On deformations of module categories over finite tensor categories

Dissertation zur Erlangung des Doktorgrades an der Fakultät für Mathematik, Informatik und Naturwissenschaften der Universität Hamburg

vorgelegt im Fachbereich Mathematik von

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Hamburg, 2023

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Datum der Verteidigung: 28.11.2023

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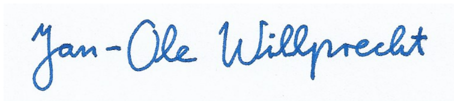

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Summary of results

In this thesis, we introduce and study infinitesimal associator deformations of a module category $\mathcal{M}$ over a monoidal category $\mathcal{C}$, including deformations of non-strict module endofunctors. We construct a cochain complex (allowing module endofunctors as coefficients), called the *associator complex*, whose cohomology controls these deformations up to equivalence. In particular, its second cohomology with trivial coefficients describes infinitesimal deformations of mixed associators of $\mathcal{M}$ while the first cohomology classifies infinitesimal deformations of module structures of a $\mathcal{C}$-module functor. We show that the associator complex admits the structure of a dg-algebra and a dg-module over the Davydov-Yetter complex of the identity functor of the monoidal category $\mathcal{C}$. We establish an isomorphism between the associator complex and the Davydov-Yetter complex of the action functor, which also respects the dg-algebra and dg-module structure. In the case, where $\mathcal{C}$ is a finite tensor category, we use techniques from relative homological algebra to realize the asssociator cohomology of a finite module category $\mathcal{M}$ (with coefficients) as a relative $\mathrm{Ext}^\bullet$ for the standard adjunction between the categories of module endofunctors and linear endofunctors of $\mathcal{M}$. In the case where $\mathcal{M}$ is an exact $\mathcal{C}$-module category, we show furthermore that the associator cohomology with coefficients is isomorphic to the relative $\mathrm{Ext}^\bullet$ of a much simpler adjunction, the one between the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}$. In particular, for trivial coefficients, the total associator cohomology is the total relative Ext group between the tensor unit of $\mathcal{C}$ and the adjoint algebra of $\mathcal{M}$. This is the main result of the thesis, which is based on another important technical result for liftings of $\mathcal{C}$-bimodule functors. We use this result first to show that the regular module category never admits associator deformations. We also apply it in explicit calculations for the case of Hopf algebras, and find non-trivial associator deformations of module categories over Sweedler's 4-dimensional Hopf algebra, including its generalizations.

Zusammenfassung der Ergebnisse

In dieser Arbeit werden infinitesimale Assoziatordeformationen einer Modulkategorie $\mathcal{M}$ über einer monoidalen Kategorie $\mathcal{C}$ eingeführt und untersucht, einschließlich Deformationen von nicht-strikten Modulendofunktoren. Wir konstruieren einen Kokettenkomplex (der Modulendofunktoren als Koeffizienten zulässt), genannt *Assoziatorkomplex*, dessen Kohomologie diese Deformationen bis auf Äquivalenz kontrolliert. Insbesondere beschreibt die zweite Kohomologie dieses Komplexes mit trivialen Koeffizienten infinitesimale Deformationen von gemischten Assoziatoren von $\mathcal{M}$, während die erste Kohomologie infinitesimale Deformationen von Modulstrukturen eines $\mathcal{C}$-Modulfunktors klassifiziert. Wir zeigen, dass der Assoziatorkomplex die Struktur einer dg-Algebra und eines dg-Moduls über dem Davydov-Yetter-Komplex des Identitätsfunktors der monoidalen Kategorie $\mathcal{C}$ besitzt. Wir konstruieren einen Isomorphismus zwischen dem Assoziatorkomplex und dem Davydov-Yetter-Komplex des Wirkungsfunktors, der auch die Struktur der dg-Algebra und des dg-Moduls respektiert. In dem Fall, dass $\mathcal{C}$ eine endliche Tensorkategorie ist, verwenden wir Techniken aus der relativen homologischen Algebra um die Assoziatorkohomologie einer endlichen Modulkategorie $\mathcal{M}$ (mit Koeffizienten) als relativen $\mathrm{Ext}^\bullet$ für die Standardadjunktion zwischen den Kategorien der Modulendofunktoren und der linearen Endofunktoren von $\mathcal{M}$ zu realisieren. Für den Fall, dass $\mathcal{M}$ eine exakte $\mathcal{C}$-Modulkategorie ist, zeigen wir außerdem, dass die Assoziatorkohomologie mit Koeffizienten isomorph zum

relativen $\mathrm{Ext}^\bullet$ einer viel einfacheren Adjunktion ist, nämlich der zwischen dem Drinfeld-Zentrum $\mathcal{Z}(\mathcal{C})$ und $\mathcal{C}$. Insbesondere ist die Assoziatorkohomologie mit trivialen Koeffizienten isomorph zum relativen $\mathrm{Ext}^\bullet$ zwischen der Tensoreinheit von $\mathcal{C}$ und der adjungierten Algebra von $\mathcal{M}$. Dies ist das Hauptresultat der Arbeit, das auf einem anderen wichtigen technischen Ergebnis für Liftings von $\mathcal{C}$-Bimodulfunktoren basiert. Wir verwenden dieses Resultat zunächst, um zu zeigen, dass die reguläre Modulkategorie keine Assoziatordeformationen zulässt. Wir wenden es außerdem in expliziten Berechnungen im Falle von Hopf-Algebren an und finden nicht-triviale Assoziatordeformationen von Modulkategorien über Sweedlers 4-dimensionaler Hopf-Algebra sowie ihrer Verallgemeinerungen.

Acknowledgements

First of all, I am deeply grateful to my advisors, Christoph Schweigert and Azat Gainutdinov. I cannot thank them enough for their guidance and support throughout this project. We had countless hours of interesting mathematical discussions and also the occasional friendly chat about non-mathematical topics, such as mushroom recipes.

I also want to thank all my colleagues from the University of Hamburg, in particular all the students and staff from the third and fourth floor of Geomatikum.
While teaching mathematics at the University of Hamburg for several years, I had the privilege of working with many awesome people: special thanks go to (in alphabetical order) Hans-Christian von Bothmer, Severin Bunk, Hubert Kiechle, Maxim Kirsebom, Susanne Koch, Janko Latschev, Alexander Lohse and Max Pitz.

I am also grateful to the other members of my committee, Susanne Koch, Natalie Neumeyer and Paul Wedrich for their time and effort.

I thank Alexei Davydov, Matthieu Faitg and Martín Mombelli for inspiring mathematical discussions and their interest in my work.

I would also like to acknowledge all the people, who keep the university running in the background like our team from the Studienbüro, the library and all the secretaries. Special thanks go to Astrid Dörhöfer, Ruth Ellebracht and Stefanie Risse.

Last but not least, I am deeply indebted to my beloved family and friends. Without their love, support and patience, I would not have been able to undertake this wonderful journey.

Contents

Introduction

In this thesis, we initiate the study of deformations of module categories over monoidal categories. In this introduction, we first explain the relevant notions and the precise setting we are working in, as well as our motivations. Then, we will summarize our main results.

Basic algebraic notions

Monoidal categories All the categories we consider in this thesis are R-linear, where R is a commutative unital ring. In our setting, a monoidal category comes with an R-bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the tensor product, as well as a unit object I. Moreover, we have left and right unitors $\mathfrak{l}$ and $\mathfrak{r}$ as well as an associator $\mathfrak{a}$, i.e., a natural family of isomorphisms

$$\mathfrak{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

in $\mathcal{C}$ for all objects $X, Y, Z \in \mathcal{C}$.

It should be appreciated that strict associativity of the tensor product, i.e., the requirement of identity of objects $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, has to be weakened to a natural isomorphism $\mathfrak{a}$. Indeed, strict associativity only rarely occurs in practice: already the tensor product of vector spaces is not strictly associative. As a consequence, monoidal categories of representation-theoretic origin are not strict. The unitors and the associator are required to satisfy two coherence conditions, the *pentagon axiom* and the *triangle axiom*, i.e., the two following diagrams commute for all $U, V, W, X \in \mathcal{C}$:

$$\begin{array}{ccc} & (U \otimes V) \otimes (W \otimes X) & \\ \overset{\mathfrak{a}_{U\otimes V,W,X}}{\nearrow}\,\cong & & \cong\,\overset{\mathfrak{a}_{U,V,W\otimes X}}{\searrow} \\ \big((U \otimes V) \otimes W\big) \otimes X & & U \otimes \big(V \otimes (W \otimes X)\big) \\ \mathfrak{a}_{U,V,W}\otimes \mathrm{id}_X \downarrow \cong & & \cong \uparrow \mathrm{id}_U \otimes \mathfrak{a}_{V,W,X} \\ \big(U \otimes (V \otimes W)\big) \otimes X & \xrightarrow[\mathfrak{a}_{U,V\otimes W,X}]{\cong} & U \otimes \big((V \otimes W) \otimes X\big) \end{array} \tag{0.1}$$

$$\begin{array}{ccc} (V \otimes I) \otimes W & \xrightarrow[\cong]{\mathfrak{a}_{V,I,W}} & V \otimes (I \otimes W) \\ & \underset{\mathfrak{r}_V \otimes \mathrm{id}_W}{\overset{\cong}{\searrow}} \quad \underset{\mathrm{id}_V \otimes \mathfrak{l}_W}{\overset{\cong}{\swarrow}} & \\ & V \otimes W & \end{array}$$

We explain the role of the pentagon axiom: two tensor products of a string of objects $(X_1, X_2, \ldots, X_n)$ of $\mathcal{C}$ that differ only by their bracketing can be related by isomorphisms built from the associator. The pentagon axiom ensures that all isomorphisms that can be built from the associator coincide. This leads to multiple tensor products which are well-defined up to unique distinguished isomorphism.

Let k be a field; k-linear monoidal categories appear in many contexts: in particular, k-linear representation categories of groups or Lie algebras over k as well as categories of bimodules over an associative k-algebra are k-linear monoidal categories. For this thesis, representation categories of bialgebras and Hopf algebras provide particularly important examples of monoidal categories: a bialgebra H over a field k has the structure of an associative unital k-algebra and of a coassociative counital k-coalgebra, where the coproduct $\Delta : H \to H \otimes H$ is required to be a morphism of unital k-algebras. The category $H - \mathbf{mod}$ of left H-modules is then a k-linear monoidal category where the tensor product $X \otimes Y$ of two H-modules X and Y is defined as the tensor product of their underlying k-vector spaces endowed with the H-action which comes from the coproduct, i.e., $h.(x \otimes y) = h'.x \otimes h''.y$ for all $h \in H, x \in X, y \in Y$. Here we use Sweedler's notation and do not write out sums over pure tensors. The monoidal category $H - \mathbf{mod}$ inherits the associator and unitors from the monoidal category of k-vector spaces. For a Hopf algebra, we moreover require the existence of an antipode which generalizes the inverse in a group algebra.

Given two R-linear monoidal categories $\mathcal{C}$ and $\mathcal{D}$, there is a notion of a *monoidal functor* $(F, \Phi) : \mathcal{C} \longrightarrow \mathcal{D}$ which consists of an R-linear functor F and a natural family $\Phi = (\Phi_{X,Y})_{X,Y \in \mathcal{C}}$ of isomorphisms $\Phi_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \xrightarrow{\cong} F(X \otimes_{\mathcal{C}} Y)$ obeying suitable coherence conditions (see Definition A.9 for details). There is also a natural notion of monoidal natural transformations between monoidal functors with the same source and target.

Module categories A monoidal category can be understood as the categorification of an algebra. Given an algebra, it is natural to study its modules. Applying the same logic to monoidal categories, we arrive at the notion of a (left) *module category*: a module category over an R-linear monoidal category $\mathcal{C}$ is an R-linear category $\mathcal{M}$ together with an R-bilinear functor $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$, called the action of $\mathcal{C}$ on $\mathcal{M}$, as well as a *mixed associator* m, which is a natural family of isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\cong} X \triangleright (Y \triangleright M)$$

in $\mathcal{M}$. The existence of a *unitor* l, i.e., a natural family of isomorphisms $l_M : I \triangleright M \xrightarrow{\cong} M$ is also required. The mixed associator and the unitor have to satisfy two coherence conditions: the *pentagon axiom*

$$\begin{array}{ccccc} & & \big((X \otimes Y) \otimes Z\big) \triangleright M & & \\ & \swarrow{\scriptstyle \mathfrak{a}_{X,Y,Z} \triangleright \mathrm{id}_M,\ \cong} & & \searrow{\scriptstyle m_{X \otimes Y, Z, M},\ \cong} & \\ \big(X \otimes (Y \otimes Z)\big) \triangleright M & & & & (X \otimes Y) \triangleright (Z \triangleright M) \\ \downarrow{\scriptstyle m_{X, Y \otimes Z, M}\ \cong} & & & & \downarrow{\scriptstyle \cong\ m_{X,Y,Z \triangleright M}} \\ X \triangleright \big((Y \otimes Z) \triangleright M\big) & & \xrightarrow[\mathrm{id}_X \triangleright m_{Y,Z,M}]{\cong} & & X \triangleright \big(Y \triangleright (Z \triangleright M)\big) \end{array} \tag{0.2}$$

and the *triangle axiom*

$$\begin{array}{ccc} (X \otimes I) \triangleright M & \xrightarrow[\cong]{m_{X,I,M}} & X \triangleright (I \triangleright M) \\ & {\scriptstyle r_X \triangleright \mathrm{id}_M}\searrow^{\cong} \quad {}^{\cong}\swarrow{\scriptstyle \mathrm{id}_X \triangleright l_M} & \\ & X \triangleright M & \end{array}$$

We briefly explain the motivation for the pentagon axiom: for a module M over an R-algebra A, one imposes the following compatibility condition on the multiplication in A and the A-action on M:

$$(a_1 \cdot a_2).m = a_1.(a_2.m)$$

This condition ensures that all ways to combine the action of A on M and the multiplication of A to obtain a morphism

$$A^{\otimes n} \otimes M \longrightarrow M$$

coincide. In the case of a module category, we obtain various functors

$$\mathcal{C}^{\times n} \times \mathcal{M} \longrightarrow \mathcal{M}$$

by combining the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ of the monoidal category $\mathcal{C}$ and the action $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$. From the associator of $\mathcal{C}$ and the mixed associator of $\mathcal{M}$, we obtain many natural isomorphisms between these functors. The pentagon axiom (0.1) of the monoidal category $\mathcal{C}$ and the pentagon axiom (0.2) of the module category $\mathcal{M}$ ensure that all these isomorphisms are identical.

Again, given two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{M}'$, there is a notion of a $\mathcal{C}$-module functor (see Definition A.17); given two $\mathcal{C}$-module functors between the same $\mathcal{C}$-module categories, there is a notion of module natural transformations (see Definition A.19). We have defined left module categories; right module categories (see Definition A.22) and bimodule categories (see Definition A.25) can be defined analogously.
We briefly comment on the importance of module categories: modules are crucial for understanding the structure of an algebra. Considering algebras with equivalent categories of modules as equivalent is the basic idea of Morita theory, which is a central tool in algebra, e.g., for the study of Brauer groups. A similar theory is emerging for monoidal categories and module categories (see, e.g., [GJS]), providing an additional motivation for the study of module categories.

Examples of module categories In the same way, any algebra is a left module over itself, every monoidal category is a module category over itself, the *regular module category.* Here the action is the tensor product and the mixed associator is the associator of the monoidal category. An important source of module categories over a given monoidal category $\mathcal{C}$ are algebras in $\mathcal{C}$: an algebra in $\mathcal{C}$ is an object $A \in \mathcal{C}$, with an associative multiplication morphism $A \otimes A \longrightarrow A$. In this terminology, an algebra in **vect**, the monoidal category of k-vector spaces, is the same as an associative k-algebra. A right A-module (internal to $\mathcal{C}$) is an object $M \in \mathcal{C}$, together with a right action $r : M \otimes A \longrightarrow M$, obeying the standard condition for a right module. If M is a right A-module, then for any $X \in \mathcal{C}$, the object $X \otimes M$ becomes a right A-module for which the right A-action is defined as the composition $(\mathrm{id}_X \otimes r) \circ \mathfrak{a}_{X,M,A}$. In this way, the category $\mathbf{mod}_{\mathcal{C}} - A$ of right A-modules becomes a left $\mathcal{C}$-module category, which inherits its mixed associator and unitor from the regular $\mathcal{C}$-module category.

This allows us to give simple examples of module categories: if G is a group, the category $\mathbf{vect}_G$ of G-graded vector spaces inherits a natural monoidal structure from the monoidal structure on the category of vector spaces. This structure can actually be twisted by a cocycle in group cohomology. A subgroup $L \subseteq G$ together with a class in group cohomology $H^2(L, \mathbb{C}^\times)$ then determines an associative algebra in $\mathbf{vect}_G$ and thus a module category. It has been shown in [O] that all (indecomposable exact) module categories over $\mathbf{vect}_G$ can be obtained in this way.

To conclude our list of examples of module categories, we describe a realization of module categories for bialgebras. We have already mentioned that every bialgebra H gives rise to a monoidal category, namely its category of modules $H - \mathbf{mod}$. A comodule algebra is by definition an algebra in the monoidal category of H-comodules and thus comes with a (coassociative) H-coaction

$$\begin{aligned} A &\longrightarrow H \otimes A \\ a &\longmapsto a_{(0)} \otimes a_{(1)} \end{aligned}$$

where we use Sweedler's notation again. The action $H - \mathbf{mod} \times A - \mathbf{mod} \longrightarrow A - \mathbf{mod}$ maps a pair (X, M) to the k-vector space $X \otimes M$, endowed with the A-action $a.(x \otimes m) = a_{(0)}.x \otimes a_{(1)}.m$. The mixed associator as well as the unitor for the module category $A - \mathbf{mod}$ over $H - \mathbf{mod}$ are again inherited from $\mathbf{vect}$. Special examples of comodule algebras are the ground field k and H itself, exhibiting $\mathbf{vect}$ and $H - \mathbf{mod}$ as module categories over $H - \mathbf{mod}$. These examples will play an important role in Section 4 and 5 of this thesis. Moreover, if H is a finite-dimensional Hopf algebra, then, by [AM, Proposition 1.19], any *exact* and *indecomposable* module category over $H - \mathbf{mod}$ is of the form $A - \mathbf{mod}$, where A is an (exact) H-comodule algebra.

Deformations

Algebraic structures over a field k amount to (collections of) k-vector spaces and maps between tensor products of such vector spaces obeying certain equations, such as (co-)associativity or (co-)commutativity. For example, an associative k-algebra amounts to a k-vector space and a k-linear map $\mu : A \otimes A \longrightarrow A$, obeying associativity

$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$$

which is a nonlinear equation on μ. To understand deformations of this multiplication, we consider an associative multiplication on A over the ring of dual numbers $k[\varepsilon]/(\varepsilon^2)$:

$$a \star b := \mu(a \otimes b) + \varepsilon \tilde{\mu}(a \otimes b)$$

with a linear map $\tilde{\mu} : A \otimes A \longrightarrow A$. We interpret $\star$ as an (*infinitesimal* or *first order*) deformation of the multiplication μ. Associativity of $\star$ is an equation, which amounts to

$$\tilde{\mu}(ab \otimes c) + \tilde{\mu}(a \otimes b)c = \tilde{\mu}(a \otimes bc) + a\tilde{\mu}(b \otimes c) \tag{0.3}$$

where the product on A is written by juxtaposition ab. We typically consider deformations up to linearized automorphisms $T(a) = a + \varepsilon g(a)$ with $g : A \longrightarrow A$ a linear map, leading to a multiplication

$$a \star_T b := T\left(T^{-1}(a) \star T^{-1}(b)\right) = \mu(a \otimes b) + \varepsilon \tilde{\mu}_T(a \otimes b)$$

with

$$\tilde{\mu}_T(a \otimes b) = \tilde{\mu}(a \otimes b) - g(a)b - ag(b) + g(ab)$$

This problem has been studied almost 60 years ago [G], leading to the following insights:

- Equation (0.3) is best interpreted by seeing $\tilde{\mu}$ as a 2-cocycle in a cochain complex of k-vector spaces, the *Hochschild complex*. Its cochain spaces are the k-vector spaces $\mathrm{Hom}_k(A^{\otimes n}, A)$ of linear maps, and the differential

  $$d^n : \mathrm{Hom}_k(A^{\otimes n}, A) \longrightarrow \mathrm{Hom}_k(A^{\otimes n+1}, A)$$

  is given by

  $$\begin{aligned} d^n(f)(a_1 \otimes \ldots \otimes a_{n+1}) =& a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1} \end{aligned}$$

  The associativity of the multiplication on A implies that the differential squares to zero.

- One then realizes that the cohomology of this complex has an algebraic meaning in lower degrees as well: the zeroth Hochschild cohomology $\mathrm{HH}^0(A, A) \cong Z(A)$ is the center of A and the first cohomology $\mathrm{HH}^1(A, A)$ is the space of derivations of A modulo inner derivations. The second cohomology $\mathrm{HH}^2(A, A)$ describes deformations of the multiplication and $\mathrm{HH}^3(A, A)$ describes obstructions.

- Finally, it turns out to be very helpful to consider more general complexes by admitting coefficients, in this case A-bimodules. Then, the *nth* cochain space is $\mathrm{Hom}_k(A^{\otimes n}, B)$, where B is an A-bimodule. It is textbook knowledge (see, e.g., [W]) that the cohomology of this cochain complex can be understood as a relative Ext:

  $$\mathrm{HH}^\bullet(A, B) = \mathrm{Ext}^\bullet_{A \otimes A^{\mathrm{op}}/k}(A, B)$$

  leading, e.g., to long exact sequences. At the same time, the theory can be related to a comonad on the category of bimodules, leading, in particular, to bar complexes.

In this thesis, we study deformations of the mixed associator of module categories which are solutions of the pentagon equation, and we observe that similar patterns emerge.

Work on which this thesis builds Similar deformation problems of coherence morphisms of categorical structures have been studied in the literature before: in particular, the study of deformations of associators of k-linear monoidal categories (and, more generally, of monoidal structures on k-linear functors) have lead to a cohomology theory, known as *Davydov-Yetter cohomology* (see, e.g., [CY] and [D1]). In this case, the cochain spaces are k-vector spaces of natural transformations. Davydov-Yetter cohomology, which we review in Section 2.1, will be an important tool in this thesis. Here, we just mention that the second Davydov-Yetter cohomology group of a monoidal functor controls infinitesimal deformations of its monoidal structure. The third Davydov-Yetter cohomology group of the identity functor controls infinitesimal deformations of the monoidal category. Recently, coefficients for the Davydov-Yetter complex have been identified in [GHS]: they are objects in the centralizer of the relevant monoidal functor. These coefficients will be crucial for our purposes as well.

We will also use in this thesis that lately, many more methods have become available for the study of Davydov-Yetter cohomology. Davydov-Yetter cohomology has been shown to be a comonad cohomology (see [GHS]) which in turn can be related to relative cohomology: specifically, the nth Davydov-Yetter cohomology of an exact k-linear monoidal functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ between finite tensor categories with coefficients $\mathrm{X}, \mathrm{Y} \in \mathcal{Z}(G)$ is isomorphic to the nth relative Ext group of X and Y (see [FGS1, Corollary 4.7]):

$$H^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \cong \mathrm{Ext}^n_{\mathcal{Z}(G),\mathcal{D}}(\mathrm{X}, \mathrm{Y}) \tag{0.4}$$

The relative Ext can then be computed with representation-theoretic methods, using specific relatively projective resolutions.

Results of this thesis

In the following, we will summarize the main results of this thesis. In Section 1, we introduce in Definition 1.1 and Definition 1.2 the notion of associator deformations and their equivalence and in Definition 1.3, deformations of module functors are introduced. Given a k-linear $\mathcal{C}$-module functor $F : \mathcal{M} \longrightarrow \mathcal{M}'$, we set up a cochain complex whose cochain spaces are k-vector spaces of natural transformations, which we call the *associator complex* (see Definition 1.7). These module functors should be seen as *coefficients*, like bimodules in the Hochschild complex or objects in centralizers for the Davydov-Yetter complex.

At this point, we should highlight the following additional result of this thesis: while studying module functors more carefully, we noticed that it is redundant to postulate the usual triangle diagram (A.12) for module functors as an axiom as is standard in the literature (see, e.g., [EGNO, Definition 7.2.1]) since it follows automatically from the pentagon axiom for module functors (Proposition A.18).

In Theorem 1.8, we show that the associator complex is set up in such a way that 2-cocycles describe associator deformations up to soft equivalence, i.e., the underlying linear functor is the identity Id and its module structure is $\mathrm{id} + \varepsilon\nu$ for some natural transformation $\nu : \mathrm{Id} \Longrightarrow \mathrm{Id}$. We also investigate the interpretation of the associator cohomology in lower degrees: in Proposition 1.11, we show that the zeroth associator cohomology with coefficients two $\mathcal{C}$-module endofunctors $(F, s), (F', s') : \mathcal{M} \longrightarrow \mathcal{M}$ is isomorphic to the vector space of $\mathcal{C}$-module transformations from (F, s) to (F', s'). In Proposition 1.12, we show that the first associator cohomology group $H^1_{ass}(F, s)$ with coefficient a $\mathcal{C}$-module endofunctor $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}$ controls infinitesimal deformations of (F, s) up to equivalence. In Section 1.4, we finally show that the obstruction theory for higher order deformations behaves as expected: if the third associator cohomology for the identity functor vanishes, then nth order deformations can be extended to $n + 1$st order by Proposition 1.15; Proposition 1.18 states a similar result for deformations of a module endofunctor (F, s) in terms of the second cohomology with coefficient (F, s).

As is standard for a bar complex, the associator complex has generally large cochain spaces; it is hard to explicitly compute cocycles or cohomology groups, even in low degrees. Since much better tools are now available for Davydov-Yetter cohomology, we relate in Section 2 associator cohomology to Davydov-Yetter cohomology. We start with the well-known fact that the structure of a $\mathcal{C}$-module category on a k-linear category $\mathcal{M}$ is encoded in a monoidal functor, the *action functor*

$$\begin{aligned} \rho_{\mathcal{M}} : \mathcal{C} &\longrightarrow \mathrm{End}(\mathcal{M}) \\ X &\longmapsto X \triangleright - \end{aligned}$$

where $\mathrm{End}(\mathcal{M})$ denotes the monoidal category of k-linear endofunctors of $\mathcal{M}$. Here, the monoidal structure of the action functor is the inverse of the mixed associator of $\mathcal{M}$. This observation allows us to establish an isomorphism between the associator complex of $\mathcal{M}$ and the Davydov-Yetter complex of the action functor of $\mathcal{M}$:

$$C^\bullet_{ass}(\mathcal{C}, \mathcal{M}) \cong C^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}}) \tag{0.5}$$

Two comments are in order: first, the isomorphism (0.5) in fact respects much more than the structure of a cochain complex. Indeed, the Davydov-Yetter complex of a monoidal functor has a natural structure of a dg-algebra (see, e.g., [BD]). We show in Section 2.4 that the Davydov-Yetter complex of a monoidal functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ is a dg-module over the dg-algebra $C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$. We also show that the associator complex $C^\bullet_{ass}(\mathcal{C}, \mathcal{M})$ naturally admits a dg-algebra structure and a dg-module structure over $C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ as well. In Proposition 2.13 and Proposition 2.15, we then show that the isomorphism (0.5) is compatible with these structures: it is an isomorphism of dg-algebras and dg-modules over $C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$.

Second, both complexes admit non-trivial coefficients. The coefficients for $C^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}})$ are objects of the centralizer $\mathcal{Z}(\rho_{\mathcal{M}})$ and the coefficients for $C^\bullet_{ass}(\mathcal{C}, \mathcal{M})$ are $\mathcal{C}$-module endofunctors of $\mathcal{M}$. Both categories are monoidally equivalent by Proposition 2.8:

$$\mathcal{Z}(\rho_{\mathcal{M}}) \cong \mathrm{End}_{\mathcal{C}}(\mathcal{M})$$

The isomorphism (0.5) extends to complexes involving non-trivial coefficients:

$$C^\bullet_{ass}\big((F, s), (F', s')\big) \cong C^\bullet_{\mathrm{DY}}\big(\rho_{\mathcal{M}}, (F, s), (F', s')\big) \tag{0.6}$$

In Section 2.4, we also address the following problem: a module M over an algebra A amounts to an algebra morphism $\rho_M : A \longrightarrow \mathrm{End}(M)$ and can thus be deformed by any algebra automorphism $\alpha : A \longrightarrow A$ to $\rho_M \circ \alpha$. For associator deformations of a $\mathcal{C}$-module category $\mathcal{M}$, by definition, we keep the action functor $\rho_{\mathcal{M}}$ fixed. Hence, we can only precompose $\rho_{\mathcal{M}}$ by a *soft* monoidal autoequivalence of $\mathcal{C}$. We explain how to obtain associator deformations of $\mathcal{M}$ from deformations of the monoidal structure of the identity functor $\mathrm{Id}_{\mathcal{C}}$.

As a first application of the isomorphism (0.6), in Section 2.5, we use a version of Ocneanu rigidity [GHS, Corollary 3.18] for monoidal functors with coefficients to obtain two rigidity results for semisimple (finite) module categories over semisimple multitensor categories and module functors between such categories.

In passing, we note in Proposition 2.5 a new interpretation of the first Davydov-Yetter cohomology group $H^1_{\mathrm{DY}}\big(G,(X,\sigma)\big)$ of a monoidal functor G with coefficient $(X,\sigma) \in \mathcal{Z}(G)$: it controls infinitesimal deformations of the half-braiding σ up to a certain equivalence.

In Section 3, we first review the relation between Davydov-Yetter cohomology and relative cohomology developed in [FGS1]. We show that the isomorphism (0.4) also holds if G is an exact monoidal functor from a finite tensor category to a finite abelian category with a right exact tensor product. This allows us in Corollary 3.4 to express associator cohomology (with coefficients) in terms of a relative Ext (with coefficients) for the adjoint pair consisting of the functor $\mathcal{U} : \mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \longrightarrow \mathrm{Rex}(\mathcal{M})$ which forgets the module structure of a linear right exact module endofunctor of $\mathcal{M}$ and the left adjoint of $\mathcal{U}$.

We now come to our deepest result. It is inspired by an open problem in Davydov-Yetter theory: is it possible to rewrite the Davydov-Yetter complex of any monoidal functor as the Davydov-Yetter complex of the identity functor with a suitable coefficient? For the forgetful functor $H-\mathbf{mod} \longrightarrow \mathbf{vect}$, where H is a finite-dimensional Hopf algebra, the answer is positive [GHS]. We show that it is possible for the action functor of any exact module category as well. Indeed, consider the following object: for any finite module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$, the end $\mathcal{A}_{\mathcal{M}} = \int_{M\in\mathcal{M}} \underline{\mathrm{Hom}}(M,M) \in \mathcal{C}$ comes with a natural half-braiding ζ and is thus an object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$. Since $\mathcal{A}_{\mathcal{M}}$ generalizes the adjoint representation of a Hopf algebra, it is known as the *adjoint algebra* of $\mathcal{M}$.

This adjoint algebra is particularly useful for *exact* module categories. Recall that a finite $\mathcal{C}$-module category $\mathcal{M}$ is called exact, if for all projective objects $P \in \mathcal{C}$ and all objects $M \in \mathcal{M}$, the object $P \triangleright M$ is projective in $\mathcal{M}$. This class of module categories is, at the moment, the best studied class of module categories. The results of [S3] indicate that for those module categories, the adjoint algebra should be useful also for questions of homological algebra.

In our main result, Theorem 3.7, we show that the associator cohomology of an exact $\mathcal{C}$-module category $\mathcal{M}$ can be related, via the right adjoint of the action functor, to the relative Ext for the adjoint pair consisting of the the forgetful functor $\mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$ and its left adjoint, at the price of introducing the adjoint algebra $\mathcal{A}_{\mathcal{M}}$ as a coefficient. Concretely, we find in Theorem 3.7:

$$H^\bullet_{ass}(\mathcal{C},\mathcal{M}) \cong \mathrm{Ext}^\bullet_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}(I,\mathcal{A}_{\mathcal{M}}) \tag{0.7}$$

Our proof uses the description of Davydov-Yetter cohomology of the action functor as a relative Ext from Corollary 3.4 as well as the lifting construction for bimodule functors established by Shimizu [S3], which we review in Section 3.4.1 and 3.4.2.
From our main result, we obtain the following rigidity result in Section 3.5: we show that for the regular $\mathcal{C}$-module category, the adjoint algebra $(\mathcal{A}_{\mathcal{C}},\zeta) \in \mathcal{Z}(\mathcal{C})$ is *relatively projective*. It follows from the isomorphism (0.7), that any finite tensor category $\mathcal{C}$, as a module category over itself, has trivial associator cohomology in degrees greater than 0.

Thus, $\mathcal{C}$ does not admit infinitesimal deformations as a module category over itself, even though $\mathcal{C}$ might have infinitesimal deformations as a monoidal category, as witnessed by Davydov-Yetter cohomology.

This finishes our summary of the first part of the thesis which is devoted to general results about associator cohomology. In the second part, we study in Section 4.1 concrete examples of module categories which are given by a comodule algebra A over a bialgebra H. It is shown in Proposition 4.4 that the associator deformation complex $C^\bullet_{ass}(H-\mathbf{mod}, A-\mathbf{mod})$ can be rewritten as a Hochschild-type complex whose cochain spaces are subspaces of $H^{\otimes n}\otimes A$ and whose differential is defined only using the H-coaction and the coproduct. In this form, it was also possible to implement the computation of associator cohomology in a computer algebra system. In Section 4.3, we describe the associator cohomology of **vect** as a module category over $H-\mathbf{mod}$.
This allows us in Section 5.1 and 5.2 to explicitly calculate the associator cohomology of **vect** as a module category in the case where H is a Taft algebra or the bosonization of an exterior algebra.

In Section 5.3, we study module categories over the finite tensor category $H-\mathbf{mod}$, where H is Sweedler's 4-dimensional Hopf algebra, a well-known non-semisimple, non-cocommutative, non-commutative complex Hopf algebra. We start with a complete list (up to isomorphism) of all four coideal subalgebras of H and study the categories of modules over them. These categories are naturally module categories over $H-\mathbf{mod}$ as we have explained above. We are then able to use comodule algebra techniques from [BM] to calculate the adjoint algebras of these module categories. The isomorphism (0.7) then allows us to calculate the associator cohomologies of the four module categories: the two non-semisimple module categories have trivial associator cohomologies, but the two semisimple module categories, **vect** and $\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod}$, each admit a 1-dimensional family of infinitesimal associator deformations. We provide an explicit 2-cocycle that generates these cohomologies, and we show that the infinitesimal deformations each admit an associated finite deformation. We thus obtain two one-parameter families of inequivalent module categories over $H-\mathbf{mod}$. Exact indecomposable module categories over Sweedler's Hopf algebra have been classified (see, e.g., [EO] and [AM]) and it turns out that our two one-parameter families recover *all* semisimple module categories over $H-\mathbf{mod}$.

1 Associator deformations

In this thesis, let k be a field. In this first section, we consider a k-linear monoidal category $\mathcal{C}$ and k-linear module categories $\mathcal{M}, \mathcal{M}'$, using the following notation:
Monoidal categories. We denote the tensor product of a monoidal category by $\otimes$ and the monoidal unit by I, and the associativity and unit constraints by $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$. These natural isomorphisms need to satisfy a pentagon and a triangle axiom (see Definition A.6). A monoidal category is called *strict*, if its associativity constraint and the unit constraints are the respective identity transformations.
Module categories. We denote the action of a monoidal category $\mathcal{C}$ on a $\mathcal{C}$-module category $\mathcal{M}$ by $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$. The (mixed) associator m is a natural family of isomorphisms $m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\cong} X \triangleright (Y \triangleright M)$ in $\mathcal{M}$. The left unitor l is a natural family of isomorphisms $l_M : I \triangleright M \xrightarrow{\cong} M$ in $\mathcal{M}$. Both m and l are required to satisfy a pentagon and a triangle axiom (see Definition A.14). As usual, a module category is called *strict* if its mixed associator and its left unitor are the respective identity transformations.

We want to study infinitesimal deformations of the mixed associator of a $\mathcal{C}$-module category $\mathcal{M}$, while keeping the monoidal structure of $\mathcal{C}$ as well as the action $\triangleright$ fixed. For this purpose, we will construct a cochain complex whose cohomology controls these deformations up to a certain notion of equivalence.

Many deformation problems have been studied in the literature. Here, we follow the notation of [LVdB, Section 4]. In this thesis, we deform categories that are k-linear. Hence, infinitesimal deformations are R_1-linear categories, where $R_1 := k[\varepsilon]/\varepsilon^2$ is the ring of dual numbers. In general, deformations of nth order are R_n-linear categories, where $R_n := k[\varepsilon]/\varepsilon^{n+1}$ for all $n \geq 1$. For any k-linear category $\mathcal{A}$, we denote by $R_n \otimes \mathcal{A}$ the R_n-linear category which has the same objects as $\mathcal{A}$ and has the induced R_n-modules $\mathrm{Hom}_{R_n \otimes \mathcal{A}}(A, B) := R_n \otimes_k \mathrm{Hom}_{\mathcal{A}}(A, B)$ as Hom spaces for all $A, B \in \mathcal{A}$. We will often suppress the tensor product in the notation of morphisms in $R_n \otimes \mathcal{A}$, i.e., we write $f + \varepsilon f^{(1)} + \ldots \varepsilon^n f^{(n)}$ instead of $1_{R_n} \otimes f + \varepsilon \otimes f + \ldots + \varepsilon^n \otimes f^{(n)}$. Given a k-linear functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$, by abuse of notation, we denote the induced R_n-linear functor by $F : R_n \otimes \mathcal{A} \longrightarrow R_n \otimes \mathcal{A}'$.
Given a k-linear monoidal category $\mathcal{C}$, the induced R_n-linear functor

$$\otimes : \quad (R_n \otimes \mathcal{C}) \times (R_n \otimes \mathcal{C}) \longrightarrow R_n \otimes \mathcal{C}$$

together with the associator $\mathfrak{a}$ as well as the left and right unitor $\mathfrak{l}$ and $\mathfrak{r}$ endow the R_n-linear category $R_n \otimes \mathcal{C}$ with the structure of an R_n-linear monoidal category. Note that the R_n-linear monoidal category $R_n \otimes \mathcal{C}$ with this monoidal structure is the trivial nth order deformation of $\mathcal{C}$ in the sense of Davydov-Yetter.

Definition 1.1. An *infinitesimal associator deformation* of a k-linear module category $\mathcal{M}$ over a k-linear monoidal category $\mathcal{C}$ is an R_1-linear, not necessarily unital, $R_1 \otimes \mathcal{C}$-module category such that: the underlying R_1-linear category is $R_1 \otimes \mathcal{M}$, the action is the induced R_1-linear functor

$$\triangleright : \quad (R_1 \otimes \mathcal{C}) \times (R_1 \otimes \mathcal{M}) \longrightarrow R_1 \otimes \mathcal{M}$$

and the associator is of the form

$$\mathbf{m}_{X,Y,M} = m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M}$$

where $m^{(1)}$ is a natural family of morphisms $m^{(1)}_{X,Y,M} : (X \otimes Y) \triangleright M \longrightarrow X \triangleright (Y \triangleright M)$ in $\mathcal{M}$. We denote this $R_1 \otimes \mathcal{C}$-module category by $(R_1 \otimes \mathcal{M}, m^{(1)})$; to the case $m^{(1)} = 0$, we refer as the *trivial associator deformation* of $\mathcal{M}$.

For brevity, we will often refer to *infinitesimal associator deformations* as *associator deformations*.

Note that the inverse of $\mathbf{m}$ is given by the natural family $\mathbf{m}^{-1} = \big(\mathbf{m}^{-1}_{X,Y,M}\big)_{X,Y \in \mathcal{C}, M \in \mathcal{M}}$ with

$$\mathbf{m}^{-1}_{X,Y,M} = m^{-1}_{X,Y,M} - \varepsilon m^{-1}_{X,Y,M} \circ m^{(1)}_{X,Y,M} \circ m^{-1}_{X,Y,M}$$

To compare associator deformations, we need to compare module categories. For this purpose, we recall the following notion: a *module functor* $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}'$ between two module categories $\mathcal{M}$ and $\mathcal{M}'$ consists of a functor F of the underlying categories as well as a natural family $s = (s_{X,M})_{X \in \mathcal{C}, M \in \mathcal{M}}$ of isomorphisms $s_{X,M} : F(X \triangleright M) \xrightarrow{\cong} X \triangleright' F(M)$ in $\mathcal{M}'$ satisfying a pentagon axiom (see Definition A.17). A module functor is called *strict* if $s = \mathrm{id}$.

Definition 1.2. An *equivalence of two associator deformations* $\mathcal{M}_1 = (R_1 \otimes \mathcal{M}, m^{(1)}_1)$ and $\mathcal{M}_2 = (R_1 \otimes \mathcal{M}, m^{(1)}_2)$ of a k-linear $\mathcal{C}$-module category $\mathcal{M}$ is an equivalence of $R_1 \otimes \mathcal{C}$-module categories, which is of the form $(\mathrm{Id}_{R_1 \otimes \mathcal{M}}, \mathrm{id} + \varepsilon t)$ where $t = (t_{X,M})_{X \in \mathcal{C}, M \in \mathcal{M}}$ is a natural family of morphisms $t_{X,M} : X \triangleright M \longrightarrow X \triangleright M$ in $\mathcal{M}$.

Analogously to Definition 1.1, we define infinitesimal deformations of *module functors*:

Definition 1.3. Let $\mathcal{M}$ and $\mathcal{M}'$ be k-linear $\mathcal{C}$-module categories. An *infinitesimal deformation* of a k-linear $\mathcal{C}$-module functor $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}'$ is the $R_1 \otimes \mathcal{C}$-module functor

$$(F, \mathbf{s}) : \quad (R_1 \otimes \mathcal{M}, 0) \longrightarrow (R_1 \otimes \mathcal{M}', 0)$$

whose module structure $\mathbf{s} = (\mathbf{s}_{X,M})_{X \in \mathcal{C}, M \in \mathcal{M}}$ is of the form

$$\mathbf{s}_{X,M} = s_{X,M} + \varepsilon s^{(1)}_{X,M}$$

where $s^{(1)} = \big(s^{(1)}_{X,M}\big)_{X \in \mathcal{C}, M \in \mathcal{M}}$ is a natural family of morphisms $s^{(1)}_{X,M} : F(X \triangleright M) \longrightarrow X \triangleright F(M)$ in $\mathcal{M}$.

In other words, an infinitesimal deformation of a $\mathcal{C}$-module functor is an $R_1 \otimes \mathcal{C}$-module functor, whose module structure is twisted by a natural transformation and whose source and target are the trivial associator deformations of the respective module categories.

Note that the inverse of $\mathbf{s}$ is given by the natural family $\mathbf{s}^{-1} = \big(\mathbf{s}^{-1}_{X,M}\big)_{X\in\mathcal{C},M\in\mathcal{M}}$ with

$$\mathbf{s}^{-1}_{X,M} = s^{-1}_{X,M} - \varepsilon s^{-1}_{X,M} \circ s^{(1)}_{X,M} \circ s^{-1}_{X,M}$$

To compare module functor deformations, we need to compare module functors. Recall from Definition A.19 that a natural transformation of $\mathcal{C}$-module functors $(F,s),(F',s') : \mathcal{M} \longrightarrow \mathcal{M}'$ is a natural transformation $\nu : F \Longrightarrow F'$ of the underlying functors such that $(\mathrm{id}_X \triangleright \nu_M) \circ s_{X,M} = s'_{X,M} \circ \nu_{X\triangleright M}$ for all $X \in \mathcal{C}, M \in \mathcal{M}$.

Definition 1.4. Two infinitesimal deformations $(F,\mathbf{s}_1),(F,\mathbf{s}_2)$ of a $\mathcal{C}$-module functor $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}'$ are called *equivalent*, if there is an isomorphism $(F,\mathbf{s}_1) \Longrightarrow (F,\mathbf{s}_2)$ of $R_1 \otimes \mathcal{C}$-module functors of the form $\mathrm{id}_F + \varepsilon\nu$ where $\nu = (\nu_M)_{M\in\mathcal{M}}$ is a natural family of morphisms $\nu_M : F(M) \longrightarrow F(M)$ in $\mathcal{M}'$.

Our goal in this section is to understand infinitesimal associator deformations and infinitesimal module functor deformations up to equivalence. However, there is another aspect of associator deformations that is of interest to us: can we use an infinitesimal associator deformation of $\mathcal{M}$, i.e., an $R_1 \otimes \mathcal{C}$-module category with underlying R_1-linear category $R_1 \otimes \mathcal{M}$, to construct a $\mathcal{C}$-module category structure of the k-linear category $\mathcal{M}$?

Definition 1.5. Let $(R_1 \otimes \mathcal{M}, m^{(1)})$ be an infinitesimal associator deformation of the k-linear $\mathcal{C}$-module category $\mathcal{M}$. The *finite (associator) deformation associated to* $(R_1 \otimes \mathcal{M}, m^{(1)})$ is a k-linear $\mathcal{C}$-module category such that: the underlying k-linear category is $\mathcal{M}$, the action and the unitor are the same as for $\mathcal{M}$ and the associator is of the form

$$\mathbf{m}^{\lambda}_{X,Y,M} = m_{X,Y,M} + \lambda m^{(1)}_{X,Y,M}$$

for some fixed scalar $\lambda \in k$. We denote this $\mathcal{C}$-module category by $\mathcal{M}^{\lambda}$.

Every associator deformation $(R_1 \otimes \mathcal{M}, m^{(1)})$ admits an associated finite deformation given by $\lambda = 0$, i.e., the one with the mixed associator $\mathbf{m}^{0}_{X,Y,M} = m_{X,Y,M}$. The trivial associator deformation $(R_1 \otimes \mathcal{M}, 0)$ always admits an associated finite deformation for each $\lambda \in k$, namely $\mathbf{m}^{\lambda}_{X,Y,M} = m_{X,Y,M}$. These two classes of associated finite deformations are trivial in the sense that they give back the $\mathcal{C}$-module category $\mathcal{M}$ that we started with. It is an open problem to classify whether an infinitesimal deformation admits a *non-trivial* associated finite deformation.

1.1 Associator deformation complex

We introduce a cochain complex whose cohomology controls associator deformations of module categories up to equivalence in the sense of Definition 1.2 as well as deformations of module functors up to equivalence in the sense of Definition 1.3. For simplicity of construction, we relegate the discussion of the case where the monoidal category, the module categories and the module functors are not necessarily strict to the appendix B.2 and give the complex in the strict case here. First, we need the following notation.

Let $\mathcal{C}$ be a k-linear strict monoidal category and let $\mathcal{M}$ and $\mathcal{M}'$ be k-linear strict $\mathcal{C}$-module categories. For any k-linear strict $\mathcal{C}$-module functor $F : \mathcal{M} \longrightarrow \mathcal{M}'$ and all $n \geq 1$, we introduce the k-multilinear functor

$$\begin{aligned}(F)^n : \quad \mathcal{C}^{\times n} \times \mathcal{M} &\longrightarrow \mathcal{M}' \\ (X_1, \ldots, X_n, M) &\longmapsto \underbrace{X_1 \rhd \ldots \rhd X_n \rhd F(M)}_{=F(X_1 \rhd \ldots \rhd X_n \rhd M)}\end{aligned}$$

where we use the convention $(F)^0 = F$.

For $n \geq 0$, the nth cochain space of the associator deformation complex of F is the k-vector space

$$C^n_{ass}(F) = \mathrm{Nat}\big((F)^n, (F)^n\big)$$

and the differential

$$\partial^n_{ass} : \quad C^n_{ass}(F) \longrightarrow C^{n+1}_{ass}(F)$$

is defined as

$$\begin{aligned}\partial^n_{ass}(f)_{X_1,\ldots,X_{n+1},M} &= \sum_{i=0}^{n+1} (-1)^i \partial^n_{ass}[i](f)_{X_1,\ldots,X_{n+1},M} \\ &= \mathrm{id}_{X_1} \rhd f_{X_2,\ldots,X_{n+1},M} + \sum_{i=1}^{n} (-1)^i f_{X_1,\ldots,X_i \otimes X_{i+1},\ldots,X_{n+1},M} \\ &\quad + (-1)^{n+1} f_{X_1,\ldots,X_n,X_{n+1} \rhd M}\end{aligned} \tag{1.1}$$

Here we have introduced the coface maps

$$\partial^n_{ass}[i] : \quad C^n_{ass}(F) \longrightarrow C^{n+1}_{ass}(F) \tag{1.2}$$

with

$$\partial^n_{ass}[i](f)_{X_1,\ldots,X_{n+1},M} := \begin{cases} \mathrm{id}_{X_1} \rhd f_{X_2,\ldots,X_{n+1},M} & \text{if } \ i = 0. \\ f_{X_1,\ldots,X_i \otimes X_{i+1},\ldots,X_{n+1},M} & \text{if } \ 1 \leq i \leq n. \\ f_{X_1,\ldots,X_n,X_{n+1} \rhd M} & \text{if } \ i = n+1. \end{cases}$$

Lemma 1.6. *The coface maps (1.2) satisfy the cosimplicial relations, which implies that ∂_{ass} is a differential.*

Proof:
We have to show that the equation

$$\partial^{n+1}_{ass}[j] \circ \partial^n_{ass}[i] = \partial^{n+1}_{ass}[i] \circ \partial^n_{ass}[j-1]$$

holds for all $n \geq 0$ and $0 \leq i < j \leq n+2$. Let $X_1, \ldots, X_{n+2} \in \mathcal{C}$, $M \in \mathcal{M}$ and $f \in C^n_{ass}(F)$. We consider five cases:

- $j = n+2, i = n+1$:

$$\begin{aligned}
&\big((\partial^{n+1}_{ass}[n+2] \circ \partial^n_{ass}[n+1])(f)\big)_{X_1,\ldots,X_{n+2},M}\\
=&\big(\partial^n_{ass}[n+1](f)\big)_{X_1,\ldots,X_{n+1},X_{n+2}\triangleright M}\\
=&f_{X_1,\ldots,X_n,X_{n+1}\triangleright(X_{n+2}\triangleright M)}\\
=&f_{X_1,\ldots,X_n,(X_{n+1}\otimes X_{n+2})\triangleright M}\\
=&\big(\partial^n_{ass}[n+1](f)\big)_{X_1,\ldots,X_n,(X_{n+1}\otimes X_{n+2})\triangleright M}\\
=&\big((\partial^{n+1}_{ass}[n+1] \circ \partial^n_{ass}[n+1])(f)\big)_{X_1,\ldots,X_{n+2},M}
\end{aligned}$$

- $j = n+2, i = 0$:

$$\begin{aligned}
&\big((\partial^{n+1}_{ass}[n+2] \circ \partial^n_{ass}[0])(f)\big)_{X_1,\ldots,X_{n+2},M}\\
=&\big(\partial^n_{ass}[0](f)\big)_{X_1,\ldots,X_{n+1},X_{n+2}\triangleright M}\\
=&\,\mathrm{id}_{X_1} \triangleright f_{X_2,\ldots,X_{n+1},X_{n+2}\triangleright M}\\
=&\,\mathrm{id}_{X_1} \triangleright (\partial^n_{ass}[n+1](f))_{X,2,\ldots,X_{n+2},M}\\
=&\big((\partial^{n+1}_{ass}[0] \circ \partial^n_{ass}[n+1])(f)\big)_{X_1,\ldots,X_{n+2},M}
\end{aligned}$$

- $0 < j < n+2, i = 0$:

$$\begin{aligned}
&\big((\partial^{n+1}_{ass}[j] \circ \partial^n_{ass}[0])(f)\big)_{X_1,\ldots,X_{n+2},M}\\
=&\big(\partial^n_{ass}[0](f)\big)_{X_1,\ldots,X_i\otimes X_{i+1},\ldots,X_{n+2},M}\\
=&\begin{cases}\mathrm{id}_{X_1\otimes X_2} \triangleright f_{X_3,\ldots,X_{n+2}\triangleright M} & \text{if} \quad j=1\\ \mathrm{id}_{X_1} \triangleright f_{X_2,\ldots,X_j\otimes X_{j+1},\ldots,X_{n+2},M} & \text{if} \quad 1<j<n+2\end{cases}\\
=&\,\mathrm{id}_{X_1} \triangleright \big(\partial^n_{ass}[j-1](f)\big)_{X_2,\ldots,X_{n+2},M}\\
=&\big((\partial^{n+1}_{ass}[0] \circ \partial^n_{ass}[j-1])(f)\big)_{X_1,\ldots,X_{n+2},M}
\end{aligned}$$

- $j = n+2, 0 < i < n+1$:

$$\begin{aligned}
&\big((\partial^{n+1}_{ass}[n+2] \circ \partial^n_{ass}[i])(f)\big)_{X_1,\ldots,X_{n+2},M}\\
=&\big(\partial^n_{ass}[i](f)\big)_{X_1,\ldots,X_{n+1},X_{n+2}\triangleright M}\\
=&f_{X_1,\ldots,X_i\otimes X_{i+1},\ldots X_{n+1},X_{n+2}\triangleright M}\\
=&\big(\partial^n_{ass}[n+1](f)\big)_{X_1,\ldots,X_i\otimes X_{i+1},\ldots,X_{n+2},M}\\
=&\big((\partial^{n+1}_{ass}[i] \circ \partial^n_{ass}[n+1])(f)\big)_{X_1,\ldots,X_{n+2},M}
\end{aligned}$$

- $0 < i < j < n+2$:

$$\begin{aligned}
&\big((\partial^{n+1}_{ass}[j] \circ \partial^n_{ass}[i])(f)\big)_{X_1,\ldots,X_{n+2},M}\\
=&\big(\partial^n_{ass}[i](f)\big)_{X_1,\ldots,X_j\otimes X_{j+1},\ldots,X_{n+2},M}\\
=&\begin{cases}f_{X_1,\ldots,X_i\otimes X_{i+1},\ldots,X_j\otimes X_{j+1},\ldots,X_{n+2},M} & \text{if} \quad i+1<j\\ f_{X_1,\ldots,X_i\otimes(X_j\otimes X_{j+1}),\ldots,X_{n+2},M} & \text{if} \quad i+1=j\end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \big(\partial^n_{ass}[j-1](f)\big)_{X_1,\dots,X_i\otimes X_{i+1},\dots,X_{n+2},M} \\
&= \big((\partial^{n+1}_{ass}[i]\circ\partial^n_{ass}[j-1])(f)\big)_{X_1,\dots,X_{n+2},M}
\end{aligned}
$$

□

The following definition introduces the central new notion of this thesis (see Definition B.9 for the case where the ingredients are not necessarily strict):

Definition 1.7. Let $\mathcal{C}$ be a k-linear strict monoidal category and let $\mathcal{M},\mathcal{M}'$ be k-linear strict $\mathcal{C}$-module categories.

1. For a k-linear strict $\mathcal{C}$-module functor $F:\mathcal{M}\longrightarrow\mathcal{M}'$, the cochain complex

$$\big(C^\bullet_{ass}(F),\partial_{ass}\big)$$

of k-vector spaces is called the *associator (deformation) complex of F*. The n-cocycles of this complex are denoted by $Z^n_{ass}(F)$ and the nth cohomology group is denoted by $H^n_{ass}(F)$.

2. In the case $F=\mathrm{Id}_{\mathcal{M}}$, we use the notation

$$\big(C^\bullet_{ass}(\mathcal{C},\mathcal{M}),\partial_{ass}\big)$$

and the complex is called the *associator (deformation) complex of $\mathcal{M}$ over $\mathcal{C}$*; its n-cocycles are denoted by $Z^n_{ass}(\mathcal{C},\mathcal{M})$ and its nth cohomology, $H^n_{ass}(\mathcal{C},\mathcal{M})$, is called the *$n$th associator (deformation) cohomology of $\mathcal{M}$ over $\mathcal{C}$*.

1.2 Second associator cohomology

As is standard in the discussion of cohomology theories, we study the interpretation of the associator deformation cohomology groups in low degrees. We start our investigation of the associator deformation complex of $\mathcal{M}$ over $\mathcal{C}$ (none of which are necessarily strict) by showing that its second cohomology group controls associator deformations up to equivalence, justifying the name of the complex.

Theorem 1.8. *Let $\mathcal{M}$ be a k-linear $\mathcal{C}$-module. A 2-cocycle $m^{(1)}\in Z^2_{ass}(\mathcal{C},\mathcal{M})$ corresponds to the associator deformation $\big(R_1\otimes\mathcal{M},m^{(1)}\big)$. Associator deformations of $\mathcal{M}$ over $\mathcal{C}$ up to equivalence in the sense of Definition 1.2 correspond to classes in $H^2_{ass}(\mathcal{C},\mathcal{M})$.*

Proof:
We divide the proof into two steps, and we will use the differential in the non-strict form as stated in equations (B.23) and (B.24). We set

$$\mathbf{m}_{X,Y,M}=m_{X,Y,M}+\varepsilon m^{(1)}_{X,Y,M}$$

for all $X,Y\in\mathcal{C}$, $M\in\mathcal{M}$. We first show that $\big(R_1\otimes\mathcal{M},m^{(1)}\big)$ is an associator deformation of $\mathcal{M}$ if and only if $m^{(1)}\in Z^2_{ass}(\mathcal{C},\mathcal{M})$: the pentagon axiom (A.6) for $\big(R_1\otimes\mathcal{M},m^{(1)}\big)$ holds if and only if the equation

$$\mathbf{m}_{X,Y,Z\triangleright M}\circ\mathbf{m}_{X\otimes Y,Z,M}=(\mathrm{id}_X\triangleright\mathbf{m}_{Y,Z,M})\circ\mathbf{m}_{X,Y\otimes Z,M}\circ(a_{X,Y,Z}\triangleright\mathrm{id}_M) \tag{1.3}$$

holds for all $X, Y, Z \in \mathcal{C}, M \in \mathcal{M}$. Using the pentagon axiom for $\mathcal{M}$ and comparing first order terms, equation (1.3) is equivalent to

$$\begin{aligned}0 =&(\mathrm{id}_X \triangleright m^{(1)}_{Y,Z,M}) \circ m_{X,Y\otimes Z,M} \circ (a_{X,Y,Z} \triangleright \mathrm{id}_M) - m_{X,Y,Z\triangleright M} \circ m^{(1)}_{X\otimes Y,Z,M}\\ &+ (\mathrm{id}_X \triangleright m_{Y,Z,M}) \circ m^{(1)}_{X,Y\otimes Z,M} \circ (a_{X,Y,Z} \triangleright \mathrm{id}_M) - m^{(1)}_{X,Y,Z\triangleright M} \circ m_{X\otimes Y,Z,M}\\ =&\partial^2_{ass}(m^{(1)})_{X,Y,Z,M}\end{aligned}$$

which proves the claim.

We now show that cohomologous 2-cocycles give equivalent associator deformations, i.e., $(R_1 \otimes \mathcal{M}, m_1^{(1)})$ and $(R_1 \otimes \mathcal{M}, m_2^{(1)})$ are equivalent deformations of $\mathcal{M}$ if and only if $m_1^{(1)}$ and $m_2^{(1)}$ differ by a coboundary. Recall from Definition 1.7 that an equivalence of the deformations $(R_1 \otimes \mathcal{M}, m_1^{(1)})$ and $(R_1 \otimes \mathcal{M}, m_2^{(1)})$ is given by a $\mathcal{C}$-module equivalence of the form

$$(\mathrm{Id}_{R_1\otimes\mathcal{M}}, \mathrm{id} + \varepsilon t) : \quad (R_1 \otimes \mathcal{M}, m^{(1)}) \longrightarrow (R_1 \otimes \mathcal{M}, m_2^{(1)})$$

The pentagon axiom (A.11) for $(\mathrm{Id}_{R_1\otimes\mathcal{M}}, \mathrm{id} + \varepsilon t)$ holds if and only if the equation

$$\begin{aligned}&\big(\mathrm{id}_X \triangleright (\mathrm{id}_{Y\triangleright M} + \varepsilon t_{Y,M})\big) \circ \big(\mathrm{id}_{X\triangleright(Y\triangleright M)} + \varepsilon t_{X,Y\triangleright M}\big) \circ \big(m_{X,Y,M} + \varepsilon(m_1^{(1)})_{X,Y,M}\big)\\ =&\big(m_{X,Y,M} + \varepsilon(m_2^{(1)})_{X,Y,M}\big) \circ \big(\mathrm{id}_{(X\otimes Y)\triangleright M} + \varepsilon t_{X\otimes Y,M}\big)\end{aligned} \tag{1.4}$$

holds for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Comparing the first order terms yields that (1.4) is equivalent to

$$\begin{aligned}(m_2^{(1)})_{X,Y,M} - (m_1^{(1)})_{X,Y,M} =&\big(\mathrm{id}_X \triangleright t_{Y,M}\big) \circ m_{X,Y,M} - m_{X,Y,M} \circ t_{X\otimes Y,M} + t_{X,Y\triangleright M} \circ m_{X,Y,M}\\ =&\partial^1_{ass}(t)_{X,Y,M}\end{aligned}$$

Note that the triangle axiom holds automatically for $(\mathrm{Id}_{R_1\otimes\mathcal{M}}, \mathrm{id} + \varepsilon t)$ by Proposition A.18.

□

Let $\mathcal{M}$ be a $\mathcal{C}$-module with unitor l. According to Definition 1.1, an infinitesimal associator deformation of $\mathcal{M}$ does not need to be unital. However, in the following, we will show that any associator deformation of $\mathcal{M}$ is equivalent to an associator deformation which has the unitor l. This means that infinitesimal associator deformations preserve the unitor of a module category up to equivalence of deformations in the sense of Definition 1.2.

Lemma 1.9. *If $m^{(1)} \in Z^2_{ass}(\mathcal{C}, \mathcal{M})$ is a 2-cocycle in the associator complex, then the following identity holds for all $X \in \mathcal{C}, M \in \mathcal{M}$:*

$$\Big(\mathrm{id}_X \triangleright \big(l_M \circ (\mathrm{id}_I \triangleright l_M) \circ m^{(1)}_{I,I,M} \circ (\mathfrak{r}_I^{-1} \triangleright \mathrm{id}_M)\big)\Big) \circ m_{X,I,M} = (\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M}$$

Proof:
Since $m^{(1)} \in Z^2_{ass}(\mathcal{C},\mathcal{M})$ is a 2-cocycle, we have $\partial^2_{ass}(m^{(1)}) = 0$. For $X, Y \in \mathcal{C}, M \in \mathcal{M}$, we calculate

$$\begin{aligned}
0 =& \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ \partial^2_{ass}(m^{(1)})_{X,Y,I,M} \\
=& \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m^{(1)}_{Y,I,M}) \circ m_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ m_{X,Y,I\rhd M} \circ m^{(1)}_{X\otimes Y,I,M} \\
&+ \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m_{Y,I,M}) \circ m^{(1)}_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ m^{(1)}_{X,Y,I\rhd M} \circ m_{X\otimes Y,I,M} \\
=& \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m^{(1)}_{Y,I,M}) \circ m_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- m_{X,Y,M} \circ (\mathrm{id}_{X\otimes Y} \rhd l_M) \circ m^{(1)}_{X\otimes Y,I,M} \\
&+ \big(id_X \rhd (\mathfrak{r}_Y \rhd \mathrm{id}_M)\big) \circ m^{(1)}_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- m^{(1)}_{X,Y,M} \circ (\mathrm{id}_{X\otimes Y} \rhd l_M) \circ m_{X\otimes Y,I,M} \\
=& \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m^{(1)}_{Y,I,M}) \circ m_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- m_{X,Y,M} \circ (\mathrm{id}_{X\otimes Y} \rhd l_M) \circ m^{(1)}_{X\otimes Y,I,M} \\
&+ m^{(1)}_{X,Y,M} \circ \underbrace{\big((\mathrm{id}_X \otimes \mathfrak{r}_Y) \rhd \mathrm{id}_M\big) \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M)}_{\overset{\text{Lemma} A.7}{=} \mathfrak{r}_{X\otimes Y} \rhd \mathrm{id}_M} \\
&- m^{(1)}_{X,Y,M} \circ (\mathfrak{r}_{X\otimes Y} \rhd \mathrm{id}_M) \\
=& \big(\mathrm{id}_X \rhd (\mathrm{id}_Y \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m^{(1)}_{Y,I,M}) \circ m_{X,Y\otimes I,M} \circ (\mathfrak{a}_{X,Y,I} \rhd \mathrm{id}_M) \\
&- m_{X,Y,M} \circ (\mathrm{id}_{X\otimes Y} \rhd l_M) \circ m^{(1)}_{X\otimes Y,I,M}
\end{aligned} \tag{1.5}$$

In the third equation of the calculation above, we have used the naturality of m for the second summand and the triangle axiom (A.7) for the third summand. In the fourth equation, we have used the naturality of $m^{(1)}$ for the third summand.

