Stratified description of the moduli spaces of Higgs bundles and connections

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Declaration

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare upon oath that I have written the present dissertation independently and have not used further resources and aids than those stated in the dissertation.

Hamburg, June 8th 2023

Duong Quy Đinh

The main results of this thesis are from my joint work with Troy Figiel and Jörg Teschner, and form parts of the following preprints

1. [10] *Classical limit of the geometric Langlands correspondence for* $SL_2(\mathbb{C})$,
   with Jörg Teschner, arXiv:2312.13393 (2023);

2. [11] *An analog of the spectral transform for holomorphic connections*,
   with Troy Figiel and Jörg Teschner, to appear;

3. [12] *Separation of variables for quantum Hitchin systems in higher genus*,
   with Jörg Teschner, in preparation.

Jörg Teschner suggested the idea of drawing an analogy between the projection of Baker-Akhiezer divisors to the base Riemann surface and apparent singularities of SL-operators, in particular regarding the degeneration of these divisors. The detailed study of how Baker-Akhiezer divisors characterize Higgs bundles in chapter 3 and the results concerning degeneration of these divisors in chapter 4 are my contribution. The results concerning SL-operators induced from holomorphic connections in subchapters 5.4 and 5.5, except definition 5.4, are my contribution. Jörg also showed me the construction of SL-operators from “building blocks” of meromorphic quadratic differentials; I worked out the details of this idea in subchapter 6.1 and used them as important ingredients of the proofs in chapter 7.

Troy Figiel did a local formal analysis of the limit upon colliding apparent singularities. The results in chapter 7 which show the existence and compute these limits in the global sense are my contribution. The suggested relation to bubbling is due to Troy.

All other main results of this thesis are my contribution.

Vladimir Roubtsov shared with Jörg Teschner and I his unpublished results [21], some of which are similar to parts of the results obtained in [10]. This thesis does not contain those results.

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Abstract

We describe how certain effective divisors, which we call Baker-Akhiezer divisors, on non-degenerate spectral curves characterize $SL_2(\mathbb{C})$-Higgs bundles. To some extent, these divisors encode the natural stratification on the Hitchin moduli space $\mathcal{M}_H$ of $SL_2(\mathbb{C})$-Higgs bundles, and their degeneration describes families of Higgs bundles that limit to lower strata. We show how apparent singularities with their accessory parameters of SL-operators are analogues of Baker-Akhiezer divisors: they also encode to some extent the natural stratification on the de Rham moduli space $\mathcal{M}_{dR}$ of irreducible $SL_2(\mathbb{C})$-connections. In addition, a collision of two simple apparent singularities can define a family of SL-operators whose limit is an SL-operator with less apparent singularities and encodes an irreducible $SL_2(\mathbb{C})$-connection in a lower stratum.

Zusammenfassung

Wir beschreiben, wie bestimmte effektive Divisoren, die wir Baker-Akhiezer Divisoren nennen, auf nicht entarteten Spektralkurven, $SL_2(\mathbb{C})$-Higgs-Bündel charakterisieren. Bis zu einem gewissen Grad kodieren diese Divisoren die natürliche Stratifikation auf dem Hitchin-Modulraum $\mathcal{M}_H$ von $SL_2(\mathbb{C})$-Higgs-Bündeln, und ihre Entartung beschreibt Familien von Higgs-Bündeln, die sich auf niedrigere Strata limitieren. Wir zeigen, wie scheinbare Singularitäten mit ihren Nebenparametern von SL-Operatoren analog zu Baker Akhiezer-Divisoren sind: Sie kodieren auch in gewissem Maße die natürliche Stratifikation auf dem de Rham Modulraum $\mathcal{M}_{dR}$ der irreduziblen $SL_2(\mathbb{C})$-Zusammenhänge. Außerdem kann eine Kollision von zwei einfachen scheinbaren Singularitäten eine Familie von SL-Operatoren definieren, deren Grenzwert ein SL-Operator mit weniger scheinbaren Singularitäten ist und einen irreduziblen $SL_2(\mathbb{C})$-Zusammenhang in einem unteren Stratum kodiert.

Contents

Chapter 1

Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. The recurring themes in this thesis are

(a) to characterize $SL_2(\mathbb{C})$-Higgs bundles on X in terms of certain effective divisors, which we will call Baker-Akhiezer divisors, on their associated spectral curves;

(b) to characterize SL-operators on X, which are natural objects that realize monodromy representations in $PSL_2(\mathbb{C})$, in terms of their apparent singularities and accessory parameters;

(c) to demonstrate that apparent singularities with their accessory parameters are the analogues of Baker-Akhiezer divisors, in particular in their degeneration behaviors.

Let $\mathcal{M}_H(\Lambda)$ be the moduli space of $SL_2(\mathbb{C})$-Higgs bundles on X with the underlying bundles having determinant Λ. Hitchin [35] [36] showed that a generic point in $\mathcal{M}_H(\Lambda)$ corresponds to the isomorphism class of a line bundle satisfying certain conditions on the spectral curve associated to the Higgs bundle, which is a double covering of X embedded into the total space of T^*X. As such line bundles can be represented by effective divisors upon adjusting the degrees [31], theme (a) is hardly surprising.

Our particular way to introduce the so-called Baker-Akhiezer divisors to characterize Higgs bundles, however, has the following important feature. It is known that $\mathcal{M}_H(\Lambda)$ admits a $\mathbb{C}^*$-action which induces a stratification on it: the top open stratum consists of Higgs bundles with stable underlying bundles, and the other strata consisting of Higgs bundles with unstable underlying bundles are determined by the degree of the destabilizing sub-line bundles [34] [35]. The input data that define a Baker-Akhiezer divisor consist of a Higgs bundle $[E, \phi] \in \mathcal{M}_H(\Lambda)$ with a non-degenerate associated spectral curve and a sub-line bundle L of E. In particular, the degree of the Baker-Akhiezer divisor defined by $(L \hookrightarrow E, \phi)$ is equal to $\deg(KL^{-2}\Lambda)$. Therefore, to some extent, Baker-Akhiezer divisors characterize the stratification on $\mathcal{M}_H(\Lambda)$ by their degrees. In fact, for the cases where L is the destabilizing subbundle, these divisors were briefly discussed in the original work of Hitchin [35], and recently revisited in [34] for different purposes.

Furthermore, one can understand how the lower strata compactify the higher ones in terms of Baker-Akhiezer divisors. Namely, one can construct a family of Higgs bundles staying in one stratum that limits to a point in a lower stratum, the corresponding Baker-Akhiezer divisors of which contain no summand equal to the pull-back of a divisor on X but limit to an effective divisor containing a summand of the form $\pi^*(x_0)$ for some $x_0 \in X$. The family of effective divisors on X defined by projecting Baker-Akhiezer divisors from the spectral curves displays a collision of two points to x_0 and a disappearance of these points at the limit.

Another important feature of Baker-Akhiezer divisors is that they have analogues in the natural objects on X that realize monodromy representations $\pi_1 \to PSL_2(\mathbb{C})$, which we will call projective monodromy representations. An SL-operator, which is a collection of local Schrödinger-like differential operators $\{\partial_{z_\alpha}^2 + q_\alpha(z_\alpha)\}$ that satisfy certain compatibility conditions upon transition among the coordinated charts $\{(U_\alpha, z_\alpha)\}$, is such an object [39] [40]. By taking the ratio of two linearly independent local solutions and analytically continuing to all of X, one obtains a projective monodromy representation. In general, to realize a projective monodromy representation, an SL-operators needs to have *apparent singularities*, which are double poles of $q_\alpha(z_\alpha)$ with specific Laurent tails and around which the projective monodromy is trivial. It turns out that apparent singularities with certain accessory parameters can be considered as the analogues of Baker-Akhiezer divisors. For example, with the input data consisting of an irreducible $SL_2(\mathbb{C})$-holomorphic connection and a sub-line bundle of the underlying bundle, one can induce an SL-operator. The positions of the apparent singularities and their accessory parameters are then induced in a very similar way to how Baker-Akhiezer divisors are defined.

The analogy between Baker-Akhiezer divisors and apparent singularities with their accessory parameters extends to their limiting behavior. Namely, a family of projective connections defined by colliding two simple apparent singularities and tuning their accessory parameters in a specified way limits to a projective connection having two less apparent singularities.

In the following, we briefly recall the relevant geometric objects and moduli spaces before summarizing the main results of this thesis.

1.1 Higgs bundles, holomorphic connections and projective connections

$SL_2(\mathbb{C})$-Higgs bundles. An $SL_2(\mathbb{C})$-Higgs bundle is a pair (E,ϕ) where E is a rank-2 holomorphic bundle and $\phi : E \to E\otimes K$, where K is the canonical line bundle of X, is an endormorphism of E twisted by holomorphic 1-forms and has zero trace. We say that ϕ is a Higgs field on E. Such an $SL_2(\mathbb{C})$-Higgs bundle is called stable (semi-stable) if all subbundles $L \hookrightarrow E$ that are ϕ-invariant, i.e. $\phi(L)\subset L\otimes K$, satisfy $2\deg(L) < \deg(\det(E))$ $\big($ respectively, $2\deg(L)\leq \deg(\det(E))$ $\big)$. In other words, an $SL_2(\mathbb{C})$-Higgs bundle (E,ϕ) with $\det(E)=\Lambda$ is stable if and only if either E is stable, or if E is destabilized by L_E then the $\mathcal{O}_X$-linear morphism

$$c_{L_E}(\phi) : L_E \hookrightarrow E \xrightarrow{\phi} E\otimes K \to L_E^{-1}\Lambda K, \tag{1.1}$$

where the last arrow is induced by the quotient of the embedding $L_E \hookrightarrow E$, is non-zero. Generalization of the composition (1.1) to the case where L is any subbundle of E (cf. (1.3)) will be a central object in this thesis.

The Hitchin moduli space $\mathcal{M}_H(\Lambda)$ of stable $SL_2(\mathbb{C})$-Higgs bundles with the underlying bundles having determinant Λ was first constructed by Hitchin [35] [36]. Let [1] $(E,\phi)\in\mathcal{M}_H(\Lambda)$ with $q=\det(\phi)\in H^0(K^2)$ non-degenerate, i.e. the zeroes of q are all simple. Associated to q is the *spectral curve* $S_q \xrightarrow{\pi} X$ defined by solving for eigen-values of ϕ. Central to the work of Hitchin is the spectral correspondence between such a Higgs bundle (E,ϕ) and the isomorphism class of the *eigen-line bundle* $\mathcal{L}$ on S_q, defined by solving for an eigen-subspace of $\pi^*(\phi)$ at each point on S_q. One can recover (E,ϕ) by taking the direct image of $\mathcal{L}\otimes\pi^*(K)$. The Hitchin fibration $h:\mathcal{M}_H(\Lambda)\to H^0(K^2)$ defined by $[E,\phi]\mapsto\det(\phi)$ equips $\mathcal{M}_H(\Lambda)$ with an integrable structure; the functions on $\mathcal{M}_H(\Lambda)$ defined by h together with a choice of basis of $H^0(K^2)$ are called (classical) Hitchin Hamiltonians.

One can define a $\mathbb{C}^*$-action on $\mathcal{M}_H(\Lambda)$ by $\lambda.[E,\phi]=[E,\lambda\phi]$ for $\lambda\in\mathbb{C}^*$. The Białynicki-Birula stratification on $\mathcal{M}_H=\mathcal{M}_H(\Lambda)$ is induced by the $\mathbb{C}^*$-action and is the decomposition $\mathcal{M}_H = W^+_{\mathcal{N}} \sqcup \big(\bigsqcup_d W^+_{N_d}\big)$, where $W^+_{\mathcal{N}}=\{[E,\phi]\in\mathcal{M}_H(\Lambda)\mid E \text{ stable}\}$ is the top stratum, and $W^+_{N_d}=\{[E,\phi]\in\mathcal{M}_H(\Lambda)\mid E \text{ destabilized by } L_E, \deg(KL_E^{-2}\Lambda)=d\}$.

[1] We will abuse the notations writing (E,ϕ) also for the point it defines in $\mathcal{M}_H(\Lambda)$ unless an emphasis on the isomorphisms that identify different Higgs bundles is needed.

$SL_2(\mathbb{C})$-connections. An $SL_2(\mathbb{C})$-holomorphic connection is a pair (F,∇) where F is a holomorphic bundle with $\det(F)\cong\mathcal{O}_X$, and $\nabla: F\to FK$ is a map of sheaves of holomorphic sections that satisfies the Leibniz rule and induces the trivial connection on $\mathcal{O}_X$. Such a holomorphic connection is equivalent to a flat $SL_2(\mathbb{C})$-connection on the underlying smooth bundle and gives rise to monodromy representation in $SL_2(\mathbb{C})$ by developing parallel frames. The automorphism group of the underlying smooth bundle acts on the set of holomorphic connections by conjugation, and we say two holomorphic connections are isomorphic if there is a smooth automorphism relating them. The de Rham moduli space $\mathcal{M}_{dR}$ as a set consists of isomorphism classes of irreducible $SL_2(\mathbb{C})$-holomorphic connections, which are those that leave no holomorphic subbundle invariant. It is known that $\mathcal{M}_{dR}$ is complex smoothex analytic space of dimension $6g-6$ [53] [9].

Simpson [54] defined a natural stratification on $\mathcal{M}_{dR}$ which is analogous to the Białynicki-Birula stratification on $\mathcal{M}_H(\mathcal{O}_X)$. It is defined by embedding both $\mathcal{M}_{dR}$ and $\mathcal{M}_H(\mathcal{O}_X)$ in the Hodge moduli space $\mathcal{M}_{Hod}$ of the so-called λ-connections, and restricting the Białynicki-Birula stratification induced by a natural $\mathbb{C}^*$-action on $\mathcal{M}_{Hod}$ to $\mathcal{M}_{dR}$ and $\mathcal{M}_H(\mathcal{O}_X)$. For $\mathcal{M}_H(\mathcal{O}_X)$, this restriction coincides with the Białynicki-Birula stratification defined on it intrinsically. For $\mathcal{M}_{dR}$, this defines the stratification $\mathcal{M}_{dR}=W^{dR}_{\mathcal{N}}\cup\left(\bigcup_d W^{dR}_{N_d}\right)$, where $W^{dR}_{\mathcal{N}}=\{[F,\nabla]\in\mathcal{M}_{dR}\mid F\text{ stable}\}$ and $W^{dR}_{N_d}=\{[F,\nabla]\in\mathcal{M}_{dR}\mid F\text{ destabilized by }L_F,\deg(KL_F^{-2})=d\}$.

Projective connections, projective structures and SL-operators. Given a representation $\pi_1(X)\to PSL_2(\mathbb{C})$, there are three equivalent types of objects that give rise to the same $PSL_2(\mathbb{C})$-monodromy up to conjugation. Since $PSL_2(\mathbb{C})$ is the automorphism group of $\mathbb{P}^1$, a natural geometric object that realizes this monodromy representation is a fiber bundle with $\mathbb{P}^1$-fibers and locally constant $PSL_2(\mathbb{C})$-valued transition functions, i.e. a *flat* $\mathbb{P}^1$*-bundle*. We call such a flat $\mathbb{P}^1$-bundle together with a choice of global, nonparallel holomorphic section a *projective connection*. The global holomorphic section of a projective connection can be represented in the local parallel frames of the flat bundle on each sufficiently small chart by a local holomorphic function valued in $\mathbb{C}\subset\mathbb{P}^1$. This defines a *projective structure* on X, namely a maximal atlas of coordinate charts the values of which are related by Möbius transformations. An object of the third type, an SL*-operator*, is a collection of local Schrödinger-like differential operators $\{\partial^2_{z_\alpha}+q_\alpha(z_\alpha)\}$ that satisfy certain compatibility conditions upon transition among the coordinate charts $\{(U_\alpha, z_\alpha)\}$: the solutions to these local differential operators transform as local sections of a line bundle of degree $1-g$, such as $K^{-1/2}$, and define via their ratio a corresponding projective structure. There is a natural notion of isomorphism among objects of each of these types.

To realize a generic projective monodromy representation in terms of an object of these types, we will need to include *apparent singularities*, which are certain distinguished points around which the projective monodromy representation is trivial. For a flat $\mathbb{P}^1$-bundle together with a choice of a global section, apparent singularities are precisely where the section is tangential to the local constant leaves (with respect to which the transition functions are locally constant). For a projective structure, apparent singularities are points where the local functions have zero derivative, i.e. where they cannot serve as local coordinates. For an SL-operator $\mathcal{D} = \{\partial_{z_\alpha}^2 + q_\alpha(z_\alpha)\}$, apparent singularities are the double poles of $q_\alpha(z_\alpha)$ with specific Laurent tails. For example, for $x \in U_\alpha$, if

$$q_\alpha(z_\alpha) = -\frac{3}{4(z_\alpha - z_\alpha(x))^2} + \frac{\nu_{x,z_\alpha}}{z_\alpha - z_\alpha(x)} - \nu_{x,z_\alpha}^2 + \mathcal{O}(z_\alpha - z_\alpha(x)), \tag{1.2}$$

then x is called an apparent singularity of $\mathcal{D}$ with multiplicity 1, and $\nu_{x,z_\alpha} \in \mathbb{C}$ is called the *accessory parameter* with respect to the local coordinate z_α. The leading coefficients of the Laurent series at apparent singularities of higher order satisfy higher order algebraic relations. Although these leading coefficients depend on the choice of local coordinates, the algebraic relations they satisfy are invariant upon a change of coordinates. These algebraic constraints guarantee that the ratios of linearly independent solutions to the local differential operators are holomorphic, with their derivatives vanishing to the right order at the apparent singularities.

Although we will occasionally refer to projective connections for geometric meaning, it is the explicit nature of SL-operators that will help us carry out computation and prove results. This approach relies on the fact that, with respect to a coordinate atlas $\{(U_\alpha, z_\alpha)\}$ subordinate to a holomorphic projective structure, i.e. one that has no apparent singularity [2], $\{q_\alpha(z_\alpha)dz_\alpha^2\}$ glue into a meromorphic quadratic differential. Chapter 6 and chapter 7 of this thesis are where this

[2] An example of a holomorphic projective structure is provided by the universal covering of X via the uniformization theorem: the realization of X as a quotient of the upper-half plane induces coordinate charts on X the value of which are related by Möbius transformations.

approach is carried out.

1.2 Summary of main results

Baker-Akhiezer divisors. Consider $(E, \phi) \in \mathcal{M}_H(\Lambda)$ with $q = \det(\phi) \in H^0(K^2)$ non-degenerate. Let us now take a subbundle L of E. On X, consider the composition

$$c_L(\phi) : L \hookrightarrow E \xrightarrow{\phi} E \otimes K \to L^{-1}\Lambda K, \tag{1.3}$$

where the last arrow is induced by the quotient of $L \hookrightarrow E$, and on S_q, consider the composition

$$\pi^*(L) \hookrightarrow \pi^*(E) \to \mathcal{L}^{-1}\pi^*(\Lambda), \tag{1.4}$$

where the last arrow is the quotient of $\mathcal{L} \hookrightarrow \pi^*(E)$. Note that $c_L(\phi) \neq 0$, as otherwise L is ϕ-invariant and the zeroes of $\det(\phi)$ have non-trivial multiplicity. We let $D = \sum_{i=1}^d \tilde{x}_i$ be the involution of the zero divisor of (1.4) and call it the *Baker-Akhiezer divisor* associated to the data $(L \hookrightarrow E, \phi)$ (definition 3.1). In fact, D depends only on the isomorphism class $[L \hookrightarrow E, \phi]$, where isomorphisms of two such data are defined as isomorphisms of the underlying bundles that commute with Higgs fields and embeddings of subbundles. We justify this terminology after the proof of proposition 3.1.

Our first results, proposition 3.1 and theorem 3.7, relate Baker-Akhiezer divisors with zero divisors of (1.3), characterize the eigen-line bundles in terms of Baker-Akhiezer divisors, and establish the correspondence, up to a square-root of $\mathcal{O}_X$, between these divisors and their defining input. We summarize the main points of these results in the following.

THEOREM 1.1. *Let $S_q \xrightarrow{\pi} X$ be the spectral curve associated to a non-degenerate quadratic differential $q \in H^0(K^2)$.*

(i) If D is the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$ with $\det(\phi) = q$, then $\pi(D)$ coincides with the zero divisor of $c_L(\phi)$, and the eigen-line bundle of (E, ϕ) is isomorphic to $\pi^*(LK^{-1}) \otimes \mathcal{O}_{S_q}(D)$.

(ii) The construction of Baker-Akhiezer divisors and remembering the line bundle defines a bi-

jection

$$\left\{ [L \hookrightarrow E, \phi] \;\middle|\; \begin{array}{l} L \text{ a subbundle of } E, \\ \det(E) = \Lambda, \\ \det(\phi) = q \end{array} \right\} \longleftrightarrow \left\{ ([L], D) \;\middle|\; \begin{array}{l} D \text{ effective on } S_q, \text{ contains} \\ \text{no pull-back of divisors on } X, \\ KL^{-2}\Lambda \cong \mathcal{O}_X(\pi(D)) \end{array} \right\}.$$

The map induced by forgetting the subbundle, i.e. $(L \hookrightarrow E, \phi) \mapsto D$*, is a* $2^{2g} : 1$ *map.*

Apparent singularities as analogues of projection of Baker-Akhiezer divisors. Given an $SL_2(\mathbb{C})$-holomorphic connection (F, ∇) and a subbundle $L \hookrightarrow F$, the analogue of (1.3) is the composition

$$c_L(\nabla) : L \hookrightarrow F \xrightarrow{\nabla} F \otimes K \to L^{-1}K, \tag{1.5}$$

A priori this composition is only $\mathbb{C}$-linear since it involves ∇, but since the morphism $F \otimes K \to L^{-1}K$ is induced from the quotient of the embedding $L \hookrightarrow F$, $c_L(\nabla)$ is overall $\mathcal{O}_X$-linear and hence is a section of KL^{-2}. Clearly $c_L(\nabla)$ is non-zero if and only if L is not invariant by ∇; in particular, if (F, ∇) is irreducible then any subbundle L would induce $c_L(\nabla) \neq 0$. In this case, the zero divisor of $c_L(\nabla)$ is an effective divisor $\mathbf{x}$ with $KL^{-2} \cong \mathcal{O}_X(\mathbf{x})$.

By choosing local flat frames of F w.r.t. ∇, one can define a flat $SL_2(\mathbb{C})$-bundle F^∇ and then projectivize to obtain a flat $PSL_2(\mathbb{C})$-bundle $\mathbb{P}(F^\nabla)$ with $\mathbb{P}^1$-fibers. By projectivizing the subbundle $L^\nabla \hookrightarrow F^\nabla$ that corresponds to $L \hookrightarrow F$, which we denote by $L^\nabla \hookrightarrow F^\nabla$, one defines a section $\mathbb{P}(L^\nabla)$ of $\mathbb{P}(F^\nabla)$. The projective monodromy representation of $\mathbb{P}(F^\nabla)$ is the composition of the monodromy representation in $SL_2(\mathbb{C})$ of F^∇ with the projection $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$, Twisting the data $(L \hookrightarrow F, \nabla)$ by a square-root of $\mathcal{O}_X$ and projectivizing would define the same projective connection $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$. We denote by $[L \hookrightarrow F, \nabla]$ the isomorphism class of such data, where isomorphisms are defined as isomorphisms of the underlying bundles that commute with the holomorphic connections and embeddings of subbundles.

A rather explicit description of $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$ is found in its corresponding SL-operator as follows. Suppose that in certain local frames adapted to L, ∇ takes the form $\partial + \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} dz$.

Then one can show that local differential operators of the form $\partial_z^2 + q(z)$ where

$$q(z) = -b(z)c(z) - \left(a(z) - \frac{c'(z)}{2c(z)}\right)^2 - \left(a(z) - \frac{c'(z)}{2c(z)}\right)'$$

define an SL-operator that depends only on (F, ∇) and the embedding $L \hookrightarrow F$. We denote this SL-operator by $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$. Then one can show furthermore that $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$ is equivalent to the projective connection $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$.

Let $\mathcal{R}_{SL_2(\mathbb{C})}$ be the set of conjugacy classes of irreducible monodromy representations in $SL_2(\mathbb{C})$. Let $\mathcal{M}^0_{\mathcal{D}}$, $\mathcal{M}^0_{(P,s)}$ and $\mathcal{M}^0_P$ be the sets of isomorphism classes of SL-operators, projective connections and flat $PSL_2(\mathbb{C})$ bundles with $\mathbb{P}^1$ fibers, respectively, whose projective monodromy representations are irreducible and lift to monodromy representations in $SL_2(\mathbb{C})$. Let $\mathcal{R}^0_{PSL_2(\mathbb{C})}$ be the set of conjugacy classes of projective monodromy representations that lift to those in $\mathcal{R}_{SL_2(\mathbb{C})}$. Denote by $\operatorname{div}(\mathcal{D})$ the effective divisor formed by the apparent singularities of the SL-operator $\mathcal{D}$ counted with multiplicity. The following theorem summarizes important results from subchapter 5.5 and shows how apparent singularities are the analogues of the projection to X of Baker-Akhiezer divisors.

THEOREM 1.2. *(i) Given an irreducible $SL_2(\mathbb{C})$-connection (F, ∇) and a subbundle $L \hookrightarrow F$, div$(\mathcal{D}_{(L \hookrightarrow F, \nabla)})$ is the zero divisor of $c_L(\nabla)$.*

(ii) The following diagram is commutative.

$$\begin{tikzcd}
 & \left\{ \begin{array}{c} [L \hookrightarrow F, \nabla] \mid \\ [F, \nabla] \in \mathcal{M}_{dR} \end{array} \right\} \arrow[r] \arrow[dl, "2^{2g}:1"'] \arrow[d, "2^{2g}:1"] & \mathcal{M}_{dR} \arrow[r] \arrow[d, "2^{2g}:1"] & \mathcal{R}_{SL_2(\mathbb{C})} \arrow[d, "2^{2g}:1"] \\
\mathcal{M}^0_{\mathcal{D}} \arrow[r, "1:1"] \arrow[rrr, bend right] & \mathcal{M}^0_{(P,s)} \arrow[r] & \mathcal{M}^0_P \arrow[r] & \mathcal{R}^0_{PSL_2(\mathbb{C})}.
\end{tikzcd}$$

Here the first two vertical arrows projectivize the corresponding data, the arrows with targets

$\mathcal{R}_{SL_2(\mathbb{C})}$ and $\mathcal{R}^0_{PSL_2(\mathbb{C})}$ evaluate the conjugacy class of the monodromy representations, and the arrows with targets $\mathcal{M}_{dR}$ and $\mathcal{M}^0_P$ forget the subbundles and global section of the respective data. All vertical arrows are surjective, with points in the same fiber, except for the last vertical arrow, differing by a twist by a square-root of $\mathcal{O}_X$.

We note that, analogous to how the degree of Baker-Akhiezer divisors encodes to some extent the stratum a Higgs bundle is contained, the number of apparent singularities, counted with multiplicity, of an SL-operator $\mathcal{D}$ can encode the stratum a holomorphic connection $[F, \nabla] \in \mathcal{M}_{dR}$ is contained if the monodromy representation of F^∇ projects to the projective monodromy representation realized by $\mathcal{D}$. In particular, if $\mathcal{D}$ has fewer than $2g-2$ apparent singularities, then F is strictly unstable.

Accessory parameters as analogues of cotangent fiber coordinates of Baker-Akhiezer divisors. While apparent singularities are analogues of the projection to X of Baker-Akhiezer divisors, their respective accessory parameters are the analogues of the cotangent fiber coordinates of Baker-Akhiezer divisors. To give a precise formulation of this statement, we will need the following genericity condition for an effective divisor $\mathbf{x}$ on X. Let $Q_{\mathbf{x}}$ be the sublinear space of $H^0(K^2)$ consisting of quadratic differentials whose zero divisors are bounded below by $\mathbf{x}$, namely $Q_{\mathbf{x}} \coloneqq \{q \in H^0(K^2) \mid \mathbf{x} \leq \mathrm{div}(q)\} \cup \{0 \in H^0(K^2)\}$. We will say that $\mathbf{x}$ is Q-generic if the dimension of $Q_{\mathbf{x}}$ has the minimal, expected value, namely

$$\dim Q_{\mathbf{x}} = \begin{cases} 3g-3-\deg(\mathbf{x}) & \text{for } \deg(\mathbf{x}) < 3g-3, \\ 0 & \text{for } \deg(\mathbf{x}) \geq 3g-3. \end{cases}$$

PROPOSITION 1.3. *(Proposition 6.5) Suppose* $\deg(\Lambda) - g$ *is odd. Let q_0 be a non-degenerate holomorphic quadratic differential, and $x'_1 + ... + x'_{3g-3}$ be a reduced Q-generic divisor. If in addition there is no exceptional divisor on the spectral curve S_{q_0} projecting to $x'_1 + ... + x'_{3g-3}$, then there exist open neighborhoods $V \subset H^0(K^2)$ of q_0, $U_r \subset X$ of x'_r and an embedding*

$$U_1 \times ... \times U_{3g-3} \times V \longrightarrow \mathcal{M}_H(\Lambda),$$
$$(\vec{x}, q) = (x_1, ..., x_{3g-3}, q) \longmapsto [E_{(\vec{x},q)}, \phi_{(\vec{x},q)}]$$

where $\det(\phi_{(\vec{x},q)}) = q$ *and* $E_{(\vec{x},q)}$ *admits a subbundle* $L_{\vec{x}}$ *with zero divisor of* $c_{L_{\vec{x}}}(\phi_{(\vec{x},q)})$ *being* $x_1 + ... + x_{3g-3}$. *Furthermore, there exist a coordinate* z_r *on* U_r *and an injective map of sets*

$$U_1 \times ... \times U_{3g-3} \times V \longrightarrow \{SL\text{-operators}\},$$
$$(\vec{x}, q) = (x_1, ..., x_{3g-3}, q) \longmapsto \mathcal{D}_{(\vec{x},q)}$$

where $\mathcal{D}_{(\vec{x},q)}$ *has simple apparent singularities* x_1, ..., x_{3g-3} *with respective accessory parameters* ν_1, ..., ν_{3g-3} *satisfying* $\nu_r^2 + q(z_r(x_r)) = 0$ *for* $r = 1, ..., 3g-3$.

