kT}]}$ with $v_iv_j \in E(G_C)$. The number of hyperedges containing a pair with codegree at most $\frac{c_0n}{9t}$ is at most

$$
\frac{c_0 n}{9t}(|V_{iT}||V_{jT}| + |V_{iT}||V_{kT}| + |V_{jT}||V_{kT}|) \leqslant \frac{c_0}{3}(1+\gamma')^2\Big(\frac{n}{t}\Big)^3 \leqslant \frac{c_0}{2}\Big(\frac{n}{t}\Big)^3\,.
$$

This assures a tight path $P_1^{(3)} \subseteq G_1^{(3)}$ with $|V(P_1^{(3)})| = \frac{c_0n}{9t} + 2$. We may always start $P_1^{(3)}$ with a ξ-connectable pair and the last pair in $P_1^{(3)} \subseteq G_1^{(3)}$ can be made ξ-connectable by possibly removing the last vertex. Thus we have a tight path $P_1^{(3)} \subseteq G_1^{(3)}$ with $|V(P_1^{(3)})| \geqslant \frac{c_0n}{9t} + 1$ and ξ-connectable ends.

For $s \geqslant 1$, consider

$$
G_{s+1}^{(3)} = G_s^{(3)}[V(G_s^{(3)}) \setminus V(P_s^{(3)})] \quad \text{and} \quad V(G_{s+1}^{(3)}) = V_{iT}^{(s+1)}\dot\cup V_{jT}^{(s+1)}\dot\cup V_{kT}^{(s+1)}\,.
$$

While $|V_{iT}^{(s+1)}| \geqslant \sqrt[4]{5\varepsilon^{1/4}}|V_{iT}|$ and similarly for $|V_{jT}^{(s+1)}|$ and $|V_{kT}^{(s+1)}|$, we have that each pair among $V_{iT}^{(s+1)}, V_{jT}^{(s+1)}, V_{kT}^{(s+1)}$ is $(\sqrt{5\varepsilon^{1/4}}, d)$-regular for some $d \geqslant d_0$. Thus the number of ξ-connectable triangles is at least

$$\begin{aligned}
&(d_0^3 - 3\sqrt{5\varepsilon^{1/4}})|V_{iT}^{(s)}||V_{jT}^{(s)}||V_{kT}^{(s)}| \\
&\qquad\geqslant (d_0^3 - 3\sqrt{5\varepsilon^{1/4}})(\sqrt[4]{5\varepsilon^{1/4}})^3(1-\gamma')^3(1-\varepsilon)^3\left(\frac{\eta}{T_0^2}\right)^3\left(\frac{n}{t}\right)^3 \\
&\qquad\geqslant \frac{d_0^3}{2}\varepsilon^3\frac{1}{2}\left(\frac{\eta}{T_0^2}\right)^3\left(\frac{n}{t}\right)^3 = c_1\left(\frac{n}{t}\right)^3.
\end{aligned}$$

Repeating the previous procedure we get a tight path $P_{s+1}^{(3)} \subseteq G_{s+1}^{(3)}$ with $|V(P_{s+1}^{(3)})| \geqslant \frac{c_1 n}{9t} + 1$ and ξ-connectable ends. Observe that each $P_{s+1}^{(3)}$ is a triangle path in $G[V_{iT}\dot\cup V_{jT}\dot\cup V_{kT}]$.

The number of disjoint triangle paths we obtain in this way is at most

$$|V(G_1^{(3)})|\frac{9t}{c_1 n} \leqslant 3(1+\gamma')\frac{n}{t}\frac{9t}{c_1 n} \leqslant \frac{27(1+\gamma')}{c_1}.$$

They cover all but at most $\sqrt[4]{5\varepsilon^{1/4}}|V_{iT}|$ vertices of V_{iT} and similarly for V_{jT}, V_{kT}.

Repeat the procedure for each triangle in $T_{\eta,\mathfrak{R}}$. We arrive at a collection $\mathcal{P}$ of disjoint triangle paths with ξ-connectable ends such that

$$|\mathcal{P}| \leqslant \frac{T_0^3}{3} \cdot \frac{27(1+\gamma')}{c_1} \leqslant c.$$

The triangle paths in $\mathcal{P}$ cover all but at most

$$\begin{aligned}
\sqrt[4]{5\varepsilon^{1/4}} \sum_{ijk=T\in T_{\eta,\mathfrak{R}}} (|V_{iT}| + |V_{jT}| + |V_{kT}|) + |V_0'| &\leqslant (\sqrt[4]{5\varepsilon^{1/4}} + 4\eta + 2\varepsilon)n \\
&\leqslant (3\sqrt[16]{\varepsilon} + 4\eta)n \leqslant \alpha n
\end{aligned}$$

vertices of G. □

2.6 Embedding spanning graphs of small bandwidth

In order to show that the good property implies the existence of spanning 3-chromatic subgraphs with bounded degree and sublinear bandwidth, we follow the approach in [20]; that is, we show that the reduced graph contains the 2nd power of a Hamiltonian cycle and use it for applying the blow-up lemma.

Theorem 2.6.1 (Blow-up Lemma (Theorem 1.4 in [18]))**.** *For all Δ, Δ_R, κ and $d > 0$ there exist $\varepsilon, \alpha > 0$ such that for every t there is n_0 such that the following holds.*

For every $n_1, \ldots, n_t$ with $n_0 \leqslant n = \sum n_i$ and $n_i \leqslant \kappa n_j$ for all $i, j \in [t]$, assume that we are given graphs R, R^* with $V(R) = [t], \Delta(R) < \Delta_R$ and $R^* \subseteq R$, and graphs G, H on $V(G) = V_1 \dot\cup \ldots \dot\cup V_t$ and $V(H) = W_1 \dot\cup \ldots \dot\cup W_t$ with

(G1) $|V_i| = n_i$ *for every* $i \in [t]$*,*

(G2) $(V_i)_{i \in [t]}$ *is* (ε, d)*-regular on* R*, and*

(G3) $(V_i)_{i \in [t]}$ *is* (ε, d)*-super-regular on* R^**.*

Further let $\Delta(H) \leqslant \Delta$, and let there be a function $f \colon V(H) \to [t]$ with $f^{-1}(i) = W_i$ and a set $X \subseteq V(H)$ with

(H1) $|X \cap W_i| \leqslant \alpha n_i$*,*

(H2) $|W_i| \leqslant n_i$ *for every* $i \in [t]$*,*

(H3) for every edge $\{u, v\} \in E(H)$ *we have* $\{f(u), f(v)\} \in E(R)$*,*

(H4) for every edge $\{u, v\} \in E(H) \setminus E(H[X])$ *we have* $\{f(u), f(v)\} \in E(R^*)$*.*

Then $H \subseteq G$.

When preparing G for the blow-up lemma, after showing that the reduced graph contains the 2nd power of a Hamiltonian cycle, we want to determine the graphs R and R^*. In our case, R will be a *ladder*.

A graph L on $3t$ vertices $V(L) = \bigcup_{i \in [t]} \{a_i, b_i, c_i\}$ is a *ladder* when $a_i b_i c_i$ are triangles and between $a_i b_i c_i$ and $a_{i+1} b_{i+1} c_{i+1}$ for $i \in [t]$, with $a_{t+1} b_{t+1} c_{t+1} = a_1 b_1 c_1$, we have all edges, except $a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}$.

Theorem 2.6.2 (Lemma for G). *For every $\mu, \delta, \zeta, \xi > 0$, there exist $\eta, \varrho, \nu > 0$, $d_0 > 0$ such that for $\varepsilon > 0$, there is $\varepsilon \geqslant \varepsilon_1 > 0$, t_1 and n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good and $\mathfrak{A}_G$ is complete, there is a partition

$$V(G) = \bigcup_{i \in [t_1]} (V_{a_i} \dot\cup V_{b_i} \dot\cup V_{c_i}),$$

and a ladder L with $V(L) = \bigcup_{i \in [t_1]} \{a_i, b_i, c_i\}$, such that

(g1) for any $i \in [t_1]$, we have $0 \leqslant ||V_{a_i}| - |V_{b_i}||, ||V_{a_i}| - |V_{c_i}||, ||V_{c_i}| - |V_{b_i}|| \leqslant 1$,

(g2) $(1 - \sqrt{d_0})\frac{n}{3t_1} \leqslant |V_{a_i}|, |V_{b_i}|, |V_{c_i}| \leqslant (1 + \sqrt{d_0})\frac{n}{3t_1}$,

(g3) G is (ε_1, d_0)-regular on $L^+ = (V(L), E(L) \cup \{a_1a_2, b_2b_3, c_3c_4\})$,

(g4) G is $(\varepsilon_1, d_0, \frac{d_0}{8})$-super-regular on L.

For preparing H, we want to determine a homomorphism that maps $V(H)$ into vertices of the ladder. While defining the homomorphism, we follow the bandwidth order of H and we watch out that the sizes of images and partition classes of $V(G)$ almost match, that is possible when the bandwidth is small enough. We get the following lemma.

Lemma 2.6.3 (Lemma for H). *For every $\alpha, \gamma' > 0$, integers t and $n_{a_i}, n_{b_i}, n_{c_i}$ ($i \in [t]$) such that $\sum_{i \in [t]}(n_{a_i} + n_{b_i} + n_{c_i}) = n$, $\frac{n}{2t} \leqslant n_{a_i} + n_{b_i} + n_{c_i} \leqslant \frac{n}{t}$ and*

$$|n_{a_i} - n_{b_i}|, |n_{a_i} - n_{c_i}|, |n_{b_i} - n_{c_i}| \leqslant 1,$$

there exists $\beta > 0$ such that the following holds.

For H on n vertices with $\chi(H) \leqslant 3$ and $bw(H) \leqslant \beta n$ and for L a ladder with $V(L) = \bigcup_{i \in [t]} \{a_i, b_i, c_i\}$, we have a homomorphism $f\colon V(H) \to V(L)$ and a set of special vertices $X \subseteq V(H)$ such that

(h1) for every $i \in [t]$, $||f^{-1}(a_i)| - n_{a_i}|, ||f^{-1}(b_i)| - n_{b_i}|, ||f^{-1}(c_i)| - n_{c_i}| \leqslant 3\gamma' \frac{n}{3t}$,

(h2) for every $i \in [t]$, $|X \cap f^{-1}(a_i)|, |X \cap f^{-1}(b_i)|, |X \cap f^{-1}(c_i)| \leqslant \frac{\alpha n}{3t}$,

(h3) for every $uv \in E(H[V(H) \setminus X])$, we have $f(u)f(v) \in E(a_ib_ic_i)$ for some $i \in [t]$.

The last step in the preparation for the blow-up lemma is to adjust the sizes of the partition classes of G according to the images of the homomorphism from H into the ladder.

Lemma 2.6.4. *Let G on n vertices with $V(G) = \bigcup_{i\in[t]}(V_{a_i}\dot\cup V_{b_i}\dot\cup V_{c_i})$ and a ladder L satisfying properties (g2), (g3), (g4) of Lemma 2.6.2 with $\varepsilon, d_0 > 0$ be given. There exists $\gamma^* > 0$ such that for any integers $m_{a_i}, m_{b_i}, m_{c_i}$ $(i \in [t])$ with $\sum_{i=1}^{t}(m_{a_i} + m_{b_i} + m_{c_i}) = n$ and*

$$||V_{a_i}| - m_{a_i}|, ||V_{b_i}| - m_{b_i}|, ||V_{c_i}| - m_{c_i}| \leqslant \gamma^* \frac{n}{3t},$$

there is $V(G) = \bigcup_{i\in[t]}(V'_{a_i}\dot\cup V'_{b_i}\dot\cup V'_{c_i})$ satisfying that G is $(2\varepsilon, d_0)$-regular on L^+, that G is $(2\varepsilon, d_0, \frac{d_0}{34})$-super-regular on the triangles $a_ib_ic_i$ of the ladder, and

$$|V'_{a_i}| = m_{a_i}, |V'_{b_i}| = m_{b_i}, |V'_{c_i}| = m_{c_i}\,.$$

These lemmas will be proven in the following sections. Now we show how they can be combined to prove our desired bandwidth result.

Theorem 2.6.5. *For every $\mu, \delta, \zeta, \xi > 0$ and positive integer Δ, there exist $\beta, \eta, \varrho, \nu > 0$ and n_0 such that the following holds.*

If G on $n \geqslant n_0$ vertices is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good and $\mathfrak{A}_G$ is complete, if H on n vertices is such that $\chi(H) \leqslant 3$, $bw(H) \leqslant \beta n$ and $\Delta(H) \leqslant \Delta$, then H is contained in G.

Proof. Let $\mu, \delta, \zeta, \xi > 0$ and Δ be given. Apply Lemma 2.6.2 and get $\eta_0, \varrho, \nu > 0$ $d_0 > 0$. Set $\Delta_R = 6, \kappa = 3$ and apply Lemma 2.6.1 to get $\varepsilon(d_0), \alpha > 0$. For $\varepsilon(d_0)$, Lemma 2.6.2 gives us $\varepsilon(d_0) \geqslant \varepsilon_1 > 0$, t, n_0. For t, Lemma 2.6.1 gives is n'_0.

We are given G on $n \geqslant \max\{n_0, n'_0\}$ vertices such that G is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good and $\mathfrak{A}_G$ is complete. According to Lemma 2.6.2, we get a partition $V(G) = \bigcup_{i\in[t]}\{V_{a_i}, V_{b_i}, V_{c_i}\}$ and ladder L. Apply Lemma 2.6.4 for G and L and get $\gamma^* > 0$. Apply Lemma 2.6.3 with

$$\gamma' = \frac{\gamma^*}{3}, \quad \gamma, \quad \alpha' = \alpha(1 - \sqrt{d_0} - 3\gamma'), \quad t, \quad \text{and} \quad n_{a_i} = |V_{a_i}|, n_{b_i} = |V_{b_i}|, n_{c_i} = |V_{c_i}|\,,$$

to get $\beta > 0$.

Let H on n vertices such that $\chi(H) \leqslant 3$, $bw(H) \leqslant \beta n$ and $\Delta(H) \leqslant \Delta$.

Lemma 2.6.3 gives us a homomorphism $f\colon V(H) \to V(L)$ and $X \subseteq V(H)$. Set

$$|f^{-1}(a_i)| = m_{a_i}, |f^{-1}(b_i)| = m_{b_i}, |f^{-1}(c_i)| = m_{c_i}\,.$$

Property (*h1*) of the homomorphism gives us that $||V_{a_i}| - m_{a_i}| \leqslant \gamma^* \frac{n}{3t}$, similarly to b_i and c_i. Lemma 2.6.4, gives us $V(G) = \bigcup_{i\in[t]}(V'_{a_i} \dot\cup V'_{b_i} \dot\cup V'_{c_i})$.

We note that $V(G) = \bigcup_{i\in[t]}(V'_{a_i} \dot\cup V'_{b_i} \dot\cup V'_{c_i})$ with $R = L$ and $R^* = \dot\bigcup_{i\in[t]} L[\{a_i, b_i, c_i\}]$ and $V(H) = \bigcup_{i\in[t]}(f^{-1}(a_i) \dot\cup f^{-1}(b_i) \dot\cup f^{-1}(c_i))$ satisfy the conditions of Lemma 2.6.1. Indeed, properties (*g3*) and (*g4*) give us (*G2*) and (*G3*). Since f is a homomorphism, we get (*H3*) and property (*h3*) gives (*H4*).

For the conditions concerning the sizes of the partition classes, note that $|V'_{a_i}| = |f^{-1}(a_i)|$ (satisfying (*H2*)). To check property (*H1*), note that (*h1*) $||f^{-1}(a_i)| - n_{a_i}|| \leqslant 3\gamma' \frac{n}{3t}$ and (*g2*) $n_{a_i} \geqslant (1 - \sqrt{d_0})\frac{n}{3t}$ give us $|f^{-1}(a_i)| \geqslant (1 - \sqrt{d_0} - 3\gamma')\frac{n}{3t}$ and (*h2*) gives us (*H1*). Moreover, assuming $\sqrt{d_0} + 3\gamma' \leqslant \frac{1}{2}$,

$$\frac{|f^{-1}(a_i)|}{|f^{-1}(b_i)|} \leqslant \frac{1 + \sqrt{d_0} + 3\gamma'}{1 - \sqrt{d_0} - 3\gamma'} \leqslant 3 = \kappa\,.$$

Therefore $H \subseteq G$. □

2.6.1 Lemma for G

When preparing G for the blow-up lemma, we take a regular partition of G such that the reduced graph inherits the properties of good graphs from G and then we find the 2nd power of a Hamiltonian cycle in the reduced graph. In order to get a ladder, given the 2nd power of a Hamiltonian cycle, we exchange each vertex for 3 independent vertices. We make every edge of the ladder super-regular, by moving vertices to the exceptional class V_0 of our regular partition. Then we need to redistribute the exceptional class V_0 using the special paths P_I in the 2nd power of the Hamiltonian cycle, thus we use the following sets.

Given $c > 0$, a graph G with $\mathcal{R}_G$ and a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_t\}$ of $V(G)$, for each $v \in V(G)$ we define

$$I_c^{\mathcal{P},G}(v) = \{i \in [t] \colon |U_v \cap V_i| \geqslant c|V_i|\}\,.$$

These are the classes that significantly intersect the robust neighbourhood of v.

When applying the Theorem 1.1.5 for the reduced graph, we will give the sets $I_c^{\mathcal{P},G}$ as inputs, but the number of such sets must be at most $2^{\kappa t}$, where t is the number of vertices in the reduced graph. For this technical reason, we first exchange the vertices in the reduced graph for independent sets of size $\frac{1}{\kappa}$. We define the *complete s-blow-up* of a graph G to be the graph G^s with $V(G^s) = \dot{\bigcup}_{v\in V(G)} V_v$, where $V_v = \{v(1), \dots, v(s)\}$ is independent and for each $uv \in E(G)$, we have $u(i)v(j) \in E(G^s)$ for all $i, j \in [s]$. We need to show that the complete s-blow-up of a good graph is also good.

Lemma 2.6.6. *Given $\mu, \delta, \zeta, \varrho, \xi, \eta, \nu > 0$, such that $\xi \geqslant \max\{4\sqrt{\varrho}, 4\nu^2\}$ and $\eta \geqslant \max\{6\nu, 6\sqrt{\varrho}\}$ and given an integer $s \geqslant 2$, if G with $\mathcal{R}_G$ is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good, then the complete s-blow-up G^s with*

$$\mathcal{R}_{G^s} = \Big\{U_{v(i)} = \bigcup_{u\in U_v} V_u \colon v(i) \in V(G^s)\Big\}$$

is $(\frac{\mu}{2}, \delta, \zeta, \frac{\varrho^4}{4}, \xi, 2\eta, \nu^2)$-good.

Proof. Let $\mu, \delta, \zeta, \varrho, \xi, \eta, \nu > 0$, an integer $s \geqslant 2$, a good graph G with $\mathcal{R}_G$ and $|V(G)| = n$ be given. Consider G^s and $\mathcal{R}_{G^s}$.

First check that $G^s[U_{v(i_0)}]$ is $\frac{\mu}{2}$-inseparable and contains at least $\delta(ns)^3$ connectable triangles. Take $X \subseteq U_{v(i_0)}$ with $|X| \leqslant \frac{|U_{v(i_0)}|}{2}$. Say $U_v = \{w_1, \dots, w_k\}$ and choose $V' = \{w_1(i_1), \dots, w_k(i_k)\}$ for some $i_1, \dots, i_k \in [s]$. We get that $G^s[V']$ is a copy of $G[U_v]$. Observe that for every $i, i' \in [|U_v|]$ and $j, j' \in [s]$ such that $w_i(j)w_{i'}(j') \in E_{G^s[U_{v(i_0)}]}(X, U_{v(i_0)} \smallsetminus X)$ we have that $w_i(j)w_{i'}(j') \in E_{G^s[V']}(X \cap V', V' \smallsetminus X)$ for $s^{|U_v|-2}$ different choices of V', thus

$$e_{G^s[U_{v(i_0)}]}(X, U_{v(i_0)} \smallsetminus X) = \frac{1}{s^{|U_v|-2}} \sum_{V'} e_{G^s[V']}(X \cap V', V' \smallsetminus X)\,.$$

Moreover,

$$\sum_{V'} |X \cap V'||V' \smallsetminus X| = s^{|U_v|-2}(|X||U_{v(i_0)} \smallsetminus X| - \sum_{w_i \in U_v} |X \cap V_{w_i}||V_{w_i} \smallsetminus X|)\,.$$

Since $|V_{w_i}| = s$, $|U_v| \geqslant 4$, $|U_{v(i_0)}| = s|U_v|$ and $|X| \leqslant \frac{|U_{v(i_0)}|}{2}$, we have that

$$\sum_{w_i \in U_v} |X \cap V_{w_i}||V_{w_i} \setminus X| \leqslant \sum_{w_i \in U_v} |X \cap V_{w_i}|s \leqslant s|X| \leqslant s|X|\frac{|U_v|}{4} = \frac{|X||U_{v(i_0)}|}{4}$$
$$\leqslant \frac{|X||U_{v(i_0)}|}{2} - \frac{|X|^2}{2} = \frac{|X||U_{v(i_0)} \setminus X|}{2}\,.$$

Since $e_{G^s[V']}(X \cap V', V' \setminus X) \geqslant \mu|X \cap V'||V' \setminus X|$, we conclude that

$$e_{G^s[U_{v(i_0)}]}(X, U_{v(i_0)} \setminus X) \geqslant \frac{\mu}{2}|X||U_{v(i_0)} \setminus X|\,.$$

We have that $G[U_v]$ contains δn^3 triangles $w_1 w_2 w_3$ and $G^s[V_{w_1} \cup V_{w_2} \cup V_{w_3}]$ contains s^3 triangles $w_1(i)w_2(i')w_3(i'')$ with $i, i', i'' \in [s]$. We have that $G^s[U_{v(i_0)}] = G^s[\bigcup_{w_i \in U_v} V_{w_i}]$, thus there are at least $\delta(ns)^3$ triangles in $G^s[U_{v(i_0)}]$. If w_1, w_2 is in U_v for ξn different $v \in V(G)$, then $w_1(i), w_2(i') \in U_{v(j)}$ for all $j \in s$ and $w_1(i)w_2(i')$ is ξ-connectable in G^s.

We check that the auxiliary graph $\mathfrak{A}_{G^s} = A(\mathcal{R}_{G^s}, \zeta)$ is $\frac{\mu}{2}$-inseparable. Let $\mathfrak{A}_G = A(\mathcal{R}_G, \zeta)$ and take $uv \in E(\mathfrak{A}_G)$. We have $|E(G[U_u]) \cap E(G[U_v])| \geqslant \zeta n^2$, for each $xy \in E(G[U_u]) \cap E(G[U_v])$, we have s^2 edges $x(j)y(j') \in E(G^s[U_{u(i)}]) \cap E(G^s[U_{v(i')}])$ for all $i, i' \in [s]$, therefore $u(i)v(i') \in E(\mathfrak{A}_{G^s})$. The graph $\mathfrak{A}_{G^s}$ is the complete s-blow-up of $\mathfrak{A}_G$ and as shown before, $\mathfrak{A}_{G^s}$ is $\frac{\mu}{2}$-inseparable.

Let $A' \subseteq V(G^s)$ with $|A'| \leqslant \nu^2 ns$ and $F' \subseteq E(G^s)$ such that $|F'| \leqslant \frac{\varrho^4}{4}(ns)^2$ and consider $X_{F'} \subseteq V(G^s)$ and $G^s_{A',F'}$ as in the definition of good graphs, we have that $|X_{F'}| \leqslant \frac{\varrho^2}{2} ns$.

Take $A \subseteq V(G)$ to be all vertices x such that $|V_x \cap A'| \geqslant \nu s$ and take $F \subseteq E(G)$ such that $xy \in F$ if and only if, $|X_{F'} \cap V_x| \geqslant \varrho s$ or $|E(V_x, V_y) \cap F'| \geqslant \varrho^2 s^2$. We have that $|A| \leqslant \frac{|A'|}{\nu s} \leqslant \nu n$ and

$$|F| \leqslant \frac{|X_{F'}|}{\varrho s} n + \frac{|F'|}{\varrho^2 s^2} \leqslant \frac{\varrho}{2} n^2 + \frac{\varrho^2}{4} n^2 \leqslant \varrho n^2\,.$$

Since G is good, we get a connectable fractional triangle factor $f_{G_{A,F}}$.