We set $Y = I$ in equation (1.5) and apply $\mathrm{id}_X \rhd l_M$ from the left to obtain the following equation:

$$\begin{aligned}
0 =& (\mathrm{id}_X \rhd l_M) \circ \big(\mathrm{id}_X \rhd (\mathrm{id}_I \rhd l_M)\big) \circ (\mathrm{id}_X \rhd m^{(1)}_{I,I,M}) \circ m_{X,I\otimes I,M} \circ (\mathfrak{a}_{X,I,I} \rhd \mathrm{id}_M) \\
&- (\mathrm{id}_X \rhd l_M) \circ m_{X,I,M} \circ (\mathrm{id}_{X\otimes I} \rhd l_M) \circ m^{(1)}_{X\otimes I,I,M} \\
=& \Big(\mathrm{id}_X \rhd \big(l_M \circ (\mathrm{id}_I \rhd l_M) \circ m^{(1)}_{I,I,M}\big)\Big) \circ m_{X,I\otimes I,M} \circ \Big(\big((\mathrm{id}_X \otimes \mathfrak{r}_I^{-1}) \circ \mathfrak{r}_{X\otimes I}\big) \rhd \mathrm{id}_M\Big) \\
&- (\mathfrak{r}_X \rhd \mathrm{id}_M) \circ (\mathrm{id}_{X\otimes I} \rhd l_M) \circ m^{(1)}_{X\otimes I,I,M} \\
=& \Big(\mathrm{id}_X \rhd \big(l_M \circ (\mathrm{id}_I \rhd l_M) \circ m^{(1)}_{I,I,M}\big)\Big) \circ \big(\mathrm{id}_X \rhd (\mathfrak{r}_I^{-1} \rhd \mathrm{id}_M)\big) \circ m_{X,I,M} \circ (\mathfrak{r}_{X\otimes I} \rhd \mathrm{id}_M) \\
&- (\mathrm{id}_X \rhd l_M) \circ (\mathfrak{r}_X \rhd \mathrm{id}_{I\rhd M}) \circ m^{(1)}_{X\otimes I,I,M} \\
=& \Big(\mathrm{id}_X \rhd \big(l_M \circ (\mathrm{id}_I \rhd l_M) \circ m^{(1)}_{I,I,M} \circ (\mathfrak{r}_I^{-1} \rhd \mathrm{id}_M)\big)\Big) \circ m_{X,I,M} \circ (\mathfrak{r}_{X\otimes I} \rhd \mathrm{id}_M) \\
&- (\mathrm{id}_X \rhd l_M) \circ m^{(1)}_{X,I,M} \circ (\mathfrak{r}_{X\otimes I} \rhd \mathrm{id}_M)
\end{aligned}$$

In the second equation, we have used Lemma A.7 on the first summand and the triangle axiom (A.7) on the second summand. In the third equation, we have used the naturality of m for the first summand and the naturality of $\mathfrak{r}$ for the second summand. In the last equation, we have used the naturality of $m^{(1)}$ and the fact that $\mathfrak{r}_{X\otimes I} = \mathfrak{r}_X \otimes \mathrm{id}_I$.

The claim follows since $\mathfrak{r}$ is an isomorphism. □

Proposition 1.10. *Let $\big(R_1 \otimes \mathcal{M}, m^{(1)}\big)$ be an associator deformation of the k-linear $\mathcal{C}$-module $\mathcal{M}$. There is a unital associator deformation $\big(R_1 \otimes \mathcal{M}, \tilde{m}^{(1)}\big)$ of $\mathcal{M}$, which has the unitor l and the deformations $\big(R_1 \otimes \mathcal{M}, m^{(1)}\big)$ and $\big(R_1 \otimes \mathcal{M}, \tilde{m}^{(1)}\big)$ are equivalent in the sense of Definition 1.2.*

Proof:
Consider the 1-cochain $t^{(1)} \in C^1_{ass}(\mathcal{C}, \mathcal{M})$ with

$$t^{(1)}_{X,M} = -(\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M} \circ (\mathfrak{r}_X^{-1} \triangleright \mathrm{id}_M) : \quad X \triangleright M \longrightarrow X \triangleright M \tag{1.6}$$

for all $X \in \mathcal{C}, M \in \mathcal{M}$. By Theorem 1.8, the associator deformations $\big(R_1 \otimes \mathcal{M}, m^{(1)}\big)$ and $\big(R_1 \otimes \mathcal{M}, m^{(1)} + \partial^1_{ass}(t^{(1)})\big)$ are equivalent in the sense of Definition 1.2. What remains to be shown is that l is a left unitor for the $R_1 \otimes \mathcal{C}$-module $\big(R_1 \otimes \mathcal{M}, m^{(1)} + \partial^1_{ass}(t^{(1)})\big)$, i.e., we have to show that the equation

$$(\mathrm{id}_X \triangleright l_M) \circ \big(m_{X,I,M} + \varepsilon(m^{(1)}_{X,I,M} + \partial^1_{ass}(t^{(1)})_{X,I,M})\big) = \mathfrak{r}_X \triangleright \mathrm{id}_M \tag{1.7}$$

holds for all $X \in \mathcal{C}, M \in \mathcal{M}$. Using the triangle axiom (A.7) for the $\mathcal{C}$-module $\mathcal{M}$, equation (1.7) reduces to

$$(\mathrm{id}_X \triangleright l_M) \circ \big(m^{(1)}_{X,I,M} + \partial^1_{ass}(t^{(1)})_{X,I,M}\big) = 0$$

We use the differential in the non-strict case as stated in (B.23):

$$\begin{aligned}
&(\mathrm{id}_X \triangleright l_M) \circ \big(m^{(1)}_{X,I,M} + \partial^1_{ass}(t^{(1)})_{X,I,M}\big)\\
=&(\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M} + (\mathrm{id}_X \triangleright l_M) \circ (\mathrm{id}_X \triangleright t^{(1)}_{I,M}) \circ m_{X,I,M}\\
&- (\mathrm{id}_X \triangleright l_M) \circ m_{X,I,M} \circ t^{(1)}_{X\otimes I,M} + (\mathrm{id}_X \triangleright l_M) \circ t^{(1)}_{X,I\triangleright M} \circ m_{X,I,M}\\
=&(\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M} + (\mathrm{id}_X \triangleright l_M) \circ (\mathrm{id}_X \triangleright t^{(1)}_{I,M}) \circ m_{X,I,M}\\
&- (\mathfrak{r}_X \triangleright \mathrm{id}_M) \circ t^{(1)}_{X\otimes I,M} + t^{(1)}_{X,M} \circ (\mathrm{id}_X \triangleright l_M) \circ m_{X,I,M}\\
=&(\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M} + (\mathrm{id}_X \triangleright l_M) \circ (\mathrm{id}_X \triangleright t^{(1)}_{I,M}) \circ m_{X,I,M}\\
&- t^{(1)}_{X,M} \circ (\mathfrak{r}_X \triangleright \mathrm{id}_M) + t^{(1)}_{X,M} \circ (\mathfrak{r}_X \triangleright \mathrm{id}_M)\\
=&(\mathrm{id}_X \triangleright l_M) \circ m^{(1)}_{X,I,M} - \Big(\mathrm{id}_X \triangleright \big(l_M \circ (\mathrm{id}_I \triangleright l_M) \circ m^{(1)}_{I,I,M} \circ (\mathfrak{r}_I^{-1} \triangleright \mathrm{id}_M)\big)\Big) \circ m_{X,I,M}\\
=&0
\end{aligned}$$

Here, we have used the triangle axiom (A.7) in the second and third equation, the naturality of $t^{(1)}$ in the third equation, the definition of $t^{(1)}$ (see (1.6)) in the second to last equation and Lemma 1.9 in the last equation. □

1.3 Zeroth and first associator cohomology

As is standard, we also investigate the associator cohomology groups in lower degrees. In this subsection, we neither assume the monoidal and module categories nor the module functors to be strict. Let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{M}$ and $\mathcal{M}'$ be k-linear $\mathcal{C}$-module categories.

Proposition 1.11. *The zeroth associator cohomology of a k-linear $\mathcal{C}$-module functor $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}$ with coefficient a k-linear $\mathcal{C}$-module functor $(F',s') : \mathcal{M} \longrightarrow \mathcal{M}$ is given by the k-vector space of module transformations*

$$H^0_{ass}\big((F,s),(F',s')\big) = \mathrm{Nat}_{\mathcal{C}}\big((F,s),(F',s')\big).$$

Proof:
Recall from Definition A.19 that $\nu \in \mathrm{Nat}_{\mathcal{C}}\big((F,s),(F',s')\big)$ if and only if $\nu : F \Longrightarrow F'$ is a natural transformation and the following diagram commutes for all $X \in \mathcal{C}, M \in \mathcal{M}$:

$$\begin{array}{ccc} F(X \triangleright M) & \xrightarrow{s_{X,M}} & X \triangleright F(M) \\ \downarrow{\scriptstyle \nu_{X\triangleright M}} & & \downarrow{\scriptstyle \mathrm{id}_X \triangleright \nu_M} \\ F'(X \triangleright M) & \xrightarrow[s'_{X,M}]{} & X \triangleright F'(M) \end{array} \tag{1.8}$$

By equation (B.21), we have that $\nu \in H^0_{ass}\big((F,s),(F',s')\big) = \ker(\partial^0_{ass})$ amounts to

$$\partial^0_{ass}(\nu)_{X,M} = (\mathrm{id}_X \triangleright \nu_M) \circ s_{X,M} - s'_{X,M} \circ \nu_{X\triangleright M} = 0$$

for all $X \in \mathcal{C}, M \in \mathcal{M}$. This is equivalent to the commutativity of diagram (1.8) which proves the claim.

□

Proposition 1.12. *Let $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}$ be a k-linear $\mathcal{C}$-module functor. A 1-cocycle $s^{(1)} \in Z^1_{ass}(F,s)$ corresponds to the infinitesimal deformation $(F,\mathbf{s})$ with*

$$\mathbf{s}_{X,M} = s_{X,M} + \varepsilon s^{(1)}_{X,M}$$

Module functor deformations of (F,s) up to equivalence in the sense of Definition 1.4 correspond to classes in $H^1_{ass}(F,s)$.

Proof:
We divide the proof into two parts, and we will use the differential in the non-strict form as stated in equations (B.21) and (B.22) for $(F,s) = (F',s')$. We first show that $(F,\mathbf{s})$ with $\mathbf{s}_{X,M} = s_{X,M} + \varepsilon s^{(1)}_{X,M}$ is an infinitesimal deformation of the $\mathcal{C}$-module functor (F,s) if and only if $s^{(1)} \in Z^1_{ass}(F,s)$. The pentagon axiom (A.11) for $(F,\mathbf{s})$ holds if and only if the equation

$$m_{X,Y,M} \circ \mathbf{s}_{X\otimes Y,M} = (\mathrm{id}_X \triangleright \mathbf{s}_{Y,M}) \circ \mathbf{s}_{X,Y\triangleright M} \circ F(m_{X,Y,M})$$

holds for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Using the pentagon axiom for the $\mathcal{C}$-module functor (F, s) in zeroth-order terms, this is equivalent to

$$\begin{aligned} 0 =& \big(\mathrm{id}_X \triangleright s^{(1)}_{Y,M}\big) \circ s_{X,Y \triangleright M} \circ F(m_{X,Y,M}) - m_{X,Y,F(M)} \circ s^{(1)}_{X \otimes Y,M} \\ &+ (\mathrm{id}_X \triangleright s_{Y,M}) \circ s^{(1)}_{X,Y \triangleright M} \circ F(m_{X,Y,M}) \\ \overset{\text{def}}{=}& \partial^1_{ass}\big(s^{(1)}\big)_{X,Y,M} \end{aligned}$$

which means $s^{(1)} \in Z^1_{ass}(F, s)$. Due to Proposition A.18, the triangle axiom holds for $(F, \mathbf{s})$.

Now we show that two deformations $(F, \mathbf{s}_1)$ and $(F, \mathbf{s}_2)$ of the $\mathcal{C}$-module functor (F, s) are equivalent via a natural isomorphism

$$\mathrm{id} + \varepsilon\nu : (F, \mathbf{s}) \Longrightarrow (F, \mathbf{s}') \tag{1.9}$$

for some $\nu \in \mathrm{End}(F)$ if and only if $s^{(1)}_1 + \partial^0_{ass}(\nu) = s^{(1)}_2$. Recall from Definition A.19 that (1.9) is a morphism of $R_1 \otimes \mathcal{C}$-module functors if and only if the equation

$$(\mathbf{s}_2)_{X,M} \circ (\mathrm{id}_{X \triangleright M} + \varepsilon\nu_{X \triangleright M}) = \big(\mathrm{id}_X \triangleright (\mathrm{id}_M + \varepsilon\nu_M)\big) \circ (\mathbf{s}_1)_{X,M} \tag{1.10}$$

holds for all $X \in \mathcal{C}, M \in \mathcal{M}$. A calculation shows that equation (1.10) is equivalent to

$$\big(s^{(1)}_1\big)_{X,M} + \underbrace{(\mathrm{id}_X \triangleright \nu_M) \circ s_{X,M} - s_{X,M} \circ \nu_{X \triangleright M}}_{=\partial^0_{ass}(\nu)_{X,M}} = \big(s^{(1)}_2\big)_{X,M}$$

which concludes the proof.

□

The interpretation of the cohomology group $H^1_{ass}\big((F, s), (F', s')\big)$ for *different* $\mathcal{C}$-module functors $(F, s), (F', s') : \mathcal{M} \longrightarrow \mathcal{M}'$ in terms of deformation theory is still unclear.

1.4 Higher order deformations and obstructions

The next question is whether deformations from Definition 1.1 and 1.3 can be extended to higher orders. We will see that obstructions to extending associator deformations of $\mathcal{M}$ over $\mathcal{C}$ live in $H^3_{ass}(\mathcal{C}, \mathcal{M})$ and obstructions to extending module functor deformations of $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}$ live in $H^2_{ass}(F, s)$.

We start with the following definitions, adapting the standard definitions to our problem:

Definition 1.13. For $n \geq 1$, an *nth order associator deformation* of a k-linear module category $\mathcal{M}$ over a k-linear monoidal category $\mathcal{C}$ is an R_n-linear $R_n \otimes \mathcal{C}$-module category such that: the underlying R_n-linear category is $R_n \otimes \mathcal{M}$, the action is the induced R_n-linear functor

$$\triangleright : \quad (R_n \otimes \mathcal{C}) \times (R_n \otimes \mathcal{M}) \longrightarrow R_n \otimes \mathcal{M}$$

and the associator is of the form

$$\mathbf{m}_{X,Y,M} = m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M} + \ldots + \varepsilon^n m^{(n)}_{X,Y,M}$$

where $m^{(i)}$ is a natural family of morphisms $m^{(i)}_{X,Y,M} : (X \otimes Y) \triangleright M \longrightarrow X \triangleright (Y \triangleright M)$ in $\mathcal{M}$ for $1 \leq i \leq n$. To the case $m^{(1)} = \ldots = m^{(n)} = 0$, we refer as the *trivial nth order associator deformation* of $\mathcal{M}$ and we denote it by $(R_n \otimes \mathcal{M}, 0)$.

In particular, the infinitesimal associator deformations from Definition 1.1 are first order associator deformations.

Definition 1.14. An *extension* of an nth order associator deformation

$$m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M} + \ldots + \varepsilon^n m^{(n)}_{X,Y,M}$$

to $n+1$st order is an $n+1$st order associator deformation of the form

$$m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M} + \ldots + \varepsilon^n m^{(n)}_{X,Y,M} + \varepsilon^{n+1} m^{(n+1)}_{X,Y,M}$$

where $m^{(n+1)}$ is a natural transformation with components $m^{(n+1)}_{X,Y,M} : (X \otimes Y) \triangleright M \longrightarrow X \triangleright (Y \triangleright M)$.

The next proposition solves the problem of extending associator deformations to higher orders:

Proposition 1.15. *If $H^3_{ass}(\mathcal{C}, \mathcal{M}) \cong 0$, then any nth order associator deformation can be extended to an $n+1$st order associator deformation.*

Proof:
Let $\hat{\mathbf{m}}$ with

$$\hat{\mathbf{m}}_{X,Y,M} = m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M} + \ldots + \varepsilon^n m^{(n)}_{X,Y,M}$$

be an nth order associator deformation of $\mathcal{M}$ and let $m^{(n+1)}$ be a natural transformation with components $m^{(n+1)}_{X,Y,M} : (X \otimes Y) \triangleright M \longrightarrow X \triangleright (Y \triangleright M)$. The natural transformation $\mathbf{m}$ with

$$\mathbf{m}_{X,Y,M} = m_{X,Y,M} + \varepsilon m^{(1)}_{X,Y,M} + \ldots + \varepsilon^{n+1} m^{(n+1)}_{X,Y,M}$$

is an extension of $\hat{\mathbf{m}}$ to $n+1$st order if and only if the pentagon axiom

$$\big(\mathrm{id}_X \triangleright \mathbf{m}_{Y,Z,M}\big) \circ \mathbf{m}_{X,Y\otimes Z,M} \circ \big(\mathfrak{a}_{X,Y,Z} \triangleright \mathrm{id}_M\big) = \mathbf{m}_{X,Y,Z\triangleright M} \circ \mathbf{m}_{X\otimes Y,Z,M}$$

holds for all $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$, which is equivalent to

$$\begin{aligned}
& m_{X,Y,Z\triangleright M} \circ m_{X\otimes Y,Z,M} + \sum_{j=1}^{n+1} \varepsilon^j m_{X,Y,Z\triangleright M} \circ m^{(j)}_{X\otimes Y,Z,M} + \sum_{i=1}^{n+1} \varepsilon^i m^{(i)}_{X,Y,Z\triangleright M} \circ m_{X\otimes Y,Z,M} \\
& + \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \varepsilon^{i+j} m^{(i)}_{X,Y,Z\triangleright M} \circ m^{(j)}_{X\otimes Y,Z,M} \\
=& \big(\mathrm{id}_X \triangleright m_{Y,Z,M}\big) \circ m_{X,Y\otimes Z,M} \circ \big(\mathfrak{a}_{X,Y,Z} \triangleright \mathrm{id}_M\big)
\end{aligned}$$

$$+\sum_{j=1}^{n+1}\varepsilon^j\big(\mathrm{id}_X\rhd m_{Y,Z,M}\big)\circ m^{(j)}_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big)$$

$$+\sum_{i=1}^{n+1}\varepsilon^i\big(\mathrm{id}_X\rhd m^{(i)}_{Y,Z,M}\big)\circ m_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big)$$

$$+\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}\varepsilon^{i+j}\big(\mathrm{id}_X\rhd m^{(i)}_{Y,Z,M}\big)\circ m^{(j)}_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big) \tag{1.11}$$

The terms of degree lower than $n+1$ on the left-hand side and right-hand side of (1.11) agree since $\hat{\mathbf{m}}$ satisfies the pentagon axiom as an nth order associator deformation. Comparing $n+1$st order terms in (1.11), we obtain

$$m_{X,Y,Z\rhd M}\circ m^{(n+1)}_{X\otimes Y,Z,M}+m^{(n+1)}_{X,Y,Z\rhd M}\circ m_{X\otimes Y,Z,M}+\sum_{i+j=n+1}m^{(i)}_{X,Y,Z\rhd M}\circ m^{(j)}_{X\otimes Y,Z,M}$$
$$=\big(\mathrm{id}_X\rhd m_{Y,Z,M}\big)\circ m^{(n+1)}_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big)$$
$$+\big(\mathrm{id}_X\rhd m^{(n+1)}_{Y,Z,M}\big)\circ m_{X,Y\otimes Z,M}\circ\big(a_{X,Y,Z}\rhd\mathrm{id}_M\big)$$
$$+\sum_{i+j=n+1}\big(\mathrm{id}_X\rhd m^{(i)}_{Y,Z,M}\big)\circ m^{(j)}_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big)$$

which is equivalent to

$$\partial^2_{ass}\big(m^{(n+1)}\big)_{X,Y,Z,M}$$
$$=\sum_{i+j=n+1}\Big[m^{(i)}_{X,Y,Z\rhd M}\circ m^{(j)}_{X\otimes Y,Z,M}-\big(\mathrm{id}_X\rhd m^{(i)}_{Y,Z,M}\big)\circ m^{(j)}_{X,Y\otimes Z,M}\circ\big(\mathfrak{a}_{X,Y,Z}\rhd\mathrm{id}_M\big)\Big]$$

for all $X,Y,Z\in\mathcal{C}, M\in\mathcal{M}$, where we have used the differential in the non-strict case as stated in (B.24). □

In a similar vein, we introduce higher order deformations of module functors:

Definition 1.16. Let $\mathcal{M}$ and $\mathcal{M}'$ be k-linear $\mathcal{C}$-module categories. An *nth order deformation* of a k-linear $\mathcal{C}$-module functor $(F,s):\mathcal{M}\longrightarrow\mathcal{M}'$ is the $R_n\otimes\mathcal{C}$-module functor

$$(F,\mathbf{s}):\quad (R_n\otimes\mathcal{M},0)\longrightarrow(R_n\otimes\mathcal{M}',0)$$

where $\mathbf{s}=(\mathbf{s}_{X,M})_{X\in\mathcal{C},M\in\mathcal{M}}$ is of the form

$$\mathbf{s}_{X,M}=s_{X,M}+\varepsilon s^{(1)}_{X,M}+\ldots+\varepsilon^n s^{(n)}_{X,M}$$

and $s^{(i)}=\big(s^{(i)}_{X,M}\big)_{X\in\mathcal{C},M\in\mathcal{M}}$ is a natural family of morphisms $s^{(i)}_{X,M}:F(X\rhd M)\longrightarrow X\rhd\ F(M)$ for all $1\le i\le n$.

Definition 1.17. An *extension* of an nth order deformation

$$s_{X,M} + \varepsilon s^{(1)}_{X,M} + \ldots + \varepsilon^n s^{(n)}_{X,M}$$

of a $\mathcal{C}$-module functor $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}'$ to $n+1$st order is an $n+1$st order deformation of the $\mathcal{C}$-module functor $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}'$, which is of the form

$$s_{X,M} + \varepsilon s^{(1)}_{X,M} + \ldots + \varepsilon^n s^{(n)}_{X,M} + \varepsilon^{n+1} s^{(n+1)}_{X,M}$$

where $s^{(n+1)}$ is a natural transformation with components $s^{(n+1)}_{X,M} : F(X \triangleright M) \longrightarrow X \triangleright F(M)$.

The next proposition solves the extension problem for deformations of module functors:

Proposition 1.18. *If $H^2_{ass}(F,s) \cong 0$, then any nth order deformation of the k-linear $\mathcal{C}$-module functor $(F,s) : \mathcal{M} \longrightarrow \mathcal{M}$ can be extended to an $n+1$st order deformation.*

Proof:
Let $(F, \hat{\mathbf{s}})$ be an nth order module functor deformation of (F,s) with

$$\hat{\mathbf{s}}_{X,M} = s_{X,M} + \sum_{i=1}^{n} \varepsilon^i s^{(i)}_{X,M}$$

and let $s^{(n+1)}$ be a natural transformation with components $s^{(n+1)}_{X,M} : F(X \triangleright M) \longrightarrow X \triangleright F(M)$. We set

$$\mathbf{s}_{X,M} = s_{X,M} + \sum_{i=1}^{n+1} \varepsilon^i s^{(i)}_{X,M}$$

and thus, $(F, \mathbf{s})$ is an extension of $(F, \hat{\mathbf{s}})$ to $n+1$st order if and only if the pentagon axiom for $(F, \mathbf{s})$ holds, i.e., if and only if

$$m_{X,Y,F(M)} \circ \mathbf{s}_{X \otimes Y, M} = (\mathrm{id}_X \triangleright \mathbf{s}_{Y,M}) \circ \mathbf{s}_{X, Y \triangleright M} \circ F(m_{X,Y,M})$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. This is equivalent to the following equation:

$$\begin{aligned} & m_{X,Y,F(M)} \circ s_{X \otimes Y, M} + \sum_{i=1}^{n+1} \varepsilon^i m_{X,Y,F(M)} \circ s^{(i)}_{X \otimes Y, M} \\ =& \big(\mathrm{id}_X \triangleright s_{Y,M}\big) \circ s_{X, Y \triangleright M} \circ F(m_{X,Y,M}) \\ &+ \sum_{j=1}^{n+1} \varepsilon^j \big(\mathrm{id}_X \triangleright s_{Y,M}\big) \circ s^{(j)}_{X, Y \triangleright M} \circ F(m_{X,Y,M}) \\ &+ \sum_{i=1}^{n+1} \varepsilon^i \big(\mathrm{id}_X \triangleright s^{(i)}_{Y,M}\big) \circ s_{X, Y \triangleright M} \circ F(m_{X,Y,M}) \\ &+ \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \varepsilon^{i+j} \big(\mathrm{id}_X \triangleright s^{(i)}_{Y,M}\big) \circ s^{(j)}_{X, Y \triangleright M} \circ F(m_{X,Y,M}) \end{aligned} \tag{1.12}$$

The terms of degree lower than $n+1$ on the left-hand side and right-hand side of (1.12) agree since $\hat{\mathbf{s}}$ satisfies the pentagon axiom as $(F, \hat{\mathbf{s}})$ is an nth order module functor deformation by assumption. Comparing $n+1$st order terms in (1.12), we obtain

$$
\begin{aligned}
&m_{X,Y,F(M)} \circ s^{(n+1)}_{X\otimes Y,M}\\
=&\big(\mathrm{id}_X \triangleright s_{Y,M}\big) \circ s^{(n+1)}_{X,Y\triangleright M} \circ F\big(m_{X,Y,M}\big) + \big(\mathrm{id}_X \triangleright s^{(n+1)}_{Y,M}\big) \circ s_{X,Y\triangleright M} \circ F(m_{X,Y,M})\\
&+ \sum_{i+j=n+1} \big(\mathrm{id}_X \triangleright s^{(i)}_{Y,M}\big) \circ s^{(j)}_{X,Y\triangleright M} \circ F(m_{X,Y,M})
\end{aligned}
$$

which is equivalent to

$$
\partial^1_{ass}\big(s^{(n+1)}\big)_{X,Y,M} = -\sum_{i+j=n+1} \big(\mathrm{id}_X \triangleright s^{(i)}_{Y,M}\big) \circ s^{(j)}_{X,Y\triangleright M} \circ F(m_{X,Y,M})
$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$.

□

These results show that the associator deformation complex, at least in low degrees, describes aspects of the deformation theory of mixed associators and module functors according to the standard paradigm.

2 Relation to the Davydov-Yetter complex

Given the importance of the associator deformation complex for the problem of deformations in the sense of Definition 1.1 and 1.3, we are looking for explicit tools to compute it. The relation to Davydov-Yetter cohomology, presented in this section, provides such tools since Davydov-Yetter cohomology is, by now, a well-studied theory (see, for instance, [GHS] and [FGS1]).

2.1 The Davydov-Yetter complex

We give a brief review of the Davydov-Yetter complex of a strict monoidal functor between strict monoidal categories. For the discussion of the non-strict case with coefficients, we refer the reader to appendix B.1.

In this section, let $\mathcal{C}$ and $\mathcal{D}$ be strict k-linear monoidal categories. By abuse of notation, we denote both the tensor product in $\mathcal{C}$ as well the tensor product in $\mathcal{D}$ by $\otimes$. Recall that a monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ consists of a k-linear functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ together with a natural family $\Phi = (\Phi_{X,Y})_{X,Y \in \mathcal{C}}$ of isomorphisms

$$\Phi_{X,Y} : G(X) \otimes G(Y) \xrightarrow{\cong} G(X \otimes Y)$$

and an isomorphism $\varphi : I \xrightarrow{\cong} G(I)$, satisfying coherence conditions (see Definition A.9). A monoidal functor is called strict, if Φ and φ are the identity, in which case they will be suppressed in our notation.

In the remainder of this section, let $G : \mathcal{C} \longrightarrow \mathcal{D}$ be a k-linear strict monoidal functor. For $n \geq 1$, we define the k-multilinear functors

$$\begin{aligned} G^{\times n} : \quad \mathcal{C}^{\times n} &\longrightarrow \mathcal{D}^{\times n} \\ (X_1, \ldots, X_n) &\longrightarrow \big(G(X_1), \ldots, G(X_n)\big) \end{aligned}$$

$$\begin{aligned} \overset{n}{\otimes} : \quad \mathcal{D}^{\times n} &\longrightarrow \mathcal{D} \\ (Y_1, \ldots, Y_n) &\longrightarrow Y_1 \otimes \ldots \otimes Y_n \end{aligned}$$

Recall that a *half-braiding for an object $X \in \mathcal{D}$ relative to a strict monoidal functor* $G : \mathcal{C} \longrightarrow \mathcal{D}$ is a natural isomorphism $\sigma^X : X \otimes G \Longrightarrow G \otimes X$ such that the diagram

$$\begin{array}{ccc} X \otimes G(V) \otimes G(W) & \xrightarrow{\sigma^X_{V \otimes W}} & G(V) \otimes X \otimes G(W) \\ \big\downarrow{\scriptstyle \sigma^X_V \otimes \mathrm{id}_{G(W)}} & \nearrow{\scriptstyle \mathrm{id}_{G(V)} \otimes \sigma^X_W} & \\ G(V) \otimes X \otimes G(W) & & \end{array}$$

commutes for all $V, W \in \mathcal{C}$ (see also Definition B.1 in the non-strict case). We can now define *the centralizer $\mathcal{Z}(G)$ of a monoidal functor G* which is the category whose objects are pairs (X, σ^X) consisting of an object $X \in \mathcal{D}$ and a half-braiding σ^X relative to G. Morphisms $g : (X, \sigma^X) \longrightarrow (Y, \sigma^Y)$ in $\mathcal{Z}(G)$ are morphisms $g : X \longrightarrow Y$ in $\mathcal{D}$, such that the following diagram commutes for all $V \in \mathcal{C}$:

$$\begin{array}{ccc} X \otimes G(V) & \xrightarrow{\sigma^X_V} & G(V) \otimes X \\ {\scriptstyle g\otimes \mathrm{id}_{G(V)}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G(V)}\otimes g} \\ Y \otimes G(V) & \xrightarrow[\sigma^Y_V]{} & G(V) \otimes Y \end{array} \tag{2.1}$$

The centralizer $\mathcal{Z}(G)$ is a monoidal category with the tensor product

$$(X, \sigma^X) \otimes (Y, \sigma^Y) = (X \otimes Y, \sigma^{X\otimes Y})$$

where $\sigma^{X\otimes Y} : X \otimes Y \otimes G \Longrightarrow G \otimes X \otimes Y$ is the half-braiding whose components are

$$\sigma^{X\otimes Y}_V := (\sigma^X_V \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma^Y_V) \tag{2.2}$$

for all $V \in \mathcal{C}$.

Example 2.1. *On any monoidal category $\mathcal{C}$, the identity functor $\mathrm{Id}_\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}$ can be endowed with the structure of a strict monoidal functor. The centralizer of the identity functor with this monoidal structure is the Drinfeld center of the monoidal category, i.e., $\mathcal{Z}(\mathrm{Id}_\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.*

It is very natural to consider coefficients for the Davydov-Yetter complex, as introduced in [GHS, Def. 3.3], that live in the centralizer $\mathcal{Z}(G)$. The *Davydov-Yetter complex* $(C^\bullet_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}), \partial_{\mathrm{DY}})$ of a strict k-linear monoidal functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ with coefficients $\mathrm{X} = (X, \sigma^X), \mathrm{Y} = (Y, \sigma^Y) \in \mathcal{Z}(G)$ has as its nth cochain space the k-vector space

$$C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) := \begin{cases} \mathrm{Hom}_\mathcal{D}(X, Y) & \text{if } \ n = 0. \\ \mathrm{Nat}\big(X \otimes (\overset{n}{\otimes} \circ G^{\times n}), (\overset{n}{\otimes} \circ G^{\times n}) \otimes Y\big) & \text{if } \ n \geq 1. \end{cases}$$

Explicitly, for $n \geq 1$, the nth cochain space $C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$ consists of natural transformations $b = (b_{X_1,\ldots,X_n})_{X_1,\ldots,X_n\in\mathcal{C}}$ whose components are morphisms in $\mathcal{D}$ of the form

$$b_{X_1,\ldots,X_n} : \quad X \otimes G(X_1) \otimes \ldots \otimes G(X_n) \longrightarrow G(X_1) \otimes \ldots \otimes G(X_n) \otimes Y$$

For $n \geq 1$, the differential $\partial^n_{\mathrm{DY}} : C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \longrightarrow C^{n+1}_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$ is defined as the alternating sum

$$\begin{aligned} \partial^n_{\mathrm{DY}}(b)_{X_1,\ldots,X_{n+1}} =& \sum_{i=0}^{n+1} \partial^n_{\mathrm{DY}}[i](b)_{X_1,\ldots,X_{n+1}} \\ =& \big(\mathrm{id}_{G(X_1)} \otimes b_{X_2,\ldots,X_{n+1}}\big) \circ \big(\sigma^X_{X_1} \otimes \mathrm{id}_{G(X_2)\ldots G(X_{n+1})}\big) \\ &+ \sum_{i=1}^{n} (-1)^i b_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \\ &+ (-1)^{n+1} \big(\mathrm{id}_{G(X_1)\ldots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \circ \big(b_{X_1,\ldots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big) \end{aligned}$$

of the coface maps

$$\partial^n_{\mathrm{DY}}[0](b)_{X_1,\dots,X_{n+1}} := \big(\mathrm{id}_{G(X_1)} \otimes b_{X_2,\dots,X_{n+1}}\big) \circ \big(\sigma^X_{X_1} \otimes \mathrm{id}_{G(X_2)\dots G(X_{n+1})}\big)$$

$$\partial^n_{\mathrm{DY}}[i](b)_{X_1,\dots,X_{n+1}} := b_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}}$$

$$\partial^n_{\mathrm{DY}}[n+1](b)_{X_1,\dots,X_{n+1}} := \big(\mathrm{id}_{G(X_1)\dots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \circ \big(b_{X_1,\dots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big)$$

Note that here and in the remainder of the thesis, we will often omit tensor products that appear in indices, i.e., we write f_{XY} instead of $f_{X\otimes Y}$ etc.
For $n = 0$, the differential is defined as

$$\partial^0_{\mathrm{DY}}(b)_{X_1} = \big(\mathrm{id}_{G(X_1)} \otimes b\big) \circ \sigma^X_{X_1} - \sigma^Y_{X_1} \circ \big(b \otimes \mathrm{id}_{G(X_1)}\big)$$

In case of trivial coefficients, i.e., $(X, \sigma^X) = (Y, \sigma^Y) = (I, \mathrm{id})$, we use the shorthand notation $C^\bullet_{\mathrm{DY}}(G)$ for the Davydov-Yetter complex of G.

The Davydov-Yetter cohomology groups in low degrees have interpretations in terms of the deformation of monoidal functors and monoidal categories (see, e.g., [GHS, Remark 3.7]):

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. An *infinitesimal deformation* of a monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ is a monoidal functor from $R_1 \otimes \mathcal{C}$ to $R_1 \otimes \mathcal{D}$ whose underlying functor is the induced R_1-linear functor G and whose monoidal structure has the components

$$\Phi_{X,Y} + \varepsilon\Phi^{(1)}_{X,Y} : \quad G(X) \otimes G(Y) \xrightarrow{\cong} G(X \otimes Y)$$

where $\Phi^{(1)}$ is a natural family of morphisms $\Phi^{(1)}_{X,M} : G(X) \otimes G(Y) \longrightarrow G(X \otimes Y)$ in $\mathcal{D}$. Two infinitesimal deformations of G are called *equivalent*, if there is an isomorphism of monoidal functors between them, which is of the form $\mathrm{id} + \varepsilon\nu$ where ν is a natural family of morphisms $\nu_X : G(X) \longrightarrow G(X)$ in $\mathcal{D}$.

Proposition 2.3. *Let $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ be a k-linear monoidal functor. A 2-cocycle $\Phi^{(1)} \in Z^2_{\mathrm{DY}}(G)$ corresponds to the infinitesimal deformation of the monoidal structure of G which has the components*

$$\Phi_{X,Y} + \varepsilon\Phi^{(1)}_{X,Y} : \quad G(X) \otimes G(Y) \xrightarrow{\cong} G(X \otimes Y)$$

Deformations of the monoidal structure of G up to equivalence in the sense of Definition 2.2 correspond to classes in $H^2_{ass}(G)$.

We conclude this section with an interpretation of the first Davydov-Yetter cohomology group with coefficients in terms of deformation theory, which was previously unknown in the literature:

Definition 2.4. Let $\mathcal{C}$ be a not necessarily strict k-linear monoidal category and let $\mathcal{D}$ be a k-linear strict monoidal category. Let $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ be a k-linear monoidal functor and let $(X, \sigma) \in \mathcal{Z}(G)$.

1. An *infinitesimal deformation of a half-braiding* $\sigma : X \otimes G \Longrightarrow G \otimes X$ *for an object* $X \in \mathcal{D}$ *relative to* G is a half-braiding $\boldsymbol{\sigma} : X \otimes G \Longrightarrow G \otimes X$ with components

$$\boldsymbol{\sigma}_V = \sigma_V + \varepsilon \tilde{\sigma}_V$$

where $\tilde{\sigma}_V \in \mathrm{Hom}_{\mathcal{D}}\big(X \otimes G(V), G(V) \otimes X\big)$, i.e., $(X, \boldsymbol{\sigma}) \in R_1 \otimes \mathcal{Z}(G)$.

2. Two infinitesimal deformations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma'}$ of a half-braiding σ for X relative to G are called *equivalent*, if there is an isomorphism in $R_1 \otimes \mathcal{Z}(G)$ of the form

$$\mathrm{id}_X + \varepsilon c : \quad (X, \boldsymbol{\sigma}) \xrightarrow{\cong} (X, \boldsymbol{\sigma'})$$

where $c \in \mathrm{End}_{\mathcal{D}}(X)$.

Proposition 2.5. *Let* $(X, \sigma) \in \mathcal{Z}(G)$. *A* 1*-cocycle* $\tilde{\sigma} \in Z^1_{\mathrm{DY}}\big(G, (X, \sigma)\big)$ *corresponds to the infinitesimal deformation* $(X, \boldsymbol{\sigma}) \in R_1 \otimes \mathcal{Z}(G)$ *with*

$$\boldsymbol{\sigma}_V = \sigma_V + \varepsilon \tilde{\sigma}_V$$

Infinitesimal deformations of σ *up to equivalence in the sense of Definition 2.4 correspond to classes in* $H^1_{\mathrm{DY}}\big(G, (X, \sigma)\big)$.

Proof:
We first show that $(X, \boldsymbol{\sigma}) \in R_1 \otimes \mathcal{Z}(G)$ if and only if $\tilde{\sigma} \in Z^1_{\mathrm{DY}}\big(G, (X, \sigma)\big)$: diagram (B.1) for $(X, \boldsymbol{\sigma})$ commutes if and only if the equation

$$(\Phi_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{G(V)} \otimes \boldsymbol{\sigma}_W) \circ (\boldsymbol{\sigma}_V \otimes \mathrm{id}_{G(W)}) = \boldsymbol{\sigma}_{VW} \circ (\mathrm{id}_X \otimes \Phi_{V,W})$$

holds for all $V, W \in \mathcal{C}$. This is equivalent to the equation

$$\begin{aligned}
&(\Phi_{V,W} \otimes X) \circ (\mathrm{id}_{G(V)} \otimes \sigma_W) \circ (\sigma_V \otimes \mathrm{id}_{G(W)}) \\
&\quad + \varepsilon(\Phi_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{G(V)} \otimes \sigma_W) \circ (\tilde{\sigma}_V \otimes \mathrm{id}_{G(W)}) \\
&\quad + \varepsilon(\Phi_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{G(V)} \otimes \tilde{\sigma}_W) \circ (\sigma_V \otimes \mathrm{id}_{G(W)}) \\
=&\sigma_{V \otimes W} \circ (\mathrm{id}_X \otimes \Phi_{V,W}) + \varepsilon \tilde{\sigma}_{V \otimes W} \circ (\mathrm{id}_X \otimes \Phi_{V,W})
\end{aligned} \tag{2.3}$$

Using the commutative diagram (B.1) for the half-braiding σ, the zeroth-order terms in (2.3) agree. Thus, equation (2.3) holds if and only if

$$\begin{aligned}
0 =&(\Phi_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{G(V)} \otimes \tilde{\sigma}_W) \circ (\sigma_V \otimes \mathrm{id}_{G(W)}) - \tilde{\sigma}_{V \otimes W} \circ (\mathrm{id}_X \otimes \Phi_{V,W}) \\
&+ (\Phi_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{G(V)} \otimes \sigma_W) \circ (\tilde{\sigma}_V \otimes \mathrm{id}_{G(W)}) \\
=&\partial^1_{\mathrm{DY}}(\tilde{\sigma})_{V,W}
\end{aligned}$$

which is equivalent to $\tilde{\sigma} \in Z^1_{\mathrm{DY}}\big(G, (X, \sigma)\big)$.

To conclude the proof, we show that two deformations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are equivalent via an isomorphism

$$\mathrm{id}_X + \varepsilon c : \quad (X, \boldsymbol{\sigma}) \xrightarrow{\cong} (X, \boldsymbol{\sigma}')$$

in $R_1 \otimes \mathcal{Z}(G)$ for some $c \in \mathrm{End}_{\mathcal{D}}(X)$ if and only if

$$\partial^0_{\mathrm{DY}}(c)_V = \tilde{\sigma}'_V - \tilde{\sigma}_V$$

for all $V \in \mathcal{C}$. Indeed, $\mathrm{id}_X + \varepsilon c$ is a morphism in $R_1 \otimes \mathcal{Z}(G)$ if and only if diagram (B.2) commutes, i.e., the following equation holds for all $V \in \mathcal{C}$:

$$\big(\mathrm{id}_{G(V)} \otimes (\mathrm{id}_X + \varepsilon c)\big) \circ \boldsymbol{\sigma}_V = \boldsymbol{\sigma}'_V \circ \big((\mathrm{id}_X + \varepsilon c) \otimes \mathrm{id}_{G(V)}\big)$$

This is equivalent to

$$\sigma_V + \varepsilon(\mathrm{id}_{G(V)} \otimes c) \circ \sigma_V + \varepsilon\tilde{\sigma}_V = \sigma_V + \varepsilon\sigma_V \circ (c \otimes \mathrm{id}_{G(V)}) + \varepsilon\tilde{\sigma}'_V \tag{2.4}$$

The first order terms in (2.4) agree and thus, equation (2.4) is equivalent to

$$\tilde{\sigma}'_V - \tilde{\sigma}_V = (\mathrm{id}_{G(V)} \otimes c) \circ \sigma_V - \sigma_V \circ (c \otimes \mathrm{id}_{G(V)}) = \partial^0_{\mathrm{DY}}(c)$$

which finishes the proof.

□

2.2 The Davydov-Yetter complex of the action functor

Since we want to establish a relation of the associator complex of a $\mathcal{C}$-module category $\mathcal{M}$ to a complex of Davydov-Yetter type, we need to encode the structure of a module category in terms of a monoidal functor. A way to do this is to consider the action functor

$$\begin{aligned} \rho_{\mathcal{M}} : \mathcal{C} &\longrightarrow \mathrm{End}(\mathcal{M}) \\ X &\longmapsto X \triangleright - \end{aligned} \tag{2.5}$$

Here we endow the k-linear category $\mathrm{End}(\mathcal{M})$ of endofunctors on $\mathcal{M}$ with its standard monoidal structure, i.e., the composition of endofunctors which we denote by $\bullet$. If $F, F', H, H' : \mathcal{M} \longrightarrow \mathcal{M}$ are endofunctors and $\nu : F \Longrightarrow F'$ and $\eta : H \Longrightarrow H'$ are natural transformations, we define the natural transformation $\nu \bullet \eta : F \bullet H \Longrightarrow F' \bullet H'$ via $(\nu \bullet \eta)_M := F'(\eta_M) \circ \nu_{H(M)} = \nu_{H'(M)} \circ F(\eta_M)$.
We endow the action functor (2.5) with the following monoidal structure:

$$(\mathfrak{m}_{X,Y})_M := m^{-1}_{X,Y,M} : \quad \underbrace{\big(\rho_{\mathcal{M}}(X) \bullet \rho_{\mathcal{M}}(Y)\big)(M)}_{=X\triangleright(Y\triangleright M)} \xrightarrow{\cong} \underbrace{\rho_{\mathcal{M}}(X \otimes Y)(M)}_{=(X\otimes Y)\triangleright M} \tag{2.6}$$

and

$$l^{-1}_M : \quad M \xrightarrow{\cong} \rho_{\mathcal{M}}(I)(M) = I \triangleright M$$

where m is the mixed associator of $\mathcal{M}$ and l is the left unitor of $\mathcal{M}$.

Example 2.6. *For a strict $\mathcal{C}$-module category $\mathcal{M}$, we will investigate the Davydov-Yetter complex of the action functor $\rho_{\mathcal{M}}$: for $n \geq 1$, the nth cochain space $C^n_{\mathrm{DY}}(\rho_{\mathcal{M}})$ consists of natural transformations $b = (b_{X_1,\dots,X_n})_{X_1,\dots,X_n\in\mathcal{C}}$ whose components*

$$b_{X_1,\dots,X_n}: \quad \rho_{\mathcal{M}}(X_1) \bullet \dots \bullet \rho_{\mathcal{M}}(X_n) \Longrightarrow \rho_{\mathcal{M}}(X_1) \bullet \dots \bullet \rho_{\mathcal{M}}(X_n) \tag{2.7}$$

are natural transformations between endofunctors of $\mathcal{M}$. The components of (2.7) in $\mathcal{M}$ are morphisms of the form

$$(b_{X_1,\dots,X_n})_M: \quad X_1 \triangleright \dots \triangleright X_n \triangleright M \longrightarrow X_1 \triangleright \dots \triangleright X_n \triangleright M \tag{2.8}$$

The differential on components in $\mathcal{M}$ reads

$$\begin{aligned}\big(\partial^n_{\mathrm{DY}}(b)_{X_1,\dots,X_{n+1}}\big)_M =& \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet b_{X_2,\dots,X_{n+1}}\big)_M + \sum_{i=1}^{n}(-1)^i (b_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ &+ (-1)^{n+1}\big(b_{X_1,\dots,X_n} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =& \,\mathrm{id}_{X_1} \triangleright (b_{X_2,\dots,X_{n+1}})_M + \sum_{i=1}^{n}(-1)^i (b_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ &+ (-1)^{n+1}(b_{X_1,\dots,X_n})_{X_{n+1}\triangleright M}\end{aligned} \tag{2.9}$$

The 0th cochain space of the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}})$ is the k-vector space of natural endotransformations of the identity functor in $\mathcal{M}$:

$$C^0_{\mathrm{DY}}(\rho_{\mathcal{M}}) = \mathrm{Nat}(\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}}) \tag{2.10}$$

and the differential is

$$\begin{aligned}\big(\partial^0_{\mathrm{DY}}(b)_{X_1}\big)_M =& (\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet b)_M - (b \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_1)})_M \\ =& \,\mathrm{id}_{X_1} \triangleright b_M - b_{X_1 \triangleright M}\end{aligned} \tag{2.11}$$

In the situation of Example 2.6, it is now easy to see from (2.8) and (2.10) that for any $n \geq 0$, the map

$$\theta^n : C^n_{ass}(\mathcal{C}, \mathcal{M}) \longrightarrow C^n_{\mathrm{DY}}(\rho_{\mathcal{M}}) \tag{2.12}$$

with

$$\big(\theta^n(f)_{X_1,\dots,X_n}\big)_M := f_{X_1,\dots,X_n,M}$$

and

$$\theta^0(f)_M := f_M$$

for all $X_1, \dots, X_n \in \mathcal{C}, M \in \mathcal{M}$ is an isomorphism of k-vector spaces.

Proposition 2.7. *Let $\mathcal{C}$ be a strict monoidal category and let $\mathcal{M}$ be a strict $\mathcal{C}$-module. The maps (2.12) combine into an isomorphism of cochain complexes*

$$C^\bullet_{ass}(\mathcal{C}, \mathcal{M}) \cong C^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}}) \tag{2.13}$$

In particular, there is an isomorphism in cohomology

$$H^\bullet_{ass}(\mathcal{C}, \mathcal{M}) \cong H^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}}) \tag{2.14}$$

Proof:
We need to show the compatibility of the isomorphisms (2.12) with the differentials, i.e., we have to show that the following diagram commutes for all $n \geq 0$:

$$\begin{array}{ccc} C^n_{ass}(\mathcal{C},\mathcal{M}) & \xrightarrow{\partial^n_{ass}} & C^{n+1}_{ass}(\mathcal{C},\mathcal{M}) \\ {\scriptstyle \theta^n}\downarrow & & \downarrow{\scriptstyle \theta^{n+1}} \\ C^n_{\mathrm{DY}}(\rho_{\mathcal{M}}) & \xrightarrow[\partial^n_{\mathrm{DY}}]{} & C^{n+1}_{\mathrm{DY}}(\rho_{\mathcal{M}}) \end{array} \tag{2.15}$$

We divide the proof into two parts: for $n \geq 1$, the following calculation shows the commutativity of diagram (2.15), where we have used (2.9) in the first equation and (1.1) in the third equation:

$$\begin{aligned} \big(\partial^n_{\mathrm{DY}}(\theta^n(f))_{X_1,\dots,X_{n+1}}\big)_M =& \, \mathrm{id}_{X_1} \triangleright \big(\theta^n(f)_{X_2,\dots,X_{n+1}}\big)_M + \sum_{i=1}^n (-1)^i \big(\theta^n(f)_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}}\big)_M \\ & + (-1)^{n+1}\big(\theta^n(f)_{X_1,\dots,X_n}\big)_{X_{n+1}\triangleright M} \\ =& \, \mathrm{id}_{X_1} \triangleright f_{X_2,\dots,X_{n+1},M} + \sum_{i=1}^n (-1)^i f_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1},M} \\ & + (-1)^{n+1} f_{X_1,\dots,X_n,X_{n+1}\triangleright M} \\ =& \big(\theta^{n+1}(\partial^n_{ass}(f))_{X_1,\dots,X_{n+1}}\big)_M \end{aligned}$$

Analogously, the following calculation shows the commutativity of diagram (2.15) for $n = 0$, where we have used (2.11) in the first equation and (1.1) in the third equation:

$$\begin{aligned} \big(\partial^0_{\mathrm{DY}}(\theta^0(f))_{X_1}\big)_M =& \, \mathrm{id}_{X_1} \triangleright \theta^0(f)_M - \theta^0(f)_{X_1\triangleright M} \\ =& \, \mathrm{id}_{X_1} \triangleright f_M - f_{X_1\triangleright M} \\ =& \big(\theta^1(\partial^0_{ass}(f))_{X_1}\big)_M \end{aligned}$$

□

We want to extend the result from Proposition 2.7 to also allow non-trivial coefficients in either of the two complexes. Coefficients for the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(\rho_{\mathcal{M}})$ of the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ live in the centralizer $\mathcal{Z}(\rho_{\mathcal{M}})$. Thus, in the following, we will investigate the monoidal category $\mathcal{Z}(\rho_{\mathcal{M}})$ more closely: an object $(F, \sigma) \in \mathcal{Z}(\rho_{\mathcal{M}})$ consists of a k-linear endofunctor $F : \mathcal{M} \longrightarrow \mathcal{M}$ together with a half-braiding $\sigma : F \bullet \rho_{\mathcal{M}}(?) \Longrightarrow \rho_{\mathcal{M}}(?) \bullet F$, i.e., a family of natural isomorphisms

$$\sigma_X : F \bullet \rho_{\mathcal{M}}(X) \Longrightarrow \rho_{\mathcal{M}}(X) \bullet F$$

each of which has components

$$(\sigma_X)_M : \quad \big(F \bullet \rho_{\mathcal{M}}(X)\big)(M) = F(X \triangleright M) \xrightarrow{\cong} X \triangleright F(M) = \big(\rho_{\mathcal{M}}(X) \bullet F\big)(M) \tag{2.16}$$

in $\mathcal{M}$. The isomorphisms (2.16) have the same source and target as the components of a $\mathcal{C}$-module structure on F. Indeed, the categories $\mathcal{Z}(\rho_{\mathcal{M}})$ and $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ are isomorphic as monoidal categories, where we endow $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ with its standard tensor product, i.e., $(F, s) \bullet (H, t) = (F \bullet H, st)$ where

$$(st)_{X,M} := s_{X,H(M)} \circ F(t_{X,M}) \tag{2.17}$$

for all $X \in \mathcal{C}, M \in \mathcal{M}$.

Proposition 2.8. *Let $\mathcal{C}$ be a monoidal category and let $\mathcal{M}$ be a $\mathcal{C}$-module category. The strict monoidal functor*

$$\begin{aligned} \Theta : \mathcal{Z}(\rho_{\mathcal{M}}) &\longrightarrow \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \\ (F, \sigma) &\longmapsto (F, \hat{\sigma}) \end{aligned}$$

is an isomorphism of monoidal categories, where

$$(\hat{\sigma})_{X,M} := (\sigma_X)_M : \quad F(X \triangleright M) \longrightarrow X \triangleright F(M)$$

for all $X \in \mathcal{C}, M \in \mathcal{M}$.

Proof:
We first show that $(F, \hat{\sigma})$ is indeed an object in $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$. Recall the monoidal structure $\mathfrak{m}$ of the action functor from (2.6). Since σ is a half-braiding, diagram (B.1) commutes which is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} (F \bullet \rho_{\mathcal{M}}(X) \bullet \rho_{\mathcal{M}}(Y))(M) & \xrightarrow{(\mathrm{id}_F \bullet \mathfrak{m}_{X,Y})_M} & (F \bullet \rho_{\mathcal{M}}(X \otimes Y))(M) \\ \downarrow{\scriptstyle (\sigma_X \bullet \mathrm{id}_{\rho_{\mathcal{M}}(Y)})_M} & & \\ (\rho_{\mathcal{M}}(X) \bullet F \bullet \rho_{\mathcal{M}}(Y))(M) & & \downarrow{\scriptstyle (\sigma_{X \otimes Y})_M} \\ \downarrow{\scriptstyle (\mathrm{id}_{\rho_{\mathcal{M}}(X)} \bullet \sigma_Y)_M} & & \\ (\rho_{\mathcal{M}}(X) \bullet \rho_{\mathcal{M}}(Y) \bullet F)(M) & \xrightarrow[(\mathfrak{m}_{X,Y} \bullet \mathrm{id}_F)_M]{} & (\rho_{\mathcal{M}}(X \otimes Y) \bullet H)(M) \end{array} \tag{2.18}$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Rewriting diagram (2.18) in terms of $\triangleright$, we obtain the commuting diagram

$$\begin{array}{ccc} F(X \triangleright (Y \triangleright M)) & \xrightarrow{F(m^{-1}_{X,Y,M})} & F((X \otimes Y) \triangleright M) \\ \downarrow{\scriptstyle \hat{\sigma}_{X, Y \triangleright M}} & & \\ X \triangleright F(Y \triangleright M) & & \downarrow{\scriptstyle \hat{\sigma}_{X \otimes Y, M}} \\ \downarrow{\scriptstyle \mathrm{id}_X \triangleright \hat{\sigma}_{Y,M}} & & \\ X \triangleright (Y \triangleright F(M)) & \xrightarrow[m^{-1}_{X,Y,F(M)}]{} & (X \otimes Y) \triangleright F(M) \end{array}$$

which is just the pentagon diagram (A.11). Since the triangle axiom is fulfilled automatically due to Proposition A.18, we have shown that $(F, \hat{\sigma}) \in \mathrm{End}_{\mathcal{C}}(\mathcal{M})$.

Next, we show that a morphism $\nu : (F, \sigma^F) \longrightarrow (H, \sigma^H)$ in $\mathcal{Z}(\rho_{\mathcal{M}})$ is also a natural transformation of $\mathcal{C}$-module functors $(F, \hat{\sigma}^F) \Longrightarrow (H, \hat{\sigma}^H)$. Since ν is a morphism in $\mathcal{Z}(\rho_{\mathcal{M}})$, the diagram

$$\begin{array}{ccc} (F \bullet \rho_{\mathcal{M}}(X))(M) & \xrightarrow{(\sigma^F_X)_M} & (\rho_{\mathcal{M}}(X) \bullet F)(M) \\ {\scriptstyle (\nu \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X)})_M} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}_{\rho_{\mathcal{M}}(X)} \bullet \nu)_M} \\ (H \bullet \rho_{\mathcal{M}}(X))(M) & \xrightarrow[(\sigma^H_X)_M]{} & (\rho_{\mathcal{M}}(X) \bullet H)(M) \end{array}$$

commutes for all $X \in \mathcal{C}, M \in \mathcal{M}$ (see (2.1)) which is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} F(X \triangleright M) & \xrightarrow{\hat{\sigma}^F_{X,M}} & X \triangleright F(M) \\ {\scriptstyle \nu_{X \triangleright M}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_X \triangleright \nu_M} \\ H(X \triangleright M) & \xrightarrow[\hat{\sigma}^H_{X,M}]{} & V \triangleright H(M) \end{array}$$

But this is just the condition that ν is also a transformation of $\mathcal{C}$-module functors (see (A.16)).

It is obvious that the functor Θ is an isomorphism of k-linear categories. To show that Θ is a strict monoidal functor, note that $\Theta(F, \sigma^F) \bullet \Theta(H, \sigma^H)$ is the endofunctor $F \bullet H$ of $\mathcal{M}$ whose $\mathcal{C}$-module structure has the components $\hat{\sigma}^F_{X,H(M)} \circ F(\hat{\sigma}^H_{X,M})$ by definition of the tensor product in $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ (see (2.17)). On the other hand, $\Theta\big((F, \sigma^F) \otimes (H, \sigma^H)\big) = \Theta(F \bullet H, \sigma^{F \bullet H})$ is the endofunctor $F \bullet H$ of $\mathcal{M}$ with the $\mathcal{C}$-module structure

$$\begin{aligned} (\sigma^{F \bullet H}_X)_M =& (\sigma^F_X \bullet \mathrm{id}_H)_M \circ (\mathrm{id}_F \bullet \sigma^H_X)_M \\ =& (\sigma^F_X)_{H(M)} \circ F\big((\sigma^H_X)_M\big) \\ =& \hat{\sigma}^F_{X,H(M)} \circ F(\hat{\sigma}^H_{X,M}) \end{aligned}$$

where we have used the definition of the tensor product in $\mathcal{Z}(\rho_{\mathcal{M}})$ (see (2.2)) in the first equation. Hence, we have shown that

$$\Theta\big((F, \sigma^F) \otimes (H, \sigma^H)\big) = \Theta(F, \sigma^F) \bullet \Theta(H, \sigma^H)$$

□

From now on, we will identify the monoidal categories $\mathcal{Z}(\rho_{\mathcal{M}})$ and $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$, i.e., we will not distinguish in the notation between k-linear endofunctors of $\mathcal{M}$ with a half-braiding relative to $\rho_{\mathcal{M}}$ and k-linear $\mathcal{C}$-module endofunctors of $\mathcal{M}$. Thus, we have shown that $\mathcal{C}$-module endofunctors of $\mathcal{M}$ appear as coefficients of the Davydov-Yetter complex of the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$.

We can identify the associator deformation complex with the Davydov-Yetter complex of the action functor in a more general setting *with coefficients* and for a *non-strict* module category over a *non-strict* monoidal category:

Theorem 2.9. *Let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{M}$ be a k-linear $\mathcal{C}$-module category, none of which are necessarily strict. Let $(F,s),(F',s'):\mathcal{M}\longrightarrow\mathcal{M}$ be k-linear $\mathcal{C}$-module endofunctors. There is an isomorphism of cochain complexes*

$$C^\bullet_{ass}\big((F,s),(F',s')\big)\cong C^\bullet_{\mathrm{DY}}\big(\rho_\mathcal{M},(F,s),(F',s')\big) \tag{2.19}$$

Proof:
To establish the isomorphism (2.19) in the general case with non-strict ingredients, we need tools from appendix B. In particular, we go the route via an auxiliary complex (see Definition B.3).
We consider the following concatenation of isomorphisms of cochain complexes from Proposition B.12 and Remark B.7:

$$C^n_{ass}\big((F,s),(F',s')\big)\xrightarrow{\vartheta_n}C^n_{\mathrm{aux}}\big(\rho_\mathcal{M},(F,s),(F',s')\big)\xrightarrow{\lambda_n}C^n_{\mathrm{DY}}\big(\rho_\mathcal{M},(F,s),(F',s')\big) \tag{2.20}$$

The isomorphism (2.20) maps an n-cochain

$$f=(f_{X_1,\dots,X_n,M})_{X_1,\dots,X_n\in\mathcal{C},M\in\mathcal{M}}\in C^n_{ass}\big((F,s),(F',s')\big)$$

in the associator complex to the n-cochain in the Davydov-Yetter complex, which has the components

$$\big((\mathfrak{m}_n)_{X_1,\dots,X_n}\big)_M\circ f_{X_1,\dots,X_n,M}\circ\big((m_n)_{X_1,\dots,X_n}\big)_M$$

(see (B.15) and (B.16) for the definition of the isomorphisms $\mathfrak{m}_n$ and m_n).
On the other hand, an n-cochain

$$b=(b_{X_1,\dots,X_n})_{X_1,\dots,X_n\in\mathcal{C}}\in C^n_{\mathrm{DY}}\big(\rho_\mathcal{M},(F,s),(F',s')\big)$$

in the Davydov-Yetter complex gets mapped to the n-cochain in the associator complex, which has the components

$$\big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_n}\big)_M\circ(b_{X_1,\dots,X_n})_M\circ\big((m_n^{-1})_{X_1,\dots,X_n}\big)_M$$

□

Corollary 2.10. *There is an isomorphism of the associator deformation complex of $\mathcal{M}$ over $\mathcal{C}$ and the Davydov-Yetter complex of the action functor*

$$C^\bullet_{ass}(\mathcal{C},\mathcal{M})\cong C^\bullet_{\mathrm{DY}}(\rho_\mathcal{M}) \tag{2.21}$$

In particular, the isomorphisms (2.19) and (2.21) induce isomorphisms in cohomology

$$H^\bullet_{ass}\big((F,s),(F',s')\big)\cong H^\bullet_{\mathrm{DY}}\big(\rho_\mathcal{M},(F,s),(F',s')\big)$$

and

$$H^\bullet_{ass}(\mathcal{C},\mathcal{M})\cong H^\bullet_{\mathrm{DY}}(\rho_\mathcal{M}) \tag{2.22}$$

respectively.

2.3 The associator deformation complex as a dg-algebra

In this subsection, let $\mathcal{C}$ denote a strict k-linear monoidal category and let $\mathcal{M}$ be a strict k-linear $\mathcal{C}$-module category. Structures of a dg-algebra or a dg-module on cochain complexes provide ways to produce new cocycles of higher degree from known ones, thus being a useful tool in the theory of deformations.

It is easy to check that the following assignment endows $C^\bullet_{ass}(\mathcal{C},\mathcal{M})$ with the structure of an associative graded unital k-algebra:

$$\begin{aligned} C^n_{ass}(\mathcal{C},\mathcal{M}) \otimes C^m_{ass}(\mathcal{C},\mathcal{M}) &\longrightarrow C^{n+m}_{ass}(\mathcal{C},\mathcal{M}) \\ f\otimes g &\longmapsto f\cdot g \end{aligned} \tag{2.23}$$

where

$$(f\cdot g)_{X_1,\dots,X_{n+m},M} := (\mathrm{id}_{X_1\dots X_n} \triangleright g_{X_{n+1},\dots,X_{n+m},M}) \circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m}\triangleright M}$$

In fact, this structure is compatible with the differential:

Proposition 2.11. *The associator deformation complex $C^\bullet_{ass}(\mathcal{C},\mathcal{M})$ together with the product (2.23) is a dg-algebra.*

Proof:
Let $m,n \geq 0$ and let $X_1,\dots,X_{n+m+1} \in \mathcal{C}, M\in\mathcal{M}$. We show that the product (2.23) satisfies the graded Leibniz rule, i.e.,

$$\partial^n_{ass}(f)\cdot g + (-1)^n f\cdot \partial^m_{ass}(g) = \partial^{n+m}_{ass}(f\cdot g) \tag{2.24}$$

for all $f\in C^n_{ass}(\mathcal{C},\mathcal{M})$ and $g\in C^m_{ass}(\mathcal{C},\mathcal{M})$. We calculate both summands of the left-hand side of (2.24) separately:

$$\begin{aligned} &\big(\partial^n_{ass}(f)\cdot g\big)_{X_1,\dots,X_{n+m+1},M} \\ =&\big(\mathrm{id}_{X_1\dots X_{n+1}} \triangleright g_{X_{n+2},\dots,X_{n+m+1},M}\big)\circ \partial^n_{ass}(f)_{X_1,\dots,X_{n+1},X_{n+2}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \\ =&\big(\mathrm{id}_{X_1\dots X_{n+1}} \triangleright g_{X_{n+2},\dots,X_{n+m+1},M}\big)\circ \big(\mathrm{id}_{X_1}\triangleright f_{X_2,\dots,X_{n+1},X_{n+2}\triangleright\dots\triangleright X_{n+m+1}\triangleright M}\big) \\ &+\sum_{i=1}^n (-1)^i \big(\mathrm{id}_{X_1\dots X_{n+1}} \triangleright g_{X_{n+2},\dots,X_{n+m+1},M}\big)\circ f_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1},X_{n+2}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \\ &+(-1)^{n+1}\big(\mathrm{id}_{X_1\dots X_{n+1}} \triangleright g_{X_{n+2},\dots,X_{n+m+1},M}\big)\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &(-1)^n\big(f\cdot\partial^m_{ass}(g)\big)_{X_1,\dots,X_{n+m+1},M} \\ =&(-1)^n\big(\mathrm{id}_{X_1\dots X_n}\triangleright \partial^m_{ass}(g)_{X_{n+1},\dots,X_{n+m+1},M}\big)\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \\ =&(-1)^n\big(\mathrm{id}_{X_1\dots X_n}\triangleright \mathrm{id}_{X_{n+1}}\triangleright g_{X_{n+2},\dots,X_{n+m+1},M}\big)\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \\ &+\sum_{i=1}^m (-1)^{i+n}\big(\mathrm{id}_{X_1\dots X_n}\triangleright g_{X_{n+1},\dots,X_{n+i}X_{n+i+1},\dots,X_{n+m+1},M}\big)\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \\ &+(-1)^{n+m+1}\big(\mathrm{id}_{X_1\dots X_n}\triangleright g_{X_{n+1},\dots,X_{n+m},X_{n+m+1}\triangleright M}\big)\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright\dots\triangleright X_{n+m+1}\triangleright M} \end{aligned} \tag{2.26}$$

Adding (2.25) and (2.26) yields the right-hand side of (2.24) as desired:

$$\begin{aligned}
&\big(\mathrm{id}_{X_1} \triangleright \mathrm{id}_{X_2\ldots X_{n+1}} \triangleright g_{X_{n+2},\ldots,X_{n+m+1},M}\big) \circ \big(\mathrm{id}_{X_1} \triangleright f_{X_2,\ldots,X_{n+1},X_{n+2}\triangleright\ldots\triangleright X_{n+m+1}\triangleright M}\big)\\
&+\sum_{i=1}^{n}\big(\mathrm{id}_{X_1\ldots X_iX_{i+1}\ldots X_{n+1}} \triangleright g_{X_{n+2},\ldots,X_{n+m+1},M}\big) \circ f_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1},X_{n+2}\triangleright\ldots\triangleright X_{n+m+1}\triangleright M}\\
&+(-1)^n\sum_{j=1}^{m}(-1)^j\big(\mathrm{id}_{X_1\ldots X_n} \triangleright g_{X_{n+1},\ldots,X_{n+j}X_{n+j+1},\ldots,X_{n+m+1},M}\big)\\
&\qquad\qquad \circ f_{X_1,\ldots,X_n,X_{n+1}\triangleright\ldots X_{n+j}X_{n+j+1}\ldots\triangleright X_{n+m+1}\triangleright M}\\
&+(-1)^{m+n+1}\big(\mathrm{id}_{X_1\ldots X_n} \triangleright g_{X_{n+1},\ldots,X_{n+m},X_{n+m+1}\triangleright M}\big) \circ f_{X_1,\ldots,X_n,X_{n+1}\triangleright\ldots\triangleright X_{n+m+1}\triangleright M}\\
=&\big(\mathrm{id}_{X_1} \triangleright (f\cdot g)_{X_2,\ldots,X_{n+m+1},M}\big) + \sum_{i=1}^{n+m}(-1)^i(f\cdot g)_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+m+1},M}\\
&+(-1)^{n+m+1}(f\cdot g)_{X_1,\ldots,X_{n+m},X_{n+m+1}\triangleright M}\\
=&\partial^{n+m}_{ass}(f\cdot g)_{X_1,\ldots,X_{n+m+1},M}
\end{aligned}$$

□

As is standard, it follows from the graded Leibniz rule that the associative graded algebra structure of the associator complex descends to cohomology: if both f and g are cocycles, then $f \cdot g$ is a cocycle. Also, if f is a coboundary and g is a cocycle or if f is a cocycle and g is a coboundary, then $f \cdot g$ is a coboundary.

