Ends of graphs

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Chapter 1

Introduction

Our topic is infinite graph theory, with our focus on the *ends* of an infinite graph (which can be informally viewed as endpoints of rays), and their role in extensions of results known for finite graphs. Often, these extensions fail, if one does not take into account the ends of the graph, but otherwise hold. In other cases, results become more interesting when ends are considered as well as vertices.

An example for the latter is the Erdős-Menger conjecture for infinite graphs (recently proved by Aharoni and Berger): we shall prove a generalization which allows for ends in the considered paths and separators. This means that in an infinite graph, we allow paths to be infinite. Moreover, considering ends on a par with vertices, we will allow these paths, then called *arcs*, to start or end in ends, and to pass through them. Similarly, the notion of a cycle will be generalized to that of a (possibly infinite) *circle*, which may pass through ends. This leads to a different notion of forests (so-called *topological forests*) in infinite graphs.

Another aspect of the ends is that since in many ways they behave like vertices, they should be attributed a *degree*. We introduce such a notion as well as a concept of *parity* for ends. For ends of finite degree the parity will coincide with the parity of the degree, while ends of infinite degree will be classified into 'even' and 'odd'. Using these concepts (arcs, circles, topological forests, degrees and parities of ends) we extend several results from finite graph theory verbatim to infinite graphs.

Formally, an *end* of an infinite graph is an equivalence class of rays, where two rays are equivalent if no finite set of vertices separates them. The origin of this notion dates back to the 1940's when it was first introduced by Hopf [27] and Freudenthal [23], later it was reintroduced independently by Halin [24]. An infinite graph G together with its ends can be viewed as a topological space $|G|$ (for locally finite graphs also known as the *Freudenthal compactification* of G); the topology we endow $|G|$ with is due to Freudenthal [22] and Jung [29].

From now on, we will view the graph G with its ends topologically rather than in the usual combinatorial way, attaching equal importance to the ends of G as to the vertices. So our analogues of paths in the topological space $|G|$ will be homeomorphic images of the unit interval, so-called *arcs*, which may start in, pass through, and end in ends. All of these topological concepts as well as some basic terminology will be introduced in detail in Chapter 2.

We adopt our topological viewpoint in Chapter 3, whose topic is a well-known conjecture of Erdős (see Nash-Williams [39]), concerning a non-trivial extension of Menger's theorem to infinite graphs. It asks whether, given an infinite graph G and sets $A, B \subseteq V(G)$, there exists a family of disjoint A–B paths $\mathcal{P}$ together with an A–B separator X consisting of a choice of one vertex from each path in $\mathcal{P}$.

A topological extension to infinite graphs of this conjecture is to consider arcs instead of paths, and to allow A, B and X to contain ends as well as vertices. It then becomes necessary to require disjointness of the closures of A and B. If the disjointness is attained, then the purely topological version can be reduced (Diestel [13]) to the following alternative natural extension, which only allows ends as starting and ending points of paths, and in the separator.

Theorem 3.1.1. [9] *Let $G = (V, E, \Omega)$ be a graph and let $A, B \subseteq V \cup \Omega$ be such that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$, the closures being taken in $|G|$. Then G satisfies the Erdős-Menger conjecture for A and B.*

We prove this extension by reducing it to the vertex version, which was recently established by Aharoni and Berger [1]. We shall further see that the condition $A \cap \overline{B} = \emptyset = \overline{A} \cap B$ cannot be dropped, not even for graphs that are poor in structure, such as trees. [9]

In the same way as paths in infinite graphs are generalized to arcs, the notion of cycles should be generalized in a way that allows them to pass through ends. This leads to a definition of a *circle* as a homeomorphic image of the unit circle in the compactified graph $|G|$. For example, a double-ray whose subrays are equivalent in some underlying graph G, forms a circle in $|G|$ if we add this end. On the other hand, viewed on its own, the double-ray has two ends, together with which it will not form a circle. Not only infinite circles will be admitted, but also certain *thin* infinite sums (these are such that no vertex or edge is repeated infinitely often). The resulting cycle space $\mathcal{C}(G)$ introduced by Diestel and Kühn [17, 18] (sometimes referred to as the *topological cycle space*) retains all the basic properties of the cycle space of a finite graph.

One of these is the characterisation of a cycle space element as the edge set of a subgraph H that has all degrees even. This characterisation does not extend to elements of the topological cycle space of an infinite graph, if we only consider degrees of vertices. To see this, consider again the example of the double-ray: it does not form a circle (together with its ends), although all vertices have even degree.

This motivates us to introduce a degree concept for the ends of an infinite graph. [12] In the same way as the degree of a vertex is the number of incident edges, the degree of an end should be related to its rays. So there seem to be two sensible notions of the degree of an end ω: the first is the *vertex-degree*, defined as the maximal cardinality of a set of vertex-disjoint rays in ω, the second is the *edge-degree*, defined as the maximal cardinality of a set of edge-disjoint rays in ω (both possibly infinite). That these maxima do indeed exist is non-trivial, but a result of Halin [25] resp. Chapter 4/ [12]. Observe that with either of these two notions the counterexample of the double ray above ceases to be one, as its ends have vertex- and edge-degree 1.

Which of the two different concepts is adequate depends on the situation. In the case of cycle space problems, the edge version is more natural, and in fact, the vertex version is not sufficient. (In Chapter 5, we will encounter a situation where the vertex-degree is appropriate and needed.) Introducing also a concept of *parity* for ends of infinite edge-degree, we show in Chapter 4 the following special case of the characterisation of the cycle space elements.

Theorem 4.1.4. **[12]** *Let G be a locally finite graph. Then $E(G) \in \mathcal{C}(G)$ if and only if every vertex and every end of G has even edge-degree.*

The definition of the edge-degree of an end in a *subgraph H* is slightly more complicated: it turns out that instead of counting ω-rays one should count arcs converging to ω. With this notion we show that the cycles of a locally finite graph are precisely those connected subgraphs in which all vertices and all ends have degree resp. edge-degree 2. This is a straightforward generalization of the fact that in a finite graph the cycles are the 2-regular connected subgraphs. [12]

In Chapter 5 (see also [44]), we gain insight into the main difference of the two degree concepts for ends. While the edge-degree is appropriate in situations where edges matter, as in questions concerning the cycle space, the vertex-degree is needed in situations where vertices play the more important role.

This becomes clear when we try to extend a well-known theorem of Mader [36] to locally finite graphs. It states that if a finite graph has average (and hence minimum) degree at least $4k + 1$, then it contains a k-connected subgraph. Now, in locally finite graphs it is necessary to require not only high minimum degree for the vertices (which alone will not force any interesting substructure, as there are infinite trees of arbitrarily high minimum degree), but also high minimum *vertex-degree* for the ends of the graph in order to obtain a highly connected subgraph. More precisely, with a minimum degree resp. vertex-degree of order k^2 in vertices and ends we are able to force a k-connected subgraph.

Theorem 5.1.2. **[44]** *Let $k \in \mathbb{N}$ and let G be an infinite locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then G has a k-connected subgraph.*

If, on the other hand, in addition to the high degrees at the vertices, we only require high *edge-degree* for the ends, Mader's theorem does not extend to infinite graphs. We exhibit a counterexample in respect to this. But, high minimum edge-degree at the ends (together with high minimum degree at the vertices) suffices to force highly edge-connected subgraphs in locally finite graphs. [44] In fact, the minimum (edge-)degree we require for a locally finite graph in order to have a k-connected subgraph is only linear in k.

Another application of the end degree concept will be given in Chapter 6 (see also [42]), where we extend Nash-Williams' arboricity theorem [38] to locally finite graphs. This states that a finite graph is the edge-disjoint union of at most k forests if no set of ℓ vertices induces more than $k(\ell - 1)$ edges. The theorem extends easily, if the usual notion of a forest is used, which is that of a graph that contains no finite cycles. But in our topological setting, considering only such forests is not appropriate. The strengthening we prove, forbids the partitioning forests (or more precisely their closures) to contain circles, i.e. requires them to be *topological forests.*

This can only be achieved by a further condition: we have to place an upper bound on the degrees of the ends of the graph. Here, again, we consider the edge-degrees of the ends, which yield a smaller restriction and are more natural in the situation (as we are dealing with topological forests, i.e. circles).

Theorem 6.1.2. [42] *Let $k \in \mathbb{N}$, and let G be a locally finite graph in which no set of ℓ vertices induces more than $k(\ell-1)$ edges. Furthermore, let every end of G have edge-degree $< 2k$. Then $|G|$ is the edge-disjoint union of at most k topological forests in $|G|$.*

Next, we shall give extensions to infinite graphs of results that concern cycles, or the cycle space. We start in Chapter 7 with the generalization to locally finite graphs of a result by Gallai (see Lovász [33]). This states that every finite graph G has a vertex partition into two parts such that each induces an element of the cycle space of G. We showthat the theorem fails for infinite graphs if the cycle space is defined as the span of the edge sets of finite cycles in G, but extends with the topological cycle space $\mathcal{C}(G)$.

Theorem 7.1.4. [8] *For every locally finite graph G there is a partition of $V(G)$ into two (possibly empty) sets V_1, V_2 such that $E(G[V_i]) \in \mathcal{C}(G)$ for both $i = 1, 2$.*

Using similar techniques we prove that if Seymour's faithful cycle cover conjecture [41] is true for finite graphs then it also holds for locally finite graphs when infinite cyles are allowed in the cover, but not otherwise. We also consider extensions of both results to certain classes of graphs with infinite degrees. [8]

The next chapter, Chapter 8, is devoted to an extension of MacLane's planarity criterion to locally finite graphs. The original version of this theorem [34] states that a finite graph is planar if and only if its cycle space has a basis $\mathcal{B}$ such that every edge is contained in at most two members of $\mathcal{B}$. Solving a problem of Wagner [46], we show that the topological cycle space allows a verbatim generalization of MacLane's criterion to locally finite graphs.

Theorem 8.1.3. [11] *Let G be a countable locally finite graph. Then, G is planar if and only if $\mathcal{C}(G)$ has a simple generating set.*

This extension then enables us to extend also Kelmans' planarity criterion [30]. Both MacLane's and Kelmans' theorem fail in infinite graphs if only finite cycles are allowed. We again prove extensions to certain classes of graphs with infinite degrees. [11]

We now turn to a question on finite graphs due to Locke [32]. He asked under which conditions the cycle space of a finite graph is spanned by its long cycles. More precisely, the question is whether there exists a smallest m, such that if in a finite graph G, every two vertices are joined by a path of length mk, where $k \in \mathbb{N}$, then the cycle space $\mathcal{C}(G)$ is generated by the cycles of length $\geq k$.

Locke proves [32] his conjecture for the case that m is allowed to depend on k: then $m \leq k$. We show that $m \leq 2$, which also holds for infinite locally finite graphs. For such, we generalize the problem to infinite k, which leads inevitably to a topological reformulation of the problem. We prove that if every two vertices are linked by an arc of infinite length (i.e. an arc that passes through an end), then the cycle space is spanned by the infinite circles (more precisely, by their edge sets, which we shall call *circuits*). Together, this amounts to the following theorem.

Theorem 9.1.2. **[10]** *Let $k \in \mathbb{N} \cup \infty$. If every two vertices of a locally finite graph G are the endvertices of an arc of length $2k$, then the circuits of length $\geq k$ generate the cycle space $\mathcal{C}(G)$ of G.*

It is easily seen by a Mengerian argument that in an infinite locally finite 2-connected graph the condition of Theorem 9.1.2. is satisfied. Thus, the topological cycle space of an infinite locally finite 2-connected graph is generated by its infinite circuits.

Chapter 2

Terminology and basic facts

2.1 Basics: rays, ends and separators

The basic terminology we use can be found in Diestel [16]. Our graphs are undirected, and, unless otherwise stated, simple. When A is a set, we write $\bigcup A$ for the union of all its elements.

Let G be a fixed infinite graph. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray are its *tails*. Two rays in a graph G are *equivalent* if no finite set of vertices separates them. As one easily observes, this condition holds if and only if there are infinitely many disjoint (finite) R_1–R_2 paths. This in turn is equivalent to the existence of a ray that meets both R_1 and R_2 infinitely often. The corresponding equivalence classes of rays are the *ends* of G. We denote the set of ends of G by $\Omega(G)$. An ω*-ray* is simply a ray of $\omega \in \Omega(G)$.

A set S of vertices or edges of G is said to *separate* a set $V' \subseteq V(G)$ from an end $\omega \in \Omega(G)$ if it meets every ω-ray that starts in V'. This is equivalent to that the (unique) component C of $G - S$ with $\omega \in \overline{C}$ is disjoint from V'. Similarily, S *separates* two ends ω and ω', if the closure of each component of $G - S$ contains at most one of ω, ω'.

For a subgraph $H \subseteq G$, the *boundary* $\partial^*_G H$ of H (or $\partial^* H$, where no confusion is possible) is the set $N(G - H)$ of all neighbours in H of vertices of $G - H$. Analogously, the *co-boundary* $\partial_G H$ of H (or ∂H) is the cut $E_G(H, G - H)$. In particular, $\partial^* G$, ∂G, $\partial^* \emptyset$, and $\partial \emptyset$ are all empty.

A *region* of G is an induced subgraph H which is connected and whose co-boundary is finite. Then $H' \subseteq H$ is a region of G if and only if it is a region of H. The region H is *even* resp. *odd* if $|\partial H|$ is even resp. odd. Note that given a subgraph $H \subseteq G$ and an end $\omega \in \Omega(G)$ with $\omega \notin \overline{H}$ its boundary $\partial^* H$ separates ω from $V(H)$. The same is true for the co-boundary ∂H.

A standard tool in infinite graph theory is König's infinity lemma (see for example Diestel [16] for a proof):

Lemma 2.1.1. *Let $W_1, W_2, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let H be a graph on their union. For every $n \geq 2$ assume that each vertex in W_n has a neighbour in W_{n-1}. Then H contains a ray $v_1 v_2 \dots$ with $v_n \in W_n$ for all n.*

2.2 The topological space $|G|$

Let us define a topology, which we call VTOP, on G together with its ends; if G is locally finite, it is known as its *Freudenthal compactification*. At the end of Chapter 7, we introduce a topology on certain classes of graphs with infinite degree, which is called ITOP. For locally finite graphs, VTOP and ITOP coincide.

We begin by viewing G itself (without ends) as the point set of a 1-complex. Then every edge is a copy of the real interval $[0, 1]$, and we give it the corresponding metric and topology. For every vertex v we take as a basis of open neighbourhoods the open stars of radius $1/n$ around v. (That is to say, for every integer $n \geq 1$ we declare as open the set of all points on edges at v that have distance less than $1/n$ from v, in the metric of that edge.)[1]

In order to extend this topology to $\Omega(G)$, we take as a basis of open neighbourhoods of a given end $\omega \in \Omega(G)$ the sets of the form

$$\hat{C}(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}(S, \omega)\,,$$

where $S \subseteq V(G)$ is a finite set of vertices, $C(S, \omega)$ is the unique component of $G - S$ in which every ray in ω has a tail, $\Omega(S, \omega)$ is the set of all ends $\omega' \in \Omega(G)$ whose rays have a tail in $C(S, \omega)$, and $\mathring{E}(S, \omega)$ is the set of all inner points of edges between S and $C(S, \omega)$. We also write $\overline{C}(S, \omega)$ for the union of $C(S, \omega)$ and $\Omega(S, \omega)$.

Let $|G|$ denote the topological space on the point set $V(G) \cup \Omega(G) \cup \bigcup E(G)$ thus defined. We shall freely view G and its subgraphs either as abstract graphs or as subspaces of $|G|$. Note that in $|G|$ every ray converges to the end of which it is an element.

Given a set $X \subseteq |G|$, put $V(X) := X \cap V$, and let $E(X)$ be the set of edges e with $e \subseteq X$. We write $\overline{X}$ for the closure of X in $|G|$. For example, the set $\overline{C}(S, \omega)$ defined above is the closure in $|G|$ of the set $C(S, \omega)$. Generally, the difference between a subgraph H and its closure $\overline{H}$ is always a set of ends of G (possibly empty). These need not correspond to ends of H and should not be confused with them. For example, if G is the 1-way infinite ladder and H consists of all the rungs, then $\overline{H} \setminus H$ consists of one point, the unique end ω of G. But H itself has no ends. Similarly, the subgraph $H' = G - E(H)$ of G consists of two disjoint rays and thus has two ends, but $\overline{H'} \setminus H' = \{\omega\}$ as before.

[1]If G is locally finite, this is the usual identification topology of the 1-complex. Vertices of infinite degree, however, have a countable neighbourhood basis in VTOP, which they do not have in the 1-complex.

2.3 Arcs, circles and topological forests

Let us first see how the notion of a path generalizes in our topological setting. A continuous image of the unit interval $[0,1]$ in $|G|$ is a *topological path*. The images of 0 and 1 are the *endpoints* of the topological path. A homeomorphic image of $[0,1]$ in $|G|$ is called an *arc* in $|G|$. Observe that this definition includes all finite paths. Analogously to ω-rays, let us say that an arc is an ω-arc, if the end ω is one of its endpoints.

Similarly, a set $C \subseteq |G|$ is a *circle* if it is homeomorphic to the unit circle. Then C includes every edge of which it contains an inner point, and the graph consisting of these edges and their endvertices is the *cycle* defined by C. Conversely, it is not hard to show [17] that $C \cap G$ is dense in C, so every circle is the closure in $|G|$ of its cycle and hence defined uniquely by it. Note that every finite cycle in G is also a cycle in this sense, but there can also be infinite cycles. The edge set of a cycle is called a *circuit*. See [17, 18] for more details on infinite cycles.

Having adapted the notion of a cycle to our topological viewpoint, we must do the same for forests and, in particular, spanning trees. The closure $\overline{H}$ in $|G|$ of a subgraph H of G is a *topological forest* if it contains no circles. A *topological spanning tree* is a path-connected topological forest in $|G|$ that contains all vertices of G (it then also contains all ends and all edges of which it contains inner points). See [19] for more details on topological spanning trees.

A fundamental property of a tree is that it contains a path between any two of its vertices. That is the reason why topological spanning trees are required to be path-connected rather than only topologically connected. The next theorem shows that this makes no difference in locally finite graphs[2].

Theorem 2.3.1 (Diestel and Kühn [19]). *If G is locally finite, then every closed connected subset of $|G|$ is path-connected.*

2.4 Degrees of ends

Let us now introduce our concepts of end degrees. As ends are equivalence classes of rays, the degree of an end should in some way be related to its rays. Also, the rays may be seen as somewhat analoguos to the incident edges of a vertex, whose number is the degree of the vertex.

Thus there are basically two possibilities how the degree notion can be extended to ends. The *vertex-degree* (also known as the *multiplicity*) of an end $\omega \in \Omega(G)$ is defined as the supremum of the cardinalities of sets of vertex-disjoint rays in ω. Similarly, the *edge-degree* of ω is the supremum of the cardinalities of sets of edge-disjoint rays in ω. These two suprema are indeed maxima: this is shown

[2] Although the topology considered in [19] is slightly different, it coincides with ours for locally finite graphs. See also the footnote following Theorem 2.5.1.

in [25] for the vertex-degree, and in Chapter 4, Lemma 4.4.5 (see also [12]), for the edge-degree (in this respect, Andreae [3] proves a similar result).

The (edge-) degree of an end in a subgraph $H \subseteq G$, and the parity of an end will be defined in Chapter 4, as they will only be needed there.

2.5 The cycle space $\mathcal{C}(G)$

Call a family $(D_i)_{i\in I}$ of subsets of $E(G)$ *thin* if no vertex of G is incident with an edge in D_i for infinitely many i. (Thus in particular, no edge lies in more than finitely many D_i.) Let the *sum* $\sum_{i\in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i, and let the *topological cycle space* $\mathcal{C}(G)$ of G be the set of all sums of (thin families of) circuits, finite or infinite. Symmetric difference as addition makes $\mathcal{C}(G)$ into an $\mathbb{F}_2$ vector space, which coincides with the usual cycle space of G when G is finite. We remark that $\mathcal{C}(G)$ is closed under taking infinite thin sums (Diestel and Kühn [17, 18]), which is not obvious from the definitions.

As with finite graphs, elements of the cycle space can be decomposed into cycles:

Theorem 2.5.1 (Diestel and Kühn [18]). *Every element of the topological cycle space $\mathcal{C}(G)$ of a graph G is the edge-disjoint union of cycles.*

We remark that, although the topology for $|G|$ considered in [17, 18, 19] is slightly larger than ours[3], Theorem 2.5.1, as well as Theorem 2.3.1 above and Theorem 2.5.2 below, are nevertheless applicable in our context. This is because the cycles in $|G|$ coincide for these topologies: as one readily checks, the identity on $|G|$ between the two spaces is bicontinuous when restricted to a circle in either space.

The orthogonality to every cut of G is another basic characterisation of the elements of the cycle space of a finite graph. The straightforward extension of this fact will serve as one of our main tools to decide whether a given set of edges is an element of the cycle space.

Theorem 2.5.2 (Diestel and Kühn [17]). *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then $Z \in \mathcal{C}(G)$ if and only if $|F \cap Z|$ is even for every finite cut F of G.*

[3]There, some more basic open sets are allowed: in the place of $\mathring{E}(S,\omega)$ we could take an arbitrary union of open half-edges from C towards S, one from every S–C edge. When G is locally finite, this yields the same topology. When G has vertices of infinite degree, it is easy to see that our topology is slightly sparser but still yields the same topological cycle space.

Chapter 3

The Erdős-Menger conjecture with ends

3.1 Introduction

Erdős conjectured (see Nash-Williams [39]) that Menger's theorem should extend to infinite graphs as follows:

Erdős-Menger Conjecture. *For every graph $G = (V, E)$ and any two sets $A, B \subseteq V$ there is a set $\mathcal{P}$ of disjoint A–B paths in G and an A–B separator X consisting of a choice of one vertex from each of the paths in $\mathcal{P}$.*

A proof of this conjecture has recently been obtained by Aharoni and Berger [1].

There is a natural extension of the Erdős-Menger conjecture in which the sets A and B may contain ends as well as vertices. Here, the A–B paths in $\mathcal{P}$ can be either finite paths linking two vertices, or rays linking a vertex to an end, or double rays linking two ends. Similarly, the separator X may contain ends (that lie in A or B), thus blocking any ray belonging (= converging) to that end. These notions will be precisely defined in the next section.

We prove the extended ends version of the conjecture by reducing it to the vertex version. Our proof uses a refinement of techniques developed by Diestel [13], where this reduction was carried out for countable graphs.

The Erdős-Menger conjecture for ends is not true for arbitrary sets A and B (of vertices and ends): a necessary condition is that the closure of A in $|G|$ does not meet B, and vice versa. This condition cannot even be dropped, if the considered graph G is a tree. An example is this respect will be given in Section 3.3.3.

The main result of this chapter is the following.

Theorem 3.1.1. **[9]** *Let $G = (V, E, \Omega)$ be a graph and let $A, B \subseteq V \cup \Omega$ be such that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$, the closures being taken in $|G|$. Then G satisfies the Erdős-Menger conjecture for A and B.*

As mentioned in the introduction, one may also consider a purely topological version of the Erdős-Menger conjecture, in which $\mathcal{P}$ is any set of A–B arcs in the space $|G|$, and the set X is required to meet every A–B arc in $|G|$. This version of the conjecture can fail unless A and B have disjoint closures in $|G|$. But in that case it can be reduced to Theorem 3.1.1 (see Diestel [13]), so the purely topological version offers nothing new.

3.2 Discussion of the ends version

Let us make clear the definitions of paths and separators, which differ slightly from the usual ones (as they may contain ends), but are vital for the precise meaning of our result. Throughout this chapter, *paths* in G can be finite paths (which contain at least one vertex), rays, double rays, or singleton sets $\{\omega\}$, where ω is an end of G. The closure of an infinite path P contains one or two ends of G. (Even if P is a double ray, its closure may contain only one end, as in the ladder example above.) We will often consider such an end as the first or last point of P, and when we say that two paths are *disjoint* then these points too shall be distinct. (The first and last point of a path $P = \{\omega\}$, of course, is ω.) For $A, B \subseteq V \cup \Omega$, a path is an *$A$–$B$ path* if its first but no other point lies in A and its last but no other point lies in B.

The union of a ray R and infinitely many disjoint paths starting on R but otherwise disjoint from R is a *comb* with *spine* R. The last points (vertices or ends) of those paths are the *teeth* of the comb. We will frequently use the following simple lemma:

Lemma 3.2.1. **[9]** *In the graph $G = (V, E, \Omega)$ let R be a ray of an end ω, and let $X \subseteq V \cup \Omega$ such that $\omega \notin X$. Then $\omega \in \overline{X}$ if and only if G contains a comb with spine R and teeth in X.* $\square$

A set $X \subseteq V \cup \Omega$ is an A–B *separator* in a subspace $T \subseteq |G|$ if every path P in T with its first point in A and its last point in B satisfies $\overline{P} \cap X \neq \emptyset$. (We express this informally by saying that "P meets X", though strictly speaking we shall mean $\overline{P}$ rather than just P.) We say that a set $Y \subseteq V \cup \Omega$ lies *on* a set $\mathcal{P}$ of disjoint A–B paths if Y consists of a choice of exactly one vertex or end from every path in $\mathcal{P}$. We say that G *satisfies the Erdős-Menger conjecture for A and B*, or that *the Erdős-Menger conjecture holds for G, A, B*, if $|G|$ contains a set $\mathcal{P}$ of disjoint A–B paths and an A–B separator on $\mathcal{P}$. (Thus, officially, we always refer to the ends version of the conjecture. But this is compatible with the traditional terminology: if neither A nor B contains an end then neither can any A–B path, so the conjecture with ends automatically defaults to the original conjecture in this case.)

The terms needed to state the main result of this chapter are now precisely defined. We shall prove the following slight strengthening of Theorem 3.1.1 which, as in the vertex case, allows the intersection of A and B itself to be non-empty:

Theorem 3.2.2. [9] *Let $G = (V, E, \Omega)$ be a graph, and let $A, B \subseteq V \cup \Omega$ be such that $A \cap (\overline{B} \setminus B) = \emptyset = (\overline{A} \setminus A) \cap B$. Then G satisfies the Erdős-Menger conjecture for A and B.*

We remark that the disjointness condition in Theorem 3.2.2 is necessary, even if the considered graph has a simple structure, e.g. is a tree; a counterexample for when the condition is violated is given in Section 3.3. The disjointness condition means that any ray whose end lies in A can be separated from B by a finite set of vertices, and vice versa with A and B interchanged. Note that this does not imply the much stronger condition that A and B can be finitely separated, in which case the proof is immediate by standard alternating path techniques (see Diestel [13]). A more typical example for the disjointness condition is to take as A and B distinct levels of vertices in a tree: if the tree is $\aleph_0$-regular, for example, it contains infinitely many disjoint paths between these levels, so A and B have disjoint closures (in fact, are closed and disjoint) but cannot be finitely separated.

3.3 Trees are not easier

Let T be obtained from the infinite binary tree by adding a copy v' of each vertex v and joining v and v' with an edge. Denote by A the set of all newly added vertices, and choose as B the set $\Omega(T)$. Observe that $\overline{A} \cap B \neq \emptyset$. In fact, this violation of the disjoint closures condition makes the Erdős-Menger conjecture fail.

Suppose that there is a set of disjoint A–B paths $\mathcal{P}$ and an A–B separator X on $\mathcal{P}$. We claim that $X \subseteq B$. Indeed, otherwise there is path $P \in \mathcal{P}$ that meets X in a vertex x. Let y be the vertex that follows x on P, and let z be the one that follows y. Then zP fails to meet X, implying that $z' \in X$, since X separates z' from B. But then the path $P' \in \mathcal{P}$ that starts in z' meets P. As $x \neq z'$, and hence $P \neq P'$, this contradicts the disjointness of the paths in $\mathcal{P}$.

We have thus shown that $X \subseteq B$. Now, as $|A|$ is countable, while $|B|$ is not, there is an end $\omega \in B$ which $\mathcal{P}$ and hence also X misses. But then we easily find an A–ω path that misses X, yielding the desired contradiction, as X is an A–B separator.

3.4 Proof of the theorem

Our aim is to reduce the ends version of the Erdős-Menger conjecture, Theorem 3.2.2, to the original vertex version as stated in the Introduction and recently proved by Aharoni and Berger.

We begin by showing that, as in the vertex case of the conjecture, we may assume without loss of generality that $A \cap B = \emptyset$. In the vertex case, one simply deletes $A \cap B$ from the graph, finds a path system and separator in $G - (A \cap B)$, and then adds the deleted vertices both to the path system (as singleton A–B paths) and to the separator, to obtain a solution for G. When $A \cap B$ is infinite, however, deleting it can result in the destruction or splitting of ends. Before we allow ourselves to assume that $A \cap B = \emptyset$, therefore, we have to make sure that this will not affect any ends in A or B. Our first lemma ensures this, and thereby reduces the stronger form of our theorem (Theorem 3.2.2) to the version stated in the introduction, Theorem 3.1.1.

Lemma 3.4.1. [9] *Let $G = (V, E, \Omega)$ be a graph, and let $A, B \subseteq V \cup \Omega$ satisfy*

$$A \cap (\overline{B} \setminus B) = \emptyset = (\overline{A} \setminus A) \cap B.$$

Then for the graph $G' := G - (A \cap B \cap V)$ there are sets $A', B' \subseteq V(G') \cup \Omega(G')$ satisfying the following conditions:

(i) *if $A \subseteq V$ then $A' \subseteq A$, and if $B \subseteq V$ then $B' \subseteq B$;*

(ii) $A' \cap \overline{B'} = \emptyset = \overline{A'} \cap B'$;

(iii) *if G' satisfies the Erdős-Menger conjecture for A' and B', then G satisfies it for A and B.*

Proof. Put $A' := A \setminus B$ and $B' := B \setminus A$, both of which are subsets of $|G|$. Consider a ray R of an end α in A' or B', say in A'. Then R has a tail in G'. Indeed, if not then there are vertices of $A \cap B \cap V \subseteq B$ in every neighbourhood of $\alpha \in A \setminus B$. Consequently, $\alpha \in A \cap (\overline{B} \setminus B)$, which is a contradiction. Similarly, two rays R_1, R_2 in G' of which R_1 is a ray of an end $\omega \in A' \cup B'$ are equivalent in G' if and only if they are equivalent in G. Indeed, if R_1 and R_2 are equivalent in G then there is a ray $R_3 \in \omega$ that meets both of R_1 and R_2 infinitely often. Now R_3 has a tail in G', showing that R_1 and R_2 are also equivalent in G'.

Thus, mapping every end of G in $A' \cup B'$ to the unique end of G' that contains tails of its rays defines a bijection between the ends in $A' \cup B'$ and certain ends in G'. Using this bijection (and a slight abuse of notation) we may view A' and B' also as subsets of $V(G') \cup \Omega(G')$. Clearly, these satisfy (i). Moreover, $A' \cap B'$ is still empty, so the disjointness assumption stated in the lemma implies (ii).

For (iii), let X' be an A'–B' separator on a set of disjoint A'–B' paths $\mathcal{P}'$ in G'. Adding to $\mathcal{P}'$ the trivial paths $\{x\}$ for all $x \in A \cap B$ yields a set $\mathcal{P}$ of disjoint A–B paths with the A–B separator $X := X' \cup (A \cap B)$ on it. □

In Lemma 3.4.5, we shall need a family of disjoint subgraphs of G (with certain properties) such that every end of A lies in the closure of one of these subgraphs. Such a family cannot always be found. But our next lemma finds instead a family of subgraphs such that the ends of A not contained in their closures form a set I that can be ignored: those ends will automatically be separated from B by any $(A \setminus I)$–B separator on a set of disjoint A–B paths.