Let $\mathbf{x} = x_1 + ... + x_d$ be a reduced effective divisor. Fix a point $[p_1, ..., p_d] \in (T^*X)^{[d]} := (T^*X)^d/S_d$, the d-fold symmetric product of the total space of the cotangent bundle of X, that projects to $\mathbf{x}$. Then the space of spectral curves that pass through p_1, ..., p_d and admits these points as effective divisors and the space of SL-operators having $\mathbf{x}$ as their apparent singularities and same respective accessory parameters are both affine spaces [3] modeled on $Q_{\mathbf{x}}$. In addition, corollaries 3.8 and 5.8 together show how analogously constrained the input data are in this case.

PROPOSITION 1.4. *(i) Two Higgs bundles* $[E_1, \phi_1]$, $[E_2, \phi_2] \in \mathcal{M}_H(\Lambda)$ *define the same point in* $(T^*X)^{[d]}$ *via the construction of Baker-Akhiezer divisor only if* $E_1 \cong E_2 \otimes N$ *for some* N *with* $N^2 \cong \mathcal{O}_X$.

(ii) Let $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be* SL*-operators whose apparent singularities are all simple and projective monodromy representations have lifts to irreducible monodromy representations in* $SL_2(\mathbb{C})$. *Then* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *have the same apparent singularities and respective accessory parameters if and only if* $\mathcal{D}_1 \sim \mathcal{D}_{(L_1 \hookrightarrow F_1, \nabla_1)}$ *and* $\mathcal{D}_2 \sim \mathcal{D}_{(L_2 \hookrightarrow F_2, \nabla_2)}$ *for some* $L_1 \cong L_2 \otimes N$, $F_1 \cong F_2 \otimes N$ *with* $N^2 \cong \mathcal{O}_X$.

The fact that the destabilizing subbundle of a strictly unstable bundle has a unique up to scaling embedding has the following analogous consequences for Higgs bundles and SL-operators (cf. corollaries 3.10 and 5.4).

PROPOSITION 1.5. *(i) On a non-degenerate spectral curve, there is no exceptional divisor of*

[3] If $\mathbf{x}$ has points of multiplicity 2, the same statement would hold for SL-operators (cf. proposition 5.7) but not for Higgs bundles and spectral curves. In this sense SL-operators encode more information than spectral curves and Higgs bundles.

degree $< 2g-2$, i.e. an effective divisor of degree $< 2g-2$ is equivalent to no other effective divisors.

(ii) Two projective connections with the same irreducible projective monodromy representation that has a lift to $SL_2(\mathbb{C})$ and the same divisor of apparent singularities of degree $< 2g-2$ are isomorphic.

Double points in Baker-Akhiezer divisors and double apparent singularities. The following propositions (propositions 4.3 and 7.6) show that one can form a double point in Baker-Akhiezer divisor and a double apparent singularity by, respectively, colliding two simple points of Baker-Akhiezer divisors and two simple apparent singularities.

PROPOSITION 1.6. *(Proposition 4.3) Let $(E, \phi) \in \mathcal{M}_H(\Lambda)$ with the associcated non-degenerate spectral curve $S \xrightarrow{\pi} X$ and L be a subbundle of E such that $c_L(\phi)$ has a double zero at $x_0 \in X$ which is not a branch point of S. Let D be the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$ and $\tilde{x}_0$ be the point with multiplicity 2 in D with $\pi(\tilde{x}_0) = x_0$. Let (U, z) be a coordinate neighborhood of x_0, where $z(x_0) = 0$, U is simply connected and contains no branch point of S. Then there exist a family of Higgs bundles $\{(E_u, \phi_u)\}_{u \in z(U)}$ and a family of line bundles $\{L_u\}_{u \in z(U)}$ of the same degree as L parameterized by U such that*

(i) $[L_0] = [L]$ in Jac$_{\deg(L)}(X)$ and $(E_0, \phi_0) = (E, \phi)$ in $\mathcal{M}_H(\Lambda)$;

(ii) for all $u \in z(U)$, E_u admits L_u as a subbundle;

(iii) for all $u \neq 0$, the Baker-Akhiezer divisor of $(L_u \hookrightarrow E_u, \phi_u)$ is $D - 2\tilde{x}_0 + \tilde{x}_+ + \tilde{x}_-$, where $\tilde{x}_\pm$ lie in the component of $\pi^{-1}(U)$ containing $\tilde{x}_0$ and are such that $z(\pi(\tilde{x}_\pm)) = \pm u$.

Furthermore, these families define embeddings $U \hookrightarrow \mathcal{M}_H(\Lambda)$ and $U \hookrightarrow$ Jac$_{\deg(L)}(X)$.

The following proposition is the analogue for SL-operators of proposition 1.6 provided that certain Q-genericity conditions are met with regard to the choice of the collision site x_0. We will also need to tune the accessory parameters of the colliding simple apparent singularities for them to form a double one at the limit.

PROPOSITION 1.7. *(Proposition 7.6) Let $\mathcal{D}$ be an SL-operator with div$(\mathcal{D}) = 2x_0 + x_3 + ... + x_d$ being Q-generic and $d \leq 3g - 3$. Then there exists a coordinate neighborhood (U, z) of x_0, where $U \subset U'$ and $z(x_0) = 0$, and a family of SL-operators $\{\mathcal{D}_u\}_{u \in z(U)}$ parameterized by U such that*

(i) $\mathcal{D}_0 = \mathcal{D}$;

(ii) *for $u \neq 0$, $\mathcal{D}_u$ has simple apparent singularities at x_3, ..., x_d and $x_\pm \in U$ with $z(x_\pm) = \pm u$;*

(iii) *for $u \neq 0$, the accessory parameters $\nu_\pm(u)$ of $x_\pm$ w.r.t. the local coordinate z, as functions of u, have simple poles at $u = 0$ and Laurent expansions $\nu_\pm(u) = \mp \frac{1}{4u} + \nu_0^{\mathcal{D}} \pm \nu' u...$, where $2\nu_0^{\mathcal{D}}$ is the accessory parameter of the double apparent singularity x_0 of $\mathcal{D}$.*

Furthermore, this family defines via taking monodromy a holomorphic map $U \to$ Hom$(\pi_1, PSL_2(\mathbb{C}))$, which is injective for $d < 2g - 2$.

We expect that if the projective monodromy representation of $\mathcal{D}$ defines a point generic enough in $\mathcal{R}^0_{PSL_2(\mathbb{C})}$, then such a family of SL-operators defines an embedding $U \hookrightarrow \mathcal{R}^0_{PSL_2(\mathbb{C})}$ which lifts to an embedding $U \hookrightarrow \mathcal{R}_{SL_2(\mathbb{C})}$.

In addition, one can check that if $\{(L_u \hookrightarrow F_u, \nabla_u)\}$ is a family of holomorphic connections together with subbundles such that $c_{L_u}(\nabla_u)$ has zero divisor $x_\pm + x_3 + ... + x_d$ for $u \neq 0$ and $2x_0 + x_3 + ... + x_d$ for $u = 0$, then the family of SL-operators $\{\mathcal{D}_u := \mathcal{D}_{(L_u \hookrightarrow F_u, \nabla_u)}\}$ has $x_\pm(u)$ as simple apparent singularities with accessory parameters of the required form (cf. example 7.2). Such a family $\{(L_u \hookrightarrow F_u, \nabla_u)\}$ can be obtained by applying the so-called "conformal limit" [9] to a family of Higgs bundles and sub-bundles provided by Proposition 1.6.

Reduction of the degree of Baker-Akhiezer divisors. Let $[E, \phi] \in \mathcal{M}_H(\Lambda)$ with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$, and L be a subbundle of E. The following proposition (proposition 4.2) shows, given a point $x_0 \in X$ that is not a branch point of S, the existence of families of Higgs bundles parameterized by a neighborhood U of x_0 such that Higgs bundles corresponding to points in $U \setminus \{x_0\}$ admit subbundles of degree $\deg(L) - 1$, but their limit at x_0 is (E, ϕ). Such a family defines an embedding $U \hookrightarrow \mathcal{M}_H(\Lambda)$.

PROPOSITION 1.8. *(Proposition 4.2) Let $(E, \phi) \in \mathcal{M}_H(\Lambda)$ with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$, and L be a subbundle of E. Given $x_0 \in X$ not a branch point of S, let (U, z)*

be a coordinate neighborhood of x_0, where $z(x_0) = 0$, U is simply connected and contains no branch point of S. Then there exist a family of Higgs bundles $\{(E_u, \phi_u)\}_{u \in z(U)}$ and a family of line bundles $\{L_u\}_{u \in z(U)}$ of degree $\deg(L) - 1$ parameterized by U such that

(i) $[L_0] = [L \otimes \mathcal{O}_X(-x_0)]$ *in Jac*$_{\deg(L)-1}(X)$ *and* $(E_0, \phi_0) = (E, \phi)$ *in* $\mathcal{M}_H(\Lambda)$;

(ii) for all $u \in z(U)$, (E_u, ϕ_u) has S as its spectral curve;

(iii) for all $u \neq 0$, E_u admits L_u as a subbundle;

(iv) for all $u \neq 0$, the Baker-Akhiezer divisor of $(L_u \hookrightarrow E_u, \phi_u)$ is $D + \tilde{x}_+ + \tilde{x}_-$, where D is the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$ and $\tilde{x}_\pm$ lie in different distinct components of $\pi^{-1}(U)$ with $z(\pi(\tilde{x}_\pm)) = \pm u$.

Furthermore, these families define embeddings $U \hookrightarrow \mathcal{M}_H(\Lambda)$ and $U \hookrightarrow$ Jac$_{\deg(L)-1}(X)$.

Proposition 1.8 in particular shows how one could limit to a Białynicki-Birula stratum from a higher stratum. For simplicity we have limited to families of Higgs bundles staying on one fixed smooth Hitchin fiber.

Given a family $\{[E_u, \phi_u]\}_{u \in z(U)}$ constructed by proposition 1.8, one can use the $\mathbb{C}^*$-action on $\mathcal{M}_H(\Lambda)$ to define a family of Higgs bundles $\{[F_u, \psi_u]\}_{u \neq 0, u \in z(U)}$ parameterized by $U \setminus \{x_0\}$, where $[F_u, \psi_u] \coloneqq u.[E_u, \phi_u]$ for each $u \neq 0$. In case E_u is unstable for all $u \in z(U)$, proposition 4.4 shows that the limit $[F_0, \psi_0] \coloneqq \lim_{u \to 0} [E'_u, \phi'_u]$ lies in the nilpotent cone [4] and admits $L_0 \cong L \otimes \mathcal{O}_X(-x_0)$ as a subbundle. Hence, the $u \to 0$ limit of $[F_u, \psi_u]$ stays in the same Białynicki-Birula stratum, in contrast to the limit $[E, \phi]$ of $[E_u, \phi_u]$. In the sense that $[E_u, \phi_u] = u^{-1}.[F_u, \psi_u]$, one might regard the limiting behavior to a lower stratum of $\{[E_u, \phi_u]\}_{u \neq 0, u \in z(U)}$ as a blow-up of the limiting behavior that remains in the same stratum of $\{[F_u, \psi_u]\}_{u \neq 0, u \in z(U)}$. [5]

Reduction of the number of apparent singularities. The analogy between Baker-Akhiezer divisors and apparent singularities and their accessory data extends to the families of the corresponding objects, the limits of which respectively have Baker-Akhiezer divisors and divisors of apparent singularities of lower degree. The following proposition (proposition 7.9) is the analogue

[4]The nilpotent cone is the fiber over 0 of the Hitchin fibration $h : \mathcal{M}_H(\Lambda) \to H^0(K^2)$.

[5]The limit $[F_0, \psi_0]$ provides an example of a theorem on the so-called very-stable Higgs bundles, those $\mathbb{C}^*$-fixed points in $\mathcal{M}_H(\Lambda)$ with associated upward flows intersecting the nilpotent cone only once, recently studied by [34].

of proposition 1.8. We will again need some Q-genericity conditions for the "collision site", and tune the accessory parameters corresponding to the colliding apparent singularities for them to "disappear" at the limit.

PROPOSITION 1.9. *(Proposition 7.9) Let $\mathcal{D}$ be an SL-operator with div$(\mathcal{D}) = x_3 + ... + x_d$ for $d \leq 3g-3$, and x_0 be a point on X such that $2x_0 + x_3 + ...x_d$ is Q-generic. Then there exists a coordinate neighborhood (U, z) of x_0 and a family of SL-operators $\{\mathcal{D}_u\}_{u \in z(U)}$ parameterized by U such that*

(i) $\mathcal{D}_0 = \mathcal{D}$;

(ii) for $u \neq 0$, $\mathcal{D}_u$ has simple apparent singularities at x_3, ..., x_d and $x_\pm \in U$ with $z(x_\pm) = \pm u$;

(iii) for $u \neq 0$, the accessory parameters $\nu_\pm(u)$ of $x_\pm$ w.r.t. the local coordinate z, as functions of u, have simple poles at $u = 0$ and Laurent expansions $\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2)$.

Furthermore, this family defines via taking monodromy a holomorphic map $U \to$ Hom$(\pi_1, PSL_2(\mathbb{C}))$, which is injective for $d < 2g-2$.

We expect that such a family would generically define an embedding $U \hookrightarrow \mathcal{R}^0_{PSL_2(\mathbb{C})}$ which lifts to an embedding $U \hookrightarrow \mathcal{R}_{SL_2(\mathbb{C})}$.

There exists a surgery of projective structures called bubbling, which takes a projective structure and a path on the surface that contains no apparent singularities as the input. This surgery cuts open the underlying Riemann surface along the chosen path and glues in a copy of $\mathbb{P}^1$ which is also cut open along the image of the path under the local function defined by the projective structure [8]. The output is another projective structure which induces the same projective monodromy representation, but is subordinate to a different Riemann surface, i.e. a different complex structure for the underlying smooth surface, and has two extra apparent singularities. Now, for $u \in z(U)$ let $\mathcal{U}_u$ be the projective structure corresponding to $\mathcal{D}_u$ constructed in proposition 1.9. We found evidences suggesting that, for $u \neq 0$, $\mathcal{U}_u$ is the output of a bubbling that produces $x_\pm(u)$ as the two extra apparent singularities.

1.3 Outlooks into geometric Langlands

One of the main motivations for the projects leading to this thesis is to understand explicitly the geometric Langlands correspondence for the case where the Lie group is $G = SL_2(\mathbb{C})$. Certain aspects of the geometric Langlands correspondence have been made explicitly in the cases where the Riemann surface has genus zero [22] or one [19] and has punctures, and we would like to emulate this success for the cases where the Riemann surface is compact and has genus ≥ 2.

We now briefly describe a formulation of the geometric Langlands correspondence and the explicit formulation in the genus zero case, before discussing how one can expect a generalization of the strategy and how the results of this thesis fit in this scheme. Several of the constructions and concepts that are not directly relevant to the content of this thesis will not be explained in details; we will refer the readers to the references in this case.

Quantization of the Hitchin system and geometric Langlands. The formulation of the geometric Langlands correspondence of our interest predicts a correspondence between flat $PSL_2(\mathbb{C})$-bundles on a Riemann surface X and objects called *Hecke-eigensheaves* on the moduli stack $\mathrm{Bun}_{SL_2(\mathbb{C})}$ of $SL_2(\mathbb{C})$-bundles on X [5] [23] [6]. Using techniques inspired by conformal field theories, Beilinson-Drinfeld [5] proved [7] this correspondence for a class of distinguished flat bundles called *opers* [4] which in our case are precisely the holomorphic projective structures, i.e. those without apparent singularities. They "quantized" the Hitchin system by constructing the quantum Hitchin Hamiltonians, which are certain differential operators that act on a line bundle on $\mathrm{Bun}_{SL_2(\mathbb{C})}$, commute with each other and have the classical Hitchin Hamiltonians as their symbols. One then can argue that the sought-after Hecke eigen-sheaves can be encoded in the eigen-functions of these quantum Hitchin Hamiltonians, with their eigen-values encoding the corresponding opers [23]. In this sense, the quantization of the Hitchin system is an essential ingredient to proving the geometric Langlands correspondence in the special case of holomorphic projective structures.

[6] For $G = SL_2(\mathbb{C})$, the Langlands dual group is ${}^LG = PSL_2(\mathbb{C})$. In general, the geometric Langlands correspondence is between Hecke eigensheaves on the moduli stack $\mathrm{Bun}_{SL_2(\mathbb{C})}$ of G-bundles and flat ${}^LG$-bundles. For brevity, we only summarize the geometric Langlands correspondence for the case $G = SL_2(\mathbb{C})$.

[7] Beilinson-Drinfeld proved the geometric Langlands correspondence for opers assuming that G is a connected simply-connected simple Lie group.

Geometric Langlands on punctured spheres. Variants of the geometric Langlands correspondence exist for the cases $g = 0$ and 1, i.e. Riemann sphere and torus with punctures. The case of the Riemann sphere with N punctures is more explicit, since one can essentially work with a coordinate z for the whole Riemann surface. In this case, an open dense set of $\mathrm{Bun}_{SL_2(\mathbb{C})}$ is isomorphic to $(\mathbb{P}^1)^{N-3}$ [22]. If the punctures are at $z = z_1, \ldots, z_N$, a projective structure or equivalently an SL-operator without apparent singularities takes the form

$$\partial_z^2 + \sum_{r=1}^{N} \frac{c_r}{(z-z_r)^2} + \sum_{r=1}^{N} \frac{\nu_r}{z-z_r}. \tag{1.6}$$

With some proper setup, the Hecke-eigensheaf corresponding to such a holomorphic projective structure can be encoded by the quantum Hitchin eigen-functions $\Psi_{\boldsymbol{\nu}}$ with eigen-values equal to the residues $\boldsymbol{\nu} = (\nu_1, ..., \nu_N)$ of the "potential" in (1.6) [22],

$$H_r \Psi_{\boldsymbol{\nu}} = \nu_r \Psi_{\boldsymbol{\nu}}, \qquad\qquad r = 1, ..., N. \tag{1.7}$$

Here $\Psi_{\boldsymbol{\nu}} = \Psi_{\boldsymbol{\nu}}(y_1, ..., y_{N-3})$ is dependent on the coordinates y_1, ..., y_{N-3} of $(\mathbb{P}^1)^{[N-3]} \subset \mathrm{Bun}_{SL_2(\mathbb{C})}$, and the quantum Hitchin Hamiltonians $H_r = H_r(\partial_{y_1}, ..., \partial_{y_{N-3}}, y_1, ..., y_{N-3})$ are second-order differential operators.

Sklyanin's separation of variables. It is known that in this case the quantum Hitchin Hamiltonians H_r can be identified with the Hamiltonians of the Gaudin model, which is a quantum integrable spin chain model [29]. Sklyanin [55] discovered a trick to rewrite (1.7) to a more solvable form: one can show that there exists an integral transform

$$\Phi_{\boldsymbol{\nu}}(x_1, .., x_{N-3}) = \int dy_1 ... dy_{N-3} \mathcal{K}(y_1, ..., y_{N-3}, x_1, ..., x_{N-3}) \Psi_{\boldsymbol{\nu}}(y_1, ..., y_{N-3}), \tag{1.8}$$

where $\mathcal{K}(y_1, ..., y_{N-3}, x_1, ..., x_{N-3})$ is an explicit integration kernel [25] [52], that satisfies

$$\left(\partial_{x_s}^2 + \sum_{r=1}^{N} \frac{c_r}{(x_s - z_r)^2} + \sum_{r=1}^{N} \frac{\nu_r}{x_s - z_r} \right) \Phi_{\boldsymbol{\nu}}(x_1, .., x_{N-3}) = 0, \qquad s = 1, ..., N-3. \tag{1.9}$$

Since (1.9) is a decoupled system of differential equations, this rewriting [8] of the eigen-value problem (1.7) is called the separation of variables for the quantum Hitchin system in this case. Remarkably, it also makes explicit the geometric Langlands correspondence: the integral transform $\Phi_{\boldsymbol{\nu}}(x_1, .., x_{N-3})$ satisfies the very differential equation defining the projective structure to which the Hecke-eigensheaf encoded in $\Psi_{\boldsymbol{\nu}}(y_1, ..., y_{N-3})$ corresponds to.

Drinfeld's construction of geometric Langlands. In [13], Drinfeld gave a construction of the geometric Langlands correspondence for the case $GL_2(\mathbb{C})$ over the function field of a curve over $\mathbb{F}_q$. Frenkel [22] has reinterpreted Drinfeld's construction in a geometric context and drew comparison with Sklyanin's separation of variables for the case $SL_2(\mathbb{C})$.

The rough idea is as follows. Let X be a compact Riemann surface of genus $g \geq 2$. Let d be a positive integer and $n = d - 2g + 2$. Let $\mathcal{N}^n$ be the moduli space of rank-2 bundles of degree n, and $\mathcal{N}^n_{2,1}$ the moduli space of rank-2 bundles of degree d admitting $\mathcal{O}_X$ as subbundles, i.e. a point of $\mathcal{N}^n_{2,1}$ is an equivalence class of an extension of the form

$$0 \to \mathcal{O}_X \to F \to \Lambda \to 0, \qquad \deg(\Lambda) = n, \tag{1.10}$$

modulo scaling [9]. Let $j^\vee : \mathcal{N}^n_{2,1} \to \mathrm{Jac}_n(X)$ be the map that sends a point in $\mathcal{N}^n_{2,1}$ that can be put in the form (1.10) to $[\Lambda] \in \mathrm{Jac}_n(X)$. It is a projection with the fiber over $[\Lambda]$ being $\mathbb{P}H^1(\Lambda^{-1})$. Its dual projection is the map $j : X^{[d]} \to \mathrm{Jac}_n(X)$, where $X^{[d]} = X^d/S_d$ is the d-fold symmetric product of X, that sends an effective divisor D of degree d on X to $[K^{-1} \otimes \mathcal{O}_X(D)]$. The fiber of j over $[\Lambda]$ is $\mathbb{P}H^0(K\Lambda)$.

$$\begin{tikzcd} & \mathcal{N}^n_{2,1} \arrow[dl, dashed, "i"'] \arrow[dr, "j^\vee"] & & X^{[d]} \arrow[dl, "j"'] \\ \mathcal{N}^n & & \mathrm{Jac}_n(X) & \end{tikzcd} \tag{1.11}$$

Let $i : \mathcal{N}^n_{2,1} \to \mathcal{N}^n$ be the rational map that picks out $[F]$ from (1.10). Its fiber over $[F]$ consists of sections of F that are nowhere-vanishing. If $n \geq 2g - 1$, then the image of i defines

[8] There are in total $N - 3$ $\mathcal{O}_{\mathrm{Bun}_{SL_2(\mathbb{C})}}$-linearly independent quantum Hitchin Hamiltonians.

[9] Scaling the embeddings of $\mathcal{O}_X$ defines different equivalence classes of extensions, but the same subbundle in F.

an open dense set [10] in $\mathcal{N}^n$.

Let $n \geq 2g-1$. Now, given a monodromy representation $\check{\rho} : \pi_1 \to GL_2(\mathbb{C})$, let $F_{\check{\rho}}$ be a flat rank-2 bundle that realizes ρ. Then one can construct a perverse sheaf $F_{\check{\rho}}^{(d)} = (\pi_* F_{\check{\rho}}^{\boxtimes d})^{S_d}$, where π is the quotient $X^d \to X^{[d]}$, on $X^{[d]}$. There exists a transformation called Radon transform [7] [44] [41] that, as j and $j^\vee$ are dual projective fibrations, induces a sheaf $\mathcal{G}_{\check{\rho}}^n$ on $\mathcal{N}_{2,1}^n$ from $F_{\check{\rho}}^{(d)}$. $\mathcal{G}_{\check{\rho}}^n$ can be shown [13] to be an irreducible perverse sheaf which is constant along the generic fiber of i, and hence is the pull-back of a perverse sheaf $\mathcal{F}_{\check{\rho}}^n$ on $\mathcal{N}^n$.

Frenkel in [22] sketched how one should expect certain analogue of $\mathcal{F}_{\check{\rho}}^n$ for the case $SL_2(\mathbb{C})$ induces the Hecke-eigensheaf corresponding to the projective monodromy representation ρ : $\pi_1 \to PSL_2(\mathbb{C})$ defined by $\check{\rho}$. In particular, Frenkel showed that in the genus zero case, Sklyanin's separation of variables (1.8) is precisely a concrete realization of the passage from perverse sheaves on $X^{[d]}$ defined by monodromy representations in $SL_2(\mathbb{C})$ to Hecke-eigensheaves that are geometric Langlands counterparts of the induced projective monodromy representations.

It is suggested in [22] [52] that, for higher genus cases, one can emulate the success in genus zero in understanding more explicitly the geometric Langlands correspondence by finding the analogue of Sklyanin's "separation of variables" $(y_1, ..., y_{N-3}) \to (x_1, ..., x_{N-3})$. The idea is to find a generalization of Sklyanin's trick that would concretely realize Drinfeld's idea.

Classical separation of variables. As a first step in this approach to geometric Langlands, one can observe that, in the genus zero case, there is a change of variables of the classical Hitchin system that inspires the integral transform (1.8). One looks at the expression of the Higgs fields in a local frame, picks out the lower-left component and its zeroes on the punctured Riemann sphere. The lower-left component in this case is a function on the punctured Riemann surface that is linear in the cotangent fiber coordinates of $(\mathbb{P}^1)^{N-3} \subset \mathrm{Bun}_{SL_2(\mathbb{C})}$ that are conjugate to y_1, ..., y_{N-3}, and the zeroes of the lower-left component are the variables x_1, ..., x_{N-3}. The change of variables from $(y_1, ..., y_{N-3})$ together with their conjugate variables to $(x_1, ..., x_{N-3})$ together with their conjugate variables at the classical level then induces a natural analogue at the quantum level,

[10] A holomorphic rank-2 bundle on X of degree $\geq 2g-1$ always has sections by Riemann-Roch theorem. The image of i then only misses the bundles that have no nowhere-vanishing sections, which form a positive-codimensional loci in $\mathcal{N}^n$ since their Baker-Akhiezer divisors induced by these sections are of positive codimension on the corresponding spectral curves (cf. appendix C).

which leads to the rewriting (1.9) of (1.7).

Returning to this thesis, on a compact Riemann surface X of genus ≥ 2, observe that in local frames adapted to a subbundle $L \hookrightarrow E$ the lower-left component of a Higgs field E is precisely $c_L(\phi)$ as defined in (1.3). By theorem 1.1, its zeroes are the projections to X of the Baker-Akhiezer divisors defined by $(L \hookrightarrow E, \phi)$. We therefore expect that, if it is indeed possible to have an explicit reformulation of the quantum Hitchin eigen-functions in terms of the $SL_2(\mathbb{C})$-operator provided by the geometric Langlands correspondence, the projection to X of Baker-Akhiezer divisors will play a role similar to the "separated variables" x_1, ..., x_{N-3}.

In fact, some results supporting this expectation have been obtained in an ongoing project [12]. The basis of these results, which is a separation of variables of classical Hitchin systems, is reported in our paper [10]. This classical separation of variables amounts to a (rational) symplectomorphism $T^*\mathcal{N}_{\Lambda,n} \to (T^*X)^{[d]}$, where $\mathcal{N}_{\Lambda,n}$ is the moduli spaces of pairs (rank-2 bundle with fixed determinant Λ, subbundle of degree n). One notes that these moduli spaces are contained in a diagram obtained by adapting Drinfeld's diagram (1.11) to the case $G = SL_2(\mathbb{C})$, i.e. fixing the determinant Λ,

$$\begin{tikzcd} & \mathcal{N}_{\Lambda,n} \arrow[dl, dashed, "i"'] \arrow[dr, "j"] & & \mathcal{N}^{\vee}_{\Lambda,n} \arrow[dl, "j^{\vee}"'] \arrow[r] & X^{[m]} \\ \mathcal{N}_{\Lambda} & & \mathrm{Pic}^d & & \end{tikzcd} . \qquad (1.12)$$

In appendix C, we have also sketched such a change of variables at the classical level for the case $G = GL_2(\mathbb{C})$.