Corollary 2.12. *Associator deformation cohomology $H^\bullet_{ass}(\mathcal{C}, \mathcal{M})$ inherits the structure of an associative graded unital k-algebra.*

Recall from [BD, Corollary 3.5] that the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G)$ of a strict monoidal functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ between strict monoidal categories $\mathcal{C}$ and $\mathcal{D}$ has a natural structure of an associative unital graded k-algebra via the cup product

$$\cup : \quad C^m_{\mathrm{DY}}(G) \otimes C^n_{\mathrm{DY}}(G) \longrightarrow C^{m+n}_{\mathrm{DY}}(G) \tag{2.27}$$

where

$$(b \cup c)_{X_1,\ldots,X_{m+n}} := \begin{cases} b \otimes c_{X_1,\ldots,X_n} & \text{if } \ m = 0, n \geq 1 \\ b_{X_1,\ldots,X_m} \otimes c & \text{if } \ m \geq 1, n = 0 \\ b_{X_1,\ldots,X_m} \otimes c_{X_{m+1},\ldots,X_{m+n}} & \text{if } \ n, m \geq 1 \end{cases}$$

and

$$b \cup c := b \otimes c$$

if $m = n = 0$. It is also known that the cup product descends to cohomology, i.e., the induced map in cohomology

$$\cup : \quad H^m_{\mathrm{DY}}(G) \otimes H^n_{\mathrm{DY}}(G) \longrightarrow H^{m+n}_{\mathrm{DY}}(G)$$

endows the Davydov-Yetter cohomology $H^\bullet_{\mathrm{DY}}(G)$ with the structure of an associative graded k-algebra. The cup product (2.27) and the product (2.23) are related by the isomorphism (2.21):

Proposition 2.13. *The isomorphism (2.13) of cochain complexes is an isomorphism of dg-algebras and thus, the induced isomorphism in cohomology (2.14) is an isomorphism of graded k-algebras.*

Proof:
Let $m, n \geq 0$ and let $f \in C^m_{ass}(\mathcal{C}, \mathcal{M})$ and $g \in C^n_{ass}(\mathcal{C}, \mathcal{M})$. The proof falls naturally into four cases: for $m = n = 0$, we have

$$\theta^0(f.g)_M = g_M \circ f_M = (f \bullet g)_M = \big(\theta^0(f) \bullet \theta^0(g)\big)_M = \big(\theta^0(f) \cup \theta^0(g)\big)_M$$

For $m = 0$ and $n \geq 1$, we calculate

$$\begin{aligned}\big(\theta^n(f.g)_{X_1,\dots,X_n}\big)_M =&(f.g)_{X_1,\dots,X_n,M}\\ =&g_{X_1,\dots,X_n} \circ f_{X_1 \triangleright \dots X_n \triangleright M}\\ =&\big(\theta^n(g)_{X_1,\dots,X_n}\big)_M \circ \big(\theta^0(f)_{\rho_\mathcal{M}(X_1)\dots\rho_\mathcal{M}(X_n)}\big)_M\\ =&\big(\theta^0(f) \bullet \theta^n(g)_{X_1,\dots,X_n}\big)_M\\ =&\big(\big(\theta^0(f) \cup \theta^n(g)\big)_{X_1,\dots,X_n}\big)_M\end{aligned}$$

and for $m \geq 1$ and $n = 0$ we have

$$\begin{aligned}\big(\theta^m(f.g)_{X_1,\dots,X_m}\big)_M =&(f.g)_{X_1,\dots,X_m,M}\\ =&(\mathrm{id}_{X_1\dots X_m} \triangleright g_M) \circ f_{X_1,\dots,X_m,M}\\ =&\big(\rho_\mathcal{M}(X_1) \bullet \dots \bullet \rho_\mathcal{M}(X_m)\big)\big(\theta^0(g)_M\big) \circ \big(\theta^m(f)_{X_1,\dots,X_m}\big)_M\\ =&\big(\theta^m(f)_{X_1,\dots,X_m} \bullet \theta^0(g)\big)_M\\ =&\big(\big(\theta^m(f) \cup \theta^0(g)\big)_{X_1,\dots,X_m}\big)_M\end{aligned}$$

The following calculation for $m, n \geq 1$ concludes the proof:

$$\begin{aligned}&\big(\theta^{m+n}(f \cdot g)_{X_1,\dots,X_{m+n}}\big)_M\\ =&(f \cdot g)_{X_1,\dots,X_{m+n},M}\\ =&\big(\mathrm{id}_{X_1\dots X_m} \triangleright g_{X_{m+1},\dots,X_{m+n},M}\big) \circ f_{X_1,\dots,X_m,X_{m+1}\triangleright\dots\triangleright X_{m+n}\triangleright M}\\ =&\rho_\mathcal{M}(X_1 \otimes \dots \otimes X_m)\big((\theta^n(g)_{X_{m+1},\dots,X_{m+n}})_M\big) \circ \big(\theta^m(f)_{X_1,\dots,X_m}\big)_{(\rho_\mathcal{M}(X_{m+1}\dots X_{m+n}))(M)}\\ =&\big(\theta^m(f)_{X_1,\dots,X_m} \bullet \theta^n(g)_{X_{m+1},\dots,X_{m+n}}\big)_M\\ =&\big(\big(\theta^m(f) \cup \theta^n(g)\big)_{X_1,\dots,X_{m+n}}\big)_M\end{aligned}$$

□

2.4 Deformations by (soft) autoequivalences

In classical algebra, it is well-known that the structure of a module M over an algebra A,

$$\begin{aligned} A \times M &\longrightarrow M \\ (a, m) &\longmapsto a.m \end{aligned}$$

can be deformed by any automorphism $\alpha : A \longrightarrow A$ of the algebra A into the following A-action on M:

$$\begin{aligned} A \times M &\longrightarrow M \\ (a, m) &\longmapsto \alpha(a).m \end{aligned}$$

Of course the structure of an A-module on M is nothing else than an algebra map

$$\begin{aligned} \rho_M : A &\longrightarrow \mathrm{End}(M) \\ a &\longmapsto a.- \end{aligned}$$

and the deformation by α is the composition $\rho_M \circ \alpha$.

We will study the analogous situation for *infinitesimal* deformations of the action functor as a monoidal functor. Recall that the composition

$$(G, \Phi) \circ (H, \Psi) : \mathcal{B} \longrightarrow \mathcal{D}$$

of monoidal functors $(H, \Psi) : \mathcal{B} \longrightarrow \mathcal{C}$ and $(G, \Phi) : \mathcal{C} \longrightarrow \mathcal{D}$ is again a monoidal functor whose monoidal structure is defined as

$$G(\Psi_{X,Y}) \circ \Phi_{H(X),H(Y)} : \quad G\big(H(X)\big) \otimes G\big(H(Y)\big) \xrightarrow{\cong} G\big(H(X \otimes Y)\big) \tag{2.28}$$

for all $X, Y \in \mathcal{C}$. Now, we will compose infinitesimal deformations of monoidal functors: as discussed in Proposition 2.3, 2-cocycles in the Davydov-Yetter complex $C^{\bullet}_{\mathrm{DY}}(G)$ of a monoidal functor G give rise to infinitesimal deformations of the monoidal structure of G. Let $\Phi^{(1)} \in Z^2_{\mathrm{DY}}(G)$, $\Psi^{(1)} \in Z^2_{\mathrm{DY}}(H)$ and consider the following infinitesimal deformations of the monoidal structures of G and H:

$$\Phi_{X,Y} + \varepsilon \Phi^{(1)}_{X,Y} : \quad G(X) \otimes G(Y) \xrightarrow{\cong} G(X \otimes Y)$$

$$\Psi_{X,Y} + \varepsilon \Psi^{(1)}_{X,Y} : \quad H(X) \otimes H(Y) \xrightarrow{\cong} H(X \otimes Y)$$

By (2.28), the composition of the deformed monoidal structures is

$$\begin{aligned} &G\big(\Psi_{X,Y} + \varepsilon \Psi^{(1)}_{X,Y}\big) \circ (\Phi_{H(X),H(Y)} + \varepsilon \Phi^{(1)}_{H(X),H(Y)}) \\ =&G(\Psi_{X,Y}) \circ \Phi_{H(X),H(Y)} + \varepsilon \Big(G(\Psi_{X,Y}) \circ \Phi^{(1)}_{H(X),H(Y)} + G\big(\Psi^{(1)}_{X,Y}\big) \circ \Phi_{H(X),H(Y)} \Big) \end{aligned}$$

Since the composition of monoidal functors is again a monoidal functor, the natural transformation which has the components

$$G(\Psi_{X,Y}) \circ \Phi^{(1)}_{H(X),H(Y)} + G\big(\Psi^{(1)}_{X,Y}\big) \circ \Phi_{H(X),H(Y)} \tag{2.29}$$

is a 2-cocycle in the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G\circ H)$ by Proposition 2.3. We now consider the case $\Phi^{(1)}=0$ and $(H,\Psi)=(\mathrm{Id}_{\mathcal{C}},\mathrm{id})$, i.e., we study infinitesimal deformations of (G,Φ) which come from infinitesimal deformations of the identity functor $(\mathrm{Id}_{\mathcal{C}},\mathrm{id})$. In this situation, the Davydov-Yetter 2-cocycle (2.29) reduces to

$$G\big(\Psi^{(1)}_{X,Y}\big)\circ\Phi_{X,Y} \tag{2.30}$$

Every infinitesimal deformation of $(\mathrm{Id}_{\mathcal{C}},\mathrm{id})$ is a *soft* autoequivalence of the monoidal category $R_1\otimes\mathcal{C}$ since its underlying R_1-linear functor is the identity. In this sense, we view the infinitesimal deformation of G by the 2-cocycle (2.30) as a deformation by the soft autoequivalence $\big(\mathrm{Id}_{R_1\otimes\mathcal{C}},\mathrm{id}+\varepsilon\Psi^{(1)}\big):R_1\otimes\mathcal{C}\longrightarrow R_1\otimes\mathcal{C}$.

From now on, let $\mathcal{C},\mathcal{D}$ be strict monoidal categories and let $(G,\Phi,\varphi):\mathcal{C}\longrightarrow\mathcal{D}$ be a monoidal functor. There is another natural way of obtaining the 2-cocycle (2.30) via the following dg-module structure of the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G)$ over the dg-algebra $\big(C^\bullet_{\mathrm{DY}}(\mathcal{C}),\cup\big)$ (see (2.27) for the definition of the cup product and (B.8) for the definition of the isomorphism Φ_n): for all $n,m\geq 0$, consider the k-linear map

$$\begin{aligned} C^m_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})\otimes C^n_{\mathrm{DY}}(G)&\longrightarrow C^{m+n}_{\mathrm{DY}}(G)\\ f\otimes g&\longmapsto f.g \end{aligned} \tag{2.31}$$

where

$$(f.g)_{X_1,\ldots,X_{m+n}}:=\begin{cases}\big(G(f_{X_1,\ldots,X_m})\otimes g\big)\circ(\Phi_m)_{X_1,\ldots,X_m} & \text{if } n=0,m\geq 1\\[2ex] G(f\otimes\mathrm{id}_{X_1\ldots X_n})\circ g_{X_1,\ldots,X_n} & \text{if } n\geq 1,m=0\\[2ex] \Phi_{X_1\ldots X_m,X_{m+1}\ldots X_{m+n}} & \\ \circ\big(G(f_{X_1,\ldots,X_m})\otimes g_{X_{m+1},\ldots,X_{m+n}}\big) & \\ \circ\big((\Phi_m)_{X_1,\ldots,X_m}\otimes\mathrm{id}_{G(X_{m+1})\ldots G(X_{m+n})}\big) & \text{if } n,m\geq 1\end{cases} \tag{2.32}$$

and

$$f.g:=(\varphi^{-1}\otimes\mathrm{id}_I)\circ(G(f)\otimes g)\circ(\varphi\otimes\mathrm{id}_I)$$

for $m=n=0$.

Remark 2.14. *For $G=\mathrm{Id}_{\mathcal{C}}$, the graded $C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$-module structure from (2.31) reduces to the cup product (2.27).*

To obtain the 2-cocycle (2.30), we consider the case $n=0,m=2$ in (2.32): since $\mathcal{D}$ is assumed to be strict, the following equation holds for all $X,Y\in\mathcal{C}$:

$$\big(\Psi^{(1)}.\,\mathrm{id}_I\big)_{X,Y}=G\big(\Psi^{(1)}_{X,Y}\big)\circ\Phi_{X,Y} \tag{2.33}$$

Let us now study the case, where G is the action functor $\rho_{\mathcal{M}}:\mathcal{C}\longrightarrow\mathrm{End}(\mathcal{M})$ of a not necessarily strict module category $\mathcal{M}$ over a strict monoidal category $\mathcal{C}$. Recall the monoidal structure of $\rho_{\mathcal{M}}$ from (2.6) and the isomorphisms $\mathfrak{m}_n$ and m_n from (B.15) and

(B.16). Let $n, m \geq 0$. We use (2.32) to endow the Davydov-Yetter complex of the action functor with the following structure of a dg-module over the dg-algebra $\big(C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}), \cup\big)$:

$$\begin{aligned} C^m_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}) \otimes C^n_{\mathrm{DY}}(\rho_{\mathcal{M}}) &\longrightarrow C^{m+n}_{\mathrm{DY}}(\rho_{\mathcal{M}}) \\ f \otimes g &\longmapsto f.g \end{aligned}$$

where

$$\big((f.g)_{X_1,\dots,X_{m+n}}\big)_M := \begin{cases} (f_{X_1\dots X_m} \triangleright g_M) \circ \big((\mathfrak{m}_m)_{X_1,\dots,X_m}\big)_M & \text{if } \ n = 0, m \geq 1 \\ \\ \big((f \otimes \mathrm{id}_{X_1\dots X_n}) \triangleright \mathrm{id}_M\big) \circ (g_{X_1,\dots,X_n})_M & \text{if } \ n \geq 1, m = 0 \\ \\ m^{-1}_{X_1\dots X_m, X_{m+1}\dots X_{m+n}} \\ \circ\big(f_{X_1,\dots,X_m} \triangleright (g_{X_{m+1},\dots,X_{m+n}})_M\big) \\ \circ\big((\mathfrak{m}_m)_{X_1,\dots,X_m}\big)_{X_{m+1}\triangleright(\dots\triangleright(X_{m+n}\triangleright M)\dots)} & \text{if } \ n, m \geq 1 \end{cases} \tag{2.34}$$

and

$$(f.g)_M := l_M \circ (f \triangleright g_M) \circ l_M^{-1}$$

for $m = n = 0$.

Similar to (2.33), we can now generate the following Davydov-Yetter 2-cocycle for the action functor $\rho_{\mathcal{M}}$ from a Davydov-Yetter 2-cocycle $\Psi^{(1)}$ of the identity functor of $\mathcal{C}$, where id denotes the identity transformation on the identity functor of $\mathcal{M}$:

$$\big((\Psi^{(1)}.\,\mathrm{id})_{X,Y}\big)_M = \big(\Psi^{(1)}_{X,Y} \triangleright \mathrm{id}_M\big) \circ m^{-1}_{X,Y,M} \tag{2.35}$$

Applying the isomorphism (2.20) to the Davydov-Yetter 2-cocycle (2.35) yields the following 2-cocycle in the associator deformation complex of $\mathcal{M}$ over $\mathcal{C}$:

$$m_{X,Y,M} \circ \big(\Psi^{(1)}_{X,Y} \triangleright \mathrm{id}_M\big) \tag{2.36}$$

By Proposition 1.8, we obtain an infinitesimal deformation of $\mathcal{M}$ with the mixed associator

$$\begin{aligned} \mathbf{m}_{X,Y,M} &= m_{X,Y,M} + \varepsilon m_{X,Y,M} \circ \big(\Psi^{(1)}_{X,Y} \triangleright \mathrm{id}_M\big) \\ &= m_{X,Y,M} \circ \big(\mathrm{id}_{(X\otimes Y)\triangleright M} + \varepsilon \Psi^{(1)}_{X,Y} \triangleright \mathrm{id}_M\big) \end{aligned}$$

If we transport the dg-module structure (2.34) along the isomorphism (2.21), we obtain the following dg-module structure on the associator complex:

$$\begin{aligned} C^m_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}) \otimes C^n_{ass}(\mathcal{C}, \mathcal{M}) &\longrightarrow C^{m+n}_{ass}(\mathcal{C}, \mathcal{M}) \\ f \otimes g &\longmapsto f.g \end{aligned}$$

where

$$(f.g)_{X_1,\dots,X_{m+n},M} := \begin{cases} \big((\mathfrak{m}_m^{-1})_{X_1,\dots,X_m}\big)_M \circ (f_{X_1\dots X_m} \triangleright g_M) \\ \circ\big((\mathfrak{m}_m)_{X_1,\dots,X_m}\big)_M \circ \big((m_m^{-1})_{X_1,\dots,X_m}\big)_M & \text{if } \ n=0, m\geq 1 \\ \\ \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_n}\big)_M \circ \big((f\otimes \mathrm{id}_{X_1\dots X_n}) \triangleright \mathrm{id}_M\big) \\ \circ(g_{X_1,\dots,X_n})_M \circ \big((m_n)_{X_1,\dots,X_n}\big)_M & \text{if } \ n\geq 1, m=0 \\ \\ \big((\mathfrak{m}_{m+n}^{-1})_{X_1,\dots,X_{m+n}}\big)_M \circ m^{-1}_{X_1\dots X_m, X_{m+1}\dots X_{m+n}} \\ \circ\big(f_{X_1,\dots,X_m} \triangleright (g_{X_{m+1},\dots,X_{m+n}})_M\big) \\ \circ\big((\mathfrak{m}_m)_{X_1,\dots,X_m}\big)_{X_{m+1}\triangleright(\dots\triangleright(X_{m+n}\triangleright M)\dots)} \\ \circ\big((m_{m+n}^{-1})_{X_1,\dots,X_{m+n}}\big)_M & \text{if } \ n,m\geq 1 \end{cases}$$

and

$$(f.g)_M := l_M \circ (f \triangleright g_M) \circ l_M^{-1}$$

for $m=n=0$.

As is standard, this promotes the isomorphism (2.21) to an isomorphism of dg-modules:

Proposition 2.15. *The isomorphism (2.21) of cochain complexes is an isomorphism of dg-modules over the dg-algebra $\big(C^\bullet_{\mathrm{DY}}(\mathrm{Id}_\mathcal{C}), \cup\big)$ and thus, the induced isomorphism in cohomology (2.22) is an isomorphism of graded $\big(H^\bullet_{\mathrm{DY}}(\mathrm{Id}_\mathcal{C}), \cup\big)$-modules.*

Obviously, other structures on the Davydov-Yetter complex of the action functor, such as brackets (see, e.g., [BD]), can be transported along the isomorphism (2.21) as well.

The remainder of this section is dedicated to the proof that the maps (2.31) indeed endow the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G)$ with the structure of a dg-module over the dg-algebra $\big(C^\bullet_{\mathrm{DY}}(\mathcal{C}), \cup\big)$.

Lemma 2.16. *For $n \geq 1$ and $1 \leq i \leq n$, the following equality holds:*

$$\begin{aligned} &(\Phi_{n+1})_{X_1,\dots,X_{n+1}} \\ =&(\Phi_n)_{X_1,\dots,X_{i-1},X_iX_{i+1},X_{i+2},\dots X_{n+1}} \circ (\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})}) \end{aligned}$$

Proof:
For $n=1$, the statement follows immediately. Let $n \geq 2$. In the following calculation, we use the definition of Φ_n (see (B.9)) in the first equation, the naturality of $\Phi_{X_i,X_{i+1}}$ in the second equation and the hexagon axiom (A.2) in the third equation:

$$\begin{aligned} &(\Phi_n)_{X_1,\dots,X_{i-1},X_iX_{i+1},X_{i+2},\dots X_{n+1}} \circ (\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})}) \\ =&\Phi_{X_1,X_2\dots X_{n+1}} \circ (\mathrm{id}_{G(X_1)} \otimes \Phi_{X_2,X_3\dots X_{n+1}}) \circ \dots \circ (\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_iX_{i+1},X_{i+2}\dots X_{n+1}}) \\ &\circ (\mathrm{id}_{G(X_1)\dots G(X_iX_{i+1})} \otimes \Phi_{X_{i+2},X_{i+3}\dots X_{n+1}}) \circ \dots \circ (\mathrm{id}_{G(X_1)\dots G(X_iX_{i+1})\dots G(X_{n-1})} \otimes \Phi_{X_n,X_{n+1}}) \\ &\circ (\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})}) \\ =&\Phi_{X_1,X_2\dots X_{n+1}} \circ (\mathrm{id}_{G(X_1)} \otimes \Phi_{X_2,X_3\dots X_{n+1}}) \circ \dots \circ (\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_iX_{i+1},X_{i+2}\dots X_{n+1}}) \end{aligned}$$

$$\circ\,(\mathrm{id}_{G(X_1)\dots G(X_{i-1})}\otimes\Phi_{X_i,X_{i+1}}\otimes\mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})})$$
$$\circ\,(\mathrm{id}_{G(X_1)\dots G(X_{i+1})}\otimes\Phi_{X_{i+2},X_{i+3}\dots X_{n+1}})\circ\dots\circ(\mathrm{id}_{G(X_1)\dots G(X_{n-1})}\otimes\Phi_{X_n,X_{n+1}})$$
$$=\Phi_{X_1,X_2\dots X_{n+1}}\circ(\mathrm{id}_{G(X_1)}\otimes\Phi_{X_2,X_3\dots X_{n+1}})\circ\dots\circ(\mathrm{id}_{G(X_1)\dots G(X_{i-1})}\otimes\Phi_{X_i,X_{i+1}\dots X_{n+1}})$$
$$\circ\,(\mathrm{id}_{G(X_1)\dots G(X_i)}\otimes\Phi_{X_{i+1},X_{i+2}\dots X_{n+1}})\circ\dots\circ(\mathrm{id}_{G(X_1)\dots G(X_{n-1})}\otimes\Phi_{X_n,X_{n+1}})$$
$$=(\Phi_{n+1})_{X_1,\dots,X_{n+1}}$$

□

Lemma 2.17. *For $n,n'\geq 1$, the following equality holds:*

$$(\Phi_{n+n'})_{X_1,\dots,X_{n+1}}=\Phi_{X_1\dots X_n,X_{n+1}\dots X_{n+n'}}\circ\big((\Phi_n)_{X_1,\dots,X_n}\otimes(\Phi_{n'})_{X_{n+1},\dots,X_{n+n'}}\big)$$

Proof:
This result follows from Lemma 2.16 together with the hexagon axiom (A.2).

□

Proposition 2.18. *The maps (2.31) endow $C^\bullet_{\mathrm{DY}}(G)$ with the structure of a dg-module over the dg-algebra $(C^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}),\cup)$.*

Proof:
We first need to show that the equation

$$(f\cup f').g=f.(f'.g)$$

holds for all $f\in C^m_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}), f'\in C^{m'}_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ and $g\in C^n_{\mathrm{DY}}(G)$. We only prove the case $n,m\geq 1$ because the other cases are completely analogous. In the following computation, we have used the hexagon diagram (A.2) as well as Lemma 2.17 in the fourth equation:

$$\big((f\cup f').g\big)_{X_1,\dots,X_{m+m'+n}}$$
$$=\Phi_{X_1\dots X_{m+m'},X_{m+m'+1}\dots X_{m+m'+n}}\circ\big(G((f\cup f')_{X_1,\dots,X_{m+m'}})\otimes g_{X_{m+m'+1},\dots,X_{m+m'+n}}\big)$$
$$\circ\,\big((\Phi_{m+m'})_{X_1\dots X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1})\dots G(X_{m+m'+n})}\big)$$
$$=\Phi_{X_1\dots X_{m+m'},X_{m+m'+1}\dots X_{m+m'+n}}\circ\big(G(f_{X_1,\dots,X_m}\otimes f'_{X_{m+1},\dots,X_{m+m'}})\otimes g_{X_{m+m'+1},\dots,X_{m+m'+n}}\big)$$
$$\circ\,\big((\Phi_{m+m'})_{X_1\dots X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1})\dots G(X_{m+m'+n})}\big)$$
$$=\Phi_{X_1\dots X_{m+m'},X_{m+m'+1}\dots X_{m+m'+n}}\circ(\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1}\dots X_{m+m'+n})})$$
$$\circ\,\big(G(f_{X_1,\dots,X_m})\otimes G(f'_{X_{m+1},\dots,X_{m+m'}})\otimes g_{X_{m+m'+1},\dots,X_{m+m'+n}}\big)$$
$$\circ\,(\Phi^{-1}_{X_1\dots X_m,X_{m+1}\dots X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1}\dots X_{m+m'+n})})$$
$$\circ\,\big((\Phi_{m+m'})_{X_1\dots X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1})\dots G(X_{m+m'+n})}\big)$$
$$=\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+m'+n}}\circ(\mathrm{id}_{G(X_1\dots X_m)}\otimes\Phi_{X_{m+1}\dots X_{m+m'},X_{m+m'+1}\dots X_{m+m'+n}})$$
$$\circ\,\big(G(f_{X_1,\dots,X_m})\otimes G(f'_{X_{m+1},\dots,X_{m+m'}})\otimes g_{X_{m+m'+1},\dots,X_{m+m'+n}}\big)$$
$$\circ\,\big((\Phi_m)_{X_1,\dots,X_m}\otimes(\Phi_{m'})_{X_{m+1},\dots,X_{m+m'}}\otimes\mathrm{id}_{G(X_{m+m'+1})\dots G(X_{m+m'+n})}\big)$$

$$
\begin{aligned}
=&\Phi_{X_1\ldots X_m, X_{m+1}\ldots X_{m+m'+n}} \circ \big(G(f_{X_1,\ldots,X_m}) \otimes (f'.g)_{X_{m+1},\ldots,X_{m+m'+n}}\big)\\
&\circ \big((\Phi_m)_{X_1,\ldots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\ldots G(X_{m+m'+n})}\big)\\
=&\big(f.(f'.g)\big)_{X_1,\ldots,X_{m+m'+n}}
\end{aligned}
$$

Now we show that the action (2.31) satisfies the graded Leibniz rule, i.e., the equation

$$
\partial_{\mathrm{DY}}^{m+n}(f.g) = \partial_{\mathrm{DY}}^{m}(f).g + (-1)^m f.\partial_{\mathrm{DY}}^{n}(g) \tag{2.37}
$$

holds for all $f \in C_{\mathrm{DY}}^m(\mathrm{Id}_{\mathcal{C}}), g \in C_{\mathrm{DY}}^n(G)$. We only prove the case $m, n \geq 1$ because the other cases are completely analogous. We first compute the left-hand side of (2.37):

$$
\begin{aligned}
&\partial_{\mathrm{DY}}^{m+n}(f.g)_{X_1,\ldots,X_{m+n+1}}\\
=&(\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}} \circ \big(\mathrm{id}_{G(X_1)} \otimes (\Phi_{m+n}^{-1})_{X_2,\ldots,X_{m+n+1}}\big) \circ \big(\mathrm{id}_{G(X_1)} \otimes (f.g)_{X_2,\ldots,X_{m+n+1}}\big)\\
&+\sum_{i=1}^{m+n} (-1)^i (\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}} \circ (\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}}^{-1} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{m+n+1})})\\
&\quad \circ (\Phi_{m+n}^{-1})_{X_1,\ldots,X_iX_{i+1},\ldots X_{m+n+1}} \circ (f.g)_{X_1,\ldots,X_iX_{i+1},\ldots X_{m+n+1}}\\
&\quad \circ (\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{m+n+1})})\\
&+(-1)^{m+n+1} (\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}} \circ \big((\Phi_{m+n}^{-1})_{X_1,\ldots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\big)\\
&\quad \circ \big((f.g)_{X_1,\ldots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\big)\\
=&(\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}} \circ \big(\mathrm{id}_{G(X_1)} \otimes (\Phi_{m+n}^{-1})_{X_2,\ldots,X_{m+n+1}}\big)\\
&\quad \circ (\mathrm{id}_{G(X_1)} \otimes \Phi_{X_2\ldots X_{m+1}, X_{m+2}\ldots X_{m+n+1}}) \circ \big(\mathrm{id}_{G(X_1)} \otimes G(f_{X_2,\ldots,X_{m+1}}) \otimes g_{X_{m+2},\ldots,X_{m+n+1}}\big)\\
&+\sum_{i=1}^{m} (-1)^i (\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}}\\
&\quad \circ (\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}}^{-1} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{m+1})} \otimes \mathrm{id}_{G(X_{m+2})\ldots G(X_{m+n+1})}\big)\\
&\quad \circ (\Phi_{m+n}^{-1})_{X_1,\ldots,X_iX_{i+1},\ldots X_{m+n+1}} \circ \Phi_{X_1\ldots X_{m+1}, X_{m+2}\ldots X_{m+n1}}\\
&\quad \circ \big(G(f_{X_1,\ldots,X_iX_{i+1},\ldots,X_{m+1}}) \otimes g_{X_{m+2},\ldots,X_{m+n+1}}\big)\\
&\quad \circ \big((\Phi_m)_{X_1,\ldots,X_iX_{i+1},\ldots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2})\ldots G(X_{m+n+1})}\big)\\
&\quad \circ (\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{m+1})} \otimes \mathrm{id}_{G(X_{m+2})\ldots G(X_{m+n+1})}\big)\\
&+\sum_{j=1}^{n} (-1)^{m+j} (\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}}\\
&\quad \circ (\mathrm{id}_{G(X_1)\ldots G(X_m)} \otimes \mathrm{id}_{G(X_{m+1})\ldots G(X_{m+j-1})} \otimes \Phi_{X_{m+j},X_{m+j+1}}^{-1} \otimes \mathrm{id}_{G(X_{m+j+2})\ldots G(X_{m+n+1})}\big)\\
&\quad \circ (\Phi_{m+n}^{-1})_{X_1,\ldots,X_m,X_{m+1},\ldots,X_{m+j}X_{m+j+1},\ldots X_{m+n+1}} \circ \Phi_{X_1\ldots X_{m+1}, X_{m+2}\ldots X_{m+n1}}\\
&\quad \circ \big(G(f_{X_1,\ldots,X_m}) \otimes g_{X_{m+1},\ldots,X_{m+j}X_{m+j+1},\ldots,X_{m+n+1}}\big)\\
&\quad \circ \big((\Phi_m)_{X_1,\ldots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\ldots G(X_{m+j}X_{m+j+1})\ldots G(X_{m+n+1})}\big)\\
&\quad \circ (\mathrm{id}_{G(X_1)\ldots G(X_m)} \otimes \mathrm{id}_{G(X_{m+1})\ldots G(X_{m+j-1})} \otimes \Phi_{X_{m+j},X_{m+j+1}} \otimes \mathrm{id}_{G(X_{m+j+2})\ldots G(X_{m+n+1})}\big)\\
&+(-1)^{m+n+1} (\Phi_{m+n+1})_{X_1,\ldots,X_{m+n+1}} \circ \big((\Phi_{m+n}^{-1})_{X_1,\ldots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\big)\\
&\quad \circ (\Phi_{X_1\ldots X_m, X_{m+1}\ldots X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}) \circ \big(G(f_{X_1,\ldots,X_m}) \otimes g_{X_{m+1},\ldots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\big)\\
&\quad \circ \big((\Phi_m)_{X_1,\ldots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\ldots G(X_{m+n+1})}\big)
\end{aligned} \tag{2.38}
$$

We now compute the right-hand side of (2.37), which gives

$$
\begin{aligned}
&\left(\partial^m_{\mathrm{DY}}(f).g\right)_{X_1,\dots,X_{m+n+1}}\\
=&\Phi_{X_1\dots X_{m+1},X_{m+2}\dots X_{m+n+1}} \circ \left(G(\partial^m_{\mathrm{DY}}(f)_{X_1,\dots,X_{m+1}}) \otimes g_{X_{m+2},\dots X_{m+n+1}}\right)\\
&\quad\circ \left((\Phi_{m+1})_{X_1,\dots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2}\dots G(X_{m+n+1}))}\right)\\
=&\Phi_{X_1\dots X_{m+1},X_{m+2}\dots X_{m+n+1}} \circ \left(G(\mathrm{id}_{X_1} \otimes f_{X_2,\dots,X_{m+1}}) \otimes g_{X_{m+2},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left((\Phi_{m+1})_{X_1,\dots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2}\dots G(X_{m+n+1}))}\right)\\
&+\sum_{i=1}^{m}(-1)^i\Phi_{X_1\dots X_{m+1},X_{m+2}\dots X_{m+n+1}} \circ \left(G(f_{X_1,\dots,X_iX_{i+1},\dots,X_{m+1}}) \otimes g_{X_{m+2},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left((\Phi_{m+1})_{X_1,\dots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2}\dots G(X_{m+n+1}))}\right)\\
&+(-1)^{m+1}\Phi_{X_1\dots X_{m+1},X_{m+2}\dots X_{m+n+1}} \circ \left(G(f_{X_1,\dots,X_m} \otimes \mathrm{id}_{X_{m+1}}) \otimes g_{X_{m+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left((\Phi_{m+1})_{X_1,\dots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2})\dots G(X_{m+n+1})}\right)
\end{aligned}
\tag{2.39}
$$

and

$$
\begin{aligned}
&(-1)^m\left(f.\partial^n_{\mathrm{DY}}(g)\right)_{X_1,\dots,X_{m+n+1}}\\
=&(-1)^m\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+n+1}} \circ \left(G(f_{X_1,\dots,X_m}) \otimes \partial^n_{\mathrm{DY}}(g)_{X_{m+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left((\Phi_m)_{X_1,\dots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+n+1})}\right)\\
=&(-1)^m\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+n+1}} \circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes (\Phi_{n+1})_{X_{m+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes \mathrm{id}_{G(X_{m+1})} \otimes (\Phi_n^{-1})_{X_{m+2},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(G(f_{X_1,\dots,X_m}) \otimes \mathrm{id}_{G(X_{m+1})} \otimes g_{X_{m+2},\dots X_{m+n+1}}\right) \circ \left((\Phi_m)_{X_1,\dots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+n+1})}\right)\\
&+\sum_{j=1}^{n}(-1)^{m+j}\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+n+1}} \circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes (\Phi_{n+1})_{X_{m+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+j-1})} \otimes \Phi^{-1}_{X_{m+j},X_{m+j+1}} \otimes \mathrm{id}_{G(X_{m+j+2})\dots G(X_{m+n+1})}\right)\\
&\quad\circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes (\Phi_n^{-1})_{X_{m+1},\dots,X_{m+j}X_{m+j+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(G(f_{X_1,\dots,X_m}) \otimes g_{X_{m+1},\dots,X_{m+j}X_{m+j+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+j-1})} \otimes \Phi_{X_{m+j},X_{m+j+1}} \otimes \mathrm{id}_{G(X_{m+j+2})\dots G(X_{m+n+1})}\right)\\
&\quad\circ \left((\Phi_m)_{X_1,\dots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+n+1})}\right)\\
&+(-1)^{m+n+1}\Phi_{X_1\dots X_m,X_{m+1}\dots X_{m+n+1}} \circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes (\Phi_{n+1})_{X_{m+1},\dots,X_{m+n+1}}\right)\\
&\quad\circ \left(\mathrm{id}_{G(X_1\dots X_m)} \otimes (\Phi_n^{-1})_{X_{m+1},\dots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\right)\\
&\quad\circ \left(G(f_{X_1,\dots,X_m}) \otimes g_{X_{m+1},\dots,X_{m+n}} \otimes \mathrm{id}_{G(X_{m+n+1})}\right) \circ \left((\Phi_m)_{X_1,\dots,X_m} \otimes \mathrm{id}_{G(X_{m+1})\dots G(X_{m+n+1})}\right)
\end{aligned}
\tag{2.40}
$$

To see why equation (2.37) holds, we will go through (2.38) term by term. The first boundary terms of (2.39) and (2.38) agree: we write $G(\mathrm{id}_{(X_1)} \otimes f_{X_2,\dots,X_{m+1}})$ as $\Phi_{X_1,X_2\dots X_{m+1}} \circ \left(\mathrm{id}_{G(X_1)} \otimes G(f_{X_2,\dots,X_{m+1}})\right) \circ \Phi^{-1}_{X_1,X_2\dots X_{m+1}}$ and apply Lemma 2.17 to the term Φ_{m+1} in (2.39) and to the terms Φ_{m+n+1} and Φ_{m+n} in (2.38).
Note that the last boundary term of (2.39) and the first boundary term of (2.40) cancel each other: we apply Lemma 2.17 to the term Φ_{m+1} in (2.39) and the term Φ_{n+1} in (2.40). We write $G(f_{X_1,\dots,X_m} \otimes \mathrm{id}_{X_{m+1}})$ as $\Phi_{X_1\dots X_m,X_{m+1}} \circ \left(G(f_{X_1,\dots,X_m}) \otimes \mathrm{id}_{G(X_{n+1})}\right) \circ \Phi^{-1}_{X_1\dots X_m,X_{m+1}}$ and finally use the hexagon axiom (A.2).
Now we look at the bulk terms of (2.38): we apply Lemma 2.17 to the term Φ_{m+n+1} and Lemma 2.16 to the term

$$\big((\Phi_m)_{X_1,\ldots,X_iX_{i+1},\ldots,X_{m+1}} \otimes \mathrm{id}_{G(X_{m+2})\ldots G(X_{m+n+1})}\big)$$
$$\circ\, (\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{m+1})} \otimes \mathrm{id}_{G(X_{m+2})\ldots G(X_{m+n+1})}\big)$$

in the sum indexed by i and we apply Lemma 2.17 to the terms Φ_{m+n+1} and Φ^{-1}_{m+n} in the sum indexed by j. This shows that the bulk terms of (2.38) agree with the bulk terms of the sum of (2.39) and (2.40).
Applying Lemma 2.17 to the terms Φ_{m+n+1} and Φ^{-1}_{m+n} in the last boundary term of (2.38), we see that it agrees with the last boundary term of (2.40). □

Corollary 2.19. *The Davydov-Yetter cohomology $H^\bullet_{\mathrm{DY}}(G)$ of a monoidal functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ inherits the structure of a graded module over the graded associative k-algebra $(H^\bullet_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}}), \cup)$.*

We summarize some of our results from this section in the following

Corollary 2.20. *Every Davydov-Yetter 2-cocycle $\Psi^{(1)} \in C^2_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ gives rise to an infinitesimal associator deformation $\big(R_1 \otimes \mathcal{M}, m^{(1)}\big)$, where $m^{(1)} \in C^2_{ass}(\mathcal{C}, \mathcal{M})$ is the associator 2-cocycle with*

$$m^{(1)}_{X,Y,M} = m_{X,Y,M} \circ \big(\Psi^{(1)}_{X,Y} \rhd \mathrm{id}_M\big) \tag{2.41}$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$.

If the Davydov-Yetter 2-cocycles $\Psi^{(1)}_1, \Psi^{(1)}_2 \in C^2_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ are cohomologous, then the associator deformations $\big(R_1 \otimes \mathcal{M}, m^{(1)}_1\big)$ and $\big(R_1 \otimes \mathcal{M}, m^{(1)}_2\big)$ with $(m^{(1)}_i)_{X,Y,M} = m_{X,Y,M} \circ \big((\Psi^{(1)}_i)_{X,Y} \rhd \mathrm{id}_M\big)$ for $i = 1, 2$, are equivalent in the sense of Definition 1.2.

Proof:
In (2.36), we have seen that (2.41) is an associator 2-cocycle. If two Davydov-Yetter 2-cocycles $\Psi^{(1)}_1, \Psi^{(1)}_2 \in C^2_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ are cohomologous, then by Proposition 2.15, the associator 2-cocycles $m^{(1)}_1$ and $m^{(1)}_2$ with $(m^{(1)}_1)_{X,Y,M} = m_{X,Y,M} \circ \big((\Psi^{(1)}_1)_{X,Y} \rhd \mathrm{id}_M\big)$ and $(m^{(1)}_2)_{X,Y,M} = m_{X,Y,M} \circ \big((\Psi^{(1)}_2)_{X,Y} \rhd \mathrm{id}_M\big)$ are cohomologous as well. Hence, by Theorem 1.8, the associator deformations $\big(R_1 \otimes \mathcal{M}, m^{(1)}_1\big)$ and $\big(R_1 \otimes \mathcal{M}, m^{(1)}_2\big)$ are equivalent in the sense of Definition 1.2.

□

2.5 Vanishing of associator cohomology

In the following, let k be an algebraically closed field of characteristic 0 and all categories and functors in this section are assumed to be k-linear. Let $\mathcal{C}$ be a strict finite multitensor category (see Definition A.12) and let $\mathcal{M}$ be a $\mathcal{C}$-module category, which is also a finite abelian category.

Having related Davydov-Yetter cohomology and associator deformation cohomology (Proposition 2.9), we can use results on Ocneanu rigidity for the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ (which amounts to a vanishing Davydov-Yetter cohomology) to obtain rigidity results for module endofunctors as well as for module categories. First, we recall the following vanishing result for Davydov-Yetter cohomology.

Proposition 2.21. [GHS, Corollary 3.18] *Let $G : \mathcal{C} \longrightarrow \mathcal{D}$ be a strict monoidal functor between strict semisimple finite multitensor categories. Then, $H^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \cong 0$ for $n > 0$ and for all $\mathrm{X}, \mathrm{Y} \in \mathcal{Z}(G)$.*

This allows us to show

Proposition 2.22. *If $\mathcal{C}$ and $\mathcal{M}$ are semisimple and $(F, s), (F', s') : \mathcal{M} \longrightarrow \mathcal{M}$ are $\mathcal{C}$-module functors, then $H^n_{ass}\big((F, s), (F', s')\big) \cong 0$ for $n > 0$.*

Proof:
Since $\mathcal{M}$ is semisimple, every k-linear endofunctor of $\mathcal{M}$ is exact. Since $\mathcal{M}$ is finite as well, Proposition A.11 yields that all k-linear endofunctors of $\mathcal{M}$ admit left and right adjoints, i.e., every object in the monoidal category $\mathrm{End}(\mathcal{M})$ has both a left and right dual. Since $\mathcal{M}$ is finite, the category $\mathrm{End}(\mathcal{M})$ is finite and thus a semisimple finite multitensor category.
Thus, the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ is a strict monoidal functor between semisimple finite multitensor categories and hence

$$H^n_{ass}\big((F, s), (F', s')\big) \cong H^n_{\mathrm{DY}}\big(\rho_{\mathcal{M}}, (F, s), (F', s')\big) \cong 0$$

for $n > 0$ where we have used Proposition 2.9 in the first isomorphism and Proposition 2.21 in the second isomorphism.

□

Corollary 2.23. *If $\mathcal{C}$ and $\mathcal{M}$ are semisimple and $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}$ is a $\mathcal{C}$-module endofunctor, then $H^1_{ass}(F, s) \cong 0$ and thus (F, s) does not admit infinitesimal deformations in the sense of Definition 1.3.*

Corollary 2.24. *If $\mathcal{C}$ and $\mathcal{M}$ are semisimple, then $H^n_{ass}(\mathcal{C}, \mathcal{M}) \cong 0$ for $n > 0$. In particular, $\mathcal{M}$ does not admit associator deformations in the sense of Definition 1.1.*

3 Associator cohomology as a relative Ext

In Section 2, we have been exploring the connection of associator cohomology and Davydov-Yetter cohomology. To gain more tools for the computation of associator cohomology, we will use methods from relative homological algebra, more precisely the theory of relative Exts which has been used successfully in Davydov-Yetter theory (see [FGS1]).

In this section, we are going to present associator cohomology as a relative Ext in Theorem 3.7, but first we review the link between Davydov-Yetter cohomology and relative Exts. In the following, we assume that k is an algebraically closed field.

3.1 Davydov-Yetter cohomology as a relative Ext

Following [M1, Chapter IX] and [FGS1, Section 2.1], we provide a brief review of some notions from relative homological algebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories and let $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive, exact and faithful functor and let $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{A}$ be its left adjoint, i.e., we have an adjunction $\mathcal{F} \dashv \mathcal{U}$ of a special type, called a *resolvent pair*.

Allowable morphisms. A morphism $f : A \longrightarrow A'$ in $\mathcal{A}$ is called *allowable*, if there is a morphism $g : \mathcal{U}(A') \longrightarrow \mathcal{U}(A)$ such that $\mathcal{U}(f) \circ g \circ \mathcal{U}(f) = \mathcal{U}(f)$.

Relatively projectives. An object $P \in \mathcal{A}$ is called *relatively projective*, if for all morphisms $h : P \longrightarrow A'$ and all exact sequences $A \xrightarrow{f} A' \longrightarrow 0$ in $\mathcal{A}$, where f is allowable, there is a morphism $h_f : P \longrightarrow A$ in $\mathcal{A}$, such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & \exists h_f \swarrow & \downarrow h & & \\ A & \xrightarrow[f]{} & A' & \longrightarrow & 0 \end{array}$$

Relatively projective resolutions. A *relative projective resolution* of an object $A \in \mathcal{A}$ is an exact sequence

$$\ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$$

in $\mathcal{A}$, where each morphism is allowable and each $P_i \in \mathcal{A}$ is relatively projective.

Similar to the characterization of projective objects in ordinary homological algebra, one can characterize relatively projective objects by exactness properties of a certain Hom functor:

Lemma 3.1. [FGS1, Lemma 2.5] *An object $P \in \mathcal{A}$ is relatively projective if and only if the k-linear functor* $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \longrightarrow$ **vect** *sends allowable short exact sequences to short exact sequences.*

The notion of a relative Ext is defined analogously to the notion of an Ext in homological algebra but replacing exact sequences with exact sequences of *allowable morphisms*:

Definition 3.2. Let $A, A' \in \mathcal{A}$ and let $\ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$ be a relatively projective resolution of A in $\mathcal{A}$. The nth cohomology of the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_0, A') \xrightarrow{d_1^*} \mathrm{Hom}_{\mathcal{A}}(P_1, A') \xrightarrow{d_2^*} \mathrm{Hom}_{\mathcal{A}}(P_2, A') \xrightarrow{d_3^*} \ldots$$

is called the *nth relative* Ext *of A and A'* and is denoted by $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(A, A')$.

We now review a connection between the Davydov-Yetter cohomology of a monoidal functor and the corresponding relative Exts established in [FGS1] and based on the previous work [GHS].

Let $\mathcal{C}$ be a strict finite tensor category and let $\mathcal{D}$ be a strict monoidal category, which is not necessarily rigid. Let $G : \mathcal{C} \longrightarrow \mathcal{D}$ be a strict monoidal functor. We first notice that we can define left dual objects in $\mathcal{D}$ for objects that lie in the image of G: for all $X \in \mathcal{C}$, we set $G(X)^* := G(X^*)$ and we define the (co)evaluation maps for $G(X)$ as the images of the (co)evaluation maps of $X \in \mathcal{C}$ under G. The zigzag axioms (see Definition A.8) follow immediately from the corresponding zigzag axioms in $\mathcal{C}$. The right dual of $G(X)$ is defined in a similar manner. In other words, while $\mathcal{D}$ might be non-rigid, the image of G is still a rigid monoidal subcategory of $\mathcal{D}$. We also note that no assumptions on the simplicity of the tensor unit in $\mathcal{D}$ were made.

From now on, we will furthermore assume that $\mathcal{D}$ is a finite abelian category with right exact tensor product, an assumption which is satisfied by the monoidal category of right exact linear endofunctors $\mathrm{Rex}(\mathcal{M})$ of a finite abelian category $\mathcal{M}$, which we will review in the following section. Moreover, we assume that the functor G is k-linear and right exact. Now, the relevant resolvent pair for the Davydov-Yetter cohomology of G is the following adjunction between the target category $\mathcal{D}$ and the centralizer of the monoidal functor G:

$$\begin{array}{c} \mathcal{Z}(G) \\ \mathcal{F}_G \left\uparrow \dashv \right\downarrow \mathcal{U} \\ \mathcal{D} \end{array} \tag{3.1}$$

Here, the left adjoint $\mathcal{F}_G : \mathcal{D} \longrightarrow \mathcal{Z}(G)$ of the forgetful functor $\mathcal{U} : \mathcal{Z}(G) \longrightarrow \mathcal{D}$ is given by

$$\mathcal{F}_G : \ V \longmapsto \big(Z_G(V), j^{Z_G(V)}\big) \tag{3.2}$$

where $Z_G(V)$ is the following coend

$$Z_G(V) := \int^{X \in \mathcal{C}} G(X)^* \otimes V \otimes G(X) \tag{3.3}$$

that admits a canonical half-braiding $j^{Z_G(V)}$ which will be reviewed below. This adjunction is discussed in detail in [GHS, Section 3.3], where the category $\mathcal{D}$ was assumed to be a finite tensor category. But this assumption is too strong for our next applications in the world of module categories, which is why we consider more general target categories of G. Fortunately, all the results from [GHS, Section 3.3] are still valid under our weaker assumptions on $\mathcal{D}$: the existence of the above coend, its (bi)monadic property, etc. Let us begin with proving the existence of the coend (3.3) as an object of $\mathcal{D}$. We start with the known result [S1, Theorem 3.6] on the existence of the following coend in $\mathcal{C} \boxtimes \mathcal{C}$:

$$A := \int^{X \in \mathcal{C}} X^* \boxtimes X \ . \tag{3.4}$$

For all $V \in \mathcal{D}$, we consider the functor

$$\begin{aligned} F_V : \quad \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{D} \\ (Y, X) &\longmapsto G(Y) \otimes_{\mathcal{D}} V \otimes_{\mathcal{D}} G(X) \\ (f, g) &\longmapsto G(f) \otimes_{\mathcal{D}} \mathrm{id}_V \otimes_{\mathcal{D}} G(g) \end{aligned}$$

where we emphasized the tensor product $\otimes_{\mathcal{D}}$ in $\mathcal{D}$ which is assumed to be k-linear and right exact in both variables. Since G is assumed to be k-linear and right exact, the above functor F_V is also k-linear and right exact in both variables for all $V \in \mathcal{D}$. Therefore, by the universal property of the Deligne product (see [D2]), there is a unique k-linear right exact functor $\tilde{F}_V : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathcal{D}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{F_V} & \mathcal{D} \\ \boxtimes \big\downarrow & \nearrow_{\exists\, \tilde{F}_V} & \\ \mathcal{C} \boxtimes \mathcal{C} & & \end{array}$$

Furthermore, using the assumption that both $\mathcal{C}$ and $\mathcal{D}$ are finite abelian categories, the k-linear right exact functor $\tilde{F}_V$ has a right adjoint by Proposition A.11, and therefore, $\tilde{F}_V$ is left adjoint. Applying this left adjoint functor $\tilde{F}_V$ to the existing coend A from (3.4) and using that left adjoint functors commute with all small colimits that exist in its source category (see, e.g., [M1, Chapter V.5]), we get that $\tilde{F}_V(A) \cong Z_G(V)$ and the coend $Z_G(V)$ from (3.3) exists for all $V \in \mathcal{D}$ under the assumptions made.

For all $V \in \mathcal{D}$, we denote the universal dinatural transformation of the coend $Z_G(V)$ by $i(V)$; its component on any $Y \in \mathcal{C}$ is the following morphism in $\mathcal{D}$:

$$i_Y(V) : \quad G(Y)^* \otimes V \otimes G(Y) \longrightarrow Z_G(V)$$

In the following, we will illustrate how to use the universal property of the coend $Z_G(V)$ (see Definition A.3) to define morphisms that have $Z_G(V)$ as a source. Let $f : V \longrightarrow W$ be a morphism in $\mathcal{D}$. It is easy to check that the family $\beta = (\beta_Y)_{Y \in \mathcal{C}}$ of morphisms

$$\beta_Y = i_Y(W) \circ (\mathrm{id}_{G(Y)^*} \otimes f \otimes \mathrm{id}_{G(Y)}) : \quad G(Y)^* \otimes V \otimes G(Y) \longrightarrow Z_G(W)$$

is a dinatural transformation. By the universal property of the coend $Z_G(V)$, there is a unique morphism $Z_G(f) : Z_G(V) \longrightarrow Z_G(W)$ in $\mathcal{D}$, such that the following diagram commutes for all $Y \in \mathcal{C}$:

$$\begin{array}{ccc} G(Y)^* \otimes V \otimes G(Y) & \xrightarrow{\mathrm{id}_{G(Y)^*} \otimes f \otimes \mathrm{id}_{G(Y)}} & G(Y)^* \otimes W \otimes G(Y) \\ {\scriptstyle i_Y(V)} \big\downarrow & & \big\downarrow {\scriptstyle i_Y(W)} \\ Z_G(V) & \underset{\exists!\ Z_G(f)}{\dashrightarrow} & Z_G(W) \end{array}$$

Using this procedure, we can define the value of Z_G on morphisms, turning it into a functor (see also Proposition A.5). Moreover, the functor Z_G is a monad on $\mathcal{D}$ (see, e.g., [DS] and [S4] for more details).
In the sequel, we will often implicitly use the universal property of the coend to define morphisms.

Next, recall that $Z_G(V)$ is equipped with a canonical half-braiding $j^{Z_G(V)}$ making it an object in the centralizer $\mathcal{Z}(G)$: for all $Y \in \mathcal{C}$, the canonical half-braiding $j^{Z_G(V)}$ is the natural family of isomorphisms

$$j_Y^{Z_G(V)}: \quad Z_G(V) \otimes G(Y) \xrightarrow{\cong} G(Y) \otimes Z_G(V)$$

where $j_Y^{Z_G(V)}$ is defined as the unique morphism such that the following diagram commutes for all $X \in \mathcal{C}$ (see also [FS, Lemma 13]):

$$\begin{array}{ccc}
G(X)^* \otimes V \otimes G(X) \otimes G(Y) & \xrightarrow{i_X(V) \otimes \mathrm{id}_{G(Y)}} & Z_G(V) \otimes G(Y) \\
\Big\downarrow{\scriptstyle \mathrm{coev}_{G(Y)} \otimes \mathrm{id}_{G(X)^* \otimes V \otimes G(X) \otimes G(Y)}} & & \Big\downarrow{\scriptstyle j_Y^{Z_G(V)}} \\
\underbrace{G(Y) \otimes G(Y)^* \otimes G(X)^* \otimes V \otimes G(X) \otimes G(Y)}_{\cong G(Y) \otimes G(X \otimes Y)^* \otimes V \otimes G(X \otimes Y)} & \xrightarrow[\mathrm{id}_{G(Y)} \otimes i_{XY}(V)]{} & G(Y) \otimes Z_G(V)
\end{array}$$

One can show that $j^{Z_F(V)}$ is indeed a half-braiding relative to G (see Definition B.2).

We are now left to check that the left adjoint to $\mathcal{U}$ is indeed given by the functor $\mathcal{F}_G$ from (3.2). This is again quite standard (see, e.g., [DS, BV, S4]) and we sketch the arguments here. For this purpose, consider the map

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}\big((Z_G(V), j^{Z_G(V)}), (W, \sigma)\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big(V, \mathcal{U}(W, \sigma)\big) \tag{3.5}$$

which sends a morphism $f : (Z_G(V), j^{Z_G(V)}) \longrightarrow (W, \sigma)$ in $\mathcal{Z}(\mathcal{C})$ to the morphism

$$V = \underbrace{I \otimes V \otimes I}_{= G(I)^* \otimes V \otimes G(I)} \xrightarrow{i_I(V)} Z_G(V) \xrightarrow{f} \mathcal{U}(W, \sigma) = W$$

in $\mathcal{D}$. The inverse of (3.5) is given by the map

$$\mathrm{Hom}_{\mathcal{C}}\big(V, \mathcal{U}(W, \sigma)\big) \longrightarrow \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}\big((Z_G(V), j^{Z_G(V)}), (W, \sigma)\big)$$

which sends a morphism $g : V \longrightarrow \mathcal{U}(W, \sigma)$ to the unique morphism $\tilde{g} : (Z_G(V), j^{Z_G(V)}) \longrightarrow (W, \sigma)$ in $\mathcal{Z}(\mathcal{C})$, such that the following diagram commutes for all $X \in \mathcal{C}$:

$$\begin{array}{ccc}
G(X)^* \otimes V \otimes G(X) & \xrightarrow{i_X(V)} & Z_G(V) \\
\Big\downarrow{\scriptstyle \mathrm{id}_{G(X)^*} \otimes g \otimes \mathrm{id}_{G(X)}} & & \\
G(X)^* \otimes W \otimes G(X) & & \Big\downarrow{\scriptstyle \exists!\ \tilde{g}} \\
\Big\downarrow{\scriptstyle \mathrm{id}_{G(X)^*} \otimes \sigma_{G(X)}} & & \\
G(X)^* \otimes G(X) \otimes W & \xrightarrow[\mathrm{ev}_{G(X)} \otimes \mathrm{id}_W]{} & I \otimes W = W
\end{array}$$

The final step in our review of the relation of Davydov-Yetter cohomology to relative Exts is to show that the Davydov-Yetter complex of G with coefficients is isomorphic to the bar complex of the monad $Z_G(-)$. This was proven in [GHS, Lemma 3.15] under the assumption that $\mathcal{D}$ is rigid, but the proof actually follows verbatim under our assumptions on the target category $\mathcal{D}$ (recall from above that the image of G is a rigid monoidal subcategory of $\mathcal{D}$). Therefore, the main result in [GHS, Theorem 3.11] is equally valid in our setting and this is the starting point of the construction in the follow-up paper [FGS1] that expresses the Davydov-Yetter cohomology of a monoidal functor G as a relative Ext of the corresponding adjunction (3.1). In [FGS1], no exactness properties of G or of the tensor product in $\mathcal{D}$ were used, after one proves the existence of the coend $Z_G(V)$ which we did above. Hence, we are able to reformulate the main result of [FGS1] in slightly more generality:

Proposition 3.3. [FGS1, Corollary 4.7] *Let $\mathcal{C}$ be a strict finite tensor category, let $\mathcal{D}$ be a finite abelian strict monoidal category with right exact tensor product and let $G : \mathcal{C} \longrightarrow \mathcal{D}$ be a right exact k-linear monoidal functor. The nth Davydov-Yetter cohomology of G with coefficients $\mathrm{X}, \mathrm{Y} \in \mathcal{Z}(G)$ is isomorphic to the nth relative* Ext *of* X *and* Y*:*

$$H^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \cong \mathrm{Ext}^n_{\mathcal{Z}(G),\mathcal{D}}(\mathrm{X}, \mathrm{Y})$$

3.2 Davydov-Yetter cohomology of the action functor

Let $\mathcal{C}$ be a finite (multi)tensor category. Recall that a finite $\mathcal{C}$-module category is a $\mathcal{C}$-module category $\mathcal{M}$, whose underlying linear category is finite abelian and the action $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ is k-linear in both variables and exact in the first variable.
If $\mathcal{M}$ is a finite $\mathcal{C}$-module category, then the action $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ is also exact in the second variable. Indeed, by [EGNO, Proposition 7.1.6], we have the following adjunctions for all $X \in \mathcal{C}$:

$$X^* \triangleright - \dashv X \triangleright - \dashv {}^*X \triangleright -$$

and thus the action is exact in the second variable by Proposition A.11. In particular, the action functor

$$\begin{aligned} \rho_{\mathcal{M}} : \mathcal{C} &\longrightarrow \mathrm{Rex}(\mathcal{M}) \\ X &\longmapsto X \triangleright - \end{aligned}$$

of a finite $\mathcal{C}$-module category $\mathcal{M}$ takes values in the category $\mathrm{Rex}(\mathcal{M})$ of right exact linear endofunctors of $\mathcal{M}$. By [S3, Lemma 3.1], the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{Rex}(\mathcal{M})$ of a finite module category is exact.

Let $\mathcal{M}$ be a strict finite module category over a strict finite multitensor category $\mathcal{C}$. We now specialize the construction from the previous subsection to the case where the monoidal functor G is the action functor $\rho_{\mathcal{M}}$. Note that the target category of the action functor, $\mathrm{Rex}(\mathcal{M})$, is not rigid in general: its tensor product is the composition of endofunctors and thus the left (respectively right) dual of $F \in \mathrm{Rex}(\mathcal{M})$ is precisely the left (respectively right) adjoint of F. But since F is only assumed to be a *right exact* k-linear functor between finite abelian categories, F always has a right adjoint but not necessarily a left adjoint (see Proposition A.11). Hence, the tensor product in $\mathrm{Rex}(\mathcal{M})$ is only right exact in both variables (see also (A.5)). This is the reason why we need the generalization of [FGS1, Corollary 4.7] (see Proposition 3.3).

Using the isomorphism of monoidal categories $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \cong \mathcal{Z}(\rho_{\mathcal{M}})$ from Proposition 2.8, the adjunction (3.1) takes the form

$$\begin{array}{c} \mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \\ \mathcal{F}_{\rho_{\mathcal{M}}} \uparrow \dashv \downarrow \mathcal{U} \\ \mathrm{Rex}(\mathcal{M}) \end{array}$$

Therefore, using first Corollary 2.10 and then Proposition 3.3 we immediately conclude:

Corollary 3.4. *Let $\mathcal{C}$ be a strict finite tensor category and let $\mathcal{M}$ be a strict finite $\mathcal{C}$-module category. The nth associator deformation cohomology of the $\mathcal{C}$-module endofunctor $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}$ with coefficient the $\mathcal{C}$-module endofunctor $(F', s') : \mathcal{M} \longrightarrow \mathcal{M}$ is isomorphic to the nth relative* Ext *of (F, s) and (F', s'):*

$$H^n_{ass}\big((F, s), (F', s')\big) \cong \mathrm{Ext}^n_{\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}),\mathrm{Rex}(\mathcal{M})}\big((F, s), (F', s')\big)$$

3.3 Associator cohomology via adjoint algebras

Let $\mathcal{C}$ be a finite tensor category. Recall that an *exact* $\mathcal{C}$-module category is a finite $\mathcal{C}$-module category $\mathcal{M}$, such that for all projective objects $P \in \mathcal{C}$ and all objects $M \in \mathcal{M}$, the object $P \triangleright M$ is projective in $\mathcal{M}$.

Let us begin with the following remark: in [S3, Corollary 7.5], it is shown that if $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then the Hochschild cohomology of $\mathcal{M}$ is isomorphic to the Ext of the tensor unit I and the associated adjoint algebra $\mathcal{A}_{\mathcal{M}}$ (which will be defined shortly below):

$$\mathrm{HH}^\bullet(\mathcal{M}) \cong \mathrm{Ext}^\bullet_{\mathcal{C}}(I, \mathcal{A}_{\mathcal{M}})$$

For exact module categories, we establish a similar description of associator cohomology but as a *relative* Ext over $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}$ in Theorem 3.7. First we need to fix some notation and collect a few facts about the internal Hom. In the following, let $\mathcal{M}$ be a finite $\mathcal{C}$-module category.

Remark 3.5. *Recall that the action $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ is exact in both variables. In particular, for all $M \in \mathcal{M}$, the functor*

$$- \triangleright M : \mathcal{C} \longrightarrow \mathcal{M}$$

admits a right adjoint

$$\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \longrightarrow \mathcal{C} \tag{3.6}$$

by Proposition A.11. For all $M \in \mathcal{M}$, let $\underline{\tau}^M = \big(\underline{\tau}^M_{X,N}\big)_{X \in \mathcal{C}, N \in \mathcal{M}}$ denote the family of natural isomorphisms

$$\underline{\tau}^M_{X,N}: \quad \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}\big(X, \underline{\mathrm{Hom}}(M,N)\big) \tag{3.7}$$

that is part of the data of the adjunction

$$- \triangleright M \dashv \underline{\mathrm{Hom}}(M, -) \tag{3.8}$$

The right adjoint (3.6) naturally extends to a functor

$$\underline{\mathrm{Hom}}(-,-): \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{C} \tag{3.9}$$

such that the isomorphisms (3.7) are also natural in M (see, e.g., [R, Proposition 4.3.6]*). We call the functor (3.9) the internal* Hom *of $\mathcal{M}$. Moreover, for all $M \in \mathcal{M}$, the action $- \triangleright M : \mathcal{C} \longrightarrow \mathcal{M}$ is a $\mathcal{C}$-module functor with module structure $m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\cong} X \triangleright (Y \triangleright M)$. Thus, by Proposition A.27, the right adjoint $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ is a $\mathcal{C}$-module functor, whose module structure we denote by $\underline{s}$. Note that by Proposition A.27, the unit*

$$\underline{\eta}^M : \mathrm{Id}_{\mathcal{C}} \Longrightarrow \underline{\mathrm{Hom}}(M, - \triangleright M)$$

and the counit

$$\underline{\epsilon}^M : \underline{\mathrm{Hom}}(M,-) \triangleright M \Longrightarrow \mathrm{Id}_{\mathcal{M}}$$

of the adjunction (3.8) are $\mathcal{C}$-module transformations for all $M \in \mathcal{M}$, where the compositions $\underline{\mathrm{Hom}}(M,-) \circ (- \triangleright M)$ and $(- \triangleright M) \circ \underline{\mathrm{Hom}}(M,-)$ are endowed with the $\mathcal{C}$-module structure from (2.17).