Lemma 3.4.2. [9] *Let $G = (V, E, \Omega)$ be a graph, and let $A, B \subseteq V \cup \Omega$ be such that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Then for every set $A_\Omega \subseteq A \cap \Omega$ there exist a set $I \subseteq A_\Omega$,* *an ordinal μ^*, and families $(G_\mu)_{\mu<\mu^*}$ and $(S_\mu)_{\mu<\mu^*}$ such that, for every $\mu < \mu^*$, the graph $G_\mu - S_\mu$ is a component of $G - S_\mu$ with S_μ as its finite set of neighbours, and*

(i) $\overline{G_\mu - S_\mu} \cap B = \emptyset$;

(ii) *if* $G_\mu \neq \emptyset$ *then* $\overline{G_\mu} \cap A_\Omega \neq \emptyset$;

(iii) $V(G_\nu \cap G_\mu) \subseteq S_\nu \cap S_\mu$ *for all* $\nu < \mu$.

Moreover,

(iv) *for every end* $\alpha \in A_\Omega \setminus I$ *there is a* $\mu < \mu^*$ *with* $\alpha \in \overline{G_\mu}$;

(v) *every* $(A \setminus I)$*–*B *separator on a set of disjoint* $(A \setminus I)$*–*B *paths is also an* A*–*B *separator.*

Proof. We construct the families $(G_\mu)_{\mu<\mu^*}$ and $(S_\mu)_{\mu<\mu^*}$ and a transfinite sequence $I_0 \subseteq I_1 \subseteq \ldots \subseteq A_\Omega$ recursively. The sets I_μ $(\mu < \mu^*)$ will serve as precursors to I. To simplify notation, we write $C_\mu := G_\mu - S_\mu$ for every μ. For the construction, we will in addition to (i)–(iii) require for every μ that

(vi) $I_\mu \cap \overline{G_\nu} = \emptyset$ for all $\nu \leq \mu$.

We start by setting $I_0, G_0, S_0 := \emptyset$. Consider the least ordinal $\mu > 0$ such that the above sets are already defined for all $\lambda < \mu$. If μ is a limit, we set

$$I_\mu := \bigcup_{\lambda<\mu} I_\lambda$$

and $G_\mu, S_\mu := \emptyset$. This choice clearly satisfies (i)–(iii) and (vi).

Suppose now that μ is a successor, $\mu = \lambda + 1$ say. If every end in $A_\Omega \setminus I_\lambda$ lies in some $\overline{G_\nu}$ with $\nu < \mu$, we set $\mu^* := \mu$ and terminate the recursion. So suppose there is an end $\alpha \in A_\Omega \setminus I_\lambda$ that lies in no earlier $\overline{G_\nu}$. Then, if possible, choose a finite vertex set S such that $C(S, \alpha)$ avoids all G_ν with $\nu < \mu$.

Such a choice of S is impossible if and only if

$$\textit{for every finite } S \subseteq V \textit{ there is a } \nu < \mu \textit{ with } C(S, \alpha) \cap G_\nu \neq \emptyset. \tag{3.1}$$

In this case we choose to ignore α, i.e. set $I_\mu := I_\lambda \cup \{\alpha\}$ and $G_\mu, S_\mu := \emptyset$. Again the requirements (i)–(iii) are clearly met, while (vi) holds by the choice of α.

Now suppose we can find S as desired. As $A \cap \overline{B} = \emptyset$, we can also find a basic open neighbourhood $\hat{C}(S', \alpha)$ of α in $|G|$ that is disjoint from B. We now define S_μ as the set of neighbours of $C(S \cup S', \alpha)$ and $G_\mu := G[S_\mu \cup C(S_\mu, \alpha)]$. Then (i) holds since $S_\mu \supseteq S'$, while (ii) holds as $\alpha \in \overline{G_\mu}$. To see (iii), first note that

$$G_\nu \cap C_\mu = \emptyset \text{ for all } \nu < \mu$$

by the choice of S. So, all we have to show is that $G_\nu \cap S_\mu \subseteq S_\nu$. Consider a vertex $v \in G_\nu \cap S_\mu$. Since S_μ is the set of neighbours of C_μ, there is a vertex $w \in C_\mu$ adjacent to v. As noted above, $w \notin G_\nu$. So v is a vertex in $G_\nu = C_\nu \cup N(C_\nu)$ with a neighbour outside G_ν, implying $v \notin C_\nu$ and hence $v \in S_\nu$, as desired.

Let us finally set $I_\mu := I_\lambda$ and verify (vi). We only need to show that $I_\mu \cap \overline{G_\mu} = \emptyset$. Suppose that intersection contains an end α'. Let $\mu' < \mu$ be minimal such that $\alpha' \in I_{\mu'}$. Then (3.1) should have been satisfied for μ' and α', but fails with $S := S_\mu$ as $C(S_\mu, \alpha') = C_\mu$, a contradiction.

Having defined I_μ, G_μ and S_μ for all $\mu < \mu^*$ so that (i)–(iii) and (vi) are satisfied, we put

$$I := \bigcup_{\mu < \mu^*} I_\mu.$$

Together with the definition of μ^* this implies (iv). Observe that from (vi) we obtain $I \cap \overline{G_\mu} = \emptyset$ for all $\mu < \mu^*$.

To establish (v) let $\mathcal{P}$ be a system of disjoint $(A \setminus I)$–B paths and X an $(A \setminus I)$–B separator on $\mathcal{P}$. Now suppose that X is not an A–B separator in $|G|$, i.e. there is a path Q from A to B that avoids X. By turning Q into a path $\tilde{Q}$ from $A \setminus I$ to B that avoids X, we will obtain a contradiction.

We may assume that Q starts at an end $\alpha \in I$. Let μ be the step at which α was added to I, i.e. let μ be minimal with $\alpha \in I_\mu$. Choose a finite vertex set S such that $\overline{C}(S, \alpha)$ is disjoint from B (this is possible, as $A \cap \overline{B} = \emptyset$). Then any path of $\mathcal{P}$ that meets $C(S, \alpha)$ must pass through S. Hence only finitely many paths of $\mathcal{P}$ can meet $C(S, \alpha)$, and so $X_\alpha := X \cap \overline{C}(S, \alpha)$ is also finite. Conditions (iii) and (iv) ensure that every end in X_α lies in exactly one $\overline{C_\lambda}$; let $\{\lambda_1, \dots, \lambda_m\}$ be the set of these λ. Then for

$$S' := S \cup (X_\alpha \cap V) \cup \bigcup_{i=0}^{m} S_{\lambda_i}$$

we have

$$\overline{C}(S', \alpha) \cap X = \emptyset.$$

Now, all we need is a point of $A \setminus I$ that lies in $\overline{C}(S', \alpha)$ (and thus can be used to change Q into the desired path). Indeed, if there is an ordinal $\lambda < \mu$ such that $G_\lambda \neq \emptyset$ and

$$C_\lambda \subseteq C(S', \alpha), \tag{3.2}$$

we can complete the proof as follows. By (ii) for λ there will be an end $\alpha' \in A$ in $\overline{C_\lambda} \subseteq \overline{C}(S', \alpha)$. Since $I \cap \overline{G_\lambda} = \emptyset$, we have $\alpha' \in A \setminus I$. Take an α'–Q path P in $\overline{C}(S', \alpha)$ with last vertex x, say. Then P avoids X, and hence so does the path $\tilde{Q} := PxQ$. Thus, $\tilde{Q}$ is as desired.

So suppose there is no ordinal $\lambda < \mu$ satisfying (3.2). Then for all $\lambda < \mu$ we have either $C_\lambda \cap C(S', \alpha) = \emptyset$ or $C_\lambda \cap S' \neq \emptyset$. As all the C_λ are disjoint by (iii), only finitely many of them meet S'; let $\lambda_{m+1}, \dots, \lambda_n$ be the corresponding ordinals.

Then
$$S'' := S' \cup \bigcup_{i=m+1}^{n} S_{\lambda_i}$$
satisfies $C(S'', \alpha) \cap C_\lambda = \emptyset$ for all $\lambda < \mu$.
However, $G_\lambda \cap C(S'', \alpha)$ cannot be empty for all $\lambda < \mu$, as this would contradict (3.1) for step μ with $S := S''$. So there exists an ordinal $\lambda < \mu$ with $S_\lambda \cap C(S'', \alpha) \neq \emptyset$. A vertex v in this intersection must have a neighbour in C_λ, which then also lies in $S' \cup C(S', \alpha)$ because $C(S'', \alpha) \subseteq C(S', \alpha)$. Thus,
$$(S' \cup C(S', \alpha)) \cap C_\lambda \neq \emptyset.$$
Since $C_\lambda \nsubseteq C(S', \alpha)$ by assumption, this implies that C_λ meets S'. But then $\lambda \in \{\lambda_{m+1}, \dots, \lambda_n\}$ and hence $S_\lambda \subseteq S''$, contradicting the fact that v lies in both S_λ and $C(S'', \alpha)$. □

For our end-to-vertex reduction we need two more lemmas.

Lemma 3.4.3 (Diestel [13]). *Let H be a subgraph of a graph G, let $S \subseteq V(H)$ be finite, and let $T \subseteq V(H) \cup \Omega(G)$ be such that $T \subseteq \overline{H}$. Then $\overline{H}$ contains a set $\mathcal{P}$ of disjoint S–T-paths and an S–T-separator (in $\overline{H}$) on $\mathcal{P}$.*

For a set T of vertices in a graph H, a *T-path* is a path that meets T only in its first and last vertex. A set of paths will be called *disjoint outside* a given subgraph $Q \subseteq H$ if distinct paths meet only in Q.

Lemma 3.4.4. [43] *Let H be a graph, $T \subseteq V(H)$ finite, and $k \in \mathbb{N}$. Then H has a subgraph H' containing T such that for every T-path $Q = s \dots t$ in H meeting $H - H'$ there are k distinct T-paths from s to t in H' that are disjoint outside Q.*

Our next lemma allows us to replace the set $A \subseteq V \cup \Omega$ in Theorem 3.1.1 with a set A' consisting only of vertices.

Lemma 3.4.5. [9] *Let $G = (V, E, \Omega)$ be a graph, and let $A, B \subseteq V \cup \Omega$ be such that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Then there exist a minor $G' = (V', E', \Omega')$ of G and sets $A' \subseteq V'$ and $B' \subseteq V' \cup \Omega'$ satisfying the following conditions:*

(i) *if $B \subseteq V$ then $B' \subseteq B$;*

(ii) *$A' \cap \overline{B'} = \emptyset = \overline{A'} \cap B'$;*

(iii) *G satisfies the Erdős-Menger-conjecture for A and B if G' satisfies it for A' and B'.*

Proof. Applying Lemma 3.4.2 with $A_\Omega := A \cap \Omega$ we obtain an ordinal μ^*, subgraphs G_μ, finite vertex sets S_μ and a set of ends $I \subseteq A$. Our aim is to change G into G' by deleting and contracting certain connected subgraphs of our graphs $G_\mu - S_\mu$.

By Lemma 3.4.2 (iii) we shall be able to do this independently for the various G_μ: for each $\mu < \mu^*$ separately, we shall find in $G_\mu - S_\mu$ a set $\mathcal{D}_1(\mu)$ of connected subgraphs to be deleted, and another set $\mathcal{D}_2(\mu)$ of connected subgraphs that will be contracted.

Fix $\mu < \mu^*$. If G_μ is empty we let $\mathcal{D}_1(\mu) = \mathcal{D}_2(\mu) = \emptyset$. Assume now that $G_\mu \neq \emptyset$. Put $A_\mu := A \cap \overline{G_\mu}$. Applying Lemma 3.4.3 to $H = G_\mu$ we find in $\overline{G_\mu}$ a finite set $\mathcal{P}$ of disjoint S_μ–A_μ paths and an S_μ–A_μ separator X_μ on $\mathcal{P}$. We write $X_\mu = U_\mu \cup O_\mu$, where $U_\mu = X_\mu \cap V$ and $O_\mu = X_\mu \cap \Omega$, both of which are finite since $|X_\mu| \leq |\mathcal{P}| \leq |S_\mu|$. Moreover,

$$U_\mu \textit{ separates } S_\mu \textit{ from } A_\mu \setminus O_\mu \textit{ in } G. \tag{3.3}$$

Indeed, every S_μ–$(A_\mu \setminus O_\mu)$ path in G lies in $\overline{G_\mu}$ and hence meets X_μ, and since it cannot meet O_μ unless it ends there, it meets X_μ in U_μ.

We define $\mathcal{D}_1(\mu)$ as the set of all the components D of $G - U_\mu$ whose closure $\overline{D}$ meets $A_\mu \setminus O_\mu$. By (3.3), these components satisfy $D \subseteq G_\mu - S_\mu$, and their neighbourhood $N(D) \subseteq U_\mu$ in G is finite. In addition,

$$\overline{D} \cap O_\mu = \emptyset \textit{ for all } D \in \mathcal{D}_1(\mu). \tag{3.4}$$

For if $\alpha \in \overline{D} \cap O_\mu$, say, and P is the S_μ–A_μ path in $\mathcal{P}$ that ends in α, then P has a tail in D. Since P does not meet $U_\mu \supseteq N(D)$, this implies $P \subseteq \overline{D}$. Consequently, $S_\mu \cap D$ is not empty as it contains at least the first vertex of P. This contradicts $D \subseteq G_\mu - S_\mu$.

Put

$$H_\mu := G_\mu - \bigcup \mathcal{D}_1(\mu).$$

Note that, as every $v \in U_\mu$ lies on a path in $\mathcal{P}$,

$$G_\mu \textit{ contains a set of disjoint } H_\mu\text{–}A_\mu \textit{ paths whose set of first points is } U_\mu. \tag{3.5}$$

By (3.3) and the definition of H_μ, we have $\overline{H_\mu} \cap A \subseteq U_\mu \cup O_\mu = X_\mu$. Since O_μ is finite, we can extend $U_\mu \cup S_\mu$ to a finite set $T_\mu \subseteq V(H_\mu)$ that separates the ends in O_μ pairwise in G. Let H'_μ be the finite subgraph of H_μ containing T_μ which Lemma 3.4.4 provides for $k := |S_\mu| + 1$, and for each $\alpha \in O_\mu$ let D_α be the component of $G - H'_\mu$ to which α belongs. Finally, we conclude our definitions for μ by setting $\mathcal{D}_2(\mu) := \{D_\alpha \mid \alpha \in O_\mu\}$.

Define for $i = 1, 2$

$$\mathcal{D}_i := \bigcup_{\mu < \mu^*} \mathcal{D}_i(\mu).$$

Observe that, by Lemma 3.4.2 (iii) and since their neighbourhoods in G are finite, the elements of $\mathcal{D}_1 \cup \mathcal{D}_2$ have pairwise disjoint closures.

Before we can define G', we first have to introduce a graph $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\Omega})$ from which we will obtain G' by deleting certain vertices. Let $\tilde{G}$ be obtained from $G - \bigcup \mathcal{D}_1$ by contracting every $D_\alpha \in \mathcal{D}_2$ to a single vertex a_α, and put

$$A^* := \{a_\alpha \mid D_\alpha \in \mathcal{D}_2\}.$$

Then for $Z := \bigcup \mathcal{D}_1 \cup \bigcup \mathcal{D}_2$ we have

$$G - Z = G \cap \tilde{G} = \tilde{G} - A^*.$$

By Lemma 3.4.2 (iii) and by (3.3), the union of the sets of paths in (3.5) for all $\mu < \mu^*$ is a set of disjoint paths. Thus, for $U := \bigcup_{\mu<\mu^*} U_\mu$

there is a set of disjoint U–A paths whose set of first points is U, and whose paths meet $\tilde{G}$ only in U. (3.6)

An important property of $\tilde{G}$ is that the ends of G in $B \cap \Omega$ correspond closely to ends of $\tilde{G}$. To establish this correspondence formally, we begin with the following observation:

Every ray of an end $\beta \in B$ has a tail in $G - Z$. (3.7)

To see this, recall that all the $D \in \mathcal{D}_1 \cup \mathcal{D}_2$ have pairwise disjoint closures, and that each of them is a connected subgraph of G whose closure contains an end or a vertex of A. Hence, a ray R of β meets only finitely many $D \in \mathcal{D}_1 \cup \mathcal{D}_2$, as we could otherwise find infinitely many disjoint R–A paths, giving $\overline{A} \cap B \neq \emptyset$ by Lemma 3.2.1 – a contradiction. Also, R meets every $D \in \mathcal{D}_1 \cup \mathcal{D}_2$ only finitely often. Indeed, D lies in G_μ for some $\mu < \mu^*$ and is thus, by Lemma 3.4.2 (i), separated from β by its finite set of neighbours $N(D)$. This establishes (3.7).

Let R_1, R_2 be two rays in $G \cap \tilde{G}$, and assume that the end of R_1 lies in B. Then R_1 and R_2 are equivalent in G if and only if they are equivalent in $\tilde{G}$. (3.8)

To prove (3.8), suppose first that R_1, R_2 are equivalent in G, i.e. belong to the same end $\beta \in B$. Then there is a ray R_3 that meets both R_1 and R_2 infinitely often, and hence ends in β. By (3.7), R_3 has a tail in $G - Z = \tilde{G} - A^*$, showing that R_1 and R_2 are equivalent also in $\tilde{G}$.

Conversely, if R_1 and R_2 are joined in $\tilde{G}$ by infinitely many disjoint paths, we can replace any vertices $a_\alpha \in \tilde{V} \setminus V = A^*$ on these paths by finite paths in D_α to obtain infinitely many disjoint R_1–R_2 paths in G. This completes the proof of (3.8).

We can now define our correspondence between the ends in B and certain ends of $\tilde{G}$. For every end $\beta \in B$ there is by (3.7) an end $\beta' \in \tilde{\Omega}$ such that $\beta \cap \beta' \neq \emptyset$. By (3.8), this end β' is unique and the map $\beta \mapsto \beta'$ is injective. Moreover,

$$\tilde{B} := (B \cap V) \cup \{\beta' \mid \beta \in B \cap \Omega\} \subseteq \tilde{V} \cup \tilde{\Omega}$$

by Lemma 3.4.2 (i). For each $\mu < \mu^*$, let

$$\tilde{A}_\mu := U_\mu \cup \{a_\alpha \mid \alpha \in O_\mu\},$$

if $G_\mu \neq \emptyset$; if $G_\mu = \emptyset$, put $A_\mu, \tilde{A}_\mu := \emptyset$. Then let

$$\tilde{A} := \left(A \setminus \left(\bigcup_{\mu<\mu^*} A_\mu \cup I \right) \right) \cup \bigcup_{\mu<\mu^*} \tilde{A}_\mu,$$

which is a subset of $\tilde{V}$ by Lemma 3.4.2 (iii),(iv). Finally, let

$$G' := \tilde{G} - (\tilde{A} \cap \tilde{B}).$$

To show the assertions (i)–(iii), we will apply Lemma 3.4.1 to the graph $\tilde{G}$ and the sets $\tilde{A}$ and $\tilde{B}$.

So, let us show that

$$(\overline{\tilde{A}} \setminus \tilde{A}) \cap \tilde{B} = \emptyset = \tilde{A} \cap (\overline{\tilde{B}} \setminus \tilde{B})$$

(with closures taken in $|\tilde{G}|$). We trivially have $\tilde{A} \cap (\overline{\tilde{B}} \setminus \tilde{B}) = \emptyset$ because $\tilde{A} \subseteq \tilde{V}$. To prove that $(\overline{\tilde{A}} \setminus \tilde{A}) \cap \tilde{B} = \emptyset$, consider an end $\beta' \in \tilde{B}$. The corresponding end $\beta \in B$ has a neighbourhood $C := \hat{C}(S, \beta)$ in $|G|$ that avoids A. By (3.6), and since S is finite, the intersection $C \cap U =: U_C$ is finite. Also, as in the proof of (3.7), C may meet only finitely many $D_\alpha \in \mathcal{D}_2$. Denote by O_C the set of the corresponding $a_\alpha \in \tilde{G}$. Adding to $S \setminus Z$ the sets U_C and O_C then yields a finite set $S' \subseteq \tilde{V}$ such that the neighbourhood $\hat{C}'(S', \beta')$ in $|\tilde{G}|$ even avoids $\tilde{A}$.

Thus, Lemma 3.4.1 is applicable and yields sets $A' \subseteq V'$ and $B' \subseteq V' \cup \Omega'$ satisfying (ii). Assertion (i) follows from the definition of $\tilde{B}$ and Lemma 3.4.1 (i).

We now prove assertion (iii) of the lemma. Suppose G' satisfies the Erdős-Menger conjecture for A' and B'. Then, by Lemma 3.4.1, there is also in $\tilde{G}$ a set $\tilde{\mathcal{P}}$ of disjoint $\tilde{A}$–$\tilde{B}$ paths and an $\tilde{A}$–$\tilde{B}$ separator $\tilde{X}$ on $\tilde{\mathcal{P}}$. In order to turn $\tilde{\mathcal{P}}$ into a set $\mathcal{P} := \{P \mid \tilde{P} \in \tilde{\mathcal{P}}\}$ of disjoint A–B paths in G, consider any $\tilde{P} \in \tilde{\mathcal{P}}$. If the first point a of $\tilde{P}$ lies in A we leave $\tilde{P}$ unchanged, i.e. set $P := \tilde{P}$. If $a \in \tilde{A} \setminus (A \cup A^*)$, then $a \in U_\mu$ for some $\mu < \mu^*$, and we let P be the union of $\tilde{P}$ with an A_μ–U_μ path in G_μ that ends in a; this can be done disjointly for different $\tilde{P} \in \tilde{\mathcal{P}}$ if we use the paths from (3.6). Moreover, the A_μ–H_μ path concatenated with $\tilde{P}$ in this way has only its last vertex in $\tilde{G}$, so it will not meet any other vertices on $\tilde{\mathcal{P}}$. Finally if $a = a_\alpha \in A^*$, we let P be obtained from $\tilde{P}$ by replacing a with a path in D_α that starts at the end α and ends at the vertex of D_α incident with the first edge of $\tilde{P}$ (the edge incident with a). In all these cases we have $P \subseteq G$, because $\tilde{P}$ has no vertex in A^* other than possibly a. And no vertex of P other than possibly its last vertex lies in B, because $B \cap V = \tilde{B} \cap \tilde{V}$ and any new initial segment of P lies in a subgraph $G_\lambda - S_\lambda$ of G which avoids B by Lemma 3.4.2 (i).

It remains to check that the paths P just defined have distinct last points in B even when the last points of the corresponding paths $\tilde{P}$ are ends. However if $\tilde{P}$

ends in $\beta' \in \tilde{B}$ then its tail $\tilde{P} - a \subseteq P \subseteq G$ is equivalent in $\tilde{G}$ to some ray in $\beta' \cap \beta$, by definition of β'. By (3.8) this implies $\tilde{P} - a \in \beta$, so the last point of P is $\beta \in B$. And since the map $\beta \mapsto \beta'$ is well defined, these last points differ for distinct P, because the corresponding paths $\tilde{P}$ have different endpoints β' by assumption.

We still need an A–B separator on $\mathcal{P}$. The only vertices $x \in \tilde{X}$ that do not lie on the path P obtained from the path $\tilde{P}$ containing x are points in A^*. So let X be obtained from $\tilde{X}$ by replacing every end $\beta' \in \tilde{X} \cap \tilde{B}$ with the corresponding end $\beta \in B$ and replacing every $a_\alpha \in \tilde{X} \cap A^*$ with the end $\alpha \in A$. Since $P \in \mathcal{P}$ starts in α if $\tilde{P}$ starts in a_α (and P ends in β if $\tilde{P}$ ends in β'), this set X consists of a choice of one point from every path in $\mathcal{P}$.

Let us then show that

$$X \textit{ is an } A\text{–}B \textit{ separator in } G. \tag{3.9}$$

Suppose there exists a path $Q \subseteq G - X$ that starts in A and ends in B. Lemma 3.4.2 (v) enables us to choose Q as a path starting in $A \setminus I$. Our aim is to turn Q into an $\tilde{A}$–$\tilde{B}$ path Q' in $\tilde{G}$ that avoids $\tilde{X}$, which contradicts the choice of $\tilde{X}$.

If Q meets $\bigcup \mathcal{D}_1$, it has a last vertex there by (3.7), in $D \in \mathcal{D}_1(\lambda)$, say. Its next vertex a lies in U_λ, by the definition of D. We then define (for the time being) Q' as the final segment aQ of Q starting at a. If Q has no vertex in $\bigcup \mathcal{D}_1$, then either the first point of Q is a vertex $a \in A \cap \tilde{A}$ (in which case we put $Q' := Q$), or Q starts at an end $\alpha \in A \setminus I$. By Lemma 3.4.2 (iv), there exists a $\lambda < \mu^*$ such that $\alpha \in \overline{G}_\lambda$, which implies $\alpha \in O_\lambda$. We make $a := a_\alpha$ the starting vertex of Q' and continue Q' along Q, beginning with the last D_α–$\tilde{G}$ edge on Q. Our assumption of $\alpha \notin X$ implies that $a_\alpha \notin \tilde{X}$, by the definition of X. Thus in the first two cases, Q' is now a path in $G - \bigcup \mathcal{D}_1$; in the third, Q' is a path in $(G - \bigcup \mathcal{D}_1)/D_\alpha$, which starts at the vertex $a \in \tilde{A}$ and avoids $\tilde{X}$.

However, Q' may still meet $\mathcal{D}_2$. And although we know from (3.7) that Q' has a last vertex in $\bigcup \mathcal{D}_2$, say in $D_{\alpha'}$, we cannot simply shorten Q' to a path $a_{\alpha'}Q'$ in $\tilde{G}$, because it may happen that $a_{\alpha'} \in \tilde{X}$. Instead, we will use Lemma 3.4.4 to replace any segments of Q' that meet some $D_\alpha \in \mathcal{D}_2$ (with $a_\alpha \neq a$) by paths through the corresponding G_μ that avoid $\tilde{X}$. As we only have to deal with a finite initial segment of Q' and the D_α are all disjoint, we are able to modify Q' step by step. Eventually, we will obtain a (walk that can be pruned to a) path Q' in $\tilde{G}$ that avoids $\tilde{X}$, yielding the desired contradiction.

So consider a segment of Q' that meets some $D_\alpha \in \mathcal{D}_2$. By definition of D_α we may assume that segment to be a T_μ-path $sQ't$ in H_μ, where μ is such that $D_\alpha \subseteq G_\mu$. By definition of H'_μ (which is a subgraph of $\tilde{G}$ by Lemma 3.4.2 (iii), i.e. no parts of H'_μ were deleted or contracted when we defined $\tilde{G}$), there are $|S_\mu| + 1$ paths from s to t in H'_μ that are disjoint outside $sQ't$. But H'_μ contains at most $|S_\mu|$ vertices from $\tilde{X}$: since these lie on disjoint paths ending in $\tilde{B}$ and S_μ separates $H'_\mu \subseteq G_\mu$ from B in G and hence from $\tilde{B}$ in $\tilde{G}$, all of these paths must meet S_μ. So one of our $|S_\mu| + 1$ s–t paths in H'_μ avoids $\tilde{X}$, and we can use this path to replace $sQ't$

on Q'. This completes the proof of (3.9). $\square$

Proof of Theorem 3.2.2: Let $G = (V, E, \Omega)$ be given, and let $A, B \subseteq V \cup \Omega$ be such that $A \cap (\overline{B} \setminus B) = \emptyset = (\overline{A} \setminus A) \cap B$. By Lemma 3.4.1, we may assume that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Applying Lemma 3.4.5 twice, first for A and then for B, we may further assume that $A \cup B \subseteq V$. Now the statement to be proved is Erdős's conjecture as stated in the Introduction, which has been proved by Aharoni and Berger [1]. $\square$

Chapter 4

Degree and parity of ends

4.1 Introduction

One of the most basic characterisations of the elements of the cycle space of a finite graph is the following (see, for example Diestel [16]):

Theorem 4.1.1. *Let H be a subgraph of a finite graph G. Then $E(H)$ is an element of the cycle space of G if and only if every vertex of G has even degree in H.*

Simple examples show that for infinite graphs it is not sufficient to consider vertex degrees. Consider, for instance, the double ray D. Since $|D|$ is homeomorphic to the unit interval, it contains no circles, hence, it follows that $\mathcal{C}(D) = \{\emptyset\}$. Thus $E(D) \notin \mathcal{C}(D)$, even though every vertex of D has degree 2 in D. The problem here seems to arise from the ends rather than the vertices of the considered graph. In this respect, Diestel and Kühn [18] raised the following problem:

Problem 4.1.2. *Characterise the circles and the elements of the cycle space of an infinite graph in purely combinatorial terms, such as vertex degrees and 'degrees of ends'.*

Now, if we use the vertex-degree defined in Chapter 2, then Theorem 4.1.1 fails for infinite graphs, as the graph G in Figure 4.1 demonstrates. The degrees resp. vertex-degrees of all vertices and ends are even, but since G contains an odd cut, its edge set is not an element of $\mathcal{C}(G)$ by Theorem 2.5.2.

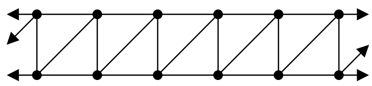

Figure 4.1: Both ends have even vertex-degree, but $E(G) \notin \mathcal{C}(G)$.

Looking more closely we see that although the vertex-degree of each end is even, their edge-degree is odd, namely three. Thus with this measure instead we would have correctly decided that $E(G) \notin \mathcal{C}(G)$.

In fact, it is not a complete surprise that for problems concerning the cycle space, which is a subspace of the edge space, the edge-degree should be the more adequate concept. In Chapter 5 we shall encounter a situation where the vertex-degree is more appropriate.

For measuring edge-degrees in subgraphs, it will be necessary to substitute 'rays' with 'arcs'; the reasons for this will be discussed in Section 4. 4.3. This notion allows us to solve the first part of Problem 4.1.2; we prove a straightforward adaption of the well-known fact that the cycles in a finite graph are exactly its 2-regular connected subgraphs.

Theorem 4.1.3. [12] *Let C be a subgraph of a locally finite graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree resp. edge-degree two in C.*

Depending on its edge-degree, an end can be assigned a parity, ie. the label 'even' or 'odd'—as long as the degree is finite. Inspired by Laviolette [31], who introduced a concept to measure the parity of vertices of infinite degree, we assign a parity also to ends of infinite edge-degree [12]. A classification of ends into even and odd ends has already been achieved by Nash-Williams [37] for the case of eulerian graphs with only finitely many ends. Our definition coincides with Nash-Williams' in these graphs but covers all locally finite graphs. Moreover, with our definition the following important special case of Problem 4.1.2 becomes true, which is the main result of this chapter.

Theorem 4.1.4. [12] *Let G be a locally finite graph. Then $E(G) \in \mathcal{C}(G)$ if and only if every vertex and every end of G has even edge-degree.*

An extension of this characterisation to arbitrary subgraphs of G would solve Problem 4.1.2 completely. We shall offer a conjecture in that respect (see Section 4.4.4).