Generalization to higher ranks. Laumon [44] suggested a generalization of Drinfeld's construction of geometric Langlands correspondence to higher ranks via a diagram that is similar to (1.11) and (1.12) but extends further to the left. One achieves this by again modeling moduli spaces of rank-r holomorphic bundles in terms of the spaces of extensions by rank-$(r-1)$ holomorphic bundles, and then applying a chain of Radon transforms [11].

It is natural to expect that this can be realized by an analogue of Sklyanin's separation of variables in higher ranks [22], and in particular, we expect that the projection to X of a general-

[11]The challenge, as pointed out in [22], is to prove that the Radon transforms are irreducible perverse sheaves.

ization of Baker-Akhiezer divisors in higher rank cases should also play the roles of the separated variables. Hausel-Hitchin [34] recently studied a variant of this generalization of Baker-Akhiezer divisors for higher ranks, albeit for different purposes.

Analytic geometric Langlands. We would like to point out an example of success in approaching geometric Langlands from the point of view of separation of variables. In [52], by supposing that there exists an integral transform of the form (1.8) that satisfies (1.9), Teschner was led to propose that the geometric Langlands counterparts of holomorphic projective connections with monodromy representations in $PSL_2(\mathbb{R})$ (up to conjugation) can be encoded by single-valued quantum Hitchin eigen-functions. It is natural to regard the single-valued quantum Hitchin Hamiltonians as the analogues of automorphic forms in the original Langlands program.

The set of holomorphic projective connections with monodromy representations in $PSL_2(\mathbb{R})$ is discrete in the moduli space of projective connections, and hence a fit interpretation of this correspondence is that one has imposed a natural quantization condition in addition to the quantization of the Hitchin system constructed by Beilinson-Drinfeld. Etingof-Frenkel-Kazhdan [15] [16] [17] further supported this interpretation of a quantization condition by showing that, for the genus zero case, the single-valued quantum Hitchin eigen-functions are automatically square-integrable, which is a necessary condition from the physicist's point of view.

Generalization to projective structures with apparent singularities. There is an unpublished construction by Beilinson-Drinfeld sketched by Frenkel in [23] which claims a generalization of the correspondence between holomorphic projective structures and Hecke-eigensheaves to general projective structures, i.e. those with apparent singularities. We expect that this can be achieved by generalizing the separation of variables techniques including the poles defining apparent singularities of the corresponding SL-operators. In particular, we expect that our work in this thesis showing the analogy between Baker-Akhiezer divisors and apparent singularities with their accessory parameters will help to make this approach explicit. As pointed out in [23], one challenge would be to show that two projective structures that are not equivalent, in particular having different apparent singularities, but yield equivalent projective monodromy representations would yield equivalent Hecke-eigensheaves [12]. One necessary condition would be that the corresponding quantum Hitchin eigen-functions have the same conjugacy classes of monodromy around their singular loci in $\mathrm{Bun}_{SL_2(\mathbb{C})}$, which consist of bundles that admit nilpotent Higgs fields.

We believe that understanding Higgs bundles and projective structures in terms of divisors that play analogous roles, i.e. Baker-Akhiezer divisors and apparent singularities as this thesis demonstrates, is the first key step in this approach to geometric Langlands.

[12] Frenkel in section 9.6 of [23] pointed out the challenge that, a priori, the constructed Hecke-eigensheaves are dependent on the choices of Borel reductions of the flat $PSL_2(\mathbb{C})$-bundle that satisfies the oper condition on the complement of a finite set of X. Such a Borel reduction is simply a section of the $PSL_2(\mathbb{C})$-bundle, and the finite set on which the oper condition fails consists of the zeroes of its differential. By theorem 1.2, different Borel reductions of a flat $PSL_2(\mathbb{C})$-bundle, whose projective monodromy representation can be lifted to $SL_2(\mathbb{C})$, are the projectivization of different subbundles of a flat $SL_2(\mathbb{C})$-bundle whose monodromy representations in $SL_2(\mathbb{C})$ is a lift.

Chapter 2

Moduli spaces of stable bundles and Higgs bundles

Throughout this thesis, X is a compact Riemann surface of genus $g \geq 2$. In this chapter we review the relevant results of moduli spaces of bundles and Higgs bundles, with an emphasis on the natural stratification on them.

2.1 Moduli spaces of bundles

Given a rank-2 holomorphic bundle E, we say a subbundle M of E is a maximal subbundle of E if $\deg(M) \geq \deg(L)$ for all other subbundles L of E. A rank-2 holomorphic bundle E with $\det(E) = \Lambda$ is called stable if

$$s(E) \coloneqq \deg(\Lambda M^{-2}),$$

where M is a maximal subbundle of E, is positive. In other words, for E being stable $s(E)$ has 0 as a strict lower bound. It follows by a theorem of Nagata [45] that $s(E) \leq g$. Hence, since $s(E) \equiv \deg(\Lambda)$ mod 2, $\max s(E) = g$ if $g \equiv \deg(\Lambda)$ mod 2 and $\max s(E) = g - 1$ otherwise.

Given a line bundle Λ on X, the moduli space of stable bundles $\mathcal{N}_\Lambda$ with fixed determinant Λ on X as a set consists of isomorphisms classes of such stable bundles. It is known that $\mathcal{N}_\Lambda$ is a smooth complex projective variety of complex dimension $3g - 3$ [49] [47]. For $s \equiv \deg(\Lambda)$ mod 2 and in the range $1 \leq s \leq g - 2$, let $\mathcal{N}_\Lambda(s) = \{[E] \in \mathcal{N}_\Lambda \mid s(E) = s\}$. Then $\mathcal{N}_\Lambda(s)$ is an irreducible algebraic variety of dimension $2g + s - 2$. The closure in $\mathcal{N}_\Lambda$ of these algebraic varieties define a natural stratification on $\mathcal{N}_\Lambda$ which we call the Segre stratification [42].

It is known that if $s(E) = g$, then the set of its maximal subbundles is of dimension 1 [42]. On the other hand, a generic bundle E with $s(E) = g - 1$ only has a finite number of maximal subbundles, and a generic bundle E with $s(E) < g - 1$ has only one maximal subbundle. If however E is strictly unstable, i.e. $s(E) < 0$, then E has a unique maximal subbundle L_E.

Regardless of the value of $s(E)$, if M is a maximal subbundle of E then its embedding into E is unique up to a scaling, i.e. $h^0(M^{-1}E) = 1$.

2.2 Moduli spaces of Higgs bundles

An $SL_2(\mathbb{C})$-Higgs bundle is a pair (E, ϕ) where E is a holomorphic rank-2 bundle and $\phi \in H^0(\text{End}_0(E) \otimes K)$ is a trace-less holomorphic endomorphism of E twisted by holomorphic one-forms.

It is known that the moduli space $\mathcal{M}_H(\Lambda)$ of $SL_2(\mathbb{C})$-stable Higgs bundles with the underlying bundles having determinant Λ has dimension $6g - 6$ [36]. Since tensoring a Higgs bundle with a line bundle keeps the parity of the degree of the determinant and leaves the Higgs fields intact in a covariant way, $\mathcal{M}_H(\Lambda) \cong \mathcal{M}_H(\Lambda')$ if and only if $\deg(\Lambda) - \deg(\Lambda')$ is even. In other words, the moduli spaces of $SL_2(\mathbb{C})$-Higgs bundles are of two isomorphism classes, defined by whether $\deg(\Lambda)$ is odd or even. We will often write $\mathcal{M}_H \equiv \mathcal{M}_H(\Lambda)$ when it is not necessary to emphasize the choice of Λ. We will also often call both (E, ϕ) and its isomorphism class a Higgs bundle, and abuse the notation by simply writing (E, ϕ) for $[(E, \phi)] \in \mathcal{M}_H$ unless an emphasis on the fact that distinct Higgs bundles can be identified via isomorphisms is called for.

2.2.1 Underlying bundles of stable Higgs bundles

For a stable bundle E of determinant Λ, any traceless Higgs field $\phi \in H^0(\text{End}_0(E) \otimes K)$ defines a stable Higgs bundle (E, ϕ) and hence a point $[E, \phi] \in \mathcal{M}_H$. One can show that a Higgs field $\phi \in H^0(\text{End}_0(E) \otimes K)$ defines a cotangent vector on the moduli space $\mathcal{N} \equiv \mathcal{N}_\Lambda$ of stable bundles with determinant Λ. Hence $T^*\mathcal{N} \subset \mathcal{M}_H$. This embedding is in fact open dense. In addition, $\mathcal{M}_H$ can be equipped with a natural symplectic structure which restricts to the canonical one on $T^*\mathcal{N}$.

Not all rank-2 unstable bundles form stable Higgs bundles. We refer to [35] for a complete classification.

PROPOSITION 2.1. [35] (E, ϕ) *is stable if and only if one of the following conditions holds*

(i) E *is stable,*

(ii) E *is strictly semi-stable and* $g > 2$*,*

(iii) $E \cong L \otimes U$ *is strictly semi-stable and* $g = 2$*, where the rank-2 bundle* U *is either decomposable or an extension of* $\mathcal{O}_X$ *by itself,*

(iv) E *is destabilized by subbundle* $L_E \hookrightarrow E$ *with* $h^0(KL_E^{-2}\Lambda) > 1$*,*

(v) $E = L_E \oplus L_E^{-1}\Lambda$ *with* $h^0(KL_E^{-2}\Lambda) = 1$.

For our purpose, it will be instructive to understand the cases of strictly unstable bundles in details. We now review the representation of Higgs bundles in terms of extensions of line bundles before discussing these cases.

Higgs bundles in terms of extensions of line bundles. Suppose E can be realized as an extension of a line bundle $L^{-1}\Lambda$ by L, i.e. there exists a s.e.s.

$$0 \to L \to E \to L^{-1}\Lambda \to 0. \tag{2.1a}$$

An extension of this form is equivalent to the data of a subbundle $L \hookrightarrow E$, or equivalently a collection of the transition functions of E of the form

$$(E)_{\alpha\beta} = \begin{pmatrix} l_{\alpha\beta} & l_{\alpha\beta}\epsilon_{\alpha\beta} \\ 0 & l_{\alpha\beta}^{-1}\lambda_{\alpha\beta} \end{pmatrix} \tag{2.1b}$$

where $l_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ are respectively transition functions of L and Λ. A Higgs field ϕ on E then defines the composition

$$c_L(\phi) : L \hookrightarrow E \xrightarrow{\phi} E \otimes K \to L^{-1}\Lambda K. \tag{2.1c}$$

In other words, the embedding $L \hookrightarrow E$ defines a map $c_L : H^0\left(\mathrm{End}_0(E) \otimes K\right) \to H^0\left(KL^{-2}\Lambda\right)$. We might later simply write $c \equiv c_L(\phi)$ when the input data $(L \hookrightarrow E, \phi)$ are clear in context. Concretely, if over an open set $U_\alpha \subset X$ the Higgs field takes the local form

$$\phi_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & -a_\alpha \end{pmatrix} \tag{2.1d}$$

in certain local frames adapted $L \hookrightarrow E$, then $\{c_\alpha\}$ glue into the global section $c_L(\phi)$ of $KL^{-2}\Lambda$. Note that $c_L(\phi) = 0$ if and only if L is ϕ-invariant.

The space of extension classes of the form (2.1a) is canonically isomorphic to $H^1(L^2\Lambda^{-1})$, which is dual to $H^0(KL^{-2}\Lambda)$ via Serre duality. One could show that for all $\phi \in H^0\left(\mathrm{End}_0(E) \otimes K\right)$ the pairing via Serre duality of the class $[E] \in H^1(L^2\Lambda^{-1})$ of the extension (2.1a) and $c_L(\phi) \in$

$H^0(KL^{-2}\Lambda)$ satisfies

$$\langle [E], c_L(\phi)\rangle = 0. \tag{2.2}$$

In other words, the image of c_L is contained in the hyperplane

$$\ker([E]) = \{c \in H^0(KL^{-2}\Lambda) \mid \langle [E], c\rangle = 0\}$$

defined by $(2.1a)$. See appendix B for the proof of this Serre duality constraint and a detailed analysis of the image of c_L.

Underlying unstable bundles. Let E be a strictly unstable bundle with determinant Λ and a destabilizing subbundle $L_E \hookrightarrow E$, i.e. $\deg(L_E^{-2}\Lambda) < 0$. If $L \hookrightarrow E$ is another subbundle, then $\deg(L) \leq \deg(L_E^{-1}\Lambda)$: otherwise the composition $L \hookrightarrow E \to L_E^{-1}\Lambda$ is zero, which is impossible by the assumption that L and L_E are different subbundles of E. It follows that L_E is the unique subbundle that destabilizes E, and furthermore its embedding into E is unique up to scaling. Hence a Higgs field ϕ on E defines a stable Higgs bundle if and only if $c_{L_E}(\phi) \in H^0(KL_E^{-2}\Lambda)$ is nonzero.

If $h^0(KL_E^{-2}\Lambda) > 1$, which is the generic case when $\deg(KL_E^{-2}\Lambda) > g$, then one can show that there are Higgs fields on E that define stable Higgs bundles, i.e. $\mathrm{im}(c_{L_E})$ contains nonzero elements of $H^0(KL_E^{-2}\Lambda)$. Indeed, one can show that E^*LK is isomorphic to the bundle of traceless Higgs fields preserving L_E (cf. appendix B), and hence c_{L_E} fits in the induced long exact sequence

$$0 \to H^0\left(E^*L_EK\right) \to H^0\left(\mathrm{End}_0(E)\otimes K\right) \overset{c_{L_E}}{\to} H^0\left(KL_E^{-2}\Lambda\right) \to H^1\left(E^*L_EK\right) \to \ldots \tag{2.3}$$

It follows from Serre duality and the fact that L_E has a unique up to scaling injection into E that $h^1\left(E^*L_EK\right) = h^0(L_E^{-1}E) = 1$. Hence if $h^0(KL_E^{-2}\Lambda) > 1$ then $\mathrm{im}(c_{L_E})$ has positive dimension.

On the other hand, if $h^0(KL_E^{-2}\Lambda) = 1$, which is the generic case when $\deg(KL_E^{-2}\Lambda) \leq g$, then it follows from the Serre duality constraint (2.2) that (E,ϕ) is stable if and only if $c_{L_E}(\phi) \neq 0$ and $E = L_E \oplus L_E^{-1}\Lambda$.

2.2.2 Spectral correspondence and integrable structure

The Hitchin fibration $h : \mathcal{M}_H \to H^0(K^2)$ associates to the isomorphism class of a Higgs bundle (E, ϕ) the quadratic differential $q = \det(\phi)$. A generic Hitchin fiber is isomorphic to an abelian variety, namely the Prym variety of the associated spectral curve. This endows $\mathcal{M}_H$ with the structure of an algebraic integrable system [36].

To see this, first note that associated to a quadratic differential $q = \det(\phi)$ is a "spectral curve" S_q embedded in the total space of T^*X. The spectral curve encodes the eigen-values of the Higgs field: concretely, if u is the coordinate of an open set $U \subset X$, v the fiber coordinate of the restriction of T^*X to U and $\phi(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & -a(u) \end{pmatrix}$ locally, then locally S_q is defined by

$$v^2 + q(u) = v^2 - a(u)^2 - b(u)c(u) = 0. \tag{2.4}$$

The morphism $S_q \xrightarrow{\pi} X$ induced by $T^*X \to X$ is a $2:1$ covering that branches at the zeroes of $\det(\phi)$. The involution σ of S_q interchanges points corresponding to the eigenvalues $v = \pm(-q(u))^{1/2}$ of $\phi(u)$. We say a quadratic differential q and its associated spectral curve S_q are non-degenerate if the zeroes of q are all simple. In this case, S_q is a smooth compact Riemann surface of genus $\tilde{g} = 4g - 3$, and in particular $\pi^*(K)$ has a canonical section defined by v that vanishes precisely at the ramification divisor R_q of S_q. A spectral curve S_q is called degenerate when q has zeroes of non-trivial multiplicity; the most degenerate case is $q = 0$, and we call the fiber $h^{-1}(0)$ the nilpotent cone.

Eigen-line bundles. In the non-degenerate case, up to isomorphism, a Higgs bundle (E, ϕ) with $q = \det(\phi)$ corresponds to a sub-line bundle $\mathcal{L}$ of $\pi^*(E)$ on S_q, defined as the kernel of the morphism $(\pi^*(\phi) - v) : \pi^*(E) \to \pi^*(E \otimes K)$ [35]. In other words at each point $p \equiv (u, v) \in S_q$, $\mathcal{L}$ as a subbundle of $\pi^*(E)$ is defined by the eigen-subspace of $\pi^*(\phi)(p)$ with the eigen-value v. Since $\pi^*(\phi)(p)$ also has $-v$ as its eigen-value, which defines $\sigma(p) \equiv (u, -v)$, one can similarly define a sub-line bundle of $\pi^*(E)$ with these eigen-values, which is nothing but $\sigma^*(\mathcal{L})$. The line bundles $\mathcal{L}$ and $\sigma^*(\mathcal{L})$ are called the eigen-line bundles of (E, ϕ); they coincide at the ramification points of $S_q \xrightarrow{\pi} X$ and satisfy

$$\mathcal{L} \otimes \sigma^*(\mathcal{L}) \cong \pi^*(\Lambda \otimes K^{-1}). \tag{2.5}$$

Conversely, given a line bundle $\mathcal{L}$ satisfying (2.5), one can show that the direct image $\pi_*(\mathcal{L} \otimes \pi^*(K))$ is a rank-2 bundle, whose determinant is isomorphic to Λ and pull-back to S_q contains $\mathcal{L}$ and $\sigma^*(\mathcal{L})$ as subbundles. A Higgs field can be constructed from the fact that, at $p \in S_q$, $\pi^*(\phi) \mid_p$ acts on $\mathcal{L} \mid_p$ and $\sigma^*(\mathcal{L}) \mid_p$ with eigen-values corresponding to p and $\sigma(p)$ respectively. By construction, the eigen-line bundles of this Higgs bundle are $\mathcal{L}$ and $\sigma^*(\mathcal{L})$.

Prym variety and integrable structure. Condition (2.5), which relates the fixed determinant of the underlying rank-2 bundles on X and the eigen-line bundles on S_q, in fact implies that the Hitchin fiber $h^{-1}(q)$ is isomorphic to the Prym variety of S_q if q has only simple zeroes.

The Prym variety Prym(S_q) is the subset of the Jacobian of S_q defined as the kernel of the norm map $\pi_* : \text{Jac}_0(S_q) \to \text{Jac}_0(X)$ that sends the equivalence class $[D]$ of degree-0 divisor D on S_q to $[\pi(D)]$. It is known that Prym$(S_q) = \{[L] \in \text{Jac}_0(S_q) \mid L \otimes \sigma^*(L) \cong \mathcal{O}_{S_q}\}$ if we regard Jac$_0(S_q)$ as the set of isomorphism classes of degree-0 line bundles on S_q. Then choosing any line bundle $\mathcal{L}_0$ that satisfies condition (2.5) allows us to define an isomorphism $h^{-1}(q) \xrightarrow{\sim}$ Prym(S_q) by $[E, \phi] \mapsto [\mathcal{L}_0{}^{-1} \otimes \mathcal{L}_{(E,\phi)}]$ where $\mathcal{L}_{(E,\phi)}$ is the eigen-line bundle of (E, ϕ). Since a line bundle satisfies (2.5) if and only if it is the eigen-line bundle of a Higgs bundle having S_q as its spectral curve, we have defined such an isomorphism simply by identifying a point in $h^{-1}(q)$ with $0 \in$ Prym(S_q).

REMARK 2.1. For S_q non-degenerate, pulling-back line bundles from X to S_q defines an embedding $\pi^* : \text{Jac}_0(X) \hookrightarrow \text{Jac}_0(S_q)$. The intersection of Prym(S_q) and the copy of Jac$_0(X)$ is the discrete set of 2^{2g} points $\{\pi^* L \mid L^{\otimes 2} \cong \mathcal{O}_X\}$.

2.2.3 Natural stratification

$\mathbb{C}^*$-fixed points. The Hitchin moduli space admits a $\mathbb{C}^*$-action defined as $\lambda.[E, \phi] = [E, \lambda\phi]$ for $\lambda \in \mathbb{C}^*$. The fixed point locus $\mathcal{M}_H^{\mathbb{C}^*}$ of the $\mathbb{C}^*$-action is a subset of the nilpotent cone $h^{-1}(0)$. Clearly $\mathcal{M}_H^{\mathbb{C}^*}$ contains the zero section $\mathcal{N} \cong \{(E, 0) \mid E \in \mathcal{N}\}$ of $T^*\mathcal{N} \subset \mathcal{M}_H$. The other

$\mathbb{C}^*$-fixed points have also been classified: for E destabilized by M, a Higgs bundle $(E, \phi) \in \mathcal{M}_H$ is $\mathbb{C}^*$-fixed if and only if $(E, \phi) \sim (E_M, \phi_c)$ where

$$E_M = M \oplus M^{-1}\Lambda, \qquad \phi_c = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \tag{2.6}$$

with some nonzero $c \in H^0(KM^{-2}\Lambda)$ [35][34]. We write

$$\mathcal{M}_H^{\mathbb{C}^*} = \mathcal{N} \cup \big(\underset{d}{\cup} N_d\big), \qquad N_d = \{[E_M, \phi_c] \mid \deg(c) = d\}, \tag{2.7}$$

where $\mathcal{N}$ and N_d are connected components of $\mathcal{M}_H^{\mathbb{C}^*}$, with $0 \leq d \leq 2g-2$ if $\deg(\Lambda)$ is even and $1 \leq d \leq 2g-1$ if $\deg(\Lambda)$ is odd.

Upward flows of $\mathbb{C}^*$-fixed points. An important property of the Hitchin fibration is properness and equivariance with respect to the $\mathbb{C}^*$-action on $\mathcal{M}_H$ [48] [34]. This implies that any point $[E, \phi] \in \mathcal{M}_H$ has a well-defined limit $\lim\limits_{\lambda \to 0}[E, \lambda\phi] \in \mathcal{M}_H$. For $[E, \phi]$ with E stable, $(E, \lambda\phi)$ is also stable for any $\lambda \in \mathbb{C}$ and hence $\lim\limits_{\lambda \to 0}[E, \lambda\phi] = [E, 0]$. For E unstable with L_E the destabilizing subbundle, since $(E, 0)$ is not a stable Higgs bundle the limit is different: we have in fact $\lim\limits_{\lambda \to 0}[E, \lambda\phi] = [E_M, \phi_c]$ as in (2.6) with $M = L_E$ and $c = c_{L_E}(\phi) = c_M(\phi_c)$. We can illustrate this by, for each $\lambda \in \mathbb{C}^*$, considering the automorphism $g_\lambda = \begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}$ of E and noting its actions on the transition functions $(E)_{\alpha\beta}$ of E in local frames adapted to L_E as

$$g^{-1} \begin{pmatrix} (L_E)_{\alpha\beta} & (L_E)_{\alpha\beta}\epsilon_{\alpha\beta} \\ 0 & (L_E)_{\alpha\beta}(\Lambda)_{\alpha\beta} \end{pmatrix} g = \begin{pmatrix} (L_E)_{\alpha\beta} & \lambda(L_E)_{\alpha\beta}\epsilon_{\alpha\beta} \\ 0 & (L_E)_{\alpha\beta}(\Lambda)_{\alpha\beta} \end{pmatrix}, \tag{2.8}$$

which at the limit $\lambda \to 0$ provides the transition functions of E_M. The action of g on the Higgs field $\lambda\phi$ is

$$g^{-1}(\lambda\phi)g = g^{-1} \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & -\lambda a \end{pmatrix} g = \begin{pmatrix} \lambda a & \lambda^2 b \\ c & -\lambda a \end{pmatrix},$$

which at the limit $\lambda \to 0$ provides ϕ_c.

We note that, regardless of whether E is stable or not, if $L_E \hookrightarrow E$ is a maximal subbundle of E, then it is also the maximal subbundle of the underlying bundle at the limit $\lim\limits_{\lambda\to 0}[E,\phi]$. For a $\mathbb{C}^*$-fixed point α, we say $W_\alpha \coloneqq \{[E,\phi]\in\mathcal{M}_H \mid \alpha = \lim\limits_{\lambda\to 0}[E,\lambda\phi]\}$ is the upward flow of α.

EXAMPLE 2.2. For E stable and $\alpha = (E,0)$, $W_{[E,0]}$ is the cotangent fiber $T^*_{[E]}\mathcal{N} \subset T^*\mathcal{N} \subset \mathcal{M}_H$. If $\deg(\Lambda)$ is even, upon choosing a spin structure $K^{1/2}$, at the other extreme is the upward flow of $\alpha = \left[K^{1/2}\oplus K^{-1/2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$. This upward flow intersects each Hitchin fiber $h^{-1}(q)$ at precisely one point defined by $\left(K^{1/2}\oplus K^{-1/2}, \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}\right)$ and is called the Hitchin section. There are 2^{2g} such Hitchin sections corresponding to 2^{2g} choices of $K^{1/2}$.

It is known that W_α is Lagrangian for any $\alpha \in \mathcal{M}_H^{\mathbb{C}^*}$, which is the generalization of the fact that $W_{[E,0]}$ is Lagrangian in $T^*\mathcal{N} \subset \mathcal{M}_H$ for E stable [9]. The decomposition

$$\mathcal{M}_H = W_{\mathcal{N}} \sqcup \big(\bigsqcup_d W_{N_d}\big),$$

where $W_F \coloneqq \bigcup\limits_{\alpha\in F} W_\alpha$ for each connected component F of $\mathcal{M}_H^{\mathbb{C}^*}$, is called the Białynicki-Birula stratification of $\mathcal{M}_H$. Note that $W_{\mathcal{N}} \cong T^*\mathcal{N}$ inherits a natural stratification from the Segre stratification on $\mathcal{N}$ and its union with $\bigsqcup\limits_d W_{N_d}$ is a natural generalization.

Chapter 3

Baker-Akhiezer divisors

In this chapter we introduce the notion of Baker-Akhiezer divisors on non-degenerate spectral curves. The input data to define these divisors consist of a Higgs bundles with non-degenerate associated spectral curve and an injection of a line bundle to the underlying rank-2 bundle. While the terminology for these divisors is new and inspired by the literature on integrable systems, the usage of these divisors is not entirely new. Hitchin in his original paper [35] already characterized Higgs bundles with underlying unstable bundles in terms of these divisors, and the recent work [34] of Hausel-Hitchin also made extensive use of them in particular in their analysis for different purposes.

3.1 Definitions and basic properties

Explicit definition of Baker-Akhiezer divisors. Let (E, ϕ) be a semi-stable Higgs bundle with an associated non-degenerate quadratic differential q, and $L \hookrightarrow E$ a subbundle. Then $c = c_L(\phi) \in H^0(KL^{-2}\Lambda)$ defined as in (2.1c) is nonzero: otherwise the zeroes of q will have multiplicity. Consider its zero divisor $\mathrm{div}(c) = \sum_i^d x_i$. At each x_i, equation (2.4) for the spectral curve $S_q \xrightarrow{\pi} X$ reduces to $v^2 - a(u(x_i))^2 = 0$. If x_i is not a branch point, then the two points in $\pi^{-1}(x_i)$ are unambiguously labeled by $v = \pm a(u(x_i))$; in this case let $\tilde{x}_i$ be the point defined by $v = -a(u(x_i))$. If x_i is a branch point then let $\tilde{x}_i$ be the ramification point $\pi^{-1}(x_i)$. We define

$$D \coloneqq \sum_{i=1}^{d} \tilde{x}_i. \tag{3.1}$$

Clearly D is dependent only on the data $(L \hookrightarrow E, \phi)$. We say D is the *Baker-Akhiezer divisor* of this data. We will write $D = D\,(L \hookrightarrow E, \phi)$ when we want to emphasize this dependence, otherwise we will simplify the notation. Inspired by [35] and [34], in definition 3.1 we will characterize these divisors in an invariant way and include the case where the injection $L \to E$ has zeroes and hence does not define a subbundle.

REMARK 3.1. Since $q = \det(\phi)$ has only simple zeroes, if a branch point of $S_q \xrightarrow{\pi} X$ is contained in $\mathrm{div}(c)$ then it must have multiplicity 1. The corresponding ramification point then has multiplicity 1 in the Baker-Akhiezer divisor D. Hence by construction D contains no part equal to the pull-back of a divisor on X [13].