Let $X \in \mathcal{C}$. By $\overline{\tau}^X = \big(\overline{\tau}^X_{Y,Z}\big)_{Y,Z \in \mathcal{C}}$ we denote the natural family of isomorphisms

$$\overline{\tau}^X_{Y,Z}: \quad \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, Z) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(Y, Z \otimes X^*) \tag{3.10}$$

that is part of the data of the adjunction

$$- \otimes X \dashv - \otimes X^* \tag{3.11}$$

(see also (A.5)). We denote the unit of the adjunction (3.11) by

$$\overline{\eta}^X : \mathrm{Id}_{\mathcal{C}} \Longrightarrow (- \otimes X) \otimes X^* \tag{3.12}$$

and the counit by

$$\overline{\epsilon}^X : (- \otimes X^*) \otimes X \Longrightarrow \mathrm{Id}_{\mathcal{C}} \tag{3.13}$$

Following, e.g., [S3, Section 2.4], for all $X, Y \in \mathcal{C}, M, N \in \mathcal{M}$, we consider the following chain of isomorphisms of k-vector spaces

$$\mathrm{Hom}_{\mathcal{C}}\big(Y, \underline{\mathrm{Hom}}(X \triangleright M, N)\big) \cong \mathrm{Hom}_{\mathcal{M}}\big(Y \triangleright (X \triangleright M), N\big)$$

$$\cong \mathrm{Hom}_{\mathcal{M}}\big((Y \otimes X) \triangleright M, N\big)$$
$$\cong \mathrm{Hom}_{\mathcal{C}}\big(Y \otimes X, \underline{\mathrm{Hom}}(M, N)\big)$$
$$\cong \mathrm{Hom}_{\mathcal{C}}\big(Y, \underline{\mathrm{Hom}}(M, N) \otimes X^*\big)$$

Via the Yoneda lemma, we obtain a natural isomorphism

$$\xi_{X,M,N} : \quad \underline{\mathrm{Hom}}(X \triangleright M, N) \xrightarrow{\cong} \underline{\mathrm{Hom}}(M, N) \otimes X^* \tag{3.14}$$

which is given explicitly as

$$\xi_{X,M,N} = \big(\overline{\tau}^{X}_{Y,\underline{\mathrm{Hom}}(M,N)} \circ \underline{\tau}^{M}_{Y\otimes X,N} \circ (m_{Y,X,M})^* \circ \big(\underline{\tau}^{X\triangleright M}_{Y,N}\big)^{-1}\big)(\mathrm{id}_Y) \tag{3.15}$$

where $Y = \underline{\mathrm{Hom}}(X \triangleright M, N)$. We also define the morphism

$$\hat{\xi}_{X,M,N} := (\xi_{X,M,N} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)} \otimes \mathrm{ev}_X)$$

Now consider the end

$$\mathcal{A}_{\mathcal{M}} := \int_{M\in\mathcal{M}} \underline{\mathrm{Hom}}(M, M)$$

in $\mathcal{C}$, whose universal dinatural transformation we denote by

$$\pi_M : \mathcal{A}_{\mathcal{M}} \longrightarrow \underline{\mathrm{Hom}}(M, M)$$

for all $M \in \mathcal{M}$. The existence of the end $\mathcal{A}_{\mathcal{M}}$ and its connection with the action functor was established in the following result by Shimizu:

Theorem 3.6. [S3, Theorem 3.4] *Let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category.*

1. *If $H \in \mathrm{Rex}(\mathcal{M})$, then the end of the functor*

$$\underline{\mathrm{Hom}}\big(-, H(-)\big) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{C}$$

exists and the functor

$$\rho^{\mathrm{ra}}_{\mathcal{M}} : \mathrm{Rex}(\mathcal{M}) \longrightarrow \mathcal{C}$$
$$H \longmapsto \int_{M\in\mathcal{M}} \underline{\mathrm{Hom}}\big(M, H(M)\big)$$

is right adjoint to the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{Rex}(\mathcal{M})$.

2. *The functor $\rho^{\mathrm{ra}}_{\mathcal{M}} : \mathrm{Rex}(\mathcal{M}) \longrightarrow \mathcal{C}$ is k-linear and left exact. If $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then $\rho^{\mathrm{ra}}_{\mathcal{M}}$ is an exact functor.*

The end $\mathcal{A}_{\mathcal{M}}$ has the structure of an algebra in $\mathcal{C}$ and is called *the adjoint algebra* of $\mathcal{M}$ (see [S3]).

Moreover, the end $\mathcal{A}_\mathcal{M}$ is naturally an object of the Drinfeld center of $\mathcal{C}$. Its half-braiding ζ is the unique natural isomorphism such that the following diagram commutes for all $X \in \mathcal{C}, M \in \mathcal{M}$ (see [S3, Lemma 3.12]):

$$\begin{array}{ccc}
\mathcal{A}_\mathcal{M} \otimes X & \xrightarrow{\pi_{X \triangleright M} \otimes \mathrm{id}_X} & \underline{\mathrm{Hom}}(X \triangleright M, X \triangleright M) \otimes X \\
 & & \downarrow{\scriptstyle \hat{\xi}_{X,M,X \triangleright M}} \\
{\scriptstyle \zeta_X} \downarrow & & \underline{\mathrm{Hom}}(M, X \triangleright M) \\
 & & \downarrow{\scriptstyle \underline{s}_{X,M}} \\
X \otimes \mathcal{A}_\mathcal{M} & \xrightarrow[\mathrm{id}_X \otimes \pi_M]{} & X \otimes \underline{\mathrm{Hom}}(M, M)
\end{array} \tag{3.16}$$

We are now ready to formulate the main result of this thesis:

Theorem 3.7. *If $\mathcal{C}$ is a strict finite tensor category and $\mathcal{M}$ is a strict exact $\mathcal{C}$-module category, then the nth associator cohomology of $\mathcal{M}$ over $\mathcal{C}$ is isomorphic to the nth relative* Ext *of the tensor unit and the adjoint algebra $\mathcal{A}_\mathcal{M}$:*

$$H^n_{ass}(\mathcal{C}, \mathcal{M}) \cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}\big((I, \mathrm{id}), (\mathcal{A}_\mathcal{M}, \zeta)\big)$$

or, equivalently, there is an isomorphism

$$H^n_{ass}(\mathcal{C}, \mathcal{M}) \cong H^n_{\mathrm{DY}}\big(\mathrm{Id}_\mathcal{C}, (I, \mathrm{id}), (\mathcal{A}_\mathcal{M}, \zeta)\big)$$

Before proving this result, we first need to introduce the so-called lifting procedure for bimodule functors – the key ingredient of the proof of Theorem 3.7.

3.4 Proof of Theorem 3.7

This section is dedicated to the proof of Theorem 3.7 as well as developing the machinery needed. Unless stated otherwise, $\mathcal{C}$ denotes a monoidal category and $\mathcal{M}$ denotes a $\mathcal{C}$-module category, none of which are assumed to be strict.

3.4.1 The lifting of bimodule functors

We recall from, e.g., [FSS1, Def. 2.12] the notion of the center of a bimodule category (see also Definition A.25), which generalizes the Drinfeld center for the regular $\mathcal{C}$-bimodule category $\mathcal{C}$.

Definition 3.8. Let $\mathcal{C}$ be a monoidal category and let $\mathcal{N}$ be a $\mathcal{C}$-bimodule category. A *balancing* is a natural family $\sigma^N = (\sigma^N_X)_{X \in \mathcal{C}}$ of isomorphisms $\sigma^N_X : N \triangleleft X \xrightarrow{\cong} X \triangleright N$, such that the identity

$$\sigma^N_I = l_N^{-1} \circ r_N \tag{3.17}$$

holds and the following diagram commutes for all $X, Y \in \mathcal{C}$:

$$\begin{array}{ccc} N \lhd (X \otimes Y) & \xrightarrow[\cong]{\sigma^N_{X\otimes Y}} & (X \otimes Y) \rhd N \\ \downarrow {\scriptstyle n_{N,X,Y}\ \cong} & & \uparrow {\scriptstyle \cong\ m^{-1}_{X,Y,N}} \\ (N \lhd X) \lhd Y & & X \rhd (Y \rhd N) \\ \downarrow {\scriptstyle \sigma^N_X \lhd \mathrm{id}_Y\ \cong} & & \uparrow {\scriptstyle \cong\ \mathrm{id}_X \rhd \sigma^N_Y} \\ (X \rhd N) \lhd Y & \xrightarrow[b_{X,N,Y}]{\cong} & X \rhd (N \lhd Y) \end{array} \tag{3.18}$$

Definition 3.9. Let $\mathcal{C}$ be a monoidal category and let $\mathcal{N}$ be a $\mathcal{C}$-bimodule category. The *balanced center* of $\mathcal{N}$ is the k-linear category $\mathcal{Z}_b(\mathcal{N})$ whose objects are pairs (N, σ), which consist of an object $N \in \mathcal{N}$ and a balancing σ. A morphism $f : (N, \sigma) \longrightarrow (N', \sigma')$ in $\mathcal{Z}_b(\mathcal{N})$ is a morphism $f : N \longrightarrow N'$ in $\mathcal{N}$, such that the following diagram commutes for all $X \in \mathcal{C}$:

$$\begin{array}{ccc} N \lhd X & \xrightarrow[\cong]{\sigma_X} & X \rhd N \\ \downarrow {\scriptstyle f \lhd \mathrm{id}_X} & & \downarrow {\scriptstyle \mathrm{id}_X \rhd f} \\ N' \lhd X & \xrightarrow[\sigma'_X]{\cong} & X \rhd N' \end{array}$$

Example 3.10. [S3, Examples 3.9 & 3.10]

1. *Any monoidal category $\mathcal{C}$ is a $\mathcal{C}$-bimodule category in the obvious way. In the special case $\mathcal{N} = \mathcal{C}$, the balanced center $\mathcal{Z}_b(\mathcal{C})$ from Definition 3.9 reduces to $\mathcal{Z}(\mathcal{C})$, the Drinfeld center of $\mathcal{C}$.*

2. *Let $\mathcal{C}$ be a finite multitensor category and let $\mathcal{M}$ be a finite $\mathcal{C}$-module category. We endow the k-linear category* $\mathrm{Rex}(\mathcal{M})$ *with the structure of a $\mathcal{C}$-bimodule category with the left action*

$$(X \rhd F)(M) := X \rhd F(M)$$

and the right action

$$(F \lhd X)(M) := F(X \rhd M)$$

for all $X \in \mathcal{C}, M \in \mathcal{M}$ and $F \in \mathrm{Rex}(\mathcal{M})$. For a natural transformation $\alpha : F \Longrightarrow F'$ and a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, the natural transformation $\alpha \lhd f = \big((\alpha \lhd f)_M\big)_{M\in\mathcal{M}} : F \lhd X \Longrightarrow F' \lhd Y$ has the components

$$(\alpha \lhd f)_M := \alpha_{Y \rhd M} \circ F(f \rhd \mathrm{id}_M) : \quad F(X \rhd M) \longrightarrow F'(Y \rhd M)$$

The mixed associator $\tilde{m}$ of the left $\mathcal{C}$-action on $\mathrm{Rex}(\mathcal{M})$ *has the components*

$$(\tilde{m}_{X,Y,F})_M := m_{X,Y,F(M)} : \quad (X \otimes Y) \rhd F(M) \xrightarrow{\cong} X \rhd \big(Y \rhd F(M)\big)$$

and the left unitor $\tilde{l}$ has the components

$$(\tilde{l}_F)_M := l_{F(M)} : \quad I \triangleright F(M) \xrightarrow{\cong} F(M)$$

The mixed associator $\tilde{n}$ of the right $\mathcal{C}$-action has the components

$$(\tilde{n}_{F,X,Y})_M := F(m_{X,Y,M}) : \quad F\big((X \otimes Y) \triangleright M\big) \xrightarrow{\cong} F\big(X \triangleright (Y \triangleright M)\big)$$

and the right unitor $\tilde{r}$ has the components

$$(\tilde{r}_F)_M := F(l_M) : \quad F(I \triangleright M) \xrightarrow{\cong} F(M)$$

for all $X, Y \in \mathcal{C}, F \in \mathrm{Rex}(\mathcal{M}), M \in \mathcal{M}$.

Let us now take a look at the balanced center $\mathcal{Z}_b\big(\mathrm{Rex}(\mathcal{M})\big)$: an object in $\mathcal{Z}_b\big(\mathrm{Rex}(\mathcal{M})\big)$ is a pair (F, σ^F) consisting of a right exact k-linear endofunctor $F : \mathcal{M} \longrightarrow \mathcal{M}$ together with a balancing σ^F, i.e., a natural family of isomorphisms $\sigma^F_X : F \triangleleft X \Longrightarrow X \triangleright F$. For each $X \in \mathcal{C}$, σ^F_X is a natural transformation whose components are isomorphisms

$$(\sigma^F_X)_M : \quad F(X \triangleright M) \xrightarrow{\cong} X \triangleright F(M)$$

in $\mathcal{M}$ for all $M \in \mathcal{M}$. Similarly to Proposition 2.8, we identify a balancing σ^F with a $\mathcal{C}$-module structure on the functor F. Here, the diagram (3.18) corresponds to the pentagon (A.11) and condition (3.17) corresponds to the triangle (A.12). We thus have an isomorphism of k-linear categories

$$\mathcal{Z}_b\big(\mathrm{Rex}(\mathcal{M})\big) \cong \mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$$

Remark 3.11. [S3, Section 3] *If we endow $\mathcal{C}$ with the structure of the regular $\mathcal{C}$-bimodule category and $\mathrm{Rex}(\mathcal{M})$ with the $\mathcal{C}$-bimodule structure from Example 3.10, then the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{Rex}(\mathcal{M})$ becomes a $\mathcal{C}$-bimodule functor, whose left $\mathcal{C}$-module structure*

$$\mathfrak{s}_{X,Y} : \quad \rho_{\mathcal{M}}(X \triangleright Y) \Longrightarrow X \triangleright \rho_{\mathcal{M}}(Y)$$

has the components

$$(\mathfrak{s}_{X,Y})_M = m_{X,Y,M} : \quad \rho_{\mathcal{M}}(X \triangleright Y)(M) = (X \otimes Y) \triangleright M \xrightarrow{\cong} \big(X \triangleright \rho_{\mathcal{M}}(Y)\big)(M) = X \triangleright (Y \triangleright M)$$

and whose right $\mathcal{C}$-module structure

$$\mathfrak{t}_{Y,X} : \quad \rho_{\mathcal{M}}(Y \triangleleft X) \Longrightarrow \rho_{\mathcal{M}}(Y) \triangleleft X$$

has the components

$$(\mathfrak{t}_{Y,X})_M = m_{Y,X,M} : \quad \rho_{\mathcal{M}}(Y \triangleleft X)(M) = (Y \otimes X) \triangleright M \xrightarrow{\cong} \big(\rho_{\mathcal{M}}(Y) \triangleleft X\big)(M) = Y \triangleright (X \triangleright M)$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$.

It is well-known that for every monoidal category $\mathcal{C}$, the Drinfeld center $\mathcal{Z}(\mathcal{C})$ admits a forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$. Analogously, for every $\mathcal{C}$-bimodule category $\mathcal{M}$, the balanced center $\mathcal{Z}_b(\mathcal{M})$ admits a forgetful functor $\mathcal{U}_{\mathcal{M}} : \mathcal{Z}(\mathcal{M}) \longrightarrow \mathcal{M}$ which sends a pair (M, σ) to the underlying object $M \in \mathcal{M}$.

In [S3, Section 3.6], Shimizu proposed a process to lift a $\mathcal{C}$-bimodule functor $Q : \mathcal{M} \longrightarrow \mathcal{N}$ to a k-linear functor $\mathcal{Z}_b(Q) : \mathcal{Z}_b(\mathcal{M}) \longrightarrow \mathcal{Z}_b(\mathcal{N})$ between the respective balanced centers, in the sense that the following diagram commutes (where $\mathcal{U}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{N}}$ denote the respective forgetful functors):

$$\begin{array}{ccc} \mathcal{Z}_b(\mathcal{M}) & \xrightarrow{\mathcal{Z}_b(Q)} & \mathcal{Z}_b(\mathcal{N}) \\ {\scriptstyle \mathcal{U}_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{U}_{\mathcal{N}}} \\ \mathcal{M} & \xrightarrow[Q]{} & \mathcal{N} \end{array}$$

Actually, there is a 2-functor

$$\mathcal{Z}_b : \mathcal{C} - \mathrm{Bimod} \longrightarrow k - \mathrm{Cat}$$

from the 2-category of k-linear $\mathcal{C}$-bimodule categories to the 2-category of k-linear categories. In the following, we give a more detailed exposition of the lifting construction than the reference [S3] which does not provide proofs of all the statements.

Lemma 3.12. *Let $\mathcal{N}$ and $\mathcal{N}'$ be $\mathcal{C}$-bimodule categories and let $(Q, s, t) : \mathcal{N} \longrightarrow \mathcal{N}'$ be a $\mathcal{C}$-bimodule functor. The assignment*

$$\begin{aligned} \mathcal{Z}_b(Q, s, t) : \mathcal{Z}_b(\mathcal{N}) &\longrightarrow \mathcal{Z}_b(\mathcal{N}') \\ \big(N, \sigma^N\big) &\longmapsto \big(Q(N), \sigma^{Q(N)}\big) \\ f &\longmapsto Q(f) \end{aligned}$$

is a k-linear functor, where $\sigma^{Q(N)}$ is the balancing with

$$\sigma_X^{Q(N)} := s_{X,N} \circ Q\big(\sigma_X^N\big) \circ t_{N,X}^{-1} : \quad Q(N) \lhd' X \xrightarrow{\cong} X \rhd' Q(N) \tag{3.19}$$

for all $X \in \mathcal{C}$.

Proof:
Let $(N, \sigma) \in \mathcal{Z}_b(\mathcal{N})$. We start by showing that $\big(Q(N), \sigma^{Q(N)}\big)$ is an object in $\mathcal{Z}_b(\mathcal{N}')$, i.e., we show that $\sigma^{Q(N)}$ is a balancing (see Definition 3.8). It is clear that $\sigma^{Q(N)}$ is a natural isomorphism as a composition of natural isomorphisms. Using the commuting triangles (A.12) and (A.21) in the last equation, we calculate

$$\begin{aligned} \sigma_I^{Q(N)} &= s_{I,N} \circ Q(\sigma_I) \circ t_{N,I}^{-1} \\ &= s_{I,N} \circ Q\big(l_N^{-1} \circ r_N\big) \circ t_{N,I}^{-1} \\ &= s_{I,N} \circ Q\big(l_N^{-1}\big) \circ Q\big(r_N\big) \circ t_{N,I}^{-1} \\ &= \big(l'_{Q(N)}\big)^{-1} \circ r'_{Q(N)} \end{aligned}$$

What remains to be shown is that the following diagram commutes for all $X, Y \in \mathcal{C}$:

$$\begin{CD}
Q(N) \lhd' (X \otimes Y) @>{\sigma^{Q(N)}_{X\otimes Y}}>{\cong}> (X \otimes Y) \rhd' Q(N)\\
@V{n'_{Q(N),X,Y}}V{\cong}V @A{\cong}A{\left(m'_{X,Y,Q(N)}\right)^{-1}}A\\
\big(Q(N) \lhd' X\big) \lhd' Y @. X \rhd' \big(Y \rhd' Q(N)\big)\\
@V{\sigma_X^{Q(N)} \lhd' \mathrm{id}_Y}V{\cong}V @A{\cong}A{\mathrm{id}_X \lhd' \sigma_Y^{Q(N)}}A\\
\big(X \rhd' Q(N)\big) \lhd' Y @>{\cong}>{b'_{X,Q(N),Y}}> X \rhd' \big(Q(N) \lhd' Y\big)
\end{CD} \tag{3.20}$$

Recall that the following diagrams commute for all $X, Y \in \mathcal{C}$ because s and t are natural isomorphisms

$$\begin{CD}
Q\big(X \rhd (N \lhd Y)\big) @>{Q(\mathrm{id}_X \rhd \sigma_Y)}>{\cong}> Q\big(X \rhd (Y \rhd N)\big)\\
@V{s_{X,N\lhd Y}}V{\cong}V @V{\cong}V{s_{X,Y\rhd N}}V\\
X \rhd' Q(N \lhd Y) @>{\cong}>{\mathrm{id}_X \rhd' Q(\sigma_Y)}> X \rhd' Q(Y \rhd N)
\end{CD} \tag{3.21}$$

$$\begin{CD}
Q\big((N \lhd X) \lhd Y\big) @>{Q(\sigma_X \lhd \mathrm{id}_Y)}>{\cong}> Q\big((X \rhd N) \lhd Y\big)\\
@V{t_{N\lhd X,Y}}V{\cong}V @V{\cong}V{t_{X\rhd N,Y}}V\\
Q(N \lhd X) \lhd' Y @>{\cong}>{Q(\sigma_X) \lhd' \mathrm{id}_Y}> Q(X \rhd N) \lhd' Y
\end{CD} \tag{3.22}$$

The calculations

$$\begin{aligned}
&t^{-1}_{X\rhd N,Y} \circ \left(s^{-1}_{X,N} \rhd' \mathrm{id}_Y\right) \circ \left(\sigma_X^{Q(N)} \lhd' \mathrm{id}_Y\right) \circ (t_{N,X} \lhd' \mathrm{id}_Y) \circ t_{N\lhd X,Y}\\
=&t^{-1}_{X\rhd N,Y} \circ \left((s^{-1}_{X,N} \circ \sigma_X^{Q(N)} \circ t_{N,X}) \rhd' \mathrm{id}_Y\right) \circ t_{N\lhd X,Y}\\
\overset{(3.19)}{=}&t^{-1}_{X\rhd N,Y} \circ \Big(\left(s^{-1}_{X,N} \circ (s_{X,N} \circ Q(\sigma_X) \circ t^{-1}_{N,X}) \circ t_{N,X}\right) \lhd' \mathrm{id}_Y\Big) \circ t_{N\lhd X,Y}\\
=&t^{-1}_{X\rhd N,Y} \circ \left(Q(\sigma_X) \lhd' \mathrm{id}_Y\right) \circ t_{N\lhd X,Y}\\
\overset{(3.22)}{=}&t^{-1}_{X\rhd N,Y} \circ t_{X\rhd N,Y} \circ Q(\sigma_X \lhd \mathrm{id}_Y)\\
=&Q(\sigma_X \lhd \mathrm{id}_Y)
\end{aligned}$$

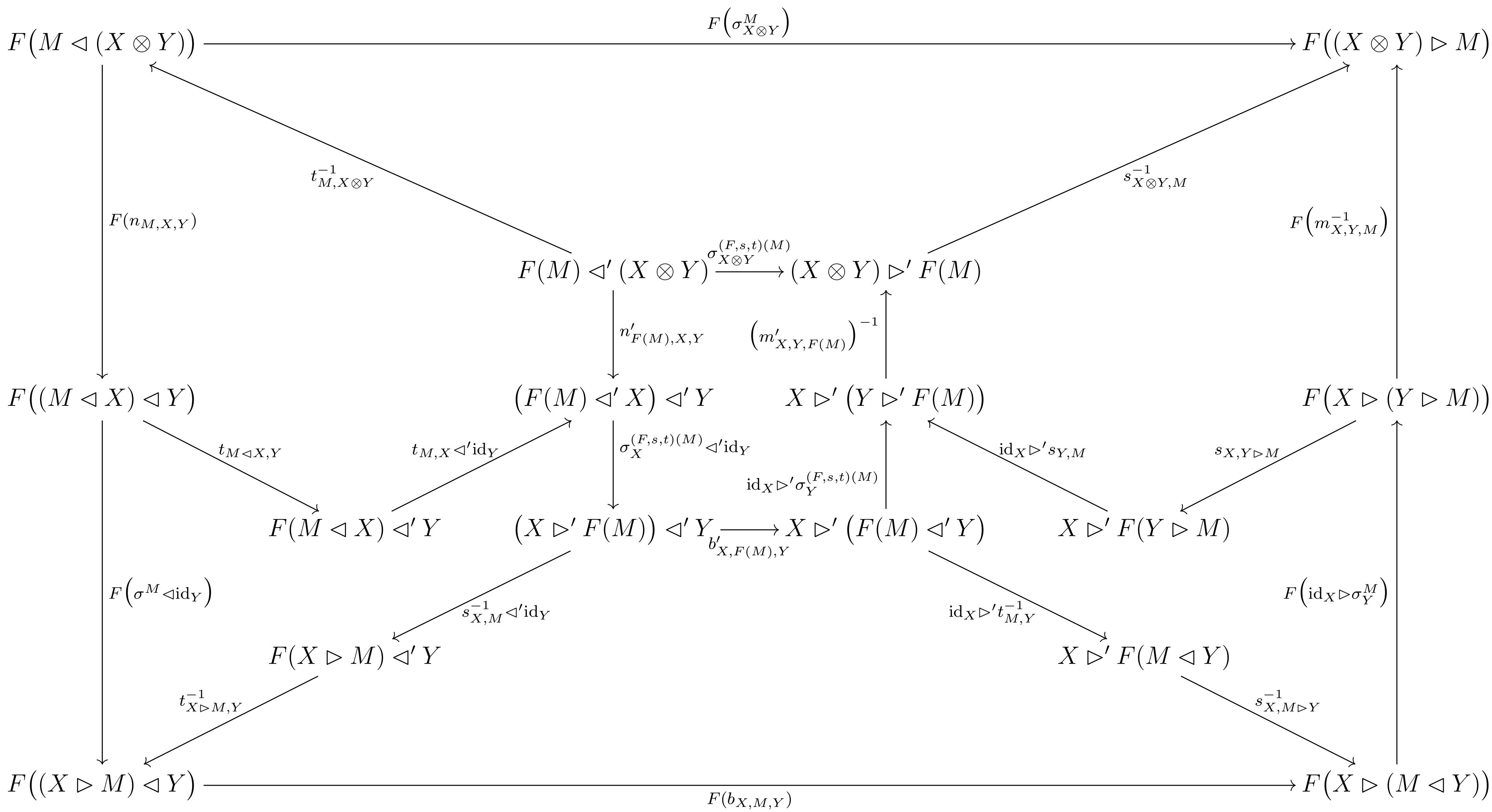
and

$$\begin{aligned}
&(\mathrm{id}_X \rhd' s_{Y,N}) \circ s_{X,Y\rhd N} \circ Q(\mathrm{id}_X \rhd \sigma_Y) \circ s^{-1}_{X,N\lhd Y} \circ \left(\mathrm{id}_X \rhd' t^{-1}_{N,Y}\right)\\
\overset{(3.21)}{=}&(\mathrm{id}_X \rhd' s_{Y,N}) \circ \left(\mathrm{id}_X \rhd' Q(\sigma_Y)\right) \circ s_{X,N\lhd Y} \circ s^{-1}_{X,N\lhd Y} \circ \left(\mathrm{id}_X \rhd' t^{-1}_{N,Y}\right)\\
=&\,\mathrm{id}_X \rhd' \left(s_{Y,N} \circ Q(\sigma_Y) \circ t^{-1}_{N,Y}\right)
\end{aligned}$$

$$\stackrel{(3.19)}{=} \mathrm{id}_X \triangleright' \sigma_Y^{Q(N)}$$

show that the lower left and the lower right hexagons in the diagram on the following page commute. Moreover, the lower middle slice commutes due to the commutativity of diagram (A.22) because (Q, s, t) is a $\mathcal{C}$-bimodule functor. The upper left pentagon commutes via the commutativity of (A.20) because (Q, t) is a right $\mathcal{C}$-module functor, and the commutativity of the upper right pentagon follows from (A.11) because (Q, s) is a left $\mathcal{C}$-module functor.

Finally, the commutativity of the outer large rectangle follows from applying Q to the commutative diagram (3.18) because σ is a balancing by assumption. Hence, the diagram in the middle commutes, which is the desired commutativity of diagram (3.20).


$F\big(M \triangleleft (X \otimes Y)\big)$
$F\big(\sigma^M_{X\otimes Y}\big)$
$F\big((X \otimes Y) \triangleright M\big)$
$t^{-1}_{M,X\otimes Y}$
$s^{-1}_{X\otimes Y,M}$
$F(n_{M,X,Y})$
$F\big(m^{-1}_{X,Y,M}\big)$
$F(M) \triangleleft' (X \otimes Y)$
$\sigma^{(F,s,t)(M)}_{X\otimes Y}$
$(X \otimes Y) \triangleright' F(M)$
$n'_{F(M),X,Y}$
$\big(m'_{X,Y,F(M)}\big)^{-1}$
$F\big((M \triangleleft X) \triangleleft Y\big)$
$\big(F(M) \triangleleft' X\big) \triangleleft' Y$
$X \triangleright' \big(Y \triangleright' F(M)\big)$
$F\big(X \triangleright (Y \triangleright M)\big)$
$t_{M\triangleleft X,Y}$
$t_{M,X} \triangleleft' \mathrm{id}_Y$
$\sigma^{(F,s,t)(M)}_X \triangleleft' \mathrm{id}_Y$
$\mathrm{id}_X \triangleright' s_{Y,M}$
$s_{X,Y\triangleright M}$
$\mathrm{id}_X \triangleright' \sigma^{(F,s,t)(M)}_Y$
$F(M \triangleleft X) \triangleleft' Y$
$\big(X \triangleright' F(M)\big) \triangleleft' Y$
$b'_{X,F(M),Y}$
$X \triangleright' \big(F(M) \triangleleft' Y\big)$
$X \triangleright' F(Y \triangleright M)$
$F\big(\sigma^M \triangleleft \mathrm{id}_Y\big)$
$s^{-1}_{X,M} \triangleleft' \mathrm{id}_Y$
$\mathrm{id}_X \triangleright' t^{-1}_{M,Y}$
$F\big(\mathrm{id}_X \triangleright \sigma^M_Y\big)$
$F(X \triangleright M) \triangleleft' Y$
$X \triangleright' F(M \triangleleft Y)$
$t^{-1}_{X\triangleright M,Y}$
$s^{-1}_{X,M\triangleright Y}$
$F\big((X \triangleright M) \triangleleft Y\big)$
$F(b_{X,M,Y})$
$F\big(X \triangleright (M \triangleleft Y)\big)$

Finally, we have to show that

$$\mathcal{Z}_b(Q,s,t)(f) = Q(f): \quad \big(Q(N), \sigma^{Q(N)}\big) \longrightarrow \big(Q(N'), (\sigma')^{Q(N')}\big)$$

is a morphism in $\mathcal{Z}_b(\mathcal{N}')$ for all $(N,\sigma),(N',\sigma') \in \mathcal{Z}_b(\mathcal{N})$ and all morphisms $f:(N,\sigma)\longrightarrow(N',\sigma')$ in $\mathcal{Z}_b(\mathcal{N})$. Hence, we have to check that the following diagram commutes for all $X \in \mathcal{C}$:

$$\begin{array}{ccc} Q(N) \triangleleft' X & \xrightarrow[\cong]{\sigma_X^{Q(N)}} & X \triangleright' Q(N) \\ {\scriptstyle Q(f)\triangleleft'\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \triangleright' Q(f)} \\ Q(N') \triangleleft' X & \xrightarrow[(\sigma')_X^{Q(N')}]{\cong} & X \triangleright' Q(N') \end{array} \tag{3.23}$$

Since $f:(N,\sigma)\longrightarrow(N',\sigma')$ is a morphism in $\mathcal{Z}_b(\mathcal{N})$, the diagram

$$\begin{array}{ccc} N \triangleleft X & \xrightarrow[\cong]{\sigma_X} & X \triangleright N \\ {\scriptstyle f\triangleleft\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \triangleright f} \\ N' \triangleleft X & \xrightarrow[\sigma'_X]{\cong} & X \triangleright N' \end{array} \tag{3.24}$$

commutes for all $X \in \mathcal{C}$ and thus applying the functor Q to diagram (3.24) yields the commutativity of the outer rectangle in the following diagram for all $X \in \mathcal{C}$:

$$\begin{array}{ccccc} Q(N \triangleleft X) & & \xrightarrow[\cong]{Q(\sigma_X)} & & Q(X \triangleright N) \\ & \nwarrow^{t^{-1}_{N,X}}_{\cong} & & \nearrow^{s^{-1}_{X,N}}_{\cong} & \\ & Q(N) \triangleleft' X & \xrightarrow[\cong]{\sigma_X^{Q(N)}} & X \triangleright' Q(N) & \\ {\scriptstyle Q(f\triangleleft\mathrm{id}_X)}\downarrow & {\scriptstyle Q(f)\triangleleft'\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X\triangleright' f} & \downarrow{\scriptstyle Q(\mathrm{id}_X\triangleright f)} \\ & Q(N') \triangleleft' X & \xrightarrow[(\sigma')_X^{Q(N')}]{\cong} & X \triangleright' Q(N') & \\ & \swarrow^{\cong}_{t^{-1}_{N',X}} & & \searrow^{\cong}_{s^{-1}_{X,N'}} & \\ Q(N' \triangleleft X) & & \xrightarrow[Q(\sigma'_X)]{\cong} & & Q(X \triangleright N') \end{array}$$

Both the small upper and lower quadrilaterals commute by definition of $\sigma^{Q(N)}$ and $(\sigma')^{Q(N')}$ and the small left and right quadrilaterals commute because s and t are natural isomorphisms. Thus, the inner rectangle commutes, which is just the desired commutativity of diagram (3.23).

□

Lemma 3.13. *Let $(Q,s,t),(Q',s',t'):\mathcal{N}\longrightarrow\mathcal{N}'$ be $\mathcal{C}$-bimodule functors. If $\nu:(Q,s,t)\Longrightarrow(Q',s',t')$ is a $\mathcal{C}$-bimodule transformation, then $\mathcal{Z}_b(\nu):\mathcal{Z}_b(Q,s,t)\Longrightarrow\mathcal{Z}_b(Q',s',t')$ with $\mathcal{Z}_b(\nu)_{(N,\sigma)}=\nu_N$ for all $(N,\sigma)\in\mathcal{Z}_b(\mathcal{N})$ is a natural transformation.*

Proof:
Let $(N, \sigma) \in \mathcal{Z}_b(\mathcal{N})$. We need to show that

$$\mathcal{Z}_b(\nu)_{(N,\sigma)} = \nu_N : \quad \mathcal{Z}_b(Q)(N,\sigma) = \big(Q(N), \sigma^{Q(N)}\big) \longrightarrow \mathcal{Z}_b(Q')(N,\sigma) = \big(Q'(N), \sigma^{Q'(N)}\big)$$

is a morphism in $\mathcal{Z}_b(\mathcal{N}')$, i.e., we have to show that the following diagram commutes for all $X \in \mathcal{C}$:

$$\begin{array}{ccc} Q(N) \lhd' X & \xrightarrow{\sigma_X^{Q(N)}} & X \rhd' Q(N) \\ \downarrow{\scriptstyle \nu_N \lhd' \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_X \rhd' \nu_N} \\ Q'(N) \lhd' X & \xrightarrow[\sigma_X^{Q'(N)}]{} & X \rhd' Q'(N) \end{array} \tag{3.25}$$

Using the definition of $\sigma^{Q(N)}$ and $\sigma^{Q'(N)}$ (see (3.19)) and that ν is a natural transformation of bimodule functors, we obtain the commutativity of the diagram

$$\begin{array}{ccccccc} Q(N) \lhd' X & \xrightarrow{t_{N,X}^{-1}} & Q(N \lhd X) & \xrightarrow{Q(\sigma_X)} & Q(X \rhd N) & \xrightarrow{s_{X,N}} & X \rhd' Q(N) \\ \downarrow{\scriptstyle \nu_N \lhd \mathrm{id}_X} & & \downarrow{\scriptstyle \nu_{N \lhd X}} & & \downarrow{\scriptstyle \nu_{X \rhd N}} & & \downarrow{\scriptstyle \mathrm{id}_X \rhd' \nu_N} \\ Q'(N) \lhd' X & \xrightarrow[t'^{-1}_{N,X}]{} & Q'(N \lhd X) & \xrightarrow[Q'(\sigma_X)]{} & Q'(X \rhd N) & \xrightarrow[s'_{X,N}]{} & X \rhd' Q'(N) \end{array}$$

and thus the desired commutativity of diagram (3.25). □

Combining Lemma 3.12 and Lemma 3.13, we get that the assignment

$$\begin{aligned} \mathcal{Z}_b : \mathbf{Fun}_{\mathcal{C},\mathcal{C}}(\mathcal{N}, \mathcal{N}') &\longrightarrow \mathbf{Fun}_k(\mathcal{Z}_b(\mathcal{N}), \mathcal{Z}_b(\mathcal{N}')) \\ (Q, s, t) &\longmapsto \mathcal{Z}_b(Q, s, t) \\ \big[\nu : (Q, s, t) \Longrightarrow (Q', s', t')\big] &\longmapsto \big[\mathcal{Z}_b(\nu) : \mathcal{Z}_b(Q, s, t) \Longrightarrow \mathcal{Z}_b(Q', s', t')\big] \end{aligned} \tag{3.26}$$

is a k-linear functor. Here, $\mathbf{Fun}_{\mathcal{C},\mathcal{C}}(\mathcal{N}, \mathcal{N}')$ denotes the k-linear category of k-linear $\mathcal{C}$-bimodule functors from $\mathcal{N}$ to $\mathcal{N}'$ and $\mathbf{Fun}_k(\mathcal{Z}_b(\mathcal{N}), \mathcal{Z}_b(\mathcal{N}'))$ denotes the k-linear category of k-linear functors from $\mathcal{Z}_b(\mathcal{N})$ to $\mathcal{Z}_b(\mathcal{N}')$.

3.4.2 The lifting and relative Exts

We will now investigate exactness properties of the lifting of bimodule functors in the finite case, i.e., $\mathcal{C}$ is assumed to be a finite tensor category and $\mathcal{N}, \mathcal{N}'$ are finite $\mathcal{C}$-bimodules. First, we show that the k-linear functor (3.26) preserves adjoints, a fact that was also used in [S3]:

Lemma 3.14. *If a $\mathcal{C}$-bimodule functor $(Q, s, t) : \mathcal{N} \longrightarrow \mathcal{N}'$ has a right adjoint $Q^{\mathrm{ra}} : \mathcal{N}' \longrightarrow \mathcal{N}$ (respectively a left adjoint $Q^{\mathrm{la}} : \mathcal{N}' \longrightarrow \mathcal{N}$), then the lifting $\mathcal{Z}_b(Q) : \mathcal{Z}_b(\mathcal{N}) \longrightarrow \mathcal{Z}_b(\mathcal{N}')$ has a right adjoint $\mathcal{Z}_b(Q)^{\mathrm{ra}} : \mathcal{Z}_b(\mathcal{N}') \longrightarrow \mathcal{Z}_b(\mathcal{N})$ (respectively a left adjoint $\mathcal{Z}_b(Q)^{\mathrm{la}} : \mathcal{Z}_b(\mathcal{N}') \longrightarrow \mathcal{Z}_b(\mathcal{N})$) and $\mathcal{Z}_b(Q)^{\mathrm{ra}} \cong \mathcal{Z}_b(Q^{\mathrm{ra}})$ (respectively $\mathcal{Z}_b(Q)^{\mathrm{la}} \cong \mathcal{Z}_b(Q^{\mathrm{la}})$).*

Proof:
We only prove the case for the right adjoint, since the proof for the left adjoint is completely analogous. Let $\eta : \mathrm{Id}_{\mathcal{N}} \Longrightarrow Q^{\mathrm{ra}} \circ Q$ and $\epsilon : Q \circ Q^{\mathrm{ra}} \Longrightarrow \mathrm{Id}_{\mathcal{N}'}$ denote the unit and the counit of the adjunction $Q \dashv Q^{\mathrm{ra}}$.
Adjoints of k-linear functors are k-linear, hence the functor Q^{ra} is k-linear and by Proposition A.27, the functor Q^{ra} admits the structure of a $\mathcal{C}$-bimodule functor and η as well as ϵ are $\mathcal{C}$-bimodule transformations. We endow the identity functors on $\mathcal{N}$ and $\mathcal{N}'$ with the identity bimodule structure. Thus, we have natural transformations

$$\mathcal{Z}_b(\eta) : \quad \mathcal{Z}_b(\mathrm{Id}_{\mathcal{N}}) = \mathrm{Id}_{\mathcal{Z}_b(\mathcal{N})} \Longrightarrow \mathcal{Z}_b(Q^{\mathrm{ra}} \circ G) = \mathcal{Z}_b(Q^{\mathrm{ra}}) \circ \mathcal{Z}_b(Q)$$

and

$$\mathcal{Z}_b(\epsilon) : \quad \mathcal{Z}_b(Q \circ Q^{\mathrm{ra}}) = \mathcal{Z}_b(Q) \circ \mathcal{Z}_b(Q^{\mathrm{ra}}) \Longrightarrow \mathcal{Z}_b(\mathrm{Id}_{\mathcal{N}'}) = \mathrm{Id}_{\mathcal{Z}_b(\mathcal{N}')}$$

that satisfy the triangle identities, i.e.,

$$\mathcal{Z}_b(\epsilon)_{\mathcal{Z}_b(Q)(N,\sigma)} \circ \mathcal{Z}_b(Q)\big(\mathcal{Z}_b(\eta)_{(N,\sigma)}\big) \stackrel{\mathsf{def}}{=} \epsilon_{Q(N)} \circ Q(\eta_N) = \mathrm{id}_{Q(N)} = \mathrm{id}_{\mathcal{Z}_b(Q)(N,\sigma)}$$

and

$$\mathcal{Z}_b(Q^{\mathrm{ra}})\big(\mathcal{Z}_b(\epsilon)_{(N,\sigma)}\big) \circ \eta_{\mathcal{Z}_b(G^{\mathrm{ra}})(N,\sigma)} \stackrel{\mathsf{def}}{=} Q^{\mathrm{ra}}(\epsilon_N) \circ \eta_{Q^{\mathrm{ra}}(N)} = \mathrm{id}_{Q^{\mathrm{ra}}(N)} = \mathrm{id}_{\mathcal{Z}_b(Q^{\mathrm{ra}})(N,\sigma)}$$

for all $(N, \sigma) \in \mathcal{Z}_b(\mathcal{N})$. Hence $\mathcal{Z}_b(Q) \dashv \mathcal{Z}_b(Q^{\mathrm{ra}})$ and $\mathcal{Z}_b(Q^{\mathrm{ra}}) \cong \mathcal{Z}_b(Q)^{ra}$ since adjoints are unique up to isomorphism. □

The lifting (3.26) also preserves left and right exact functors:

Lemma 3.15. *Let $\mathcal{N}$ and $\mathcal{N}'$ be finite $\mathcal{C}$-bimodule categories, such that $\mathcal{Z}_b(\mathcal{N})$ and $\mathcal{Z}_b(\mathcal{N}')$ are finite abelian categories. If $Q : \mathcal{N} \longrightarrow \mathcal{N}$ is a left/right exact $\mathcal{C}$-bimodule functor, then the lifting $\mathcal{Z}_b(Q) : \mathcal{Z}_b(\mathcal{N}) \longrightarrow \mathcal{Z}_b(\mathcal{N}')$ is a left/right exact k-linear functor.*

Proof:
We only prove the case where Q is assumed to be a left exact functor, since the proof for the right exact case is completely analogous. Since $\mathcal{N}$ and $\mathcal{N}'$ are finite abelian categories, the k-linear left exact functor Q admits a left adjoint $Q^{\mathrm{la}} : \mathcal{N}' \longrightarrow \mathcal{N}$ by Proposition A.11. Proposition A.27 yields that Q^{la} admits the structure of $\mathcal{C}$-bimodule functor.
It follows from Lemma 3.14 that $\mathcal{Z}_b(Q)$ has the left adjoint $\mathcal{Z}_b(Q^{\mathrm{la}})$. Hence, the k-linear functor $\mathcal{Z}_b(Q) : \mathcal{Z}_b(\mathcal{N}) \longrightarrow \mathcal{Z}_b(\mathcal{N}')$ is left exact by Proposition A.11 since $\mathcal{Z}_b(\mathcal{N})$ and $\mathcal{Z}_b(\mathcal{N}')$ are finite abelian categories by assumption. □

Let $\mathcal{N}, \mathcal{N}'$ be $\mathcal{C}$-bimodule categories and let $(Q, s, t) : \mathcal{N} \longrightarrow \mathcal{N}'$ be a $\mathcal{C}$-bimodule functor. We consider the situation where $\mathcal{U} : \mathcal{Z}_b(\mathcal{N}) \longrightarrow \mathcal{N}$ and $\mathcal{U}' : \mathcal{Z}_b(\mathcal{N}') \longrightarrow \mathcal{N}'$ are forgetful functors and $\mathcal{F}, \mathcal{F}'$ are their respective left adjoints:

$$\begin{array}{ccc} \mathcal{Z}_b(\mathcal{N}) & \xrightarrow{\ \mathcal{Z}_b(Q)\ } & \mathcal{Z}_b(\mathcal{N}') \\ \mathcal{F} \uparrow \dashv \downarrow \mathcal{U} & & \mathcal{F}' \uparrow \dashv \downarrow \mathcal{U}' \\ \mathcal{N} & \xrightarrow[\ Q\ ]{} & \mathcal{N}' \end{array} \tag{3.27}$$

Lemma 3.16. *In the situation of (3.27), the lifting $\mathcal{Z}_b(Q) : \mathcal{Z}_b(\mathcal{N}) \longrightarrow \mathcal{Z}_b(\mathcal{N}')$ preserves allowable morphisms.*

Proof:
Let $\big(X, \sigma^X\big), \big(Y, \sigma^Y\big) \in \mathcal{Z}_b(\mathcal{N})$ and let $f : \big(X, \sigma^X\big) \longrightarrow \big(Y, \sigma^Y\big)$ be an allowable morphism in $\mathcal{Z}_b(\mathcal{N})$. Hence, there is a morphism $g : Y \longrightarrow X$ in $\mathcal{N}$, such that $\mathcal{U}(f) \circ g \circ \mathcal{U}(f) = \mathcal{U}(f)$, i.e.,

$$f \circ g \circ f = f \tag{3.28}$$

To show that $\mathcal{Z}_b(Q)(f) : \mathcal{Z}_b(Q)\big(X, \sigma^X\big) \longrightarrow \mathcal{Z}_b(Q)\big(Y, \sigma^Y\big)$ is an allowable morphism, we consider the morphism $Q(g) : Q(Y) \longrightarrow Q(X)$ in $\mathcal{N}'$. The calculation

$$\begin{aligned}
\mathcal{U}'\big(\mathcal{Z}_b(Q)(f)\big) \circ Q(g) \circ \mathcal{U}'\big(\mathcal{Z}_b(Q)(f)\big) &= Q(f) \circ Q(g) \circ Q(f) \\
&= Q(f \circ g \circ f) \\
&\overset{(3.28)}{=} Q(f) \\
&= \mathcal{U}'\big(\mathcal{Z}_b(Q)(f)\big)
\end{aligned}$$

shows that $\mathcal{Z}_b(Q)(f) : \mathcal{Z}_b(Q)\big(X, \sigma^X\big) \longrightarrow \mathcal{Z}_b(Q)\big(Y, \sigma^Y\big)$ is indeed an allowable morphism in $\mathcal{Z}_b(\mathcal{N}')$. □

Lemma 3.17. *In the situation of (3.27), let $\mathcal{N}$ and $\mathcal{N}'$ be finite $\mathcal{C}$-bimodule categories, such that $\mathcal{Z}_b(\mathcal{N})$ and $\mathcal{Z}_b(\mathcal{N}')$ are finite abelian categories. If $(Q, s, t) : \mathcal{N} \longrightarrow \mathcal{N}'$ is a $\mathcal{C}$-bimodule functor which has an exact right adjoint $Q^{\mathrm{ra}} : \mathcal{N}' \longrightarrow \mathcal{N}$, then $\mathcal{Z}_b(Q)$ preserves relatively projective objects.*

Proof:
Let $(P, \sigma) \in \mathcal{Z}_b(\mathcal{N})$ be relatively projective. We want to apply Lemma 3.1 to show that the object $\mathcal{Z}_b(Q)(P, \sigma) \in \mathcal{Z}(\mathcal{N}')$ is relatively projective.

There is a natural isomorphism of k-linear functors

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}\big(\mathcal{Z}_b(Q)(P, \sigma), -\big) &\cong \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big((P, \sigma), \mathcal{Z}_b(Q^{\mathrm{ra}})(-)\big) \\
&\cong \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big((P, \sigma), -\big) \circ \mathcal{Z}_b(Q^{\mathrm{ra}})(-)
\end{aligned}$$

Since $Q^{\mathrm{ra}} : \mathcal{N} \longrightarrow \mathcal{N}'$ is a $\mathcal{C}$-bimodule functor by Proposition A.27, the functor $\mathcal{Z}_b(Q^{\mathrm{ra}}) : \mathcal{Z}_b(\mathcal{N}') \longrightarrow \mathcal{Z}_b(\mathcal{N})$ preserves allowable morphisms by Lemma 3.16. Moreover, $\mathcal{Z}_b(Q^{\mathrm{ra}})$ is an exact functor by Lemma 3.15 since Q^{ra} is an exact functor by assumption. Hence, $\mathcal{Z}_b(Q^{\mathrm{ra}})$ preserves allowable short exact sequences.

Since $(P, \sigma) \in \mathcal{Z}_b(\mathcal{N})$ is relatively projective, the functor $\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big((P, \sigma), -\big)$ sends allowable short exact sequences in $\mathcal{Z}_b(\mathcal{N})$ to short exact sequences of k-vector spaces by Lemma 3.1. Therefore, $\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}\big(\mathcal{Z}_b(Q)(P, \sigma), -\big) \cong \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big((P, \sigma), -\big) \circ \mathcal{Z}_b(Q^{\mathrm{ra}})(-)$ sends allowable short exact sequences in $\mathcal{Z}_b(\mathcal{N}')$ to short exact sequences of k-vector spaces. Thus, the object $\mathcal{Z}_b(Q)(P, \sigma) \in \mathcal{Z}_b(\mathcal{N}')$ is relatively projective by Lemma 3.1.

□

Proposition 3.18. *In the situation of (3.27), let $\mathcal{N}$ and $\mathcal{N}'$ be finite $\mathcal{C}$-bimodule categories, such that $\mathcal{Z}_b(\mathcal{N})$ and $\mathcal{Z}_b(\mathcal{N}')$ are finite abelian categories. If $(Q,s,t):\mathcal{N}\longrightarrow\mathcal{N}'$ is an exact $\mathcal{C}$-bimodule functor which has an exact right adjoint $Q^{\mathrm{ra}}:\mathcal{N}'\longrightarrow\mathcal{N}$, then there is an isomorphism of k-vector spaces*

$$\mathrm{Ext}^n_{\mathcal{Z}_b(\mathcal{N}'),\mathcal{N}'}\big(\mathcal{Z}_b(Q)(N,\sigma),(N',\sigma')\big)\cong\mathrm{Ext}^n_{\mathcal{Z}_b(\mathcal{N}),\mathcal{N}}\big((N,\sigma),\mathcal{Z}_b(Q^{\mathrm{ra}})(N',\sigma')\big) \tag{3.29}$$

for all $n\geq 0$.

Proof:
For all $(N,\sigma)\in\mathcal{Z}_b(\mathcal{N})$ and all $(N',\sigma')\in\mathcal{Z}_b(\mathcal{N}')$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}\big(\mathcal{Z}_b(Q)(N,\sigma),(N',\sigma')\big)\cong\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big((N,\sigma),\mathcal{Z}_b(Q^{\mathrm{ra}})(N',\sigma')\big) \tag{3.30}$$

that is part of the adjunction $\mathcal{Z}_b(Q)\dashv\mathcal{Z}_b(Q^{\mathrm{ra}})$. Consider a relatively projective resolution of $(N,\sigma)\in\mathcal{Z}_b(\mathcal{N})$ (we suppress the balancings in the notation):

$$\ldots\longrightarrow P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}N\longrightarrow 0 \tag{3.31}$$

Recall from Lemma 3.16 that $\mathcal{Z}_b(Q)$ preserves allowable morphisms. By assumption, Q is exact and thus $\mathcal{Z}_b(Q)$ is exact by Lemma 3.15. Also, Q has an exact right adjoint Q^{ra} by assumption and therefore $\mathcal{Z}_b(Q)$ preserves relatively projective objects by Lemma 3.17. Hence, applying $\mathcal{Z}_b(Q)$ to the relatively projective resolution (3.31) yields a relatively projective resolution

$$\ldots\longrightarrow\mathcal{Z}_b(Q)(P_2)\xrightarrow{\mathcal{Z}_b(Q)(d_2)}\mathcal{Z}_b(Q)(P_1)\xrightarrow{\mathcal{Z}_b(Q)(d_1)}\mathcal{Z}_b(Q)(P_0)\xrightarrow{\mathcal{Z}_b(Q)(d_0)}\mathcal{Z}_b(Q)(N)\longrightarrow 0 \tag{3.32}$$

of $\mathcal{Z}_b(Q)(N,\sigma)\in\mathcal{Z}_b(\mathcal{N}')$. We obtain the following ladder diagram by applying the functor $\mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}(-,N')$ to (3.32) and the vertical isomorphisms are (3.30):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}\big(\mathcal{Z}_b(Q)(P_0),N'\big) & \xrightarrow{(\mathcal{Z}_b(Q)(d_1))^*} & \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N}')}\big(\mathcal{Z}_b(Q)(P_1),N'\big) & \longrightarrow & \ldots \\
 & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big(P_0,\mathcal{Z}_b(Q^{\mathrm{ra}})(N')\big) & \xrightarrow[d_1^*]{} & \mathrm{Hom}_{\mathcal{Z}_b(\mathcal{N})}\big(P_1,\mathcal{Z}_b(Q^{\mathrm{ra}})(N')\big) & \longrightarrow & \ldots
\end{array} \tag{3.33}$$

Since the isomorphisms (3.30) are natural, the ladder (3.33) commutes. Hence, the vertical isomorphisms combine into an isomorphism of cochain complexes. The induced isomorphism in cohomology is the desired isomorphism (3.29).

□

Remark 3.19. *Actually, Lemma 3.17 and Proposition 3.18 do not require Q^{ra} to be exact, but just to preserve allowable short exact sequences.*

To conclude the proof of Theorem 3.7, we now specialize to the case where $\mathcal{C}$ is a strict finite tensor category and $\mathcal{M}$ is a strict exact $\mathcal{C}$-module category. Recall the $\mathcal{C}$-bimodule structure of the k-linear category $\mathrm{Rex}(\mathcal{M})$ and the isomorphism of k-linear categories $\mathcal{Z}_b\big(\mathrm{Rex}(\mathcal{M})\big) \cong \mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$ from Example 3.10 as well as the $\mathcal{C}$-bimodule structure of the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{Rex}(\mathcal{M})$ from Remark 3.11.

Thus, $\mathcal{N} = \mathcal{C}$ and $\mathcal{N}' = \mathrm{Rex}(\mathcal{M})$ are strict $\mathcal{C}$-bimodules and $Q = \rho_{\mathcal{M}}$ is an exact strict $\mathcal{C}$-bimodule functor (see also [S3, Lemma 3.1 & Section 3.4]). Pictorially, we have the following situation:

$$\begin{array}{ccc} \mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}_b(\mathcal{C}) & \xrightarrow{\mathcal{Z}_b(\rho_{\mathcal{M}})} & \mathcal{Z}_b\big(\mathrm{Rex}(\mathcal{M})\big) \cong \mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \\ \mathcal{F} \uparrow \dashv \downarrow \mathcal{U} & & \mathcal{F}' \uparrow \dashv \downarrow \mathcal{U}' \\ \mathcal{C} & \xrightarrow[\rho_{\mathcal{M}}]{} & \mathrm{Rex}(\mathcal{M}) \end{array}$$

Proof of Theorem 3.7:
If $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then $\rho_{\mathcal{M}}^{\mathrm{ra}} : \mathrm{Rex}(\mathcal{M}) \longrightarrow \mathcal{C}$ is an exact functor by Theorem 3.6. Thus, the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{Rex}(\mathcal{M})$ is a $\mathcal{C}$-bimodule functor with an exact right adjoint. We calculate:

$$\begin{aligned} H^n_{ass}(\mathcal{C}, \mathcal{M}) &\cong \mathrm{Ext}^n_{\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}),\mathrm{Rex}(\mathcal{M})}\big((\mathrm{Id}_{\mathcal{M}}, \mathrm{id}), (\mathrm{Id}_{\mathcal{M}}, \mathrm{id})\big) \\ &\cong \mathrm{Ext}^n_{\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}),\mathrm{Rex}(\mathcal{M})}\big(\mathcal{Z}_b(\rho_{\mathcal{M}})(I, \mathrm{id}), (\mathrm{Id}_{\mathcal{M}}, \mathrm{id})\big) \\ &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}\big((I, \mathrm{id}), \mathcal{Z}_b(\rho^{\mathrm{ra}}_{\mathcal{M}})(\mathrm{Id}_{\mathcal{M}}, \mathrm{id})\big) \\ &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}\big((I, \mathrm{id}), (\mathcal{A}_{\mathcal{M}}, \zeta)\big) \\ &\cong H^n_{\mathrm{DY}}\big(\mathrm{Id}_{\mathcal{C}}, (I, \mathrm{id}), (\mathcal{A}_{\mathcal{M}}, \zeta)\big) \end{aligned}$$

Here we have used Corollary 3.4 in the first isomorphism. In the second isomorphism, we have used that $(\mathrm{Id}_{\mathcal{M}}, \mathrm{id}) \cong \mathcal{Z}_b(\rho_{\mathcal{M}})(I, \mathrm{id})$ via the inverse of the unitor. In the third isomorphism, we have used Theorem 3.7. In the second to last isomorphism, we have used the definition of the adjoint algebra and the last isomorphism comes from Proposition 3.3.

□

3.5 Associator deformations of the regular module category

Let $\mathcal{C}$ be a strict finite tensor category. We present an application of Theorem 3.7 to investigate associator deformations of the regular $\mathcal{C}$-module category $\mathcal{C}$.

Previously, we have studied the left adjoint of the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$. Let us now take a look at the right adjoint

$$\begin{array}{c} \mathcal{Z}(\mathcal{C}) \\ \mathcal{U} \big\downarrow \dashv \big\uparrow \mathcal{R} \\ \mathcal{C} \end{array}$$

The right adjoint $\mathcal{R} : \mathcal{C} \longrightarrow \mathcal{Z}(\mathcal{C})$ of the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$ is given by

$$\mathcal{R} : \ V \longmapsto \big(R(V), h^{R(V)}\big)$$

where $R(V)$ is the following end

$$R(V) := \int_{X\in\mathcal{C}} X \otimes V \otimes X^*$$

whose universal dinatural transformation we denote by

$$\pi'_Y : \quad \int_{X\in\mathcal{C}} X \otimes V \otimes X^* \longrightarrow Y \otimes V \otimes Y^*$$

The end $R(V)$ admits a canonical half-braiding $h^{R(V)}$ (see, e.g., [S2, Section 3.2]), which we will explain in the case $V = I$. Let $\zeta' := h^{R(I)}$. For each $X \in \mathcal{C}$, we define $\zeta'_X : R(I) \otimes X \longrightarrow X \otimes R(I)$ to be the unique morphism in $\mathcal{C}$, such that the following diagram commutes for all $Y \in \mathcal{C}$:

$$\begin{array}{ccc} R(I) \otimes X & \xrightarrow{\pi'_{X\otimes Y}\otimes \mathrm{id}_X} & X \otimes Y \otimes (X \otimes Y)^* \otimes X \\ & & \big\downarrow {\scriptstyle \xi'_{X\otimes Y, X, Y,}\otimes \mathrm{id}_X} \\ {\scriptstyle \zeta'_X} \big\downarrow & & X \otimes Y \otimes Y^* \otimes X^* \otimes X \\ & & \big\downarrow {\scriptstyle \mathrm{id}_{X\otimes Y\otimes Y^*}\otimes \mathrm{ev}_X} \\ X \otimes R(I) & \xrightarrow[\mathrm{id}_X\otimes \pi'_Y]{} & X \otimes Y \otimes Y^* \end{array} \tag{3.34}$$

Here, analogously to (3.14), for all $X, Y, Z \in \mathcal{C}$, we obtain an isomorphism

$$\xi'_{Z,X,Y} : \quad Z \otimes (X \otimes Y)^* \xrightarrow{\cong} Z \otimes Y^* \otimes X^*$$

from the Yoneda lemma by considering the following chain of natural isomorphisms of k-vector spaces for all $W \in \mathcal{C}$:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(W, Z \otimes (X \otimes Y)^*) &\cong \mathrm{Hom}_{\mathcal{C}}(W \otimes X \otimes Y, Z) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(W \otimes X, Z \otimes Y^*) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(W, Z \otimes Y^* \otimes X^*) \end{aligned}$$

Recall the natural isomorphism $\overline{\tau}$ from (3.10). Explicitly, we have

$$\xi'_{Z,X,Y} = \big(\overline{\tau}^{X}_{Z\otimes(X\otimes Y)^*, Z\otimes Y^*} \circ \overline{\tau}^{Y}_{Z\otimes(X\otimes Y)^*\otimes X, Z} \circ \big(\overline{\tau}^{X\otimes Y}_{Z\otimes(X\otimes Y)^*, Z}\big)^{-1}\big)(\mathrm{id}_{Z\otimes(X\otimes Y)^*}) \tag{3.35}$$

for all $X, Y, Z \in \mathcal{C}$.

We want to show that the object $\mathcal{R}(I) = (\int_{X\in\mathcal{C}} X \otimes X^*, \zeta')$ and the adjoint algebra $(\mathcal{A}_{\mathcal{C}}, \zeta) = (\int_{X\in\mathcal{C}} \underline{\mathrm{Hom}}(X,X), \zeta)$ of the $\mathcal{C}$-module category $\mathcal{C}$ are isomorphic in the Drinfeld center $\mathcal{Z}(\mathcal{C})$. First of all, note that we have a natural isomorphism $\gamma : \underline{\mathrm{Hom}}(-,-) \Longrightarrow - \otimes (-)^*$ of functors whose components we denote by

$$\gamma_{X,Y} : \underline{\mathrm{Hom}}(X,Y) \xrightarrow{\cong} Y \otimes X^*$$

for all $X, Y \in \mathcal{C}$. Recall the unit and counit of the adjunctions (3.8) and (3.11) from Remark 3.5 as well as (3.12) and (3.13). Explicitly, we have (see also [R, Proposition 4.4.1] and the proof of Lemma A.21):

$$\gamma_{X,Y} = (\underline{\epsilon}^{X}_{Y} \otimes \mathrm{id}_{X^*}) \circ \overline{\eta}^{X}_{\underline{\mathrm{Hom}}(X,Y)}$$

for all $X, Y \in \mathcal{C}$. By the functoriality of the end (see Proposition A.5), there exists a unique isomorphism

$$\tilde{\gamma} : \quad \int_{X\in\mathcal{C}} \underline{\mathrm{Hom}}(X,X) \xrightarrow{\cong} \int_{X\in\mathcal{C}} X \otimes X^* \tag{3.36}$$

in $\mathcal{C}$, such that the diagram

$$\begin{array}{ccc} \int_{X\in\mathcal{C}} \underline{\mathrm{Hom}}(X,X) & \xrightarrow{\pi_Y} & \underline{\mathrm{Hom}}(Y,Y) \\ {\scriptstyle \exists!\ \tilde{\gamma}} \downarrow & & \downarrow {\scriptstyle \gamma_{Y,Y}} \\ \int_{X\in\mathcal{C}} X \otimes X^* & \xrightarrow[\pi'_Y]{} & Y \otimes Y^* \end{array}$$

commutes for all $Y \in \mathcal{C}$, i.e.,

$$\gamma_{Y,Y} \circ \pi_Y = \pi'_Y \circ \tilde{\gamma}. \tag{3.37}$$

What remains to be shown is that the isomorphism $\tilde{\gamma}$ is an isomorphism in $\mathcal{Z}(\mathcal{C})$, i.e.,

$$\zeta'_X \circ (\tilde{\gamma} \otimes \mathrm{id}_X) = (\mathrm{id}_X \otimes \tilde{\gamma}) \circ \zeta_X \tag{3.38}$$

for all $X \in \mathcal{C}$. To prove this, we first need the following lemma:

Lemma 3.20. *The following diagram commutes for all $X, Y \in \mathcal{C}$:*

$$\begin{array}{ccc} X \otimes Y \otimes (X\otimes Y)^* \otimes X & \xleftarrow[\gamma_{X\otimes Y, X\otimes Y} \otimes \mathrm{id}_X]{} & \underline{\mathrm{Hom}}(X\otimes Y, X\otimes Y) \otimes X \\ {\scriptstyle \xi'_{X\otimes Y, X, Y} \otimes \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle \xi_{X,Y,X\otimes Y} \otimes \mathrm{id}_X} \\ X \otimes Y \otimes Y^* \otimes X^* \otimes X & \xleftarrow[\gamma_{Y, X\otimes Y} \otimes \mathrm{id}_{X^*\otimes X}]{} & \underline{\mathrm{Hom}}(Y, X\otimes Y) \otimes X^* \otimes X \end{array} \tag{3.39}$$

Proof:
We first show that the following diagram commutes for all $V, X, Y \in \mathcal{C}$:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}\big(V, \underline{\mathrm{Hom}}(X \otimes Y, X \otimes Y)\big) & \xrightarrow[\cong]{(\gamma_{X\otimes Y, X \otimes Y})_*} & \mathrm{Hom}_{\mathcal{C}}\big(V, X \otimes Y \otimes (X \otimes Y)^*\big) \\
\Big(\underline{\tau}^{X\otimes Y}_{V, X\otimes Y}\Big)^{-1} \Big\downarrow \cong & & \cong \Big\downarrow \Big(\overline{\tau}^{X\otimes Y}_{V, X\otimes Y}\Big)^{-1} \\
\mathrm{Hom}_{\mathcal{C}}(V \otimes X \otimes Y, X \otimes Y) & = & \mathrm{Hom}_{\mathcal{C}}(V \otimes X \otimes Y, X \otimes Y) \\
\underline{\tau}^{Y}_{V\otimes X, X\otimes Y} \Big\downarrow \cong & & \cong \Big\downarrow \overline{\tau}^{Y}_{V\otimes X, X\otimes Y} \\
\mathrm{Hom}_{\mathcal{C}}\big(V \otimes X, \underline{\mathrm{Hom}}(Y, X \otimes Y)\big) & \xrightarrow[\cong]{(\gamma_{Y, X \otimes Y})_*} & \mathrm{Hom}_{\mathcal{C}}(V \otimes X, X \otimes Y \otimes Y^*) \\
\overline{\tau}^{X}_{V, \underline{\mathrm{Hom}}(Y, X\otimes Y)} \Big\downarrow \cong & & \cong \Big\downarrow \overline{\tau}^{X}_{V, X\otimes Y \otimes Y^*} \\
\mathrm{Hom}_{\mathcal{C}}\big(V, \underline{\mathrm{Hom}}(Y, X \otimes Y) \otimes X^*\big) & \xrightarrow[(\gamma_{Y, X\otimes Y} \otimes \mathrm{id}_{X^*})_*]{\cong} & \mathrm{Hom}_{\mathcal{C}}(V, X \otimes Y \otimes Y^* \otimes X^*)
\end{array} \tag{3.40}$$

By [R, Proposition 4.4.1], we have

$$\gamma_{X, Y \otimes X} \circ \underline{\eta}^X_Y = \overline{\eta}^X_Y \tag{3.41}$$

for all $X, Y \in \mathcal{C}$. For all $X, V, W \in \mathcal{C}$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, W)$, we calculate

$$\begin{aligned}
\big((\gamma_{X,W})_* \circ \underline{\tau}^X_{V,W}\big)(g) =& (\gamma_{X,W})_* \circ \big(\underline{\mathrm{Hom}}(X, g) \circ \underline{\eta}^X_V\big) \\
=& \gamma_{X,W} \circ \underline{\mathrm{Hom}}(X, g) \circ \underline{\eta}^X_V \\
=& (g \otimes \mathrm{id}_{X^*}) \circ \gamma_{X, V \otimes X} \circ \underline{\eta}^X_V \\
=& (g \otimes \mathrm{id}_{X^*}) \circ \overline{\eta}^X_V \\
=& \overline{\tau}^X_{V,W}(g)
\end{aligned}$$

Here, we have used the naturality of γ in the third equation and (3.41) in the fourth equation. Hence, the two upper rectangles in diagram (3.40) commute and the bottom rectangle commutes by the naturality of $\overline{\tau}^X$.