For each triangle $xyz \in T_{G_{A,F}}$, we get s^3 triangles in $G^s[V_x \cup V_y \cup V_z]$. Note that $x \notin A$ implies $|V_x \cap A'| < \nu s$ and x not being isolated in $G_{A,F}$ implies $|X_{F'} \cap V_x| < \varrho s$, thus $|V_x \cap V(G^s_{A',F'})| \geqslant (1 - \nu - \varrho)s$, similarly for V_y and V_z. Moreover, $xy \notin F$ gives us $|E(V_x, V_y) \cap F'| \leqslant \varrho^2 s^2$, similarly for xz, yz. Therefore,

$$|T_{G^s_{A',F'}[V_x,V_y,V_z]}| \geqslant s^3 - 3(\nu + \varrho + \varrho^2)s^3\,. \tag{2.6.1}$$

For $x(i)y(i')z(i'') \in T_{G^s_{A',F'}[V_x,V_y,V_z]}$, set

$$f_{G^s_{A',F'}}(x(i)y(i')z(i'')) = \frac{f_{G_{A,F}}(xyz)}{s^2},\ \text{if } xyz \in T_{G_{A,F}}$$
$$f_{G^s_{A',F'}}(x(i)y(i')z(i'')) = 0,\ \text{if } xyz \notin T_{G_{A,F}}\,.$$

We check the total weight in a vertex $x_0(i) \in V_{x_0}$.

$$\sum_{x_0(i)y(i')z(i'')\in T_{G^s_{A',F'}}} f_{G^s_{A',F'}}(x_0(i)y(i')z(i'')) \leqslant s^2 \sum_{x_0yz\in T_{G_{A,F}}} \frac{f_{G_{A,F}}(x_0yz)}{s^2} \leqslant 1\,.$$

For the total weight, consider $\nu \leqslant \frac{1}{6}$ and observe that

$$|A'| \leqslant |A|s + (n - |A|)\nu s \leqslant (|A| + \nu n)s \quad \text{and} \quad |A| \leqslant \nu n \leqslant \Big(\frac{1}{2} - 2\nu\Big)n\,.$$

Then, using (2.6.1) and considering $\nu \leqslant \frac{1}{6}$ and $\nu + \varrho + \varrho^2 \leqslant \frac{\eta}{2}$,

$$\begin{aligned}
W(f_{G^s_{A',F'}}) &\geqslant \Big(\frac{n}{3} - \eta(n - |A|)\Big)(1 - 3(\nu + \varrho + \varrho^2))s \\
&\geqslant \frac{ns}{3} - \eta ns + \eta|A|s - (\nu + \varrho + \varrho^2)ns \\
&\geqslant \frac{ns}{3} - 2\eta ns + 2\eta|A|s + \frac{\eta}{2}ns - \eta|A|s \\
&\geqslant \frac{ns}{3} - 2\eta ns + 2\eta|A|s + \frac{\eta}{2}ns - \Big(\frac{1}{2} - 2\nu\Big)\eta ns \\
&\geqslant \frac{ns}{3} - 2\eta ns + 2\eta|A|s + 2\eta\nu ns \\
&\geqslant \frac{ns}{3} - 2\eta(ns - (|A| + \nu n)s) \geqslant \frac{ns}{3} - 2\eta(ns - |A'|)\,.
\end{aligned}$$

We check that $f_{G^s_{A',F'}}$ is a ξ-connectable fractional triangle factor. If a triangle $x(i)y(i')z(i'')$ in $G^s_{A',F'}$ is such that $f_{G^s_{A',F'}}(x(i)y(i')z(i'')) > 0$, then $f_{G_{A,F}}(xyz) > 0$ and we may assume that xy is ξ-connectable in G. Thus $x, y \in U_v$ for ξn different $v \in V(G)$ and, for every $j \in [s]$, we have $V_x, V_y \subseteq U_{v(j)}$. Thus $x(i), y(i') \in U_{v(j)}$ for ξns different $v(j)$. □

We are now able to prepare G for our application of the blow-up lemma.

Proof of Lemma 2.6.2. Let $\mu, \delta, \zeta, \xi > 0$ be given. Take

$$\mu' = \frac{\mu^3(\sqrt{\zeta})^4}{4}, \quad \delta' = \frac{\delta}{2}, \quad \xi' = \frac{\xi}{8}, \quad \zeta' = \frac{\zeta}{2}.$$

Apply Theorem 2.2.1 with $\frac{\mu'}{2}, \delta', \zeta', \xi', c(\delta) = \frac{\delta}{4}$ and $c_\varrho \geqslant 16\sqrt[8]{4}$. Get

$$\kappa, \quad \alpha, \quad 2\eta', \quad \frac{\varrho'^4}{4} > 0 \quad \text{with} \quad 2\eta', \xi' \geqslant c_\varrho \Big(\frac{\varrho'^4}{4}\Big)^{\frac{1}{8}}.$$

Set

$$k_1 = \gamma = M = \min\Big\{\frac{\mu\sqrt{\zeta}}{8}, \frac{\xi^4}{16^4}, \frac{\zeta}{14}, \frac{\delta^2}{8^2}, \Big(\frac{\eta'}{8\sqrt{3}} - \frac{\sqrt{\varrho'}}{\sqrt{3}}\Big)^4\Big\} \quad \text{and} \quad \gamma' = \frac{\gamma^2}{2^{\frac{4}{\gamma}+4}}.$$

Then take

$$d_0 = \min\Big\{\frac{\mu^3\sqrt{\zeta}^4}{8}, \frac{M}{8}, \sqrt{\frac{10^4}{4\cdot 14\cdot 36}\Big\lceil\frac{1}{\kappa}\Big\rceil^{-1}}\Big\}.$$

Let $\varepsilon > 0$. Take

$$\varepsilon_0 = \min\Big\{\Big(\frac{\varepsilon}{5}\Big)^8, \Big(\frac{d_0}{10}\Big)^4\Big\}, \quad t_0 = \frac{1}{\varepsilon_0} \quad \text{and} \quad \varepsilon_1 = 5\sqrt[8]{\varepsilon_0}.$$

Apply the Regularity Lemma, get T and n_0'.

Take

$$\eta = \frac{\eta'}{2} \quad \text{and} \quad \varrho \geqslant \varrho' + 3\sqrt{M}.$$

Take $\nu' \leqslant \min\{\frac{\eta'}{6}, \frac{\sqrt{\xi'}}{2}\}$ and $\nu = \nu' + 3\sqrt[4]{\varepsilon_0}$.

Let G on large enough n vertices with robust neighbourhoods $\mathcal{R}_G$ be $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good. By the Regularity Lemma there is a partition $\mathcal{P} = \{V_0, \dots, V_t\}$ of $V(G)$ with $t_0 \leqslant t \leqslant T$ which is an ε_0-regular partition for G.

We delete edges of $E(G)$ in two steps. First delete $E_1 \subseteq E(G)$ containing the edges inside partition sets, incident to V_0 and between irregular pairs, then

$$|E_1| \leqslant \binom{|V_i|}{2}t + |V_0|n + |V_i|^2\varepsilon_0 t^2 \leqslant \Big(\frac{1}{2t} + \varepsilon_0 + \varepsilon_0\Big)n^2 \leqslant 3\varepsilon_0 n^2.$$

Let $Y_1 \subseteq V(G)$ be the vertices incident to at least $\frac{Mn}{2}$ edges in E_1,

$$|Y_1| \leqslant \frac{6\varepsilon_0}{M}n \leqslant \sqrt{\varepsilon_0}n.$$

Secondly delete $E_2 \subseteq E(G) \setminus E_1$ containing the edges between pairs with density smaller than d_0. We have that $|E_2| \leqslant d_0 n^2$.

Observe that if (V_i, V_j) is (ε_0, d_{ij})-regular with $d_{ij} < d_0$ and $X_{ij} \subseteq V_i$ are the vertices with at least $\frac{M}{4}|V_j|$ neighbours in V_j, then

$$|X_{ij}|\frac{M}{4}|V_j| \leqslant e(X_{ij}, V_j) \leqslant d_0|X_{ij}||V_j| + \varepsilon_0|V_i||V_j| \quad \text{thus} \quad |X_{ij}| \leqslant \frac{\varepsilon_0}{\frac{M}{4} - d_0}|V_i|\,.$$

Let $X_i = \bigcup X_{ij}$ with the union over all $j \in [t]$ such that (V_i, V_j) is (ε_0, d_{ij})-regular, but $d_{ij} < d_0$. Let X_i^* be the vertices $v \in X_i$ such that $v \in X_{ij}$ for at least $\frac{M}{4}t$ different $j \in [t]$. Then

$$|X_i^*| \leqslant \frac{\sum_{j\in[t]} \frac{\varepsilon_0}{\frac{M}{4}-d_0}\frac{n}{t}}{\frac{M}{4}t} \leqslant \frac{\varepsilon_0}{(\frac{M}{4} - d_0)\frac{M}{4}}\frac{n}{t}\,.$$

Let Y_2 be the vertices incident to at least $\frac{Mn}{2}$ edges in E_2. We show that $Y_2 \subseteq \bigcup_{i\in[t]} X_i^*$. For $v \in V_i \setminus X_i$, we have that

$$|\{vw\colon w \in N_G(v)\} \cap E_2| < \frac{M}{4}\frac{n}{t}t \leqslant \frac{Mn}{4}\,,$$

thus $(V_i \setminus X_i) \nsubseteq Y_2$. For $v \in X_i \setminus X_i^*$, we have that

$$|\{vw\colon w \in N_G(v)\} \cap E_2| < \frac{n}{t}\frac{M}{4}t + \frac{M}{4}\frac{n}{t}t \leqslant \frac{Mn}{2}\,,$$

thus $(X_i \setminus X_i^*) \nsubseteq Y_2$. We have that,

$$|Y_2| \leqslant \sum_{i\in[t]} |X_i^*| \leqslant \frac{\varepsilon_0 n}{(\frac{M}{4} - d_0)\frac{M}{4}} \leqslant \frac{\varepsilon_0 n}{\frac{M^2}{32}} \leqslant \sqrt{\varepsilon_0}n\,.$$

Let $E' = E_1 \dot\cup E_2$ and $Y = Y_1 \cup Y_2$, we have that if v is incident to at least Mn edges in E', then $v \in Y$. Moreover,

$$|E'| \leqslant \Big(3\varepsilon_0 + d_0\Big)n^2 \quad \text{and} \quad |Y| \leqslant 2\sqrt{\varepsilon_0}n\,.$$

We move the classes V_j such that $|V_j \cap Y| \geqslant \sqrt[4]{\varepsilon_0}|V_j|$ to V_0 and get the partition $\mathcal{P}' = \{V_0', V_1, \ldots, V_{t'}\}$, such that $|V_0'| \leqslant \varepsilon_0 n + 2\sqrt[4]{\varepsilon_0}n$ and $(1 - 2\sqrt[4]{\varepsilon_0})t \leqslant t' \leqslant t$. Set $t_1 = \lceil\frac{1}{\kappa}\rceil t'$.

Let $G' = (V(G) \setminus V_0', E(G) \setminus E')$ and $\mathfrak{R} = R(\varepsilon_0, d_0, \mathcal{P}')$ with $\mathcal{R}_{\mathfrak{R}} = \{U_i : i \in [t']\}$, where we take U_i in the following way.

Consider the bipartite graph $H_i = (V_i \setminus Y \dot\cup [t'], E_i)$ where $\{v, j\} \in E_i$ when $|N_{G'}(v) \cap U_v \cap V_j| \geqslant k_1 |V_j|$. We show that there is $V_i' \subseteq V_i \setminus Y$ and $U_i \subseteq [t']$ such that

(1) $|V_i'| \geqslant (1-\gamma)|V_i \setminus Y|$,

(2) if $v \in V_i'$ then $|N_{H_i}(v) \cap U_i| \geqslant |N_{H_i}(v)| - \gamma t$,

(3) if $j \in U_i$ then $|N_{H_i}(j) \cap V_i'| \geqslant \frac{\gamma^2}{2^{\frac{4}{\gamma}+4}} |V_i \setminus Y|$.

Observe that if $v \in V_i \setminus Y$, then

$$|U_v \cap N_{G'}(v)| \geqslant 2\sqrt{\zeta}n - Mn - |V_0' \setminus V_0| \geqslant \sqrt{\zeta}n\,.$$

If $B(v) \subseteq [t']$ are the classes j such that $|N_{G'}(v) \cap U_v \cap V_j| \geqslant k_1 |V_j|$, then

$$\sqrt{\zeta}n \leqslant |U_v \cap N_{G'}(v)| \leqslant |B(v)|\frac{n}{t} + t'k_1\frac{n}{t} \quad \text{and} \quad |B(v)| \geqslant t(\sqrt{\zeta} - k_1)\,. \qquad (2.6.2)$$

That is $|N_{H_i}(v)| \geqslant t(\sqrt{\zeta} - k_1)$.

We set a sequence of sets $B_1, \ldots, B_{k_0}, B_{k_0+1} \subseteq [t']$ with $|B_k| \geqslant \frac{\gamma}{2}t'$ for $k \in [k_0]$ and $|B_{k_0+1}| < \frac{\gamma}{2}t'$ and we also set a sequence $A_1, \ldots, A_{k_0} \subseteq (V_i \setminus Y)$ such that

$$B_1 = \left\{ j \in [t'] : |N_{H_i}(j)| < \left(\frac{\gamma}{4}\right)^2 |V_i \setminus Y| \right\},$$

$$A_1 = \left\{ v \in V_i \setminus Y : |N_{H_i}(v) \cap B_1| > \frac{\gamma}{4}t' \right\},$$

$$B_2 = \left\{ j \in [t'] \setminus B_1 : |N_{H_i}(j) \cap (V_i \setminus (Y \cup A_1))| < \left(\frac{\gamma}{8}\right)^2 |V_i \setminus Y| \right\},$$

and for $3 \leqslant k \leqslant k_0 + 1$,

$$A_{k-1} = \left\{ v \in V_i \setminus (Y \cup A_1 \cup \cdots \cup A_{k-2}) : |N_{H_i}(v) \cap B_{k-1}| > \frac{\gamma}{2^k}t' \right\},$$

$$B_k = \left\{ j \in [t'] \setminus (B_1 \cup \cdots \cup B_{k-1}) : |N_{H_i}(j) \cap (V_i \setminus (Y \cup A_1 \cup \cdots \cup A_{k-1}))| < \left(\frac{\gamma}{2^{k+1}}\right)^2 |V_i \setminus Y| \right\}.$$

Since $|B_k| \geqslant \frac{\gamma}{2}t'$ for $k \in [k_0]$, we have that $k_0 \leqslant \frac{2}{\gamma}$.

For $k \in [k_0+1]$, let $B'_k = B_1 \cup \cdots \cup B_k$ and $A'_{k-1} = A_1 \cup \cdots \cup A_{k-1}$ ($A'_0 = \varnothing$), we take $V'_i = V_i \setminus (Y \cup A'_{k_0})$ and $U_i = [t'] \setminus B'_{k_0+1}$.

For (1), observe that for $k \in [k_0]$,

$$\frac{\gamma}{2^{k+1}} t'|A_k| \leqslant e(A_k, B_k) \leqslant e(V_i \setminus (Y \cup A'_{k-1}), B_k) \leqslant |B_k| \Big(\frac{\gamma}{2^{k+1}}\Big)^2 |V_i \setminus Y| \leqslant \Big(\frac{\gamma}{2^{k+1}}\Big)^2 t' |V_i \setminus Y|\,,$$

$$\text{thus} \quad |A_k| \leqslant \frac{\gamma}{2^{k+1}} |V_i \setminus Y|\,.$$

Therefore,

$$\sum_{k \in [k_0]} |A_k| \leqslant \sum_{k=1}^{\infty} \frac{\gamma}{2^{k+1}} |V_i \setminus Y| \leqslant \frac{\gamma}{2} |V_i \setminus Y|\,.$$

For (2), observe that for any $v \in V_i \setminus (Y \cup A'_{k_0})$,

$$\sum_{k \in [k_0]} |N_{H_i}(v) \cap B_k| \leqslant \sum_{k \in [k_0]} \frac{\gamma}{2^{k+1}} t' \leqslant \frac{\gamma}{2} t'\,.$$

Since $|B_{k_0+1}| < \frac{\gamma}{2} t'$, we have that $|N_{H_i}(v) \cap U_i| \geqslant |N_{H_i}(v)| - (\frac{\gamma}{2} + \frac{\gamma}{2})t$.

For (3), consider $j \in [t'] \setminus B'_{k_0+1}$. Then $j \in [t'] \setminus B'_{k_0}$ and by our choice of B_{k_0+1}, we have that $|N_{H_i}(j) \cap (V_i \setminus (Y \cup A'_{k_0}))| \geqslant (\frac{\gamma}{2^{k_0+2}})^2 |V_i \setminus Y|$. Since $k_0 \leqslant \frac{2}{\gamma}$, we have that $|N_{H_i}(j) \cap V'_i| \geqslant \frac{\gamma^2}{2^{\frac{4}{\gamma}+4}} |V_i \setminus Y|$.

For any $i \in [t']$ and $v \in V'_i$, let

$$U'_v = U_v \cap N_{G'}(v) \cap \bigcup_{j \in U_i} V_j\,.$$

Then,

$$|U_v \setminus U'_v| \leqslant |N_G(v) \setminus N_{G'}(v)| + |N_{G'}(v) \cap U_v \cap \bigcup_{j \notin U_{i_0}} V_j| \leqslant (M + 2\sqrt[4]{\varepsilon_0} + k_1 + \gamma)n$$

$$|E(G[U_v]) \setminus E(G'[U'_v])| \leqslant (M + 2\sqrt[4]{\varepsilon_0} + k_1 + \gamma + 3\varepsilon_0 + d_0)n^2 \leqslant 4Mn^2\,.$$

We want to show that U_{i_0} is μ'-inseparable. First we consider the minimum degree $\delta(\mathfrak{R}[U_{i_0}])$. Let $j_0 \in U_{i_0}$, $v \in N_{H_{i_0}}(j_0) \cap V'_{i_0}$ and $w \in (U_v \cap V_{j_0}) \setminus Y$. Since U_v

is μ-inseparable $|N_{G[U_v]}(w)| \geqslant \mu(|U_v|-1)$. We have that,

$$\begin{aligned}|N_{G'[U'_v]}(w)| &\geqslant \mu(|U_v|-1) - |U_v \setminus U'_v| - |\{uw\colon uw \in E'\}| \\ &\geqslant \mu|U_v| - \mu - (M + 2\sqrt[4]{\varepsilon_0} + k_1 + \gamma + M)n \\ &\geqslant \mu|U_v| - \frac{3(M+k_1)}{2\sqrt{\zeta}} 2\sqrt{\zeta}n \geqslant \frac{\mu|U_v|}{2}\,. \end{aligned} \tag{2.6.3}$$

Therefore,

$$|N_{\mathfrak{R}[U_{i_0}]}(j_0)| \geqslant \frac{\mu|U_v|}{2}\frac{t}{n} \geqslant \mu\sqrt{\zeta}t\,.$$

For any subset $X \subseteq U_{i_0}$ such that $|X| \leqslant \frac{\mu\sqrt{\zeta}}{2}t$, we have that

$$e_{\mathfrak{R}[U_{i_0}]}(X, U_{i_0} \setminus X) \geqslant |X|\frac{\mu\sqrt{\zeta}}{2}t \geqslant \frac{\mu\sqrt{\zeta}}{2}|X||U_{i_0} \setminus X|\,.$$

Now consider a partition $X \subseteq U_{i_0}$, $\bar{X} = U_{i_0} \setminus X$ such that

$$|X|, |\bar{X}| \geqslant \frac{\mu\sqrt{\zeta}}{2}t\,.$$

For $v \in V'_{i_0}$, take the following partition of U'_v,

$$X_v = U_v \cap N_{G'}(v) \cap \bigcup_{i \in X} V_i \quad \text{and} \quad \bar{X}_v = U_v \cap N_{G'}(v) \cap \bigcup_{i \in \bar{X}} V_i\,.$$

For a first case, assume there is $v \in V'_{i_0}$ such that,

$$|X_v|, |\bar{X}_v| \geqslant \mu^2(\sqrt{\zeta})^2|U'_v| \geqslant \mu^2(\sqrt{\zeta})^3 n\,.$$

Assume wlog that $|X_v| \leqslant |\bar{X}_v|$, thus $|U_v \setminus X_v| \geqslant \sqrt{\zeta}n$ and since U_v is μ-inseparable we have that $e_G(X_v, U_v \setminus X_v) \geqslant \mu|X_v||U_v \setminus X_v|$ and

$$\begin{aligned}e_{G'}(X_v, \bar{X}_v) &\geqslant \mu|X_v||U_v \setminus X_v| - |U_v \setminus U'_v||X_v| - |E'| \\ &\geqslant (\mu\sqrt{\zeta} - M - 2\sqrt[4]{\varepsilon_0} - k_1 - \gamma)n|X_v| - (3\varepsilon_0 + d_0)n^2 \\ &\geqslant \Big(\frac{\mu^3(\sqrt{\zeta})^4}{2} - (3\varepsilon_0 + d_0)\Big)n^2\end{aligned}$$

We have that

$$e_{\mathfrak{R}[U_{i_0}]}(X, \bar{X}) \geqslant \frac{\mu^3(\sqrt{\zeta})^4}{4}t^2 \geqslant \frac{\mu^3(\sqrt{\zeta})^4}{4}|X||\bar{X}|\,.$$

For a second case, we have that for every $v \in V'_{i_0}$ either

$$|X_v| < \mu^2(\sqrt{\zeta})^2|U'_v| \leqslant \mu^2(\sqrt{\zeta})^2 n \quad \text{or} \quad |\bar{X}_v| < \mu^2(\sqrt{\zeta})^2|U'_v|\,.$$

Consider $V'_{i_0}(X) \subseteq V'_{i_0}$ such that $v \in V'_{i_0}(X)$ implies $|\bar{X}_v| < \mu^2(\sqrt{\zeta})^2|U'_v|$ and $v \in V'_{i_0}(\bar{X})$ implies $|X_v| < \mu^2(\sqrt{\zeta})^2|U'_v|$; note that $V'_{i_0}(\bar{X}) = V'_{i_0} \smallsetminus V'_{i_0}(X)$. Since $\mathfrak{A}_G$ is complete, if $V'_{i_0}(X), V'_{i_0}(\bar{X}) \neq \varnothing$, then there is $uv \in E(\mathfrak{A}_G)$ with $u \in V'_{i_0}(X)$ and $v \in V'_{i_0}(\bar{X})$. We have that $|U_u \cap U_v| \geqslant 2\sqrt{\zeta}n$ and we derive a contradiction, since,

$$\sqrt{\zeta}n \leqslant 2\sqrt{\zeta}n - 2(M + 2\sqrt[4]{\varepsilon_0} + k_1 + \gamma)n \leqslant |U'_u \cap U'_v| < 2\mu^2(\sqrt{\zeta})^2 n\,.$$

If say $V'_{i_0}(\bar{X}) = \varnothing$, then for every $j \in \bar{X}$, property $(h3)$ gives us $\gamma'|V_{i_0} \smallsetminus Y|$ vertices in $v \in V'_{i_0}$ such that $|N_{G'}(v) \cap U_v \cap V_j| \geqslant k_1|V_j|$, thus $|U'_v \cap (V_j \smallsetminus Y)| \geqslant (k_1 - \sqrt[4]{\varepsilon_0})|V_j|$. Take $w \in U'_v \cap (V_j \smallsetminus Y)$, we have by (2.6.3), that $|N_{G'[U'_v]}(w)| \geqslant \mu\sqrt{\zeta}n$.

Since $|\bar{X}_v| \leqslant \mu^2(\sqrt{\zeta})^2 n$,

$$|N_{G'[U'_v]}(w) \cap X_v| \geqslant \mu\sqrt{\zeta}n - \mu^2(\sqrt{\zeta})^2 n \geqslant \frac{\mu\sqrt{\zeta}}{2}n\,.$$

Thus $e_{\mathfrak{R}[U_{i_0}]}(\{j\}, X) \geqslant \frac{\mu\sqrt{\zeta}}{2}t$. Going over all $j \in \bar{X}$, we get

$$e_{\mathfrak{R}[U_{i_0}]}(X, \bar{X}) \geqslant \frac{\mu\sqrt{\zeta}}{2}|\bar{X}||X|\,.$$

If $ab \in E(G')$ is ξ-connectable in G, then it is in U_v for $\xi n - |V'_0| \geqslant \frac{\xi}{2}n$ vertices in $V(G')$. Therefore, there are at least $\frac{\xi}{4}t$ classes $I(ab) \subseteq [t']$ such that for $i \in I(ab)$, at least $\frac{\xi}{4}\frac{n}{t} - (\sqrt[4]{\varepsilon_0} + \gamma)\frac{n}{t} \geqslant \frac{\xi}{8}\frac{n}{t}$ vertices $v \in V'_i$ are such that $ab \in E(G[U_v])$.