We introduce and discuss our parity concept as well as the edge-degree notion for subgraphs in Sections 4.4.2, 4.4.3, and 4.4.4. Theorem 4.1.4 will be proved in Section 4.4.5. In Section 4.4.6, we show Theorem 4.1.3 and other results, and in the last section, we briefly discuss an alternative notion of parity.

4.2 Parity

Our edge-degree concept clearly divides the ends of finite edge-degree into even and odd ends, but how are we to deal with ends of infinite edge-degree? We may not simply treat them as odd ends, since the edge set of the infinite grid obviously is an element of its cycle space but the only end of the grid has infinite edge-degree.

On the other hand, classifying all ends of infinite edge-degree as even is not any better: consider the graph G in Figure 4.2. All vertices have even degree and both ends have infinite edge-degree, but G has an odd cut (which together with Theorem 2.5.2 implies that $E(G) \notin \mathcal{C}(G)$).

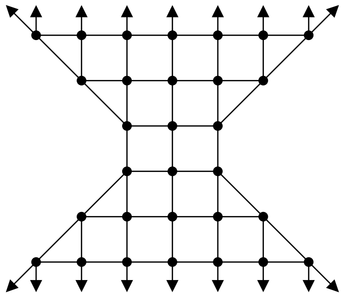

Figure 4.2: Both ends have infinite edge-degree, but $E(G) \notin \mathcal{C}(G)$.

Consequently, the edge-degree, if infinite, is not sufficiently fine enough to determine the parity of an end. For an adequate refinement we will use the following characterisation of ends with even finite edge-degree.

Lemma 4.2.1. **[12]** *In a locally finite graph G let $\omega \in \Omega(G)$ have finite edge-degree k. Then the following statements are equivalent:*

(i) *k is even;*

(ii) *there is a finite $S \subseteq V(G)$ such that for every finite set $S' \supseteq S$ of vertices the maximal number of edge-disjoint ω-rays starting in S' is even.*

Proof. Consider a set $\mathcal{R}$ of edge-disjoint ω-rays of maximal cardinality $|\mathcal{R}| = k$, and let U be the set of starting vertices of $\mathcal{R}$. Then, for every finite set $S' \supseteq U$, $\mathcal{R}$ has maximal cardinality among all sets of edge-disjoint ω-rays starting in S'. Thus, putting $S := U$, we deduce that (i) implies (ii). Also, (ii) implies (i), which we see by choosing $S' = S \cup U$. $\square$

Observe that as every finite set $S \subseteq V(G)$ gives (essentially) rise to a neighbourhood $\hat{C}(S, \omega)$ of ω, condition (ii) in Lemma 4.2.1 can be alternatively formulated using these neighbourhoods, or using regions whose closures contain ω:

(ii′) There is a region A of G with $\omega \in \overline{A}$ such that for every region $B \subseteq A$ of G with $\omega \in \overline{B}$ the maximal number of edge-disjoint rays of ω starting outside B is even.

This motivates the following definition [12] of the parity of an end: an end ω of a locally finite graph is said to be *even* if ω satisfies (ii) of Lemma 4.2.1. Otherwise

ω is *odd.* Thus, ω is odd if and only if for all finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ such that the maximal number of edge-disjoint ω-rays starting in S' is odd. By Lemma 4.2.1, an end ω of finite edge-degree is even if and only if $d(\omega)$ is even.

Observe that our notion of parity is not symmetric. Indeed, roughly speaking, while an even end ω has a neighbourhood, inside which there will always be even maximal sets of ω-rays, an odd end ω' just allows arbitrarily 'close' sets S in which start odd maximal sets of ω'-rays.

Let us turn back to the examples that motivated our struggle for a concept of parity, the infinite grid, and the graph G from Figure 4.2. Their ends turn out to have the expected even resp. odd degree. Indeed, for the infinite grid we can choose $S = \emptyset$, and for G it suffices for S to separate the two ends of G. Then $|E(S, C)|$ is odd for any infinite component C of $G - S$, and so is $|E(S', C')|$ for any $S' \supseteq S$ and infinite component C' of $G - S'$ (because all vertex degrees are even).

4.3 Edge-degrees in subgraphs

It is not possible to extend our edge-degree notion literally to subgraphs H of G. The are two obstacles (as also observed in [12]).

First, we cannot simply measure the edge-degrees of the ends of H (as opposed to those of G). This is not surprising as H is embedded in the space $|G|$. If H is a double ray, for instance, then (viewed as a graph on its own and not as a subgraph) it has two ends, each of which has edge-degree 1. On the other hand, the tails of H may lie in the same end of G, in which case $\overline{H}$ is a circle in $|G|$. Thus the ends contained in $\overline{H}$ should have edge-degree 2 in H, not 1. Therefore, we only consider ends of G (and not of H).

Second, even taking that into account, the literal extension to subgraphs fails: Consider the bold subgraph of the graph in Figure 4.3, and let ω be the end of G "to the right". Then, if we count the edge-disjoint ω-rays that lie in H, we find that apart from tail-equivalence there is only one ω-ray. But as $\overline{H}$ is a circle, we would expect the end to have edge-degree 2. In contrast, if we consider the

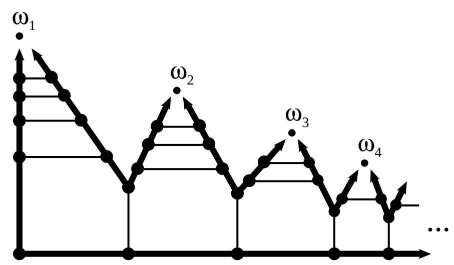

Figure 4.3: In subgraphs, counting edge-disjoint rays is not enough.

maximal number of edge-disjoint ω-*arcs* in $\overline{H}$ instead of counting edge-disjoint ω-rays in H we obtain the desired edge-degree 2. Counting arcs will indeed turn out to be successful, and the following proposition, which we shall prove in the next section, shows that in G it makes actually no difference whether we count rays or arcs:

Proposition 4.3.1. **[12]** *Let G be a locally finite graph, and let $\omega \in \Omega(G)$. Then for every finite $S \subseteq V(G)$ the maximal number of edge-disjoint ω-rays starting in S equals the maximal number of edge-disjoint ω-arcs starting in S.*

Hence, for a subgraph H of a locally finite graph G, and $\omega \in \Omega(G)$, we define, analogously to the definition of $d(\omega)$ given above, the edge-degree of ω in H as

$$d_H(\omega) := \sup\{|\mathcal{R}| : \mathcal{R} \text{ is a set of edge-disjoint } \omega\text{-arcs in } \overline{H}\} \in \mathbb{N} \cup \{\infty\}.$$

We note that the supremum is attained (see Lemma 4.4.5). Further, observe that $d(\omega) = d_G(\omega)$. Indeed, suppose otherwise, ie. $d(\omega) < d_G(\omega)$. So, in particular, $d(\omega)$ is finite. For a set of $d(\omega)+1$ edge-disjoint ω-arcs, let $S \subseteq V(G)$ be a choice of exactly one vertex from each of the arcs. Then, by Proposition 4.3.1, there are also $d(\omega)+1$ edge-disjoint ω-rays starting in S, a contradiction.

The parity of an end in H is defined as follows:

Definition 4.3.2. **[12]** *An end ω of G is* even in H *if there is a finite $S \subseteq V(G)$ such that for every finite $S' \subseteq V(G)$ with $S' \supseteq S$ the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S' is even. Otherwise, ω is* odd in H.

Note that by Proposition 4.3.1, the definition of parity is consistent with the one given previously. Furthermore, it can be seen similarly as in the proof of Lemma 4.2.1 that for an end ω with finite edge-degree in H, ω has even edge-degree in H if and only if $d_H(\omega)$ is even.

A complete solution of Problem 4.1.2 requires an analogon of Theorem 4.1.4 for subgraphs H of G. The forward direction of such an analogon can be proved easily with the same methods as used for Theorem 4.1.4. Furthermore, if G has only countably many ends the problem is not overly difficult (Proposition 4.7.3). In view of this, and in view of Theorems 4.1.3 and 4.1.4, and two more results in Section 4.6, which demonstrate that the edge-degrees of the ends behave in many aspects similar to the degrees of vertices, we offer the following conjecture:

Conjecture 4.3.3. **[12]** *Let H be a subgraph of a locally finite graph G. Then $E(H) \in \mathcal{C}(G)$ if and only if every vertex has even degree in H, and every end has even edge-degree in H.*

4.4 A cut criterion

In this section we prove Proposition 4.3.1. The other result of this section is Corollary 4.4.7, which yields a criterion for the parity of an end in terms of cut cardinalities. Let us start with a simple lemma that shows how we can construct a topological path by piecing together infinitely many arcs.

Lemma 4.4.1. **[12]** *Let G be a locally finite graph, and let for $n \in \mathbb{N}$, $\phi_n : [0,1] \to |G|$ be a homeomorphism such that if $A_n := \phi_n([0,1])$ it holds that:*

(i) *$A_n \cap A_m \subseteq V(G) \cup \Omega(G)$ for $n \neq m$; and*

(ii) *$\phi_n(1) = \phi_{n+1}(0)$ for all n.*

Then there is an $x \in |G|$ such that $\bigcup_{n=1}^{\infty} A_n \cup \{x\}$ is a topological path from $\phi_1(0)$ to x.

Proof. Instead of the ϕ_n let us consider compositions with suitable homeomorphisms $\phi'_n : [1-2^{-(n-1)}, 1-2^{-n}] \to A_n$. Together the ϕ'_n define, by (ii), a continuous function $\phi' : [0,1) \to |G|$. As $|G|$ is compact, the sequence $\phi_1(0) = \phi'(1/2), \phi_2(0) = \phi'(3/4), \ldots$ has an accumulation point x. We claim that $\phi : [0,1] \to |G|$ defined by $\phi(s) := \phi'(s)$ for $s \in [0,1)$ and by $\phi(1) := x$ is continuous. Let a neighbourhood V of x be given, and note that because of (i) and (ii), none of the $\phi_n(0)$ is an inner point of an edge, and thus x is an end. Then there is a basic open neighbourhood $\hat{C}(S,x) \subseteq V$ that contains all but finitely many of the $\phi_n(0)$. By (i), only finitely many of the A_n meet the finite cut $\partial C(S,x)$. So, there is an N such that $A_n \subseteq \hat{C}(S,x)$ for $n \geq N$. Consequently, $\phi^{-1}(V)$ contains the open set $(1-2^{-N}, 1]$, and thus is a neighbourhood of 1 in $[0,1]$. $\square$

Menger's theorem applied to the line graph implies that between any two finite edge sets E_1, E_2 in a graph there are as many edge-disjoint E_1–E_2 paths as the minimal number of edges needed in order to separate E_1 and E_2. The following lemma generalises this result to arcs. Let us say that an arc A is an E_1–E_2 arc if it has exactly one edge in E_1, exactly one in E_2, and these are incident with an endpoint of A.

Lemma 4.4.2. **[12]** *Let H be a subgraph of a locally finite graph G. Let $E_1, E_2 \subseteq E(H)$ be finite. Then the maximal number of edge-disjoint E_1–E_2 arcs in $\overline{H} \subseteq |G|$ equals the minimum k such that there is a finite set $X \subseteq E(G)$ separating E_1 from E_2 in G with $k = |X \cap E(H)|$.*

Proof. Let S be a finite vertex set such that $E_1 \cup E_2 \subseteq E(G[S])$, let $v_1, v_2, \ldots$ be an enumeration of $V(G)$, and put $G_n := G[S \cup \{v_1, \ldots, v_n\}]$ for $n \in \mathbb{N}$. Let $\mathcal{L}_n$ be the set of all sets M satisfying

(i) M is a set of pairwise edge-disjoint subgraphs of H;

(ii) for each $L \in M$ there is an E_1–E_2 path P with $P \cap G_n = L$; and

(iii) $|M| \geq k$.

Let us show that $\mathcal{L}_n$ is non-empty for each n. Contract each component of $G - G_n$ to a vertex (keeping parallel edges but deleting loops), and denote the resulting finite graph by $\tilde{G}_n$. Let $\tilde{H}_n$ be the subgraph of $\tilde{G}_n$ that consists of the edges in $E(H) \cap E(\tilde{G}_n)$ together with the incident vertices. Now, let $\tilde{M}$ be a set of edge-disjoint E_1–E_2 paths in $\tilde{H}_n$ of maximal cardinality. By Menger's theorem applied to the line graph, there is a finite set $\tilde{X} \subseteq E(\tilde{H})$ of cardinality $|\tilde{M}|$ that separates E_1 from E_2 in $\tilde{H}_n$. Then $X := \tilde{X} \cup (E(\tilde{G}_n) \setminus E(\tilde{H}_n))$ separates E_1 from E_2 in $\tilde{G}_n$, and $|X \cap E(H)| = |\tilde{M}|$. Next, observe that X is also an E_1–E_2 separator in G, implying $|\tilde{M}| = |X \cap E(H)| \geq k$. Put $M := \{\tilde{L} \cap G_n : \tilde{L} \in \tilde{M}\}$, and note that this choice satifies (i) and (iii). Furthermore, by replacing each vertex of $\tilde{G}_n - G$ that $\tilde{L}$ meets with a path through the respective component of $G - G_n$, we easily find for the corresponding $L \in M$ a path such that (ii) is satisfied. Thus, $\mathcal{L}_n \neq \emptyset$.
Define a graph on the vertex set $\bigcup_{n=1}^{\infty} \mathcal{L}_n$ with edges $M_n M_{n+1}$ for $M_n \in \mathcal{L}_n$ and $M_{n+1} \in \mathcal{L}_{n+1}$ if $M_n = \{L \cap G_n : L \in M_{n+1}\}$. As every $M \in \mathcal{L}_{n+1}$ has a neighbour in $\mathcal{L}_n$ we may apply the infinity lemma 2.1.1, which yields a ray $M_1 M_2 \ldots$ with $M_n \in \mathcal{L}_n$ for all $n \geq 1$. For each $L_1 \in M_1$ there is a sequence $L_1, L_2, \ldots$ with $L_n \in M_n$ and $L_n = L_{n+1} \cap G_n$. More precisely, there are, by (iii), k such sequences so that their respective unions $L^1 = \bigcup L_n^1 \ldots, L^k = \bigcup L_n^k$ are mutually edge-disjoint. Furthermore, $L^i \subseteq \overline{H}$, and $L^i \cap G_n = L_n^i \in M_n$ for $i = 1, \ldots, k$.
We claim that each of the L^i contains an E_1–E_2 arc. Indeed, consider an $i \in \{1, \ldots, k\}$, and let v and w be the endvertices of the path P for which $P \cap G_1 = L_1^i$. Introduce a new vertex z to G, and link it to v and w. Denote by Q the resulting v–w path vzw. We want to show that $E(L^i \cup Q)$ is an element of the cycle space of $G' := G \cup Q$. To this end, let F be a finite cut of G', and choose n large enough so that $F \subseteq E(G_n \cup Q)$. Denote by P_n the v–w path with $P_n \cap G_n = L_n^i$, which exists by (ii). Then

$$\begin{aligned} F \cap E(L^i \cup Q) &= F \cap (E(G_n \cap L^i) \cup Q) \\ &= F \cap (E(G_n \cap L_n^i) \cup Q) = F \cap E(P_n \cup Q). \end{aligned}$$

The last intersection is an even set as $E(P_n \cup Q)$ is a circuit in G', and hence, $E(L^i \cup Q)$ an element of the cycle space of G', by Theorem 2.5.2. Thus, by Theorem 2.5.1 there is a circle D with $Q \subseteq D \subseteq L^i \cup Q$. Hence, $A^i := D \setminus (Q \setminus \{v, w\})$ is an arc from v to w. Therefore, there are k edge-disjoint E_1–E_2 arcs $A^1, \ldots A^k$ in $\overline{H}$.
Since there is a finite set $X \subseteq E(G)$ separating E_1 from E_2 in G such that $k = |X \cap E(H)|$, there cannot be more than $k + 1$ arcs in $\overline{H}$ connecting E_1 and E_2, as each of them meets X. □

Corollary 4.4.3. [12] *Let G be a locally finite graph, let H be a subgraph, and let $C_1 \supseteq C_2$ be regions of G. Then the maximal number of edge-disjoint $\partial C_1 \cap E(H)$–$\partial C_2 \cap E(H)$ arcs in $\overline{H}$ equals the minimum k such that there is a region D with $C_1 \supseteq D \supseteq C_2$ and $|\partial D \cap E(H)| = k$.*

Proof. By Lemma 4.4.2, the maximal number of edge-disjoint $\partial C_1 \cap E(H)$–$\partial C_2 \cap E(H)$ arcs in $\overline{H}$ equals the minimal $k' \in \mathbb{N}$ such that there is a finite $X \subseteq E(G)$ with $|X \cap E(H)| = k'$ which separates $\partial C_1 \cap E(H)$ from $\partial C_2 \cap E(H)$ in G. Any such X with $|X|$ minimal gives rise to a region D as above, hence $k = k'$. □

We obtain a Mengerian criterion:

Lemma 4.4.4. [12] *Let G be a locally finite graph, let H be a subgraph, let $\omega \in \Omega(G)$, and let $S \subseteq V(G)$ be finite. Then the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S equals the minimum $|F \cap E(H)|$ over all cuts F separating S from ω.*

Proof. First, choose a region $C_1 \subseteq C_0 := C(S, \omega)$ with $\omega \in \overline{C_1}$ such that $|\partial C_1 \cap E(H)|$ is minimal among all regions $C \subseteq C_0$ with $\omega \in \overline{C}$. To prove the assertion it suffices to find a set of edge-disjoint ω-arcs in $\overline{H}$ starting in S, where each of the arcs uses exactly one edge from $\partial C_1 \cap E(H)$.

Next, observe that the closure of one of the components C of $C_1 - N(G - C_1)$ contains ω. Choose C_2 such that $|\partial C_2 \cap E(H)|$ is minimal among all regions $C \subseteq C_1$ satisfying $C \cup N(C) \subseteq C_1$ and $\omega \in \overline{C}$. Continuing in this manner, we obtain regions C_i with $\omega \in \overline{C_i}$ that satisfy for $i \geq 1$:

$$C_{i+1} \cup N(C_{i+1}) \subseteq C_i; \text{ and} \tag{4.1}$$

$$|\partial C_i \cap E(H)| \leq |\partial D \cap E(H)| \text{ for every region } D \text{ with } C_i \supseteq D \supseteq C_{i+1}. \tag{4.2}$$

We claim that

> *there exists a set $\mathcal{A}$ of edge-disjoint ω-arcs in $\overline{H}$ starting outside C_1 such that $|\partial C_1 \cap E(H)| = |\mathcal{A}|$.* (4.3)

Indeed, because of (4.2), Corollary 4.4.3 yields for every $i \geq 1$ a set $\mathcal{P}_i$ of edge-disjoint $(\partial C_i \cap E(H))$–$(\partial C_{i+1} \cap E(H))$ arcs in $\overline{H}$, with $|\mathcal{P}_i| = |\partial C_i \cap E(H)|$. Hence, for any edge $e \in \partial C_1 \cap E(H)$ there is an $A_1 \in \mathcal{P}_1$ that starts in e. The arc A_1 ends in an edge $e' \in \partial C_2 \cap E(H)$, and thus there exists an arc $A_2 \in \mathcal{P}_2$ such that $A_1 \cap A_2 = e'$. In this manner we find a sequence $A_1, A_2, \ldots$ so that A_i and A_{i+1} overlap in exactly one edge of $\partial C_{i+1} \cap E(H)$. As each $A_i \in \mathcal{P}_i$ is an $(\partial C_i \cap E(H))$–$(\partial C_{i+1} \cap E(H))$ arc, A_i and A_j are disjoint for $|i - j| > 1$. Thus, by deleting the last vertex and all inner points of the last edge in each A_i, we obtain a sequence of edge-disjoint arcs to which we may apply Lemma 4.4.1. This yields an $x \in |G|$ together with an arc $A_e \subseteq \bigcup_{i=1}^{\infty} A_i \cup \{x\}$ in $\overline{H}$ that starts in e and ends in x, which uses exactly one edge in each $\partial C_i \cap E(H)$.

Suppose $x \neq \omega$. Then, there exists a finite set $T \subseteq V(G)$ that separates ω and x. By (4.1), there is an $N \in \mathbb{N}$ such that $T \cap V(C_N) = \emptyset$. Thus, C_N is completely contained in one component of $G - T$, and only one of ω, x lies in $\overline{C_N}$, a contradiction. So, we obtain for each e an ω-arc A_e, and all these arcs are edge-disjoint since for each i the arcs in $\mathcal{P}_i$ are.

Finally, using Corollary 4.4.3 we lengthen the A_e in order to obtain a set of edge-disjoint ω-arcs which start in S (this is possible by the minimal choice of C_1). Note that each of these arcs indeed uses exactly one edge from $\partial C_1 \cap E(H)$. $\square$

Similarly we can show that for a given subgraph H of G and an end $\omega \in \Omega(G)$ there indeed exists a maximal set of edge-disjoint ω-arcs in $\overline{H}$.

Lemma 4.4.5. **[12]** *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$ such that $d_H(\omega) = \infty$. Then there is an infinite set of edge-disjoint ω-arcs in $\overline{H}$.*

Proof. As in the proof of Lemma 4.4.4 we define a sequence of regions C_i satisfying (4.1) and (4.2). Again, Corollary 4.4.3 yields for each $i \in \mathbb{N}$ a set of edge-disjoint $(\partial C_{i-1} \cap E(H))$–$(\partial C_i \cap E(H))$ arcs, which we piece together with the help of Lemma 4.4.1 to obtain a set of edge-disjoint ω-arcs in $\overline{H}$ (whose union contains all edges of all of the $\partial C_i \cap E(H)$). Note that this set is indeed infinite, since $d_H(\omega) = \infty$, and hence by Lemma 4.4.4, we may assume $\partial C_{i-1} \cap E(H) < \partial C_i \cap E(H)$ for all $i \in \mathbb{N}$. $\square$

We finally prove Proposition 4.3.1, which we restate:

Proposition 4.4.1. **[12]** *Let G be a locally finite graph, and let ω be an end of G. Then for every finite set $S \subseteq V(G)$ the maximal number of edge-disjoint ω-rays starting in S equals the maximal number of edge-disjoint ω-arcs starting in S.*

Proof. By Lemma 4.4.4, the maximal number of edge-disjoint ω-arcs starting in S equals the minimal cardinality of a finite cut that separates S from ω. This minimal cardinality, on the other hand, equals the maximal number of edge-disjoint ω-rays starting in S: there clearly cannot be more ω-rays, and conversely, using Menger's theorem applied to the line graph, we can piece together the rays we need along minimal separating cuts, in a similar fashion as in Lemma 4.4.4. $\square$

Two further results that follow immediately from Lemma 4.4.4 are characterisations of edge-degrees of ends in terms of cut cardinalities:

Corollary 4.4.6. **[12]** *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$. Then $d_H(\omega) = k \in \mathbb{N}$ if and only if k is the smallest integer such that every finite $S \subseteq V(G)$ can be separated from ω with a finite cut that shares exactly k edges with $E(H)$.*

Corollary 4.4.7. **[12]** *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$. Then ω has even edge-degree in H if and only if there is a finite $S \subseteq V(G)$ such that for every finite $S' \subseteq V(G)$ with $S' \supseteq S$ it holds: if $F \subseteq E(G)$ is a finite cut separating S' and ω with $|F \cap E(H)|$ minimal, then $|F \cap E(H)|$ is even.*

4.5 Proof of Theorem 4.1.4

This proof can also be found in [12]. The forward direction follows from Theorem 2.5.2, which ensures that every finite cut of G is even, and thus together with Corollary 4.4.7 implies the assertion.

For the backward direction suppose that $E(G) \notin \mathcal{C}(G)$. Observe that we may assume G to be connected, which means in particular that G is countable. We shall find a sequence $C_1 \supseteq C_2 \supseteq \dots$ of regions of G that satisfy

(i) $\partial_G C_n$ is an odd cut, for $n \geq 1$;

(ii) $C_n \cup N(C_n) \subseteq C_{n-1}$, for $n \geq 2$; and

(iii) if D is a region of G with $C_{n-1} \supseteq D \supseteq C_n$ then $|\partial_G D| \geq |\partial_G C_{n-1}|$, for $n \geq 2$,

Then G has an odd end, contradicting the assumption, as desired. Indeed, by piecing together paths in the C_n, we see that there is a ray R which has a tail in every C_n. Let ω be the end with $R \in \omega$, and consider any finite $S \subseteq V(G)$. Choose I large enough such that $C_I \subseteq C(S, \omega)$, which is possible by (ii). So, every cut that separates $S' := N(C_I)$ from ω has cardinality at least $|\partial_G C_I|$, by (iii). Thus by (i) and Corollary 4.4.7, ω has odd edge-degree.

For our construction, we need a further condition for $n \geq 1$. Let us call a region C of a graph H a *k-region* if $|\partial_H C| = k$.

(iv) for every k-region $D \subseteq C_n$ of G with $k < |\partial_G C_n|$ there is an $\ell \in \mathbb{N}$ and even regions $K_1, \dots, K_\ell \subseteq C_n$ such that $|\partial_G K_i| \leq k$ for all i, and $V(D) \subseteq \bigcup_{i=1}^{\ell} V(K_i)$.

This condition, of course, is trivially satisfied if k is even.

As $E(G) \notin \mathcal{C}(G)$, Theorem 2.5.2 ensures the existence of odd regions in G. Choose any odd region C_1 such that $\partial_G C_1$ has minimal cardinality. This choice satisfies (i) and (iv), and that is all we required for $n = 1$.

Now, suppose the C_i to be defined for $i \leq n$. In order to find a suitable C_{n+1}, we shall contract certain even k-regions D of G (contained in C_n) for which $k < |\partial_G C_n|$. In the resulting minor, which has only big cuts, we will choose a small odd cut, which in G induces the desired region C_{n+1}.

We will construct this minor in several steps. More precisely, for each even integer $m < |\partial_G C_n|$ we define a minor G^m of $G =: G^0$, which will have the properties:

(a) G^m is obtained from G^{m-2} by contracting disjoint infinite m-regions K of G^{m-2} with $E(K) \subseteq E(C_n)$ for $m \geq 2$; and

(b) $|E(D) \cap E(G^m)| < \infty$ for every k-region D of G with $D \subseteq C_n$ and $k \leq m$.

Observe that, by (a) and as all vertices of G are even, all vertices of G^m have even degree too. We claim that (b) together with (iv) implies for $m < |\partial_G C_n| - 2$:

(c) every k-region D of G^m with $E(D) \subseteq E(C_n)$ and $k \leq m+1$ is finite.

Indeed, consider a k-region D of G^m with $E(D) \subseteq E(C_n)$ and $k \leq m+1$. By uncontracting, we obtain from D a region D' of G with $\partial_G D' = \partial_{G^m} D$ and $E(D) \subseteq E(D')$. Since by assumption $m < |\partial_G C_n| - 2$, we get that $k < |\partial_G C_n|$. Then (iv) implies that there is a finite set $\mathcal{K}$ of regions $K \subseteq C_n$ such that their union contains all vertices of D' (and thus also all but finitely many edges of D'). Each $K \in \mathcal{K}$ is an ℓ-region with even $\ell \leq k = m+1$. As $m+1$ is odd we get $\ell \leq m$, and hence by (b), that $E(K) \cap E(G^m)$ is finite. Thus $|E(D') \cap E(G^m)| < \infty$, and hence D is finite. This establishes (c).

As G is connected, $G^0 = G$ obviously satisfies (b), which is all we required for $m = 0$. So, assume $m \geq 2$, and G^i to be constructed for all even $i < m$. We define a sequence $(L_j)_{j \in \mathbb{N}}$ of (not necessarily induced) subgraphs of G^{m-2}; by contracting the components of their union L we obtain G^m.

Consider an enumeration $R_1, R_2, \ldots$ of all infinite m-regions of G^{m-2} with $E(R_i) \subseteq E(C_n)$ (such an enumeration is possible since $E(G^{m-2}) \subseteq E(G)$ is countable). Put $L_1 := R_1$, and let for $j > 1$,

$$L_j := L_{j-1} \cup R_j \;\; \textit{if} \;\; \partial_{G^{m-2}} R_j \cap E(L_{j-1}) = \emptyset \tag{4.4}$$

and $L_j := L_{j-1}$ otherwise. Note that in the former case each component of L_{j-1} is either contained in R_j or disjoint from R_j. Thus, by induction on j, every component K of L_j is an infinite m-region of G^{m-2}.

Put $L := \bigcup_{j \in \mathbb{N}} L_j$, and consider a component K of L. Certainly, K is an infinite induced subgraph in G^{m-2} with $E(K) \subseteq E(C_n)$. We claim that $k := |\partial_{G^{m-2}} K| = m$. Clearly, $k \leq m$ as otherwise there would already be a component $K' \subseteq K$ of some L_j with $|\partial_{G^{m-2}} K'| > m$, which is impossible. On the other hand, $k \geq m$, by (c) for $m-2$; thus $k = m$, as desired. We now obtain G^m from G^{m-2} by contracting the components of L to one vertex each (keeping multiple edges but deleting loops). Obviously, G^m satisfies (a).

Before we show the validity of (b), let us prove for all even $i < m < |\partial_G C_n|$ that it holds that:

($*$) for every region $D \subseteq C_n$ of G with $\partial_G D \subseteq E(G^i)$ there is a (possibly empty) induced subgraph $D' \subseteq C_n$ that satisfies

  (I) there are finitely many regions $K_1, \ldots, K_\ell$ of G each of which contracts to a vertex of degree $\leq i+2$ in G^{i+2} such that $V(D) \setminus V(D') \subseteq \bigcup_{j=1}^{\ell} V(K_j)$;

  (II) if there is a region C of G with $D \subseteq C$ and $\partial_G C \subseteq E(G^{i+2})$, then also $D' \subseteq C$;

  (III) $|\partial_G D'| \leq |\partial_G D|$; and

  (IV) $\partial_G D' \subseteq E(G^{i+2})$.

Observe that D' has at most $|\partial_G D'|$ components (since G is connected). Each of these is a region of G with properties (II)–(IV).

Let us now show $(*)$. Given D as above, choose an induced subgraph $\tilde{D} \subseteq C_n$ such that $|\partial_G \tilde{D} \setminus E(G^{i+2})|$ is minimal among all induced subgraphs that satisfy (I), (II), (III) and $\partial_G \tilde{D} \subseteq E(G^i)$ (which is possible as D itself has these four properties). If $|\partial_G \tilde{D} \setminus E(G^{i+2})| = 0$, we may put $D' := \tilde{D}$, so suppose otherwise. Then, by (a), there is an $(i+2)$-region K of G^i with $E(K) \subseteq E(C_n)$, which is contracted to a vertex in G^{i+2} and for which holds that $\partial_G \tilde{D} \cap E(K) \neq \emptyset$. Denote by $\tilde{D}^i$ the image of $\tilde{D}$ in G^i, ie. the induced subgraph of G^i with $\partial_{G^i} \tilde{D}^i = \partial_G \tilde{D}$ and $E(\tilde{D}) \cap E(G^i) = E(\tilde{D}^i)$.

