

Analysis of Natural Function Spaces and Dynamics on Noncompact Manifolds under Symmetry

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Introduction

The main goal of this thesis is to discuss the foundations of dynamical systems, whose state space is a space of maps defined on a noncompact domain and whose dynamics are compatible with the symmetries of this domain.

The most important example of such a dynamical system is the motion of a fluid in an unbounded domain $M \subset (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ with nontrivial isometry group $\text{Isom}(M)$. For example, the dynamics of an inviscid incompressible fluid in an unbounded domain M are governed by the Euler equations

$$\begin{aligned} \frac{\partial u}{\partial t}(t) + \nabla_{u(t)}u(t) &= -\text{grad } p(t) \\ \text{div } u(t) &= 0 \quad , \quad u(t) \parallel \partial M \end{aligned}$$

for the vector field $u(t)$ and the pressure function $p(t)$ of the fluid. These equations are compatible with the symmetry of M , because for every element $g \in \text{Isom}(M)$ of the isometry group and every solution $(u(t), p(t))$ also $(Tg \circ u(t) \circ g^{-1}, p(t) \circ g^{-1})$ is a solution. Thus the dynamics of an inviscid incompressible fluid fit into the general setting of our discussion: The state space is a space of maps on a noncompact domain - here the space of divergence free vector fields on M parallel to the boundary - and the dynamics are compatible with the symmetry. Also the dynamics of all other kinds of fluids like incompressible fluids with constant viscosity governed by the Navier-Stokes equations or ideal compressible isentropic fluids fit into this setting.

A mathematical rigorous treatment of such dynamical systems with symmetry obviously requires to specify, which spaces of maps are used. For example, continuously differentiable maps endowed with the topology of uniform convergence in all derivatives up to a certain order may be used, or Sobolev maps endowed with the Sobolev topology. Now the first task is to assure that the partial differential equations modeling the dynamical system really have a solution within the chosen space of maps. This thesis discusses mainly those systems, which are generated by ordinary differential equations on an infinite-dimensional space of maps and hence solvable.

For example, the Euler equations are such a system. Indeed, the motion of a fluid can not only be modeled by its velocity vector field $u(t)$, but also by its particle map $\eta(t)$. If at time 0 a particle is at the point $m \in M$, then within the time t the particle moves to the point $\eta(t)(m)$. Obviously the vector field u and the particle map η are related: If u is given, then η is the solution operator of the time-dependent ordinary differential equation $\dot{m}(t) = u(t)(m)$ on M , and conversely η determines u by $u(t) := \dot{\eta}(t) \circ \eta(t)^{-1}$. Note further that the property $\text{div } u(t) = 0$ of solutions u of the Euler equations translates into the property that the map $\eta(t)$ on M preserves the volume form (hence the fluid is really incompressible). [Ebin, Marsden] established for compact domains M the local existence of Sobolev solution $u(t) \in H^s(TM)$ by proving that the Euler equations - viewed in terms of the volume form preserving particle map η - are geodesic equations and thus ordinary differential equations on the infinite dimensional space $\text{Diff}_{\text{Vol}}^s(M)$ of volume form preserving Sobolev diffeomorphisms on M w.r.t. the H^0 -metric $\int_M \langle X, Y \rangle dm$. On noncompact domains M it is much

more difficult to prove an analogous result. Using the assumption of bounded geometry and a spectral condition [Eichhorn,Schmid] have shown that the Euler equations are geodesic equations on $\text{Diff}_{\text{Vol}}^s(M)$ also in the noncompact case.

However, regarding pattern formation on noncompact manifolds, solutions within the class of Sobolev vector fields are not really interesting, as such solutions vanish at infinity and thus do not allow patterns typical for noncompact domains, like e.g. upwinding spiral flows in a cylinder. But even if solutions within another space of mappings could be established, which are not vanishing at infinity, there still remains a problem: For noncompact M the symmetry does generally not act continuously on Banach manifolds of mappings. For example, consider the space $C_b(\mathbb{R}^2, \mathbb{R}^2)$ of uniformly continuous maps on the Euclidean \mathbb{R}^2 endowed with the topology of uniform convergence, where the isometries g of \mathbb{R}^2 act by composition $(g, \eta) \mapsto g \circ \eta$. Then a rotation by an arbitrary small angle does generally not sent η to a map which is arbitrarily near to η . In fact, the distance between $\eta = \text{Id}$ and its rotation by an arbitrary small angle $\phi \neq 0$ is always

$$\sup_{x,y} \left\| \begin{pmatrix} \cos \phi & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \infty$$

Thus for all $\phi \neq 0$ the map $\eta = \text{Id}$ is sent to a map in a different connected component of the space of $C_{\text{unif}}(\mathbb{R}^2, \mathbb{R}^2)$. Hence the action of the symmetry group is not continuous and even not strongly continuous, i.e. also $g \mapsto g\eta$ is not continuous. The same is true on the space $\text{Diff}^s(M)$ and its tangential bundle $T\text{Diff}^s(M)$, although the induced action $Tg \circ u \circ g^{-1}$ on the tangential space $T_{\text{Id}}\text{Diff}^s(M) = H^s(TM)$ is continuous.

The noncontinuity of the symmetry on configuration spaces like $\text{Diff}^s(M)$ is caused by the fact that the symmetry acts by composition, but in the case of a noncompact domain composition and evaluation are not continuous for nearly all choices of Banach manifolds of maps. Indeed, for a locally compact topological space M the natural topology on a space of continuous maps on M is the topology of uniform convergence on compact subsets, as this topology is the coarsest topology such that composition and evaluation are continuous. But endowed with this topology the space $C(M, \mathbb{R}^n)$ is merely a complete locally convex topological vector space instead of a Banach space.

Thus there are two ways to proceed, when discussing the foundations of dynamical systems under symmetry on noncompact domains: Either Banach manifolds of maps can be used, where the analysis is well developed, but a discontinuous action of the symmetry (which causes problems in the discussion of pattern formation) and the absence of some patterns (like spirals in Sobolev spaces) has to be accepted. Or instead, manifolds of maps modeled over complete locally convex topological vector spaces can be used, where the symmetry group acts continuously, but the analysis is not so well developed and there is a lack of theorems, which guarantee the solvability of application relevant equations.

The first point of view is adopted e.g. by [Wulff] and [Sandstede, Scheel, Wulff]. The main aim of this thesis is to lay the foundations for the second approach to dynamical systems on noncompact domains under symmetry. Thus a main task is to develop the analysis on manifolds of mappings, which are modeled over locally convex topological vector spaces. This is not an easy task: Contrary to the category of normable spaces

the category of locally convex topological vector spaces is not tensorial closed, and thus there is no natural space of continuous linear maps between locally convex topological vector space. A negative consequence is that it is not clear how to define continuously differentiable maps. However, by using a tensorial closed category of vector spaces endowed with a slightly more general topological structure than a locally convex topology, this problem can be solved and a sufficient differential calculus can be developed.

But analysis requires more than just a differential calculus: Differential equations must be solved, an inverse function theorem is needed, and other theorems of classical analysis must be transferred to the new setting. This is also not an easy task, as there even are differential equations with continuous linear right hand side, which are locally not solvable, so that a precise discussion is needed. Here our choice of the tensorial closed category is helpful, because it guarantees that our continuously differential maps $f : X \rightarrow Y$ are locally Lipschitz continuous. A generalization of the contraction mapping principle then allows to characterize by growing conditions those initial values, where a differential equations with locally Lipschitz continuous right hand side can locally be solved. Finally manifolds modeled on complete locally convex topological vector spaces are considered. Here it is important that manifolds are locally not merely identified with open subsets, but with more general subsets like dense intersections of balls. This is necessary, as for example the exponential mapping on a noncompact manifold is usually only bijective on a neighbourhood of the zero section, but the vectorfields with values in this neighbourhood form generally not an open set.

After having laid the foundations, in the second part of this thesis fluid dynamical systems and pattern formation on noncompact manifolds are discussed. From the proof of [Ebin,Marsden] and [Eichhorn,Schmid], which show that the Euler equations are geodesic (and hence Hamiltonian) equations on the group of volume form preserving diffeomorphisms, it is concluded that the Euler- or Navier-Stokes equations can not be solved for manifolds of mappings modeled over locally convex spaces like the local Sobolev space $H_{loc}^s(M)$, because the equations are nonlocal essentially due to incompressibility and/or viscosity. But it can be argued that other fluid dynamical equations like those modeling inviscous compressible fluids can be solved using local spaces, as they have a finite velocity of propagation. Finally the methods of pattern formation under symmetry in the Banach case and in the locally convex case are compared.

Part I

Analysis on Natural Spaces of Maps

Natural spaces of maps deserve a central role in the discussion of pattern formation under symmetry, because usually the symmetry acts by composition on spaces of maps, but the natural spaces of maps are exactly those induced by composition. The most general setting, where it is possible to endow sets of maps with a natural structure, is the setting of a **tensorial or cartesian closed category**. After the definition and discussion of such categories, **general topological and uniform structures** on a set are introduced and the cartesian closedness of the categories associated to such structures is discussed. This discussion is extended to vector spaces endowed with a compatible topological structure and thus to **linear analysis**. There the main result is that the category of locally convex topological vector spaces, which is not tensorial closed, can be enlarged to a very similar tensorial closed category. Finally the natural spaces of continuous linear maps in this tensorial closed category allow to develop a satisfying **nonlinear analysis** on locally convex topological vector spaces and on manifolds modeled over these spaces.

1 Categorical Preliminaries

A category is an abstraction of the behaviour shown by maps under composition. As categories are so intimately related to the composition of maps, it is not surprising that the notion of a natural space of maps is a categorial notion. To define natural spaces of maps, another categorial notion is needed, namely that of a tensor product. Thus after having introduced categories, functors and natural transformations, tensor categories and natural spaces of maps in tensor categories are discussed.

1.1 Categories

A class \mathfrak{C} endowed with a partial multiplication $\circ : D(\circ) \subset \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ which is associative and has units is called a category. More precisely, call elements of \mathfrak{C} morphisms, denote morphisms by letters f, g, \dots , use the notation fg instead of $f \circ g$ and say “ fg exists” instead of $(f, g) \in D(\circ)$. Then a partial multiplication is said to be associative if the existence of the product $(fg)h$ (resp. $f(gh)$, resp. fg and gh) implies the existence of the product $f(gh)$ (resp. $(fg)h$, resp. $(fg)h$ and $f(gh)$) and the validity of $(fg)h = f(gh)$. Further a morphism e is called a unit if the existence of fe implies $fe = f$ and the existence of ef implies $ef = f$. Units are usually denoted by $\text{Id}_X, \text{Id}_Y, \dots$ and the capital letters X, Y, \dots used to distinguish units are called the objects associated to the units. With this notation a partial multiplication is said to have units if to every f there are units Id_X and Id_Y such that $f \text{Id}_X$ and $\text{Id}_Y f$ exist. For every morphism f such units Id_X, Id_Y are automatically unique, and in analogy to the case of maps between sets write $f : X \rightarrow Y$ and call X the domain and Y the codomain of f . For $f : X \rightarrow Y$ and $g : Y' \rightarrow Z$ the existence of gf is equivalent to $Y = Y'$.

Denote by $\mathfrak{C}(X, Y)$ the class of morphisms with domain X and codomain Y . A category \mathfrak{C} is called locally small if $\mathfrak{C}(X, Y)$ is a set for every pair of units X, Y , and usually categories are assumed to be locally small. A morphism $f : X \rightarrow Y$ is called a monomorphism if $fg = fh$ implies $g = h$, an epimorphism if $gf = hf$ implies $g = h$, and an isomorphism if there is a $g : Y \rightarrow X$ such that $gf = \text{Id}_X$ and $fg = \text{Id}_Y$. In the last case the morphism g is automatically unique, is called the inverse of f and is denoted by f^{-1} . Two objects X and Y are called isomorphic if there is an isomorphism $f : X \rightarrow Y$, and in this case $X \cong Y$ is written. Isomorphism is an equivalence relation on the class $\text{Ob}(\mathfrak{C})$ of objects in \mathfrak{C} .

Let (\mathfrak{C}, \circ) be a category, then the opposite (or dual) category \mathfrak{C}^{op} is defined as the class \mathfrak{C} endowed with the partial multiplication $f \circ^{op} g := g \circ f$. This construction allows to define the dual notion to each categorial notion. For example, a monomorphism in \mathfrak{C}^{op} is an epimorphism in \mathfrak{C} , and that's why the notion of an epimorphism is called dual to the notion of a monomorphism.

Define to categories $\mathfrak{C}, \mathfrak{D}$ the product category $\mathfrak{C} \times \mathfrak{D}$ as the product class endowed with the partial multiplication $(f, g)(f', g') = (ff', gg')$. Particular $(f, g)(f', g')$ exists iff ff' and gg' exist. An object 0 of a category \mathfrak{C} is called initial if to every object X there is exactly one morphism from 0 to X . Dually an object 1 is called terminal if to every object X there is exactly one morphism from X to 1 .

At last some examples of categories shall be given. Loosely spoken, if sets endowed with a certain kind of structure are considered, then the class of structure preserving maps form a category, whose partial multiplication is given by composition and whose objects can be identified with the structured sets. For example, the maps between sets (with no structure) form the categories \mathfrak{Set} , the continuous maps between topological spaces form a category \mathfrak{Top} , the group homomorphisms form a category \mathfrak{Grp} , the linear maps between vector spaces over a field K form a category \mathfrak{Vec}_K and so on. Also the maps between categories which respect partial multiplication and preserve units form a category \mathfrak{Cat} , whose morphisms are called functors. Other examples of categories are

- Groups G , where the morphisms are given by the elements of G and the multiplication is not partially but totally, so that there is only one unit - the neutral element of the group.
- Partially ordered sets P , where pairs $(x, y) \in P \times P$ with $x \leq y$ are considered to be morphisms and the partial multiplication is defined by $(x, y)(y', z) := (x, z)$ whenever $y = y'$ is valid.

1.2 Functors

As already mentioned above, a functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ between categories $\mathfrak{C}, \mathfrak{D}$ is a map such that the existence of fg in \mathfrak{C} implies the existence of $F(f)F(g)$ in \mathfrak{D} as well as the validity of $F(fg) = F(f)F(g)$, and units e of \mathfrak{C} must be mapped to units $F(e)$ of \mathfrak{D} . A functor F from \mathfrak{C} to \mathfrak{D} is called covariant, while functors from \mathfrak{C}^{op} to \mathfrak{D} are called contravariant.

Example: For every locally small category \mathfrak{C} the map $\mathfrak{C}(\cdot, \cdot) : \mathfrak{C}^{op} \times \mathfrak{C} \rightarrow \mathbf{Set}$ defined by $(X, Y) \mapsto \mathfrak{C}(X, Y)$ and $(f : W \rightarrow X, g : Y \rightarrow Z) \mapsto (\mathfrak{C}(X, Y) \ni h \mapsto g \circ h \circ f \in \mathfrak{C}(W, Z))$ is a functor into the category of all sets. It is called the morphism functor on \mathfrak{C} .

1.2.1 Natural Transformations

Let $F, G : \mathfrak{C} \rightarrow \mathfrak{D}$ be functors. A map $\alpha : \mathfrak{C} \rightarrow \mathfrak{D}$ is called a natural transformation from F to G and is denoted by $\alpha : F \rightarrow G$, if the existence of fg implies the existence of $\alpha(f)F(g)$ and $G(f)\alpha(g)$ as well as the equalities $\alpha(fg) = \alpha(f)F(g) = G(f)\alpha(g)$. Natural transformations are uniquely determined by their values on objects. Indeed, every map $\alpha : \text{Ob}(\mathfrak{C}) \rightarrow \mathfrak{D}$ with the property that $f : C \rightarrow C'$ implies the existence of $\alpha(C')F(f)$, $G(f)\alpha(C)$ and the equality $\alpha(C')F(f) = G(f)\alpha(C)$ can be extended to a unique natural transformation by $\alpha(f) := \alpha(C')F(f) = G(f)\alpha(C)$.

Let $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$ be natural transformations, then define a new natural transformation $\beta \cdot \alpha : F \rightarrow H$ by $(\beta \cdot \alpha)(f) := \beta(C')\alpha(f) = \beta(f)\alpha(C)$ for a morphism $f : C \rightarrow C'$ in \mathfrak{C} . Instead of $\beta \cdot \alpha$ shortly write $\beta\alpha$. The class of all natural transformations between functors from \mathfrak{C} to \mathfrak{D} endowed with the partial multiplication \cdot is again a category. It shall be called the category of functors from \mathfrak{C} to \mathfrak{D} and is denoted by $\mathfrak{D}^{\mathfrak{C}}$. The units of this category are exactly the natural transformations of the form $G : G \rightarrow G, g \mapsto G(g)$ with a functor G . Thus the units can be identified with functors, and hence the name category of functors from \mathfrak{C} to \mathfrak{D} is justified.

A natural transformation $\alpha : F \rightarrow G$ in this category is an isomorphism iff every $\alpha(C) : F(C) \rightarrow G(C), C \in \text{Ob}(\mathfrak{C})$, is an isomorphism in \mathfrak{D} . Functors F and G are called naturally isomorphic if there is such an isomorphism $\alpha : F \rightarrow G$. Equivalence classes of functors w.r.t. the equivalence relation given by natural isomorphism are said to be functors determined merely up to natural isomorphism. The image $F(C)$ of an object C is not well-defined for an up to natural isomorphism determined functor F , but its the equivalence class $[F(C)]$ w.r.t. isomorphism in \mathfrak{D} is well-defined.

1.2.2 Adjoint Functors

A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is said to be left adjoint to the functor $G : \mathfrak{D} \rightarrow \mathfrak{C}$, and conversely G is said to be right adjoint to F , if there is a natural transformation $\eta : \text{Id}_{\mathfrak{C}} \rightarrow G \circ F$, called the unit of the adjunction, and a natural transformation $\epsilon : F \circ G \rightarrow \text{Id}_{\mathfrak{D}}$, called the counit of the adjunction, such that $(G \circ \epsilon)(\eta \circ G) = G$ and $(\epsilon \circ F)(F \circ \eta) = F$ are valid. Further, if the natural transformations η and ϵ are isomorphisms, then $F : \mathfrak{C} \leftrightarrow \mathfrak{D} : G$ is called an equivalence between the categories \mathfrak{C} and \mathfrak{D} , which are called equivalent in this case.

To better understand adjunction and equivalence, note that for locally small categories a functor F is left adjoint to G iff there is a natural isomorphism $\mathfrak{D}(F(\cdot), \cdot) \cong \mathfrak{C}(\cdot, G(\cdot))$. Thus for adjoint functors F, G morphisms from $F(C)$ to D correspond uniquely to morphisms from C to $G(D)$ in a way that respects composition. Hence a left adjoint F to G is something like a weak left inverse to G .

1.3 Limits

Let \mathcal{I} be a category and let $F : \mathcal{I} \rightarrow \mathcal{C}$ be a functor. Call \mathcal{I} an index category, denote objects in \mathcal{I} by i, j, \dots and morphisms from i to j by ϕ_{ij} . A pair consisting of an object $L \in \text{Ob}(\mathfrak{C})$ and morphisms $f_i : L \rightarrow F(i)$ is called the limit of F , if $F(\phi_{ij}) \circ f_i = f_j$ holds and if to every other pair (L', f'_i) with $F(\phi_{ij}) \circ f'_i = f'_j$ there is a unique map $h : L' \rightarrow L$ so that $f'_i = f_i \circ h$ factors. More formally limits can be defined as terminal objects in some diagram category, and this equivalent definition shows that a limit (L, f_i) is unique up to diagram isomorphy. Further the notion of a limit can be dualized, and the result is called a colimit.

Product and Coproduct Let $\mathcal{I} = \cdot \cdot$ consist of two objects and no nontrivial morphisms. Then a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ merely chooses two objects A, B of \mathcal{C} . The limit of F is called the product and is denoted by $A \times B$, while the morphisms are called projections and are denoted by $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$. The dual of the notion of a product is a coproduct, it is denoted by $A + B$ and the morphisms $\iota_1 : A \rightarrow A + B$, $\iota_2 : B \rightarrow A + B$ are called inclusions.

Pullback and Pushout Let $\mathcal{I} = \cdot \rightarrow \cdot \leftarrow \cdot$ consist of three objects and two nontrivial morphisms. A functor $F : \mathcal{I} \rightarrow \mathcal{C}$ merely renders this form by choosing morphisms f, g of the form $A \xrightarrow{f} C \xleftarrow{g} B$. The limit of F is denoted by $A_f \times_g B$ and is called the pullback, while the morphisms again are called projection and are denoted¹ by $\pi_1 : A_f \times_g B \rightarrow A$ and $\pi_2 : A_f \times_g B \rightarrow B$. The dual of a pullback is a pushout.

Direct Limits Let \mathcal{I} be a partially ordered set which is directed upwards, i.e. to every i, j there is a k with $i, j \leq k$. Consider \mathcal{I} as a category, then a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ chooses an upward directed family $C_i \rightarrow C_j$, $i \leq j$, of objects in \mathcal{C} . The colimit of F is called the direct limit of the C_i (ATTENTION: The direct limit is in fact a colimit, only historically it is called a limit) and is denoted by $\lim_i C_i$, while the morphisms $C_i \rightarrow \lim_i C_i$ are called inclusions. The dual of direct limits are the projective limits.

Preserving Limits A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is said to preserve the limits defined for the index category \mathcal{I} , if for every limit (L, f_i) in \mathcal{C} of a functor $F' : \mathcal{I} \rightarrow \mathcal{C}$ the pair $(F(L), F(f_i))$ is the limit of the functor $F \circ F'$ in \mathcal{D} . It can be proved that left adjoints preserve arbitrary colimits, while right adjoints preserve arbitrary limits.

¹Note that the third morphism $A_f \times_g B \rightarrow C$ does not need a special symbol because it is $f \circ \pi_1 = g \circ \pi_2$.

1.4 Forgetful Functors

A functor $V : \mathfrak{C} \rightarrow \mathfrak{D}$ is called faithful or a forgetful functor, if for every $X, Y \in \text{Ob}(\mathfrak{C})$ the map $V_{X,Y} : \mathfrak{C}(X, Y) \rightarrow \mathfrak{D}(V(X), V(Y))$ is injective. If V is a forgetful functor and a left adjoint $F : \mathfrak{D} \rightarrow \mathfrak{C}$ exists, then $F(D)$ is called the free object generated by D w.r.t. the forgetful functor.

For example, consider the forgetful functor $V : \mathbf{AbGrp} \rightarrow \mathbf{Set}$ which sends a homomorphism h of abelian groups to the map h , i.e. V does nothing. Then there is a left adjoint F to V , which sends a set X to the abelian group $G = \{\sum_{i=1}^m n_i x_i \mid n_i \in \mathbb{Z}, x_i \in X\}$ and a map $f : X \rightarrow Y$ to the homomorphism $\sum_{i=1}^m n_i x_i \mapsto \sum_{i=1}^m n_i f(x_i)$ of groups. Or consider the forgetful functor $V : \mathbf{Top} \rightarrow \mathbf{Set}$, whose left adjoint endows a set with the discrete topology while it leaves maps unchanged.

Initial and Final Objects Let $V : \mathfrak{C} \rightarrow \mathfrak{D}$ be a forgetful functor. Suppose that a morphism $g : V(X') \rightarrow V(X)$ of \mathfrak{D} lies in the image of $V_{X',X}$ if and only if for every i the morphism $g_i \circ g$ lies in the image of V_{X',X_i} . Then the object $X \in \text{Ob}(\mathfrak{C})$ is called an initial object to the family $g_i : V(X) \rightarrow V(X_i)$. Note that every g_i itself lies in the image of V_{X,X_i} because of $g_i \circ V(\text{Id}_X) = g_i \circ \text{Id}_{V(X)} = g_i$, so that there are unique morphisms $f_i : X \rightarrow X_i$ in \mathfrak{C} with $V(f_i) = g_i$.

Dually an object X is called a final object to the family $g_i : V(X_i) \rightarrow V(X)$ of morphisms, if it is equivalent that $g : V(X) \rightarrow V(X')$ lies in the image of $V_{X,X'}$ and that every $g \circ g_i$ lies in the image of $V_{X_i,X'}$.

Say that a category has initial resp. final objects w.r.t. the forgetful functor V , if to every family $g_i : Y \rightarrow V(X_i)$ resp. $g_i : V(X_i) \rightarrow Y$ of morphisms there is an initial resp. final object X to the family g_i that satisfies $V(X) = Y$. Such an object X is automatically unique up to isomorphism.

If a category has initial objects w.r.t. the forgetful functor $V : \mathfrak{C} \rightarrow \mathfrak{D}$ and if certain limits exist in \mathfrak{D} , then these limits also exist in \mathfrak{C} . Indeed, if $F : \mathcal{I} \rightarrow \mathfrak{D}$ is a functor and (Y, g_i) is the limit of $V \circ F$ in \mathfrak{D} , then the initial object X determined up to isomorphism by $V(X) = Y$ and the family $g_i = V(f_i)$ is the limit of F in \mathfrak{C} . An analogous result also holds for colimits.

1.5 Tensor Products

Let \mathfrak{C} be a category. A functor $\cdot \otimes \cdot : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ (which maybe is defined only up to natural isomorphism) is called a tensor product on \mathfrak{C} if the functors $\cdot \otimes (\cdot \otimes \cdot)$ and $(\cdot \otimes \cdot) \otimes \cdot$ are naturally isomorphic. In other words, \otimes is called a tensor product if it is associative in the sense that $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ holds naturally in A, B, C . An object E with $E \otimes A \cong A \cong A \otimes E$ naturally in A is called a unit of the tensor product. Units of tensor products are unique up to isomorphism because of $E \cong E \otimes E' \cong E'$ for two units E, E' , and a morphism $E \rightarrow C$ is called a tensor point of C . Further a tensor product is called symmetric if there is a natural isomorphism $A \otimes B \cong B \otimes A$.

A pair consisting of a category \mathfrak{C} and a symmetric tensor product \otimes having a unit is called a tensor category. A tensor category is called tensorial closed if there is a right adjoint functor $\cdot : \mathfrak{C}^{op} \times \mathfrak{C} \rightarrow \mathfrak{C}$ (which maybe is defined only up to natural isomorphism) to the tensor product \otimes . Then the functor \cdot is called the morphism functor² and the image B^A of objects (A, B) is called the object of morphisms from A to B . This notion is justified because of the natural bijections

$$\mathfrak{C}(A, C^B) \cong \mathfrak{C}(A \otimes B, C)$$

so that especially the tensor points of B^A correspond to the morphisms from A to B .

Let \mathfrak{C} be a tensorial closed category, then the universal property of the counit $\text{ev}_{Y,X} : Y^X \otimes X \rightarrow Y$ (which is here called evaluation) assures the existence of a natural transformation $\text{comp}_{Z,Y,X} : Y^X \otimes Z^Y \rightarrow Z^X$ (called inner composition) satisfying

$$\epsilon_{Z,X} \circ (\text{comp}_{Z,Y,X} \otimes \text{Id}_X) = \epsilon_{Z,Y} \circ (\text{Id}_{Z^Y} \otimes \epsilon_{Y,X}) \quad .$$

Note that comp mimics the (outer) partial multiplication of the category within the category, as $\text{comp} \circ (f', g') = (f \circ g)'$ holds, where $h' : E \rightarrow Y^X$ denotes the tensor point corresponding to the morphism $h : X \rightarrow Y$. Indeed, if the counit $\text{ev}_{Y,X}$ is written as if it were the evaluation $(f, x) \mapsto f(x)$ of maps in Y^X , then the equation defining comp can be written as $\text{comp}(g, f)(x) = f(g(x))$. Moreover it can be proven that comp is associative.

1.5.1 The Product as a Tensor Product

Let a category \mathfrak{C} have finite products, then \times is a symmetric tensor product with the terminal object 1 being the unit, and thus (\mathfrak{C}, \times) is a tensor category. Such tensor categories are called cartesian, and if the morphism functor exists, the category is called cartesian closed instead of tensorial closed. Further the tensor points of an object C , i.e. the morphisms from 1 to C , are called merely points instead of tensor points.

Example: *The category \mathbf{Set} is cartesian closed. The morphism functor is nothing else than $\mathbf{Set}(\cdot, \cdot)$, as $\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$ naturally via the natural isomorphism $f(a, b) \mapsto (a \mapsto (b \mapsto f(b, a)))$. Because the terminal object 1 in \mathbf{Set} is the set consisting of one element, the points of a set correspond to the elements of the set.*

²Note that also the functor $\mathfrak{C}(\cdot, \cdot)$ from $\mathfrak{C}^{op} \times \mathfrak{C}$ to \mathbf{Set} is called the morphism functor, but usually this ambiguity does not cause problems, as B^A is mostly the set $\mathfrak{C}(A, B)$ endowed with a structure so that it becomes an object of \mathfrak{C} .

Example: The category \mathfrak{Top} of topological spaces and continuous maps endowed with the product \times is not cartesian closed. To obtain a cartesian closed category of spaces there are two possibilities: On the one hand the notion of a topological structure can be weakened, and on the other hand only special topological spaces can be allowed.

Indeed, if the notion of a space is weakened by using a convergence relation between filters and points instead of open sets, then the so defined category of convergence spaces (and also the stronger category of limit spaces) is cartesian closed. The space Y^X of morphisms from X to Y is nothing else than the set $C(X, Y)$ of continuous functions from X to Y , endowed with the convergences $\mathcal{F} \rightarrow f$ of filters \mathcal{F} on $C(X, Y)$ to functions $f \in C(X, Y)$ iff for every convergence $\mathcal{G} \rightarrow x$ on X the image filter $\text{ev}(\mathcal{F} \times \mathcal{G})$ under the evaluation map ev converges to $f(x)$. Further for separated locally compact topological spaces X and arbitrary topological spaces Y the convergence space Y^X is even a topological space, namely $C(X, Y)$ endowed with the compact-open topology.

On the other hand it is possible to consider certain subclasses of topological spaces to obtain a cartesian closed category. For example, the compactly generated spaces X are such a class. They are defined to be those separated topological spaces for which a set $A \subset X$ is closed iff $A \cap K$ is closed for every compact set $K \subset X$. In this category a product $X \times Y$ is not the usual product of topological spaces but its „Kelley-fication“, and a space Y^X of morphisms is the „Kelley-fication“ of the set $C(X, Y)$ endowed with the compact-open topology.

Example: In the category \mathfrak{Cat} of all categories the functor $\cdot : \mathfrak{Cat}^{op} \times \mathfrak{Cat} \rightarrow \mathfrak{Cat}$ given by $(\mathfrak{C}, \mathfrak{D}) \mapsto \mathfrak{D}^{\mathfrak{C}}$, $(F : \mathfrak{C}' \rightarrow \mathfrak{C}, G : \mathfrak{D} \rightarrow \mathfrak{D}') \mapsto (\mathfrak{D}^{\mathfrak{C}} \ni \alpha \mapsto G \circ \alpha \circ F \in \mathfrak{D}'^{\mathfrak{C}'})$ is right adjoint to $\times : \mathfrak{Cat} \times \mathfrak{Cat} \rightarrow \mathfrak{Cat}$.

In fact, if $F : \mathfrak{C} \times \mathfrak{D} \rightarrow \mathfrak{E}$ is a functor, then $F(ff', gg') = F(f, g)F(f', g')$ holds and especially for $f : C \rightarrow C'$ also $F(f, gg') = F(f, g)F(C, g') = F(C', g)F(f, g')$ is valid. Thus $\alpha_f : g \mapsto F(f, g)$ is a natural transformation between the functors $g \mapsto F(C, g)$ and $g \mapsto F(C', g)$ from \mathfrak{D} to \mathfrak{E} . Hence $\hat{F} : f \mapsto \alpha_f$ is a functor from \mathfrak{C} to $\mathfrak{E}^{\mathfrak{D}}$ because of

$$\begin{aligned} \hat{F}(ff')(g) &= F(ff', g) = F(f, D')F(f', g) = F(f, D')\alpha_{f'}(g) = \\ &= \alpha_f(D')\alpha_{f'}(g) = (\alpha_f \cdot \alpha_{f'})(g) = (\hat{F}(f) \cdot \hat{F}(f'))(g) \end{aligned}$$

for every $g : D \rightarrow D'$, i.e. $\hat{F}(ff') = \hat{F}(f) \cdot \hat{F}(f')$ is valid.

Conversely to a functor $H : \mathfrak{C} \rightarrow \mathfrak{E}^{\mathfrak{D}}$ the map defined by $\tilde{H} : (f, g) \mapsto H(f)(g)$ is a functor from $\mathfrak{C} \times \mathfrak{D}$ into \mathfrak{E} . Because of $\hat{F} = F$ and $\hat{H} = H$ there is a bijection $\mathfrak{Cat}(\mathfrak{C} \times \mathfrak{D}, \mathfrak{E}) \cong \mathfrak{Cat}(\mathfrak{C}, \mathfrak{E}^{\mathfrak{D}})$. This bijection is natural and thus proves that \cdot is right adjoint to \times . In retrospect this justifies to use the symbol $\mathfrak{D}^{\mathfrak{C}}$ for the category of functors from \mathfrak{C} to \mathfrak{D} . Indeed, it is the object of morphisms from \mathfrak{C} to \mathfrak{D} in the tensor category (\mathfrak{Cat}, \times) .

1.5.2 The Pointwise Tensor Product

If \mathfrak{C} is a concrete category, i.e. there is a forgetful functor V into \mathfrak{Set} , then the pointwise evaluation $\text{ev}_y : \mathfrak{C}(Y, Z) \rightarrow V(Z)$, $h \mapsto V(h)(y)$, is defined for every $y \in V(Y)$. If there are initial objects Z^Y to the family ev_y ³, then up to natural isomorphy they define a bifunctor \cdot from $\mathfrak{C}^{op} \times \mathfrak{C}$ to \mathfrak{C} acting on morphisms by $(h^f)(g) := h \circ g \circ f$. Now, if there is a left adjoint \otimes to this bifunctor \cdot , then it is a tensor product⁴ and shall be called the pointwise tensor product. Further, if the pointwise tensor product has a unit and is symmetric, then trivially (\mathfrak{C}, \otimes) is tensorial closed.

Example: *In the category \mathfrak{Set} of maps between sets the pointwise tensor product exists and is identical with the usual product. In fact, trivially a map $g : A \rightarrow \mathfrak{Set}(B, C)$ is a map, if each $\text{ev}_b \circ g$ is a map for $b \in B$. Thus $\cdot := \mathfrak{Set}(\cdot, \cdot)$ is the bifunctor induced by the pointwise evaluations, and clearly this bifunctor is right adjoint to the usual product \times of sets.*

Example: *In the category \mathfrak{Top} of topological spaces and continuous maps the pointwise evaluations induce on $C(Y, Z)$ the pointwise convergence of maps. The tensor product exists, it assigns to topological spaces the set $X \times Y$ endowed with the final topology generated by the maps $\iota_y : x \mapsto (x, y)$ and $\iota_x : y \mapsto (x, y)$.*

Indeed, for three topological spaces X, Y, Z a map $\tilde{f} : X \rightarrow \mathfrak{Set}(Y, Z)$ has its image in the subset $C(Y, Z) \subset \mathfrak{Set}(Y, Z)$ and is continuous w.r.t. pointwise convergence, if all the maps $x \mapsto \tilde{f}(x)(y)$ and $y \mapsto \tilde{f}(x)(y)$ are continuous. Thus the map $\theta_{X, Y, Z} : f(x, y) \mapsto (x \mapsto (y \mapsto f(x, y)))$ is a bijection between $C(X \otimes Y, Z)$ and $C(X, C(Y, Z))$, where $C(Y, Z)$ is endowed with the pointwise convergence. There remains to prove the naturality of θ : For continuous maps $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ and $h : Z \rightarrow Z'$ as well as $\alpha : X' \otimes Y' \rightarrow Z$ the equality

$$\begin{aligned} \theta(h \circ \alpha \circ f \otimes g) &= x \mapsto (y \mapsto h(\alpha(f(x), g(y)))) = \\ &= x \mapsto (h \circ \theta(\alpha)(f(x)) \circ g) = C(g, h) \circ \theta(\alpha) \circ f \end{aligned}$$

holds and thus θ defines a natural transformation.

Finally let us discuss the pointwise tensor product in the category of modules over commutative rings. Note that the tensorproduct of vector spaces (modules over fields) and of abelian groups (modules over \mathbb{Z}) is a subexample thereof.

³In other words, the set $\mathfrak{C}(Y, Z)$ can be made to a unique object Z^Y of \mathfrak{C} by requiring the universal property, that a map $g : V(X) \rightarrow \mathfrak{C}(Y, Z)$ is induced by a morphism f , i.e. $g = V(f)$, iff $\text{ev}_y \circ g : V(X) \rightarrow V(Z)$ is induced by a morphism for every $y \in V(Y)$.

⁴Indeed, associativity holds because of $\mathfrak{C}(A \otimes (B \otimes C), D) \cong \mathfrak{C}(A, \mathfrak{C}(B \otimes C, D)) \cong \mathfrak{C}(A, \mathfrak{C}(B, \mathfrak{C}(C, D))) \cong \mathfrak{C}(A \otimes B, \mathfrak{C}(C, D)) \cong \mathfrak{C}((A \otimes B) \otimes C, D)$, where no notational difference is made between sets of maps and the initial objects to such sets.

Example: The pointwise evaluation on $\mathfrak{Mod}_R(M, N) =: \text{Hom}(M, N)$ induce the pointwise addition $(\phi + \psi)(m) := \phi(m) + \psi(m)$ and pointwise scalar multiplication $(r\phi)(m) := r\phi(m)$ of homomorphisms. In this way $\text{Hom}(M, N)$ is turned into a module N^M which is initial w.r.t. the pointwise evaluations, and a bifunctor is defined $(M, N) \mapsto N^M$. To construct the pointwise tensor product \otimes , note that a map $f : L \times M \rightarrow N$ induces a homomorphism $l \mapsto (m \mapsto f(l, m))$ from L to N^M iff $f(l + l', m) = f(l, m) + f(l', m)$, $f(l, m + m') = f(l, m) + f(l, m')$ and $f(rl, m) = rf(l, m) = f(l, rm)$ are valid, i.e. iff f is bilinear. Thus, to have a natural isomorphy $\text{Hom}(L \otimes M, N) \cong \text{Hom}(L, N^M)$, a module $L \otimes M$ is needed which has the property that the bilinear maps from $L \times M$ into some N uniquely correspond to the linear maps from $L \otimes M$ to N in a natural way.

To construct such a module $L \otimes M$, use the free R -module $F = \bigoplus_{(l,m) \in L \times M} R$ generated by the set $L \times M$ and impose on it the relations that a bilinear map has to fulfill. This can be done by factoring out a submodule: Write elements of F as formal linear combinations $\sum_i r_i(l_i, m_i)$, form the submodule G of F generated by the elements

$$\begin{aligned} (l + l', m) - (l, m) - (l', m), & (l, m + m') - (l, m) - (l, m'), \\ (rl, m) - r(l, m), & (l, rm) - r(l, m) \end{aligned} \quad (1)$$

set $L \otimes M := F/G$ and denote elements $(l, m) + G$ of $L \otimes M$ by $l \otimes m$.

Then the module $L \otimes M$ really has the requested universal property: As F is the module freely generated by $L \times M$, every map $f : L \times M \rightarrow N$ into a R -module N uniquely extends to a homomorphism $\tilde{f} : F \rightarrow N$, and especially this holds for bilinear maps $f : L \times M \rightarrow N$. The extension \tilde{f} of such a bilinear map $f \in \text{Hom}(L, N^M)$ particularly satisfies $G \subset \text{Ker}(\tilde{f})$ because of

$$\begin{aligned} f(l + l', m) = f(l, m) + f(l', m), & f(l, m + m') = f(l, m) + f(l, m'), \\ f(rl, m) = rf(l, m) = f(l, rm) \end{aligned} \quad (2)$$

and thus also $\hat{f} : F/G \rightarrow N$, $w + G \mapsto \hat{f}(w)$, is well-defined. Now the assignment $\theta : \text{Hom}(L, N^M) \ni f \mapsto \hat{f} \in \text{Hom}(L \otimes M, N)$ is the natural bijection searched for. Indeed, if $\hat{f} = \hat{f}'$ for $f, f' \in \text{Hom}(L, N^M)$, then \hat{f} and \hat{f}' have the same values on the generating set $\{l \otimes m \in L \otimes M \mid (l, m) \in L \times M\}$ of the module $L \otimes M$. But f and f' contain G in their kernel, because they are homomorphisms from L to N^M or equivalently bilinear. Hence f and f' (viewed as bilinear maps from $L \times M$ to N) have the same values on $L \times M$, i.e. they define the same map. This proves the injectivity of the map θ . To show its surjectivity, observe that a homomorphism $\hat{f} : L \otimes M \rightarrow N$ induces by $f(l, m) := \hat{f}(l \otimes m)$ a map on $L \times M$, which satisfies (2) because the elements of (1) lie in G , and thus f is a bilinear map or equivalently a homomorphism from L to N^M . Hence θ is bijective. Finally define the map $\tau : L \times M \rightarrow L \otimes M$ by $(l, m) \mapsto l \otimes m$ and note that $\theta^{-1}(\hat{f}) = f \circ \tau$ holds. Thus the bijection θ is really natural, as it is defined by the universal property that to every bilinear map $f : L \times M \rightarrow N$ there is a unique linear map $\hat{f} : L \otimes M \rightarrow N$ with $f = \hat{f} \circ \tau$.

2 Topological Structures

In analysis a precise notion of convergence is needed. But what kind of objects on a set X are those which can converge? The topological answer to this question is that filters on X are the objects which can converge. Thereby a proper filter \mathcal{F} on X is a set of subsets of X such that

- $U \in \mathcal{F}$ and $U \subset V$ imply $V \in \mathcal{F}$,
- $U, V \in \mathcal{F}$ imply $U \cap V \in \mathcal{F}$,
- $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$

hold. While the third condition only eliminates the improper filters \emptyset and $\mathcal{P}(X)$, the first and second condition assure that the sets in a filter are directed downwards. This downward directedness of a filter models the intuitive notion of a converging (or contracting) family of sets and justifies why filters are regarded as the right objects which can converge.

Let us gather some facts about filters. Filters can be defined by filter bases, i.e. sets \mathcal{FB} of subsets of X with the property that to every $U, V \in \mathcal{FB}$ there is a $W \in \mathcal{FB}$ with $W \subset U \cap V$. Indeed, for a filter base \mathcal{FB} on X the set of all $U \subset X$ for which there is a $V \in \mathcal{FB}$ with $V \subset U$ forms a filter \mathcal{F} , called the filter generated by the filter base. A special case of a filter base are the endpieces $\{\{x_n | n \geq N\} | N \in \mathbb{N}\}$ of a sequence x_n . Thus every sequence induces a filter, and only this filter is important for the convergence of the sequence. For example, the sequences x_n and x_{n+k} , $k \in \mathbb{N}$ fixed, are generally different, but induce the same filters and have the same convergence properties.

Further the set of all filters is partially ordered by inclusion. A filter \mathcal{F} is called coarser than \mathcal{G} (and \mathcal{G} is called finer than \mathcal{F}) if $\mathcal{F} \subset \mathcal{G}$ holds. Clearly the improper filters \emptyset resp. $\mathcal{P}(X)$ are the finest resp. coarsest filters. But in topology only proper filters are considered, and while $\{X\}$ is the coarsest proper filter, the finest proper filters are called ultrafilters. Trivially for every $x \in X$ the filter $\mathcal{F}_x := \{U \subset X | x \in U\}$ is an ultrafilter, but the lemma of Zorn guarantees that to every filter \mathcal{F} an ultrafilter \mathcal{G} containing \mathcal{F} can be found. Hence generally there also exist nontrivial ultrafilters. For two proper filters \mathcal{F}, \mathcal{G} , the infimum always exists and is the filter $\mathcal{F} \cap \mathcal{G}$, while the supremum exists iff $F \cap G \neq \emptyset$ holds for every $F \in \mathcal{F}, G \in \mathcal{G}$, and in this case it equals $\{F \cap G | F \in \mathcal{F}, G \in \mathcal{G}\}$.

Filters can also be transported by maps. Every map $f : X \rightarrow Y$ between sets X, Y induces a map of the corresponding filter sets by restricting the map $(f^{-1})^{-1}$ between the double power sets $\mathcal{P}(\mathcal{P}(X))$ and $\mathcal{P}(\mathcal{P}(Y))$ to the set of all filters on X . The image of a filter \mathcal{F} on X under this map is shortly denoted by $f(\mathcal{F})$ and given by $\{V \subset Y | f^{-1}(V) \in \mathcal{F}\}$ (or equivalently generated by the filter base $\{f(U) | U \in \mathcal{F}\}$). It has the properties that proper filters are mapped to proper filters, that $f(\mathcal{F}_x) = \mathcal{F}_{f(x)}$ holds, that $\mathcal{F} \subset \mathcal{G}$ implies $f(\mathcal{F}) \subset f(\mathcal{G})$ and that $f(\mathcal{F} \cap \mathcal{G}) = f(\mathcal{F}) \cap f(\mathcal{G})$ is valid even for arbitrary intersections. Also the preimage of a proper filter \mathcal{G} on Y under $(f^{-1})^{-1}$

shortly denoted by $f^{-1}(\mathcal{G})$ is defined, but it is proper again if and only if $f^{-1}(V) \neq \emptyset$ holds for all $V \in \mathcal{G}$, and in this case it is generated by the filter base $\{f^{-1}(V) | V \in \mathcal{G}\}$.

Till now proper filters and their properties have been discussed. However, analysis needs a precise notion for the convergence of a proper filter to a point. A relation \rightarrow between proper filters on X and elements of X specifies such a notion and is called a convergence relation. For a fixed convergence relation \rightarrow on X , a filter \mathcal{F} is called convergent to $x \in X$, if the relation $\mathcal{F} \rightarrow x$ holds. A convergence relation \rightarrow is called finer than \rightarrow' (and \rightarrow' coarser than \rightarrow) if $\mathcal{F} \rightarrow x$ implies $\mathcal{F} \rightarrow' x$. Clearly this notion defines a partial order on the set of all convergence relations, which is merely the usual order of relations given by \subset , and thus arbitrary suprema and infima exist.

With this concept of convergence, a map $f : X \rightarrow Y$ between sets X, Y endowed with convergence relations $\rightarrow_X, \rightarrow_Y$ is called continuous at $x \in X$, if $\mathcal{F} \rightarrow_X x$ implies $f(\mathcal{F}) \rightarrow_Y f(x)$, and continuous, if it is continuous at every $x \in X$. Trivially the composition $f \circ g$ of continuous maps f, g is again continuous.