The following proposition describes the eigen-line bundles $\mathcal{L}$ and $\sigma^*(\mathcal{L})$ of (E, ϕ) in terms of L and D. The result has a straightforward generalization to the case $GL_2(\mathbb{C})$. Hausel-Hitchin [34] has noted similar results in the case where E is unstable.

PROPOSITION 3.1. *Let (E, ϕ) be an $SL_2(\mathbb{C})$-Higgs bundle with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$. Let L be a subbundle of E and D the Baker-Akhiezer divisor of the data $(L \hookrightarrow E, \phi)$. Then*

$$\mathcal{L} \cong \pi^* \left(K^{-1}L\right) \otimes \mathcal{O}_S(D), \qquad \sigma^*(\mathcal{L}) \cong \pi^* \left(K^{-1}L\right) \otimes \mathcal{O}_S(\sigma(D)). \tag{3.2}$$

Proof. We will abuse the notations by using the same notations to denote the local functions on X and their pull-backs on S; this in particular applies to components of ϕ and transition functions of E. In the local frames of $\pi^*(E)$ adapted to the pull-back of (2.1a) from X to S, one can check that local sections of the form $\begin{pmatrix} v + a(u) \\ c(u) \end{pmatrix}$ are eigen-vectors of $\pi^*\phi$ with eigen-value v, and hence are local sections of $\mathcal{L} \hookrightarrow \pi^*(E)$. As we transit from one component in $\pi^{-1}(U_\alpha)$ to one intersecting component in $\pi^{-1}(U_\beta)$, these local sections transform as

$$\begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix} \mapsto (E)^{-1}_{\alpha\beta} \begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix} = \begin{pmatrix} l_{\alpha\beta} & l_{\alpha\beta}\epsilon_{\alpha\beta} \\ 0 & l^{-1}_{\alpha\beta}\lambda_{\alpha\beta} \end{pmatrix}^{-1} \begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix}. \tag{3.3}$$

Noting the transformation of the Higgs field (B.2), we can rewrite (3.3) as

$$1. \begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix} \mapsto l^{-1}_{\alpha\beta} k_{\alpha\beta} \begin{pmatrix} v_\beta + a_\beta \\ c_\beta \end{pmatrix} = l_{\beta\alpha} k^{-1}_{\beta\alpha} \begin{pmatrix} v_\beta + a_\beta \\ c_\beta \end{pmatrix}. \tag{3.4}$$

where $k_{\alpha\beta}$ is the transition function of K, and $v_\alpha = k_{\alpha\beta} v_\beta$ (since they are fiber coordinates of

[13] The pull-back to S of a branch point on X, regarded as a divisor on X, takes multiplicity into account and so has multiplicity 2.

K). Note that $\begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix}$ vanishes only at D, and hence can serve as a local frame of $\mathcal{L}$ if U_α contains no point of D. In other words, away from neighborhoods of $\tilde{x}_i$, $\mathcal{L}$ has the same transition functions as $\pi^*\left(LK^{-1}\right)$.

To get a local frame of $\mathcal{L}$ on a component of $\pi^{-1}(U_\alpha)$ containing $\tilde{x}_i$, we can quotient out from $\begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix}$ the minimum of zero multiplicities of $v_\alpha + a_\alpha$ and c_α at $\tilde{x}_i$. If $x_i \in U_\alpha$ is not a branch point, (2.4) implies that this is the multiplicity of x_i in div(c). If $x_i \in U_\alpha$ is a branch point and has multiplicity 1 in div(c), then $\tilde{x}_i$ is a simple zero of $v + a_\alpha \equiv v + \pi^*(a_\alpha)$ and a double zero of $c_\alpha \equiv \pi^*(c_\alpha)$ on S. In either case, we can quotient out from $\begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix}$ precisely the multiplicity of x_i in div(c) to construct a local frame of $\mathcal{L}$. This explains the correction $\mathcal{O}_S(D)$ to $\pi^*\left(LK^{-1}\right)$ in (3.2). $\square$

REMARK 3.2. Since $\begin{pmatrix} v_\alpha + a_\alpha \\ c_\alpha \end{pmatrix}$ resembles Baker-Akhiezer functions in the integrable system literature [2] we are inspired to associate the terminology "Baker-Akhiezer" to its zero divisor D.

EXAMPLE 3.3. Let q be a non-degenerate quadratic differential. Recall that the intersection of the Hitchin fiber $h^{-1}(q)$ with the Hitchin section corresponding to the spin structure $K^{1/2}$ is defined by $\left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}\right)$. The Baker-Akhiezer divisors of the data defined by this Higgs bundle and taking $K^{1/2}$ and $K^{-1/2}$ as subbundles are respectively the trivial divisor and the ramification divisor R_q on $S_q \xrightarrow{\pi_q} X$. Since $\mathcal{O}_{S_q}(R_q) \cong \pi_q^*(K)$, it follows from (3.2) that the eigen-line bundle is isomorphic to $\pi_q^*(K^{-1/2})$ either way.

Following the discussion on isomorphisms between generic Hitchin fibers and Prym varieties, one can define the isomorphism $I_{q,K^{1/2}} : h^{-1}(q) \xrightarrow{\sim} \text{Prym}(S_q)$ that identifies $\left[K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}\right]$ with $0 \in \text{Prym}(S_q)$. If $[E, \phi] \in h^{-1}(q)$ and D is the Baker-Akhiezer divisor of some data $(L \hookrightarrow E, \phi)$, then $I_{q,K^{1/2}}$ sends $[E, \phi]$ to the isomorphism class of the line bundle $\pi_q^*(LK^{-1/2}) \otimes \mathcal{O}_{S_q}(D)$. In particular, the intersection of $h^{-1}(q)$ with another Hitchin section corresponding to another spin structure $K^{1/2}\mathcal{O}_X^{1/2}$ is mapped to the isomorphism class of $\pi^*(\mathcal{O}_X^{1/2})$. Hence $I_{q,K^{1/2}}$

maps points on the 2^{2g} Hitchin sections surjectively to $\mathrm{Prym}(S_q) \cap \pi^*(\mathrm{Jac}_0(X))$.

Formal definition of Baker-Akhiezer divisors. We now give an invariant and slightly more general definition of Baker-Akhiezer divisors. This characterization of these divisors has featured in [35] [34].

Suppose (E, ϕ) is a Higgs bundle with non-degenerate spectral curve $S \xrightarrow{\pi} X$. The eigen-line bundle $\mathcal{L}$ of (E, ϕ) is a subbundle of $\pi^*(E)$ and hence defines an extension

$$0 \to \mathcal{L} \to \pi^*(E) \to \mathcal{L}^{-1}\pi^*(\Lambda) \to 0. \tag{3.5}$$

Let $L \to E$ be an injection which possibly has zeroes. We will in particular denote by "$L \hookrightarrow E$" an injection which has no zero, i.e. an embedding that makes L into a subbundle of E. Consider the composition

$$\pi^*(L) \to \pi^*(E) \to \mathcal{L}^{-1}\pi^*(\Lambda). \tag{3.6}$$

The support of the zero divisor of this composition consists of the pull-back of the support of the zero divisor of $L \to E$ and points where $\pi^*(L)$ coincides with $\mathcal{L}$ as subbundles of $\pi^*(E)$.

Suppose $L \hookrightarrow E$ has no zero, and hence E can be realized as an extension of the form (2.1a). We claim that the zero divisor of (3.6) is $\sigma(D)$ where D is defined as in (3.1). Indeed, if a local frame of $\mathcal{L}$ takes the form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in some local frame of $\pi^*(E)$ adapted to the pull-back of (2.1a), then an $SL_2(\mathbb{C})$-change of local frames of $\pi^*(E)$ from one adapted to $\pi^*(L) \hookrightarrow \pi^*(E)$ to one adapted to $\mathcal{L} \hookrightarrow \pi^*(E)$ can take the form $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}^{-1} = \begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix}$. One then can take $\begin{pmatrix} y_2 \\ -x_2 \end{pmatrix}$ as a local frame of $\pi^*(L)$ in certain local frame of $\pi^*(E)$ adapted to $\mathcal{L} \hookrightarrow \pi^*(E)$, and so locally the composition (3.6) can be modeled as $1 \mapsto \begin{pmatrix} y_2 \\ -x_2 \end{pmatrix} \mapsto -x_2$. It then follows from the explicit construction of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in the proof of proposition 3.1 that indeed formula (3.1) gives the involution of the zero divisor of (3.6).

DEFINITION 3.1. Given a Higgs bundle (E, ϕ) with non-degenerate spectral curve $S \xrightarrow{\pi} X$ and an injection $L \to E$, *the Baker-Akhiezer divisor* associated to these data is the involution of the zero divisor of the composition $\pi^*(L) \to \pi^*(E) \to \mathcal{L}^{-1}\pi^*(\Lambda)$.

The cases where $L \to E$ has zeroes is a straightforward generalization. Indeed, if $L \to E$ has B as its zero divisor, then there exists a subbundle $L(B) := L \otimes \mathcal{O}_X(B) \hookrightarrow E$ such that its composition with the canonical injection of sheaves $L \xrightarrow{s_B} L(B)$ defines $L \to E$. The Baker-Akhiezer divisors of $(L \to E, \phi)$ is equal to $\pi^*(B)$ plus that of $(L(B) \hookrightarrow E, \phi)$, with the latter containing no part equal to the pull-back of a divisor on X (cf. remark 3.1). The proof of the following proposition, which generalizes proposition 3.1, is straightforward.

PROPOSITION 3.2. *Let D be the Baker-Akhiezer divisor of $(L \to E, \phi)$ on a non-degenerate spectral curve $S \xrightarrow{\pi} X$. Then*

(a) D contains $\pi^*(B)$ for some effective divisor B on X if and only if $L \to E$ vanishes at B, counted with multiplicity. In particular, D contains no part equal to the pull-back of a divisor on X if and only if L is a subbundle of E, and in this case D is given by (3.1);

(b) the eigen-line bundle $\mathcal{L}$ of (E, ϕ) is isomorphic to $\pi^*\left(LK^{-1}\right) \otimes \mathcal{O}_S(D)$;

(c) D satisfies $\mathcal{O}_X(\pi(D)) \cong KL^{-2}\Lambda$, where $\Lambda = \det(E)$.

Anti-symmetrization of Baker-Akhiezer divisors. The following proposition shows that, while the construction of Baker-Akhiezer divisors depends not only on Higgs bundles but also on injections from line bundles, the anti-symmetrization of these divisors are invariants of the Higgs bundles themselves.

PROPOSITION 3.3. *If D and D' are Baker-Akhiezer divisors of (E, ϕ) on $S \xrightarrow{\pi} X$ induced respectively by injections $L \to E$ and $L' \to E$, then $D - \sigma(D) \sim D' - \sigma(D')$. In particular, $D - \sigma(D)$ represents $(\sigma^*(\mathcal{L}))^{-1}\mathcal{L}$ where $\mathcal{L}$ is the eigen-line bundle of (E, ϕ).*

Proof. Applying (3.2) to express $\mathcal{L}$ and $\sigma^*(\mathcal{L})$ each in terms of D and D', one can check that

$$\mathcal{O}_S\left(D - \sigma(D)\right) \cong \left(\sigma^*(\mathcal{L})\right)^{-1}\mathcal{L} \cong \mathcal{O}_S\left(D' - \sigma(D')\right). \tag{3.7}$$

$\square$

REMARK 3.4. The anti-symmetrization $D - \sigma(D)$ of a Baker-Akhiezer divisor D on S, which represents the line bundle $(\sigma^*(\mathcal{L}))^{-1}\mathcal{L}$, can be expressed in another way. Following example 3.3, consider the isomorphism $I_{q,K^{1/2}} : h^{-1}(q) \xrightarrow{\sim} \text{Prym}(S_q)$ defined by identifying $0 \in \text{Prym}(S_q)$ with the intersection of $h^{-1}(q)$ with the Hitchin section corresponding to a spin structure $K^{1/2}$. For (E, ϕ) a Higgs bundle with $\det(\phi) = q$ and D its Baker-Akhiezer divisor induced by an injection $L \to E$, we have $I_{q,K^{1/2}}([E,\phi]) = [\pi_q^*(LK^{-1/2}) \otimes \mathcal{O}_{S_q}(D)]$ and

$$2I_{q,K^{1/2}}([E,\phi]) = [\pi_q^*\left(L^2K^{-1}\right) \otimes \mathcal{O}_{S_q}(2D)] = [D - \sigma(D)] \tag{3.8}$$

where we have used property (c) of proposition 3.2. It is in the sense that (3.8) is an invariant of (E, ϕ) that we may claim that the isomorphism $I_{q,K^{1/2}}$ is somewhat "canonical" (there are still 2^{2g} such "canonical" isomorphisms corresponding to 2^{2g} distinct spin structures on X though).

3.2 Inverse construction

In the following we will show that, given the data (q, D) where q is a quadratic differential with simple zeroes and D an effective divisor on the spectral curve $S_q \xrightarrow{\pi} X$, we can construct the data $(L \to E, \phi)$ that defines D as its Baker-Akhiezer divisor. The solutions to this inverse problem are not unique, since twisting one solution with a line bundle defines another. For solutions that define Higgs bundle in the same moduli space $\mathcal{M}_H(\Lambda)$, it will be clear shortly that they differ only by a twist by a square-root of the trivial line bundle.

It is instructive to see first the existence and uniqueness up to isomorphism of the "normalized" solutions, i.e. $L = \mathcal{O}_X$, and in the case where the injection is an embedding, i.e. $L = \mathcal{O}_X$ is a subbundle, via an explicit construction. A more abstract proof can be found in the discussion following theorem 8.1 in [35].

PROPOSITION 3.4. *Let q be a non-degenerate quadratic differential and $D = \tilde{x}_1 + ... + \tilde{x}_d$ an effective divisor on $S_q \xrightarrow{\pi} X$ that does not contain the pull-back of an effective divisor on X. Then* *there exist a rank-2 bundle E' that arises as an extension of $\Lambda' = K^{-1} \otimes \mathcal{O}_X(\pi(D))$ by $\mathcal{O}_X$,*

$$0 \to \mathcal{O}_X \to E' \to \Lambda' \to 0, \tag{3.9}$$

and a holomorphic Higgs field ϕ' on E' such that D is the Baker-Akhiezer divisor of $(\mathcal{O}_X \hookrightarrow E', \phi')$. The Higgs bundle (E', ϕ') is unique up to isomorphism, and the embedding $\mathcal{O}_X \hookrightarrow E'$ is unique up to scaling.

Proof. We first note that the uniqueness statement of the proposition would follow from the existence statement and the properties of Baker-Akhiezer divisors. Indeed, if D is simultaneously the Baker-Akhiezer divisor of $(\mathcal{O}_X \hookrightarrow E', \phi')$ and $(\mathcal{O}_X \hookrightarrow E'', \phi'')$, then by proposition 3.1 both (E', ϕ') and (E'', ϕ'') are isomorphic to the direct image of $\mathcal{O}_{S_q}(D)$. To show uniqueness up to scaling of the embeddings, observe that two embeddings $i_1, i_2 : \mathcal{O}_X \hookrightarrow E'$ define the same 1-dimensional subspaces in the fibers of E over the zero divisor of the composition $\mathcal{O}_X \overset{i_1}{\hookrightarrow} E' \to \Lambda'$, where the surjection is the quotient of i_2. On the other hand, if they define the same Baker-Akhiezer divisor D, then $\pi^*(i_1)$ and $\pi^*(i_2)$ define the same 1-dimensional subspaces in the fibers of $\pi^*(E)$ over D, and hence i_1 and i_2 define the same 1-dimensional subspaces in the fibers of E over $\pi(D)$. which is of degree $\deg(K\Lambda') > \deg(\Lambda')$. Hence if D has no point with non-trivial multiplicity, the composition $\mathcal{O}_X \overset{i_1}{\hookrightarrow} E' \to \Lambda'$ must vanish, which occurs if and only if i_1 and i_2 are scalings of each other. By an argument analogous to the discussion leading to definition 3.1, one could show that this statement also holds with multiplicity counted.

To prove the existence statement, we now construct (E', ϕ') on an explicit covering. Let $\mathbf{x} \coloneqq \pi(D) = x_1 + ... + x_d$, where $x_i = \pi(\tilde{x}_i)$. Let $\mathbf{p} = p_1 + ... + p_m$ and $\mathbf{q} = q_1 + ... + q_n$ be effective divisors such that $\Lambda' \cong \mathcal{O}_X(\mathbf{p} - \mathbf{q})$, and each p_j, q_k has multiplicity 1 in $\mathbf{p} + \mathbf{q} + \mathbf{x}$. Let (U_i, u_i) be a small coordinate neighborhood of x_i with $u_i(x_i) = 0$, and (U_{p_j}, z_{p_j}), (U_{q_k}, z_{q_k}) be similarly defined coordinate neighborhoods of p_j, q_k respectively. W.l.o.g. assume that these neighborhoods do not intersect each other. Consider a covering of X defined by

$$\left\{X', U_1, ..., U_d, U_{p_1}, ..., U_{p_m}, U_{q_1}, ..., U_{q_n}\right\}$$

where $X' = X \setminus \{\text{supp}(\mathbf{x}+\mathbf{p}+\mathbf{q})\}$. Our ansatz for the transition functions of E' are

$$(E')_{U_i X'} = \begin{pmatrix} 1 & \epsilon_i/u_i \\ 0 & 1 \end{pmatrix}, \quad (E')_{U_{p_j} X'} = \begin{pmatrix} 1 & 0 \\ 0 & z_{p_j} \end{pmatrix}, \quad (E')_{U_{q_k} X'} = \begin{pmatrix} 1 & 0 \\ 0 & z_{q_k}^{-1} \end{pmatrix} \tag{3.10}$$

where $\epsilon_i = \epsilon_i(u_i)$ is a holomorphic function of u_i on $U_i \cap X' = \{u_i \neq 0\}$. Let A be a holomorphic differential and $C \in \Omega_{-\mathbf{x}-\mathbf{q}+\mathbf{p}}$ a meromorphic differential that vanishes at $\mathbf{x}+\mathbf{q}$, counted with multiplicity, and has simple poles at p_1, ..., p_m [14]. With $B = \left(q - A^2\right)/C$ being a holomorphic differential on X', our ansatz for ϕ' is that it takes the local form $\begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ on X'. Then ϕ' of this form would be regular on all of X if its local forms

$$\begin{pmatrix} A + (\epsilon_i/u_i)C & B - 2(\epsilon_i/u_i)A - (\epsilon_i/u_i)^2 C \\ C & -A - (\epsilon_i/u_i)C \end{pmatrix}, \quad \begin{pmatrix} A & z_{p_j}^{-1} B \\ z_{p_j} C & -A \end{pmatrix}, \quad \begin{pmatrix} A & z_{q_k} B \\ z_{q_k}^{-1} C & -A \end{pmatrix} \tag{3.11}$$

on U_i, U_{p_j} and U_{q_k} respectively are regular. In addition, it follows from the explicit construction (3.1) of Baker-Akhiezer divisors that, if the function $-A(u_i) - \frac{\epsilon_i}{u_i} C(u_i)$ evaluated at $u_i = 0$ is equal to the square-root of $q(u_i) \mid_{u_i=0}$ that determines $\tilde{x}_i \in \pi^{-1}(x_i) = \{\tilde{x}_i, \sigma(\tilde{x}_i)\}$, then this would give the Higgs bundle (E', ϕ') we seek. With the ansatz $\epsilon_i(u_i) = \epsilon_i^0 / u_i^{|x_i|-1}$, where $\epsilon_i^0 \in \mathbb{C}$ and $|x_i|$ is the multiplicity of x_i in $\mathbf{x}$, we can solve this condition, now a linear one, for a unique $\epsilon_i^0 \in \mathbb{C}$. This determines the tuple $\boldsymbol{\epsilon} = (\epsilon_1(u_1), ..., \epsilon_d(u_d))$, which determines E' as an extension of Λ' by $\mathcal{O}_X$. A direct check shows that with the chosen A, B, C and $\boldsymbol{\epsilon}$, the expressions in (3.11) are automatically regular. $\square$

REMARK 3.5.

i. The construction of (E', ϕ') in the proof of proposition 3.4 gives an explicit description of the push-forward of $\mathcal{O}_{S_q}(D)$.

ii. The fact that A can be any holomorphic differential gives us some degrees of freedom to adjust $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_d)$, i.e. constructing equivalent extensions of Λ' by $\mathcal{O}_X$, and in particular elements in the isomorphism class of E'. This reflects the fact that $\dim \text{Ext}\left(\Lambda', \mathcal{O}_X\right) =$

[14] By construction C is unique up to scaling and is identified with the section of $K\Lambda'$ having $\mathbf{x}$ as its zero divisor via the isomorphism $\Omega_{-\mathbf{x}-\mathbf{q}+\mathbf{p}} \xrightarrow{\sim} K\Lambda' \cong \mathcal{O}_X(\mathbf{x})$.

$h^1(\Lambda'^{-1}) = h^0(K\Lambda')$ is smaller than $d = \deg(K\Lambda')$.

As an example, for any $h \le g$ consider a subset $\{x_{j_1}, ..., x_{j_h}\} \subset \{x_1, ..., x_d\}$ such that $\dim\{\omega \in H^0(K) \mid x_{j_1} + ...x_{j_h} \le \text{div}(\omega)\} = g - h$ (a generic situation). Then we can choose A such that $-A(u_{j_k}(x_{j_k}))$ is equal to the square-root of $q(u_{j_k}(x_{j_k}))$ that determines $\tilde{x}_{j_k}$. This fixes $\epsilon_{j_k} = 0$ for $k = 1, ..., h$. We see explicitly here how the underlying bundle of a Higgs bundle with a generic Baker-Akhiezer divisor of degree $\le g$ must be split (cf. the last case in proposition 2.1).

We have assumed that D does not contain the pull-back of any effective divisor on X in proposition 3.4 to construct E' with an embedding $\mathcal{O}_X \hookrightarrow E'$. On the other hand, if D contains the pull-back of some effective divisor on X, then there exists some effective divisor B on X such that $D - \pi^*(B)$ does not contain the pull-back of any effective divisor on X. It is straightforward to generalize proposition 3.4 to this case.

PROPOSITION 3.5. *Let D be an effective divisor on a non-degenerate spectral curve $S_q \xrightarrow{\pi} X$, and suppose that B is an effective divisor on X such that $D - \pi^*(B)$ does not contain the pull-back of any effective divisor. Then there exists a Higgs bundle (E', ϕ'), where E' can be realized as an extension of $\Lambda' = K^{-1} \otimes \mathcal{O}_X(\pi(D) - B)$ by $\mathcal{O}_X$, such that D is the Baker-Akhiezer divisor of the data $\left(\mathcal{O}_X(-B) \xrightarrow{s_B} \mathcal{O}_X \hookrightarrow E', \phi'\right)$. The Higgs bundle (E', ϕ') is unique up to isomorphism, and the embedding $\mathcal{O}_X \hookrightarrow E'$ is unique up to scaling.*

Proof. The proposition follows from applying proposition 3.4 to $D - \pi^*(B)$ and noting that $\mathcal{O}_X(-B) \xrightarrow{s_B} \mathcal{O}_X$ has B as its zero divisor. $\square$

Propositions 3.4 and 3.5 give the inverse construction of the Baker-Akhiezer divisors in the "normalized" situation where the line bundle is $\mathcal{O}_X$ and the determinant bundle $\Lambda' = K^{-1} \otimes \mathcal{O}_X(\pi(D))$ is determined by D. The following proposition is concerned with the situation where the determinant bundle Λ is fixed, i.e. we work on a fixed moduli space of $SL_2(\mathbb{C})$-Higgs bundles $\mathcal{M}_H(\Lambda)$.

PROPOSITION 3.6. *Given an effective divisor D on a non-degenerate $SL_2(\mathbb{C})$-spectral curve $S_q \xrightarrow{\pi} X$ and line bundles L, Λ on X satisfying $KL^{-2}\Lambda \cong \mathcal{O}_X(\pi(D))$, there exists a unique up to isomorphism $SL_2(\mathbb{C})$-Higgs bundle (E, ϕ) with $\det(E) \cong \Lambda$, and a unique up to scaling* *injection $L \to E$ such that D is the Baker-Akhiezer divisor of $(L \to E, \phi)$. In particular, $L \to E$ is a subbundle if and only if D contains no pull-back of an effective divisor on X.*

Proof. Apply proposition 3.5 and tensor with L. □

Let us define an isomorphism class $[L \to E, \phi]$ of the input data of Baker-Akhiezer divisors by saying that two representative data are isomorphic if there are isomorphisms of the underlying bundles and line bundles that commute with the injections and Higgs fields [15]. Clearly Baker-Akhiezer divisors defined by isomorphic data coincide. The following theorem summarizes the invertible properties of the construction of BA-divisors.

THEOREM 3.7. *Consider the moduli space $\mathcal{M}_H(\Lambda)$ of $SL_2(\mathbb{C})$-Higgs bundles on X with the underlying bundles of determinant Λ, and a non-degenerate quadratic differential q and spectral curve $S_q \xrightarrow{\pi} X$. Then the construction of Baker-Akhiezer divisors and remembering the line bundle defines a bijection*

$$\left\{ [L \to E, \phi] \;\middle|\; \begin{array}{l} \det(E) = \Lambda, \\ \det(\phi) = q \end{array} \right\} \longleftrightarrow \left\{ ([L], D) \;\middle|\; \begin{array}{l} D \textit{ effective on } S_q \\ KL^{-2}\Lambda \cong \mathcal{O}_X(\pi(D)) \end{array} \right\}.$$

In particular, this bijection restricts to a bijection in the cases of subbundles

$$\left\{ [L \hookrightarrow E, \phi] \;\middle|\; \begin{array}{l} L \textit{ a subbundle of } E, \\ \det(E) = \Lambda, \\ \det(\phi) = q \end{array} \right\} \longleftrightarrow \left\{ ([L], D) \;\middle|\; \begin{array}{l} D \textit{ effective on } S_q, \textit{ contains} \\ \textit{no pull-back of divisors on } X, \\ KL^{-2}\Lambda \cong \mathcal{O}_X(\pi(D)) \end{array} \right\}.$$

The map induced by forgetting the subbundle, i.e. $[L \hookrightarrow E, \phi] \mapsto D$, is a $2^{2g} : 1$ map.

Proof. The bijective property follows from the inverse construction of Baker-Akhiezer divisors (cf. proposition 3.6). The 2^{2g} covering property follows from the fact that twisting input data by a square-root of $\mathcal{O}_X$ do not change the induced Baker-Akhiezer divisors. These twists exhaust all possible input data of a Baker-Akhiezer divisor since its projection to X determines the line bundle L up to such a twist. □

[15]Since scalings are isomorphisms of line bundles, scaling the injections from line bundles to rank-2 bundles will define the same isomorphism class $[L \to E, \phi]$.

3.3 Discussion and some applications

Families of Higgs fields inducing the same point in $T^*X^{[d]}$. A Baker-Akhiezer divisor of degree d on a spectral curve defines a point in the symmetric product $(T^*X)^{[d]}$ of T^*X via the inclusion of the spectral curve to the total space of T^*X. Given $d < 3g-3$ and $\mathbf{p} = [p_1, ..., p_d] \in (T^*X)^{[d]}$, there exists a positive-dimensional family of spectral curves each of which goes through $p_1, ..., p_d$ and admits an effective divisor defined by these points. For example, if the divisor $\mathbf{x}$ on X defined by projecting $\mathbf{p}$ is reduced, this family of spectral curves is an affine space modeled over $Q_{\mathbf{x}}$. Hence one same point $\mathbf{p} \in (T^*X)^{[d]}$ corresponds to Baker-Akhiezer divisors on different spectral curves associated to non-isomorphic Higgs bundles. The underlying bundles, however, are constrained.

COROLLARY 3.8. *Two Higgs bundles $[E_1, \phi_1]$, $[E_2, \phi_2] \in \mathcal{M}_H(\Lambda)$ define the same point in $(T^*X)^{[d]}$ via the construction of Baker-Akhiezer divisors only if $E_1 \cong E_2 \otimes N$ where $N^2 \cong \mathcal{O}_X$.*

Proof. For $i \in \{1, 2\}$, let D_i be the Baker-Akhiezer divisors on S_{q_i} of the data $(L_i \hookrightarrow E_i, \phi_i)$ such that D_1 and D_2 induce the same point $\mathbf{p} = [p_1, ..., p_d] \in (T^*X)^{[d]}$. Tensoring $(L_i \hookrightarrow E_i, \phi_i)$ with L_i^{-1} does not change the Baker-Akhiezer divisors and brings us to the situation where proposition 3.4 can be applied: D_i then is the unique up to isomorphism Baker-Akhiezer divisor of some data $(\mathcal{O}_X \hookrightarrow E_i', \phi_i')$. In particular, if we choose the same differentials A and C in the proof of proposition 3.4 in constructing (E_1', ϕ_1') and (E_2', ϕ_2'), we can construct E_1' and E_2' out of the same data $(\epsilon_1(u_1), ..., \epsilon_2(u_2))$. Hence $E_1' = E_2'$. The proposition now follows from the fact that $E_i' = E_i \otimes L_i^{-1}$ and $L_1^2 \cong L_2^2$. $\square$

Recall that a bundle E, $\det(E) = \Lambda$, is not maximally stable if it admits a subbundle $L_E \hookrightarrow E$ satisfying $\deg\left(L_E^{-2}\Lambda\right) < g-1$, i.e. $\deg\left(KL_E^{-2}\Lambda\right) < 3g-3$. Hence a Higgs bundle (E, ϕ) with E not maximally stable will induce Baker-Akhiezer divisors of degree $< 3g-3$ if we choose a maximal subbundle of E for the input data. The following corollary follows immediately from the above propositions and corollary.