Suppose that one of $|E_{G^i}(K \cap \tilde{D}^i, \tilde{D}^i \setminus K)|$, $|E_{G^i}(K \setminus \tilde{D}^i, G^i - (\tilde{D}^i \cup K))|$ is smaller than or equal to $|E_{G^i}(K \cap \tilde{D}^i, K \setminus \tilde{D}^i)|$. Then, putting either $\hat{D}^i = \tilde{D}^i \setminus K$ or $\hat{D}^i = G^i[\tilde{D}^i \cup K]$ we get

$$|\partial_{G^i} \hat{D}^i| \leq |\partial_{G^i} \tilde{D}^i| = |\partial_G \tilde{D}|. \tag{4.5}$$

Observe that $\partial_{G^i} \hat{D}^i \cap E(K) = \emptyset$, and denote by $\hat{D}$ the induced subgraph of G that we obtain from $\hat{D}^i$ by uncontracting. Then $\partial_G \hat{D} = \partial_G \hat{D}^i$ has fewer edges outside $E(G^{i+2})$ than $\partial_G \tilde{D}$. We claim that this contradicts the minimal choice of $\tilde{D}$. Indeed, $\partial_G \hat{D} \subseteq E(G^i)$, and also (I) and (III) hold for $\hat{D}$: the latter by (4.5), and for the former observe that each K_i either still contracts to a vertex of degree $\leq i+2$ in G^{i+2} or is contained in a region which does so. Adding K to these regions, we obtain the K_i as desired for (I).

To see that $\hat{D}$ satisfies (II), consider a region $C \supseteq D$ with $\partial_G C \subseteq E(G^{i+2})$. Observe that $\tilde{D} \subseteq C$ because $\tilde{D}$ satisfies (II). Now, since $\partial_G C \cap E(K) = \emptyset$, either $E(K) \subseteq E(C)$ or $E(K) \subseteq E(G - C)$ because K is connected. The latter case is impossible, as $\partial_G \tilde{D} \cap E(K) \neq \emptyset$. Hence, $E(K) \subseteq E(C)$, and thus, as $\hat{D}^i \subseteq G^i[\tilde{D}^i \cup K]$, we get $\hat{D} \subseteq C$, as desired. Note also that $\hat{D} \subseteq C_n$, as $\partial_G C_n \subseteq E(G^{i+2})$ by (a).

We may therefore assume that

$$|E_{G^i}(K \cap \tilde{D}^i, K \setminus \tilde{D}^i)| < |E_{G^i}(K \cap \tilde{D}^i, \tilde{D}^i \setminus K)|, |E_{G^i}(K \setminus \tilde{D}^i, G^i - (\tilde{D}^i \cup K))|,$$

and thus $|\partial_{G^i}(K \cap \tilde{D}^i)|, |\partial_{G^i}(K \setminus \tilde{D}^i)| < |\partial_{G^i} K|$. As K is infinite and as $K \cap \tilde{D}^i$ and $K \setminus \tilde{D}^i$ have only finitely many components (since G is connected), one of these components, say K', is infinite. Now, K' is a region of G^i with $\partial_{G^i} K' \subseteq \partial_{G^i}(K \cap \tilde{D}^i)$ or $\partial_{G^i} K' \subseteq \partial_{G^i}(K \setminus \tilde{D}^i)$. In both cases, $|\partial_{G^i} K'| < |\partial_{G^i} K| = i+2$. Because $E(K') \subseteq E(K) \subseteq E(C_n)$ and $i \leq m-2 < |\partial_G C_n| - 2$, this contradicts (c). We have thus shown $(*)$.

Let us prove that G^m also satisfies (b). For this, consider a region $D \subseteq C_n$ of G with $|\partial_G D| \leq m$, and suppose that $E(D) \cap E(G^m)$ is infinite. Assume D to be chosen among all such regions such that i is maximal with $\partial_G D \subseteq E(G^i)$. Now, if $i < m$, then $(*)$ yields a subgraph D'. By (I), all but finitely many of the edges in $E(D) \cap E(G^m)$ lie in $E(D')$. Since D' has only finitely many components, there is one, C say, such that $E(C) \cap E(G^m)$ is infinite. Since, by (III), $|\partial_G C| \leq |\partial_G D|$, and since, by (IV), $\partial_G C \subseteq E(G^{i+2})$, we obtain a contradiction to the choice of D. Thus, we may assume that $i = m$, ie. $\partial_G D \subseteq E(G^m)$.

Therefore, by performing the according contractions we obtain from D an infinite region $\tilde{D}$ of G^{m-2} such that $\partial_{G^{m-2}}\tilde{D} = \partial_G D$ and $E(\tilde{D}) \subsetneq E(C_n)$. Because of (c) for $m-2$ and because of $|\partial_G D| \leq m$, we get $|\partial_{G^{m-2}}\tilde{D}| = m$. Hence, the region $\tilde{D}$ appears in the enumeration $R_1, R_2, \ldots$ used in the construction of G^m, ie. there is a j with $\tilde{D} = R_j$. Since $\partial_{G^{m-2}}R_j \subseteq E(G^m)$ it follows from (4.4) that $E(R_j) \subseteq E(L_j) \subseteq E(L)$. Thus, in G^m all edges of $\tilde{D} = R_j$ are contracted, and consequently, $E(D) \cap E(G^m) = E(R_j) \cap E(G^m) = \emptyset$, a contradiction to $|E(D) \cap E(G^m)| = \infty$.

Having constructed G^m for all $m \leq M := |\partial_G C_n| - 1$, we finally find the region C_{n+1}. Observe that, by (a), $\partial_G C_n$ is a cut of G^M, and that the cut F of G^M that consists of those edges in $E(G^M) \cap E(C_n)$ that in G^M are adjacent to $\partial_G C_n$ has odd cardinality (because by (a), all vertices of G^M are even). Thus, since F is also a cut of G, there exists a region C of G with $\partial_G C \subseteq E(G^M)$ that satisfies (i) and (ii) for $n+1$.

We claim that

$$\textit{any region } C \textit{ of } G \textit{ that for } n+1 \textit{ satisfies, (i), (ii) and } \partial_G C \subseteq E(G^M) \textit{ also satisfies (iii).} \tag{4.6}$$

Indeed, consider a k-region D of G with $C_n \supseteq D \supseteq C$. Then $E(C) \cap E(G^M)$ is infinite, by (i). (For this, contract all edges in $E(G^M) \setminus (E(C) \cup \partial_G C)$, and recall that a finite graph always has an even number of odd vertices.) From (b) for $m = M$ it follows that $k \geq M + 1 = |\partial_G C_n|$, as desired for (iii).

Now, choose C_{n+1} such that $\partial_G C_{n+1}$ has minimal (odd) cardinality among all regions satisfying (i), (ii) and (iii) for $n+1$.

To see (iv) for $n+1$, consider a k-region $D \subseteq C_{n+1}$ with $k < |\partial_G C_{n+1}|$. If $k \leq M$, then we can apply (iv) for n, so suppose $k > M$. Furthermore, we may assume that k is odd, as otherwise we can choose $\ell := 1$ and $K_1 := D$. Then D satisfies (i) and (ii) for $n+1$, and we should have chosen D as C_{n+1}, if not (iii) and thus, by (4.6), also $\partial_G D \subseteq E(G^M)$ fails for D. Repeated use of $(*)$, where we apply $(*)$ in each step to every component of the subgraph D' obtained in the previous step, yields a subgraph D^* of C_n such that each of its components $K_1, K_2, \ldots, K_\ell$ has properties (II)–(IV) for $m = M$. In particular, (II) implies for $1 \leq i \leq \ell$ that $K_i \subseteq C_{n+1}$. Let $\{K_{\ell+1}, \ldots, K_L\}$ be the set of all regions that arose as one of the K_i in one of the applications of $(*)$. Then $V(D) \subseteq \bigcup_{i=1}^{L} V(K_i)$, by (I).

By (III), $|\partial_G K_i| \leq k$ for $i = 1, \ldots, \ell$. Now, if there is an $j \in \{1, \ldots, \ell\}$ such that $|\partial_G K_j|$ is odd, then $K_j \subseteq C_{n+1}$ satisfies (i), (ii), and, by (IV) and (4.6), also (iii) for $n+1$, contradicting the choice of C_{n+1}. So, $|\partial_G K_i|$ is even and $\leq k$ for $i = 1, \ldots, L$ (for $i > \ell$ this follows from (I) and $k > M$). As $\partial_G C_{n+1} \subseteq E(G^M)$, and thus $\partial_G C_{n+1} \cap \bigcup_{i=\ell+1}^{L} E(K_i) = \emptyset$, and as $K_1, \ldots, K_\ell \subseteq C_{n+1}$, each of the $K_1, \ldots, K_L$ either lies completely in C_{n+1} or is disjoint from it. Together with $D \subseteq C_{n+1}$ this implies that $V(D) \subseteq \bigcup_{K \in \mathcal{K}} V(K)$ for $\mathcal{K} := \{K_i : K_i \subseteq C_{n+1}$ and

$1 \leq i \leq L\}$, which proves (iv) for $n+1$. This completes the proof of the theorem.

4.6 Properties of edge-degree and parity

As an indication that the edge-degree and the parity of an end indeed behave as expected, we extend three basic properties of the degree in finite graphs to edge-degrees and parity in locally finite graphs. At the end of this section, however, we present two examples where edge-degrees of ends differ in their behaviour from degrees of vertices.

The number of odd vertices in a finite graph is always even. We prove the following easy analogon.

Proposition 4.6.1. **[12]** *Let G be a locally finite graph. Then the number of odd vertices and ends in G is even or infinite.*

Proof. Suppose that the set $\mathcal{O}$ of odd vertices and ends has odd cardinality. Observe that there is a finite set $S \subseteq V(G)$ that contains all vertices of $\mathcal{O}$ and separates the ends in $\mathcal{O}$ pairwisely. By Corollary 4.4.7, there is for each end $\omega \in \mathcal{O}$ an odd region $A_\omega \subseteq G - S$ with $\omega \in \overline{A_\omega}$. Observe that the A_ω are pairwise disjoint. So, contracting each A_ω to a vertex a_ω we arrive at a graph G' that has an odd number of odd vertices, and in which all ends have even edge-degree. Now, consider two copies of G' and add all edges vv', where v is an odd vertex of G' and v' its copy. The resulting graph has an odd cut, but no odd vertices or ends, a contradiction to Theorems 4.1.4 and 2.5.2. □

Dirac [20] observed that if a finite graph has minimum vertex degree $k \geq 2$ then it contains a circuit of length $k+1$. This becomes false for infinite graphs: an easy counterexample is the k-regular infinite tree. But the tree ceases to be a counterexample if a minimum degree is also imposed on the ends, and indeed, then Dirac's result extends to locally finite graphs:

Theorem 4.6.2. **[12]** *Let G be a locally finite graph, and let $H \subseteq G$ be a subgraph so that every vertex and every end $x \in \overline{H}$ has degree resp. edge-degree at least $k \geq 2$ in H. Then there is a circuit $C \subseteq E(H)$ of G with length $\geq k+1$.*

First, note that the theorem is best possible, even for infinite graphs. Indeed, consider disjoint copies $G_1, G_2, \ldots$ of K^{k+1}. Identify a vertex in G_1 with a vertex in G_2. Then identify a different vertex in G_2 with a vertex of G_3 and so on. In the resulting graph the minimum vertex degree is k, and it is easy to see that the single end has edge-degree k too, but there is no circuit of length greater than $k+1$. See Figure 4.4 for an example with $k=2$.

Next, let us remark that the long circuit provided by the theorem may be infinite, and indeed the result becomes false if we require finite circuits. To see this, consider a k-regular tree H with root r. Let G be the graph obtained by adding an edge

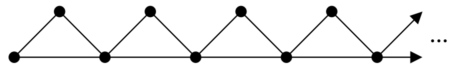

Figure 4.4: Theorem 4.6.2 is best possible for $k = 2$.

between any two vertices which the same distance to r. Then G has a single end, which has infinite edge-degree in H, but H does not contain any finite circuits.

For the proof of Theorem 4.6.2, we need the following lemma, which can be found in Hall and Spencer [26, p. 208].

Lemma 4.6.3. *Every topological path with distinct endpoints x, y in a Hausdorff space X contains an arc between x and y.*

Proof of Theorem 4.6.2. First, observe that if H has an infinite block then H contains two disjoint rays that are equivalent in H (and thus also in G). By linking these by a path in H we obtain a double ray whose edge set is an infinite circuit of G.

Therefore, we may assume that every block of H is finite. Next, suppose that there is a block B of H that contains at most one vertex v with $d_B(v) < k$. Pick a longest path in B. One of the endvertices has at least k neighbours on that path, and hence there is a finite circuit of length $\geq k + 1$ in B.

So, every block B of H is finite and contains at least two vertices of H with degree $< k$ in B, which then are cutvertices of H. Now, replace every block B of H by a tree $T \subseteq B$ whose leaves are exactly the cutvertices of H incident with B. Then every vertex of the resulting forest $H' \subseteq H$ has degree ≥ 2 as every block contains two cutvertices.

Assume that $E(H')$ does not contain infinite circuits, and let $v_1, v_2, \ldots$ be an enumeration of $V(H')$. We will inductively construct for $n \in \mathbb{N}$ homeomorphisms $\phi_n : [0, 1] \to \overline{H'} \subseteq |G|$. Choosing b_0 as any vertex in H' and putting $A_n := \phi_n([0, 1])$, we require that for $n \geq 1$ both $a_n := \phi_n(0)$ and $b_n := \phi_n(1)$ are vertices, and satisfy:

(i) $a_n = b_{n-1}$ for $n \geq 2$;

(ii) $A_m \cap A_n = \emptyset$ for $1 \leq m \leq n - 2$ and $A_{n-1} \cap A_n = \{b_{n-1}\}$;

(iii) there is a cutvertex v incident with two blocks B, B' of H such that $d_B(v) < k$ and such that A_n contains two edges incident with v, one in $E(B)$ and the other in $E(B')$ (let us call any arc with that property *deficient*); and

(iv) if there is a topological path in $\overline{H'}$ from b_{n-1} to v_n that is edge-disjoint from $B_{n-1} := \bigcup_{i=1}^{n-1} A_i$, then $v_n \in A_n$.

Note that for $n \geq 1$, B_n is a topological path.

In order to construct ϕ_n, assume $\phi_1, \dots, \phi_{n-1}$ to be defined already. First, suppose there is a topological path as required by (iv). By Lemma 4.6.3, either b_{n-1} and v_n are the endpoints of an arc A that is edge-disjoint from B_{n-1}, or $b_{n-1} = v_n$, in which case we put $A := \{v_n\}$. We claim that $A \cap B_{n-1} = \{b_{n-1}\}$. Indeed, otherwise let v be the vertex with $b_{n-1}v \subseteq A$. Then $A \cup B_{n-1}$ contains a topological path from v to b_{n-1} that avoids all inner points of $b_{n-1}v$, and hence, by Lemma 4.6.3, also a b_{n-1}–v arc A'. Thus, $A' \cup b_{n-1}v \subseteq A \cup B_{n-1} \subseteq \overline{H'}$ is a circle, contradicting our assumption.

We now lengthen A so that it also satisfies (iii). Because every vertex has degree ≥ 2 in H', and because $\overline{H'}$ does not contain any circles, v_n has a neighbour in $H' \setminus A \cup B_{n-1}$. Continuing in this way, we obtain a v_n–B path in H' that meets $A \cup B_{n-1}$ only in v_n, where B is a block of H which is adjacent to the block that contains v_n. As $B \cap \overline{H'}$ is connected and as $\overline{H'}$ does not contain any circles, B is disjoint from $A \cup B_{n-1}$. So, since B has a cutvertex b with $d_B(b) < k$, there is a deficient path $P \subseteq H'$ that starts in v_n and is otherwise disjoint from $A \cup B_{n-1}$. Thus, we easily find a homeomorphism $\phi_n : [0,1] \to A \cup P$ which satisfies (i)–(iv).

So suppose there is no topological path as in (iv). Again we find a deficient path $P \subseteq H'$ starting in b_{n-1} which is disjoint from $B_{n-1} \setminus \{b_{n-1}\}$, and the respective homeomorphism $\phi_n : [0,1] \to P$ has properties (i)–(iv).

This process yields a set of arcs A_n, to which we apply Lemma 4.4.1. We obtain an $x \in |G|$, which is necessarily an end, such that $A^* := \bigcup_{n=1}^{\infty} A_n \cup \{x\}$ is a topological path from b_0 to x.

The end x has edge-degree k in H, and hence there are k edge-disjoint arcs $R_1, \dots, R_k \subseteq \overline{H}$ that start in x. Each of the R_i meets $A^* \setminus \{x\}$ in every neighbourhood of x. Indeed, suppose there is a neighbourhood U of x and an index j such that $R_j \cap U$ is disjoint from $A^* \setminus \{x\}$. Since R_j is continuous, there is a subarc of R_j which starts in x and is completely contained in U. Pick a vertex v_m on this subarc, and denote by R the subarc of R_j between x and v_m. Then $\bigcup_{n=m-1}^{\infty} A_n \cup R$ clearly is a topological path from b_{m-1} to v_m which is edge-disjoint from B_{m-1}, a contradiction to (iv) as $v_m \notin A^* \supseteq A_m$.

Let $\phi : [0,1] \to A^*$ be a continuous function with range A^* and $\phi(1) = x$. Choose an $s \in [0,1)$ such that each of the R_i hits A^* in a $\phi(r_i)$ with $r_i < s$. Because of (iii), we may assume that $v := \phi(s)$ is a cutvertex incident with two blocks B, B' of H such that $d_B(v) < k$ and such that A^* contains two edges incident with v, one in $E(B)$ and the other in $E(B')$. Not all of the k arcs R_i can go through the cut $F := E_H(v, B - v)$ of H, which has cardinality $d_B(v) < k$; so assume R_j does not contain any edge of F. Let uw be the (unique) edge in $E(A) \cap F$, and assume $\phi^{-1}(u) \leq \phi^{-1}(w)$. Then $(A \cup R_j) \setminus uw \cup \{u, w\}$ contains a topological path from w to u (simply run from w to x along A, then from x to $\phi(r_j)$ along R_j and finally from $\phi(r_j)$ to u along A). Therefore, there is also an arc $R \subseteq (A \cup R_j) \setminus uw \cup \{u, w\}$ with endpoints u and w, by Lemma 4.6.3. Consequently, $R \cup uw \subseteq \overline{H}$ is a circle. Since $E(R)$ is disjoint from F and every B–$(B'-v)$ path in H has to go through F, $|E(R)|$ is infinite. Thus, $E(R \cup uw) \subseteq E(H)$ is an infinite circuit, as desired. $\square$

In a finite graph the cycles are exactly the connected 2-regular subgraphs. We extend this characterisation to locally finite graphs.

Theorem 4.6.3. [12] *Let C be a subgraph of a locally finite graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree resp. edge-degree two in C.*

Proof. If $\overline{C}$ is a circle, then it is clearly topologically connected and every vertex and every end $x \in \overline{C}$ has degree resp. edge-degree two in C.
For the converse direction, Theorem 4.6.2 implies that there is a circle $D \subseteq \overline{C}$. Suppose there exists a point $z \in \overline{C} \setminus D$. Theorem 2.3.1 yields an arc $A \subseteq \overline{C}$ that starts at z and ends in D. As both A and D are closed, A has a first point in D, ie. a point x such that the subarc A' of A between z and x meets D only in x. Thus, there are three edge-disjoint arcs in C with common endpoint x, two in D and the arc A'. So, x is either a vertex or an end and has degree resp. edge-degree at least 3 in C, a contradiction. Thus, $\overline{C} = D$. □

Let us now turn to two areas in which the edge-degree of ends differs from the degree of vertices. The two examples we exhibit can also be found in [12].

For a subgraph H of a graph G, deleting $E(H)$ reduces the degree of a vertex $v \in V(G)$ by its degree in H, ie. $d_G(v) = d_H(v) + d_{G-E(H)}(v)$. Although for an end ω it clearly holds that $d_G(\omega) \geq d_H(\omega) + d_{G-E(H)}(\omega)$, equality is in general not ensured. Consider the $4 \times \infty$-grid, which has a single end. As depicted in Figure 4.5, the removal of (the edge set of) a ray R leads to a decrease of the edge-degree from 4 to any of $3, 2, 1$ or 0, depending on how R is chosen. Similarly, deleting a circuit can lead to an odd decrease in the edge-degree.

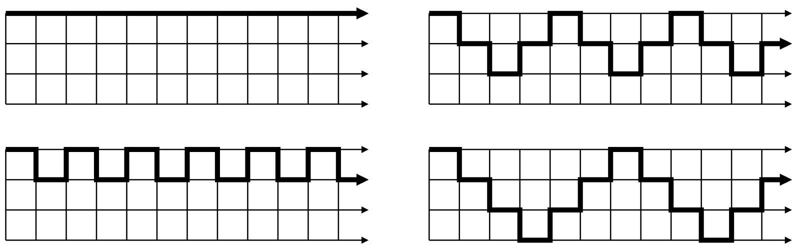

Figure 4.5: Removal of a ray lets the edge-degree decrease by 1 or more.

The second area where considering the edge-degree differs from the degree is in its behaviour concerns extremal results. A classical theorem by Mader [35], for instance, states that high average degree forces a finite graph to contain a large complete minor. This, however, fails for locally finite graphs even if every end has high edge-degree. Figure 4.6 indicates how for every $k \geq 5$ a planar k-regular graph with a single end of infinite edge-degree can be constructed. Being planar, such a graph can never contain even a K^5 as a minor.

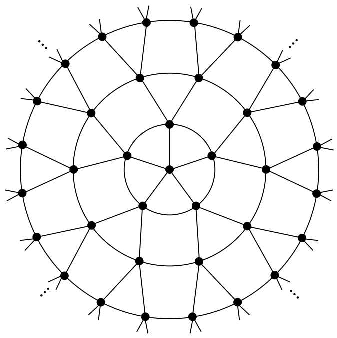

Figure 4.6: High degree in all vertices and high edge-degree in the single end but planar.

4.7 Weakly even ends

Finally, let us briefly discuss an alternative parity concept, which arises from the observation [12] that (ii) of Lemma 4.2.1 is equivalent to:

(iii) for every finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ of vertices such that the maximal number of edge-disjoint ω-rays starting in S' is even.

Lemma 4.2.1 (ii) was our main motivation for our definition of an even end. In the same vein, (iii) leads to the following alternative definition of parity, which differs only in that the quantifiers are exchanged:

Definition 4.7.1. **[12]** *Let H be a subgraph of a locally finite graph G. Call ω* weakly even *in H if for every finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ of vertices such that the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S' is even. Otherwise, ω is* strongly odd *in H.*

Observe that an even end is weakly even, and that a strongly odd end is odd. For ends of finite edge-degree the two parity concepts are equivalent; this can be seen in a similar way as the equivalence of (ii) and (iii). For ends of infinite edge-degree, however, this need not be true: consider a ray $v_1v_2\ldots$, and replace each edge v_iv_{i+1} by i (subdivided) parallel edges. The obtained graph has a single end, which is both odd and weakly even.

This construction only works because there are odd vertices present. But could an odd end exist in a graph that has all vertices even and all ends weakly even? Or, on the contrary:

Problem 4.7.2. **[12]** *Does Theorem 4.1.4 remain true if we substitute "even ends" by "weakly even ends"?*

We have been unable to settle the problem. However, we can answer both this question and Conjecture 4.3.3 positively for locally finite graphs with only countably many ends:

Proposition 4.7.3. **[12]** *Let G be a locally finite graph with only countably many ends, and let H a subgraph. Then $E(H) \in \mathcal{C}(G)$ if and only if every vertex has even degree in H and if every end has weakly even edge-degree in H.*

Proof. The forward direction follows immediately from Theorem 2.5.2 and Corollary 4.4.7. For the backward direction, suppose $E(H) \notin \mathcal{C}(G)$, which by Theorem 2.5.2 means that G has a finite cut F with $|F \cap E(H)|$ odd. Let $\omega_1, \omega_2, \ldots$ be an enumeration of $\Omega(G)$. We successively define a sequence $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots$ of finite sets of disjoint regions $A \subseteq G - F$ of G with $|\partial A \cap E(H)|$ even and such that for each ω_i with $i \leq n$ there is an $A \in \mathcal{A}_n$ with $\omega_i \in \overline{A}$. Put $\mathcal{A}_0 := \emptyset$. In order to define the set $\mathcal{A}_n$ first check whether there is an $A \in \mathcal{A}_{n-1}$ such that $\omega_n \in \overline{A}$, in which case we put $\mathcal{A}_n := \mathcal{A}_{n-1}$. Otherwise consider the (finite) set S of all neighbours of each $A \in \mathcal{A}_{n-1}$ and of the endvertices of the edges in F. As ω_n is weakly even, Lemma 4.4.4 yields a region $B \subseteq G - F$ with $\omega_n \in \overline{B}$ and $A \cap B = \emptyset$ for all $A \in \mathcal{A}_{n-1}$. Put $\mathcal{A}_n := \mathcal{A}_{n-1} \cup \{B\}$. Finally, contracting all the disjoint regions $A \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$ to a vertex each yields a finite graph with all vertex degrees even in H that has a cut F with $|F \cap E(H)|$ odd, a contradiction. $\square$

Chapter 5

Forcing highly connected subgraphs

5.1 Introduction

In a finite graph, high average degree forces the existence of a highly connected subgraph:

Theorem 5.1.1 (Mader [36]). *Any finite graph G of average degree at least $4k$ has a k-connected subgraph.*

In infinite graphs, there is no adequate notion of the 'average degree'. So for an extension of the theorem to infinite graphs we must replace 'average degree' with 'minimum degree'.

As in finite graphs, we define a graph G to be *k-connected* if $|G| > k$ and no set of fewer than k vertices separates G. Then, simply requiring high minimum degree for the vertices is not enough, as the counterexample of the infinite r-regular tree T^r demonstrates. Now, since an infinite tree has rather 'thin' ends, this suggests, as conjectured by Diestel [15], that a minimum degree condition has to be imposed also on the ends of the graph.

So, let us require the ends of the graph to have high minimum degree as well, in the sense that their *vertex-degree* is high. Then the T^r ceases to be a counterexample, as each of its ends has vertex-degree 1. And indeed, with this further condition on the vertex-degrees of the ends, highly connected subgraphs can be forced in locally finite graphs. This will be our main theorem in this chapter.

Theorem 5.1.2. **[44]** *Let $k \in \mathbb{N}$ and let G be an infinite locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then G has a k-connected subgraph.*

It follows from the following stronger result, which we prove in Section 5.5.4:

Theorem 5.1.3. [44] *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then every infinite region of G has a k-connected region.*

What happens if we weaken the condition on the ends, and only require high *edge-degree* instead of high vertex-degree? It turns out that this is not enough, i.e. high edge-degrees at the ends and high degrees at the vertices together are not sufficient to force highly connected subgraphs, or even highly connected minors, in infinite graphs. Indeed, in Section 5.5.3 we exhibit for all $r \in \mathbb{N}$ a locally finite graph of minimum degree and minimum edge-degree r that has no 4-connected subgraph and no 6-connected minor.

But, the assumption of high edge-degree does suffice to force highly edge-connected subgraphs in locally finite graphs (where a subgraph H is *k-edge-connected* if $|H| > 1$ and no set of fewer than k edges separates H). Moreover, such can be found in every infinite region:

Theorem 5.1.4. [44] *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $4k+1$ and each end has edge-degree at least $2k-1$. Then every infinite region of G has a k-edge-connected region.*

We remark that in general, it is not possible to force *finite* highly (edge-) connected subgraphs in infinite graphs by assuming high minimum degree and vertex- (or edge-) degree. Neither can we force *infinite* highly (edge-) connected subgraphs (see discussion after Corollary 5.2.2).

5.2 Forcing highly edge-connected subgraphs

We start by proving our second result, Theorem 5.1.4, which is easier. For this, we need a lemma.

Lemma 5.2.1. [44] *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $\delta_V \geq 4k + 1$. Then every finite non-empty region C of G with $|\partial C| \leq \frac{\delta_V}{2}$ has a k-connected subgraph.*

Proof. Let $v \in V(C)$, and set ∂C. By assumption, v has degree at least δ_V in G, and thus degree at least $\delta_V - |\partial C| \geq |\partial C|$ in C. Hence C contains more than $|\partial C|$ vertices, and therefore has average degree $d(C) \geq \delta_V - 1 \geq 4k$. Thus Theorem 5.1.1 yields a k-connected subgraph of C. □

Theorem 5.1.4. [44] *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $4k+1$ and each end has edge-degree at least $2k-1$. Then every infinite region of G has a k-edge-connected region.*

Proof. Let C be an infinite region of G, and assume that C has no finite k-edge-connected subgraph. We prove that then C has an infinite k-edge-connected region H.

First, suppose that for every infinite region C' of C there is a non-empty region $C'' \subseteq C' - \partial^* C'$ of C such that $|\partial C''| < 2k-1$. Then any such C'' is infinite, by Lemma 5.2.1 and by the assumption that C contains no finite k-edge-connected (and thus in particular no finite k-connected) subgraph. Hence there exists a sequence $C =: C_0, C_1, \ldots$ of infinite regions of C such that for $i \geq 1$

(i) $C_i \subseteq C_{i-1} - \partial^* C_{i-1}$; and

(ii) $|\partial C_i| < 2k-1$.

Now, as each of the C_i is connected, there is a sequence $(P_i)_{i\in\mathbb{N}}$ of $\partial^* C_i$–$\partial^* C_{i+1}$ paths such that for $i \geq 1$ the path P_{i+1} starts in the last vertex of P_i. By (i), the paths P_i are non-trivial, and hence their union $P := \bigcup_{i=1}^{\infty} P_i$ is a ray which has a tail in each of the C_i. Let ω be the end of G that contains P. As, by assumption, ω has edge-degree at least $2k-1$, there is a family $\mathcal{R}$ of $2k-1$ edge-disjoint ω-rays in G. For each ray $R \in \mathcal{R}$ let n_R denote the distance its starting vertex has to $\partial^* C_1$. Set $n := \max\{n_R : R \in \mathcal{R}\}$. Then by (i), all of the $2k-1$ disjoint rays in $\mathcal{R}$ start outside C_{n+1}. But each ray in $\mathcal{R}$ is equivalent to P, and hence eventually enters C_{n+1}, a contradiction as $|\partial C_{n+1}| < 2k-1$ by (ii).

Hence, there is an infinite region C' of C so that for each non-empty region $C'' \subseteq C' - \partial^* C'$ of C holds that

$$|\partial C''| \geq \; 2k-1. \tag{5.1}$$

Observe that as G is locally finite, there exist regions $\subseteq C' - \partial^* C'$ of C which are infinite: take, for example, any infinite component of $C' - \partial^* C'$. Now, choose an infinite region $H \subseteq C' - \partial^* C'$ of C with $|\partial H|$ minimal. By (5.1), ∂H consists of at least $2k-1$ edges.

We claim that H is the desired k-edge-connected region of C. Indeed, suppose otherwise. Then (here we need that H is non-trivial), H has a cut F with $|F| < k$. We may assume that F is a minimal cut, i.e. leaves only two components D, D' in $H - F$. One of the two, say D, is infinite. Then, by the choice of H, the cut $\partial D \subseteq F \cup \partial H$ contains at least $|\partial H|$ edges. Hence, D is incident with all but at most $|F|$ edges of ∂H. Thus $D' \subseteq C' - \partial^* C'$ is a (non-empty) region of C with

$$|\partial D'| \leq |\partial H| - |\partial H \cap \partial D| + |F| \leq 2|F| < 2k-1,$$

a contradiction to (5.1). $\square$

Theorem 5.1.4 is best possible in the sense that high edge-degree is not sufficient to force highly connected subgraphs, as we shall see in the next section. Furthermore, it has two interesting corollaries.