However, to get a convenient notion of convergence and continuity, \rightarrow is required to have certain properties. If such properties are formulated by axioms, a convergence relation on a set X satisfying these axioms is called a topological structure on X . Different axioms define different sorts of topological structures, and the properties, which such different topological structures have, are discussed in the following paragraphs.

2.1 Convergence Spaces

The weakest useful topological structure on a set X is given by a convergence relation satisfying

- $\mathcal{F} \rightarrow x$ and $\mathcal{F} \subset \mathcal{G}$ imply $\mathcal{G} \rightarrow x$,
- $\mathcal{F}_x \rightarrow x$ for all $x \in X$.

A set X endowed with such a convergence relation is called a convergence space.

The forgetful functor from the category of convergence spaces to the category of spaces endowed with a convergence relation has a left adjoint. Indeed, the left adjoint assigns to set X endowed with a convergence relation \rightarrow the finest coarser convergence relation \rightarrow' which turns X into a convergence space. Note that \rightarrow' is defined by the convergences $\mathcal{F} \rightarrow' x$ of the original convergence relation, by the convergences $\mathcal{F}_x \rightarrow' x$ and by the convergences of all finer filters according to the first property above. Because maps preserve trivial ultrafilters and the order, this convergence relation \rightarrow' has the property that a map $f : X \rightarrow Y$ from (X, \rightarrow) into a convergence space Y is continuous iff it is continuous as a map from (X, \rightarrow') into Y , and thus left adjointness has been proved. Hence every set endowed with a convergence relation can be made to a convergence space in a natural way.

The category of continuous maps between convergence spaces has initial and final objects w.r.t. the forgetful functor into the category of sets. Indeed, the initial convergence

relation to a family $f_i : X \rightarrow X_i$ of maps from a set X into convergence spaces X_i is the finest convergence relation on X such that all f_i are continuous. It is given by $\mathcal{F} \rightarrow x$ iff $f_i(\mathcal{F}) \rightarrow f_i(x)$ holds for every i , and thus a map $f : X' \rightarrow X$ from a convergence space X' to X is continuous, iff all maps $f_i \circ f$ are continuous. By analogy the final convergence relation to a family $f_i : X_i \rightarrow X$ of maps from convergence spaces X_i into a set X is the coarsest convergence relation on X such that the maps f_i are continuous. It is generated by the convergences $f_i(\mathcal{F}_i) \rightarrow f_i(x_i)$ where $\mathcal{F}_i \rightarrow x_i$ are arbitrary converging filters in X_i . Thus a map $f : X \rightarrow X'$ in a convergence space X' is continuous iff $f \circ f_i$ is continuous for every i . The existence of initial and final objects implies the existence of limits and colimits within the category of convergence spaces, see 1.4.

Moreover, the category of convergence spaces is cartesian closed. To see this, let us first consider the product. Up to isomorphy it is given by the set $\prod_i X_i$ endowed with the initial convergence relation w.r.t. the projections $\pi_i : \prod_i X_i \rightarrow X_i$. With the product $\prod_i \mathcal{F}_i$ of filters \mathcal{F}_i on X_i defined to be the coarsest filter \mathcal{F} on $\prod_i X_i$ with $\pi_i(\mathcal{F}) = \mathcal{F}_i$ for every i , the convergence relation on $\prod_i X_i$ is just the one generated by $(\prod_i \mathcal{F}_i) \rightarrow (x_i)$ to arbitrary converging filters $\mathcal{F}_i \rightarrow x_i$. Now let Y, Z be convergence spaces and define the convergence space Z^Y as the set $C(Y, Z)$ of continuous maps from Y to Z endowed with the convergence relation given by $\mathcal{F} \rightarrow f$ iff the image of $\mathcal{F} \times \mathcal{G}$ under the evaluation $\text{ev} : C(Y, Z) \times Y \rightarrow Z$ converges to $f(y)$ for every filter $\mathcal{G} \rightarrow y$ on Y , i.e. $\text{ev}(\mathcal{F} \times \mathcal{G}) \rightarrow f(y)$. This convergence relation really endows $C(Y, Z)$ with the structure of a convergence space:

- If $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F} \rightarrow f$, then for every $\mathcal{G} \rightarrow y$ the convergence $\text{ev}(\mathcal{F} \times \mathcal{G}) \rightarrow f(y)$ holds, and because of $\text{ev}(\mathcal{F} \times \mathcal{G}) \subset \text{ev}(\mathcal{F}' \times \mathcal{G})$ also $\text{ev}(\mathcal{F}' \times \mathcal{G}) \rightarrow f(y)$ and thus $\mathcal{F}' \rightarrow f$ is valid.
- The equation $\text{ev}(\mathcal{F}_f \times \mathcal{G}) = f(\mathcal{G})$ is valid for every $\mathcal{G} \rightarrow y$, and thus continuity $f(\mathcal{G}) \rightarrow f(y)$ of f implies $\text{ev}(\mathcal{F}_f \times \mathcal{G}) \rightarrow f(y)$, i.e. $\mathcal{F}_f \rightarrow f$.

A map $f : X \times Y \rightarrow Z$ is continuous iff $\hat{f} : x \mapsto (y \mapsto f(x, y)) : X \rightarrow Z^Y$ is continuous. Indeed, the equation $f = \text{ev} \circ (\hat{f} \times \text{Id}_Y)$ holds, and thus on the one hand the continuity of \hat{f} implies the continuity of f , because ev and Id_Y are continuous. On the other hand for a continuous f also every map $f_x := f(x, \cdot)$ is continuous, and thus \hat{f} really maps X into $C(Y, Z)$. Further \hat{f} itself is continuous, because by the definition of the convergence relation on $C(Y, Z)$ a map g from a space X into Z^Y is continuous iff $\text{ev} \circ (g \times \text{Id}_Y)$ is continuous. Hence $g := \hat{f}$ and the equation $f = \text{ev} \circ (\hat{f} \times \text{Id}_Y)$ yield that the continuity of f implies the continuity of \hat{f} .

2.2 Limit Spaces

A convergence space X is called a limit space, if it additionally has the property that $\mathcal{F} \rightarrow x$ and $\mathcal{G} \rightarrow x$ imply $\mathcal{F} \cap \mathcal{G} \rightarrow x$. Especially a filter \mathcal{F} in a limit space converges to x iff the coarser filter $\mathcal{F} \cap \mathcal{F}_x$ consisting of the sets $U \in \mathcal{F}$ with $x \in U$ converges to x . Thus, for testing continuity in x , it is sufficient to consider only filters whose sets contain x . Again there is a left adjoint to the forgetful functor from the category of

limit spaces to the category of convergence spaces, given by the convergences $\mathcal{F} \rightarrow' x$ whenever there are filters \mathcal{F}_i , $i = 1, \dots, n$, with $\mathcal{F}_i \rightarrow x$ and $\mathcal{F} = \bigcap_{i=1}^n \mathcal{F}_i$. Thus to every convergence space (X, \rightarrow) there is naturally a associated limit space (X, \rightarrow') .

The category of limit spaces is cartesian closed, because for a limit space Z the convergence space Z^Y is automatically a limit space. Indeed, let $\mathcal{F} \rightarrow f$ and $\mathcal{F}' \rightarrow f$, then for every $\mathcal{G} \rightarrow y$ the convergences $\text{ev}(\mathcal{F} \times \mathcal{G}) \rightarrow f(y)$ and $\text{ev}(\mathcal{F}' \times \mathcal{G}) \rightarrow f(y)$ hold. As Z is a limit space, also $\text{ev}(\mathcal{F} \times \mathcal{G}) \cap \text{ev}(\mathcal{F}' \times \mathcal{G}) \rightarrow f(y)$ is valid. Hence $\text{ev}(\mathcal{F} \cap \mathcal{F}') \times \mathcal{G} \rightarrow f(y)$ holds because of $\text{ev}(\mathcal{F} \times \mathcal{G}) \cap \text{ev}(\mathcal{F}' \times \mathcal{G}) = \text{ev}(\mathcal{F} \times \mathcal{G}) \cap (\mathcal{F}' \times \mathcal{G})$ (preimages preserve intersections) and $(\mathcal{F} \times \mathcal{G}) \cap (\mathcal{F}' \times \mathcal{G}) = (\mathcal{F} \cap \mathcal{F}') \times \mathcal{G}$.

2.3 Pretopological Spaces

If a limit space X has the additional property that the convergence of an arbitrary family of filters $\mathcal{F}_i \rightarrow x$ implies $\bigcap_i \mathcal{F}_i \rightarrow x$, then X is called a pretopological space. In a pretopological space a filter \mathcal{F} converges to x if it is finer than the neighbourhood filter $\mathcal{U}(x)$ defined by $\mathcal{U}(x) := \bigcap_{\mathcal{F} \rightarrow x} \mathcal{F}$. Sets $U \in \mathcal{U}(x)$ are called neighbourhoods of x and especially neighbourhoods of x always contain x . A pretopological space can be defined much easier than a limit space, as only to every point x a filter $\mathcal{U}(x)$ of sets containing x must be specified. And also continuity of a map $f : X \rightarrow Y$ between pretopological spaces X, Y can be tested much easier, as only $f(\mathcal{U}_X(x))$ has to be finer than $\mathcal{U}_Y(f(x))$ for every x . Equivalently f is continuous iff to every neighbourhood V of $f(x)$ in Y the preimage $f^{-1}(V)$ is a neighbourhood of x in X , or iff to every neighbourhood V of $f(x)$ in Y there is a neighbourhood U of x in X with $f(U) \subset V$.

Again to every limit space (X, \rightarrow) there is a naturally associated pretopological space $(X, \mathcal{U}(x))$, given by the neighbourhood filters $\mathcal{U}(x) := \bigcap_{\mathcal{F} \rightarrow x} \mathcal{F}$. But contrary to the category of convergence or limit spaces, the category of pretopological spaces is not cartesian closed, see e.g. [Sioen]. Observe also that the argument used to prove the cartesian closedness of the category of limit spaces cannot be generalized to pretopological spaces, as $(\mathcal{F} \times \mathcal{G}) \cap (\mathcal{F}' \times \mathcal{G}) = (\mathcal{F} \cap \mathcal{F}') \times \mathcal{G}$ is not valid for arbitrary instead of finite intersections. Indeed, $G \cap G'$ is an element of the filter \mathcal{G} , but not an arbitrary intersection $\bigcap_i G_i$.

2.4 Topological Spaces

A pretopological space X is called topological if every neighbourhood U of x contains a neighbourhood U' of x such that U is a neighbourhood of every point $y \in U'$. In a topological space a set $U \subset X$ is called open if it is a neighbourhood of all its points $x \in U$, and complements of open sets are called closed. Instead of specifying a neighbourhood filter for each point, a topological space can also be defined by specifying a system of open sets $U \subset X$ having the properties, that \emptyset and X are open, that for open U, U' also $U \cap U'$ is open and that for open U_i also $\bigcup_i U_i$ is open. Such a system \mathcal{T} of open sets is called a topology on X . Like filters, topologies can also be generated by bases, i.e. collections of sets U such that to every U, U' there is U'' with $U'' \subset U \cap U'$. If

a topology is given, the set X can be made uniquely to a topological space whose open sets are those contained in the given topology. For a topology \mathcal{T} the neighbourhood filters on X are given by $\mathcal{U}(x) := \{U' \mid \exists U \in \mathcal{T} : x \in U \subset U'\}$, so that a set U is a neighbourhood of x iff it has an open subset containing x . Further a map $f : X \rightarrow Y$ between topological spaces is continuous iff the preimages of open sets are open.

To every pretopological space X there again is a naturally associated topology, where those sets U are open which contain a neighbourhood of every $x \in U$. And like the category of pretopological spaces also the category of topological spaces is not cartesian closed, e.g. the exponent $\mathbb{R}^{\mathbb{R}^{\mathbb{N}}}$ does not exist as a topological space, see [Preuss] and the references therein. Instead it can be proved that the limit space Y^X is a topological space for every topological space Y iff X is core-compact, see [Escardo, Heckmann, Theorem 5.3]. In other words, exactly the core-compact topological spaces are exponentiable. Hereby X is called core-compact, iff every neighbourhood V of a point $x \in X$ contains a neighbourhood U of x so that every open covering of V contains a finite covering of U .

2.5 Properties of Spaces with Topological Structure

Hierarchy of topological structures The left adjoints to the forgetful functors between categories to different topological structures allow to use notions, which are defined for a stronger topological structure only, also in a space equipped with a weaker topological structure. For example, a set in a convergence space X is called open if it is open in the topological space naturally associated to X . Furthermore many topological notions can be formulated intrinsically in the category of convergence spaces. For example, a set $U \subset X$ in a convergence space is open w.r.t. the naturally associated topology iff $\mathcal{F} \rightarrow x$ implies $U \in \mathcal{F}$ for every $x \in U$. And an inner point x of a set A is nothing else than a point such that $\mathcal{F} \rightarrow x$ implies $A \in \mathcal{F}$.

Limit Points Let X be a convergence space. A point x is called a limit point of a filter \mathcal{F} if there is a finer filter which converges to x . Further x is called a limit point of a set $A \subset X$ if it is a limit point of the filter $\{U \subset X \mid A \subset U\}$ generated by A , or equivalently if there is a filter on A whose image under the inclusion $A \rightarrow X$ converges to x . The set of all limit points of A is called the closure of A and is denoted by \overline{A} . Instead of formulating axioms for \rightarrow , also axioms for $A \mapsto \overline{A}$ could be formulated to define different types of topological structures.

Definition of Limits Let X, Y be convergence spaces and let $f : X \rightarrow Y$ be a map. Instead of saying that for every filter $\mathcal{F} \rightarrow x$ on X the filter $f(\mathcal{F})$ converges to y , usually $\lim_{h \rightarrow x} f(h) = y$ is written. Thus $\lim_{h \rightarrow x} f(h) = f(x)$ is equivalent to the continuity of f at x . However, when $f : D \subset X \rightarrow Y$ is a map defined on a subspace D of X and $x \in X$ is a limit point of D but not contained in D , this symbolism is not sufficient to discuss the convergence of $f(h)$ for $h \rightarrow x$. To extend \lim in such a way, write $\lim_{D \ni h \rightarrow x} f(h) = y$ if x is a limit point of D and if the filter $f(\mathcal{F})$ converges to y for every filter \mathcal{F} on D with $\iota(\mathcal{F}) \rightarrow x$, where $\iota : D \rightarrow X$ denotes the inclusion and such a filter \mathcal{F} on D exists as x is a limit point of D .

Compactness A convergence space is called compact, if every filter has a finer convergent filter, or equivalently if every filter has a limit point. Every separated compact topological space is T_4 , see e.g. [Schubert, I.8.2, Satz 1].

A (pre-)topological space is called locally compact, if for every point the compact neighbourhoods form a base of the neighbourhood filter. A separated topological space is core-compact iff it is locally compact, see [Escardo, Heckmann, after Theorem 5.3]. The equivalence of core-compactness and exponentiability implies that for a separated topological space X and every topological space Y the limit space Y^X is a topological space iff X is locally compact. In this case the topology on Y^X is called the compact-open topology and is generated by the sets $\{f \in Y^X \mid f(K) \subset U\}$ with $K \subset X$ compact and $U \subset Y$ open.

A separated topological space X is called compactly generated if $A \subset X$ being closed is equivalent $A \cap K$ being closed for every compact $K \subset X$. Especially a separated X is compactly generated, if for every $M \subset X$ and every limit point x of M there is a compact set K such that x is a limit point of $M \cap K$. Thus separated locally compact spaces and first countable separated spaces are compactly generated. Contrary to the category \mathfrak{Top} of all topological spaces, the category \mathfrak{CGTop} of compactly generated separated topological spaces is cartesian closed. However, the product and the exponential are not the same as e.g. in the category of limit spaces and thus \mathfrak{CGTop} is not a cartesian closed subcategory of \mathfrak{Lim} , see [Steenrod]. More precisely denote by $k(\cdot)$ the Kelley-fication⁵, then the product in \mathfrak{CGTop} is the Kelley-fication $X \times_{cg} Y := k(X \times Y)$ of the usual product, and the space of maps Y^{cgZ} is the Kelley-fication of the set $C(Y, Z)$ endowed with the compact open topology. Then the natural isomorphism $C(X \times_{cg} Y, Z) \cong C(X, Y^{cgZ}) \cong C(X, k(C^{co}(Y, Z)))$ can be proven, and thus the category is cartesian closed.

Homeomorphisms While the multiplication \circ in the group $\text{Homeo}(X) \subset X^X$ of continuous and continuous invertible maps $f : X \rightarrow X$ is automatically continuous for a convergence or limit space X , the inversion $i : \text{Homeo}(X) \rightarrow \text{Homeo}(X)$ does not need to be continuous, see e.g. [Bourbaki, X.3, Exercise 17b]. But often a continuous inversion is needed, as then $\text{Homeo}(X)$ is a convergence or limit group. This can be forced by endowing $\text{Homeo}(X)$ with a finer convergence structure than the one induced by the inclusion $\iota : \text{Homeo}(X) \rightarrow X^X$. Clearly the coarsest convergence structure which makes $\text{Homeo}(X)$ to a convergence group is the initial w.r.t. the two maps $i, \iota : \text{Homeo}(X) \rightarrow X^X$.

The question arises under which conditions on a space X the inversion $i : \text{Homeo}(X) \rightarrow X^X$ is automatically continuous. For example, on a separated locally compact space X the inversion i is automatically continuous if X is locally connected. In this case the compact-open topology on $\text{Homeo}(X)$ and the initial topology w.r.t. $i, \text{Id} : \text{Homeo}(X) \rightarrow X^X$ are the same, see [Bourbaki, X.3, Exercise 17a].

⁵The Kelley-fication $k(\cdot)$ is nothing else than the right adjoint to the forgetful functor from \mathfrak{CGTop} to \mathfrak{Top} . It thus has the universal property that for a compact generated space X a map into Y is continuous iff it is continuous into $k(Y)$. The Kelley-fication endows a separated topological space X with the compactly generated topology where an $A \subset X$ is closed iff $A \cap C$ is closed for all compact $C \subset X$ in the original topology of X .

3 Uniform Structures

Topological structures on a set X allow to speak about convergence, but not about uniform convergence and related notions like Cauchy filters or completeness. That's why uniform structures are introduced in analysis. A uniform structure on a set X is a chosen set of proper filters on $X \times X$, and the elements \mathcal{U} of this chosen set of filters are called uniformities on X . Like convergence relations also uniform structures are ordered by \subset , have arbitrary infima and suprema, and a uniformity is called finer than another uniformity if it is a subset.

Every uniform structure induces a topological structure on X by setting $\mathcal{F} \rightarrow x$ whenever $\mathcal{F} \times \mathcal{F}_x$ is a uniformity. Note that $\mathcal{F} \times \mathcal{F}_x$ is the image of \mathcal{F} under the injection $\iota_x : X \rightarrow X \times X, x' \mapsto (x', x)$.

Further call a map $f : X \rightarrow Y$ between sets X, Y endowed with uniform structures uniformly continuous if for every uniformity \mathcal{U} on X the filter $(f \times f)(\mathcal{U})$ is a uniformity on Y . Uniformly continuous maps are obviously continuous w.r.t. the induced convergence relation, as $(f \times f)(\mathcal{F} \times \mathcal{F}_x) = f(\mathcal{F}) \times \mathcal{F}_{f(x)}$ is valid. Thus there is a forgetful functor from the category of sets endowed with a uniform structure to the category of sets endowed with a topological structure. Corresponding to the hierarchy of topological structures there is also a hierarchy of uniform structures which is discussed in the following paragraphs.

3.1 Pre- and Semiuniform Convergence Spaces

A set X endowed with a uniform structure is called a preuniform convergence space if

- a filter \mathcal{V} on $X \times X$, which is finer than a uniformity \mathcal{U} , is itself a uniformity,
- for every $x \in X$ the filter $\mathcal{F}_{(x,x)}$ is a uniformity.

Obviously the induced topological structure on a preuniform convergence space X is that of a convergence space.

The category of preuniform convergence spaces has initial and final objects w.r.t. the forgetful functor into the category of sets, and thus also arbitrary limits and colimits, see 1.4. It is further cartesian closed: The natural space Z^Y of maps is given by the set $C_{unif}(Y, Z)$ of uniformly continuous maps endowed with the uniform structure where a filter \mathcal{U} is a uniformity if for every uniformity \mathcal{V} on Y the image $(\text{ev} \times \text{ev})(\mathcal{U} \times \mathcal{V})$ under the evaluation $\text{ev} : (f, y) \mapsto f(y)$ is a uniformity on Z . The proof is an analogy to the proof that the category of convergence spaces is cartesian closed.

Note that the induced topology depends on the choice of the injection ι_x , because $x' \mapsto (x', x)$ as well as $x' \mapsto (x, x')$ could have been used to define the convergence relation. To eliminate this possibility of choice it seems useful not to consider preuniform convergence spaces, but semiuniform convergence spaces. These spaces are defined by additionally requiring the property that for a uniformity \mathcal{U} also the inverse $\mathcal{U}^{-1} := \{U^{-1} | U \in \mathcal{F}\}$ is a uniformity, where U^{-1} is defined by $U^{-1} := \{(y, x) | (x, y) \in U\}$. Such spaces still induce generally only convergence spaces, and again their category is cartesian closed.

3.2 Pre-, Semi- and Uniform Limit spaces

Let X be a preuniform resp. semiuniform convergence space, then X is called a preuniform resp. semiuniform limit space, if for uniformities \mathcal{U}, \mathcal{V} also the intersection $\mathcal{U} \cap \mathcal{V}$ is a uniformity. The topological structure induced by a pre- or semiuniform limit space is that of a limit space because of

$$(\mathcal{F} \cap \mathcal{G}) \times \mathcal{F}_x = (\mathcal{F} \times \mathcal{F}_x) \cap (\mathcal{G} \times \mathcal{F}_x) \quad .$$

Again the category of pre- or semiuniform limit spaces is cartesian closed.

Furthermore a semiuniform limit space is called a uniform limit space if for uniformities \mathcal{U}, \mathcal{V} the composition $\mathcal{U} \circ \mathcal{V}$ generated by the sets $U \circ V$ (where $U \in \mathcal{U}, V \in \mathcal{V}$ and $U \circ V := \{(x, z) | \exists y \in X : (x, y) \in U \text{ and } (y, z) \in V\}$) is also a uniformity, unless this composition $\mathcal{U} \circ \mathcal{V}$ is not the improper filter $\mathcal{P}(X \times X)$ ⁶. The topological structure induced by a uniform limit space is still generally only that of a limit space, and again the category formed by those spaces is cartesian closed.

3.3 Pre- and Semiuniform Spaces

If a preuniform resp. semiuniform limit space X has the property that arbitrary intersections of uniformities are uniformities, then X is simply called a preuniform resp. semiuniform space. Let \mathcal{U} be the intersection over all uniformities on X , then a filter on $X \times X$ is a uniformity on X if it is finer than \mathcal{U} . The elements $U \in \mathcal{U}$ are called entourages and they automatically contain the diagonal $\Delta := \{(x, x) | x \in X\}$. Thus a set X can be easily made to a preuniform space by choosing a filter, whose elements contain the diagonal Δ , or can be made to a semiuniform space by additionally requiring that with an entourage U also the inverse U^{-1} is an entourage.

The topological structure induced by a pre- or semiuniform space is that of a pretopological space, the neighbourhood filter of a point $x \in X$ is given by $\mathcal{U}(x) := \{\{x' | (x', x) \in U\} | U \in \mathcal{U}\}$ because $\mathcal{U}(x)$ is the preimage $\iota_x^{-1}(\mathcal{U})$. Like in the topological regime also the categories of pre- or semiuniform spaces are not cartesian closed.

3.4 Uniform Spaces

A uniform space is a semiuniform space having the property that the composition $\mathcal{U} \circ \mathcal{V}$ of uniformities \mathcal{U}, \mathcal{V} is itself a uniformity unless it is not the improper filter $\mathcal{P}(X \times X)$. To turn a set X into a uniform space by specifying a filter of entourages, an entourage U has to contain Δ , with U also U^{-1} has to be an entourage and to every entourage U there has to be an entourage V with $V \circ V \subset U$.

A uniform space X induces a topology on X . Indeed, denote to an entourage U and to a point x the set $\{x' | (x', x) \in U\}$ by $U(x)$, then every neighbourhood of x has the form $U(x)$ for an entourage U . Now choose to U an entourage V with $V \circ V \subset U$, then

⁶Note that $\mathcal{U} \circ \mathcal{V}$ is the improper filter iff there are $U \in \mathcal{U}$ und $V \in \mathcal{V}$ with $U \circ V = \emptyset$.

$y \in V(x)$ implies $V(y) \subset (V \circ V)(x) \subset U(x)$. Thus in every neighbourhood $U(x)$ there is a neighbourhood $V(x)$ such that $U(x)$ is a neighbourhood of all the points $y \in V(x)$, and this is exactly the property required for a topological space. Finally note, that the category of uniform spaces is not cartesian closed.

3.5 Properties of Spaces with Uniform Structure

First let's mention that on a set X different uniform structures can induce the same topological structure. Thus spaces endowed with a uniform structure are not automatically isomorphic if they are topologically isomorphic. Further there are topological structures which cannot be induced by any uniform structure. Therefore call a space endowed with a topological structure uniformisable if its topological structure is induced by a uniform structure.

Uniformisability A topological space is uniformisable iff it is completely regular ⁷ Thus uniform spaces have much to do with real numbers, in fact, it can be shown that the uniform structure of every uniform space X can be generated by pseudometrics, see [Schubert, II.2.7, Satz 2]. Hereby a pseudometric is a map $d : X \times X \rightarrow [0, \infty)$ with $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$, and it generates a filter \mathcal{U}_d of entourages on X by choosing the sets $\{(x, y) | d(x, y) \leq \epsilon\}$, $\epsilon > 0$, as entourages. Then the uniform structure of every uniform space is the supremum of such uniformities \mathcal{U}_{d_i} to certain pseudometrics d_i .

Cauchy Filters A filter \mathcal{F} on a preuniform convergence space X is called a Cauchy filter if $\mathcal{F} \times \mathcal{F}$ is a uniformity. The image of a Cauchy filter under a uniformly continuous map (but not generally under a continuous map) is itself a Cauchy filter. Let U be an entourage in a preuniform space, then a set $A \subset X$ is called small of order U , if $x, x' \in A$ implies $(x, x') \in U$. A filter \mathcal{F} on X is a Cauchy filter if it contains arbitrary small sets, i.e. to every entourage $V \in \mathcal{U}$ there is a set $A \in \mathcal{F}$ which is small of order V .

Completeness A preuniform convergence space is called complete if every Cauchy filter converges. To every separated uniform space X there is up to isomorphy a unique completion, i.e. a complete separated uniform limit space \overline{X} and a uniformly continuous map $\iota : X \rightarrow \overline{X}$ with the universal property that to every other uniformly continuous map $f : X \rightarrow Y$ into a complete separated uniform space there is a unique map $\hat{f} : \hat{X} \rightarrow Y$ with $f = \hat{f} \circ \iota$. In other words, the forgetful functor from the category of complete separated uniform spaces into the category of separated uniform spaces has a left adjoint. For arbitrary limit spaces there seem to exist different types of completions, see [Reed].

⁷A space with topological structure is called completely regular, if to every $x \in X$ and every closed set $A \subset X$ not containing x there is a continuous function $f : X \rightarrow [0, 1]$ (or to \mathbb{R}) with $x \in f^{-1}(\{0\})$ and $A \subset f^{-1}(\{1\})$.

4 Linear Analysis

Let X be a real or complex vector space. A convergence relation \rightarrow on X is called compatible if addition and scalar multiplication are continuous. A vector space X together with a compatible convergence relation \rightarrow is called a convergence, limit, pretopological or topological vector space provided that \rightarrow turns X into a convergence, limit, pretopological or topological space. Further a compatible convergence relation on a vector space X induces a uniform structure by assigning to every filter $\mathcal{F} \rightarrow 0$ the uniformity $\hat{\mathcal{F}}$ generated by $\{(x, y) | x - y \in U\} | U \in \mathcal{F}\}$. This induced uniform structure is of the same type as the convergence relation, and in turn it induces the original topological structure on X . Thus, on a vector space endowed with a compatible convergence relation, it makes sense to speak about uniformly continuous maps, Cauchy filters and completeness. Additionally to the mentioned compatible topological structures on a vector space, there are others of which a locally convex topology is certainly the most well-known. These compatible topological structures are discussed in the first paragraph.

Now linear analysis is the mathematical field which studies vector spaces with a compatible topological structure and continuous linear maps between them. In other words, linear analysis examines categories of continuous linear maps between vector spaces endowed with a compatible topological structure. Clearly such a study strongly depends on the categories under consideration, because basic constructions like tensor products and natural spaces of maps are defined by universal properties and hence depend on the used category. Therefore in the second resp. third paragraph tensor products resp. natural spaces of maps and their properties shall be discussed. Finally the fourth paragraph presents main results of linear analysis like an inversion theorem, and discusses the solvability of linear differential equations.

4.1 Vector Spaces endowed with a Compatible Topological Structure

Before other compatible topological structures on vector spaces are discussed, let us give a summary of the usual compatible topological structures on vector spaces. A compatible convergence relation on a vector space is uniquely determined by the set of filters that converge to 0, because translations are continuous. Conversely, it is possible to define a compatible convergence relation on a vector space by specifying a set \mathfrak{F} of filters which are assumed to converge to 0. However, this set of filters must satisfy certain axioms to guarantee the continuity of addition and scalar multiplication.

So for example, to define a convergence relation turning a vector space X into a convergence vector space, only a set \mathfrak{F} of filters on X satisfying

$$(CVS-1) \quad \mathfrak{F} + \mathfrak{F} \subset \mathfrak{F},$$

$$(CVS-2) \quad \mathcal{U}_{\mathbb{R}}(0)\mathfrak{F} \subset \mathfrak{F},$$

$$(CVS-3) \quad \lambda\mathfrak{F} \subset \mathfrak{F} \text{ for all } \lambda \in \mathbb{R},$$

(CVS-4) $\mathcal{U}_{\mathbb{R}}(0)x \in \mathfrak{F}$ for all $x \in X$,

and being directed (i.e. if $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in \mathfrak{F}$) must be specified. Indeed, there exists exactly one convergence relation \rightarrow turning X into a convergence vector space such that $\mathcal{F} \rightarrow 0$ is equivalent to $\mathcal{F} \in \mathfrak{F}$.

In the same way limit vector spaces can be defined by specifying such a set \mathfrak{F} of filters that in addition is closed under intersections (i.e. $\mathcal{F}, \mathcal{G} \in \mathfrak{F}$ imply $\mathcal{F} \cap \mathcal{G} \in \mathfrak{F}$), and pretopological vector spaces can be defined by specifying a filter \mathcal{U} generating \mathfrak{F} via $\mathfrak{F} = \{\mathcal{F} | \mathcal{U} \subset \mathcal{F}\}$ and satisfying

(TVS-A-1) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V + V \subset U$,

(TVS-A-2) for every $U \in \mathcal{U}$ there is a $\lambda > 0$ and a $V \in \mathcal{U}$ such that $[-\lambda, \lambda]V \subset U$,

(TVS-A-3) for every $U \in \mathcal{U}$ and every λ there is a $V \in \mathcal{U}$ such that $\lambda V \subset U$,

(TVS-A-4) for every $U \in \mathcal{U}$ and every x there is a $\lambda > 0$ such that $[-\lambda, \lambda]x \subset U$.

Note that every pretopological vector space is automatically a topological vector space, as the first condition assures to every neighbourhood U of 0 a neighbourhood V of 0 with $V + V \subset U$, and thus U is a neighbourhood of every $x \in V$ because $x + V \subset V + V \subset U$ holds. Hence there is no difference between pretopological vector spaces and topological vector spaces.

The axioms for the neighbourhoods of 0 in a (pre)topological vector space X can also be written in a more common form by calling a set U absorbing, if for every $x \in X$ there is a λ with $\lambda x \in U$, and balanced, if $[-1, 1]U = U$ holds. Then every compatible topology can be defined by specifying a filter base \mathcal{UB} consisting of balanced and absorbing sets such that

(TVS-B-1) to every $U \in \mathcal{UB}$ there is a $V \in \mathcal{UB}$ with $V + V \subset U$,

(TVS-B-2) to every $U \in \mathcal{UB}$ and every $\lambda \neq 0$ there is a $V \in \mathcal{UB}$ with $V \subset \lambda U$.

For the reader's convenience the equivalence of the axioms is proved here ⁸.

Now let us study other types of compatible topological structures on vector spaces. The most well-known is that of a locally convex topology, but let us start with weaker compatible topological structures, as these play a major role in the discussion of natural spaces of maps.

⁸Proof: On the one hand, let \mathcal{U} be the filter generated by \mathcal{UB} . As 0 is contained in all sets of the filter base \mathcal{UB} , the axioms (TVS-B) for \mathcal{UB} are equivalent to the axioms (TVS-A-1) and (TVS-A-3) for \mathcal{U} , while (TVS-A-2) and (TVS-A-4) are satisfied because every set in \mathcal{UB} is balanced and absorbing. Conversely, a filter \mathcal{U} fulfilling (TVS-A) has a base \mathcal{UB} consisting of balanced and absorbing sets. Indeed, (TVS-A-4) guarantees that every $U \in \mathcal{U}$ is absorbing and (TVS-A-3) with $\lambda = 0$ guarantees $0 \in U$. Moreover (TVS-A-2) yields to every $U \in \mathcal{U}$ a $\lambda > 0$ and a $V \in \mathcal{U}$ with $\bigcup_{0 < |r| \leq \lambda} rV \subset U$. Now rV lies for every $r \neq 0$ itself in \mathcal{U} , because (TVS-A-3) applied to $V, \frac{1}{r}$ yields a $V' \in \mathcal{U}$ with $\frac{1}{r}V' \subset V$, i.e. $V' \subset rV$ and hence $rV \in \mathcal{U}$. As further $0 \in V$, the set $\bigcup_{0 < |r| \leq \lambda} rV$ is balanced, is an element of \mathcal{U} and is contained in U . Thus the balanced sets form a filter base.

Equable and Pseudotopological Limit Vector Spaces Equable and pseudotopological limit vector spaces are limit vector spaces where the conditions (CVS-123) are satisfied not only by the set \mathfrak{F} of filters convergent to 0, but also by filters \mathcal{F} themselves that generate \mathfrak{F} . To be more precise, call a filter \mathcal{F} equable if the condition $\mathcal{F} = \mathcal{U}_{\mathbb{R}}(0)\mathcal{F}$ related to (CVS-2) is satisfied, and pseudotopological if \mathcal{F} is equable and additionally the conditions $\mathcal{F} = \mathcal{F} + \mathcal{F}$ (related to (CVS-1)) as well as $\mathcal{F} = \lambda\mathcal{F}$ for all $\lambda \neq 0$ (related to (CVS-3)) are satisfied. Now a limit vector space X is called equable, if to every filter $\mathcal{G} \rightarrow 0$ there is an equable filter $\mathcal{F} \rightarrow 0$ with $\mathcal{F} \subset \mathcal{G}$, and X is called pseudotopological if there is a pseudotopological filter $\mathcal{F} \rightarrow 0$ with $\mathcal{F} \subset \mathcal{G}$.

To every limit vector space X there is a coarsest finer equable resp. pseudotopological compatible limit space structure on X . It is generated by all equable filters \mathcal{F} resp. pseudotopological filters which converge to 0 in the original limit space structure, it is denoted by X^{equ} resp. X^{pstop} and it is called the equablification resp. pseudotopologisation of X . Note that these constructions define the right adjoints to the forgetful functor from the category of equable resp. pseudotopological limit spaces to the category of all limit spaces. Thus a continuous linear map $f : X \rightarrow Y$ from an equable resp. pseudotopological limit vector space X into a limit vector space Y stays continuous if instead of Y the equablification Y^{equ} resp. pseudotopologisation Y^{pstop} is used.

The pseudotopological limit vector spaces are exactly the direct limits of topological vector spaces within the category of limit vector spaces. Indeed, a pseudotopological filter \mathcal{F} generates a topology on the subspace $X_{\mathcal{F}} := \{x \in X \mid \forall U \in \mathcal{F} \exists \lambda \in \mathbb{R} : \lambda x \in U\}$ of those points which are absorbed by each $U \in \mathcal{F}$. Really, the axioms TVS-A123 automatically hold because \mathcal{F} is pseudotopological, and the fourth condition $\mathcal{F} \subset \mathcal{U}(0)x$ is valid exactly for the points $x \in X_{\mathcal{F}}$. Thus $X = \lim_{\mathcal{F}} X_{\mathcal{F}}$ holds as a limit space, where each $X_{\mathcal{F}}$ is endowed with the topology generated by \mathcal{F} and \mathcal{F} runs through the directed set of all pseudotopological filters $\mathcal{F} \rightarrow 0$ on X . Hence the category of pseudotopological limit vector spaces is identical with the category of direct limits of topological vector spaces, where the direct limits are formed within the category of limit vector spaces.

Local Convexity A subset C of a vector space X is called convex if $x_i \in C$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$ imply $\sum_{i=1}^n \lambda_i x_i \in C$. In other words, a subset C of X is called convex if every convex combination of points in C lies again in C , and this condition needs to be tested only for pairs of points⁹. Now a filter is called locally convex if it has a base consisting of convex sets, and a limit vector space is called locally convex if to every convergent filter \mathcal{G} there is a locally convex convergent filter $\mathcal{F} \subset \mathcal{G}$. Especially a topological vector space X is locally convex if the neighbourhood filter of zero is locally convex, i.e. in every neighbourhood of zero there is a convex neighbourhood.

Again to every limit vector space there is a coarsest finer compatible locally convex limit space structure generated by the locally convex filters which converge to 0 in the original

⁹More precisely, if every convex combination of n points $x_i \in C$, $n \geq 2$, lies in C , then also every convex combination of $n + 1$ points does. Indeed, let x_{n+1} be another point in C and let $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$ be arbitrary. Assume without restriction $\lambda_{n+1} \neq 1$. Then $\sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} = 1$ holds, so that $x := \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} x_i \in C$ is valid by assumption. But also $\sum_{i=1}^{n+1} \lambda_i x_i = (1 - \lambda_{n+1})x + \lambda_{n+1}x_{n+1} \in C$ is valid by assumption and thus the induction step has been proved.

limit space structure on X . It is denoted by X^{con} and is called the convexification of X . As usual, a continuous linear map from a locally convex limit vector space X into a limit vector space Y stays continuous when viewed as map from X into Y^{con} , because the image of a convex set under a linear map f is a convex set and thus the image $f(\mathcal{F})$ of a locally convex filter \mathcal{F} is again locally convex, as it is generated by the sets $f(C)$, $C \in \mathcal{F}$ convex.

Note that local convexity can be combined with the notion of an equable or pseudotopological limit space. Especially the locally convex pseudotopological limit vector spaces are exactly the direct limits of locally convex topological spaces within the category of limit vector spaces. Finally let us summarize facts about locally convex topological vector spaces X . As a compatible topology on a vector space can be defined by a filter base consisting of balanced and absorbing sets U which satisfy (TVS-B), in order to define a locally convex topology merely the convexity of every U must be additionally postulated. A balanced and convex set is called absolutely convex, thus a locally convex topology can be defined by absolutely convex and absorbing subsets U which satisfy (TVS-B). However, every locally convex topology can be defined in an easier way by using pseudonorms instead of filter bases. Indeed, every locally convex topology is generated by pseudonorms¹⁰, i.e. it is the initial topology w.r.t. a family of pseudonorms, as on the one hand a topology generated by pseudonorms is locally convex, and on the other hand to every open absolutely convex neighbourhood U of zero the Minkowski functional $p_U(x) := \inf\{\lambda > 0 | x \in \lambda U\}$ can be defined. This functional satisfies $0 \leq p_U(x) < \infty$ because U is absorbing, p_U is a pseudonorm because U is absolutely convex, and p_U is continuous because openness of U implies $rU = \{x | p(x) < r\}$, i.e. $x - y \in rU$ implies $|p_U(x) - p_U(y)| \leq p_U(x - y) < r$ and thus the continuity of p_U . The equality $U = \{x | p(x) < 1\}$ also guarantees that the initial topology w.r.t. all p_U , U open convex balanced neighbourhood of 0, is exactly the original locally convex topology. Hence every locally convex topology is generated by pseudonorms.

The description of a locally convex topology via pseudonorms also allows to define the left adjoint to the forgetful functor from the category of locally convex to the category of all topological or limit vector spaces. It endows a topological or limit vector space X with the initial topology w.r.t. those pseudonorms, which are continuous for the original topological structure of X . Further the continuity of linear maps between locally convex topological vector spaces can be formulated using pseudonorms: The set of continuous pseudonorms on a locally convex topological vector space X can be ordered by $p \leq q$ whenever there is a constant $C \in \mathbb{R}^+$ with $p(x) \leq Cq(x)$ for all $x \in X$, or equivalently whenever to every $\epsilon > 0$ there is a $\delta > 0$ such that $q(x) \leq \delta$ implies $p(x) \leq \epsilon$. Two continuous pseudonorms p, q are equivalent w.r.t. this order if there are $c, C \in \mathbb{R}^+$ with $cp(x) \leq q(x) \leq Cp(x)$, i.e. p, q are equivalent as pseudometrics. Let Γ_X be a set of continuous pseudonorms p on X which generates the topology of X and is directed, i.e. to every continuous pseudonorm q on X there is a $p \in \Gamma_X$ with $q \leq p$. Note that every generating set Γ_X of pseudonorms can be made into a directed set by adding the continuous pseudonorm $\max(p_1, \dots, p_n)$ for $p_1, \dots, p_n \in \Gamma_X$. Now a linear

¹⁰A map $p : X \rightarrow \mathbb{R}_0^+$ is called a pseudonorm if $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda|p(x)$ are valid for all $x, y \in X$ and $\lambda \in \mathbb{R}$.

map $A : X \rightarrow Y$ is continuous iff to every pseudonorm $q \in \Gamma_Y$ there is a pseudonorm $p \in \Gamma_X$ and a constant $C \in \mathbb{R}_0^+$ such that $q(Ax) \leq Cp(x)$. Or in other words, there has to be a map $\sigma : \Gamma_Y \rightarrow \Gamma_X$ such that the smallest numbers $q^{\sigma(q)}(A) \in [0, \infty]$ defined by $q(Ax) \leq q^{\sigma(q)}(A)\sigma(q)(x)$ satisfy $q^{\sigma(q)}(A) < \infty$ for all $q \in \Gamma_Y$.

4.2 Tensor Products

Let \mathcal{C} be the category of all continuous linear maps between vector spaces endowed with a certain compatible topological structure. It would be nice, if the analysis of \mathcal{C} automatically included the analysis of all continuous multilinear maps between the spaces in \mathcal{C} . This would be the case, if there existed a tensor product \otimes on \mathcal{C} with the universal property that continuous bilinear maps from $X \times Y$ to Z correspond to continuous linear maps from $X \otimes Y$ to Z .

Fortunately, always a functor \otimes with this universal property exists and is called the topological tensor product¹¹. On objects it is the algebraic tensor product $X \otimes Y$ endowed with the finest compatible topological structure coarser than the final topological structure induced by the canonical bilinear map $\tau : X \times Y \rightarrow X \otimes Y$. Indeed, then $X \otimes Y$ is an object of \mathcal{C} with the property that the continuous bilinear maps $B : X \times Y \rightarrow Z$ correspond via $B = A \circ \tau$ to the continuous linear maps $A : X \otimes Y \rightarrow Z$ ¹².

However, the so constructed functor \otimes does not need to have the properties of a tensor product for all categories \mathcal{C} (although it is always called the topological tensor product because of its universal property that continuous bilinear maps correspond to continuous linear maps on the tensor product). Namely, \otimes is not automatically associative, especially for the category of all topological vector spaces. This is shown in the next few paragraphs, which describe the topological tensor product in different categories.

Let's start with the category of limit vector spaces. There the topological structure on the tensor product is given by the convergences

¹¹In literature, \otimes is often called the projective topological tensor product to distinguish it from topological tensor products, where the continuous linear maps from $X \otimes Y$ to Z correspond to bilinear maps that are not continuous with respect to the product structure on $X \times Y$, but with respect to some other structure. For example, on the set $X \times Y$ the final topological structure induced by $\iota_x : y \mapsto (x, y)$ and $\iota_y : x \mapsto (x, y)$ could also be used. Then bilinear maps are continuous iff they are separately continuous in each component, and the related topological tensor product is called injective. However, in this thesis only the projective tensor product is used and thus the adjective "projective" is suppressed.

¹²More precisely, if on the one hand A is continuous on $X \otimes Y$, then A is also continuous w.r.t the finer final structure induced by τ and thus $B := A \circ \tau$ is continuous. On the other hand, if B is continuous, then also the linear map A given by the universal property $B = A \circ \tau$ of the algebraic tensor product is continuous w.r.t. the final topological structure induced by τ . However, A is moreover continuous for the finest compatible structure coarser than the final structure induced by τ due to the linearity of A and the compatible topological structure on Z . In fact, the initial topological structure induced by A is coarser than the final one induced by τ (because A is continuous w.r.t. this final one) and automatically compatible (because $A \circ +_{X \otimes Y} = +_Z \circ (A \times A)$ as well as $A \circ \cdot_{X \otimes Y} = \cdot_Z \circ (\text{Id}_{\mathbb{R}} \times A)$ hold and thus $+_{X \otimes Y}$ as well as $\cdot_{X \otimes Y}$ are continuous). Hence A is also continuous w.r.t. the finest compatible topological structure coarser than the final one induced by τ .

$$\sum_{i=1}^n \tau(\mathcal{F}_i \times \mathcal{G}_i) \rightarrow \sum_{i=1}^n x_i \otimes y_i$$

for $\mathcal{F}_i \rightarrow x_i, \mathcal{G}_i \rightarrow y_i$. Note that these convergences on $X \otimes Y$ really define the finest compatible limit space structure coarser than the final limit space structure induced by τ . In fact, the final limit space structure induced by τ is given by the convergences $\tau(\mathcal{F} \times \mathcal{G}) \rightarrow x \otimes y$ and is thus not compatible, as it does not regard those elements, which can not be represented by a singleton $x \otimes y$. Obviously, for the category of limit vector spaces \otimes is associative and thus really a tensor product. The same is true for equable, pseudotopological and locally convex limit vector spaces, but not for topological vector spaces.