COROLLARY 3.9. *Let $d < 3g-3$. Then for any point $\mathbf{p} = [p_1, ..., p_d] \in (T^*X)^{[d]}$ there exists a bundle E which is not maximally stable and a positive-dimensional family of Higgs fields on E, such that the Baker-Akhiezer divisors of these data using a maximal subbundle of E all define* $\mathbf{p}$.

Conversely, if E is not maximally stable, then any Higgs field on E is contained in a positive-dimensional family of Higgs fields on E, such that the Baker-Akhiezer divisors of these data using a maximal subbundle of E all define the same point in $(T^*X)^{[d]}$ for some $d < 3g - 3$.

Exceptional divisors on non-degenerate spectral curves. Suppose $\dim\{L \hookrightarrow E\} \geq 2$, i.e. there exists at least 2 linearly independent embeddings from L to E. Then any Higgs field ϕ on E induces a positive-dimensional family of effective divisors on $S_{\det(\phi)}$ all of which are equivalent via the construction of Baker-Akhiezer divisors. If $\deg(KL^{-2}\Lambda) \leq 4g - 3$, the genus of a non-degenerate spectral curve, then these Baker-Akhiezer divisors are exceptional divisors. Conversely, by theorem 3.7, an exceptional divisor D on a non-degenerate spectral curve S, i.e. $\deg(D) < 4g - 3$, $\dim|D| \geq 1$, that contains no pull-back of an effective divisor on X induces a family of embeddings $\{L \hookrightarrow E\}$ of dimension ≥ 2.

COROLLARY 3.10. *On a non-degenerate spectral curve there exists no exceptional divisor of degree $< 2g - 2$.*

Proof. An exceptional divisor of degree $< 2g - 2$ implies the existence of a strictly unstable holomorphic rank-2 bundle E with destabilizing subbundle L_E such that $\dim\{L_E \hookrightarrow E\} \geq 2$, which is impossible. $\square$

Caustics and theta divisor revisited. Consider the locus in a Hitchin fiber $h^{-1}(q)$ defined by Higgs bundles with stable underlying bundles. The projection from this locus to the moduli space $\mathcal{N}_\Lambda$ of stable bundles, defined by forgetting the Higgs fields, is a local diffeomorphism at a generic point. It fails to be a diffeomorphism at the locus where $h^{-1}(q)$ is tangential to the fibers of $T^*\mathcal{N}_\Lambda \subset \mathcal{M}_H(\Lambda)$. The projection to $\mathcal{N}_\Lambda$ of this locus, i.e. $\{[E] \in \mathcal{N}_\Lambda \mid T^*_{[E]}\mathcal{N}_\Lambda \text{ is tangential to } h^{-1}(q)\}$, is called a caustic formed by $h^{-1}(q)$.

Hitchin [35] showed that for a stable bundle E, the two Lagrangian submanifolds $T^*_{[E]}\mathcal{N}_\Lambda$ and $h^{-1}(q)$ of $\mathcal{M}_H(\Lambda)$ are tangential to each other at $[E, \phi]$ if and only if the line bundle

$$\pi^*(K)(\sigma^*(\mathcal{L}))^{-1}\mathcal{L},$$

where $\mathcal{L}$ is the eigen-line bundle of (E, ϕ), has a non-zero section. This is a non-generic condition since $\pi^*(K)(\sigma^*(\mathcal{L}))^{-1}\mathcal{L}$ is of degree $\tilde{g}-1$. For D a Baker-Akhiezer divisor of (E,ϕ) induced by some injection $L \to E$, since the ramification divisor R_q satisfies $\mathcal{O}_{S_q}(R_q) \cong \pi^*(K)$, it follows from proposition 3.3 that we can reformulate this condition by requiring

$$R_q + D - \sigma(D) \tag{3.12}$$

to be effective on S_q. By Riemann's theorem the image of these divisors under the Abel-Jacobi map $S_q^{[\tilde{g}-1]} \to \text{Jac}^{\tilde{g}-1}(S_q) \cong \text{Jac}(S_q)$ is a translation of the theta divisor. [16] Since the theta divisor is of codimension 1 in Jac(S_q), caustics formed by a smooth Hitchin fiber is of codimension 1 in $\mathcal{N}_\Lambda$.

Since $\pi_q^*(K)$ has g linearly independent sections that are pull-backed from X in addition to the canonical section that vanishes at R_q, we have $\dim|R_q| = h^0(R_q) - 1 \geq g$. Let $D^\sigma < D$ be the σ-invariant part of D, i.e. $p < D^\sigma$ if either p is a ramification point or $p + \sigma(p) < D$. Then $D - \sigma(D) = D' - \sigma(D')$ where $D' = D - D^\sigma$. It is then easy to see that if D is sufficiently σ-invariant, i.e. the degree of $D - D^\sigma$ is sufficiently low, then $R_q + D - \sigma(D)$ is effective. In fact, one could check that if $\deg(D - D^\sigma) \leq g$ then

$$h^0(R_q + D - \sigma(D)) \geq g - \deg(D - D^\sigma) + 1 > 0. \tag{3.13}$$

EXAMPLE 3.6. Let q be a non-degenerate quadratic differential, and $\mathbf{x} = x_1 + ... + x_h <$ div(q) a divisor on X of degree $h \geq 2g-3$, each point of which is a branch point of S_q. Let $\tilde{\mathbf{x}} = \tilde{x}_1 + ...\tilde{x}_h$ where $\tilde{x}_i \in S_q$ is the ramification point corresponding to x_i. For $d \leq 3g-3$, if D is the Baker-Akhiezer divisor of degree d induced by a Higgs bundle (E,ϕ) together with a subbundle $L \hookrightarrow E$ and is such that $\tilde{\mathbf{x}} = D^\sigma < D$, then $\deg(D - D^\sigma) \leq g$ and hence $R_q + D - \sigma(D)$ is effective. For fixed data $(q, \mathbf{x}, d)$, it follows from proposition 3.6 that by varying $D - \tilde{\mathbf{x}}$ we can obtain 2^{2g} families of Higgs bundles of dimension $(d-h)$, a generic Higgs bundle of which has stable underlying bundle and hence projects to a point in the caustics formed by $h^{-1}(q)$ in $\mathcal{N}_\Lambda$.

[16] Note also the line bundle $\mathcal{O}_{S_q}(R_q+D-\sigma(D))$ is the image of $(E,\phi) \in h^{-1}(q)$ under the composition of $I_{q,K^{1/2}}$: $h^{-1}(q) \xrightarrow{\sim} \text{Prym}(S_q)$ (cf. remark 3.4) with the map $\text{Prym}(S_q) \to \text{Jac}^{\tilde{g}-1}(S_q)$ defined by $[L] \mapsto [\pi^*(K) \otimes L^2]$.

Intersections of Hitchin fibers with $\mathbb{C}^*$-orbits in the unstable strata. The intersection of a generic Hitchin fiber with a generic cotangent fiber of the moduli space of stable bundles is a discrete set of 2^{3g-3} points. A point at which a Hitchin fiber is tangential to a cotangent fiber of the moduli space of stable bundles can be regarded as a double intersection point of these two Lagrangian subspaces of $\mathcal{M}_H$; hence the phenomena of caustics can be regarded as an enumerative problem.

The generalization of the cotangent fiber of a stable bundle is the upward flow W_α^+ for α a $\mathbb{C}^*$-fixed point in $\mathcal{M}_H$. Recall from the previous chapter that if E is destabilized by L_E, then a stable Higgs bundle (E,ϕ) defines a point in W_α^+ where $\alpha = (E_M, \phi_c)$ is defined in (2.6),

$$E_M = M \oplus M^{-1}\Lambda, \qquad \phi_c = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \tag{3.14}$$

with $M = L_E$ and $c = c_{L_E}(\phi)$. The following proposition, which characterizes the intersection points of W_α^+ with a smooth Hitchin fiber, follows from theorem 3.7 and the fact that an unstable bundle has a unique destabilizing subbundle. A more general version of this result that applies to arbitrary ranks was established by Hausel-Hitchin [34].

PROPOSITION 3.11. *Let $\alpha = (E_M, \phi_c)$ be a $\mathbb{C}^*$-fixed point in $\mathcal{M}_H$ and q a non-degenerate quadratic differential. Then $W_\alpha^+ \cap h^{-1}(q)$ is in 1-1 correspondence with*

$$\left\{ D \text{ effective on } S_q \;\middle|\; \begin{array}{l} \pi(D) = div(c), D \text{ contains no} \\ \text{pull-back of a divisor on } X \end{array} \right\},$$

where the correspondence is defined by constructing Baker-Akhiezer divisors using the destabilizing subbundle M. In particular, generically $\left|W_\alpha^+ \cap h^{-1}(q)\right| = 2^{\deg(c)}$, which corresponds to the case where no point in c has non-trivial multiplicity or is a branch point of S_q.

Chapter 4

Degeneration of Baker-Akhiezer divisors

In this chapter we discuss certain types of families of Higgs bundles in $\mathcal{M}_H \equiv \mathcal{M}_H(\Lambda)$, the underlying bundles of which admit subbundles of the same degree, that limit to a Higgs bundle with the underlying bundle admitting a subbundle of higher degree. These types of families are understood most easily in terms of the corresponding Baker-Akhiezer divisors, so we start from this perspective.

4.1 Reduction of the degree of Baker-Akhiezer divisors

Let $(E,\phi) \in \mathcal{M}_H$ be a Higgs bundle with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$ with involution σ, $L \hookrightarrow E$ a subbundle and D the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$. Recall from proposition 3.2 that D does not contain any summand being the pull-back of a divisor on X and satisfies $L^2 \cong K\Lambda \otimes \mathcal{O}_X(-\pi(D))$, and the eigen-line bundle of (E,ϕ) is isomorphic to $\pi^*(LK^{-1}) \otimes \mathcal{O}_S(D)$.

Consider a sufficiently small, simply connected open set $U \subset X$ that does not contain a branch point of S. Consider an effective divisor $\tilde{\mathbf{x}} = \tilde{x}_+ + \tilde{x}_-$ where $\tilde{x}_\pm$ lie in distinct components of $\pi^{-1}(U)$ and are such that $\tilde{x}_+ \neq \sigma(\tilde{x}_-)$. In other words, for $x_\pm = \pi(\tilde{x}_\pm)$ we have $x_+ \neq x_-$. By theorem 3.7, $D + \tilde{\mathbf{x}}$ is the Baker-Akhiezer divisor of some Higgs bundle in $\mathcal{M}_H$ induced by some subbundle which is of degree $\deg(L)-1$ and is a square-root of $K\Lambda \otimes \mathcal{O}_X(-\pi(D+\tilde{\mathbf{x}}))$. For each $\tilde{\mathbf{x}}$, there are 2^{2g} such square-roots. Varying $\tilde{\mathbf{x}}$ then induces 2^{2g} families of line bundles of degree $\deg(L)-1$. If U is sufficiently small, these families define distinct subsets in $\mathrm{Jac}_{\deg(L)-1}(X)$, each of which, given a point $x_0 \in U$, contains upon closure a unique square-root of $K\Lambda \otimes (-\pi(D) - 2x_0)$ as the limit of $\tilde{\mathbf{x}} \to \pi^*(x_0)$. One particular family would contain $L(-x_0)$ upon closure. Let $L_{\tilde{\mathbf{x}}}$ be the square-root of $K\Lambda \otimes \mathcal{O}_X(-\pi(D+\tilde{\mathbf{x}}))$ contained in that family, i.e.

$$\lim_{\tilde{\mathbf{x}} \to \pi^*(x_0)} [L_{\tilde{\mathbf{x}}}] = [L \otimes \mathcal{O}_X(-x_0)] \in \mathrm{Jac}_{\deg(L)-1}(X) \tag{4.1a}$$

for any $x_0 \in U$.

Let $(L_{\tilde{\mathbf{x}}} \hookrightarrow E_{\tilde{\mathbf{x}}}, \phi_{\tilde{\mathbf{x}}})$ be the data that induces $D + \tilde{\mathbf{x}}$ as its Baker-Akhiezer divisor. The eigen-line bundle of $(E_{\tilde{\mathbf{x}}}, \phi_{\tilde{\mathbf{x}}})$ is isomorphic to $\pi^*(L_{\tilde{\mathbf{x}}}K^{-1}) \otimes \mathcal{O}_S(D + \tilde{\mathbf{x}})$. It follows from (4.1a) that as $\tilde{\mathbf{x}} \to \pi^*(x_0)$, this family of line bundles limits to $\pi^*(LK^{-1}) \otimes \mathcal{O}_S(D)$, which is the eigen-line bundle of (E, ϕ). In other words,

$$\lim_{\tilde{\mathbf{x}} \to \pi^*(x_0)} (E_{\tilde{\mathbf{x}}}, \phi_{\tilde{\mathbf{x}}}) = (E, \phi) \in \mathcal{M}_H \tag{4.1b}$$

for any $x_0 \in U$. This is clear in terms of Baker-Akhiezer divisors: for any $x_0 \in U$, $D + \pi^*(x_0)$ is the Baker-Akhiezer divisor induced by (E, ϕ) via the injection $L \otimes \mathcal{O}_X(-x_0) \xrightarrow{s_{x_0}} L \hookrightarrow E$ which vanishes at x_0. The following proposition summarizes our discussion.

PROPOSITION 4.1. *Let $(E, \phi) \in \mathcal{M}_H$ with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$, and L a subbundle of E. Let $U \subset X$ be a simply connected open set that contains no branch point of S. Then there exist embeddings*

$$\begin{aligned} \mathbf{U}' &\longrightarrow \mathcal{M}_H, & \mathbf{U}' &\longrightarrow Jac_{\deg(L)-1}(X), \\ \mathbf{x} &\longmapsto (E_{\mathbf{x}}, \phi_{\mathbf{x}}), & \mathbf{x} &\longmapsto [L_{\mathbf{x}}], \end{aligned} \tag{4.2}$$

where $\mathbf{U}' := \{\mathbf{x} = (x_+, x_-) \in U^2 \mid x_+ \neq x_-\}$, such that

(a) for all $\mathbf{x} \in \mathbf{U}'$, $E_{\mathbf{x}}$ admits $L_{\mathbf{x}}$ as a subbundle;

(b) for all $\mathbf{x} \in \mathbf{U}'$, the Baker-Akhiezer divisor of $(L_{\mathbf{x}} \hookrightarrow E_{\mathbf{x}}, \phi_{\mathbf{x}})$ is $D + \tilde{x}_+ + \tilde{x}_-$, where $\tilde{x}_\pm \in \pi^{-1}(x^\pm)$ respectively and lie in distinct components of $\pi^{-1}(U)$, and D is the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$;

(c) for all $x_0 \in U$, $\lim_{x_+ \to x_0 \leftarrow x_-} [L_{\mathbf{x}}] = [L \otimes \mathcal{O}_X(-x_0)]$ in $Jac_{\deg(L)-1}(X)$ and $\lim_{x_+ \to x_0 \leftarrow x_-} (E_{\mathbf{x}}, \phi_{\mathbf{x}}) = (E, \phi)$ in $\mathcal{M}_H$.

In chapter 7 where we analogously analyze the collision limit of apparent singularities of SL-operators, we will find it convenient for calculation to specialize to the case where U is equipped with a coordinate z with $z(x_0) = 0$ for some $x_0 \in U$ and $z(x_\pm) = \pm u$. It is instructive to now spell out the specialization of proposition 4.1 to this case in order to later see the analogy between colliding points in the projection of Baker-Akhiezer divisors and apparent singularities.

PROPOSITION 4.2. *Let $(E, \phi) \in \mathcal{M}_H$ with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$, and L be a subbundle of E. Given $x_0 \in X$ not a branch point of S, let (U, z) be a coordinate neighborhood of x_0, where $z(x_0) = 0$, U is simply connected and contains no branch point of S. Then there exist a family of Higgs bundles $\{(E_u, \phi_u)\}_{u \in z(U)}$ and a family of line bundles $\{L_u\}_{u \in z(U)}$ of degree $\deg(L) - 1$ parameterized by U such that*

(i) $[L_0] = [L \otimes \mathcal{O}_X(-x_0)]$ *in* $Jac_{\deg(L)-1}(X)$ *and* $(E_0, \phi_0) = (E, \phi)$ *in* $\mathcal{M}_H$*;*

(ii) for all $u \in z(U)$, (E_u, ϕ_u) *has* S *as its spectral curve;*

(iii) for all $u \neq 0$, E_u *admits* L_u *as a subbundle;*

(iv) for all $u \neq 0$*, the Baker-Akhiezer divisor of* $(L_u \hookrightarrow E_u, \phi_u)$ *is* $D + \tilde{x}_+ + \tilde{x}_-$*, where* D *is the Baker-Akhiezer divisor of* $(L \hookrightarrow E, \phi)$ *and* $\tilde{x}_\pm$ *lie in different distinct components of* $\pi^{-1}(U)$ *with* $z(\pi(\tilde{x}_\pm)) = \pm u$*.*

Furthermore, these families define embeddings $U \hookrightarrow \mathcal{M}_H$ *and* $U \hookrightarrow Jac_{\deg(L)-1}(X)$*.*

We also note that if $\deg(D) < 2g - 2$, these families limit to lower Bialynicki-Birula strata, i.e. the strata of Higgs bundles with increasingly unstable underlying bundles. The limits of these families compactify the open dense locus consisting of Higgs bundles with stable underlying bundles into the whole Prym variety.

4.2 Double point in Baker-Akhiezer divisors

In the above discussion, we have analyzed families of Baker-Akhiezer divisors whose limits contain a summand of the form $\pi^*(x_0)$ for some $x_0 \in X$. The following proposition is the counterpart of proposition 4.2 for the case where the limit contains a double point that projects to $2x_0$. In this case, unlike in proposition 4.2, the underlying bundle of the corresponding Higgs bundle at the limit admits a subbundle of the same degree as in the family.

PROPOSITION 4.3 (PROPOSITION 1.8). *Let $(E, \phi) \in \mathcal{M}_H$ with the associcated non-degenerate spectral curve $S \xrightarrow{\pi} X$ and L be a subbundle of E such that $c_L(\phi)$ has a double zero at $x_0 \in X$*

which is not a branch point of S. Let D be the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$ and $\tilde{x}_0$ be the point with multiplicity 2 in D with $\pi(\tilde{x}_0) = x_0$. Let (U, z) be a coordinate neighborhood of x_0, where $z(x_0) = 0$, U is simply connected and contains no branch point of S. Then there exist a family of Higgs bundles $\{(E_u, \phi_u)\}_{u \in z(U)}$ and a family of line bundles $\{L_u\}_{u \in z(U)}$ of the same degree as L parameterized by U such that

(i) $[L_0] = [L]$ *in* $Jac_{\deg(L)}(X)$ *and* $(E_0, \phi_0) = (E, \phi)$ *in* $\mathcal{M}_H$*;*

(ii) for all $u \in z(U)$, E_u admits L_u as a subbundle;

(iii) for all $u \neq 0$, the Baker-Akhiezer divisor of $(L_u \hookrightarrow E_u, \phi_u)$ is $D - 2\tilde{x}_0 + \tilde{x}_+ + \tilde{x}_-$, where $\tilde{x}_\pm$ lie in the component of $\pi^{-1}(U)$ containing $\tilde{x}_0$ and are such that $z(\pi(\tilde{x}_\pm)) = \pm u$.

Furthermore, these families define embeddings $U \hookrightarrow \mathcal{M}_H$ and $U \hookrightarrow Jac_{\deg(L)}(X)$.

Proof. The proposition follows from the discussion preceding proposition 4.1 by requiring $\tilde{x}_\pm$ now to be contained in the same component of $\pi^{-1}(U)$ containing $\tilde{x}_0$. □

4.3 Local model and scaling of families of Higgs bundles

In the following, we consider a family of Higgs bundles obtained by scaling a family of Higgs bundles in proposition 4.2.

PROPOSITION 4.4. *Let $[E, \phi] \in \mathcal{M}_H$ with associated non-degenerate spectral curve $S \xrightarrow{\pi} X$, L be a subbundle of E, and $x_0 \in X$ which is not a branch point of S and not a zero of $c_L(\phi)$. Suppose E is destabilized by a subbundle L_E with $\deg(L_E^{-2}\Lambda) < 2g-4$. Let (U, z) be a coordinate neighborhood of x_0 and $\{[E_u, \phi_u]\}_{u \in z(U)}$ a family of Higgs bundles constructed by proposition 4.2 with $(E, \phi) = (E_0, \phi_0)$. Consider the family of Higgs bundle $\{[F_u, \psi_u]\}_{u \in z(U), u \neq 0}$ defined by $[F_u, \psi_u] \coloneqq u.[E_u, \phi_u]$ for $u \neq 0$.*

Then the limit $[F_0, \psi_0] \coloneqq \lim_{u \to 0} [F_u, \psi_u]$ exists, lies in the nilpotent cone and is not $\mathbb{C}^*$-invariant. The underlying bundle F_0 is destabilized by $L_0 \cong L \otimes \mathcal{O}_X(-x_0)$, and $c_{L_0}(\psi_0)$ has a double zero at x_0.

Let $\lambda = u^{-1}$ for $u \neq 0$. As we can write $[E_u, \phi_u] = \lambda.[F_u, \psi_u]$ for $\lambda < \infty$, we can think of the family $\{[E_u, \phi_u]\}_{u \in z(U), z \neq 0}$ as a blow-up of the family $\{F_u, \psi_u\}_{u \in z(U), u \neq 0}$. Note that while the latter stays in the same Białynicki-Birula stratum at the limit, the former limits to a point in a lower stratum. This is how one might understand the compactification of the Białynicki-Birula strata which completes a generic Hitchin fiber into a Prym variety involves certain "going to infinity" ingredients.

In the following, we will prove proposition 4.4 by constructing an explicit local model for $\{[E_u, \phi_u]\}_{u\in z(U), z\neq 0}$. The main ingredients of the construction are the Hecke transformations of bundles and Higgs bundles, which have invariant definitions and work for higher rank cases as well. We refer to [34] for a modern introduction of these ingredients. Since our goal is to have an explicit local model, we will however not discuss these transformations in their invariant forms.

Modifying bundles. Let $(E, \phi) \in \mathcal{M}_H$ with non-degenerate spectral curve $S_q \xrightarrow{\pi} X$ associated to $q = \det(\phi)$. Let L be a subbundle of E, and $x_0 \in X$ such that x_0 is not a zero of $c_L(\phi)$ and not a branch point of S_q. Let (W', z) be a coordinate neighborhood of x_0, where $z(x_0) = 0$, W' is simply-connected, contains no zero of q and of $c_L(\phi)$. Choose an atlas $\mathcal{U}'$ on X that contains W' as a chart, and define E in terms of transition functions on this atlas w.r.t. local frames adapted to $L \hookrightarrow E$, i.e. the transition functions are of upper triangular form as in (2.1b)

To define E_u for $u \neq 0$ in terms of transition functions, we first refine the atlas $\mathcal{U}'$ as follows. Let U and V be simply-connected neighborhoods of x_0 such that $U \subsetneq V \subsetneq W'$, and w.l.o.g. assume that V intersects no other elements of $\mathcal{U}'$. We define a refinement $\mathcal{U}$ of $\mathcal{U}'$ by refining W' into $W \cup V$, where $W = W' \setminus U$. The transition function of E on this new atlas upon transiting between W and V is the identity.

Now, for each nonzero $u \in z(U)$, let $x_\pm(u)$ be points U defined by $z(x_\pm(u)) = \pm u$. Furthermore, given $\vec{\epsilon} = (\epsilon_+, \epsilon_-) \in \mathbb{C}^2$, we define the bundle $E'_{u,\vec{\epsilon}}$ with transition functions on $\mathcal{U}$ that are the same as those of E except

$$\left(E'_{u,\vec{\epsilon}}\right)_{WV} = \begin{pmatrix} z-u & \epsilon_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+u & \epsilon_- \\ 0 & 1 \end{pmatrix}. \tag{4.3}$$

Then $E'_{u,\vec{\epsilon}}$ admits $L'_u := L \otimes \mathcal{O}_X(-x_+(u) - x_-(u))$ as a subbundle with $L^{-1}\Lambda$ the quotient bundle, i.e. it fits in the s.e.s.

$$0 \to L'_u \to E'_{u,\vec{\epsilon}} \to L^{-1}\Lambda \to 0.$$

REMARK 4.1. Compared with the notions of Hecke transformations of bundles [34], one can see that $E'_{u,\vec{\epsilon}}$ is the result of two consecutive Hecke transformations from E at x_+ and then at x_-. Hecke transformations of a bundle at a point $p \in X$ are defined using the choices of a subspace of the fiber of that bundle at p. The parameter $\epsilon_+ \in \mathbb{C}$ in (4.3), for example, encodes such a choice of 1-dimensional subspaces of $E\mid_{x_+}$ w.r.t. a local frame of $E\mid_U$ adapted to $L\mid_U$. The limit $\epsilon_+ \to \infty$ corresponds to the choice of the subspace of $E\mid_{x_+(u)}$ defined by $L\mid_{x_+(u)}$.

Let $E_{u,\vec{\epsilon}} = E'_{u,\vec{\epsilon}} \otimes N_u$ where N_u is a square-root of $\mathcal{O}_X(x_+ + x_-)$ such that $\lim\limits_{u\to 0}[N_u] = [\mathcal{O}_X(x_0)]$ in the corresponding Picard component. Then $E_{u,\vec{\epsilon}}$ has the same determinant as that of E and admits $L_u := L'_u \otimes N_u$ as a subbundle. The transition functions on $\mathcal{U}$ of $E_{u,\vec{\epsilon}}$ in local frames adapted to the embedding $L_u \hookrightarrow E_{u,\vec{\epsilon}}$ are the same as those of $E'_{u,\vec{\epsilon}}$ in local frames adapted to L'_u up to an "abelian" twist by the transition functions of N_u.

Modifying Higgs bundles. Let D be the Baker-Akhiezer divisor of $(L \hookrightarrow E, \phi)$. We now explain that to specific values of $\epsilon_\pm \in \mathbb{C}$, one can define Higgs fields $\phi_{u,\vec{\epsilon}}$ on $E_{u,\vec{\epsilon}}$ such that $\det(\phi_{u,\vec{\epsilon}}) = q = \det(\phi)$ and the projection to X of the Baker-Akhiezer divisor associated to $\left(L_u \hookrightarrow E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}}\right)$ is

$$\pi(D) + x_+(u) + x_-(u). \tag{4.4}$$

Suppose $\phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ in local frames adapted to $L \hookrightarrow E$. Away from V, in local frames adapted to L_u, we can let $\phi_{u,\vec{\epsilon}}$ take the same form as ϕ there. It follows from the transition function (4.3) of $E_{u,\vec{\epsilon}}$ that on V, $\phi_{u,\vec{\epsilon}}$ takes the form

$$\phi_{u,\vec{\epsilon}}\mid_V = \begin{pmatrix} a_u(z) & b_u(z) \\ c_u(z) & -a_u(z) \end{pmatrix}, \tag{4.5a}$$

where

$$a_u(z) = a(z) - \epsilon_-(z-u)c(z) - \epsilon_+c(z), \tag{4.5b}$$

$$b_u(z) = \frac{-\epsilon_-^2(z-u)^2c(z) + 2\epsilon_-(z-u)(a(z)-\epsilon_+c(z)) - \epsilon_+^2c(z) + 2\epsilon_+a(z) + b(z)}{(z-u)(z+u)}, \tag{4.5c}$$

$$c_u(z) = (z-u)(z+u)c. \tag{4.5d}$$

The regularity condition of $b_u(z)$ at $x = \pm u$ implies

$$\begin{aligned} c_+\epsilon_+^2 - 2a_+\epsilon_+ - b_+ &= 0, \\ 4u^2c_-\epsilon_-^2 + 4u\left(a_- - \epsilon_+c_-\right)\epsilon_- + \left(c_-\epsilon_+^2 - 2a_-\epsilon_+ - b_-\right) &= 0, \end{aligned} \tag{4.6}$$

where by $a_\pm$, $b_\pm$ and $c_\pm$ we mean the evaluation of $a_u(z)$, $b_u(z)$ and $c_u(z)$ at $z = \pm u$. The solutions of (4.6) are

$$\epsilon_{+,\pm}(u) = \frac{a_+ \pm \sqrt{-q_+}}{c_+}, \qquad \epsilon_{-,\pm}(u, \epsilon_+(u)) = \frac{-a_- + \epsilon_{+,\pm}(u)c_- \pm \sqrt{-q_-}}{2uc_-} \tag{4.7}$$

where $q_\pm = q(z)\mid_{z=\pm u} = -a_\pm{}^2 - b_\pm c_\pm$. Here we have chosen a square-root of $-q(z)$, which is equivalent to marking a component of $\pi^{-1}(W)$.