Corollary 5.2.2. [44] *Let $k \in \mathbb{N}$ and let C be an infinite region of a locally finite graph G which has minimum degree $4k+1$ at the vertices and minimum edge-degree $2k-1$ at the ends. Then C has either infinitely many disjoint finite k-edge-connected regions or an infinite k-edge-connected region.*

Proof. Take an inclusion-maximal set $\mathcal{D}$ of disjoint finite k-edge-connected regions of C (which exists by an easy application of Zorn's Lemma), and assume that $|\mathcal{D}| < \infty$. Since $C' := C - \bigcup_{D\in\mathcal{D}} D \subseteq C$ is an infinite region of G, we may use Theorem 5.1.4 to obtain a k-edge-connected region H of C. Then H is infinite by the choice of $\mathcal{D}$. □

The two configurations of Corollary 5.2.2 of which one necessarily appears need not both exist. Indeed, for given $r \in \mathbb{N}$, it is easy to construct an infinite locally finite graph G which has minimum degree and vertex- (and thus edge-) degree r but no infinite 3-edge-connected subgraph. We obtain G from the $r \times \mathbb{N}$ grid by joining each vertex to r disjoint copies of K^{r+1}. Any infinite subgraph of G which is at least 2-edge-connected is also a subgraph of the $r \times \mathbb{N}$ grid, and hence is at most 2-edge-connected.

On the other hand, there are also locally finite graphs of high minimum degree and vertex-degree that have no finite highly edge-connected subgraphs. For given $r \in \mathbb{N}$, add some edges to each level S_i of the r-regular tree T^r so that in the obtained graph $\tilde{T}^r$ each S_i induces a path. The only end of $\tilde{T}^r$ has infinite vertex- and edge-degree, and the vertices of $\tilde{T}^r$ have degree at least r. Now, for every finite subgraph H of $\tilde{T}^r$ there is last level of $\tilde{T}^r$ that contains a vertex v of H. Then v has degree at most 3 in H, and hence, H is not 4-edge-connected.

Our second corollary of Theorem 5.1.4 describes how the graph G decomposes into subgraphs that either are highly edge-connected or only have subgraphs that send many edges to the outside:

Corollary 5.2.3. [44] *Let $k \in \mathbb{N}$, and let G be a locally finite graph with minimum degree $4k+1$ at the vertices and minimum edge-degree $2k-1$ at the ends. Then there is a countable set $\mathcal{D}$ of disjoint k-edge-connected regions of G such that $|\partial H| \geq \max\{2k, |H|\}$ for each subgraph H of $G - \bigcup_{D\in\mathcal{D}} D$.*

Proof. Let $\mathcal{D}$ be an inclusion-maximal set $\mathcal{D}$ of disjoint k-edge-connected regions of G (which exists by Zorn's Lemma). Since G is locally finite and we may assume it to be connected, G is countable, and therefore $\mathcal{D}$ is countable.

Observe that it suffices to show $|\partial H| \geq \max\{2k, |H|\}$ for induced connected subgraphs H of $G - \bigcup_{D\in\mathcal{D}} D$, and consider such an H. If H is infinite, then Theorem 5.1.4 and the (maximal) choice of $\mathcal{D}$ imply that H is not a region of G, i.e. that $|\partial H|$ is infinite, as desired.

So assume that H is finite. Then in particular, H is a region of G, and thus Lemma 5.2.1 ensures that $|\partial H| \geq 2k$. Also, $|\partial H| \geq |H|$, as otherwise H has average degree $d(H) \geq \delta_V - 1 \geq 4k$, and hence H has a k-edge-connected subgraph by Theorem 5.1.1, contradicting the choice of $\mathcal{D}$. $\square$

5.3 High edge-degree is not enough

For given $r \in \mathbb{N}$ we will construct a locally finite graph G_r of minimum degree r (at the vertices) and minimum edge-degree $\geq r$ at the ends that has no 4-connected subgraph and no 6-connected minor. This construction can also be found in [44].

We start with an infinite rooted tree T_r in which each vertex sends r edges to the next level. The graph G_r will be obtained from T_r in the following manner. Let S_0 consist of the root of T_r and for $i \geq 1$ denote by S_i the i-th level of T_r. Now, successively for $i \geq 1$, we add some vertices to S_i, which results in an enlarged ith level S'_i, and then add some edges between $S'_i - S_i$ and S_{i+1}. For this, consider those subsets of S_i whose elements have the same neighbour in S_{i-1}. For each maximal such set S, fix an enumeration $s_1, s_2, \dots, s_r$ of S, and add $r-1$ new vertices $v_1^S, v_2^S, \dots, v_{r-1}^S$ to S_i. Denote by S'_i the set thus obtained from S_i. Then for each

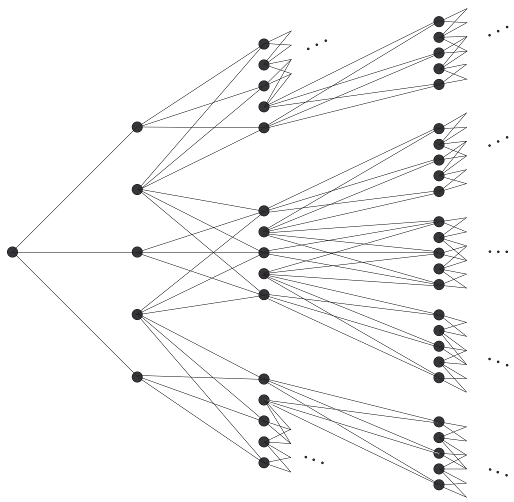

Figure 5.1: The graph G_3.

$j \leq r-1$ and each S as above add all edges between v_j^S and $N_{S_{i+1}}(\{s_j, s_{j+1}\})$. This yields a graph G_r on the disjoint union of sets $S'_1, S'_2, \dots$ as depicted in Figure 5.1 for $r = 3$.

Lemma 5.3.1. [44] *G_r has minimum degree r at the vertices and minimum edge-degree $\geq r$ at the ends.*

For the proof, we need a lemma which follows immediately from Corollary 4.4.6:

Lemma 5.3.2. *If a locally finite graph G has no cuts of cardinality $< k \in \mathbb{N}$, then each of its ends has edge-degree at least k.*

Proof of Lemma 5.3.1. By construction, G_r has mimimum degree r at the vertices. To see that the ends of G_r have edge-degree at least r, we use Lemma 5.3.2; hence, it suffices to show that G_r has no cuts of cardinality less than r. This will be done by proving inductively for $n \in \mathbb{N}$ that the vertices in $\bigcup_{i=0}^{n} S_i'$ cannot be separated in G_r by less than r edges. The assertion clearly holds for $n = 0$, as $S_0' = S_0$ consists of only one vertex. So suppose $n > 0$. By assumption, S_{n-1}' cannot be separated in G_r by less that r edges, and by construction, S_{n-1} cannot be separated in G_r by less than r edges from any of the maximal subsets S of S_n whose elements have the same neighbour in S_{n-1}. Hence, we only need to show that no such S together with the corresponding $v_1^S, v_2^S, \ldots, v_{r-1}^S \in S_n' - S_n$ can be separated in G_r by less than r edges. But this is easy: any two vertices of $S \cup \{v_1^S, v_2^S, \ldots, v_{r-1}^S\}$ are connected by r edge-disjoint paths in $G_r[S_n' \cup S_{n+1}]$. □

Observe that every finite set A of vertices can be separated from any end ω by at most three vertices (namely by the neighbours of the unique component of $G_r - S_i'$ that contains a ray in ω, where i is large enough so that $A \subseteq S_i'$). Hence, each end of G_r has vertex-degree at most 3.

In fact, Theorem 5.1.3 ensures that every graph of high minimum degree (at the vertices) has either an end of small vertex-degree or a highly connected subgraph. We shall see now that the latter is not the case for G_r.

Lemma 5.3.3. [44] *G_r has no 4-connected subgraph.*

Proof. Suppose G_r has a 4-connected subgraph H, and let $i \in \mathbb{N}$ so that $V(H) \cap S_i' \neq \emptyset$. Now, if there is vertex $v \in V(H) - S_{i+1}'$, then it can be separated in G_r (and thus also in H) from $V(H) \cap S_i'$ by at most three vertices (namely by the neighbours of the component of $G_r - S_{i+1}'$ that contains v). So, as H is 4-connected, $V(H) - S_{i+1}'$ must be empty. Hence, H is finite, implying that there is a maximal $j \in \mathbb{N}$ such that $V(H) \cap S_j' \neq \emptyset$. But then by construction of G_r, any vertex in $V(H) \cap S_j'$ has degree at most three in H, a contradiction as H is 4-connected. □

It is slightly more difficult to prove that G_r has no highly connected minor.

Lemma 5.3.4. [44] *G_r has no 6-connected minor.*

Proof. Suppose that G_r has a 6-connected minor M. Then there is an $n \in \mathbb{N}$ so that each branch-set of M has a vertex in $\bigcup_{i=0}^{n} S_i'$. Furthermore, since M is 6-connected, each separator $T \subseteq \bigcup_{i=0}^{n} S_i'$ of G_r with $|T| \leq 5$ leaves a component C of $G_r - T$ such that $V(C) \cup T$ meets one and hence every branch-set of M. So as each S_i' can be separated in G_r from any component of $G - S_i'$ by at most three vertices, there is an $i < n$ such that each branch-set of M meets $S_i' \cup S_{i+1}'$.

Moreover, there is a maximal set S of neighbours in S_{i+1} of the same vertex in S_i such that each branch-set of M has a vertex in $S' := S \cup N_{S'_i}(S) \cup \{v_1^S, v_2^S, \ldots, v_r^S\}$. Then $|S' \cap S'_i| \leq 3$.

We claim that M is also a minor of the finite graph G'_r (see Figure 5.2) which

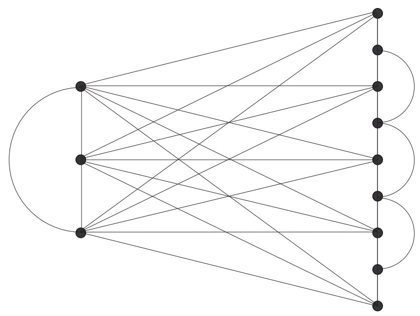

Figure 5.2: The graph G'_5 for $|S' \cap S'_i| = 3$.

is obtained from $G_r[S']$ by adding an edge between every two vertices that are neighbours of the same component of $G_r - S'$. Indeed, each component C of $G_r - S'$ has at most three neighbours in S'. Hence, since M is 6-connected, C meets only (if at all) those branch-sets of M that also meet $N_{S'}(C)$. It is easy to see that M is still a minor of the graph we obtain from G_r by deleting C and adding all edges between vertices in $N_{S'}(C)$. Arguing analogously for the other components of $G_r - S'$, we see that M is also a minor of G'_r.

As $|S' \cap S'_i| \leq 3$, all but at most 3 branch-sets of M in G'_r have all their vertices in $|S' \cap S'_{i+1}|$. Then these give rise to a 3-connected minor of $G'_r - S'_i$. But each non-trivial block of $G'_r - S'_i$ is a triangle and hence has no 3-connected minor, yielding the desired contradiction. □

Note that the two latter results are best possible, since G_r has a 3-connected subgraph, the K^4, and a 5-connected minor, the K^6.

5.4 Forcing highly connected subgraphs

We finally prove the main result of this chapter, which we restate:

Theorem 5.1.3. [44] *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then every infinite region of G has a k-connected region.*

Proof. Let C be an infinite region of G, and assume that C has no finite k-connected subgraph. We shall then find an infinite region H of C which is k-

connected. Set $\delta_V := 6k^2 - 5k + 3$ and $\delta_\Omega := 6k^2 - 9k + 4$. Note that we may assume that $k > 1$.

First, suppose that for every infinite region C' of C there is a region $C'' \subseteq C' - \partial^* C'$ of C such that $|\partial^* C''| < \delta_\Omega$ and $V(C'') \neq \partial^* C''$. Observe that each such C'' is infinite, as otherwise $C'' - \partial^* C''$ has minimum degree $\delta(C'' - \partial^* C'') \geq \delta_V - \delta_\Omega + 1 \geq 4k$. Then Theorem 5.1.1 yields a finite k-connected subgraph of $C'' \subseteq C$, contradicting our assumption. Hence, there exists a sequence $C =: C_1, C_2, \ldots$ of infinite regions of C such that $C_i \subseteq C_{i-1} - \partial^* C_{i-1}$ and $|\partial^* C_i| < \delta_\Omega$ for all $i > 1$.

As in the proof of Theorem 5.1.4, we see that there is an end $\omega \in \Omega(G)$ that has a ray R such that each of the C_i contains a tail of R. As ω has vertex-degree at least δ_Ω, there are δ_Ω disjoint ω-rays in G. The starting vertices of these lie at finite distance to $\partial^* C_1$, hence, since $C_i \subseteq C_{i-1} - \partial^* C_{i-1}$ for $i > 1$, there is an $n \in \mathbb{N}$ so that all of the δ_Ω disjoint ω-rays start outside C_n. But (being equivalent to R) each of these rays eventually enters C_n, a contradiction because $|\partial^* C_n| < \delta_\Omega$.

Hence, there is an infinite region C' of C such that

$$|\partial^* C''| \geq \delta_\Omega \textit{ for each region } C'' \subseteq C' - \partial^* C' \textit{ of } C \textit{ with } V(C'') \neq \partial^* C''. \tag{5.2}$$

For a region $H \subseteq C' - \partial^* C'$ of C write

$$\Sigma_H := \sum_{v \in V(H)} \max\{0, \delta_V - d_H(v)\},$$

and choose an infinite region $H \subseteq C' - \partial^* C'$ of C such that $k|\partial^* H| + \Sigma_H$ is minimal. Observe that this sum is finite, since all vertices of H but those in $\partial^* H$ have degree $\geq \delta_V$ in H, and it is possible to choose $H \subseteq C' - \partial^* C'$ with $|\partial^* H| < \infty$ because G is locally finite. Then $|\partial^* H| \geq \delta_\Omega$ by (5.2).

Assume that there is a vertex $v \in V(H)$ that has degree at most $2k - 1$ in H. Then $d_{H-v}(w) = d_H(w) - 1$ for each of the at most $2k - 1$ neighbours w of v in H, and $d_{H-v}(w') = d_H(w')$ for all other vertices w' in H. Therefore,

$$\begin{aligned} k|\partial^*(H - v)| + \Sigma_{H-v} &\leq k|\partial^* H| + k(2k-2) + \Sigma_H + (2k-1) - (\delta_V - d_H(v)) \\ &\leq k|\partial^* H| + \Sigma_H + 2k(k+1) - \delta_V \\ &< k|\partial^* H| + \Sigma_H. \end{aligned}$$

So any infinite component of $H - v$ is a better choice than H, a contradiction. We thus have shown that

$$d_H(v) \geq 2k \textit{ for all } v \in V(H). \tag{5.3}$$

We shall now prove that H is the desired k-connected region of C. Indeed, suppose otherwise. Then H has a separator T of cardinality $< k$, which we may assume to be a minimal separator. Note that each such separator leaves a component D of $H - T$ such that $H - D$ is an infinite region of C. We claim that T and D can be chosen such that for $H' := H - D$

$$d_{H'}(v) \geq 2 \textit{ for each vertex } v \in T. \tag{5.4}$$

Indeed, choose a separator T of minimal cardinality in H and a component D of $H - T$ such that the number of vertices in T that have degree at most 1 in H' is minimal. Suppose that there is a $v \in T$ so that $d_{H'}(v) \leq 1$. Then the minimality of T implies that $d_{H'}(v) = 1$, and that the neighbour w of v in H' does not lie in T. By (5.3), w has degree at least $2k \geq 3$ in H. Hence, since $w \notin T$, also $d_{H'}(w) \geq 3$. Thus the number of vertices in $T' := T \setminus \{v\} \cup \{w\}$ that have degree at most 1 in $H - (D \cup \{v\})$ is smaller than the number of vertices in T that have degree at most 1 in H'. Now, the minimality of $|T|$ ensures that T' is a minimal separator of H, and has minimal cardinality. Furthermore, $D \cup \{v\}$ is a component of $H - T'$ (as T is a minimal separator and hence v sends an edge to D), and $H - (D \cup \{v\})$ is infinite (as H' is), a contradiction to the choice of T. This establishes (5.4).

We claim that

$$|V(D) \cap \partial^* H| \geq \delta_\Omega - |T|. \tag{5.5}$$

Then we obtain for the infinite region $H' \subseteq C' - \partial^* C'$ of C that

$$\begin{aligned} |\partial^* H'| &\leq |\partial^* H| - |V(D) \cap \partial^* H| + |T| \\ &\leq |\partial^* H| - \delta_\Omega + 2|T|. \end{aligned}$$

Furthermore, by (5.4),

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H + \sum_{v \in T} \max\{0, \delta_V - d_{H'}(v)\} \\ &\leq \Sigma_H + (\delta_V - 2)|T|, \end{aligned}$$

and so

$$\begin{aligned} k|\partial^* H'| + \Sigma_{H'} &\leq k|\partial^* H| + \Sigma_H - k\delta_\Omega + (\delta_V + 2k - 2)|T| \\ &\leq k|\partial^* H| + \Sigma_H - k\delta_\Omega + (6k^2 - 3k + 1)(k - 1) \\ &< k|\partial^* H| + \Sigma_H, \end{aligned}$$

contradicting the choice of H.

It remains to show the validity of (5.5). Suppose otherwise, i.e. that $|V(D) \cap \partial^* H| < \delta_\Omega - |T|$. Then for the region $\tilde{D} := G[V(D) \cup T] \subseteq C' - \partial^* C'$ of C holds that

$$|\partial^* \tilde{D}| = |T \cup (V(D) \cap \partial^* H)| \leq |T| + |V(D) \cap \partial^* H| < \delta_\Omega.$$

Hence by (5.2), $V(\tilde{D}) = \partial^* \tilde{D}$, implying that $V(D) \subseteq \partial^* H$. In particular, $|D| < \delta_\Omega - |T|$. So each vertex $v \in V(D)$ has degree at most $|D \cup T - \{v\}| \leq \delta_\Omega - 1 = \delta_V - 4k$ in H. Then $\delta_V - d_H(v) \geq 4k$, and thus

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H - \sum_{v \in V(D)} \max\{0, \delta_V - d_H(v)\} + \sum_{v \in T} (d_H(v) - d_{H'}(v)) \\ &\leq \Sigma_H - 4k|D| + |T||D| \\ &< \Sigma_H. \end{aligned}$$

On the other hand, (5.3) ensures that $|D| \geq k$. So

$$|\partial^* H'| \leq |\partial^* H| - |D| + |T| < |\partial^* H|,$$

and thus

$$k|\partial^* H'| + \Sigma_{H'} < k|\partial^* H| + \Sigma_H,$$

a contradiction to the choice of H. This completes the proof of (5.5), and hence the proof of the theorem. $\square$

Theorem 5.1.3 has two corollaries. The proof of the first is analogous to that of Corollary 5.2.2.

Corollary 5.4.1. [44] *Let $k \in \mathbb{N}$ and let C be an infinite region of a locally finite graph of minimum degree $6k^2 - 5k + 3$ at the vertices and minimum vertex-degree $6k^2 - 9k + 4$ at the ends. Then C has either infinitely many disjoint finite k-connected regions or an infinite k-connected region.*

Again, these two configurations need not both exist, as the examples following Corollary 5.2.2 illustrate. (For this, observe that if a graph has no highly edge-connected subgraph then it clearly has no highly connected subgraph.)
The second corollary of Theorem 5.1.3 is an analogon of Corollary 5.2.3.

Corollary 5.4.2. [44] *Let $k \in \mathbb{N}$, and let G be a locally finite graph with minimum degree $\delta_V \geq 6k^2 - 5k + 3$ at the vertices and minimum vertex-degree $\delta_\Omega \geq 6k^2 - 9k + 4$ at the ends. Then there is a countable set $\mathcal{D}$ of disjoint k-connected regions of G such that $|\partial^* H| \geq \max\{\delta_\Omega, \frac{k-1}{k}|H| + 1\}$ for each non-empty subgraph H of $G - \bigcup_{D \in \mathcal{D}} D$.*

Proof. Similarly as in the proof of Corollary 5.2.3, take an inclusion-maximal set $\mathcal{D}$ of disjoint k-connected regions of G, which then is countable.
Observe that we only need to consider induced connected non-empty subgraphs H of $G - \bigcup_{D \in \mathcal{D}} D$. So let H be a such. If H is infinite, then Theorem 5.1.3 and the choice of $\mathcal{D}$ imply that H is not a region, i.e. that $|\partial^* H|$ is infinite, as desired.
So assume that H is finite. Then $|\partial^* H| \geq \delta_\Omega$, as otherwise $H - \partial^* H$ has minimum degree $d(H - \partial^* H) \geq \delta_V - \delta_\Omega + 1 \geq 4k$, and hence H has a k-connected subgraph by Theorem 5.1.1, contradicting the choice of $\mathcal{D}$.
Also, $|\partial^* H| > \frac{k-1}{k}|H|$. Indeed, suppose otherwise. Then H has average degree

$$d(H) \geq \frac{\delta_V |H - \partial^* H| + |\partial^* H|}{|H|} \geq \delta_V - (\delta_V - 1)\frac{|\partial^* H|}{|H|} \geq \frac{\delta_V + k - 1}{k} \geq 4k,$$

since we may assume that $k \geq 2$. Thus Theorem 5.1.1 yields a k-connected subgraph of H, a contradiction to the choice of $\mathcal{D}$. $\square$

Chapter 6

Arboricity

6.1 Introduction

A criterion for the smallest number of acyclic subgraphs of a finite graph whose union contains the entire graph is given by Nash-Williams' arboricity theorem:

Theorem 6.1.1 (Nash-Williams [38]). *Let $k \in \mathbb{N}$, and let G be a finite multigraph in which no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Then G is the edge-disjoint union of at most k forests.*

Theorem 6.1.1 easily extends to locally finite graphs by compactness, if a forest is defineded as a graph that contains no *finite* cycles. However, in our setting the forests of an appropriate infinite analogue of Nash-Williams' theorem should not be allowed to contain infinite cycles either, i.e. be *topological forests.* Such a result would be much stronger. So much so, in fact, that without additional constraints it is false.

Indeed, consider the infinite ladder in which each rung except the first has been subdivided and all other edges duplicated (see Figure 6.1). This multigraph satis-

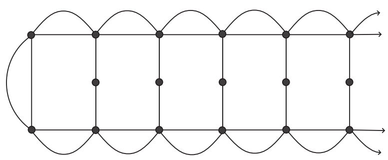

Figure 6.1: Every two forests partitioning the multigraph contain infinite cycles

fies the condition of Theorem 6.1.1 for $k = 2$, because it is an edge-disjoint union of two (ordinary) forests, but clearly, every two such forests must each contain a double ray. That double ray forms an infinite cycle as its closure contains the graph's single end.

We can easily generalise this counterexample to arbitrary $k \in \mathbb{N}$ and simple graphs. Simply replace each of the subgraphs of the form $(\{v, w\}, \{vw, vw\})$ with a simple finite graph H that is the union of k edge-disjoint spanning trees (for example $H = K^{2k}$), identifying v, w with distinct vertices of H. Then, as before, G is an edge-disjoint union of k ordinary forests, and hence satisfies Nash-Williams' condition that no set of ℓ vertices spans more than $k(\ell - 1)$ edges. But any partition of G into k forests induces such a partition in each copy of H, ie. into spanning trees of H. Each of these contains a v–w path, so each of our k forests contains a double ray and thus an infinite cycle. These counterexamples are due to Bruhn and Diestel (unpublished).

In order to generalise Theorem 6.1.1 to topological forests, we thus need to impose some further conditions. One natural way to do this is to require local sparseness not only for all finite subgraphs (as in Nash-Williams' condition) but also around ends, eg. by placing an upper bound on their edge-degrees. Observe that this is a weaker condition than the same restriction on the vertex-degree of the ends. Also, it is more natural: the arboricity theorem deals with forests, i.e. in the considered subgraphs we forbid cycles.

Further note that the counterexamples above each have an end of edge-degree $\geq 2k$. It is not difficult to construct others whose end has edge-degree exactly $2k$. Just choose H such that it contains a vertex of degree k and identify this vertex with v.

The following conjecture of Diestel [15], whose proof is the main result of this chapter, is therefore best possible in this sense:

Theorem 6.1.2. [42] *Let $k \in \mathbb{N}$, and let G be a locally finite graph in which no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Further, let every end of G have edge-degree $< 2k$. Then $|G|$ is the edge-disjoint union of at most k topological forests in $|G|$.*

Although, as we have seen, the bound of $2k$ in Theorem 6.1.2 cannot be reduced, the theorem has no direct converse: a partition into k topological forests does not force all edge-degrees to be small. The $\mathbb{N} \times \mathbb{N}$ grid, for example, is an edge-disjoint union of two topological forests (its horizontal vs. its vertical edges), but its unique end has infinite edge-degree.

6.2 Finitely many small cuts cut off all ends

Consider a locally finite graph G, a finite set $S \subseteq V(G)$ and an end $\omega \in \Omega(G)$ of finite degree $< k$. Then Corollary 4.4.6 yields a cut of cardinality $< k$ that separates S from ω, and therefore induces a region $K_\omega \subseteq G - S$ whose closure contains ω.

Now, instead of ω consider a finite set of ends, each of which has edge-degree $< k$. Again we want to find regions K_ω of small co-boundary whose closures contain the respective end ω. These can easily be chosen disjoint, because there is a finite set $S' \supseteq S$ which separates our ends pairwisely.

Finally, consider the set of *all* ends of edge-degree $< k$. Cleary, there is still a (possibly infinite) set of regions $K \subseteq G - S$ with $|\partial K| < k$ such that every such end lies in the closure of one of them. But are we still able to choose these regions disjoint? The next lemma gives a positive answer to this question.

Lemma 6.2.1. **[42]** *Let $k \in \mathbb{N}$, let G be a locally finite graph and let $S \subseteq V(G)$ be finite. Then there is a set $\mathcal{K}$ of disjoint regions $K \subseteq G - S$ of G with $|\partial K| < k$, such that for every $\omega \in \Omega(G)$ with $d(\omega) < k$ there is a $K \in \mathcal{K}$ with $\omega \in \overline{K}$.*

Proof. If S is empty, then $\mathcal{K} := \{G\}$ is as desired, so assume $S \neq \emptyset$. We use induction on k to prove the existence of a set $\mathcal{K}^k$ of disjoint regions of G such that for all $K \in \mathcal{K}^k$:

(i) $K \subseteq G - S$ and $|\partial K| < k$;

(ii) there is no finite set $\mathcal{H}$ such that $V(K) = \bigcup_{H \in \mathcal{H}} V(H)$ and $|\partial H| < |\partial K|$ for all $H \in \mathcal{H}$.

In addition, we require for all regions $K' \subseteq G - S$:

(iii) if $|\partial K'| < k$ then $E(K') - \bigcup_{K \in \mathcal{K}^k} E(K)$ is finite.

We claim that then $\mathcal{K}^k$ is the desired set $\mathcal{K}$ of the lemma. Indeed, consider an end $\omega \in \Omega(G)$ with $d(\omega) < k$, and let $R \in \omega$. By Corollary 4.4.6, there is a region $K' \subseteq G - S$ of G with $\omega \in \overline{K'}$ such that $|\partial K'| < k$. By (iii), $E(K') - \bigcup_{K \in \mathcal{K}^k} E(K)$ is finite. Hence, R has only finitely many of its edges outside $\bigcup_{K \in \mathcal{K}^k} E(K)$. Thus, since the $K \in \mathcal{K}$ are pairwise disjoint, there is a $K \in \mathcal{K}$ such that R has a tail in K, implying $\omega \in \overline{K}$, as desired.

Put $\mathcal{K}^1 := \emptyset$, a choice which trivially satisfies (i) and (ii), and also (iii) because $S \neq \emptyset$ and we may suppose G to be connected (thus, the only $K' \subseteq G - S$ with $|\partial K'| < 1$ is $K' = \emptyset$). So assume we already found a set $\mathcal{K}^{k-1}$ satisfying (i)–(iii); we show the existence of the set $\mathcal{K}^k$.

Let $H_1, H_2, \ldots$ be an enumeration of all regions $H \subseteq G - S$ of G with $|\partial H| < k$ and $E(H) - \bigcup_{K \in \mathcal{K}^{k-1}} E(K)$ infinite (there are only countably many such regions as $E(G)$ is countable). From the H_i and $\mathcal{K}^{k-1}$ we construct a sequence of subgraphs $L_i \subseteq G - S$ as follows. Put $L_0 := \bigcup_{K \in \mathcal{K}^{k-1}} K$, and let for $i \in \mathbb{N}$

$$L_i := L_{i-1} \cup H_i \ \ \textit{if} \ \ \partial H_i \cap E(L_{i-1}) = \emptyset$$

and $L_i := L_{i-1}$ otherwise. It is easily shown by induction that for all $i \in \mathbb{N}$ each component of L_i is a region that sends less than k edges to the rest of G.

Now, put $L := \bigcup_{i=1}^{\infty} L_i$ and let $\mathcal{K}^k$ be the set of the components of L. Note that $\bigcup_{K\in\mathcal{K}^k} K = L \subseteq G - S$ and that the $K \in \mathcal{K}^k$ are induced subgraphs of G. Furthermore, $|\partial K| < k$ for each $K \in \mathcal{K}^k$, as otherwise there would already have been a component K' of L_i with $|\partial K'| \geq k$ for some $i \in \mathbb{N}$ (just choose i such that L_i contains at least k vertices which are incident with edges in ∂K plus finite paths that connect these vertices pairwisely). Thus, $\mathcal{K}^k$ is a set of disjoint regions for which (i) holds.

Let us show (ii). Suppose there are a $K \in \mathcal{K}^k$ and a finite set $\mathcal{H}$ such that $V(K) = \bigcup_{H\in\mathcal{H}} V(H)$ and $|\partial H| < |\partial K|$ for all $H \in \mathcal{H}$. By (i), $|\partial K| < k$, thus, $|\partial H| < k-1$ for all $H \in \mathcal{H}$. Then (iii) for $k-1$ yields that $E(H) - \bigcup_{K'\in\mathcal{K}^{k-1}} E(K')$ is finite for all $H \in \mathcal{H}$. As $|\mathcal{H}| < \infty$ and ∂H is bounded for all $H \in \mathcal{H}$, also $E(K) - \bigcup_{H\in\mathcal{H}} E(H)$ is finite, implying that $E(K) - \bigcup_{K'\in\mathcal{K}^{k-1}} E(K')$ is finite. Hence K contains none of the H_i used in the construction of $\mathcal{K}^k$, and thus $K = K'$ for some $K' \in \mathcal{K}^{k-1}$, contradicting (ii) for $k-1$.

Finally, we prove (iii). Suppose there is a region $K' \subseteq G - S$ of G with $|\partial K'| < k$ such that $E(K') - \bigcup_{K\in\mathcal{K}^k} E(K) = E(K') - E(L)$ is infinite. Assume K' is chosen with $|\partial K' \cap E(L)|$ minimal. Because $K' = H_j$ for some $j \in \mathbb{N}$, there is a $K \in \mathcal{K}^k$ that contains edges of $\partial K'$, as otherwise $\partial K' \cap E(L_{j-1}) = \emptyset$, resulting in $K' \subseteq L_j \subseteq L$, which contradicts our assumption that $E(K') - E(L)$ is infinite.