Let $V, V' \in \mathcal{C}$ and let $f : V' \longrightarrow V$ be a morphism in $\mathcal{C}$. The following diagram commutes for all $X, Y \in \mathcal{C}$ due to the naturality of $\underline{\tau}$ and $\overline{\tau}$:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}\big(V, X\otimes Y\otimes (X\otimes Y)^*\big) & \xrightarrow{\quad f^* \quad} & \mathrm{Hom}_{\mathcal{C}}\big(V', X\otimes Y\otimes (X\otimes Y)^*\big) \\
\big(\overline{\tau}^{X\otimes Y}_{V,X\otimes Y}\big)^{-1} \Big\downarrow \cong & & \cong \Big\downarrow \big(\overline{\tau}^{X\otimes Y}_{V',X\otimes Y}\big)^{-1} \\
\mathrm{Hom}_{\mathcal{C}}(V\otimes X\otimes Y, X\otimes Y) & & \mathrm{Hom}_{\mathcal{C}}(V'\otimes X\otimes Y, X\otimes Y) \\
\overline{\tau}^{Y}_{V\otimes X,X\otimes Y} \Big\downarrow \cong & & \cong \Big\downarrow \overline{\tau}^{Y}_{V'\otimes X,X\otimes Y} \\
\mathrm{Hom}_{\mathcal{C}}(V\otimes X, X\otimes Y\otimes Y^*) & & \mathrm{Hom}_{\mathcal{C}}(V'\otimes X, X\otimes Y\otimes Y^*) \\
\overline{\tau}^{X}_{V,X\otimes Y\otimes Y^*} \Big\downarrow \cong & & \cong \Big\downarrow \overline{\tau}^{X}_{V',X\otimes Y\otimes Y^*} \\
\mathrm{Hom}_{\mathcal{C}}(V, X\otimes Y\otimes Y^*\otimes X^*) & \xrightarrow[\quad f^* \quad]{} & \mathrm{Hom}_{\mathcal{C}}(V', X\otimes Y\otimes Y^*\otimes X^*)
\end{array}
\tag{3.42}
$$

In particular, for $V=\underline{\mathrm{Hom}}(X\otimes Y, X\otimes Y)$, $V' = X\otimes Y\otimes (X\otimes Y)^*$ and $f=\gamma^{-1}_{X\otimes Y,X\otimes Y}$, the commutativity of (3.42) yields

$$
\begin{aligned}
&(\gamma_{V',V'})^* \circ \overline{\tau}^{X}_{V',X\otimes Y\otimes Y^*} \circ \overline{\tau}^{Y}_{V'\otimes X,X\otimes Y} \circ \big(\overline{\tau}^{X\otimes Y}_{V',X\otimes Y}\big)^{-1} \circ (\gamma^{-1}_{V',V'})^* \\
=&\overline{\tau}^{X}_{V,X\otimes Y\otimes Y^*} \circ \overline{\tau}^{Y}_{V\otimes X,X\otimes Y} \circ \big(\overline{\tau}^{X\otimes Y}_{V,X\otimes Y}\big)^{-1}
\end{aligned}
\tag{3.43}
$$

We are now ready to calculate

$$
\begin{aligned}
&(\gamma_{Y,X\otimes Y}\otimes \mathrm{id}_{X^*})\circ \xi_{X,Y,X\otimes Y} \\
=&(\gamma_{Y,X\otimes Y}\otimes \mathrm{id}_{X^*})\circ \Big(\overline{\tau}^{X}_{V,\underline{\mathrm{Hom}}(Y,X\otimes Y)} \circ \underline{\tau}^{Y}_{V\otimes X,X\otimes Y} \circ \big(\underline{\tau}^{X\otimes Y}_{V,X\otimes Y}\big)^{-1}\Big)(\mathrm{id}_V) \\
=&\Big((\gamma_{Y,X\otimes Y}\otimes \mathrm{id}_{X^*})_* \circ \Big(\overline{\tau}^{X}_{V,\underline{\mathrm{Hom}}(Y,X\otimes Y)} \circ \underline{\tau}^{Y}_{V\otimes X,X\otimes Y} \circ \big(\underline{\tau}^{X\otimes Y}_{V,X\otimes Y}\big)^{-1}\Big)(\mathrm{id}_V) \\
=&\Big(\overline{\tau}^{X}_{V,X\otimes Y\otimes Y^*} \circ \overline{\tau}^{Y}_{V\otimes X,X\otimes Y} \circ \big(\overline{\tau}^{X\otimes Y}_{V,X\otimes Y}\big)^{-1} \circ (\gamma_{X\otimes Y,X\otimes Y})_*\Big)(\mathrm{id}_V) \\
\overset{(3.43)}{=}&\Big((\gamma_{V',V'})^* \circ \overline{\tau}^{X}_{V',X\otimes Y\otimes Y^*} \circ \overline{\tau}^{Y}_{V'\otimes X,X\otimes Y} \circ \big(\overline{\tau}^{X\otimes Y}_{V',X\otimes Y}\big)^{-1} \circ (\gamma^{-1}_{V',V'})^* \circ (\gamma_{X\otimes Y,X\otimes Y})_*\Big)(\mathrm{id}_V) \\
=&\Big(\overline{\tau}^{X}_{V',X\otimes Y\otimes Y^*} \circ \overline{\tau}^{Y}_{V'\otimes X,X\otimes Y} \circ \big(\overline{\tau}^{X\otimes Y}_{V',X\otimes Y}\big)^{-1}\Big)(\mathrm{id}_{V'}) \circ \gamma_{V',V'} \\
=&\xi'_{X\otimes Y,X,Y}\circ \gamma_{V',V'}
\end{aligned}
$$

Here we have used the definition of ξ (see (3.15)) in the first equation and the commutativity of diagram (3.40) in the third equation. In the last equation, we have used the definition of ξ' (see (3.35)).

Tensoring the equation

$$
(\gamma_{Y,X\otimes Y}\otimes \mathrm{id}_{X^*})\circ \xi_{X,Y,X\otimes Y} = \xi'_{X\otimes Y,X,Y}\circ \gamma_{V',V'}
$$

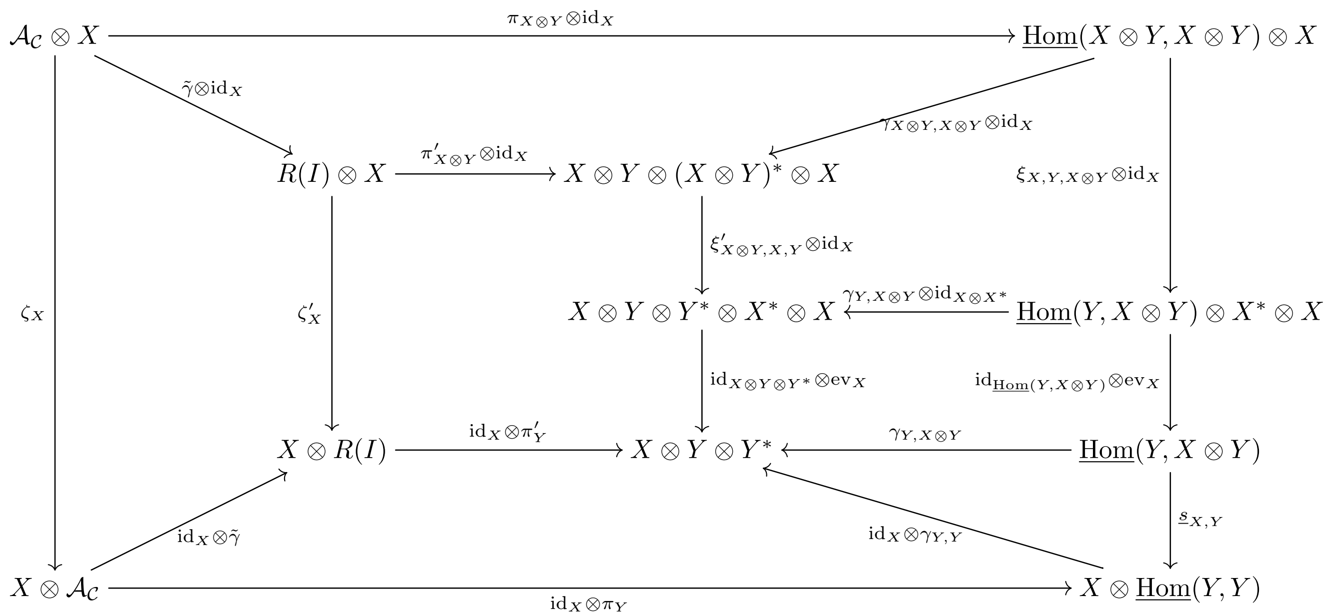
with id_X from the right now yields the desired commutativity of diagram (3.39).

□

Lemma 3.21. *The isomorphism (3.36) is an isomorphism in the Drinfeld center $\mathcal{Z}(\mathcal{C})$:*

$$
\tilde{\gamma}: \quad (\mathcal{A}_{\mathcal{C}}, \zeta) \xrightarrow{\cong} (R(I), \zeta')
\tag{3.44}
$$

Proof:
We need to show that $\tilde{\gamma}$ is an isomorphism in the category $\mathcal{Z}(\mathcal{C})$, i.e., that (3.38) holds for all $X \in \mathcal{C}$. For this purpose, consider the following diagram:



(3.45)

By [S3, Lemma 3.12] (see also (3.16)), the half-braiding ζ is the unique natural isomorphism, such that the large outer diagram in (3.45) commutes for all $X, Y \in \mathcal{C}$. By [S2, Section 3.2]) (see also (3.34)), the half-braiding ζ' is the unique natural isomorphism, such that the central inner diagram commutes for all $X, Y \in \mathcal{C}$. The quadrilateral on the top right commutes by Lemma 3.20 and the triangle on the bottom right commutes by Lemma A.21. The quadrilateral in the middle on the right side commutes due to the naturality of ev_X. The quadrilaterals on the top and bottom commute by (3.37).

We conclude that the isomorphism

$$(\mathrm{id}_X \otimes \tilde{\gamma}^{-1}) \circ \zeta'_X \circ (\tilde{\gamma} \otimes \mathrm{id}_X)$$

makes the large outer diagram commute for all $X \in \mathcal{C}$. Hence, it follows from the uniqueness of ζ that

$$\zeta_X = (\mathrm{id}_X \otimes \tilde{\gamma}^{-1}) \circ \zeta'_X \circ (\tilde{\gamma} \otimes \mathrm{id}_X)$$

for all $X \in \mathcal{C}$, which is the desired equation (3.38). □

Theorem 3.22. *For every strict finite tensor category $\mathcal{C}$, the nth associator cohomology of the regular $\mathcal{C}$-module category vanishes for $n > 0$:*

$$H^n_{ass}(\mathcal{C}, \mathcal{C}) \cong 0$$

In particular, the regular $\mathcal{C}$-module category does not admit associator deformations in the sense of Definition 1.1.

Proof:
Consider the following chain of isomorphisms of vector spaces:

$$\begin{aligned} H^n_{ass}(\mathcal{C},\mathcal{C}) &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C})}(I,\mathcal{A}_{\mathcal{C}}) \\ &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C})}(I,\mathcal{R}(I)) \\ &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C})}(I,{}^*\mathcal{F}(I^*)) \\ &\cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C})}(\mathcal{F}(I^*),I) \\ &\cong 0 \end{aligned}$$

The first isomorphism comes from Theorem 3.7, the second isomorphism is (3.44). For the third isomorphism, we have used that the functors $\mathcal{R}$ and ${}^*\mathcal{F}(?^*)$ are isomorphic (see [S1, Lemma 2.5]). In the fourth isomorphism, we have used a variant of [FGS1, Corollary 3.3], namely the isomorphism

$$\mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}(X,{}^*Y\otimes Z) \cong \mathrm{Ext}^n_{\mathcal{Z}(\mathcal{C}),\mathcal{C}}(Y\otimes X,Z)$$

applied to the case $X = Z = I$ and $Y = \mathcal{F}(I^*)$. Finally, the last isomorphism follows because $\mathcal{F}(I^*)$ is relatively projective.

□

4 Module categories for Hopf algebras

Let k be an algebraically closed field. Unadorned tensor products are tensor products over k-vector spaces. In this section, unless otherwise specified, let H be a finite-dimensional k-bialgebra with coproduct $\Delta : H \longrightarrow H \otimes H$ and counit $\epsilon : H \longrightarrow k$. We use Sweedler's notation for the coproduct, i.e., we write $\Delta(h) = h' \otimes h''$ for all $h \in H$.
Let $\mathfrak{l} = (\mathfrak{l}_V)_{V\in\mathbf{vect}}$ denote the left unitor in **vect**, the monoidal category of finite-dimensional k-vector spaces. Its components are the following isomorphisms:

$$\begin{aligned} \mathfrak{l}_V : \quad k \otimes V &\xrightarrow{\cong} V \\ \kappa \otimes v &\longmapsto \kappa.v \end{aligned} \tag{4.1}$$

Let us also recall the associator $\mathfrak{a} = (\mathfrak{a}_{X,Y,Z})_{X,Y,Z\in\mathbf{vect}}$ in **vect**:

$$\begin{aligned} \mathfrak{a}_{X,Y,Z} : \quad (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) \\ (x \otimes y) \otimes z &\longmapsto x \otimes (y \otimes z) \end{aligned} \tag{4.2}$$

In this section, we turn to the case where the monoidal category is given by $H-\mathbf{mod}$, the category of finite-dimensional modules over H, and we investigate several classes of module categories over this monoidal category from the standpoint of associator deformations. For this purpose, recall the following monoidal structure on the category $H-\mathbf{mod}$: the tensor product of two H-modules X and Y is the tensor product of the underlying k-vector spaces $X\otimes Y$ endowed with the H-action coming from the coproduct, i.e., $h.(x\otimes y) = h'.x\otimes h''.y$ for all $h \in H, x \in X, y \in Y$. The mixed associator is inherited from the category of k-vector spaces. Indeed, for all $X, Y, Z \in H-\mathbf{mod}$, the map (4.2) is H-linear: for $h \in H, x \in X, y \in Y, z \in Z$ we have

$$h.((x \otimes y) \otimes z) = (h')'.x \otimes (h')''.y \otimes h''.z \tag{4.3}$$

and

$$h.(x \otimes (y \otimes z)) = h'.x \otimes (h'')'.y \otimes (h'')''.z \tag{4.4}$$

By the coassociativity of the coproduct, the right-hand sides of (4.3) and (4.4) agree. The left and right unitor are inherited from **vect** as well. They are H-linear by the counitality of the coproduct.

4.1 Comodule algebras

Recall that a (left) H-coaction on a k-vector space V is a k-linear map $\delta : V \longrightarrow H \otimes V$, such that the coassociativity condition $(\mathrm{id}_H \otimes \delta)\circ\delta = (\Delta\otimes \mathrm{id}_V)\circ\delta$ and the counitality condition $\mathfrak{l}_V \circ (\epsilon\otimes \mathrm{id}_V)\circ\delta = \mathrm{id}_V$ hold. A (left) H-comodule algebra is a finite-dimensional k-algebra A with an H-coaction $\delta : A \longrightarrow H \otimes A$ which is a map of k-algebras, i.e., $\delta(1_A) = 1_H \otimes 1_A$ and $\delta(a \cdot b) = \delta(a)\cdot\delta(b)$ for all $a, b \in A$. In other words, an H-comodule algebra is an algebra object in the monoidal category of H-comodules.
We use Sweedler's notation for the coaction, i.e., we write $\delta(a) = a_{(0)} \otimes a_{(1)} \in H \otimes A$ for all $a \in A$.

Example 4.1. *We give some standard examples for comodule algebras:*

1. *Any bialgebra H is an H-comodule algebra with the coproduct as coaction.*
2. *The ground field k is an H-comodule algebra over a k-bialgebra H with the coaction $\delta(\kappa) = 1_H \otimes \kappa$ for all $\kappa \in k$.*
3. *A (left) coideal subalgebra is a subalgebra $A \subseteq H$ of H, such that $\Delta(A) \subseteq H \otimes A$. The restriction of the coproduct $\Delta|_A : A \longrightarrow H \otimes A$ endows A with the structure of an H-comodule algebra. The bialgebra H is an example of a coideal subalgebra.*
4. *Specifically, the coideal subalgebra $k\langle 1_H \rangle \subseteq H$ generated by the unit 1_H of H becomes an H-comodule algebra with the coaction $\delta(1_H) = 1_H \otimes 1_H$, called the trivial comodule algebra.*

For the remainder of this section, let A be a left H-comodule algebra. Comodule algebras provide examples of module categories over the k-linear monoidal category $\mathcal{C} = H - \mathbf{mod}$. Indeed, the k-linear category $\mathcal{M} = A - \mathbf{mod}$ of finite-dimensional A-modules is a $\mathcal{C}$-module category with the following module structure: the action of $\mathcal{C}$ on $\mathcal{M}$ is given by the functor $\mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ that maps a pair (X, M) to the k-vector space $X \otimes M$, which we endow with the following A-module structure:

$$a.(x \otimes m) = a_{(0)}.x \otimes a_{(1)}.m \tag{4.5}$$

for all $a \in A, x \in X, m \in M$. The mixed associator is inherited from **vect**. Indeed, if $X, Y \in H - \mathbf{mod}, M \in A - \mathbf{mod}$, then the map (4.2) is a morphism of A-modules: the A-action on $(X \otimes Y) \otimes M$ is defined as

$$a.\big((x \otimes y) \otimes m\big) = a'_{(0)}.x \otimes a''_{(0)}.y \otimes a_{(1)}.m \tag{4.6}$$

and the A-action on $X \otimes (Y \otimes M)$ reads

$$a.\big(x \otimes (y \otimes m)\big) = a_{(0)}.x \otimes (a_{(1)})_{(0)}.y \otimes (a_{(1)})_{(1)}.m \tag{4.7}$$

The right-hand sides of (4.6) and (4.7) agree by the coassociativity of the H-coaction on A. The unitor is inherited from **vect** as well: for any A-module V, the map (4.1) is a morphism of A-modules by the counitality of the H-coaction on A.

Lemma 4.2. *Let $n \geq 0$. Any natural family $f = (f_{X_0,\ldots,X_n,M})_{X_0,\ldots,X_n \in H-\mathbf{mod}, M \in A-\mathbf{mod}}$ of morphisms*

$$f_{X_0,\ldots,X_n,M} : \quad X_0 \otimes \ldots \otimes X_n \otimes M \longrightarrow X_0 \otimes \ldots \otimes X_n \otimes M$$

is uniquely determined by its value

$$f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H) \in H^{\otimes n+1} \otimes A$$

and

$$f_{X_0,\ldots,X_n,M}(x_0 \otimes \ldots \otimes x_n \otimes m) \in X_0 \otimes \ldots \otimes X_n \otimes M$$

is given by the componentwise left action of $f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_A) \in H^{\otimes n+1} \otimes A$ on $x_0 \otimes \ldots \otimes x_n \otimes m \in X_0 \otimes \ldots \otimes X_n \otimes M$.

Proof:
Let $X_0, \ldots, X_n \in H - \mathbf{mod}, M \in A - \mathbf{mod}$. For $x_i \in X_i$ and $m \in M$, we consider the H-linear map

$$\begin{aligned} \bar{x}_i : H &\longrightarrow X_i \\ h &\longmapsto h.x_i \end{aligned}$$

and the A-linear map

$$\begin{aligned} \bar{m} : A &\longrightarrow M \\ a &\longmapsto a.m \end{aligned}$$

Since f is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} H^{\otimes n} \otimes A & \xrightarrow{f_{H,\ldots,H,A}} & H^{\otimes n} \otimes A \\ \downarrow{\scriptstyle \bar{x}_1 \otimes \ldots \otimes \bar{x}_n \otimes \bar{m}} & & \downarrow{\scriptstyle \bar{x}_1 \otimes \ldots \otimes \bar{x}_n \otimes \bar{m}} \\ X_1 \otimes \ldots \otimes X_n \otimes M & \xrightarrow[f_{X_1,\ldots,X_n,M}]{} & X_1 \otimes \ldots \otimes X_n \otimes M \end{array} \tag{4.8}$$

and the calculation

$$\begin{aligned} f_{X_1,\ldots,X_n,M}(x_1 \otimes \ldots \otimes x_n \otimes m) &= \big(f_{X_1,\ldots,X_n,M} \circ (\bar{x}_1 \otimes \ldots \otimes \bar{x}_n \otimes \bar{m})\big)(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ &\overset{(4.8)}{=} \big((\bar{x}_1 \otimes \ldots \otimes \bar{x}_n \otimes \bar{m}) \circ f_{H,\ldots,H,A}\big)(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ &= (\bar{x}_1 \otimes \ldots \otimes \bar{x}_n \otimes \bar{m})\big(f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A)\big) \end{aligned}$$

proves the claim. □

4.1.1 Reformulation of the associator complex

Recall that for $n \geq 1$, an n-cocycle $f \in C^n_{ass}(H - \mathbf{mod}, A - \mathbf{mod})$ is a natural family of A-module morphisms

$$f_{X_1,\ldots,X_n,M} : \quad X_1 \otimes \ldots \otimes X_n \otimes M \longrightarrow X_1 \otimes \ldots \otimes X_n \otimes M \tag{4.9}$$

For concatenations of multiple coproducts after a coaction, we use the following shorthand notation: for $n \geq 1$ and $a \in A$, we set

$$\delta^{(n)}(a) := {a_{(0)}}^{(1)} \otimes \ldots \otimes {a_{(0)}}^{(q)} \otimes a_{(1)} \in H^{\otimes n} \otimes A \tag{4.10}$$

and $\delta^{(0)}(a) := a$. Let $X_1, \ldots, X_n \in H - \mathbf{mod}, M \in A - \mathbf{mod}$. The A-action on $X_1 \otimes \ldots \otimes X_n \otimes M$ is then given by the componentwise action (see also [Y, Section 7.3])

$$a.(x_1 \otimes \ldots \otimes x_n \otimes m) = \delta^{(n)}(a).(x_1 \otimes \ldots \otimes x_n \otimes m)$$

Our goal in this section is to rewrite the associator deformation complex of the module category $A - \mathbf{mod}$ over the monoidal category $H - \mathbf{mod}$ in terms of data of the comodule algebra A and the k-bialgebra H. For this purpose, we introduce the cochain complex

$$C^\bullet_{alg}(H, A)$$

whose nth cochain space is the k-vector space that consists of elements $r \in H^{\otimes n} \otimes A$ that satisfy the condition

$$\delta^{(n)}(a) \cdot r = r \cdot \delta^{(n)}(a) \tag{4.11}$$

for all $a \in A$ (see also (4.10)). The differential

$$\partial^n_{alg}: \quad C^n_{alg}(H, A) \longrightarrow C^{n+1}_{alg}(H, A)$$

is defined as the alternating sum

$$\begin{aligned}\partial^n_{alg}(h_1 \otimes \ldots \otimes h_n \otimes a) = &\, 1_H \otimes h_1 \otimes \ldots \otimes h_n \otimes a \\ &+ \sum_{i=1}^{n} (-1)^i h_1 \otimes \ldots \otimes \Delta(h_i) \otimes \ldots \otimes h_n \otimes a \\ &+ (-1)^{n+1} h_1 \otimes \ldots \otimes h_n \otimes \delta_(a)\end{aligned}$$

for all $n \geq 0$ and $h_1, \ldots, h_n \in H, a \in A$.

Lemma 4.3. *For all $n \geq 0$, the k-linear map*

$$\begin{aligned}\alpha_n: \quad C^n_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) &\xrightarrow{\cong} C^n_{alg}(H, A) \\ f &\longmapsto f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A)\end{aligned} \tag{4.12}$$

is an isomorphism of k-vector spaces.

Proof:
Let $n \geq 0$ and let $f \in C^n_{ass}(H - \mathbf{mod}, A - \mathbf{mod})$. First, note that f is a natural family of morphisms (4.9) and hence, by Lemma 4.2, f is uniquely determined by its value $f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \in H^{\otimes n} \otimes A$.

We now show that the element $f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \in H^{\otimes n} \otimes A$ satisfies condition (4.11). Indeed, right multiplication with an element of $H^{\otimes n} \otimes A$ is an A-linear map $H^{\otimes n} \otimes A \longrightarrow H^{\otimes n} \otimes A$ and thus, by naturality of f, the following diagram commutes for all $a \in A$:

$$\begin{array}{ccc} H^{\otimes n} \otimes A & \xrightarrow{f_{H,\ldots,H,A}} & H^{\otimes n} \otimes A \\ \Big\downarrow{\cdot \delta^{(n)}(a)} & & \Big\downarrow{\cdot \delta^{(n)}(a)} \\ H^{\otimes n} \otimes A & \xrightarrow[f_{H,\ldots,H,A}]{} & H^{\otimes n} \otimes A \end{array} \tag{4.13}$$

Hence, the following calculation shows that $f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \in C^n_{alg}(H, A)$:

$$\begin{aligned} f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \cdot \delta^{(n)}(a) &= \big((\cdot \delta^{(n)}(a)) \circ f_{H,\ldots,H,A}\big)(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ &\overset{(4.13)}{=} \big(f_{H,\ldots,H,A} \circ (\cdot \delta^{(n)}(a))\big)(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ &= f_{H,\ldots,H,A}\big(a_{(0)}{}^{(1)} \otimes \ldots \otimes a_{(0)}{}^{(q)} \otimes a_{(1)}\big) \\ &= f_{H,\ldots,H,A}\big(a.(1_H \otimes \ldots \otimes 1_H \otimes 1_A)\big) \end{aligned}$$

$$=a.f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A)$$
$$=\delta^{(n)}(a) \cdot f_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A)$$

□

Proposition 4.4. *The isomorphisms of k-vector spaces (4.12) combine into an isomorphism of cochain complexes*

$$C^\bullet_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) \cong C^\bullet_{alg}(H, A)$$

Proof:
We have to show that the following diagram commutes for all $n \geq 0$:

$$\begin{array}{ccc} C^{n+1}_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) & \xrightarrow[\cong]{\alpha_{n+1}} & C^{n+1}_{alg}(H, A) \\ \uparrow \partial^n_{ass} & & \uparrow \partial^n_{alg} \\ C^n_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) & \xleftarrow[\alpha_n^{-1}]{\cong} & C^n_{alg}(H, A) \end{array}$$

For $h_1, \ldots, h_n \in H$ and $a \in A$, we calculate

$$\begin{aligned} &(\alpha_{n+1} \circ \partial^n_{ass} \circ \alpha_n^{-1})(h_1 \otimes \ldots \otimes h_n \otimes a) \\ =&\partial^n_{ass}(\underbrace{\alpha_n^{-1}(h_1 \otimes \ldots \otimes h_n \otimes a)}_{=:f})_{H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ =&(\mathrm{id}_H \otimes f_{H,\ldots,H,A})(1_H \otimes \ldots \otimes 1_H \otimes 1_A) + \sum_{i=1}^n (-1)^i f_{H,\ldots,H\otimes H,\ldots,H,A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ &+ (-1)^{n+1} f_{H,\ldots,H,H\otimes A}(1_H \otimes \ldots \otimes 1_H \otimes 1_A) \\ =&1_H \otimes h_1.1_H \otimes \ldots \otimes h_n.1_H \otimes a.1_A \\ &+ \sum_{i=1}^n (-1)^i h_1.1_H \otimes \ldots \otimes h_i.(1_H \otimes 1_H) \otimes \ldots \otimes h_n.1_H \otimes a.1_A \\ &+ (-1)^{n+1} h_1.1_H \otimes \ldots \otimes h_n.1_H \otimes a.(1_H \otimes 1_A) \\ =&\partial^n_{alg}(h_1 \otimes \ldots \otimes h_n \otimes a) \end{aligned}$$

which proves the claim. □

We give an interpretation of the 0th cohomology of the associator deformation complex $C^\bullet_{ass}(H - \mathbf{mod}, A - \mathbf{mod})$ in terms of the center and the H-coinvariants of the comodule algebra A:

Corollary 4.5. *The* 0*th associator cohomology of* $A - \mathbf{mod}$ *as a module category over* $H - \mathbf{mod}$ *is*

$$H^0_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) \cong Z(A) \cap A^{coH}$$

where $A^{coH} := \{a \in A \mid \delta(a) = 1_H \otimes a\}$ *denotes the* H*-coinvariants of* A.

Proof:
Proposition 4.4 yields the isomorphism $H^0_{ass}(H-\mathbf{mod}, A-\mathbf{mod}) \cong H^0\big(C^*_{alg}(H,A)\big)$ and we have $a \in H^0\big(C^*_{alg}(H,A)\big)$ if and only if $\partial^0_{alg}(a) = 0$, i.e., $\delta(a) = 1_H \otimes a$.

□

Thus, for $\mathcal{C} = H - \mathbf{mod}$ and $\mathcal{M} = A - \mathbf{mod}$, we know from Proposition 1.11 and Corollary 4.5 that

$$\mathrm{Nat}_{\mathcal{C}}\big((\mathrm{Id}_{\mathcal{M}}, \mathrm{id}), (\mathrm{Id}_{\mathcal{M}}, \mathrm{id})\big) \cong H^0_{ass}(\mathcal{C}, \mathcal{M}) \cong Z(A) \cap A^{coH}$$

4.2 Exact module categories

In this section, let H be a finite-dimensional Hopf algebra with antipode $S : H \longrightarrow H$.

Generally, calculating associator cohomology is a hard task but if H as well as A are low-dimensional, then the cochain spaces in the complex $C^\bullet_{alg}(H, A)$, at least in low degrees, are low-dimensional as well and the reformulation of the associator complex presented in Proposition 4.4 is quite useful. But Proposition 4.4 is not very helpful if one wants to calculate the dimension of the associator cohomology $H^n_{ass}(H - \mathbf{mod}, A - \mathbf{mod})$ for sufficiently large $n \geq 0$. However, if $A - \mathbf{mod}$ is an *exact* module category over $H - \mathbf{mod}$, there is a more conceptual way to calculate the associator cohomology $H^\bullet_{ass}(H - \mathbf{mod}, A - \mathbf{mod})$ using the relative Ext formulation from Theorem 3.7:

$$H^\bullet_{ass}(H - \mathbf{mod}, A - \mathbf{mod}) \cong \mathrm{Ext}^\bullet_{\mathcal{Z}(H-\mathbf{mod}),H-\mathbf{mod}}(k, \mathcal{A}_{A-\mathbf{mod}}) \tag{4.14}$$

We should comment on the validity of the application of Theorem 3.7 in this non-strict setting where the associator of the monoidal category $H - \mathbf{mod}$ and the monoidal structure of the action functor are induced by the (non-trivial) associator (4.2) of **vect**. Note that by [JS], any monoidal functor admits a strictification, which is a strict monoidal functor between strict monoidal categories. In [FGS2] it is shown that the Davydov-Yetter cohomology of a monoidal functor (with coefficients) is isomorphic to the Davydov-Yetter cohomology of its strictification. Applying this result to the action functor $\rho_{A-\mathbf{mod}} : H - \mathbf{mod} \longrightarrow \mathrm{Rex}(A - \mathbf{mod})$ allows us to use Theorem 3.7 also in this non-strict setting.

For the remainder of this section, let A be an H-comodule algebra, such that $\mathcal{M} = A - \mathbf{mod}$ is an exact module category over $\mathcal{C} = H - \mathbf{mod}$. Comodule algebras with this property are called *exact* comodule algebras (see also [AM]). If A is a semisimple comodule algebra, then A is exact, since in that case all objects in $A - \mathbf{mod}$ are projective. If $A \subseteq H$ is a (left) coideal subalgebra (see Example 4.1), then A is exact by [AM, Proposition 1.20 (ii)]. Exact comodule algebras also play an important role in the study of indecomposable exact module categories over finite-dimensional Hopf algebras:

Proposition 4.6. [AM, Proposition 1.19] *For any indecomposable exact module category $\mathcal{M}$ over $H - \mathbf{mod}$ for a finite-dimensional Hopf algebra H, there is an exact comodule algebra A, such that $\mathcal{M}$ and $A - \mathbf{mod}$ are equivalent as module categories over $H - \mathbf{mod}$.*

To calculate the relative Ext on the right-hand side of (4.14), we do not need to work in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ directly. We can just as well work in the category of Yetter-Drinfeld modules over H or in the category of modules over the Drinfeld double of H since they are all equivalent as monoidal categories. Let us recall these notions in the following.
A Yetter-Drinfeld module over H is an H-module as well as an H-comodule, such that the H-action $. : H \otimes X \longrightarrow X$ and the H-coaction $\delta : X \longrightarrow H \otimes X$ satisfy the compatibility condition

$$\delta(h.x) = h' x_{(0)} S(h''') \otimes h''.x_{(1)}$$

for all $h \in H, x \in X$. It is well-known that the following is an equivalence of monoidal categories

$$\begin{aligned} {}^H_H\mathcal{YD} - \mathbf{mod} &\xrightarrow{\simeq} \mathcal{Z}(H - \mathbf{mod}) \\ X &\longrightarrow (X, \sigma^X) \end{aligned}$$

where X keeps its H-module structure and σ^X is the half-braiding $\sigma^X_V : X \otimes V \xrightarrow{\cong} V \otimes X$ with $\sigma^X_V(x \otimes v) = x_{(0)} v \otimes x_{(1)}$ for all $V \in H - \mathbf{mod}, v \in V, x \in X$.

The Drinfeld double of H has the underlying k-vector space

$$D(H) := H^* \otimes H \tag{4.15}$$

For $h, l \in H, f, g \in H^*$, consider the following multiplication in $D(H)$:

$$\begin{aligned} (\epsilon \otimes h) \cdot (f \otimes 1_H) =& f(S(h')?h''') \otimes h'' \\ (f \otimes 1_H) \cdot (\epsilon \otimes h) =& f \otimes h \\ (\epsilon \otimes h) \cdot (\epsilon \otimes l) =& \epsilon \otimes hl \\ (f \otimes 1_H) \cdot (g \otimes 1_H) =& (f * g) \otimes 1_H \end{aligned}$$

where $*$ denotes the multiplication in the algebra H^*, i.e., $(f * g)(h) = f(h'')g(h')$ for all $h \in H$. With these conventions, both H and H^* can be embedded into $D(H)$ as subalgebras via $\epsilon \otimes H \subseteq D(H)$ and $H^* \otimes 1_H \subseteq D(H)$. As is well-known, modules over the Drinfeld double $D(H)$ are just another way to look at Yetter-Drinfeld modules over H. Indeed, the following is an equivalence of monoidal categories

$${}^H_H\mathcal{YD} - \mathbf{mod} \xrightarrow{\simeq} D(H) - \mathbf{mod} \tag{4.16}$$

where a Yetter-Drinfeld module X gets mapped to the $D(H)$-module with the same H-module structure and the H^*-action $f.x = f(x_{(0)})x_{(1)}$ for all $f \in H^*, x \in X$.

Let $\mathcal{D} \in \{{}^H_H\mathcal{YD} - \mathbf{mod}, D(H) - \mathbf{mod}\}$. By definition, to calculate the relative Ext from (4.14), we need a relatively projective resolution of the tensor unit k in the monoidal category $\mathcal{D}$:

$$\ldots \xrightarrow{d_3} P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d} k \longrightarrow 0$$

to which we apply the functor $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{A}_{\mathcal{M}})$, which gives us

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_0, \mathcal{A}_{\mathcal{M}}) \xrightarrow{d_0^*} \mathrm{Hom}_{\mathcal{D}}(P_1, \mathcal{A}_{\mathcal{M}}) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathcal{D}}(P_2, \mathcal{A}_{\mathcal{M}}) \xrightarrow{d_2^*} \dots \tag{4.17}$$

and finally calculate the cohomology of the complex (4.17). To do this, we need to better understand the adjoint algebra $\mathcal{A}_{\mathcal{M}}$ as an object in $\mathcal{D}$. The following lemma provides us with a way of explicitly computing the adjoint algebra $\mathcal{A}_{\mathcal{M}}$ as an object in ${}^H_H\mathcal{YD} - \mathbf{mod}$:

Lemma 4.7. [BM] *Consider A as a regular A-bimodule. We endow the k-vector space $H \otimes A$ with the right A-module structure $(h \otimes a).b = h \otimes ab$, where $h \in H, a, b \in A$ and the usual left A-action (see (4.5)). As a k-vector space, the adjoint algebra $\mathcal{A}_{\mathcal{M}}$ of the module category $\mathcal{M} = A - \mathbf{mod}$ over $H - \mathbf{mod}$ is the space of A-bimodule morphisms* $\mathrm{Hom}_{A-\mathrm{Bimod}}(H \otimes A, A)$, *i.e., maps $f : H \otimes A \longrightarrow A$ such that*

$$f(h \otimes ab) = f(h \otimes a)b \tag{4.18}$$

and

$$f(b_{(0)}h \otimes b_{(1)}a) = bf(h \otimes a)$$

for all $h \in H, a, b \in A$. The Yetter-Drinfeld module structure of $\mathcal{A}_{\mathcal{M}}$ is given by the H-action

$$(x.f)(h \otimes a) = f(hx \otimes a)$$

and the H-coaction

$$\delta(f)(h \otimes a) = S(h')f(h'' \otimes 1_A)_{(0)}h''' \otimes f(h'' \otimes 1_A)_{(1)}a$$

where $h, x \in H, a \in A, f \in \mathcal{A}_{\mathcal{M}}$.

Note that (4.18) is equivalent to the condition that

$$f(h \otimes a) = f(h \otimes 1_A)a$$

for all $h \in H, a \in A$.

An alternative description of the adjoint algebra is given in the following lemma:

Lemma 4.8. [BM, Lemma 4.6] *As a k-vector space, the adjoint algebra $\mathcal{A}_{\mathcal{M}}$ of the module category $\mathcal{M} = A - \mathbf{mod}$ over $H - \mathbf{mod}$ is the subspace of elements $\sum_j f_j \otimes a_j \in H^* \otimes A$, such that*

$$\sum_j f_j(b_{(0)}h)a_i b_{(1)} = \sum_j f_j(h)ba_i \tag{4.19}$$

for all $b \in A, h \in H$. Let $\sum_j f_i \otimes a_j \in \mathcal{A}_{\mathcal{M}}, h \in H$. The H-action on $\mathcal{A}_{\mathcal{M}}$ is given by

$$h.(\sum_j f_j \otimes a_j) = \sum_j f_(?h) \otimes a_j \tag{4.20}$$

and the H-coaction is

$$\delta\big(\sum_j f_j \otimes a_j\big) = \sum_j S(?')(a_j)_{(0)}?''' \otimes f_j(?'') \otimes (a_j)_{(1)} \tag{4.21}$$

If one wants to work with modules over the Drinfeld double $D(H)$ instead, it is now straightforward to use the equivalence (4.16) to obtain the H^*-action on $\mathcal{A}_{\mathcal{M}}$ and understand the adjoint algebra as an object in the category $D(H) - \mathbf{mod}$. Indeed, the H^*-action on $\mathcal{A}_{\mathcal{M}}$ coming from the H-coaction (4.21) is just

$$f.\big(\sum_j f_j \otimes a_j\big) = \sum_j f\big(S(?')f_j(?'')(a_j)_{(0)}?'''\big) \otimes (a_j)_{(1)} \tag{4.22}$$

for $f \in H^*, \sum\limits_j f_j \otimes a_j \in \mathcal{A}_{\mathcal{M}}$.

4.3 The module category vect

In Section 4.1, we have considered a class of examples of module categories over the monoidal category $H - \mathbf{mod}$, which came from comodule algebras. We now want to study **vect** as a module category over $H - \mathbf{mod}$. Indeed, an H-module X acts on a k-vector space V via $X \triangleright V = U(X) \otimes V$, where $U : H - \mathbf{mod} \longrightarrow \mathbf{vect}$ is the forgetful functor which sends an H-module to its underlying k-vector space. Note that we could also consider k as an H-comodule algebra (see Example 4.1) and use the structure of $k - \mathbf{mod} = \mathbf{vect}$ as a module category over $H - \mathbf{mod}$ described in (4.5). It is easy to see that this structure of **vect** as module category over $H - \mathbf{mod}$ coincides with the one coming from the forgetful functor U.

We want to study the associator cohomology of **vect** as a module category over $H - \mathbf{mod}$. We use the connection between the Davydov-Yetter complex of the action functor

$$\begin{aligned} \rho_{\mathbf{vect}} : \quad H - \mathbf{mod} &\longrightarrow \mathrm{Rex}(\mathbf{vect}) \\ X &\longmapsto U(X) \otimes - \end{aligned}$$

and the associator deformation complex of **vect** over $H - \mathbf{mod}$ (see Corollary 2.10). In the examples we will be looking at, the Davydov-Yetter cohomologies of the forgetful functor have been calculated. Thus, we also need to relate the Davydov-Yetter complexes of the forgetful functor U and $\rho_{\mathbf{vect}}$.

Proposition 4.9. *There is an isomorphism*

$$C^\bullet_{ass}(H - \mathbf{mod}, \mathbf{vect}) \cong C^\bullet_{\mathrm{DY}}(U)$$

of the associator deformation complex of the module category **vect** *over the monoidal category* $H - \mathbf{mod}$ *and the Davydov-Yetter complex of the forgetful functor* $U : H - \mathbf{mod} \longrightarrow \mathbf{vect}$. *In particular, this yields an isomorphism of the cohomologies*

$$H^\bullet_{ass}(H - \mathbf{mod}, \mathbf{vect}) \cong H^\bullet_{\mathrm{DY}}(U)$$

Proof:
For all $n \geq 0$, we define an isomorphism of k-vector spaces

$$\alpha_n : \quad C^n_{\mathrm{DY}}(\rho_{\mathbf{vect}}) \xrightarrow{\cong} C^n_{\mathrm{DY}}(U)$$

which maps a n-cochain $b = \big((b_{X_1,\ldots,X_n})_V\big)_{X_1,\ldots,X_n\in H-\mathbf{mod},V\in\mathbf{vect}} \in C^n_{\mathrm{DY}}(\rho_{\mathbf{vect}})$ to its component on the ground field, i.e., $V = k$:

$$\alpha_n(b)_{X_1,\ldots,X_n} := (b_{X_1,\ldots,X_n})_k$$

We show that the α_n combine into an isomorphism of cochain complexes:

$$\begin{aligned}
\partial^n_{\mathrm{DY}}\big(\alpha_n(b)\big)_{X_1,\ldots,X_{n+1}} =& \,\mathrm{id}_{U(X_1)} \otimes \alpha_n(b)_{X_2,\ldots,X_{n+1}} + \sum_{i=1}^{n}(-1)^i\alpha_n(b)_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \\
&+ (-1)^{n+1}\alpha_n(b)_{X_1,\ldots,X_n} \otimes \mathrm{id}_{U(X_{n+1})} \\
=& \,\mathrm{id}_{X_1} \otimes (b_{X_2,\ldots,X_{n+1}})_k + \sum_{i=1}^{n}(-1)^i(b_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}})_k \\
&+ (-1)^{n+1}(b_{X_1,\ldots,X_n})_k \otimes \mathrm{id}_{X_{n+1}} \\
=& \,\mathrm{id}_{X_1} \otimes (b_{X_2,\ldots,X_{n+1}})_k + \sum_{i=1}^{n}(-1)^i(b_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}})_k \\
&+ (-1)^{n+1}(b_{X_1,\ldots,X_n})_{X_{n+1}\otimes k} \\
=& \big(\partial^n_{\mathrm{DY}}(b)_{X_1,\ldots,X_{n+1}}\big)_k \\
=& \,\alpha_{n+1}\big(\partial^n_{\mathrm{DY}}(b)\big)_{X_1,\ldots,X_{n+1}}
\end{aligned}$$

In the third equality above, we have used that $(b_{X_1,\ldots,X_n})_{X_{n+1}} = (b_{X_1,\ldots,X_n})_k \otimes \mathrm{id}_{X_{n+1}}$. To see why this equation holds, consider the monoidal equivalence

$$G' : \quad \mathrm{Rex}(\mathbf{vect}) \xrightarrow{\simeq} \mathbf{vect},\ T \longmapsto T(k)$$

Its quasi-inverse is

$$G : \quad \mathbf{vect} \xrightarrow{\simeq} \mathrm{Rex}(\mathbf{vect}),\ V \longmapsto V \otimes -$$

We thus have $G\big(G'(b_{X_1,\ldots,X_n})\big)_{X_{n+1}} = G\big((b_{X_1,\ldots,X_n})_k\big)_{X_{n+1}} = (b_{X_1,\ldots,X_n})_k \otimes \mathrm{id}_{X_{n+1}}$.

Finally, consider the following chain of isomorphisms of cochain complexes, where the first isomorphism comes from Corollary 2.10:

$$C^\bullet_{ass}(H - \mathbf{mod}, \mathbf{vect}) \cong C^\bullet_{\mathrm{DY}}(\rho_{\mathbf{vect}}) \cong C^\bullet_{\mathrm{DY}}(U)$$

□

5 Examples

5.1 Bosonization of exterior algebras

Consider the 2^{l+1}-dimensional $\mathbb{C}$-algebra given by the presentation

$$B_l := \mathbb{C}\left\langle x_1, \ldots, x_l, g \mid \forall i, j \in \{1, \ldots, l\} : x_i x_j = -x_j x_i,\ g x_i = -x_i g,\ g^2 = 1 \right\rangle \tag{5.1}$$

The following coproduct, counit and antipode endow B_l with the structure of a Hopf algebra:

$$\begin{aligned} \Delta(x_i) &= 1 \otimes x_i + x_i \otimes g, & \Delta(g) &= g \otimes g \\ \epsilon(x_i) &= 0, & \epsilon(g) &= 1 \\ S(x_i) &= g x_i, & S(g) &= g \end{aligned}$$

We want to use Proposition 4.9 to study the associator cohomology of **vect** over the finite tensor category $B_l - \mathbf{mod}$. We begin by gathering some results about the Davydov-Yetter cohomology of the tensor category $B_l - \mathbf{mod}$. Its dimensions are well-known:

Proposition 5.1. [GHS, Theorem 5.1] *For all $n \geq 0$, the dimension of the nth Davydov-Yetter cohomology of the tensor category $B_l - \mathbf{mod}$ is given by*

$$\dim\big(H^n_{\mathrm{DY}}(B_l - \mathbf{mod})\big) = \begin{cases} \binom{l+n-1}{n} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

It has been shown that the Davydov-Yetter cohomology of the tensor category $B_l - \mathbf{mod}$ is isomorphic to the Davydov-Yetter cohomology of the forgetful functor $U : B_l - \mathbf{mod} \longrightarrow \mathbf{vect}$.

Proposition 5.2. [FGS1, Proposition 5.4] *For all $n \geq 0$, there is an isomorphism between the nth Davydov-Yetter cohomology of the tensor category $B_l - \mathbf{mod}$ and the nth Davydov-Yetter cohomology of the forgetful functor U:*

$$H^n_{\mathrm{DY}}(B_l - \mathbf{mod}) \cong H^n_{\mathrm{DY}}(U)$$

We are now ready to investigate the dimensions of the associator cohomology of **vect** over $B_l - \mathbf{mod}$.

Corollary 5.3. *For all $n \geq 0$, the dimension of the nth associator cohomology of the $\mathbb{C}$-linear module category **vect** over the tensor category $B_l - \mathbf{mod}$ is given by*

$$\dim\big(H^n_{ass}(B_l - \mathbf{mod}, \mathbf{vect})\big) = \begin{cases} \binom{l+n-1}{n} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In particular, it follows from Corollary 5.3 that

$$\dim\big(H^2_{ass}(B_l - \mathbf{mod}, \mathbf{vect})\big) \neq 0$$

Thus, by Proposition 1.8, the $\mathbb{C}$-linear module category **vect** over $B_l - \mathbf{mod}$ does admit infinitesimal associator deformations. Moreover, by Proposition 1.15, there are no obstructions to extending these deformations to higher orders (in the sense of Definition 1.13) since $H^3_{ass}(B_l - \mathbf{mod}, \mathbf{vect}) \cong 0$.

5.2 Taft algebras

Let $m \geq 2$. For $q \in \mathbb{C}$ a primitive mth root of unity, consider the Taft algebra, i.e., the m^2-dimensional $\mathbb{C}$-algebra given by the presentation

$$T_q := \mathbb{C}\langle x, g \mid gx = qxg,\ x^m = 0,\ g^m = 1\rangle \tag{5.2}$$

The following coproduct, counit and antipode endow T_q with the structure of a Hopf algebra:

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes g, & \Delta(g) &= g \otimes g \\ \epsilon(x) &= 0, & \epsilon(g) &= 1 \\ S(x) &= -xg^{-1}, & S(g) &= g^{-1} \end{aligned}$$

We want to use Proposition 4.9 to calculate the associator cohomology of **vect** over the finite tensor category $T_q - \mathbf{mod}$. As in the previous subsection, we use a known result about the Davydov-Yetter cohomology of the tensor category $T_q - \mathbf{mod}$.

Proposition 5.4. [FGS1, Section 5.3.1] *For all $n \geq 0$, there is an isomorphism*

$$H^n_{\mathrm{DY}}(T_q - \mathbf{mod}) \cong H^n_{\mathrm{DY}}(U) \cong \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

of $\mathbb{C}$-vector spaces.

We are now ready to calculate the associator cohomology of **vect** over $T_q - \mathbf{mod}$.

Corollary 5.5. *For all $n \geq 0$, there is an isomorphism*

$$H^n_{ass}(T_q - \mathbf{mod}, \mathbf{vect}) \cong \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

of $\mathbb{C}$-vector spaces.

In particular, it follows from Corollary 5.5 that

$$H^2_{ass}(T_q - \mathbf{mod}, \mathbf{vect}) \cong \mathbb{C}$$

Thus, by Proposition 1.8, the $\mathbb{C}$-linear module category **vect** over $T_q - \mathbf{mod}$ does admit infinitesimal associator deformations. Moreover, by Proposition 1.15, there are no obstructions to extending these deformations to higher orders (in the sense of Definition 1.13) since $H^3_{ass}(T_q - \mathbf{mod}, \mathbf{vect}) \cong 0$.

5.3 Sweedler's Hopf algebra

We consider Sweedler's 4-dimensional Hopf algebra, i.e., the non-semisimple, non-commutative $\mathbb{C}$-algebra given by the presentation

$$H := \mathbb{C}\left\langle x, g \mid gx = -xg,\ x^2 = 0,\ g^2 = 1_H \right\rangle$$

The following coproduct, counit and antipode endow H with the structure of a non-cocommutative Hopf algebra:

$$\begin{aligned} \Delta(x) &= 1_H \otimes x + x \otimes g, & \Delta(g) &= g \otimes g \\ \epsilon(x) &= 0, & \epsilon(g) &= 1 \\ S(x) &= gx, & S(g) &= g \end{aligned}$$

Note that H is equal to the Hopf algebra B_1 from (5.1) and to the Taft algebra T_{-1} from (5.2). In this section, we want to study associator deformations of module categories over the finite tensor category $\mathcal{C} := H - \mathbf{mod}$. For this purpose, consider the following complete list (up to isomorphism) of coideal subalgebras of H:

- the (semi)simple 1-dimensional trivial coideal subalgebra $\mathbb{C}\langle 1_H \rangle$; note that

  $$\mathbb{C}\langle 1_H \rangle - \mathbf{mod} \cong \mathbf{vect}$$

  as $\mathcal{C}$-module categories

- the semisimple 2-dimensional coideal subalgebra $\mathbb{C}\langle 1_H, g \rangle$; note that

  $$\mathbb{C}\langle 1_H, g \rangle - \mathbf{mod} \cong \mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$$

  as $\mathcal{C}$-module categories

- the non-semisimple 2-dimensional coideal subalgebra $\mathbb{C}\langle 1_H, gx \rangle$
- the Hopf algebra H itself is a 4-dimensional non-semisimple coideal subalgebra and $H - \mathbf{mod}$ is just the regular $\mathcal{C}$-module category $\mathcal{C}$.

We will study associator deformations of the above-mentioned $\mathcal{C}$-module categories. The following is the main result of this section:

Theorem 5.6. *The $\mathcal{C}$-module categories* $\mathbf{vect}$ *and* $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$ *each admit a 1-dimensional family of infinitesimal associator deformations. For each $\lambda \in \mathbb{C}$, these infinitesimal deformations admit an associated finite deformation* $\mathbf{vect}^\lambda$ *and* $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda$ *in the sense of Definition 1.5. The $\mathcal{C}$-module categories $\mathcal{C}$ and* $\langle 1_H, gx \rangle - \mathbf{mod}$ *do not admit infinitesimal associator deformations.*

We will use comodule algebra techniques to prove the above theorem. In the following, we will provide a list of H-comodule algebras from which we will construct $\mathcal{C}$-module categories and study their associator deformations. We will compute the respective associator cohomologies using the strategy described in Section 4.2.

- For each $\xi \in \mathbb{C}$, consider the semisimple 4-dimensional $\mathbb{C}$-algebra given by the presentation

$$A_\xi := \mathbb{C}\left\langle \hat{x}, \hat{g} \mid \hat{g}\hat{x} = -\hat{x}\hat{g},\ \hat{x}^2 = \hat{1},\ \hat{g}^2 = \hat{1} \right\rangle \tag{5.3}$$

The following coaction endows each A_ξ with the structure of an H-comodule algebra:

$$\delta(\hat{x}) = 1_H \otimes \hat{x} + \xi x \otimes \hat{g}, \qquad \delta(\hat{g}) = g \otimes \hat{g}$$

There is a single simple A_ξ-module, namely the 2-dimensional A_ξ-module S with basis $\{a_+, a_-\}$. The A_ξ-action on S is defined as $\hat{g}.a_\pm = \pm a_\pm$ and $\hat{x}.a_\pm = a_\mp$.

- We can view the coideal subalgebra $\mathbb{C}\langle 1_H, g\rangle$ as part of a family of semisimple H-comodule algebras: indeed, for each $\xi \in \mathbb{C}$, consider the 2-dimensional $\mathbb{C}$-algebra given by the presentation

$$D_\xi := \mathbb{C}\left\langle \hat{w} \mid \hat{w}^2 = \hat{1} \right\rangle \tag{5.4}$$

The following coaction endows each D_ξ with the structure of an H-comodule algebra:

$$\delta(\hat{w}) = g \otimes \hat{w} + \xi g x \otimes \hat{1} \tag{5.5}$$

Note that we have $D_0 \cong \mathbb{C}\langle 1_H, g\rangle$ as H-comodule algebras. Each algebra D_ξ comes with two simple modules: the 1-dimensional simple D_ξ-modules J_+ and J_- have the basis $\{j_+\}$ and $\{j_-\}$ respectively. The D_ξ-action on $J_\pm$ is defined as $\hat{w}.j_\pm = \pm j_\pm$.

- Finally, we consider H as a comodule algebra over itself with the coproduct as coaction. It has two simple modules: the 1-dimensional simple H-modules I_+ and I_- have the basis $\{i_+\}$ and $\{i_-\}$ respectively. The H-action on them is defined as $g.i_\pm = \pm i_\pm$ and $x.i_\pm = 0$.

5.3.1 Calculation of adjoint algebras

In [GHS, Lemma 5.7], the authors construct a relatively projective resolution of the tensor unit in the category $D(H) - \mathbf{mod}$ of modules over the Drinfeld double of H (see also (4.15)). We want to use this relatively projective resolution to calculate the associator cohomology of the $\mathcal{C}$-module categories $A_\xi - \mathbf{mod}$, $D_\xi - \mathbf{mod}$ and $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$ as discussed in Section 4.2. For this purpose, we first collect some useful facts about the category $D(H) - \mathbf{mod}$. In the following, let t^* denote the dual basis element of a basis element $t \in \{1_H, g, x, gx\}$ of H.

It is convenient to introduce the *generators*

$$h := 1^* - g^* \quad \text{and} \quad y := x^* + (gx)^* \tag{5.6}$$

of the algebra H^* because then it will suffice to define $D(H)$-module structures only on the generators $g, x, h, y \in D(H)$. Indeed, we have the following relations:

$$\frac{1}{2}(h^2 + h) = 1_H^*, \qquad \frac{1}{2}(h^2 - h) = g^*$$

$$\frac{1}{2}(y * h + y) = x^*, \qquad \frac{1}{2}(y * h - y) = (gx)^*$$

Remark 5.7. *In the following, we list some $D(H)$-modules, which we will use frequently.*

- *We denote the tensor unit of $D(H)-\mathbf{mod}$ by I. It is 1-dimensional and has the following $D(H)$-module structure, where $t \in \mathsf{I}$:*

$$\begin{aligned} g.t &= -t, & x.t &= 0 \\ h.t &= t, & y.t &= 0 \end{aligned}$$

- *In* [GHS, Section 5], *the 2-dimensional relatively projective $D(H)$-modules $\mathsf{C}_\pm$ and $\mathsf{A}_\pm$ are introduced. The module C_+ has the basis $\{1_H^*, x^*\}$ and has the following $D(H)$-module structure:*

$$\begin{aligned} g.1_H^* &= 1_H^*, & x.1_H^* &= 0 \\ h.1_H^* &= 1_H^*, & y.1_H^* &= x^* \\ g.x^* &= -x^*, & x.x^* &= 0 \\ h.x^* &= -x^*, & y.x^* &= 0 \end{aligned}$$

- *The module C_- has the basis $\{g^*, (gx)^*\}$ and the $D(H)$-action is defined as:*

$$\begin{aligned} g.g^* &= -g^*, & x.g^* &= 0 \\ h.g^* &= -g^*, & y.g^* &= (gx)^* \\ g.(gx)^* &= (gx)^*, & x.(gx)^* &= 0 \\ h.(gx)^* &= (gx)^*, & y.(gx)^* &= 0 \end{aligned}$$

- *The $D(H)$-module A_+ has the basis $\{\alpha_+, \beta_+\}$ and A_- has the basis $\{\alpha_-, \beta_-\}$, where the $D(H)$-action is defined as follows:*

$$\begin{aligned} g.\alpha_\pm &= \pm\,\alpha_\pm, & x.\alpha_\pm &= \beta_\pm \\ h.\alpha_\pm &= \mp\,\alpha_\pm, & y.\alpha_\pm &= 0 \\ g.\beta_\pm &= \mp\,\beta_\pm, & x.\beta_\pm &= 0 \\ h.\beta_\pm &= \pm\,\beta_\pm, & y.\beta_\pm &= 2\alpha_\pm \end{aligned}$$

- *The $D(H)$-module W_ξ^+ has the basis $\{v_+, w_+\}$ and the $D(H)$-module W_ξ^- has the basis $\{v_-, w_-\}$, where the $D(H)$-action is defined as follows:*

$$\begin{aligned} g.v_\pm &= -v_\pm, & x.v_\pm &= w_\pm \\ h.v_\pm &= -v_\pm, & y.v_\pm &= \pm\frac{\xi^2}{2}w_\pm \\ g.w_\pm &= w_\pm, & x.w_\pm &= 0 \\ h.w_\pm &= w_\pm, & y.w_\pm &= 0 \end{aligned}$$

We are now ready to calculate the adjoint algebras of the module categories $A_\xi - \mathbf{mod}, D_\xi - \mathbf{mod}$ and $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$:

Proposition 5.8. *As a $D(H)$-module, the adjoint algebra $\mathcal{A}_{A_\xi-\mathbf{mod}}$ decomposes into the direct sum*

$$\mathcal{A}_{A_\xi-\mathbf{mod}} \cong \mathsf{A}_+ \oplus \mathsf{W}_\xi^- \tag{5.7}$$

Proof:
We use Lemma 4.8 to calculate the adjoint algebra $\mathcal{A}_{A_\xi-\mathbf{mod}}$. First, we need to understand $\mathcal{A}_{A_\xi-\mathbf{mod}}$ as a complex vector space. An element $v \in \mathcal{A}_{A_\xi-\mathbf{mod}}$ is an element of $H^* \otimes A_\xi$, i.e., it is of the form

$$\begin{aligned} v =&\big(\alpha_1 1_H^* + \alpha_2 g^* + \alpha_3 x^* + \alpha_4 (gx)^*\big) \otimes \hat{1} + \big(\beta_1 1_H^* + \beta_2 g^* + \beta_3 x^* + \beta_4 (gx)^*\big) \otimes \hat{g} \\ &+ \big(\gamma_1 1_H^* + \gamma_2 g^* + \gamma_3 x^* + \gamma_4 (gx)^*\big) \otimes \hat{x} + \big(\delta_1 1_H^* + \delta_2 g^* + \delta_3 x^* + \delta_4 (gx)^*\big) \otimes \hat{g}\hat{x} \end{aligned}$$

for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $1 \leq i \leq 4$. and it satisfies the condition (4.19). Hence, the following equation needs to hold for all $l \in H, a \in A_\xi$:

$$\begin{aligned} &\big(\alpha_1 1_H^*(a_{(0)}l) + \alpha_2 g^*(a_{(0)}l) + \alpha_3 x^*(a_{(0)}l) + \alpha_4 (gx)^*(a_{(0)}l)\big) a_{(1)} \\ &+\big(\beta_1 1_H^*(a_{(0)}l) + \beta_2 g^*(a_{(0)}l) + \beta_3 x^*(a_{(0)}l) + \beta_4 (gx)^*(a_{(0)}l)\big) \hat{g} a_{(1)} \\ &+\big(\gamma_1 1_H^*(a_{(0)}l) + \gamma_2 g^*(a_{(0)}l) + \gamma_3 x^*(a_{(0)}l) + \gamma_4 (gx)^*(a_{(0)}l)\big) \hat{x} a_{(1)} \\ &+\big(\delta_1 1_H^*(a_{(0)}l) + \delta_2 g^*(a_{(0)}l) + \delta_3 x^*(a_{(0)}l) + \delta_4 (gx)^*(a_{(0)}l)\big) \hat{g}\hat{x} a_{(1)} \\ &=\big(\alpha_1 1_H^*(l) + \alpha_2 g^*(l) + \alpha_3 x^*(l) + \alpha_4 (gx)^*(l)\big) a \\ &+\big(\beta_1 1_H^*(l) + \beta_2 g^*(l) + \beta_3 x^*(l) + \beta_4 (gx)^*(l)\big) a\hat{g} \\ &+\big(\gamma_1 1_H^*(l) + \gamma_2 g^*(l) + \gamma_3 x^*(l) + \gamma_4 (gx)^*(l)\big) a\hat{x} \\ &+\big(\delta_1 1_H^*(l) + \delta_2 g^*(l) + \delta_3 x^*(l) + \delta_4 (gx)^*(l)\big) a\hat{g}\hat{x} \end{aligned} \tag{5.8}$$

It is sufficient to check equation (5.8) on basis elements, i.e., $l \in \{1_H, g, x, gx\}, a \in \{\hat{1}, \hat{g}, \hat{x}, \hat{g}\hat{x}\}$. We thus obtain the following conditions on the coefficients of v:

$$\begin{aligned} &\alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_4, \quad \beta_1 = \beta_2, \quad \gamma_1 = -\gamma_2, \quad \gamma_3 = -\gamma_4, \quad \delta_1 = -\delta_2 \\ &\beta_3 = \beta_4 = \delta_3 = \delta_4 = 0, \quad 2\beta_1 + \xi\gamma_3 = 0, \quad 2\delta_1 + \xi\alpha_3 = 0 \end{aligned}$$

Hence, the following is a basis of $\mathcal{A}_{A_\xi-\mathbf{mod}}$:

$$\begin{aligned} b_1 =&1_H^* \otimes \hat{1} + g^* \otimes \hat{1} \\ b_2 =&x^* \otimes \hat{1} + (gx)^* \otimes \hat{1} - \frac{\xi}{2} 1_H^* \otimes \hat{g}\hat{x} + \frac{\xi}{2} g^* \otimes \hat{g}\hat{x} \\ b_3 =&1_H^* \otimes \hat{x} - g^* \otimes \hat{x} \\ b_4 =&\frac{\xi}{2} 1_H^* \otimes \hat{g} + \frac{\xi}{2} g^* \otimes \hat{g} + x^* \otimes \hat{x} - (gx)^* \otimes \hat{x} \end{aligned}$$

We are now going to calculate the Yetter-Drinfeld module structure of $\mathcal{A}_{A_\xi-\mathbf{mod}}$, starting with the H-action. By (4.20), we have

$$g.b_1 = 1_H^*(?g) \otimes \hat{1} + g^*(?g) \otimes \hat{1}$$

Since

$$1_H^*(hg) = \begin{cases} 0 & \text{if } h \in \{1_H, x, gx\} \\ 1_H & \text{if } h = g \end{cases}$$

and

$$g^*(hg) = \begin{cases} 0 & \text{if } h \in \{g, x, gx\} \\ 1_H & \text{if } h = 1_H \end{cases}$$

we obtain $1_H^*(?g) = g^*$ and $g^*(?g) = 1_H^*$ and thus

$$g.b_1 = b_1$$

Similarly, using (4.20), we calculate that

$$\begin{aligned} x.b_1 =& 0 & & \\ g.b_2 =& -b_2, & x.b_2 =& b_1 \\ g.b_3 =& -b_3, & x.b_3 =& 0 \\ g.b_4 =& b_4, & x.b_4 =& b_3 \end{aligned}$$

Now we calculate the H-comodule structure of $\mathcal{A}_{A_\xi-\mathbf{mod}}$. According to (4.21), we have

$$\delta(b_1) = S(?')?''' \otimes 1_H^*(?'') + S(?')?''' \otimes g^*(?'')$$

But $S(l')l''' \otimes 1_H^*(l'') + S(l')?''' \otimes g^*(l'') = 0$ for $l \in \{x, gx\}$ and

$$S(g)g \otimes 1_H^*(g) + S(g)g \otimes g^*(g) = S(1_H)1_H \otimes 1_H^*(1_H) + S(1_H)1_H \otimes g^*(1_H) = 1_H \otimes \hat{1}$$

Therefore, we get

$$\delta(b_1) = 1_H \otimes b_1$$

and similarly

$$\begin{aligned} \delta(b_2) =& g \otimes b_2 - \frac{\xi^2}{2} gx \otimes b_1 \\ \delta(b_3) =& 1_H \otimes b_3 + 2x \otimes b_4 \\ \delta(b_4) =& g \otimes b_4 \end{aligned}$$

To understand $\mathcal{A}_{A_\xi-\mathbf{mod}}$ as a $D(H)$-module, all we need is to calculate the H^*-action corresponding to the H-coaction from above. We use the description from (4.22):

$$1_H^*.b_1 = 1_H^*(1_H)b_1 = b_1, \quad g^*.b_1 = x^*.b_1 = (gx)^*.b_1 = 0$$

Using the generators h and y of H^* (see (5.6)), we obtain:

$$h.b_1 = b_1, \quad y.b_1 = 0$$

Similarly, we calculate the H^*-action on the remaining basis elements of $\mathcal{A}_{A_\xi-\mathbf{mod}}$:

$$\begin{aligned} h.b_2 =& -b_2, & y.b_2 =& -\frac{\xi^2}{2} b_1 \\ h.b_3 =& b_3, & y.b_3 =& 2b_4 \\ h.b_4 =& -b_4, & y.b_4 =& 0 \end{aligned}$$

Finally, identifying b_1 with w_-, b_2 with v_-, b_3 with β_+ and b_4 with α_+ proves the desired decomposition (5.7). □

The proofs of the following two propositions are analogous to the one from Proposition 5.8.