Hence, for each nonzero $u \in z(U)$, if $\epsilon_\pm$ take the values given in (4.7) then $\phi_{u,\vec{\epsilon}}$ given by (4.5) is a holomorphic Higgs field on $E_{u,\vec{\epsilon}}$ that satisfies condition (4.4). Let us denote the Baker-Akhiezer divisor of $(L_u \hookrightarrow E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}})$ by $D_u = D + \tilde{x}_+ + \tilde{x}_-$ where $\pi(\tilde{x}_\pm) = x_\pm$. As we have marked a component of $\pi^{-1}(W)$ by choosing a square-root $\sqrt{-q(z)}$, we can be more specific about $\tilde{x}_\pm$ by observing that

$$(a_u(u), a_u(-u)) = \begin{cases} \left(\mp\sqrt{-q(u)}, \pm\sqrt{-q(-u)}\right) & \text{for } \vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\pm}) \\ \left(\mp\sqrt{-q(u)}, \mp\sqrt{-q(-u)}\right) & \text{for } \vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\mp}) \end{cases}. \tag{4.8}$$

It follows in particular that if $\vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\pm})$ then $\tilde{x}_\pm$ lie in different components of $\pi^{-1}(W)$. In this case, the family $\{(E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}})\}_{u\in z(U), u\neq 0}$ yields $\lim_{u\to 0}[E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}}] = [E, \phi]$ in $\mathcal{M}_H(\Lambda)$, and hence defines an example of proposition 4.2. On the other hand, if $\vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\mp})$ then $\tilde{x}_\pm$ lie in the same component of $\pi^{-1}(W)$: the family $\{(E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}})\}_{u\in z(U),u\neq 0}$ then extends to an example of proposition 4.3.

REMARK 4.2. For the case $\vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\pm})$, we have $\epsilon_-(u) \overset{u\to 0}{\to} \infty$. However,

$$\lim_{u\to 0} (u\epsilon_-) = \frac{\sqrt{-q(x_0)}}{c(x_0)} \tag{4.9}$$

is well-defined.

REMARK 4.3. The Higgs bundles $(E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}})$, where $\vec{\epsilon}$ take the values given in (4.7), can be induced by applying Hecke transformations to (E, ϕ) and tensoring with N_u to ensure $\det(E_{u,\vec{\epsilon}}) = \det(E)$). Indeed, the regularity condition (4.6) with solutions (4.7) implies precisely that $E'_{u,\vec{\epsilon}} = E_{u,\vec{\epsilon}} \otimes N_u^{-1}$ is the result of two consecutive Hecke transformations from E at x_+ and then x_-. These Hecke transformations for Higgs bundles are defined with the choice of the subspace of the fibers at $x_\pm$ being the eigen-spaces defined by the Higgs fields [34]. For $\vec{\epsilon} = (\epsilon_{+,\pm}, \epsilon_{-,\pm})$, the limit $\epsilon_-(u) \overset{u\to 0}{\to} \infty$ reflects that fact that as $u \to 0$, the eigen-space over x_- defined by the Higgs field limits to the subspace defined by the subbundle, to which our local frames are adapted (cf. remark 4.1).

Proof of proposition 4.4. By the above discussion, the family $\{[E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}}]\}_{u\in z(U),u\neq 0}$ for $\vec{\epsilon} = (\epsilon_{+,+}, \epsilon_{-,+})$ extends to a family constructed by proposition 4.2. Plugging $\epsilon_{+,+}(u)$ into $\epsilon_{-,+}(u)$ in (4.7), with $\lambda = u^{-1}$, we rewrite the local form (4.5) of $\phi_{u,\vec{\epsilon}}$ on V as

$$a_u(z) = \lambda\left[ua(z) - (u\epsilon_-)(z-u)c(z) - u\epsilon_+ c(z)\right], \tag{4.10a}$$

$$\begin{aligned} b_u(z) = \frac{\lambda^2}{z^2-u^2}\big[&-(z-u)^2 c(z)(u\epsilon_-)^2 + 2u^2\epsilon_-(z-u)(a(z)-\epsilon_+c(z)) \\ &-u^2\left(\epsilon_+^2 c(z) - 2\epsilon_+ a(z) - b(z)\right)\big], \end{aligned} \tag{4.10b}$$

$$c_u(z) = (z-u)(z+u)c. \tag{4.10c}$$

We now regard $[F_u, \psi_u] \in \mathcal{M}_H(\Lambda)$ as defined by the condition $[E_{u,\vec{\epsilon}}, \phi_{u,\vec{\epsilon}}] = \lambda.[F_u, \psi_u]$. By our assumption on the degree of the destabilizing subbundle $L = L_E$ of E, the two isomorphic bundles

F_u and $E_{u,\vec{\epsilon}}$ are both destabilized by L_u and are strictly unstable. Hence it has an automorphism of the form $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$. This automorphism together with the scaling that defines $[F_u, \psi_u]$ allows us to write the local form on V of ψ_u as

$$\psi_u \mid_V = \begin{pmatrix} \lambda^{-1} a_u(z) & \lambda^{-2} b_u(z) \\ (z-u)(z+u)c & -\lambda^{-1} a_u(z) \end{pmatrix} \tag{4.11}$$

and as $\begin{pmatrix} ua & u^2 b \\ c & -ua \end{pmatrix}$ outside V (recall $\phi_{u,\vec{\epsilon}}$ takes the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ there.) Noting that $u\epsilon_-(u)$ has a well-defined $u \to 0$ limit (cf. remark 4.2), we can now compute the limit $[F_0, \psi_0] = \lim_{u\to 0} [F_u, \psi_u]$ directly. We see that F_0 is destabilized by $L_0 := L \otimes \mathcal{O}_X(-x_0)$, and ψ_0 w.r.t. local frames adapted to L_0 takes the form

$$\begin{pmatrix} -\lim\limits_{u\to 0} (u\epsilon_-)\, zc & -\lim\limits_{u\to 0} (u\epsilon_-)^2\, c \\ z^2 c & \lim\limits_{u\to 0} (u\epsilon_-)\, zc \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{-q(x_0)}}{c(x_0)} zc & -\left(\frac{\sqrt{-q(x_0)}}{c(x_0)}\right)^2 c \\ z^2 c & \frac{\sqrt{-q(x_0)}}{c(x_0)} zc \end{pmatrix}. \tag{4.12}$$

on V and $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ outside V. In particular, ψ_0 is nilpotent and $c_{L_0}(\psi_0)$ has a double zero at x_0.
□

We note that (F_0, ψ_0) provides an example of a theorem by Hausel-Hitchin on "very-stable Higgs bundle" [34]. These are the $\mathbb{C}^*$-fixed points in $\mathcal{M}_H$ whose upward flows intersect the nilpotent cone only there and nowhere else. Since ψ_0 is nilpotent, the $\mathbb{C}^*$-fixed point (cf. (2.6))

$$\lim_{k\to 0} k.[F_0, \psi_0] = [E_{L_0}, \phi_{c_0}] = \left[L_0 \oplus L_0^{-1}\Lambda, \begin{pmatrix} 0 & 0 \\ c_0 & 0 \end{pmatrix} \right],$$

where $c_0 = c_{L_0}(\psi_0)$, provides an example of non-very-stable Higgs bundles. The theorem of Hausel-Hitchin says that we can indeed detect this simply by looking at the zero divisor of $c_0 \in H^0(KL_0^{-2}\Lambda)$: a $\mathbb{C}^*$-fixed point $[E_M, \phi_{c'}]$ is very-stable if and only if the zero divisor of c' is reduced.

Chapter 5

Holomorphic connections, projective connections, projective structures and SL-operators

5.1 Projective connections and projective structures

In the following, we recall the notions of projective connections and projective structures, two geometric objects that naturally realize monodromy representations in $PSL_2(\mathbb{C}) \cong PGL_2(\mathbb{C})$.

DEFINITION 5.1. A projective connection on X is a pair (P, s) where P is a flat $PSL_2(\mathbb{C})$-bundle on X with $\mathbb{P}^1$-fibers and $s : X \to P$ is a global holomorphic section which is not parallel w.r.t. the flat structure of P. The points where $ds = 0$ are called apparent singularities; the order of an apparent singularity is the order of the zero of ds at that point.

We denote by div$((P, s))$ the divisor of apparent singularities, counted with multiplicity, of a projective connection (P, s). Two projective connections (P_1, s_1) and (P_2, s_2) are isomorphic if there exists an isomorphism $P_1 \to P_2$ of holomorphic fiber bundles that commutes with s_1 and s_2. Clearly if (P_1, s_1) is isomorphic to (P_2, s_2), then div$((P_1, s_1)) =$ div$((P_2, s_2))$.

DEFINITION 5.2. A projective structure is a maximal atlas $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of X together with local holomorphic functions $\{w_\alpha : U_\alpha \to \mathbb{C}\}$ the values of which are related by Möbius transformations, i.e. for all $x \in U_\alpha \cap U_\beta$,

$$w_\beta(x) = \frac{a_{\beta\alpha} w_\alpha(x) + b_{\beta\alpha}}{c_{\beta\alpha} w_\alpha(x) + d_{\beta\alpha}}, \qquad \begin{pmatrix} a_{\beta\alpha} & b_{\beta\alpha} \\ c_{\beta\alpha} & d_{\beta\alpha} \end{pmatrix} \in SL_2(\mathbb{C}). \tag{5.1}$$

Apparent singularities are where $dw_\alpha = 0$; the order of an apparent singularity is the order of the zero of dw_α at that point.

Note that w_α can serve as a local coordinate on U_α if and only if U_α contains no apparent singularity. It is clear that a projective structure is determined once a local holomorphic function $w_\alpha : U_\alpha \to \mathbb{C}$ is determined. Given a point x and the analytic continuation $[\gamma].w_\alpha$ of w_α

along a closed path γ containing x, the composition along γ of the Möbius transformations maps $([\gamma].w_\alpha)(x)$ back to $w_\alpha(x)$. In other words, the composition along a closed path of the Möbius transformations is the inverse of the $PSL_2(\mathbb{C})$-action defined by analytic continuation. The holomorphic function $\tilde{w}_\alpha : \tilde{X} \to \mathbb{P}^1$, where $\tilde{X}$ is the universal cover of X, defined by analytically continuing w_α is called a developing map of the projective structure.

Given a projective connection (P, s), by letting s be represented by local holomorphic functions on charts of a sufficiently refined atlas of X, one can define local holomorphic functions the values of which are related by constant $PSL_2(\mathbb{C})$-valued transition functions. A maximal atlas together with these local holomorphic functions define a projective structure. Conversely, from the local holomorphic functions of a projective structure one can define a section of a flat bundle with $\mathbb{P}^1$-fibers, the constant $PSL_2(\mathbb{C})$-valued transition functions of which are the Möbius transformations. In other words, a projective structure is equivalent to an isomorphism class of projective connections.

5.2 SL-operators

DEFINITION 5.3. Let N be a line bundle on X, defined via transition functions $(N)_{\alpha\beta}$ over a coordinate covering $\mathcal{U} = \{(U_\alpha, z_\alpha)\}_{\alpha\in\mathcal{I}}$ of X. An SL-operator $\mathcal{D}$ on N for the coordinate covering $\mathcal{U}$ is a collection of meromorphic differential operators $\{\mathcal{D}_\alpha = \partial^2_{z_\alpha} + q_\alpha(z_\alpha)\}_{\alpha\in\mathcal{I}}$ such that $f_\alpha(z_\alpha)$ is a solution to $\mathcal{D}_\alpha$ if and only if $(N)_{\beta\alpha} f_\alpha$ is a solution to $\mathcal{D}_\beta \mid_{U_\alpha \cap U_\beta}$. The points at which $\left(\frac{f_{\alpha,1}}{f_{\alpha,2}}\right)' = 0$, where $f_{\alpha,1}$ and $f_{\alpha,2}$ are two linearly independent solutions of $\mathcal{D}_\alpha$, are called apparent singularities, with the order of the apparent singularity defined to be the order of the zero of $\left(\frac{f_{\alpha,1}}{f_{\alpha,2}}\right)'$. Two SL-operators on N for two coordinate coverings are considered equivalent if their union is also an SL-operator. An SL-operator on N is an equivalence class of SL-operators on N for different coverings.

We denote by $\mathrm{div}(\mathcal{D})$ the divisor formed by apparent singularities, counted with multiplicity, of the SL-operator $\mathcal{D}$.

REMARK 5.1. It was shown in [39] that an SL-operator exists on a line bundle N if and only if $\deg(N) = 1 - g$, such as $N \cong K^{-1/2}$. Such an SL-operator then can be regarded as a differential operator $N \to NK^2$ whose principal symbol is 1 and subprincipal symbol is 0 [24].

It is clear from the definition of SL-operators on N that, if N' is a flat line bundle with constant transition functions, the same collection of local differential operators would define SL-operators on both N and $N \otimes N'$. Hence the specific line bundle on which these local differential operators act is not of interest to us. From now on, we will fix $N = K^{-1/2}$ and by an SL-operator we will mean an SL-operator on $K^{-1/2}$.

We now elaborate on the explicit forms of an SL-operator before showing its equivalence to the notions of projective connections and projective structures.

Transformation rules of local differential operators. Similar to projective connections and projective structures, an SL-operator is determined once we know its local form $\mathcal{D}_\alpha = \partial_{z_\alpha}^2 + q_\alpha(z_\alpha)$ over one coordinate open set (U_α, z_α). This is because on the overlap $U_\alpha \cap U_\beta$,

$$2q_\beta(z_\beta(z_\alpha))(z'_\beta(z_\alpha))^2 = 2q_\alpha(z_\alpha) - \{z_\beta, z_\alpha\} \tag{5.2}$$

where $\{g(z), z\} := \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$ is the Schwarzian derivative of a function $g(z)$. To see this, note that $\{w(z_\alpha), z_\alpha\} = 2q_\alpha(z_\alpha)$ where $w := \frac{f_{\alpha,1}}{f_{\alpha,2}}$ is the ratio of two linearly independent solutions to $\mathcal{D}_\alpha$ [58]. On the overlap $U_{\alpha\beta}$, since $f_{\beta,1} = l_{\beta\alpha} f_{\alpha,1}$ and $f_{\beta,2} = l_{\beta\alpha} f_{\alpha,2}$ are solutions to $\mathcal{D}_\beta$, we also have $\{w(z_\beta), z_\beta\} = 2q_\beta(z_\beta)$ as $w = \frac{f_{\beta,1}}{f_{\beta,2}}$. Then (5.2) follows from the transformations upon a change of coordinates $z_\alpha \mapsto z_\beta(z_\alpha)$ of a Schwarzian derivative

$$\{g(z_\beta(z_\alpha)), z_\alpha\} = (z'_\beta)^2 \{g(z_\beta), z_\beta\} + \{z_\beta, z_\alpha\}.$$

There are two important consequences of (5.2) that we will exploit heavily in this work. One is that if an SL-operator $\mathcal{D}$ has no apparent singularity on U_α and hence the ratio w_α of two linearly independent local solutions can serve as a local coordinate, then $\mathcal{D}$ takes the simple form $\partial_{w_\alpha}^2$ in terms of the local coordinate w_α. Another is that in case the change of coordinate $z_\alpha \mapsto z_\beta(z_\alpha)$ is a Möbius transformation, which occurs if and only if $\{z_\beta, z_\alpha\} = 0$, it follows from (5.2) that $\{q_\alpha(z_\alpha)\}$ glue into a quadratic differential whose local forms are $q_\alpha(z_\alpha) dz_\alpha^2$.

Hence if the coordinate covering $\{(U_\alpha, w_\alpha)\}$ is a holomorphic projective structure, i.e. one that has no apparent singularity, then the collection of local differential operators $\{\partial_{w_\alpha}^2\}$ defines an SL-operator. Any other collection of local differential operators $\{\partial_{w_\alpha}^2 + q_\alpha(w_\alpha)\}$ defines an

SL-operator if and only if $\{q_\alpha(z_\alpha)dz_\alpha^2\}$ glue into a meromorphic quadratic differential.

Local forms near apparent singularities. It follows from (5.2) that, given a local coordinate z, the SL-operator takes the form $D_z = \partial_z^2 + \frac{1}{2}\{w(z), z\}$ for w being the ratio of two linearly independent solutions. Suppose now $z = 0$ is an apparent singularity. The Laurent tail of $q(z) = \frac{1}{2}\{w(z), z\}$ at $z = 0$ depends on the order of this apparent singularity, i.e. the order of the zero of w' at $z = 0$. If it is a simple apparent singularity, i.e. $w(z) = \sum_{k\geq 0} w_k z^k$ with $w_1 = 0$ and $w_2 \neq 0$, then

$$D_z = \partial_z^2 - \frac{3}{4z^2} + \frac{\mu}{z} + q_0 + \mathcal{O}(z) \tag{5.3a}$$

where $\mu = -\frac{3w_3}{4w_2}$ and $q_0 = -(\frac{3w_3}{4w_2})^2$. Hence

$$\mu^2 + q_0 = 0. \tag{5.3b}$$

On the other hand, if $z = 0$ is a double apparent singularity, i.e. $w = \sum_{k\geq 0} w_k z^k$ with $w_1 = w_2 = 0$ and $w_3 \neq 0$, then

$$D_z = \partial_z^2 - \frac{2}{z^2} + \frac{2\nu}{z} + q_0 + q_1 z + \mathcal{O}(z^2) \tag{5.4a}$$

where

$$\nu = -\frac{2w_4}{3w_3}, \qquad q_0 = \frac{4w_4^2 - 15w_3w_5}{9w_3^2}, \qquad q_1 = \frac{4(8w_4^3 - 15w_3w_4w_5)}{27w_3^3}.$$

One can check that

$$\nu^3 + q_0\nu + \frac{q_1}{2} = 0. \tag{5.4b}$$

Although the specific coefficients of the Laurent tails (5.3a) and (5.4a) depend on the coordinate z via the expansion of $w(z)$, the polynomial constraints (5.3b) and (5.4b) they satisfy are invariant upon a change of coordinates.

Using the Frobenius method one can obtain the Laurent expansions of the solutions around an apparent singularity. If $z = 0$ is a simple apparent singularity, solutions to D_z of the form (5.3) have the form

$$z^{-1/2}\left(F_0+\mu F_0 z+F_2 z^2+\mathcal{O}(z)\right),$$

where F_0 and F_2 are the free parameters on which all higher order coefficients depend. If $z=0$ is a double apparent singularity, solutions to D_z of the form (5.4) have the form

$$z^{-1}\left[F_0+\lambda F_0 z+\left(\lambda^2+\frac{q_0}{2}\right)z^2+F_3 z^3+\mathcal{O}(z^4)\right],$$

where F_0 and F_3 are the free parameters on which all higher order coefficients depend. One can check explicitly that the Wronskian $W(f_1,f_2)$ of two linearly independent solutions f_1 and f_2 of these forms is holomorphic at $z=0$, and hence the derivative of the ratio of two such solutions

$$\left(\frac{f_1}{f_2}\right)'=\frac{W(f_1,f_2)}{f_2^2} \tag{5.5}$$

admits $z=0$ as its simple and double zeroes, respectively.

Correspondence to projective structures. A projective structure $\{U_\alpha,w_\alpha\}_{\alpha\in\mathcal{I}}$ gives rise to an SL-operator as follows. On each U_α containing no apparent singularity we define a differential operator $\mathcal{D}_\alpha:=\partial_{w_\alpha}^2$, and on each U_γ containing some apparent singularities we use a local coordinate z_γ and define the differential operator $\mathcal{D}_\gamma=\partial_{z_\gamma}^2+\frac{1}{2}\{w_\gamma(z_\gamma),z_\gamma\}$.

On the other hand, given an SL-operator $\{\mathcal{D}_\alpha\}$ on a line bundle N over a coordinate covering $\mathcal{U}=\{(U_\alpha,z_\alpha)\}$, one can define a projective structure by taking the ratios $w_\alpha=\frac{f_{\alpha,1}}{f_{\alpha,2}}$ of two linearly independent solutions to $\mathcal{D}_\alpha$ with Wronskian $W(f_{\alpha,1},f_{\alpha,2})=1$. It follows from (5.5) that w_α is a holomorphic function $U_\alpha\to\mathbb{C}$, and dw_α vanishes at the apparent singularity of the SL-operator with the order equal to the order of the apparent singularities of $\mathcal{D}_\alpha$. The local functions w_α and w_β defined this way are related by a Möbius transformation since $f_{\beta,1}$ and $f_{\beta,2}$ are linear combinations of $f_{\alpha,1}$ and $f_{\alpha,2}$ scaled by $(N)_{\beta\alpha}$. Hence a projective structure is equivalent to an isomorphism class of SL-operators, with the positions and order of apparent singularities matched. The projective monodromy representation $\pi_1(X)\to PSL_2$ that are inherent in the notions of a projective connection and projective structure is realized in an SL-operator via the ratios w_α of local solutions. Note that these ratios of local solutions are holomorphic at apparent singularities, and hence the projective monodromy representations do not detect these singularities.

EXAMPLE 5.2. On a compact Riemann surface X of genus $g \geq 2$, the most distinguished projective connection is induced by the uniformization theorem, which realizes X as a quotient of the upper-half plane and equips on it a distinguished maximal coordinate atlas $\{(U_\alpha, x_\alpha)\}$ from the upper-half plane. We call this the uniformizing projective structure of X. Since nowhere $dx_\alpha = 0$, the projective structure $\{(U_\alpha, x_\alpha)\}$ has no apparent singularity. The corresponding SL-operator takes the form $\partial^2_{x_\alpha}$ in each U_α. This uniformizing projective structure is distinguished in the sense that the projective coordinates x_α all take values in the upper half-plane.

It follows from (5.2) that if q is a holomorphic quadratic differential, then $\mathcal{D}_\alpha \coloneqq \partial^2_{x_\alpha} + q(x_\alpha)$ glue into an SL-operator that has no apparent singularity. These form the space of holomorphic projective connections, i.e. those that have no apparent singularities, which is an affine space modeled over $H^0(K^2) \cong \mathbb{C}^{3g-3}$. The isomorphism class of each such SL-operator is equivalent to a maximal coordinate atlas that is also a projective structure.

REMARK 5.3. Although we do not need to be specific about the line bundle on which an SL-operator acts and hence can assume it to be $K^{-1/2}$, given a coordinate atlas subordinate to a holomorphic projective structure $\{(U_\alpha, x_\alpha)\}$, one can in fact define explicitly the transition functions of a line bundle N of degree $1-g$ as follows. If U_α and U_β both contain no apparent singularity and $w_\beta = g_{\beta\alpha}.w_\alpha = \frac{a_{\beta\alpha}w_\alpha + b_{\beta\alpha}}{c_{\beta\alpha}w_\alpha + d_{\beta\alpha}}$ with $g_{\beta\alpha} = \begin{pmatrix} a_{\beta\alpha} & b_{\beta\alpha} \\ c_{\beta\alpha} & d_{\beta\alpha} \end{pmatrix} \in SL_2$, then we define $(N)_{\alpha\beta} \coloneqq c_{\beta\alpha}w_\alpha + d_{\beta\alpha}$. Then $(N)_{\alpha\beta}$ defined this way satisfy the cocyle conditions. More crucially, $(N)_{\alpha\beta}$ and $(N)_{\alpha\beta}w_\beta$ are linear combinations of 1 and w_α and hence are solutions to $\mathcal{D}_\alpha \mid_{U_{\alpha\beta}}$: hence if f_β is a solution of $\mathcal{D}_\beta$ then $(N)_{\alpha\beta}f_\beta$ is a solution of $\mathcal{D}_\alpha \mid_{U_\alpha \cap U_\beta}$.

For U_γ containing some simple apparent singularities and U_α not containing any apparent singularities, it is slightly more complicated to define $(N)_{\alpha\gamma}$. First, observe that upon analytic continuation $(N)_{\alpha\gamma}$ has monodromy -1 around a simple apparent singularity contained in U_γ. This is because, around each such apparent singularity, a solution f_γ to $\mathcal{D}_\gamma$ has monodromy -1, while $(N)_{\alpha\gamma}\, f_\gamma$ has trivial monodromy upon analytic continuation since it is a linear combination of 1 and $w_\alpha = \frac{a_{\alpha\gamma}w_\gamma + b_{\alpha\gamma}}{c_{\alpha\gamma}w_\gamma + d_{\alpha\gamma}}$, two solutions to $\partial^2_{w_\alpha}$. One way to make sure the transition functions of N involving U_γ satisfy the cocyle condition is to define a covering $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ such that U_γ contains an even number of apparent singularities, counted with multiplicity, and these apparent singularities are not contained in any other open set $U_{\alpha\neq\gamma}$. Under these conditions, if $f_{\gamma,1}$ and $f_{\gamma,2}$ are two solutions to $\mathcal{D}_\gamma$ with $W(f_{\gamma,1}, f_{\gamma,2}) = 1$ and $\frac{f_{\gamma,1}}{f_{\gamma,2}} = w_\gamma = \frac{a_{\gamma\alpha}w_\alpha + b_{\gamma\alpha}}{c_{\gamma\alpha}w_\alpha + d_{\gamma\alpha}}$, we define $(N)_{\alpha\gamma} = (c_{\gamma\alpha}w_\alpha + d_{\gamma\alpha})\, f'_{\gamma,1} - (a_{\gamma\alpha}w_\alpha + b_{\gamma\alpha})\, f'_{\gamma,2}$. Then $(N)_{\alpha\gamma}f_{\gamma,1}$ and $(N)_{\alpha\gamma}f_{\gamma,2}$ are linear combinations of w_α and 1 and hence are solutions of $\mathcal{D}_\alpha$ as desired. These local differential operators satisfy the transformation rules (5.2), and we hence have defined an SL-operator on N over the coordinate covering $\{U_\alpha\}_{\alpha\in\mathcal{I}}$.

Linearization. Let us from now on again fix $N = K^{-1/2}$. It is well-known that a local differential operator $\partial^2_{z_\alpha} + q_\alpha(z_\alpha)$ is equivalent to the local linear differential operator

$$\partial_{z_\alpha} + \begin{pmatrix} 0 & -q_\alpha(z_\alpha) \\ 1 & 0 \end{pmatrix}. \tag{5.6}$$

A solution $f_\alpha(z_\alpha)$ to $\partial^2_{z_\alpha} + q_\alpha(z_\alpha)$ defines a solution $\begin{pmatrix} -f'_\alpha \\ f_\alpha \end{pmatrix}$ to (5.6). Given an SL-operator $\mathcal{D} = \{\mathcal{D}_\alpha = \partial^2_{z_\alpha} + q_\alpha(z_\alpha)\}$, extending the linearization (5.6) to all of X defines a meromorphic connection $\nabla_{\mathcal{D}}$ on a holomorphic bundle F_{op}, which is the unique up to scaling non-trivial extension of $N = K^{-1/2}$ by $K^{1/2}$.

To see this, note that given two solutions f_α and f_β to $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ that represent the same section of N, i.e. $f_\alpha = (N)_{\alpha\beta}f_\beta(z_\beta)$, the transition function $(F_{op})_{\alpha\beta}$ maps the flat section $\begin{pmatrix} -f'_\beta(z_\beta) \\ f_\beta(z_\beta) \end{pmatrix}$ into $\begin{pmatrix} -f'_\alpha(z_\alpha) \\ f_\alpha(z_\alpha) \end{pmatrix}$. It follows that

$$(F_{op})_{\alpha\beta} = \begin{pmatrix} z'_\beta (N)_{\alpha\beta} & -(N)'_{\alpha\beta} \\ 0 & (N)_{\alpha\beta} \end{pmatrix} \tag{5.7}$$

where the derivatives of z_β and $(N)_{\alpha\beta}$ are with respect to z_α. Since the canonical line bundle K

can be characterized by transition functions of the form $z'_\beta(z_\alpha)$, (5.7) defines F_{op} as an extension

$$0 \to KN \cong K^{1/2} \to F_{op} \to N = K^{-1/2} \to 0. \tag{5.8}$$

One can check that if we set $(N)_{\alpha\beta} = \Big(z'_\beta(z_\alpha)\Big)^{-1/2}$, then upon conjugation with (5.7) the local forms of the connection transform as $\partial_{z_\alpha} + \begin{pmatrix} 0 & -q_\alpha(z_\alpha) \\ 1 & 0 \end{pmatrix} \mapsto \partial_{z_\beta} + \begin{pmatrix} 0 & -q_\beta(z_\beta) \\ 1 & 0 \end{pmatrix}$ with $q_\beta(z_\alpha)$ and $q_\alpha(z_\alpha)$ following the transformation rules (5.2). In other words, we have defined F_{op} such that there exists a connection $\nabla_{\mathcal{D}}$ that on U_α takes the form $\partial_{z_\alpha} + \begin{pmatrix} 0 & -q_\alpha(z_\alpha) \\ 1 & 0 \end{pmatrix}$ in certain local frames adapted to $K^{1/2} \hookrightarrow F_{op}$.