There the topology on the tensor product $X \otimes Y$ is given by the neighbourhoods $\sum_{k \in \mathbb{N}} \tau(U_k \times V_k)$ of zero, where U_k, V_k are sequences of zero-neighbourhoods in X, Y . Indeed, if B is a continuous bilinear from $X \times Y$ to Z , then the linear map A on $X \otimes Y$ defined by $B = A \circ \tau$ is also continuous: Choose to a zero-neighbourhood $W \subset Z$ a sequence W_k of zero-neighbourhoods such that $\sum_{k \in \mathbb{N}} W_k \subset W$, let U_k, V_k be zero-neighbourhoods in X, Y with $B(U_k \times V_k) \subset W_k$ and let $U := \sum_{k \in \mathbb{N}} \tau(U_k \times V_k)$. Then

$$A(U) = \sum_{k \in \mathbb{N}} A(\tau(U_k \times V_k)) = \sum_{k \in \mathbb{N}} B(U_k \times V_k) = \sum_{k \in \mathbb{N}} W_k \subset W$$

is valid, and thus A is continuous. Hence $X \otimes Y$ endowed with the above defined topology is really the topological tensor product in the category of topological vector spaces. But it is generally not associative, e.g. [Glöckner, Theorem 1] proved that for $X = Y = \mathbb{R}^{\mathbb{N}}$ the natural isomorphism $(X \otimes X) \otimes X \rightarrow X \otimes (X \otimes X)$ is not continuous.

Contrary to the category of topological vector spaces, where the topological tensor product is generally not a tensor product due to the lack of associativity, in the category of locally convex topological vector spaces or normable vector spaces the functor \otimes is associative. Because this fact is well-known, a proof is omitted and only the description of the locally convex topology on $X \otimes Y$ is recalled: It is generated by the pseudonorms $(p \otimes q)(u) := \inf\{\sum_i p(x_i)q(y_i) \mid u = \sum_i x_i \otimes y_i\}$, where p, q denote continuous pseudonorms on X, Y .

4.3 Natural Spaces of Maps

In linear analysis the same problems can be encountered as in general topology, namely the categories of topological and locally convex topological vector spaces are not tensorial closed. Thus let us start again in the category of limit vector spaces to discuss the existence of natural spaces of continuous linear maps.

Let X, Y be two limit vector spaces, then there is a coarsest limit space structure on the set $L(X, Y)$ of continuous linear maps from X to Y making the bilinear evaluation $\text{ev} : L(X, Y) \times X \rightarrow Y$ continuous. The convergence $\mathcal{F} \rightarrow A$ is valid w.r.t. this limit space structure iff $\text{ev}(\mathcal{F} \times \mathcal{G}) \rightarrow Ax$ holds for every filter $\mathcal{G} \rightarrow x$ on X . In this way

$L(X, Y)$ becomes a limit vector space¹³, and the functor $(X, Y) \mapsto L(X, Y)$ is the right adjoint to the topological tensor product with the evaluation (considered as a linear map from $L(X, Y) \otimes X$ to Y) as counit. Thus the category of limit vector spaces is tensorial closed. For topological vector spaces X, Y the convergence $\mathcal{F} \rightarrow A$ in the limit vector space $L(X, Y)$ holds, if and only if to every $x \in X$ and every neighbourhood V of $Ax \in Y$ there is a neighbourhood U of x and a set $W \in \mathcal{F}$ such that $A'(U) \subset V$ for all $A' \in W$.

Also the categories of equable, pseudotopological and locally convex limit vector spaces or combinations thereof are tensorial closed. The natural space of maps is the equablification $L(X, Y)^{equ}$, pseudotopologisation $L(X, Y)^{pstop}$ or convexification $L(X, Y)^{con}$ of the natural space of maps $L(X, Y)$ within the category of limit vector spaces. Indeed, let $A : X \rightarrow L(Y, Z)$ be a continuous linear map from an equable resp. pseudotopological resp. locally convex limit vector space X into the natural limit vector space $L(X, Y)$, then the universal properties of equablification, pseudotopologisation and convexification guarantee that $A : X \rightarrow L(X, Y)^{equ|pstop|con}$ is continuous. Thus the natural isomorphy $L(X \otimes Y, Z) \cong L(X, L(Y, Z)) \cong L(X, L(Y, Z)^{equ|pstop|con})$ holds and proves tensorial closedness.

But in general, even for locally convex topological vector spaces X, Y none of the so defined limit vector spaces $L(X, Y)$ or $L(X, Y)^{equ|pstop|con}$ is a topological vector space. In fact, these spaces cannot be topological vector spaces, because on the set $L(X, Y)$ generally there exists no vector space topology making the evaluation continuous. Thus while in general topology there is always a topology making the evaluation continuous (e.g. the discrete topology), but generally there is no such coarsest topology, in linear analysis it is even worse: Generally there is no compatible topology on $L(X, Y)$ making the evaluation continuous. Especially the category of locally convex topological vector spaces is not tensorial closed, in contrast to the category of normable spaces, where the usual norm topology on $L(X, Y)$ has this desired property.

To outline a proof of these statements, recall that a linear map $A : X \rightarrow Y$ between locally convex topological vector spaces X, Y is continuous exactly if to every continuous pseudonorm q on Y there exists a continuous pseudonorm p on X and a constant $C \in \mathbb{R}_0^+$ with $q(Ax) \leq Cp(x)$ for all $x \in X$. For practical purposes denote for a linear map $A : X \rightarrow Y$ and pseudonorms p on X , q on Y by $q^p(A) \in [0, \infty]$ the minimum of all constants C with $q(Ax) \leq Cp(x)$ for all $x \in X$. Then a linear map A is continuous if and only if to every continuous pseudonorm q on Y there exists a continuous pseudonorm p on X with $q^p(A) < \infty$, or with the notation $\sigma(q) := p$ equivalently, if a map σ from a directed and generating set of all continuous pseudonorms on Y to a directed and generating set of all continuous pseudonorms on X exists with $q^{\sigma(q)}(A) < \infty$ for all q on Y .

Observe that q^p for continuous pseudonorms p on X and q on Y has itself all properties of a pseudonorm on the space $L(X, Y)$ of continuous linear maps from X to Y except that it can assign the value ∞ . This defect mainly causes the fact that the category

¹³Remark that scalar multiplication and addition of maps are defined pointwisely and so their continuity is implied by the continuity of ev.

of locally convex topological vector spaces is not tensorial closed. Indeed, consider the locally convex pseudotopological limit vector space $L(X, Y)^{con, pstop}$ associated to X, Y . As X, Y are locally convex topological vector spaces, there is an easy way to describe $L(X, Y)^{con, pstop}$ as a direct limit of locally convex topological spaces: Define to a map σ from a directed and generating set of continuous pseudonorms on Y to a directed and generating set of continuous pseudonorms on X the set $L_\sigma(X, Y)$ of those continuous linear maps $A \in L(X, Y)$ with $q^{\sigma(q)}(A) < \infty$ for all continuous pseudonorms q on Y and endow $L_\sigma(X, Y)$ with the locally convex topology generated by the pseudonorms $q^{\sigma(q)}$ (which obviously do not assign the value ∞ on $L_\sigma(X, Y)$). Let the set of all such maps σ be ordered by $\sigma \leq \sigma'$ if and only if $\sigma(q) \leq \sigma'(q)$ for all q on Y . Then the direct limit $\lim_\sigma L_\sigma(X, Y)$ can be formed in the category of limit vector spaces. As a set $\lim_\sigma L_\sigma(X, Y)$ equals $L(X, Y)$ ¹⁴, and a filter \mathcal{F} converges to A if and only if there is a σ such that the trace¹⁵ \mathcal{F}_σ of \mathcal{F} on $L_\sigma(X, Y)$ exists and converges to A in $L_\sigma(X, Y)$.

To prove that $L(X, Y)^{con, pstop}$ and $\lim_\sigma L_\sigma(X, Y)$ are the same limit vector spaces, note that due to the inequality $q(Ax) \leq q^{\sigma(q)}(A)\sigma(q)(x)$ the evaluation map is also continuous when viewed as a map $ev : (\lim_\sigma L_\sigma(X, Y)) \times X \rightarrow Y$. Hence the limit space structure $(\lim_\sigma L_\sigma(X, Y))$ is finer than $L^{con, pstop}(X, Y)$. Further a linear map $A : Z \rightarrow L(X, Y)^{con, pstop}$ from a topological vector space Z is continuous iff it is continuous as a map into $\lim_\sigma L_\sigma(X, Y)$ ¹⁶. Thus also a linear map $A : \lim_i Z_i \rightarrow L(X, Y)^{con, pstop}$ from a direct limit of locally convex topological spaces Z_i (within the category of limit vector spaces) and therefore from an arbitrary locally convex pseudotopological limit vector space is continuous iff it is continuous as a map into $\lim_\sigma L_\sigma(X, Y)$. Especially the map $Id_{L(X, Y)} : \lim_\sigma L_\sigma(X, Y) \rightarrow L(X, Y)^{con, pstop}$ is an isomorphism of limit vector spaces, and hence $L(X, Y)^{con, pstop} = \lim_\sigma L_\sigma(X, Y)$ holds.

Now the statement can be proved that for a locally convex but nonnormable topological vector space X and a separated locally convex vector space Y there is in general no vector space topology on $L(X, Y)$ making the evaluation continuous. In fact, if there would be such a topology \mathcal{T} , then the linear map $Id : (L(X, Y), \mathcal{T}) \rightarrow \lim_\sigma L_\sigma(X, Y)$ would be continuous because $L^{con, pstop}(X, Y) = \lim_\sigma L_\sigma(X, Y)$ would be coarser than $(L(X, Y), \mathcal{T})$. But $Id : L(X, Y) \rightarrow \lim_\sigma L_\sigma(X, Y)$ only could be continuous if there were be a σ and a neighbourhood $W \subset L(X, Y)$ of 0 such that $Id(W) \subset L_\sigma(X, Y)$. Because W would be absorbing as a neighbourhood of 0 in a topological vector space this would imply $Id(L(X, Y)) \subset L_\sigma(X, Y)$, i.e. $L(X, Y) = L_\sigma(X, Y)$. However, for nonnormable X and separated $Y \neq \{0\}$ there is no σ with $L(X, Y) = L_\sigma(X, Y)$ ¹⁷.

¹⁴Recall that a linear map A is continuous if and only if to every continuous pseudonorm q on Y there exists a continuous pseudonorm $\sigma(q)$ on X such that $q^{\sigma(q)}(A) < \infty$.

¹⁵A filter \mathcal{F} on X is said to have a trace on the subset $A \subset X$, if $U \cap A \neq \emptyset$ holds for all $U \in \mathcal{F}$, and the induced proper filter $\mathcal{F}_A := \{U \cap A | U \in \mathcal{F}\}$ is called the trace of \mathcal{F} on A .

¹⁶One direction is trivial because $\lim_\sigma L_\sigma(X, Y)$ is finer than $L(X, Y)^{con, pstop}$. To show the other direction, let A be continuous w.r.t. $L(X, Y)^{con, pstop}$, then the bilinear map $\hat{A} : Z \times X \rightarrow Y$ is continuous, too. Hence to every q on Y there is a $\sigma(q)$ on X , a neighbourhood $W \subset Z$ of zero and a constant $C_q < \infty$ such that $z \in W$ implies $q(\hat{A}(z, x)) \leq C_q \sigma(q)(x)$, i.e. $q^{\sigma(q)}(\hat{A}(z, \cdot)) < \infty$ for all $z \in W$. Because W is absorbing and A linear this implies $q^{\sigma(q)}(\hat{A}(z, \cdot)) < \infty$ for all $z \in Z$, hence $A(Z) \subset L_\sigma(X, Y)$ and clearly A (restricted to have values in $L_\sigma(X, Y)$) is continuous. Therefore it is continuous as a map into $\lim_\sigma L_\sigma(X, Y)$.

¹⁷If there would be a σ with $L(X, Y) = L_\sigma(X, Y)$, then $q^\sigma(q)(A) < \infty$ would hold for all $A \in L(X, Y)$

Thus the assumption of having a compatible topology on $L(X, Y)$, which makes the evaluation continuous, has been contradicted.

Note further that for a normable vector space X the limit space $L(X, Y)^{con, pstop} = \lim_{\sigma} L_{\sigma}(X, Y)$ is in fact a locally convex topological vector space¹⁸, and if also Y is normable, then the direct limit is identical with the usual normable space $L(X, Y)$. Thus to generalize linear analysis on normed spaces in a way that includes all locally convex topological vector spaces, the best category to work in is the category of locally convex pseudotopological limit vector spaces, or equivalently the category of direct limits of locally convex topological vector spaces (within the category of all limit vector spaces).

Bornological Locally Convex Topological Vector Spaces Although the category of all locally convex topological vector spaces is not tensorial closed, a subcategory may be tensorial closed even if it contains nonnormable spaces. This is no contradiction to the previous results, because in a subcategory the choice of a space is restricted and therefore the product as well as the exponential can differ from the usual one. The bornological locally convex topological vector spaces¹⁹ form such a subcategory, like e.g. the category of compactly generated separated topological spaces is cartesian closed although the category of all topological spaces is not. The role of the compact-open topology is played here by the bounded-open topology $L^{bo}(X, Y)$ ²⁰, while the bornologification $b(\cdot)$ ²¹ is used instead of the Kelley-fication. With these tools it can be proved that the category of bornological locally convex topological vector spaces is a tensorial closed category with tensor product given by $X \otimes_{bor} Y = b(X \otimes Y)$, and the natural space of maps is $L^{bor}(X, Y) = b(L^{bo}(X, Y))$. The category of bornological locally

and all continuous pseudonorms q on Y . Thus for $q \neq 0$ every $x' \in X'$ would be bounded on the 0-neighbourhood $U_q := \{x \in X | \sigma(q)(x) \leq 1\}$, because for an y with $q(y) \neq 0$ the continuous linear map $A : x \mapsto x'(x)y$ would satisfy $|x'(x)q(y)| = q(Ax) \leq C\sigma(q)(x)$ with a $C < \infty$. Hence x' would satisfy $|x'(x)| \leq \frac{C}{q(y)} < \infty$ for all $x \in U_q$, so that there would exist a bounded neighbourhood of 0 in X , which is only possible for normable X .

¹⁸Choose as generating sets of continuous pseudonorms on X the set $\{\|\cdot\|_X\}$, then there is only one σ (it maps every continuous pseudonorm on Y to $\|\cdot\|_X$). Therefore $L(X, Y)$ equals the locally convex topological vector space $L_{\sigma}(X, Y)$ and hence can be topologized by the pseudonorms $q^{\|\cdot\|_X}$, q continuous pseudonorm on Y , which do not assign the value ∞ .

¹⁹A locally convex topological vector space is called bornological, if every absolutely convex subset that absorbs each bounded set is already a 0-neighbourhood. Recall that the topology of a locally convex topological vector space is generated by the open absolutely convex sets and that a subset B is called bounded if it is absorbed by each 0-neighbourhood, i.e. for every 0-neighbourhood U there exists a r such that $B \subset [0, r]U$.

²⁰The bounded-open topology on $L(X, Y)$ is generated by the sets $\{A \in L(X, Y) | A(B) \subset U\}$ with $B \subset X$ bounded and $0 \in U \subset Y$ open. It is the same as the topology of uniform convergence on bounded sets. Hence for normed spaces it is the usual norm topology on the space of maps.

²¹The bornologification $b(\cdot)$ is the right adjoint to the forgetful functor from the category of bornological to the category of all locally convex topological spaces. It refines the topology of a locally convex topological space X to the finest locally convex topology with the same bounded sets as X or equivalently to the topology generated by all bounded pseudonorms (i.e. those pseudonorms which map bounded sets w.r.t. the original topology on X to bounded sets in \mathbb{R}_0^+). Bornologification has the universal property that a map $A : X \rightarrow Y$ from a bornological locally convex topological vector space X into a locally convex topological vector space Y is continuous iff it is continuous into $b(Y)$.

convex vector spaces is used in the convenient calculus developed by [Fröhlicher,Kriegl] and [Kriegl,Michor], also see appendix A.

Completeness On a limit vector space every continuous linear map A is automatically uniformly continuous w.r.t. the induced uniform structure. Indeed, if \mathcal{F} is a filter converging to 0 and $\hat{\mathcal{F}}$ denotes its induced uniformity, then

$$(A(\hat{\mathcal{F}})) \subset ((A \times A)(\hat{\mathcal{F}}))$$

is valid. Especially the continuous linear map $\text{ev} : L(X, Y) \otimes X \rightarrow Y$ is uniformly continuous, although the associated bilinear map is not. It is an important fact that the completeness of Y implies the completeness of the natural space of maps $L(X, Y)$ ²² and its equablification, pseudotopologisation or convexification²³.

4.4 Theorems of Linear Analysis

As we are mainly interested in locally convex topological vector spaces, we usually work in the tensorial closed category of locally convex pseudotopological limit vector spaces or equivalently in the category of direct limits of locally convex topological spaces (within the category of limit vector spaces), and usually we assume such spaces to be separated and complete.

Note that a linear map $A : \lim X_i \rightarrow \lim Y_j$ between direct limits of locally convex topological vector spaces is continuous, if for every i there is a j such that the restriction $A|_{X_i}^{Y_j}$ to X_i into Y_j is continuous. Hence in most cases it is sufficient to prove a theorem for maps between locally convex topological vector spaces, as it induces a theorem between the direct limits of such spaces.

²²Proof: The evaluation map $\text{ev} : L(X, Y) \otimes X \rightarrow Y$ is uniformly continuous and hence maps Cauchy filters to Cauchy filters. Let \mathcal{F} be a Cauchy filter on $L(X, Y)$ and let \mathcal{G} be a filter converging to x on X , then $\text{ev}(\mathcal{F} \otimes \mathcal{G})$ is a Cauchy filter on Y and converges by completeness. Its limit $y \in Y$ is independent of \mathcal{G} (Let $\mathcal{G}' \rightarrow x$ be another filter, then $\text{ev}(\mathcal{F} \times (\mathcal{G} \cap \mathcal{G}'))$ converges and is coarser than both $\text{ev}(\mathcal{F} \times \mathcal{G})$ and $\text{ev}(\mathcal{F} \times \mathcal{G}')$, so that both filters must converge to the same y .) and thus allows to define a map $A : X \rightarrow Y$, which is linear (Let $\mathcal{G} \rightarrow x$ and $\mathcal{G}' \rightarrow x'$, then the filter $\text{ev}(\mathcal{F} \times (\mathcal{G} + \mathcal{G}'))$ converges on the one hand to $A(x + x')$ and because of bilinearity on the other hand to $Ax + Ax'$, so that additivity has been shown; analogously for homogeneity) and continuous ($\lim_{x' \in \mathcal{G}} Ax' = \lim_{x' \in \mathcal{G}} \lim_{A' \in \mathcal{F}} A'x' = Ax$ holds for $\mathcal{F} \rightarrow A$, $\mathcal{G} \rightarrow x$) Now in $L(X, Y)$ per definition this implies the convergence of \mathcal{F} to A and thus completeness.

²³Completeness of $L(X, Y)^{\text{equ|pstop|con}}$ is implied by the fact that the equablification, pseudotopologisation or convexification of a complete limit vector space is again complete. Indeed, let X be complete and let \mathcal{F} be a Cauchy filter on $X^{\text{equ|pstop|con}}$, i.e. $\mathcal{F} \times \mathcal{F}$ is a uniformity on $X^{\text{equ|pstop|con}}$. Then there is an equable resp. pseudotopological resp. locally convex filter \mathcal{G} converging to 0 in X with $\hat{\mathcal{G}} \subset \mathcal{F} \times \mathcal{F}$. Therefore $\mathcal{F} \times \mathcal{F}$ is also a uniformity on X , and completeness of X implies the convergence of \mathcal{F} to some x . Without restriction assume $x \in U$ for every $U \in \mathcal{F}$, then the relation $\mathcal{G} + x = \hat{\mathcal{G}}(x) \subset (\mathcal{F} \times \mathcal{F})(x) = \mathcal{F}$ implies that \mathcal{F} is also finer than $\mathcal{G} + x$ for the equable resp. pseudotopological resp. locally convex filter $\mathcal{G} \rightarrow 0$, and thus \mathcal{F} converges also to x in $X^{\text{equ|pstop|con}}$.

Invertibility Let $T : X \rightarrow Y$ be a continuous linear map between complete and separated locally convex vector spaces X, Y with a continuous inverse $T^{-1} \in L_\tau(Y, X)$. Assume that $A \in L(X, Y)$ satisfies $p^{\tau(p)}(T^{-1})\tau(p)^p(T - A) < 1$ for all continuous pseudonorms p on X , then A is also continuously invertible. Indeed, because of

$$A = T \circ (\text{Id}_X - T^{-1} \circ (T - A))$$

it is sufficient to prove that the linear operator $\text{Id}_X - T^{-1} \circ (T - A)$ on X has a continuous inverse. Set $B := T^{-1} \circ (T - A)$, then the assumption implies $p^p(B) \leq p^{\tau(p)}(T^{-1})\tau(p)^p(T - A) < 1$ for all continuous pseudonorms p on X . As $p^p(\sum_n B^n) \leq \sum_n p^p(B)^n$ is valid and the right hand side is a Cauchy series for every p due to $p^p(B) < 1$, also the series $\sum_n B^n$ is a Cauchy series and converges in $L_{\text{Id}}(X)$ by completeness. Because of

$$(\text{Id}_X - B) \left(\sum_{n=0}^k B^n \right) = \text{Id}_X - B^{k+1} \rightarrow \text{Id}_X$$

for $k \rightarrow \infty$ the limit $\sum_{n=0}^{\infty} B^n$ is the inverse $(\text{Id}_X - B)^{-1}$, so that A is continuously invertible with inverse $A^{-1} = (\sum_{n=0}^{\infty} B^n) \circ T^{-1}$. Thus every operator in the set $\{A \in L(X, Y) | \forall p : \tau(p)^p(T - A) < p^{\tau(p)}(T^{-1})^{-1}\}$ is continuously invertible.

Let us clarify the structure of this set: To every continuously invertible operator T there is an invertible σ mapping continuous pseudonorms on Y to continuous pseudonorms on X such that $T \in L_\sigma(X, Y)$ and $T^{-1} \in L_{\sigma^{-1}}(Y, X)$. For example, $\sigma(q)(\cdot) := q(T\cdot)$ defines such a map with $\sigma^{-1}(p)(\cdot) := p(T^{-1}\cdot)$ as its inverse, and $q^{\sigma(q)}(T) = 1 = p^{\sigma^{-1}(p)}(T^{-1})$ holds for all continuous pseudonorms p on X , q on Y . Thus let $\tau := \sigma^{-1}$ and $r_q := \sigma(q)^q(T^{-1})^{-1}$, then $r_q > 0$ and all elements of the set $B_{r_q}(T) = \{A \in L_\sigma(X, Y) | \forall q : q^{\sigma(q)}(T - A) < r_q\}$ are invertible.

Note that $B_{r_q}(T)$ is in general not a neighbourhood of T , as it is the intersection of infinitely many and not finitely many open q -balls $B_{r_q}(T) := \{A \in L_\sigma(X, Y) | q^{\sigma(q)}(T - A) < r_q\}$ around T in $L_\sigma(X)$. This situation is typical of the analysis on locally convex vector spaces X developed here: Usually the presented theorems do not guarantee the existence of a neighbourhood or equivalently a finite intersection of balls, but only the existence of an intersection of infinitely many balls around a point to a radius family, which is calculated by the theorem. In the worst case, the calculated radius family enforces the intersection of balls only to contain the point around which it is defined, and then the theorem is trivial and not applicable. But in better cases the calculated radius family assures that the intersection of balls is large enough, e.g. that $B_{r_q}(T) - T$ spans a dense subset, so that the theorem can be applied to the problem considered. Thus it is essential that the radius family is computed by the theorem, because this is the only way to decide whether the theorem can be applied to a concrete problem or not. An intersection of balls is from now on shortly called an i-ball.

This point of view on analysis is different from the point of view presented in other books about analysis on locally convex topological vector spaces, where theorems use stronger assumptions to prove stronger results, e.g. the existence of neighbourhoods instead of i-balls. However, it is often not easy to verify the strong assumptions of such

theorems in a concrete case. The advantage of the approach presented here is that it is easy to decide whether a theorem is applicable to a concrete problem, the disadvantage is that the theorems are not formulated in a way that guarantees their applicability to a certain class of maps a priori. For example, the inverse mapping theorem proved later in this thesis guarantees the existence of an inverse to a continuously differentiable map $f : X \rightarrow Y$ on an i -ball around a point $x \in X$, whenever f has a continuously invertible derivative at x , and computes the radius family of the i -ball, while the famous Nash-Moser inversion theorem proved e.g. in [Kriegl, Michor, 51.17] can be applied to the class of all tame smooth mappings f between tame Fréchet spaces with a tame smooth linear inverse $(\tilde{x}, h) \rightarrow Df(\tilde{x})^{-1}h$ on a neighbourhood around x ²⁴ and guarantees the existence of an inverse to f on a neighbourhood of x .

Intersections of Balls Let X be a separated locally convex topological vector space. A closed i -ball $B_{r_q}(0)$ around 0 to a radius family $0 < r_q \leq \infty$ is the intersection of closed q -balls $B_{r_q}^q(0) := \{x \mid q(x) \leq r_q\}$ with radiuses r_q , where q runs through a generating set of the continuous pseudonorms on X . Note that every closed i -ball $B_{r_q}(0)$ is a closed absolutely convex subset of X . It is possible that a closed i -ball $B_{r_q}(0)$ is merely the set $\{0\}$, e.g. if q runs through a directed generating set of continuous pseudonorms and $\lim_q r_q = 0$ is valid. But it is also possible that $B_{r_q}(0)$ is a neighbourhood of zero or even the whole space X , e.g. if $r_q = \infty$ for all large q or even all q . The largest closed i -balls are surely the neighbourhoods of zero, but there are also smaller i -balls, which deserve to be called large.

For example, those closed i -balls could be called large, which absorb the whole space X . However, for Fréchet or Banach spaces, such closed i -balls are automatically neighbourhoods of zero. Indeed, call a locally convex space X barreled²⁵, if every closed absolutely convex subset which absorbs X contains automatically a neighbourhood of zero. Thus if X is barreled and $B_{r_q}(0)$ is so large that it absorbs the whole space X , then it automatically contains a neighbourhood of zero. A topological space is called a Baire space if every countable intersection of dense open sets is automatically dense. The theorem of Baire says that completely metrizable spaces are Baire spaces, and [Saxon] has proved that a topological vector space is a Baire space iff every absorbing closed balanced subset is a neighbourhood of zero. Especially every linear Baire space is barreled, and hence for Fréchet or Banach spaces X a closed i -ball $B_{r_q}(0)$, which is not a neighbourhood of zero, does not absorb the whole set X .

²⁴A Fréchet space X is called graded if it is provided with a fixed increasing family p_n of generating continuous pseudonorms. A linear map $A : X \rightarrow Y$ between such Fréchet spaces $(X, p_n), (Y, q_n)$ is called tame of degree d and base b if for every $n \geq b$ there is a $C_n < \infty$ with $q_n(Ax) \leq C_n p_{n+d}(x)$. Note that our map σ replaces the grading and that the conditions formulated in terms of σ could be formulated in terms of the tameness constants C_n . A Fréchet space (X, p_n) is called tame if it is a tame direct summand of the space S of (very) fast falling sequences in a Banach space, i.e. the injection $\iota : X \rightarrow S$ and the projection $\pi : S \rightarrow X$ are tame. A continuous nonlinear map $f : X \rightarrow Y$ between graded Fréchet spaces is called tame of degree d and base b if locally for every $n \geq b$ there is a $C_n < \infty$ with $q_n(f(x)) \leq C_n(1 + p_{n+d}(x))$. A smooth mapping is called tame smooth if every derivative $(x, h) \mapsto D^k f(x)h$ is tame.

²⁵Bourbaki and Dieudonné showed that the class of barreled spaces is the largest class of locally convex vector spaces for which the uniform boundedness theorem holds, thus barreled spaces are interesting per se.

Thus beginning with neighbourhoods of zero, the next smaller class of closed i -balls in barrelled spaces, like e.g. Fréchet spaces, consists of those closed i -balls, which do not absorb the whole space X but only a dense subspace. In this case $B_{r_q}(0)$ is obviously not a neighbourhood of zero. However, within the class of closed i -balls which absorb a dense subspace, also larger and smaller i -balls can be distinguished. For example, a closed i -ball can be a thick set or not: A subset of a topological space is called meagre, if it is a countable union of nowhere dense sets, where a set is called nowhere dense, if its closure has an empty interior. The Baire property is equivalent to the statement that a meagre set has no inner points. An infinitesimal analogue of a meagre set in a locally convex space is that of a non-norming set. A subset of a locally convex space is called norming, if its closed absolutely convex hull contains a neighbourhood of zero, and else it is called non-norming. A subset is called thin, if it is a countable union of non-norming sets, and else it is called thick. Now a metrizable locally convex space X is barrelled iff it is Baire-like, i.e. iff every increasing sequence A_n of closed absolutely convex subsets with $\bigcup_n A_n = X$ has an element containing a neighbourhood of zero. Thus a subset of a Fréchet or Banach space is thick, iff it spans a dense and barrelled subspace.

Hence those closed i -balls, which span a dense and barrelled subspace, are thick sets, and can be distinguished from those smaller closed i -balls, which are not thick sets. But maybe the thick closed i -balls are also automatically neighbourhoods of zero? To exclude this case, let us ask, whether a Fréchet space has always a nontrivial dense and barrelled subspace. This problem is solved positively for all nonnormable Fréchet spaces, but for Banach spaces the answer is open, as it is equivalent to the so called separable quotient problem: Is there always a subspace Y of a Banach space X such that X/Y is infinite-dimensional and separable? See [Nygaard] for a discussion.

However, often we consider even smaller closed i -balls, which do not span a dense and barrelled but only a dense subspace. Such closed i -balls are called dense closed i -balls. For example, consider the Fréchet space $C(\mathbb{R})$ endowed with the topology of uniform convergence on compact sets and the closed i -ball $B_1(0) := \{x \mid \forall t \in \mathbb{R} : |x(t)| \leq 1\}$, then $B_1(0)$ is obviously no neighbourhood of zero. But as it contains all functions with compact support whose values are smaller than 1, the span of $B_1(0)$ is a dense subspace. The main property of dense closed i -balls in complete locally convex spaces X is that they are large enough to recover the whole space X by scaling and completion, and that's why it is often good enough to consider dense closed i -balls instead of neighbourhoods of zero.

Finally note that $\lim_q r_q = 0$ implies $B_{r_q}(0) = \{0\}$ for every directed generating family of pseudonorms q . Thus directed families are not appropriate to define large closed i -balls. Instead, to define large closed i -balls, generating families of pseudonorms q having the property that only finitely many pseudonorms of the family intersect²⁶ should be used. For example, on $C(\mathbb{R})$ use the generating family $p_k(x) := \sup_{t \in [k, k+1]} |x(t)|$ of continuous pseudonorms, then the closed i -ball $B_{r_k}(0)$ spans a dense subset of $C(\mathbb{R})$ for every radius family r_k , as $B_{r_k}(0)$ contains the functions of compact support.

²⁶Two pseudonorms p, q are said to be disjoint, if $\overline{p^{-1}(0) + q^{-1}(0)} = X$ holds, or equivalently, if $r \leq p$ and $r \leq q$ implies $r = 0$, and they are said to intersect, if they are not disjoint.

Linear Differential Equations Let $A : X \rightarrow X$ be a continuous linear map on a complete separated locally convex topological vector space X . Contrary to the case of Banach spaces the differential equation $\dot{x} = Ax$ does not need to be solvable for all initial values $x(0)$ and all times. Indeed, the solution should be $\exp(tA)x(0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x(0)$, but this series does not need to converge: If $A \in L_{\sigma}(X)$, then convergence can merely be proved for those initial values $x(0)$ and times t , for which the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\prod_{m=0}^{n-1} \sigma^m(p)^{\sigma^{m+1}(p)}(A) \right) \sigma^n(p)(x(0))$$

converges. Thus - depending on the growth of $\sigma^n(p)(x(0))$ - a linear differential equation on a locally convex space could have no solution, a finite-time solution, or like in the Banach case an all-time solution.

Example: Define a continuous linear map $A : X \rightarrow X$ on the space X of fast falling both sided sequences x_k , $k \in \mathbb{Z}$, endowed with the pseudonorms $p_n(x) := \sup_k (1 + |k|)^n |x_k|$ by $(Ax)_k := a(k)x_k$ with a polynomial $a(k) = a_d k^d + \dots + a_0$ of degree d . From

$$p_n(Ax_k) = \sup_k (1 + |k|)^n |a(k)x_k| \leq \left(\sup_k \frac{|a(k)|}{(1 + |k|)^d} \right) \left(\sup_k (1 + |k|)^{n+d} |x_k| \right) = p_n^{\sigma(p_n)}(A) p_{n+d}(x_k)$$

and $\lim_{|k| \rightarrow \infty} \frac{|a(k)|}{1 + |k|^d} = |a_d|$ it can be deduced that A lies in $L_{\sigma}(X)$ for $\sigma(p_n) := p_{n+d}$ and satisfies $p_n^{\sigma(p_n)}(A) = \sup_k \frac{|a(k)|}{(1 + |k|)^d} < \infty$.

The possible solution of the differential equation $\dot{x} = Ax$ to the initial value $x(0)$ is $x_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} a(k)^n x_k(0) = \exp(ta(k))x_k(0)$, but this sequence $x_k(t)$ does not need to be fast falling for all t . For example, let $a(k) = k^2$ and $x_k(0) = \exp(-\epsilon k^2)$, then $x_k(t) = \exp(k^2(t - \epsilon))$ is a fast falling sequence only for $t < \epsilon$. A similar example, where no solution to an initial value exists for all times $t > 0$, can be found in [Kriegel, Michor, 32.12]. Note that these examples are not academic but relevant, as multiplication with polynomials in a space of sequences is intimately connected to differentiation in a space of functions. In fact, by Fourier transformation the space X of fast falling sequences is isomorphic to the space $C^{\infty}(S^1, \mathbb{R})$ endowed with the uniform convergence in arbitrary high derivatives, and under this identification the above defined operator A with $a(k) = k^2$ becomes the negative Laplacian $-\Delta$, i.e. solving $u_t = -\Delta u$ in $C^{\infty}(S^1, \mathbb{R})$ is equivalent to solving $\dot{x} = Ax$ in the space X of fast falling sequences. Thus it is not astonishing that in general there are no solutions, as the differential equation $u_t = -\Delta u$ is not well-posed forward in time. However, note also that we are mainly interested in the case where the different pseudonorms arise through convergence on compact subsets of a noncompact space, where such problems are rarer.

Uniform Boundedness Principle Let X be a complete and separated locally convex topological vector space, and let $H \subset L(X, Y)$ be a set of continuous linear maps from X into a locally convex topological vector space Y . Let q be a continuous pseudonorm on Y and assume $\sup_{T \in H} q(Tx) < \infty$ for each $x \in X$, then there is generally no continuous pseudonorm p on X with $\sup_{T \in H} q^p(T) < \infty$, i.e. in general the uniform boundedness principle is not valid.

But if the topology of X is generated by a countable set of pseudonorms p_n and thus X is completely metrizable, then X is called a Fréchet space and the uniform boundedness principle holds. Indeed, fix a continuous pseudonorm q on Y . Each set $M_C := \{x \in X \mid \forall T \in H : q(Tx) \leq C\}$ is closed, because $M_C = \bigcap_{T \in H} \{x \mid q(Tx) \leq C\}$ is an intersection of closed sets. Further due to the assumption $\bigcup_{C \in \mathbb{N}} M_C = X$ is valid. By Baire's category theorem there is a continuous pseudonorm p on X , a positive $\epsilon > 0$ and a point $x_0 \in X$ such that $\{x \mid p(x - x_0) \leq \epsilon\} \subset M_C$. In other words, $p(x - x_0) \leq \epsilon$ implies $q(Tx) \leq C$ for all $T \in H$. Thus $p(x) \leq \epsilon$ implies $q(Tx) \leq q(T(x + x_0)) + q(Tx_0) \leq C + C_{x_0}$ for all $T \in H$, i.e. $q(Tx) \leq \frac{C + C_{x_0}}{\epsilon} p(x)$ holds for all $T \in H$ and all $x \in X$. Hence to each q the existence of a p with $\sup_{T \in H} q^p(T) < \infty$ has been proved.

Especially the pointwise limit of continuous linear maps between Fréchet spaces is a continuous linear map. Indeed, if \mathcal{F} is a filter on $L(X, Y)$ with $\mathcal{F}x \rightarrow Tx$ for every $x \in X$, then trivially T is linear. Further for every continuous pseudonorm q on Y and every $x \in X$ there is a $U \in \mathcal{F}$ such that $\sup_{T' \in U} q(T'x) < \infty$. Thus there is a continuous pseudonorm p on X with $\sup_{T' \in U} q^p(T') < \infty$ by the uniform boundedness principle, so that $q^p(T) < \infty$ is valid and hence T is continuous.

For Fréchet spaces not only the uniform boundedness principle is valid, but also the open mapping theorem and the closed graph theorem can be proved like in the Banach case, because the Baire property is valid. As a consequence, an algebraic decomposition $X = A \oplus B$ of a Fréchet space X into closed subspaces A, B is automatically a topological decomposition, i.e. there exist continuous linear projections onto A and B ²⁷. But as in the Banach case not every closed subspace A has a closed complement B .

In the more general case of a complete and separated locally convex topological vector space, not every algebraic decomposition is automatically a topological decomposition. However, finite-dimensional subspaces A always admit closed complements and continuous projections: Due to its completeness a finite-dimensional subspace A is automatically closed in X . Choose continuous linear functionals $x' := (x'_1, \dots, x'_n)$ such that $x'|_A : A \rightarrow \mathbb{R}^n$ is an isomorphism, then the closed subspace defined by $B := \bigcap_{i=1}^n \text{Ker}(x'_i)$ is a complement to A and there is a continuous projection onto A . Indeed, by using the inclusion $\iota : A \rightarrow X$ define the map $Px := \iota((x'|_A)^{-1}(x'(x)))$ on X . Then P is a continuous linear map satisfying $P(X) = A$, $Px = x$ for $x \in A$ and $x - Px \in B$ for all $x \in X$ (because of $x'(x - Px) = x'(x) - x'(x) = 0$). Thus P is a continuous linear projection onto A .

²⁷For Banach or Fréchet space the open mapping theorem is valid, so that algebraic decompositions into closed subspaces are automatically topological: If $X = A \oplus B$ is algebraically valid, then due to the closedness of A, B the space $A \oplus B$ endowed with the pseudonorms $p(a, b) := p(a) + p(b)$, p pseudonorm on X , is a Banach or Fréchet space. Further the map $A \oplus B \ni (a, b) \mapsto a + b \in X$ is linear, bijective and continuous due to the triangle inequality $p(a + b) \leq p(a) + p(b) = p(a, b)$, so that it has a continuous inverse by the open mapping theorem. Thus $A \oplus B$ and X are also topologically isomorphic by $(a, b) \mapsto a + b$. As a consequence, to every continuous pseudonorm p there is a continuous pseudonorm q with $p(a) + p(b) \leq Cq(a + b)$, hence also the projections $a + b \mapsto a$ and $a + b \mapsto b$ are continuous.

Pseudo-Banach and pseudo-Hilbert Spaces A pseudo-Banach space is a complete and separated locally convex topological vector space $X = \lim_i X_i$, which is a projective limit of Banach spaces $(X_i, \|\cdot\|_i)$ such that the restrictions $|_j : X_i \rightarrow X_j$ ($j < i$) are contractions, i.e. $\|x_i|_j\|_j \leq \|x_i\|_i$ is satisfied for all $j \leq i$ and $x_i \in X_i$. If all X_i are Hilbert spaces, then X is called a pseudo-Hilbert space. For example, the space $L^2_{loc}(M)$ of locally square integrable functions on a manifold M is the projective limit $\lim_{\Omega \subset M} L^2(\Omega)$ of the Hilbert spaces $L^2(\Omega)$, where Ω runs through the precompact domains in M . Because of $\int_{\Omega} |x|^2 \leq \int_{\Omega'} |x|^2$ for $\Omega \subset \Omega'$ the restrictions $|_{\Omega'} : L^2(\Omega') \rightarrow L^2(\Omega)$ are contractions, so that $L^2_{loc}(\Omega)$ is a pseudo-Hilbert space.

Denote by $|_i : X \rightarrow X_i$ the restriction of X onto X_i , then the topology on X is generated by the pseudonorms $p_i(x) := \|x|_i\|_i$. In the case of a pseudo-Hilbert space these pseudonorms satisfy the parallelogram equality $p_i(x+y)^2 + p_i(x-y)^2 = 2p_i(x)^2 + 2p_i(y)^2$, or equivalently are induced by pseudo-scalar products $p_i(\cdot, \cdot)$ via $p_i(x) = \sqrt{p_i(x, x)}$. However, not only topological properties of X are fixed by its representation as a projective limit of Hilbert spaces, but also the geometry of X is determined due to the fact that the restrictions were required to be contractions.

Indeed, the geometry of a Banach or Hilbert space is not determined by its topological structure only, but depends on the choice of the norm or scalar product, as in general the orthogonal projections²⁸ are different for equivalent but different norms or scalar products. Now the requirement that for a pseudo-Hilbert space the restrictions are contractions allows to define the norm function $p(x) := \lim_i p_i(x)$ on X with values in $[0, \infty]$, because the net²⁹ $i \mapsto p_i(x)$ on \mathbb{R} is monotone increasing due to the contraction property $p_j(x) \leq p_i(x)$ for $j \leq i$, and thus $\lim_i p_i(x)$ converges either to some number in \mathbb{R}_0^+ or to infinity.

The norm function p on X with values in $[0, \infty]$ is the starting point for the discussion of geometry on a pseudo-Hilbert space: Let K be a subset of X . An element $Px \in K$ is called a best approximation of x by elements of K if $p(x - Px) = \inf_{y \in K} p(x - y) < \infty$ is valid. Obviously $\inf_{y \in K} p(x - y)$ is smaller than infinity only, if there is an element $y \in K$ with $p(x - y) < \infty$. If there is no such $y \in K$, then every element of K approximates x as bad as every other element of K . Let us say in this case that x cannot be approximated by elements of K .

Theorem 4.1 *Let X be a pseudo-Hilbert space, let $x \in X$ and let K be a closed and convex subset of X . Then either x cannot be approximated by elements of K or there is a unique best approximation Px of x by elements of K .*

Proof: Let $d := \inf_{y \in K} p(x - y) < \infty$ and let $y_n \in K$ be a sequence with $p(x - y_n) \rightarrow d$ for $n \rightarrow \infty$, then

$$p(y_n - y_m)^2 = 2p(x - y_m)^2 + 2p(x - y_n)^2 - 4p(x - \frac{y_n + y_m}{2})^2 \rightarrow 0 \quad (3)$$

²⁸Call a projection P on a Banach space orthogonal if $\|P\| = 1$. For Hilbert spaces $\|P\| = 1$ is equivalent to $P^* = P$ and thus to orthogonality in the usual sense.

²⁹A map f from a directed set A into a set X is called a net on X , and every subset of the form $\{f(a) | a \geq a_0\} \subset X$ is called an endpiece of the net f . The end pieces of a net generate a filter on X , and a net on a space X endowed with a convergence relation is called convergent if the filter generated by its endpieces converges.

is valid for $n, m \rightarrow \infty$ by the parallelogram equality, the convergences $p(x - y_m)^2$, $p(x - y_n)^2 \rightarrow d^2$ and the inequality $p(x - \frac{y_n + y_m}{2})^2 \geq d^2$. This inequality holds because the convexity of K implies $\frac{y_n + y_m}{2} \in K$ and hence $p(x - \frac{y_n + y_m}{2})^2 \geq d^2$. Thus by the above formula the sequence y_n is a Cauchy sequence, because $p_i(y_n - y_m) \leq p(y_n - y_m) \rightarrow 0$ is valid for $n, m \rightarrow \infty$ and every i . Hence the sequence y_n has a limit Px by completeness of X , and Px lies in K because of $y_n \in K$ and the closedness of K . Moreover Px satisfies $p(x - Px) = d$, as $p_i(x - Px) = \lim_n p_i(x - y_n) \leq \lim_n p(x - y_n) = d$ is valid for all i , and thus on the one hand $\lim_p p(x - Px) \leq d$ holds, while on the other hand $p(x - Px) \geq d$ is valid because of $Px \in K$. Finally the best approximation is unique, because if there were another best approximation Px' , then

$$\begin{aligned} p(x - \frac{1}{2}(Px + Px'))^2 &< p(x - \frac{1}{2}(Px + Px'))^2 + p(\frac{1}{2}(Px - Px'))^2 = \\ &\frac{1}{2}p(x - Px)^2 + \frac{1}{2}p(x - Px')^2 = d^2 \end{aligned}$$

would be valid by the parallelogram equality and the fact that there is an index i with $0 \neq p_i(Px - Px') \leq p(Px - Px')$. Thus $\frac{1}{2}(Px + Px') \in K$ would be a better approximation of x by elements of K , in contradiction to $d := \inf_{y \in K} p(y - x)$. \square

Thus whenever K is closed, convex and contains an element $y \in K$ with $\lim_p p(y - x) < \infty$, there is a unique best approximation Px of x by elements of K , and if K does not contain such an element y , then obviously every element of K is a bad approximation of x .

On Hilbert spaces best approximations can be used to characterize orthogonal projections. For pseudo-Hilbert spaces X an analogous result can be proved: Let K be a closed and convex subset of X and assume that for $x \in X$ there is a $y \in K$ such that $p(x - y) < \infty$. Denote by Px the best approximation of x by elements of K , then $p(x - Px, y - Px) \leq 0$ holds for all $y \in K$ with $p(y - Px) < \infty$, because

$$\begin{aligned} p(x - Px)^2 &\leq p(x - ((1 - t)Px + ty))^2 = p(x - Px + t(Px - y))^2 = \\ &p(x - Px)^2 + 2tp(x - Px, Px - y) + t^2p(y - Px)^2 \end{aligned}$$

implies $p(x - Px, y - Px) \leq \frac{t}{2}p(y - Px)^2$ for all $t \in [0, 1]$. Conversely, if $x' \in K$ is such that $p(x - x') < \infty$ and $p(x - x', y - x') \leq 0$ are valid for all $y \in K$ with $p(y - x') < \infty$, then

$$\begin{aligned} p(x - y)^2 &= p((x - x') - (y - x'))^2 = \\ &p(x - x')^2 - 2p(x - x', y - x') + p(y - x')^2 \geq p(x - x')^2 \end{aligned}$$

holds for all $y \in K$ with $p(y - x') < \infty$, i.e. x' is the best approximation. Thus the best approximation can be characterized as the unique element Px of K with $p(x - Px) < \infty$ and $p(x - Px, y - Px) \leq 0$ for all $y \in K$ with $p(y - Px) < \infty$.