Hence $\partial K' \cap E(K) \neq \emptyset$, but $\partial K \cap E(L) = \emptyset$, implying that $|\partial(K \cup K') \cap E(L)|, |\partial(K' - K) \cap E(L)| < |\partial K' \cap E(L)|$. As G is connected, $K' - K$ has only finitely many components, one of which is a region $K'' \subseteq G - S$ such that $E(K'') - E(L)$ is infinite and $|\partial K'' \cap E(L)| < |\partial K' \cap E(L)|$. Also $K \cup K'$ is such a region. Thus, the choice of K' ensures that $|\partial(K \cup K')|$, $|\partial(K' - K)| \geq k$.

Then

$$\begin{aligned}
|\partial(K \cap K')| + k &\leq |\partial(K \cap K')| + |\partial(K \cup K')| \\
&= |E(K \cap K', K - K')| + |E(K \cap K', K' - K)| \\
&\quad + |E(K - K', G - (K \cup K'))| \\
&\quad + |E(K' - K, G - (K \cup K'))| \\
&\quad + 2|E(K \cap K', G - (K \cup K'))| \\
&= |\partial K| + |\partial K'| - 2|E(K - K', K' - K)| \\
&\leq |\partial K| + |\partial K'| \\
&< |\partial K| + k,
\end{aligned}$$

and similarly

$$\begin{aligned}
|\partial(K - K')| + k &\leq |\partial(K - K')| + |\partial(K' - K)| \\
&\leq |\partial K| + |\partial K'| \\
&< |\partial K| + k.
\end{aligned}$$

Thus, $|\partial(K \cap K')|$, $|\partial(K - K')| < |\partial K|$. But then each of the finitely many components of $K \cap K'$ and of $K - K'$ sends $< |\partial K|$ edges to the rest of G, while $V(K) = V(K \cap K') \cup V(K - K')$, a contradiction to (ii). $\square$

If, as is the case in Theorem 6.1.2, $d(\omega)$ is bounded for all $\omega \in \Omega(G)$, the set $\mathcal{K}$ from Lemma 6.2.1 has to be finite:

Lemma 6.2.2. [42] *Let $k \in \mathbb{N}$, let G be a locally finite graph, and let $S \subseteq V(G)$ be finite. Suppose that every $\omega \in \Omega(G)$ has edge-degree $< k$. Then there is a finite number of disjoint regions $K_1, K_2, \ldots, K_n \subseteq G - S$ with $|\partial K_i| < k$ for all $i = 1, \ldots, n$ such that for every $\omega \in \Omega(G)$ there is an $i \leq n$ with $\omega \in \overline{K_i}$.*

For the proof, we need a standard lemma:

Lemma 6.2.3 (Diestel [16]). *A locally finite connected graph G contains for every infinite set $U \subseteq V(G)$ a ray R together with infinitely many disjoint U–$V(R)$ paths.*

Proof of Lemma 6.2.2. Lemma 6.2.1 supplies us with a set $\mathcal{K}$ of disjoint regions K of G that have the desired properties. If we can show that $\mathcal{K}$ is finite, we are done. So suppose otherwise, and let $U \subseteq V(G)$ contain exactly one vertex of each $K \in \mathcal{K}$. Then, as we may assume G is connected, Lemma 6.2.3 yields a ray R together with infinitely many disjoint U–$V(R)$ paths. As the end ω that contains R lies in the closure of one of the $K \in \mathcal{K}$, a tail of R is contained in the region K, which contradicts the existence of infinitely many disjoint U–$V(R)$ paths. $\square$

6.3 Arboricity for locally finite graphs

In our proof of Theorem 6.1.2 we shall successively define certain finite sets $S_1 \subseteq S_2 \subseteq \ldots$ of vertices, together with partitions of $E(G[S_i])$. In order to extend the partition of $E(G[S_i])$ to a partition of $E(G[S_{i+1}])$, we want to use Theorem 6.1.1 on the graph $\tilde{G}$ obtained from $G[S_{i+1}]$ by contracting S_i to a vertex, which we can do if the arboricity condition holds for $\tilde{G}$. The following lemma ensures that there is a way to choose the S_i so that it does:

Lemma 6.3.1. [42] *Let $k \in \mathbb{N}$, and let G be a locally finite graph in which no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Then for every finite $S \subseteq V(G)$ there is a finite $S' \subseteq V(G)$ with $S' \supseteq S$ such that $||G[X]|| + |E(X, S')| \leq k|X|$ for each $X \subseteq V(G - S')$.*

Proof. Put $S_0 := S$, and for $i \geq 1$ successively define S_i as $S_{i-1} \cup X_i$ if there is an $X_i \subseteq V(G - S_{i-1})$ such that $||G[X_i]|| + |E(X_i, S_{i-1})| > k|X_i|$. Observe that then X_i is finite. Either the process stops at some $I \in \mathbb{N}$ in which case we put $S' := S_I$ and are done, or we obtain an infinite sequence $S_0 \subseteq S_1 \subseteq \ldots$ together with the corresponding X_i. In the latter case, consider for $n := k|S|$ the set S_n. By choice of the X_i,

$$||G[S_n]|| \geq \sum_{i=1}^{n}(k|X_i|+1) = k(\sum_{i=1}^{n}|X_i| + |S|) = k|S_n| > k(|S_n|-1),$$

contradicting our assumption that the arboricity condition holds for G. $\square$

We define for a vertex $v \in V(G)$ and for $i \in \mathbb{N}$ the set $N_i(v)$ to be the set of all vertices with distance i to v (thus, in particular, $N_1(v) = N(v)$). For the set $\{N_i(x) : x \in X\}$, where $X \subseteq V(G)$ and $i \in \mathbb{N}$, we write $N_i(X)$.

Proof of Theorem 6.1.2. We successively define for all $i \in \mathbb{N}$ finite sets $S_i \subseteq V(G)$, together with k edge-disjoint forests $F_1^i, \ldots, F_k^i$, such that

(i) $S_{i-1} \cup N(S_{i-1}) \subseteq S_i$, for $i \geq 2$;

(ii) $F_j^{i-1} \subseteq F_j^i$, for $j = 1, \ldots, k$ and $i \geq 2$;

(iii) $\bigcup_{j=1}^{k} E(F_j^i) = E(G[S_i])$; and

(iv) If $C \subseteq G$ is a cycle so that $C \cap G[S_i] \subseteq F^i_{i \bmod k}$, then $V(C) \cap S_{i-1} = \emptyset$, for $i \geq 2$.

We claim that the (by (ii) well-defined) unions $\bigcup_{i=1}^{\infty} F_1^i, \ldots, \bigcup_{i=1}^{\infty} F_k^i$ are the desired topological forests. Indeed, (i) and (iii) ensure that their edge sets partition $E(G)$. Suppose that there is a $j \in \{1, \ldots, k\}$ so that $\bigcup_{i=1}^{\infty} F_j^i$ contains an infinite cycle C of G. Let v be a vertex in $V(C)$. By (i), we can choose $i \geq 2$ so that $v \in S_{i-1}$ and $j = i \mod k$. This contradicts (iv).
A further condition is needed to make the successive choice of the forests F_j^i possible. We require that for $i \in \mathbb{N}$

(v) $||G[X]|| + |E(X, S_i)| \leq k|X|$ for every $X \subseteq V(G - S_i)$.

Let S_1 be any one-elemented subset of $V(G)$ and put $F_1^1, \ldots, F_k^1 := \emptyset$; this choice obviously satisfies (iii) and (v), which is all we required for $i = 1$. So suppose $i \geq 2$, and that $S_\ell, F_1^\ell, \ldots, F_k^\ell$ are already defined for $\ell < i$ and satisfy (i)–(v). Since S_{i-1} is finite, Lemma 6.2.2 yields a finite number of regions $K_1, \ldots, K_n \subseteq G - S_{i-1}$ of G such that $|\partial K_m| < 2k$ for all $m = 1, \ldots, n$ and such that every end of G lies in the closure of one of the K_m. Then $T := V(G - \bigcup_{m=1}^{n} K_m)$ has only finitely many components, none of which may contain a ray. Thus T is finite, hence, as $|\bigcup_{m=1}^{n} \partial K_m| < \infty$, also $S := T \cup \bigcup_{m=1}^{n} N(G - K_m)$ is finite. So Lemma 6.3.1 yields a finite $S' \supseteq S$. Put $S_i := S'$ and observe that conditions (i) and (v) are satisfied.
In order to define the forests $F_1^i, \ldots, F_k^i$, we consider the multigraph $\tilde{G}$ obtained from $G[S_i]$ by contracting S_{i-1} to the vertex s_{i-1}, keeping multiple edges but

deleting loops (if necessary, we first make S_{i-1} connected by adding some extra edges). Note that $K_m \cap G[S_i] = K_m \cap \tilde{G} \subseteq \tilde{G}$ and furthermore, as $\partial K_m \subseteq E(G[S_i])$, also $\partial K_m \subseteq E(\tilde{G})$. Condition (v) for $i-1$ (together with the arboricity condition for G) implies that in the finite multigraph $\tilde{G}$, no set of ℓ vertices induces more than $k(\ell-1)$ edges. Hence, by Theorem 6.1.1 there is a partition of $E(\tilde{G})$ into the edge sets of k forests $\tilde{F}_1, \dots, \tilde{F}_k \subseteq \tilde{G}$. Let $I := i \mod k$ and assume the $\tilde{F}_j$ are chosen so that $|E(\tilde{F}_I) \cap \bigcup_{m=1}^n \partial K_m|$ is minimal.

We claim that for $m = 1, \dots, n$:

all edges in $E(\tilde{F}_I) \cap \partial K_m$ are incident with the same component of $\tilde{F}_I \cap K_m$. (6.1)

Then the partition of $E(\tilde{G})$ into $E(\tilde{F}_1), \dots, E(\tilde{F}_k)$ corresponds to a partition of $E(G[S_i]) - E(G[S_{i-1}])$ into the edge sets of k forests $F_1, \dots, F_k \subseteq G[S_i]$. Put $F_j^i := F_j^{i-1} \cup F_j$ for $j = 1, \dots, k$, and observe that F_j^i is a forest since F_j^{i-1} as well as F_j is acyclic, and any cycle meeting both contains a subgraph that corresponds to a cycle of $\tilde{F}_j$. This choice satisfies (ii) and (iii). In order to see (iv), let $C \subseteq G$ be a cycle with $C \cap G[S_i] \subseteq F_I^i$, and suppose $V(C) \cap S_{i-1} \neq \emptyset$. By Theorem 2.5.2, C meets each ∂K_m in an even number of edges. For every two edges in $E(C) \cap \partial K_m$ there is a path in $F_I^i \cap K_m$ that connects their endvertices in K_m, because of (6.1). So, if $E(C) \cap \partial K_m \neq \emptyset$, we can substitute $C \cap K_m$ with the union of these paths. Doing so successively for all m, we obtain a finite subgraph of the forest F_I^i, that has only vertices of degree ≥ 2, and thus contains a cycle, which is impossible. This establishes (iv).

So, let us prove (6.1). Consider an $m \in \{1, \dots, n\}$. As otherwise (6.1) is clearly satisfied for m, suppose that $|E(\tilde{F}_I) \cap \partial K_m| \geq 2$. Because $|\partial K_m| < 2k$, there is then a $j \in \{1, \dots, k\}$ such that $|E(\tilde{F}_j) \cap \partial K_m| \leq 1$. We may assume that there indeed is an edge $e \in E(\tilde{F}_j) \cap \partial K_m$, as otherwise taking any edge from $E(\tilde{F}_I) \cap \partial K_m$ and adding it to $\tilde{F}_j$ clearly yields a better choice of the forests $\tilde{F}_1, \dots, \tilde{F}_k$. Let $e = vw$ with $v \in V(K_m)$ and $w \in V(\tilde{G} - K_m)$.

Now, consider the graph $\tilde{F}$ obtained from $\tilde{F}_I$ by contracting the components of $\tilde{F}_I \cap K_m$ and of $\tilde{F}_I - K_m$, deleting loops. Then $E(\tilde{F}) = E(\tilde{F}_I) \cap \partial K_m$; furthermore, $\tilde{F}$ is a forest, as $\tilde{F}_I$ is one. Let $\tilde{v} \in V(\tilde{F})$ be the vertex whose branch-set in $\tilde{F}_I$ contains v. Choose $X \subseteq V(\tilde{F})$ with $\tilde{v} \in X$ such that the branch-set of each $x \in X$ lies in K_m and that every non-trivial component of $\tilde{F}$ has exactly one vertex in X. Now, put $E_1 := E(X, N_1(X)) \cup E(N_2(X), N_3(X)) \cup \dots$ and $E_2 := E(N_1(X), N_2(X)) \cup E(N_3(X), N_4(X)) \cup \dots$; these two sets clearly partition $E(\tilde{F})$. Observe that in $\tilde{G}$

each component of $\tilde{F}_I - K_m$ is adjacent to at most one edge of E_1, (6.2)

and

each component of $\tilde{F}_I \cap K_m$ is adjacent to at most one edge of $E_2 \cup e$. (6.3)

Put $H_\ell := \tilde{F}_\ell$ for $\ell \in \{1, \ldots, k\} \setminus \{I, j\}$ and let H_I, H_j be subgraphs of $\tilde{G}$ with

$$E(H_I) := E(\tilde{F}_I - K_m) \cup E_1 \cup E(\tilde{F}_j \cap K_m),$$

$$E(H_j) := E(\tilde{F}_j - K_m) \cup E_2 \cup e \cup E(\tilde{F}_I \cap K_m).$$

We claim that H_I and H_j are forests. Indeed, any cycle in H_I contains edges of E_1, and thus a path in $\tilde{F}_I - K_m$ that connects two edges of E_1, which is impossible, by (6.2). On the other hand, any cycle in H_j must contain edges of $E_2 \cup e$, and thus a path in $\tilde{F}_I \cap K_m$ that connects two edges of $E_2 \cup e$, a contradiction to (6.3). Hence, as $E(H_1), \ldots, E(H_k)$ clearly partition $E(\tilde{G})$, and $E(\tilde{F}_I \cap \bigcup_{m=1}^n \partial K_m) = E(H_I \cap \bigcup_{m=1}^n \partial K_m) \cup E_2$, the choice of $\tilde{F}_1, \ldots, \tilde{F}_k$ implies that $E_2 = \emptyset$. Suppose there is an edge $e' \in E_1 = E_1 \cup E_2 = E(\tilde{F}_I) \cap \partial K_m$, with endvertex x in K_m, such that there is no v–x path in $\tilde{F}_I \cap K_m$. Then put $H'_I := (V(H_I), E(H_I) - \{e'\})$ and $H'_j := (V(H_j), E(H_j) + \{e'\})$. Observe that H'_j is a forest, as any cycle in H'_j contains both e and e', and thus a v–x path in $\tilde{F}_I \cap K_m$, which is impossible. But since H'_I has less edges in $\bigcup_{m=1}^n \partial K_m$ than $\tilde{F}_I$, this contradicts the choice of $\tilde{F}_1, \ldots, \tilde{F}_k$.

So every edge in $E(\tilde{F}_I) \cap \partial K_m$ is incident with the component of $\tilde{F}_I - K_m$ that contains v, establishing (6.1). $\square$

Chapter 7

Cycle-cocycle partitions

7.1 Introduction

By a result of Gallai (see Lovász [33]), every finite graph has a 'cycle-cocycle' partition of its edge set induced by a bipartition of its vertex set:

Theorem 7.1.1. *Every finite graph G admits a vertex partition into (possibly empty) sets V_1, V_2 such that both $E(G[V_1])$ and $E(G[V_2])$ are elements of the cycle space of G.*

Let us see what happens if we disallow infinite cycles in infinite graphs. Then, Gallai's theorem does not extend to infinite graphs. Indeed, a partition into finite sums of finite cycles does not exist, for instance, when G is an infinite disjoint union of triangles.

One way to deal with the problem is to look for an equivalent reformulation of Theorem 7.1.1 and extend that. For example:

Theorem 7.1.2. *Every locally finite graph G admits a vertex partition into (possibly empty) sets V_1, V_2 such that in both $G[V_1]$ and $G[V_2]$ all vertex degrees are even.*

(The proof of Theorem 7.1.2 is an easy exercise in compactness. It is also an immediate corollary of Theorem 7.1.4 below.)

However, the requirement that all vertex degrees of a subgraph H of a finite graph G should be even is only one equivalent reformulation among many of saying that $E(H)$ lies in the cycle space of G. Another is that H should be an edge-disjoint union of cycles (and isolated vertices). But if in this latter reformulation we only allow finite cycles, then Theorem 7.1.1 no longer extends to infinite graphs:

Example 7.1.3. [8] *The graph G shown in Figure 7.1 has a unique vertex partition into two induced even-degree subgraphs. One of these is edgeless, the other a double ray.*

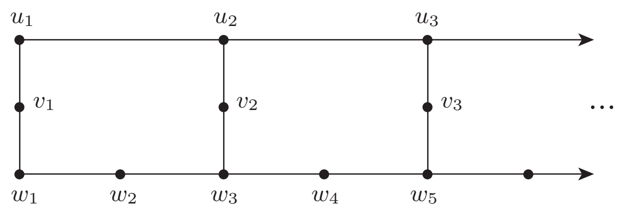

Figure 7.1: A graph with no bipartition into edge-disjoint unions of finite cycles

Proof. Consider any partition (V_1, V_2) of $V(G)$. Note that if two vertices x, y (such as u_1 and w_1) have a common neighbour z (such as v_1) not adjacent to any other vertex, then x and y must lie in the same partition class: otherwise, z would have degree 1 in its partition class. Thus if $u_1 \in V_1$, say, we deduce inductively that $w_1, w_3, w_5, \ldots \in V_1$ and hence also $u_2, u_3, u_4, \ldots \in V_1$. But $u_2, u_3, u_4 \ldots$ must not have degree 3 in $G[V_1]$, so $v_2, v_3, v_4, \ldots \in V_2$. Finally, v_1 lies in V_1 because u_2 does, so inductively $w_2, w_4, \ldots \in V_1$.
Thus, V_2 is the independent set $\{v_2, v_3, \ldots\}$, while V_1 consists of the remaining vertices, which span a double ray. $\square$

We will see in this chapter that, despite Example 7.1.3, Theorem 7.1.2 is not the strongest possible extension of Theorem 7.1.1. Indeed, the double ray $G[V_1]$ in Figure 7.1 forms an infinite cycle in the topological cycle space $\mathcal{C}(G)$. So for our space $\mathcal{C}(G)$, the graph of Figure 7.1 is no longer a counterexample to Theorem 7.1.1. And indeed, we have the following extension of Theorem 7.1.1 to infinite graphs, which implies Theorem 7.1.2 but is quite a bit stronger:

Theorem 7.1.4. [**8**] *For every locally finite graph G there is a partition of $V(G)$ into two (possibly empty) sets V_1, V_2 such that $E(G[V_i]) \in \mathcal{C}(G)$ for both $i = 1, 2$.*

We shall prove Theorem 7.1.4 in Section 7.7.2. In Section 7.7.3 we use similar techniques to extend the *cycle double cover conjecture* and Seymour's *faithful cycle cover conjecture* to locally finite graphs: if these conjectures are true for finite graphs, they also hold for locally finite graphs with our notion of an infinite topological cycle space. (The latter conjecture fails unless infinite cycles are admitted; for the former we have been unable to decide whether infinite cycles are really needed.) In Section 7.7.4 we generalize our results to graphs with infinite degrees, as far as this can be reasonably expected.

7.2 Cycle-cocycle partitions

The purpose of this section is to prove Theorem 7.1.4. This proof will also serve as a model for other proofs later in this chapter, which will refer to this proof and skip the corresponding details.

Our proof of Theorem 7.1.4 will be a compactness proof, but we shall need the non-trivial Theorem 2.5.2 from [17] to make this possible. Recall that while Theorem 7.1.2 has a straightforward compactness proof, the naïve extension of Theorem 7.1.1 to locally finite graphs does not (and is in fact false). The reason is, roughly speaking, that having all degrees even is a 'local' property of finite subsets $S \subseteq V(G)$ (one that S will satisfy in every large enough induced subgraph or in none), while inducing part of an element of the (combinatorial) cycle space based on finite cycles is not: the sequence of finite cycles $C_n = P_n + e_n$, for example, where the $P_n = v_{-n}v_{-(n-1)} \dots v_{n-1}v_n$ are nested paths and e_n is the edge $v_{-n}v_n$, 'tends' for $n \to \infty$ to the double ray $D = \dots v_{-1}v_0v_1 \dots$ whose edge set does not lie in the combinatorial cycle space of $\bigcup_{n\in\mathbb{N}} C_n$. However, D is an infinite cycle in $\bigcup_{n\in\mathbb{N}} C_n$, and more generally it turns out that all such 'limits' of finite cycles in a graph G are elements of $\mathcal{C}(G)$ (though not necessarily single infinite cycles).

We shall cast our compactness proof in terms of König's infinity lemma 2.1.1 as stated in Chapter 2.

Proof of Theorem 7.1.4. By treating the components of G separately, we may assume that G is connected. Hence, being locally finite, G is countable. Let $v_1, v_2, \dots$ be an enumeration of $V(G)$. For $n \in \mathbb{N}$ set $S_n := \{v_1, \dots, v_n\}$, and define W_n as the set of all quadruples $(V_1, V_2, \mathcal{E}_1, \mathcal{E}_2)$ such that

(i) (V_1, V_2) is a partition of S_n into two (possibly empty) sets; and

(ii) for $i = 1, 2$, $\mathcal{E}_i$ is a partition of $E(G[V_i])$ such that for each $E \in \mathcal{E}_i$ there is a finite cycle $C \subseteq G - V_{3-i}$ with $E(C \cap G[V_i]) = E$.

Each set W_n is clearly finite. It is non-empty by Theorem 7.1.1 applied to $G[S_n]$; recall that every element of the cycle space of a finite graph is a disjoint union of edge sets of cycles, which we can take as the partition sets for $\mathcal{E}_1$ and $\mathcal{E}_2$.

Let us define a graph H on $\bigcup_{n=1}^{\infty} W_n$. For $n \geq 2$, let $(V_1, V_2, \mathcal{E}_1, \mathcal{E}_2) \in W_n$ be adjacent to $(V_1', V_2', \mathcal{E}_1', \mathcal{E}_2') \in W_{n-1}$ if and only if, for both $i = 1, 2$,

(iii) $V_i' \subseteq V_i$;

(iv) for each $E' \in \mathcal{E}_i'$ there is an $E \in \mathcal{E}_i$ such that $E \cap E(G[V_i']) = E'$.

Observe that for $n \geq 2$ every vertex in W_n has a neighbour in W_{n-1}.

By the infinity lemma (2.1.1), there is a ray $v_1v_2 \dots$ in H with $(V_1^n, V_2^n, \mathcal{E}_1^n, \mathcal{E}_2^n) := v_n \in W_n$ for all n. Clearly, $V_1 := \bigcup_{n=1}^{\infty} V_1^n$ and $V_2 := \bigcup_{n=1}^{\infty} V_2^n$ form a partition of $V(G)$. By (iv), there is for every non-empty element E_i^n of a set $\mathcal{E}_i^n$ a unique ascending chain $E_i^n \subseteq E_i^{n+1} \subseteq \dots$ with $E_i^m \in \mathcal{E}_i^m$ for all $m \geq n$. Let $\mathcal{E}_i$ be the set consisting of the unions of such ascending chains, $i = 1, 2$. By (ii), the sets in $\mathcal{E}_i$ are disjoint and cover all of $E(G[V_i])$. Thus, $\mathcal{E}_i$ is a partition of $E(G[V_i])$.

We shall use Theorem 2.5.2 to show that all the sets $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ are elements of $\mathcal{C}(G)$; since disjoint unions are thin sums (as G is locally finite), this will imply that $\bigcup \mathcal{E}_1$ and $\bigcup \mathcal{E}_2$ too are elements of $\mathcal{C}(G)$. Let $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ be given, and write $E_n := E \cap E(G[V_i^n])$ for each n.

Consider a finite cut F of G. Choose n large enough that $F \subseteq E(G[S_n])$. By (ii), there is a finite cycle $C \subseteq G - V_{3-i}^n$ with $E(C \cap G[V_i^n]) = E_n$. Then

$$F \cap E = F \cap E(G[S_n]) \cap E = F \cap E_n = F \cap E(C \cap G[V_i^n]) = F \cap E(C).$$

Since C is a cycle, the last intersection is even. Hence $E \in \mathcal{C}(G)$ by Theorem 2.5.2, as desired. □

7.3 Related problems

Another problem concerning cycles is the well-known cycle double cover conjecture, which states that every bridgeless finite graph has a cycle double cover. (A *cycle double cover* of a graph G is a family of cycles such that each edge of G lies on exactly two of those cycles.) Using the same techniques as in the proof of Theorem 7.1.4 one can show that if the cycle double cover conjecture is true for finite graphs then it also holds for locally finite graphs, possibly with infinite cycles. However, we have been unable to construct an example where infinite cycles are really needed.

The situation is different for the following related conjecture of Seymour, which extends with infinite cycles but fails with finite cycles only. For a graph G and a map $p : E(G) \to \mathbb{N}$ ($\ni 0$) a *faithful cycle cover* of (G, p) is a family of cycles such that every edge $e \in G$ lies on exactly $p(e)$ of those cycles. Such a map p is *admissible* if $p(F) = \sum_{f \in F} p(f)$ is even and $p(e) \le p(F)/2$ for every finite cut F and every edge $e \in F$. We call p *even* if all its values $p(e)$ are even numbers. If (G, p) is to have a faithful cycle cover, then obviously p has to be admissible, and we shall see below that for some G it has to be even. Since the constant map with value 2 is admissible for bridgeless graphs, the following *faithful cycle cover conjecture* extends the cycle double cover conjecture:

Conjecture 7.3.1 (Seymour [41]). *Let G be a finite graph, and p an even admissible map. Then (G, p) has a faithful cycle cover.*

Unlike the cycle double cover conjecture, we know that Conjecture 7.3.1 fails for locally finite graphs unless we allow infinite cycles. Here is a simple example. Let G be the double (= two-way infinite) ladder, and let p assign 0 to every rung and 2 to all the other edges. By our current definition of admissibility (which requires $p(e) \le p(F)/2$ only for finite cuts F), the function p is admissible. But G contains no finite cycle that avoids all rungs, so (G, p) has no faithful cover consisting of finite cycles. (It does, however, have a faithful cover consisting of two copies of the infinite cycle spanned by the edges for which $p = 2$.)

The above example is no longer a counterexample to the infinite analogue of Conjecture 7.3.1 if we require of an admissible map p that it satisfies $p(e) \le p(F)/2$

also for infinite cuts F (and edges $e \in F$): if e is any edge with $p(e) = 2$ and R is a maximal ray in the subgraph of $G - e$ spanned by all its remaining edges with $p = 2$, then e and the edges with $p = 0$ incident with R form an infinite cut F such that $p(e) = p(F)$. Thus, p is no longer admissible, and we no longer have a contradiction.

Our next example, however, shows that strengthening the definition of 'admissible' as above is not enough to make Conjecture 7.3.1 true for locally finite graphs—if only finite cycles are admitted. Consider the ladder G shown in Figure 7.2 and the admissible map $p: E(G) \to \mathbb{N}$ defined by $p(e_i) = p(e'_i) = 2i$ and $p(f_i) = 2$ for all i. (Since $p(e) > 0$ for all e, we trivially have $p(e) \le p(F)/2$ also for infinite cuts F.) Suppose there is a faithful cycle cover which contains a finite cycle D. Obviously, D contains exactly two rungs f_m, f_n, with $m < n$, say. Let $\mathcal{C}$ be the subfamily of the cover consisting of those cycles which pass through the edge e_n. Each but at most one (which might go through f_n) of the cycles in $\mathcal{C}$ must use the edge e_{n-1}. Thus, at least $|\mathcal{C}| - 1 = 2n - 1$ cycles of the cover meet the edge e_{n-1}, contradicting $p(e_{n-1}) = 2n - 2$. Therefore, the only faithful cycle cover that (G, p) can have (and which is easily seen to exist) must be one consisting of infinite cycles.

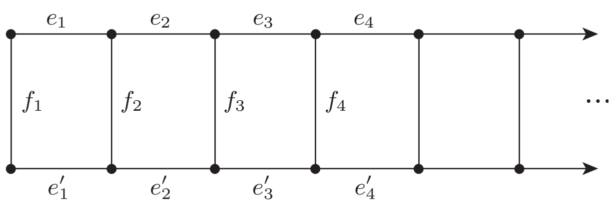

Figure 7.2: The unique faithful cycle cover consists of infinite cycles only

As soon as we allow infinite cycles, however, Conjecture 7.3.1 does extend to locally finite graphs:

Theorem 7.3.2. [8] *Let G be a locally finite graph and $p\colon E(G) \to \mathbb{N}$ an even admissible map. If Conjecture 7.3.1 is true then (G, p) has a faithful cycle cover.*

Proof. We sketch how the proof of Theorem 7.1.4 has to be amended for Theorem 7.3.2. As before, we may assume that G is connected. Let $v_1, v_2, \dots$ be an enumeration of its vertices, and set $G_n := G[\{v_1, \dots, v_n\}]$. We define W_n as the set of all families $\mathcal{E}$ of edge sets $E \subseteq E(G_n)$ such that

(i) every edge $e \in G_n$ lies in exactly $p(e)$ members of $\mathcal{E}$; and

(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subseteq G$ with $E(C \cap G_n) = E$.

The sets W_n are nonempty. Indeed, consider the multigraph obtained by contracting the components of $G - G_n$ to one vertex each, keeping parallel edges but deleting loops. Subdividing the parallel edges we obtain a simple finite graph G'_n.

The map p induces an even and admissible map on G'_n, for which there is a faithful cycle cover by assumption. It is easy to see that the corresponding edges in G satisfy (i) and (ii).

The rest of the proof is analogous to that of Theorem 7.1.4: applying the infinity lemma to an auxiliary graph H, we obtain a family of elements of $\mathcal{C}(G)$ such that every edge e lies on exactly $p(e)$ members of this family. By Theorem 2.5.1, we can modify this into a faithful cover consisting of single cycles. Therefore, if the faithful cycle cover conjecture holds for finite graphs, it is also true for locally finite graphs. □

Conjecture 7.3.1 requires p to be even, and indeed if p is allowed to assume odd values the conjecture becomes false: take the Petersen graph, and give p the value 2 on a perfect matching and 1 on all other edges.

Take any subgraph of an infinite graph G, and contract some—possibly infinitely many—of its edges; the resulting graph will be called a *minor* of G. Then the following result, whose finite version is a theorem of Alspach, Goddyn and Zhang [2], can be proved like Theorem 7.3.2.

Theorem 7.3.3. [8] *Let G be a locally finite graph not containing the Petersen graph as a minor, and let $p : E(G) \to \mathbb{N}$ be any admissible map (even or not). Then (G, p) has a faithful cycle cover.*

7.4 Graphs with infinite degrees

Theorem 7.1.4 does not extend to arbitrary graphs with vertices of infinite degree. For example, consider the graph G obtained by joining a vertex v_0 to every vertex of a ray $R := v_1v_2v_3\ldots$. Suppose there is a partition as in Theorem 7.1.4, and assume that $v_0 \in V_1$. By the definition of thin sums, no element of $\mathcal{C}(G)$ can have infinitely many edges incident with v_0. So there is a maximal $n \geq 0$ with $v_n \in V_1$. But then v_{n+1} has degree 1 in $G[V_2]$, a contradiction.