EXAMPLE 5.4. If $\mathcal{D} = \{\mathcal{D}_\alpha\}$ has no apparent singularity, then in particular $\nabla_{\mathcal{D}}$ is a holomorphic connection on F_{op}. Such a holomorphic connection $(F_{op}, \nabla_{\mathcal{D}})$ is called an oper. Let $\{(U_\alpha, w_\alpha)\}$ be a coordinate atlas induced by ratios of local solutions to $\mathcal{D}$. If $w_\beta = \frac{a_{\beta\alpha} w_\alpha + b_{\beta\alpha}}{c_{\beta\alpha} w_\alpha + d_{\beta\alpha}}$ then the transition function of $N = K^{-1/2}$ can be taken to be $(N)_{\alpha\beta} = c_{\beta\alpha} w_\alpha + d_{\beta\alpha}$. It follows that the transition of F_{op} takes the form $(F_{op})_{\alpha\beta} = \begin{pmatrix} (c_{\beta\alpha} w_\alpha + d_{\beta\alpha})^{-1} & -c_{\beta\alpha} \\ 0 & c_{\beta\alpha} w_\alpha + d_{\beta\alpha} \end{pmatrix}$. On each U_α the connection $\nabla_{\mathcal{D}}$ takes the form $\partial_{w_\alpha} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and the flat sections $\begin{pmatrix} -1 \\ w_\alpha \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span a local flat frame. Gauge transformation of the form $g_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & w_\alpha \end{pmatrix}$ switch between local frames adapted to $K^{1/2} \hookrightarrow F_{op}$ and these local flat frames, with respect to which the flat structure of $(F_{op}, \nabla_{\mathcal{D}})$ can be characterized by constant transition functions

$$(E_{op, \nabla_{\mathcal{D}}})_{\alpha\beta} = g_\alpha^{-1} \, (F_{op})_{\alpha\beta} \, g_\beta = \begin{pmatrix} d_{\beta\alpha} & b_{\beta\alpha} \\ c_{\beta\alpha} & a_{\beta\alpha} \end{pmatrix}. \tag{5.9}$$

Note that one can write

$$w_\alpha = \frac{d_{\beta\alpha} w_\beta - b_{\beta\alpha}}{(-c_{\beta\alpha}) w_\beta + a_{\beta\alpha}}$$

and hence (5.9) gives the transition functions of the coordinate atlas $\{(U_\alpha, -w_\alpha)\}$, which is subordinate to the same projective structure as $\{(U_\alpha, w_\alpha)\}$.

REMARK 5.5. If (U_γ, w_γ) contains some apparent singularities, then $\nabla_{\mathcal{D}}$ is meromorphic on U_γ and takes the form $\partial_{z_\gamma} + \begin{pmatrix} 0 & -\frac{1}{2}\{w_\gamma, z_\gamma\} \\ 1 & 0 \end{pmatrix}$. Similar to the discussion in remark 5.3, to define transition functions of F_{op} that satisfy the cocyle conditions, one can require U_γ to contain an even number of apparent singularities and that these apparent singularities are not contained in any other open set $U_{\alpha\neq\gamma}$. An explicit transition function can be defined by requiring that, given solutions $f_{\gamma,1}$ and $f_{\gamma,2}$ to $\mathcal{D}_\gamma$ such that $W(f_{\gamma,1}, f_{\gamma,2}) = 1$ and $w_\gamma = \frac{f_{\gamma,1}}{f_{\gamma,1}}$, the columns of the matrix $(F_{op})_{\beta\gamma} \begin{pmatrix} -f'_{\gamma,2} & -f'_{\gamma,1} \\ f_{\gamma,2} & f_{\gamma,1} \end{pmatrix}$ are flat sections of $\partial_{z_\beta} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This implies $(F_{op})_{\beta\gamma} \begin{pmatrix} -f'_{\gamma,2} & -f'_{\gamma,1} \\ f_{\gamma,2} & f_{\gamma,1} \end{pmatrix} = \begin{pmatrix} -c_{\gamma\beta} & -a_{\gamma\beta} \\ c_{\gamma\beta}w_\beta + d_{\gamma\beta} & a_{\gamma\beta}w_\beta + b_{\gamma\beta} \end{pmatrix}$, and hence the transition function of F_{op} w.r.t. local frames adapted to (5.8) is

$$(F_{op})_{\beta\gamma} = \begin{pmatrix} -c_{\gamma\beta}f_{\gamma,1} + a_{\gamma\beta}f_{\gamma,2} & -c_{\gamma\beta}f'_{\gamma,1} + a_{\gamma\beta}f'_{\gamma,2} \\ 0 & (c_{\gamma\beta}w_\beta + d_{\gamma\beta})\, f'_{\gamma,1} - (a_{\gamma\beta}w_\beta + b_{\gamma\beta})\, f'_{\gamma,2} \end{pmatrix}. \tag{5.10}$$

Note that this is compatible to the discussion in remark 5.3.

Not any SL-operator $\mathcal{D}$ can be linearized into a meromorphic connection $\nabla_{\mathcal{D}}$ of the above form on F_{op}. For example, if $\mathcal{D}$ has an odd number of apparent singularities, then we cannot always group an even number of apparent singularities to apply (5.10); this is an expression of this failure to linearize. However, if the projective monodromy of $\mathcal{D}$ has a lift to SL_2, then such a linearization exists. Indeed, as will be shown in the following, there exists an $SL_2(\mathbb{C})$-holomorphic connection (F, ∇) together with a subbundle $L \hookrightarrow F$ such that its projectivization gives a projective connection equivalent to $\mathcal{D}$. The number of apparent singularities will be even in this case.

5.3 Holomorphic connections and monodromy representations to $SL_2(\mathbb{C})$

Let F^s be a smooth rank-2 bundle on X with a fixed isomorphism between its determinant line bundle $\det(F^s)$ and the trivial line bundle. We will let F^s be the underlying smooth objects for the following holomorphic objects.

An $SL_2(\mathbb{C})$-holomorphic connection ∇ on an $SL_2(\mathbb{C})$-holomorphic bundle F is a $\mathbb{C}$-linear map $\nabla : F \to FK$ satisfying the Leibniz rule $\nabla(fs) = \partial f \otimes s + f\nabla s$ for any local holomorphic function f and local section s of F, such that the induced connection on $\det(F^s)$ is the trivial connection $\partial : f \mapsto f(z)dz$. Since there is no $(2,0)$-form on a Riemann surface, (F, ∇) is automatically a flat connection and gives rise to a monodromy representation $\check{\rho}_{(F,\nabla)} : \pi_1 \to SL_2(\mathbb{C})$ via developing local parallel frames. In this case, we say (F, ∇) realizes the monodromy representation $\check{\rho} = \check{\rho}_{(F,\nabla)}$.

By changing to local frames of F that consist of local parallel sections and using a new set of transition functions with respect to these local flat frames, one can define a flat bundle F^{∇} with constant $SL_2(\mathbb{C})$-valued transition functions, the composition of which along a closed path γ is $\check{\rho}_{(F,\nabla)}([\gamma])$. Since the transition functions of F^{∇} are constant, ∂ is a well-defined holomorphic connection on F^{∇}. The changes of local frames define an isomorphism $F \xrightarrow{f} F^{\nabla}$ such that $\nabla = f^{-1} \circ \partial \circ f$, and we say (F, ∇) and (F^{∇}, ∂) are isomorphic as holomorphic connections. In general, for two $SL_2(\mathbb{C})$-holomorphic connections (F_1, ∇_1) and (F_2, ∇_2), we say they are isomorphic if there exists an smooth automorphism of the underlying bundle F^s relating ∇_1 and ∇_2.

On the other hand, given a monodromy representation $\check{\rho} : \pi_1(X) \to SL_2(\mathbb{C})$, it is straightforward to define a flat rank-2 bundle $F^{\check{\rho}}$ with constant $SL_2(\mathbb{C})$-valued transition functions that realizes $\check{\rho}$. Clearly a holomorphic connection (F, ∇) realizes $\check{\rho}$ if and only (F, ∇) is isomorphic to $(F^{\check{\rho}}, \partial)$ as holomorphic connections.

Moduli spaces. Let $\mathcal{C}_{F^s}$ be the set of flat connections on the smooth rank-2 bundle F^s that induces the trivial connection on the trivial line bundle $\det(F_s)$. Then $\mathcal{C}_{F^s}$ is an infinite dimensional affine space modeled on $\Omega^1(X, \mathrm{End}_0(F_s))$, where $\mathrm{End}_0(F_s)$ is the bundle of traceless endomorphisms of F_s. It has a complex structure induced by $SL_2(\mathbb{C})$ [58] and admits an action by conjugation from the gauge group $\mathcal{G}(F_s)$ of smooth automorphisms of F_s that act trivially on $\det(F_s)$. We define the de Rham moduli space by

$$\mathcal{M}_{dR} = \mathcal{C}_{F^s}^{irr} / \mathcal{G}(F_s),$$

where $\mathcal{C}_{F^s}^{irr} \subset \mathcal{C}_{F^s}$ is the subspace of irreducible flat connections, i.e. ones that leave no smooth subbundle of F^s invariant. It is known that $\mathcal{M}_{dR}$ is a smooth complex analytic space of complex dimension $6g-6$ [53] [9]. By the above discussion, $\mathcal{M}_{dR}$ is also the moduli space of irreducible $SL_2(\mathbb{C})$-holomorphic connections, i.e. ones that leave no holomorphic subbundle of the underlying $SL_2(\mathbb{C})$-holomorphic bundle invariant.

Consider the set $\mathrm{Hom}(\pi_1, SL_2(\mathbb{C}))$ of homomorphisms from π_1 to $SL_2(\mathbb{C})$. It has the structure of an affine variety, and admits an action from $SL_2(\mathbb{C})$ by conjugation. We define the $SL_2(\mathbb{C})$-representation variety by

$$\mathcal{R}_{SL_2(\mathbb{C})} = \mathrm{Hom}^{irr}(\pi_1, SL_2(\mathbb{C}))//SL_2(\mathbb{C})$$

where $\mathrm{Hom}^{irr}(\pi_1, SL_2(\mathbb{C}))$ is the subspace of irreducible monodromy representations and the double slash indicates invariant theoretic quotient [53] [58]. It is known that $\mathcal{R}_{SL_2(\mathbb{C})}$ is an irreducible affine variety of complex dimension $6g-6$. Taking the monodromy representation of an irreducible $SL_2(\mathbb{C})$-holomorphic connection gives an irreducible monodromy representation in $SL_2(\mathbb{C})$, and this defines a homeomorphism $\mathcal{M}_{dR} \to \mathcal{R}_{SL_2(\mathbb{C})}$ [58].

Simpson's stratification on $\mathcal{M}_{dR}$. There exists a natural stratification on $\mathcal{M}_{dR}$ that is very similar to the stratification on $\mathcal{M}_H$ described in chapter 2. Simpson [54] defines this stratification in terms of λ-connections, which we summarize as follows.

A λ-connection is a triple $(\lambda, F, \nabla_\lambda)$, where $\lambda \in \mathbb{C}$, F is a holomorphic bundle, and $\nabla_\lambda : F \to FK$ is a map between sheaves of holomorphic sections satisfying a λ-scaled Leibniz rule $\nabla_\lambda(fs) = \lambda \partial f \otimes s + f \nabla_\lambda s$ for any local holomorphic function f and local section s of F. Hence a $(\lambda = 0)$-connection is a Higgs bundle and a $(\lambda = 1)$-connection is a holomorphic connection. The Hodge moduli space $\mathcal{M}_{Hod}$ of irreducible λ-connections has a projection $\mathcal{M}_{Hod} \to \mathbb{C}$ that picks out the factor λ, with the fibers over 0 and 1 being $\mathcal{M}_H(\mathcal{O}_X)$ and $\mathcal{M}_{dR}$ respectively. This projection is equivariant w.r.t. the $\mathbb{C}^*$-action on $\mathcal{M}_{Hod}$ defined by $t.[\lambda, F, \nabla_\lambda] = [t\lambda, F, t\nabla_{t\lambda}]$ for $t \in \mathbb{C}^*$. The set $\mathcal{M}_{Hod}^{\mathbb{C}^*}$ of $\mathbb{C}^*$-fixed points in $\mathcal{M}_{Hod}$ are the same as the set of $\mathbb{C}^*$-fixed points on

$\mathcal{M}_H(\mathcal{O}_X) \subset \mathcal{M}_{Hod}$ (cf. (2.7)), i.e.

$$\mathcal{M}_{Hod}^{\mathbb{C}^*} = \mathcal{M}_H^{\mathbb{C}^*}(\mathcal{O}_X) = \mathcal{N} \cup \left(\underset{d}{\cup} N_d \right).$$

It is known that $\lim\limits_{t \to 0} t.[F, \nabla]$ exists and is contained in $\mathcal{M}_{Hod}^{\mathbb{C}^*}$ for all $[F, \nabla] \in \mathcal{M}_{dR} \subset \mathcal{M}_{Hod}$, where we have regarded the holomorphic connection $[F, \nabla]$ as a $(\lambda = 1)$-connection. If F is stable then $\lim\limits_{t \to 0} t.[F, \nabla] = [F, 0] \in \mathcal{N}$. If F is destabilized by L_F, consider

$$c_{L_F}(\nabla) : L_F \hookrightarrow F \xrightarrow{\nabla} FK \to {L_F}^{-1}K. \tag{5.11}$$

Then it is known that $\lim\limits_{t \to 0} t.[F, \nabla] = [E_{L_F}, \phi_c] \in N_d$ for $c = c_{L_F}(\nabla) \in H^0(KL_F^{-2})$ and $d = \deg(\mathrm{div}(c)) = \deg(KL_F^{-2})$ (recall the definitions of these Higgs bundles in (2.6)). For $\alpha \in \mathcal{M}_{Hod}^{\mathbb{C}^*}$, let $W_\alpha^{dR} \subset \mathcal{M}_{dR}$ consist of all points $[F, \nabla]$ with $\lim\limits_{t \to 0} t.[F, \nabla] = \alpha$. The Simpson's stratification on $\mathcal{M}_{dR}$ is the decomposition

$$\mathcal{M}_{dR} = W_{\mathcal{N}}^{dR} \cup \left(\underset{d}{\cup} W_{N_d}^{dR} \right)$$

where $W_{\mathcal{N}}^{dR} = \underset{\alpha \in \mathcal{N}}{\cup} W_\alpha^{dR}$ and $W_{N_d}^{dR} = \underset{\alpha \in N_d}{\cup} W_\alpha^{dR}$. We note that, similar to the stratification on $\mathcal{M}_H$, the degree $d = \deg(KL^{-2})$ of the zero divisor of $c_L(\nabla) \in H^0(KL^{-2})$ for a subbundle L of maximal degree of F tells which stratum a point $[F, \nabla] \in \mathcal{M}_{dR}$ is in: if $0 \leq d \leq 2g - 2$ then $[F, \nabla] \in W_{N_d}^{dR}$, and if $d > 2g - 2$ then $[F, \nabla] \in W_{\mathcal{N}}^{dR}$.

Note that, upon fixing a spin structure $K^{1/2}$, W_0 is a Hitchin section in $\mathcal{M}_H(\mathcal{O}_X)$ and W_0^{dR} is the space of opers in $\mathcal{M}_{dR}$ (cf. example 5.4). The underlying bundles $K^{1/2} \oplus K^{-1/2}$ and F_{op} are the most unstable ones defining the objects in the corresponding moduli spaces.

REMARK 5.6. That $W_\alpha \subset \mathcal{M}_H$ and $W_\alpha^{dR} \subset \mathcal{M}_{dR}$ are biholomorphic via the so-called "conformal limit" is proved for the case α being a stable Higgs bundle by [9]. The first example of this biholomorphism is for the case where α is the intersection of a Hitchin section with the nilpotent cone, i.e. a biholomorphism between a Hitchin section and a space of opers. This was first conjectured by Gaiotto [27] and proved by Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke [14].

5.4 From holomorphic connections to projective connections

Lifts of projective monodromy representations. Given a projective monodromy representation $\rho : \pi_1 \to PSL_2(\mathbb{C})$, we say $\check{\rho} : \pi_1 \to SL_2(\mathbb{C})$ is a lift of ρ if ρ is equal to the composition of $\check{\rho}$ with the projection $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$. Concretely, it means that for generators $A_{i=1,...,g}$, $B_{j=1,...,g}$ of π_1 representing a basis of cycles of X, one can find $\check{\rho}(A_i), \check{\rho}(B_j) \in SL_2(\mathbb{C})$ that projects to $\rho(A_i), \rho(B_j) \in PSL_2(\mathbb{C})$ and satisfy

$$[\check{\rho}(A_g), \check{\rho}(B_g)] \dots [\check{\rho}(A_1), \check{\rho}(B_1)] = 1. \tag{5.12}$$

A projective monodromy representation ρ, if it has a lift to $SL_2(\mathbb{C})$, has in total 2^{2g} lifts modulo conjugation: these lifts correspond to the freedom to choose the sign $\{\pm\}$ for the lifts of $\rho(A_i)$ and $\rho(B_j)$ to $SL_2(\mathbb{C})$, since these signs cancel after taking the commutator in (5.12). On the other hand, two monodromy representations $\check{\rho}_1, \check{\rho}_2 : \pi_1 \to SL_2(\mathbb{C})$ are lifts of the same projective monodromy representation if and only if $\check{\rho}_1([C]) = \pm\check{\rho}_2([C])$ for any generator $C \in \{A_1, ..., A_g, B_1, ..., B_g\}$. In practice, we will care about lifts of projective monodromy representations modulo conjugation.

LEMMA 5.1. *Two $SL_2(\mathbb{C})$-holomorphic connections (F_1, ∇_1), (F_2, ∇_2) have monodromy representations $\check{\rho}_1, \check{\rho}_2 : \pi_1 \to SL_2(\mathbb{C})$ that are lifts of the same projective monodromy representation up to conjugation if and only if $F_1 \cong F_2 \otimes N$ for some line bundle N with $N^2 \cong \mathcal{O}_X$.*

Proof. The condition $\check{\rho}_1([\gamma]) = \pm\check{\rho}_2([\gamma])$ for any closed path γ is equivalent to the fact that the two flat bundles $F_1^{\nabla_1}$ and $F_2^{\nabla_2}$ are such that $F_1^{\nabla_1} \cong F_2^{\nabla_2} \otimes N$ where N is a flat line bundle with constant transition functions valued in $\{\pm 1\}$. This occurs if and only if $N^2 \cong \mathcal{O}_X$. □

Projectivization. Given a holomorphic connection (F, ∇), we can projectivize the $SL_2(\mathbb{C})$-flat bundle F^∇ to obtain a $PSL_2(\mathbb{C})$-flat bundle $\mathbb{P}(F^\nabla)$, the $\mathbb{P}^1$-fibers of which are the projectivization of the fibers of F^∇. The constant $PSL_2(\mathbb{C})$-valued transition functions of $\mathbb{P}(F^\nabla)$ are induced by the constant $SL_2(\mathbb{C})$-valued transition functions of F^∇ via the projection $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$. Let $L \hookrightarrow F$ be a subbundle that is not invariant by ∇, and $L \cong L^\nabla \hookrightarrow F^\nabla$ the corresponding subbundle induced by the isomorphism $F \to F^\nabla$. Then $\mathbb{P}(L^\nabla)$ defines a section of $\mathbb{P}(F^\nabla)$ and hence a projective connection. Conversely, if (P, s) is a projective connection with projective monodromy representation which has a lift to $SL_2(\mathbb{C})$, then (P, s) is equivalent to $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$ for some holomorphic connection (F, ∇) and subbundle L of F.

Irreducible projective monodromy representations. We say a projective monodromy representation $\rho : \pi_1 \to PSL_2(\mathbb{C})$ is reducible if $\rho(\pi_1) \subset PSL_2(\mathbb{C}) \equiv Aut(\mathbb{P}^1)$ has a fixed point on $\mathbb{P}^1$, and irreducible if it is not reducible. Clearly if ρ has a lift to $SL_2(\mathbb{C})$, it is irreducible if and only if its lift is irreducible.

Similar to how irreducible monodromy representations in $SL_2(\mathbb{C})$ are in 1-1 correspondence with irreducible flat $SL_2(\mathbb{C})$-connections, irreducible projective monodromy representation are in 1-1 correspondence with flat $PSL_2(\mathbb{C})$-bundles with $\mathbb{P}^1$-fibers that have no global parallel section. Furthermore, if an irreducible projective monodromy representation ρ has a lift $\check{\rho}$ in $SL_2(\mathbb{C})$ which is realized by an irreducible holomorphic connection (F, ∇), then $P = \mathbb{P}(F^\nabla)$ realizes ρ.

Inducing apparent singularities. Given an $SL_2(\mathbb{C})$-holomorphic connection (F, ∇) and a subbundle L of F, the composition

$$c_L(\nabla) : L \hookrightarrow F \xrightarrow{\nabla} FK \to L^{-1}K, \tag{5.13}$$

where the last arrow is induced by the quotient $F \to L^{-1}$ of the embedding $L \hookrightarrow F$, is nonzero if L is not invariant by ∇. Hence in particular if (F, ∇) is irreducible then for all subbundle L of F we have $c_L(\nabla) \neq 0$. A priori, unlike (2.1c) which is $\mathcal{O}_X$-linear, (5.13) is only $\mathbb{C}$-linear since it involves ∇. However, since $FK \to L^{-1}K$ is induced from the quotient map of $L \hookrightarrow F$, overall (5.13) is $\mathcal{O}_X$-linear.

The following proposition shows that the zero divisor of (5.13) is the loci of apparent singularities. In this sense, (5.13) is the analogue of (2.1c), and apparent singularities are the analogues of the projection to X of Baker-Akhiezer divisors.

PROPOSITION 5.2. *Suppose (F, ∇) is a holomorphic connection and $L \hookrightarrow F$ a subbundle not invariant by ∇. Then the divisor of apparent singularities div$((\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla)))$ of the projective connection $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$ coincides with the zero divisor of $c_L(\nabla)$.*

Proof. Observe that if the embedding $L^\nabla \hookrightarrow F^\nabla$ is generated by local sections of the form

$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$, where i_1 and i_2 have no common zero, then upon projectivization $\mathbb{P}(L^\nabla)$ takes the form i_1/i_2. The apparent singularities of $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$ hence are the zeroes of $i_1' i_2 - i_1 i_2'$, counted with multiplicity. If in a neighborhood i_1 (or i_2) is nowhere-vanishing, then $\begin{pmatrix} i_1 & 0 \\ i_2 & 1/i_1 \end{pmatrix}$ (or $\begin{pmatrix} i_1 & -1/i_2 \\ i_2 & 0 \end{pmatrix}$, respectively) switches between the local flat frames of (F^∇, ∂) and local frames adapted to $L^\nabla \hookrightarrow F^\nabla$, in which ∇ takes the form

$$\partial + \begin{pmatrix} \frac{i_1'}{i_1} & 0 \\ i_1 i_2' - i_1' i_2 & -\frac{i_1'}{i_1} \end{pmatrix} \qquad (\text{or } \partial + \begin{pmatrix} \frac{i_2'}{i_2} & 0 \\ i_1 i_2' - i_1' i_2 & -\frac{i_2'}{i_2} \end{pmatrix}, \text{respectively}). \tag{5.14}$$

Since the lower-left component locally represents the composition (5.13), the proposition follows. □

Let (F_1, ∇_1) and (F_2, ∇_2) be two irreducible $SL_2(\mathbb{C})$-holomorphic connections with monodromy representations that project to the same projective monodromy up to conjugation. By lemma 5.1, $F_1 \cong F_2 \otimes N$ where $N^2 \cong \mathcal{O}_X$. If L_2 is a subbundle of F_2, then there exists a subbundle $L_1 \cong L_2 \otimes N$ of F_1 such that $c_{L_1}(\nabla_1)$ is identified with $c_{L_2}(\nabla_2)$ via the isomorphism $KL_1^{-2} \cong KL_2^{-2}$. In particular, the projective connections $(\mathbb{P}(F_1^{\nabla_1}), \mathbb{P}(L_1^{\nabla_1}))$ and $(\mathbb{P}(F_2^{\nabla_2}), \mathbb{P}(L_2^{\nabla_2}))$ are isomorphic.

We say two data $(L_1 \hookrightarrow F_1, \nabla_1)$ and $(L_2 \hookrightarrow F_2, \nabla_2)$ are isomorphic if there is an isomorphism $L_1 \xrightarrow{\sim} L_2$ that commutes with an isomorphism $F_1 \xrightarrow{\sim} F_2$ which makes (F_1, ∇_1) isomorphic to (F_2, ∇_2). The projective connections $(\mathbb{P}(F_1^{\nabla_1}), \mathbb{P}(L_1^{\nabla_1}))$ and $(\mathbb{P}(F_2^{\nabla_2}), \mathbb{P}(L_2^{\nabla_2}))$ clearly are isomorphic if and only if $(L_1 \hookrightarrow F_1, \nabla_1)$ and $(L_2 \hookrightarrow F_2, \nabla_2)$ are isomorphic.

Let $\mathcal{R}^0_{PSL_2(\mathbb{C})}$ be the set of conjugacy classes of projective monodromy representations that are irreducible and have lifts to $SL_2(\mathbb{C})$. Let $\mathcal{M}^0_{(P,s)}$ and $\mathcal{M}^0_P$ be the set of isomorphism classes of projective connections and flat $PSL_2(\mathbb{C})$-bundles with $\mathbb{P}^1$-fibers respectively, whose projective monodromy representations define points in $\mathcal{R}^0_{PSL_2(\mathbb{C})}$.

The proof of the following proposition follows from lemma 5.1 and the above discussion.

Proposition 5.3. *The following diagram, where the first two vertical arrows are defined by* *projectivizing the corresponding data, is commutative.*

$$\begin{tikzcd}
\left\{ \begin{array}{c} (L \hookrightarrow F, \nabla) \mid \\ {[F,\nabla] \in \mathcal{M}_{dR}} \end{array} \right\} / \sim \arrow[r] \arrow[d, "2^{2g}:1"] & \mathcal{M}_{dR} \arrow[r] \arrow[d, "2^{2g}:1"] & \mathcal{R}_{SL_2(\mathbb{C})} \arrow[d, "2^{2g}:1"] \\
\mathcal{M}^0_{(P,s)} \arrow[r] & \mathcal{M}^0_P \arrow[r] & \mathcal{R}^0_{PSL_2(\mathbb{C})}
\end{tikzcd}$$

All vertical arrows are surjective, with points in the same fiber of the first two vertical arrows differing by a twist of a flat line bundle whose square is $\mathcal{O}_X$.

EXAMPLE 5.7. Let $\mathcal{D}$ be an SL-operator having no apparent singularity, and $\left(K^{1/2} \hookrightarrow F_{op}, \nabla_{\mathcal{D}}\right)$ the linearization data of $\mathcal{D}$. We claim that the projective connection defined by projectivizing these data $(P, s) = \left(\mathbb{P}(F_{op}^{\nabla_{\mathcal{D}}}), \mathbb{P}\left((K^{1/2})^{\nabla_{\mathcal{D}}}\right)\right)$ is equivalent to the projective structure defined by $\{(U_\alpha, w_\alpha)\}$, where w_α are ratios of local solutions to $\mathcal{D}$. As discussed in example 5.4, the flat structure of P is characterized by projecting the SL_2-constant transition functions (5.9). On (U_α, w_α) and in the local flat frame that differs from a frame adapted to $K^{1/2}$ by the change of basis $g_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & w_\alpha \end{pmatrix}$, the generator of $(K^{1/2})^{\nabla_{\mathcal{D}}}$ takes the form $g_\alpha^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_\alpha \\ -1 \end{pmatrix}$. The section s is hence locally represented by the local function $-w_\alpha$. Hence the projective connection (P, s) corresponds to the coordinate atlas $\{(U_\alpha, -w_\alpha)\}$, which can also be obtained by locally solving $\mathcal{D}$. Note that this is consistent with the SL_2-constant transition functions (5.9) of $F_{op}^{\nabla_{\mathcal{D}}}$.