Moreover, if K is a closed subspace, the last condition can be replaced by $p(x - Px, y) = 0$ for all $y \in K$ with $p(y) < \infty$, choose simply $\pm y + Px$ instead of y . To a subspace $K \subset X$ define its orthogonal complement K^\perp by $z \in K^\perp$ whenever $p(z) < \infty$ and $p(z, y) = 0$ is valid for all $y \in K$ with $p(y) < \infty$. Then K^\perp is a subspace (however in general it is

not closed in X) and the operator P , which to x assigns its best approximation Px by elements of K , is characterized by $x - Px \in K^\perp$. Note that $K \cap K^\perp = \{0\}$ holds, as $y \in K \cap K^\perp$ implies $p(y) < \infty$ and $p(y, y) = 0$, i.e. $y = 0$. Thus $K + K^\perp$ is a direct sum and the largest subspace of X whose elements have a best approximation by elements of K .

The operator P from $K + K^\perp$ to K is a projection because of $P(Px) = Px$ and linear because of $x + x' - (Px + Px') = (x - Px) + (x' - Px') \in K^\perp$, i.e. $P(x + x') = Px + Px'$, and $\lambda x - \lambda Px \in K^\perp$, i.e. $P(\lambda x) = \lambda Px$. However, P is continuous only w.r.t. the topology on $K + K^\perp$, where points x, x' with $p(x - x') = \infty$ lie in different components and each component is the complete metric space endowed with the metric $d(x, x') := p(x - x')$ ³⁰. Note that each component contains a copy of K^\perp , that the component containing 0 is in fact a Hilbert space, and thus also every other component is a Hilbert manifold.

Spectral Theory Let X be a complete and separated locally convex topological vector space over the field \mathbb{C} of complex numbers. A value $\lambda \in \mathbb{C}$ is said to be an element of the resolvent set of an operator $T \in L(X)$, if $\lambda - T$ is continuously invertible³¹, and else λ is said to lie in the spectrum $\Sigma(T)$ of T . Contrary to the case of Banach spaces, in general the spectrum $\Sigma(T) \subset \mathbb{C}$ of an operator T on a complete and separated locally convex topological vector space may be unbounded or empty, and might not be closed, see [Maeda] for corresponding examples. However, the spectral radius $r(T) := \sup_{\lambda \in \Sigma(T)} |\lambda|$ can generally be estimated by the convergence radius of the Neumann series.

Let $T \in L(X)$ be an operator, then the convergence radius $r_N(T) \in [0, \infty]$ of the Neumann series $\lambda \mapsto \sum_n T^n / \lambda^n$ in $L(X)$ is given by the formula $r_N(T) = \inf\{|\lambda| \mid T^n / \lambda^n \rightarrow 0\}$ or equivalently by $r_N(T) = \sup_q \inf_p \limsup_n \sqrt[p]{q^p(T^n)}$ ³². Indeed, $|\lambda| > r_N(T)$ implies the convergence of the Neumann series: Choose $h < 1$ such that $|\lambda|h > r_N(T)$, then $T^n / (\lambda h)^n \rightarrow 0$ in some $L_\tau(X)$ by definition of $r_N(T)$, so that to every continuous pseudonorm q on X and every $\epsilon > 0$ there is a $N \in \mathbb{N}$ with $q^{r(q)}(T^n / \lambda^n) \leq h^n \epsilon$ for all

³⁰Indeed, let x, x' be in the same component. The equality $p(x - x')^2 = p(P(x - x'))^2 + 2p(P(x - x'), (x - x') - P(x - x')) + p((x - x') - P(x - x'))^2$ is valid, and as $p((x - x') - P(x - x'), P(x - x')) = 0$ holds because of $p(P(x - x')) \leq p(Px - x) + p(x - x') + p(x' - Px') < \infty$ ((Px, x) , (x, x') and (x', Px') lie in the same component) and due to the orthogonality characterization, the equations $p(P(x - x'))^2 \leq p(x - x')^2$ and $p((x - x') - P(x - x'))^2 \leq p(x - x')^2$ are valid, i.e. P and $\text{Id} - P$ are continuous w.r.t. the metric $d(x, x') = p(x - x')$ on each component. Finally each component is complete, because a Cauchy sequence x_n in a component is also a Cauchy sequence in X due to $p_i(x_n - x_m) \leq p(x_n - x_m)$ for each i . Now completeness of X implies that x_n has a limit x in X , and this limit lies in the same component as all x_m because of $p_i(x - x_m) = \lim_n p_i(x_n - x_m) \leq \lim_n p(x_n - x_m) \leq C_m$ for all i with a constant C_m independent of i , i.e. $p(x - x_m) \leq C_m < \infty$ is valid.

³¹If T is an operator on a Fréchet space X , then invertibility implies continuous invertibility by the open mapping theorem, thus a value λ lies in the resolvent set if $\lambda - T$ is bijective.

³²Proof:

$$\begin{aligned} \inf\{|\lambda| \mid T^n / \lambda^n \rightarrow 0\} &= \inf\{|\lambda| \mid \forall q \exists p : q^p(T^n) / |\lambda|^n \rightarrow 0\} = \\ \sup_q \inf_p \inf\{|\lambda| \mid \lim_n q^p(T^n) / |\lambda|^n = 0\} &= \sup_q \inf_p \limsup_n \sqrt[p]{q^p(T^n)} \end{aligned}$$

$n > N$. Further $\sum_n h^n$ is convergent because of $h < 1$, so that to every $\epsilon > 0$ there is an N such that $\sum_{n=m}^{\infty} h^k \leq 1$ for all $m > N$. Thus to every q and $\epsilon > 0$ there is an N such that

$$q^{\tau(q)} \left(\sum_{n=m}^{m'} T^n / \lambda^n \right) \leq \sum_{n=m}^{m'} q^{\tau(q)} (T^n / \lambda^n) \leq \sum_{n=m}^{\infty} h^n \epsilon \leq \epsilon$$

for all $m, m' > N$. Hence the Neumann series $\sum_n T^n / \lambda^n$ is Cauchy and converges in $L_\tau(X)$ for $|\lambda| > r_N(T)$. On the other hand, convergence of $\sum_n T^n / \lambda^n$ implies $T^n / \lambda^n \rightarrow 0$ and thus $|\lambda| \geq r_N(T)$, so that $r_N(T)$ is really the convergence radius of the Neumann series.

Now $|\lambda| > r_N(T)$ implies the existence of the limit $1/\lambda \sum_{n=0}^{\infty} T^n / \lambda^n$, and this limit is the inverse of $\lambda - T$. Indeed,

$$(\lambda - T) \left(1/\lambda \sum_{n=0}^m T^n / \lambda^n \right) = \sum_{n=0}^m T^n / \lambda^n - \sum_{n=1}^{m+1} T^n / \lambda^n = 1 - \frac{T^{m+1}}{\lambda^{m+1}} \rightarrow 1$$

holds for $m \rightarrow \infty$, so that by continuity of composition the limit is right-inverse to $\lambda - T$, and by commutativity of the operators $\lambda - T$ and T also left-inverse. Thus every λ with $|\lambda| > r_N(T)$ is a resolvent value of T and $r_N(T) \geq r(T)$ is valid.

Hence the convergence radius $r_N(T)$ of the Neumann series bounds the spectrum of T , but contrary to the Banach case $r_N(T) = \infty$ and $r_N(T) > r(T)$ are possible. In the Banach case the finiteness of $r_N(T)$ can be deduced from $r_N(T) \leq \|T\|$. This inequality can be generalized as follows: Suppose that $T \in L_\sigma(X)$ and that τ bounds σ^n for all $n \in \mathbb{N}$, i.e. $C_n := \sigma^n(p)^{\tau(p)}(\text{Id}) < \infty$ is valid for all $n \in \mathbb{N}$. Then $p^{\tau(p)}(A) \leq p^{\sigma^n(p)}(A) C_n$ holds for every operator A . Thus $p^{\tau(p)}(T^n) \leq C_n \prod_{k=0}^{n-1} \sigma^k(p)^{\sigma^{k+1}(p)}(T)$ is valid, and hence $r_N(T)$ can be estimated by

$$r_N(T) \leq \inf \left\{ |\lambda| \mid \left(C_n \prod_{k=0}^{n-1} \sigma^k(p)^{\sigma^{k+1}(p)}(T) \right) / \lambda^n \rightarrow 0 \right\} .$$

For values λ satisfying the growth condition on the right hand side, the operator $(\lambda - T)^{-1}$ lies in $L_\tau(X)$, as the growth condition implies the convergence $T^n / \lambda^n \rightarrow 0$ in $L_\tau(X)$. But there are operators $T \in L_\sigma(X)$, for which σ^n cannot be bounded by any τ , or for which the growth of the numerator on the right hand side is larger than every exponential growth $n \mapsto \lambda^n$, and for such operators T the finiteness $r_N(T) < \infty$ cannot be deduced.

Example: The operator $T : x(t) \mapsto ax(-t)$, $a \neq 0$, on $X := C(\mathbb{R}, \mathbb{C})$ satisfies $T \in L_\sigma(X)$ for $\sigma(\sup_K) = \sup_{-K}$ and $p^{\sigma(p)}(T) = |a|$ is valid for all $p = \sup_K$. Now $\sigma^n(\sup_K)(x) \leq \sup_{K \cup -K}(x) =: \tau(\sup_K)$ holds and $|a|^n / |\lambda|^n$ converges to zero for all $|\lambda| > |a|$. Thus the spectrum of T is contained in the circle of radius $|a|$. Indeed, $(Tx)(t) = ax(-t) = \lambda x(t)$ is valid for $\lambda = \pm a$ and even resp. odd functions $x(t)$, so that in this case the Neumann bound of the spectrum by a convergence radius is the best possible.

Example: The operator $T : x(t) \mapsto ax(t+1)$, $a > 0$, on $X = C(\mathbb{R}, \mathbb{C})$ satisfies $T \in L_\sigma(X)$ for $\sigma(\sup_K) = \sup_{K+1}$ and $p^{\sigma(p)}(T) = a$ is valid for all $p = \sup_K$. But

as there is no pseudonorm, which bounds \sup_{K+n} for every n , our method yields no bound of the spectrum of T . In fact, $(Tx)(t) = ax(t+1) = \lambda x(t)$ is valid for every $x(t) := c(t)(\frac{\lambda}{a})^t$ with a 1-periodic function c , thus every $\lambda > 0$ is an eigenvalue and the spectrum of T is unbounded.

However, $\sigma^n(\sup_{[a,b]})(x) \leq \sup_{[a,+\infty)}(x)$ is valid on the space of those $x \in C(\mathbb{R}, \mathbb{C})$ which are bounded at $+\infty$, and T restricted to this space has a spectrum bounded by $|a|$, because the series $|a|^n/|\lambda|^n$ converges to zero for all $|\lambda| > |a|$. Indeed, for $\lambda > a$ the functions $t \mapsto (\frac{\lambda}{a})^t$ are not bounded at $+\infty$.

As in the case of Banach spaces, the operator norm of T can be chosen arbitrarily near to $r_N(T)$. Indeed, let $R > r_N(T)$, then $T^n/R^n \rightarrow 0$ in some L_τ . Thus the supremum $\tau'(p)(x) := \sup_{n \geq 0} p(T^n/R^n x)$ exists for every continuous pseudonorm p and every $x \in X$, and $\tau'(p)$ is a continuous pseudonorm on X , because the inequality $\tau'(p)(x) \leq (\sup_{n \geq 0} p^{\tau(p)}(T^n/R^n)) \tau(p)(x)$ is valid with the continuous pseudonorm $\tau(p)$. Finally from $p(x) \leq \tau'(p)(x)$ ($n = 0$) and

$$\begin{aligned} p(Tx) \leq \tau'(p)(Tx) &= \sup_{n \geq 0} p(T^n/R^n Tx) = R \sup_{n \geq 0} p(T^{n+1}/R^{n+1}x) \leq \\ &R \sup_{n \geq 0} p(T^n/R^n x) = R\tau'(p)(x) \end{aligned}$$

the inequality $p^{\tau'(p)}(T) \leq R$ can be deduced for all p .

The most important property of the resolvent $\lambda' \mapsto (\lambda' - T)^{-1}$ on Banach spaces is its analyticity. Let us clarify, on which parts of the resolvent set the resolvent is also analytic in the more general case of a complete and separated locally convex topological vector space X .

A value λ in the resolvent set of T is called analytic, if it satisfies $r_N((\lambda - T)^{-1}) < \infty$, because this condition implies the analyticity of the resolvent $\lambda' \mapsto (\lambda' - T)^{-1} \in L(X)$ in λ . Indeed, let λ' satisfy $r_N((\lambda - T)^{-1}) < 1/|\lambda - \lambda'|$, then there is a $h < 1$ such that $r_N((\lambda - T)^{-1}) < h/|\lambda - \lambda'|$. Thus $((\lambda - \lambda')(\lambda - T)^{-1})^n/h^n \rightarrow 0$ in some $L_\tau(X)$, and again due to the convergence of $\sum_n h^n$, to every continuous pseudonorm q and every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $q^{\tau(q)}(\sum_{n=m}^{m'} ((\lambda - \lambda')(\lambda - T)^{-1})^n) \leq \epsilon$ for all $m, m' > N$. Hence $\sum_n ((\lambda - T)^{-1})^n (\lambda - \lambda')^n$ is Cauchy and converges in the complete space $L_\tau(X)$ for every $|\lambda - \lambda'| < 1/r_N((\lambda - T)^{-1})$, so that this series defines an analytic function depending on λ' near λ . As the limit of this series is $(\text{Id} - (\lambda - \lambda')(\lambda - T)^{-1})^{-1}$ and as the equality $(\lambda' - T) = (\lambda - T)(\text{Id} - (\lambda - \lambda')(\lambda - T)^{-1})$ is valid, also $\lambda' \mapsto (\lambda' - T)^{-1} \in L_\tau(X)$ is an analytic function within the circle of radius $1/r_N((\lambda - T)^{-1})$ around λ .

Consequently the analytic part of the resolvent set is an open subset of \mathbb{C} and the resolvent is an analytic function on it. Denote by $r_A(T)$ the supremum of all absolute values $|\lambda|$ for λ lying in the complement of the analytic part of the resolvent set, then $r_A(T) \geq r_N(T)$ is valid. Indeed, $f: \lambda \mapsto (\lambda - T)^{-1}$ is an analytic function on $\{\lambda \mid |\lambda| > r_A(T)\}$. As $(\lambda - T)^{-1}$ is represented by the series $1/\lambda \sum_n T^n/\lambda^n$ having convergence radius $r_N(T)$, but every representing series of an analytic function on $\{\lambda \mid |\lambda| > r_A(T)\}$ automatically converges outside the circle with radius $r_A(T)$, the inequality $r_A(T) \geq r_N(T)$ can be concluded.

Analyticity of the resolvent can be used to prove the existence of spectral projections for decompositions of the spectrum of T into two disjoint closed parts $\Sigma(T) = \Sigma_1 \cup \Sigma_2$, which are separated by the analytic part of the resolvent set, i.e. there is a cycle γ within the analytic part of the resolvent set, which once winds around points of Σ_1 and zero times around points of Σ_2 . Then the continuous linear operator $P := \frac{1}{2\pi i} \int_{\gamma} (\lambda - T)^{-1} d\lambda$ is well-defined and does not depend on the choice of γ . Indeed, Riemannian integration with values in (complete and separated) locally convex limit vector spaces like $L(X)$ has been defined before, the integrand $(\lambda - T)^{-1}$ is analytic, and by Cauchy's theorem the integral is independent of a concrete choice of γ . As in the Banach case, now the properties of P can be proved, see e.g. [Lanford, Proposition 2.3.1]: P satisfies $P^2 = P$ and thus is a projection called the spectral projection onto the generalized eigenspace $P(X)$ of Σ_1 , P commutes with every operator commuting with T , T leaves invariant the closed decomposition $X = P(X) \oplus \text{Ker}(P)$, and T restricted to $P(X)$ resp. $\text{Ker}(P)$ has the spectrum Σ_1 resp. Σ_2 .

Finally, if T is an operator on a real space X , consider its complexification $T^{\mathbb{C}}$ on the complexified space $X^{\mathbb{C}}$. If $\Sigma(T) = \Sigma_1 \cup \Sigma_2$ is a decomposition of the spectrum of T into disjoint closed sets, which are separated by the analytic part of the resolvent set and are invariant under conjugation, then the spectral projection commutes with conjugation and thus is the complexification of a projection P on the real space X . Thus also in the real case there are spectral projections.

5 Nonlinear Analysis

In this chapter basic theorems of nonlinear analysis on locally convex vector spaces are proved. First we discuss integration and differentiation. Afterwards the contraction mapping principle is generalized to locally convex topological vector spaces. As a consequence we can prove the local existence and uniqueness of solutions of differential equations to initial values, which satisfy a growth condition, and an implicit function theorem on intersections of balls (shortly: on i -balls). Finally manifolds modeled over dense i -balls in locally convex vector spaces are defined.

5.1 Integration

The aim of this section is to discuss the integration of maps having values in a vector space X endowed with a compatible topological structure. On the one hand the componentwise Lebesgue integral of maps on a measure space (M, μ) with values in \mathbb{R}^n is generalized to an integral of maps with values in a separated locally convex topological vector space. On the other hand Riemannian integration on complete and separated locally convex limit vector spaces is mentioned. Both integrals coincide for continuous maps on compact intervals with values in a complete separated locally convex topological vector space, and thus in many parts of this thesis it is not important which notion of an integral is used.

5.1.1 Lebesgue integration of maps with values in separated locally convex topological vector spaces

For a separated locally convex topological vector space X the map $j : X \rightarrow (X')^*$, $x \mapsto (x' \mapsto x'(x))$, into the algebraic dual $(*)$ of the topological dual $(')$ of X is injective by the theorem of Hahn-Banach: There are enough continuous linear functionals to separate points in X . For which locally convex topologies on X' the map j is onto $X'' \subset (X')^*$ and therefore surjective as a map $j : X \rightarrow X''$, is answered by the following theorem of Mackey-Arens.

Theorem 5.1 *Let X be a separated locally convex topological vector space. For a locally convex topology on X' the map $j : X \rightarrow X''$ is surjective iff the topology on X' is finer than the weak topology $\sigma(X', j(X))$ and coarser than the Mackey topology $\tau(X', j(X))$.*

Proof: Use theorem [Heuser, 70.3] for the dual system $(X', j(X))$. □

Therefore the Mackey topology $\tau(X', j(X))$ is the finest locally convex topology on X' with $X'' = j(X)$. A basis of its neighbourhood filter at zero is given by the polar sets

$$K^o := \{x' \mid \sup_{x \in K} |x'(x)| \leq 1\}$$

to weakly compact³³ and absolutely convex³⁴ sets $K \subset X$. Note that the absolute convex hull $cb(K)$ of a (weakly) compact set $K \subset X$ is in general not relatively (weakly) compact, so that there are less absolutely convex and (weakly) compact sets than merely (weakly) compact sets. However, for Fréchet spaces X the absolutely convex hull of a compact set is again relatively compact.

Now let (M, μ) be a measure space. The following definition of a X -valued integral generalizes the componentwise integral of maps with values in $X = \mathbb{R}^n$ ³⁵: A map $f : M \rightarrow X$ is called integrable if for every $x' \in X'$ the function $x' \circ f : M \rightarrow \mathbb{R}$ is integrable and $x' \mapsto \int_M (x' \circ f) d\mu$ is continuous as a linear mapping from X' to \mathbb{R} with respect to the Mackey topology on X' . Thus for an integrable map f the element

$$\int_M f d\mu := j^{-1} \left(x' \mapsto \int_M (x' \circ f) d\mu \right)$$

of X exists and is called the integral of f . Because the Mackey topology on X' is the finest with $X'' = j(X)$, the so defined class of integrable X -valued functions is the largest possible class which can be obtained by componentwise integration.

By construction the formula $x'(\int_M f d\mu) = \int_M (x' \circ f) d\mu$ holds for all $x' \in X'$, and also $A(\int_M f d\mu) = \int_M (A \circ f) d\mu$ is valid for every continuous linear map A between separated locally convex topological vector spaces. Further for a topological space M with Borel measure μ a continuous function f from M to X is integrable over a compact set $K \subset M$, if the absolutely convex hull of $f(K)$ is relatively compact. Indeed, let $f : M \rightarrow X$ be continuous, let K be compact and let the closure $\overline{cb(f(K))}$ of the absolutely convex hull of the compact set $f(K)$ be compact. Obviously the continuous real valued function $x' \circ f|_K$ on the compact set K is integrable for every $x' \in X'$. To show continuity of $x' \mapsto \int_M (x' \circ f_K) d\mu$ w.r.t. the Mackey topology, note that the compact set $\overline{cb(f_K(M))}$ is especially weakly compact³⁶. Now let the filter \mathcal{F} on X' converge to x' in the Mackey topology, then to every ϵ there exists a set $U \in \mathcal{F}$ with $\sup_{x \in \overline{cb(f_K(M))}} |y'(x) - x'(x)| \leq \epsilon$ for all $y' \in U$, because the Mackey topology is the topology of uniform convergence on weakly compact and absolute convex subsets of X . Therefore

$$\left| \int_K (y' \circ f) d\mu - \int_K (x' \circ f) d\mu \right| \leq \int_K |(y' - x') \circ f| d\mu \leq \mu(K)\epsilon$$

³³A set $K \subset X$ is called weakly compact iff it is compact with respect to the weak topology $\sigma(X, X')$ on X .

³⁴A set $K \subset X$ is called absolutely convex iff it is convex and balanced, i.e. $x, y \in K$ and $0 \leq \lambda \leq 1$ imply $(1 - \lambda)x + \lambda y \in K$ and $\pm \lambda x \in K$.

³⁵For a function $f : M \rightarrow \mathbb{R}^n$ the componentwise integral is defined by

$$\int_M f d\mu = \left(\int_M (e'_1 \circ f) d\mu, \dots, \int_M (e'_n \circ f) d\mu \right)$$

where e'_i denotes the dual basis to the standard basis e_i of \mathbb{R}^n .

³⁶The weak topology on X is coarser than the primary topology, hence every open covering in the weak topology is an open covering in the primary topology. Thus for a compact set $K \subset X$ w.r.t. the primary topology, in every covering with weakly open sets a finite covering can be found, so that K is also weakly compact.

is valid and implies the convergence of the image of \mathcal{F} under the map $y' \mapsto \int_K (y' \circ f) d\mu$ to $\int_K (x' \circ f) d\mu$. Thus the continuity of this map w.r.t. the Mackey topology on X' is proved. The same argumentation even shows that weakly continuous functions $f : M \rightarrow X$ are integrable over a compact set $K \subset M$, if the absolutely convex hull $cb(f(K))$ of the weakly compact set $f(K)$ is relatively weakly compact.

Further recall that for Fréchet spaces X the absolutely convex hull of the compact set $f(K) \subset X$ is automatically relatively compact, thus the condition is not required and hence every continuous map f into a Fréchet space is integrable over compact sets. Finally let's prove that the property $|\int_M f d\mu| \leq \int_M |f| d\mu$ of the real valued integral generalises to the X -valued case.

Lemma 5.2 *Let p be a continuous seminorm on X . Then $p(\int_M f d\mu) \leq \int_M (p \circ f) d\mu$ holds.*

Proof: For the element $\int_M f d\mu$ of X there exists a $x' \in X'$ with $x'(\int_M f d\mu) = p(\int_M f d\mu)$ and $|x'(x)| \leq p(x)$ for all $x \in X$ by the theorem of Hahn-Banach. With such a continuous linear functional x' the inequality

$$p(\int_M f d\mu) = |x'(\int_M f d\mu)| = |\int_M (x' \circ f) d\mu| \leq \int_M |x' \circ f| d\mu \leq \int_M (p \circ f) d\mu \quad .$$

holds. □

Integration of maps with values in reflexive Banach spaces For reflexive Banach spaces there exists an easier derivable, but not so powerful theory of integration. Note that a Banach space X is reflexive iff the topology induced by the norm on X' is coarser than the Mackey topology. In this case the definition

$$\int_M f d\mu := j^{-1} \left(x' \mapsto \int_M (x' \circ f) d\mu \right)$$

can certainly be used for functions f with integrable $x' \circ f$ for all $x' \in X'$ and norm-continuous mapping $x' \mapsto \int_M (x' \circ f) d\mu$. Both conditions are satisfied, if the real valued function $\|f\|$ is integrable, because

$$|\int_M (x' \circ f) d\mu| \leq \int_M |x' \circ f| d\mu \leq \left(\int_M \|f\| d\mu \right) \|x'\| \quad .$$

holds. However, note that the integrability of $\|f\|$ is a stronger condition than the one given above, even for a reflexive Banach spaces.

5.1.2 Riemannian integration on complete and separated locally convex limit vector spaces

While in the former paragraphs integration w.r.t. a measure has been generalized to functions with values in separated locally convex topological vector spaces, Riemannian integration shall also be considered for the reader's convenience. Let X be a complete and separated locally convex limit vector space. For a function f on a compact interval $[a, b]$ with values in X , for subdivisions $Z = \{a = t_0 < \dots < t_n = b\}$ with fineness $|Z| := \max_{i=0, \dots, n-1} (t_{i+1} - t_i)$ and for $\delta > 0$ define the subsets $S_Z := \sum_{i=0}^{n-1} (t_{i+1} - t_i) f([t_i, t_{i+1}])$ and $S(\delta) := \bigcup_{|Z| \leq \delta} S_Z$ of X . Obviously $(S(\delta))_{\delta > 0}$ is a filter base, and for continuous f the so defined filter is, in fact, a Cauchy filter w.r.t. the induced uniform convergence structure on X .

Indeed, recall that a Cauchy filter on a limit vector space is a filter \mathcal{F} such that $\mathcal{F} \times \mathcal{F}$ is a uniformity. As the induced uniform limit structure on X is generated by filters convergent to zero and the limit vector space X is locally convex, a filter \mathcal{F} is Cauchy exactly iff a locally convex filter $\mathcal{G} \rightarrow 0$ exists such that for every $U \in \mathcal{G}$ there is a $V \in \mathcal{F}$ with $V - V \subset U$. Now let $\mathcal{G} \rightarrow 0$ be an arbitrary locally convex filter. Choose to a convex set $U \in \mathcal{G}$ a $\delta > 0$ with $f(\xi) - f(\zeta) \in U$ for $|\xi - \zeta| \leq \delta$ (such a δ exists because the continuous map f on the compact set $[a, b]$ is automatically uniformly continuous). Then for $\xi_i, \zeta_i \in [t_i, t_{i+1}]$ the relation

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) (f(\xi_i) - f(\zeta_i)) \in \sum_{i=0}^{n-1} (t_{i+1} - t_i) U = U$$

holds for all subdivisions Z with $|Z| \leq \delta$, because U is convex. Thus $S(\delta) - S(\delta) \subset U$ is valid, and hence the filter generated by $(S(\delta))_{\delta > 0}$ is a Cauchy filter.

Therefore call a function f on $[a, b]$ with values in a complete and separated locally convex limit vector space X Riemann integrable whenever $(S(\delta))_{\delta > 0}$ generates a Cauchy filter, and denote its limit by the symbol $\int_a^b f$, called the Riemannian integral of f over the interval $[a, b]$. If X is not complete, the limit $\int_a^b f$ can still be considered as an element of the completion of X . However, for arbitrary limit vector spaces there seem are different types of completions, see [Reed].

The Riemannian integral is linear and the mean value theorem holds: For a continuous $f : [a, b] \rightarrow X$ and a convex set $C \subset X$ with $f([a, b]) \subset C$ the relation $\int_a^b f \in (b - a)C$ holds. Also the inequality $p(\int_a^b f) \leq \int_a^b p \circ f$ is valid for every continuous pseudonorm p on X . Further for complete separated locally convex topological vector spaces X the Riemannian integral of a continuous map f on a compact interval and the Lebesgue integral coincide. Indeed, $\int_{[a, b]} x' \circ f d\lambda = \int_a^b x' \circ f$ holds for the \mathbb{R} -valued Lebesgue and Riemannian integral, and as $x'(\int_a^b f) = \int_a^b x' \circ f$ is valid according to the equality $x'(S_Z) = \sum (t_{i+1} - t_i) (x' \circ f)([t_i, t_{i+1}])$ and the continuity of x' , also the X -valued Lebesgue and Riemannian integral coincide.

5.2 Differentiation

There are various notions of differentiable maps on limit vector spaces. Some of them shall be discussed in this section. As before, the focus lies on locally convex topological vector spaces and the tensorial closed category of locally convex pseudotopological limit vector spaces.

Partially Differentiable Maps

A continuous map c from an interval containing zero into a limit space M with $c(0) = m$ is called a curve in M through m . Note that zero could be the left endpoint resp. the right endpoint resp. an inner point of the interval I , and thus there are three different types of curves through m . Depending on the type of the curve c through m , define its germ at zero as the equivalence class of c w.r.t. the equivalence relation on the set of all curves of the same type, where $c \sim d$ if there is a neighbourhood I of zero in \mathbb{R}_0^+ resp. \mathbb{R}_0^- resp. \mathbb{R} with $c|_I = d|_I$. The germ at zero of a curve c through m is shortly called an inner resp. outer resp. bothsided direction at $m \in M$.

Omit the adjectives inner, outer or bothsided, whenever the type of a direction is not important, and just say “the direction c at m ” instead of “the direction at m represented by the curve c ”. To a direction c and a scaling parameter $s \in \mathbb{R}$ define the scaled direction $sc : t \mapsto c(st)$. Note that $(s, c) \mapsto sc$ is an operation³⁷ of (\mathbb{R}, \cdot) on the set of all directions at a certain point. A negative scaling parameter s maps inner directions to outer directions, and conversely. Further note that every curve $c : I \rightarrow M$ on an interval I with $r \in I$ induces the direction $t \mapsto c(r + t)$ at the point $c(r) \in M$.

Now let us specify certain properties of directions in a separated limit vector space X , where the earlier defined properties will be weaker than the following. A direction c in X is called locally Lipschitz continuous, if there is a neighbourhood I of zero such that for every continuous pseudonorm p ³⁸ there exists a constant L with $p(c(s) - c(t)) \leq L|s - t|$ for all $s, t \in I$. The direction c is called differentiable, if the limit

$$\dot{c} := \lim_{0 \neq t \rightarrow 0} \frac{c(t) - c(0)}{t}$$

exists, and \dot{c} is called the derivative of c . Note that the differentiation $c \mapsto \dot{c}$ is homogeneous. If a direction can be represented by a curve c , which induces differentiable directions at all points $c(t)$ for t in some neighbourhood of zero, then the direction represented by c is called differentiable. If further the derivatives $\dot{c}(t)$ of the induced directions at $c(t)$ form a continuous map $t \mapsto \dot{c}(t)$ on a neighbourhood of zero, then the direction is called continuously differentiable or a C^1 -direction. If moreover $t \mapsto \dot{c}(t)$ is not only continuous but locally Lipschitz continuous, then the direction is called locally Lipschitz differentiable or a Lip^1 -direction. By regarding $t \mapsto \dot{c}(t)$ as new direction in X ,

³⁷An operation of (\mathbb{R}, \cdot) on a set C is a map $\mathbb{R} \times C \ni (s, c) \mapsto sc \in C$ such that $(ss')c = s(s'c)$, $1c = c$ and $0c = 0d$ is valid for all $c, d \in C$, $s, s' \in \mathbb{R}$.

³⁸The notion of a continuous pseudonorm makes sense also for a limit vector space, although usually we consider continuous pseudonorms on locally convex topological vector spaces only.

these notions can be iterated to define k -times differentiable, C^k - and Lip^k -directions. Hereby higher order derivatives are denoted by $c^{(k)}$. Note that every C^{k+1} -direction on a locally convex topological vector space is also a Lip^k -direction³⁹. A direction c is called smooth or a C^∞ -direction, if it is a C^k -direction (or a Lip^k -direction) for every $k \in \mathbb{N}$, and analytical or a C^ω -direction, if $\sum_{k=1}^\infty t^k c^{(k)}$ converges on some neighbourhood of zero. Finally a direction of the form $t \mapsto \sum_{k=1}^n t^k x_k$ is called polynomial and especially directions of the form $t \mapsto x + th$ are called linear.

A map $f : M \rightarrow Y$ from a space M into a separated limit vector spaces Y is called partially differentiable at m resp. partially differentiable resp. partially C^k resp. ... into the direction c , if the direction $f \circ c$ is differentiable at $f(m)$ resp. differentiable resp. C^k resp. ..., and its derivative $\frac{\partial f}{\partial c}(m) := (f \circ c)'$ is called the partial derivative of f into the direction c at m . Analogously higher order derivatives $\frac{\partial^k f}{\partial c^k}(m) := (f \circ c)^{(k)}$ are defined. Trivially the chain rule is valid, i.e. if $g : L \rightarrow M$ is a map and c is a direction at $l \in L$, then a map $f : M \rightarrow Y$ is partially differentiable at $g(l)$ into the direction $g \circ c$, iff $f \circ g$ is partially differentiable at l into the direction c , and $\frac{\partial(f \circ g)}{\partial c}(m) = \frac{\partial f}{\partial(g \circ c)}(g(m))$ is valid. Moreover, as differentiation of directions is homogeneous, also partial differentiation $\frac{\partial f}{\partial sc}(a) = s \frac{\partial f}{\partial c}(a)$ is homogeneous.

From now on assume that M is a subset of a separated limit vector space X . To distinguish directions in X and M , a direction c in M is called tangential to M . A tangential direction c is called differentiable resp. C^k resp. ..., if its prolongation to X is differentiable resp. C^k resp. Derivatives $\dot{c} \in X$ of tangential directions are called tangential vectors, and we denote the set of all tangential vectors at $m \in M$ by $T_m M$. Note that at the moment $T_m M$ does not need to be a linear subspace of X , as M does not need to be a nice subset of M like, for example, an open set.

A map $f : X \supset M \rightarrow Y$ is called differentiable resp. C^k resp. ... along differentiable resp. C^k - resp. ...-directions at m , if f is partially differentiable resp. partially C^k resp. partially ... into all differentiable resp. C^k - resp. ...-directions, which are tangential to M at m .

In classical analysis a map $f : X \supset M \rightarrow Y$ is partially differentiable at an inner point $m \in M$, if f is differentiable at m along linear directions⁴⁰, as this condition is equivalent to the existence of the limits

$$\frac{\partial f}{\partial h}(m) := \lim_{0 \neq t \rightarrow 0} \frac{f(m + th) - f(m)}{t} .$$

³⁹Continuity of $t \mapsto c^{(k+1)}(t)$ on some neighbourhood of zero implies that for every continuous pseudonorm p on X the function $p(c^{(k+1)})$ is locally bounded near every t and hence bounded on compact intervals. Thus for a C^{k+1} direction there is a compact interval I containing zero such that for every continuous pseudonorm p there exists a constant C with $\sup_{h \in I} p(c^{(k+1)}(h)) \leq C$. Now the mean value theorem implies that the inequality $p(c^{(k)}(s) - c^{(k)}(t)) \leq (\sup_{r \in I} p(c^{(k+1)}(r))) |s - t|$ is valid for every continuous pseudonorm p on X and $s, t \in I$, hence $c^{(k)}$ is locally Lipschitz continuous.

⁴⁰Note that all linear directions at m are tangential to M , because m is required to be an inner point of M .

For such maps f the derivative $Df(m) : h \mapsto \frac{\partial f}{\partial h}(m)$ is well-defined at inner points $m \in M$, and in classical analysis f is called Gateux-differentiable at m , if $Df(m)$ is a continuous linear map from X to Y .

However, for maps $f : X \supset M \rightarrow Y$, which are differentiable (or C^k or ...) along differentiable (or C^k - or ...) directions at m , it makes more sense to consider the map $T_m f : \dot{c} \mapsto \frac{\partial f}{\partial c}(m)$ from the tangential space $T_m M \subset X$ to Y . This map is well-defined, if every partial derivative $\frac{\partial f}{\partial c}(m)$ depends on the tangential vector \dot{c} of the direction c only. A map f with this property could be called Gateux-differentiable at m , if $T_m f$ had a continuous linear extension to the linear subspace $\text{Span}(T_m M) \subset X$ ⁴¹. But as closed subspaces have much better properties than arbitrary subspaces, let us call a map f Gateux-differentiable at m , if f is differentiable along differentiable directions at m and has a well-defined derivative $T_m f$, which can be extended to a continuous linear on the closed linear subspace $\overline{\text{Span}(T_m M)} \subset X$. Note that this extension is automatically unique, as Y is separated. Further for a complete space Y the existence of a continuous linear extension to $\overline{\text{Span}(T_m M)}$ implies the existence of a continuous linear extension to $\overline{\text{Span}(T_m M)}$ ⁴². Usually for a Gateux-differentiable map f the continuous linear extension is again denoted by $T_m f$ and is called the Gateux-derivative of f , while $T_m M$ denotes the closed linear subspace generated by the original set of tangential vectors.

Due to our requirement, that a Gateux-differentiable map is differentiable along differentiable (and not only linear) directions and that the partial derivative into a direction c depends on \dot{c} only, the chain rule is already valid on the level of Gateux-differentiable maps: If $g : W \supset L \rightarrow M \subset X$ and $f : X \supset M \rightarrow Y$ are Gateux-differentiable at $l \in L$ resp. $g(l) \in M$, then $f \circ g$ is Gateux-differentiable at l and $T_l(f \circ g) = T_{g(l)}f \circ T_l g$ holds. Indeed,

$$\frac{\partial(f \circ g)}{\partial c}(l) = \frac{\partial f}{\partial(g \circ c)}(g(l)) = T_{g(l)}f(g \circ \dot{c}) = T_{g(l)}f\left(\frac{\partial g}{\partial c}(l)\right) = T_{g(l)}f(T_l g(\dot{c}))$$

is valid for all tangential directions c at $l \in L$, thus the continuous linear map $T_{g(l)}f \circ T_l g$ is the extension looked for.

But maybe a map, which transports a certain kind of directions, is automatically Gateux-differentiable? In general this is not true for maps, which transport differentiable or C^k -directions⁴³. On the contrary, a map f , which is Lip^k along Lip^k -

⁴¹Here linear extensions have to be considered, because generally $T_m M$ is not a linear subspace.

⁴²Proof: The continuous linear map $T_m f$ on the linear subspace $\text{Span}(T_m M)$ of X is automatically uniformly continuous, so that it maps Cauchy filters on $\text{Span}(T_m M)$ to Cauchy filters on Y . Let \mathcal{F} be a filter on $\text{Span}(T_m M)$, whose prologation to X converges to x . Then especially \mathcal{F} is a Cauchy filter, so that $T_m f(\mathcal{F})$ converges to a point $y \in Y$, as Y is complete. Further the point y does not depend on the filter \mathcal{F} , but merely on its limit x : For every other filter $\mathcal{G} \rightarrow x$ on $\text{Span}(T_m M)$ the Cauchy filter $T_m f(\mathcal{F} \cap \mathcal{G}) = T_m f(\mathcal{F}) \cap T_m f(\mathcal{G})$ on Y is convergent, and its limit is again the point y , as the finer filter $T_m f(\mathcal{F})$ converges to y . Thus also $T_m f(\mathcal{G})$ converges to y , as $T_m f(\mathcal{F} \cap \mathcal{G})$ is coarser and converges to y . Hence $x \mapsto y$ defines the continuous linear extension of $T_m f$ to the closed subspace $\overline{\text{Span}(T_m M)}$.

⁴³For example, [Kriegel, Michor, 3.3] discuss the map

$$f(x, y) := \frac{1}{x^2 + y^2}(x^3 - 3xy^2, 3x^2y - y^3) \quad ,$$

directions, has automatically a Gateux-derivative for a large class of vector spaces, the so-called convenient vector spaces. In fact, the convenient calculus of [Kriegl,Michor] and [Fröhlicher,Kriegl] proves that for convenient vector spaces X, Y a map f on a c^∞ -open set $M \subset X$ ⁴⁴ is Lip^k along Lip^k -directions, iff the partial derivatives form a derivative $Tf : M \rightarrow L(X, Y)$, $m \mapsto (\dot{c} \mapsto \frac{\partial f}{\partial c}(m))$, which is Lip^{k-1} along Lip^{k-1} directions, see also A. Hereby $L(X, Y)$ denotes the natural space of maps in the tensorial closed category of bornological locally convex vector spaces and is itself a convenient vector space, see 4.3. Thus by considering convenient vector spaces X, Y to be endowed with their bornological topology, a map which is Lip^k along Lip^k -directions has automatically a Gateux-derivative which is Lip^{k-1} along Lip^{k-1} -directions.

However, as in classical analysis on Banach spaces, maps with a Gateux-derivative do not have to be continuous. Moreover convenient calculus does not help to prove generalizations of existence theorems on Banach spaces. This is the reason why in this thesis a nonlinear analysis using C^k -maps w.r.t. the tensorial closed category of locally convex pseudotopological limit vector spaces is developed, where existence theorems can be proved in a Banach-like style due to the fact that C^1 -maps are automatically locally Lipschitz continuous.

But before we start to develop this analysis, let us discuss, under which conditions a Gateux-differentiable map on a non-open set M has a derivative defined on the whole space X . Indeed, to have a large class of manifolds modeled over locally convex spaces, we have to consider C^k -maps on non-open sets M , as such sets are needed as domains of charts. But to define a C^1 -map as a Gateux-differentiable map with continuous derivative into the natural space $L(X, Y)$ of continuous linear maps, the derivative has to be defined on the whole space X even for a non-open set M . Therefore the set M should have the property that its tangential space $T_m M$ is at every point m the whole space X , or equivalently that the original set $T_m M$ of tangential vectors satisfies $\text{Span}(T_m M) = X$. Call such a set M tangentially dense at m . Open subsets M of X are tangentially dense at every $m \in M$, but there are much smaller sets M , which are tangentially dense at every $m \in M$, e.g. tangentially open or linearly dense sets.

To discuss such sets, let us introduce some notions: Denote by $\mathcal{U}(0)$ the neighbourhood filter of 0 in \mathbb{R}_0^+ , \mathbb{R}_0^- or \mathbb{R} , depending on the type of the considered directions, tangential spaces and derivatives. The topology generated by the convergences $c(\mathcal{U}(0)) \rightarrow x$ for every tangential differentiable direction c at x is called the tangential topology on X , and obviously this topology is finer than the original topology of X . A set $M \subset X$ is called tangentially open if it is open w.r.t. the tangential topology on X , or equivalently, if every direction c tangential to X at a point $m \in M$ is also tangential to M . If M

which satisfies $f(t(x, y)) = tf(x, y)$ and can be continuously extended to \mathbb{R}^2 by $f(0, 0) := (0, 0)$. The map f is partially (continuously) differentiable along (continuously) differentiable directions, and the derivative $Df(0, 0)$ is well-defined and continuous due to the validity of

$$\frac{\partial f}{\partial c}(0, 0) = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(c(0))}{t} = \lim_{t \rightarrow 0} f\left(\frac{c(t)}{t}\right) = f(\dot{c})$$

for (continuously) differentiable directions c at $(0, 0)$. But obviously $Df(0, 0) : \dot{c} \mapsto f(\dot{c})$ is not linear.
⁴⁴ c^∞ -open sets are defined in A and have the property $T_m M = X$ for all $m \in M$.

is a neighbourhood of the point m w.r.t. the tangential topology on X , then m is called a tangentially inner point of M . Obviously every set M is tangentially dense at tangentially inner points $m \in M$, in fact, the original set of tangential vectors is the whole space X . Note that the tangential topology is intimately related to the c^∞ -topology defined by [Kriegel, Michor] (see A), as the c^∞ -topology is given by the same convergences, but instead of all tangential differentiable directions only the smooth directions are used. In the same way, the linear topology on X can be defined by the convergences $c(\mathcal{U}(0)) \rightarrow x$ for linear directions $c : t \mapsto x + th$, $h \in X$, at x , and linearly open sets M can be considered. Further a set M can be called linearly dense at $m \in M$, if the linear directions at $m \in M$ span a dense subspace of X , i.e. if there is a dense set of vectors h in X such that $x + Ih \subset M$ is valid for some interval containing zero. Trivially for a subset M of X the implications open \Rightarrow tangentially open \Rightarrow linearly open \Rightarrow linearly dense \Rightarrow tangentially dense are valid. Thus there are a lot of sets M , which are not open, but where every tangential space $T_m M$ is the whole space X .

Differentiable Maps

If a map $f : X \supset M \rightarrow N \subset Y$ is Gateux-differentiable at a point m where M is tangentially dense, then the continuous affine linear map $m' \mapsto f(m) + T_m f(m' - m)$ is a good approximation of f along tangential differentiable directions at m , but not uniformly on a whole neighbourhood of m in M . That's why differentiable maps are introduced.

Fix a tensorial closed category of vector spaces endowed with a compatible topological structure. A map $f : X \supset M \rightarrow Y$ is called differentiable at a tangentially dense point $m \in M$ (in the sense of approximation with continuous linear maps) if there is a map $\Delta : M \rightarrow L(X, Y)$ into the natural space $L(X, Y)$ of continuous linear maps such that $f(m') - f(m) = \Delta(m')(m' - m)$ holds for all $m' \in M$ and Δ is continuous at m . As continuity of Δ at m depends on the topological structure of $L(X, Y)$ and thus on the chosen category, also differentiability depends on the chosen category.

A differentiable map f is automatically continuous and Gateux-differentiable at m with derivative $T_m f = \Delta(m)$. Indeed, $m' \mapsto \Delta(m')(m' - m)$ is continuous and has the value 0 at m , so that $\lim_{m' \rightarrow m} f(m') - f(m) = 0$ holds. Further if c is a tangential differentiable direction at m , then f is partially differentiable into the direction c with $\frac{\partial f}{\partial c}(m) = \Delta(m)(\dot{c})$, because $f(c(t)) - f(c(0)) = \Delta(c(t))(c(t) - m)$ implies

$$\frac{\partial f}{\partial c}(m) = \lim_{0 \neq t \rightarrow 0} \frac{f(c(t)) - f(c(0))}{t} = \lim_{0 \neq t \rightarrow 0} \Delta(c(t)) \left(\frac{c(t) - c(0)}{t} \right) = \Delta(m)(\dot{c})$$

due to the continuity of Δ in both arguments and the homogeneity of each continuous linear map $\Delta(c(t))$. Thus f is Gateux-differentiable at m with derivative $T_m f = \Delta(m)$.