The number of pairs (ab, v), where $ab \in E(G[U_v]) \smallsetminus E(G'[U'_v])$ and $v \in V(G')$ is at most $4Mn^3$. Therefore, the set E_{loss} of connectable edges that are in $E(G[U_v]) \smallsetminus E(G'[U'_v])$ for at least $4\sqrt{M}n$ different $v \in V(G')$ has size

$$|E_{loss}|4\sqrt{M}n \leqslant 4Mn^3 \quad \text{and} \quad |E_{loss}| \leqslant \sqrt{M}n^2\,.$$

Thus we have that for any ξ-connectable edge $ab \in E(G') \smallsetminus E_{loss}$ the number of classes $i \in I(ab)$ that do not contain any $v \in V_i'$ such that $ab \in E(G'[U_v'])$ is at most

$$\frac{4\sqrt{M}n}{\frac{\xi}{4}\frac{n}{t}} \leqslant \frac{16\sqrt{M}}{\xi}t \leqslant \sqrt[4]{M}t\,.$$

Therefore if $a \in V_{j_0}$, $b \in V_{j_1}$, then $j_0 j_1$ is in at least $(\frac{\xi}{4} - \sqrt[4]{M})t \geqslant \frac{\xi}{8}t$ different robust neighbourhoods U_i for $i \in [t']$.

Let $v \in V_{i_0}'$, we have that $G[U_v]$ contains δn^3 different ξ-connectable triangles. Excluding the triangles with an edge in $E(G[U_v]) \setminus E(G'[U_v'])$ or in E_{loss} and projecting the remaining ones into $\mathfrak{R}[U_{i_0}]$, the number of $\frac{\xi}{8}$-connectable triangles in $\mathfrak{R}[U_{i_0}]$ is at least

$$(\delta - 4M - \sqrt{M})n^3\frac{t^3}{n^3} \geqslant \frac{\delta}{2}t^3\,.$$

Next we want that $\mathfrak{A}_{\mathfrak{R}} = A(\mathcal{R}_{\mathfrak{R}}, \zeta')$ is inseparable and we start with the following observation. Consider $uv \in E(\mathfrak{A}_G)$ such that $u \in V_i'$, $v \in V_j'$ for $i, j \in [t']$, we have $|E(G[U_u]) \cap E(G[U_v])| \geqslant \zeta n^2$ and

$$\begin{aligned}|E(G'[U_u']) \cap E(G'[U_v'])| &\geqslant \zeta n^2 - |U_u \setminus U_u'|n - |U_v \setminus U_v'|n - |E'| \\ &\geqslant \zeta n^2 - (2M + 4\sqrt[4]{\varepsilon_0} + 2k_1 + 2\gamma + 3\varepsilon_0 + d_0)n^2 \geqslant \frac{\zeta}{2}n^2\,.\end{aligned}$$

Projecting these edges to $\mathfrak{R}$, we get $|E(\mathfrak{R}[U_i]) \cap E(\mathfrak{R}[U_j])| \geqslant \frac{\zeta}{2}t^2$ and $ij \in E(\mathfrak{A}_{\mathfrak{R}})$.

Take $B \subseteq V(\mathfrak{A}_{\mathfrak{R}})$ with $|B| \leqslant \frac{t'}{2}$ and $B_G = \bigcup_{i \in B} V_i$. Since $\mathfrak{A}_G$ is μ-inseparable, then $e_{\mathfrak{A}_G}(B_G, V(G) \setminus B_G) \geqslant \mu|B_G||V(G) \setminus B_G|$. The set B_G contains all the vertices in the classes of B and $V(G) \setminus B'$ contains V_0' and the other classes. Consider the edges in $E_{\mathfrak{A}_G}(B_G, V(G) \setminus B_G)$, ignore those ending in V_0' or in $V_i \setminus V_i'$ for all $i \in [t']$ and project the remaining ones into $E_{\mathfrak{A}_{\mathfrak{R}}}(B, [t'] \setminus B)$, since $|V_0'| \leqslant (\varepsilon_0 + 2\sqrt[4]{\varepsilon_0})n$ and

$|[t'] \setminus B| \geqslant \frac{t'}{2}$ and $|V_i \setminus V'_i| \leqslant (\sqrt[4]{\varepsilon_0} + \gamma)\frac{n}{t}$, we get

$$\begin{aligned} e_{\mathfrak{A}_{\mathfrak{R}}}(B, [t'] \setminus B) \geqslant & \Big(\mu |B_G||(V(G) \setminus V'_0) \setminus B_G| - (1-\mu)|B_G||V'_0| \\ & - |B||V_i \setminus V'_i||(V(G) \setminus V'_0) \setminus B_G| - |[t'] \setminus B||V_i \setminus V'_i||B_G| \Big) \frac{t^2}{n^2} \\ \geqslant & \Big(\mu |B||[t'] \setminus B| \frac{n^2}{t^2} - |V'_0||B|\frac{n}{t} \\ & - (|B||[t'] \setminus B|\frac{n}{t} + |[t'] \setminus B||B|\frac{n}{t})(\sqrt[4]{\varepsilon_0} + \gamma)\frac{n}{t} \Big) \frac{t^2}{n^2} \\ \geqslant & \mu |B||[t'] \setminus B| - 4(\varepsilon_0 + 2\sqrt[4]{\varepsilon_0})|B|\frac{t'}{2} - 2(\sqrt[4]{\varepsilon_0} + \gamma)|B||[t'] \setminus B| \\ \geqslant & \big(\mu - (4\varepsilon_0 + 10\sqrt[4]{\varepsilon_0} + 2\gamma)\big)|B||[t'] \setminus B| \geqslant \frac{\mu}{2}|B||[t'] \setminus B| \,. \end{aligned}$$

Thus $\mathfrak{A}_{\mathfrak{R}}$ is $\frac{\mu}{2}$-inseparable.

Finally we show that $\mathfrak{R}$ satisfies property iii. of good graphs. Let $A \subseteq [t']$ with $|A| \leqslant \nu' t'$ and $F \subseteq E(\mathfrak{R})$ with $|F| \leqslant \varrho' t'^2$. Take $A' = \bigcup_{i \in A} V_i \cup V'_0$, then $|A'| \leqslant \nu n$ and take $F' = \bigcup_{ij \in F} E(V_i, V_j) \cup E' \cup E_{loss}$, then

$$|F'| \leqslant (\varrho' + 3\varepsilon_0 + d_0 + \sqrt{M})n^2 \leqslant \varrho n^2 \,.$$

Then $G_{A',F'}$ contains a ξ-connectable fractional triangle factor $f_{G_{A',F'}}$ with

$$W(f_{G_{A',F'}}) \geqslant \frac{n}{3} - \eta(n - |A'|) \,.$$

For each $ijk \in T(\mathfrak{R}_{A,F})$, set

$$f_{\mathfrak{R}_{A,F}}(ijk) = \sum_{uvw \in T(G_{A',F'}[V_i \cup V_j \cup V_k])} f_{G_{A',F'}}(uvw)\frac{t}{n} \,.$$

The total weight in $i_0 \in V(\mathfrak{R}_{A,F})$ is

$$\sum_{i_0 jk \in T(\mathfrak{R}_{A,F})} \sum_{uvw \in T(G_{A',F'}[V_{i_0} \cup V_j \cup V_k])} f_{G_{A',F'}}(uvw)\frac{t}{n} \leqslant |V_{i_0}|\frac{t}{n} \leqslant 1 \,.$$

For each $uvw \in T(G_{A',F'})$ with $u \in V_i, v \in V_j, w \in V_k$, if $i, j, k \notin X_F$, then

$ijk \in T(\mathfrak{R}_{A,F})$. Using that $|A'| \geqslant |A|\frac{n}{t}$, $\frac{\eta}{2} \geqslant \sqrt{\varrho'}$ and $\nu' \leqslant \frac{1}{2}$, the total weight is

$$\begin{aligned}
W(f_{\mathfrak{R}_{A,F}}) &= \sum_{ijk \in T(\mathfrak{R}_{A,F})} \sum_{uvw \in T(G_{A',F'}[V_i,V_j,V_k])} f_{G_{A',F'}}(uvw)\frac{t}{n} \\
&\geqslant \frac{t}{n} \sum_{T(G_{A',F'})} f_{G_{A',F'}}(uvw) - \frac{t}{n}|X_F|\frac{n}{t} \\
&\geqslant \frac{t}{n}\Big(\frac{n}{3} - \eta(n - |A'|)\Big) - |X_F| \\
&\geqslant \frac{t}{3} - \eta t - \sqrt{\varrho'}t + \eta|A'|\frac{t}{n} \\
&\geqslant \frac{t}{3} - \big(\eta + \sqrt{\varrho'} + \eta\nu'\big)t + \eta\nu' t + \eta|A| \\
&\geqslant \frac{t}{3} - 2\eta t + 2\eta|A| \geqslant \frac{t}{3} - \eta'(t - |A|)\,.
\end{aligned}$$

We are left to check that this fractional triangle factor is connectable. If $f_{\mathfrak{R}_{A,F}}(ijk) > 0$, then there is $uvw \in T_{G_{A',F'}[V_i,V_j,V_k]}$ such that $f_{G_{A',F'}}(uvw) > 0$ and one of the edges is ξ-connectable in G, say uv with $u \in V_i$, $v \in V_j$. Since $V_0' \subseteq A'$ and $E' \subseteq F'$, we have that $uv \in E(G')$ and since $E_{loss} \subseteq F'$, we have that $ij \in E(\mathfrak{R})$ is a $\frac{\xi}{8}$-connectable edge in $\mathfrak{R}$. We showed that $\mathfrak{R}$ is $(\mu', \delta', \zeta', \varrho', \xi', \eta', \nu')$-good.

Now let $w \in V(G)$, we address the number of connectable triangles with all vertices in $I^{\mathcal{P},G}_{\delta/8}(w)$. The number of triangles in U_w with a vertex outside $\bigcup_{i \in I^{\mathcal{P},G}_{\delta/8}(w)} V_i$ is at most

$$(t - |I^{\mathcal{P},G}_{\delta/8}(w)|)\frac{\delta}{8}\frac{n}{t} \cdot n^2 + |V_0'|n^2 \leqslant \frac{\delta}{4}n^3\,.$$

We have that $|E'|n + |E_{loss}|n \leqslant \frac{\delta}{4}n^3$, thus we have a set $T_I(w)$ of ξ-connectable triangles with vertices in $(U_w \cap \bigcup_{i \in I^{\mathcal{P},G}_{\delta/8}(w)} V_i)$ and edges in $E(G') \setminus E_{loss}$ and $|T_I(w)| \geqslant \frac{\delta}{2}n^3$. The projections of the triangles in $T_I(w)$ into $\mathfrak{R}$ gives us at least $\frac{\delta}{2}t^3$ different $\frac{\xi}{8}$-connectable triangles in $\mathfrak{R}$ with all vertices in $I^{\mathcal{P},G}_{\delta/8}(w)$.

Apply Lemma 2.5.1 to $\mathcal{P}$ with ε_0, $\gamma_1 = \gamma_2 = 0$, $\gamma' \leqslant \frac{\varepsilon_0}{56}$, t', $s(i) = \lceil\frac{1}{\kappa}\rceil$ $(i \in [t])$ and for $j \in [s(i)]$, $f_i(j) = \lceil\frac{1}{\kappa}\rceil^{-1}$. We get

$$\mathcal{P}' = \{V_0'\} \cup \{V_{ij} \colon i \in [t'], j \in [\lceil 1/\kappa \rceil]\}\,,$$

such that

1. $(1-\gamma')\lceil\frac{1}{\kappa}\rceil^{-1}\frac{n}{t} \leqslant |V_{ij}| \leqslant (1+\gamma')\lceil\frac{1}{\kappa}\rceil^{-1}\frac{n}{t}$,

2. if $i \in I^{\mathcal{P},G}_{\frac{\delta}{8}}(v)$, then $ij \in I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v)$ for all $j \in [\lceil 1/\kappa \rceil]$.

Since $|V_{ij}| \geqslant (1-\gamma')\lceil \frac{1}{\kappa} \rceil^{-1}|V_i|$, if $(V_i, V_{i'})$ is (ε_0, d_0)-regular, then for

$$\varepsilon' = \frac{\varepsilon_0}{(1-\gamma')^2}\left\lceil \frac{1}{\kappa} \right\rceil^2,$$

each $(V_{ij}, V_{i'j'})$ is (ε', d_0)-regular.

The reduced graph $\mathfrak{R}' = R(\varepsilon', d_0, \mathcal{P}')$ is a complete $\lceil \frac{1}{\kappa} \rceil$-blow-up of $\mathfrak{R}$. Take

$$\mathcal{R}_{\mathfrak{R}'} = \left\{ U_{i(j)}^{\mathfrak{R}'} = \bigcup_{k \in U_i^{\mathfrak{R}}} \{k(1), \ldots, k(\lceil 1/\kappa \rceil)\} \colon i \in [t], j \in [\lceil 1/\kappa \rceil] \right\},$$

and Lemma 2.6.6 gives us that $\mathfrak{R}'$ is $(\frac{\mu'}{2}, \delta', \zeta', \frac{\varrho'^4}{4}, \xi', 2\eta', \nu'^2)$-good.

We have that $I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v) = \bigcup_{i \in I^{\mathcal{P},G}_{\frac{\delta}{8}}(v)} \{i(1), \ldots, i(\lceil 1/\kappa \rceil)\}$, thus

$$|\{I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v) \colon v \in V(G)\}| = |\{I_1, \ldots, I_m\}| \leqslant 2^{t'} \leqslant 2^{\kappa |V(\mathfrak{R}')|}.$$

Each triangle in $\mathfrak{R}$ gives us $\lceil \frac{1}{\kappa} \rceil^3$ different triangles in $\mathfrak{R}'$. If $xx' \in E(\mathfrak{R})$ is in $\xi' t'$ different U_i, then for $j, j' \in [\lceil \frac{1}{\kappa} \rceil]$ the edge $x(j)x'(j') \in E(\mathfrak{R}')$ is in $U^{\mathfrak{R}'}_{i(j'')}$ for $j'' \in [\lceil \frac{1}{\kappa} \rceil]$, therefore $x(j)x'(j')$ is ξ'-connectable in $\mathfrak{R}'$. Since there are at least $\frac{\delta}{2}t^3$ connectable triangles in $I^{\mathcal{P},G}_{\frac{\delta}{8}}(v)$, we get at least $\frac{\delta}{2}|V(\mathfrak{R}')|^3$ distinct ξ'-connectable triangles in $\mathfrak{R}'$ with vertices in $I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v)$.

Theorem 2.2.1 gives us that $\mathfrak{R}'$ contains the second power of a Hamiltonian cycle $C^2 = x_1 \ldots x_{|V(\mathfrak{R}')|}$, with a segment $P_6 = x_1 \ldots x_6$, which is the third power of a path and another segment P_I such that for all $i \in [m]$, we have that P_I contains $\alpha t' \lceil 1/\kappa \rceil$ different triangle paths on 5 vertices, all in I_i.

Apply Lemma 2.5.1 to $\mathcal{P}'$ with

$$\varepsilon', \quad \gamma_1 = \gamma_2 = \gamma', \quad \gamma'' = \gamma' \leqslant \frac{\varepsilon'}{56}, \quad s(i) = 3 \quad \text{and} \quad f_i(j) = \frac{1}{3},$$

get

$$\mathcal{P}'' = \{V_0\} \cup \{V_{ij_1}, V_{ij_2}, V_{ij_3} \colon i \in [t], j \in [\lceil 1/\kappa \rceil]\} \quad \text{and} \quad V_{ij} = V_{ij_1} \dot\cup V_{ij_2} \dot\cup V_{ij_3},$$

such that

1. $(1-2\gamma')\lceil \frac{1}{\kappa} \rceil^{-1}\frac{n}{3t} \leqslant |V_{ij_a}| \leqslant (1+2\gamma')\lceil \frac{1}{\kappa} \rceil^{-1}\frac{n}{3t}$, for $a \in [3]$,

2. if $ij \in I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v)$, then $ij_1, ij_2, ij_3 \in I^{\mathcal{P}'',G}_{\frac{(1-2\gamma')}{(1+2\gamma')}\frac{\delta}{8}}(v)$.

Since $|V_{ij_a}| \geqslant \frac{(1-2\gamma')}{3}\lceil\frac{1}{\kappa}\rceil^{-1}|V_i|$, if $(V_i, V_{i'})$ is (ε_0, d_0)-regular, then for

$$\varepsilon'' = \frac{9\varepsilon_0}{(1-2\gamma')^2}\left\lceil\frac{1}{\kappa}\right\rceil^2,$$

each $(V_{ij_a}, V_{i'j'_b})$ is (ε'', d_0)-regular for $a, b \in [3]$.

We get that $\mathfrak{R}'' = R(\varepsilon'', d_0, \mathcal{P}'')$ is the complete 3-blow-up of $\mathfrak{R}'$ and contains the complete 3-blow-up of the second power of a Hamiltonian cycle $C^{2(3)}$. We rename the classes of $\mathcal{P}''$ according the second power of a Hamiltonian cycle $C^2 = x_1 \dots x_{|V(\mathfrak{R}')|}$, is such way that,

$$\mathcal{P}'' = \{V_0\} \cup \{V_{a_i}, V_{b_i}, V_{c_i} : i \in [t'\lceil 1/\kappa\rceil]\},$$

where,

$$\begin{aligned} V_{x_i} &= V_{a_{i-2}} \dot\cup V_{a_{i-1}} \dot\cup V_{a_i}, \text{ for } i \equiv 1 \pmod 3, \\ V_{x_i} &= V_{b_{i-2}} \dot\cup V_{b_{i-1}} \dot\cup V_{b_i}, \text{ for } i \equiv 2 \pmod 3, \\ V_{x_i} &= V_{c_{i-2}} \dot\cup V_{c_{i-1}} \dot\cup V_{c_i}, \text{ for } i \equiv 0 \pmod 3, \end{aligned}$$

with $a_0 = a_{t'\lceil 1/\kappa\rceil}, a_{-1} = a_{t'\lceil 1/\kappa\rceil - 1}, b_0 = b_{t'\lceil 1/\kappa\rceil}$ (See Figure 2.6.1). We get that $V(L) = \bigcup_{i \in t'\lceil 1/\kappa\rceil}\{a_i, b_i, c_i\}$ are the vertices of a ladder in $\mathfrak{R}''$. The segment P_6 gives us the edges a_1a_2, b_2b_3, c_3c_4 and we get $L^+ \subseteq \mathfrak{R}''$.

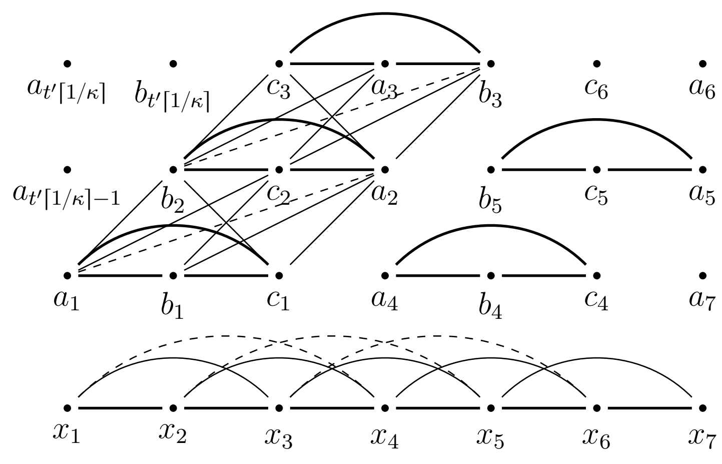


Figure 2.6.1: $L^+ \subseteq \mathfrak{R}''$.

We make all edges in L^+ super-regular by moving vertices to V_0. If (V_{a_i}, V_{b_i}) is (ε'', d)-regular, let $X \subseteq V_{a_i}$ be the vertices that have less than $\frac{d_0}{2}|V_{b_i}|$ neighbours in V_{b_i}, then $|X| \leqslant \frac{2\varepsilon''}{d_0}|V_{a_i}|$, since

$$|X|\frac{d_0}{2}|V_{b_i}| \geqslant e(X, V_{b_i}) \geqslant d_0|X||V_{b_i}| - \varepsilon''|V_{a_i}||V_{b_i}|\,.$$

We have degree $d_{L^+}(a_i) \leqslant 7$ and for each neighbour of a_i we move X to V_0, let V'_{a_i} be the new set. We have that for any V_{a_i} (similarly for V_{b_i} or V_{c_i}), since $\varepsilon'' \leqslant 36\lceil\frac{1}{\kappa}\rceil^2\varepsilon_0$ and by our choices of d_0 and ε_0,

$$|V'_{a_i}| \geqslant \big(1 - \frac{14\varepsilon''}{d_0}\big)|V_{a_i}| \geqslant \big(1 - \frac{d_0}{4}\big)|V_{a_i}|\,.$$

In fact, for each class V_{a_i}, we move exactly $\frac{d_0}{4}(1+2\gamma')\lceil\frac{1}{\kappa}\rceil^{-1}\frac{n}{3t}$ vertices to V_0. Thus,

$$(1 - 3\gamma' - \frac{d_0}{4})\Big\lceil\frac{1}{\kappa}\Big\rceil^{-1}\frac{n}{3t} \leqslant |V'_{a_i}| \leqslant (1+2\gamma')\Big\lceil\frac{1}{\kappa}\Big\rceil^{-1}\frac{n}{3t}\,.$$

If (V_{a_i}, V_{b_j}) is (ε'', d)-regular, then (V'_{a_i}, V'_{b_j}) is (ε^*, d)-regular where, taking $C = 36\lceil\frac{1}{\kappa}\rceil^2$ and assuming without loss of generality that $\sqrt{\varepsilon_0} \leqslant \varepsilon \leqslant \frac{1}{56C}$, we have

$$\frac{\varepsilon''}{\big(1 - \frac{d_0}{4}\big)^2} \leqslant \frac{C\varepsilon_0}{1 - \frac{d_0}{2}} \leqslant 2C\varepsilon_0 \leqslant \frac{\sqrt{\varepsilon_0}}{28} = \varepsilon^*\,.$$

For any $a_ib_j \in E(L^+)$ and $v \in V'_{a_i}$, we have that

$$|N_G(v) \cap V'_{b_j}| \geqslant \Big(\frac{d_0}{2} - \frac{d_0}{4}\Big)|V_{b_i}| \geqslant \frac{d_0}{4}|V_{b_i}|\,. \tag{2.6.4}$$

The new set V'_0 is such that

$$\begin{aligned}|V'_0| &\leqslant \varepsilon_0 n + 2\sqrt[4]{\varepsilon_0}n + \frac{14\varepsilon''}{d_0}n\\ &\leqslant \varepsilon_0 n + 2\sqrt[4]{\varepsilon_0}n + \frac{\sqrt[4]{\varepsilon_0}}{56}n = 3\sqrt[4]{\varepsilon_0}n\,.\end{aligned}$$

We redistribute the vertices in V'_0 keeping the super-regularity in $E(L)$ and the regularity on $E(L^+) \smallsetminus E(L)$. For that we use the special segment $P_I \subseteq C^2 \subseteq \mathfrak{R}'$. For any $I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v)$ there are at least $\alpha t'\lceil 1/\kappa\rceil$ different triangle paths $x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5} \subseteq P_I$ with all vertices in $I^{\mathcal{P}',G}_{\frac{(1-\gamma')}{(1+\gamma')}\frac{\delta}{8}}(v)$.