The problem here is that no element of the topological cycle space is allowed to have a vertex of infinite degree. Indeed if we weaken our concept of infinite sums, forbidding only those where some *edge* lies in infinitely many of the summands (i.e. making no restrictions on vertices), our counterexample ceases to be one: for $V_1 := \{v_3, v_6, v_9, \ldots\}$, the set $V_2 := V \setminus V_1$ induces an element of the cycle space. Of course, there was a good reason for forbidding these sums: summing up the triangles $v_0v_1v_2v_0, v_0v_2v_3v_0, v_0v_3v_4v_0, \ldots$ yields the ray v_0v_1R, which should then also be a member of the cycle space. But this is not unreasonable: as v_0 cannot be separated finitely from the ray R, this ray may be seen as converging to v_0. Indeed, although R does not converge to v_0, it nearly does: VTOP cannot separate its end from v_0 by two disjoint open sets. If we adjust our topology so that R does converge against v_0, by identifying v_0 with the end containing R, the ray v_0v_1R becomes a cycle as desired.

Let us make that precise. We say that a vertex v *dominates* an end ω in G if there is ray $R \in \omega$ and an infinite set of v–R paths that meet pairwise only in v. Assuming that

$$\textit{every end of } G \textit{ is dominated by at most one vertex,} \tag{7.1}$$

we now identify each vertex with all the ends it dominates, to obtain space $\tilde{G}$ whose (quotient) topology we denote by ITop. Note that, by (7.1), the vertices of G remain distinct in this identification. The identification space $\tilde{G}$ is Hausdorff (unlike $|G|$, when G has a dominated end), and compact (Diestel [14]) if G is 2-connected and satisfies condition (7.2) below. See Diestel and Kühn [19] for more on ITop.[1]

To obtain a cycle space which retains the natural properties of the topological cycle space of a locally finite graph, we have to impose another restriction on our graph G. Indeed, consider two vertices x and y that are linked by infinitely many independent paths. Then we can generate each of these paths P as a sum of cycles, so P should be in our cycle space. To avoid this, we require the following:

$$\textit{No two vertices of } G \textit{ are joined by infinitely many independent paths.} \tag{7.2}$$

Note that (7.2) implies (7.1). As before, we define as *cycles* those subgraphs of G whose closure in $\tilde{G}$ is homeomorphic to the unit circle, and the *topological cycle space* $\mathcal{C}(\tilde{G})$ of $\tilde{G}$ is defined as the span of all sums of cycles such that no edge appears in infinitely many of the summands. For the rest of this section, we assume that the graphs G we consider satisfy (7.2), and that all cycles are defined with respect to $\tilde{G}$.

Theorem 7.1.4 now extends to graphs with infinite degrees, as follows:

Theorem 7.4.1. [8] *Let G be a graph satisfying* (7.2)*. Then there is a partition of $V(G)$ into two (possibly empty) sets V_1, V_2 such that $E(G[V_i]) \in \mathcal{C}(\tilde{G})$ for both $i = 1, 2$.*

For the proof of Theorem 7.4.1 we may assume G to be 2-connected, because the topological cycle space of a graph is the direct product of the topological cycle spaces of its blocks. (Recall that vertices are now allowed to lie in infinitely many summands as long as no edge does.) Then G is countable (Diestel and Kühn [19]). We now proceed exactly as in the proof of Theorem 7.1.4, except that instead of Theorem 2.5.2 we use the following analogous result:

Lemma 7.4.2 (Diestel and Kühn [19]). *Let G be a graph satisfying* (7.2)*. Then $\mathcal{C}(\tilde{G})$ consists of precisely those sets of edges that meet every finite cut in an even number of edges.*

[1] Here, we obtain ITop from VTop, which is slightly sparser than the topology Top from which ITop is derived in [19]. However, it is not difficult to see that both topologies yield the same cycle space. In particular, Lemma 7.4.2 is still applicable.

Our results of Section 7.3 can also be extended to graphs with infinite degrees, but we require the following strengthening of (7.2):

No two vertices of G are joined by infinitely many edge-disjoint paths. (7.3)

This is indeed stronger than (7.2), see (Diestel and Kühn [19]).

We need another lemma.

Lemma 7.4.3. [8] *Let G be a 2-connected multigraph satisfying* (7.3)*, and let U be a finite set of vertices in G. Then we can contract edges of G, deleting loops but keeping any multiple edges that arise, so that no two vertices from U are identified, the multigraph H obtained has only finitely many edges and vertices, and every cut of H is also a cut of G.*

Proof. First, note that the set $\mathcal{K}$ of components of $G - U$ is finite. Indeed, as G is 2-connected, every component C of $G - U$ has distinct neighbours u, v in U. If $\mathcal{K}$ is infinite, then infinitely many $C \in \mathcal{K}$ are joined to the same two vertices u, v (because U is finite), so these are linked by infinitely many independent paths. This contradicts (7.3).

Next, consider a component $C \in \mathcal{K}$. For every two vertices $u, v \in U$ that both send infinitely many edges to C there is a finite cut $F_{u,v} \subseteq E(C)$ separating $N(u) \cap V(C)$ from $N(v) \cap V(C)$ in C, because of (7.3). Let F_C be the union of all such cuts $F_{u,v}$. Note that F_C is finite, as there are only finitely many pairs u, v. Then the set $\mathcal{K}_C$ of components of $C - F_C$ is also finite, and so is $\mathcal{K}' := \bigcup_{C \in \mathcal{K}} \mathcal{K}_C$. Each $D \in \mathcal{K}'$ sends only finitely many edges to $G - U - D$, and at most one vertex in U sends infinitely many edges to D. If such a vertex exists, we denote it by u_D.

In G, contract every $D \in \mathcal{K}'$ to a vertex v_D, keeping parallel edges but deleting loops. If two vertices of the resulting multigraph are joined by infinitely many edges, then these are u_D and v_D for some $D \in \mathcal{K}'$. In a second step, we now contract all these edges $u_D v_D$, again keeping parallel edges. We obtain a finite multigraph H in which no two vertices from U are identified. (Note in particular that the edge set of H is finite, despite the parallel edges that arose in the contraction.) Since we did not delete any edges except loops, every cut of H is also a cut of G. □

Theorem 7.4.4. [8] *Let G be a graph satisfying* (7.3)*, and let $p \colon E(G) \to \mathbb{N}$ be an even admissible map. If Conjecture 7.3.1 is true then* (G, p) *has a faithful cycle cover.*

Proof. Consider a block B of G. Every cut of B is a cut of G, so the restriction of p to B is an even admissible map on B. As $\mathcal{C}(G)$ is the direct product of the topological cycle spaces of the blocks of G, we may therefore assume G to be 2-connected. (Note that p assigns zero to bridges, so we need not cover these.) Then G is countable (Diestel and Kühn [19]).

Consider an enumeration $v_1, v_2, \ldots$ of $V(G)$, and set $G_n := G[\{v_1, \ldots, v_n\}]$. Define W_n as the set of all families $\mathcal{E}$ of sets $E \subseteq E(G_n)$ such that

(i) every edge $e \in E(G_n)$ lies in exactly $p(e)$ members of $\mathcal{E}$; and

(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subseteq G$ with $E(C \cap G_n) = E$.

Let us show that the sets W_n are not empty. Apply Lemma 7.4.3 with $U = \{v_1, \ldots, v_n\}$, and denote the multigraph H obtained by G'_n. Since every cut of G'_n is also one of G, the map p induces an admissible even map p'_n on G'_n. By subdividing edges we obtain from G'_n a simple graph G''_n with admissible even map p''_n (induced by p'_n). Then by assumption there is a faithful cycle cover of (G''_n, p''_n). Every cycle in that cover can be extended to a finite cycle in G. The family of these cycles then satisfies (i) and (ii), thus proving $W_n \neq \emptyset$.

The rest of the proof is again analogous to that of Theorem 7.1.4, since every element of $\mathcal{C}(\tilde{G})$ is an edge-disjoint union of cycles by Theorem 2.5.1. □

Using the same techniques as above, we can also extend Theorem 7.3.3:

Theorem 7.4.5. [8] *Let G be a graph that satisfies* (7.3) *and does not contain the Petersen graph as a minor, and let $p : E(G) \to \mathbb{N}$ be any admissible map. Then (G, p) has a faithful cycle cover.*

Chapter 8

MacLane's planarity criterion

8.1 Introduction

A set (or family) $\mathcal{E}$ of edge sets $E \subseteq E(G)$ is called *simple*, if every edge of G lies in at most two elements of $\mathcal{E}$. MacLane's planarity criterion states:

Theorem 8.1.1 (MacLane [34]). *A finite graph G is planar if and only if its cycle space has a simple generating set.*

Wagner [46] raised the question if MacLane's result could be extended so that it characterises planar graphs which are infinite. Rather than modifying the planarity criterion, Thomassen [45] describes all infinite graphs that satisfy MacLane's condition. For this, recall that a *vertex accumulation point*, abbreviated VAP, of a plane graph Γ is a point p of the plane such that every neighbourhood of p contains an infinite number of vertices of Γ.

Theorem 8.1.2 (Thomassen [45]). *Let G be an infinite 2-connected graph. Then G has a VAP-free embedding in the plane if and only if there is a simple set of finite circuits that generate all finite circuits.*

Bonnington and Richter [5] also provide a generalization of MacLane's theorem using the *even cycle space* $\mathcal{Z}(G)$, defined as the set of all subgraphs of G with all vertex degrees even. With this space they investigate which graphs have an embedding with k VAPs.

Our main result in this chapter is a verbatim generalization of MacLane's theorem to locally finite graphs:

Theorem 8.1.3. **[11]** *Let G be a countable locally finite graph. Then, G is planar if and only if $\mathcal{C}(G)$ has a simple generating set.*

This, together with Tutte's generating theorem for locally finite graphs proved by Bruhn [6], enables us to extend also Kelmans' planarity criterion [30] to locally finite graphs.

Theorem 8.1.4. **[11]** *Let G be a locally finite* 3*-connected graph. If G is planar then every edge appears in exactly two peripheral circuits. Conversely, if every edge appears in at most two peripheral circuits then G is planar.*

We discuss Theorem 8.1.3 in Section 8.8.2. In Section 8.8.3 we investigate some properties of simple generating sets. The main result of the chapter will be proved in the course of Sections 8.8.4 and 8.8.5. In Section 8.8.6 we extend Kelmans' planarity criterion to locally finite graphs. Finally, we briefly discuss in Section 8.8.2 extension to certain classes of non-locally finite graphs.

8.2 Infinite circuits in generating sets

First let us make the notion of a generating set more precise. A *generating set* of the topological cycle space will be a set $\mathcal{F} \subseteq \mathcal{C}(G)$ such that every element of $\mathcal{C}(G)$ can be written as a thin sum of elements of $\mathcal{F}$. Thus, in contrast to a generating set in the vector space sense we allow (thin) infinite sums. There are two reasons for this. First, thin sums are integral to the topological cycle space of an infinite graph, so it seems unnatural to forbid them. Second, MacLane's criterion is false if we insist that every $Z \in \mathcal{C}(G)$ is a finite sum of elements of a simple subset of $\mathcal{C}(G)$, as we shall see in Proposition 8.3.2.
To show that, in a certain sense, Theorem 8.1.3, is as strong as possible, we need the following theorem, which is of interest on its own. It will be proved in Section 4. For a circle $C \subseteq |G|$, call the circuit $E(C)$ *peripheral* if the subgraph $C \cap G$ of the graph G is induced and non-separating.

Theorem 8.2.1. **[11]** *Let G be a* 3*-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of circuits. Then every element of $\mathcal{F}$ is a peripheral circuit.*

First note that because of Theorem 2.5.1, if the topological cycle space has a simple generating set then it also has a simple generating set consisting of circuits.

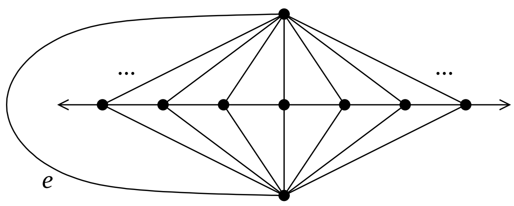

Figure 8.1: A planar graph whose topological cycle space has no simple generating set.

Theorem 8.1.3 is formulated for locally finite graphs, and indeed it is false for arbitrary infinite graphs. Indeed, consider the 3-connected graph G in Figure 8.1, which is not locally finite. By Theorem 8.2.1 and the remark following it, we may assume that a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consists of peripheral circuits (finite or infinite). In particular, no circuit which contains the edge e is in $\mathcal{F}$. But then such a circuit cannot be generated by any sum of circuits of $\mathcal{F}$. Thus, there is no simple generating set of $\mathcal{C}(G)$, but G is clearly planar.

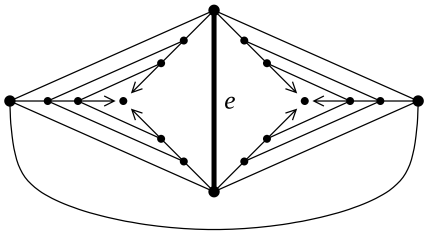

Figure 8.2: A locally finite graph without a simple generating set of *finite* circuits.

Infinite circuits are inevitable in a certain sense: there is not always in a planar graph a simple generating set comprised of only finite circuits. Consider the graph G in Figure 8.2, and suppose there is a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consisting of finite circuits. Since G is 3-connected, every $C \in \mathcal{F}$ is, by Theorem 8.2.1, a peripheral circuit. Now, if a finite circuit C contains the edge e then the the subgraph consisting of the edges in C with their incident vertices is clearly separating, and thus C not peripheral. Consequently, the circuits in $\mathcal{F}$ are not even sufficient to generate every finite circuit (namely any one containing e).

8.3 Simple generating sets

As a tool, we introduce the notion of a 2-basis. For this, let $\mathcal{B} \subseteq \mathcal{C}(G)$ be a simple generating set of the topological cycle space of G. We call $\mathcal{B}$ a 2-*basis* of $\mathcal{C}(G)$ if for every element $Z \in \mathcal{C}(G)$ there is a unique (thin) subset of $\mathcal{B}$, henceforth denoted by $\mathcal{B}_Z$, with $Z = \sum_{B \in \mathcal{B}_Z} B$. Observe that in a finite graph the 2-bases are exactly the simple bases of $\mathcal{C}(G)$, and thus conform with the traditional definition of a 2-basis in a finite graph.

Since we have left linear algebra with our definition of a 2-basis (allowing thin infinite sums), it is not clear if the properties usually expected of a basis are still retained. One of these, which we shall need later on, is that a generating set always contains a basis. For simple sets this is true:

Lemma 8.3.1. **[11]** *Let G be a 2-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$. If $\mathcal{F}$ is not a 2-basis, then for any $Z \in \mathcal{F}$ the set $\mathcal{F} \setminus \{Z\}$ is a 2-basis of $\mathcal{C}(G)$.*

Proof. Observe first that it suffices to check the uniqueness required in the definition of a 2-basis for the empty set: a simple generating subset $\mathcal{B}$ of $\mathcal{C}(G)$ is a 2-basis if and only if for every $\mathcal{B}' \subseteq \mathcal{B}$ with $\sum_{B\in\mathcal{B}'} B = \emptyset$ it follows that $\mathcal{B}' = \emptyset$.
Let us assume there is a non-empty set $\mathcal{D} \subsetneq \mathcal{F}$ with $\sum_{B\in\mathcal{D}} B = \emptyset$. Since G is 2-connected every edge of G appears in a finite circuit, and thus in at least one element of $\mathcal{F}$. But as $\mathcal{F}$ is simple and $\sum_{B\in\mathcal{D}} B = \emptyset$ no edge of G can lie in an element of $\mathcal{D}$ and at the same time in an element of $\mathcal{F} \setminus \mathcal{D}$.
So, $E_1 := \bigcup \mathcal{D}$ and $E_2 := \bigcup(\mathcal{F} \setminus \mathcal{D})$ define a partition of $E(G)$ (note that both sets are non-empty). Because G is 2-connected there is, by Menger's theorem, for any two edges a finite circuit through both of them. Therefore, there is a circuit D which shares an edge e_1 with E_1 and another edge e_2 with E_2. Let $\mathcal{D}' \subseteq \mathcal{F}$ be such that $D = \sum_{B\in\mathcal{D}'} B$. Then $D' := \sum_{B\in\mathcal{D}\cap\mathcal{D}'} B \subseteq D$, since for any edge $e \in D' \setminus D$ both $\mathcal{D}' \setminus \mathcal{D}$ and $\mathcal{D} \cap \mathcal{D}'$ have an element which contains e; thus $e \in E_1 \cap E_2$, which is impossible. Therefore, D' is a subset of the circuit D, and thus either $D' = \emptyset$ or $D' = D$. Since $e_1 \in D'$ the former case is impossible; the latter, however, is so too, as $D' \subseteq E_1$ cannot contain $e_2 \in E_2$, a contradiction.
We thus have shown:

$$\sum_{B\in\mathcal{D}} B = \emptyset \text{ for } \mathcal{D} \subseteq \mathcal{F} \text{ implies } \mathcal{D} = \emptyset \text{ or } \mathcal{D} = \mathcal{F}.$$

So, if $\mathcal{F}$ is not a 2-basis, then none of its subsets but itself generates the empty set. In particular, $\mathcal{F}$ is thin. For any $Z \in \mathcal{F}$,

$$Z = \sum_{B\in\mathcal{F}\setminus\{Z\}} B,$$

thus, the thin simple set $\mathcal{F} \setminus \{Z\}$ certainly generates the topological cycle space. It also is a 2-basis, as none of its non-empty subsets generates the empty set. □

With our definition of a generating set, which allows infinite sums, we shall show that MacLane's criterion holds for locally finite graphs. Since, in a vector space context, one usually allows only finite sums for a generating set, there is one obvious question: Does Theorem 8.1.3 remain true if we consider simple generating sets in the vector space sense? The answer is a strikingly clear no:

Proposition 8.3.2. **[11]** *There is no locally finite 2-connected infinite graph in which the topological cycle space has a simple generating set in the vector space sense (i.e. allowing only finite sums).*

Proof. Suppose there is such a graph G so that $\mathcal{C}(G)$ has a simple set $\mathcal{A} \subseteq \mathcal{C}(G)$ which generates every $Z \in \mathcal{C}(G)$ through a finite sum. We determine the cardinality of $\mathcal{C}(G)$ in two ways.
First, since $\mathcal{A}$ is simple, every of the countably many edges of G lies in at most two elements of $\mathcal{A}$. Therefore, $\mathcal{A}$ is a countable set, and thus, $\mathcal{C}(G)$ also.

Second, there is, by Lemma 8.3.1, a 2-basis $\mathcal{B} \subseteq \mathcal{A}$. As $\mathcal{C}(G)$ is an infinite set (since G is infinite and 2-connected), so is $\mathcal{B}$. Hence, there are distinct $B_1, B_2, \ldots \in \mathcal{B}$. Also, as G is locally finite and $\mathcal{B}$ simple, all subsets of $\mathcal{B}$ are thin. Therefore, all the sums

$$\sum_{i \in I} B_i \text{ for } I \subseteq \mathbb{N}$$

are distinct elements of $\mathcal{C}(G)$. Since the power set of $\mathbb{N}$ has uncountable cardinality, it follows that $\mathcal{C}(G)$ is uncountable, a contradiction. $\square$

The rest of this section is devoted to the proof of Theorem 8.2.1, which we restate:

Theorem 8.3.1. **[11]** *Let G be a 3-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of circuits. Then every element of $\mathcal{F}$ is a peripheral circuit.*

A basic tool when dealing with finite circuits are bridges, see for instance Bondy and Murty [4]. As our circuits may well be infinite, we need an adaption of the notion of a bridge, which we introduce together with a number of related results before proving the theorem.

Definition 8.3.3 (Bruhn [6]). *Let $C \subseteq |G|$ be a circle in a graph G. We call the closure B of a topological component of $|G| \setminus C$ a* bridge *of C. The points in $B \cap C$ are called the* attachments *of B in C.*

There is a close relationship between bridges and peripheral circuits. Indeed, in a 3-connected graph a circuit D is peripheral if and only if the circle $\overline{D}$ has a single bridge (see Bruhn [6]).

For the subgraph $H := C \cap G$, the following can be shown: a set $B \subseteq |G|$ is a bridge of C if and only if it is induced by a chord of H or if there is a component K of $G - H$ such that B is the closure of K plus the edges between K and H together with the incident vertices. Thus, our definition coincides with the traditional definition of a bridge in a finite graph.

Lemma 8.3.4 (Bruhn [6]). *Let $C \subseteq |G|$ be a circle in a graph G, and let B be a bridge of C. Let x be an attachment of B. Then:*

(i) *x is a vertex or an end;*

(ii) *if x is an end then every neighbourhood of x contains attachments of B that are vertices;*

(iii) *every edge of which B contains an inner point lies entirely in B; and*

(iv) *either B is induced by a chord of C or the subgraph $(B \cap G) - V(C)$ is non-empty and connected.*

We define a *residual arc* of the bridge B in the circle C to be the closure of a topological component of $C \setminus B$. Note that if B has at least two attachments every residual arc is indeed an arc (if not then the circle C itself is a residual arc, and it is the only one).

Lemma 8.3.5 (Bruhn [6]). *Let G be a 2-connected graph, and let $C \subseteq |G|$ be a circle with a bridge B. Then:*

(i) *the endpoints of a residual arc L of B in C are attachments of B; and*

(ii) *for a point $x \in C \setminus B$ there is exactly one residual arc L of B in C containing x.*

We say a bridge B of C *avoids* another bridge B' of C if there is a residual arc of B that contains all attachments of B'. Otherwise, they *overlap*. Note that overlapping is a symmetric relation. Two bridges B and B' of C are called *skew* if C contains four (distinct) points v, v', w, w' in that cyclic order such that v, w are attachments of B and v', w' attachments of B'. Clearly, if two bridges B and B' are skew, they overlap. On the other hand, in a 3-connected graph, overlapping bridges are either skew or 3-*equivalent*, i.e. they both have only three attachments which are the same:

Lemma 8.3.6. **[11]** *Let G be a 3-connected graph. Let $C \subseteq |G|$ be a circle, and let B and B' be two overlapping bridges of C. Then B and B' are either skew or 3-equivalent.*

Proof. First, if either B or B' is induced by a chord, it is easy to see, that they are skew because they overlap. Thus, by Lemma 8.3.4 (iv), we may assume that each of the bridges has three attachments. Next, assume that $B \cap C = B' \cap C$. If $|B \cap C| = 3$ then B and B' are 3-equivalent, otherwise they are clearly skew.

So, suppose there is an attachment u of B with $u \notin B'$. The attachment u is contained in a residual arc L of B'. Its endpoints u', v' are attachments of B'. Since B and B' are overlapping, not all all attachments of B may lie in L. Thus, there is an attachment $v \in C \setminus L$ of B. Then, the sequence u, u', v, v' shows that B and B' are skew. □

For a set $X \subseteq |G|$, an X-path is a path that starts in X, ends in X and is otherwise disjoint from X.

Lemma 8.3.7. **[11]** *Let B and B' be two skew bridges of a circle $C \subseteq |G|$ in a graph G. Then there are two disjoint C-paths $P = u \dots v$ and $P' = u' \dots v'$ such that u, u', v, v' appear in that order on C.*

Proof. Since B and B' are skew there are points x, x', y, y' appearing in that cyclic order on C such that x, y are attachments of B and x', y' are attachments of B'. If x is a vertex put $u := x$. If not, then there is a whole arc $A \subseteq C$ around x disjoint from any of the other points. In A we find, by Lemma 8.3.4 (ii), an attachment u of B that is a vertex. Doing the same for x', y and y', if necessary, we end up with vertices u, u', v, v' appearing in that cyclic order on C such that $u, v \in B$ and $u', v' \in B'$. As $(B \cap G) - V(C)$ is connected, by Lemma 8.3.4 (iv), we find an u–v path P through B, and analogously an u'–v' path P' through B'. Since bridges meet only in attachments, P and P' are disjoint. $\square$

We need that in a 3-connected graph, for any circle, there are always two overlapping bridges (if there is more than one bridge at all). For this, we define for a circle C in the graph G the *overlap graph* of C in G as the graph on the bridges of C such that two bridges are adjacent if and only if they overlap. The next lemma ensures that there are always overlapping bridges.

Lemma 8.3.8 (Bruhn [6]). *For every circle C in a 3-connected graph G the overlap graph of C in G is connected.*

The next simple lemma will be used repeatedly in the proof of Theorem 8.2.1.

Lemma 8.3.9. [11] *Let G be a 3-connected graph, and let $\mathcal{B}$ be a 2-basis of $\mathcal{C}(G)$ consisting of circuits. Let C and D be circuits in G such that $\overline{C} \cap \overline{D}$ is an arc. Suppose that $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Then, either $\mathcal{B}_C \subseteq \mathcal{B}_D$ or $\mathcal{B}_D \subseteq \mathcal{B}_C$.*

Proof. Put $K := \sum_{B \in \mathcal{B}_C \cap \mathcal{B}_D} B$ and consider an edge $e \notin C \cup D$. Then both $\mathcal{B}_C$ and $\mathcal{B}_D$ contain either both or none of the at most two circuits $B \in \mathcal{B}$ with $e \in B$. Thus, both or none of them is in $\mathcal{B}_C \cap \mathcal{B}_D$, and hence $e \notin K$. Therefore, K is an element of the topological cycle space contained in $C \cup D$. These are precisely $\emptyset$, C, D and $C + D$ (since $\overline{C} \cap \overline{D}$ is an arc). Note that $K \neq \emptyset$ as $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Also, $K \neq C + D$, since otherwise

$$\mathcal{B}_C \cap \mathcal{B}_D = \mathcal{B}_K = \mathcal{B}_{C+D} = \mathcal{B}_C \triangle \mathcal{B}_D,$$

which is impossible. Consequently, we obtain either $K = C$ and thus, $\mathcal{B}_C \subseteq \mathcal{B}_D$, or $K = D$ and $\mathcal{B}_D \subseteq \mathcal{B}_C$. $\square$

Proof of Theorem 8.2.1. Note that it suffices to prove the theorem for a 2-basis $\mathcal{B}$. Indeed, if $\mathcal{F}$ is not a 2-basis, consider two distinct elements Z_1 and Z_2 of $\mathcal{F}$. By Lemma 8.3.1, both $\mathcal{F} \setminus \{Z_1\}$ and $\mathcal{F} \setminus \{Z_2\}$ are a 2-basis of $\mathcal{C}(G)$, and, if Theorem 8.2.1 holds for these, it clearly also holds for $\mathcal{F}$.

Consider a non-peripheral circuit C. Then, the circle $\overline{C}$ has more than one bridge [6]. Two of these, B and B' say, are, by Lemma 8.3.8, overlapping. By Lemma 8.3.6, they are either skew or 3-equivalent. We show that $C \notin \mathcal{B}$ for each of the two cases.

(i) Suppose that B and B' are skew. By Lemma 8.3.7, there are two disjoint $\overline{C}$-paths $P = u \dots v$ and $P' = u' \dots v'$ such that u, u', v, v' appear in this order on

$\overline{C}$. Denote by $L_{uu'}, L_{u'v}, L_{vv'}, L_{v'u}$ the closures of the topological components of $\overline{C} \setminus \{u, u', v, v'\}$ such that x, y are the endpoints of L_{xy}. Define the circuits

$$\begin{aligned} C_1 &:= E(L_{uu'} \cup L_{u'v} \cup P), & C_2 &:= E(L_{vv'} \cup L_{v'u} \cup P), \\ D_1 &:= E(L_{u'v} \cup L_{vv'} \cup P') \quad \text{and} & D_2 &:= E(L_{v'u} \cup L_{uu'} \cup P'). \end{aligned}$$

Observe that $C_1 + C_2 = C = D_1 + D_2$, and additionally, that $\overline{C_i} \cap \overline{D_j}$ is an arc for any $i, j \in \{1, 2\}$.
Suppose $C \in \mathcal{B}$. Since

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} = \mathcal{B}_{C_1+C_2} = \mathcal{B}_C = \{C\},$$

not both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{C_2}$ may contain C. As the same holds for D_1 and D_2 we may assume that

$$C \notin \mathcal{B}_{C_1} \text{ and } C \notin \mathcal{B}_{D_1}. \tag{8.1}$$

Consider an edge $e \in C_1 \cap D_1 \subseteq C$. Both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ must contain a circuit which contains e. By (8.1), this cannot be C. Therefore, and since $\mathcal{B}$ is simple, $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ contain the same circuit K with $e \in K$. Consequently, $\mathcal{B}_{C_1} \cap \mathcal{B}_{D_1} \neq \emptyset$, and applying Lemma 8.3.9 we may assume that

$$\mathcal{B}_{C_1} \subseteq \mathcal{B}_{D_1}. \tag{8.2}$$

Now, consider an edge $e' \subseteq L_{uu'}$, hence $e \in C_1 \cap D_2$. There is a circuit $K' \in \mathcal{B}_{C_1}$ with $e' \in K' \neq C$. By (8.2), $K' \in \mathcal{B}_{D_1}$, but since e' lies in $L_{uu'}$ we have $e' \notin D_1$. Thus, $\mathcal{B}_{D_1}$ also contains the other circuit in $\mathcal{B}$ that contains e', which is C, a contradiction to (8.1). Therefore, $C \notin \mathcal{B}$.
(ii) Suppose that B and B' are 3-equivalent. Let v_1, v_2, v_3 be their attachments, which then are vertices (by Lemma 8.3.4 (ii)). Then there is a vertex $x \in V(B \setminus C)$ and three x–$\overline{C}$ paths $P_i = x \dots v_i \subseteq B$, $i = 1, 2, 3$ whose interiors are pairwise disjoint. Let $Q_i = y \dots v_i$ be analogous paths in B'. The closures of the topological component of $\overline{C} \setminus \{v_1, v_2, v_3\}$ are three arcs; denote by $L_{i,i+1}$ the one that has v_i and v_{i+1} as endpoints (where indices are taken mod 3). For $i = 1, 2, 3$, define the circuits

$$C_i := E(L_{i,i+1} \cup P_i \cup P_{i+1}) \text{ and } D_i := E(L_{i,i+1} \cup Q_i \cup Q_{i+1}).$$

Note that $C_1 + C_2 + C_3 = C = D_1 + D_2 + D_3$.
Now suppose $C \in \mathcal{B}$. As

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} \triangle \mathcal{B}_{C_3} = \mathcal{B}_{C_1+C_2+C_3} = \mathcal{B}_C = \{C\},$$

either C lies in all of the $\mathcal{B}_{C_i}$ or in only one of them, in $\mathcal{B}_{C_3}$, say. In both cases, we have $C \notin \mathcal{B}_{C_1+C_2}$. We obtain the same result for the D_i: either C lies in all of the $\mathcal{B}_{D_i}$ or in only one of them. In any case, we can define D as either D_1 or $D_2 + D_3$ such that $C \notin \mathcal{B}_D$. Put $D' := C + D$, and note that $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$.

Then, since $C_1 + C_2$ shares an edge in C with D, and neither $\mathcal{B}_{C_1+C_2}$ nor $\mathcal{B}_D$ contains C, we have $\mathcal{B}_{C_1+C_2} \cap \mathcal{B}_D \neq \emptyset$. Applying Lemma 8.3.9, we obtain that one of the two sets $\mathcal{B}_{C_1+C_2}, \mathcal{B}_D$ is contained in the other.