Further the chain rule is valid for differentiable maps: If $g : L \rightarrow M$ and $f : M \rightarrow N$ are differentiable at $l \in L$ resp. $g(l) \in M$, then $f \circ g$ is also differentiable at l with derivative $T_l(f \circ g) = T_{g(l)} f \circ T_l g$. Indeed,

$$f(g(l')) = f(g(l)) + \Delta_f(g(l'))(g(l') - g(l)) = f(g(l)) + (\Delta_f(g(l')) \circ \Delta_g(l'))(l' - l)$$

is valid and $\Delta_{f \circ g}(l) := \Delta_f(g(l')) \circ \Delta_g(l')$ is again continuous at l into the natural space of continuous linear maps, as the composition of continuous linear maps is continuous and g is continuous at l . Thus $f \circ g$ is differentiable, and the equality $T_l(f \circ g) = T_{g(l)}f \circ T_lg$ is implied by

$$\Delta_{f \circ g}(l) = \lim_{l' \rightarrow l} \Delta_f(g(l')) \circ \Delta_g(l') = T_{g(l)}f \circ T_lg \quad .$$

It is usually not easy to test, whether a map f is differentiable (in the sense of approximation with continuous linear maps) at a tangentially dense point m , as it is not easy to find a map Δ with the requested properties. It would be much easier to test differentiability, if apart from topological constructions only the supposed derivative $T_m f$ were used in the definition of differentiability, because $T_m f$ can be computed a priori by calculating partial derivatives. Such a notion of differentiability can be obtained by considering convergence properties of the remainder and is called differentiability (in the sense of remainder convergence).

For a differentiable map f at m (in the sense of approximation with continuous linear maps) the remainder $R(m') := f(m') - f(m) - T_m f(m' - m) = (\Delta(m') - T_m f)(m' - m)$ has usually special convergence properties, as $A(m') := \Delta(m') - T_m f$ is continuous with value $0 \in L(X, Y)$ at m and convergence in the natural space $L(X, Y)$ is usually much stronger than merely convergence in $C(X, Y)$ for a chosen tensorial closed category of vector spaces endowed with a compatible topological structure.

Example: *In the category of normable vector spaces the continuity of a map $A : X \supset M \rightarrow L(X, Y)$ at a point $m \in M$ with value $A(m) = 0$ implies the equality $\lim_{m' \neq m \rightarrow m} \frac{\|A(m')(m' - m)\|}{\|m' - m\|} = 0$ ⁴⁵. Thus the remainder $R(m')$ of a differentiable map f (in the sense of approximation with continuous linear maps) has the convergence property*

$$\lim_{m' \neq m \rightarrow m} \frac{\|R(m')\|}{\|m' - m\|} = 0 \quad (4)$$

But why should we not use this consequence of differentiability (in the sense of approximation with continuous linear maps) within the category of normable vector spaces to define a new notion of differentiability (in the sense of remainder convergence)? A map $f : X \supset M \rightarrow N \subset Y$ is called differentiable (in the sense of remainder convergence) at a point m where M is tangentially dense, if there is a continuous linear map $T_m f : X \rightarrow Y$ such that the remainder $R(m') := f(m') - f(m) - T_m f(m' - m)$ has the convergence property (4). In fact, for inner points $m \in M$ this is the usual notion of Fréchet differentiability in classical analysis.

More generally, fix a convergence property for a map $R : X \supset M \rightarrow Y$ at a point m where M is tangentially dense, and assume that having this convergence property implies $R(m) = 0$ as well as the Gateaux-differentiability of R at m with derivative 0. Usually such convergence properties are derived by considering the convergence of

⁴⁵Proof: If $A : X \supset M \rightarrow L(X, Y)$ is continuous at m with value $A(m) = 0$, then to every $\epsilon > 0$ there is a neighbourhood U of m such that $\frac{\|A(m')h\|}{\|h\|} \leq \|A\| \leq \epsilon$ holds for all $m' \in U$ and $h \neq 0$, especially for those $h \neq 0$ with $h = m' - m$.

$R(m') := A(m')(m' - m)$ at $m \in M$ within some tensorial closed category, where $A : X \supset M \rightarrow L(X, Y)$ is a continuous map into the natural space $L(X, Y)$ of continuous linear maps with value $A(m) = 0$ at m .

Now a map $f : X \supset M \rightarrow N \subset Y$ is called differentiable (in the sense of remainder convergence) at a point $m \in M$ where M is tangentially dense, if there is a continuous linear map $T_m f : X \rightarrow Y$ such that the remainder $R(m') := f(m') - f(m) - T_m f(m' - m)$ has the fixed convergence property at $m \in M$. As then R is Gateux-differentiable at m with derivative 0, the map f is automatically Gateux-differentiable at m with derivative $T_m f$.

For example, in the category of limit vector spaces $A(m') \rightarrow 0$ in $L(X, Y)$ for $m' \rightarrow m$ is equivalent to $\lim_{(m', x') \rightarrow (m, x)} A(m')x' \rightarrow 0$. Now if c is a tangential direction at m , then

$$\lim_{0 \neq t \rightarrow 0} \frac{R(c(t))}{t} = \lim_{0 \neq t \rightarrow 0} A(c(t)) \frac{c(t) - c(0)}{t} = 0$$

holds for the map $R(m') = A(m')(m' - m)$. Thus the derived convergence property is $\lim_{0 \neq t \rightarrow 0} \frac{R(c(t))}{t} = 0$ for all tangential differentiable directions c at m . Using this convergence property, a map f is differentiable (in the sense of remainder convergence) at a point m where M is tangentially dense, if there is a continuous linear map $T_m f : X \rightarrow Y$ with $T_m f(\dot{c}) = \frac{\partial f}{\partial c}(m)$ for all tangential differentiable directions c at m , i.e. differentiability is the same as Gateux-differentiability. This is not surprising, as in the category of all limit vector spaces the topological structure of $L(X, Y)$ is induced by $L(X, Y) \subset C(X, Y)$ and thus convergence in $L(X, Y)$ is not stronger than convergence in $C(X, Y)$. Note that also for this weak notion of differentiability the chain rule is valid, as it is valid for Gateux-differentiable maps, but continuity of f at m can not be deduced. In [Keller, 1.2] convergence properties of remainders in other tensorial closed categories are derived, e.g. in the category of bornological locally convex topological vector spaces or the category of equable limit vector spaces.

Finally, let us consider the category of pseudotopological locally convex limit vector spaces or equivalently the category of direct limits of locally convex topological vector spaces (within the category of limit vector spaces) we are mainly interested in. Let X, Y be locally convex topological vector spaces and let $A : X \supset M \rightarrow \lim_{\sigma} L_{\sigma}(X, Y)$ be continuous with value 0 at m , then there is a σ mapping continuous pseudonorms on Y to continuous pseudonorms on X such that $\lim_{m' \rightarrow m} q^{\sigma(q)}(A(m')) = 0$ holds for all continuous pseudonorms q on Y . Especially for every pair (q, ϵ) (q a continuous pseudonorm on Y and $\epsilon > 0$) there is a pair (p, δ) (p a continuous pseudonorm on X and $\delta > 0$) such that $p(m' - m) \leq \delta$ implies $q(A(m')(m' - m)) \leq \epsilon p(m' - m)$. Thus the derived convergence property is the condition that to every (q, ϵ) there has to be a (p, δ) such that $p(m' - m) \leq \delta$ implies $q(R(m')) \leq \epsilon p(m' - m)$, or symbolically that to every q there is a p with $\lim_{m' \rightarrow m} \frac{q(R(m'))}{p(m' - m)} = 0$. As required, a map R satisfying this convergence property has the value 0 and is Gateux-differentiable with derivative 0 at m .

The notion of differentiability induced by this convergence property of the remainder has many good properties. In fact, a differentiable map $f : X \supset M \rightarrow N \subset Y$ is automatically continuous at $m \in M$, as the convergence property implies $q(R(m')) \leq \epsilon$

for $p(m' - m) \leq \min(\delta, 1)$, thus R and hence $f(m') = f(m) + T_m f(m' - m) + R(m')$ is continuous at m . Also the chain rule is valid, i.e. if g and f are differentiable, then also $g \circ f$ is differentiable (in the sense of remainder convergence). Indeed, the equation

$$f(g(l+h)) = f(g(l) + T_l g(h) + R_g(h)) = f(g(l)) + T_{g(l)} f(T_l g(h)) + T_{g(l)} f(R_g(h)) + R_f(h')$$

is valid, where $h' := T_l g(h) + R_g(h)$ has been defined. Now to every pseudonorm r there is a pseudonorm p such that $\lim_{h \rightarrow 0} \frac{r(T_{g(l)} f(R_g(h)))}{p(h)} = 0$ holds ⁴⁶, and to every r there is a p such that $\lim_{h \rightarrow 0} \frac{r(R_f(h'))}{p(h)} = 0$ is valid ⁴⁷. Hence $R_{f \circ g}(h) := f(g(l+h)) - f(g(l)) - T_{g(l)} f(T_l g(h))$ has the desired remainder property $\lim_{l' \rightarrow l} \frac{r(R_{f \circ g}(l'))}{p(l'-l)} = 0$.

C^k -Maps

Fix again a tensorial closed category of vector spaces endowed with a compatible topological structure. For a Gateux-differentiable map $f : X \supset M \rightarrow Y$ on a tangentially dense set M the derivative $T_m f$ at each $m \in M$ is defined on the whole space X , and hence f induces a map $Tf : M \rightarrow L(X, Y)$ into the natural space $L(X, Y)$ of continuous linear maps. If this map Tf is continuous, then f is called continuously differentiable or a C^1 -map. Especially the map $Tf : M \times X \rightarrow Y$ is continuous, because the topological structure of $L(X, Y)$ is finer than the one induced by the inclusion $L(X, Y) \subset C(X, Y)$, and thus the continuous map $Tf : M \rightarrow L(X, Y)$ yields a continuous map $Tf : M \times X \rightarrow Y$. However, as a C^1 -map is merely required to be Gateux-differentiable, it is not obvious, whether a C^1 -map is also continuous or differentiable.

Let us merely discuss the case we are mainly interested in, where X is a locally convex topological vector space. [Keller] proves for different categories - like e.g. the category of limit vector spaces and the category of equable limit vector spaces - that a C^1 -map on an open subset M of a locally convex topological vector space X is continuous and differentiable (in the sense of remainder convergence) ⁴⁸. Let us summarize his arguments and generalize them to tangentially dense subsets $M \subset X$. Generally $f(c(1)) - f(c(0)) = \int_0^1 T_{c(t)} f(\dot{c}(t)) dt$ is valid for every differentiable curve $c : [0, 1] \rightarrow M$, but it is not possible to conclude from this equation the continuity or differentiability of f . In fact, even if $c(0), c(1)$ are connected by a short path in X , the subset M could have a fractal structure so that a path in M between $c(0)$ and $c(1)$ could be forced to be long. But if M is locally convex, i.e. the convex neighbourhoods of m in M generate the neighbourhood filter of m in the relative topology

⁴⁶To r choose q with $r^q(T_{g(l)} f) < \infty$, to q choose p with $\lim_{h \rightarrow 0} \frac{q(R_g(h))}{p(h)} = 0$.

⁴⁷To r choose q such that $\lim_{h' \rightarrow 0} \frac{r(R_f(h'))}{q(h')} = 0$, to q choose p such that $\frac{q(h')}{p(h)}$ is bounded on a neighbourhood of 0. The last choice is possible because $h' = T_l g(h) + R_g(h)$ and thus for every ϵ there is a neighbourhood U of 0 such that $h \in U$ implies $q(h') \leq q^p(T_l g)p(h) + \epsilon p(h)$.

⁴⁸The C^1 -maps of [Keller] are assumed to be defined on an open set M and are only required to be partially differentiable into linear directions. But the chain rule proved by [Keller] guarantees that a C^1 -map is also (continuously) differentiable along (continuously) differentiable directions, thus the only difference to our setting is the openness of M .

of $M \subset X$, then a continuously differentiable map f on M is continuous. Indeed, $\lim_{h \rightarrow 0} f(m+h) - f(m) = \int_0^1 (\lim_{h \rightarrow 0} T_{m+th} f(h)) dt = 0$ is valid, because on the one hand for a convex neighbourhood $U \subset M$ of m with $m+h \in U$ the line $c(t) = m+th$, $t \in [0, 1]$, is contained in U , and because on the other hand the derivative Tf is continuous in both arguments so that limit and integration commute. Thus f is continuous in this case, and $f(m+h) - f(m) = (\int_0^1 T_{m+th} f dt)h$ shows that f is also differentiable (in the sense of approximation by continuous linear maps, and hence also in the weaker sense of remainder convergence). Moreover the composition $f \circ g$ of two continuously differentiable maps is continuously differentiable, i.e. the chain rule is valid for C^1 -maps on locally convex sets. Indeed, $T(f \circ g) = T_{g(\cdot)} f \circ T.g$ holds by the chain rule for Gateaux-differentiable maps and is itself continuous by the continuity of g and the continuity of the composition \circ of continuous linear maps.

We want to make explicit these arguments again for the category we are mainly interested in, the category of pseudotopological locally convex limit vector spaces or equivalently direct limits of locally convex topological vector spaces: Let f be a continuously differentiable map on a locally convex and linearly dense subset M of a separated locally convex topological vector space X into Y . To prove that f is continuous and differentiable, we have to show that there is a map σ between the sets of continuous pseudonorms on Y resp. X such that for every continuous pseudonorm q on Y and every $\epsilon > 0$ there is a neighbourhood U of m in M with $q(R(h)) \leq \epsilon \sigma(q)(h)$ for all $m+h \in U$. Now there is a convex neighbourhood $U \subset M$ such that $T_U f \subset L_\sigma(X, Y)$, because Tf is a continuous map into the natural space of maps $\lim_\sigma L_\sigma(X, Y)$, and by making U smaller it can also be guaranteed that $q^{\sigma(q)}(T_{m+th} f - T_m f) \sigma(q)(h) \leq \epsilon$ holds for all h with $m+h \in U$. The remainder satisfies $R(h) = \int_0^1 (T_{m+th} f - T_m f) h dt$, and thus the inequality

$$q(R(h)) \leq \sup_{0 \leq t \leq 1} q^{\sigma(q)}(T_{m+th} f - T_m f) \sigma(q)(h) \leq \epsilon \sigma(q)(h)$$

holds for all h with $m+h \in U$. Hence f is differentiable in the sense of remainder convergence.

The most important property of C^1 -maps w.r.t. the tensorial closed category of locally convex pseudotopological limit vector spaces is that such C^1 -maps are locally Lipschitz continuous. Call a map f locally Lipschitz continuous, if to every point $m_0 \in M$ there is a neighbourhood $U \subset M$ and a σ mapping continuous pseudonorms on Y to continuous pseudonorms on X such that $q(f(m) - f(m')) \leq L_q \sigma(q)(m - m')$ holds with a nonnegative number $L_q < \infty$ for all continuous pseudonorms q on Y and all $m, m' \in U$. The smallest of such numbers L_q is denoted by $q^{\sigma(q)}(f)(U)$. Now let f be continuously differentiable, then to a point x_0 there is a convex neighbourhood U and a σ such that $T_U f \subset L_\sigma(X, Y)$. But on a convex set U the mean value theorem is valid, i.e. for every $m \in U$, $y' \in Y'$ and $h \in X$ with $m+h \in U$ there exists a $r = r(m, y', h) \in (0, 1)$ such that $y'(f(m+h) - f(m)) = y'(T_{m+rh} f(h))$ holds⁴⁹. Further, as to every continuous pseudonorm q on Y , $m \in U$ and $h \in X$ with $m+h \in U$ there exists a $y' \in Y'$ with

⁴⁹Proof: Applying the mean value theorem on \mathbb{R} to the function $t \mapsto y'(f(m+th))$ guarantees the existence of a number $r \in (0, 1)$ with $y'(f(m+h) - f(m)) = y'(T_{m+rh} f(h))$

$y'(f(m+h) - f(m)) = q(f(m+h) - f(m))$ and $y'(y) \leq q(y)$ by the Hahn-Banach theorem, the mean value theorem implies the existence of a $r = r(q, m, h) \in (0, 1)$ with

$$q(f(m+h) - f(m)) = y'(T_{m+rh}f(h)) \leq q(T_{m+rh}f(h)) \quad . \quad (5)$$

Because of $m+rh \in U$ also $T_{m+rh}f \in L_\sigma(X, Y)$ holds and thus

$$q(f(m+h) - f(m)) \leq \left(\sup_{\xi \in U} q^{\sigma(q)}(T_\xi f) \right) \sigma(q)(h)$$

is valid. By making U smaller, for a single q it is possible to choose the Lipschitz constants arbitrarily near to $q^{\sigma(q)}(T_{m_0}f)$ due to the continuity of Tf .

Remark that the Lipschitz continuity of a map f implies, that f has a really strong local character. In fact, $q(f(m) - f(m'))$ depends merely on the contribution of m, m' to the value $\sigma(q)(m - m')$. For example, an operator F on $C(\mathbb{R}, \mathbb{R})$, for which the value $F(x)(t)$ at t depends on the values of $x(\cdot)$ on the whole set \mathbb{R} and not only on the values of $x(\cdot)$ on a compact subset of \mathbb{R} , can not be a C^1 -map, as this property contradicts the existence of a compact set K such that $|F(x)(t) - F(y)(t)| \leq \sup_K |x - y|$.

Higher Order Derivatives

Let f be a C^1 -map and suppose that also $Tf : X \supset M \rightarrow L(X, Y)$ is a C^1 -map. Then f is called a C^2 -map and

$$T^2f := T(Tf) : X \supset M \rightarrow L(X, L(X, Y)) = L^2(X, Y) \cong L(X \otimes X, Y)$$

is a continuous map into the continuous bilinear maps on X . More generally call $f : X \supset M \rightarrow Y$ a C^k -map if $Tf : X \supset M \rightarrow L(X, Y)$ exists and is a C^{k-1} -map, while the k -th derivative of f is defined by $T^k(f) := T(T^{k-1}f) : X \supset M \rightarrow L^k(X, Y)$.

Note that the k -th derivative $T^k f(m)$ of a C^k -map f at $m \in M$ is a symmetric k -linear map. Indeed, $T_m^2 f(x_1, x_2) = T_m^2 f(x_2, x_1)$ has to be proved. By defining $A : \mathbb{R}^2 \rightarrow X$, $A(e_1) := m + x$, $A(e_2) := m + x'$, it suffices to prove $\frac{\partial \phi}{\partial x \partial y} = \frac{\partial \phi}{\partial y \partial x}$ for the function $\phi := x' \circ f \circ A : \mathbb{R}^2 \rightarrow \mathbb{R}$ and all $x' \in X'$. But this is a standard result of classical analysis, and thus symmetry is proved also in the more general case of locally convex topological vector spaces.

5.3 The Contraction Mapping Principle

Let X be a uniform space and $T : X \rightarrow X$ an operator. A point $x \in X$ with $Tx = x$ is called a fixed point of T . To prove a generalisation of the contraction mapping principle, recall that the uniform structure on X is generated by the uniformly continuous pseudometrics (see [Schubert, II.2.7, Satz 2]). Let d, δ be such uniformly continuous pseudometrics on X and denote by $d^\delta(T) \in [0, \infty]$ the smallest number with $d(Tx, Ty) \leq d^\delta(T)\delta(x, y)$ for all $x, y \in X$. An operator T is called Lipschitz continuous if there is a set D of pseudometrics generating the uniform structure of X and a map

$\sigma : D \rightarrow D$ such that $d^{\sigma(d)}(T) < \infty$ holds for all $d \in D$. Clearly a Lipschitz continuous operator is uniformly continuous. A Lipschitz continuous operator T is called a strict contraction if the series

$$\sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(x, y)$$

converges for all $d \in D$ and every $x, y \in X$.

Note that for a metric space (X, d) both notions are generalisations of the usual definitions: The set $D := \{d\}$ generates the uniform structure of X , so that the usual notion of Lipschitz continuity is obtained by setting $\sigma(d) := d$ and $L := d^d(T) < \infty$, while being a strict contraction is equivalent to the convergence of $\sum_{k=0}^{\infty} L^k d(x, y)$ and thus to $L < 1$.

Theorem 5.3 *Let X be a complete and separated uniform space. Then every strict contraction $T : X \rightarrow X$ has a unique fixed point, the limit of the sequence $T^n x_0$ starting at an arbitrary $x_0 \in X$.*

Proof: Let $\sigma : D \rightarrow D$ be the map of pseudometrics associated to the strict contraction T and choose an arbitrary pseudometric $d \in D$. Obtain from

$$d(T^{k+1}x_0, T^k x_0) \leq \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(Tx_0, x_0)$$

the inequality

$$\begin{aligned} d(T^n x_0, T^m x_0) &\leq \sum_{k=m}^{n-1} d(T^{k+1}x_0, T^k x_0) \leq \\ &\sum_{k=m}^{n-1} \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(Tx_0, x_0) \end{aligned}$$

for $n > m$. As T is a strict contraction and $d \in D$, the last term is a partial sum of a convergent series. Hence for every $\epsilon > 0$ there is an N such that this last term is smaller than ϵ for every $n > m > N$. Thus for every $\epsilon > 0$ there is an N with $d(T^n x_0, T^m x_0) \leq \epsilon$ for all $n, m \geq N$. As D generates the uniform structure of X and $d \in D$ is arbitrarily chosen, the sequence $T^n x_0$ is a Cauchy sequence. The completeness of X now yields the convergence of the Cauchy sequence $T^n x_0$ to a limit $x \in X$.

This point x is a fixed point of T , because the continuity of T implies $x = \lim_n T^n x_0 = T(\lim_n T^n x_0) = Tx$. Moreover every other fixed point of T equals x . Indeed, let y be another fixed point of T , then to every $d \in D$ and every $\epsilon > 0$ there is an N such that $d(x, y) = d(T^n x, T^m y) \leq \sum_{k=m}^{n-1} \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(x, y)$ is smaller than ϵ for all $n > m > N$, so that $d(x, y) = 0$ holds for every $d \in D$ and hence the separateness implies $x = y$. \square

Corollary 5.4 *Let X be a complete and separated locally convex vector space, let $A \subset X$ be a closed subset and let $T : A \rightarrow A$ be a strict contraction⁵⁰. Then there exists a unique fixed point $x \in A$ of T .*

The proof of the contraction mapping principle also shows that a fixed point already exists whenever the series

$$\sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(Tx_0, x_0)$$

converges for some x_0 and all $d \in D$. However, in this case the fixed point x obtained as the limit of $T^n x_0$ can be guaranteed to be unique only in the subspace of all y for which the series

$$\sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \sigma^l(d)^{\sigma^{l+1}(d)}(T) \right) \sigma^k(d)(x, y)$$

converges for every $d \in D$.

Example: Denote by $C(\mathbb{R}, \mathbb{R})$ the space of continuous functions endowed with the uniform structure of uniform convergence on compact subsets and consider the linear operator $(Tx)(t) := \frac{1}{2}x(t+1)$. Let $p_K := \max_K |\cdot|$ and $\sigma(p_K) := p_{K+1}$, then the inequality $p_K(Tx - Ty) \leq \frac{1}{2}\sigma(p_K)(x - y)$ is valid, so that T is Lipschitz continuous with $p_K^{\sigma(p_K)}(T) = \frac{1}{2}$. However, the series $\sum_{k=0}^{\infty} \frac{1}{2^k} p_{K+k}(x)$ converges only for function $x(t)$ growing more slowly than 2^t , so that a unique solution only exists in the subspace $A := \{x | \forall K : \sum_{k=0}^{\infty} \frac{1}{2^k} p_{K+k}(x) < \infty\}$. Indeed, $Tx = x$ has the solutions $c(t)2^t$ with 1-periodic c and the only solution in A is the constant function 0.

5.4 Existence and uniqueness of solutions of differential equations

Being prepared by the theory of integration, differentiation and the contraction mapping principle, let us now turn to our main question, the unique solvability of differential equations in complete separated locally convex vector spaces X .

Solutions of the initial value problem $\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0$, correspond to fixed points of the operator $(Tx)(t) := x_0 + \int_{t_0}^t f(s, x(s))ds$ on $C(I, X)$, where I is some interval having t_0 as an inner point. Indeed, a fixed point $x \in C(I, X)$ of T is differentiable and satisfies $x(t_0) = x_0$ as well as $\dot{x}(t) = f(t, x(t))$, while a (differentiable) solution $x(t)$ of the initial value problem has the fixed point property by the ordinary fundamental theorem of calculus.

Let us try to find conditions which imply that T is a strict contraction: To a radius family $r_q > 0$ indexed by a generating set of continuous pseudonorms q on X denote by

⁵⁰As the uniform structure of $A \subset X$ is generated by continuous pseudonorms, a map T is a strict contraction, if there is a map σ on a set D of pseudonorms p which generates the topology such that with the smallest number $p^{\sigma(p)}(T)$ defined by $p(Tx - Ty) \leq p^{\sigma(p)}(T)\sigma(p)(x - y)$ the series $\sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \sigma^l(p)^{\sigma^{l+1}(p)}(T) \right) \sigma^k(p)(x - y)$ converges for all $p \in D$ and $x, y \in A$

$B_{r_q}(x_0) := \{x | \forall q : q(x - x_0) \leq r_q\}$ the closed i-ball around x_0 . Suppose that the map f is q -bounded on $B_{r_q}(x_0)$ by

$$M_q := \sup_{s \in I, x \in B_{r_q}(x_0)} q(f(s, x)) < \infty$$

and has the Lipschitz constant

$$L_q := \sup_{s \in I} q^{\sigma(q)}(f(s, \cdot)|_{B_{r_q}(x_0)}) < \infty$$

w.r.t. a map σ on the chosen generating set of continuous pseudonorms. Further assume that there is a T_0 with $[t_0 - T_0, t_0 + T_0] \subset I$ and $0 < T_0 \leq r_q/M_q$ such that the series $\sum_{k=0}^{\infty} T_0^k \left(\prod_{l=0}^{k-1} L_{\sigma^l(q)} \right) \sigma^k(q)(x - y)$ converges for all $x, y \in B_{r_q}(x_0)$.

Then the operator T maps $C([t_0 - T_0, t_0 + T_0], B_{r_q}(x_0))$ into itself, as $y \in B_{r_q}(x_0)$ implies

$$q((Ty)(t) - x_0) \leq \int_{t_0}^t q(f(s, y(s))) ds \leq |t - t_0| M_q \leq T_0 M_q \leq r_q$$

for all $t \in [t_0 - T_0, t_0 + T_0]$. Further define $\hat{q}(x(\cdot)) := \sup_{t \in [t_0 - T_0, t_0 + T_0]} q(x(t))$, then the inequality

$$\begin{aligned} q((Tx - Ty)(t)) &\leq \int_{t_0}^t q(f(s, x(s)) - f(s, y(s))) ds = \\ &\int_{t_0}^t q^{\sigma(q)}(f(s, \cdot)|_{B_{r_q}(x_0)}) \sigma(q)(x(s) - y(s)) ds \leq \\ &|t - t_0| L_q \sigma(\hat{q})(x - y) \end{aligned}$$

implies $\hat{q}(Tx - Ty) \leq T_0 L_q \sigma(\hat{q})(x - y)$ for all $x, y \in B_{r_q}(x_0)$. Thus the convergence of the above defined series assures that T is a strict contraction on the closed i-ball $B_{r_q}(x_0)$. Hence T has a unique fixed point, the local solution looked for.

Theorem 5.5 *Let X be a complete and separated locally convex vector space, let $x_0 \in X$ be a point of X and let I be an interval with inner point t_0 . Suppose that on a closed i-ball $B_{r_q}(x_0)$ to a radius family r_q the map $f : I \times X \rightarrow X$ is bounded by M_q and has Lipschitz constants L_q w.r.t. a mapping σ on a generating set of continuous pseudonorms q on X . Further assume that there is a $0 < T_0 \leq r_q/M_q$ such that the series*

$$\sum_{k=0}^{\infty} T_0^k \left(\prod_{l=0}^{k-1} L_{\sigma^l(q)} \right) \sigma^k(q)(x - y)$$

converges for all $x, y \in B_{r_q}(x_0)$. Then there exists a unique solution $x : [t_0 - T_0, t_0 + T_0] \rightarrow X$ of the initial value problem $\dot{x}(t) = f(t, x(t))$, $x(t_0) := x_0$.

Although on a first glance the assumptions of this theorem seem to be rather restrictive, they are not. Indeed, practically the theorem can be used as follows:

- Calculate the Lipschitz constants of f near (s, x) . If f is continuously differentiable, this can be done by calculating the derivatives $T_{(s,x)}f$ and the values $q^p(T_{(s,x)}f)$ on a neighbourhood of (s, x) for continuous pseudonorms p, q on X .
- Choose an interval I and a radius family $r_q > 0$ such that the positive numbers r_q/M_q are bounded away from zero, where M_q denotes the q -bound $M_q := \sup_{s \in I, x \in B_{r_q}(x_0)} q(f(s, x))$ of f on the ball $B_{r_q}(x_0)$. If it is difficult to compute M_q , the estimate $M_q \leq \sup_{s \in I} q(f(s, x_0)) + L_q r_{\sigma(q)}$ can be used.
- As r_q/M_q is bounded away from zero, there is a T_0 with $0 < T_0 \leq r_q/M_q$ for all q . Now try to enforce the convergence of the series

$$\sum_{k=0}^{\infty} T_0^k \left(\prod_{l=0}^{k-1} L_{\sigma^l(q)} \right) \sigma^k(q)(x - y)$$

for $x, y \in B_{r_q}(x_0)$ by choosing a small T_0 or by repeating the second step with smaller I and smaller family r_q , but pay attention that r_q/M_q stays bounded away from zero.

- If it is not possible to enforce the series to be convergent, there may be no solution starting from $x(t_0) = x_0$.

Thus to conclude, it is possible to prove locally the existence and uniqueness of solutions of differential equations to those initial values, which satisfy a growth condition determined by the map f in form of the above series, however if f itself has a large growth, there may be no local solutions starting at a certain initial value.

Example: Let $S : \mathbb{R} \rightarrow \mathbb{R}$ be locally Lipschitz. It is equivalent to solve the differential equation $\dot{x} = S \circ x$ on $C(\mathbb{R}, \mathbb{R})$ starting at the initial value x_0 , or to solve the differential equation $\dot{x} = S(x)$ on \mathbb{R} for all initial values $x_0(m)$. If S and/or x_0 grow so fast that the solutions of $\dot{x} = S(x)$ to the initial values $x_0(m)$ have an arbitrary short time of existence, then there is no local solution of $\dot{x} = S \circ x$ in $C(\mathbb{R}, \mathbb{R})$ starting from the initial value x_0 . But if e.g. S is globally Lipschitz and x_0 is globally bounded, then there is a solution. Indeed, $\sup_K |S(x(m)) - S(x_0(m))| \leq L \sup_K |x(m) - x_0(m)|$ (so that $\sigma = \text{Id}$ can be chosen) and $\sup_K |S(x(m))| \leq M$ are valid with constants L, M on the i -ball $B_1(x_0) := \{x \mid \sup_{\mathbb{R}} |x(m) - x_0(m)| \leq 1\}$ ⁵¹. Thus the existence time of a solution starting at $x_0(m)$ is independent from m , which is expressed in our context by the convergence of

$$\sum_{k=0}^{\infty} T_0^k L^k \sup_K |x(m) - y(m)|$$

for all K whenever $T_0 < \min(L, \frac{1}{M})$.

⁵¹If x_0 is bounded, then also $S \circ x_0$ is bounded, so that $|S(x(m))| \leq |S(x(m)) - S(x_0(m))| + |S(x_0(m))| \leq L \cdot 1 + B =: M$ is valid for all $x \in B_1(x_0)$.

5.5 Inverse Mapping Theorem

Theorem 5.6 *Suppose that a continuously differentiable map $f : X \rightarrow Y$ has a continuously invertible derivative $T_{x_0}f$ at the point x_0 and that $T_U f \subset L_\sigma(X, Y)$ holds on a convex neighbourhood U of x_0 with an invertible σ such that $T_{x_0}f^{-1} \in L_{\sigma^{-1}}(X, Y)$. Then f is invertible on an i -ball $B_{r_q}(f(x_0))$ with continuously differentiable inverse f^{-1} , and the radius family r_q is estimated in the proof.*

Proof: The map f has an inverse near x_0 , iff the map $h : X \rightarrow X$ defined by $h(x) := T_{x_0}^{-1}(f(x + x_0) - f(x_0))$ has an inverse near 0. Note that $h(0) = 0$ as well as $T_0h = \text{Id}$ hold, and that $Th : (U - x_0) \rightarrow L_{\text{Id}}(X)$ is continuous.

Solving $h(x) = y$ is equivalent to solving $x = y + x - h(x)$, so that a fixed point of the map $g_y(x) = y + g(x)$ must be found, where g is defined by $g(x) := x - h(x)$. Note that $g(0) = 0$, $T_0g = 0$ and that $Tg : (U - x_0) \rightarrow L_{\text{Id}}(X)$ is continuous. Thus for every p a neighbourhood of zero can be found, on which the inequality $p^p(T_xg) \leq \frac{1}{2}$ is valid. The intersection of all such neighbourhoods contains a closed i -ball $B_{r_p}(0) := \{x | p(x) \leq r_p\}$, on which the inequality $p^p(T_xg) \leq \frac{1}{2}$ holds for all p , and the radius family r_p can be estimated by this condition.

By the mean value theorem $p(g(x)) = p(g(x) - g(0)) \leq \frac{1}{2}p(x - 0) \leq \frac{1}{2}r_p$ is valid for $x \in B_{r_p}(0)$. Thus $y \in B_{r_p/2}(0)$ and $x \in B_{r_p}(0)$ imply $p(g_y(x)) \leq p(y) + p(g(x)) \leq r_p$. Hence g_y maps the i -ball $B_{r_p}(0)$ into itself for every $y \in B_{r_p/2}(0)$. Further g_y has the same Lipschitz constant $\frac{1}{2}$ as g on $B_{r_p}(0)$, and is a strict contraction on $B_{r_p}(0)$ for every $y \in B_{r_p/2}(0)$, because $\sum_{k=0}^{\infty} \frac{1}{2^k} r_p = 2r_p$ for every p .

Thus for every $y \in B_{r_p/2}(0)$ there is a unique fixed point $x = g_y(x)$ in $B_{r_p}(0) := \{x | q(x) \leq r_q\}$, i.e. a unique solution of $h(x) = y$. Hence h has an inverse $h^{-1} : B_{r_p/2}(0) \rightarrow h^{-1}(B_{r_p/2}(0)) \subset B_{r_p}(0)$. Further $h^{-1}(B_{r_p/2}(0))$ contains an i -ball, as i -balls are intersections of finite balls, preimages of such finite balls contain open sets and thus finite balls, and preimages preserve arbitrary intersections. Hence h is a bijection between i -balls, whose radius family can be estimated by the above construction.

Further $p(g_y(x) - g_y(x')) \leq \frac{1}{2}p(x - x')$ implies the inequality $p(x - x') - p(h(x) - h(x')) \leq \frac{1}{2}p(x - x')$ and thus $p(x - x') \leq 2p(h(x) - h(x'))$, i.e. $p(h^{-1}(y) - h^{-1}(y')) \leq 2p(y - y')$. Hence h^{-1} is Lipschitz continuous on $B_{r_p/2}(0)$. Moreover $p^p(\text{Id} - T_xh) = p^p(T_xg) \leq \frac{1}{2} < 1 = (p^p(\text{Id}))^{-1}$ holds, so that T_xh^{-1} exists by the invertibility theorem of linear analysis, is an element of $L_{\text{Id}}(X)$ for each $x \in B_{r_p}(0)$ and satisfies $p^p(T_xh^{-1}) \leq 2$. Now let $y_1 = h(x_1)$, $y_2 = h(x_2) \in B_{r_p/2}(0)$, then

$$\begin{aligned} p(h^{-1}(y_1) - h^{-1}(y_2) - T_{x_2}h^{-1}(y_1 - y_2)) &= \\ p(T_{x_2}h^{-1}(T_{x_2}h(x_1 - x_2) - h(x_1) - h(x_2))) &\leq \\ 2p(h(x_1) - h(x_2) - T_{x_2}h(x_1 - x_2)) & \end{aligned}$$

is valid. From this inequality and the estimation $p(x_1 - x_2) \leq 2p(y_1 - y_2)$ there can be deduced that h^{-1} is differentiable on $B_{r_p/2}(0)$ and has the derivative $T(h^{-1}) = Th \circ h^{-1}$. \square

5.6 Manifolds

A C^k -manifold modeled over a Banach or Hilbert space is defined to be a set M endowed with a maximal atlas of charts onto open subsets of a Banach or Hilbert space, where the chart changes are C^k -mappings. Such an atlas induces a topology on M , and usually this topology is assumed to be separated and paracompact, so that M admits continuous partitions of unity. Further M is usually supposed also to admit C^k -partitions of unity⁵², so that there is a Finslerian or Riemannian metric⁵³ on M . The choice of such a Riemannian or Finslerian metric turns M into a metric space (M, d) , and often (M, d) is supposed to be complete w.r.t. the chosen metric.

An analogous definition of manifolds modeled over complete and separated locally convex topological vector spaces has serious problems. First of all, the notion of a C^k -mapping depends on the chosen tensorial closed category extending the category of locally convex topological vector spaces. Here we choose the category of locally convex pseudotopological limit vector space as extension. Second, open sets in a locally convex topological vector space are really large sets, thus it is desirable to model manifolds on more general subsets. Non-open sets have also been used to model manifolds in other theories of differentiation, e.g. [Kriegel, Michor, Chapter VI] model manifolds over c^∞ -open sets related to their convenient calculus. We choose dense i-balls to model manifolds on locally convex spaces, because on these subsets the notion of a C^k -mapping is still well-defined: Derivatives are continuous linear maps on the whole original locally convex topological vector space and C^1 -maps are locally Lipschitz continuous, as dense i-balls are locally convex and tangentially dense. Further the inverse mapping theorem suggests to use dense i-balls to model manifolds.

However, using i-balls as chart domains prohibits the use of continuous deformations in charts. In fact, if subsets of a manifold are identified with dense i-balls instead of open subsets, then a continuous deformation in a chart generally runs out of the chart domain and thus cannot be transported back into the manifold. But note that our local existence theorem guarantees that a solution of a differential equation starting at x_0 stays in an i-ball $B_{r_q}(x_0)$ for some time. Thus if the radius family of the chart domain is such that it contains the i-ball $B_{r_q}(x_0)$, then a local solution of a differential equations can be transported back to the manifold. Hence it is still desirable to have large chart domains, i.e. i-balls to radius family which do not fall too fast, as then in a concrete case it is easier to prove that some local deformation stays in a chart domain and does not run out of it.

Further a manifold should have an induced topology and it should be possible to endow a manifold with a uniform structure. But if merely dense i-balls are identified with subsets of a manifold M , then there is no appropriate way to induce a topology on M . For example, if we consider the identity on the locally convex space $C(\mathbb{R}, \mathbb{R})$ restricted to the

⁵²However, not every Banach space X has the property that a C^k -manifold modeled over X admits C^k -partitions of unity. But if the norm of the Banach space is C^k away from zero, than C^k -partitions of unity exist.

⁵³A Finslerian metric is a norm and a Riemannian metric is a scalar product on each tangential space $T_m M$, assigned in a differentiable way.

i-ball $\sup_{\mathbb{R}} |x(r)| \leq 1$, then there are many sequences, which converge to zero but have no trace on this i-ball. For example, the sequence of mappings $x_n(r) := \frac{1}{n}r$ converges to 0 in $C(\mathbb{R}, \mathbb{R})$, but no x_n satisfies $\sup_{\mathbb{R}} |x_n(r)| \leq 1$. Thus if subsets of M are merely identified with dense i-balls, then the notion of convergence in the modeling locally convex topological vector space would be very different from the notion of convergence defined by $x_m \rightarrow x$, if all x_m lie in a chart domain and converge to the image of x in this chart domain.

Here we overcome these difficulties by considering merely projective limits of Banach manifolds. More precisely, suppose that M_i is a family of Banach manifolds indexed by elements of a directed set I , and that each M_i is endowed with a Finslerian or Riemannian metric on the tangential bundle $\pi_i : TM_i \rightarrow M_i$ with corresponding distance d_i and corresponding exponential map $\exp_i : TM_i \rightarrow M_i$. Let $|_i^j : TM_j \rightarrow TM_i$ be vector bundle maps over uniformly continuous and differentiable maps $|_i^j : M_j \rightarrow M_i$ (denoted by the same symbol), which satisfy $|_i^j \circ |_j^k = |_i^k$ and $|_i^i = \text{Id}$. Thus $(i, j) \mapsto |_i^j$ is a functor from the directed set I (regarded as category) into the category of vector bundle maps over uniformly continuous and differentiable maps, and hence the projective limits $M := \lim_i M_i$ and $TM := \lim_i TM_i$ can be formed within the category of uniform spaces resp. vector bundles over uniform spaces. Denote shortly by $|_i$ the restriction $|_i^j$ from some M_j to M_i whenever $i \leq j$, and suppose that π_i and \exp_i are compatible with the restrictions $|_i$ in the sense that $|_i \circ \pi_j = \pi_i \circ |_j$ and $|_i \circ \exp_j = \exp_i \circ |_j$ are valid for $i \leq j$. Then, due to the universal property of projective limits and the compatibility of $|_i$ with π and \exp , unique maps $\pi : TM \rightarrow M$ and $\exp : TM \rightarrow M$ can be defined by $|_i \circ \pi = \pi_i \circ |_i$ and $|_i \circ \exp = \exp_i \circ |_i$ for all i . In this way, TM becomes a vector bundle over M via the map π , however the fibers $T_m M := \lim_i T_{m|_i} M_i$ are only projective limits of Banach spaces and thus in general locally convex topological vector spaces. Further let \exp_i have local injectivity radius $r_i(m_i)$ at m_i , then \exp is injective at m on the i-ball $\{v \in T_m M | v|_i T_{m|_i} M_i \Rightarrow \|v|_i\|_i < r_i(m|_i)\}$.

Now we have a uniform structure on M , a tangential bundle TM on M with projection π and charts \exp on i-balls. But still we do not know whether the chart changes $\exp_{m'}^{-1} \circ \exp_m$ are differentiable in our sense. Here the following lemma often helps.

Lemma 5.7 *A map $f : \lim X_i \rightarrow \lim Y_j$ is Lipschitz, iff for every j there is an i and a Lipschitz map $f_{ij} : X_i \rightarrow Y_j$ satisfying $\pi_j \circ f = f_{ij} \circ \pi_i$.*

Proof: One direction is trivial: If for every j there is an i and a Lipschitz map $f_{ij} : X_i \rightarrow Y_j$ satisfying $\pi_j \circ f = f_{ij} \circ \pi_i$, then

$$\begin{aligned} q_j(f(x) - f(x')) &= q_j(\pi_j(f(x)) - \pi_j(f(x'))) = q_j(f_{ij}(\pi_i(x)) - f_{ij}(\pi_i(x'))) \leq \\ &L_{ij} p_i(\pi_i(x) - \pi_i(x')) = L_{ij} p_i(x - x') \end{aligned}$$

is valid and thus f is Lipschitz. For the other direction, note that f is Lipschitz if all $g := \pi_j \circ f : \lim X_i \rightarrow Y_j$ are Lipschitz. Now g is Lipschitz if there is an i such that $q_j(g(x) - g(x')) \leq L p_i(x - x')$. Define $f_{ij} : X_i \rightarrow Y_j$ by $g = f_{ij} \circ \pi_i$, then f_{ij} is well-defined, since $\pi_i(x) = \pi_i(x')$ implies $p_i(x - x') = 0$ and thus $q_j(g(x) - g(x')) = 0$, i.e. $g(x) = g(x')$. Further obviously f_{ij} has Lipschitz constant $\leq L$. \square

For example, let M be a noncompact, second countable and separated finite-dimensional manifold, and let N be a Riemannian manifold with metric d_N , tangential projection π_N and exponential map \exp_N . Then the space of continuous maps from M to N endowed with the topology of uniform convergence on compact subsets is defined as a uniform space by the projective limit $C(M, N) = \lim_K (C(K, N), \sup_K d_N(\cdot, \cdot))$, where K runs through all compact subsets of M and $|_K$ is the restriction of maps. However, each $C(K, N)$ is not only a uniform space, but a Banach manifold:

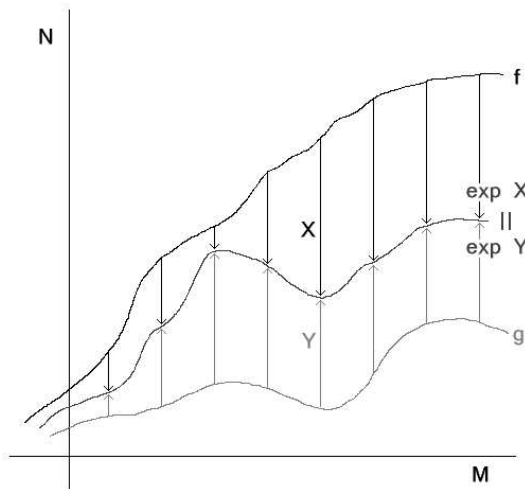


Figure 1: Charts at f, g and Chart-Changes

The tangential space at $g \in C(K, N)$ is the space $C(g^*TN)$ of continuous vector fields $X : K \rightarrow TN$ over g , i.e. $X(k) \in T_{g(k)}N$, and a chart at g is given by $\exp_g : C(g^*TN) \ni X \mapsto \exp_N \circ X \in C(K, N)$. Thus we can endow $C(M, N)$ with the structure defined above. In this way $C(M, TN)$ is the tangential bundle of $C(M, N)$, the projection π is $X \mapsto \pi_{TN} \circ X$ and the exponential map \exp is $X \mapsto \exp_{TN} \circ X$. Further the i -ball, where \exp is injective, is dense, as the vector fields with compact support are dense. Moreover, the chart changes $(\exp_h^{-1} \circ \exp_f) : C(f^*TN) \rightarrow C(h^*TN)$, $f, h \in C(M, N)$, are given by composition of vector fields $X : M \rightarrow TN$ over f with $\exp_{h(\cdot)}^{-1} \circ \exp_{f(\cdot)}$. Now for C^k -maps h, f and restricted to compact domains this composition is C^k by the ordinary omega lemma.

Thus if there is a covering of M by a uniformly locally finite atlas of normal charts, then especially $\exp_{h(\cdot)}^{-1} \circ \exp_{f(\cdot)}$ is Lipschitz continuous by the above lemma. But we have also a candidate for the derivative, namely the composition with the derivative, and thus the chart changes $\exp_{h(\cdot)}^{-1} \circ \exp_{f(\cdot)}$ are really differentiable in our sense.

This example $C(M, N)$ is a typical example of a manifold of mappings on a noncompact manifold M modeled over a locally convex topological vector space. In the next chapter we will see, how other manifolds of mappings can be defined in the same way and can be used in fluid dynamics.