Proposition 5.9. *As a $D(H)$-module, the adjoint algebra $\mathcal{A}_{D_\xi-\mathbf{mod}}$ decomposes into the direct sum*

$$\mathcal{A}_{D_\xi-\mathbf{mod}} \cong \mathsf{A}_- \oplus \mathsf{W}_\xi^+$$

Proposition 5.10. *As a $D(H)$-module, the adjoint algebra $\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle-\mathbf{mod}}$ decomposes into the direct sum*

$$\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle-\mathbf{mod}} \cong \mathsf{A}_- \oplus \mathsf{C}_-$$

In particular, $\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle-\mathbf{mod}} \in D(H) - \mathbf{mod}$ is relatively projective.

5.3.2 Calculation of associator cohomology

Lemma 5.11. *Recall the definition of the $D(H)$-modules $\mathsf{A}_\pm, \mathsf{C}_\pm$ and $\mathsf{W}_\xi^\pm$ from Remark 5.7. We have the following isomorphisms of $\mathbb{C}$-vector spaces:*

$$\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-) \cong \mathbb{C}, \quad \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^-) \cong 0 \tag{5.9}$$

$$\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^+) \cong \mathbb{C}, \quad \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^+) \cong 0 \tag{5.10}$$

$$\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{A}_-) \cong \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{A}_-) \cong 0 \tag{5.11}$$

$$\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{A}_+) \cong \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{A}_+) \cong 0 \tag{5.12}$$

Proof:
We first show that $\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-) \cong \mathbb{C}$. Let $f \in \mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-)$. The morphism f is uniquely determined by its values on basis elements, hence we make the following general ansatz for f: there are $a_1, a_2, b_1, b_2 \in \mathbb{C}$, such that

$$f(1_H^*) = a_1 v_1 + a_2 v_2, \quad f(x^*) = b_1 v_1 + b_2 v_2$$

The $D(H)$-linearity of f gives us conditions on a_1, a_2, b_1, b_2. We start with the action of g:

$$-a_1 v_1 + a_2 v_2 = a_1 g.v_1 + a_2 g.v_2 = g.f(1_H^*) = f(g.1_H^*) = f(1_H^*) = a_1 v_1 + a_2 v_2$$

This gives us $a_1 = 0$. The action of y yields

$$0 = a_2 y.v_2 = y.f(1_H^*) = f(y.1_H^*) = f(x^*) = b_1 v_1 + b_2 v_2$$

and thus $b_1 = b_2 = 0$. The remaining linearity conditions on f do not give new conditions on the coefficients a_1, a_2, b_1, b_2. Therefore, the complex vector space $\mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-)$ has the basis $\{f\}$, where f is the morphism that sends 1_H^* to v_1 and x^* to 0.

We now show that $\mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^-) \cong 0$. Let $l \in \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^-)$. Again, l is determined by its values on basis elements and we make the following general ansatz for l: there are $c_1, c_2, d_1, d_2 \in \mathbb{C}$, such that

$$l(g^*) = c_1 v_1 + c_2 v_2, \quad l\big((gx)^*\big) = d_1 v_1 + d_2 v_2$$

As above, we obtain conditions on the coefficients c_1, c_2, d_1, d_2 from the $D(H)$-linearity of l. The action of g yields

$$-c_1v_1 + c_2v_2 = c_1g.v_1 + c_2g, v_2 = g.l(g^*) = l(g.g^*) = l(-g^*) = -c_1v_1 - c_2v_2$$

and thus $c_2 = 0$. From the action of x we get

$$c_1v_1 = c_1x.v_1 = x.l(g^*) = l(x.g^*) = l(0) = 0$$

and thus $c_1 = 0$. Finally, the action of y gives us

$$0 = y.0 = y.l(g^*) = l(y.g^*) = l\big((gx)^*\big) = d_1v_1 + d_2v_2$$

Hence $d_1 = d_2 = 0$ but this means that $l = 0$ and $\mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^-) \cong 0$.

The computation of the isomorphisms in (5.10), (5.11) and (5.12) is completely analogous.

□

Proposition 5.12. *For all $n \geq 0$, we have*

$$H^n_{ass}(H - \mathbf{mod}, A_\xi - \mathbf{mod}) \cong H^n_{ass}(H - \mathbf{mod}, D_\xi - \mathbf{mod}) \cong \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

as $\mathbb{C}$-vector spaces. Hence, both $A_\xi - \mathbf{mod}$ and $D_\xi - \mathbf{mod}$ each admit a 1-dimensional family of unobstructed associator deformations.

Proof:
To calculate the associator cohomology $H^n_{ass}(H-\mathbf{mod}, A_\xi-\mathbf{mod})$, we will use the strategy explained in Section 4.2. For this purpose, we consider the following relatively projective resolution of the tensor unit $\mathsf{I} \in D(H) - \mathbf{mod}$, which was constructed in [GHS, Lemma 5.7]:

$$\ldots \longrightarrow \mathsf{C}_- \longrightarrow \mathsf{C}_+ \longrightarrow \mathsf{C}_- \longrightarrow \mathsf{C}_+ \longrightarrow \mathsf{I} \longrightarrow 0$$

Hence, we need to calculate the cohomology of the following cochain complex of $\mathbb{C}$-vector spaces

$$0 \to \mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathcal{A}_{A_\xi-\mathbf{mod}}) \to \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathcal{A}_{A_\xi-\mathbf{mod}}) \to \mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathcal{A}_{A_\xi-\mathbf{mod}}) \to \ldots$$

By Proposition 5.8, we have an isomorphism

$$\mathcal{A}_{A_\xi-\mathbf{mod}} \cong \mathsf{A}_+ \oplus \mathsf{W}_\xi^-$$

of $D(H)$-modules, which induces an isomorphism of Hom spaces:

$$\mathrm{Hom}_{D(H)}(\mathsf{C}_\pm, \mathcal{A}_{A_\xi-\mathbf{mod}}) \cong \mathrm{Hom}_{D(H)}(\mathsf{C}_\pm, \mathsf{A}_+) \oplus \mathrm{Hom}_{D(H)}(\mathsf{C}_\pm, \mathsf{W}_\xi^-) \cong \mathrm{Hom}_{D(H)}(\mathsf{C}_\pm, \mathsf{W}_\xi^-)$$

Here we have used (5.12) in the second isomorphism. Therefore, it suffices to calculate the cohomology of the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-) \longrightarrow \mathrm{Hom}_{D(H)}(\mathsf{C}_-, \mathsf{W}_\xi^-) \longrightarrow \mathrm{Hom}_{D(H)}(\mathsf{C}_+, \mathsf{W}_\xi^-) \longrightarrow \dots \tag{5.13}$$

But by (5.9), the cochain complex (5.13) is isomorphic to the following cochain complex:

$$0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow \dots$$

Hence, we have

$$H^n_{ass}(H-\mathbf{mod}, A_\xi - \mathbf{mod}) \cong \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 0$.

Analogously, using the decomposition

$$\mathcal{A}_{D_\xi - \mathbf{mod}} \cong \mathsf{A}_- \oplus \mathsf{W}_\xi^+$$

from Proposition 5.9 and the isomorphisms (5.10) and (5.11), we obtain the desired isomorphism for $H^\bullet_{ass}(H-\mathbf{mod}, D_\xi - \mathbf{mod})$. □

Proposition 5.13. *For all $n \geq 1$, we have*

$$H^n_{ass}(H-\mathbf{mod}, H-\mathbf{mod}) \cong H^n_{ass}(H-\mathbf{mod}, \mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}) \cong 0$$

and thus, neither $H-\mathbf{mod}$ nor $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$ admit infinitesimal associator deformations.

Proof:
The isomorphism $H^n_{ass}(H-\mathbf{mod}, H-\mathbf{mod}) \cong 0$ for $n \geq 1$ follows immediately from Theorem 3.22.

For $H^n_{ass}(H-\mathbf{mod}, \mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod})$ note that the adjoint algebra $\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}}$ is relatively projective by Proposition 5.10 and hence the right dual ${}^*\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}}$ is relatively projective since duals of relatively projective objects are relatively projective (see [FGS1, Proposition 3.2]). To conclude the proof, consider for $n \geq 1$ the following chain of isomorphisms, where we have used Theorem 3.7 in the first isomorphism and [FGS1, Corollary 3.3] in the second isomorphism:

$$\begin{aligned} H^n_{ass}(H-\mathbf{mod}, D-\mathbf{mod}) &\cong \mathrm{Ext}^n_{D(H)-\mathbf{mod}, H-\mathbf{mod}}(\mathsf{I}, \mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}}) \\ &\cong \mathrm{Ext}^n_{D(H)-\mathbf{mod}, H-\mathbf{mod}}({}^*\mathcal{A}_{\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}}, \mathsf{I}) \\ &\cong 0 \end{aligned}$$

□

Let us conclude this subsection with a review of which parts of the proof of Theorem 5.6, we have finished already: recall that $H = T_{-1}$ (see (5.2)). Hence, we know from Corollary 5.5 that **vect** as a $\mathcal{C}$-module category admits a one-parameter family of infinitesimal deformations. Since $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod} \cong D_0 - \mathbf{mod}$ as $\mathcal{C}$-module categories, Proposition 5.12 yields that $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$ admits a one-parameter family of infinitesimal associator deformations as well. In Proposition 5.13, we have shown that both $\mathcal{C}$ as a regular $\mathcal{C}$-module category and $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$ do not admit infinitesimal associator deformations.

5.3.3 Finite deformations

Let $\mathcal{C}$ denote the finite tensor category $H - \mathbf{mod}$. From [GHS, Remark 5.8], it is known that the natural family $\Psi^{(1)} = \big(\Psi^{(1)}_{V,W}\big)_{V,W\in\mathcal{C}}$ with

$$\begin{aligned}\Psi^{(1)}_{V,W}: \quad V\otimes W &\longrightarrow V\otimes W\\ v\otimes w &\longmapsto x.v\otimes xg.w\end{aligned}$$

is a 2-cocycle in the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(\mathrm{Id}_\mathcal{C})$, whose cohomology class generates $H^2_{\mathrm{DY}}(\mathrm{Id}_\mathcal{C})$. Hence, by Proposition 2.3, the natural family $\Psi = (\Psi_{V,W})_{V,W\in\mathcal{C}}$ with

$$\Psi_{V,W} = \mathrm{id}_{V\otimes W} + \varepsilon\Psi^{(1)}_{V,W} \tag{5.14}$$

is a non-trivial infinitesimal deformation of the monoidal structure of the identity functor $(\mathrm{Id}_\mathcal{C}, \mathrm{id})$. By Corollary 2.20, the infinitesimal deformation (5.14) gives rise to an infinitesimal associator deformation $\big(R_1\otimes\mathbf{vect}, m^{(1)}\big)$, where $m^{(1)}\in C^2_{ass}(\mathcal{C},\mathcal{M})$ is the associator 2-cocycle with

$$m^{(1)}_{V,W,M} = m_{V,W,M}\circ\big(\Psi^{(1)}_{V,W}\triangleright \mathrm{id}_M\big)$$

Therefore, the natural family $\mathbf{m} = (\mathbf{m}_{V,W,M})_{V,W\in\mathcal{C},M\in\mathbf{vect}}$ with

$$\begin{aligned}\mathbf{m}_{V,W,M}\big((v\otimes w)\otimes m\big) =& m_{V,W,M}\big((v\otimes w)\otimes m\big) + \varepsilon\big(m_{V,W,M}\circ\big(\Psi^{(1)}_{V,W}\triangleright\mathrm{id}_M\big)\big)\big((v\otimes w)\otimes m\big)\\ =& v\otimes(w\otimes m) + \varepsilon x.v\otimes(xg.w\otimes m)\end{aligned}$$

for all $v\in V, w\in W, m\in M$, is the mixed associator of the $R_1\otimes\mathcal{C}$-module category $\big(R_1\otimes\mathbf{vect}, m^{(1)}\big)$.

In the following, we will show that the infinitesimal deformations (5.14) and $\big(R_1\otimes\mathbf{vect}, m^{(1)}\big)$ can be promoted to finite deformations (see also Definition 1.5):

Proposition 5.14. *For all $\lambda\in\mathbb{C}$, the natural family $\Psi^\lambda = \big(\Psi^\lambda_{V,W}\big)_{V,W\in\mathcal{C}}$ of isomorphisms*

$$\begin{aligned}\Psi^\lambda_{V,W} = \mathrm{id}_{V\otimes W} + \lambda\Psi^{(1)}_{V,W}: \quad V\otimes W &\xrightarrow{\cong} V\otimes W\\ v\otimes w &\longmapsto v\otimes w + \lambda x.v\otimes xg.w\end{aligned}$$

and the isomorphism $\mathrm{id}_\mathbb{C}:\mathbb{C}\xrightarrow{\cong}\mathbb{C}$ endow the identity functor $\mathrm{Id}_\mathcal{C}$ with the structure of a monoidal functor $\big(\mathrm{Id}_\mathcal{C}, \Psi^\lambda, \mathrm{id}_\mathbb{C}\big):\mathcal{C}\longrightarrow\mathcal{C}$.

Proof:
Let $V, W, Z \in \mathcal{C}$ and $v \in V, w \in W, z \in Z$. We check that $\big(\mathrm{Id}_{\mathcal{C}}, \Psi^{\lambda}\big)$ satisfies the hexagon axiom (A.2):

$$\begin{aligned}
&\big(\Psi^{\lambda}_{V,W\otimes Z} \circ (\mathrm{id}_V \otimes \Psi^{\lambda}_{W,Z}) \circ \mathfrak{a}_{V,W,Z}\big)\big((v\otimes w)\otimes z\big)\\
=&\big(\Psi^{\lambda}_{V,W\otimes Z} \circ (\mathrm{id}_V \otimes \Psi^{\lambda}_{W,Z})\big)\big(v\otimes (w\otimes z)\big)\\
=&\big(\Psi^{\lambda}_{V,W\otimes Z}\big)\big(v\otimes (w\otimes z) + \lambda v\otimes (x.w\otimes xg.z)\big)\\
=&v\otimes (w\otimes z) + \lambda v\otimes (x.w\otimes xg.z) + \lambda x.v\otimes \big(xg.(w\otimes z)\big) + \lambda^2 x.v\otimes \big(xg.(x.w\otimes xg.z)\big)\\
=&v\otimes (w\otimes z) + \lambda v\otimes (x.w\otimes xg.z) + \lambda x.v\otimes (g.w\otimes xg.z) + \lambda x.v\otimes (xg.w\otimes z)\\
=&\mathfrak{a}_{V,W,Z}\big((v\otimes w)\otimes z + \lambda(v\otimes x.w)\otimes xg.z + \lambda(x.v\otimes g.w)\otimes xg.z + \lambda(x.v\otimes xg.w)\otimes z\big)\\
=&\mathfrak{a}_{V,W,Z}\big((v\otimes w)\otimes z + \lambda(x.v\otimes xg.w)\otimes z + \lambda\big(x.(v\otimes w)\big)\otimes xg.z\\
&+ \lambda^2\big(x.(x.v\otimes xg.w)\big)\otimes xg.z\big)\\
=&\big(\mathfrak{a}_{V,W,Z}\circ \Psi^{\lambda}_{V\otimes W,Z}\big)\big((v\otimes w)\otimes z + \lambda(x.v\otimes xg.w)\otimes z\big)\\
=&\big(\mathfrak{a}_{V,W,Z}\circ \Psi^{\lambda}_{V\otimes W,Z}\circ(\Psi^{\lambda}_{V,W}\otimes \mathrm{id}_Z)\big)\big((v\otimes w)\otimes z\big)
\end{aligned}$$

Let $\kappa \in \mathbb{C}$. The following calculation shows that the unit axiom (A.3) holds:

$$\big(\Psi^{\lambda}_{\mathbb{C},V}\circ(\mathrm{id}_{\mathbb{C}}\otimes \mathrm{id}_V)\big)(\kappa\otimes v) = \Psi^{\lambda}_{\mathbb{C},V}(\kappa\otimes v) = \kappa\otimes v + \lambda \underbrace{x.\kappa}_{=0}\otimes xg.v = (\mathfrak{l}_V^{-1}\circ \mathfrak{l}_V)(\kappa\otimes v)$$

The unit axiom (A.4) is satisfied since

$$\big(\Psi^{\lambda}_{V,\mathbb{C}}\circ(\mathrm{id}_V\otimes \mathrm{id}_{\mathbb{C}})\big)(v\otimes\kappa) = \Psi^{\lambda}_{V,\mathbb{C}}(v\otimes\kappa) = v\otimes\kappa + \lambda x.v\otimes \underbrace{xg.\kappa}_{=0} = (\mathfrak{r}_V^{-1}\circ \mathfrak{r}_V)(v\otimes\kappa)$$

It is easy to see that the inverse of Ψ^{λ} is given by the natural family $\big(\Psi^{\lambda}\big)^{-1} = \big(\big(\Psi^{\lambda}\big)^{-1}_{V,W}\big)_{V,W\in\mathcal{C}}$ of isomorphisms

$$\begin{aligned}
\big(\Psi^{\lambda}\big)^{-1}_{V,W} = \mathrm{id}_{V\otimes W} - \lambda\Psi^{(1)}_{V,W} : \quad V\otimes W &\xrightarrow{\cong} V\otimes W\\
v\otimes w &\longmapsto v\otimes w - \lambda x.v\otimes xg.w
\end{aligned}$$

□

Corollary 5.15. *For all $\lambda \in \mathbb{C}$, the infinitesimal associator deformation $\big(R_1\otimes\mathbf{vect}, m^{(1)}\big)$ admits an associated finite deformation $\mathbf{vect}^{\lambda}$, whose mixed associator is the natural family $\mathbf{m}^{\lambda} = (\mathbf{m}^{\lambda}_{V,W,M})_{V,W,\in\mathcal{C},M\in\mathbf{vect}}$ of isomorphisms*

$$\begin{aligned}
\mathbf{m}^{\lambda}_{V,W,M} : \quad (V\otimes W)\otimes M &\xrightarrow{\cong} V\otimes (W\otimes M)\\
(v\otimes w)\otimes m &\longmapsto v\otimes(w\otimes m) + \lambda x.v\otimes (xg.w\otimes m)
\end{aligned} \tag{5.15}$$

and whose unitor is inherited from the $\mathcal{C}$-module category $\mathbf{vect}$.

Proof:
Let $\lambda \in \mathbb{C}$. We can either directly check the pentagon axiom (A.6) and the triangle axiom (A.7) for the mixed associator $\mathbf{m}^\lambda$ or use Proposition 5.14, which is the path we are going to take. Consider the following composition of monoidal functors

$$\mathcal{C} \xrightarrow{(\mathrm{Id}_\mathcal{C},\Psi^\lambda)} \mathcal{C} \xrightarrow{(\rho_{\mathbf{vect}},\Phi)} \mathrm{Rex}(\mathbf{vect}) \tag{5.16}$$

where $(\rho_{\mathbf{vect}}, \Phi)$ is the action functor of the $\mathcal{C}$-module category $\mathbf{vect}$ with its standard monoidal structure and $(\mathrm{Id}_\mathcal{C}, \Psi^\lambda)$ is the deformed identity functor from Proposition 5.14. The underlying linear functor of the composition (5.16) is still equal to $\rho_{\mathbf{vect}}$ and the monoidal structure Φ^λ of the composition is, according to (2.28), given by

$$(\Phi^\lambda_{V,W})_M = \rho_{\mathbf{vect}}(\Psi^\lambda_{V,W})_M \circ (\Phi_{V,W})_M = (\Psi^\lambda_{V,W} \otimes \mathrm{id}_M) \circ m^{-1}_{V,W,M}$$

for all $V, W \in \mathcal{C}, M \in \mathbf{vect}$. By Proposition A.16, the monoidal structure Φ^λ on $\rho_{\mathbf{vect}}$ corresponds to the following $\mathcal{C}$-module structure on $\mathbf{vect}$, where $v \in V, w \in W, m \in M$:

$$\begin{aligned}\big((\Psi^\lambda_{V,W} \otimes \mathrm{id}_M) \circ m^{-1}_{V,W,M}\big)^{-1}\big((v \otimes w) \otimes m\big) =& \big(m_{V,W,M} \circ ((\Psi^\lambda_{V,W})^{-1} \otimes \mathrm{id}_M)\big)\big((v \otimes w) \otimes m\big)\\ =& v \otimes (w \otimes m) - \lambda x.v \otimes (xg.w \otimes m)\end{aligned}$$

Of course, the sign can be absorbed into the parameter λ, which concludes the proof.

□

By abuse of notation, we will denote the cocycles that generate the cohomologies $H^2_{ass}(H-\mathbf{mod}, D_\xi-\mathbf{mod})$ and $H^2_{ass}(H-\mathbf{mod}, A_\xi-\mathbf{mod})$ by $m^{(1)}$ as well.

Proposition 5.16. *The natural family*

$$m^{(1)} = (m^{(1)}_{V,Z,M})_{V,Z\in\mathcal{C},M\in D_\xi-\mathbf{mod}} \tag{5.17}$$

with

$$\begin{aligned} m^{(1)}_{V,Z,M}: \quad & (V \otimes Z) \otimes M \xrightarrow{\cong} V \otimes (Z \otimes M)\\ & (v \otimes z) \otimes m \longmapsto x.v \otimes (xg.z \otimes m)\end{aligned}$$

is a 2-cocycle in the associator complex $C^\bullet_{ass}(H-\mathbf{mod}, D_\xi-\mathbf{mod})$. Moreover, the cohomology class $[m^{(1)}]$ is not trivial and hence the set $\{[m^{(1)}]\}$ is a basis of the 1-dimensional $\mathbb{C}$-vector space $H^2_{ass}(H-\mathbf{mod}, D_\xi-\mathbf{mod})$.

Proof:
We use the isomorphism of cochain complexes $C^\bullet_{ass}(H-\mathbf{mod}, D_\xi-\mathbf{mod}) \cong C^\bullet_{alg}(H, D_\xi)$ from Lemma 4.3. We first show that the element $x \otimes xg \otimes \hat{1} \in H^{\otimes 2} \otimes D_\xi$ is a 2-cocycle in the complex $C^\bullet_{alg}(H, D_\xi)$. Recall from (4.10) that

$$\delta^{(2)}(\hat{w}) = g \otimes g \otimes \hat{w} + \xi g \otimes gx \otimes \hat{1} + \xi gx \otimes 1_H \otimes \hat{1}$$

It is easy to see that

$$\delta^{(2)}(\hat{w})\cdot(x\otimes xg\otimes\hat{1})=(x\otimes xg\otimes\hat{1})\cdot\delta^{(2)}(\hat{w})$$

and hence $x\otimes xg\otimes\hat{1}\in C^2_{alg}(H,D_\xi)$ (see (4.11)). The 2-cocycle condition for $x\otimes xg\otimes\hat{1}$ is easy to check as well:

$$\begin{aligned}\partial^2_{alg}(x\otimes xg\otimes\hat{1}) =&1_H\otimes x\otimes xg\otimes\hat{1}-\Delta(x)\otimes xg\otimes\hat{1}+x\otimes\Delta(xg)\otimes\hat{1}\\ &-x\otimes xg\otimes 1_H\otimes\hat{1}\\ =&1_H\otimes x\otimes xg\otimes\hat{1}-1_H\otimes x\otimes xg\otimes\hat{1}-x\otimes g\otimes xg\otimes\hat{1}\\ &+x\otimes g\otimes xg\otimes\hat{1}+x\otimes xg\otimes 1_H\otimes\hat{1}-x\otimes xg\otimes 1_H\otimes\hat{1}\\ =&0\end{aligned}$$

We now show that $x\otimes xg\otimes\hat{1}$ is not a coboundary in the complex $C^\bullet_{alg}(H,D_\xi)$. For this purpose, consider a general element in $b\in H\otimes D_\xi$, i.e., b is of the form

$$\begin{aligned}b =&a_{1,1}1_H\otimes\hat{1}+a_{2,1}g\otimes\hat{1}+a_{3,1}x\otimes\hat{1}+a_{4,1}gx\otimes\hat{1}\\ &+a_{1,2}1_H\otimes\hat{w}+a_{2,2}g\otimes\hat{w}+a_{3,2}x\otimes\hat{w}+a_{4,2}gx\otimes\hat{w}\end{aligned}$$

for some $a_{i,j}\in\mathbb{C}$. The element b is a 1-cochain in the complex $C^\bullet_{alg}(H,D_\xi)$ if and only if the condition (4.11) holds, i.e.,

$$\delta(\hat{w})\cdot b=b\cdot\delta(\hat{w})$$

which yields the following conditions on the $a_{i,j}$:

$$a_{3,1}=a_{3,2}=0,\quad a_{4,1}-\xi a_{2,2}=0,\quad a_{4,2}-\xi a_{2,1}=0$$

We now calculate $\partial^1_{alg}(b)$:

$$\begin{aligned}\partial^1_{alg}(b) =&\partial^1_{alg}\big(a_{1,1}1_H\otimes\hat{1}+a_{2,1}g\otimes\hat{1}+a_{4,1}gx\otimes\hat{1}+a_{1,2}1_H\otimes\hat{w}+a_{2,2}g\otimes\hat{w}+a_{4,2}gx\otimes\hat{w}\big)\\ =&a_{1,1}1_H\otimes 1_H\otimes\hat{1}+a_{2,1}1_H\otimes g\otimes\hat{1}-a_{2,1}g\otimes g\otimes\hat{1}+a_{2,1}g\otimes 1_H\otimes\hat{1}\\ &+a_{4,1}1_H\otimes gx\otimes\hat{1}-a_{4,1}g\otimes gx\otimes\hat{1}+a_{1,2}1_H\otimes g\otimes\hat{w}+a_{1,2}\xi 1_H\otimes gx\otimes\hat{1}\\ &+a_{2,2}1_H\otimes g\otimes\hat{w}+a_{2,2}\xi g\otimes gx\otimes\hat{1}+a_{4,2}1_H\otimes gx\otimes\hat{w}-a_{4,2}g\otimes gx\otimes\hat{w}\\ &-a_{4,2}gx\otimes 1_H\otimes\hat{w}+a_{4,2}gx\otimes g\otimes\hat{w}+a_{4,2}\xi gx\otimes gx\otimes\hat{1}\end{aligned}$$

Setting $\partial^1_{alg}(b)=x\otimes xg\otimes\hat{1}$, we obtain the following conditions on the $a_{i,j}$:

$$\begin{aligned}&a_{1,1}=a_{2,1}=a_{4,2}=0\\ &a_{4,1}+a_{1,2}\xi=0,\quad -a_{4,1}+a_{2,2}\xi=0,\quad 0=-1\end{aligned}$$

which are obviously impossible. Hence, the 2-cocycle $x\otimes xg\otimes\hat{1}\in C^2_{alg}(H,D_\xi)$ is not a coboundary. To conclude the proof, recall that the 2-cocycle $x\otimes xg\otimes\hat{1}\in C^2_{alg}(H,D_\xi)$ corresponds to the 2-cocycle (5.17) by Lemma 4.3. In particular, the 2-cocycle (5.17) is not a coboundary and by Proposition 5.12 the $\mathbb{C}$-vector space $H^2_{ass}(H-\mathbf{mod},D_\xi-\mathbf{mod})$ is 1-dimensional.

□

The proofs of the following three results are analogous to the proofs of Corollary 5.15 and Proposition 5.16.

Corollary 5.17. *Let $m^{(1)}$ be the 2-cocycle from Proposition 5.16 (see (5.17)). For all $\lambda \in \mathbb{C}$, the infinitesimal associator deformation $\big(R_1 \otimes D_\xi - \mathbf{mod}, m^{(1)}\big)$ admits an associated finite deformation $(D_\xi - \mathbf{mod})^\lambda$, whose mixed associator is the natural family $\mathbf{m}^\lambda = (\mathbf{m}^\lambda_{V,W,M})_{V,W,\in\mathcal{C},M\in D_\xi-\mathbf{mod}}$ of isomorphisms*

$$\begin{aligned} \mathbf{m}^\lambda_{V,W,M}: \quad & (V \otimes W) \otimes M \xrightarrow{\cong} V \otimes (W \otimes M) \\ & (v \otimes w) \otimes m \longmapsto v \otimes (w \otimes m) + \lambda x.v \otimes (xg.w \otimes m) \end{aligned} \tag{5.18}$$

and whose unitor is inherited from the $\mathcal{C}$-module category $D_\xi - \mathbf{mod}$.

Proposition 5.18. *The natural family*

$$m^{(1)} = (m^{(1)}_{V,Z,M})_{V,Z\in\mathcal{C},M\in A_\xi-\mathbf{mod}} \tag{5.19}$$

with

$$\begin{aligned} m^{(1)}_{V,Z,M}: \quad & (V \otimes Z) \otimes M \xrightarrow{\cong} V \otimes (Z \otimes M) \\ & (v \otimes z) \otimes m \longmapsto x.v \otimes (xg.z \otimes m) \end{aligned}$$

is a 2-cocycle in the associator complex $C^\bullet_{ass}(H - \mathbf{mod}, A_\xi - \mathbf{mod})$. Moreover, the cohomology class $[m^{(1)}]$ is not trivial and hence the set $\{[m^{(1)}]\}$ is a basis of the 1-dimensional $\mathbb{C}$-vector space $H^2_{ass}(H - \mathbf{mod}, A_\xi - \mathbf{mod})$.

Corollary 5.19. *Let $m^{(1)}$ be the 2-cocycle from Proposition 5.18 (see (5.19)). For all $\lambda \in \mathbb{C}$, the infinitesimal associator deformation $\big(R_1 \otimes A_\xi - \mathbf{mod}, m^{(1)}\big)$ admits an associated finite deformation $(A_\xi - \mathbf{mod})^\lambda$, whose mixed associator is the natural family $\mathbf{m}^\lambda = (\mathbf{m}^\lambda_{V,W,M})_{V,W,\in\mathcal{C},M\in A_\xi-\mathbf{mod}}$ of isomorphisms*

$$\begin{aligned} \mathbf{m}^\lambda_{V,W,M}: \quad & (V \otimes W) \otimes M \xrightarrow{\cong} V \otimes (W \otimes M) \\ & (v \otimes w) \otimes m \longmapsto v \otimes (w \otimes m) + \lambda x.v \otimes (xg.w \otimes m) \end{aligned}$$

and whose unitor is inherited from the $\mathcal{C}$-module category $A_\xi - \mathbf{mod}$.

It is now a natural question to classify the $\mathcal{C}$-module categories $\mathbf{vect}^\lambda$ and $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda$ with respect to the parameter λ:

Proposition 5.20. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$. The $\mathcal{C}$-modules $\mathbf{vect}^{\lambda_1}$ and $\mathbf{vect}^{\lambda_2}$ are equivalent if and only if $\lambda_1 = \lambda_2$.*

Proof:
First, we construct an equivalence $F : \mathbf{vect}^{\lambda_1} \xrightarrow{\simeq} \mathbf{vect}^{\lambda_2}$ of the underlying $\mathbb{C}$-linear categories. Note that any equivalence of linear categories sends simple objects to simple objects and an equivalence between semisimple linear categories is uniquely determined by its value on simple objects. Obviously, the category of $\mathbb{C}$-vector spaces only has one isomorphism class of simple objects. Thus, we know that $F(\mathbb{C}) \cong \mathbb{C}$. By Lemma A.20,

we may assume that F is the identity functor on **vect**.

We want to promote F to a $\mathcal{C}$-module functor. Let s denote the $\mathcal{C}$-module structure of F. By Lemma 4.2, s is uniquely determined by its value $s_{H,\mathbb{C}}(1_H \otimes 1) \in H \otimes \mathbb{C}$ and for all $V \in H-\mathbf{mod}, M \in \mathbf{vect}$, the element $s_{V,M}(v \otimes m) \in V \otimes M$ is given by the componentwise left action of $s_{H,\mathbb{C}}(1_H \otimes 1)$ on $v \otimes m \in V \otimes M$. Since $s_{H,\mathbb{C}}(1_H \otimes 1) \in H \otimes \mathbb{C}$, the most general form it can take is

$$s_{H,\mathbb{C}}(1_H \otimes 1) = \mu_1 1_H \otimes 1 + \mu_2 g \otimes 1 + \mu_3 x \otimes 1 + \mu_4 gx \otimes 1$$

for some $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{C}$. With this general ansatz, we will now calculate the pentagon (A.11) to obtain conditions on $\mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2$ for $(F, s) : \mathbf{vect}^{\lambda_1} \xrightarrow{\simeq} \mathbf{vect}^{\lambda_2}$ to be an equivalence of $\mathcal{C}$-module categories. Let $V, W \in H - \mathbf{mod}, M \in \mathbf{vect}$ and $v \in V, w \in W, m \in M$. The right path of the pentagon reads

$$\begin{aligned}
&(\mathbf{m}^{\lambda_2}_{V,W,M} \circ s_{V \otimes W,M})\big((v \otimes w) \otimes m\big) \\
=&\mathbf{m}^{\lambda_2}_{V,W,M}\Big(\mu_1(v \otimes w) \otimes m + \mu_2(g.v \otimes g.w) \otimes m + \mu_3(v \otimes x.w) \otimes m + \mu_3(x.v \otimes g.w) \otimes m \\
&+ \mu_4(g.v \otimes xg.w) \otimes m + \mu_4(xg.v \otimes w) \otimes m\Big) \\
=&\mu_1 v \otimes (w \otimes m) + \mu_2 g.v \otimes (g.w \otimes m) + \mu_3 v \otimes (x.w \otimes m) + \mu_3 x.v \otimes (g.w \otimes m) \\
&+ \mu_4 g.v \otimes (xg.w \otimes m) + \mu_4 xg.v \otimes (w \otimes m) + \lambda_2\mu_1 x.v \otimes (xg.w \otimes m) \\
&+ \lambda_2\mu_2 xg.v \otimes (x.w \otimes m)
\end{aligned}$$

The left path of the pentagon reads

$$\begin{aligned}
&\big((\mathrm{id}_V \otimes s_{W,M}) \circ s_{V,W \otimes M} \circ \mathbf{m}^{\lambda_1}_{V,W,M}\big)\big((v \otimes w) \otimes m\big) \\
=&\big((\mathrm{id}_V \otimes s_{W,M}) \circ s_{V,W \otimes M}\big)\Big(v \otimes (w \otimes m) + \lambda_1 x.v \otimes (xg.w \otimes m)\Big) \\
=&(\mathrm{id}_V \otimes s_{W,M})\Big(\mu_1 v \otimes (w \otimes m) + \mu_2 g.v \otimes (w \otimes m) + \mu_3 x.v \otimes (w \otimes m) \\
&+ \mu_4 xg.v \otimes (w \otimes m) + \lambda_1\mu_1 x.v \otimes (xg.w \otimes m) + \lambda_1\mu_2 g.(x.v) \otimes (xg.w \otimes m)\Big) \\
=&\mu_1^2 v \otimes (w \otimes m) + \mu_1\mu_2 v \otimes (g.w \otimes m) + \mu_1\mu_3 v \otimes (x.w \otimes m) + \mu_1\mu_4 v \otimes (xg.w \otimes m) \\
&+ \mu_2\mu_1 g.v \otimes (w \otimes m) + \mu_2^2 g.v \otimes (g.w \otimes m) + \mu_2\mu_3 g.v \otimes (x.w \otimes m) \\
&+ \mu_2\mu_4 g.v \otimes (xg.w \otimes m) \\
&+ \mu_3\mu_1 x.v \otimes (w \otimes m) + \mu_3\mu_2 x.v \otimes (g.w \otimes m) + \mu_3^2 x.v \otimes (x.w \otimes m) \\
&+ \mu_3\mu_4 x.v \otimes (xg.w \otimes m) \\
&+ \mu_4\mu_1 xg.v \otimes (w \otimes m) + \mu_4\mu_2 xg.v \otimes (g.w \otimes m) + \mu_4\mu_3 xg.v \otimes (x.w \otimes m) \\
&+ \mu_4^2 xg.v \otimes (xg.w \otimes m) \\
&+ \lambda_1\mu_1^2 x.v \otimes (xg.w \otimes m) + \lambda\mu_1\mu_2 x.v \otimes \big(g.(xg.w) \otimes m\big) \\
&+ \lambda_1\mu_2\mu_1 g.(x.v) \otimes (xg.w \otimes m) + \lambda_1\mu_2^2 g.(x.v) \otimes \big(g.(xg.w) \otimes m\big)
\end{aligned}$$

Now we set $V = W = H$ and $M = \mathbb{C}$ as well as $v = w = 1_H$ and $m = 1$. Comparing coefficients of the basis vectors of $H^{\otimes 2} \otimes \mathbb{C}$ in the pentagon equation above, we obtain the following system of nonlinear equations:

$$\begin{aligned}\mu_1^2 =&\mu_1, \quad \mu_2^2 = \mu_2, \quad \mu_3^2 = \lambda_1\mu_1\mu_2, \quad \mu_4^2 = \lambda_1\mu_1\mu_2\\ \lambda_2\mu_1 =&\mu_3\mu_4 + \lambda_1\mu_1^2, \quad \lambda_2\mu_2 = \mu_4\mu_3 + \lambda_1\mu_2^2\\ \mu_3 =&\mu_1\mu_3 = \mu_2\mu_3, \quad \mu_4 = \mu_1\mu_4 = \mu_2\mu_4\\ 0 =&\mu_1\mu_2 = \mu_1\mu_3 = \mu_1\mu_4 = \mu_2\mu_3 = \mu_2\mu_4\end{aligned}$$

The above system has the following two non-zero solutions:

$$\lambda_1 = \lambda_2, \quad \mu_1 = 1, \quad \mu_2 = \mu_3 = \mu_4 = 0 \tag{5.20}$$

$$\lambda_1 = \lambda_2, \quad \mu_2 = 1, \quad \mu_1 = \mu_3 = \mu_4 = 0 \tag{5.21}$$

The first solution (5.20) as well as the second solution (5.21) each turn s into an isomorphism, which is moreover an involution. Since s is determined by its value $s_{H,\mathbb{C}}(1_H \otimes 1) \in H \otimes \mathbb{C}$, the linear equivalence F admits a $\mathcal{C}$-module structure if and only if $\lambda_1 = \lambda_2$.

□

Thus, we have constructed a one-parameter family of non-equivalent $\mathcal{C}$-module categories with the same underlying $\mathbb{C}$-linear category **vect** and the same $\mathcal{C}$-action.

Proposition 5.21. *Let $\nu_1, \nu_2 \in \mathbb{C}$. The $\mathcal{C}$-modules $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_1}$ and $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_2}$ are equivalent if and only if $\nu_1 = \nu_2$.*

Proof:
Recall the simple objects $J_\pm$ in $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$: $J_\pm$ has the basis $\{j_\pm\}$ and the $\mathbb{C}[\mathbb{Z}_2]$-action given by $\hat{w}.j_\pm = \pm j_\pm$.

This proof is similar to the one for Proposition 5.20. However, this time there are, up to natural isomorphism, two linear autoequivalences of $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$. The first one maps J_+ to J_+ and J_- to J_- and the second one maps J_+ to J_- and J_- to J_+. By Lemma A.20, it suffices to study one representative of each of the isomorphism classes of linear equivalences and show that they admit a $\mathcal{C}$-module structure if and only if $\nu_1 = \nu_2$. The representatives we choose are the identity functor and the functor

$$\begin{aligned}F: \quad (\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_1} &\longrightarrow (\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_2}\\ M &\longmapsto M \otimes J_-\\ f &\longmapsto f \otimes \mathrm{id}_{J_-}\end{aligned}$$

We start with the discussion of the $\mathcal{C}$-module structure $s = (s_{V,M})_{V\in\mathcal{C},M\in\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod}}$ of the linear equivalence F. By an argument which is analogous to the one in Lemma 4.2, we can make the following general ansatz for s: there are $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that for all $V \in \mathcal{C}, M \in \mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$, the morphism

$$s_{V,M}: \quad \underbrace{(V \otimes M) \otimes J_-}_{=F(V\otimes M)} \longrightarrow \underbrace{V \otimes (M \otimes J_-)}_{=V\otimes F(M)}$$

is given by

$$s_{V,M}\big((v \otimes m) \otimes j_-\big) =av \otimes (m \otimes j_-) + bg.v \otimes (m \otimes j_-) + cx.v \otimes (m \otimes j_-)$$

$$+ dgx.v \otimes (m \otimes j_-) + \alpha v \otimes (\hat{w}.m \otimes j_-) + \beta g.v \otimes (\hat{w}.m \otimes j_-)$$
$$+ \gamma x.v \otimes (\hat{w}.m \otimes j_-) + \delta gx.v \otimes (\hat{w}.m \otimes j_-)$$

where $v \in V, m \in M$. Since s is a family of (iso)morphisms in the category $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$, the following linearity condition has to be satisfied for all $V \in \mathcal{C}, M \in \mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$, $v \in V, m \in M$:

$$s_{V,M}\big(\hat{w}.\big((v \otimes m) \otimes j_-\big)\big) = \hat{w}.s_{V,M}\big((v \otimes m) \otimes j_-\big) \tag{5.22}$$

From (5.22) we get the condition that

$$c = d = \gamma = \delta = 0 \tag{5.23}$$

Recall that (F, s) satisfies the pentagon axiom (A.11) if and only if the following equation holds for all $V, Z \in H - \mathbf{mod}, M \in \mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$:

$$\mathbf{m}^{\nu_2}_{V,Z,F(M)} \circ s_{V \otimes Z,M} = (\mathrm{id}_V \otimes s_{Z,M}) \circ s_{V,Z \otimes M} \circ F(\mathbf{m}^{\nu_1}_{V,Z,M})$$

From the pentagon axiom, we obtain the following conditions on $a, b, \alpha, \beta, \nu_1, \nu_2$:

$$\begin{aligned}
&a = a^2 + \alpha\beta, \quad b = b^2 + \alpha\beta, \quad \alpha = a\alpha + b\alpha, \quad \beta = b\beta + a\beta \\
&\nu_2 a = \nu_1 a^2 + \nu_1\alpha\beta, \quad \nu_2 = \nu_1 b^2 + \nu_1\alpha\beta, \quad \nu_2\alpha = \nu_1 a\alpha + \nu_1 b\alpha, \quad \nu_2\beta = \nu_1 b\beta + \nu_1 a\beta \\
&0 = ab + \alpha^2 = a\beta + a\alpha = ab + \beta^2 = b\alpha + b\beta \\
&0 = \nu_1 ab + \nu_1\alpha^2 = \nu_1 a\beta + \nu_1 a\alpha = \nu_1 ab + \nu_1\beta^2 = \nu_1 b\alpha + \nu_1 b\beta
\end{aligned} \tag{5.24}$$

This nonlinear system of equations has the following four non-zero solutions:

$$\begin{aligned}
&b = 1 - a, \quad \alpha = -\sqrt{a^2 - a}, \quad \beta = \sqrt{a^2 - a}, \quad \nu_1 = \nu_2 \\
&b = 1 - a, \quad \alpha = \sqrt{a^2 - a}, \quad \beta = -\sqrt{a^2 - a}, \quad \nu_1 = \nu_2 \\
&a = \alpha = \beta = 0, \quad b = 1, \quad \nu_1 = \nu_2 \\
&a = 1, \quad b = \alpha = \beta = 0, \quad \nu_1 = \nu_2
\end{aligned}$$

From this we see that if $\nu_1 \neq \nu_2$, then there are no solutions and hence no $\mathcal{C}$-module structures on F.

Let us now study the identity functor on $\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod}$. By Lemma 4.2 we can make the following general ansatz for a $\mathcal{C}$-module structure $s' = (s'_{V,M})_{V \in \mathcal{C}, M \in \mathbb{C}[\mathbb{Z}_2]-\mathbf{mod}}$ of the identity functor: there are $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that for all $V \in \mathcal{C}, M \in \mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$, the morphism

$$s'_{V,M} : V \otimes M \longrightarrow V \otimes M$$

is given by

$$\begin{aligned}
s'_{V,M}(v \otimes m) =& av \otimes m + bg.v \otimes m + cx.v \otimes m + dgx.v \otimes m \\
&+ \alpha v \otimes \hat{w}.m + \beta g.v \otimes \hat{w}.m + \gamma x.v \otimes \hat{w}.m + \delta gx.v \otimes \hat{w}.m
\end{aligned}$$

for all $v \in V, m \in M$. Again, from the linearity condition on s', we obtain (5.23) and the pentagon axiom leads to the same system of equations (5.24). Hence, the $\mathcal{C}$-module categories $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_1}$ and $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^{\nu_2}$ are equivalent if and only if $\nu_1 = \nu_2$.

□

Similar to the case for **vect**, we have constructed a one-parameter family of non-equivalent $\mathcal{C}$-module categories with the same underlying $\mathbb{C}$-linear category $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$ and the same $\mathcal{C}$-action. This concludes the proof of Theorem 5.6.

5.3.4 Conclusion

In the preceding subsection, we have constructed the $\mathbb{C}$-linear $\mathcal{C}$-module categories $\mathbf{vect}^\lambda$ and $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$ for each $\lambda \in \mathbb{C}$. One can now ask whether these $\mathcal{C}$-module categories again admit (infinitesimal) associator deformations. To answer this question, we will realize both $\mathbf{vect}^\lambda$ and $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$ as categories of modules over H-comodule algebras, starting with $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$. For this purpose, recall for each $\xi \in \mathbb{C}$ the H-comodule algebra D_ξ from (5.4).

Proposition 5.22. *Let $\lambda, \xi \in \mathbb{C}$. If $\lambda = \frac{\xi^2}{4}$, then the functor*

$$\begin{aligned}(F^{\lambda,\xi}, s): \quad D_\xi - \mathbf{mod} &\xrightarrow{\cong} (\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda \\ M &\longmapsto M \\ f &\longmapsto f\end{aligned}$$

is an isomorphism of $\mathcal{C}$-module categories, where $s = (s_{V,M})_{V\in\mathcal{C}, M\in D_\xi-\mathbf{mod}}$ is the natural family of isomorphisms

$$\begin{aligned}s_{V,M}: \quad F^{\lambda,\xi}(V\otimes M) &\xrightarrow{\cong} V\otimes F^{\lambda,\xi}(M) \\ v\otimes m &\longmapsto g.v\otimes m + \frac{\xi}{2} gx.v\otimes \hat{w}.m\end{aligned}$$

In particular, for all $\lambda\in\mathbb{C}$, there is an isomorphism of $\mathcal{C}$-module categories

$$D_{2\sqrt{\lambda}} - \mathbf{mod} \cong (\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$$

Proof:
First, note that the inverse of s is given by the natural family $s^{-1} = (s^{-1}_{V,M})_{V\in\mathcal{C}, M\in D_\xi-\mathbf{mod}}$ with

$$\begin{aligned}s^{-1}_{V,M}: \quad V\otimes F^{\lambda,\xi}(M) &\xrightarrow{\cong} F^{\lambda,\xi}(V\otimes M) \\ v\otimes m &\longmapsto g.v\otimes m + \frac{\xi}{2} gx.v\otimes \hat{w}.m\end{aligned}$$

Let $V, Z\in\mathcal{C}, M\in D_\xi - \mathbf{mod}$ and $v\in V, z\in Z, m\in M$. We need to show that $(F^{\lambda,\xi}, s)$ satisfies the pentagon axiom (A.11). We calculate both paths of the pentagon separately. The right path is:

$$\begin{aligned}&\big(\mathbf{m}^\lambda_{V,Z,F^{\lambda,\xi}(M)} \circ s_{V\otimes Z,M}\big)\big((v\otimes z)\otimes m\big) \\ =&\mathbf{m}^\lambda_{V,Z,F^{\lambda,\xi}(M)}\big((g.(v\otimes z)\otimes m + \frac{\xi}{2}(gx.(v\otimes z))\otimes \hat{w}.m\big) \\ =&\mathbf{m}^\lambda_{V,Z,F^{\lambda,\xi}(M)}\big((g.v\otimes g.z)\otimes m + \frac{\xi}{2}(g.v\otimes gx.z)\otimes\hat{w}.m + \frac{\xi}{2}(gx.v\otimes z)\otimes \hat{w}.m\big) \\ =&g.v\otimes(g.z\otimes m) + \frac{\xi}{2} g.v\otimes(gx.z\otimes\hat{w}.m) + \frac{\xi}{2} gx.v\otimes(z\otimes\hat{w}.m) \\ &+\lambda xg.v\otimes(x.z\otimes m)\end{aligned}$$

The left path of the pentagon diagram (A.11) is:

$$\big((\mathrm{id}_V\otimes s_{Z,M})\circ s_{V,Z\otimes M}\circ F^{\lambda,\xi}(m_{V,Z,M})\big)\big((v\otimes z)\otimes m\big)$$

$$
\begin{aligned}
&=\big((\mathrm{id}_V \otimes s_{Z,M}) \circ s_{V,Z\otimes M}\big)\big(v \otimes (z \otimes m)\big)\\
&=(\mathrm{id}_V \otimes s_{Z,M})\big(g.v \otimes (z \otimes m) + \frac{\xi}{2} gx.v \otimes \hat{w}.(z \otimes m)\big)\\
&=(\mathrm{id}_V \otimes s_{Z,M})\big(g.v \otimes (z \otimes m) + \frac{\xi}{2} gx.v \otimes (g.z \otimes \hat{w}.m) + \frac{\xi^2}{2} gx.v \otimes (gx.z \otimes m)\big)\\
&=g.v \otimes (g.z \otimes m) + \frac{\xi}{2} g.v \otimes (gx.z \otimes \hat{w}.m) + \frac{\xi}{2} gx.v \otimes (z \otimes \hat{w}.m)\\
&\quad + \frac{\xi^2}{4} gx.v \otimes (gxg.z \otimes m) + \frac{\xi^2}{2} gx.v \otimes (x.z \otimes m)\\
&=g.v \otimes (g.z \otimes m) + \frac{\xi}{2} g.v \otimes (gx.z \otimes \hat{w}.m) + \frac{\xi}{2} gx.v \otimes (z \otimes \hat{w}.m)\\
&\quad + \frac{\xi^2}{4} gx.v \otimes (x.z \otimes m)
\end{aligned}
$$

By assumption, $\lambda = \frac{\xi^2}{4}$ and thus both paths of the pentagon agree.

Finally, the following calculation shows that the map $s_{V,M}$ is a morphism in the category $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda$:

$$
\begin{aligned}
s_{V,M}\big(\hat{w}.(v \otimes m)\big) &= s_{V,M}(g.v \otimes \hat{w}.m + \xi gx.v \otimes m)\\
&= g^2.v \otimes \hat{w}.m + \xi g^2 x.v \otimes m + \frac{\xi}{2} gxg.v \otimes \hat{w}^2.m\\
&= v \otimes \hat{w}.m + \xi x.v \otimes m - \frac{\xi}{2} x.v \otimes m\\
&= v \otimes \hat{w}.m + \frac{\xi}{2} x.v \otimes m\\
&= g^2.v \otimes \hat{w}.m + \frac{\xi}{2} g^2 x.v \otimes \hat{w}^2.m\\
&= \hat{w}.(g.v \otimes m + \frac{\xi}{2} gx.v \otimes \hat{w}.m)\\
&= \hat{w}.s_{V,M}(v \otimes m)
\end{aligned}
$$

□

Let $\lambda \in \mathbb{C}$. Using Proposition 5.22 in the first isomorphism and Proposition 5.12 in the second one, we can now calculate the associator cohomology of $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda$ for all $n \geq 0$:

$$
H^n_{ass}(H - \mathbf{mod}, (\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda) \cong H^n_{ass}(H - \mathbf{mod}, D_{2\sqrt{\lambda}} - \mathbf{mod}) \cong \begin{cases} \mathbb{C} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
$$

Therefore, for all $\lambda \in \mathbb{C}$, the $\mathcal{C}$-module category $(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda$ again admits a one-parameter family of infinitesimal deformations. However, from the discussion in Proposition 5.16 and Corollary 5.17, we know that each of these infinitesimal deformations by the 2-cocycle $m^{(1)}$ admits an associated finite deformation for all $\nu \in \mathbb{C}$, where the mixed associator is of the form

$$
m^\nu_{V,W,M}\big((v \otimes w) \otimes m\big) = v \otimes (w \otimes m) + \lambda x.v \otimes (xg.w \otimes m) + \nu x.v \otimes (xg.w \otimes m)
$$

for all $V, W \in \mathcal{C}, M \in (\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda, v \in V, w \in W, m \in M$. But the $\mathcal{C}$-module category with this mixed associator is of just $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^{\lambda+\nu}$ and hence, still a member of the family $((\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda)_{\lambda\in\mathbb{C}}$.

Let us now study associator deformations of $\mathbf{vect}^\lambda$. Recall for each $\xi \in \mathbb{C}$ the H-comodule algebra A_ξ from (5.3) and the simple A_ξ-module S.

Proposition 5.23. *Let $\lambda, \nu, \xi \in \mathbb{C}$. If $\lambda = \frac{\xi^2}{4}+\nu$, then the equivalence of linear categories*

$$\begin{aligned} F^{\lambda,\nu,\xi}: \quad \mathbf{vect}^\lambda &\xrightarrow{\simeq} (A_\xi - \mathbf{mod})^\nu \\ M &\longmapsto M \otimes S \\ f &\longmapsto f \otimes \mathrm{id}_S \end{aligned}$$

can be promoted to an equivalence of $\mathcal{C}$-module categories. In particular, for all $\lambda \in \mathbb{C}$, we have

$$\mathbf{vect}^\lambda \simeq (A_0 - \mathbf{mod})^\lambda$$

and

$$\mathbf{vect}^\lambda \simeq A_{2\sqrt{\lambda}} - \mathbf{mod} \tag{5.25}$$

as $\mathcal{C}$-module categories.