First assume that $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_D$, and consider an edge $e \in C$ that lies in both $C_1 + C_2$ and D'. Such an edge exists since $D' = D_1$ or $D' = D_2 + D_3$. Since $C \notin \mathcal{B}_{C_1+C_2}$, e lies in a circuit $K \neq C$ in $\mathcal{B}_{C_1+C_2}$, and thus also $K \in \mathcal{B}_D$. On the other hand, $e \in C \in \mathcal{B}_{D'}$ contradicts $e \in D'$.

So, we may assume that $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2}$. Because $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$ we even have $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \cap \mathcal{B}_{D'}$. Thus, by Lemma 8.3.9, either $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'}$ or $\mathcal{B}_{D'} \subseteq \mathcal{B}_{C_1+C_2}$. The latter is impossible as $C \notin \mathcal{B}_{C_1+C_2}$. Therefore, we obtain

$$\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}.$$

Now, from $C \notin \mathcal{B}_{C_1+C_2}$ follows that $\mathcal{B}_{C_1+C_2} = \mathcal{B}_D$, contradicting $C_1 + C_2 \neq D$. Thus, $C \notin \mathcal{B}$. □

8.4 The backward implication

In this section, we show the backward implication of Theorem 8.1.3, namely that if the topological cycle space has a simple generating set then G is planar. But first, let us remark that it is sufficient to show Theorem 8.1.3 for 2-connected graphs. Indeed, the Kuratowski planarity criterion for countable graphs below asserts that a countable graph is planar if and only if its blocks are planar.

Theorem 8.4.1 (Dirac and Schuster [21]). *Let G be a countable graph. Then, G is planar if and only if G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.*

The backward direction will follow from the next lemma.

Lemma 8.4.2. [11] *Let G be a 2-connected graph such that $\mathcal{C}(G)$ has a 2-basis, and let $H \subseteq G$ be a finite 2-connected subgraph. Then $\mathcal{C}(H)$ has a 2-basis.*

Proof. Let $\mathcal{B}$ be the 2-basis of $\mathcal{C}(G)$. Since H is finite, there are $Z \in \mathcal{C}(H)$ with a non-empty generating set $\mathcal{B}_Z \subseteq \mathcal{B}$ which is $\subseteq$-minimal among all B_Z with $Z \in \mathcal{C}(H)$. Let us denote these by $Z_1, \ldots, Z_k$.

Consider a $D \in \mathcal{C}(H)$ with $\mathcal{B}_D \cap \mathcal{B}_{Z_i} \neq \emptyset$ for some i. We claim that $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$. First, note that

$$C := \sum_{B \in \mathcal{B}_D \cap \mathcal{B}_{Z_i}} B \subseteq E(H).$$

Indeed, consider an edge $e \notin E(H)$. Since $Z_i, D \subseteq E(H)$, and since $\mathcal{B}$ is simple, e either lies on exactly two or on none of the elements of $\mathcal{B}_{Z_i}$, and the same holds for $\mathcal{B}_D$. Furthermore, if e lies on two elements of $\mathcal{B}_{Z_i}$ and on two of $\mathcal{B}_D$, these must be the same. So, $e \notin C$.

Therefore, $C \subseteq E(H)$, and thus $C \in \mathcal{C}(H)$. As $\mathcal{B}_C \subseteq \mathcal{B}_{Z_i}$ we obtain, by the minimality of $\mathcal{B}_{Z_i}$, that $C = Z_i$. Consequently, $\mathcal{B}_{Z_i} = \mathcal{B}_C \subseteq \mathcal{B}_D$, as claimed.

This result also implies $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for all $1 \leq i < j \leq k$. Thus, every edge of H appears in at most two of the Z_i. Furthermore, we claim that $\{Z_1, \ldots, Z_k\}$ is a generating set for $\mathcal{C}(H)$. Then, $\{Z_1, \ldots, Z_k\}$ contains a 2-basis of $\mathcal{C}(H)$, and we are done.

So consider a $D \in \mathcal{C}(H)$, and let I denote the set of those indices i with $\mathcal{B}_{Z_i} \cap \mathcal{B}_D \neq \emptyset$. We may assume $I = \{1, \ldots, k'\}$ for a $k' \leq k$. Then, by $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$ and $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for $i, j \in I$, it follows that $\mathcal{B}_D$ is the disjoint union of the sets $\mathcal{B}_{Z_1}, \mathcal{B}_{Z_2}, \ldots, \mathcal{B}_{Z_{k'}}$ and

$$\mathcal{B}' := \mathcal{B}_D \setminus \bigcup_{i=1}^{k'} \mathcal{B}_{Z_i}.$$

Consequently,

$$\sum_{B \in \mathcal{B}'} B = \sum_{B \in \mathcal{B}_D} B + \sum_{B \in \mathcal{B}_{Z_1}} B + \ldots + \sum_{B \in \mathcal{B}_{Z_{k'}}} B = D + Z_1 + \ldots + Z_{k'} \subseteq E(H)$$

as all the summands lie in H. Now, if $\mathcal{B}' \neq \emptyset$ then there is a $Z \in \mathcal{C}(H)$ with a non-empty and minimal $\mathcal{B}_Z \subseteq \mathcal{B}'$ which then must be one of the Z_i, a contradiction. Thus, $\mathcal{B}'$ is empty and we have $D = \sum_{i=1}^{k'} Z_i$. □

For the backward implication of Theorem 8.1.3, we use the well-known fact that the cycle space of every subdivision of K_5 or of $K_{3,3}$ fails to have a 2-basis (see, for instance, Diestel [16]).

Lemma 8.4.3. [11] *Let G be a locally finite 2-connected graph such that $\mathcal{C}(G)$ has a simple generating set. Then G is planar.*

Proof. Suppose not. Then G contains, by Theorem 8.4.1, a subdivision H of K_5 or of $K_{3,3}$ as subgraph. By Lemma 8.3.1, $\mathcal{C}(G)$ has a 2-basis. Then, by Lemma 8.4.2, $\mathcal{C}(H)$ also has a 2-basis, which is impossible. □

8.5 The forward implication

To show the forward implication of Theorem 8.1.3, i.e. that the topological cycle space of a planar graph has a simple generating set, we proceed as in the finite case: we embed our graph G in the sphere and then show that the set of the face boundaries' edge sets is a simple generating set. So, our first priority is to ensure that every face is indeed bounded by a circle of $|G|$. As for the backward direction we may assume that G is 2-connected.

This, however, is certainly not the case when a VAP of the embedded graph coincides with a vertex or an inner point of an edge. To avoid this problem we consider topological embeddings of the space $|G|$ in the sphere (rather than graph embeddings of G), which, in our context, is no restriction:

Theorem 8.5.1 (Richter and Thomassen [40]). *Let G be a locally finite 2-connected planar graph. Then $|G|$ embeds in the sphere.*

We call a topological space 2-*connected* if it is connected and remains so after the deletion of any point. Thus, any embedding of the (standard) compactification $|G|$ of a 2-connected graph G in the sphere clearly is 2-connected. Note that also, any such embedding is compact if G is locally finite and connected. A *face* of a compact subset K of the sphere is a component of the complement of K. A face boundary $\partial f \subseteq K$ of a face f is simply the boundary of f. If K is the image of $|G|$ under an embedding, then it can be shown in a similar way as for finite plane graphs (see for instance Diestel [16]) that if an inner point of an edge lies in a face boundary then the whole edge lies in it.

Theorem 8.5.2 (Richter and Thomassen [40]). *Every face of a compact 2-connected locally connected subset of the sphere is bounded by a simple closed curve.*

Another result of Richter and Thomassen [40] states that $|G|$ is locally connected if G is locally finite and connected[1]. As a simple closed curve by definition is homeomorphic to the unit circle, we obtain:

Corollary 8.5.3.[11] *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \to S^2$. Then the face boundaries of $\varphi(|G|)$ are circles of $|G|$.*

Showing the forward implication, we now complete the proof of Theorem 8.1.3:

Lemma 8.5.4.[11] *Let G be a locally finite 2-connected planar graph. Then, $\mathcal{C}(G)$ has a simple generating set.*

Proof. By Theorem 8.5.1, $|G|$ has an embedding $\varphi : |G| \to S^2$ in the sphere. Put $\Gamma := \varphi(|G|)$. We show that the set $\mathcal{F}$ which we define to consist of the edge sets of the face boundaries of Γ, is a simple generating set of $\mathcal{C}(G)$. Certainly, $\mathcal{F}$ is simple, and, by Corollary 8.5.3, a subset of $\mathcal{C}(G)$. So, we only have to prove that every element of the topological cycle space is the sum of certain elements of $\mathcal{F}$.

Fix a face f^* of Γ. First, consider a circuit C in G. Then for the circle $\overline{C}$, $\varphi(\overline{C})$ is homeomorphic to the unit circle and, thus, bounds two faces (by the Jordan-curve theorem). Let f_C be the face not containing f^*. As G is 2-connected, every edge e lies on a finite circuit, and therefore on the boundaries of exactly two faces of Γ, which we denote by f_e and f'_e. Hence, the set

$$\mathcal{B}^C := \{E(\partial f) \,:\, f \subseteq f_C \text{ is a face of } \Gamma\}$$

[1]They show this to be true for all *pointed* compactifications of G, which are those obtained from the standard compactification by identifying some ends.

is thin. Moreover, as we have $f_e, f'_e \subseteq f_C$ or $f_e, f'_e \nsubseteq f_C$ if and only if $e \notin C$, it follows that

$$\sum_{B \in \mathcal{B}^C} B = C. \tag{8.3}$$

Now, consider an arbitrary element Z of the topological cycle space. By definition, there is a thin family $\mathcal{D}$ of circuits with $Z = \sum_{C \in \mathcal{D}} C$. If none of the elements of $\mathcal{F}$ appears in $\mathcal{B}^C$ for infinitely many $C \in \mathcal{D}$, then the family $\mathcal{B}$, which we define to be the (disjoint) union of all $\mathcal{B}^C$ with $C \in \mathcal{D}$, is thin (since every edge lies on exactly two face boundaries). Then, $Z = \sum_{B \in \mathcal{B}} B$, and we are done. Therefore, if $F(\Gamma)$ is the set of faces of Γ, it suffices to show that the set

$$F := \{f \in F(\Gamma) \, : \, f \subseteq f_C \text{ for infinitely many } C \in \mathcal{D}\}$$

is empty.

So suppose $F \neq \emptyset$. By definition of f_C, we have $f^* \nsubseteq f_C$ for all $C \in \mathcal{D}$, and thus also $F \neq F(\Gamma)$. Hence, there is an edge e such that one of its adjacent faces, say f_e, lies in F and the other, f'_e, in $F(\Gamma) \setminus F$. Then, $E(\partial f_e)$ appears in infinitely many $\mathcal{B}^C$ while e lies on only finitely many $C \in \mathcal{D}$. Thus, also $E(\partial f'_e)$ lies in infinitely many $\mathcal{B}^C$, which implies $f'_e \in F$, a contradiction. □

8.6 Kelmans' planarity criterion

For finite 3-connected graphs there is another well-known planarity criterion, namely Kelmans' criterion. It follows from MacLane's criterion together with Tutte's theorem. The latter has been shown by Bruhn for locally finite graphs:

Theorem 8.6.1 (Bruhn [6]). *Let G be a locally finite 3-connected graph. Then the peripheral circuits generate the topological cycle space.*

We know from Corollary 8.5.3 that the face boundaries of a locally finite 2-connected planar graph are circles. When G is 3-connected then, as for finite graphs (see Diestel [16]), the Jordan curve theorem implies that these circles are precisely the closures in $|G|$ of the peripheral circuits of G:

Lemma 8.6.2. *Let G be a locally finite 3-connected graph with an embedding $\varphi : |G| \to S^2$ in the sphere. Then, the face boundaries are precisely the closures in $\varphi(|G|)$ of the peripheral circuits of G.* □

Now, we easily obtain a verbatim generalization of Kelmans' criterion for locally finite graphs.

Theorem 8.1.4. [11] *Let G be a locally finite 3-connected graph. If G is planar then every edge appears in exactly two peripheral circuits. Conversely, if every edge appears in at most two peripheral circuits then G is planar.*

Proof. If G is planar then there is, by Theorem 8.5.1, also an embedding of $|G|$, in which, by Lemma 8.6.2, the closure of every peripheral circuit is a face boundary. Since G is 2-connected every edges lies in exactly two face boundaries, hence in exactly two peripheral circuits of G.

For the backward implication let $\mathcal{F}$ be the set of all peripheral circuits of G, which then is simple. Thus, $\mathcal{F}$ is, by Theorem 8.6.1, a simple generating set, and hence G planar, by Theorem 8.1.3. □

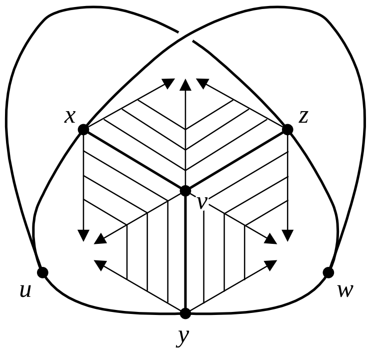

Figure 8.3: Infinite circuits are necessary for Kelmans' criterion

As MacLane's planarity criterion, Kelmans', too, fails when infinite circuits are prohibited. Indeed, there are 3-connected non-planar graphs in which every edge lies on at most two finite peripheral circuits. The graph G shown in Figure 8.3 is such an example. It consists of a $K_{3,3}$ (bold) to which three disjoint infinite 3-ladders are added. First observe that any finite peripheral circuit that contains edges of $G - \{u, w\}$ cannot contain any edge incident with either one of u, w, as otherwise it also contains (the edges of) a finite $\{x, y, z\}$–$\{x, y, z\}$ path in $G - \{u, w\}$, and thus is separating. Therefore, every finite peripheral circuit of G has either none or all of its edges incident with $\{u, w\}$; in the latter case, it is a circuit of $G[u, w, x, y, z]$.

Now, assume that there is an edge of G that appears in three finite peripheral circuits. All of these circuits then lie either in $G-\{u, w\}$ or in $G[u, w, x, y, z]$, where they are also peripheral. Now, it is easy to check that none of the edges of the finite graph $G[u, w, x, y, z]$ lies on three peripheral circuits, and by Theorem 8.1.4 this is also impossible for any edge of the planar 3-connected graph $G - \{u, w\}$. This shows that Kelmans' criterion fails if only finite circuits are admitted.

8.2 Graphs with infinite degrees

Let us briefly return to non-locally finite graphs. We have seen in Section 8.2 that MacLane's planarity criterion fails for arbitrary infinite graphs. In this section, we prove the criterion for graphs G that satisfy

$$\textit{no two vertices are joined by infinitely many edge-disjoint paths,} \qquad (*)$$

using the space $\tilde{G}$, which has the topology ITop, as defined in Section 7.4.
With $\mathcal{C}(\tilde{G})$, MacLane's planarity criterion holds:

Theorem 8.2.1.[11] *Let G be a countable graph satisfying $(*)$. Then G is planar if and only if $\mathcal{C}(\tilde{G})$ has a simple generating set.*

The backward direction of Theorem 8.2.1 can be shown in exactly the same way as for locally finite graphs (detailed in Section 8.4).
For the forward direction, we need another tool. Let G be a multigraph satisfying $(*)$, and let G^* be another multigraph with a bijection ${}^*\colon E(G) \to E(G^*)$. Call G^* a *dual* of G if the following holds for every set $F \subseteq E(G)$: F is a circuit in $\tilde{G}$ if and only if $F^* := \{f^* : f \in F\}$ is a minimal non-empty cut in G^*. Using $\mathcal{C}(\tilde{G})$, Bruhn and Diestel [7] showed the following analogon of Whitney's planarity criterion:

Theorem 8.2.2 (Bruhn and Diestel [7]). *Let G be a countable graph satisfying $(*)$.*

(i) *G has a dual if and only if G is planar.*

(ii) *If G^* is a dual of G and $F \subseteq E(G)$, then $F \in \mathcal{C}(\tilde{G})$ if and only if F^* is a cut in G^*.*

Now, the forward direction of Theorem 8.2.1 follows as in the finite case. Since G is planar, there is, by Theorem 8.2.2, a dual G^* of G. We claim that $\mathcal{B} := \{F : F^* = E(v) \text{ for some } v \in V(G^*)\}$ is a simple generating set of $\mathcal{C}(\tilde{G})$. Certainly, $\mathcal{B}$ is simple, and, by Theorem 8.2.2, each of its elements is a member of the topological cycle space. In addition, any $Z \in \mathcal{C}(\tilde{G})$ can be generated by $\mathcal{B}$. Indeed, Theorem 8.2.2 implies that Z^* is a cut in G^*. Let (A, B) be the corresponding partition of $V(G^*)$, i.e. $Z^* = E_{G^*}(A, B)$. Then,

$$Z^* = \sum_{v \in A} E(v), \text{ and therefore } Z = \sum_{F \in \mathcal{B}'} F,$$

where $\mathcal{B}' := \{F : F^* = E(v),\ v \in A\} \subseteq \mathcal{B}$. This completes the forward direction of Theorem 8.2.1.

Chapter 9

Long circuits generate the cycle space

9.1 Introduction

Locke [32] conjectured that there is a constant m so that the cycle space of any finite graph, in which every two vertices can be joined by a path of length at least mk, is generated by the cycles of length $\geq k$. To motivate the conjecture he shows that if between any two vertices there is a path of length $\geq (k-1)^2 + 1$ then the circuits of length $\geq k$ generate the cycle space.

We prove that the conjecture is true, even for locally finite graphs. Moreover, we show that the constant can be chosen as $m = 2$.

Theorem 9.1.1. *Let $k \in \mathbb{N}$. If every two vertices of a locally finite graph G are the endvertices of a path of length $2k$, then the cycles of length $\geq k$ generate the cycle space $\mathcal{C}(G)$ of G.*

On the other hand, m has to be at least 1. Indeed, otherwise choose k large enough so that $mk + 1 < k$. Consider the graph G which is obtained by identifying an edge of two cycles of length $mk+1$, i.e. that consists of a cycle of length $2mk$ plus an additional edge (that joins two vertices of maximal distance). Then every two vertices of G are linked by a path of length mk, but, as $mk+1 < k$, the only cycle of length $\geq k$ does not generate $\mathcal{C}(G)$.

We then consider a natural extension of Theorem 9.1.1, which allows k also to be infinite. It is in fact a topological version of Locke's conjecture. Therefore, we have to consider arcs instead of paths. We define the *length* of an arc as the number of edges it contains.

Theorem 9.1.2. [10] *Let $k \in \mathbb{N} \cup \infty$. If every two vertices of a locally finite graph G are the endvertices of an arc of length $2k$, then the cycles of length $\geq k$ generate the cycle space $\mathcal{C}(G)$ of G.*

Observe that the possibly more natural condition that every two *points* of $|G|$ are joined by an arc of length $2k$, is at once implied by the condition we chose.

Since an infinite locally finite graph is 2-connected if and only if every two vertices of an locally finite graph G are joined by an arc of infinite length[1], Theorem 9.1.2 for $k = \infty$ is equivalent to the following:

Theorem 9.1.3. **[10]** *The cycle space of an infinite locally finite 2-connected graph is generated by its infinite circuits.*

9.2 Locke's conjecture with finite k

Let us replace the original condition of Theorem 9.1.1 with a more natural one.

Lemma 9.2.1. **[10]** *If every two vertices of a locally finite graph G are the endvertices of a path of length $2k$, then every edge lies on a cycle of length $\geq 2k$.*

Proof. Just take the union of the edge considered with the long path connecting its endvertices. □

The reverse is true if G is connnected. Hence, the following theorem is equivalent to Theorem 9.1.1.

Theorem 9.2.2. **[10]** *If every edge of a locally finite graph connected G lies on some cycle of length $2k$, then the cycles of length $\geq k$ generate the cycle space $\mathcal{C}(G)$ of G.*

Proof. Suppose otherwise. Then G has a cycle which cannot be generated by cycles of length $\geq k$. Choose such a C, together with a cycle C' of length $\geq 2k$, and a path $Q_C \subseteq C \cap C'$ so that $|E(C) - E(Q_C)|$ is minimal. Then, as there is a cycle of length $\geq 2k$ through every edge, C and C' share at least one edge. Consequentely, Q_C is non-trivial. Moreover, C' contains a C-path (since C and C' cannot be identical).

Assume that C' contains only one C-path P. Then $C + C'$ cannot be generated with cycles of length $\geq k$ (because by assumption, we cannot generate C with these). In particular, $C + C'$ has length $< k$. But then, $C' = C + (C + C')$ has length $< 2k$, a contradiction.

Thus, C' contains at least two C-paths. For any such C-path P, denote by x_P and y_P its starting vertex resp. its endvertex. Let C_P be the x_P–y_P path in C that is edge-disjoint from Q_C. Suppose that $C_P \cup E(P)$ cannot be generated by cycles of length $\geq k$. Then $C_P \cup E(P)$, C' and P are a better choice than C, C' and

[1] Indeed, if G is 2-connected, then in any two vertices start disjoint ω-rays for any end $\omega \in \Omega(G)$. For the other implication, note that every arc linking two vertices of the same block lies entirely in that block. Hence, G has only one block, i.e. it is 2-connected.

Q_C, because $|E(C_P \cup E(P)) \setminus E(P)| = |E(C_P)| < |E(C) \setminus E(Q_C)|$, where the last inequality is due to the fact that C' contains a second C-path. This contradicts our choice of C.
So for every C-path $P \subseteq C'$ the corresponding cycle $C_P \cup E(P)$ can be generated with cycles of length $\geq k$. Thus, also $C'' := C' + \sum_{P \in \mathcal{P}} (C_P \cup E(P))$ can be generated with cycles of length $\geq k$, where we sum over the set $\mathcal{P}$ of all C-paths $P \subseteq C'$. Clearly, $C'' \subseteq C$, and so, as C is a cycle and $C'' \neq \emptyset$ since $E(Q_C) \subseteq C''$, it follows that $C = C''$. This, again, contradicts the choice of C. $\square$

9.3 Locke's conjecture with infinite k

Let us now allow k to be infinite. As shown in the introduction, we only need to prove Theorem 9.1.3, which together with Theorem 9.1.1 yields a proof of Theorem 9.1.2. Define a *double ray-circuit* to be a circuit which is the edge set of a double ray. Then, Theorem 9.1.3 is an immediate corollary of the following theorem.

Theorem 9.3.1. **[10]** *Let G be an infinite 2-connected locally finite graph. Then there is a thin set of double ray-circuits that generates the cycle space $\mathcal{C}(G)$.*

Diestel and Kühn [17] show that there is always a thin generating set consisting of finite circuits. Thus, Theorem 9.3.1 may be understood as a converse of that fact. A spanning tree T is *end-faithful* if for an arbitrary root r, T has exactly one ray starting in r in every end of G.

Theorem 9.3.2 (Diestel and Kühn [17]). *Let G be a locally finite connected graph, and let T be any spanning tree of G. Then every element of the cycle space $\mathcal{C}(G)$ is generated by fundamental circuits of T if and only if T is end-faithful. Also, if T is end-faithful, then the set of fundamental circuits of T is a thin set.*

Since every countable graph has a normal spanning tree (Jung [28]), and since every normal spanning tree is end-faithful, there is always a thin set of finite circuits that generates the cycle space in a locally finite graph.

Lemma 9.3.3. **[10]** *Let C be a finite circuit in a graph G, and let D be a double ray-circuit which shares an edge with C. Then there are double ray-circuits $D_1, D_2 \subseteq C \cup D$ such that $D_1 + D_2 = C$.*

Proof. As C is finite, there are two vertices $v, w \in V(C)$ that are linked by an arc D' with $E(D') \subseteq D$ which is internally disjoint from C. Choose a v–w path P in C, and put $D_1 := E(D' \cup P)$ and $D_2 := D_1 + C$. $\square$

Lemma 9.3.4. **[10]** *Let G be an infinite 2-connected locally finite graph. Then there is for every edge vw of G a double ray D containing vw such that $E(D)$ is a double ray-circuit.*

Proof. Choose an end $\omega \in \Omega(G)$. Put $S_1 := \{v, w\}$ and for $i = 2, 3, \dots$ choose S_i with minimal cardinality subject to $S_i \cup C(S_i, \omega) \subseteq C(S_{i-1}, \omega)$. (Thus, S_i separates S_{i-1} from the end ω.) Then S_i is a minimal S_i–S_{i+1} separator (which for $i = 1$ is ensured by the 2-connectivity of G and for $i = 2, 3, \dots$ by the minimal choice of the S_i). Thus for each i Menger's Theorem yields $|S_i|$ disjoint S_i–S_{i+1} paths, the union of which contains two disjoint rays in ω one starting in v and the other in w. Using these we obtain the desired double ray D. $\square$

Proof of Theorem 9.3.1. Fix a vertex u in G. By Lemma 9.3.4, there is for each $e \in E(G)$ a double ray-circuit D_e containing e; let us assume D_e to be chosen such that $d(u, D_e)$ is maximal (which is possible as the endvertices of e are at finite distance from u). We claim that

$$\text{the set } \mathcal{D} := \{E(D_e) : e \in E(G)\} \text{ is thin.} \tag{9.1}$$

Assume (9.1) to be true. By Theorem 9.3.2 there is a thin set $\mathcal{C}$ of finite circuits that generates the cycle space. For each $C \in \mathcal{C}$ pick an edge $e^C \in C$. Lemma 9.3.3 yields two double ray-circuits $D_1^C, D_2^C \subseteq C \cup D_{e^C}$ with $D_1^C + D_2^C = C$. As both $\mathcal{C}$ and $\mathcal{D}$ are thin, the set $\{D_1^C, D_2^C : C \in \mathcal{C}\}$ is a thin set that generates the cycle space of G (since $\mathcal{C}$ is a generating set).

So, suppose that (9.1) does not hold. Then, there is an edge e^* which is met by infinitely many of the D_e. Denote by S the set of all vertices of G with distance $\leq d(u, e^*) + 1$ to u. Then $G - S$ has a component K that contains an infinite number of edges e with $e^* \in D_e$. If we can show that

$$\text{the number of blocks of } K \text{ is finite,} \tag{9.2}$$

we are done: then, there is an infinite block B^∞ of K that contains an edge e with $e^* \in D_e$. Lemma 9.3.4 yields a double ray-circuit containing e in B^∞, which, having a greater distance to u than D_e has, contradicts the choice of D_e.

Since G is 2-connected, there is, by Menger's theorem, for every block B of K an $N_K(S)$–$N_K(S)$ path P_B in K containing an edge of B. Assume that K has an infinite number of blocks. Then there are two vertices in $N_K(S)$ that are the endvertices of P_B for infinitely many blocks B of K. We fix B as one of these blocks, and then choose B' as another which is disjoint from the finite path P_B. This yields two finite blocks B, B' of K such that P_B and $P_{B'}$ have the same endvertices and $P_B \cap E(B') = \emptyset$.

Let x be the last vertex on $P_{B'}$ that also lies on P_B before $P_{B'}$ enters B', and let y be the next vertex on $P_{B'}$ that also lies on P_B. Then, $E(xP_{B'}y \cup xP_By)$ is a circuit of K that meets both $E(B')$ and $E(K) \setminus E(B')$, and thus contains a cutvertex of B'. This yields the desired contradiction, as a cutvertex may not lie on a circuit. $\square$

9.4 Graphs with infinite degrees

Note that Theorem 9.1.2 becomes false for graphs with infinite degrees. Indeed, observe that every infinite circuit that contains an edge of one of the triangles in Figure 9.1 is incident with both x and y and is disjoint from any other triangle. Thus, the union of the (edge-sets of the) triangles, which is certainly an element of the cycle space, cannot be generated by a set of infinite circuits. Finally, note, that every two vertices can be joined by an arc of infinite length.

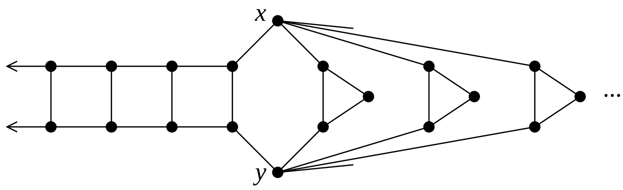

Figure 9.1: Result false for graphs with infinite degrees

Finally, let us remark that in a non-locally finite graph also the finite circuits may not be sufficient to generate the cycle space (although this is still true in any countable graph). Let $v_1, v_2, \dots$ be some distinguished vertices, and let there be a double ray $D^r = \dots w_{-1}^r w_0^r w_1^r \dots$ for every $r \in \mathbb{R}$. Join v_n to w_{-n}^r and to w_n^r for all n and r. Suppose $Z := \sum_{r\in\mathbb{R}} D^r = \bigcup_{r\in\mathbb{R}} D^r \in \mathcal{C}(G)$ is the sum of finite circuits. Since Z is uncountable the sum contains uncountably many distinct finite circuits, each of which is incident with one of the v_n. Thus, there is a v_n which is incident with infinitely many of the summands, a contradiction.

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Zusammenfassung

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ENDS OF GRAPHS
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Die vorliegende Arbeit behandelt Themen der unendlichen Graphentheorie. Im Mittelpunkt steht dabei das schon von Hopf [27] und Freudenthal [22] eingeführte Konzept der *Enden* eines Graphen. Unter Einbeziehung der Enden lassen sich Resultate der endlichen Graphentheorie auf unendliche Graphen übertragen, die andernfalls scheitern. Auch in anderen Fällen lohnt es, die Ecken und Enden eines unendlichen Graphen als gleichberechtigt zu betrachten.
Wir erlauben daher *unendliche Wege* und *unendliche Kreise*, die durch Enden 'hindurchlaufen': genauer gesagt sind dies homöomorphe Bilder des Einheitsintervalls bzw. des Einheitskreises, unter Verwendung der natürlichen Topologie auf dem Graphen zusammen mit seinen Enden (für lokal endliche Graphen ist dies deren Freudenthal-Kompaktifizierung). Unendliche Kreise und der daraus resultierende Zyklenraum $\mathcal{C}(G)$ unendlicher Graphen wurden von Diestel und Kühn[17, 18] eingeführt.
Analog zum Gradbegriff für Ecken entwickeln wir einen *Gradbegriff für Enden*, der globale Forderungen wie z.B. hohen Minimalgrad auch für unendliche Graphen erlaubt. Desweiteren definieren wir die *Parität* bei unendlichen Grad.
Diese Anpassungen der Standardbegriffe ermöglichen die wortwörtliche Übertragung folgender Ergebnisse auf unendliche Graphen:

- Charakterisierung der Graphen G mit $E(G) \in \mathcal{C}(G)$ als solche, die überall geraden Grad haben,
- Erzwingung hochzusammenhängender Teilgraphen durch hohen Minimalgrad (im Endlichen ein Satz von Mader),
- Nash-Williams' Arborizitätssatz (allerdings mit einer zusätzlichen Beschränkung des Endengrades),
- Gallai's Satz,

- MacLane's Plättbarkeitskriterium,
- Erzeugung des Zyklenraums durch lange Kreise (Locke).

Eines der wichtigsten Resultate ist

- Die Endenversion der Erdős-Menger Vermutung.

Diese bekannte Vermutung von Erdős erweitert den Satz von Menger auf unendliche Graphen, und ist in der Eckenversion kürzlich von Aharoni und Berger bewiesen worden. In der Endenversion sind neben Ecken auch Enden in den zu verbindenden Mengen sowie dem Trenner erlaubt.

Lebenslauf

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