Manifolds defined by directional and functional structures The category of C^k -manifolds and C^k -maps does not have nice categorial properties like cartesian closedness even for $k = \infty$ ⁵⁴. To obtain cartesian closedness, the notion of a manifold has to be considerably generalized by considering directional and functional structures instead of atlases of charts, see [Fröhlicher, Kriegel].

⁵⁴By [Michor, 4.9], for noncompact M there is no way to make $C^\infty(M, N)$ into a manifold using charts on open subsets of a topological vector space, unless the topology on $C^\infty(M, N)$ is finer than the \mathfrak{D} -topology. And with the fine- \mathfrak{D} -topology $C^\infty(M, N)$ becomes a manifold modeled on open subsets of the locally convex topological vector spaces $C_c^\infty(f^*TN)$, but this topology is so fine that C^∞ -maps f and g lie in the same connected component, only if they are identical outside a set of compact support.

A space M endowed with a set \mathcal{C} of distinguished directions is called a directionally structured space, if \mathcal{C} contains the constant directions $c : t \mapsto m$ at all points $m \in M$ and is closed under scaling, i.e. $c \in \mathcal{C}$ implies $sc \in \mathcal{C}$ for every $s \in \mathbb{R}$. Define the bundle $\pi : \mathcal{C} \rightarrow M$, which to every direction $c \in \mathcal{C}$ assigns its base point $c(0) \in M$, and denote the fibers $\pi^{-1}(\{m\})$ of π at m by $\mathcal{C}(m)$. A map $f : M \rightarrow N$ between directionally structured spaces M, N is called a morphism of directionally structured spaces at $m \in M$, if $c \in \mathcal{C}_M(m)$ implies $f \circ c \in \mathcal{C}_N(f(m))$. The category of directionally structured spaces has initial and final objects⁵⁵. Often a directionally structured space (M, \mathcal{C}) is additionally endowed with a homogeneous bundle map $J : \mathcal{C} \rightarrow JM$ from π into a bundle JM over M called the jet bundle, where each fiber $J_m M$ carries an operation of (\mathbb{R}, \cdot) . In this case, $f : M \rightarrow N$ is called a morphism at $m \in M$, if additionally for every $c \in \mathcal{C}_M(m)$ the value $J(f \circ c) \in J_n N$ depends on $Jc \in J_m M$ only, so that the map $c \mapsto f \circ c$ induces a map $J_m f : Jc \mapsto J(f \circ c)$ from $J_m M$ to $J_n N$. If further the bundle $JM \rightarrow M$ has an additional structure, e.g. the structure of a vector bundle, then Jf is also required to be compatible with the additional structure, e.g. Jf is required to be a vector bundle map.

Dually to directions and directionally structured spaces, functionals and functionally structured spaces are defined. The germ of a continuous function $\phi : M \rightarrow \mathbb{R}$ with $\phi(m) = 0$ at a point $m \in M$ is called a functional at m . Hereby the germ of ϕ at m is the equivalence class of ϕ w.r.t. the equivalence relation defined by $\phi \sim \psi$, if there is a neighbourhood U of m with $\phi|_U = \psi|_U$ ⁵⁶. Again identify the continuous function ϕ with the functional induced by ϕ at m , define to a functional ϕ at m the scaled functional $(s\phi)(m') := s\phi(m')$, and consider positive and negative functionals dually to inner and outer directions. Further observe that every continuous function $\phi : M \rightarrow \mathbb{R}$ induces the functional $\phi - \phi(m)$ at $m \in M$.

A space M endowed with a distinguished set \mathcal{F} of functionals⁵⁷ is called a functionally structured space, if \mathcal{F} contains the zero functional at every m and is closed under scaling. A map $f : M \rightarrow N$ between functionally structured spaces is called a morphism at m , if $\phi \in \mathcal{F}_N(f(m))$ implies $\phi \circ f \in \mathcal{F}_M(m)$. The category of functionally structured spaces has initial and final objects⁵⁸. Often functionally structured spaces are additionally endowed with jet bundles J^*M over M , and then morphisms are required to induce compatible bundle maps $J^*f : J^*N \rightarrow J^*M$.

The combination of directionally and functionally structured spaces is the notion of an \mathcal{S} -structured space M , where \mathcal{S} is a fixed set of germs of functions h on \mathbb{R} at zero with

⁵⁵The initial structure on M w.r.t. $f_i : M \rightarrow M_i$ is given by the initial topological structure and those directions c in M with $f_i \circ c \in \mathcal{C}_{M_i}$ for all i . The final structure on M w.r.t. $f_i : M_i \rightarrow M$ is given by the final topological structure, the directions $f_i \circ c$, $c \in \mathcal{C}_{M_i}$, and the constant directions on M .

⁵⁶If M is a convergence or limit space, then a neighbourhood of m is a set $U \in \mathcal{U}(m)$ in the neighbourhood filter $\mathcal{U}(m) = \bigcap_{\mathcal{G} \rightarrow m} \mathcal{G}$ induced by the topological structure on M at m .

⁵⁷In this section the symbol \mathcal{F} is reserved for functional structures and is not used for filters.

⁵⁸The initial structure on M w.r.t. $f_i : M \rightarrow M_i$ is given by the initial topological structure, the functionals $\phi \circ f_i$, $\phi \in \mathcal{F}_{M_i}$, and the zero functional at every m . The final structure on M w.r.t. $f_i : M_i \rightarrow M$ is given by the final topological structure and those functionals with $\phi \circ f \in \mathcal{F}_{M_i}$ for all i .

$h(0) = 0$, and on M a directional structure \mathcal{C} as well as a functional structure \mathcal{F} are given, which determine each other by the requirements

- $c \in \mathcal{C}(m)$ iff $\phi \circ c \in \mathcal{S}$ for all $\phi \in \mathcal{F}(m)$,
- $\phi \in \mathcal{F}(m)$ iff $\phi \circ c \in \mathcal{S}$ for all $c \in \mathcal{C}(m)$.

A map $f : M \rightarrow N$ is a morphism, if $c \in \mathcal{C}_M$ implies $f \circ c \in \mathcal{C}_N$, or equivalently $\phi \in \mathcal{F}_N$ implies $\phi \circ f \in \mathcal{F}_M$, or equivalently $c \in \mathcal{C}_M(m)$ and $\phi \in \mathcal{F}_N(f(m))$ imply $\phi \circ f \circ c \in \mathcal{S}$. The category of \mathcal{S} -structured spaces has initial and final objects⁵⁹. Further this category is cartesian closed, if the set $\mathcal{S}(\mathbb{R}, \mathbb{R})$ of those functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(\cdot + t) - f(t) \in \mathcal{S}$ for all $t \in \mathbb{R}$ becomes an \mathcal{S} -structured space by allowing those directions $c : \mathbb{R} \rightarrow \mathcal{S}(\mathbb{R}, \mathbb{R})$, which are morphisms from $\mathbb{R} \times \mathbb{R}$ to \mathbb{R} when interpreted as maps $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. This is the case for the set $\mathcal{S} = \text{Lip}^k(\mathbb{R}, \mathbb{R})$ of Lip^k -maps on \mathbb{R} . However note that for convenient vector spaces X, Y there is a difference between the convenient vector space $\text{Lip}^k(X, Y)$ and the Lip^k -space $\text{Lip}^k(X, Y)$, see [Fröhlicher, Kriegl, 4.5], so that the category of convenient vector spaces and Lip^k -maps is cartesian closed only if $k = \infty$.

⁵⁹An arbitrary set \mathcal{C} of directions on M generates a \mathcal{S} -structure on M by $\phi \in \mathcal{F}_M$, iff $\phi \circ c \in \mathcal{S}$ for all $c \in \mathcal{C}$, and $c \in \mathcal{C}_M$, iff $\phi \circ c \in \mathcal{S}$ for all $\phi \in \mathcal{F}_M$. In an analogous way also an arbitrary set \mathcal{F} of functionals on M generates a \mathcal{S} -structure. Now the initial structure on M w.r.t. $f_i : M \rightarrow M_i$ is generated by the initial directional structure \mathcal{C}_M , while the final structure on M w.r.t. $f_i : M_i \rightarrow M$ is generated by the final functional structure \mathcal{F}_M .

Part II

Dynamical Systems and Pattern Formation under Symmetry

6 Manifolds of Mappings and Geometrical Fluid Dynamics

In this chapter geometric fluid dynamics - especially on noncompact domains - is discussed. Thereby we restrict ourselves to domains which are manifolds of bounded geometry, possibly with boundary, because on such domains the Sobolev embedding theorems (and other constructions used later) are valid. After an introduction to bounded geometry, manifold of mappings like the local and global Sobolev spaces of maps on a domain of bounded geometry are defined, and especially the group of diffeomorphisms within such manifolds of mappings is studied. As an application the Euler equations are interpreted as ordinary differential equations with continuously differentiable right hand side on such manifolds of mappings: The Euler equations are geodesic equations on the manifold $\text{Diff}_{Vol}(M)$ of volume form preserving global Sobolev diffeomorphisms w.r.t. the H^0 -metric, and other fluid dynamical equations can be characterized in a similar way as Hamiltonian equations on manifolds of mappings. Finally we discuss, which fluid dynamical equations can be interpreted as Hamiltonian equations on local spaces of mappings. The conclusion is that the Euler and Navier-Stokes equations can not be interpreted as geodesic equations on such local spaces of mappings, as they have nonlocal properties due to incompressibility and viscosity, while the hyperbolic equations modeling compressible fluids can be interpreted as ordinary differential equations on local spaces, as they are local equations due to their finite velocity of propagation. Thus the analysis developed in the first part can be used to study pattern formation in compressible inviscous fluids but not in incompressible viscous fluids.

6.1 Bounded Geometry

For all vector bundles E over a n -dimensional manifold M the local spaces $C^k(E)$ and $H_{loc}^s(E)$ of sections of E can be defined in terms of a covering by charts and the corresponding norms in \mathbb{R}^n . They are independent of the choice of the covering by charts, as merely properties of sections on compact subsets are required. But to define the global spaces $C_b^k(E)$ and $H^s(E)$ for noncompact manifolds M may cause problems, as the local definition in a chosen collection of charts may generally depend on the choice of this collection and may differ from the definition in terms of a Riemannian metric on E and M . This is not the case for manifolds and vector bundles of bounded geometry, as on such manifolds there are distinguished uniformly locally finite covering by charts.

Define for a topological vector bundle $p : E \rightarrow M$ over a manifold M in analogy to uniform structures the notion of entourages: The entourages shall form a filter of sets $V \subset E$ such that

- every entourage V contains the zero section,
- with V also $-V$ is an entourage,
- for every entourage V there is an entourage W with $W + W \subset V$,
- the topology on E generated by the neighbourhoods $V|_U$ of 0 , $V \subset E$ an entourage and $U \subset M$ a neighbourhood of m , is the original topology on E .

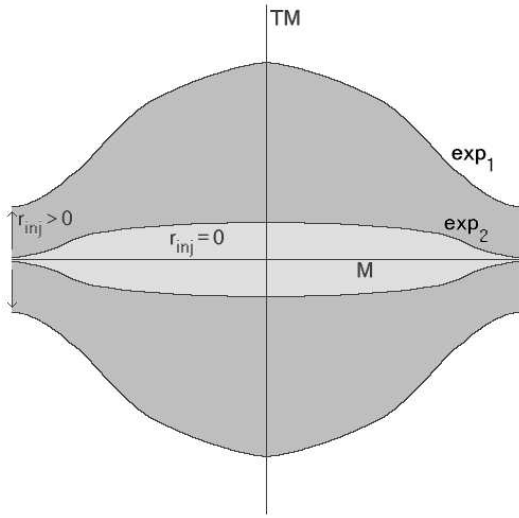


Figure 2: Injectivity Radiuses of Different Metrics

On every topological vector bundle trivially the neighbourhoods of the zero section form a filter of entourages. But for a Riemannian metric on E (or more generally for a Finslerian) also the sets of the form $\{v \mid \|v\| \leq \epsilon\}$, $\epsilon > 0$, generate a filter of entourages, which generally is not identical with the filter of neighbourhoods of the zero section, as such neighbourhoods can become arbitrary small at infinity. More precisely, every neighbourhood U of the zero section contains sets of the form $\{v \mid \|v\|_m \leq h(m)\}$ with a continuous function $h : M \rightarrow \mathbb{R}^+$. Thus call a neighbourhood U arbitrary small at infinity, if $\inf_{m \in M} h(m) = 0$ holds for all such sets contained in U . Obviously neighbourhoods of the zero section that are arbitrary small at infinity do not contain sets of the form $\{v \mid \|v\|_m \leq \epsilon\}$, $\epsilon > 0$.

These different notions of entourages become especially important for the tangential bundle TM of a Riemannian manifold M . There the Riemannian metric on M induces beneath the uniform structure on TM generated by the entourages $\{v \mid \|v\| \leq \epsilon\}$ also a uniform structure on M via the distance function

$$d(m, m') := \inf \left\{ \int_0^1 \|\dot{c}(t)\|_{c(t)} dt \mid c \in C^1([0, 1], M) : c(0) = m, c(1) = m' \right\}.$$

Now recall that for every Riemannian manifold $(\pi, \exp) : TM \rightarrow M \times M$ is a diffeomorphism from a neighbourhood of the zero section onto a neighbourhood of the diagonal in $M \times M$. But these neighbourhoods do not need to be entourages induced by the Riemannian metric (i.e. on TM of the form $\{v \mid \|v\| \leq \epsilon\}$ and on $M \times M$ of the form $\{(x, y) \mid d(x, y) \leq \epsilon'\}$) and (π, \exp) does generally not need to be uniformly continuous (i.e. for every entourage W on M there is an entourage V on TM such that $(\pi, \exp)(V) \subset W$).

A manifold is said to have bounded geometry⁶⁰ of order k , if

- the injectivity radius $r_{inj}(M)$ of M is positive,
- the Riemannian curvature tensor R is uniformly bounded up to order k , i.e. $\|\nabla^i R\| \leq C$ for all $0 \leq i \leq k$.

If M has boundary ∂M , assume further that

- the boundary has a normal collar, i.e. there is an $r > 0$ such that the normal collar map $N_1 := [0, r) \times \partial M \rightarrow M$, $(t, m) \mapsto \exp(t\nu_m)$ is a diffeomorphism onto its image, where ν_m denotes the unit inward normal vector at m ,
- the injectivity radius $r_{inj}(\partial M)$ is positive and instead of $r_{inj}(M) > 0$ it is required, that \exp is a diffeomorphism on $\{v_m \in T_m M \mid \|v_m\|_m < r\}$ for a radius $r > 0$ and all $m \in M \setminus N_{\frac{1}{3}}$, where N_s denotes the image of $[0, sr) \times \partial M$ under the normal collar map,
- the second fundamental form l is uniformly bounded up to order k , i.e. $\|\nabla^i l\| \leq C$ for all $0 \leq i \leq k$.

By a theorem of [Green] on every finite-dimensional manifold M there exists a complete metric such that M is of bounded geometry of arbitrary high order k . Usually we do assume that a manifold has bounded geometry of such a high order k that we do not have to care about the best possible values of k in theorems. For example, the Sobolev embedding theorem $H^s \subset C_b^k$ can be proved for differentiability order s smaller than the boundary order k , see [Eichhorn, Schmid, Proposition 2.6].

A consequence of the boundedness of the curvature tensor is that on a manifold of bounded geometry the derivative $d\exp$ of the exponential map has uniformly bounded differentials up to order k . Especially (π, \exp) is globally Lipschitz continuous for $k \geq 1$ and thus uniformly continuous.

Instead of $\|\nabla^i R\| \leq C$ for $i = 0, \dots, k$ equivalently it is possible to require that in normal coordinates the metric tensor g and its inverse are uniformly bounded up to order k , see [Schick, A.1].

A manifold of bounded geometry has for all small enough $r > 0$ a countable covering with normal charts $\kappa_i : \mathbb{R}^n \supset B_r(0) \rightarrow N(m_i, r) := \{\exp(v) \mid v \in T_{m_i} M, \|v\| \leq r\} \subset M$ and a subordinate partition of unity ϵ_i such that

- the points m_i lie either on the boundary ∂M or are bounded away from the boundary, i.e. $m_i \in M \setminus N_{\frac{2}{3}}$,

⁶⁰There are also weaker notions of bounded geometry. For example, if the Ricci curvature tensor is bounded from below and the manifold has a positive injectivity radius, then also the Sobolev embeddings are valid, see e.g. [Hebey, Theorem 3.5, Proposition 3.6], and the manifold is said to have Ricci bounded geometry.

- the covering has uniformly finite multiplicity, i.e. for all $m \in M$ the number of indices i with $N(m, r) \cap N(m_i, r) \neq \emptyset$ is bounded by a constant independent of m .
- the functions ϵ_i are bounded up to order k in normal coordinates.

Further the volume $\text{Vol}(B_m(r))$ of balls $B_m(r)$ is uniformly bounded from below by a nonnegative monotone function $V(r)$ with $V(r) \rightarrow \infty$ for $r \rightarrow \infty$, see [Schick, 3.20+3.22].

On a manifold of bounded geometry, the C_b^k -norm

$$\|f\|_{C_b^k(M)} := \sum_i \|\epsilon_i(f \circ \kappa_i)\|_{C_b^k}$$

and the H^s -norm

$$\|f\|_{H^s(M)} := \sum_i \|\epsilon_i(f \circ \kappa_i)\|_{H^s}$$

of a function $f : M \rightarrow \mathbb{R}$, defined in terms of normal charts κ_i of a uniformly finite covering by normal charts and a subordinated partition of unity ϵ_i , are independent of the choice of the covering and the partition of unity. In the same way, the spaces $C_b^k(E)$ resp. $H^s(E)$ of sections of a bundle E over M of bounded geometry⁶¹ are defined as those C^k -sections ϕ with $\|\phi\|_{C_b^k(M)} < \infty$ resp. by the completion of $C_b^s(E)$ w.r.t. the H^s -norm. The definition of these spaces is independent of the particular choice of a covering and a subordinate partition of unity, and these spaces of sections have the usual properties well-known for compact manifolds, see [Schick, 3.25]:

- $C_b^k(E)$ are Banach spaces, $H^s(E)$ are Hilbert spaces,
- the norms $\|\alpha\|_{C_b^k} := \sum_{i=0}^k \sup_{m \in M} \|\nabla^i \alpha_m\|_m$ resp. $\|\alpha\|_{H^s(\Lambda^k M)}^2 = \sum_{i=0}^s \int_M \|\nabla^i \alpha_m\|_m^2 dm$ defined in terms of a Riemannian metric are equivalent to the norms defined in charts,
- the embedding theorem $H^s(E) \rightarrow C_b^k(E)$ for $s > \frac{\dim(E)}{2} + k$ is valid.
- the norms $\|\alpha\|_{C_b^k} := \sum_{i=0}^k \sup_{m \in M} \|\nabla^i \alpha_m\|_m$ resp. $\|\alpha\|_{H^s(\Lambda^k M)}^2 = \sum_{i=0}^s \int_M \|\nabla^i \alpha_m\|_m^2 dm$ defined in terms of a Riemannian metric are equivalent to the norms defined in charts,
- for differential forms, i.e. $E = \Lambda^k M$, also

$$(\alpha, \beta)_{H^s(\Lambda^k M)}^2 = \sum_{i=1}^s \int_M ((d + \delta)^i \alpha, (d + \delta)^i \beta)$$

yields an equivalent norm on $H^s(E)$, where d is the differential, δ the codifferential, $*$ the Hodge operator on forms and $(\alpha, \beta) = \alpha \wedge * \beta$.

⁶¹A bundle E is said to have bounded geometry of order k , if over each normal chart of M a trivialisation of the bundle is chosen such that the transition functions are uniformly bounded up to order k in normal charts. If E has a Riemannian metric, then the duality mapping $E \mapsto E'$ is required to be uniformly bounded with uniformly bounded inverse up to order k , or equivalently the metric tensor and its inverse have to be bounded uniformly.

- the linear embedding $H^s \rightarrow H^t$, $s > t$, is continuous (but generally not compact),
- let F be another bundle of bounded geometry over a l' -codimensional submanifold N of M , then differential operators $L : C^\infty(E) \rightarrow C^\infty(F)$ of order l , which have uniformly bounded coefficients up to order l , extend to continuous linear operators $L : H^s(E) \rightarrow H^{s-l-\frac{l'}{2}}(F)$ for $s > l + \frac{l'}{2}$,
- the spaces $H^s(E)$ and $H^{-s}(E')$ are dual to each other

6.2 Manifolds of Mappings

Let M, N be finite-dimensional complete smooth oriented Riemannian manifolds which are not assumed to be compact but merely locally compact or - in other words - open manifolds. The aim of this section is to describe how certain sets of maps $f : M \rightarrow N$ can be defined and can be turned into manifolds.

Let Map be a functor⁶² which associates to every finite dimensional smooth Riemannian vector bundle E over a finite dimensional smooth Riemannian manifold M a Banach space $\text{Map}(E)$ of sections of E over M containing $C^\infty(E) \subset \text{Map}(E)$. For example, C_k^b and H^s are such functors.

To associate a manifold $\text{Map}(M, N)$ of mappings between M and N to such a functor, let $\exp_N : TN \rightarrow N$ be the exponential map to the Riemannian metric with norm $\|\cdot\|_N$ on TN . Note that \exp_N is a local addition, i.e. $\exp_N(0_n) = n$ holds for all $n \in N$ and with the projection $\pi_N : TN \rightarrow N$ of the tangential bundle the map $(\pi_N, \exp_N) : TN \rightarrow N \times N$ is a diffeomorphism from some neighbourhood U of the zero section in TN onto some neighbourhood V of the diagonal in $N \times N$. A vector field X over f ⁶³ is called U -near to the zero section if $X(m) \in U$ is valid for all $m \in M$, and a map $x : M \rightarrow N$ is called V -near to f if $(f(m), x(m)) \in V$ is valid for all $m \in M$. Denote by U_f the vector fields $X(m) \in \text{Map}(f^*TM)$ over f which stay U -near to the zero section and by V_f the smooth maps x which stay V -near to f . Then $U_f \cap C^\infty(f^*TN)$ and V_f can be identified via the bijection $\phi_f : X \mapsto \exp_N \circ X$, whose inverse is the map $\phi_f^{-1} : x \mapsto (\pi_N, \exp_N)^{-1} \circ (f, x)$.

To construct the manifold $\text{Map}(M, N)$ via these charts, transfer the uniform structure of the linear space $\text{Map}(f^*TN) \cap C^\infty(f^*TN)$ ⁶⁴ to a uniform structure on $C^\infty(M, N)$ by defining the base

$$V_\epsilon := \{(f, x) | \exists X \in C^\infty(f^*TN) : x = \exp_N \circ X, \quad \|X\| \leq \epsilon\}$$

for $\epsilon > 0$. But notice that it still has to be verified, whether V_ϵ is really the base of a uniform structure. After this is done, $\text{Map}(M, N)$ can be defined as the completion of

⁶²In [Palais] a precise description of those functors Map which are convenient for constructing manifolds of mappings can be found.

⁶³A vectorfield X over f is a map $X : M \rightarrow TN$ which satisfies $\pi_N \circ X = f$, or equivalently a section of the pullback f^*TN of the bundle TN by f .

⁶⁴The Riemannian metric and its associated Levy-Cita connection on the tangential bundle TN over N are transported to the bundle f^*TN over M via pullback, thus f^*TN is a Riemannian vector bundle over M and hence $\text{Map}(f^*TN)$ is defined.

$C^\infty(M, N)$ w.r.t. this uniform structure. Then the charts ϕ_f can be uniquely continued to charts from U_f to $\overline{V}_f \subset \text{Map}(M, N)$, as they are uniform maps⁶⁵ due to the definition of $V_{p,\epsilon}$ and thus map Cauchy filters to Cauchy filters.

Now we have constructed charts around $f \in \text{Map}(M, N)$ with values in $\text{Map}(f^*TN)$, but for different points $f, g \in \text{Map}(M, N)$ these charts map into different spaces $\text{Map}(f^*TN)$, $\text{Map}(g^*TN)$, and also the differentiability of the chart changes $\phi_g^{-1} \circ \phi_f$ has not yet been verified. Thus to conclude finally that $\text{Map}(M, N)$ is a differentiable manifold, it remains to verify that the chart changes $\phi_g^{-1} \circ \phi_f$ are diffeomorphisms. Especially in this case their linearization is an isomorphism between $\text{Map}(f^*TN)$ and $\text{Map}(g^*TN)$, i.e. up to isomorphy these locally convex vector spaces are identical, and thus $\text{Map}(M, N)$ is modeled at each point over the same linear space. The chart changes are given by $\phi_g^{-1} \circ \phi_f : X \mapsto (\pi_N, \exp_N)^{-1} \circ (g, \exp_N \circ X)$ for a vectorfield X over f . But a better way to describe chart changes is as follows: Let $x \in V_f \cap V_g$, let $X \in U_f$ be the vector field over f with $\exp_N \circ X = x$ and let $Y \in U_g$ be the vector field over g with $\exp_N \circ Y = x$. Then the vector field $(\phi_g^{-1} \circ \phi_f)(X)$ over g is obtained at the point m by transporting the vector $X(m)$ over $f(m)$ parallel along the curve $[0, 1] \ni s \mapsto \exp_{f(m)} sX(m)$ to the vector $P_f(X(m))$ over $x(m) = \exp_{f(m)} X(m)$, and afterwards by transporting the vector $P_f(X(m))$ over $x(m) = \exp_{g(m)} Y(m)$ parallel along the curve $[0, 1] \ni s \mapsto \exp_{g(m)}(1-s)Y(m)$ to the vector $P_{-g}(P_f(X(m))) = (\phi_g^{-1} \circ \phi_f)(X)(m)$.

Thus to establish continuity and differentiability of the chart changes, generally it has to be controlled how \exp_N , composition and parallel transport act on vector fields and their derivatives, what requires especially the estimation of Jacobi vector fields.

Global C_b^k - and $W^{k,p}$ -spaces on noncompact manifolds While for compact manifolds M and $\text{Map} = C_b^k$ resp. $\text{Map} = W^{k,p}$ the above construction goes through without problems, for noncompact M it is difficult to control parallel transport, but possible in the case of bounded geometry.

Let us have a closer look how the global spaces are constructed for noncompact manifolds. To a Riemannian vector bundle E over a manifold M the space $C_b^k(E)$ is defined as the set of C^k -sections $X : M \rightarrow E$ endowed with the norm $\|X\|_{C_b^k(E)} := \sum_{i=0}^k \sup_{m \in M} \|\nabla^i X\|_N$. To construct the manifold $C_b^k(M, N)$ via completion, we use only those smooth maps f whose differentials are bounded globally up to order k , i.e. $\sum_{i=0}^{k-1} \sup_M \|\nabla^i df\| < \infty$, instead of all C^∞ -maps, as else it is not possible to estimate the Jacobi fields. Therefore we have to assure, that the charts $\phi_f(X) = \exp_N \circ X$ really map smooth C_b^k -vector fields X over f to such bounded smooth maps. This is the case for manifolds of bounded geometry, as on such manifolds the exponential map \exp_N has bounded derivatives $\nabla^i d \exp_n$ up to the boundedness order. Now the charts are well-defined and it must be proved that the sets V_ϵ really generate a uniform structure. This has been done by [Eichhorn, Schmid] for C_b^k - and also for the global Sobolev $W^{k,p}$ -spaces. In the proof lengthy calculations are needed which estimate Jacobi vectorfields, exponential maps and curvature terms globally by usage of the assumptions of bounded geometry. Finally the differentiability of chart changes has to be verified. Here the local ω -lemma is used: If $h \in C^{\infty, m+k}(\mathbb{R}^i, \mathbb{R}^j)$ and $f \in C_b^k(U, \mathbb{R}^i)$ (U an open subset of

⁶⁵Indeed, $\phi : \bigcup_f \text{Map}(f^*TN) \rightarrow \text{Map}(M, N)$ maps $\{(f, X_f) \mid \|X_f\| \leq \epsilon\}$ onto V_ϵ .

some \mathbb{R}^l), then the map $\omega_h : f \mapsto h \circ f$ from $C_b^k(U, \mathbb{R}^i)$ to $C_b^k(U, \mathbb{R}^j)$ is a C^m -map with differential $D(\omega_h) = \omega_{Dh}$. Now apply the local ω -lemma in the domains of a uniform covering by normal charts of M to the chart change

$$(\phi_g^{-1} \circ \phi_f)(X) = \omega_{\exp_{g(m)}^{-1} \circ \exp_{f(m)}}(X)$$

for $f, g \in C^{\infty, k}$. Using the fact that the derivatives $d(\exp_{g(m)}^{-1} \circ \exp_{f(m)})$ are globally bounded up to the boundedness order b , so that $\exp_{g(m)}^{-1} \circ \exp_{f(m)}$ is a $C^{\infty, b+1}$ -map, it can be concluded that the chart changes $\omega_{\exp_{g(m)}^{-1} \circ \exp_{f(m)}}$ are C^{b+1-k} -maps.

Doing the same for the global Sobolev spaces proves the main theorems [Eichhorn, Schmid, 4.20+5.2]: For complete noncompact manifolds M, N of bounded geometry up to order b , $b \geq k$ and in the case of Sobolev spaces additionally $k > \frac{\dim N}{p} + 1$ the spaces $C_b^k(M, N)$ resp. $W^{k,p}(M, N)$ are C^{b-k+1} -manifolds. Note that both spaces decompose into connected components, as maps which differ by vector fields with unbounded resp. not-globally-integrable derivatives smaller than k can not belong to the same component.

Local C^k - and $W_{loc}^{k,p}$ -spaces As already introduced in section 5.6, the local spaces are defined as the projective limits $C^k(M, N) = \lim_K C_b^k(K, N)$ and $W_{loc}^{k,p}(M, N) = \lim_K W^{k,p}(K, N)$ in the category of uniform spaces. Charts around a point $f \in C^k(M, N)$ resp. $f \in W_{loc}^{k,p}(M, N)$ are given by $X \mapsto \exp_N \circ X$ for $X \in C^k(f^*TN)$ resp. $X \in W_{loc}^{k,p}(f^*TN)$. These charts are defined generally merely on dense i -balls, and in the case of a manifold of bounded geometry we have a uniformly finite covering by normal charts, so that the chart changes from a chart at f to a chart at h are in fact differentiable, if h, f have more regularity than k . But as these more regular maps lie dense, we have enough charts to cover the whole space of mappings.

Manifolds with Boundary If N has boundary, then a space of maps into N is generally not a manifold, as $H^s(M, N)$ has infinite dimensional corners, see [Marsden, Ratiu, 2]. However, N can be embedded into a manifold \tilde{N} without boundary such that N and ∂N are submanifolds of \tilde{N} . Then the space $H^s(M, \tilde{N})$ is again a smooth manifold.

6.3 Diffeomorphism Groups

The last section has discussed the manifold structure of spaces of maps on noncompact manifolds. In this section we want to discuss, whether the groups $\text{Diff}(M)$ of orientation preserving diffeomorphisms associated to these spaces of maps are submanifolds and some kind of Lie groups.

Local case For a diffeomorphism $f : M \rightarrow M$ in $C^k(M)$ there is always a dense i -ball around f , which consists of diffeomorphisms only. In fact, let $\lambda(m)$ be the absolutely smallest eigenvalue of $T_m f$ w.r.t. some chosen covering of M in charts. Disturb f by

a vectorfield over f which is so small that at every m the eigenvalues of the disturbed map do not become zero, then the disturbed map is again a diffeomorphism. Obviously here vectorfields which are small enough and have compact support can be used. As these span a dense subspace, there is a dense infinite ball around f consisting of diffeomorphisms. Thus the diffeomorphisms form a submanifold, however not modeled on an open set but on an infinite dense ball, and in fact the smaller the eigenvalues of Df are, the smaller is this i-ball. Recall that the smaller the i-ball is, the more difficult is it to prove that local deformations, as e.g. to solve a differential equations starting at f , do not leave the chart domain.

The composition $(f, g) \mapsto f \circ g$ of diffeomorphisms is continuous, but differentiable by the ω -lemma only if the diffeomorphism f is of a better differentiability order. The inversion is continuous, as M is a locally connected space, see the paragraph about inversion in 2.5, but again differentiability depends on the differentiability of the map which shall be inverted.

In $W_{loc}^{k,p}(M, M)$ the notion of a diffeomorphism makes sense for $k > \frac{\dim(M)}{p} + 1$, as by Sobolevs embedding theorem all such maps are C^1 -maps and thus classical diffeomorphisms. Also here the set of diffeomorphisms is a submanifold modeled on infinite dense balls around f , whose width depends on the smallest eigenvalues of f . Note that in the Sobolev setting composition makes sense for Sobolev diffeomorphisms, but not for arbitrary Sobolev maps, as the composition of such maps does not need to be a map from the same Sobolev class. But for Sobolev diffeomorphism composition and inversion are well-defined and continuous, however differentiability again depends on a better differentiability order of the diffeomorphisms.

Global case In $C_b^k(M)$ it is required that for a diffeomorphism f the absolutely smallest eigenvalue $\lambda(m)$ of $T_m f$ is bounded globally on M by a constant $\lambda > 0$ in some uniformly finite covering by normal chart. Then these diffeomorphisms form an open submanifold $\text{Diff}_b^k(M)$ in the usual sense of Banach manifolds, as now there is not only a dense infinite ball but an open neighbourhood of f consisting of diffeomorphisms, namely those maps $g = \exp_N \circ X$, $X \in C_b^k(f^*TN)$ for which X is so small that to eigenvalue of f becomes 0 under the disturbance by X . Thus every connected component of $\text{Diff}_b^k(M)$ is again a C^{b-m+1} -Banach manifold. Especially the component of identity is mapped to itself under composition and thus the diffeomorphisms form a topological group, however differentiability is again only possible for diffeomorphisms of a better differentiability order.

For the group $\text{Diff}^r(M)$, $r > \frac{\dim(M)}{2} + 1$, of H^r -Sobolev diffeomorphism on M the same result is valid. But there is another problem, as the group is defined as the completion of C^∞ -maps with bounded derivatives up to order r . But then $\text{Diff}^{r+s}(M)$ is generally not dense in $\text{Diff}^r(M)$, as $\text{Diff}^{r+s}(M)$ is the completion of those C^∞ -maps with bounded derivative up to order $r + s$, but $\text{Diff}^r(M)$ is defined as the completion of the C^∞ -maps with bounded derivative only up to order r . That's why the group $\text{Diff}^{\infty,r}$ should be used, which is obtained by completing those smooth maps which are bounded in all

derivatives. However, on the component of identity $\text{Diff}_{\text{Id}}^{\infty,r}(M) = \text{Diff}_{\text{Id}}^r(M)$ holds⁶⁶. Proofs of these statements can be found in [Eichhorn, Section 6].

Boundary In the case of boundary $\text{Diff}(M, M)$ should be considered as a subset of $\text{Map}(M, \tilde{M})$ where \tilde{M} is a manifold containing M and ∂M as submanifolds. Further there are different types of diffeomorphism groups, and it has to be proven for each type that $\text{Diff}(M, M)$ is a submanifold of $\text{Map}(M, \tilde{M})$.

Two types of such diffeomorphism groups are most naturally, namely the group of those diffeomorphism which leave ∂M invariant, i.e. $\eta(\partial M) \subset \partial M$, and the group of those diffeomorphisms which leave ∂M pointwisely fixed, i.e. $\eta(m) = m$ for all $m \in \partial M$. The proof that these diffeomorphism groups are submanifolds, whose tangential space at η consists of the vector fields over η tangent to the boundary resp. zero on the boundary is contained in [Ebin,Marsden, Section 6] for compact manifolds and Sobolev diffeomorphism, but the same proof is valid for C_b^k -diffeomorphisms, noncompact manifolds of bounded geometry and the local spaces.

Note that there are also other useful types of diffeomorphism groups on manifolds with boundary, e.g. those diffeomorphisms which leave the boundary invariant and satisfy a free-slip condition at the boundary, see [Shkoller].

6.4 The Group of Volume Form preserving Diffeomorphisms

Not only those subgroups of the group of orientation preserving diffeomorphism which preserve in some sense the boundary, but also the subgroups which preserve additionally a given form on the manifold are important. Here we mainly consider the subgroup of those diffeomorphisms η which preserve the orientation and the volume form μ associated to the Riemannian metric on the manifold M , i.e. $\eta^*\mu = \mu$ is valid. But in the same way diffeomorphisms which preserve other volume forms than the one associated to the Riemannian metric or a symplectic form can be considered.

Denote the set of orientation and volume preserving diffeomorphisms by $\text{Diff}_{\text{Vol}}(M)$. For compact manifolds (with boundary) and Sobolev diffeomorphism it is shown in [Ebin,Marsden, Sections 4+8] by using Hodge theory that $\text{Diff}_{\text{Vol}}(M)$ is a submanifold of $\text{Diff}(M)$. In [Eichhorn,Schmid, Theorem 3.3] for noncompact M of bounded geometry (without boundary, but the arguments generalize to manifolds with boundary) and global Sobolev spaces the same is shown under the assumption that the essential spectrum $\sigma_{\text{ess}}(\Delta|_{\text{Ker}(\Delta)^\perp})$ of the Laplace-Beltrami operator Δ acting on functions and restricted to the complement of its kernel is bounded away from zero. Let us briefly review how these results can be proved.

⁶⁶Those diffeomorphisms which differ from Id by a H^r -vector field are arbitrary near to Id at infinity and thus especially bounded in all derivatives, as Id is bounded in all derivatives.

6.4.1 Hodge Theory

Let M be a n -dimensional complete oriented Riemannian manifold and denote by $C^\infty(\Lambda^k M)$ the set of smooth k -forms on M . Further let d be the differential which maps k -forms to $(k+1)$ -forms, denote by $*$ the Hodge star operator which maps k -forms to $(n-k)$ -forms, let $\delta := (-1)^{n(k+1)+1} * d*$ be the codifferential which maps k -forms to $(k-1)$ -forms, and let $\Delta := (d + \delta)^2 = d\delta + \delta d$ be the Laplace-Beltrami operator which maps k -forms to k -forms. Note that $*$, δ and Δ depend on the choice of the Riemannian metric and orientation on M . For $M = \mathbb{R}^n$ the Laplace-Beltrami operator on functions is in normal coordinates the usual Laplacian $-\sum_{i=1}^n \frac{\partial^2}{\partial x^i}$.

Define the H^0 -pseudoscalar product $(\alpha, \beta)_K := \int_K \alpha \wedge * \beta$ of k -forms α, β over a compact n -dimensional submanifold $K \subset M$ with smooth boundary, or more generally of a k -form α and a l -form β over a compact $k+n-l$ -dimensional submanifold K with smooth boundary. Then the differential d and the codifferential δ satisfy the equation

$$(d\alpha, \beta)_K = (\alpha, \delta\beta)_K + (\alpha, \beta)_{\partial K}$$

for smooth $k-1$ -forms α , smooth k -forms β and compact K with smooth boundary by the theorem of Stokes. Indeed, integrate

$$d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^{k-1} \alpha \wedge d * \beta = d\alpha \wedge * \beta - \alpha \wedge * \delta \beta$$

over K to obtain

$$(\alpha, \beta)_{\partial K} = \int_{\partial K} \alpha \wedge * \beta = \int_K d(\alpha \wedge * \beta) = (d\alpha, \beta)_K - (\alpha, \delta\beta)_K \quad .$$

Further partial integration remains valid for the whole noncompact manifold M instead of K and smooth H^1 -forms α, β , i.e. $\alpha, \beta, d\alpha, d\beta$ are smooth and square-integrable ⁶⁷.

Denote by $(\alpha, \beta)_{K,s} := \sum_{i=0}^s ((d + \delta)^i \alpha, (d + \delta)^i \beta)_K$ the H^s -pseudoscalar product of forms, and define the local and global Sobolev spaces $H_{loc}^s(\Lambda^k M)$ resp. $H^s(\Lambda^k M)$ as the completion of $C_c^\infty(\Lambda^k M)$ w.r.t. the pseudoscalar products $(\cdot, \cdot)_{K,s}$ resp. the scalar product $(\cdot, \cdot)_{M,s}$.

Compact Manifolds On compact manifolds the differential d , the codifferential δ and the Laplacian Δ can be extended to operators $d : H^{s+1}(\Lambda^{k-1} M) \rightarrow H^s(\Lambda^k M)$, $\delta : H^{s+1}(\Lambda^{k+1} M) \rightarrow H^s(\Lambda^k M)$ and $\Delta : H^{s+2}(\Lambda^k M) \rightarrow H^s(\Lambda^k M)$ on the Sobolev

⁶⁷See e.g. [Lott, Lemma 1] for manifolds without boundary, but the arguments are also valid for manifolds with boundary. Indeed, if ϕ_i is a sequence of compactly supported functions with $\phi_i \rightarrow 1$ and $d\phi \rightarrow 0$, then

$$\lim_i \int_M \phi_i d(\omega \wedge \eta) = \lim_i \int_{\partial M} \phi_i (\omega \wedge \eta) - \int_M d\phi_i \wedge \omega \wedge \eta \rightarrow \int_{\partial M} \omega \wedge \eta$$

is valid.

spaces. The Laplacian is a uniformly elliptic differential operator⁶⁸ and an essentially self-adjoint operator on the Hilbert space $H^s(\Lambda^k M)$ ⁶⁹. The kernel of Δ in H^s is usually denoted by \mathcal{H} and its elements are called harmonic forms. By regularity theory the harmonic forms are automatically smooth. As the Laplacian has further a compact resolvent⁷⁰, its spectrum consists of non-negative discrete eigenvalues whose corresponding eigenspaces are pairwise orthogonal and finite-dimensional. Especially \mathcal{H}^s is finite dimensional.

On a compact manifold M without boundary the formula $(\Delta\alpha, \alpha)_M = (d\alpha, d\alpha)_M + (\delta\alpha, \delta\alpha)_M$ is valid, thus being a harmonic form is equivalent to $d\alpha = 0$ and $\delta\alpha = 0$, and the Laplacian is nonnegative⁷¹. The Hodge decomposition theorem now says that the orthogonal complement of \mathcal{H} is the image of Δ :

$$H^s(\Lambda^k M) = \Delta(H^{s+2}(\Lambda^k M)) \oplus \mathcal{H} = d(H^{s+1}(\Lambda^{k-1} M)) \oplus \delta(H^{s+1}(\Lambda^{k+1} M)) \oplus \mathcal{H}$$

is an orthogonal decomposition of $H^s(\Lambda^k M)$ in the H^0 - and also in the H^s -norm, see e.g. [Warner, 6]. Denote by \mathcal{C}^s the forms $\alpha \in H^{s+1}(\Lambda^{k+1} M)$ which satisfy $\delta\alpha = 0$ and are called coclosed, then the Hodge decomposition theorem implies also the validity of the orthogonal decomposition

$$H^s(\Lambda^k M) = d(H^{s+1}(\Lambda^{k-1}(M))) \oplus \mathcal{C}^s \quad (6)$$

Now let M be a compact manifold with smooth boundary ∂M . Note that by the trace theorem⁷² every form in $H^s(\Lambda^k M)$ can be restricted to a $H^{s-\frac{l}{2}}$ -form on an l -dimensional submanifold of M . Especially H^s -forms on M , $s \geq 1$, can be restricted to H^0 -forms on the boundary ∂M . A form α is called tangential to ∂M if its normal part $n\alpha := \iota^*(\ast\alpha)$ vanishes, and it is called normal to the boundary if its tangential part $t\alpha := \iota^*(\alpha)$ vanishes. Hereby $\iota : \partial M \rightarrow M$ denoted the inclusion of the boundary and the pullbacks are well-defined by the trace theorem.

On a compact manifold M with boundary the Laplace operator together with its boundary condition is still uniformly elliptic and essentially self-adjoint. But the formula $(\Delta\alpha, \alpha)_M = (d\alpha, d\alpha)_M + (\delta\alpha, \delta\alpha)_M$ is not valid anymore, and thus a harmonic form

⁶⁸A differential operator L of order l on k -forms is called elliptic at a point $m \in M$ if its symbol at m defined by $\sigma_{L,m}(\xi)(v) := L(\phi^l \alpha)(m)$ for a $v \in \Lambda_m^k M$, a k -form α with $\alpha(m) = v$ and a function ϕ with $\phi(m) = 0$ and $d\phi(m) = \xi \in T_m^* M$ is an isomorphism $\sigma_L(\xi) : \Lambda_m^k M \rightarrow \Lambda_m^k M$ for each $0 \neq \xi \in T_m^* M$. Equivalently in coordinates a differential operator $L(m) = P_l(m, D) + \dots + P_0(m, D)$ of order l is elliptic at $m \in M$, if the matrix $P_l(m, \xi)$ obtained by inserting $\xi \in \mathbb{R}^n$ instead of the partial differentials $D = (\frac{\partial}{\partial x_1}|_m, \dots, \frac{\partial}{\partial x_n}|_m)$ is non-singular at the point m . A differential operator L is called uniformly elliptic if the symbol of L is bounded uniformly on M , i.e. there is a constant C with $\|\sigma_{L,m}(\xi)^{-1}\| \|\xi\|^l \leq C$ for all $m \in M$ and $\xi \in T_m^* M$. Note that on a compact manifold every elliptic operator is uniformly elliptic, because the positive function $m \mapsto \|\sigma_{L,m}(\xi)^{-1}\|$ on the compact space M is bounded away from 0.

⁶⁹A densely defined operator A on a Hilbert space is called self-adjoint if $A^* = A$ and essentially self-adjoint if $A^* = A^{**}$ or equivalently if A has a self-adjoint extension.

⁷⁰The map $(A - \lambda \text{Id})^{-1}$ defined for $\lambda \notin \sigma(A)$ is called the resolvent of A . An operator is called compact if the image of a bounded sequence has a convergent subsequence.