Proof:
We will show that the natural family of isomorphisms $s = (s_{V,M})_{V\in\mathcal{C}, M\in A_\xi-\mathbf{mod}}$ with

$$\begin{aligned} s_{V,M}: \quad \underbrace{F^{\lambda,\nu,\xi}(V\otimes M)}_{(V\otimes M)\otimes S} &\xrightarrow{\cong} \underbrace{V\otimes F^{\lambda,\nu,\xi}(M)}_{=V\otimes(M\otimes S)} \\ (v\otimes m)\otimes a &\longmapsto \frac{1}{2}v\otimes\big(m\otimes(\hat{1}-\hat{x}).a\big) + \frac{1}{2}g.v\otimes\big(m\otimes(\hat{1}+\hat{x}).a\big) \\ &\qquad - \frac{\xi}{4}x.v\otimes\big(m\otimes(\hat{g}-\hat{g}\hat{x}).a\big) - \frac{\xi}{4}gx.v\otimes\big(m\otimes(\hat{g}+\hat{g}\hat{x}).a\big) \end{aligned}$$

is a $\mathcal{C}$-module structure on the linear equivalence functor $F^{\lambda,\nu,\xi}$. The inverse of s is given by the natural family $s^{-1} = (s^{-1}_{V,M})_{V\in\mathcal{C}, M\in A_\xi-\mathbf{mod}}$ with

$$\begin{aligned} s^{-1}_{V,M}: \quad \underbrace{V\otimes F^{\lambda,\nu,\xi}(M)}_{=V\otimes(M\otimes S)} &\xrightarrow{\cong} \underbrace{F^{\lambda,\nu,\xi}(V\otimes M)}_{=(V\otimes M)\otimes S} \\ v\otimes(m\otimes a) &\longmapsto \frac{1}{2}(v\otimes m)\otimes(\hat{1}-\hat{x}).a + \frac{1}{2}(g.v\otimes m)\otimes(\hat{1}+\hat{x}).a \\ &\qquad - \frac{\xi}{4}(x.v\otimes m)\otimes(\hat{g}+\hat{g}\hat{x}).a + \frac{\xi}{4}(gx.v\otimes m)\otimes(\hat{g}-\hat{g}\hat{x}).a \end{aligned}$$

Let $V, W \in \mathcal{C}, M \in A_\xi - \mathbf{mod}$ and $v \in V, w \in W, m \in M$. We need to show that $(F^{\lambda,\nu,\xi}, s)$ satisfies the pentagon axiom (A.11). We calculate both paths of the pentagon separately. The path along the right is

$$
\begin{aligned}
&(\mathbf{m}^{\nu}_{V,W,F^{\lambda,\nu,\xi}(M)}\circ s_{V\otimes W,M})\big(((v\otimes w)\otimes m)\otimes a\big)\\
=&\mathbf{m}^{\nu}_{V,W,F^{\lambda,\nu,\xi}(M)}\Big(\frac{1}{2}(v\otimes w)\otimes(m\otimes a)-\frac{1}{2}(v\otimes w)\otimes(m\otimes\hat{x}.a)+\frac{1}{2}g.(v\otimes w)\otimes(m\otimes a)\\
&+\frac{1}{2}g.(v\otimes w)\otimes(m\otimes\hat{x}.a)+\frac{\xi}{4}x.(v\otimes w)\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}x.(v\otimes w)\otimes(m\otimes\hat{g}.a)\\
&-\frac{\xi}{4}gx.(v\otimes w)\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}gx.(v\otimes w)\otimes(m\otimes\hat{g}.a)\Big)\\
=&\mathbf{m}^{\nu}_{V,W,F^{\lambda,\nu,\xi}(M)}\Big(\frac{1}{2}(v\otimes w)\otimes(m\otimes a)-\frac{1}{2}(v\otimes w)\otimes(m\otimes\hat{x}.a)+\frac{1}{2}(g.v\otimes g.w)\otimes(m\otimes a)\\
&-\frac{1}{2}(g.v\otimes g.w)\otimes(m\otimes\hat{x}.a)+\frac{\xi}{4}(v\otimes x.w)\otimes(m\otimes\hat{g}\hat{x}.a)+\frac{\xi}{4}(x.v\otimes g.w)\otimes(m\otimes\hat{g}\hat{x}.a)\\
&-\frac{\xi}{4}(v\otimes x.w)\otimes(m\otimes\hat{g}.a)-\frac{\xi}{4}(x.v\otimes g.w)\otimes(m\otimes\hat{g}.a)-\frac{\xi}{4}(g.v\otimes gx.w)\otimes(m\otimes\hat{g}\hat{x}.a)\\
&-\frac{\xi}{4}(gx.v\otimes w)\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}(g.v\otimes gx.w)\otimes(m\otimes\hat{g}.a)-\frac{\xi}{4}(gx.v\otimes w)\otimes(m\otimes\hat{g}.a)\Big)\\
=&\frac{1}{2}v\otimes\big(w\otimes(m\otimes a)\big)-\frac{1}{2}v\otimes\big(w\otimes(m\otimes\hat{x}.a)\big)+\frac{1}{2}g.v\otimes\big(g.w\otimes(m\otimes a)\big)\\
&+\frac{1}{2}g.v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)+\frac{\xi}{4}v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)+\frac{\xi}{4}x.v\otimes\big(g.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&-\frac{\xi}{4}v\otimes\big(x.w\otimes(m\otimes\hat{g}.a)\big)-\frac{\xi}{4}x.v\otimes\big(g.w\otimes(m\otimes\hat{g}.a)\big)-\frac{\xi}{4}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&-\frac{\xi}{4}gx.v\otimes\big(w\otimes(m\otimes\hat{g}\hat{x}.a)\big)-\frac{\xi}{4}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}.a)\big)-\frac{\xi}{4}gx.v\otimes\big(w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{1}{2}\nu x.v\otimes\big(gx.w\otimes(m\otimes a)\big)+\frac{1}{2}\nu x.v\otimes\big(gx.w\otimes(m\otimes\hat{x}.a)\big)\\
&-\frac{1}{2}\nu gx.v\otimes\big(x.w\otimes(m\otimes a)\big)-\frac{1}{2}\nu gx.v\otimes\big(x.w\otimes(m\otimes\hat{x}.a)\big)
\end{aligned}
$$

The left path of the pentagon (A.11) is

$$
\begin{aligned}
&\big((\mathrm{id}_V\otimes s_{W,M})\circ s_{V,W\otimes M}\circ F^{\lambda,\nu,\xi}(\mathbf{m}^{\lambda}_{V,W,M})\big)\big(((v\otimes w)\otimes m)\otimes a\big)\\
=&\big((\mathrm{id}_V\otimes s_{W,M})\circ s_{V,W\otimes M}\big)\big((v\otimes(w\otimes m))\otimes a+\lambda(x.v\otimes(xg.w\otimes m))\big)\\
=&(\mathrm{id}_V\otimes s_{W,M})\Big(\frac{1}{2}v\otimes((w\otimes m)\otimes a)-\frac{1}{2}v\otimes((w\otimes m)\otimes\hat{x}.a)\\
&+\frac{1}{2}g.v\otimes((w\otimes m)\otimes a)+\frac{1}{2}g.v\otimes((w\otimes m)\otimes\hat{x}.a)+\frac{\xi}{4}x.v\otimes((w\otimes m)\otimes\hat{g}\hat{x}.a)\\
&-\frac{\xi}{4}x.v\otimes((w\otimes m)\otimes\hat{g}.a)-\frac{\xi}{4}gx.v\otimes((w\otimes m)\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}gx.v\otimes((w\otimes m)\otimes\hat{g}.a)\\
&+\frac{1}{2}\nu x.v\otimes((xg.w\otimes m)\otimes a)-\frac{1}{2}\nu x.v\otimes((xg.w\otimes m)\otimes\hat{x}.a)\\
&+\frac{1}{2}\nu gx.v\otimes((xg.w\otimes m)\otimes a)+\frac{1}{2}\nu x.v\otimes((xg.w\otimes m)\otimes\hat{x}.a)\Big)\\
=&\frac{1}{4}v\otimes\big(w\otimes(m\otimes a)\big)-\frac{1}{4}v\otimes\big(w\otimes(m\otimes\hat{x}.a)\big)+\frac{1}{4}v\otimes\big(g.w\otimes(m\otimes a)\big)\\
&+\frac{1}{4}v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)+\frac{\xi}{8}v\otimes x.w\otimes\big(m\otimes\hat{g}\hat{x}.a\big)-\frac{\xi}{8}v\otimes\big(x.w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{\xi}{8}v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)-\frac{\xi}{8}v\otimes\big(gx.w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{1}{4}v\otimes\big(w\otimes(m\otimes\hat{x}.a)\big)+\frac{1}{4}v\otimes\big(w\otimes(m\otimes\hat{x}^2.a)\big)-\frac{1}{4}v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)\\
&-\frac{1}{4}v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)-\frac{\xi}{8}v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}^2.a)\big)+\frac{\xi}{8}v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\xi}{8}v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}^2.a)\big)+\frac{\xi}{8}v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&+\frac{1}{4}g.v\otimes\big(w\otimes(m\otimes a)\big)-\frac{1}{4}g.v\otimes\big(w\otimes(m\otimes\hat{x}.a)\big)+\frac{1}{4}g.v\otimes\big(g.w\otimes(m\otimes a)\big)\\
&+\frac{1}{4}g.v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)+\frac{\xi}{8}g.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)-\frac{\xi}{8}g.v\otimes\big(x.w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{\xi}{8}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)-\frac{\xi}{8}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}.a)\big)\\
&+\frac{1}{4}g.v\otimes\big(w\otimes(m\otimes\hat{x}.a)\big)-\frac{1}{4}g.v\otimes\big(w\otimes(m\otimes\hat{x}^2.a)\big)+\frac{1}{4}g.v\otimes\big(g.w\otimes(m\otimes\hat{x}.a)\big)\\
&+\frac{1}{4}g.v\otimes\big(g.w\otimes(m\otimes\hat{x}^2.a)\big)+\frac{\xi}{8}g.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}^2.a)\big)-\frac{\xi}{8}g.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&-\frac{\xi}{8}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}^2.a)\big)-\frac{\xi}{8}g.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&+\frac{\xi}{8}x.v\otimes\big(w\otimes(m\otimes\hat{g}\hat{x}.a)\big)-\frac{\xi}{8}x.v\otimes\big(w\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)\big)+\frac{\xi}{8}x.v\otimes\big(g.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&+\frac{\xi}{8}x.v\otimes\big(g.w\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)\big)+\frac{\xi^2}{16}x.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}\hat{x}.a)\big)-\frac{\xi^2}{16}x.v\otimes\big(x.w\otimes(m\otimes\hat{g}^2\hat{x}.a)\big)\\
&-\frac{\xi^2}{16}x.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}\hat{x}.a)\big)-\frac{\xi^2}{16}x.v\otimes\big(gx.w\otimes(m\otimes\hat{g}^2\hat{x}.a)\big)\\
&-\frac{\xi}{8}x.v\otimes\big(w\otimes(m\otimes\hat{g}.a)\big)+\frac{\xi}{8}x.v\otimes\big(w\otimes(m\otimes\hat{x}\hat{g}.a)\big)-\frac{\xi}{8}x.v\otimes\big(g.w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{\xi}{8}x.v\otimes\big(g.w\otimes(m\otimes\hat{x}\hat{g}.a)\big)-\frac{\xi^2}{16}x.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)\big)+\frac{\xi^2}{16}x.v\otimes\big(x.w\otimes(m\otimes\hat{g}^2.a)\big)\\
&+\frac{\xi^2}{16}x.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)\big)+\frac{\xi^2}{16}x.v\otimes\big(gx.w\otimes(m\otimes\hat{g}^2.a)\big)\\
&-\frac{\xi}{8}gx.v\otimes\big(w\otimes(m\otimes\hat{g}\hat{x}.a)\big)+\frac{\xi}{8}gx.v\otimes\big(w\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)\big)-\frac{\xi}{8}gx.v\otimes\big(g.w\otimes(m\otimes\hat{g}\hat{x}.a)\big)\\
&-\frac{\xi}{8}gx.v\otimes\big(g.w\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)\big)-\frac{\xi^2}{16}gx.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}\hat{x}.a)\big)+\frac{\xi^2}{16}gx.v\otimes\big(x.w\otimes(m\otimes\hat{g}^2\hat{x}.a)\big)\\
&+\frac{\xi^2}{16}gx.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}\hat{x}.a)\big)+\frac{\xi^2}{16}gx.v\otimes\big(gx.w\otimes(m\otimes\hat{g}^2\hat{x}.a)\big)\\
&-\frac{\xi}{8}gx.v\otimes\big(w\otimes(m\otimes\hat{g}.a)\big)+\frac{\xi}{8}gx.v\otimes\big(w\otimes(m\otimes\hat{x}\hat{g}.a)\big)-\frac{\xi}{8}gx.v\otimes\big(g.w\otimes(m\otimes\hat{g}.a)\big)\\
&-\frac{\xi}{8}gx.v\otimes\big(g.w\otimes(m\otimes\hat{x}\hat{g}.a)\big)-\frac{\xi^2}{16}gx.v\otimes\big(x.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)\big)+\frac{\xi^2}{16}gx.v\otimes\big(x.w\otimes(m\otimes\hat{g}^2.a)\big)\\
&+\frac{\xi^2}{16}gx.v\otimes\big(gx.w\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)\big)+\frac{\xi^2}{16}gx.v\otimes\big(gx.w\otimes(m\otimes\hat{g}^2.a)\big)\\
&+\frac{\lambda}{4}x.v\otimes\big(xg.w\otimes(m\otimes a)\big)-\frac{\lambda}{4}x.v\otimes\big(xg.w\otimes(m\otimes\hat{x}.a)\big)+\frac{\lambda}{4}x.v\otimes\big(gxg.w\otimes(m\otimes a)\big)\\
&+\frac{\lambda}{4}x.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}.a)\big)\\
&-\frac{\lambda}{4}x.v\otimes\big(xg.w\otimes(m\otimes\hat{x}.a)\big)-\frac{\lambda}{4}x.v\otimes\big(xg.w\otimes(m\otimes\hat{x}^2.a)\big)+\frac{\lambda}{4}x.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}.a)\big)\\
&+\frac{\lambda}{4}x.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}^2.a)\big)\\
&+\frac{\lambda}{4}gx.v\otimes\big(xg.w\otimes(m\otimes a)\big)-\frac{\lambda}{4}gx.v\otimes\big(xg.w\otimes(m\otimes\hat{x}.a)\big)+\frac{\lambda}{4}gx.v\otimes\big(gxg.w\otimes(m\otimes a)\big)\\
&+\frac{\lambda}{4}gx.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}.a)\big)
\end{aligned}
$$

$$+\frac{\lambda}{4}gx.v\otimes\big(xg.w\otimes(m\otimes\hat{x}.a)\big)-\frac{\lambda}{4}gx.v\otimes\big(xg.w\otimes(m\otimes\hat{x}^2.a)\big)+\frac{\lambda}{4}gx.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}.a)\big)$$
$$+\frac{\lambda}{4}gx.v\otimes\big(gxg.w\otimes(m\otimes\hat{x}^2.a)\big)$$

Using the assumption $\lambda=\frac{\xi^2}{4}+\nu$, one can see that both paths of the pentagon do indeed agree. Finally, we check the A_ξ-linearity of $s_{V,M}$. For $\hat{g}$, we have

$$\begin{aligned}
s_{V,M}\big(\hat{g}.\big((v\otimes m)\otimes a\big)\big)=&s_{V,M}\big((v\otimes m)\otimes\hat{g}.a\big)\\
=&\frac{1}{2}v\otimes(m\otimes\hat{g}.a)-\frac{1}{2}v\otimes(m\otimes\hat{x}\hat{g}.a)+\frac{1}{2}g.v\otimes(m\otimes\hat{g}.a)\\
&+\frac{1}{2}g.v\otimes(m\otimes\hat{x}\hat{g}.a)+\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)-\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}^2.a)\\
&-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}\hat{x}\hat{g}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}^2.a)\\
=&\frac{1}{2}g.v\otimes(m\otimes\hat{g}.a)-\frac{1}{2}g.v\otimes(m\otimes\hat{g}\hat{x}.a)+\frac{1}{2}g^2.v\otimes(m\otimes\hat{g}.a)\\
&+\frac{1}{2}g^2.v\otimes(m\otimes\hat{g}\hat{x}.a)+\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}^2\hat{x}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}^2.a)\\
&-\frac{\xi}{4}g^2x.v\otimes(m\otimes\hat{g}^2\hat{x}.a)-\frac{\xi}{4}g^2x.v\otimes(m\otimes\hat{g}^2.a)\\
=&\hat{g}.\big(\frac{1}{2}v\otimes(m\otimes a)-\frac{1}{2}v\otimes(m\otimes\hat{x}.a)+\frac{1}{2}g.v\otimes(m\otimes a)\\
&+\frac{1}{2}g.v\otimes(m\otimes\hat{x}.a)+\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}.a)\\
&-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}.a)\big)\\
=&\hat{g}.s_{V,M}\big((v\otimes m)\otimes a\big)
\end{aligned}$$

and for $\hat{x}$ we calculate

$$\begin{aligned}
s_{V,M}\big(\hat{x}.\big((v\otimes m)\otimes a\big)\big)=&s_{V,M}\big((v\otimes m)\otimes\hat{x}.a\big)\\
=&\frac{1}{2}v\otimes(m\otimes\hat{x}.a)-\frac{1}{2}v\otimes(m\otimes\hat{x}^2.a)+\frac{1}{2}g.v\otimes(m\otimes\hat{x}.a)\\
&+\frac{1}{2}g.v\otimes(m\otimes\hat{x}^2.a)+\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}\hat{x}^2.a)-\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}\hat{x}.a)\\
&-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}\hat{x}^2.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}\hat{x}.a)
\end{aligned}\tag{5.26}$$

and

$$\begin{aligned}
\hat{x}.s_{V,M}\big((v\otimes m)\otimes a\big)=&\hat{x}.\big(\frac{1}{2}v\otimes(m\otimes a)-\frac{1}{2}v\otimes(m\otimes\hat{x}.a)+\frac{1}{2}g.v\otimes(m\otimes a)\\
&+\frac{1}{2}g.v\otimes(m\otimes\hat{x}.a)+\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}x.v\otimes(m\otimes\hat{g}.a)\\
&-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}\hat{x}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{g}.a)\big)\\
=&\frac{1}{2}v\otimes(m\otimes\hat{x}.a)+\frac{\xi}{2}x.v\otimes(m\otimes\hat{g}.a)-\frac{1}{2}v\otimes(m\otimes\hat{x}^2.a)\\
&-\frac{\xi}{2}x.v\otimes(m\otimes\hat{g}\hat{x}.a)+\frac{1}{2}g.v\otimes(m\otimes\hat{x}.a)+\frac{\xi}{2}xg.v\otimes(m\otimes\hat{g}.a)
\end{aligned}$$

$$+\frac{1}{2}g.v\otimes(m\otimes\hat{x}^2.a)+\frac{\xi}{2}xg.v\otimes(m\otimes\hat{g}\hat{x}.a)+\frac{\xi}{4}x.v\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)$$
$$-\frac{\xi}{4}x.v\otimes(m\otimes\hat{x}\hat{g}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{x}\hat{g}\hat{x}.a)-\frac{\xi}{4}gx.v\otimes(m\otimes\hat{x}\hat{g}.a) \tag{5.27}$$

It is now easy to see that the right-hand sides of (5.26) and (5.27) agree. □

Let $\lambda \in \mathbb{C}$. Using (5.25) in the first isomorphism and Proposition 5.12 in the second one, we can now calculate the associator cohomology of $\mathbf{vect}^\lambda$ for all $n \geq 0$:

$$H^n_{ass}(H-\mathbf{mod},\mathbf{vect}^\lambda)\cong H^n_{ass}(H-\mathbf{mod},A_{2\sqrt{\lambda}}-\mathbf{mod})\cong\begin{cases}\mathbb{C} & \text{if } n \text{ is even.}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Thus, for each $\lambda \in \mathbb{C}$, we find that $\mathbf{vect}^\lambda$ admits a one-parameter family of infinitesimal deformations. Analogously to the discussion before Proposition 5.23, we find that finite deformations of $\mathbf{vect}^\lambda$ exist and are again of the form $\mathbf{vect}^{\lambda+\nu}$ for all $\nu \in \mathbb{C}$.

We have seen above that finite deformations of the $\mathcal{C}$-module categories $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$ and $\mathbf{vect}^\lambda$ do not lead us out of the families $((\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda)_{\lambda\in\mathbb{C}}$ and $(\mathbf{vect}^\lambda)_{\lambda\in\mathbb{C}}$ respectively. In the following, we will explain why it is not surprising that we do not see new module categories here: indecomposable exact module categories over Taft algebras have been classified by Etingof and Ostrik [EO] by studying *simple from the right algebras* in the finite tensor category $T_q-\mathbf{mod}$. In [M2, Section 8.1], the same classification result was proven using comodule algebra techniques and explicit realizations of the module categories are given as categories of modules over H-comodule algebras (recall from (4.5) that any H-comodule algebra gives rise to a $\mathcal{C}$-module category). Note that we use different conventions from the authors [EO] and [M2], namely we use the opposite coalgebra structure in the definition of the Taft algebra. As a consequence, the monoidal structure considered in [EO] and [BM] is the monoidally opposite one to ours, and left module categories in [EO] and [BM] are in bijection to our right module categories and vice versa. However, H admits a braiding; indeed, recall for all $\mu \in \mathbb{C}$ the following R-matrix

$$R_\mu=\frac{1}{2}(1_H\otimes 1_H+1_H\otimes g+g\otimes 1_H-g\otimes g)+\frac{\mu}{2}(x\otimes x+x\otimes gx+gx\otimes gx-gx\otimes x)$$

with inverse

$$R_\mu^{-1}=\frac{1}{2}(1_H\otimes 1_H+1_H\otimes g+g\otimes 1_H-g\otimes g)+\frac{\mu}{2}(x\otimes x+gx\otimes x+gx\otimes gx-x\otimes gx)$$

from [K, Chapter VIII.2]. Each braiding yields a bijection between left and right $\mathcal{C}$-module categories so that we can use the following classification result:

Theorem 5.24. [EO, Theorem 4.10] *Up to equivalence of module categories, the following is a complete list of indecomposable exact $\mathcal{C}$-module categories:*

- *there are two one-parameter families of semisimple module categories. The members of one family each have one simple object, the members of the other family each have two simple objects.*

- *there are two non-semisimple module categories, one of them has one simple object and the other one has two simple objects.*

To conclude this section, we make the two following observations:

1. Above, we have constructed families $(\mathbf{vect}^\lambda)_{\lambda\in\mathbb{C}}$ and $\big((\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda\big)_{\lambda\in\mathbb{C}}$ of semisimple $\mathcal{C}$-module categories. For each $\lambda \in \mathbb{C}$, the category $\mathbf{vect}^\lambda$ has one simple object, namely $\mathbb{C}$, and the category $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$ has the two simple objects J_+ and J_-. We have also encountered two non-semisimple $\mathcal{C}$-module categories: $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$ and the regular $\mathcal{C}$-module $\mathcal{C}$. Recall that $\mathbb{C}\langle 1_H, gx\rangle - \mathbf{mod}$ has a single simple object, the 1-dimensional module on which gx acts by 0, and $\mathcal{C}$ has two simple objects I_+ and I_-.

   In other words, we were able to recover *all* indecomposable exact $\mathcal{C}$-module categories starting from associator deformations of module categories which we constructed from the four coideal subalgebras of H.

2. From Proposition 4.6 we know that for any $\lambda \in \mathbb{C}$, we can also represent the $\mathcal{C}$-module category $\mathbf{vect}^\lambda$ as a category of modules over an exact H-comodule algebra, where the mixed associators are inherited from $\mathbf{vect}$. Indeed, we have seen in Proposition 5.23 and Proposition 5.22 that

   $$\mathbf{vect}^\lambda \simeq A_{2\sqrt{\lambda}} - \mathbf{mod}$$

   and

   $$(\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod})^\lambda \simeq D_{2\sqrt{\lambda}} - \mathbf{mod}$$

   for all $\lambda \in \mathbb{C}$. Therefore, in our examples, we were able to trade the module category of the rather complicated H-comodule algebra $A_{2\sqrt{\lambda}}$ with the mixed associator inherited from $\mathbf{vect}$ for the module category of the trivial H-comodule algebra $\mathbb{C}$ under the cost of introducing a non-trivial mixed associator (5.15) on $\mathbf{vect}$. The same is true for $(\mathbb{C}[\mathbb{Z}_2]-\mathbf{mod})^\lambda$: we were able to trade the more complicated coaction of $D_{2\sqrt{\lambda}}$ (see (5.5)) under the cost of introducing a non-trivial mixed associator (5.18) on $\mathbb{C}[\mathbb{Z}_2] - \mathbf{mod}$.

A Background

For the convenience of the reader, we collect some algebraic and categorical notions in this part of the appendix, to which we refer frequently throughout the main text. In the appendix, let k be a field and let R be a commutative unital ring.

A.1 Ends and coends

In Section 3, we study functors which are defined via certain universal objects, called ends and coends. In this section, we quickly review the necessary definitions and results, following [M1, Chapter IX]. In the following, let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition A.1. Let $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and let $X \in \mathcal{D}$. A *dinatural transformation* $\alpha : S \stackrel{\cdot\cdot}{\longrightarrow} X$ *from* S *to* X is a family $\alpha = (\alpha_C)_{C\in\mathcal{C}}$ of morphisms $\alpha_C : S(C, C) \longrightarrow X$ in $\mathcal{D}$, such that the following diagram commutes for all $B, C \in \mathcal{C}$ and all morphisms $f : B \longrightarrow C$ in $\mathcal{C}$:

$$\begin{array}{ccc} S(C,B) & \xrightarrow{S(\mathrm{id}_C, f)} & S(C,C) \\ {\scriptstyle S(f,\mathrm{id}_B)}\downarrow & & \downarrow{\scriptstyle \alpha_C} \\ S(B,B) & \xrightarrow[\alpha_B]{} & X \end{array}$$

Analogously, a *dinatural transformation* $\beta : X \stackrel{\cdot\cdot}{\longrightarrow} S$ *from* X *to* S is a family $\beta = (\beta_C)_{C\in\mathcal{C}}$ of morphisms $\beta_C : X \longrightarrow S(C, C)$ in $\mathcal{D}$, such that the following diagram commutes for all $B, C \in \mathcal{C}$ and all morphisms $f : B \longrightarrow C$ in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{\beta_B} & S(B,B) \\ {\scriptstyle \beta_B}\downarrow & & \downarrow{\scriptstyle S(\mathrm{id}_B, f)} \\ S(C,C) & \xrightarrow[S(f,\mathrm{id}_C)]{} & S(B,C) \end{array}$$

Definition A.2. An *end* of a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ is an object $E \in \mathcal{D}$ together with a dinatural transformation $\pi : E \stackrel{\cdot\cdot}{\longrightarrow} S$, which satisfies the following universal property: for every object $X \in \mathcal{D}$ and every dinatural transformation $\beta : X \stackrel{\cdot\cdot}{\longrightarrow} S$ there is a unique morphism $h : X \longrightarrow E$ in $\mathcal{D}$, such that $\beta_C = \pi_C \circ h$ for all $C \in \mathcal{C}$. In this case, we call π the *universal dinatural transformation* and we write $E \cong \int_{C\in\mathcal{C}} S(C, C)$.

Dually, we define the *coend* of a functor:

Definition A.3. A *coend* of a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ is an object $Z \in \mathcal{D}$ together with a dinatural transformation $\omega : S \stackrel{\cdot\cdot}{\longrightarrow} Z$, which satisfies the following universal property: for every object $X \in \mathcal{D}$ and every dinatural transformation $\alpha : S \stackrel{\cdot\cdot}{\longrightarrow} X$ there is a unique morphism $g : Z \longrightarrow X$ in $\mathcal{D}$, such that $\alpha_C = g \circ \omega_C$ for all $C \in \mathcal{C}$. In this case, we call ω the *universal dinatural transformation* and we write $Z \cong \int^{C\in\mathcal{C}} S(C, C)$.

Example A.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. The end of the functor* $\mathrm{Hom}_{\mathcal{D}}(F-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$ *exists and is given by*

$$\int_{C\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}\big(F(C), G(C)\big) \cong \mathrm{Nat}(F, G)$$

with the universal dinatural transformation

$$\tilde{\pi} : \quad \mathrm{Nat}(F,G) \stackrel{\cdot\cdot}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}\big(F(-), G(-)\big)$$

with components

$$\begin{aligned} \tilde{\pi}_C : \quad \mathrm{Nat}(F,G) &\longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(C), G(C)\big) \\ \lambda &\longmapsto \lambda_C \end{aligned}$$

for all $C \in \mathcal{C}$.

Ends behave well with respect to natural transformations:

Proposition A.5. [M1, Chapter IX.7, Proposition 1] *Let $S, S' : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be functors whose ends exist. Let π, π' denote the respective universal dinatural transformations. If $\gamma : S \Longrightarrow S'$ is a natural transformation, then there exists a unique morphism*

$$\int_{C\in\mathcal{C}} \gamma_{C,C} : \quad \int_{C\in\mathcal{C}} S(C,C) \longrightarrow \int_{C\in\mathcal{C}} S'(C,C)$$

in $\mathcal{D}$, such that the following diagram commutes for all $B \in \mathcal{C}$:

$$\begin{array}{ccc} \int_{C\in\mathcal{C}} S(C,C) & \xrightarrow{\pi_B} & S(B,B) \\ {\scriptstyle \exists!\ \int_{C\in\mathcal{C}} \gamma_{C,C}} \downarrow & & \downarrow {\scriptstyle \gamma_{B,B}} \\ \int_{C\in\mathcal{C}} S'(C,C) & \xrightarrow[\pi'_B]{} & S'(B,B) \end{array}$$

Moreover, if $S'' : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ is a functor whose end exists and if $\gamma' : S \Longrightarrow S'$ and $\gamma'' : S' \Longrightarrow S''$ are natural transformations, then

$$\int_{C\in\mathcal{C}} (\gamma''_{C,C} \circ \gamma'_{C,C}) = \int_{C\in\mathcal{C}} \gamma''_{C,C} \circ \int_{C\in\mathcal{C}} \gamma'_{C,C}$$

Of course, an analogous statement to Proposition A.5 also holds for coends, i.e., coends are functorial as well.

A.2 Monoidal categories and monoidal functors

Definition A.6. [EGNO, Definition 2.2.8] A *monoidal category* is a category $\mathcal{C}$ together with the following data:

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the tensor product
- a fixed object $I \in \mathcal{C}$, called the unit of the monoidal category
- a natural family $\mathfrak{a} = (\mathfrak{a}_{U,V,W})_{U,V,W \in \mathcal{C}}$ of isomorphisms $\mathfrak{a}_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$
- a natural family $\mathfrak{l} = (\mathfrak{l}_V)_{V \in \mathcal{C}}$ of isomorphisms $\mathfrak{l}_V : I \otimes V \xrightarrow{\cong} V$
- a natural family $\mathfrak{r} = (\mathfrak{r}_W)_{W \in \mathcal{C}}$ of isomorphisms $\mathfrak{r}_W : W \otimes I \xrightarrow{\cong} W$

The above data are required to satisfy the *pentagon axiom* and the *triangle axiom*, i.e., the following diagrams commute for all $U, V, W, X \in \mathcal{C}$:

$$\begin{array}{ccccc}
 & & (U \otimes V) \otimes (W \otimes X) & & \\
 & \overset{\mathfrak{a}_{U\otimes V,W,X}}{\underset{\cong}{\nearrow}} & & \overset{\mathfrak{a}_{U,V,W\otimes X}}{\underset{\cong}{\searrow}} & \\
\big((U \otimes V) \otimes W\big) \otimes X & & & & U \otimes \big(V \otimes (W \otimes X)\big) \\
\mathfrak{a}_{U,V,W}\otimes \mathrm{id}_X \big\downarrow \cong & & & & \cong \big\uparrow \mathrm{id}_U \otimes \mathfrak{a}_{V,W,X} \\
\big(U \otimes (V \otimes W)\big) \otimes X & & \xrightarrow[\mathfrak{a}_{U,V\otimes W,X}]{\cong} & & U \otimes \big((V \otimes W) \otimes X\big)
\end{array}$$

$$\begin{array}{ccc}
(V \otimes I) \otimes W & \xrightarrow[\cong]{\mathfrak{a}_{V,I,W}} & V \otimes (I \otimes W) \\
 & {\scriptstyle \mathfrak{r}_V \otimes \mathrm{id}_W,\ \cong} \searrow \quad \swarrow {\scriptstyle \cong,\ \mathrm{id}_V \otimes \mathfrak{l}_W} & \\
 & V \otimes W &
\end{array} \tag{A.1}$$

The family $\mathfrak{a}$ is called the *associator*; the families $\mathfrak{l}$ and $\mathfrak{r}$ are called the *left* and *right unitor* respectively.
A monoidal category is called *R-linear* (or just *linear*), if its underlying category is R-linear and the tensor product is R-linear in both arguments.

Similarly to the triangle axiom (A.1) for the left unitor, there is an analogous commutative triangle diagram for the right unitor in a monoidal category:

Lemma A.7. [EGNO, Proposition 2.2.4] *Let $\mathcal{C}$ be a monoidal category with associator $\mathfrak{a}$ and right unitor $\mathfrak{r}$. The following diagram commutes for all $V, W \in \mathcal{C}$:*

$$\begin{array}{ccc}
(V \otimes W) \otimes I & \xrightarrow[\cong]{\mathfrak{a}_{V,W,I}} & V \otimes (W \otimes I) \\
 & {\scriptstyle \mathfrak{r}_{V\otimes W},\ \cong} \searrow \quad \swarrow {\scriptstyle \cong,\ \mathrm{id}_V \otimes \mathfrak{r}_W} & \\
 & V \otimes W &
\end{array}$$

Definition A.8. [EGNO, Definition 2.10.1 & Definition 2.10.2] A *left dual* of an object $X \in \mathcal{C}$ is an object $X^* \in \mathcal{C}$, if there are morphisms $\mathrm{ev}_X : X^* \otimes X \longrightarrow I$ and $\mathrm{coev}_X : I \longrightarrow X \otimes X^*$, such that

$$\mathfrak{l}_X \circ (\mathrm{id}_X \otimes \mathrm{ev}_X) \circ \mathfrak{a}_{X,X^*,X} \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \circ \mathfrak{r}_X^{-1} = \mathrm{id}_X$$

$$\mathfrak{l}_{X^*} \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ \mathfrak{a}^{-1}_{X^*,X,X^*} \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X) \circ \mathfrak{r}^{-1}_{X^*} = \mathrm{id}_{X^*}$$

A *right dual* of an object $X \in \mathcal{C}$ is an object ${}^*X \in \mathcal{C}$, if there are morphisms $\tilde{\mathrm{ev}}_X : X \otimes {}^*X \longrightarrow I$ and $\widetilde{\mathrm{coev}}_X : I \longrightarrow {}^*X \otimes X$, such that

$$\mathfrak{l}_X \circ (\tilde{\mathrm{ev}}_X \otimes \mathrm{id}_X) \circ \mathfrak{a}^{-1}_{X,{}^*X,X} \circ (\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X) \circ \mathfrak{r}_X^{-1} = \mathrm{id}_X$$

$$\mathfrak{l}_{{}^*X} \circ (\mathrm{id}_{{}^*X} \otimes \tilde{\mathrm{ev}}_X) \circ \mathfrak{a}_{{}^*X,X,{}^*X} \circ (\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{{}^*X}) \circ \mathfrak{r}^{-1}_{{}^*X} = \mathrm{id}_{{}^*X}$$

The morphisms $\mathrm{ev}_X, \tilde{\mathrm{ev}}_X$ are called *evaluation* and the morphisms $\mathrm{coev}_X, \widetilde{\mathrm{coev}}_X$ are called *coevaluation*.

A monoidal category in which every object admits a left and a right dual is called *rigid*.

In the following, let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{D}$ be monoidal categories. The structure morphisms in $\mathcal{C}'$ will be denoted by $\mathfrak{a}', \mathfrak{l}'$ and $\mathfrak{r}'$. By abuse of notion, we denote both the tensor product in $\mathcal{C}$ and the tensor product in $\mathcal{C}'$ and $\mathcal{D}$ by $\otimes$.

Definition A.9. [EGNO, Definition 2.4.1] A *monoidal functor* $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor $G : \mathcal{C} \longrightarrow \mathcal{C}'$ together with a natural family $\Phi = (\Phi_{X,Y})_{X,Y \in \mathcal{C}}$ of isomorphisms

$$\Phi_{X,Y} : G(X) \otimes G(Y) \xrightarrow{\cong} G(X \otimes Y)$$

and an isomorphism

$$\varphi : I \xrightarrow{\cong} G(I)$$

such that the following diagrams commute for all $X, Y, Z \in \mathcal{C}$:

$$\begin{array}{ccc} \big(G(X) \otimes G(Y)\big) \otimes G(Z) & \xrightarrow[\cong]{\mathfrak{a}'_{G(X),G(Y),G(Z)}} & G(X) \otimes \big(G(Y) \otimes G(Z)\big) \\ \downarrow{\scriptstyle \Phi_{X,Y} \otimes \mathrm{id}_{G(Z)}}\ \cong & & \cong\ \downarrow{\scriptstyle \mathrm{id}_{G(X)} \otimes \Phi_{Y,Z}} \\ G(X \otimes Y) \otimes G(Z) & & G(X) \otimes G(Y \otimes Z) \\ \downarrow{\scriptstyle \Phi_{X \otimes Y, Z}}\ \cong & & \cong\ \downarrow{\scriptstyle \Phi_{X, Y \otimes Z}} \\ G\big((X \otimes Y) \otimes Z\big) & \xrightarrow[G(\mathfrak{a}_{X,Y,Z})]{\cong} & G\big(X \otimes (Y \otimes Z)\big) \end{array} \tag{A.2}$$

$$\begin{array}{ccc} I \otimes G(X) & \xrightarrow[\cong]{\mathfrak{l}'_{G(X)}} & G(X) \\ \downarrow{\scriptstyle \varphi \otimes \mathrm{id}_{G(X)}}\ \cong & & \cong\ \downarrow{\scriptstyle G(\mathfrak{l}_X)^{-1}} \\ G(I) \otimes G(X) & \xrightarrow[\Phi_{I,X}]{\cong} & G(I \otimes X) \end{array} \tag{A.3}$$

$$\begin{array}{ccc} G(X)\otimes I & \xrightarrow[\cong]{\mathfrak{r}'_{G(X)}} & G(X) \\ {\scriptstyle \mathrm{id}_{G(X)}\otimes\varphi}\downarrow\cong & & \cong\downarrow{\scriptstyle G(\mathfrak{r}_X)^{-1}} \\ G(X)\otimes G(I) & \xrightarrow[\Phi_{X,I}]{\cong} & G(X\otimes I) \end{array} \tag{A.4}$$

A monoidal functor is called *R-linear* (or just *linear*), if its underlying functor is R-linear.

We now gather some definitions and useful facts about finite categories.

Definition A.10. A *finite abelian category* is a k-linear category $\mathcal{A}$ with the property that there is a finite-dimensional k-algebra A, such that there is a k-linear equivalence between $\mathcal{A}$ and $A-\mathbf{mod}$.

Linear functors between finite abelian categories admit several useful criteria regarding adjoints:

Proposition A.11. [DSPS, Proposition 1.7 & Corollary 1.9] *Let $\mathcal{A}$ and $\mathcal{B}$ be finite abelian categories. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a k-linear functor, then the following holds:*

1. *F has a left adjoint if and only if F is left exact.*
2. *F has a right adjoint if and only if F is right exact.*

Definition A.12. A *finite multitensor category* is a rigid monoidal category $\mathcal{C}$, whose underlying category is a finite abelian category and whose tensor product $\otimes : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ is k-linear in both variables.

In a rigid monoidal category $\mathcal{C}$, for every $X \in \mathcal{C}$, we have the following adjunctions by [EGNO, Proposition 2.10.8]:

$$X^* \otimes - \dashv X \otimes - \dashv {}^*X \otimes - \quad \text{and} \quad - \otimes {}^*X \dashv - \otimes X \dashv - \otimes X^* \tag{A.5}$$

It follows from Proposition A.11 that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ in a finite multitensor category is exact in both variables.

Definition A.13. A *finite tensor category* is a finite multitensor category $\mathcal{C}$, whose tensor unit I is a simple object, i.e., $\mathrm{End}_{\mathcal{C}}(I) \cong k$.

Finite tensor categories naturally arise in the representation theory of finite-dimensional Hopf algebras: recall that the category $H-\mathbf{mod}$ of finite-dimensional modules of a finite-dimensional Hopf algebra is a finite tensor category. Its tensor unit is the ground field k, which is a simple object in $H-\mathbf{mod}$.

A.3 (Bi)module categories and (bi)module functors

Definition A.14. [EGNO, Definition 7.1.2] A *left $\mathcal{C}$-module category* is a category $\mathcal{M}$ together with the following data:

- a functor $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$, called the action of $\mathcal{C}$ on $\mathcal{M}$
- a natural family $m = (m_{X,Y,M})_{X,Y \in \mathcal{C}, M \in \mathcal{M}}$ of isomorphisms $m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\cong} X \triangleright (Y \triangleright M)$
- a natural family $l = (l_M)_{M \in \mathcal{M}}$ of isomorphisms $l_M : I \triangleright M \xrightarrow{\cong} M$

The above data are required to satisfy the *pentagon axiom* and the *triangle axiom*, i.e., the following diagrams commute for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{array}{ccccc} & & \big((X \otimes Y) \otimes Z\big) \triangleright M & & \\ & \swarrow^{\mathfrak{a}_{X,Y,Z} \triangleright \mathrm{id}_M}_{\cong} & & \searrow^{m_{X \otimes Y, Z, M}}_{\cong} & \\ \big(X \otimes (Y \otimes Z)\big) \triangleright M & & & & (X \otimes Y) \triangleright (Z \triangleright M) \\ \downarrow{\scriptstyle m_{X, Y \otimes Z, M}\,\cong} & & & & \downarrow{\scriptstyle \cong\, m_{X,Y,Z \triangleright M}} \\ X \triangleright \big((Y \otimes Z) \triangleright M\big) & & \xrightarrow[\mathrm{id}_X \triangleright m_{Y,Z,M}]{\cong} & & X \triangleright \big(Y \triangleright (Z \triangleright M)\big) \end{array} \tag{A.6}$$

$$\begin{array}{ccc} (X \otimes I) \triangleright M & \xrightarrow[\cong]{m_{X,I,M}} & X \triangleright (I \triangleright M) \\ & {\scriptstyle \mathfrak{r}_X \triangleright \mathrm{id}_M}\searrow^{\cong} \quad {}^{\cong}\swarrow{\scriptstyle \mathrm{id}_X \triangleright l_M} & \\ & X \triangleright M & \end{array} \tag{A.7}$$

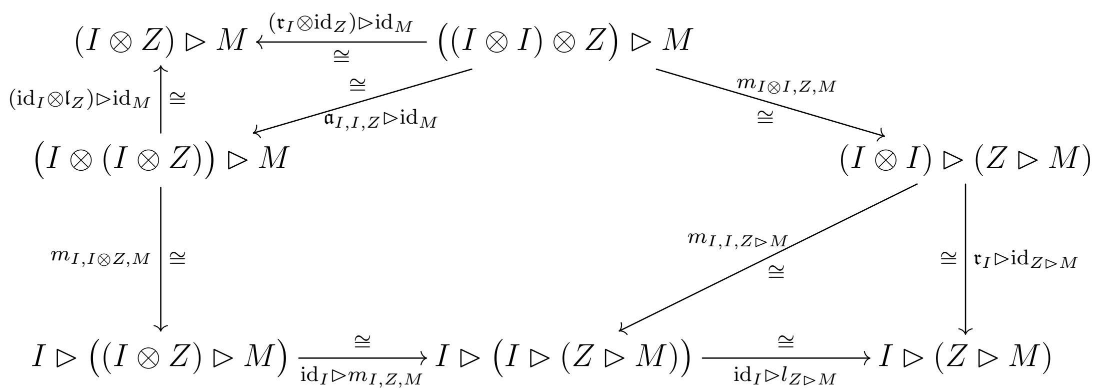
The family m is called the *(mixed) associator* and the family l is called the *(left) unitor*. We sometimes call $(\triangleright, m, l)$ the *$\mathcal{C}$-module structure* of the $\mathcal{C}$-module category $\mathcal{M}$.
A $\mathcal{C}$-module category $\mathcal{M}$ is called *R-linear* (or just *linear*), if its underlying category is R-linear and $\mathcal{C}$ is an R-linear monoidal category and the action $\triangleright$ is R-linear in both arguments.

In addition to the triangle axiom (A.7), there is another triangle diagram in every module category that always commutes:

Lemma A.15. *If $\mathcal{M}$ is a $\mathcal{C}$-module category, then the following diagram commutes for all $Z \in \mathcal{C}$:*

$$\begin{array}{ccc} (I \otimes Z) \triangleright M & \xrightarrow[\cong]{m_{I,Z,M}} & I \triangleright (Z \triangleright M) \\ & {\scriptstyle \mathfrak{l}_Z \triangleright \mathrm{id}_M}\searrow^{\cong} \quad {}^{\cong}\swarrow{\scriptstyle l_{Z \triangleright M}} & \\ & Z \triangleright M & \end{array} \tag{A.8}$$

Proof:
To show that (A.8) commutes, consider the commutative diagram



(A.9)

The pentagon in the middle is just the pentagon (A.6) of the $\mathcal{C}$-module $\mathcal{M}$ for $X = Y = I$ and the triangles on the top left and the bottom right are the triangle axioms of the monoidal category $\mathcal{C}$ (A.1) and the $\mathcal{C}$-module $\mathcal{M}$ (A.7) respectively. Going along the two paths from $I \triangleright ((I \otimes Z) \triangleright M)$ to $I \triangleright (Z \triangleright M)$ in the commuting diagram (A.9), the following calculation proves the commutativity of (A.8) as desired:

$$\begin{aligned}
&\mathrm{id}_I \triangleright (l_{Z\triangleright M} \circ m_{I,Z,M}) \\
&\overset{(A.9)}{=} (\mathfrak{r}_I \triangleright \mathrm{id}_{Z\triangleright M}) \circ \underbrace{m_{I\otimes I,Z,M} \circ \big((\mathfrak{r}_I^{-1} \otimes \mathrm{id}_Z) \triangleright \mathrm{id}_M\big)}_{=\left(\mathfrak{r}_I^{-1} \triangleright (\mathrm{id}_Z \triangleright \mathrm{id}_M) \circ m_{I,Z,M}\right)} \circ \underbrace{\big((\mathrm{id}_I \otimes \mathfrak{l}_Z) \triangleright \mathrm{id}_M\big) \circ m^{-1}_{I,I\otimes Z,M}}_{= m^{-1}_{I,Z,M} \circ \left(\mathrm{id}_I \triangleright (\mathfrak{l}_Z \triangleright \mathrm{id}_M)\right)} \\
&= \mathrm{id}_I \triangleright (\mathfrak{l}_Z \triangleright \mathrm{id}_M)
\end{aligned}$$

□

The structure of a $\mathcal{C}$-module category $\mathcal{M}$ can be encoded in a monoidal functor from $\mathcal{C}$ to $\mathrm{End}(\mathcal{M})$ and vice versa. More precisely:

Proposition A.16. [EGNO, Proposition 7.1.3] *The assignment*

$$\begin{aligned}
\{\mathcal{C}\text{-module structures on } \mathcal{M}\} &\longrightarrow \{\text{monoidal functors } \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})\} \\
(\triangleright, m, l) &\longmapsto (G, \Phi, \varphi)
\end{aligned}$$

is a bijection, where a $\mathcal{C}$-module structure $(\triangleright, m, l)$ corresponds to the monoidal functor

$$\begin{aligned}
(G, \Phi, \varphi): \quad \mathcal{C} &\longrightarrow \mathrm{End}(\mathcal{M}) \\
X &\longmapsto X \triangleright -
\end{aligned} \tag{A.10}$$

with $(\Phi_{X,Y})_M = m^{-1}_{X,Y,M}$ and $\varphi_M = l^{-1}_M$ for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. A monoidal functor $(G, \Phi, \varphi): \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ corresponds to the $\mathcal{C}$-module structure $(\triangleright, m, l)$ with the action

$$\begin{aligned}
\triangleright: \quad \mathcal{C} \times \mathcal{M} &\longrightarrow \mathrm{End}(\mathcal{M}) \\
(X, M) &\longmapsto G(X)(M)
\end{aligned}$$

and the mixed associator $m_{X,Y,M} = (\Phi^{-1}_{X,Y})_M$ and the unitor $l_M = \varphi^{-1}_M$ for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$.

Proof:
We sketch why the functor (A.10) is monoidal. Since the monoidal category $\mathrm{End}(\mathcal{M})$ is strict, the pentagon (A.6) of the $\mathcal{C}$-module $\mathcal{M}$ is equivalent to the hexagon (A.2) of (A.10). We need to show that the two rectangles (A.3) and (A.4) commute. We only show the commutativity of one of the diagrams, since the other one is analogous. Due to the strictness of the monoidal category $\mathrm{End}(\mathcal{M})$, the commutativity of the rectangle (A.3) reduces to the commutativity of the triangle

$$\begin{array}{ccc} & G(X) & \\ {}^{\varphi \bullet \mathrm{id}_{G(X)}}\swarrow & & \searrow^{G(\mathfrak{l}_X)^{-1}} \\ G(I) \bullet G(X) & \xrightarrow[\Phi_{I,X}]{} & G(I \otimes X) \end{array}$$

for all $X \in \mathcal{C}$ which is equivalent to the commutativity of diagram (A.8) for all $X \in \mathcal{C}$, $M \in \mathcal{M}$. $\square$

Definition A.17. Let $\mathcal{M}$ and $\mathcal{M}'$ be left $\mathcal{C}$-module categories. A *(left) $\mathcal{C}$-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (F, s), where $F : \mathcal{M} \longrightarrow \mathcal{M}'$ is a functor and $s = (s_{X,M})_{X \in \mathcal{C}, M \in \mathcal{M}}$ is a natural family of isomorphisms $s_{X,M} : F(X \triangleright M) \xrightarrow{\cong} X \triangleright' F(M)$, such that the *pentagon axiom* holds, i.e., the following diagram commutes for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{array}{ccccc} & & F\big((X \otimes Y) \triangleright M\big) & & \\ & {}^{F(m_{X,Y,M})}_{\cong}\swarrow & & \searrow^{s_{X \otimes Y, M}}_{\cong} & \\ F\big(X \triangleright (Y \triangleright M)\big) & & & & (X \otimes Y) \triangleright' F(M) \\ {\scriptstyle s_{X, Y \triangleright M}}\downarrow{\scriptstyle \cong} & & & & {\scriptstyle \cong}\downarrow{\scriptstyle m'_{X,Y,F(M)}} \\ X \triangleright' F(Y \triangleright M) & & \xrightarrow[\mathrm{id}_X \triangleright' s_{Y,M}]{\cong} & & X \triangleright' \big(Y \triangleright' F(M)\big) \end{array} \tag{A.11}$$

The functor F is called the underlying functor of the $\mathcal{C}$-module functor (F, s) and s is called the $\mathcal{C}$-module structure of (F, s).
A $\mathcal{C}$-module functor $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}'$ is called *R-linear* (or just *linear*), if both $\mathcal{M}$ and $\mathcal{M}'$ are R-linear $\mathcal{C}$-module categories and if the underlying functor F is R-linear.

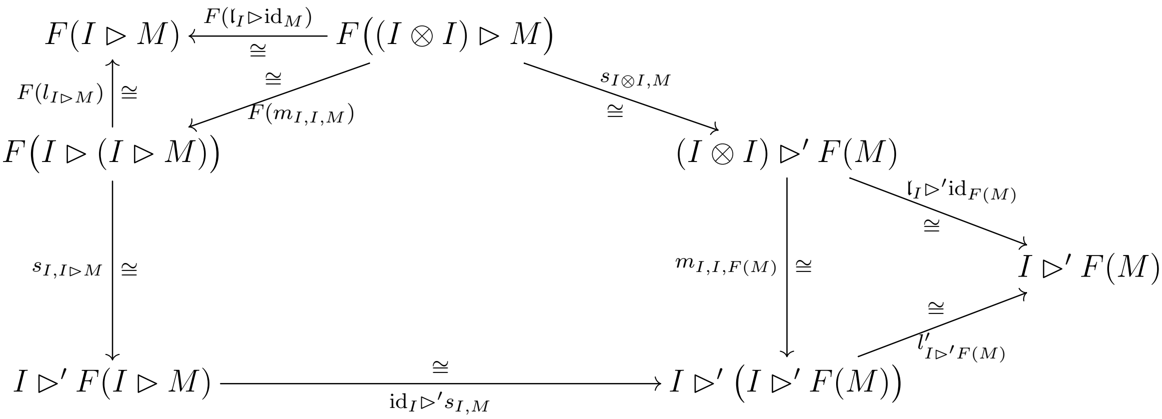
In contrast to many references and textbooks, we do not require a triangle axiom in the definition of a module functor: the commutativity of the usual triangle (A.12) follows from the pentagon (A.11) as we show in Proposition A.18. Our proof follows a similar strategy as the one found in [JS, Proposition 1.1].

Proposition A.18. *If $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}'$ is a $\mathcal{C}$-module functor, then the diagram*

$$\begin{array}{ccc} F(I \triangleright M) & \xrightarrow[\cong]{s_{I,M}} & I \triangleright' F(M) \\ & {}_{F(l_M)}^{\cong}\searrow \quad \swarrow^{\cong}_{l'_{F(M)}} & \\ & F(M) & \end{array} \tag{A.12}$$

commutes for all $M \in \mathcal{M}$.

Proof:
Consider the following commutative diagram, where the central pentagon is the pentagon (A.11) of the $\mathcal{C}$-module functor (F, s) for $X = Y = I$, the triangle in the top left comes from applying F to the commutative triangle (A.8) and the bottom right diagram is the commutative triangle (A.8):



(A.13)

We go the path from $F(I \triangleright M)$ to $I \triangleright' F(M)$ along the top of diagram (A.13):

$$(\mathfrak{l}_I \triangleright' \mathrm{id}_{F(M)}) \circ s_{I \otimes I, M} \circ F(\mathfrak{l}_I^{-1} \triangleright \mathrm{id}_M) = (\mathfrak{l}_I \triangleright' \mathrm{id}_{F(M)}) \circ (\mathfrak{l}_I^{-1} \triangleright' \mathrm{id}_{F(M)}) \circ s_{I,M} = s_{I,M} \tag{A.14}$$

Going from $F(I \triangleright M)$ to $I \triangleright' F(M)$ along the path at the bottom of diagram (A.13) yields

$$\begin{aligned} l'_{I \triangleright F(M)} \circ (\mathrm{id}_I \triangleright' s_{I,M}) \circ s_{I, I \triangleright M} \circ F(l^{-1}_{I \triangleright M}) &= s_{I,M} \circ l'_{F(I \triangleright M)} \circ (\mathrm{id}_I \triangleright' F(l_M^{-1})) \circ s_{I,M} \\ &= s_{I,M} \circ F(l_M^{-1}) \circ l'_{F(M)} \circ s_{I,M} \end{aligned} \tag{A.15}$$

By the commutativity of diagram (A.13), the right-hand sides of (A.14) and (A.15) agree and thus $F(l_M) = l'_{F(M)} \circ s_{I,M}$ since $s_{I,M}$ is an isomorphism. □

Definition A.19. [EGNO, Definition 7.2.2] Let $\mathcal{M}$ and $\mathcal{M}'$ be left $\mathcal{C}$-module categories. Let (F, s) and (F', s') be left $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{M}'$. A *$\mathcal{C}$-module transformation* between the left $\mathcal{C}$-module functors (F, s) and (F', s') is a natural transformation $\nu : F \Longrightarrow F'$, such that the following diagram commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{array}{ccc} F(X \triangleright M) & \xrightarrow[\cong]{s_{X,M}} & X \triangleright' F(M) \\ \downarrow{\scriptstyle \nu_{X \triangleright M}} & & \downarrow{\scriptstyle \mathrm{id}_X \triangleright' \nu_M} \\ F'(X \triangleright M) & \xrightarrow[s'_{X,M}]{\cong} & X \triangleright' F'(M) \end{array} \tag{A.16}$$

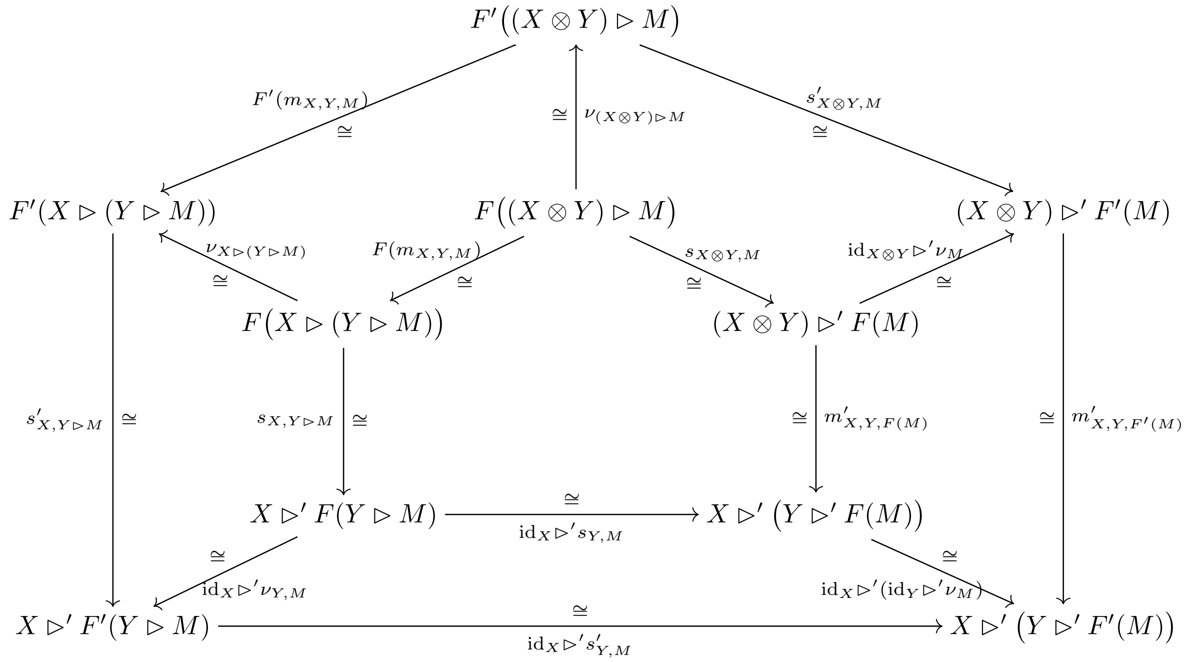
The set of all $\mathcal{C}$-module transformations from (F, s) to (F', s') is denoted by $\mathrm{Nat}_{\mathcal{C}}\big((F, s), (F', s')\big)$.

Lemma A.20. *Let $\mathcal{M}$ and $\mathcal{M}'$ be linear $\mathcal{C}$-module categories. Let $F, F' : \mathcal{M} \longrightarrow \mathcal{M}'$ be linear functors and let $\nu : F \Longrightarrow F'$ be an isomorphism of linear functors. If s is a $\mathcal{C}$-module structure on F, then the natural family $s' = (s'_{X,M})_{X \in \mathcal{C}, M \in \mathcal{M}}$ of isomorphisms*

$$s'_{X,M} = (\mathrm{id}_X \rhd \nu_M) \circ s_{X,M} \circ \nu^{-1}_{X \rhd M} \tag{A.17}$$

is a $\mathcal{C}$-module structure on F' and $\nu : (F, s) \Longrightarrow (F'.s')$ becomes an isomorphism of $\mathcal{C}$-module functors. In particular, if $(F, s) : \mathcal{M} \longrightarrow \mathcal{M}'$ is an equivalence of $\mathcal{C}$-module categories, then $(F', s') : \mathcal{M} \longrightarrow \mathcal{M}'$ is an equivalence of $\mathcal{C}$-module categories.

Proof:
We need to show that (F', s') satisfies the pentagon axiom (A.11). Consider the following diagram:



The central pentagon commutes since (F, s) is a $\mathcal{C}$-module functor. The top left quadrilateral commutes by naturality of ν, the right quadrilateral commutes by naturality of m' and the left, bottom and top right quadrilaterals commute by definition of s' (see (A.17)). Hence, the outer pentagon also commutes, which is the desired pentagon axiom for (F', s').

By definition of s', diagram (A.16) commutes and thus $\nu : (F, s) \Longrightarrow (F'.s')$ is an isomorphism of $\mathcal{C}$-module functors. □

Lemma A.21. *Let $\mathcal{C}$ be a monoidal category and let $\mathcal{M}$ be a $\mathcal{C}$-module category. We endow $\mathcal{C}$ with the structure of the regular $\mathcal{C}$-module category. If $(L, \mathrm{id}) : \mathcal{C} \longrightarrow \mathcal{M}$ is a strict $\mathcal{C}$-module functor with right adjoints $(R, s), (R', \mathrm{id}) : \mathcal{M} \longrightarrow \mathcal{C}$, where (R', id) is strict and the counit $\epsilon : L \circ R \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ of the adjunction $L \dashv R$ is a $\mathcal{C}$-module transformation, then $(R, s) \cong (R', \mathrm{id})$ as $\mathcal{C}$-module functors.*

Proof:
Let $\eta' : \mathrm{Id}_{\mathcal{M}} \Longrightarrow R' \circ L$ denote the unit and let $\epsilon' : L \circ R' \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ denote the counit of the adjunction $L \dashv R'$. Let $\eta : \mathrm{Id}_{\mathcal{M}} \Longrightarrow R \circ L$ denote the unit of the adjunction $L \dashv R$. It is well-known (see, e.g., the analogous [R, Proposition 4.4.1]) that there is a natural isomorphism $\gamma : R \Longrightarrow R'$ whose components on $M \in \mathcal{M}$ are given by

$$\gamma_M = R'(\epsilon_M) \circ \eta'_{R(M)} : \quad R(M) \longrightarrow R'(M) \tag{A.18}$$

For each $M \in \mathcal{M}$, the inverse of γ_M is the morphism

$$\gamma_M^{-1} = R(\epsilon'_M) \circ \eta_{R'(M)} : \quad R'(M) \longrightarrow R(M)$$

Now we show that $\gamma : (R, s) \Longrightarrow (R', \mathrm{id})$ is a morphism of $\mathcal{C}$-module functors, i.e., the following diagram commutes for all $X \in \mathcal{C}, M \in \mathcal{M}$:

$$\begin{array}{ccc} R(X \triangleright M) & \xrightarrow{s_{X,M}} & X \otimes R(M) \\ {\scriptstyle \gamma_{X \triangleright M}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_X \otimes \gamma_M} \\ R'(X \triangleright M) & = & X \otimes R'(M) \end{array} \tag{A.19}$$

Indeed, the following calculation shows the commutativity of diagram (A.19):

$$\begin{aligned} \gamma_{X \triangleright M} =& R'(\epsilon_{X \triangleright M}) \circ \eta'_{R(X \triangleright M)} \\ =& R'(\mathrm{id}_X \triangleright \epsilon_M) \circ R'\big(L(s_{X,M})\big) \circ \eta'_{R(X \triangleright M)} \\ =& R'(\mathrm{id}_X \triangleright \epsilon_M) \circ \eta'_{X \otimes R(M)} \circ s_{X,M} \\ =& \big(\mathrm{id}_X \otimes R'(\epsilon_M)\big) \circ (\mathrm{id}_X \otimes \eta'_{R(M)}) \circ s_{X,M} \\ =& (\mathrm{id}_X \otimes \gamma_M) \circ s_{X,M} \end{aligned}$$

Here we have used (A.18) in the first and last equality. In the second equality, we have used the assumption that $\epsilon : L \circ R \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ is a $\mathcal{C}$-module transformation. In the third equality, we have used the naturality of η' and in the fourth equality, we have used the strictness of (L, id) and (R', id). □

Definition A.22. A *right $\mathcal{C}$-module category* is a category $\mathcal{M}$ together with the following data:

- a functor $\triangleleft : \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{M}$
- a natural family $n = (n_{M,Y,X})_{M \in \mathcal{M}, X, Y \in \mathcal{C}}$ of isomorphisms $n_{M,Y,X} : M \triangleleft (Y \otimes X) \xrightarrow{\cong} (M \triangleleft Y) \triangleleft X$
- a natural family $r = (r_M)_{M \in \mathcal{M}}$ of isomorphisms $r_M : M \triangleleft I \xrightarrow{\cong} M$

The above data are required to satisfy the *pentagon axiom* and the *triangle axiom*, i.e., the following diagrams commute for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{tikzcd}[column sep=small] & M \triangleleft \big(Z \otimes (Y \otimes X)\big) \arrow[dl, "\mathrm{id}_M \triangleleft \mathfrak{a}^{-1}_{Z,Y,X}"', "\cong"] \arrow[dr, "n_{M,Z,Y\otimes X}", "\cong"'] & \\ M \triangleleft \big((Z \otimes Y) \otimes X\big) \arrow[d, "n_{M,Z\otimes Y,X}"', "\cong"] & & (M \triangleleft Z) \triangleleft (Y \otimes X) \arrow[d, "n_{M\triangleleft Z,Y,X}", "\cong"'] \\ \big(M \triangleleft (Z \otimes Y)\big) \triangleleft X \arrow[rr, "\cong", "n_{M,Z,Y} \triangleleft \mathrm{id}_X"'] & & \big((M \triangleleft Z) \triangleleft Y\big) \triangleleft X \end{tikzcd}$$

$$\begin{tikzcd} M \triangleleft (I \triangleleft X) \arrow[rr, "n_{M,I,X}", "\cong"'] \arrow[dr, "\cong", "\mathrm{id}_M \triangleleft \mathfrak{l}_X"'] & & (M \triangleleft I) \triangleleft X \arrow[dl, "\cong"', "\mathfrak{r}_M \triangleleft \mathrm{id}_X"] \\ & M \triangleleft X & \end{tikzcd}$$

Definition A.23. Let $\mathcal{M}$ and $\mathcal{M}'$ be right $\mathcal{C}$-module categories. A *right $\mathcal{C}$-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (G, t), where $G : \mathcal{M} \longrightarrow \mathcal{M}'$ is a functor and $t = (t_{M,X})_{X\in\mathcal{C}, M\in\mathcal{M}}$ is a natural family of isomorphisms $t_{M,X} : G(M \triangleleft X) \xrightarrow{\cong} G(M) \triangleleft' X$, such that the *pentagon axiom* holds, i.e., the following diagram commutes for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{tikzcd}[column sep=small] & G\big(M \triangleleft (Y \otimes X)\big) \arrow[dl, "G(n_{M,Y,X})"', "\cong"] \arrow[dr, "t_{M,Y\otimes X}", "\cong"'] & \\ G\big((M \triangleleft Y) \triangleleft X\big) \arrow[d, "t_{M\triangleleft Y,X}"', "\cong"] & & G(M) \triangleleft' (Y \otimes X) \arrow[d, "n'_{G(M),Y,X}", "\cong"'] \\ G(M \triangleleft Y) \triangleleft X \arrow[rr, "\cong", "t_{M,Y} \triangleleft' \mathrm{id}_X"'] & & \big(G(M) \triangleleft' Y\big) \triangleleft' X \end{tikzcd} \tag{A.20}$$

Linear right module categories and linear right module functors are defined analogously to their left module versions in Definition A.14 and A.17.

We can also deduce a triangle diagram for right module functors, like we did for left module functors in Proposition A.18.

Proposition A.24. *If $(G, t) : \mathcal{M} \longrightarrow \mathcal{M}'$ is a right $\mathcal{C}$-module functor, then the diagram*

$$\begin{tikzcd} G(M \triangleleft I) \arrow[rr, "t_{M,I}", "\cong"'] \arrow[dr, "\cong", "G(r_M)"'] & & G(M) \triangleleft I \arrow[dl, "\cong"', "r_{G(M)}"] \\ & G(M) & \end{tikzcd} \tag{A.21}$$

commutes for all $M \in \mathcal{M}$.

Definition A.25. [EGNO, Definition 7.1.7] *A* $(\mathcal{C},\mathcal{D})$*-bimodule category* is a category $\mathcal{M}$ that is both a left $\mathcal{C}$-module category and a right $\mathcal{D}$-module category together with a natural family $b = (b_{X,M,Z})_{X,Z\in\mathcal{C},M\in\mathcal{M}}$ of isomorphisms

$$b_{X,M,Z} : (X \triangleright M) \triangleleft Z \xrightarrow{\cong} X \triangleright (M \triangleleft Z)$$

in $\mathcal{M}$, such that the following diagrams commute for all $X, Y \in \mathcal{C}$, $W, Z \in \mathcal{D}$ and $M \in \mathcal{M}$:

$$\begin{tikzcd}
& \big((X\otimes Y)\triangleright M\big)\triangleleft Z \arrow[dl, "m_{X,Y,M}\triangleleft \mathrm{id}_Z"', "\cong"] \arrow[dr, "b_{X\otimes Y,M,Z}", "\cong"'] & \\
\big(X\triangleright(Y\triangleright M)\big)\triangleleft Z \arrow[d, "b_{X,Y\triangleright M,Z}"', "\cong"] & & (X\otimes Y)\triangleright(M\triangleleft Z) \arrow[d, "\cong"', "m_{X,Y,M\triangleleft Z}"] \\
X\triangleright\big((Y\triangleright M)\triangleleft Z\big) \arrow[rr, "\cong", "\mathrm{id}_X\triangleright b_{Y,M,Z}"'] & & X\triangleright\big(Y\triangleright(M\triangleleft Z)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
& X\triangleright\big(M\triangleleft(W\otimes Z)\big) \arrow[dl, "\mathrm{id}_X\triangleright n_{M,W,Z}"', "\cong"] \arrow[dr, "b^{-1}_{X,M,W\otimes Z}", "\cong"'] & \\
X\triangleright\big((M\triangleleft W)\triangleleft Z\big) \arrow[d, "b^{-1}_{X,M\triangleleft W,Z}"', "\cong"] & & (X\triangleright M)\triangleleft(W\otimes Z) \arrow[d, "\cong"', "n_{X\triangleright M,W,Z}"] \\
\big(X\triangleright(M\triangleleft W)\big)\triangleleft Z \arrow[rr, "\cong", "b^{-1}_{X,M,W}\triangleleft \mathrm{id}_Z"'] & & \big((X\triangleright M)\triangleleft W\big)\triangleleft Z
\end{tikzcd}$$

A $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$ is called *R-linear* (or just *linear*), if $\mathcal{M}$ is an R-linear left $\mathcal{C}$-module category and an R-linear right $\mathcal{D}$-module category.

Definition A.26. Let $\mathcal{M}$ and $\mathcal{M}'$ be $(\mathcal{C},\mathcal{D})$-bimodule categories. A $(\mathcal{C},\mathcal{D})$*-bimodule functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a triple (F, s, t), where (F, s) is a left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{M}'$ and (F, t) is a right $\mathcal{D}$-module functor from $\mathcal{M}$ to $\mathcal{M}'$, such that the following diagram commutes for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $M \in \mathcal{M}$:

$$\begin{tikzcd}
F\big((X\triangleright M)\triangleleft Y\big) \arrow[r, "F(b_{X,M,Y})", "\cong"'] \arrow[d, "t_{X\triangleright M,Y}"', "\cong"] & F\big(X\triangleright(M\triangleleft Y)\big) \arrow[d, "\cong"', "s_{X,M\triangleleft Y}"] \\
F(X\triangleright M)\triangleleft' Y \arrow[d, "s_{X,M}\triangleleft'\mathrm{id}_Y"', "\cong"] & X\triangleright' F(M\triangleleft Y) \arrow[d, "\cong"', "\mathrm{id}_X\triangleright' t_{M,Y}"] \\
\big(X\triangleright' F(M)\big)\triangleleft' Y \arrow[r, "\cong", "b'_{X,F(M),Y}"'] & X\triangleright'\big(F(M)\triangleleft' Y\big)
\end{tikzcd} \tag{A.22}$$

The functor F is called the *underlying functor* of the $(\mathcal{C}, \mathcal{D})$-bimodule functor (F, s, t).
A $(\mathcal{C}, \mathcal{D})$-bimodule functor $(F, s, t) : \mathcal{M} \longrightarrow \mathcal{M}'$ is called *R-linear* (or just *linear*), if both $\mathcal{M}$ and $\mathcal{M}'$ are R-linear $(\mathcal{C}, \mathcal{D})$-bimodule categories and if (F, s) and (F, t) are R-linear left and right module functors respectively.

Proposition A.27. [DSPS, Corollary 2.13] *Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories and let $\mathcal{N}$ and $\mathcal{N}'$ be $(\mathcal{C}, \mathcal{D})$-bimodule categories. Let $Q : \mathcal{N} \longrightarrow \mathcal{N}'$ be a $(\mathcal{C}, \mathcal{D})$-bimodule functor. If the underlying functor Q has a left (respectively right) adjoint, then the left (respectively right) adjoint has the structure of a $(\mathcal{C}, \mathcal{D})$-bimodule functor and the unit and the counit of the adjunction are natural transformations of $(\mathcal{C}, \mathcal{D})$-bimodule functors.*

B The non-strict case

We want to study associator deformations in settings where the monoidal category and the module category are not necessarily strict, for instance, when the module category comes from a (quasi-)comodule algebra over a Hopf algebra. Since we are not aware of a strictification result which allows to strictify a monoidal category $\mathcal{C}$ together with a module category $\mathcal{M}$ over $\mathcal{C}$ simultaneously, we need to generalize the associator complex from Section 1.1 to allow non-strict ingredients.
We also present a version of the Davydov-Yetter complex *with coefficients* of a not necessarily strict monoidal functor, where only the target category is assumed to be strict. This allows us to study deformations of the action functor $\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ in the most general setting, since the monoidal category $\mathrm{End}(\mathcal{M})$ is always strict. Finally, we provide all the necessary results to prove Theorem 2.9.

In this section, let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{D}$ be a k-linear strict monoidal category. Let $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ be a k-linear monoidal functor.