To each $v \in V'_0$ we associate such a triangle path $P_5(v) = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}$ satisfying that at most $\frac{|V'_0|}{\alpha t'\lceil 1/\kappa\rceil}$ different vertices $v_i, v_j \in V'_0$ have $P_5(v_i) = P_5(v_j)$. Move v to either $V'_{a_{i_2}}$ or $V'_{b_{i_2}}$ or $V'_{c_{i_2}}$, making sure that the sets after the redistribution of V_0 have sizes as even as possible. That is, for each triangle $a_ib_ic_i$ in the ladder L, we have

$$||V_{a_i}| - |V_{b_i}||, ||V_{a_i}| - |V_{c_i}||, ||V_{b_i}| - |V_{c_i}|| \leqslant 1\,.$$

Let $V''_{a_i}, V''_{b_i}, V''_{c_i}$ be the partition sets after redistributing V'_0. We have that, since wlog $\sqrt[8]{\varepsilon_0} \leqslant \varepsilon \leqslant \frac{\alpha}{36}$ and $|V_{a_i}| \geqslant \frac{n}{12t'}\lceil\frac{1}{\kappa}\rceil^{-1}$

$$|V''_{a_i}| \leqslant |V'_{a_i}| + \frac{3\sqrt[4]{\varepsilon_0}n}{\alpha t'}\left\lceil\frac{1}{\kappa}\right\rceil^{-1} \leqslant |V_{a_i}| + \frac{36\sqrt[4]{\varepsilon_0}}{\alpha}|V_{a_i}| \leqslant (1+\sqrt[8]{\varepsilon_0})|V_{a_i}|\,. \tag{2.6.5}$$

We have

$$\begin{array}{rcccl}
(1-3\gamma'-\frac{d_0}{4})\left\lceil\frac{1}{\kappa}\right\rceil^{-1}\frac{n}{3t} & \leqslant & |V''_{a_i}| & \leqslant & (1+3\gamma'+\sqrt[8]{\varepsilon_0})\left\lceil\frac{1}{\kappa}\right\rceil^{-1}\frac{n}{3t} \\
(1-\sqrt{d_0})\left\lceil\frac{1}{\kappa}\right\rceil^{-1}\frac{n}{3t} & \leqslant & |V''_{a_i}| & \leqslant & (1+\sqrt{d_0})\left\lceil\frac{1}{\kappa}\right\rceil^{-1}\frac{n}{3t}\,.
\end{array}$$

See Figure 2.6.1 and observe that for any $u \in V''_{a_i} \setminus V'_{a_i}$ (or V''_{b_i} or V''_{c_i}) and any edge in $E(L)$ incident to a_i, say a_ib_j, we have, using that $d_0 \leqslant \frac{\delta}{16}$ and $\gamma' \leqslant \frac{\varepsilon_0}{56}$,

$$|N_G(u) \cap V''_{b_j}| \geqslant \frac{(1-2\gamma')}{(1+2\gamma')}\frac{\delta}{8}|V_{b_j}| - \frac{d_0}{4}|V_{b_j}| \geqslant \frac{d_0}{2}|V_{b_j}|\,. \tag{2.6.6}$$

Considering (2.6.4), (2.6.5) and (2.6.6), we have that for any edge in $E(L)$, say a_ib_j, if $v \in V''_{a_i}$, then

$$|N_G(v) \cap V''_{b_j}| \geqslant \frac{d_0}{4}|V_{b_j}| \geqslant \frac{d_0}{4(1+\sqrt[8]{\varepsilon_0})}|V''_{b_j}| \geqslant \frac{d_0}{8}|V''_{b_j}|\,.$$

Moreover for any edge in $E(L^+)$, say a_ib_j, we have that (V'_{a_i}, V'_{b_j}) is (ε^*, d)-regular and Observation 2.6.7 with (2.6.5) and $|V'_{a_i}| \geqslant \frac{|V_{a_i}|}{2}$ gives us that (V''_{a_i}, V''_{b_j}) is (ε_1, d)-regular with

$$\varepsilon^* + 4\sqrt[8]{\varepsilon_0} \leqslant 5\sqrt[8]{\varepsilon_0} = \varepsilon_1\,.$$ □

Observation 2.6.7. *If (V_i, V_j) is an (ε, d)-regular pair and V'_i, V'_j are such that $|V'_i \setminus V_i| \leqslant \alpha|V_i|, |V'_j \setminus V_j| \leqslant \beta|V_j|$, then (V'_i, V'_j) is an $(\varepsilon+\alpha+\beta, d)$-regular pair.*

Take $X \subseteq V_i'$, $Y \subseteq V_j'$. Then,

$$
\begin{aligned}
e(X,Y) &\leqslant e(X \cap V_i, Y \cap V_j) + e(X \setminus V_i, Y) + e(Y \setminus V_j, X) \\
&\leqslant d|X \cap V_i||Y \cap V_j| + \varepsilon|V_i||V_j| + |V_i' \setminus V_i||V_j| + |V_j' \setminus V_j||V_i| \\
&\leqslant d|X \cap V_i||Y \cap V_j| + (\varepsilon + \alpha + \beta)|V_i||V_j| \\
&\leqslant d|X||Y| + (\varepsilon + \alpha + \beta)|V_i'||V_j'|\,.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
e(X,Y) &\geqslant d|X \cap V_i||Y \cap V_j| - \varepsilon|V_i||V_j| \\
&\geqslant d(|X| - \alpha|V_i|)(|Y| - \beta|V_j|) - \varepsilon|V_i||V_j| \\
&\geqslant d|X||Y| - (\varepsilon + d\alpha + d\beta)|V_i||V_j| \\
&\geqslant d|X||Y| - (\varepsilon + \alpha + \beta)|V_i'||V_j'|\,.
\end{aligned}
$$

2.6.2 Lemma for H

Here we discuss the mapping of the vertices of H into the ladder, needed for our application of the blow-up lemma.

Proof of Lemma 2.6.3. We are given $\alpha, \gamma' > 0$, integers t and $n_{a_i}, n_{b_i}, n_{c_i}$ $(i \in [t])$, such that $\sum_{i \in [t]}(n_{a_i} + n_{b_i} + n_{c_i}) = n$, $\frac{n}{2t} \leqslant n_{a_i} + n_{b_i} + n_{c_i} \leqslant \frac{n}{t}$ and

$$|n_{a_i} - n_{b_i}|, |n_{a_i} - n_{c_i}|, |n_{b_i} - n_{c_i}| \leqslant 1\,. \tag{2.6.7}$$

Take

$$\beta = \min\left\{\frac{\gamma'}{12t(t+1)}, \frac{\alpha\gamma'}{18t^2}, \frac{(\gamma')^2}{24t^2}\right\}.$$

Let H be a 3-chromatic graph on n vertices with $\mathrm{bw}(H) \leqslant \beta n$ and let L^+ be a ladder with additional edges,

$$V(L^+) = \bigcup_{i \in [t]} \{a_i, b_i, c_i\}, \quad E(L^+) = E(L) \cup \{a_1a_2, b_2b_3, c_3c_4\}\,.$$

We now set the homomorphism $f\colon V(H) \to V(L^+)$ and special set $X \subseteq V(H)$.

Fix a 3-colouring $V(H) = V_1 \dot\cup V_2 \dot\cup V_3$ of H and let $u_1 \dots u_n$ be an ordering of $V(H)$ in bandwidth order. We split the ordered $V(H)$ into consecutive sets U_{ij},

$i \in [t]$, $j \in [\frac{1}{\gamma'}]$ in the following order

$$U_{11},\ \ldots,\ U_{1\frac{1}{\gamma'}},\ U_{21},\ \ldots,\ U_{t\frac{1}{\gamma'}}\,.$$

Let

$$N_i = n_{a_i} + n_{b_i} + n_{c_i}\,,$$

and set the size $|U_{ij}| = \gamma' N_i$ for every $j \in [\frac{1}{\gamma'}]$ (we omit ceilings and floors, since anyway we work with slightly unbalanced classes). We have that $\sum_{j\in[\frac{1}{\gamma'}]} |U_{ij}| = N_i$ and the vertices in $\bigcup_{j\in[\frac{1}{\gamma'}]} U_{ij}$ will be mapped into $\{a_i, b_i, c_i\}$.

Let

$$U_{ij}^{(1)} = U_{ij} \cap V_1, \quad U_{ij}^{(2)} = U_{ij} \cap V_2 \quad \text{and} \quad U_{ij}^{(3)} = U_{ij} \cap V_3\,.$$

We map each $U_{ij}^{(1)}, U_{ij}^{(2)}, U_{ij}^{(3)}$ to a different vertex $\{a_i, b_i, c_i\}$. We set auxiliary mappings, $g\colon [t] \times [\frac{1}{\gamma'}] \to \bigcup_{i\in[t]}\{a_i, b_i, c_i\}^2$, where

$$g(ij) \in \{(a_i, b_i), (a_i, c_i), (b_i, a_i), (b_i, c_i), (c_i, a_i), (c_i, b_i)\}$$

and $g'\colon V(H) \to \bigcup_{i\in[t]}\{a_i, b_i, c_i\}$. To set g' consider $v \in V(H)$, if $v \in U_{ij}$ and $g(ij) = (\delta_1, \delta_2)$, we have

- if $v \in U_{ij}^{(1)}$, then $g'(v) = \delta_1$,
- if $v \in U_{ij}^{(2)}$, then $g'(v) = \delta_2$,
- if $v \in U_{ij}^{(3)}$, then $g'(v) = \{a_i, b_i, c_i\} \setminus \{\delta_1, \delta_2\}$.

Note that $|U_{ij}^{(1)}|, |U_{ij}^{(2)}|, |U_{ij}^{(3)}|$ might be different. For $i \in [t]$, let A_i be the number of vertices $v \in \bigcup_{j\in[\frac{1}{\gamma'}]} U_{ij}$ such that $g'(v) = a_i$, similarly define B_i, C_i. We make sure that

$$|A_i - B_i|, |A_i - C_i|, |B_i - C_i| \leqslant \gamma' N_i \tag{2.6.8}$$

using the following procedure to define g. Let $i \in [t]$ and suppose we have $g(ij)$ for $j \in [j_0]$ ($j_0 \leqslant \frac{1}{\gamma'} - 1$) and $A_i(j_0)$, $B_i(j_0)$, $C_i(j_0)$ are the number of vertices that have been associated to a_i, b_i and c_i and they differ by at most $\gamma' N_i$. Say

$$A_i(j_0) \leqslant B_i(j_0) \leqslant C_i(j_0)\,,$$

then for setting $g(i(j_0+1))$, take the largest among $|U^{(1)}_{i(j_0+1)}|, |U^{(2)}_{i(j_0+1)}|, |U^{(3)}_{i(j_0+1)}|$ and associate it to a_i, the second larger to b_i and the smallest to c_i. If the relations $A_i(j_0+1) \leqslant B_i(j_0+1) \leqslant C_i(j_0+1)$ are kept, since we associated more vertices to a_i than to b_i and more to b_i than to c_i, we have

$$|A_i(j_0+1) - B_i(j_0+1)| \leqslant |A_i(j_0) - B_i(j_0)| \leqslant \gamma' N_i\,,$$

similarly for $|B_i(j_0+1) - C_i(j_0+1)|$ and $|A_i(j_0+1) - C_i(j_0+1)|$. If any relation changes, say $A_i(j_0+1) \geqslant B_i(j_0+1)$, then

$$\begin{aligned}|A_i(j_0+1) - B_i(j_0+1)| &\leqslant \max_{x,y\in[3]} \big||U^{(x)}_{i(j_0+1)}| - |U^{(y)}_{i(j_0+1)}|\big| \\ &\leqslant |U_{i(j_0+1)}| \leqslant \gamma' N_i\,.\end{aligned}$$

If $uv \in E(H[U_{ij}])$, then u, v are in different colour classes and $g'(u)g'(v) \in E(a_ib_ic_i)$ and we would have a homomorphism if not by the edges between different U_{ij} and $U_{i'j'}$. Owing to the small bandwidth of H, there might be edges only between $U_{ij}U_{i(j+1)}$ (for $j \in [\frac{1}{\gamma'} - 1]$) or $U_{i\frac{1}{\gamma'}}U_{(i+1)1}$ (where $U_{(t+1)1} = U_{11}$). We remap some vertices of U_{ij} to set a homomorphism $f\colon V(H) \to V(L^+)$. The vertices we redistribute are part of the special set $X \subseteq V(H)$.

We set $X = \bigcup_{i\in[t],j\in[\frac{1}{\gamma'}]} X_{ij}$, where X_{ij} is taken in the following way. Let $i \in [t]$, $j \geqslant 2$ and assume $g(i(j-1)) = (a_i, b_i)$, that is

$$g'(U^{(1)}_{i(j-1)}) = a_i, \quad g'(U^{(2)}_{i(j-1)}) = b_i, \quad g'(U^{(3)}_{i(j-1)}) = c_i\,.$$

For $j = 1$, assume $g((i-1)\frac{1}{\gamma'}) = (a_{i-1}, b_{i-1})$ (where $g(0\frac{1}{\gamma'}) = g(t\frac{1}{\gamma'})$), that is

$$g'(U^{(1)}_{(i-1)\frac{1}{\gamma'}}) = a_{i-1}, \quad g'(U^{(2)}_{(i-1)\frac{1}{\gamma'}}) = b_{i-1}, \quad g'(U^{(3)}_{(i-1)\frac{1}{\gamma'}}) = c_{i-1}\,.$$

Now consider the following case, in which the three colours in U_{ij} are rotated compared to the colours of $U_{i(j-1)}$ (or $U_{(i-1)\frac{1}{\gamma'}}$), the other cases are similarly resolved. We have

$$g(ij) = (b_i, c_i), \text{ that is } g'(U^{(1)}_{ij}) = b_i,\ g'(U^{(2)}_{ij}) = c_i,\ g'(U^{(3)}_{ij}) = a_i\,.$$

We consider the following triangle walk on $3\ell+2$ vertices in L^+ (see Figure 2.6.2)

$$(a_i c_i b_i)\dots(a_2 c_2 \boldsymbol{b_2})\boldsymbol{b_3}\boldsymbol{c_3}(\boldsymbol{c_4} b_4 a_4)\dots(c_i b_i a_i)\,.$$

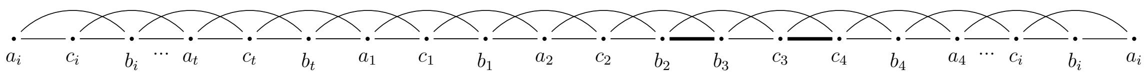


Figure 2.6.2: Triangle walk in L^+.

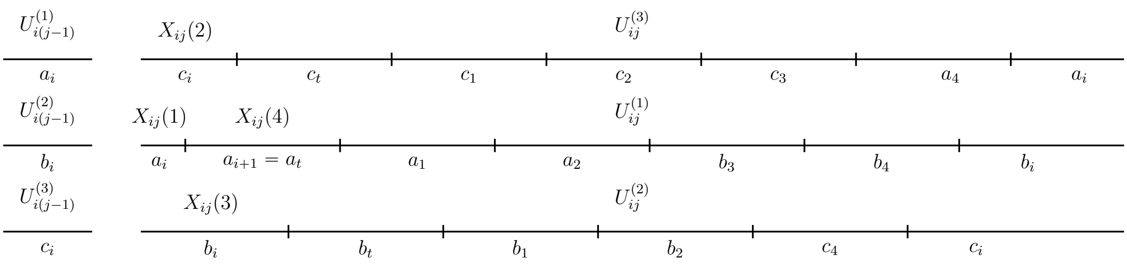


Figure 2.6.3: Setting $f(X)$.

If $i \in [t] \smallsetminus \{3\}$, then $\ell = t$. For $i = 3$, we have $\ell = 2t$.

Say $U_{ij} = u_1 \dots u_{\gamma' N_i}$ in bandwidth order. We include in X_{ij} the following vertices (see Figure 2.6.3).

- $X_{ij}(1)$ containing the vertices of $U^{(1)}_{ij}$ among the first βn vertices in U_{ij}.

- $X_{ij}(2)$ containing the vertices of $U^{(3)}_{ij}$ among the first $2\beta n$ vertices in U_{ij}.

- $X_{ij}(3m)$, for $m \in [t-1]$, containing the vertices in $U^{(2)}_{ij}$ among

$$u_{(m-1)3\beta n+1} \quad \dots \quad u_{m3\beta n}\,.$$

- $X_{ij}(3m+1)$, for $m \in [t-1]$, containing the vertices in $U^{(1)}_{ij}$ among

$$u_{\beta n+(m-1)3\beta n+1} \quad \dots \quad u_{\beta n+m3\beta n}\,.$$

- $X_{ij}(3m+2)$, for $m \in [t-1]$, containing the vertices in $U^{(3)}_{ij}$ among

$$u_{2\beta n+(m-1)3\beta n+1} \quad \dots \quad u_{2\beta n+m3\beta n}\,.$$

We set $f(U_{ij}^{(3)} \setminus X_{ij}) = a_i$, $f(U_{ij}^{(1)} \setminus X_{ij}) = b_i$, $f(U_{ij}^{(2)} \setminus X_{ij}) = c_i$ and we map the edges incident to X_{ij} into edges of L^+ in the following way, all indices being taken $\pmod t$.

- $f(X_{ij}(1)) = a_i$, $f(X_{ij}(2)) = c_i$;
- for $m \in [t-i+3]$, set $f(X_{ij}(3m)) = b_{i+m-1}$;
- for $m \in [\ell-1] \setminus [t-i+3]$, set $f(X_{ij}(3m)) = c_{4+m-(t-i+3)-1}$;
- for $m \in [t-i+2]$, set $f(X_{ij}(3m+1)) = a_{i+k}$;
- for $m \in [\ell-1] \setminus [t-i+2]$, set $f(X_{ij}(3m+1)) = b_{3+m-(t-i+2)-1}$;
- for $m \in [t-i+3]$, set $f(X_{ij}(3m+2)) = c_{i+m}$;
- for $m \in [\ell-1] \setminus [t-i+3]$, set $f(X_{ij}(3m+2)) = a_{4+m-(t-i+3)-1}$.

We observe that, for $k \in [t]$, we have

$$|X_{ij} \cap f^{-1}(a_k)|, |X_{ij} \cap f^{-1}(b_k)|, |X_{ij} \cap f^{-1}(c_k)| \leqslant 6\beta n\,, \tag{2.6.9}$$

since the worst case is when $i = 3$ and each a_k, b_k, c_k gets at most $6\beta n$ vertices from X_{ij}.

By our choice of β, we have $|U_{ij}| = \gamma' N_i \geqslant 6\beta n(t+1)$, so for any v among the last βn vertices in U_{ij}, we have $f(v) = g'(v)$. It is not hard to check that f is indeed a homomorphism between $V(H)$ and L^+.

Since $X = \bigcup_{i\in[t], j\in[\frac{1}{\gamma'}]} X_{ij}$, using (2.6.9) and our choice of β, we have that for any $k \in [t]$,

$$|X \cap f^{-1}(a_k)| \leqslant \sum_{i\in[t], j\in[\frac{1}{\gamma'}]} |X_{ij} \cap f^{-1}(a_k)| \leqslant \frac{6\beta t n}{\gamma'} \leqslant \frac{\alpha n}{3t}\,,$$

and similarly for $|X \cap f^{-1}(b_k)|, |X \cap f^{-1}(c_k)|$.

We observe that f satisfies property (*h1*). We have that (2.6.7) implies

$$\left|n_{a_i} - \frac{N_i}{3}\right| \leqslant \left|\frac{n_{a_1}}{3} - \frac{n_{a_1}}{3}\right| + \left|\frac{n_{a_1}}{3} - \frac{n_{b_1}}{3}\right| + \left|\frac{n_{a_1}}{3} - \frac{n_{c_1}}{3}\right| \leqslant \frac{2}{3}\,.$$

Thus $|n_{a_i} - \frac{N_i}{3}|, |n_{b_i} - \frac{N_i}{3}|, |n_{c_i} - \frac{N_i}{3}| \leqslant 1$. By (2.6.8) and $|A_i| + |B_i| + |C_i| = N_i$ we have that

$$\left||(g')^{-1}(a_i)| - \frac{N_i}{3}\right| \leqslant \left|\frac{A_i}{3} - \frac{B_i}{3}\right| + \left|\frac{A_i}{3} - \frac{C_i}{3}\right| \leqslant \frac{2\gamma' N_i}{3} .$$

Therefore

$$\left||(g')^{-1}(a_i)| - n_{a_i}\right| \leqslant \frac{2\gamma' N_i}{3} + 1 .$$

We have that

$$\left||(g')^{-1}(a_i)| - |f^{-1}(a_i)|\right| \leqslant \frac{6\beta tn}{\gamma'} .$$

Thus

$$\left||(f')^{-1}(a_i)| - n_{a_i}\right| \leqslant \frac{2\gamma' N_i}{3} + 1 + \frac{6\beta tn}{\gamma'} \leqslant \frac{3\gamma' n}{3t} . \qquad \square$$

2.6.3 Adjusting the partition of $V(G)$

The last step in the preparation of G and H is to match the sizes of classes in the partition of $V(G)$ to the sizes of the pre-images of the homomorphism of H into the ladder. For that we use the following lemma.

Proof of Lemma 2.6.4. We are given $\varepsilon, d_0 > 0$ and integer t. We are given a graph G on n vertices with a partition $V(G) = \bigcup_{i\in[t]}(V_{a_i} \dot\cup V_{b_i} \dot\cup V_{c_i})$ and a ladder L, such that properties (*g2*), (*g3*) and (*g4*) of Lemma 2.6.2 hold. Set $\gamma' \leqslant \frac{\varepsilon^2}{t}$ and we are given integers $m_{a_i}, m_{b_i}, m_{c_i}$ (for $i \in [t]$) such that $\sum_{i\in[t]}(m_{a_i} + m_{b_i} + m_{c_i}) = n$ and

$$|m_{a_i} - |V_{a_i}||, |m_{b_i} - |V_{b_i}||, |m_{c_i} - |V_{c_i}|| \leqslant \gamma' \frac{n}{3t} .$$

For

$$x_a = \sum_{i\in[t]} |V_{a_i}| - \sum_{i\in[t]} m_{a_i}, \quad x_b = \sum_{i\in[t]} |V_{b_i}| - \sum_{i\in[t]} m_{b_i}, \quad x_c = \sum_{i\in[t]} |V_{c_i}| - \sum_{i\in[t]} m_{c_i} ,$$

we have that $|x_a|, |x_b|, |x_c| \leqslant \gamma' \frac{n}{3} \leqslant \varepsilon^2 \frac{n}{3t}$. We get a new partition $V(G) = \bigcup_{i\in[t]}(V'_{a_i} \dot\cup V'_{b_i} \dot\cup V'_{c_i})$ in the following way. We have that $x_a + x_b + x_c = 0$, assume for example that $x_a > 0, x_b > 0, x_c < 0$, then use a_1a_2 and b_2b_3 to move vertices from V_{a_1} to V_{c_2} and from V_{b_2} to V_{c_3} and get

$$\sum_{i\in[t]} |V'_{a_i}| = \sum_{i\in[t]} m_{a_i}, \quad \sum_{i\in[t]} |V'_{b_i}| = \sum_{i\in[t]} m_{b_i}, \quad \sum_{i\in[t]} |V'_{c_i}| = \sum_{i\in[t]} m_{c_i} .$$

We had to move at most $\varepsilon^2 \frac{n}{3t}$ vertices and (V_{a_1}, V_{a_2}) being (ε, d)-regular ensures at least $\frac{2\varepsilon}{d_0}|V_{a_1}|$ vertices with at least $\frac{d_0}{2}|V_{a_2}|$ neighbours in V_{a_2} and for every $xy \in E(L^+)$ with $x \in \{a_1, b_2, c_2, c_3\}$ the new pair (V'_x, V'_y) is $(2\varepsilon, d)$-regular. Thus, the edges in $E(L^+)$ correspond to $(2\varepsilon, d_0)$-regular pairs and the edges $a_i b_i c_i$ correspond to $(2\varepsilon, d_0, \frac{d_0}{8})$-super-regular pairs.

We want to achieve $|V^*_{a_i}| = m_{a_i}$ and for that, set $x_1 = |V'_{a_1}| - m_{a_1}$. Next if $x_1 > 0$ move x_1 vertices from V'_{a_1} to V'_{a_2}; if $x_1 < 0$, move the x_1 vertices in the opposite direction. Set $V^*_{a_1}$ and V''_{a_2} as the sets V'_{a_1} and V'_{a_2} after moving x_1 vertices. Note that

$$|x_1| \leqslant \varepsilon^2 \frac{n}{3t} + \frac{\varepsilon^2 n}{3t^2} \leqslant 2\varepsilon^2 \frac{n}{3t}\,.$$

Let $i_0 \leqslant t-2$ and assume we have sets $V^*_{a_i}$ for $i \leqslant i_0$ and $V''_{a_{i_0+1}}$ such that $|V^*_{a_i}| = m_{a_i}$ and

$$||V''_{a_{i_0+1}}| - |V_{a_{i_0+1}}|| \leqslant \varepsilon^2 \frac{n}{3t} + i_0 \varepsilon^2 \frac{n}{3t^2} \leqslant 2\varepsilon^2 \frac{n}{3t}\,.$$

Set $x_{i_0+1} = |V''_{a_{i_0+1}}| - m_{a_{i_0+1}}$,

$$|x_{i_0+1}| \leqslant \varepsilon^2 \frac{n}{3t} + (i_0+1)\varepsilon^2 \frac{n}{3t^2} \leqslant 2\varepsilon^2 \frac{n}{3t}\,.$$

We move x_{i_0+1} vertices from $V''_{a_{i_0+1}}$ to $V'_{a_{i_0+2}}$, if $x_{i_0+1} > 0$ and the other way around, if $x_{i_0+1} < 0$. After moving x_{i_0+1} vertices, we have $V^*_{a_{i_0+1}}$ and $V''_{a_{i_0+2}}$. We set $V^*_{a_t} = V''_{a_t}$. Do a similar procedure to achieve $|V^*_{b_i}| = m_{b_i}$ and $|V^*_{c_i}| = m_{c_i}$, for $i \in [t]$.