⁷¹An operator A on a Hilbert space is called non-negative if $(Ax, x) \geq 0$

⁷²See [Evans, 5] or [Ebin, Marsden, at the end of 2].

does not need to be closed and coclosed. To still obtain a correspondence, decompose the boundary $\partial M = \partial M_D \dot{\cup} \partial M_N$ into submanifolds where on ∂M_D Dirichlet boundary conditions and on ∂M_N Neumann boundary conditions are assumed. These conditions up to a certain order can be written in the form

$$\begin{aligned} \alpha|_{\partial M_D} = 0 \quad \delta\alpha|_{\partial M_D} = 0 \quad \delta d\alpha|_{\partial M_D} = 0 \quad \dots \\ *\alpha|_{\partial M_N} = 0 \quad *d\alpha|_{\partial M_N} = 0 \quad *d\delta\alpha|_{\partial M_N} = 0 \quad \dots \end{aligned}$$

Now the harmonic forms which satisfy these boundary conditions up to the second order correspond exactly to the closed and coclosed forms which are tangential to ∂M_N and normal to ∂M_D , see [Schick, 5.8]. Denote these forms by \mathcal{H}^s and mark by lower indices ${}_t$ or ${}_n$ that only tangential or normal forms are considered, then the Hodge decomposition

$$H^s(\Lambda^k M) = d(H_n^{s+1}(\Lambda^{k-1} M)) \oplus \delta(H_t^{s+1}(\Lambda^{k+1} M)) \oplus \mathcal{H}^s$$

is valid, where the fact has been used that for an H^{s+1} -form α there is a closed H^{s+1} -form β with $t\beta = 0$, $n\beta = n\alpha$, and similar there is a coclosed H^{s+1} -form β with $t\beta = t\alpha$ and $n\beta = 0$, see [Ebin,Marsden, after Lemma 7.2]. Further the analogue

$$H^s(\Lambda^k M) = d(H^{s+1}(\Lambda^{k-1} M)) \oplus \mathcal{C}_t^s$$

of the decomposition (6) is valid. Indeed, by the formula $(d\alpha, \beta) = (\alpha, \delta\beta) + \int_{\partial M} \alpha \wedge *\beta$ a coclosed form β is tangential, i.e. satisfies $*\beta|_{\partial M} = 0$, if and only if $(d\alpha, \beta) = 0$ holds for all α .

Noncompact Manifolds Now let M be a noncompact manifold of bounded geometry with or without boundary. An elaborate discussion of such manifolds can be found in [Schick], a summary of this discussion has been given in 6.1. Especially a result of this discussion is that every bounded differential operator L of order l extends to a continuous linear operator $H^{s+l} \rightarrow H^s$ by [Schick, 3.25(5)], and the differential, codifferential and Laplacian are such operators by [Schick, 5.13,5.14]. Further the Laplacian is uniformly elliptic and essentially self-adjoint, and also the regularity theory is valid for a uniformly elliptic operator. Thus the Hodge decomposition

$$H^s(\Lambda^k M) = \overline{d(H_n^{s+1}(\Lambda^{k-1} M))} \oplus \overline{\delta(H_t^{s+1}(\Lambda^{k+1} M))} \oplus \mathcal{H}^s \quad .$$

is valid ([Schick, 5.10]), where instead of tangentiality and normality, i.e. the 0-th boundary conditions, it is also possible to assume all boundary values being zero ([Schick, 5.19]). But in general it is not possible to conclude that the image of d , δ or Δ is closed. However, under the spectral assumption $\inf \sigma_{ess}(\Delta|_{\ker(\Delta)^\perp}) > 0$ these images are closed, see [Eichhorn,Schmid, Proof of Theorem 3.3].

Note that the spectral assumption $\inf \sigma_{ess}(\Delta|_{\ker(\Delta)^\perp}) > 0$ is not valid for arbitrary manifolds. For example, on \mathbb{R}^n and for $k = 0$ the kernel of Δ consists of those square-integrable functions with $df = 0$ and thus is trivial, but the spectrum of Δ is $[0, \infty)$, as by Fourier transformation the Laplacian is the multiplication operator given by $(a_\xi) \mapsto (|\xi|^2 a_\xi)$.

6.4.2 $\text{Diff}_{\text{Vol}}(M)$ is a submanifold of $\text{Diff}(M)$

Now let us outline the proof that $\text{Diff}_{\text{Vol}}^s(M)$ is a submanifold of $\text{Diff}^s(M)$. First note that the preimage of $\{\mu\}$ under the map $\psi : \text{Diff}^s(M) \rightarrow H^{s-1}(\Lambda^n M)$, $\eta \mapsto \eta^*(\mu)$, is exactly the subgroup of volume form preserving diffeomorphism. Next consider only diffeomorphisms η in the component $\text{Diff}_{\text{Id}}^s(M)$ of the identity in $\text{Diff}^s(M)$, and note that ψ maps the component $\text{Diff}_{\text{Id}}^s(M)$ into the subset $\mu + dH^s(\Lambda^{n-1}M)$ of $H^{s-1}(\Lambda^n M)$, see [Eichhorn,Schmid, Theorem 3.2]. But by Hodge theory on compact manifolds or by the spectral assumption $\inf \sigma_{\text{ess}}(\Delta|_{\text{Ker}(\Delta)^\perp}) > 0$ on noncompact manifolds of bounded geometry the set $\mu + dH^s(\Lambda^{n-1}M)$ is a closed affine subspace of $H^{s-1}(\Lambda^n M)$ and thus especially a closed submanifold. Further the value μ is a regular value of ψ . Indeed, it suffices to show that the differential $T_{\text{Id}}\psi$ of ψ at Id is onto $dH^s(\Lambda^{n-1}M)$, as then also $T_\eta\psi(X) = \eta^*(L_{X \circ \eta^{-1}}\mu)$ is surjective, because η^* and right multiplication with η^{-1} are isomorphisms. Therefore observe that $T_{\text{Id}}\psi(X) = L_X\mu = di_X\mu + i_Xd\mu = di_X\mu$ is valid (because of $d\mu = 0$). Now by nondegeneracy of μ the map $X \mapsto i_X\mu$ is an isomorphism $H^s(TM) \rightarrow H^s(\Lambda^{n-1}M)$, thus $X \mapsto di_X\mu$ is onto $dH^s(\Lambda^{n-1}M)$, and hence $T_{\text{Id}}\psi$ is really surjective. This proves that $\psi^{-1}(\{\mu\}) \cap \text{Diff}_{\text{Id}}^s(M)$ is a submanifold of $\text{Diff}_{\text{Id}}^s(M)$, and by right multiplication also the other components are submanifolds.

Especially the tangential space of $\text{Diff}_{\text{Id}}^s(M)$ at Id is the space of divergence free H^s -vector fields, as the tangential space is the kernel of $T_{\text{Id}}\psi$, i.e. the space of all X with $di_X\mu = L_X\mu = 0$, and a vector field is divergence-free if $L_X\mu = 0$, or equivalently $di_X\mu = 0$, or equivalently $\delta X^\flat = 0$ (i.e. the one-form X^\flat associated to X is coclosed) is valid.

6.5 Fluid Dynamical Equations are Geometric Equations

Now we come to the reason why we studied manifolds of mappings: Fluid dynamical equations can be interpreted as Hamiltonian equations on manifolds of mappings, and especially the Euler equations are geodesic equations w.r.t. the H^0 -metric on the group $\text{Diff}_{\text{Vol}}(M)$ of volume form preserving diffeomorphisms. Let us give a brief summary of the important facts. During this summary have in mind, that there are different possibilities to describe a fluid: Either by a particle map $\eta(t) \in \text{Diff}(M)$ (a particle in the fluid, which is at time t' at the point m , moves to the point $\eta(t)(m)$ at time t), by the velocity field $\dot{\eta}(t) \in T_{\eta(t)}\text{Diff}(M)$ of the fluid in body coordinates or by the velocity vector field $u(t) \in T_{\text{Id}}\text{Diff}(M)$ of the fluid in spatial coordinates. The last description in terms of the spatial velocity vector field is probably the most popular, as e.g. the famous Euler equations

$$\begin{aligned} \frac{\partial u}{\partial t}(t) + \nabla_{u(t)}u(t) &= -\text{grad } p(t) \\ \text{div } u(t) &= 0 \quad , \quad u(t) \parallel \partial M \end{aligned}$$

for an inviscous incompressible fluid are formulated in terms of $u(t)$. The particle map $\eta(t)$ and the velocity vectorfield $u(t)$ are related by $u(t) = \dot{\eta}(t) \circ \eta(t)^{-1}$, and conversely η is the solution operator generated by the time-dependent differential equation $\dot{m} =$

$u(t)(m)$. In our discussion we will see that the description of a fluid in terms of the particle map has many advantages, as the corresponding equations on a manifold of mappings are merely ordinary differential equations with continuously differentiable right hand side and can be solved by the usual existence theorem. Now let us start summarizing the results of [Ebin,Marsden] and [Eichhorn,Schmid].

The Riemannian metric $\langle \cdot, \cdot \rangle$ on a manifold M induces the H^0 -Riemannian metric $(X, Y) := \int_M \langle X, Y \rangle \mu$ on $\text{Diff}^s(M)$. However, this is only a weak Riemannian metric, as the tangential space is the space of H^s -vector fields endowed with the H^s -norm and not with the H^0 -norm. But although the Riemannian metric is weak, the connector $K : TTM \rightarrow TM$ on M induces a connector \bar{K} on $\text{Diff}^s(M)$ by $\bar{K}(V) := K \circ V$, where $V \in T\text{Diff}^s(M) \cong \{V \in H^s(M, TTM) | \pi_{TTM} \circ V \in T\text{Diff}^s(M)\}$. In the compact case all manifolds $\text{Diff}^s(M)$ are smooth and thus \bar{K} is also smooth by the ω -lemma, as K is smooth. However, in the noncompact case the manifolds $\text{Diff}^s(M)$ have only a finite differentiability order depending on the order of bounded geometry M . Thus it must either be assumed that M has an infinite boundedness order, so that the manifolds and also \bar{K} is smooth, or else a work-around has to be established to guarantee that \bar{K} has a sufficient high differentiability order (see [Eichhorn,Schmid, Proposition 4.3 + Corollary 4.4]).

Using the connector \bar{K} , a linear connection $\bar{\nabla}$ on $\text{Diff}^s(M)$ can be defined by $\bar{\nabla}_X Y := \bar{K} \circ TY \circ X$. To show that $\bar{\nabla}$ is compatible with the H^0 -metric, torsion-free and thus really the Levi-Cita connection to the H^0 -metric, the equations $\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y]$ (torsion-freeness) and $X(Y, Z) = (\bar{\nabla}_X Y, Z) + (Y, \bar{\nabla}_X Z)$ (compatibility) are first established for right-invariant vectorfields X, Y, Z on Diff^s , and afterwards the formulas are generalized to arbitrary vector fields ([Ebin,Marsden, p.129, Step 1-3], [Eichhorn,Schmid, Lemma 4.3-4.7]). Hereby in the noncompact case it is necessary to use the group $\text{Diff}^{\infty, s}$ obtained by the completion of those C^∞ -maps, whose derivatives are bounded in all orders of differentiability instead only up to order s . Indeed, the proof establishes torsion-freeness and compatibility for H^{s+r} -vector fields and then uses denseness, but generally $\text{Diff}^{r+s}(M)$ is not dense in $\text{Diff}^s(M)$, only $\text{Diff}^{\infty, r+s}$ is dense in $\text{Diff}^{\infty, s}$.

Thus although the H^0 -metric is only weak, a connector and a linear connection to the H^0 -metric exist on $\text{Diff}^s(M)$ in the compact case and on $\text{Diff}^{\infty, s}(M)$ in the non-compact case. Further it is also easy to describe the geodesics on the diffeomorphism groups: Denote the spray on M by $Z : TM \rightarrow TTM$, then the spray to the H^0 -metric on $\text{Diff}^s(M)$ is given by $\bar{Z}(X) = Z \circ X$. Thus the geodesic equation is $\ddot{\eta}(t) = Z \circ \dot{\eta}(t)$ and the geodesic to the initial value $X \in T\text{Diff}^s(M)$ is simply $\eta : t \mapsto (m \mapsto \gamma_{X_m}(t))$, where γ_{X_m} denotes the geodesic within the manifold M to the initial value X_m . Note that the geodesic equation in terms of the spatial velocity vector field $u(t)$ has the form $\frac{\partial u}{\partial t}(t) + \nabla_{u(t)} u(t) = 0$, because every spray to a linear connection ∇ satisfies $Z \circ u = Tu \cdot u - (\nabla_u u)^l$ for a vector field u (where $(\cdot)^l$ denotes the vertical lift from TM to TTM). Thus

$$\begin{aligned} Tu(t) \cdot u(t) - (\nabla_{u(t)} u(t))^l &= Z \circ u(t) = Z \circ \dot{\eta}(t) \circ \eta(t)^{-1} = \ddot{\eta}(t) \circ \eta(t)^{-1} = \\ (u(t) \dot{\circ} \eta(t)) \circ \eta(t)^{-1} &= (\dot{u}(t) \circ \eta(t) + Tu(t) \cdot \dot{\eta}(t)) \circ \eta(t)^{-1} = \dot{u} + Tu(t) \cdot u(t) \end{aligned}$$

is valid and implies $\frac{\partial u}{\partial t}(t) + \nabla_{u(t)}u(t) = 0$.

Let us now consider the geometry w.r.t. the H^0 -metric on the submanifold $\text{Diff}_{\text{Vol}}^s(M)$. This metric has the linear connection $\bar{\nabla}^{\text{Vol}} = P \circ \bar{\nabla}$, where $P_{\text{Id}} : T_{\text{Id}} \text{Diff}(M) \rightarrow T_{\text{Id}} \text{Diff}_{\text{Vol}}(M)$ assigns to each vector field $X = \text{grad } p + P_{\text{Id}}X$ its divergence-free part $P_{\text{Id}}X$ obtained by the Hodge decomposition $H^s(\Lambda^1 M) = d(H^{s+1}(\Lambda^0 M)) \oplus \mathcal{C}_t$. An equivalent way to define P_{Id} is $P_{\text{Id}}X := X - \text{grad } p$, where p denotes the solution of the Poisson equation $\Delta p = \text{div}(X)$ on M with inhomogeneous Neumann boundary conditions $\frac{\partial p}{\partial \nu} = nX$ on ∂M (nX is the part of X normal to the boundary ∂M). Indeed, then $\text{div}(P_{\text{Id}}X) = \text{div}(X) - \text{div}(\text{grad } p) = \text{div}(X) - \Delta p = 0$ and $nP_{\text{Id}}X = nX - n \text{grad } p = 0$ are valid, i.e. $P_{\text{Id}}X$ is divergence-free and parallel to the boundary. Note that this later description is the one which can directly be used in the noncompact setting, where due to the spectral assumption $\inf \sigma_{\text{ess}}(\Delta|_{\text{Ker}(\Delta)^\perp}) > 0$ and because of $\text{Ker}(\Delta) \cap L^2(M) = 0$ (as $\text{Vol}(M) = \infty$ for manifolds of bounded geometry) the Laplace operator Δ has a continuous inverse. To conclude finally that $\bar{\nabla}^\mu = P \circ \bar{\nabla}$ is the linear connection associated to the H^0 -metric on the group of volume-preserving diffeomorphism, first it has to be assured that the extension $P_\eta(V) = P_{\text{Id}}(V \circ \eta^{-1}) \circ \eta$ of P_{Id} to the whole tangential bundle $T \text{Diff}^s(M)$ is at least C^2 , see [Ebin,Marsden, Appendix A] or [Eichhorn,Schmid, Proposition 4.9], before torsion-freeness and compatibility of $\bar{\nabla}^\mu$ can be proved. In the noncompact case it is hereby again necessary to use $\text{Diff}_{\text{Vol}}^{\infty,s}(M)$ instead of $\text{Diff}^s(M)$. To establish the differentiability of P is in fact the main step, as it assures that the geodesic spray $\bar{Z}^{\text{Vol}}(X) = TP(Z \circ X)$ to the H^0 -metric on $\text{Diff}_{\text{Vol}}^s(M)$ is a C^1 -map and thus the geodesic equations are solvable by the ordinary existence theorem.

Finally let us show that the geodesic equations $\ddot{\eta}(t) = \bar{Z}(\dot{\eta}(t))$ really are the Euler equations in terms of the velocity vector field $u(t)$. Indeed, using again $Z \circ u = Tu \cdot u - (\nabla_u u)^l$ as well as $TP((Y)^l) = (P(Y))^l$, $TP(Tu \cdot \dot{\eta}) = T(P_{\text{Id}}u) \cdot \dot{\eta}$ (see [Ebin,Marsden, Proposition 14.1]) and the divergence-freeness $P_{\text{Id}}u(t) = u(t)$ of $u(t) = \dot{\eta}(t) \circ \eta(t)^{-1}$, the equality

$$\begin{aligned} (Tu(t) \cdot u(t) - (P_{\text{Id}}(\nabla_{u(t)}u(t)))^l) \circ \eta(t) &= T(P_{\text{Id}}u(t)) \cdot \dot{\eta}(t) - (P_{\text{Id}}(\nabla_{u(t)}u(t)))^l \circ \eta(t) = \\ TP((Tu(t) \cdot u(t) - (\nabla_{u(t)}u(t))^l) \circ \eta(t)) &= TP(Z \circ u(t) \circ \eta(t)) = TP(Z \circ \dot{\eta}(t)) = \ddot{\eta}(t) = \\ (u(t) \circ \dot{\eta}(t)) &= (\dot{u}(t) \circ \eta(t) + Tu(t) \cdot \dot{\eta}(t)) = (\dot{u}(t) + Tu(t) \cdot u(t)) \circ \eta(t) \end{aligned}$$

can be obtained. Thus with the Hodge decomposition $P_{\text{Id}}(\nabla_{u(t)}u(t)) = \nabla_{u(t)}u(t) + \text{grad } p$ the Euler equations

$$\begin{aligned} \frac{\partial u}{\partial t}(t) + \nabla_{u(t)}u(t) &= -\text{grad } p(t) \\ \text{div } u(t) &= 0 \quad , \quad u(t) \parallel \partial M \end{aligned}$$

are equivalent to the geodesic equations on $\text{Diff}^s(M)$.

6.5.1 Other Fluid Dynamical Equations

Having characterized the Euler equations geometrically as geodesic equations on the group of volume preserving diffeomorphisms, let us ask whether other kinds of fluid

dynamical equations are geometrical. This question can be answered by giving different directions:

- The Euler equations could be coupled with other equations via the Trotter product. So for example the Navier-Stokes equations arise via coupling the geodesic flow Φ^t on $T\text{Diff}_{\text{Vol}}(M)$ with the semigroup U^t generated on $T_{\text{Id}}\text{Diff}_{\text{Vol}}^s(m) = \{u \in H^s(TM) \mid \text{div}(u) = 0, u|_{\partial M}\}$ by $\dot{u} =: \nu\Delta u$ ⁷³ in M and prolonged to $T\text{Diff}_{\text{Vol}}(M)$ by right multiplication according to the Trotter product $\Psi^t := \lim_{n \rightarrow \infty} (\Phi^{t/n} \circ U^{t/n})^n$. Then Ψ^t is generated by the sum of the vector fields which generate Φ^t and U^t . Thus for the concrete case where $\nu\Delta$ is the generator, the curve $\Psi^t u_0 \circ \eta^{-1}$ (where η denotes the base point of $\Psi^t u_0 \in T_\eta \text{Diff}_{\text{Vol}}^s(M)$) solves the Navier-Stokes equations⁷⁴

$$\dot{u}(t) + \nabla_{u(t)} u(t) = \nu\Delta u(t) + \text{grad } p(t) \quad \text{div}(u(t)) = 0 \quad .$$

But instead of $\nu\Delta$ also other generators could be used, e.g. those describing external forces like gravity.

- Instead of the H^0 -metric other metrics can be considered, e.g. metrics equivalent to the H^1 -metric. For example, the averaged Euler equations are characterized as geodesic equations on the group of volume-form preserving diffeomorphisms w.r.t. the H^1 -equivalent metric

$$\langle u, v \rangle_{L^2} + 2\alpha^2 \langle \text{Def}(u), \text{Def}(v) \rangle_{L^2}$$

for $u, v \in T_{\text{Id}}\text{Diff}_{\text{Vol}}^s(M)$, where $\text{Def}(u) = \frac{1}{2}L_u g = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the deformation tensor, see [Shkoller].

- Instead of the group of volume preserving diffeomorphisms one could go back to the whole group $\text{Diff}(M)$ of diffeomorphism and could, for example, consider Hamiltonian equations on semi-direct products with the group $\text{Diff}(M)$: Let G be a Lie group and let V be a representation space of G . Further let $H_v : T^*G \rightarrow \mathbb{R}$ be a family of Hamiltonians depending on the parameter $v \in V$ which is left invariant under the action of the stabilizer $G_v := \{g \mid gv = v\}$ of v in V . Then the family of Hamiltonians H_v induces a Hamiltonian H on the dual of the Lie algebra of the semi-direct product group $G \rtimes V$ (having multiplication $(g, v)(g', v') = (gg', gv' + v)$) by $H : T_e^*G \rtimes V^* \rightarrow \mathbb{R}$, $H((T_e L_g)^* \alpha_g, g^* v) = H_v(\alpha_g)$, and the solutions correspond, see [Marsden, Ratiu, Weinstein, 3.4]. Hereby $\alpha_g \in T_g^*G$, L_g is left multiplication with g and g^* denotes the dual of the linear map $g : V \rightarrow V$ given by the representation of G on V , and in fact the whole construction is Lie-Poisson reduction on semi-direct products.

⁷³Here Δ denotes the negative of the Laplace-Beltrami operator applied to the one-form u^\flat induced by the vector field u , and ν is a constant.

⁷⁴However, here we consider only manifolds without boundary, as solving $\dot{u} = \nu\Delta u$ under the boundary condition $\frac{\partial u}{\partial \nu} = 0$ on ∂M is problematically, see [Marsden, Ratiu, 15.6.ii].

To apply this construction to the dynamics of ideal compressible isentropic fluids, let $G = \text{Diff}^s(M)$ act on the space $H^s(M)$ of functions on M by $(\eta, \rho) := \rho \circ \eta^{-1}$ and consider the family of Hamiltonians

$$H_\rho(X^\flat) := \frac{1}{2} \int_M \langle X_{\eta(m)}, X_{\eta(m)} \rangle \rho(\eta(m)) dm + \int_M \rho(m) w(\rho(m) \circ J(\eta)^{-1}(m)) dm \quad ,$$

where dm denotes the Riemannian volume form on M , the function $w : \mathbb{R} \rightarrow \mathbb{R}$ models the thermodynamical inner energy at a given density value, and $J(\eta)$ is the Jacobian of the diffeomorphism η (which is the base point of X) defined by $\eta^*(\rho dm) J(\eta) = \rho dm$. Then the above construction yields the equations

$$\dot{u}(t) + \nabla_{u(t)} u(t) = -\frac{1}{\rho(t)} \text{grad } p(t) \quad \dot{\rho}(t) + \text{div}(\rho(t)u(t)) = 0 \quad p(t) = \rho(t)^2 w'(\rho(t))$$

for $(u, \rho) \in T_{\text{Id}} \text{Diff}(M) \times V$, and thus the usual model for the dynamics of an ideal compressible isentropic fluid.

6.6 Fluid Dynamics on Local Spaces

Now let us discuss what happens when instead of global Sobolev spaces the local Sobolev spaces are used. The diffeomorphism group $\text{Diff}_{loc}^s(M)$ is again a submanifold of $H_{loc}^s(M, M)$, not modeled on an open subset but merely on a dense i -ball. However, the geodesics induced by the H^0 -metric (the norm function associated to the H_{loc}^s -metrics on the pseudo-Hilbert space $H_{loc}^s(M, M)$, see 4.4), i.e. the solutions of the equation $\ddot{\eta} = \bar{Z}(\dot{\eta})$ with the spray $\bar{Z}(X) := Z \circ X$ on $\text{Diff}_{loc}^s(M)$ induced by the spray Z on M , stay for local spaces the same as for global spaces: They are given by $t \mapsto (m \mapsto \gamma_{X_m}(t))$, where $\gamma_{X_m}(t)$ denotes the geodesic on M to the initial value X_m over $\eta(m)$. But if the derivative $T\eta$ of the base point $\eta \in \text{Diff}_{loc}^s(M)$ of X has small eigenvalues $|\lambda|$ or the vector field X is large at infinity, then the geodesic $\eta(t)$ could instantly loose the property of being a diffeomorphism. Thus although $\eta(0)$ is a diffeomorphism, the geodesic $\eta(t)$ does not need to be a diffeomorphism for any $t \neq 0$. However, the only problem produced by this defect in fluid dynamics is the impossibility to define the velocity vector field $u(t) = \eta(t) \circ \eta(t)^{-1}$ of the fluid in spatial coordinates: The particle map $\eta(t)$ in $H^s(M, M)$ (instead of $\text{Diff}^s(M)$) and the vector field $\dot{\eta}$ in body coordinates stay well-defined. The loose of the diffeomorphism property is highly related to the occurrence of discontinuities like shocks in $u(t)$.

Further note that the spray \bar{Z} is smooth not only as a map on $TH^s(M, M)$ but also as a map on $TH_{loc}^s(M, M)$: Due to the standing assumption of bounded geometry, the spray Z on M is globally Lipschitz in all derivatives up to the boundedness order (which is assumed to be larger than s), and thus like in the example of 5.4 the ordinary differential equation $\dot{X} = Z \circ X$ can be solved locally at least for all bounded H_{loc}^s -vector fields X .

Thus the pressureless fluid dynamics on $H_{loc}^s(M, M)$ to the equation $\frac{\partial u}{\partial t}(t) + \nabla_{u(t)} u(t) = 0$ (in 1D this is Burger's equation) can be formulated also rigorously for local Sobolev spaces. But also if Z is not the geodesic spray, but an arbitrary second order equation

S on M , then $\dot{X} = S \circ X$ can be solved in the same way by applying the local existence theorem. Especially $\frac{\partial u}{\partial t}(t) + \nabla_{u(t)} u(t) = f$ can be solved in this way with a H^s -vectorfield f on M . Note that all these equations given by composition with S are in fact equations with a finite propagation of velocity and can be solved by the method of characteristics. But while discontinuities can arise in the vectorfield $u(t) = \dot{\eta}(t) \circ \eta(t)^{-1}$, on the level of the particle map $\eta \in H_{loc}^s(M)$ this problem can not appear: Even if η is not invertible anymore, it still solves the second order equation and is well-defined.

Further also if the second order equation is not given by composition but by $\ddot{\eta} = \bar{S}(\dot{\eta})$ with a continuously differentiable right hand side $\bar{S} : TH_{loc}^s(M, M) \rightarrow TTH_{loc}^s(M, M)$, the problem can be attacked by using our existence theorem on locally convex spaces. Note that \bar{S} is continuously differentiable if the map $K \circ \bar{S} : TH_{loc}^s(M, M) \rightarrow TH_{loc}^s(M, M)$ between locally convex topological vector spaces is continuously differentiable, as on the other component of $TTH_{loc}^s(M, M)$ the equality $T\pi \circ \bar{S} = \text{Id}_{TH_{loc}^s}$ with the smooth map $\text{Id}_{TH_{loc}^s}$ is valid due to the second order property of \bar{S} .

In fact, the requirement is essentially that \bar{S} is local, i.e. the value of $\bar{S}(X)$ on a compact set is determined by the values of X on a maybe larger, but still compact set, and not by values of X outside every compact set. But incompressibility or inviscosity of equations are non-local properties, and that's why the Euler or Navier-Stokes equations can not be solved in the local spaces. The cause for this defect is that the projection $X \mapsto PX$ of vector-fields onto its divergence-free part is not local: $PX(m)$ does not depend on the values of X in a compact set around m but globally on all values of X . In fact, we have seen in section 4.4 that projections on pseudo-Hilbert spaces $X = \lim_i (X_i, p_i)$ are in general not continuous w.r.t. the original topology, but only w.r.t. the metric induced by $p := \lim_i p_i$. Here this metric is the H^s -metric and thus P is in general not continuous w.r.t. the H_{loc}^s -pseudo-norms but only w.r.t. the H^s -norm. Or in other words: The solution of the Poisson equation⁷⁵ $\Delta p = f$ does generally not have the property that if f has compact support, then also p has compact support. For example, Green's function, which solves the equation for the delta distribution at some point as right hand side, has generally not compact support. Thus as long as we are not on a manifold M where Green's function has compact support, dynamics of incompressible fluids can not be modeled by geodesics in a local diffeomorphism group.

Thus to conclude: Although the most popular examples of systems with pattern formation, the Euler- or Navier-Stokes-equations, can not be discussed using local Sobolev spaces, the fluid dynamical equations describing the dynamics of inviscous compressible fluids can be formulated using local Sobolev spaces. This is due to the fact that these equations have a finite velocity of propagation and thus are local, but our calculus on locally convex vector spaces exactly requires locality of C^1 -maps.

⁷⁵A short summary about properties of the Poisson equation on arbitrary manifolds can be found in B.

7 Pattern Formation under Symmetry

This final chapter discusses the general theory of pattern formation under symmetry in dynamical systems, whose state space is a manifold of mappings on a noncompact domain, modeled over a Banach space or a complete locally convex topological vector space.

Let X be such a manifold of mappings on a domain M and let Φ^t be a (at least locally defined) flow on X . For example, the flow Φ^t could be the solution of the Euler or Navier-Stokes equations on a compact or noncompact manifold M , where Φ^t is (at least locally) defined on the Hilbert space $X = \{u \in H^s(TM) \mid \operatorname{div}(u) = 0, u|_{\partial M} = T_{\operatorname{Id}} \operatorname{Diff}^s(M)\}$ or on the whole tangential bundle $X = T \operatorname{Diff}^s(M)$. Or Φ^t could be a flow generated by fluid dynamical equations with finite propagation of velocity on a manifold of mappings X modeled over a locally convex topological vector space. Another important class of examples consists of flows generated by reaction-diffusion equations $\dot{x} = Ax + f(x)$ on a domain M , where $x : M \rightarrow \mathbb{R}^N$ models the density of N species, which react and move under diffusion.

Suppose further that the isometry group $G = \operatorname{Isom}(M)$ on M induces via composition an operation on the manifold X of mappings, and assume Φ^t to be equivariant w.r.t. this action, i.e. $\Phi^t(gx) = g(\Phi^t x)$ is valid. For example, the isometry group G on M induces the operation $(g, \eta) \mapsto g \circ \eta$ on the configuration space $\operatorname{Diff}_{\operatorname{Vol}}^s(M)$ of the Euler or Navier-Stokes equations, the operation $(g, U) \mapsto Tg \circ U$ on the state space $T \operatorname{Diff}_{\operatorname{Vol}}^s(M)$ and the operation $(g, u) \mapsto Tg \circ u(t) \circ g^{-1}$ on the reduced state space $T_{\operatorname{Id}} \operatorname{Diff}_{\operatorname{Vol}}^s(M)$. On these manifolds of mappings X again every $g \in G$ is an isometry of X because of $\int_M \|Tg \circ u(t) \circ g^{-1}\|^2 dm = \int_M \|u\|^2 dm$, and the flow Φ^t generated by the Euler or Navier-Stokes equations is equivariant, as the H^0 -metric is invariant and the Laplace-Beltrami operator Δ is equivariant under isometries on M . The same is true for other Hamiltonian equations modeling fluids like e.g. ideal compressible isentropic fluids. Or let the group $G = \operatorname{Isom}(M)$ of isometries on M act on the densities $x \in X = L^2(M, \mathbb{R}^N)$ or $X = BC_{\operatorname{unif}}(M, \mathbb{R}^N)$ of a reaction diffusion system $\dot{x} = Ax + f(x)$ by $(g, x) \mapsto x \circ g^{-1}$, and suppose that A is the Laplacian or another G -invariant operator, then again every g is an isometry of X ⁷⁶ and the generated flow Φ^t is equivariant.

But note that for global spaces of mappings on a noncompact manifold M the induced operation of the isometry group is often not strongly continuous⁷⁷, continuous or differentiable. Indeed, the topology of uniform convergence on compact sets is the finest topology on a space of continuous mappings on a noncompact domain M such that composition and evaluation are continuous. But for noncompact M the global spaces of mappings do not have a coarser topology than the topology of uniform convergence on compact sets. Thus, for example, the operation $(g, \eta) \mapsto g \circ \eta$ on the Hilbert manifold $\operatorname{Diff}^s(M)$ is not strongly continuous, in fact, η and $g \circ \eta$ even do not belong to the same connected component unless $g = \operatorname{Id}$, as Id and g do not differ by an H^s -vector field unless $g = \operatorname{Id}$. However, the induced operation on the Hilbert space $T_{\operatorname{Id}} \operatorname{Diff}_{\operatorname{Vol}}^s(M)$

⁷⁶Indeed, $\sup_M \|x \circ g^{-1}\| = \sup_M \|x\|$ is valid for $x \in C_{\operatorname{unif}}(M, \mathbb{R}^N)$, and $\int_M \|x(g^{-1}m)\|^2 dm = \int_M \|x(m)\|^2 dm$ holds for $x \in L^2(M, \mathbb{R}^N)$ are valid.

⁷⁷An action is called strongly continuous if $g \mapsto gx$ is continuous for every $x \in X$.

of divergence-free vector fields is strongly continuous, because

$$\int_M \|Tg(u \circ g^{-1}) - u\|^2 dm \leq \int_K \|Tg(u \circ g^{-1}) - u\|^2 + \int_{M \setminus K} \|Tg(u \circ g^{-1}) - u\|^2 dm \leq \epsilon_1 \int_M \|u\|^2 dm + \epsilon_2$$

is valid, whenever K is so large that

$$\int_{M \setminus K} \|Tg(u \circ g^{-1}) - u\|^2 dm \leq \int_{M \setminus K} \|Tg(u \circ g^{-1})\|^2 + \|u\|^2 dm < \epsilon_2$$

and g is so near to the neutral element Id that $\|Tg(u \circ g^{-1}) - u\|^2 \leq \epsilon_1 \|u\|^2$ on K . But this strongly continuous operation has other defects, namely for noncompact M and G there are generally no nontrivial finite-dimensional invariant subspaces⁷⁸ and the operation is neither continuous nor differentiable. Further, while the induced operation on $H^s(TM)$ is strongly continuous, as H^s -vectorfields vanish at infinity, the induced operation is usually not strongly continuous on spaces like $BC_{unif}(M, \mathbb{R})$. For example, consider $M = \mathbb{R}^2$, $G = E(2)$, $f(x, y) = \cos(x)$ and a rotation g about an angle ϕ , then $\sup_{\mathbb{R}^2} |(f \circ g) - f| = \sup_{x,y} |\cos(\cos \phi x + \sin(\phi)y) - \cos(x)| = 2$ is valid for all $\phi \neq 0$, so that the operation cannot be strongly continuous.

To the contrary, for the local spaces of mappings on a noncompact domain M the induced operation of the isometry group is continuous, as the topology of these spaces is finer than the topology of uniform convergence on compact subsets. In fact, this was the starting point of this thesis. Moreover, the operation is often differentiable, e.g. on $C^k(M, M)$ or $H_{loc}^s(M, M)$, $s > \dim(M)/2 + 1$, the induced action $(g, \eta) \mapsto g \circ \eta$ is smooth by the omega-lemma, as g is smooth. Also there are usually nontrivial finite-dimensional invariant subspaces, e.g. in $C(\mathbb{R}^n, \mathbb{R})$ the polynomials of degree smaller than a fixed k are invariant under $E(n)$. Thus the properties of the induced operation of the isometry group strongly depend on the used spaces and are much better for local spaces of maps than for global spaces. This should be reflected in a discussion of pattern formation.

Finally assume that the flow Φ^t depends on external parameters λ . For example, in the case of the Navier-Stokes equations the velocity vector field at the boundary could depend on a parameter λ . By changing into a frame moving with the boundary velocity vectorfield, the boundary data often can be made homogeneous, but then - from the Hamiltonian point of view - the right hand side of the Hamiltonian second-order equation $\dot{U} = \bar{S}_\lambda(U)$ depends on the parameter λ , as an extra force term arises due to the change of the frame. Such a system is the famous Taylor-Couette experiment, where

⁷⁸For example, consider $M = \mathbb{R}^n$ and let V be an invariant subspace of the complexified space $H^s(\mathbb{R}^n, \mathbb{C})$, then $E(n)$ acts on V by isometries. Because of $\frac{d}{dt}u(m + te_i)|_{t=0} = \frac{\partial u}{\partial x_i}(m)$ the translations into direction e_i are generated by the operator $\frac{\partial}{\partial x_i}$. As V is invariant, every $\frac{\partial}{\partial x_i}$ restricts to a linear operator on V . As all these operators commute, they can be diagonalized simultaneously. But if V were finite-dimensional, the operators would have common eigenfunctions $v \in V$. However, the common eigenfunctions of the operators $\frac{\partial}{\partial x_i}$ are $m \mapsto \exp(\langle b, m \rangle)$, $b \in \mathbb{C}^n$, and these function do not belong to $H^s(\mathbb{R}^n, \mathbb{C})$, so that a contradiction is obtained.

a fluid between two rotating cylinders is observed, while the rotational velocity of the cylinders depends on a parameter and can be changed, see [Chossat,Iooss].

We are interested in the bifurcations of patterns, which occur in such dynamical systems, when the parameter is varied. Hereby patterns are subsets S of the state space, which are invariant under the flow and under the symmetry. The simplest patterns are relative equilibria, i.e. flow invariant group orbits $S = Gs_0$, and relative periodic orbits $S = \Phi^{\mathbb{R}}Gs_0$, i.e. S is up to symmetry a periodic orbit, or more formally, there is a group element $g \in G$ and a time $T > 0$ such that $\Phi^T(s_0) = gs_0$ holds. For example, in the Taylor-Couette experiment spiral flows, wavy vortices and other interesting patterns can develop. Many of these patterns discussed in literature are simply relative equilibria or relative periodic orbits, and all of them are essentially finite-dimensional, i.e. they seem to be determined by a finite number of parameters.

Therefore in a mathematical study of bifurcation, one tries to reduce the bifurcation problem from the original infinite-dimensional state-space given by a manifold of mappings, to a finite-dimensional state-space given by a finite-dimensional submanifold of the infinite-dimensional state space, which is invariant under the flow and the induced action of the isometry group. This can be done via Lyapunov-Schmidt reduction or center-manifold reduction, but on Banach spaces in both reduction schemes problems arise due to the discontinuity and non-differentiability of the induced operation. We discuss these two methods of reduction in the Banach case, and generalize center-manifold reduction to the case of locally convex spaces. Having obtained a finite-dimensional manifold by reduction, bifurcations under symmetry of the original system can be discussed by using skew-product flows on this finite-dimensional manifold, and the results of this discussion are summarized in the final section.

7.1 Lyapunov-Schmidt Reduction on Banach Spaces

Bifurcations of relative equilibria and relative periodic orbits in reaction-diffusion equations on noncompact domains with symmetry first have been studied rigorously by [Wulff] using Lyapunov-Schmidt reduction. Let us summarize her results.

Let Φ^t be the parameter-dependent equivariant C^k -flow of a reaction-diffusion equation $\dot{x} = Ax + f(x, \lambda)$ on a Banach space X endowed with an operation of a finite-dimensional Lie group G by isometries $g : X \rightarrow X$, where A is a sectorial operator, f is C^k and $A + f$ is equivariant. Suppose that S_0 is a relative equilibrium or a relative periodic orbit of Φ^t at the parameter λ_0 . Choose $s_0 \in S_0$ and let $T > 0$ be arbitrary in the case of a relative equilibrium resp. in the case of a relative periodic orbit let $T := \inf\{t > 0 \mid \exists g \in G : \Phi^t(s) = gs\}$ be the minimal time, at which the group orbit and the time orbit through s_0 intersect. Further denote by g_0 the group element with $\Phi^T(s_0) = g_0s_0$.

We want to study, whether S_0 can be continued w.r.t. the parameter λ , i.e. we look for a family S_λ of relative equilibria resp. relative periodic orbits with $S_{\lambda_0} = S_0$. To obtain

such a family, we have to discuss the solvability of the equation

$$H(s, g, \lambda) = \begin{pmatrix} g^{-1}\Phi_\lambda^T(s) - s \\ x'_i(s - s_0) \end{pmatrix} = 0$$

near (s_0, g_0, λ_0) . Here the x'_i define a transversal section to Gs_0 at s_0 , so that the decomposition $X = T_{s_0}(Gs_0) \oplus \{x|x'_i(x) = 0\}$ is valid. The equations $x'_i(s - s_0) = 0$ guarantee that solutions s of $H = 0$ are not merely translations of s_0 along the group orbit. Note that a transversal section x'_i to Gs_0 at s_0 exists, whenever Gs_0 is a C^1 -submanifold of X . But the G -operation on X is generally not differentiable, so that $g \mapsto gs_0$ is not automatically a C^1 -map. Therefore it has to be required that for s_0 the subset Gs_0 is a submanifold. This requirement is essentially a regularity requirement, as e.g. for $X = C^r(M, \mathbb{R}^N)$ the map $g \mapsto s_0 \circ g^{-1}$ is C^k by the omega lemma, if s_0 is C^{k+r} .

Observe that the solvability of $H = 0$ cannot be discussed directly by the ordinary implicit function theorem, because the G -action is not differentiable and thus H is not differentiable in the G -component. The work-around provided by [Wulff] is the following method:

Assume the spectral hypothesis that the values $|\mu| \geq 1$ form a spectral set⁷⁹ of the spectrum of the operator $L := g^{-1}D\Phi_\lambda^T(s_0)$ obtained by linearizing the Poincare map $g^{-1}\Phi^T$ at s_0 , and suppose that the corresponding spectral projection P has a finite-dimensional image E^{cu} , which is called the center-unstable eigenspace.

Restrict the flow to the space $Y := \{x \in X | gx \text{ is a } C^0\text{-map in } g\}$, i.e. on the space where G acts strongly continuous. Then a scale of Banach spaces Y_j is inductively defined by $Y_j := \{u \in Y_{j-1} | \forall \xi \in \mathcal{G} : \xi u \in Y_{j-1}\}$ equipped with the norm $\|u\|_{Y_{j-1}} + \sup_{\xi \in \mathcal{G}, |\xi|=1} \|\xi u\|_{Y_{j-1}}$, where \mathcal{G} denotes the Lie algebra of G and ξu is a short notation for $\frac{d}{dt} \exp(t\xi)u|_{t=0}$. Note that Y_j consists of those elements for which $g \mapsto gx$ has a derivative in Y_{j-1} . In the same way the scale $Y'_j \subset X'$ is defined.

Because of equivariance the equation $\xi\Phi^t(u) = D\Phi^t(\xi u)$ is valid, and thus the C^k -flow on Y restricts to a C^{k-j} -flow on Y_j . Now [Wulff, Lemma 4.4] proves that there is a projector \hat{P} near P in the $L(Y)$ -norm such that $g\hat{P}$ and $\hat{P}g$ are C^k in g , and further from the spectral hypothesis it can be concluded that $s_0 \in Y_1$ holds, see [Wulff, Lemma 4.5], which yields inductively C^k -regularity of Gs_0 .

The idea is then to solve instead of $H = 0$ first the fixed point problem

$$y = \Pi(y, q, g, \lambda) := (1 - \hat{P})g^{-1}\Phi^T(y + q, \lambda)$$

for $y \in (1 - \hat{P})Y$ and $q \in \hat{P}(Y)$ near $(q_0 := \hat{P}(s_0), g_0, \lambda_0)$ by Banach's contraction mapping principle on the scale of Banach spaces $(1 - \hat{P})Y_j$ to obtain a solution $y(q, g, \lambda)$, which is C^{k-j} in its variables when y is considered as function into Y_j . After such an y has been found, the reduced equation

$$H_{red}(q, g, \lambda) = \begin{pmatrix} \hat{P}g^{-1}\Phi_\lambda^T(y(q, g, \lambda) + q) - q \\ x'_i(y(q, g, \lambda) + q - s_0) \end{pmatrix} = 0$$

⁷⁹A subset of the spectrum of an operator on a Banach space is called a spectral set if it is a union of connected components or equivalently open and closed.

can now be solved by the ordinary implicit function theorem, as this equation is C^1 in the variable g . Using this method, bifurcation of relative equilibria or relative periodic orbits S can be observed directly in the equation $H_{red} = 0$, and thus the bifurcation problem can be solved, although the action of the symmetry is not continuous or differentiable on the original Banach space.

7.2 Center-Manifold Reduction

Instead of Lyapunov-Schmidt reduction, which allows to observe bifurcations directly in a reduced finite-dimensional equation, also center-manifold reduction can be used to reduce the complexity of the bifurcation problem: Let S be a pattern of an equivariant dynamical system Φ^t on a manifold X endowed with an operation of a Lie group G , i.e. S is a flow- and G -invariant submanifold of X . To such a pattern S equivariant center-manifold reduction tries to construct a finite-dimensional locally flow- and G -invariant C^k -manifold $M_{cu} \subset X$, which is locally attracting and contains all time-orbits staying close to S for $t \rightarrow -\infty$. Such a manifold M_{cu} is called a center-unstable manifold of S and determines the long-time behaviour of the original dynamical system near S completely in the sense that M_{cu} locally attracts all orbits near S .

The Banach Case

Again let us consider a reaction-diffusion equation $\dot{u} = -Au + f(u)$ on a Banach space X with a sectorial operator A and a C^{k+2} -function f . Assume that the finite-dimensional noncompact Lie group G acts by isometries on the Banach space X and leaves the equation invariant, so that the flow Φ^t on X generated by the reaction-diffusion equation is equivariant. Then the following main theorem of [Sandstede, Scheel, Wulff] guarantees the existence of center-unstable manifolds.

Theorem 7.1 *Let $S = Gs_0$ be a relative equilibrium of the flow Φ^t on the Banach space X . Choose $\xi \in \mathcal{G}$ such that $\Phi^t(s_0) = \exp(t\xi)s_0$ and denote by $G_{s_0} := \{g | gs_0 = s_0\} \subset G$ the isotropy subgroup of s_0 . Suppose that $\{\lambda | |\lambda| \geq 1\}$ is a spectral set for the linear operator $\exp(-\xi)D\Phi^1(s_0)$ such that the associated generalized eigenspace E_{cu} , the image of the associated spectral projection P , is finite-dimensional⁸⁰. Finally require the technical assumptions*

- $g \mapsto gs_0$ is C^{k+2}
- for $\epsilon > 0$ there is a $\delta > 0$ such that $\|gs_0 - s_0\| \geq \delta$ for all g with $\text{dist}(g, G_{s_0}) \geq \epsilon$.
- $g \mapsto gv$ is C^{k+1} for any $v \in E_{cu}$
- $g \mapsto g^{-1}Pg \in L(X)$ is C^{k+1}

⁸⁰Especially, if X does not admit finite-dimensional invariant subspaces of noncompact subgroups of G , the isotropy subgroup G_{s_0} has to be compact, as E_{cu} is finite-dimensional and invariant under G_{s_0} .

Then there is a G -invariant manifold $M_{cu} \subset Y$ which is locally backward-invariant under the map Φ^t for any $t \geq 0$, locally exponentially attracting and contains all solutions, which stay close to S for all backward times. The manifold M_{cu} and the action of G on M_{cu} are of class C^{k+1} .