For $n \geq 1$, we define the two k-multilinear functors

$$\begin{aligned} G^{\times n} : \quad \mathcal{C}^{\times n} &\longrightarrow \mathcal{D}^{\times n} \\ (X_1, \ldots, X_n) &\longrightarrow \big(G(X_1), \ldots, G(X_n)\big) \end{aligned}$$

$$\begin{aligned} \overset{n}{\otimes} : \quad \mathcal{D}^{\times n} &\longrightarrow \mathcal{D} \\ (Y_1, \ldots, Y_n) &\longrightarrow Y_1 \otimes \ldots \otimes Y_n \end{aligned}$$

Coefficients for the Davydov-Yetter complex live in the centralizer of a monoidal functor, hence we recall this notion here in our general setting:

Definition B.1. Let $X \in \mathcal{D}$. A *half-braiding relative to* G is a natural isomorphism $\sigma^X : X \otimes G \Longrightarrow G \otimes X$ such that the following diagram commutes for all $V, W \in \mathcal{C}$:

$$\begin{array}{ccc} X \otimes G(V) \otimes G(W) & \xrightarrow[\cong]{\mathrm{id}_X \otimes \Phi_{V,W}} & X \otimes G(V \otimes W) \\ \downarrow{\scriptstyle \sigma^X_V \otimes \mathrm{id}_{G(W)}} \cong & & \\ G(V) \otimes X \otimes G(W) & & \cong \downarrow{\scriptstyle \sigma^X_{V \otimes W}} \\ \downarrow{\scriptstyle \mathrm{id}_{G(V)} \otimes \sigma^X_W} \cong & & \\ G(V) \otimes G(W) \otimes X & \xrightarrow[\Phi_{V,W} \otimes \mathrm{id}_X]{\cong} & G(V \otimes W) \otimes X \end{array} \tag{B.1}$$

Definition B.2. The *centralizer* $\mathcal{Z}(G, \Phi, \varphi)$ of the monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ is the category whose objects are pairs (X, σ^X) where $X \in \mathcal{D}$ and σ^X is a half-braiding relative to G and whose morphisms $f : (X, \sigma^X) \longrightarrow (Y, \sigma^Y)$ are morphisms $f : X \longrightarrow Y$ in $\mathcal{D}$, such that the following diagram commutes for all $V \in \mathcal{C}$:

$$\begin{array}{ccc} X \otimes G(V) & \xrightarrow[\cong]{\sigma^X_V} & G(V) \otimes X \\ \downarrow{\scriptstyle f \otimes \mathrm{id}_{G(V)}} & & \downarrow{\scriptstyle \mathrm{id}_{G(V)} \otimes f} \\ Y \otimes G(V) & \xrightarrow[\sigma^Y_V]{\cong} & G(V) \otimes Y \end{array} \tag{B.2}$$

We sometimes write $\mathcal{Z}(G)$ instead of $\mathcal{Z}(G, \Phi, \varphi)$ if the monoidal structure is clear from the context.

To simplify the proofs in the following subsections, we introduce the auxiliary complex of a k-linear monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ with coefficients $\mathrm{X} = (X, \sigma^X) \in \mathcal{Z}(G)$ and $\mathrm{Y} = (Y, \sigma^Y) \in \mathcal{Z}(G)$: for $n \geq 1$, the nth cochain space of the auxiliary complex consists of natural transformations β, whose components are morphisms in $\mathcal{D}$ of the form:

$$\beta_{X_1,\dots,X_n} : \quad X \otimes G(X_1) \otimes \dots \otimes G(X_n) \longrightarrow G(X_1) \otimes \dots G(X_n) \otimes Y \tag{B.3}$$

and the differential

$$\partial^n_{\mathrm{aux}} : \quad C^n_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y}) \longrightarrow C^{n+1}_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y})$$

is defined as the alternating sum over the following coface maps:

$$\partial^n_{\mathrm{aux}}[0](\beta)_{X_1,\dots,X_{n+1}} := \big(\mathrm{id}_{G(X_1)} \otimes \beta_{X_2,\dots,X_{n+1}}\big) \circ \big(\sigma^X_{X_1} \otimes \mathrm{id}_{G(X_2)\dots G(X_{n+1})}\big) \tag{B.4}$$

$$\begin{aligned}\partial^n_{\mathrm{aux}}[i](\beta)_{X_1,\dots,X_{n+1}} := &\big(\mathrm{id}_{G(X_1)\dots G(X_{i-1})} \otimes \Phi^{-1}_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})Y}\big) \\ &\circ \beta_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}} \\ &\circ \big(\mathrm{id}_{XG(X_1)\dots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\dots G(X_{n+1})}\big)\end{aligned} \tag{B.5}$$

$$\partial^n_{\mathrm{aux}}[n+1](\beta)_{X_1,\dots,X_{n+1}} := \big(\mathrm{id}_{G(X_1)\dots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \circ \big(\beta_{X_1,\dots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big) \tag{B.6}$$

More precisely:

Definition B.3. Let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{D}$ be a k-linear strict monoidal category. The *auxiliary complex* of a k-linear monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ with coefficients $\mathrm{X} = (X, \sigma^X), \mathrm{Y} = (Y, \sigma^Y) \in \mathcal{Z}(G)$ is denoted by $\big(C^\bullet_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y}), \partial_{\mathrm{aux}}\big)$, where the nth cochain space is the k-vector space

$$C^n_{\mathrm{aux}}\big((G, \Phi, \varphi), \mathrm{X}, \mathrm{Y}\big) := \begin{cases} \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } \ n = 0 \\ \mathrm{Nat}\big(X \otimes (\overset{n}{\otimes} \circ G^{\times n}), (\overset{n}{\otimes} \circ G^{\times n}) \otimes Y\big) & \text{if } \ n \geq 1 \end{cases}$$

and the differential is defined as

$$\partial^n_{\mathrm{aux}}(\beta)_{X_1,\dots,X_{n+1}} = \sum_{i=0}^{n+1} (-1)^i \partial^n_{\mathrm{aux}}[i](\beta)_{X_1,\dots,X_{n+1}}$$

for $n \geq 1$ and for $n = 0$ we set

$$\partial^0_{\mathrm{aux}}(\beta)_{X_1} = \big(\mathrm{id}_{G(X_1)} \otimes \beta\big) \circ \sigma^X_{X_1} - \sigma^Y_{X_1} \circ \big(\beta \otimes \mathrm{id}_{G(X_1)}\big) \tag{B.7}$$

Note that the bracketing of the tensor products in the source and target of the components of the n-cochains (B.3) agrees. This will make our calculations easier, but it also means that the auxiliary complex is not directly applicable to our deformation problems; in non-strict settings, however, this effect is of course invisible.

Lemma B.4. *The maps (B.4), (B.5), (B.6) satisfy the cosimplicial relations and thus, the pair* $\big(C^\bullet_{\mathrm{aux}}(G,\mathrm{X},\mathrm{Y}),\partial_{\mathrm{aux}}\big)$ *from Definition B.3 is a cochain complex of* k*-vector spaces.*

Proof:
We need to show that the coface maps satisfy the cosimplicial relations

$$\partial^{n+1}_{\mathrm{aux}}[j]\circ\partial^{n}_{\mathrm{aux}}[i]=\partial^{n+1}_{\mathrm{aux}}[i]\circ\partial^{n}_{\mathrm{aux}}[j-1]$$

for all $n\geq 0$ and $0\leq i<j\leq n+2$. We only prove the case $j=n+2$ and $i=n+1$ since the other cases are completely analogous:

$$\begin{aligned}
&\big(\partial^{n+1}_{\mathrm{aux}}[n+2]\circ\partial^{n}_{\mathrm{aux}}[n+1]\big)(f)_{X_1,\dots,X_{n+2}}\\
=&\big(\mathrm{id}_{G(X_1)\dots G(X_{n+1})}\otimes\sigma^Y_{X_{n+2}}\big)\circ\big((\partial^n_{\mathrm{aux}}[n+1](\beta))_{X_1,\dots,X_{n+1}}\otimes\mathrm{id}_{G(X_{n+2})}\big)\\
=&\big(\mathrm{id}_{G(X_1)\dots G(X_{n+1})}\otimes\sigma^Y_{X_{n+2}}\big)\circ\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\sigma^Y_{X_{n+1}}\otimes\mathrm{id}_{G(X_{n+2})}\big)\\
&\circ\big(\beta_{X_1,\dots,X_n}\otimes\mathrm{id}_{G(X_{n+1})}\otimes\mathrm{id}_{G(X_{n+2})}\big)\\
\overset{(B.1)}{=}&\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\Phi^{-1}_{X_{n+1},X_{n+2}}\otimes\mathrm{id}_Y\big)\circ\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\sigma^Y_{X_{n+1}\otimes X_{n+2}}\big)\\
&\circ\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\mathrm{id}_Y\otimes\Phi_{X_{n+1},X_{n+2}}\big)\circ\big(\beta_{X_1,\dots,X_n}\otimes\mathrm{id}_{G(X_{n+1})}\otimes\mathrm{id}_{G(X_{n+2})}\big)\\
=&\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\Phi^{-1}_{X_{n+1},X_{n+2}}\otimes\mathrm{id}_Y\big)\circ\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\sigma^Y_{X_{n+1}X_{n+2}}\big)\\
&\circ\big(\beta_{X_1,\dots,X_n}\otimes\mathrm{id}_{G(X_{n+1}X_{n+2})}\big)\circ\big(\mathrm{id}_{XG(X_1)\dots G(X_n)}\otimes\Phi_{X_{n+1},X_{n+2}}\big)\\
=&\big(\mathrm{id}_{G(X_1)\dots G(X_n)}\otimes\Phi^{-1}_{X_{n+1},X_{n+2}}\otimes\mathrm{id}_Y\big)\circ\partial^n_{\mathrm{aux}}[n+1](\beta)_{X_1,\dots,X_n,X_{n+1}\otimes X_{n+2}}\\
&\circ\big(\mathrm{id}_{XG(X_1)\dots G(X_n)}\otimes\Phi_{X_{n+1},X_{n+2}}\big)\\
=&\big(\partial^{n+1}_{\mathrm{aux}}[n+1]\circ\partial^n_{\mathrm{aux}}[n+1]\big)(\beta)_{X_1,\dots,X_{n+2}}
\end{aligned}$$

□

B.1 Davydov-Yetter complex

In [GHS, Definition 3.3], the Davydov-Yetter complex with coefficients is defined for a k-linear strict monoidal functor $G:\mathcal{C}\longrightarrow\mathcal{D}$ between k-linear strict monoidal categories $\mathcal{C}$ and $\mathcal{D}$. We will generalize the complex by keeping $\mathcal{D}$ strict but allowing $\mathcal{C}$ to be a not necessarily strict monoidal category and $(G,\Phi,\varphi):\mathcal{C}\longrightarrow\mathcal{D}$ to be a not necessarily strict monoidal functor. Before we define the Davydov-Yetter complex in this setting, we fix the following notation: for $n\geq 1$, we define the k-multilinear functor

$$\begin{aligned}
\otimes^n:\quad &\mathcal{C}^{\times n}\longrightarrow\mathcal{C}\\
(X_1,\dots,X_n)&\longrightarrow X_1\otimes\big(X_2\otimes(\dots\otimes(X_{n-1}\otimes X_n)\dots)\big)
\end{aligned}$$

as well as the rebracketing isomorphism

$$\Phi_n:\quad \overset{n}{\otimes}\circ G^{\times n}\Longrightarrow G\circ\otimes^n$$

whose components

$$(\Phi_n)_{X_1,\dots,X_n}:\quad G(X_1)\otimes\dots\otimes G(X_n)\longrightarrow G\big(X_1\otimes\big(X_2\otimes(\dots\otimes(X_{n-1}\otimes X_n)\dots)\big)\big) \tag{B.8}$$

are the following isomorphisms in $\mathcal{D}$:

$$(\Phi_n)_{X_1,\ldots,X_n} := \begin{cases} \mathrm{id}_{G(X_1)} & \text{if } \ n=1. \\ \Phi_{X_1,X_2(\ldots(X_{n-1}X_n)\ldots)} \circ (\mathrm{id}_{G(X_1)} \otimes \Phi_{X_2,X_3(\ldots(X_{n-1},X_n)\ldots)}) & \\ \circ \ldots \circ (\mathrm{id}_{G(X_1)\ldots G(X_{n-3})} \otimes \Phi_{X_{n-2},X_{n-1}X_n}) & \\ \circ(\mathrm{id}_{G(X_1)\ldots G(X_{n-2})} \otimes \Phi_{X_{n-1},X_n}) & \text{if } \ n \geq 2. \end{cases} \tag{B.9}$$

For $n \geq 1$, the nth cochain space of the Davydov-Yetter complex of a k-linear and not necessarily strict monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ with coefficients $\mathrm{X} = (X, \sigma^X), \mathrm{Y} = (Y, \sigma^Y) \in \mathcal{Z}(G)$ is the k-vector space

$$\mathrm{Nat}\big(X \otimes (\overset{n}{\otimes} \circ G^{\times n}), (G \circ \otimes^n) \otimes Y\big)$$

that consists of natural transformations b whose components are morphisms in $\mathcal{D}$ of the form

$$b_{X_1,\ldots,X_n} : \quad X \otimes G(X_1) \otimes \ldots \otimes G(X_n) \longrightarrow G\big(X_1 \otimes \big(X_2 \otimes (\ldots \otimes (X_{n-1} \otimes X_n)\ldots)\big)\big) \otimes Y \tag{B.10}$$

For $n \geq 1$ and $0 \leq j \leq n$, we consider the coface maps

$$\partial^n_{ass}[j] : \quad C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \longrightarrow C^{n+1}_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$$

with

$$\begin{aligned} \partial^n_{\mathrm{DY}}[0](b)_{X_1,\ldots,X_{n+1}} :=& \big((\Phi_{n+1})_{X_1,\ldots,X_{n+1}} \otimes \mathrm{id}_Y\big) \circ \big(\mathrm{id}_{G(X_1)} \otimes (\Phi_n^{-1})_{X_2,\ldots,X_{n+1}} \otimes \mathrm{id}_Y\big) \\ &\circ \big(\mathrm{id}_{G(X_1)} \otimes b_{X_2,\ldots,X_{n+1}}\big) \circ \big(\sigma^X_{X_1} \otimes \mathrm{id}_{G(X_1)\ldots G(X_{n+1})}\big) \end{aligned}$$

$$\begin{aligned} \partial^n_{\mathrm{DY}}[i](b)_{X_1,\ldots,X_{n+1}} :=& \big((\Phi_{n+1})_{X_1,\ldots,X_{n+1}} \otimes \mathrm{id}_Y\big) \\ &\circ \big(\mathrm{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi^{-1}_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{n+1})Y}\big) \\ &\circ \big((\Phi_n^{-1})_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \otimes \mathrm{id}_Y\big) \circ b_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \\ &\circ \big(\mathrm{id}_{XG(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \mathrm{id}_{G(X_{i+2})\ldots G(X_{n+1})}\big) \end{aligned}$$

$$\begin{aligned} \partial^n_{\mathrm{DY}}[n+1](b)_{X_1,\ldots,X_{n+1}} :=& \big((\Phi_{n+1})_{X_1,\ldots,X_{n+1}} \otimes \mathrm{id}_Y\big) \circ \big(\mathrm{id}_{G(X_1)\ldots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \\ &\circ \big((\Phi_n^{-1})_{X_1,\ldots,X_n} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{G(X_{n+1})}\big) \circ \big(b_{X_1,\ldots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big) \end{aligned}$$

where $1 \leq i \leq n$, $X_1, \ldots, X_{n+1} \in \mathcal{C}$ and $b \in \mathrm{Nat}\big(X \otimes (\overset{n}{\otimes} \circ G^{\times n}), (G \circ \otimes^n) \otimes Y\big)$.

Definition B.5. Let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{D}$ be a k-linear strict monoidal category. The *Davydov-Yetter complex* of a k-linear monoidal functor $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ with coefficients $\mathrm{X} = (X, \sigma^X), \mathrm{Y} = (Y, \sigma^Y) \in \mathcal{Z}(G)$ is denoted by $\big(C^\bullet_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}), \partial_{\mathrm{DY}}\big)$, where the nth cochain space is the k-vector space

$$C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) := \begin{cases} \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } \ n = 0 \\ \mathrm{Nat}\big(X \otimes (\overset{n}{\otimes} \circ G^{\times n}), (G \circ \otimes^n) \otimes Y\big) & \text{if } \ n \geq 1 \end{cases}$$

and the differential is defined as

$$\partial^n_{\mathrm{DY}}(b)_{X_1,\ldots,X_{n+1}} = \sum_{i=0}^{n+1} \partial^n_{\mathrm{DY}}[i](b)_{X_1,\ldots,X_{n+1}}$$

for $n \geq 1$ and for $n = 0$ we set

$$\partial^0_{\mathrm{DY}}(b)_{X_1} = \big(\mathrm{id}_{G(X_1)} \otimes b\big) \circ \sigma^X_{X_1} - \sigma^Y_{X_1} \circ \big(b \otimes \mathrm{id}_{G(X_1)}\big) \tag{B.11}$$

The space of n-cocycles is denoted by $Z^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$. If $(X, \sigma^X) = (Y, \sigma^Y)$, we also write $C^\bullet_{\mathrm{DY}}(G, (X, \sigma^X))$ instead of $C^\bullet_{\mathrm{DY}}\big(G, (X, \sigma^X), (X, \sigma^X)\big)$ etc.

Instead of showing directly that the differential ∂_{DY} squares to zero, we establish an isomorphism of the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$ and the auxiliary complex $C^\bullet_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y})$ as follows: recall from (B.3) that for $n \geq 1$, an n-cochain β in the auxiliary complex is a natural transformation with components

$$\beta_{X_1,\ldots,X_n} : \quad X \otimes G(X_1) \otimes \ldots \otimes G(X_n) \longrightarrow G(X_1) \otimes \ldots G(X_n) \otimes Y$$

Postcomposing $\beta_{X_1,\ldots,X_n}$ with the rebracketing isomorphism $(\Phi_n)_{X_1,\ldots,X_n} \otimes \mathrm{id}_Y$ (see B.8) gives a morphism in $\mathcal{D}$ of the form

$$\begin{aligned}\big((\Phi_n)_{X_1,\ldots,X_n} \otimes \mathrm{id}_Y\big) \circ \beta_{X_1,\ldots,X_n} : \quad & X \otimes G(X_1) \otimes \ldots \otimes G(X_n) \\ & \longrightarrow G\big(X_1 \otimes \big(X_2 \otimes (\ldots \otimes (X_{n-1} \otimes X_n) \ldots)\big)\big) \otimes Y,\end{aligned}$$

i.e., an n-cochain in the Davydov-Yetter complex $C^\bullet_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$ (see (B.10)). Indeed, we have an isomorphism of k-vector spaces

$$\lambda_n : \quad C^n_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y}) \xrightarrow{\cong} C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \tag{B.12}$$

with

$$\lambda_n(\beta)_{X_1,\ldots,X_n} := \big((\Phi_n)_{X_1,\ldots,X_n} \otimes \mathrm{id}_Y\big) \circ \beta_{X_1,\ldots,X_n}$$

for $n \geq 1$ and

$$\lambda_0(\beta) := \beta$$

The inverse of λ_0 is obvious and for $n \geq 1$, the inverse

$$\lambda_n^{-1} : \quad C^n_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y}) \xrightarrow{\cong} C^n_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y})$$

is constructed using Φ_n^{-1} as follows:

$$\lambda_n^{-1}(b)_{X_1,\ldots,X_n} = \big((\Phi_n^{-1})_{X_1,\ldots,X_n} \otimes \mathrm{id}_Y\big) \circ b_{X_1,\ldots,X_n}$$

Lemma B.6. *Let $\mathcal{C}$ be a k-linear monoidal category and let $\mathcal{D}$ be a k-linear strict monoidal category. Let $(G, \Phi, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ be a k-linear monoidal functor and let $\mathrm{X}, \mathrm{Y} \in \mathcal{Z}(G)$. The isomorphisms (B.12) of k-vector spaces combine into an isomorphism of cochain complexes*

$$C^\bullet_{\mathrm{aux}}(G, \mathrm{X}, \mathrm{Y}) \cong C^\bullet_{\mathrm{DY}}(G, \mathrm{X}, \mathrm{Y})$$

Proof:
We show that the isomorphisms (B.12) are compatible with the differentials, i.e., we show that the following diagram commutes for all $n \geq 0$:

$$\begin{array}{ccc} C^n_{\text{aux}}(G,\mathrm{X},\mathrm{Y}) & \xrightarrow{\partial^n_{\text{aux}}} & C^{n+1}_{\text{aux}}(G,\mathrm{X},\mathrm{Y}) \\ \lambda_n^{-1}\uparrow\cong & & \cong\downarrow\lambda_{n+1} \\ C^n_{\text{DY}}(G,\mathrm{X},\mathrm{Y}) & \xrightarrow[\partial^n_{\text{DY}}]{} & C^{n+1}_{\text{DY}}(G,\mathrm{X},\mathrm{Y}) \end{array} \tag{B.13}$$

The case $n = 0$ is clear since by definition, we have $\partial^0_{\text{aux}} = \partial^0_{\text{DY}}$ (see (B.7) and (B.11)).

For $n = 1$, the following calculation shows the commutativity of diagram (B.13), where we have used that $\lambda_1 = \text{id}$ and $\Phi_2 = \Phi$:

$$\begin{aligned}\big(\lambda_2 \circ \partial^1_{\text{aux}} \circ \lambda_1^{-1}\big)(b)_{X_1,X_2} =&(\Phi_{X_1,X_2} \otimes \text{id}_Y) \circ \partial^1_{\text{aux}}(b)_{X_1,X_2} \\ =&(\Phi_{X_1,X_2} \otimes \text{id}_Y) \circ (\text{id}_{G(X_1)} \otimes b_{X_2}) \circ (\sigma^X_{X_1} \otimes \text{id}_{G(X_2)}) \\ &- (\Phi_{X_1,X_2} \otimes \text{id}_Y) \circ (\Phi^{-1}_{X_1,X_2} \otimes \text{id}_Y) \circ b_{X_1X_2} \circ (\text{id}_X \otimes \Phi_{X_1,X_2}) \\ &+ (\Phi_{X_1,X_2} \otimes \text{id}_Y) \circ (\text{id}_{G(X_1)} \otimes \sigma^Y_{X_2}) \circ (b_{X_1} \otimes \text{id}_{G(X_2)}) \\ =&\partial^1_{\text{DY}}(b)_{X_1,X_2}\end{aligned}$$

For $n \geq 2$, we have

$$\big(\lambda_{n+1} \circ \partial^n_{\text{aux}} \circ \lambda_n^{-1}\big)(b)_{X_1,\ldots,X_{n+1}} = \big((\Phi_{n+1})_{X_1,\ldots,X_{n+1}} \otimes \text{id}_Y\big) \circ \partial^n_{\text{aux}}\big(\lambda_n^{-1}(b)\big)_{X_1,\ldots,X_{n+1}} \tag{B.14}$$

and we calculate the right-hand side of (B.14) for each coface map separately: the first boundary term reads

$$\begin{aligned}\partial^n_{\text{aux}}[0]\big(\lambda_n^{-1}(b)\big)_{X_1,\ldots,X_{n+1}} =&\big(\text{id}_{G(X_1)} \otimes \lambda_n^{-1}(b)_{X_2,\ldots,X_{n+1}}\big) \circ \big(\sigma^X_{X_1} \otimes \text{id}_{G(X_2)\ldots G(X_{n+1})}\big) \\ =&\big(\text{id}_{G(X_1)} \otimes \big((\Phi_n^{-1})_{X_2,\ldots,X_{n+1}} \otimes \text{id}_Y\big)\big) \circ (\text{id}_{G(X_1)} \otimes b_{X_2,\ldots,X_{n+1}}) \\ &\circ \big(\sigma^X_{X_1} \otimes \text{id}_{G(X_2)\ldots G(X_{n+1})}\big)\end{aligned}$$

and thus we get

$$\big((\Phi_{n+1})_{X_1,\ldots,X_{n+1}} \otimes \text{id}_Y\big) \circ \partial^n_{\text{aux}}[0]\big(\lambda_n^{-1}(b)\big)_{X_1,\ldots,X_{n+1}} = \partial^n_{\text{DY}}[0](b)_{X_1,\ldots,X_{n+1}}$$

For $1 \leq i \leq n$, we have

$$\begin{aligned}\partial^n_{\text{aux}}[i]\big(\lambda_n^{-1}(b)\big)_{X_1,\ldots,X_{n+1}} =&\big(\text{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi^{-1}_{X_i,X_{i+1}} \otimes \text{id}_{G(X_{i+2})\ldots G(X_{n+1})Y}\big) \\ &\circ \lambda_n^{-1}(b)_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \\ &\circ \big(\text{id}_{XG(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \text{id}_{G(X_{i+2})\ldots G(X_{n+1})}\big) \\ =&\big(\text{id}_{G(X_1)\ldots G(X_{i-1})} \otimes \Phi^{-1}_{X_i,X_{i+1}} \otimes \text{id}_{G(X_{i+2})\ldots G(X_{n+1})Y}\big) \\ &\circ \big((\Phi_n^{-1})_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \otimes \text{id}_Y\big) \\ &\circ (b_{X_1,\ldots,X_iX_{i+1},\ldots,X_{n+1}} \otimes \text{id}_{G(X_{n+1})}) \\ &\circ \big(\text{id}_{XG(X_1)\ldots G(X_{i-1})} \otimes \Phi_{X_i,X_{i+1}} \otimes \text{id}_{G(X_{i+2})\ldots G(X_{n+1})}\big)\end{aligned}$$

and hence we obtain

$$\big((\Phi_{n+1})_{X_1,\dots,X_{n+1}} \otimes \mathrm{id}_Y\big) \circ \partial^n_{\mathrm{aux}}[i]\big(\lambda_n^{-1}(b)\big)_{X_1,\dots,X_{n+1}} = \partial^n_{\mathrm{DY}}[i](b)_{X_1,\dots,X_{n+1}}$$

Finally, the second boundary term is

$$\begin{aligned}\partial^n_{\mathrm{aux}}[n+1]\big(\lambda_n^{-1}(b)\big)_{X_1,\dots,X_{n+1}} =& \big(\mathrm{id}_{G(X_1)\dots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \circ \big(\lambda_n^{-1}(b)_{X_1,\dots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big)\\ =& \big(\mathrm{id}_{G(X_1)\dots G(X_n)} \otimes \sigma^Y_{X_{n+1}}\big) \circ \big(\big((\Phi_n^{-1})_{X_1,\dots,X_n} \otimes \mathrm{id}_Y\big) \otimes \mathrm{id}_{G(X_{n+1})}\big)\\ &\circ \big(b_{X_1,\dots,X_n} \otimes \mathrm{id}_{G(X_{n+1})}\big)\end{aligned}$$

and thus we have

$$\big((\Phi_{n+1})_{X_1,\dots,X_{n+1}} \otimes \mathrm{id}_Y\big) \circ \partial^n_{\mathrm{aux}}[n+1]\big(\lambda_n^{-1}(b)\big)_{X_1,\dots,X_{n+1}} = \partial^n_{\mathrm{DY}}[n+1](b)_{X_1,\dots,X_{n+1}}$$

which finishes the proof.

□

We now turn to the case where the monoidal functor is the action functor

$$\rho_{\mathcal{M}} : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$$

of a k-linear $\mathcal{C}$-module category $\mathcal{M}$. Recall the monoidal structure $\mathfrak{m}$ of the action functor from (2.6). For $n \geq 1$, the rebracketing isomorphism (B.8) on components in $\mathcal{M}$ is the following isomorphism

$$\begin{aligned}\big((\mathfrak{m}_n)_{X_1,\dots,X_n}\big)_M :\quad & X_1 \triangleright \big(\dots \triangleright (X_n \triangleright M)\dots\big)\\ &\xrightarrow{\cong} \big(X_1 \otimes \big(X_2 \otimes (\dots \otimes (X_{n-1} \otimes X_n)\dots)\big)\big) \triangleright M \end{aligned} \tag{B.15}$$

with

$$\big((\mathfrak{m}_n)_{X_1,\dots,X_n}\big)_M = \begin{cases} \mathrm{id}_{X_1 \triangleright M} & \text{if } n = 1.\\ m^{-1}_{X_1, X_2(\dots(X_{n-1}X_n)\dots), M} \circ (\mathrm{id}_{X_1} \triangleright m^{-1}_{X_2, X_3(\dots(X_{n-1},X_n)\dots),M}) & \\ \circ \dots \circ \big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{n-3}} \triangleright m^{-1}_{X_{n-2}, X_{n-1}X_n, M})\dots)\big) & \\ \circ\big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{n-2}} \triangleright m^{-1}_{X_{n-1},X_n,M})\dots)\big) & \text{if } n \geq 2. \end{cases}$$

Remark B.7. *For $n \geq 1$, the isomorphism*

$$\lambda_n :\quad C^n_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big) \xrightarrow{\cong} C^n_{\mathrm{DY}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big)$$

sends an n-cocycle β that has the components

$$\begin{aligned}(\beta_{X_1,\dots,X_n})_M :\quad & F\big(X_1 \triangleright \big(\dots\big(X_{n-1} \triangleright (X_n \triangleright M)\big)\big)\dots\big)\\ &\longrightarrow X_1 \triangleright \big(\dots\big(X_{n-1} \triangleright \big(X_n \triangleright F'(M)\big)\big)\dots\big)\end{aligned}$$

to the n-cocycle in the Davydov-Yetter complex with components

$$\big(\lambda_n(\beta)_{X_1,\dots,X_n}\big)_{F'(M)} = \big((\mathfrak{m}_n)_{X_1,\dots,X_n}\big)_M \circ (\beta_{X_1,\dots,X_n})_M$$

The inverse

$$\lambda_n^{-1}: \quad C^n_{\mathrm{DY}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big) \xrightarrow{\cong} C^n_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big)$$

sends an n-cocycle b that has the components

$$\begin{aligned}(b_{X_1,\dots,X_n})_M: \quad & F\big(X_1 \triangleright \big(\dots\big(X_{n-1} \triangleright (X_n \triangleright M)\big)\big)\dots\big) \\ & \longrightarrow \big(X_1 \otimes \big(X_2 \otimes (\dots \otimes (X_{n-1} \otimes X_n)\dots)\big)\big) \triangleright F'(M)\end{aligned}$$

to the n-cocycle in the auxiliary complex which has the components

$$\big(\lambda_n^{-1}(b)_{X_1,\dots,X_n}\big)_M = \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_n}\big) \circ (b_{X_1,\dots,X_n})_M$$

Remark B.8. *Recall from Proposition 2.8 that we identify the monoidal categories $\mathcal{Z}(\rho_{\mathcal{M}})$ and $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$. Here, we list the coface maps of the cochain complex $C^\bullet_{\mathrm{DY}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big)$ for $n \geq 1$:*

$$\begin{aligned}\big(\partial^n_{\mathrm{DY}}[0](b)_{X_1,\dots,X_{n+1}}\big)_M =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}} \bullet \mathrm{id}_{F'}\big)_M \circ \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet (\mathfrak{m}_n^{-1})_{X_2,\dots,X_{n+1}} \bullet \mathrm{id}_{F'}\big)_M \\ & \circ \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet b_{X_2,\dots,X_{n+1}}\big)_M \circ \big(s_{X_1,-} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}}\big)_{F'(M)} \circ \big(\mathrm{id}_{X_1} \triangleright \big((\mathfrak{m}_n^{-1})_{X_2,\dots,X_{n+1}}\big)_{F'(M)}\big) \\ & \circ \big(\mathrm{id}_{X_1} \triangleright (b_{X_2,\dots,X_{n+1}})_M\big) \circ s_{X_1, X_2 \triangleright (\dots \triangleright (X_{n+1} \triangleright M)\dots)}\end{aligned}$$

$$\begin{aligned}\big(\partial^n_{\mathrm{DY}}[i](b)_{X_1,\dots,X_{n+1}}\big)_M =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}} \bullet \mathrm{id}_{F'}\big)_M \\ & \circ \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}^{-1}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})F'}\big)_M \\ & \circ \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}} \bullet \mathrm{id}_{F'}\big)_M \circ (b_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ & \circ \big(\mathrm{id}_{F\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}}\big)_{F'(M)} \\ & \circ \big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{i-1}} \triangleright m_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright F'(M)))\dots)})\dots)\big) \\ & \circ \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}}\big)_{F'(M)} \circ (b_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ & \circ F\big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{i-1}} \triangleright m^{-1}_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright M)\dots)})\dots)\big)\end{aligned}$$

$$\begin{aligned}\big(\partial^n_{\mathrm{DY}}[n+1](b)_{X_1,\dots,X_{n+1}}\big)_M =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}} \bullet \mathrm{id}_{F'}\big)_M \circ \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_n)} \bullet s'_{X_{n+1},-}\big)_M \\ & \circ \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_n} \bullet \mathrm{id}_{F'} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{n+1})}\big)_M \circ \big(b_{X_1,\dots,X_n} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =& \big((\mathfrak{m}_{n+1})_{X_1,\dots,X_{n+1}}\big)_{F'(M)} \circ \big(\mathrm{id}_{X_1} \triangleright \big(\dots \triangleright (\mathrm{id}_{X_n} \triangleright s'_{X_{n+1},M})\dots\big)\big) \\ & \circ \big((\mathfrak{m}_n^{-1})_{X_1,\dots,X_n}\big)_{F'(X_{n+1}\triangleright M)} \circ (b_{X_1,\dots,X_n})_{X_{n+1}\triangleright M}\end{aligned}$$

For $n = 0$, the differential is defined as

$$\begin{aligned}\big(\partial^0_{\mathrm{DY}}(b)_{X_1}\big)_M =& \big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet b\big)_M \circ s_{X_1,M} - s'_{X_1,M} \circ \big(b \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_1)}\big)_M \\ =& (\mathrm{id}_{X_1} \triangleright b_M) \circ s_{X_1,M} - s'_{X_1,M} \circ b_{X_1 \triangleright M}\end{aligned}$$

B.2 Associator deformation complex

Before we introduce the associator deformation complex in the non-strict case, we need the following notation: for $n \geq 1$, we define the k-multilinear functors

$$\begin{aligned} {}^{n}\otimes : \quad & \mathcal{C}^{\times n} \longrightarrow \mathcal{C} \\ (X_1, \ldots, X_n) & \longrightarrow \big(\ldots (X_1 \otimes X_2) \otimes \ldots\big) \otimes X_n \end{aligned}$$

$$\begin{aligned} \triangleright^{n} : \quad & \mathcal{C}^{\times n} \times \mathcal{M} \longrightarrow \mathcal{M} \\ (X_1, \ldots, X_n, M) & \longrightarrow X_1 \triangleright \big(X_2 \triangleright (\ldots \triangleright (X_n \triangleright M) \ldots)\big) \end{aligned}$$

where we use the convention that $\triangleright^0 = \mathrm{Id}_{\mathcal{M}}$.

For any k-linear endofunctor $F : \mathcal{M} \longrightarrow \mathcal{M}$ and all $n \geq 1$, we introduce the two k-multilinear functors

$$\begin{aligned} {}^{n}(F) : \quad & \mathcal{C}^{\times n} \times \mathcal{M} \longrightarrow \mathcal{M} \\ (X_1, \ldots, X_n, M) & \longrightarrow F\big(\big((\ldots (X_1 \otimes X_2) \otimes \ldots) \otimes X_n\big) \triangleright M\big) \end{aligned}$$

$$\begin{aligned} (F)^{n} : \quad & \mathcal{C}^{\times n} \times \mathcal{M} \longrightarrow \mathcal{M} \\ (X_1, \ldots, X_n, M) & \longrightarrow X_1 \triangleright \big(\ldots \big(X_{n-1} \triangleright \big(X_n \triangleright F(M)\big)\big) \ldots\big) \end{aligned}$$

encoding two different ways of putting parentheses in multiple actions. For $n = 0$ we use the convention ${}^{0}(F) = (F)^{0} = F$.

For all $n \geq 0$, we define the natural isomorphism m_n whose components are the following isomorphisms in $\mathcal{M}$:

$$\begin{aligned} \big((m_n)_{X_1,\ldots,X_n}\big)_M : \quad & F\big(X_1 \triangleright (\ldots \triangleright (X_n \triangleright M) \ldots)\big) \\ & \xrightarrow{\cong} F\big(((\ldots (X_1 \otimes X_2) \otimes \ldots) \otimes X_n) \triangleright M\big) \end{aligned} \tag{B.16}$$

where

$$(m_0)_M := \mathrm{id}_{F(M)}, \qquad \big((m_1)_{X_1}\big)_M := \mathrm{id}_{F(X_1 \triangleright M)}$$

and

$$\begin{aligned} \big((m_n)_{X_1,\ldots,X_n}\big)_M := {} & F\big(m^{-1}_{(\ldots(X_1X_2)\ldots)X_{n-1},X_n,M}\big) \circ F\big(m^{-1}_{(\ldots(X_1X_2)\ldots)X_{n-2},X_{n-1},X_n \triangleright M}\big) \circ \ldots \\ & \circ F\big(m^{-1}_{X_1X_2,X_3,X_4 \triangleright (\ldots \triangleright (X_n \triangleright M)\ldots)}\big) \circ F\big(m^{-1}_{X_1,X_2,X_3 \triangleright (\ldots \triangleright (X_n \triangleright M)\ldots)}\big) \end{aligned}$$

Note that the isomorphisms (B.16) satisfy the following recursion formula for $n \geq 2$:

$$\big((m_{n+1})_{X_1,\ldots,X_{n+1}}\big)_M = F\big(m^{-1}_{(\ldots(X_1X_2)\ldots)X_n,X_{n+1},M}\big) \circ \big((m_n)_{X_1,\ldots,X_n}\big)_{X_{n+1} \triangleright M} \tag{B.17}$$

We are ready to define the associator deformation complex: for $n \geq 0$, the nth cochain space is defined as the k-vector space

$$C^n_{ass}\big((F, s), (F', s')\big) := \mathrm{Nat}\big({}^{n}(F), (F')^{n}\big)$$

In other words, the nth cochain space of the associator deformation complex

$$C^{\bullet}_{ass}\big((F,s),(F',s')\big)$$

consists of natural transformations, whose components are morphisms in $\mathcal{M}$ that are of the form

$$\begin{aligned} f_{X_1,\dots,X_n,M}: \quad & F\big(\big(\big(\dots(X_1\otimes X_2)\otimes\dots\big)\otimes X_n\big)\triangleright M\big) \\ & \longrightarrow X_1\triangleright\big(\dots\big(X_{n-1}\triangleright\big(X_n\triangleright F'(M)\big)\big)\dots\big) \end{aligned} \tag{B.18}$$

for $X_1,\dots,X_n\in\mathcal{C}, M\in\mathcal{M}$.

For $n\geq 1$ and $0\leq j\leq n$, we define the coface maps

$$\partial^n_{ass}[j]: \quad C^n_{ass}\big((F,s),(F',s')\big)\longrightarrow C^{n+1}_{ass}\big((F,s),(F',s')\big) \tag{B.19}$$

where

$$\begin{aligned} \partial^n_{ass}[0](f)_{X_1,\dots,X_{n+1},M} :=& (\mathrm{id}_{X_1}\triangleright f_{X_2,\dots,X_{n+1},M})\circ\big(\mathrm{id}_{X_1}\triangleright((m_n)_{X_2,\dots,X_{n+1}})_M\big) \\ & \circ s_{X_1,X_2\triangleright(\dots\triangleright(X_{n+1}\triangleright M)\dots)}\circ\big((m^{-1}_{n+1})_{X_1,\dots X_{n+1}}\big)_M \end{aligned}$$

$$\begin{aligned} \partial^n_{ass}[i](f)_{X_1,\dots,X_{n+1},M} :=& (\mathrm{id}_{X_1}\triangleright(\dots\triangleright(\mathrm{id}_{X_{i-1}}\triangleright m_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright F'(M))\dots)})\dots)) \\ & \circ f_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1},M}\circ((m_n)_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ & \circ F(\mathrm{id}_{X_1}\triangleright(\dots\triangleright(\mathrm{id}_{X_{i-1}}\triangleright m^{-1}_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright M)\dots)})\dots)) \\ & \circ((m^{-1}_{n+1})_{X_1,\dots X_{n+1}})_M \end{aligned}$$

$$\begin{aligned} \partial^n_{ass}[n+1](f)_{X_1,\dots,X_{n+1},M} :=& (\mathrm{id}_{X_1}\triangleright(\dots\triangleright(\mathrm{id}_{X_n}\triangleright s'_{X_{n+1},M})\dots))\circ f_{X_1,\dots,X_n,X_{n+1}\triangleright M} \\ & \circ F(m_{(\dots(X_1X_2)\dots)X_n,X_{n+1},M}) \end{aligned}$$

for $1\leq i\leq n$ and $X_1,\dots,X_{n+1}\in\mathcal{C}, M\in\mathcal{M}$.

Definition B.9.

1. Let $\mathcal{C}$ be k-linear a monoidal category and let $\mathcal{M}$ be a k-linear $\mathcal{C}$-module category. The *associator (deformation) complex* of a k-linear $\mathcal{C}$-module endofunctor $(F,s):\mathcal{M}\longrightarrow\mathcal{M}$ with coefficient a k-linear $\mathcal{C}$-module endofunctor $(F',s'):\mathcal{M}\longrightarrow\mathcal{M}$ is denoted by $\big(C^{\bullet}_{ass}\big((F,s),(F',s')\big),\partial_{ass}\big)$, where the nth cochain space is the k-vector space

$$C^n_{ass}\big((F,s),(F',s')\big),\partial_{ass}\big)=\mathrm{Nat}\big({}^n(F),(F')^n\big)$$

and the differential is defined as

$$\partial^n_{ass}(f)_{X_1,\dots,X_{n+1},M}=\sum_{i=0}^{n+1}\partial^n_{ass}[i](f)_{X_1,\dots,X_{n+1},M}$$

for $n\geq 1$ and for $n=0$ we set

$$\partial^n_{ass}(f)_{X,M}=(\mathrm{id}_X\triangleright f_M)\circ s_{X,M}-s'_{X,M}\circ f_{X\triangleright M} \tag{B.20}$$

The space of n-cocycles is denoted by $Z^n_{ass}\big((F,s),(F',s')\big)$ and the *nth associator (deformation) cohomology* group is denoted by $H^n_{ass}\big((F,s),(F',s')\big)$. If $(F,s)=(F',s')$, we also write $C^{\bullet}_{ass}(F,s)$ instead of $C^{\bullet}_{ass}\big((F,s),(F,s)\big)$ etc.

2. In the case $(F, s) = (F', s') = (\mathrm{Id}_{\mathcal{M}}, \mathrm{id})$, the associator deformation complex is denoted by

$$(C^{\bullet}_{ass}(\mathcal{C}, \mathcal{M}), \partial_{ass})$$

and is called the *associator (deformation) complex of $\mathcal{M}$ over $\mathcal{C}$*; its n-cocycles are denoted by $Z^n_{ass}(\mathcal{C}, \mathcal{M})$ and its nth cohomology, $H^n_{ass}(\mathcal{C}, \mathcal{M})$, is called the *$n$th associator (deformation) cohomology of $\mathcal{M}$ over $\mathcal{C}$.*

Remark B.10. *We explicitly state the differential of the associator deformation complex in the first three degrees. For $\nu \in C^0_{ass}\big((F, s), (F', s')\big), t \in C^1_{ass}\big((F, s), (F', s')\big)$, $f \in C^2_{ass}\big((F, s), (F', s')\big)$ and $X_1, X_2, X_3 \in \mathcal{C}, M \in \mathcal{M}$ we have:*

$$\partial^0_{ass}(\nu)_{X_1,M} =(\mathrm{id}_{X_1} \rhd \nu_M) \circ s_{X_1,M} - s'_{X_1,M} \circ \nu_{X_1 \rhd M} \tag{B.21}$$

$$\begin{aligned}\partial^1_{ass}(t)_{X_1,X_2,M} =&(\mathrm{id}_{X_1} \rhd t_{X_2,M}) \circ s_{X_1,X_2 \rhd M} \circ F(m_{X_1,X_2,M}) - m_{X_1,X_2,F'(M)} \circ t_{X \otimes Y,M}\\ &+ (\mathrm{id}_{X_1} \rhd s'_{X_2,M}) \circ t_{X_1,X_2 \rhd M} \circ F(m_{X_1,X_2,M})\end{aligned} \tag{B.22}$$

$$\begin{aligned}\partial^2_{ass}(f)_{X_1,X_2,X_3,M} =&(\mathrm{id}_{X_1} \rhd f_{X_2,X_3,M}) \circ \big(\mathrm{id}_{X_1} \rhd (m_2)_{X_2,X_3}\big)_M \circ s_{X_1,X_2 \rhd (X_3 \rhd M)}\\ &\circ \big((m_3)_{X_1,X_2,X_3}\big)_M\\ &- m_{X_1,X_2,X_3 \rhd M} \circ f_{X_1 \otimes X_2,X_3,M} \circ \big((m_2)_{X_1 \otimes X_2,X_3}\big)_M \circ F(m^{-1}_{X_1,X_2,X_3 \rhd M})\\ &\circ \big((m_3)_{X_1,X_2,X_3}\big)_M\\ &+ (\mathrm{id}_{X_1} \rhd m_{X_2,X_3,F'(M)}) \circ f_{X_1,X_2 \otimes X_3,M} \circ \big((m_2)_{X_1,X_2 \otimes X_3}\big)_M\\ &\circ F(\mathrm{id}_{X_1} \rhd m^{-1}_{X_2,X_3,M}) \circ \big((m_3)_{X_1,X_2,X_3}\big)_M\\ &- \big(\mathrm{id}_{X_1} \rhd (\mathrm{id}_{X_2} \rhd s'_{X_3,M})\big) \circ f_{X_1,X_2,X_3 \rhd M} \circ F(m_{X_1 \otimes X_2,X_3,M})\\ =&(\mathrm{id}_X \rhd f_{X_2,X_3,M}) \circ \big(\mathrm{id}_X \rhd F(m^{-1}_{X_2,X_3,M})\big) \circ s_{X_1,X_2 \rhd (X_3 \rhd M)}\\ &\circ F(m_{X_1,X_2,X_3 \rhd M}) \circ F(m_{X_1 \otimes X_2,X_3,M})\\ &- m_{X_1,X_2,X_3 \rhd F'(M)} \circ f_{X_1 \otimes X_2,X_3,M}\\ &+ (\mathrm{id}_{X_1} \rhd m_{X_2,X_3,F'(M)}) \circ f_{X_1,X_2 \otimes X_3,M} \circ F(m^{-1}_{X_1,X_2 \otimes X_3,M})\\ &\circ F(\mathrm{id}_{X_1} \rhd m^{-1}_{X_2,X_3,M}) \circ F(m_{X_1,X_2,X_3 \rhd M}) \circ F(m_{X_1 \otimes X_2,X_3,M})\\ &- \big(\mathrm{id}_{X_1} \rhd (\mathrm{id}_{X_2} \rhd s'_{X_3,M})\big) \circ f_{X_1,X_2,X_3 \rhd M} \circ F(m_{X_1 \otimes X_2,X_3,M})\\ =&(\mathrm{id}_X \rhd f_{X_2,X_3,M}) \circ \big(\mathrm{id}_X \rhd F(m^{-1}_{X_2,X_3,M})\big) \circ s_{X_1,X_2 \rhd (X_3 \rhd M)}\\ &\circ F(\mathrm{id}_{X_1} \rhd m_{X_2,X_3,M}) \circ F(m_{X_1,X_2 \otimes X_3,M}) \circ F(\mathfrak{a}_{X_1,X_2,X_3} \rhd \mathrm{id}_M)\\ &- m_{X_1,X_2,X_3 \rhd F'(M)} \circ f_{X_1 \otimes X_2,X_3,M}\\ &+ (\mathrm{id}_{X_1} \rhd m_{X_2,X_3,F'(M)}) \circ f_{X_1,X_2 \otimes X_3,M} \circ F(\mathfrak{a}_{X_1,X_2,X_3} \rhd \mathrm{id}_M)\\ &- \big(\mathrm{id}_{X_1} \rhd (\mathrm{id}_{X_2} \rhd s'_{X_3,M})\big) \circ f_{X_1,X_2,X_3 \rhd M} \circ F(m_{X_1 \otimes X_2,X_3,M})\\ =&(\mathrm{id}_X \rhd f_{X_2,X_3,M}) \circ s_{X_1,(X_2 \otimes X_3) \rhd M} \circ F(m_{X_1,X_2 \otimes X_3,M}) \circ F(\mathfrak{a}_{X_1,X_2,X_3} \rhd \mathrm{id}_M)\\ &- m_{X_1,X_2,X_3 \rhd F'(M)} \circ f_{X_1 \otimes X_2,X_3,M}\\ &+ (\mathrm{id}_{X_1} \rhd m_{X_2,X_3,F'(M)}) \circ f_{X_1,X_2 \otimes X_3,M} \circ F(\mathfrak{a}_{X_1,X_2,X_3} \rhd \mathrm{id}_M)\\ &- \big(\mathrm{id}_{X_1} \rhd (\mathrm{id}_{X_2} \rhd s'_{X_3,M})\big) \circ f_{X_1,X_2,X_3 \rhd M} \circ F(m_{X_1 \otimes X_2,X_3,M})\end{aligned}$$

In the above calculation, we have used the pentagon axiom (A.6) for the first and third summand in the third equation and the naturality of s for the first summand in the fourth equation.
In particular, for $\nu \in C^0_{ass}(\mathcal{C},\mathcal{M}), t \in C^1_{ass}(\mathcal{C},\mathcal{M}),\ f \in C^2_{ass}(\mathcal{C},\mathcal{M})$ we have:

$$\partial^0_{ass}(\nu)_{X_1,M} = (\mathrm{id}_{X_1} \triangleright \nu_M) - \nu_{X_1 \triangleright M}$$

$$\begin{aligned}\partial^1_{ass}(t)_{X_1,X_2,M} =& (\mathrm{id}_{X_1} \triangleright t_{X_2,M}) \circ m_{X_1,X_2,M} - m_{X_1,X_2,M} \circ t_{X_1\otimes X_2,M} \\ &+ t_{X_1,X_2\triangleright M} \circ m_{X_1,X_2,M}\end{aligned} \tag{B.23}$$

$$\begin{aligned}\partial^2_{ass}(f)_{X_1,X_2,X_3,M} =& (\mathrm{id}_{X_1} \triangleright f_{X_2,X_3,M}) \circ m_{X_1,X_2\otimes X_3,M} \circ (\mathfrak{a}_{X_1,X_2,X_3} \triangleright \mathrm{id}_M) \\ &- m_{X_1,X_2,X_3\triangleright M} \circ f_{X_1\otimes X_2,X_3,M} \\ &+ (\mathrm{id}_{X_1} \triangleright m_{X_2,X_3,M}) \circ f_{X_1,X_2\otimes X_3,M} \circ (\mathfrak{a}_{X_1,X_2,X_3} \triangleright \mathrm{id}_M) \\ &- f_{X_1,X_2,X_3\triangleright M} \circ m_{X_1\otimes X_2,X_3,M}\end{aligned} \tag{B.24}$$

It would now be standard to verify that the maps (B.19) satisfy the cosimplicial relations. But we will not go this route, since it is more convenient to prove that the associator complex $C^\bullet_{ass}\big((F,s),(F',s')\big)$ is isomorphic to the auxiliary complex $C^\bullet\big(\rho_\mathcal{M},(F,s),(F',s')\big)$.

Recall that for $n \geq 1$, the nth cochain space of the auxiliary complex $C^\bullet_{\mathrm{aux}}\big(\rho,(F,s),(F',s')\big)$ consists of natural transformations, whose components are morphisms in $\mathcal{M}$ of the form

$$\begin{aligned}(\beta_{X_1,\dots,X_n})_M :\quad & F\big(X_1 \triangleright \big(\dots\big(X_{n-1} \triangleright (X_n \triangleright M)\big)\big)\dots\big) \\ & \longrightarrow X_1 \triangleright \big(\dots\big(X_{n-1} \triangleright \big(X_n \triangleright F'(M)\big)\big)\dots\big)\end{aligned}$$

Note that the morphisms $(\beta_{X_1,\dots,X_n})_M$ and (B.18) have the same target and differ only in the bracketing of their source. Thus, if we precompose an n-cocycle $f \in C^n_{ass}\big((F,s),(F',s')\big)$ with the rebracketing m_n (see (B.16)), we obtain an n-cocycle in the auxiliary complex $C^\bullet_{\mathrm{aux}}\big(\rho_\mathcal{M},(F,s),(F',s')\big)$. Indeed, for all $n \geq 0$, the map

$$\vartheta_n :\quad C^n_{ass}\big((F,s),(F',s')\big) \xrightarrow{\cong} C^n_{\mathrm{aux}}\big(\rho_\mathcal{M},(F,s),(F',s')\big) \tag{B.25}$$

which sends a natural transformation f to the natural transformation with components

$$\big(\vartheta_n(f)_{X_1,\dots,X_n}\big)_M := f_{X_1,\dots,X_n,M} \circ \big((m_n)_{X_1,\dots,X_n}\big)_M$$

is an isomorphism of k-vector spaces. Its inverse

$$\vartheta_n^{-1} :\quad C^n_{\mathrm{aux}}\big(\rho_\mathcal{M},(F,s),(F',s')\big) \xrightarrow{\cong} C^n_{ass}\big((F,s),(F',s')\big)$$

is constructed using m_n^{-1} as follows:

$$\big(\vartheta_n^{-1}(\beta)_{X_1,\dots,X_n}\big)_M = (\beta_{X_1,\dots,X_n})_M \circ \big((m_n^{-1})_{X_1,\dots,X_n}\big)_M$$

We will show that the isomorphisms (B.25) are compatible with the differentials.

Remark B.11. *In the following, we list the coface maps of the cochain complex* $C^\bullet_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big)$ *for* $n \geq 1$*:*

$$\begin{aligned}\big(\partial^n_{\mathrm{aux}}[0](\beta)_{X_1,\dots,X_{n+1}}\big)_M =&\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet \beta_{X_2,\dots,X_{n+1}}\big)_M \circ \big(s_{X_1,-} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_2)\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =&\big(\mathrm{id}_{X_1} \rhd (\beta_{X_2,\dots,X_{n+1}})_M\big) \circ s_{X_1, X_2 \rhd (\dots \rhd (X_{n+1} \rhd M)\dots)}\end{aligned} \tag{B.26}$$

$$\begin{aligned}\big(\partial^n_{\mathrm{aux}}[i](\beta)_{X_1,\dots,X_{n+1}}\big)_M =&\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}^{-1}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})F'}\big)_M \\ &\circ (\beta_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ &\circ \big(\mathrm{id}_{F\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =&\big(\mathrm{id}_{X_1} \rhd (\dots \rhd (\mathrm{id}_{X_{i-1}} \rhd m_{X_i,X_{i+1},X_{i+2}\rhd(\dots\rhd(X_{n+1}\rhd F'(M))\dots)}) \dots)\big) \\ &\circ (\beta_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}})_M \\ &\circ F\big(\mathrm{id}_{X_1} \rhd (\dots \rhd (\mathrm{id}_{X_{i-1}} \rhd m^{-1}_{X_i,X_{i+1},X_{i+2}\rhd(\dots\rhd(X_{n+1}\rhd M)\dots)}) \dots)\big)\end{aligned} \tag{B.27}$$

$$\begin{aligned}\big(\partial^n_{\mathrm{aux}}[n+1](\beta)_{X_1,\dots,X_{n+1}}\big)_M =&\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_n)} \bullet s'_{X_{n+1},-}\big)_M \circ \big(\beta_{X_1,\dots,X_n} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\ =&\big(\mathrm{id}_{X_1} \rhd \big(\dots \rhd (\mathrm{id}_{X_n} \rhd s'_{X_{n+1},M}) \dots\big)\big) \circ (\beta_{X_1,\dots,X_n})_{X_{n+1}\rhd M}\end{aligned} \tag{B.28}$$

For $n = 0$*, the differential reads*

$$\begin{aligned}\big(\partial^0_{\mathrm{aux}}(\beta)_{X_1}\big)_M =&\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet \beta\big)_M \circ s_{X_1,M} - s'_{X_1,M} \circ \big(\beta \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_1)}\big)_M \\ =&(\mathrm{id}_{X_1} \rhd \beta_M) \circ s_{X_1,M} - s'_{X_1,M} \circ \beta_{X_1 \rhd M}\end{aligned} \tag{B.29}$$

Proposition B.12. *Let* $\mathcal{C}$ *be a* k*-linear monoidal category and let* $\mathcal{M}$ *be a* k*-linear* $\mathcal{C}$*-module category. Let* $(F,s), (F',s') : \mathcal{M} \longrightarrow \mathcal{M}$ *be* k*-linear* $\mathcal{C}$*-module endofunctors. The isomorphisms (B.25) of* k*-vector spaces combine into an isomorphism of cochain complexes*

$$C^\bullet_{ass}\big((F,s),(F',s')\big) \cong C^\bullet_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big)$$

Proof:
We show the compatibility of the isomorphisms (B.25) with the differentials, i.e., we show that the following diagram commutes for all $n \geq 0$:

$$\begin{array}{ccc} C^n_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big) & \xrightarrow{\partial^n_{\mathrm{aux}}} & C^{n+1}_{\mathrm{aux}}\big(\rho_{\mathcal{M}}, (F,s), (F',s')\big) \\ \vartheta_n \uparrow \cong & & \cong \downarrow \vartheta^{-1}_{n+1} \\ C^n_{ass}\big((F,s),(F',s')\big) & \xrightarrow[\partial^n_{ass}]{} & C^{n+1}_{ass}\big((F,s),(F',s')\big) \end{array}$$

We begin with the case $n = 0$ (see (B.20) and (B.29)):

$$(\vartheta_1^{-1} \circ \partial^0_{\mathrm{aux}} \circ \vartheta_0)(f)_{X_1,M} = \partial^0_{\mathrm{aux}}(f)_{X_1,M} = \partial^0_{ass}(f)_{X_1,M}$$

Let now $n \geq 1$ and $f \in C^n_{ass}\big((F,s),(F',s')\big)$. First we apply the definition of the isomorphism ϑ^{-1}_{n+1} and the differential $\partial^n_{\mathrm{aux}}$:

$$\begin{aligned}
&\big(\vartheta^{-1}_{n+1} \circ \partial^n_{aux} \circ \vartheta_n\big)(f)_{X_1,\dots,X_{n+1},M} \\
=&\big((\partial^n_{aux}(\vartheta_n(f)))_{X_1,\dots,X_n}\big)_M \circ \big((m^{-1}_{n+1})_{X_1,\dots X_{n+1}}\big)_M \\
=&\Big[\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)} \bullet \vartheta_n(f)_{X_2,\dots,X_{n+1}}\big)_M \circ \big(s_{X_1,-} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_2)\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\
&+\sum_{i=1}^{n}(-1)^i\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}^{-1}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})F'}\big)_M \\
&\quad\circ \big(\vartheta_n(f)_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}}\big)_M \circ \big(\mathrm{id}_{F\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_{i-1})} \bullet \mathfrak{m}_{X_i,X_{i+1}} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{i+2})\dots\rho_{\mathcal{M}}(X_{n+1})}\big)_M \\
&+(-1)^{n+1}\big(\mathrm{id}_{\rho_{\mathcal{M}}(X_1)\dots\rho_{\mathcal{M}}(X_n)} \bullet s'_{X_{n+1},-}\big)_M \circ \big(\vartheta_n(f)_{X_1,\dots,X_n} \bullet \mathrm{id}_{\rho_{\mathcal{M}}(X_{n+1})}\big)_M\Big] \\
&\quad\circ \big((m^{-1}_{n+1})_{X_1,\dots X_{n+1}}\big)_M
\end{aligned} \tag{B.30}$$

We now use the equations (B.26), (B.27) and (B.28) to rewrite the summands in (B.30), which yields

$$\begin{aligned}
&\Big[(\mathrm{id}_{X_1} \triangleright f_{X_2,\dots,X_{n+1},M}) \circ \big(\mathrm{id}_{X_1} \triangleright ((m_n)_{X_2,\dots,X_{n+1}})_M\big) \circ s_{X_1,X_2\triangleright(\dots\triangleright(X_{n+1}\triangleright M)\dots)} \\
&+\sum_{i=1}^{n}(-1)^i\big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{i-1}} \triangleright m_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright F'(M))\dots)}) \dots)\big) \\
&\quad\circ f_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1},M} \circ \big((m_n)_{X_1,\dots,X_iX_{i+1},\dots,X_{n+1}}\big)_M \\
&\quad\circ F\big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_{i-1}} \triangleright m^{-1}_{X_i,X_{i+1},X_{i+2}\triangleright(\dots\triangleright(X_{n+1}\triangleright M)\dots)}) \dots)\big) \\
&+(-1)^{n+1}\big(\mathrm{id}_{X_1} \triangleright (\dots \triangleright (\mathrm{id}_{X_n} \triangleright s'_{X_{n+1},M}) \dots)\big) \circ f_{X_1,\dots,X_n,X_{n+1}\triangleright M} \circ \big((m_n)_{X_1,\dots,X_n}\big)_{X_{n+1}\triangleright M}\Big] \\
&\quad\circ \big((m^{-1}_{n+1})_{X_1,\dots X_{n+1}}\big)_M \\
=&\partial^n_{ass}(f)_{X_1,\dots,X_{n+1},M}
\end{aligned}$$

In the last equation, we have used the recursion formula (B.17) for the second boundary term.

□

References

[AM] Nicolás Andruskiewitsch and Juan Martín Mombelli, *On module categories over finite-dimensional Hopf algebras*, J. Algebra **314** (2007), no. 1, 383–418, DOI 10.1016/j.jalgebra.2007.04.006. MR2331768

[BD] Michael Batanin and Alexei Davydov, *Cosimplicial monoids and deformation theory of tensor categories*, J. Noncommut. Geom. **17** (2023), no. 4, 1167–1229, DOI 10.4171/jncg/512. MR4653784

[BM] Noelia Bortolussi and Martín Mombelli, *The character algebra for module categories over Hopf algebras*, Colloq. Math. **165** (2021), no. 2, 171–197, DOI 10.4064/cm8170-5-2020. MR4262891

[BV] Alain Bruguières and Alexis Virelizier, *Quantum double of Hopf monads and categorical centers*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1225–1279, DOI 10.1090/S0002-9947-2011-05342-0. MR2869176

[CY] Louis Crane and David N. Yetter, *Deformations of (bi)tensor categories*, Cahiers Topologie Géom. Différentielle Catég. **39** (1998), no. 3, 163–180 (English, with French summary). MR1641842

[D1] Alexei Davydov, *Twisting of monoidal structures*, preprint, arXiv:q-alg/9703001v1.

[D2] Pierre Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195 (French). MR1106898

[DS] Brian Day and Ross Street, *Centres of monoidal categories of functors*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 187–202, DOI 10.1090/conm/431/08273. MR2342829

[DSPS] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder, *The balanced tensor product of module categories*, Kyoto J. Math. **59** (2019), no. 1, 167–179, DOI 10.1215/21562261-2018-0006. MR3934626

[EGNO] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR3242743

[ENO] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), no. 2, 581–642, DOI 10.4007/annals.2005.162.581. MR2183279

[EO] Pavel Etingof and Viktor Ostrik, *Finite tensor categories*, Mosc. Math. J. **4** (2004), no. 3, 627–654, 782–783, DOI 10.17323/1609-4514-2004-4-3-627-654 (English, with English and Russian summaries). MR2119143

[FGS1] Matthieu Faitg, Azat M. Gainutdinov, and Christoph Schweigert, *Davydov-Yetter cohomology and relative homological algebra*, accepted in Selecta Mathematica, arXiv:2202.12287v3 [math.QA].

[FGS2] ———, *Coefficient theorem for deformation theory of tensor functors*, in preparation.

[FS] Jürgen Fuchs and Christoph Schweigert, *Internal natural transformations and Frobenius algebras in the Drinfeld center*, Transform. Groups **28** (2023), no. 2, 733–768, DOI 10.1007/s00031-021-09678-5. MR4586249

[FSS1] Jürgen Fuchs, Gregor Schaumann, and Christoph Schweigert, *A trace for bimodule categories*, Appl. Categ. Structures **25** (2017), no. 2, 227–268, DOI 10.1007/s10485-016-9425-3. MR3638361

[FSS2] ———, *Eilenberg-Watts calculus for finite categories and a bimodule Radford S^4 theorem*, Trans. Amer. Math. Soc. **373** (2020), no. 1, 1–40, DOI 10.1090/tran/7838. MR4042867

[G] Murray Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103, DOI 10.2307/1970484. MR0171807

[GHS] Azat M. Gainutdinov, Jonas Haferkamp, and Christoph Schweigert, *Davydov-Yetter cohomology, comonads and Ocneanu rigidity*, Adv. Math. **414** (2023), Paper No. 108853, 48, DOI 10.1016/j.aim.2022.108853. MR4536121

[GJS] César Galindo, David Jaklitsch, and Christoph Schweigert, *Equivariant Morita theory for graded tensor categories*, Bull. Belg. Math. Soc. Simon Stevin **29** (2022), no. 2, 145–171, DOI 10.36045/j.bbms.210720. MR4552232

[JS] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78, DOI 10.1006/aima.1993.1055. MR1250465

[K] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995. MR1321145

[LVdB] Wendy Lowen and Michel Van den Bergh, *Deformation theory of abelian categories*, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5441–5483, DOI 10.1090/S0002-9947-06-03871-2. MR2238922

[M1] Saunders MacLane, *Homology*, 1st ed., Die Grundlehren der mathematischen Wissenschaften, Band 114, Springer-Verlag, Berlin-New York, 1967. MR0349792

[M2] Martín Mombelli, *Module categories over pointed Hopf algebras*, Math. Z. **266** (2010), no. 2, 319–344, DOI 10.1007/s00209-009-0571-2. MR2678630

[O] Victor Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups **8** (2003), no. 2, 177–206, DOI 10.1007/s00031-003-0515-6. MR1976459

[R] Emily Riehl, *Category theory in context*, 1st ed., Aurora: Dover modern math originals, Dover Publications Inc., Mineola, New York, 2016.

[S1] Kenichi Shimizu, *On unimodular finite tensor categories*, Int. Math. Res. Not. IMRN **1** (2017), 277–322, DOI 10.1093/imrn/rnv394. MR3632104

[S2] ———, *Integrals for finite tensor categories*, Algebr. Represent. Theory **22** (2019), no. 2, 459–493, DOI 10.1007/s10468-018-9777-5. MR3921367

[S3] ———, *Further results on the structure of (co)ends in finite tensor categories*, Appl. Categ. Structures **28** (2020), no. 2, 237–286, DOI 10.1007/s10485-019-09577-7. MR4077241

[S4] ———, *Ribbon structures of the Drinfeld center of a finite tensor category*, Kodai Math. J. **46** (2023), no. 1, 75–114, DOI 10.2996/kmj46106. MR4560990

[W] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

[Y] Donald Yau, *Deformation bicomplex of module algebras*, Homology Homotopy Appl. **10** (2008), no. 1, 97–128. MR2369024