Each class is involved in 2 movements and for $\alpha \in \{a, b, c\}$,

$$|V_{\alpha_i} \cap V^*_{\alpha_i}| \geqslant |V_{\alpha_i}| - 4\varepsilon^2 \frac{n}{3t} \quad \text{and}$$

$$(1 - \sqrt{d_0} - \gamma')\frac{n}{3t} \leqslant |V_{\alpha_i}| - \gamma' \frac{n}{3t} \quad \leqslant \quad |V^*_{\alpha_i}| \quad \leqslant \quad |V_{\alpha_i}| + \gamma' \frac{n}{3t} \leqslant (1 + \sqrt{d_0} + \gamma')\frac{n}{3t}\,.$$

For $\alpha \in \{a, b, c\}$ and $\beta \in \{a, b, c\} \setminus \{\alpha\}$, we have that the pairs $(V_{\alpha_i}, V_{\beta_{i+1}})$ are super regular, therefore, at least $|V'_{\alpha_i} \cap V''_{\alpha_i}| \geqslant |V_{\alpha_i}| - 2\varepsilon^2 \frac{n}{3t} \geqslant x_i$ vertices in V''_{α_i} have at least

$$\begin{aligned}
\frac{d_0}{8}|V_{\beta_{i+1}}| - |V_{\beta_{i+1}} \setminus V^*_{\beta_{i+1}}| &\geqslant \frac{d_0}{8}|V_{\beta_{i+1}}| - |V_{\beta_{i+1}}| + |V_{\beta_{i+1}} \cap V^*_{\beta_{i+1}}| \\
&\geqslant \frac{d_0}{8}(1-\sqrt{d_0})\frac{n}{3t} - 4\varepsilon^2\frac{n}{3t} \geqslant \frac{d_0}{17}\frac{n}{3t} \\
&\geqslant \frac{d_0}{17(1+\sqrt{d_0}+\gamma')}|V^*_{\beta_{i+1}}| \geqslant \frac{d_0}{34}|V^*_{\beta_{i+1}}|
\end{aligned}$$

neighbours in $V^*_{\beta_{i+1}}$. Therefore it is always possible to move vertices from V''_{α_i} into $V'_{\alpha_{i+1}}$ and keep the degree condition in the triangle $a_{i+1}b_{i+1}c_{i+1}$. Similarly, we can move vertices from $V'_{\alpha_{i+1}}$ into $V''_{\alpha_{i+1}}$ and keep the degree condition in the triangle $a_ib_ic_i$.

Moreover if $(V_{\alpha_i}, V_{\beta_i})$ is a (ε, d)-regular pair and $X \subseteq V^*_{\alpha_i}$, $Y \subseteq V^*_{\beta_i}$, then for $X' = X \cap V_{\alpha_i}$ and $Y' = Y \cap V_{\beta_i}$, we have

$$\begin{aligned}
|X'| &\geqslant |X| - |V^*_{\alpha_i} \setminus V_{\alpha_i}| \geqslant |X| - |V^*_{\alpha_i}| + |V_{\alpha_i} \cap V^*_{\alpha_i}| \\
&\geqslant |X| - |V^*_{\alpha_i}| + |V_{\alpha_i}| - 4\varepsilon^2\frac{n}{3t} \geqslant |X| - (\gamma' + 4\varepsilon^2)\frac{n}{3t}\,.
\end{aligned}$$

Thus,

$$e(X,Y) \leqslant d|X'||Y'| + \varepsilon|V_{\alpha_i}||V_{\beta_i}| + |X||Y \setminus Y'| + |Y||X \setminus X'| \leqslant d|X||Y| + 2\varepsilon|V^*_{\alpha_i}||V^*_{\beta_i}|$$

and

$$e(X,Y) \geqslant d|X'||Y'| - \varepsilon|V_{\alpha_i}||V_{\beta_i}| \geqslant d|X||Y| - 2\varepsilon|V^*_{\alpha_i}||V^*_{\beta_i}|\,.$$

Therefore all edges in $E(L)$ correspond to $(2\varepsilon, d_0)$-regular pairs and the edges in the triangles $a_ib_ic_i$ correspond to $(2\varepsilon, d_0, \frac{d_0}{34})$-super-regular pairs. □

2.6.4 Generalizing previous results

In this section, we show that the conditions needed for a graph to be good are satisfied by graphs with the minimum degree condition from the Bandwidth Theorem 1.1.2, thus Theorem 1.1.14 generalizes this previous result.

We believe that (ϱ, d)-dense, μ-inseparable graphs are also good and with a relaxation on the condition $\mathfrak{A}_G$ being complete, the bandwidth result for tripartite graphs shall also be generalized; this proof is not included.

Theorem 2.6.8. *For $\varepsilon > 0$, there exists n_0 such that if G is a graph on $n \geqslant n_0$ vertices with minimum degree $\delta(G) \geqslant \frac{2}{3}n + \varepsilon n$, then G is $(\mu, \delta, \zeta, \varrho, \xi, \eta, \nu)$-good with*

$$\mu = \frac{2\varepsilon}{3}, \quad \delta = \frac{\varepsilon}{9}, \quad \zeta = \frac{\varepsilon}{2}, \quad \varrho = \frac{\varepsilon^2}{16}, \quad \xi = \frac{1}{3}, \quad \eta = \frac{\varepsilon}{2} \quad \textit{and} \quad \nu = \frac{\varepsilon}{4}\,.$$

Proof. Let G with $\delta(G) \geqslant \frac{2}{3}n + \varepsilon n$ be given, set $U_v \subseteq N(v)$ with $|U_v| = \frac{2}{3}n + \varepsilon n$. First we show that $G[U_v]$ is $\frac{2\varepsilon}{3}$-inseparable. Indeed, for any $x \in V(G)$, we have

$$|U_x \cap U_v| \geqslant 2\left(\frac{2}{3} + \varepsilon\right)n - n \geqslant \left(\frac{1}{3} + 2\varepsilon\right)n\,. \tag{2.6.10}$$

Take $A \subseteq U_v$ and $B = U_v \setminus A$, assume $|A| \leqslant \frac{|U_v|}{2}$. Then for any $x \in A$

$$|N(x) \cap B| \geqslant \left(\frac{1}{3} + 2\varepsilon\right)n - |A| \geqslant \left(\frac{1}{3} + 2\varepsilon\right)n - \left(\frac{1}{3} + \frac{\varepsilon}{2}\right)n \geqslant \frac{3\varepsilon}{2}n.$$

We have that $e(A, B) \geqslant |A|\frac{3\varepsilon}{2}n \geqslant \frac{3\varepsilon}{2}|A||B|$.

The number of triangles in $G[U_v]$ is at least $3\varepsilon^3 n^3$. Indeed, take $x \in U_v$ and $y \in U_v \cap N(x)$. We have that

$$|N(y) \cap (U_v \cap N(x))| \geqslant \left(\frac{2}{3} + \varepsilon + \frac{1}{3} + 2\varepsilon - 1\right)n \geqslant 3\varepsilon n\,. \tag{2.6.11}$$

Then x is in at least $|U_v \cap N(x)|\frac{3\varepsilon}{2}n$ triangles in $N(v)$. The total number of triangles in U_v is at least

$$\left(\frac{2}{3} + \varepsilon\right)\left(\frac{1}{3} + 2\varepsilon\right)\frac{1}{2}\varepsilon n^3 \geqslant \frac{\varepsilon}{9}n^3\,.$$

Note that if $xv \in E(G)$, equation (2.6.10) gives us that xv is $(\frac{1}{3} + 2\varepsilon)$-connectable.

Note that (2.6.11) gives for every $v, x \in V(G)$, that

$$|E(U_v \cap U_x)| \geqslant \left(\frac{1}{3} + 2\varepsilon\right)\frac{3\varepsilon}{2}n^2 \geqslant \frac{\varepsilon}{2}n^2\,,$$

thus $A = A(\mathcal{R}, \frac{\varepsilon}{2})$ is complete and, in particular, $\frac{3\varepsilon}{2}$-inseparable.

Finally we show that G has a fractional triangle factor f with $W(f) = \frac{n}{3}$. Consider f a maximal fractional triangle factor of G and let A be the set of unsaturated vertices. We show that A is an independent set. Otherwise consider $x, y \in A$ such that $xy \in E(G)$. Let the total weight in x be $f(x) = 1 - \varepsilon$

and in y be $f(y) = 1 - \varepsilon'$. If there is a triangle uvw with $u, v \in N(x) \cap N(y)$ and $f(uvw) > 0$, then take $\alpha = \min\{f(uvw), \varepsilon, \varepsilon'\}$. Set a new weight function f' which is the same as f except

- $f'(uvw) = f(uvw) - \alpha/2$,
- $f'(xyu) = f(xyu) + \alpha/2$,
- $f'(xyv) = f(xyv) + \alpha/2$.

The weights $f'(u) = f(u)$, $f'(v) = f(v)$, $f'(x) = 1 - \varepsilon + \alpha$, $f'(y) = 1 - \varepsilon' + \alpha$ are at most 1 and $W(f') = W(f) + \alpha/2$, contradicting the maximality of f. Then all triangles with positive weight can have at most one vertex in $N(x) \cap N(y)$.

Note that all vertices in $N(x) \cap N(y)$ must be saturated. Otherwise if $z \in N(x) \cap N(y)$ has $f(z) = 1 - \varepsilon''$, take $\alpha = \min\{\varepsilon, \varepsilon', \varepsilon''\}$ and set $f'(xyz) = f(xyz) + \alpha$. Then $W(f) \geqslant |N(x) \cap N(y)| \geqslant (1/3 + 2\varepsilon)n$, which is a contradiction.

If A is independent and $x \in A$, we have that $N(x)$ contains only saturated vertices. If uvw is a triangle in $N(x)$, then $f(uvw) = 0$, otherwise take $\alpha = \min\{2\varepsilon/3, f(uvw)\}$ and set

- $f'(xuv) = f(xuv) + \alpha/2$,
- $f'(xuw) = f(xuw) + \alpha/2$,
- $f'(xvw) = f(xvw) + \alpha/2$,
- $f'(uvw) = f(uvw) - \alpha$.

The weights $f'(u) = f(u)$, $f'(v) = f(v)$, $f'(w) = f(w)$ and $f'(x) = 1 - \varepsilon + 3\alpha/2$ are at most 1 and $W(f') = W(f) + \alpha/2$, contradicting the maximality of f. Thus, adding the weights of the vertices in $N(x)$, we count the weight of a triangle at most twice and we have that $W(f) \geqslant |N(x)|/2 \geqslant (1/3 + \varepsilon/2)n$, which is a contradiction. This gives us that all vertices must be saturated.

If we are given $A \in V(G)$ with $|A| \leqslant \frac{\varepsilon}{4}n$ and $F \in E(G)$ with $|F| \leqslant \frac{\varepsilon^2}{16}n^2$, we get $|X_F| \leqslant \frac{\varepsilon}{4}n$ and if $v \in V(G_{A,F})$, then

$$N_{G_{A,F}}(v) = N_G(v) \setminus (A \cup X_F \cup \{w \colon vw \in F\}) \,.$$

Since $v \notin X_F$, we have that $|N_{G_{A,F}}(v)| \geqslant (\frac{2}{3} + \varepsilon - 3(\frac{\varepsilon}{4}))n$. By the argument above, we have that $G_{A,F}$ contains a fractional triangle factor $f_{G_{A,F}}$ such that $W(f_{G_{A,F}}) \geqslant \frac{n}{3} - \frac{\varepsilon n}{6} \geqslant \frac{n}{3} - \frac{\varepsilon}{2}(n - |A|)$. $\square$

3. The size-Ramsey number of powers of bounded degree trees

3.1 Auxiliary results

In this section we state a few results which will be needed in the proof of our main theorem. The first lemma guarantees that, in a graph G that has edges between large subsets of vertices, there exists a long "transversal" path along a constant number of large subsets of vertices of G. Denote by $e_G(X, Y)$ the number of edges between two disjoint sets X and Y in a graph G.

Lemma 3.1.1 ([21, Lemma 3.5])**.** *For every integer $\ell \geqslant 1$ and every $\gamma > 0$ there exists $d_0 = 2 + 4/(\gamma(\ell+1))$ such that the following holds for any $d \geqslant d_0$. Let G be a graph on dn vertices such that for every pair of disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geqslant \gamma n$ we have $e_G(X, Y) > 0$. Then for every family $V_1, \ldots, V_\ell \subseteq V(G)$ of pairwise disjoint sets each of size at least γdn, there is a path $P_n = (x_1, \ldots, x_n)$ in G with $x_i \in V_j$ for all $1 \leqslant i \leqslant n$, where $j \equiv i \pmod{\ell}$.*

We will also use the classical Chernoff's inequality and Kővári–Sós–Turán theorem.

Theorem 3.1.2 (Chernoff's inequality)**.** *Let $0 < \varepsilon \leqslant 3/2$. If X is a sum of independent Bernoulli random variables then*

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon\mathbb{E}[X]) \leqslant 2 \cdot e^{-(\varepsilon^2/3)\mathbb{E}[X]}.$$

Theorem 3.1.3 (Kővári–Sós–Turán [64])**.** *Let $k \geqslant 1$ and let G be a bipartite graph with x vertices in each vertex class. If G contains no copy of $K_{2k,2k}$, then G has at most $4x^{2-1/(2k)}$ edges.*

3.2 Bijumbledness, expansion and embedding of trees

In this section we provide the necessary tools to obtain the desired monochromatic embedding of a power of a tree in the proof of Theorem 1.2.1. We start by defining the expanding property of a graph.

Property 3.2.1 (Expanding)**.** *A graph G is (n, a, b)-*expanding *if for all $X \subseteq V(G)$ with $|X| \leqslant a(n-1)$, we have $|N_G(X)| \geqslant b|X|$.*

Here $N_G(X)$ is the set of neighbours of X, i.e. all vertices in $V(G)$ that share an edge with some vertex from X. The following embedding result due to Friedman and Pippenger [41] guarantees the existence of copies of bounded degree trees in expanding graphs.

Lemma 3.2.2. *Let n and Δ be positive integers and G a non-empty graph. If G is $(n, 2, \Delta+1)$-expanding, then G contains any n-vertex tree with maximum degree Δ as a subgraph.*

Owing to Lemma 3.2.2, we are interested in graph properties that guarantee expansion. One such property is bijumbledness, defined below.

Property 3.2.3 (Bijumbledness)**.** *A graph G on N vertices is (p, ϑ)-*bijumbled *if, for all disjoint sets X and $Y \subseteq V(G)$ with $\vartheta/p < |X| \leqslant |Y| \leqslant pN|X|$, we have $\big|e_G(X,Y) - p|X||Y|\big| \leqslant \vartheta\sqrt{|X||Y|}$.*

We remark that, in the definition above, we restrict our sets X and Y not to be too small; such a restriction is not usually imposed when defining bijumbledness, but we have to do so here for certain technical reasons.

Note that bijumbledness immediately implies that

$$\text{for all disjoint sets } X,\, Y \subseteq V(G) \text{ with } |X|,\, |Y| > \vartheta/p \text{ we have } e_G(X,Y) > 0. \tag{3.2.1}$$

Moreover, a simple averaging argument guarantees that in a (p, ϑ)-bijumbled graph G on N vertices we have

$$\left| e(G) - p\binom{N}{2} \right| \leqslant \vartheta N. \tag{3.2.2}$$

We now state the first main novel ingredient in the proof of our main result (Theorem 1.2.1). The following lemma ensures that in a sufficiently large graph we get an expanding subgraph with appropriate parameters or we get reasonably large disjoint subsets of vertices that span no edges between them. This result was inspired by [76, Theorem 1.5]. Furthermore, we remark that similar results have been proved in [77, 78].

Lemma 3.2.4. *Let $f \geqslant 0$, $D \geqslant 0$, $\ell \geqslant 2$ and $\eta > 0$ be given and let $A = (\ell - 1)(D + 1)(\eta + f) + \eta$.*

If G is a graph on at least An vertices, then

1. *there is a non-empty set $Z \subseteq V(G)$ such that $G[Z]$ is (n, f, D)-expanding, or*
2. *there exist $V_1, \dots, V_\ell \subseteq V(G)$ such that $|V_i| \geqslant \eta n$ for $1 \leqslant i \leqslant \ell$ and $e_G(V_i, V_j) = 0$ for $1 \leqslant i < j \leqslant \ell$.*

Proof. Let us assume that 1 does not hold. Since G is not (n, f, D)-expanding, we can take $V_1 \subseteq V(G)$ of maximum size satisfying that $|V_1| \leqslant (\eta + f)n$ and $|N_G(V_1)| < D|V_1|$. We claim that $|V_1| \geqslant \eta n$. Assume, for the sake of contradiction that $|V_1| < \eta n$. Let

$$W_1 = V(G) \setminus (V_1 \cup N_G(V_1)).$$

Then $|W_1| > An - (D + 1)\eta n > 0$. Applying that 1 does not hold, we get $X \subseteq W_1$ such that $|X| \leqslant f(n-1)$ and $|N_{G[W_1]}(X)| < D|X|$. Note that $N_G(X) \subseteq N_{G[W_1]}(X) \cup N_G(V_1)$. Thus

$$\begin{aligned} |N_G(X \dot\cup V_1)| &= |N_{G[W_1]}(X) \cup N_G(V_1)| \\ &< D(|X| + |V_1|). \end{aligned}$$

Also $|X \dot\cup V_1| \leqslant (\eta + f)n$, deriving a contradiction to the maximality of V_1.

Let $1 \leqslant k \leqslant \ell - 2$ and suppose we have $(V_1, \dots, V_k)$ such that

1. $|V_i| \geqslant \eta n$, for $1 \leqslant i \leqslant k$;
2. $e(V_i, V_j) = 0$, for $1 \leqslant i < j \leqslant k$;
3. $|\bigcup_{i=1}^{k} (V_i \cup N_G(V_i))| < k(D + 1)(\eta + f)n$.

We can increase this sequence in the following way. Let $W_k = V(G) \smallsetminus \bigcup_{i=1}^{k}(V_i \cup N_G(V_i))$ and note that

$$\begin{aligned}|W_k| &\overset{3}{\geqslant} An - (\ell-2)(D+1)(\eta+f)n \\ &\geqslant (D+1)(\eta+f)n + \eta n \\ &> 0.\end{aligned}$$

Since 1 does not hold, there exists $V_{k+1} \subseteq W_k$ of maximum size with $|V_{k+1}| \leqslant (\eta+f)n$ such that $|N_{G[W_k]}(V_{k+1})| < D|V_{k+1}|$. Note that $e_G(V_i, V_{k+1}) \leqslant e_G(V_i, W_{k+1}) = 0$, for every $1 \leqslant i \leqslant k$. Therefore we have that 2 holds for the sequence $(V_1, \ldots, V_{k+1})$. Furthermore, note that

$$N_G(V_{k+1}) \subseteq \bigcup_{i=1}^{k} N_G(V_i) \cup N_{G[W_k]}(V_{k+1})\,. \tag{3.2.3}$$

This gives us 3 for the sequence $(V_1, \ldots, V_{k+1})$, since

$$\begin{aligned}\left|\bigcup_{i=1}^{k+1}(V_i \cup N_G(V_i))\right| &\overset{(3.2.3)}{=} \left|\bigcup_{i=1}^{k}(V_i \cup N_G(V_i)) \cup V_{k+1} \cup N_{G[W_k]}(V_{k+1})\right| \\ &< (k+1)(D+1)(\eta+f)n.\end{aligned}$$

To see that $(V_1, \ldots, V_{k+1})$ satisfies 1, define

$$W_{k+1} = V(G) \smallsetminus \bigcup_{i=1}^{k+1}(V_i \cup N_G(V_i)) \overset{(3.2.3)}{=} W_k \smallsetminus (V_{k+1} \cup N_{G[W_k]}(V_{k+1})).$$

Assume that $|V_{k+1}| < \eta n$ and derive a contradiction as before.

Therefore, when $k = \ell - 2$, we generate a sequence $(V_1, \ldots, V_{\ell-1})$ with the properties required by 2. To complete the sequence, note that 3 gives that $|W_{\ell-1}| \geqslant \eta n$ and set $V_\ell = W_{\ell-1}$. $\square$

As a corollary of the previous lemma, we get the following lemma that says that sufficiently large bijumbled graphs contain a non-empty expanding subgraph.

Lemma 3.2.5 (Bijumbledness implies expansion)**.** *Let f, ϑ, D and $c \geqslant 1$ be positive numbers with $c \geqslant 4(D+2)\vartheta$ and $a \geqslant 2(D+1)f$. If G is a $(c/(an), \vartheta)$-bijumbled graph with an vertices, then there exists a non-empty subgraph H of G that is*

(n, f, D)*-expanding.*

Proof. Let $p = c/(an)$ and let G be a (p, ϑ)-bijumbled graph. Suppose for a contradiction that no subgraph of G is (n, f, D)-expanding. We apply Lemma 3.2.4 with $\ell = 2$ and $\eta = 2\vartheta a/c$. Note that, since $a \geqslant 2(D+1)f$ and $c \geqslant 4(D+2)\vartheta$, from the choice of η we have

$$\begin{aligned} a &\geqslant (D+1)f + \frac{a}{2} \geqslant (D+1)f + \frac{2(D+2)\vartheta a}{c} \\ &\geqslant (D+1)f + (D+2)\eta = (D+1)(f+\eta) + \eta. \end{aligned}$$

Then, we get two disjoint sets $V_1, V_2 \subseteq V(G)$ with $|V_1| = |V_2| = \eta n > \vartheta/p$ such that $e_G(V_1, V_2) = 0$. On the other hand, by (3.2.1), we have $e_G(V_1, V_2) > 0$, a contradiction. Therefore, there is some subgraph of G that is (n, f, D)-expanding. $\square$

The next lemma is crucial for embedding the desired power of a tree. Let G be a graph and $\ell \geqslant r$ be positive integers. An (ℓ, r)*-blow-up* of G is a graph obtained from G by replacing each vertex of G by a clique of size ℓ and for every edge of G arbitrarily adding a complete bipartite graph $K_{r,r}$ between the cliques corresponding to the vertices of this edge.

Lemma 3.2.6 (Embedding lemma for powers of trees)**.** *Given positive integers k and Δ, there exists r_0 such that the following holds for every n-vertex tree T with maximum degree Δ. There is a tree $T' = T'(T, k)$ on at most $n+1$ vertices and with maximum degree at most Δ^{2k} such that for every graph J with $T' \subseteq J$ and any (ℓ, r)-blow-up J' of J with $\ell \geqslant r \geqslant r_0$ we have $T^k \subseteq J'$.*

Proof. Given positive integers k, Δ, take $r_0 = \Delta^{4k}$. Let T be an n-vertex tree with maximum degree Δ. Let x_0 be any vertex in $V(T)$ and consider T as rooted at x_0. For each vertex $v \in V(T)$, let $D(v)$ denote the set of *descendants* of v in T (including v itself). Let $D^i(v)$ be the set of vertices $u \in D(v)$ at distance at most i from v in T.

Let T' be a tree with vertex set consisting of a special vertex x^* and the vertices $x \in V(T)$ such that the distance between x and x_0 is a multiple of $2k$. The edge set of T' consists of the edge x^*x_0 and the pairs of vertices $x, y \in V(T') \setminus \{x^*\}$ for which $x \in D^{2k}(y)$ or $y \in D^{2k}(x)$. That is,

$$V(T') = \{x \in V(T) \colon \mathrm{dist}_T(x_0, x) \equiv 0 \pmod{2k}\} \cup \{x^*\}$$
$$E(T') = \left\{xy \in \binom{V(T') \smallsetminus \{x^*\}}{2} \colon x \in D^{2k}(y) \text{ or } y \in D^{2k}(x)\right\} \cup \{x^* x_0\}.$$

In particular, note that $\Delta(T') \leqslant \Delta^{2k}$ and $|V(T')| \leqslant n+1$. Let us consider T' as a tree rooted at x^*.

Now suppose that J is a graph such that $T' \subseteq J$ and J' is an (ℓ, r)-blow-up of J with $\ell \geqslant r \geqslant r_0$. Our goal is to show that $T^k \subseteq J'$. First, since J' is an (ℓ, r)-blow-up of J, there is a collection $\{K(x) : x \in V(J)\}$ of disjoint ℓ-cliques in J' such that for each edge $xy \in E(J)$, there is a copy of $K_{r,r}$ between the vertices of $K(x)$ and $K(y)$. Let us denote by $K(x, y)$ such copy of $K_{r,r}$.