The second technical assumption also guarantees that the action of G on the center-unstable manifold M_{cu} is proper⁸¹. The technical assumptions can be justified rigorously in many cases for reaction-diffusion systems, so that the essential assumption is merely the finite-dimensionality of the center-unstable eigenspace E_{cu} .

The Locally Convex Case

Let us indicate, how center-manifold reduction can be generalized to complete and separated locally convex topological vector spaces, first for fixed points of a map, and then for patterns S like relative equilibria and relative periodic orbits in analogy to the approach of [Sandstede, Scheel, Wulff]. Such a generalization is possible, because the existence of center-manifolds can be proved by essentially using merely Lipschitz continuous maps and the contraction mapping principle, which both have been generalized to locally convex spaces in chapter 5.

The proof for the fixed point case presented below is similar to the Banach proofs, see e.g. [Lanford, Chapter 8]. Let $F : X \rightarrow X$ be a C^1 -map (w.r.t. the tensorial closed category of locally convex pseudotopological limit vector spaces), on the complete and separated locally convex topological vector space X and consider the discrete dynamical system generated by F . Suppose that F has the fixed point 0 ⁸² and that the linearized map $DF(0) : X \rightarrow X$ at the fixed point has a center-unstable eigenspace, i.e. $\Sigma_{cu} := \{\lambda \in \Sigma(Df(0)) \mid |\lambda| \geq 1\}$ is a spectral set⁸³ of $DF(0)$. Denote the image of the corresponding spectral projection by E_{cu} and call it the generalized center-unstable eigenspace of $DF(0)$, while its closed complement - the kernel of the spectral projection - is denoted by E_s and is called the generalized stable eigenspace of $DF(0)$. Both E_{cu} and E_s are invariant under $DF(0)$, the spectrum of $DF(0)$ restricted to E_{cu} lies outside or on the boundary of the unit disk, the spectrum of $DF(0)$ restricted to E_s lies inside a disk of radius smaller than one, and an analytic part of the spectrum lies between both boundary circles.

⁸¹An operation of G on a manifold M is called proper, if the map $h : (g, m) \mapsto (m, gm)$ is proper, i.e. preimages of compact sets under h are compact. Especially every isotropy subgroup G_m is compact because of $G_m \times m = h^{-1}(\{m, m\})$.

⁸²If F has the fixed point x_0 , consider $F(x_0 + \cdot) - x_0$ as new map.

⁸³A closed subset $\Sigma_1 \subset \Sigma(T)$ is called a spectral set of an operator T , if its complement $\Sigma_2 := \Sigma(T) \setminus \Sigma_1$ is also closed in $\Sigma(T)$ (or equivalently Σ_1 is a union of connected components of $\Sigma(T)$) and is separated from Σ_2 by the analytic part of the resolvent set of $DF(0)$, see the paragraph about spectral theory in 4.4. In the Banach case, a subset Σ_1 is a spectral set iff it is a union of connected components, as all parts of the resolvent set are automatically analytic.

Theorem 7.2 *Let X be a complete and separated locally convex topological vector space admitting C^1 -cut-off functions and consider a C^1 -map F . Then there exists a backward F -invariant⁸⁴ locally defined Lipschitz-manifold M_{cu} at zero given by the graph of a Lipschitz continuous map $H : E_{cu} \supset B_{\epsilon_p}(0) \rightarrow E_s$ on an i -ball $B_{\epsilon_p}(0)$ around zero satisfying $H(0) = 0$. The manifold M_{cu} is called a center-unstable manifold of the fixed point 0 of F .*

As in the Banach case, if X admits C^k -cut-off functions, for a C^k -map F the manifold M_{cu} and the map H can be chosen to be C^k , see e.g. [Lanford, 8.3] for the Banach proof. However, we merely prove the Lipschitz case, as the C^k -case can be proved in complete analogy to the Banach case, but requires lengthy estimates.

Proof: There is a neighbourhood U of the fixed point 0 and a map σ on the set of continuous pseudonorms on X such that $DF : U \rightarrow L_\sigma(X)$. As E_{cu} and E_s are invariant under $DF(0)$, the linear map $DF(0)$ has the form $DF(0) = A \oplus B$, where $A : E_{cu} \rightarrow E_{cu}$ has spectrum on the boundary or outside of the unit circle and $B : E_s \rightarrow E_s$ has spectrum inside a circle of radius strictly smaller than 1, while an analytic part of the resolvent set of $DF(0)$ lies between these circles. Thus σ can be chosen such that $q^{\sigma(q)}(B) < 1$ and $\sigma(q)^q(A^{-1})q^{\sigma(q)}(B) < 1$ for all continuous pseudonorms q on X .

Write $F = A \oplus B + (f, g)$, where the nonlinear parts $f : X \rightarrow E_{cu}$ and $g : X \rightarrow E_s$ have values $f(0) = 0_{cu}$, $g(0) = 0_s$ and are locally Lipschitz continuous w.r.t. σ with vanishing local Lipschitz constants⁸⁵ at zero. In fact, for a C^1 -map F the nonlinear parts are not only locally Lipschitz continuous but C^1 , however here we only want to prove the Lipschitz case.

Thus we search for a backward F -invariant and locally defined Lipschitz manifold $M_{cu} = \{(x_{cu}, H(x_{cu})) \mid p(x_{cu}) \leq \epsilon_p\}$ given by the graph of a Lipschitz continuous map H on an i -ball $B_{\epsilon_p}(0)$. In the following we will see that backward F -invariance of M_{cu} is equivalent to H being a fixed point of the graph transform operator T , and we prove the existence of H by using the contraction mapping principle for T on the space \mathcal{H} of all maps $H : E_{cu} \mapsto E_s$ satisfying $H(0) = 0$ and having Lipschitz constants $p^p(H) \leq 1$. This space \mathcal{H} is a closed subset of the complete and separated locally convex space of globally Lipschitz continuous maps $H : E_{cu} \rightarrow E_s$ satisfying $H(0) = 0$ and endowed with the pseudonorms $p^p(H)$ ⁸⁶.

However, to prove that the graph transform T is a contraction, certain inequalities have to be satisfied. These inequalities can be satisfied by requiring that f, g are not only locally small with small Lipschitz constants at zero, but globally small with small Lipschitz constants. This requirement can be reached by changing the original nonlinear parts f, g outside an i -ball $B_{\epsilon_p}(0)$, which is possible if Lipschitz continuous (or C^k -) cut-

⁸⁴A set S is called backward F -invariant, if $S \subset F(S)$ is valid.

⁸⁵The local Lipschitz constants $q^{\sigma(q)}(f)(x)$ of a map f at a point x w.r.t. σ are defined by $q^{\sigma(q)}(f)(x) := \inf_U q^{\sigma(q)}(f|_U)$, where U runs through the neighbourhoods of x . Consequently the global Lipschitz constants of f w.r.t. σ are $q^{\sigma(q)}(f) = \sup_x q^{\sigma(q)}(f)(x)$.

⁸⁶The space is separated, as $p^p(H) = 0$ for all p implies $p(H(x)) = p(H(x) - H(0)) \leq p^p(H)p(x) = 0$ for all $x \in X$. Further it is complete, as $p^p(H_n - H_m) \rightarrow 0$ implies $p(H_n(x) - H_m(x)) \rightarrow 0$, so that $H_n(x)$ is a Cauchy sequence and converges in X to give a map H , which itself is Lipschitz because of $p^p(H_n) \leq p^p(H) + p^p(H_n - H) \rightarrow p^p(H)$.

off functions on X exist. From now on denote the changed nonlinear parts by f, g , and have in mind that the Lipschitz constants of these changed nonlinear parts can become arbitrary small, so that certain inequalities can be satisfied. Now let us begin with the main part of the proof:

Backward invariance of M_{cu} means $M_{cu} \subset F(M_{cu})$, or equivalently that to every x_{cu} there is a ξ_{cu} such that $(x_{cu}, H(x_{cu})) = F(\xi_{cu}, H(\xi_{cu}))$. Thus to establish backward invariance of M_{cu} , we have to find an H such that for every x_{cu} there is a ξ_{cu} with

$$x_{cu} = A\xi_{cu} + f(\xi_{cu}, H(\xi_{cu})) \quad (7)$$

$$H(x_{cu}) = BH(\xi_{cu}) + g(\xi_{cu}, H(\xi_{cu})) \quad (8)$$

Now to every x_{cu} and every H with $p^p(H) \leq 1$ the first equation (7) has a unique solution $\xi_{cu}(x_{cu}, H)$. Indeed, $p^p(H) \leq 1$ implies $q^{\sigma(q)}(f(\cdot, H(\cdot))) \leq q^{\sigma(q)}(f)$, so provided that the inequality $\sigma(q)^q(A^{-1})q^{\sigma(q)}(f) < 1$ is valid (which can be satisfied for small $q^{\sigma(q)}(f)$), lemma 7.3 implies the global invertibility of the map $\xi_{cu} \mapsto x_{cu} = A\xi_{cu} + f(\xi_{cu}, H(\xi_{cu}))$. Further the lemma estimates the global Lipschitz constants of the inverse map $x_{cu} \mapsto \xi_{cu}(x_{cu}, H)$ in the x_{cu} -component by

$$\begin{aligned} \sigma(q)^q(\xi_{cu}(\cdot, H)) &\leq (\sigma(q)^q(A^{-1})^{-1} - q^{\sigma(q)}(f))^{-1} = \\ &= \frac{\sigma(q)^q(A^{-1})}{1 - \sigma(q)^q(A^{-1})q^{\sigma(q)}(f)} \quad . \end{aligned}$$

The second equation (8) is a fixed point problem $H = TH$ on the space \mathcal{H} , where the operator T is given by $(TH)(\cdot) := BH(\xi_{cu}(\cdot, H)) + g(\xi_{cu}(\cdot, H), H(\xi_{cu}(\cdot, H)))$. We show that T is a contraction:

- The operator T maps functions H with $p^p(H) \leq 1$ into functions TH with $p^p(TH) \leq 1$, provided that the inequality $(q^{\sigma(q)}(B) + q^{\sigma(q)}(g) + q^{\sigma(q)}(f)) \sigma(q)^q(A^{-1}) \leq 1$ is valid (which can be satisfied for small $q^{\sigma(q)}(g), q^{\sigma(q)}(f)$ because of $\sigma(q)^q(A^{-1})q^{\sigma(q)}(B) < 1$). Indeed, $q^q(TH) \leq (q^{\sigma(q)}(B) + q^{\sigma(q)}(g))\sigma(q)^q(\xi_{cu}(\cdot, H))$ holds due to $\sigma(q)^{\sigma(q)}(H) \leq 1$, and hence

$$q^q(TH) \leq (q^{\sigma(q)}(B) + q^{\sigma(q)}(g)) \frac{\sigma(q)^q(A^{-1})}{1 - q^{\sigma(q)}(f)\sigma(q)^q(A^{-1})} \leq 1$$

is valid, where (7.2) and the assumption $(q^{\sigma(q)}(B) + q^{\sigma(q)}(g) + q^{\sigma(q)}(f)) \sigma(q)^q(A^{-1}) \leq 1$ have been used. Further obviously $H(0) = 0$ implies $TH(0) = 0$, so that T maps \mathcal{H} into itself.

- The operator T is a contraction on \mathcal{H} . Indeed, write

$$(TH)(x_{cu}) = G(\xi_{cu}(x_{cu}, H), H(\xi_{cu}(x_{cu}, H))) \quad ,$$

where $G(\xi_{cu}, x_s) := Bx_s + g(\xi_{cu}, x_s)$. Let $\xi = \xi_{cu}(x_{cu}, H)$ and $\xi' = \xi_{cu}(x_{cu}, H')$, then the equality

$$(TH - TH')(x_{cu}) = G(\xi, H(\xi)) - G(\xi, H'(\xi)) + G(\xi, H'(\xi)) - G(\xi', H'(\xi'))$$

is valid. Now

$$q(G(xi, H(\xi)) - G(\xi, H'(\xi))) \leq q^{\sigma(q)}(G(\xi, \cdot))\sigma(q)^{\sigma(q)}(H - H')\sigma(q)(\xi) \leq q^{\sigma(q)}(G(\xi, \cdot))\sigma(q)^{\sigma(q)}(H - H')\sigma(q)^q(\xi_{cu}(\cdot, H))q(x_{cu})$$

and $q(G(xi, H'(\xi)) - G(\xi', H'(\xi'))) \leq q^{\sigma(q)}(G)\sigma(q)(\xi - \xi')$ are valid. To estimate $\sigma(q)(\xi - \xi')$, note that by equation (7) $A\xi + f(\xi, H(\xi)) = x_{cu} = A\xi' + f(\xi', H'(\xi'))$ and thus also $A\xi + f(\xi, H(\xi)) - A\xi' - f(\xi', H'(\xi')) = f(\xi', H'(\xi')) - f(\xi', H(\xi'))$ holds. Hence

$$\begin{aligned} \sigma(q)(\xi - \xi') &\leq \sigma(q)^q(\xi_{cu}(\cdot, H))q(A\xi + f(\xi, H(\xi)) - A\xi' - f(\xi', H(\xi'))) = \\ &\sigma(q)^q(\xi_{cu}(\cdot, H))q(f(\xi', H'(\xi')) - f(\xi', H(\xi'))) \leq \\ &\sigma(q)^q(\xi_{cu}(\cdot, H))q^{\sigma(q)}(f)\sigma(q)^{\sigma(q)}(H - H')\sigma(q)(\xi') \end{aligned}$$

is valid, and because of $\sigma(q)(\xi') \leq \sigma(q)^q(\xi_{cu}(\cdot, H))q(x_{cu})$ we obtain $\sigma(q)(\xi - \xi') \leq \sigma(q)^q(\xi_{cu}(\cdot, H))^2q^{\sigma(q)}(f)q(x_{cu})\sigma(q)^{\sigma(q)}(H - H')$.

All these inequalities together imply

$$\begin{aligned} q(TH - TH')(x_{cu}) &\leq (q^{\sigma(q)}(G(\xi, \cdot)) + q^{\sigma(q)}(G)\sigma(q)^q(\xi_{cu}(\cdot, H))q^{\sigma(q)}(f)) \\ &\sigma(q)^q(\xi_{cu}(\cdot, H))\sigma(q)^{\sigma(q)}(H - H')q(x_{cu}) \end{aligned}$$

and thus $(q^q)^{\sigma(q)^{\sigma(q)}}(T) < 1$ is valid provided that

$$(q^{\sigma(q)}(G(\xi, \cdot)) + q^{\sigma(q)}(G)\sigma(q)^q(\xi_{cu}(\cdot, H))q^{\sigma(q)}(f)) \sigma(q)^q(\xi_{cu}(\cdot, H)) \leq 1 \quad .$$

The last inequality can be satisfied for small Lipschitz constants of the nonlinear parts f, g . In fact, $q^{\sigma(q)}(G(\xi, \cdot))$ is arbitrary near to $q^{\sigma(q)}(B)$ for small $q^{\sigma(q)}(g)$, $\sigma(q)^q(\xi_{cu}(\cdot, H))$ is arbitrary near to $\sigma(q)^q(A^{-1})$ for small $q^{\sigma(q)}(f)$ and $\sigma(q)^q(A^{-1})q^{\sigma(q)}(B) < 1$ holds.

Thus if the Lipschitz constants of the changed nonlinear parts are so small that the needed inequalities are satisfied, then T is a contraction and by the contraction mapping principle there is a fixed point H . Now the whole graph $\{(x_{cu}, H(x_{cu})) \mid x_{cu} \in E_{cu}\}$ of the fixed point H is backward invariant for the map with changed nonlinear parts, but as the changed nonlinear parts coincide with the original nonlinear parts f, g inside of the i -ball, we obtain a locally defined and backward-invariant Lipschitz manifold $M_{cu} = \{(x_{cu}, H(x_{cu})) \mid p(x_{cu}) \leq \epsilon_p\}$ for the original map F . \square

Lemma 7.3 *Let $A \in L_\sigma(X)$ be a continuously invertible operator on the complete and separated locally convex topological vector space X with inverse $A^{-1} \in L_{\sigma^{-1}}(X)$, and let $F : X \rightarrow X$ be a (globally) Lipschitz continuous map w.r.t. σ such that $\sigma(q)^q(A^{-1})q^{\sigma(q)}(F) < 1$ for all continuous pseudonorms q on X . Then $A+F$ is invertible and its inverse has Lipschitz constants $\sigma(q)^q((A+F)^{-1}) \leq (\sigma(q)^q(A^{-1})^{-1} - q^{\sigma(q)}(F))^{-1}$.*

Proof: For every y the equation $(A + F)(x) = y$ has a unique solution x . Indeed, let $x_0 := A^{-1}y$, then $x = x_0 + v$ is a solution of $(A + F)(x) = y$, if v is a solution of $v = -A^{-1}F(x_0 + v)$. Because the operator $Tv := -A^{-1}F(x_0 + v)$ has Lipschitz constants $\sigma(q)^{\sigma(q)}(T) \leq \sigma(q)^q(A^{-1})q^{\sigma(q)}(F) < 1$, the contraction mapping theorem can be applied. Thus T has a unique fixed point v , and hence $A + F$ is invertible. Finally the estimate of the Lipschitz constants of $(A + F)^{-1}$ follows from

$$q((A + F)(x) - (A + F)(x')) \geq q(Ax - Ax') - q(F(x) - F(x')) \geq (\sigma(q)^q(A^{-1})^{-1} - q^{\sigma(q)}(F)) \sigma(q)(x - x') \quad .$$

□

Thus center-unstable manifolds of fixed-points of maps exist also in the case of locally convex spaces. Further the existence of G -invariant center-unstable manifolds of relative equilibria and relative periodic orbits in dynamical systems under symmetry can be obtained in analogy to the Banach proof [Sandstede, Scheel, Wulff, 3.4]. However, in the locally convex case it does not make sense to use the norm $\sup_{x_{cu}} p(H(x_{cu}))$ on \mathcal{H} , but the Lipschitz norm has to be used, and contrary to the Banach case the technical assumptions on the G -action are usually automatically satisfied, as the action is usually continuous and differentiable, which makes the proof easier.

7.3 Pattern formation

Assume that by center-manifold reduction a finite-dimensional center-unstable C^{k+1} -manifold M of a relative equilibrium or a relative periodic orbit S with a proper C^{k+1} -action has been obtained. Because G acts proper, there is a Riemannian metric on M such that G acts by isometries on M , and bifurcations of S can be studied within the finite-dimensional Riemannian manifold M , see e.g. [Merker], where also the proofs of the theorems below can be found.

The main tool to study bifurcations of patterns in M is an equivariant tubular neighbourhood $TS^\perp \rightarrow S$ around S , whose existence can be proved for relative equilibria and relative periodic orbits S . Further this tubular neighbourhood can be parametrized in a convenient way by skew-products: If X has a $G \times H$ -action and Y has merely an H -action, then denote by $X \times_H Y$ the quotient $(X \times Y)/H$ and call this space, which has the G -action $g(H(x, y)) = H(gx, y)$, the skew product of X and Y . Especially if $X = G$, consider the Lie group G as endowed with the $G \times H$ -action $((g, h), \tilde{g}) \mapsto g\tilde{g}h^{-1}$. In the following K always denotes the isotropy subgroup $\{g \in G | gs_0 = s_0\}$ of some fixed $s_0 \in S$.

Theorem 7.4 *The equivariant tubular neighbourhood of a relative equilibrium S has an equivariant parametrization*

$$TS^\perp \cong G \times_K V \quad ,$$

where the compact subgroup K acts linearly on the vector space $V \cong T_{s_0}S^\perp$.

If $S = G\Phi^{\mathbb{R}}(s_0)$ is a relative periodic orbit, then there is a minimal time $T > 0$ such that $\Phi^T(s_0) = gs_0$ for some group element $g \in G$, and there is a minimal $P \in \mathbb{N} \cup \{\infty\}$ such that $\Phi^{PT}(s_0) = s_0$ (set $P = \infty$ if there is no such $P < \infty$; this can only happen in the case of a noncompact G). Denote further by L the set $\{g|\exists t : \Phi^t(s_0) = gs_0\}$, then $L/K \cong \mathbb{Z}/P\mathbb{Z}$ is valid via a homomorphism Θ .

Theorem 7.5 *The equivariant tubular neighbourhood of a relative periodic orbit S has an equivariant parametrization*

$$TS^{\perp} \cong G \times_L (\mathbb{R}/P\mathbb{Z} \times V) \quad ,$$

if the vector bundle $TS^{\perp}/K \rightarrow S/K \cong \mathbb{R}/P\mathbb{Z}$ ($=\mathbb{R}$ in the case $P = \infty$) is trivializable (non-Möbius case), and an equivariant parametrization

$$TS^{\perp} \cong G \times_L (\mathbb{R}/2P\mathbb{Z} \times_{\mathbb{Z}_2} V)$$

in the case the vector bundle $TS^{\perp} \rightarrow \mathbb{R}/P\mathbb{Z}$ is not trivialisable (Möbius case, not possible if $P = \infty$).

Hereby L acts by vector bundle maps on the vector bundle $\mathbb{R}/P\mathbb{Z} \times V \rightarrow \mathbb{R}/P\mathbb{Z}$ resp. $\mathbb{R}/2P\mathbb{Z} \times_{\mathbb{Z}_2} V \rightarrow \mathbb{R}/P\mathbb{Z}$ over the L -action $(l, r + P\mathbb{Z}) = r + \Theta(l) + P\mathbb{Z}$ on $\mathbb{R}/P\mathbb{Z}$. Especially the K -action on V is linear.

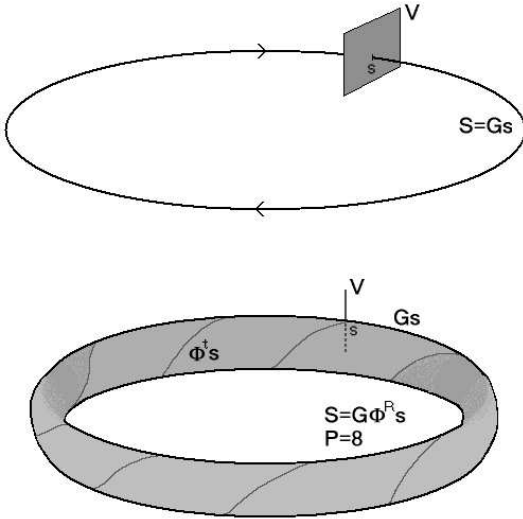


Figure 3: Parametrization of Tubular Neighbourhoods of Relative Equilibria and Relative Periodic Orbits

Using these equivariant parametrizations of tubular neighbourhoods of a relative equilibrium resp. relative periodic orbit, the flow on the tubular neighbourhood can be lifted into the total spaces $G \times V$ resp. $G \times \mathbb{R}/P\mathbb{Z} \times V$. For a relative equilibrium the flow in the lift has the form

$$\begin{aligned} \dot{g} &= Tg \cdot a(v) \\ \dot{v} &= \phi(v) \end{aligned}$$

with vectorfields a and ϕ satisfying the equivariance conditions

$$\begin{aligned} a : V &\rightarrow T_e G, & a(kv) &= Tk \cdot a(v) \cdot Tk^{-1} \\ \phi : V &\rightarrow V, & \phi(kv) &= k\phi(v), & \phi(0) &= 0 \end{aligned}$$

In this skew-product form of the flow, changes of the pattern S can be easily observed and classified: Changes in the G -component correspond to not-type-changes of the pattern. S stays a relative equilibrium, but may change its appearance. So an inward meandering spiral may become a drifting spiral and then an outward meandering spiral, depending on the sign of a in some component, but the type of S is unchanged. Contrary, bifurcations of the equilibrium 0 in the V component correspond to type-changing bifurcations of S . As on V merely the compact group K acts linear by isometries, here

the usual theorems of equivariant bifurcations under compact symmetry can be used to solve the bifurcation problem and to observe e.g. Hopf bifurcations, which generate relative periodic orbits.

Near a relative periodic orbit the lifted vector field on $G \times \mathbb{R}/P\mathbb{Z} \times V$ has the form

$$\begin{aligned}\dot{g} &= Tg \cdot a(r, v) \\ \dot{r} &= \psi(r, v) \\ \dot{v} &= \phi(r, v)\end{aligned}$$

with vector fields a , ψ and ϕ which are equivariant according to

$$\begin{aligned}a(l(r, v)) &= Tl \cdot a(r, v) \cdot Tl^{-1} \\ \psi(l(r, v)) &= \psi(r, v), \quad \psi(r, 0) \neq 0 \\ \phi(l(r, v)) &= l \cdot \phi(r, v), \quad \phi(r, 0) = 0\end{aligned}$$

Again, not-type changing bifurcations can be observed in the G -component, while type-changing bifurcations like period doubling or bifurcations to relative periodic tori can be observed in the $\mathbb{R}/P\mathbb{Z} \times V$ -component, using the usual equivariant bifurcation lemmata for the compact subgroup K acting linearly on V by isometries.

Conclusion

Starting from the observation that on a noncompact domain M the isometries induce a noncontinuous and nondifferentiable action on nearly all Banach manifolds of maps on M and that - as a consequence - the study of infinite dimensional dynamical systems on noncompact manifolds with symmetry becomes complicated, we began to explore manifolds of maps modeled over locally convex spaces. We discussed natural spaces of maps and developed a natural calculus on manifolds modeled over locally convex spaces, based on the tensorial closed category of locally convex pseudotopological limit vector spaces. The C^1 -maps of this calculus have the good property to be locally Lipschitz continuous. This enabled us to generalize well-known theorems of nonlinear analysis on Banach manifolds to manifolds modeled over locally convex spaces. For example, the solvability of ordinary differential equations with C^1 -differentiable right hand side was proved for initial values satisfying certain growth conditions.

In the second part we discussed, which fluid dynamical equations on noncompact domains can be interpreted as ordinary differential equations on manifolds of maps modeled over locally convex spaces. While incompressible or viscous problems do not allow the use of such manifolds due to their nonlocality, compressible inviscous fluid dynamical equations can be interpreted as ordinary differential equations on such spaces of maps due to their locality, and thus the formerly developed theory can be applied. Finally we discussed pattern formation under symmetry in such infinite dimensional dynamical systems. There the differentiability of the action induced by the isometries on manifolds of maps on M modeled over locally convex spaces helps to avoid many difficulties appearing in the Banach case.

A A comparison between Convenient Calculus and Analysis on Natural Spaces of Maps

The convenient calculus developed in [Fröhlicher,Kriegl] and [Kriegl,Michor] has its origin in the fact that the notion of a Lipschitz curve does not depend on the topology of a locally convex topological vector space X , but merely on its dual X' or equivalently on its bornology, and that for the so called convenient vector spaces also the notion of a Lip^k -curve does merely depend on the bornology.

Call a curve $c : \mathbb{R} \rightarrow X$ into a locally convex topological vector space X locally Lipschitz continuous at t , if there is a neighbourhood I of t in \mathbb{R} such that $B := \left\{ \frac{c(s)-c(s')}{s-s'} \mid s-s' \in I \right\}$ is bounded⁸⁷. A curve c is called locally Lipschitz continuous, if it is locally Lipschitz continuous at every t . Because the boundedness of a subset does not depend on the topology of X but on the dual X' of X only, also the notion of a locally Lipschitz continuous curve depends on the dual X' or equivalently the bornology of X only⁸⁸. Thus in the discussion of locally Lipschitz continuous curves we can use without restriction instead of the original space X its bornologification $b(X)$, which is the vector space X endowed with the finest locally convex topology having the same bounded sets as the original locally convex vector space X . Hence we can work in the tensorial closed category of bornological locally convex vector spaces discussed in section 4.3.

Recall that a curve c is called a C^1 -curve, if the limit $\dot{c}(t) := \lim_{h \rightarrow 0} \frac{c(t+h)-c(t)}{h}$ exists for every t and $t \mapsto \dot{c}(t)$ is a continuous curve. This notion can be iterated by calling c a C^k -curve, if the derivatives $\dot{c}(t)$ exist and form a C^{k-1} -curve. Further call c a Lip^k -curve, if c is a C^k -curve and all derivatives up to order k are locally Lipschitz continuous. However, while being locally Lipschitz continuous does not depend on the topology of X but only on its bornology, being a C^k - or Lip^k -curve does generally depend on the topology of X , as the existence of the limits $\dot{c}(t) := \lim_{h \rightarrow 0} \frac{c(t+h)-c(t)}{h}$ depends on the topology and not merely on the bornology of X .

Let us ask, for which kind of spaces also the notion of a Lip^k -curve depends on the bornology only. Call a curve c a scalarly Lip^k -curve, if $x' \circ c$ is a Lip^k -curve on \mathbb{R} for each $x' \in X'$. Note that being a scalarly Lip^k -curve does not depend on the topology of X , but only on the dual X' or equivalently the bornology of X . Now the key result [Kriegl,Michor, 2.1] is that difference quotients of scalarly Lip^k -curves up to order k are automatically Cauchy sequences in the Mackey limit structure⁸⁹ on X . Hereby

⁸⁷A set $B \subset X$ is called bounded, if it is absorbed by any neighbourhood U of zero, i.e. there is a λ such that $\lambda B \subset U$.

⁸⁸The collection of all bounded subsets of X is called the bornology of X . Let B be bounded w.r.t. a topology on X which is compatible with the duality (X, Y) , then B is also bounded w.r.t. all other topologies on X compatible with the duality (X, Y) . In fact, a subset $B \subset X$ is bounded, iff it is scalarly bounded, i.e. $x'(B)$ is bounded for every $x' \in X'$. As boundedness can be tested scalarly, a curve c is locally Lipschitz continuous at t , if there is a neighbourhood I of t in \mathbb{R} such that for every continuous pseudonorm p on X there exists a constant L with $p(c(s) - c(s')) \leq L|s - s'|$ (which is the usual definition of locally Lipschitz continuous curves).

⁸⁹The Mackey limit structure is also called the structure of quasibounded convergence in [Keller] and turns X into a limit vector space. Its associated topology is called the Mackey closure topology and should not be changed with the Mackey topology $\tau(X, X')$ used in duality theory. The Mackey

the Mackey limit structure on X is generated by the convergences $\mathcal{F} \rightarrow 0$ of filters \mathcal{F} , which are finer than $\mathcal{U}_{\mathbb{R}}(0) \cdot B$ for some bounded set B in X and the neighbourhood filter $\mathcal{U}_{\mathbb{R}}(0)$ of zero in \mathbb{R} . Thus for a Mackey complete space X , i.e. a locally convex space X for which every Cauchy filter w.r.t. the Mackey limit structure converges, the limits of the difference quotients of scalarly Lip^k -curves up to order k exist. Using these limits as candidates for the derivatives, [Kriegl,Michor, 2.3] proves that every scalarly Lip^k -curve in a Mackey complete space X is automatically a Lip^k -curve. Thus in a Mackey complete space X also the notion of a Lip^k -curve does not depend on the topology of X , but merely on the dual X' or equivalently the bornology of X .

Mackey complete spaces X are also called convenient. Being a convenient vector space can be characterized by many other useful properties instead of Mackey completeness, e.g. by the existence of an anti-derivative C to every smooth curve c , see [Kriegl,Michor, 2.14]. Instead of discussing such properties characterizing convenient vector spaces, let us rather discuss Lip^k -maps between convenient vector spaces X, Y : A map $f : X \supset U \rightarrow Y$ on a c^∞ -open set U is called a Lip^k -map, if it maps Lip^k -curves to Lip^k -curves. The main result [Fröhlicher,Kriegl, 4.3.27] (or [Kriegl,Michor, 12.8]) regarding Lip^k -maps is that f is a Lip^k -map, iff it has a Lip^{k-1} -derivative $Df : U \rightarrow L(X, Y)$, where $L(X, Y)$ is the natural space of maps within the tensorial closed category of bornological locally convex vector spaces. This natural space $L(X, Y)$ is the bornologification of the topology of uniform convergence on bounded subsets on $L(X, Y)$, see 4.3.

Thus Lip^k -maps have a linear derivative, and obviously the chain rule is valid. Further the space $\text{Lip}^k(U, Y)$ of Lip^k -maps from a c^∞ -open subset U of a convenient vector space X to a convenient vector space Y itself can be turned into a convenient vector space, and the category of convenient vector spaces and Lip^k -maps with these spaces of maps has a lot of properties of a cartesian closed category. In fact, if $k = \infty$, the category itself is cartesian closed, and if $k < \infty$, then the category of Lip^k maps between Lip^k -spaces (see 5.6) is cartesian closed, where the natural spaces of maps differ only slightly from a convenient vector space.

However, while the categorial properties of Lip^k -maps are really satisfying, Lip^k -maps are generally not continuous w.r.t. the locally convex topology. Further convenient calculus allows to discuss properties of geometric structures on infinite dimensional manifolds modeled over convenient vector spaces instead of Banach spaces, but it does not help to generalize existence theorems on Banach manifolds to more general manifolds. This is the main difference to the approach in this thesis, where the tensorial closed category of locally convex pseudotopological vector spaces instead of the category of bornological locally convex topological vector spaces is used to develop a calculus on locally convex spaces. In this setting, the C^1 -maps are locally Lipschitz continuous, and this property helps to prove existence theorems in a Banach-like style, contrary to the convenient calculus.

closure topology is generally not a vector space topology, because it is not compatible with addition, and is identical with the the final topology on X w.r.t. all Lip^k -curves in X , $k \in \mathbb{N} \cup \{\infty\}$ fixed, called the c^∞ -topology on X , see [Kriegl,Michor, 2.13].

B Solving the Poisson equation on Noncompact Manifolds

In this section the solvability of the Poisson equation $\Delta u = f$ on a complete oriented Riemannian manifold M is discussed for the Laplacian⁹⁰ Δ acting on functions. In the case of boundary, additionally the Neumann boundary condition $\frac{\partial u}{\partial \nu} = g$ on ∂M is imposed, where ν denotes the normal vector field to ∂M . Note that the Neumann boundary condition makes sense only if g and f satisfy the compatibility condition $\int_M f = \int_{\partial M} g$ because of

$$\int_M f = \int_M \Delta u = \int_{\partial M} \frac{\partial u}{\partial \nu} = \int_{\partial M} g \quad .$$

First let us discuss whether a Green's function G to Δ exists. A function G on $(M \times M) \setminus \{(m, m) | m \in M\}$ is called a Green's function if it satisfies $\Delta_{m'} G(m, m') = \delta_m(m')$ and $G(m, m') = G(m', m)$. A manifold is called nonparabolic, if it admits a positive Green's function G , and in this case there is also a unique minimal positive Green's function. This minimal positive Green's function can be constructed by $G(m, m') := \frac{1}{2} \int_0^\infty p(t, m, m') dt$ using the heat kernel $p(t, m, m')$, which is the smallest positive fundamental solution to the heat equation $\dot{p}(t, m, m') = -\frac{1}{2} \Delta_{m'} p(t, m, m')$ satisfying $\lim_{t \searrow 0} p(t, m, \cdot) = \delta_m$. However, the integral $\frac{1}{2} \int_0^\infty p(t, m, m') dt$ is finite only if M is non-parabolic. An equivalent condition for non-parabolicity is that Brownian motion - the Markov process defined by using p as transition density - is not recurrent⁹¹. Note that non-parabolicity can also be formulated in terms of capacity, proper massive sets or non-constant positive bounded superharmonic functions, see [Grigor'yan, Theorem 5.1]. Further on a manifold M with boundary it can be assumed that the heat kernel satisfies the Neumann boundary condition $\frac{\partial_{m'} p(t, m, m')}{\partial \nu} = 0$, or equivalently that Brownian motion is reflected at the boundary. In the non-parabolic case then also the minimal positive Green's function satisfies the Neumann boundary condition $\frac{\partial_{m'} G(m, m')}{\partial \nu} = 0$.

An alternative way to construct Green's function uses an exhausting sequence $M = \bigcup_i \Omega_i$, see [Li] or the original reference [Li, Tam]: Let each Ω_i be a precompact domain such that $m \in \Omega_i \subset \Omega_{i+1}$ and $M = \bigcup_i \Omega_i$ are valid. Let $G_i(m, m')$ be the solution of $\Delta_{m'} G_i(m, m') = \delta_m(m')$ in Ω_i satisfying the Dirichlet boundary condition $G_i(m', m) = 0$ on $\partial \Omega_i$. By the maximum principle $G_i(m, \cdot) \leq G_j(m, \cdot)$ is valid, and thus $G_i(m, \cdot)$ increases monotonically. If the limit $G(m, \cdot)$ of this functions is finite, then it is the minimal positive Green's function and M is nonparabolic. If the limit is infinite, then a Green's function G can still be obtained as the limit of the sequence $G_i(m, \cdot) - a_i$, where $a_i := \sup_{\partial B_m(1)} G_i(m, \cdot)$. However, now G changes its sign and M is parabolic. Further G may be not unique (it could depend on the exhaustion) and in the case of

⁹⁰In this section, the Laplacian on functions is the negative of the Laplace-Beltrami operator, to be in convenience with the case \mathbb{R}^n .

⁹¹Brownian motion is called recurrent, if for any nonvoid open $\Omega \subset M$ and any $m \in M$ there is with probability one a time sequence $t_k \rightarrow \infty$ such that the random walk starting at m runs through Ω at the times t_k . Thus in a recurrent Markov process a random walk runs infinitely times through every open set.

boundary there may exist no Green's function which satisfies the Neumann boundary conditions. For example, if M is a bounded domain in \mathbb{R}^n , then the Neumann boundary conditions would imply the contradiction

$$1 = \int_M \Delta G(m, \cdot) = \int_{\partial M} \frac{\partial G(m, \cdot)}{\partial \nu} = 0 \quad .$$

This problem can be solved by requiring an inhomogeneous Neumann boundary condition $\frac{\partial_{m'} G(m, m')}{\partial \nu} = \frac{1}{\text{Vol}(\partial M)}$ or more general $\frac{\partial_{m'} G(m, m')}{\partial \nu} = h(m')$ with a function h independent of m such that $\int_{\partial M} h(m') = 1$ holds, because then G satisfies the compatibility condition: The mass 1 generated by $G(m, \cdot)$ at m is allowed to flow off via the boundary.

On the one hand now theorems can be obtained which deduce non-parabolicity from curvature assumptions, volume growth conditions or inequalities. For example, if M has non-negative Ricci curvature, or has the volume doubling property $C \text{Vol}(B_m(r)) \geq \text{Vol}(B_m(2r))$ and admits a Poincare inequality $\int_{B_m(r)} |f - \bar{f}| \leq Cr \left(\int_{B_m(2r)} |\nabla f|^2 \right)^{\frac{1}{2}}$ for $f \in H^1(B_m(2r))$ (where $\bar{f} := \frac{1}{\text{Vol}(B_m(r))} \int_{B_m(r)} f$ denotes the middle value of f on $B_m(r)$), then M is non-parabolic if and only if $\int_0^\infty \frac{t}{\text{Vol}(B_t(m))} dt < \infty$ holds for a point $m \in M$. Note for example that \mathbb{R}^n is parabolic for $n = 2$ and nonparabolic for $n \geq 3$. Further in the case of nonnegative Ricci curvature also Green's function can be estimated.

On the other hand it can be discussed under which conditions a solution of $\Delta u = f$, $\frac{\partial u}{\partial \nu} = g$, can be obtained via

$$u(m) = \int_M G(m, m') f(m') dm' + \int_{\partial M} G(m, m') g(m') dm' + \text{const} \quad ,$$

Note that f and g have to satisfy some growth conditions assuring the existence of the integrals. Such growth conditions are obtained for example in [Tam, Theorem 1.1 + 1.2, Corollary 1.2]: Let M be a noncompact manifold without boundary having nonnegative Ricci curvature, let f be locally Hölder continuous and define $k(m, t) := \frac{1}{\text{Vol}(B_m(t))} \int_{B_m(t)} |f|$. Suppose that $\int_0^\infty k(m, t) dt < \infty$ is valid for some $m \in M$, and further suppose that M is nonparabolic and its Green's function satisfies an estimate $\sigma^{-1} \frac{d^2(m, m')}{\text{Vol}(B_m(d(m, m')))} \leq G(m, m') \leq \sigma \frac{d^2(m, m')}{\text{Vol}(B_m(d(m, m')))}$. Then the Poisson equation with right hand side f has a solution u . If $f \geq 0$, then u can be estimated. For arbitrary f still ∇u can be estimated, e.g. if $\int k(m, t) dt$ is uniformly bounded, then $\sup_M |\nabla u| < \infty$. Note also that the nonparabolicity assumption and the Green's function estimate can be replaced by assuming the existence of some $1 > \delta > 0$ and a function $0 \leq h(t) = o(t)$ such that $\int_0^t sk(m, s) \leq h(t)$ is valid for all m and $t \geq \delta d(m, 0)$ where $0 \in M$ is a fixed point, see [Tam, Theorem 1.1 + 1.2, Corollary 1.2]

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Abstract

The main goal of this doctoral thesis is to discuss the foundations of dynamical systems, whose state space is a space of maps defined on a noncompact domain and whose dynamics are compatible with the symmetries of this domain.

Obviously, a mathematical rigorous treatment of such dynamical systems requires to specify, which spaces of maps are used, e.g. Sobolev spaces. However, regarding pattern formation on noncompact manifolds under symmetry, solutions of dynamical equations within the class of Sobolev vector fields do not include typical patterns with noncompact symmetries, as they vanish at infinity. But even if solutions within other Banach spaces of maps can be established, the problem remains that for noncompact manifolds the symmetry group in general does not act continuously on Banach spaces of maps, as it acts by composition, but composition and evaluation are not continuous.

Therefore, in this thesis locally convex spaces of maps like the local Sobolev spaces are used to model dynamical systems, where composition, evaluation and thus also the symmetry action are continuous. However, as the analysis of such natural function spaces is not far developed in literature, a main task of this thesis is to extend the analysis to such spaces, and to provide theorems used in the study of dynamical equations. Contrary to the category of normable spaces, the category of locally convex spaces is not tensorial closed, and thus there is no natural space of continuous linear maps between locally convex spaces. A problem is that it is not clear how to define continuously differentiable maps. However, by using a tensorial closed category of vector spaces endowed with a slightly more general topological structure than a locally convex topology, this problem can be solved and a sufficient differential calculus can be developed.

But analysis requires more than just a differential calculus: Differential equations must be solved, an inverse function theorem is needed, and other theorems of classical analysis must be transferred to the new setting. However, on locally convex spaces there exist locally not solvable differential equations with continuous linear right hand side, so that a precise discussion is needed. Also here our choice of the tensorial closed category is helpful, because it guarantees that continuously differential maps are locally Lipschitz continuous, so that solvability of differential equations can be characterized by growing conditions. Finally, also manifolds modeled on complete locally convex topological vector spaces are considered.

After having laid the analytic foundations, in the second part of this thesis fluid dynamical systems and pattern formation on noncompact manifolds are discussed. It is shown that fluid dynamical equations like those modeling inviscous compressible fluids can be modeled using natural spaces of maps, the pattern formation under symmetry in the Banach case and in the locally convex case is compared, and methods to obtain the bifurcation equation in the locally convex case are developed.

Jochen Merker, “*Analysis natürlicher Funktionenräume und Dynamik auf nichtkompakten Mannigfaltigkeiten unter Symmetrie*“, 2005.

Zusammenfassung

Das Hauptziel dieser Doktorarbeit ist es, die Grundlagen dynamischer Systeme zu entwickeln, deren Zustandsraum ein Raum von Abbildungen auf einem nichtkompakten Gebiet und deren Evolution kompatibel mit den Symmetrien des Gebietes ist.

Ein rigoroses Studium solch dynamischer Systeme erfordert natürlich, den benutzten Raum von Abbildungen festzulegen, z.B. einen Sobolev-Raum. Für die Untersuchung von Musterbildung unter Symmetrie auf nichtkompakten Mannigfaltigkeiten ist die Wahl eines Sobolev-Raumes aber ungünstig, da die Differenz von Mustern mit unterschiedlicher nichtkompakter Symmetrie nicht im Unendlichen verschwindet und somit durch kein Sobolev-Vektorfeld repräsentiert werden kann. Auch die Benutzung anderer Banachräume ist keine Alternative, da für nichtkompakte Mannigfaltigkeiten i.a. die Symmetrie nicht stetig auf Banachräumen operiert, denn Komposition von und das Einsetzen in Abbildungen ist dort nicht stetig.

Deswegen werden in dieser Arbeit lokalkonvexe Räume von Abbildungen wie lokale Sobolev-Räume benutzt, auf denen die Komposition, die Evaluation und somit auch die Symmetrie-Operation stetig ist. Jedoch ist die Analysis solch natürlicher Funktionenräume bisher nicht so weit entwickelt, daß man dynamische Systeme mit analytischen Methoden untersuchen kann. Dort Abhilfe zu schaffen, ist eines der Hauptziele der Arbeit. Nun ist im Gegensatz zur Kategorie der normierbaren Räume die Kategorie der lokalkonvexen Räume leider nicht tensoriell abgeschlossen, so daß der Raum der stetigen linearen Abbildungen zwischen lokalkonvexen Räumen nicht in natürlicher Weise mit einer Topologie versehen werden kann. Insbesondere erschwert dies die Definition von stetig differenzierbaren Abbildungen. Dieses Problem wird hier gelöst, indem eine tensoriell abgeschlossene Kategorie von Räumen mit einer nur wenig allgemeineren Struktur als einer lokalkonvexen Topologie konstruiert wird. Diese erlaubt dann die Entwicklung eines leistungsfähigen Differentialkalküls.

Aber Analysis ist mehr als ein Differentialkalkül: Man muß Differentialgleichungen lösen können, benötigt einen Satz über implizite Funktionen und auch andere Sätze der klassischen Banachraum-Analysis. Aber z.B. existieren auf vollständigen lokalkonvexen Räumen stetige lineare Differentialgleichungen, die lokal nicht lösbar sind, so daß eine präzise Diskussion nötig ist. Auch hier hilft die betrachtete Kategorie weiter, denn stetig differenzierbare Abbildungen sind automatisch lokal Lipschitz stetig, so daß man die Lösbarkeit von Anfangswertaufgaben durch Wachstumsbedingungen charakterisieren kann. Ebenso wird die Modellierung von Mannigfaltigkeiten über vollständigen lokalkonvexen Räumen diskutiert.

Nachdem die analytischen Grundlagen gelegt sind, wird im zweiten Teil der Arbeit Fluidynamik und Musterbildung auf nichtkompakten Mannigfaltigkeiten diskutiert. Es wird gezeigt, daß inviskose kompressible Fluide mittels natürlicher Funktionenräume modelliert werden können, es wird die Musterbildung unter Symmetrie im Banach- und im lokalkonvexen Fall miteinander verglichen, und Methoden zur Gewinnung der Bifurkationsgleichung im lokalkonvexen Fall werden entwickelt.

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