For each $x \in V(T') \smallsetminus \{x^*\}$, let $D^+(x) = D^{k-1}(x)$ and $D^-(x) = D^{2k-1}(x) \smallsetminus D^{k-1}(x)$. In order to fix the notation, it helps to think in $D^+(x)$ and $D^-(x)$ as the *upper* and *lower half of close descendants* of x, respectively. We denote by x^+ the parent of x in T'. Suppose that there exists an injective map $\varphi : V(T) \to V(J')$ such that for every $x \in V(T') \smallsetminus \{x^*\}$, we have

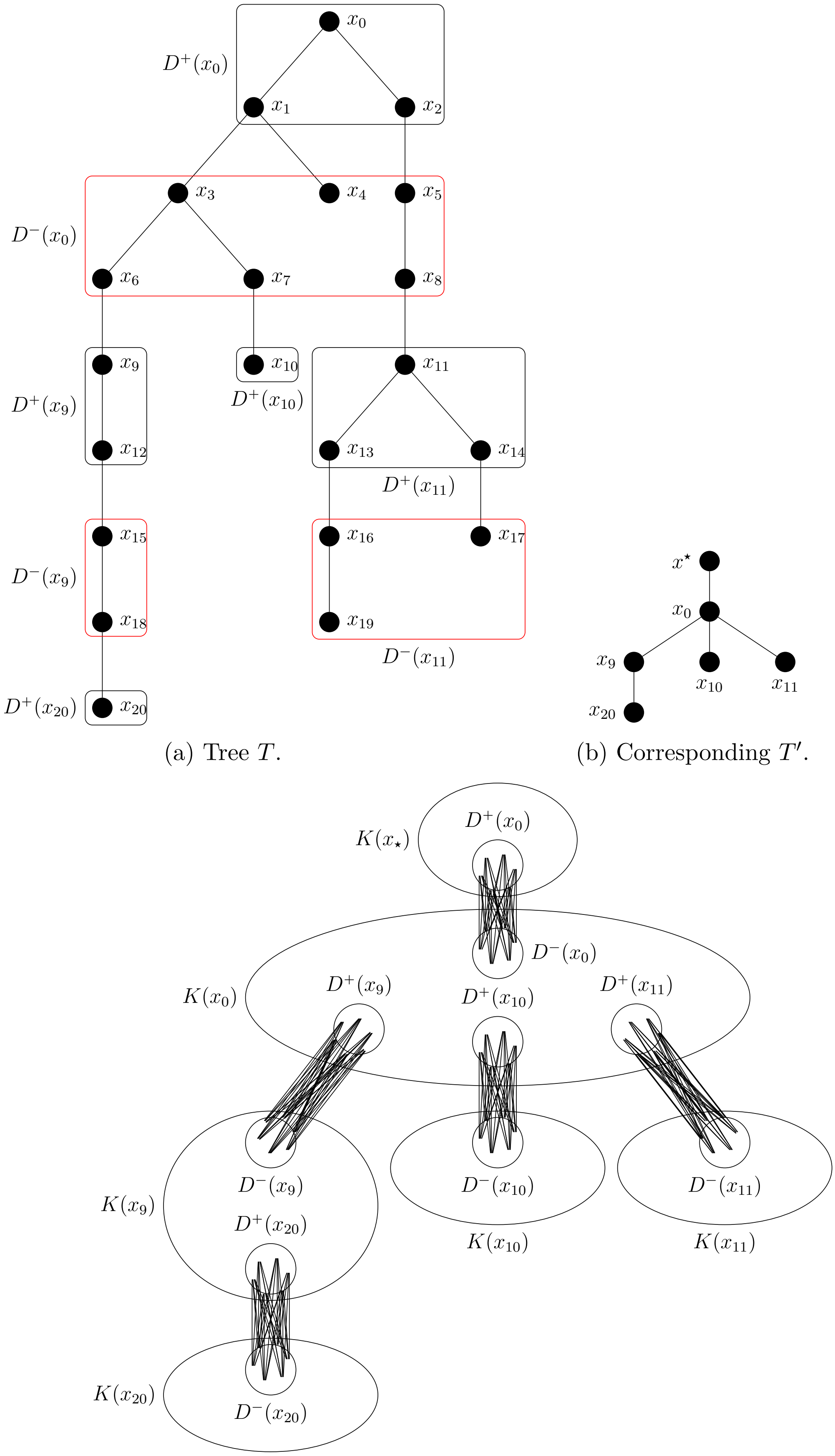
1. $\varphi(D^+(x)) \subseteq K(x, x^+) \cap K(x^+)$;

2. $\varphi(D^-(x)) \subseteq K(x, x^+) \cap K(x)$.

Then we claim that such map is in fact an embedding of T^k into J'. Figure 3.2.1 should help to visualize the concepts developed so far.

Claim 3.2.7. *If $\varphi : V(T) \to V(J')$ is an injective map such that for all $x \in V(T') \smallsetminus \{x^*\}$ the properties* (1) *and* (2) *hold, then φ is an embedding of T^k into J'.*

Proof. We want to show that if u and v are distinct vertices in T at distance at most k, then $\varphi(u)\varphi(v)$ is an edge in J'. Let $\tilde{u}$ and $\tilde{v}$ be vertices in $V(T') \smallsetminus \{x^*\}$ with $u \in D^{2k-1}(\tilde{u})$ and $v \in D^{2k-1}(\tilde{v})$. If $\tilde{u} = \tilde{v}$, then by properties (1) and (2), we have $\varphi(u)$ and $\varphi(v)$ adjacent in J', once all the vertices in $\varphi(D^{2k-1}(\tilde{u}))$ are adjacent in J' either by edges from $K(\tilde{u})$, $K(\tilde{u}^+)$ or $K(\tilde{u}, \tilde{u}^+)$. If $\tilde{u} = \tilde{v}^+$, then we must have $u \in D^-(\tilde{u})$ and $v \in D^+(\tilde{v})$ and properties (1) and (2) give us $\varphi(u), \varphi(v) \in K(\tilde{u})$. Analogously, if $\tilde{v} = \tilde{u}^+$, then $v \in D^-(\tilde{v})$ and $u \in D^+(\tilde{u})$ and properties (1) and (2)



(a) Tree T.

(b) Corresponding T'.

(c) Embedding T^k into an (ℓ, r)-blow-up of T'.

Figure 3.2.1: Illustration of the concepts and notation used throughout the proof of Lemma 3.2.6 when $\Delta = 3$ and $k = 2$.

imply that $\varphi(u), \varphi(v) \in K(\tilde{v})$. If $\tilde{u}^+ = \tilde{v}^+$ (with $\tilde{u} \neq \tilde{v}$), then we have $u \in D^+(\tilde{u})$ and $v \in D^+(\tilde{v})$ and property (1) give us $\varphi(u), \varphi(v) \in K(\tilde{u}^+)$.

Therefore we may assume that $\tilde{u}$ and $\tilde{v}$ are at distance at least 2 in T' and do not share a parent. But this implies that

$$\min\{\mathrm{dist}_T(x, y) : x \in D^{2k-1}(\tilde{u}), y \in D^{2k-1}(\tilde{v})\} \geqslant 2k + 1,$$

contradicting the fact that u and v are at distance at most k in T. □

We conclude the proof by showing that such a map exists.

Claim 3.2.8. *There is an injective map $\varphi : V(T) \to V(J')$ for which* (1) *and* (2) *hold for every $x \in V(T') \smallsetminus \{x^*\}$.*

Proof. We just need to show that for every $x \in V(T')$, there is enough room in $K(x)$ and in $K(x, x^+)$ to guarantee that (1) and (2) hold. In order to do so, $K(x)$ should be large enough to accommodate the set

$$D^-(x) \cup \bigcup_{\substack{y \in V(T') \\ y^+ = x}} D^+(y). \tag{3.2.4}$$

Since T' has maximum degree at most Δ^{2k} and T has maximum degree Δ, we have that the set in (3.2.4) has at most Δ^{4k} vertices. Since $|K(x)| = \ell \geqslant r_0 = \Delta^{4k}$, the set $K(x)$ is indeed large enough to accommodate the set in (3.2.4). Finally, since $|K(x, x^+) \cap K(x)| = |K(x, x^+) \cap K(x)| = r \geqslant r_0 = \Delta^{4k}$ the set $K(x, x^+)$ is also large enough to accommodate $D^-(x)$ or $D^+(x)$ as in properties (1) and (2). □

□

We end this section discussing a graph property that needs to be inherited by some subgraphs when running the induction in the proof of Theorem 1.2.1.

Definition 3.2.9. For positive numbers n, a, b, c, ℓ and ϑ, let $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ denote the class of all graphs G with the following properties, where $p = c/(an)$.

(*i*) $|V(G)| = an$,

(*ii*) $\Delta(G) \leqslant b$,

(*iii*) G has no cycles of length at most 2ℓ,

(*iv*) G is (p, ϑ)-bijumbled.

Only mild conditions on a, b, c, ℓ and ϑ are necessary to guarantee the existence of a graph in $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ for sufficiently large n. These conditions can be seen in (*i*)–(*iii*) in Definition 3.2.10 below. In order to keep the induction going in our main proof we also need a condition relating k and Δ, which represents, respectively, the power of the tree T we want to embed and the maximum degree of T (see (*iv*) in the next definition).

Definition 3.2.10. A 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ is *good* if

(*i*) $a \geqslant 3$,

(*ii*) $c \geqslant \vartheta\ell$,

(*iii*) $b \geqslant 9c$,

(*iv*) $\ell \geqslant 21\Delta^{2k}$.

Next we prove that conditions (*i*)–(*iii*) in Definition 3.2.10 together with $\vartheta \geqslant 32\sqrt{c}$ are enough to guarantee that there are graphs in $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ as long as n is large enough. We remark that next lemma is stated for a good 7-tuple, but condition (*iv*) of Definition 3.2.10 is not necessary and, therefore, also Δ and k are irrelevant.

Lemma 3.2.11. *If $(a, b, c, \ell, \vartheta, \Delta, k)$ is a good 7-tuple with $\vartheta \geqslant 32\sqrt{c}$, then for sufficiently large n the family $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ is non-empty.*

Proof. Let $(a, b, c, \ell, \vartheta, \Delta, k)$ be a good 7-tuple with $\vartheta \geqslant 32\sqrt{c}$ and let n be sufficiently large. Put $N = an$ and let $G^* = G(3N, p)$ be the binomial random graph with $3N$ vertices and edge probability $p = c/N$. From Chernoff's inequality (Theorem 3.1.2) we know that almost surely

$$e(G^*) \leqslant 2p\binom{3N}{2} \leqslant 9cN. \tag{3.2.5}$$

From [49, Lemma 8], we know that almost surely G^* is $(p, e^2\sqrt{6p(3N)})$-bijumbled, i.e. the following holds almost surely: for all disjoint sets X and $Y \subseteq V(G^*)$ with

$e^2\sqrt{18N}/\sqrt{p} < |X| \leqslant |Y| \leqslant p(3N)|X|$, we have

$$\big|e_{G^*}(X,Y) - p|X||Y|\big| \leqslant (e^2\sqrt{6})\sqrt{p(3N)|X||Y|}. \tag{3.2.6}$$

The expected number of cycles of length at most 2ℓ in G^* is given by $\mathbb{E}(C_{\leqslant 2\ell}) = \sum_{i=3}^{2\ell} \mathbb{E}(C_i)$, where C_i is the number of cycles of length i. Then,

$$\mathbb{E}(C_{\leqslant 2\ell}) = \sum_{i=3}^{2\ell} \binom{3an}{i} \frac{(i-1)!}{2} p^i \leqslant \sum_{i=3}^{2\ell} (3c)^i \leqslant 2\ell(3c)^{2\ell}.$$

Then, from Markov's inequality, we have

$$\mathbb{P}\big(C_{\leqslant 2\ell} \geqslant 4\ell(3c)^{2\ell}\big) \leqslant \frac{1}{2}. \tag{3.2.7}$$

Since (3.2.5) and (3.2.6) hold almost surely and the probability in (3.2.7) is at most 1/2, for sufficiently large n there exists a $(p, e^2\sqrt{18c})$-bijumbled graph G' with $3N$ vertices that contains less than $4\ell(3c)^{2\ell}$ cycles of length at most 2ℓ and $e(G') \leqslant 2p\binom{3N}{2} \leqslant 9cN$. Then, by removing $4\ell(3c)^{2\ell}$ vertices we obtain a graph G'' with no such cycles such that

$$|V(G'')| = 3an - 4\ell(3c)^{2\ell} \geqslant 2an \quad \text{and} \quad e(G'') \leqslant 9cN.$$

To obtain the desired graph G in $\mathcal{P}_n(a,b,c,\ell,\vartheta)$, we repeatedly remove vertices of highest degree in G'' until N vertices are left, obtaining a subgraph $G \subseteq G''$ such that $\Delta(G) \leqslant 9c \leqslant b$, as otherwise we would have deleted more than $e(G'')$ edges. Note that deleting vertices preserves the bijumbledness. Therefore, for all disjoint sets X and $Y \subseteq V(G)$ with $e^2\sqrt{18N}/\sqrt{p} < |X| \leqslant |Y| \leqslant p(3N)|X|$ we have

$$\big|e_G(X,Y) - p|X||Y|\big| \leqslant (e^2\sqrt{6})\sqrt{p(3N)|X||Y|} \leqslant (32\sqrt{pN})\sqrt{|X||Y|} \leqslant \vartheta\sqrt{|X||Y|}. \tag{3.2.8}$$

We obtained a graph G on N vertices and maximum degree $\Delta(G) \leqslant b$ such that G contains no cycles of length at most 2ℓ and is (p,ϑ)-bijumbled, for $p = c/N$. Therefore, the proof of the lemma is complete. □

3.3 Proof of the main result

We derive Theorem 1.2.1 from Proposition 3.3.1 below. Before continuing, given an integer $\ell \geqslant 1$, let us define what we mean by a *sheared complete blow-up* $H\{\ell\}$ of a graph H: this is any graph obtained by replacing each vertex v in $V(H)$ by a complete graph $C(v)$ with ℓ vertices, and by adding all edges *but a perfect matching* between $C(u)$ and $C(v)$, for each $uv \in E(H)$. We also define the *complete blow-up* $H(\ell)$ of a graph H analogously, but by adding all the edges between $C(u)$ and $C(v)$, for each $uv \in E(H)$.

Proposition 3.3.1. *For all integers $k \geqslant 1$, $\Delta \geqslant 2$, and $s \geqslant 1$ there exists r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \vartheta_s, \Delta, k)$ with $\vartheta_s \geqslant 32\sqrt{c_s}$ for which the following holds. If n is sufficiently large and $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \vartheta_s)$ then, for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, we have*

$$G^{r_s}\{\ell_s\} \to (T^k)_s.$$

Theorem 1.2.1 follows from Proposition 3.3.1 applied to a certain subgraph of a random graph.

Proof of Theorem 1.2.1. Fix positive integers k, Δ and s and let T be an n-vertex tree with maximum degree Δ. Proposition 3.3.1 applied with parameters k, Δ and s gives r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \vartheta_s, \Delta, k)$ with $\vartheta_s \geqslant 32\sqrt{c_s}$.

Let n be sufficiently large. By Lemma 3.2.11, since $\vartheta_s \geqslant 32\sqrt{c_s}$, there exists a graph $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \vartheta_s)$. Let χ be an arbitrary s-colouring of $E(G^{r_s}\{\ell_s\})$. Then, Proposition 3.3.1 gives that $G^{r_s}\{\ell_s\} \to (T^k)_s$. Since $|V(G)| = a_s n$, the maximum degree of G is bounded by the constant b_s, and since r_s and ℓ_s are constants, we have $e(G^{r_s}\{\ell_s\}) = O_{k,\Delta,s}(n)$, which concludes the proof of Theorem 1.2.1. $\square$

The proof of Proposition 3.3.1 follows by induction in the number of colours. Before we give this proof, let us state the results for the base case and the induction step.

Lemma 3.3.2 (Base Case)**.** *For all integers $h \geqslant 1$, $k \geqslant 1$ and $\Delta \geqslant 2$ there is an integer r and a good 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ with $\vartheta \geqslant 2^{h-1}32\sqrt{c}$ such that if n is sufficiently large, then the following holds for any $G \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$. For any n-vertex tree T with $\Delta(T) \leqslant \Delta$, the graph $G^r\{\ell\}$ contains a copy of T^k.*

Lemma 3.3.3 (Induction Step)**.** *For any positive integers $\Delta \geqslant 2$, $s \geqslant 2$, k, r, $h \geqslant 1$ and any good 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ with $\vartheta \geqslant 2^h 32\sqrt{c}$, there is a positive integer r' and a good 7-tuple $(a', b', c', \ell', \vartheta', \Delta, k)$ with $\vartheta' \geqslant 2^{h-1}32\sqrt{c'}$ such that the following holds. If n is sufficiently large then for any graph $G \in \mathcal{P}_n(a', b', c', \ell', \vartheta')$ and any s-colouring χ of $E(G^{r'}\{\ell'\})$*

(i) *there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$ for any n-vertex tree T with $\Delta(T) \leqslant \Delta$, or*

(ii) *there is $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.*

Now we are ready to prove Proposition 3.3.1.

Proof of Proposition 3.3.1. Fix integers $k \geqslant 1$, $\Delta \geqslant 2$ and $s \geqslant 1$ and define $h_i = s-i$ for $1 \leqslant i \leqslant s$. Let r_1 and a good 7-tuple $(a_1, b_1, c_1, \ell_1, \vartheta_1, \Delta, k)$ with $\vartheta_1 \geqslant 2^{h_1}32\sqrt{c_1}$ be given by Lemma 3.3.2 applied with s, k and Δ.

We will prove the proposition by induction on the number of colours $i \in \{1, \dots, s\}$ with the additional property that if the colouring has i colours, then $\vartheta_i \geqslant 2^{h_i}32\sqrt{c_i}$.

Lemma 3.3.2 implies that for sufficiently large n, if $G \in \mathcal{P}_n(a_1, b_1, c_1, \ell_1, \vartheta_1)$, then $G^{r_1}\{\ell_1\} \to (T^k)_1$. Therefore, since $\vartheta_1 \geqslant 2^{h_1}32\sqrt{c_1}$, if $i = 1$, we are done.

Assume $2 \leqslant i \leqslant s$ and suppose the statement holds for $i-1$ colours with the additional property that $\vartheta_{i-1} \geqslant 2^{h_{i-1}}32\sqrt{c_{i-1}}$, where r_{i-1} and a good 7-tuple $(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \vartheta_{i-1}, \Delta, k)$ are given by the induction hypothesis. Therefore, for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, we know that for a sufficiently large n

$$H^{r_{i-1}}\{\ell_{i-1}\} \to (T^k)_{i-1} \quad \text{for any} \quad H \in \mathcal{P}_n(a_{i-1}, b_{s-1}, c_{i-1}, \ell_{i-1}, \vartheta_{i-1}). \tag{3.3.1}$$

Note that since $i \leqslant s$, we have $h_{i-1} = s-(i-1) \geqslant 1$. Then, since $\vartheta_{i-1} \geqslant 2^{h_{i-1}}32\sqrt{c_{i-1}}$, we can apply Lemma 3.3.3 with parameters

$$\Delta, s, k, r_{i-1}, h_{i-1} \quad \text{and} \quad (a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \vartheta_{i-1}, \Delta, k)\,,$$

obtaining

$$r_i \quad \text{and} \quad (a_i, b_i, c_i, \ell_i, \vartheta_i, \Delta, k)\,,$$

with $\vartheta_i \geqslant 2^{h_i}32\sqrt{c_i}$.

Let $G \in \mathcal{P}_n(a_i, b_i, c_i, \ell_i, \vartheta_i)$ and let n be sufficiently large. Now let χ be an arbitrary i-colouring of $E(G^{r_i}\{\ell_i\})$. From Lemma 3.3.3, we conclude that either (i) there is a monochromatic copy of T^k in $G^{r_i}\{\ell_i\}$ for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, in which case the proof is finished, or (ii) there exists a graph $H \in \mathcal{P}_n(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \vartheta_{i-1})$ such that $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$ and $H^{r_{i-1}}\{\ell_{i-1}\}$ is coloured with at most $i-1$ colours under χ. In case (ii), the induction hypothesis (3.3.1) implies that we find the desired monochromatic copy of T^k in $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$. $\square$

Lemma 3.3.2 follows by proving that for a good 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ with $\vartheta \geqslant 2^{h-1}32\sqrt{c}$, large graphs G in $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ are expanding (using Lemma 3.2.5). Then, we use Lemma 3.2.2 to conclude that G contains the desired tree T. After this step we greedily find an embedding of T^k in $G\{\ell\}^k$.

Proof of the base case (Lemma 3.3.2). Let $h \geqslant 1$, $k \geqslant 1$ and $\Delta \geqslant 2$ be integers. Let

$$r = k, \quad \ell = 21\Delta^{2k}, \quad \vartheta = 4^h 256\ell, \quad c = \vartheta\ell, \quad b = 9c$$

and put $D = \Delta + 1$. Note that $\vartheta \geqslant 2^{h-1}32\sqrt{c}$ and let

$$a \geqslant 4(D+1).$$

Since $\ell \geqslant 4(\Delta + 3)$, we have $c \geqslant 4(D+2)\vartheta$. From the lower bounds on c and a we know that we can use the conclusion of Lemma 3.2.5 applying it with $f = 2$, ϑ, $D = \Delta + 1$ and c.

Note that from our choice of constants, $(a, b, c, \ell, \vartheta, \Delta, k)$ is a good tuple. Let n be sufficiently large and let T be a tree on n vertices with $\Delta(T) \leqslant \Delta$. Let $G \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$. From Lemma 3.2.5 we know that G has an $(n, 2, \Delta + 1)$-expanding subgraph and, therefore, from Lemma 3.2.2 we conclude that G contains a copy of T. Clearly, the graph G^k contains a copy of T^k. It remains to prove that the graph $G^k\{\ell\}$ also contains a copy of T^k.

Let $\{v_1, \ldots, v_n\}$ be the vertices of T_n and denote by T_j the subgraph of T induced by $\{v_1, \ldots, v_j\}$. Given a vertex $v \in V(G)$, let $C(v)$ denote the ℓ-clique in $G^k\{\ell\}$ that corresponds to v. Suppose that for some $1 \leqslant j < k$ we have embedded T_j^k in $G^k\{\ell\}$ where, for each $1 \leqslant i \leqslant j$, the vertex v_i was mapped to some $w_i \in C(v_i)$.

By the definition of $G^k\{\ell\}$, every neighbour v of v_{j+1} in G^k is adjacent to all but one vertex of $C(v_{j+1})$. Therefore, since $\Delta(T^k) \leqslant \Delta^k$ and $|C(v_{j+1})| = \ell \geqslant \Delta^k + 1$, we may thus find a vertex $w_{j+1} \in C(v_{j+1})$ such that w_{j+1} is adjacent in $G^k\{\ell\}$ to every w_i with $1 \leqslant i \leqslant j$ such that $v_i v_{j+1} \in E(T^k_{j+1})$. From that we obtain a copy of T^k_{j+1} in $G^k\{\ell\}$ where $w_i \in C(v_i)$ for $1 \leqslant i \leqslant j+1$. Therefore, starting with any vertex w_1 in $C(v_1)$, we may obtain a copy of T^k in $G^k\{\ell\}$ inductively, which proves the lemma. $\square$

The core of the proof of Theorem 1.2.1 is the induction step (Lemma 3.3.3). We start by presenting a sketch of its proof.

Sketch of the induction step (Lemma 3.3.3). We start by fixing suitable constants r', a', b', c', ℓ' and ϑ'. Let n be sufficiently large and let $G \in \mathcal{P}_n(a', b', c', \ell', \vartheta')$ be given. Consider an arbitrary colouring χ of the edges of a sheared complete blow-up $G^{r'}\{\ell'\}$ of $G^{r'}$ with s colours. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

First, note that, by Ramsey's theorem, if ℓ' is large then each ℓ'-clique $C(v)$ of $G^{r'}\{\ell'\}$ contains a large monochromatic clique. Let us say that blue is the most common colour of these monochromatic cliques. Let these blue cliques be $C'(v) \subseteq C(v)$. Then we consider a graph $J \subseteq G^{r'}$ induced by the vertices v corresponding to the blue cliques $C'(v)$ and having only the edges $\{u, v\}$ such that there is a blue copy of a large complete bipartite graph under χ in the bipartite graph induced between the blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$.

Then, by Lemma 3.2.4 applied to J, either there is a set $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is expanding, or there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. In the first case, Lemma 3.2.6 guarantees that there is a tree T' such that, if $T' \subseteq J[Z]$, then there is a blue copy of T^k in $G^{r'}\{\ell'\}$. To prove that $T' \subseteq J[Z]$, we recall that $J[Z]$ is expanding and use Lemma 3.2.2. This finishes the proof of the first case.

Now let us consider the second case, in which there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. The idea is to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and, moreover, $H^r\{\ell\}$ does not have any blue edge. For that we first obtain a path Q in G with vertices $(x_1, \ldots, x_{2a\ell n})$ such that $x_i \in V_j$ for all i where $i \equiv j \bmod \ell$. Then we partition Q into $2an$ paths $Q_1, \ldots, Q_{2an}$ with ℓ vertices each, and consider an auxiliary graph H' on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ with $Q_iQ_j \in E(H')$ if and only $E_G(V(Q_i), V(Q_j)) \neq \varnothing$. To ensure that H' inherits properties from G we use that there can bet at most one edge between Q_i and Q_j in G, because there are no cycles of length less than 2ℓ in G.

We obtain a subgraph $H'' \subseteq H'$ by choosing edges of H' uniformly at random with a suitable probability p. Then, successively removing vertices of high degree, we obtain a graph $H \subseteq H''$ with $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$. It now remains to find a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. To do so, we first observe that the paths $Q_i \in V(H')$ give rise to ℓ-cliques in $G^{r'}$ $(r' \geqslant \ell)$. One can then prove that there is a copy of $H^r\{\ell\}$ in $G^{r'}$ that avoids the edges of J. By applying the Lovász local lemma we can further deduce that there is a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. □

Proof of the induction step (Lemma 3.3.3). We start by fixing positive integers $\Delta \geqslant 2$, $s \geqslant 2$, k, r, h and a good 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ with

$$\vartheta \geqslant 2^h 32\sqrt{c}.$$

Recall that from the definition of good 7-tuple, we have

$$b \geqslant 9c.$$

Let d_0 be obtained from Lemma 3.1.1 applied with ℓ and $\gamma = 1/(2\ell)$ (note that $d_0 \leqslant 10$). Further let

$$a'' = \ell(\Delta^{2k} + 2)(2a \cdot d_0 + 2).$$

Notice that a'' is an upper bound on the value A given by Lemma 3.2.4 applied with $f = 2$, $D = \Delta^{2k} + 1$, ℓ and $\eta = 2a \cdot d_0$.

Let r_0 be given by Lemma 3.2.6 on input Δ and k. We may assume r_0 is even. Furthermore, let

$$t = \max\{r_0, \left(40(\ell b^{r+1} + \ell)\right)^{r_0}\} \quad \text{and} \quad \ell' = \max\{4s\ell^2, r_s(t)\},$$

where $r_s(t) = r(t, \ldots, t) = r(K_t, \ldots, K_t)$ denotes the s-colour Ramsey number for cliques of order t. Let $a' = \ell' a$ and note that $a'/s \geqslant 2a''$ because $\ell \geqslant 21\Delta^{2k}$. Define constants c^*, c' and r' as follows.

$$c^* = 2\ell' c, \quad c' = \frac{\ell'}{2\ell^2} c^* = \frac{\ell'^2}{\ell^2} c, \quad r' = \ell r. \tag{3.3.2}$$

Put

$$b' = 9c' \quad \text{and} \quad \vartheta' = \frac{c^*}{4c\ell}\vartheta = \frac{\ell'}{2\ell}\vartheta$$

Claim 3.3.4. *$(a', b', c', \ell', \vartheta', \Delta, k)$ is a good 7-tuple and $\vartheta' \geqslant 2^{h-1} 32\sqrt{c'}$.*

Proof. We have to check all conditions in Definition 3.2.10. Clearly $a' \geqslant 3$, $b' \geqslant 9c'$ and $\ell' \geqslant \ell \geqslant 21\Delta^{2k}$. Below we prove that the other conditions hold

- $c' \geqslant \vartheta' \ell'$:
$$c' = \frac{\ell'^2}{\ell^2} c \geqslant \frac{\ell'^2}{\ell}\vartheta = 2\vartheta'\ell' > \vartheta'\ell'.$$
- $\vartheta' \geqslant 2^{h-1} 32\sqrt{c'}$:
$$\vartheta' = \frac{\ell'}{2\ell}\vartheta \geqslant \frac{\ell'}{2\ell} 2^h 32\sqrt{c} = 2^{h-1} 32\sqrt{c'}.$$

□

Let G be a graph in $\mathcal{P}_n(a', b', c', \ell', \vartheta')$. Assume

$$N_G = a'n \quad \text{and} \quad p_G = c'/N_G$$

and let T be an arbitrary tree with n vertices and maximum degree Δ and consider an arbitrary s-colouring $\chi\colon E(G^{r'}\{\ell'\}) \longrightarrow [s]$ of the edges of $G^{r'}\{\ell'\}$. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

By Ramsey's theorem (see, for example, [22]), since $\ell' \geqslant r_s(t)$, each ℓ'-clique $C(w)$ in $G^{r'}\{\ell'\}$ (for $w \in V(G)$) contains a monochromatic clique of size at least t. Without lost of generality, let us assume that most of those monochromatic cliques are blue. Let $W \subseteq V(G)$ be the set of vertices w such that there is a blue t-clique $C'(w) \subseteq C(w)$. We have

$$|W| \geqslant \frac{|V(G)|}{s} = \frac{a'n}{s} \geqslant 2a''n. \tag{3.3.3}$$

Define J as the subgraph of $G^{r'}$ with vertex set W and edge set

$$E(J) = \Big\{uv \in E(G^{r'}[W]) : \text{ there is a blue copy of } K_{r_0,r_0} \text{ in } G^{r'}\{\ell'\}[C'(u), C'(v)]\Big\}.$$

That is, J is the subgraph of $G^{r'}$ induced by W and the edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by $G^{r'}\{\ell'\}$ between the vertex sets of the blue cliques $C'(u)$ and $C'(v)$.

We now apply Lemma 3.2.4 with $f = 2$, $D = \Delta^{2k} + 1$, ℓ, and $\eta = 2a \cdot d_0$ to the graph J (notice that $|V(J)| \geqslant 2a''n$ is large enough so we can apply Lemma 3.2.4), splitting the proof into two cases:

1 there is $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding,

2 there exist $V_1, \ldots, V_\ell \subseteq V(J)$ such that $|V_i| \geqslant 2ad_0n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i, V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$.

In case $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, we first notice that Lemma 3.2.6 applied to the graph $J[Z]$ implies the existence of a tree $T' = T'(T, \Delta, k)$ of maximum degree at most Δ^{2k} with at most $n+1$ vertices such that if $J[Z]$ contains T', then $T^k \subseteq J'$ for any (r_0, r_0)-blow-up J' of J. But since $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, Lemma 3.2.2 implies that $J[Z]$ contains a copy of T'. Therefore, the graph $G^{r'}\{\ell'\}$ contains a blue copy of T^k, as we can consider J' as the subgraph of $G^{r'}\{\ell'\}$ containing only edges inside the blue cliques $C'(u)$ (which have size $t \geqslant r_0$) and the edges of the complete blue bipartite graphs K_{r_0,r_0} between the blue cliques $C'(u)$. This finishes the proof of the first case.

We may now assume that there are subsets $V_1, \ldots, V_\ell \subseteq V(J)$ with $|V_i| \geqslant 2ad_0n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i, V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$. We want to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and contains no blue edges.

Let $J' = J[V_1 \cup \cdots \cup V_\ell]$, $G' = G[V_1 \cup \cdots \cup V_\ell]$ and note that $|V(G')| = |V(J')| \geqslant d_0 \cdot 2a\ell n$, where we recall that d_0 is the constant obtained by applying Lemma 3.1.1 with ℓ and $\gamma = 1/(2\ell)$. We want to use the assertion of Lemma 3.1.1 to obtain a transversal path of length $2a\ell n$ in G' and so we have to check the conditions adjusted to this parameter.

First note, that we have $|V_i| \geqslant 2ad_0n \geqslant \gamma d_0 \cdot 2a\ell n$ for $1 \leqslant i \leqslant \ell$. Moreover, since G' is an induced subgraph of G and $G \in \mathcal{P}_n(a', b', c', \ell, \vartheta')$, we know by (3.2.1) that for all $X, Y \subseteq V(G')$ with $|X|, |Y| > \vartheta' a' n/c'$ we have $e_{G'}(X, Y) > 0$. Observe that $\vartheta' a' n/c' < an = \gamma \cdot 2a\ell n$ once $a' = \ell' a$ and $c' > \vartheta' \ell'$. Therefore, we may use Lemma 3.1.1 to conclude that G' contains a path $P_{2a\ell n} = (x_1, \ldots, x_{2a\ell n})$ with $x_i \in V_j$ for all i, where $j \equiv i \pmod{\ell}$.

We split the obtained path $P_{2a\ell n}$ of G' into consecutive paths $Q_1, \ldots, Q_{2an}$ each on ℓ vertices. More precisely, we let $Q_i = (x_{(i-1)\ell+1}, \ldots, x_{i\ell})$ for $i = 1, \ldots, 2an$. The following auxiliary graph is the base of our desired graph $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$.

H' is the graph on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ such that $Q_i Q_j \in E(H')$ if and only if there is an edge in G between the vertex sets of Q_i and Q_j.

Claim 3.3.5. *$H' \in \mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\vartheta')$.*

Proof. We verify the conditions of Definition 3.2.9. Since H' has $2an$ vertices, condition (i) clearly holds. Since $\Delta(G) \leqslant b'$ and for any $Q_i \in V(H')$ we have $|Q_i| = \ell$ (as a subset of $V(G)$), there are at most $\ell b'$ edges in G with an endpoint in Q_i. Then, $\Delta(H') \leqslant \ell b'$.

For condition (iii), recall that any vertex of H' corresponds to a path on ℓ vertices in G. Thus, a cycle of length at most 2ℓ in H' implies the existence of a cycle of length at most $2\ell^2$ in G. Since $2\ell' \geqslant 2\ell^2$ and G has no cycles of length at most $2\ell'$, we conclude that H' contains no cycle of length at most 2ℓ, which verifies condition (iii).

Let $N_{H'} = 2an$ and

$$p_{H'} = \frac{c^*}{N_{H'}} = \frac{c^*}{2an}. \tag{3.3.4}$$

Let us verify condition (iv), i.e., we shall prove that H' is $(p_{H'}, \ell\vartheta')$-bijumbled.

Consider arbitrary sets X and Y of $V(H')$ with $\ell\vartheta'/p_{H'} < |X| \leqslant |Y| \leqslant p_{H'} N_{H'} |X|$. For simplicity, we may assume that $X = \{Q_1, \ldots, Q_x\}$ and $Y = \{Q_{x+1}, \ldots, Q_{x+y}\}$. Let $X_G = \bigcup_{j=1}^{x} Q_j \subseteq V(G)$ and $Y_G = \bigcup_{j=x+1}^{x+y} Q_j \subseteq V(G)$. Note that $|X_G| = \ell|X|$ and $|Y_G| = \ell|Y|$. As there are no cycles of length smaller than 2ℓ in G, we only have at most one edge between the vertex sets of Q_i and Q_j. Therefore we have

$$e_{H'}(X, Y) = e_G(X_G, Y_G). \tag{3.3.5}$$

We shall prove that $|e_{H'}(X,Y) - p_{H'}|X||Y|| \leqslant \ell\vartheta'\sqrt{|X||Y|}$. From the choice of c', we have

$$p_{H'}|X||Y| = \frac{c^*}{2an}|X||Y| = \frac{c'}{a'n}\ell|X|\ell|Y| = \frac{c'}{a'n}|X_G||Y_G| = p_G|X_G||Y_G|. \tag{3.3.6}$$

From the choice of ϑ', c', and $p_{H'}$, since $\ell\vartheta'/p_{H'} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, we obtain

$$\frac{\vartheta'}{p_G} < |X_G| \leqslant |Y_G| \leqslant p_G N_G |X_G|.$$

Combining (3.3.6) with (3.3.5) and the fact that G is (p_G, ϑ')-bijumbled, we get that

$$|e_{H'}(X,Y) - p_{H'}|X||Y|| = |e_G(X_G,Y_G) - p_G|X_G||Y_G|| \leqslant \vartheta'\sqrt{|X_G||Y_G|} = \ell\vartheta'\sqrt{|X||Y|}. \tag{3.3.7}$$

Therefore, H' is $(p_{H'}, \ell\vartheta')$-bijumbled, which verifies condition (iv). □

The parameters for $\mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\vartheta')$ are tightly fitted such that we can find the following subgraph of H'.

Claim 3.3.6. *There exists $H \subseteq H'$ such that $H \in \mathcal{P}_n(a,b,c,\ell,\vartheta)$.*

Proof. We first obtain $H'' \subseteq H'$ by picking each edge of H' with probability

$$p = \frac{2c}{c^*} = \frac{1}{\ell'}$$

independently at random. Note that $p \leqslant 1/2$.

From (3.2.2), we get

$$e(H') \leqslant p_{H'}\binom{2an}{2} + \ell\vartheta' 2an \leqslant (c^* + 2\ell\vartheta')an \leqslant (c^* + 2\ell\frac{c'}{\ell'})an \leqslant 2c^*an$$

From Chernoff's inequality, we then know that almost surely we have

$$e(H'') \leqslant 2p \cdot e(H') \leqslant 2 \cdot \left(\frac{2c}{c^*}\right) \cdot 2c^*an \leqslant 8acn \leqslant abn. \tag{3.3.8}$$

Let $N_{H''} = 2an$ and

$$p_{H''} = p \cdot p_{H'} = \frac{c}{an}.$$

We shall prove that H'' is $(p_{H''}, \vartheta)$-bijumbled almost surely. For that, we will first prove by using Chernoff's inequality (Theorem 3.1.2) that, for any disjoint sets X and Y of $V(H')$ with $\vartheta/p_{H''} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, we have

$$|e_{H''}(X,Y) - p \cdot e_{H'}(X,Y)| \leqslant \frac{\vartheta}{2}\sqrt{|X||Y|}. \tag{3.3.9}$$

Note that for such sets X and Y, since $|X| > \vartheta/p_{H''} \geqslant \ell\vartheta'/p_{H'}$, we can use (3.3.7).

Since $|X|, |Y| > \vartheta/p_{H''}$, we have $\sqrt{|X||Y|} > \vartheta an/c$. From $\sqrt{|X||Y|} > \vartheta an/c$, we obtain that $\ell'\vartheta < \frac{2\ell' c\sqrt{|X||Y|}}{2an}$ from which we can conclude that $2\ell\vartheta' < p_{H'}\sqrt{|X||Y|}$. Thus, we get $\ell\vartheta'\sqrt{|X||Y|} < p_{H'}|X||Y|/2$. Therefore, combining this with (3.3.7) we have

$$\frac{p_{H'}|X||Y|}{2} < e_{H'}(X,Y) < 2p_{H'}|X||Y|. \tag{3.3.10}$$

Let $\varepsilon = \vartheta\sqrt{|X||Y|}/(2p \cdot e_{H'}(X,Y))$ and note that from (3.3.10) we have $\varepsilon < 1$. Since $\vartheta \geqslant 10\sqrt{c}$, also from (3.3.10) we obtain

$$\frac{\varepsilon^2 p \cdot e_{H'}(X,Y)}{3} = \frac{|X||Y|\ell'\vartheta^2}{12 \cdot e_{H'}(X,Y)} > 4an.$$

Therefore, by using Chernoff's inequality, since there are at most 2^{4an} choices of pairs of sets $\{X, Y\}$, almost surely we have that for any disjoint subsets X and Y of vertices of H'' with $\vartheta/p_{H''} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, inequality (3.3.9) holds.

Observe that $p_{H''}N_{H''}|X| = 2c|X| \leqslant c^*|X| = p_{H'}N_{H'}|X|$. Therefore, H'' is almost surely $(p_{H''}, \vartheta)$-bijumbled, as by (3.3.7) and (3.3.9) we get

$$\begin{aligned}
|e_{H''}(X,Y) - p_{H''}|X||Y|| &\leqslant |e_{H''}(X,Y) - p \cdot e_{H'}(X,Y)| + |p \cdot e_{H'}(X,Y) - p_{H''}|X||Y|| \\
&\overset{(3.3.9)}{\leqslant} \frac{\vartheta}{2}\sqrt{|X||Y|} + p(|e_{H'}(X,Y) - p_{H'}|X||Y||) \\
&\overset{(3.3.7)}{\leqslant} \frac{\vartheta}{2}\sqrt{|X||Y|} + \frac{\ell\vartheta'}{\ell'}\sqrt{|X||Y|} \\
&= \vartheta\sqrt{|X||Y|}.
\end{aligned}$$

Therefore, there exists a $(p_{H''}, \vartheta)$-bijumbled graph H'' as above. We fix such a graph and construct the desired graph H from this H'' by sequentially removing the an vertices of highest degree. Notice that H has maximum degree at most b, otherwise this would imply that H'' has more than abn edges, contradicting (3.3.8). Since H is a subgraph of H', and H' does not contain cycles of length at most 2ℓ, the same holds for H. Finally, since deleting vertices preserves the bijumbledness property, we conclude that $H \in \mathcal{P}_n(a, b, c, \ell, \vartheta)$. $\square$

Recall that J is the subgraph of $G^{r'}$ induced by W, with $|W| \geqslant a'n/s$ and edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by the vertex sets of blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. Furthermore, recall that there are subsets $V_1, \ldots, V_\ell \subseteq V(J)$ with $|V_i| \geqslant 2ad_0n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i, V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$, and we defined $J' = J[V_1 \cup \cdots \cup V_\ell]$ and $G' = G[V_1 \cup \cdots \cup V_\ell]$. Lastly, recall that $Q_i = (x_{(i-1)\ell+1}, \ldots, x_{i\ell})$ for $i = 1, \ldots, 2an$, where the vertices x_i belong to G'. Assume, without loss of generality, $V(H) = \{Q_1, \ldots, Q_{an}\}$. In what follows, when considering the graph $H^r(\ell)$, the ℓ-clique corresponding to Q_i is composed of the vertices $x_{(i-1)\ell+1}, \ldots, x_{i\ell}$, and hence one can view $V\big(H^r(\ell)\big)$ as a subset of $V(G')$.

Claim 3.3.7. *$H^r(\ell) \subseteq G^{r'}$. Moreover, $G^{r'}$ contains a copy of $H^r\{\ell\}$ that avoids the edges of J.*

Proof. We will prove that $H^r(\ell) \subseteq G^{r'}$ where $Q_1, \ldots, Q_{an} \subseteq V(J)$ are the ℓ-cliques of $H^r(\ell)$. Suppose that Q_i and Q_j are at distance at most r in the graph H. Without loss of generality, let $Q_i = Q_1$ and $Q_j = Q_m$ for some $m \leqslant r$. Moreover, let $(Q_1, Q_2, \ldots, Q_m)$ be a path in H. Note that there exist vertices $u_1, \ldots, u_{m-1}$ and $u'_2, \ldots, u'_m$ in $V(G')$ such that $u_1 \in Q_1$, $u'_m \in Q_m$, $u_j, u'_j \in Q_j$ for all $j = 2, \ldots, m-1$ and $\{u_i, u'_{i+1}\}$ is an edge of G' for $i = 1, \ldots, m-1$.

Let $u'_1 \in Q_1$ and $u_m \in Q_m$ be arbitrary vertices. Since for any j, the set Q_j is spanned by a path on ℓ vertices in G', it follows that u_j and u'_j are at distance at most $\ell - 1$ in G' for all $1 \leqslant j \leqslant m$. Therefore, u'_1 and u_m are at distance at most $(\ell-1)m+(m-1) < \ell r \leqslant r'$ in G' and hence u'_1u_m is an edge in $G[V_1 \cup \ldots \cup V_\ell]^{r'} \subseteq G^{r'}$. Since the vertices u'_1 and u_m were arbitrary, we have shown that if Q_i and Q_j are adjacent in H^r (i.e., Q_i and Q_j are at distance at most r in H) then (Q_i, Q_j) gives a complete bipartite graph $C(Q_i, Q_j)$ in $G^{r'}$. Moreover, taking $i = j$ we see that each Q_i in $G^{r'}$ must be complete. This implies that $H^r(\ell)$ is a subgraph of $G^{r'}$.

For the second part of the claim we consider which of the edges of this copy of $H^r(\ell)$ can also be edges of J. Recall from the definition of J' that we found subsets $V_1, \ldots, V_\ell \subseteq J$ such that no edge of J lies between different parts. Moreover each set $Q_i \subseteq J$ takes precisely one vertex from each set $V_1, \ldots, V_\ell$. It follows that each Q_i is independent in J. Now let us say we have $x \in Q_i$ and $y \in Q_j$ $(i \neq j)$ that are adjacent in J. We can not have x and y in different parts of the partition $\{V_1, \ldots, V_\ell\}$. Thus x and y lie in the same part. Therefore edges from J between Q_i and Q_j must form a matching. Then we can find a copy of $H^r\{\ell\}$ that avoids J by removing a matching between the l-cliques from $H^r(\ell)$. □

To complete the proof of Lemma 3.3.3, we will embed a copy of the graph $H^r\{\ell\} \subseteq G^{r'}$ found in Claim 3.3.7 in $G^{r'}\{\ell'\}$ in such a way that $H^r\{\ell\}$ uses at most $s-1$ colours.

Claim 3.3.8. *$G^{r'}\{\ell'\}$ contains a copy of $H^r\{\ell\}$ with no blue edges.*

Proof. Recall that each vertex u in J corresponds to a clique $C'(u) \subseteq G^{r'}\{\ell'\}$ of size t and that this clique is monochromatic in blue in the original colouring χ of $E(G^{r'}\{\ell'\})$. Recall also that if an edge $\{u, v\}$ of $G^{r'}[W]$ is not in J, then there is no blue copy of K_{r_0,r_0} in the bipartite graph between $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. By the Kővári–Sós–Turán theorem (Theorem 3.1.3), there are at most $4t^{2-1/r_0}$ blue edges between $C'(u)$ and $C'(v)$. Recall further that $C'(u)$ and $C'(v)$ are, respectively, subcliques of the ℓ'-cliques $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$. Since $\{u, v\}$ is an edge of $G^{r'}$, there is a complete bipartite graph with a matching removed between $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$ and so there is a complete bipartite graph with at most a matching removed for $C'(u)$ and $C'(v)$. It follows that there are at least

$$t^2 - t - 4t^{2-1/r_0}$$

non-blue edges between $C'(u)$ and $C'(v)$.

Using the copy of $H^r\{\ell\} \subseteq G^{r'}$ avoiding edges of J obtained in Claim 3.3.7 as a 'template', we will embed a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. For each vertex $u \in V(H^r\{\ell\}) \subseteq V(J)$ we will pick precisely one vertex from $C'(u) \subseteq G^{r'}\{\ell'\}$ in our embedding. The argument proceeds by the Lovász Local Lemma.

For each $u \in V(H^r\{\ell\}) \subseteq V(J)$ let us choose $x_u \in C'(u)$ uniformly and independently at random. Let $e = \{u, v\}$ be an edge of our copy of $H^r\{\ell\}$ in $G^{r'}$ that is not in J. As pointed out above, we know that there are at least $t^2 - t - 4t^{2-1/r_0}$ non-blue edges between $C'(u)$ and $C'(v)$. Letting A_e be the event that $\{x_u, x_v\}$ is a blue edge or a non-edge in $G^{r'}\{\ell'\}$, we have that

$$\mathbb{P}[A_e] \leqslant \frac{t + 4t^{2-1/r_0}}{t^2} \leqslant 5t^{-1/r_0}.$$

The events A_e are not independent, but we can define a dependency graph D for the collection of events A_e by adding an edge between A_e and A_f if and only if $e \cap f \neq \varnothing$. Then, $\Delta = \Delta(D) \leqslant 2\Delta(H^r\{\ell\}) \leqslant 2(b^{r+1}\ell + \ell)$. From our choice of t we get that

$$4\Delta\mathbb{P}[A_e] \leqslant 40(b^{r+1}\ell + \ell^2)t^{-1/r_0} \leqslant 1$$

for all e. Then the Local Lemma [4, Lemma 5.1.1] tells us that $\mathbb{P}\big[\bigcap_e \bar{A}_e\big] > 0$, and hence a simultaneous choice of the x_u's ($u \in V(H^r\{\ell\})$) is possible, as required. This concludes the proof of Claim 3.3.8. $\square$

The proof of Lemma 3.3.3 is now complete. $\square$

3.4 Concluding remarks

To construct our graphs we need that $\mathcal{P}_n(a, b, c, \ell, \vartheta)$ is non-empty given a good 7-tuple $(a, b, c, \ell, \vartheta, \Delta, k)$ with $\vartheta \geqslant 32\sqrt{c}$. We prove this in Lemma 3.2.11 using the binomial random graph. Alternatively, it is possible to replace this by using explicit constructions of high girth expanders. For example, the Ramanujan graphs constructed by Lubotzky, Phillips, and Sarnak [72] can be used to prove Lemma 3.2.11.

We now discuss further connections between powers of trees and graph parameters related to treewidth. As pointed out in the introduction, every graph with maximum degree and bounded treewidth is contained in some bounded power of a bounded degree tree and vice versa. This implies that Corollary 1.2.2 is equivalent to Theorem 1.2.1. For bounded degree graphs, bounded treewidth is equivalent to bounded cliquewidth and also to bounded rankwidth [52]. Therefore, Corollary 1.2.2 also holds with treewidth replaced by any of these parameters. Finally, an obvious direction for further research is to investigate the size-Ramsey number of powers T^k of trees T when k and $\Delta(T)$ are no longer bounded.

4. Random perturbation of sparse graphs

4.1 Hamiltonicity

We will prove the following proposition that will be sufficient to prove Theorem 1.3.1 together with known results on Hamilton cycles in $G(n,p)$.

Proposition 4.1.1. *Let $\alpha = \alpha(n) : \mathbb{N} \to (0,1)$ such that $\alpha = \omega(n^{-\frac{1}{6}})$, and let $\beta = \beta(\alpha) = -(6+o(1))\log(\alpha)$. Then a.a.s. $G_\alpha \cup G(n, \frac{\beta}{n})$ is Hamiltonian.*

Proof of Theorem 1.3.1. Let $\alpha, \beta > 0$ such that $\beta = -(6+o(1))\log(\alpha)$. If $\alpha = O(n^{-\frac{1}{6}})$, we have $\beta \geqslant (1+o(1))\log n$ and we can infer that a.a.s. there is a Hamiltonian cycle in $G(n, \frac{\beta}{n})$ (this follows from an improvement on the result concerning the threshold for Hamiltonicity [58]). On the other hand, if $\alpha = \omega(n^{-\frac{1}{6}})$, then we apply Proposition 4.1.1 to a.a.s. get the Hamilton cycle. □

Proof of Proposition 4.1.1. To prove the proposition we apply the following strategy. We first find a long path in $G(n,p)$ alone. Then, by considering the union with G_α, we obtain a reservoir structure for each vertex that allows us to extend the length of the path iteratively. Finally, we will also be able to close this path into a cycle on all vertices. W.l.o.g. we can assume that $\alpha < \frac{1}{10}$.

Let $P = p_1, \dots, p_\ell$ be the longest path that we can find in $\mathcal{G}_1 = G(n, \frac{\beta-1}{n})$ and let $V' = \{v_1, \dots, v_k\} = V(\mathcal{G}_1) \setminus \{p_1, ..., p_\ell\}$ be the leftover. By Lemma 1.3.3, we get a.a.s. that

$$k = |V'| = n - \ell \leqslant (1-o(1))\beta \exp(1-\beta)n.. \tag{4.1.1}$$

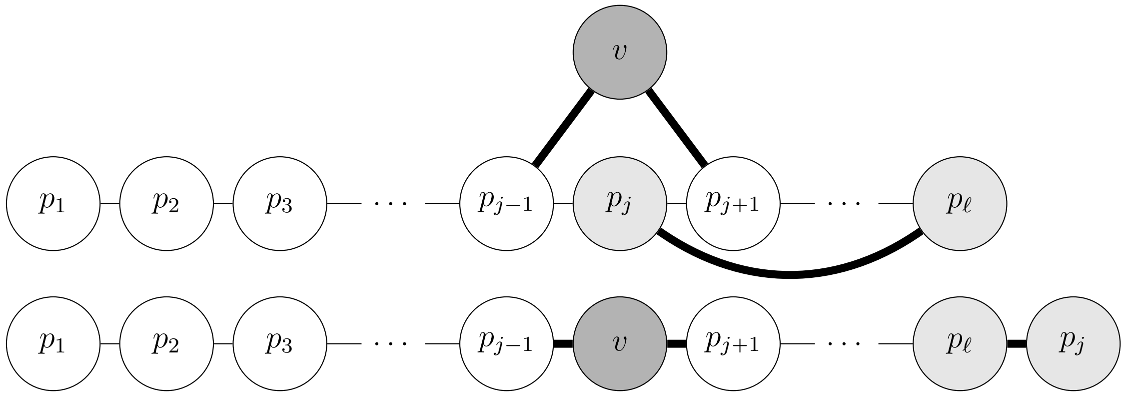


Figure 4.1.1: The top shows a path $P = p_1, \ldots, p_\ell$ and the left-over vertex v. Thin edges belong to the random graph, thick edges can be found in G_α. The bottom shows the graph after absorbing v using that $p_j \in B(p_\ell, v)$.

Next, let P' be a collection of vertices of P, where we take every other vertex of the path, excluding the last, that is

$$P' = \{p_i : i \equiv 0 \pmod 2\} \setminus \{p_\ell\}. \tag{4.1.2}$$

In the following, we will ensure certain absorbing structures that do not overlap, such that the leftover can be absorbed. Consider the union $G_\alpha \cup \mathcal{G}_1$. The following absorbing structure is the key to the argument.

Definition 4.1.2. For any vertices $u, v \in V(G_\alpha \cup \mathcal{G}_1)$ let

$$B(u, v) = \{x \in N_{G_\alpha}(u) \cap P' : N_P(x) \subseteq N_{G_\alpha}(v)\}. \tag{4.1.3}$$

If for some $v \in V'$ there is a $p_j \in B(p_\ell, v)$ we can proceed as follows (see Figure 4.1.1). By definition we have $p_{j-1}, p_{j+1} \in N_{G_\alpha}(v)$ and $p_j \in N_{G_\alpha}(p_\ell) \cap P'$. Then p_j can be replaced by v in the path P and then readded to the path P after p_ℓ. We get the path $\tilde{P} = p_1, \ldots, p_{j-1}, v, p_{j+1}, \ldots, p_\ell, p_j$, where $\tilde{P} \subseteq P \cup G_\alpha$.

To iterate this argument we show that a.a.s. for any pair of vertices u and v, the set $B(u, v)$ is large enough.

Claim 4.1.3. *We have a.a.s. that $|B(u, v)| \geqslant \frac{\alpha^3 n}{4}$ for any $u, v \in V(G_\alpha \cup \mathcal{G}_1)$.*

Proof. Let u, v be arbitrary vertices in $V = V(G_\alpha \cup \mathcal{G}_1)$. The set $B(u, v)$ is uniformly distributed over P', because $G(n, \frac{\beta-1}{n})$ is sampled independently of the deterministic graph G_α. Then by definition

$$\mathbb{E}[|B(u,v)|] \geqslant \frac{9}{10}\alpha^3|P'| \geqslant \frac{2}{5}\alpha^3(1-(1-o(1))\beta\exp(1-\beta)n \geqslant \frac{\alpha^3 n}{3}\,. \tag{4.1.4}$$

An immediate consequence of $B(u,v)$ being uniformly settled over $G(n, \frac{\beta-1}{n})$ is that $|B(u,v)| \sim Bin(|P'|, \alpha^3)$. It follows from (4.1.4) and the Chernoff bound that there is a sufficiently small, but constant, $\delta > 0$ s.t.

$$\begin{aligned}\mathbb{P}\big(|B(u,v)| < \frac{\alpha^3 n}{4}\big) &\leqslant \mathbb{P}\big(|B(u,v)| < (1-\delta)\mathbb{E}[|B(u,v)|]\big)\\ &\leqslant \exp(-\frac{\delta^2}{8}\alpha^3 n) < \exp(-\sqrt{n})\,.\end{aligned} \tag{4.1.5}$$

The claim follows from a union bound over all $\binom{n}{2}$ choices for u,v and (4.1.5). □

We now have everything at hand to absorb the leftover vertices $V' = \{v_1, \ldots, v_{|V'|}\}$ into a path $\tilde{P}$ of length $n-2$, we leave two vertices of V' out of $\tilde{P}$ for closing the cycle. Set $P_0 = P$ and for every $u, v \in V(G_\alpha \cup \mathcal{G}_1)$, let $B_0(u,v) = B(u,v)$. For $0 \leqslant i \leqslant |V'|-1$, assume we have $P_i = u_{i,1}\ldots u_{i,\ell+i}$ with $V_{P_i} = V(P) \cup \{v_1, \ldots, v_i\}$ and for every u,v, we have $B_i(u,v)$ with $|B_i(u,v)| \geqslant \frac{\alpha^3 n}{8}$. To get P_{i+1} take $u_{i,j} \in B_i(u_{i,\ell+i}, v_{i+1})$ and switch it with v_{i+1}. Then for every u, v set $B_{i+1}(u,v) = B_i(u,v) \smallsetminus u_{i,j}$. We have

$$|B(u,v) \smallsetminus B_i(u,v)| \leqslant i \leqslant |V'| \leqslant \beta\exp(1-\beta)n < \frac{\alpha^3}{8}n\,,$$

where the last inequality holds for our choice of $\beta = -(6+o(1))\log(\alpha)$, with $\alpha < \frac{1}{10}$. Set $\tilde{P} = P_{|V'|-2}$.

We have found a path $\tilde{P} = p_1, \ldots, p_{n-2}$ and we are left with two vertices $v_{|V'|-1}, v_{|V'|}$ that are not on the path. We observe that it is possible to close the Hamilton cycle by absorbing $v_{|V'|-1}$ and $v_{|V'|}$ if there is an edge between $A := B_{|V'|}(p_1, v_{|V'|-1})$ and $B := B_{|V'|}(p_{n-2}, v_{|V'|})$. Indeed, we then have w.l.o.g. $i < j$ such that $p_i \in A$, $p_j \in B$, and $p_ip_j \in E(G_\alpha \cup \mathcal{G}_1)$. By definition of A and B we can then obtain the Hamilton cycle

$$p_i, p_1, \ldots, p_{i-1}, v_{|V'|-1}, p_{i+1}, \ldots, p_{j-1}, v_{|V'|}, p_{j+1}, \ldots, p_{n-2}, p_j.$$

It remains to prove that we have an edge between A and B. For this we take

$\mathcal{G}_2 = G(n, \frac{1}{n})$. Since $|A|, |B| \geqslant \frac{\alpha^3 n}{8}$, we get

$$\mathbb{E}[e_{\mathcal{G}_2}(A, B)] \geqslant \frac{1}{n} \cdot \left(\frac{\alpha^3 n}{16}\right)^2 = \omega(1), \tag{4.1.6}$$

as $\alpha = \omega(n^{-1/6})$. Together with Chernoff's inequality this implies that a.a.s. $e_{\mathcal{G}_2}(A, B) > 0$. Since the union of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be coupled as a subgraph of $G(n, \frac{\beta}{n})$, this implies that a.a.s. there is a Hamilton cycle in $G_\alpha \cup G(n, \frac{\beta}{n})$. $\square$

Theorem 1.3.2 can be proven similarly. Moreover, a better constant can be obtained by adapting the definition of $B(u, v)$ to the setup of perfect matchings and then proving that a.a.s. $|B(u, v)| \geqslant \frac{\alpha^2 n}{4}$.

4.2 Bounded degree trees

Theorem 1.3.6 shows that an almost spanning embeddings in the random graph implies a spanning embedding in the union $G_\alpha \cup G(n, \frac{\beta}{n})$. The proof is very similar to the proof for Hamilton cycles and we will skip some details.

Proof of Theorem 1.3.6. Let G_α be given and $\mathcal{G} = G(n, \frac{\beta}{n})$. Let T be an arbitrary tree on n vertices with maximum degree Δ. Denote by T_ε the tree obtained from T by the following construction.

1. Set $T_0 = T$.
2. In every step i, check whether T_i has at most $(1 - \varepsilon)n$ vertices.
   - If this is the case, set $T_\varepsilon = T_i$ and finish the process.
   - Otherwise, create T_{i+1} by deleting one leaf of T_i.

Let $L = V(T) \setminus V(T_\varepsilon)$ be the leftover.

Then

$$|V(T_\varepsilon)| \leqslant (1 - \varepsilon)n, \qquad |L| \leqslant \varepsilon n + 1, \qquad \text{and} \qquad V(T) = V(T_\varepsilon) \cup L.$$

Let $I \subseteq V(T_\varepsilon)$ be independent and such that for every $v \in I$, we have $N_T(v) \subseteq V(T_\varepsilon)$. Observe that there exists such an I with $|I| \geqslant \frac{(1-\Delta\varepsilon)n}{\Delta+1}$.

By assumption we a.a.s. have an embedding T'_ε of T_ε into $\mathcal{G}$ and we denote by I' the image of I under this embedding. We adapt Definition 4.1.2 and define for any two vertices u, v

$$B_T(u,v) = \{x \in N_{G_\alpha}(u) \cap I' \colon N_{T'_\varepsilon}(x) \subset N_{G_\alpha}(v)\}.$$

As before, if we want to embed a vertex w that is a neighbour of an already embedded vertex u in T_ε and v is an available vertex, we can do it if $B_T(u,v)$ is non-empty. More precisely, with $x \in B_T(u,v)$, we can re-embed the vertex embedded into x to v and then embed w into x, and obtain a valid embedding of $T[V(T_\varepsilon) \cup \{u\}$. Analogously to Claim 4.1.3 we get the following.

Claim 4.2.1. *We a.a.s. have* $|B_T(u,v)| \geqslant \frac{\alpha^{\Delta+1} n}{4(\Delta+1)}$ *for any* $u, v \in V(G_\alpha \cup \mathcal{G})$.

Therefore, we can iteratively add leaves to T_ε to obtain an embedding of T into $G_\alpha \cup \mathcal{G}$. Since in every step we lose at most one vertex from each $B_T(u,v)$ this works as long as

$$|L| \leqslant \varepsilon n + 1 < |B_T(u,v)|,$$

which holds by Claim 4.2.1 and the assumption on ε and α. □

Appendix

English Summary

In this thesis we focus on problems in Extremal and Probabilistic Combinatorics. The thesis is organized in three parts, each concerning a problem regarding sufficient conditions for a host graph under different settings to contain some cycle-like subgraph.

In the first and main part, we want to embed spanning 3-chromatic graphs H with small linear bandwidth and bounded maximum degree into graphs $G = (V, E)$. For this kind of spanning subgraph, the condition we require from G is a common generalisation of the bandwidth theorem from Böttcher, Schacht and Taraz, which requires minimum degree $\delta(G) \geqslant (2/3 + o(1))|V|$ and a previous joint work with Ebsen et al., which requires uniform density $d > 0$ in linear sized subsets of vertices and density at least $\mu > 0$ in every cut. These two previous results are incomparable. On the one hand, the latter result applies to sparser graphs G, since d and μ can be arbitrarily small. On the other hand, the degree condition of the bandwidth theorem does not ensure the uniform density condition. Here we relax this notion of uniform density by requiring instead a robust almost perfect fractional triangle factor in G and thus obtaining a common generalisation of both results. This and more general results were shown independently in a recent work of Richard Lang and Nicolás Sanhueza-Matamala.

In the second part, we study the following parameter. Given a positive integer s, the *s-colour size-Ramsey number* of a graph H is the smallest integer m such that there exists a graph G with m edges and the property that, in any colouring of $E(G)$ with s colours, there is a monochromatic copy of H. We prove that, for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n. As a corollary we obtain that the s-colour size-Ramsey number of n-vertex graphs with bounded treewidth and bounded degree is linear in n, which answers a question raised by Kamčev, Liebenau, Wood and Yepremyan.

In the third part, we are interested in the model of randomly perturbed graphs that consider the union of a deterministic n-vertex graph G_α with minimum degree αn and the binomial random graph $G(n, p)$. This model was introduced by Bohman, Frieze, and Martin and for Hamilton cycles their result bridges the gap between

Dirac's theorem and the works of Posá and Koršunov on the threshold in $G(n,p)$. We extend this result in $G_\alpha \cup G(n,p)$ to sparser graphs with $\alpha = o(1)$. More precisely, for any $\varepsilon > 0$ and $\alpha \colon \mathbb{N} \to (0,1)$ we show that a.a.s. $G_\alpha \cup G(n, \beta/n)$ is Hamiltonian, where $\beta = -(6+\varepsilon)\log(\alpha)$. If $\alpha > 0$ is a fixed constant this gives the aforementioned result by Bohman, Frieze, and Martin and if $\alpha = O(1/n)$ the random part $G(n,p)$ is sufficient for ensuring a Hamiltonian cycle. We also discuss embeddings of bounded degree trees and other spanning structures in this model, which lead to interesting questions on almost spanning embeddings into $G(n,p)$.

Deutsche Zusammenfassung

Diese Dissertation konzentriert sich auf Probleme der Extremalen und Probabilistischen Kombinatorik. Sie ist in drei Teile gegliedert, die sich jeweils mit einem Problem befassen, das die hinreichenden Bedingungen dafür betrifft, dass ein Graph unter verschiedenen Bedingungen einen kreisartigen Untergraph enthält.

Im ersten Teil wollen wir 3-chromatische Graphen H mit kleiner linearer Bandweite und beschränktem Maximalgrad in Graphen $G = (V, E)$ einbetten. Für diese Art von aufspannenden Untergraphen ist die Bedingung, die wir von G verlangen, eine Verallgemeinerung des Bandweitensatzes von Böttcher, Schacht und Taraz, der einen Minimalgrad $\delta(G) \geqslant (2/3 + o(1))|V|$ erfordert, und einer früheren gemeinsamen Arbeit mit Ebsen et al., die eine gleichmäßige Dichte $d > 0$ in linear großen Teilmengen von Knoten und eine Dichte von mindestens $\mu > 0$ in jedem Schnitt erfordert. Diese beiden früheren Ergebnisse sind nicht vergleichbar. Einerseits gilt das letztere Ergebnis für dünne Graphen G, da d und μ beliebig klein sein können. Andererseits gewährleistet die Gradbedingung des Bandweitensatzes die Bedingung der einheitlichen Dichte nicht. Hier schwächen wir den Begriff der gleichmäßigen Dichte ab, indem wir stattdessen einen robusten fast perfekten Dreiecksfaktor in G verlangen und so eine gemeinsame Verallgemeinerung beider Ergebnisse erhalten. Dieses und allgemeinere Ergebnisse wurden unabhängig voneinander in einer aktuellen Arbeit von Richard Lang und Nicolás Sanhueza-Matamala gezeigt.

Im zweiten Teil untersuchen wir den folgenden Parameter. Für eine positive ganze Zahl s ist die *s-size-Ramseyzahl* eines Graphen H die kleinste ganze Zahl m, bei der es einen Graphen G mit m Kanten und der Eigenschaft gibt, dass es in jeder Färbung von $E(G)$ mit s Farben, eine monochromatische Kopie von H existiert. Wir beweisen, dass für beliebige positive ganze Zahlen k und s die s-size-Ramseyzahl der $k$ten Potenz eines beliebigen Baumes auf n Ecken mit beschränktem Grad linear in n ist. Als Korollar erhalten wir, dass die s-size-Ramseyzahl von Graphen mit beschränkter Baumbreite und beschränktem Grad linear in der Anzahl der Ecken ist, was eine Frage von Kamčev, Liebenau, Wood und Yepremyan beantwortet.

Im dritten Teil interessieren wir uns für das Modell der zufällig augmentierten Graphen, welches die Vereinigung eines deterministischen Graphen G_α mit Minimalgrad αn und des binomischen Zufallsgraphen $G(n, p)$ betrachtet. Dieses Modell wurde von Bohman, Frieze und Martin eingeführt. Für Hamiltonkreise schließt ihr Ergebnis die Lücke zwischen Diracs Theorem und den Arbeiten von Posá und Koršunov über den Schwellenwert in $G(n, p)$. Wir erweitern dieses Ergebnis in $G_\alpha \cup G(n, p)$ auf dünne Graphen mit $\alpha = o(1)$. Genauer gesagt zeigen wir für jedes $\varepsilon > 0$ und $\alpha \colon \mathbb{N} \to (0, 1)$, dass a.f.s. $G_\alpha \cup G(n, \beta/n)$ Hamiltonisch ist, wobei $\beta = -(6 + \varepsilon) \log(\alpha)$. Wenn $\alpha > 0$ eine feste Konstante ist, ergibt sich das bereits erwähnte Ergebnis von Bohman, Frieze und Martin, und wenn $\alpha = O(1/n)$, ist der Zufallgraph $G(n, p)$ ausreichend, um einen Hamiltonkreis zu gewährleisten. Wir diskutieren auch Einbettungen von Bäumen mit beschränktem Grad und andere aufspannende Strukturen in diesem Modell, die zu interessanten Fragen über fast aufspannende Einbettungen in $G(n, p)$ führen.

Publications related to this dissertation

Articles

[EMR+20] O. Ebsen, G. S Maesaka, C. Reiher, M. Schacht, and B. Schülke, *Embedding spanning subgraphs in uniformly dense and inseparable graphs*, Random Structures & Algorithms **57** (2020), no. 4, 1077–1096. ↑1.1, 1.1, 2.3, 4.2

[BKM+21] S. Berger, Y. Kohayakawa, G. S. Maesaka, T. Martins, W. Mendonça, G. O. Mota, and O. Parczyk, *The size-ramsey number of powers of bounded degree trees*, Journal of the London Mathematical Society **103** (2021), no. 4, 1314–1332. ↑4.2

[HKMM+20] M. Hahn-Klimroth, G. S Maesaka, Y. Mogge, S. Mohr, and O. Parczyk, *Random perturbation of sparse graphs*, arXiv preprint arXiv:2004.04672 (2020). ↑4.2

Extended Abstracts

[BKM+19] S. Berger, Y. Kohayakawa, G. Maesaka, T. Martins, W. Mendonça, G. Mota, and O. Parczyk, *The size-ramsey number of powers of bounded degree trees*, Acta Mathematica Universitatis Comenianae **88** (2019), no. 3, 451–456. ↑4.2

Declaration of Contributions

This thesis is a combination of three results. Each result is joint work with different research groups and here I intend to specify my contributions in each of them.

The result presented in Chapter 2, concerning a generalization for the Theorems 1.1.2 and 1.1.6, is my main contribution. This problem was suggested to me by my advisor Mathias Schacht. I was already familiar to the topic, because of my participation on the paper "Embedding spanning subgraphs in uniformly dense and inseparable graphs" [EMR$^+$20] and later I started to work on the generalization. I worked alone under the guidance of my advisor, and we discussed the progress on a regular basis.

The paper "The size-Ramsey number of powers of bounded degree trees" [BKM$^+$21] is the result of a two-months visit from Sören Berger and myself to Brazil. At IMPA we met Taísa Martins, Walner Mendonça and two visitors, Guilherme Mota and Olaf Parczyk. The problem was presented by Guilherme Mota and initial ideas were discussed. Later, the visitors moved to the University of São Paulo, where Yoshiharu Kohayakawa joined with valuable insights. I had a normal share of work, first following ideas or questioning them, then writing and proofreading. I presented this result at Eurocomb 2019 [BKM$^+$19].

The last work presented in this thesis is based on the article "Random perturbation of sparse graphs" [HKMM$^+$20] was initiated during a workshop in Cuxhaven organized by the Hamburg University of Technology. Each participant presented a problem and some of these were chosen to be worked on in groups. Our group members were Max Hahn-Klimroth, Yannick Mogge, Samuel Mohr, Olaf Parczyk and myself. The problem was suggested by Olaf Parczyk. I had a normal contribution in discussing the initial ideas for the absorption method in the proof and then proofreading the final note.

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Acknowledgements

It was a great beginning to be able to visit Brazil and work with excitement with Taísa, Walner, Guilherme, Olaf, Sören and Yoshiharu. I am also grateful for my peers at the University of Hamburg, especially my advisor Mathias, who quickly included me in their research and for the group at the Hamburg University of Technology, especially Anusch; our workshops together were always a motivational peak for me.

The middle period of this work coincided with the COVID-19 pandemic and my strong feelings of not belonging. Therefore, I am grateful for the welcoming green paths next to water streams that spread all over Hamburg. The ladybugs, who are everywhere, inspired me to trust I am part of wherever I am.

I thank my family, Célia, Jorge, Tati, Vítor, for I always feel refilled with confidence after talking to them. I hope to share more of mathematics with them someday, since they were always there supporting this “madness”.

I thank my friends. The ones that, given the distance, I had to call: Lie, Luiz, Pedro, Ludy, Rangel and T19. The ones I met in Hamburg: Yannick, Attila, Zsuzsi, Babi, Ann-Kathrin, Nina, Richard, Oliver, Simon, Alena, Konoka, Masa, Herbert, Solveig, Henriette. The four-legged friends: Mynghau, Foxy and Michi. They are very unique and relate to me in most different ways. My best friend in Hamburg, who is now sitting next to me, is Julian and I thank him for exactly that.

If I am able to believe in this work, it is because of the journey I had to take so that this could become concrete. Mathematics has given me ways to follow and now I will make way for the mathematics in me. I am looking forward to it.

Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt zu haben. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.