REMARK 5.8. Given an irreducible projective monodromy representation $\rho : \pi_1 \to PSL_2(\mathbb{C})$, we can ask how many projective connections with different sets of apparent singularities can realize ρ. If ρ has a lift $\check{\rho}$ in $SL_2(\mathbb{C})$, then it can be realized by the flat bundle $\mathbb{P}(F^\nabla)$ where (F, ∇) is an $SL_2(\mathbb{C})$-holomorphic connection realizing $\check{\rho}$. Different subbundles of F upon projectivization define different sections of $\mathbb{P}(F^\nabla)$, with the maximal subbundle(s) of F defining the section(s) of $\mathbb{P}(F^\nabla)$ with the minimal number of apparent singularities. In particular, for g odd and $\deg(F) = 0$, if F is maximally stable, i.e. $s(F) = g - 1$, then it has exactly 2^g maximal subbundles. In other words, for g odd, a sufficiently generic projective monodromy representation is realized by exactly 2^g projective connections up to isomorphism [33], all of which have $3g - 3$

apparent singularities counted with multiplicity.

COROLLARY 5.4. *Two projective connections with the same irreducible projective monodromy representations up to conjugation that have lifts to $SL_2(\mathbb{C})$ and the same divisors of apparent singularities of degree $< 2g-2$ are isomorphic.*

Proof. Two such projective connections are the projectivization of a holomorphic connection (F,∇) and subbundles L_1, L_2 of F such that $c_{L_1}(\nabla)$, $c_{L_2}(\nabla)$ both vanish at $\mathbf{x}$ and are identified via the isomorphism $KL_1^{-2} \cong \mathcal{O}_X(\mathbf{x}) \cong KL_2^{-2}$. But $\deg(\mathbf{x}) < 2g-2$, therefore L_1 and L_2 destabilize F and must be the same subbundle of F. $\square$

5.5 From holomorphic connections to SL-operators

Associated SL-operators. Given the initial data $(L \hookrightarrow F, \nabla)$, we want to have a concrete construction of an SL-operator that corresponds up to equivalence to the projective connection $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$. To this end, suppose $\nabla = \partial_z + \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}$ in certain local frames adapted to L. Consider the local differential operator $\partial_z^2 + q(z)$ where

$$q(z) = -b(z)c(z) - \left(a(z) - \frac{c'(z)}{2c(z)}\right)^2 - \left(a(z) - \frac{c'(z)}{2c(z)}\right)'. \tag{5.15}$$

We claim that local differential operators of this form define an SL-operator, the isomorphism class of which depends only on the isomorphism class of the data $(L \hookrightarrow F, \nabla)$. To see this, we need to show that $q(z)$, which is defined from the local form of ∇ in some specific local frames of F adapted to L, is invariant upon a change between local frames adapted to L and transforms appropriately upon a change of coordinates. To see this, let $g(z)$ be a gauge transformation of upper-triangular form and determinant equal to 1: it defines a change between two local frames of F adapted to L. An explicit calculation shows that, if $g^{-1}\nabla g = \partial_z + \begin{pmatrix} a_2(z) & b_2(z) \\ c_2(z) & -a_2(z) \end{pmatrix}$, then the function defined by replacing $a(z)$, $b(z)$, $c(z)$ by $a_2(z)$, $b_2(z)$, $c_2(z)$ respectively in (5.15) is equal to $q(z)$. In fact, this shows that if $(L' \hookrightarrow F', \nabla')$ is another data isomorphic to $(L \hookrightarrow F, \nabla)$, we would obtain the same local meromorphic function defined by (5.15).

For the transformation of $q(z)$ upon a change of coordinates $z \mapsto w$, first note that the components $a(z)$, $b(z)$, $c(z)$ in the affine part of the local expression of ∇ are local holomorphic one-forms. Hence what we need to show is $q(z)$ and

$$q_w(w) = -b_w(w)c_w(w) - \left(a_w(w) - \frac{\partial_w c_w(w)}{2c_w(w)}\right)^2 - \partial_w \left(a_w(w) - \frac{\partial_w c_w(w)}{2c_w(w)}\right)$$

where

$$a_w(z) = a(z(w))z'(w), \qquad b_w(z) = b(z(w))z'(w), \qquad c_w(z) = c(z(w))z'(w),$$

satisfy the transformation rule $2q(z(w))z'(w)^2 - 2q_w(w) = \{w, z\}$. An explicit calculation shows that this indeed is the case, and hence $\{\partial_z^2 + q(z)\}$ define an SL-operator.

Now suppose $z = 0$ is a simple zero of $c(z)$. Then the Laurent expansion of (5.15)

$$q(z) = \frac{-3}{4}\frac{1}{z^2} + \left[a(0) - \frac{c''(0)}{4c'(0)}\right]\frac{1}{z} - a(0)^2 + \frac{a(0)c''(0)}{2c'(0)} - \left(\frac{c''(0)}{4c'(0)}\right)^2 + \mathcal{O}(z). \tag{5.16}$$

makes $z = 0$ an apparent singularity. While apparent singularities can be regarded as the analogues of the projection to X of Baker-Akhiezer divisors, the accessory parameter

$$a(0) - \frac{c''(0)}{4c'(0)} \tag{5.17}$$

can be regarded as the analogue of the coordinate $-a(0)$ of the point in the Baker-Akhiezer divisor projecting to $z = 0$. The Laurent expansion of $q(z)$ also makes a double zero of $c_L(\nabla)$ a double apparent singularity. This enables us to make the following definition.

DEFINITION 5.4. Given an irreducible $SL_2(\mathbb{C})$-holomorphic connection (F, ∇) and a subbundle $L \hookrightarrow F$, the associated SL-operator $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$ takes the local form $\partial_z^2 + q(z)$ where $q(z)$ is defined in (5.15). The divisor of apparent singularities $\mathrm{div}(\mathcal{D}_{(L \hookrightarrow F, \nabla)})$ coincides with the zero divisor of $c_L(\nabla)$.

PROPOSITION 5.5. *Let (F, ∇) be an irreducible $SL_2(\mathbb{C})$-holomorphic connection and $L \hookrightarrow F$ a subbundle. Then the associated SL-operator $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$ is equivalent to the isomorphism class of the projective connection* $(\mathbb{P}(F^\nabla), \mathbb{P}(L^\nabla))$.

Proof. Suppose (U, w) is a coordinate chart subordinate to a projective structure corresponding to $\mathcal{D}_{(L\hookrightarrow F,\nabla)}$, such that U contains no apparent singularity. We will show that the section $\mathbb{P}(L^{\nabla})$ of $\mathbb{P}(F^{\nabla})$ is locally represented by the function $-w$ on U, and this suffices to prove the proposition. To this end, suppose $\nabla = \partial_w + \begin{pmatrix} a(w) & b(w) \\ c(w) & -a(w) \end{pmatrix}$ in some local frames adapted to L, and, upon choosing a square-root of $c(w)$, observe that the holomorphic gauge transformation

$$G = \begin{pmatrix} c(w)^{-1/2} & 0 \\ 0 & c(w)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & a(w) - \frac{c'(w)}{2c(w)} \\ 0 & 1 \end{pmatrix},$$

puts ∇ into the form

$$G^{-1}\left[\partial_w + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] G = \partial_w + \begin{pmatrix} 0 & -q(w) \\ 1 & 0 \end{pmatrix},$$

where $q(w)$ is defined by the same formula as in (5.15). Since $\partial_w^2 + q(w)$ is the local differential operator representing $\mathcal{D}_{(L\hookrightarrow F,\nabla)}$ and since w is a developing map of this local differential operator, we have $q(w) = 0$. Then $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ w \end{pmatrix}$ are two local parallel sections that can define a local frame of F. In this local parallel frame, the generator of L takes the form

$$\begin{pmatrix} 0 & -1 \\ 1 & w \end{pmatrix}^{-1} G^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c(w)^{1/2} \begin{pmatrix} w \\ -1 \end{pmatrix},$$

which defines the local function $-w$ upon projectivization. $\square$

REMARK 5.9. Let $(L_2 \hookrightarrow F_2, \nabla_2)$ be the data obtained from $(L_1 \hookrightarrow F_1, \nabla_1)$ by twisting by a square-root of $\mathcal{O}_X$. Similar to how the projective connections $\left(\mathbb{P}(F_1^{\nabla_1}), \mathbb{P}(L_1^{\nabla_1})\right)$ and $\left(\mathbb{P}(F_2^{\nabla_2}), \mathbb{P}(L_2^{\nabla_2})\right)$ are isomorphic, we also have $\mathcal{D}_{(L_1\hookrightarrow F_1,\nabla_1)}$ is equivalent to $\mathcal{D}_{(L_2\hookrightarrow F_2,\nabla_2)}$. There exists in total 2^{2g} such data, which differ from each other by a twist by a square-root of $\mathcal{O}_X$, that define the same SL-operator up to equivalence.

Inverse construction. The following proposition is the analogue of proposition 3.4; the proof also follows a similar strategy.

PROPOSITION 5.6. *Let $\mathcal{D}$ be an SL-operator with simple apparent singularities x_1, ..., x_d, where d is even. Then given a line bundle L satisfying $KL^{-2} \cong \mathcal{O}_X(\mathbf{x})$, where $\mathbf{x} = x_1 + ... + x_d$, there exists a unique up to isomorphism holomorphic connection (F, ∇) such that F admits L as a subbundle and $\mathcal{D}$ is equivalent to $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$.*

Proof. Choose $x_0 \in X$ and effective divisors $\mathbf{p} = p_1 + ...p_m$, $\mathbf{q} = q_1 + ... + q_n$ such that $L \cong \mathcal{O}_X(\mathbf{q} - \mathbf{p})$ and $\mathbf{x} + x_0 + \mathbf{p} + \mathbf{q}$ is a reduced effective divisor. Let U_r, U_{p_j}, U_{q_k} be small neighborhoods with respective coordinates z_r, z_{p_j}, z_{q_k} that vanish at x_r, p_j, q_k respectively. Let ν_1, ..., $\nu_d \in \mathbb{C}$ be the accessory parameters of the apparent singularities x_1, ..., x_d of $\mathcal{D}$ w.r.t. local coordinate z_1, ..., z_d.

We will define F in terms of its transition functions on the covering

$$\{X', U_0, U_1, ..., U_d, U_{p_1}, ..., U_{p_m}, U_{q_1}, ..., U_{q_n}\},$$

where $X' = X \setminus \{\text{supp}(\mathbf{x} + x_0 + \mathbf{p} + \mathbf{q})\}$, with the ansatz

$$(F)_{U_r X'} = \begin{pmatrix} 1 & \epsilon_r/z_r \\ 0 & 1 \end{pmatrix}, \quad (F)_{U_{p_j} X'} = \begin{pmatrix} z_{p_j}^{-1} & 0 \\ 0 & z_{p_j} \end{pmatrix}, \quad (F)_{U_{q_k} X'} = \begin{pmatrix} z_{q_k} & 0 \\ 0 & z_{q_k}^{-1} \end{pmatrix},$$

for the transition functions of F. If ∇ takes the form $\partial + \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ on X', where A, B and C are meromorphic differentials on X that are holomorphic on X', then it takes the local form

$$\partial_{z_r} + \begin{pmatrix} A + (\epsilon_r/z_r)C & B - 2(\epsilon_r/z_r)A - (\epsilon_r/z_r)^2 C \\ C & -A - (\epsilon_r/z_r)C \end{pmatrix} + \begin{pmatrix} 0 & \epsilon_r/z_r^2 \\ 0 & 0 \end{pmatrix} \tag{5.18}$$

on U_r, and

$$\partial_{z_{p_j}} + \begin{pmatrix} A & z_{p_j}^{-2}B \\ z_{p_j}^{2}C & -A \end{pmatrix} + \begin{pmatrix} z_{p_j}^{-1} & 0 \\ 0 & -z_{p_j}^{-1} \end{pmatrix} \quad \text{and} \quad \partial_{z_{q_k}} + \begin{pmatrix} A & z_{q_k}^{2}B \\ z_{q_k}^{-2}C & -A \end{pmatrix} + \begin{pmatrix} -z_{q_k}^{-1} & 0 \\ 0 & z_{q_k}^{-1} \end{pmatrix} \tag{5.19}$$

on U_{p_j} and U_{q_k} respectively. To prove the proposition, we will now show that there exist meromorphic differentials A, B, C and a tuple $(\epsilon_0, \epsilon_1, ..., \epsilon_d) \in \mathbb{C}^{d+1}$ (which defines F as an extension of L^{-1} by L) such that (i) the diagonal components in (5.18) and (5.19) are regular, (ii) $\mathcal{D}$ locally takes the form $\partial_z + q_{\mathcal{D}}(z)$ with $q_{\mathcal{D}}(z)$ defined as in (5.15), and (iii) ν_r is the accessory parameter at x_r via (5.17).

To this end, let $C \in \Omega_{2\mathbf{p}-2\mathbf{q}} \cong H^0(KL^{-2})$ be a (unique up to scaling) meromorphic differential with $\text{div}(C) = \mathbf{x} - 2\mathbf{p} + 2\mathbf{q}$. Let A have simple poles at x_0, p_1, ..., p_m, q_1, ..., q_n with residues

$$\operatorname*{Res}_{p_j} A = -1 = -\operatorname*{Res}_{q_k} A, \qquad \operatorname*{Res}_{x_0} A = n - m = \deg(L),$$

and is holomorphic elsewhere: such a meromorphic differential exists as the sum of its residues vanishes. We define F via the tuple $(\epsilon_0, \epsilon_1, ..., \epsilon_d) \in \mathbb{C}^{d+1}$ by requiring

$$\epsilon_0 C(z_0(x_0)) + (n-m) = 0,$$

$$A(z_r)\mid_{z_r=0} + \epsilon_r C'(z_r)\mid_{z_r=0} - \frac{C''(z_r)\mid_{z_r=0}}{4C'(z_r)\mid_{z_r=0}} = \nu_r \text{ for } r = 1, ..., d.$$

The first condition ensures regularity of the diagonal components of (5.18) at x_0, while the second condition ensures that ν_r is the accessory parameter at x_r via (5.17).

Let (V, z) be a coordinate open subset of X', and suppose $\mathcal{D}$ can be represented by $\partial_z + q_{\mathcal{D}}(z)$ on V. Let B be defined by analytic continuation from $\frac{-q_{\mathcal{D}}(z)}{C(z)} - \frac{1}{C(z)}\left(A(z) - \frac{C'(z)}{2C(z)}\right)^2 - \frac{1}{C(z)}\left(A(z) - \frac{C'(z)}{2C(z)}\right)'$ to all of X. Noting that $q_{\mathcal{D}}(z)$ upon analytic continuation to $(U_r \setminus \{x_r\}, z_r)$ will have simple apparent singularity at x_r with accessory parameters ν_r, one can check explicitly that the expressions in (5.18) are regular with such A, B and C for $r = 1$, ..., d. A similar check can be done at x_0 by noting that $\epsilon_0 C(z_0) \mid_{z_0=0} = -\operatorname*{Res}_{x_0} A$, and at p_j (q_k) by noting that there C has a double pole (zero, respectively). Hence we have defined a holomorphic connection (F, ∇) and an embedding $L \hookrightarrow F$. By construction, $q_{\mathcal{D}}$ is equal to the local meromorphic function defined as in (5.15) with A, B and C as the input data, and so $\mathcal{D} = \mathcal{D}_{(L \hookrightarrow F, \nabla)}$. The uniqueness statement follows from the fact that we have chosen a line bundle L satisfying $KL^{-2} \cong \mathcal{O}_X(\mathbf{x})$ among 2^{2g} of them. $\square$

EXAMPLE 5.10. As an application of proposition 5.6 and a consistency check, let $\mathcal{D}$ be an SL-operator having no apparent singularity. Then $\mathcal{D} \sim \mathcal{D}_{(L \hookrightarrow F, \nabla)}$ where $L \cong K^{1/2}$. Since $\epsilon_0 = \deg(K^{1/2}) = g - 1$ is nonzero, F is a non-trivial extension of $K^{-1/2}$ by $K^{1/2}$. Hence (F, ∇) is an oper. The fact that F is a non-trivial extension of $K^{-1/2}$ is consistent with the well-known fact that a holomorphic bundle admits a holomorphic connection if and only if all of its indecomposable factors are of degree 0.

SL-operators with the same apparent singularities and accessory parameters. Let $\mathbf{x} = x_1 + ... + x_d$ be an effective divisor such that the multiplicity of x_i, $x_i \leq \mathbf{x}$, is at most 2. For $d < 3g - 3$, in general there are more than one SL-operator with the divisor of apparent singularities being $\mathbf{x}$ and the same accessory parameters. Indeed, let us fix a coordinate atlas subordinate to a holomorphic projective structure, and suppose the atlas contains the coordinate neighborhoods (U_1, z_1), ..., (U_d, z_d) of x_1, ..., x_d. Suppose $\mathcal{D} = \{\partial_{z_\alpha}^2 + q(z_\alpha)\}$, where $\{q(z_\alpha) dz_\alpha^2\}$ glue into a meromorphic quadratic differential, is an SL-operator with div$(\mathcal{D}) = \mathbf{x}$ and respective accessory parameters ν_1, ..., ν_d w.r.t. the coordinates z_1, ..., z_d. Then given a holomorphic quadratic differential $\Delta q \in H^0(K^2)$ with $\mathbf{x} \leq$ div(Δq), $\{\partial_{z_\alpha}^2 + q(z_\alpha) + \Delta q(z_\alpha)\}$ defines another SL-operator with apparent singularities x_1, ..., x_d and respective accessory parameters ν_1, ..., ν_d.

Conversely, two SL-operators that share the same simple apparent singularities and accessory parameters define a holomorphic quadratic differential that vanishes at the simple apparent singularities. It is straightforward to see that if x_i has multiplicity 2 in $\mathbf{x}_{\mathcal{D}}$ and hence is a double apparent singularity, this statement still holds. Let us now recall, given an effective divisor $\mathbf{x}$ on X, the linear space $Q_{\mathbf{x}} < H^0(K^2)$ of quadratic differentials with zero divisors bounded below by $\mathbf{x}$, namely $Q_{\mathbf{x}} \coloneqq \{q \in H^0(K^2) \mid \mathbf{x} \leq \text{div}(q)\} \cup \{0 \in H^0(K^2)\}$. The following proposition summarizes the above discussion.

PROPOSITION 5.7. *Let $\mathbf{x} = x_1 + ... + x_d$ be an effective divisor such that the multiplicity of x_i, $x_i \leq \mathbf{x}$, is at most 2. Then the set*

$$\{ \mathcal{D} \mid div(\mathcal{D}) = \mathbf{x} \textit{ with the same accessory parameters} \}$$

is an affine space modeled on $Q_{\mathbf{x}} < H^0(K^2)$.

The following is the analogue of corollary 3.8.

COROLLARY 5.8. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be SL-operators whose apparent singularities are all simple and projective monodromy representations have lifts to $SL_2(\mathbb{C})$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same apparent singularities and respective accessory parameters if and only if $\mathcal{D}_1 \sim \mathcal{D}_{(L_1 \hookrightarrow F_1, \nabla_1)}$ and $\mathcal{D}_2 \sim \mathcal{D}_{(L_2 \hookrightarrow F_2, \nabla_2)}$ where $L_1 \cong L_2 \otimes N$, $F_1 \cong F_2 \otimes N$ for some square-root N of $\mathcal{O}_X$. In particular, one can choose $L_1 \cong L_2$ and $F_1 \cong F_2$.*

Proof. Similar to the proof of corollary 3.8, the key point is to observe that in the proof of proposition 5.6, the positions of the apparent singularities and their accessory parameters define a unique rank-2 holomorphic bundle realized as the extension of line bundles. $\square$

Varying apparent singularities. In general, there exist different SL-operators having different apparent singularities that realize the same projective monodromy. Indeed, an irreducible $SL_2(\mathbb{C})$-holomorphic connection (F, ∇) together with different subbundles of F induce different isomorphism classes of SL-operators with different apparent singularities. However, in case the number of apparent singularities, counted with multiplicity, is less than $2g-2$, there exists a unique subbundle that is maximal and destabilizes F. Hence in this case, given a projective monodromy representation with lift to $SL_2(\mathbb{C})$, the isomorphism class of the SL-operators realizing it is unique (cf. corollary 5.4).

A diagrammatic summary. Let $\mathcal{M}^0_{\mathcal{D}}$ be the set of equivalence classes of SL-operators whose projective monodromy representations have lifts to irreducible monodromy representations in

$SL_2(\mathbb{C})$. It follows from our above discussion that the following diagram is commutative.

$$\begin{tikzcd}
& \left\{\begin{array}{c}[L \hookrightarrow F, \nabla] \mid \\ [F,\nabla] \in \mathcal{M}_{dR}\end{array}\right\} \arrow[r] \arrow[dl, "2^{2g}:1"'] \arrow[d, "2^{2g}:1"] & \mathcal{M}_{dR} \arrow[r] \arrow[d, "2^{2g}:1"] & \mathcal{R}_{SL_2(\mathbb{C})} \arrow[d, "2^{2g}:1"] \\
\mathcal{M}^0_{\mathcal{D}} \arrow[r, "1:1"] \arrow[rrr, bend right] & \mathcal{M}^0_{(P,s)} \arrow[r] & \mathcal{M}^0_P \arrow[r] & \mathcal{R}^0_{PSL_2(\mathbb{C})}
\end{tikzcd}$$

Here the first two vertical arrows assign to the isomorphism class of the data $(L \hookrightarrow F, \nabla)$ the equivalence class of $\mathcal{D}_{(L \hookrightarrow F, \nabla)}$ and the isomorphism class of $(\mathbb{P}(F^{\nabla}), \mathbb{P}(L^{\nabla}))$. The arrow $\mathcal{M}^0_{\mathcal{D}} \to \mathcal{M}^0_P$ is the equivalence between the notion of SL-operators and projective connections. The first horizontal arrow of the first line and the second horizontal arrow of the second line respectively forget the subbundles of the rank-2 bundles and the sections of flat $PSL_2(\mathbb{C})$-bundles with $\mathbb{P}^1$-fibers. The arrows with targets $\mathcal{R}_{SL_2(\mathbb{C})}$ and $\mathcal{R}^0_{PSL_2(\mathbb{C})}$ compute monodromy representations. All vertical arrows are surjective, where points in the same fiber of the first two vertical arrows differ by a twist of a square-root of $\mathcal{O}_X$.

Chapter 6

Meromorphic quadratic differentials and SL-operators

In this and the next chapters, we will fix a holomorphic projective structure $\{(U_\alpha, z_\alpha)\}_{\alpha\in\mathcal{I}}$ and work with SL-operators on this maximal coordinate atlas. In these coordinate charts, an SL-operator $\mathcal{D}$ with apparent singularities x_1, ..., x_d takes the form $\{\mathcal{D}_\alpha = \partial^2_{z_\alpha} + q_{\mathcal{D}}(z_\alpha)\}_{\alpha\in\mathcal{I}}$ where $\{q_{\mathcal{D}}(z_\alpha)dz_\alpha^2\}$ define a meromorphic quadratic differential $q_{\mathcal{D}}$ that has poles precisely at the apparent singularities x_1, ..., x_d. In addition, recall from (5.3) and (5.4) the Laurent tails of $q_{\mathcal{D}}$ at the apparent singularities, namely

$$q_{\mathcal{D}}(z) = -\frac{3}{4(z-z(x))^2} + \frac{\nu_z}{z-z(x)} + q_{z,0} + \mathcal{O}(z-z(x)) \tag{6.1a}$$

$$\text{with } \nu_z^2 + q_{z,0} = 0 \tag{6.1b}$$

if $x \in U$ is a simple apparent singularity, and

$$q_{\mathcal{D}}(z) = -\frac{2}{(z-z(x))^2} + \frac{\mu_z}{z-z(x)} + q_{z,0} + q_{z,1}(z-z(x)) + +\mathcal{O}((z-z(x))^2) \tag{6.2a}$$

$$\text{with } \mu_z^3 + 4q_{z,0}\mu_z + 4q_{z,1} = 0 \tag{6.2b}$$

if $x \in U$ is a double apparent singularity.

6.1 Meromorphic quadratic differentials

Laurent expansion at double poles of quadratic differentials. In the following, we demonstrate a consistency check that upon a change of coordinates defined by a Möbius transformation, while the coefficients of the Laurent expansion of a quadratic differential at a double pole change, the constraints (6.1b) and (6.2b) are invariant.

In general, we can compute the leading terms of the transformation of the local expression of a quadratic differential at a double pole x upon a general change of coordinates $z(w) \to w$ as

$$
\begin{aligned}
q(z)dz^2 &= dz^2 \sum_{k \geq -2} q_k (z - z(x))^k \\
&= dw^2 \left[\frac{q_{-2}}{(w - w(x))^2} + \frac{(z''(x)/z'(x))q_{-2} + z'(x)q_{-1}}{w - w(x)} + \mathcal{O}(1) \right], \qquad (6.3)
\end{aligned}
$$

where $z'(x) := z'(w) \mid_{w=w(x)}$. We see that the leading coefficient at a double pole of a meromorphic quadratic differential is invariant upon any change of coordinates (this invariant is sometimes called the "quadratic residue" in the literature [56]). On the other hand, the *accessory parameters* $\nu_{\alpha,i}$ in (6.1a) and $\mu_{\alpha,i}$ in (6.2a) transform non-trivially, depending on $z(w)$.

In our case, however, since we work on coordinate charts induced by a fixed holomorphic projective structure, the relevant changes of coordinates are Möbius transformations. Combined with the specific values of the quadratic residues as in (6.1a) and (6.2a), the constraints (6.1b) and (6.2b) on these coefficients are invariant, as we now demonstrate. Consider a Möbius transformation $z_\alpha(z_\beta) = \frac{Az_\beta + B}{Cz_\beta + D}$. W.l.o.g. suppose x has coordinate $z_\alpha(x) = 0$ in U_α and $z_\beta(x) = -B/A$ with $A \neq 0$ in U_β. If $q_\alpha(z_\alpha) = \sum_{k \geq -2} q_{\alpha,k} z_\alpha^k$ then $q_\beta(z_\beta) = \sum_{k \geq -2} q_{\beta,k} \left(z_\beta - z_\beta(x)\right)^k$ where

$$
\begin{aligned}
q_{\beta,-2} &= q_{\alpha,-2}, & q_{\beta,0} &= (3A^2C^2)q_{\alpha,-2} - (3A^3C)q_{\alpha,-1} + A^4 q_{\alpha,0}, \\
q_{\beta,-1} &= -(2AC)q_{\alpha,-2} + A^2 q_{\alpha,-1}, & q_{\beta,1} &= -(4A^3C^3)q_{\alpha,-2} + (6A^4C^2)q_{\alpha,-1} - (4A^5C)q_{\alpha,0} \\
& & &\quad + A^6 q_{\alpha,1}. \qquad (6.4)
\end{aligned}
$$

For generic values of $q_{\alpha,-2}$, the constraints (6.1b) and (6.2b) will not be invariant. But one can check that

$$
\begin{aligned}
&q_{\beta,-1}^2 + q_{\beta,0} = A^4 \left(q_{\alpha,-1}^2 + q_{\alpha,0}\right) && \text{if } q_{\alpha,-2} = \frac{-3}{4}, \\
&q_{\beta,-1}^3 + 4q_{\beta,0}q_{\beta,-1} + 4q_{\beta,1} = A^6 \left(q_{\alpha,-1}^3 + 4q_{\alpha,0}q_{\alpha,-1} + 4q_{\alpha,1}\right) && \text{if } q_{\alpha,-2} = -2.
\end{aligned}
$$

The invariance of these constraints upon a Möbius transformation of coordinates is consistent with the fact that these constraints can be derived directly from the transformation rules (5.2) as in (5.3) and (5.4). Our computations merely demonstrate this explicitly.

Building blocks of meromorphic quadratic differentials. For each $x \in X$, there exists quadratic differentials that have poles at x and is holomorphic everywhere else. We will be interested in those that have simple and double poles at x. We denote the space of all such meromorphic quadratic differentials as $\Omega_x^{\otimes 2}$ and $\Omega_{2x}^{\otimes 2}$ respectively. Clearly $\Omega_x^{\otimes 2} \cong H^0\left(K_X^2 \otimes \mathcal{O}_X(x)\right)$ and $\Omega_{2x}^{\otimes 2} \cong H^0\left(K_X^2 \otimes \mathcal{O}_X(2x)\right)$. In particular, elements of $\Omega_x^{\otimes 2}$ ($\Omega_{2x}^{\otimes 2}$) that are holomorphic at x and hence on all of X correspond to sections of $K_X^2 \otimes \mathcal{O}_X(x)$ (respectively, $K_X^2 \otimes \mathcal{O}_X(2x)$) that admit x as a zero of multiplicity at least 1 (respectively, 2).

In the following we discuss, given a local coordinate z of p, certain quadratic differentials that have simple and double pole at x with specific Laurent tails in the expansion w.r.t. z. We will think of these specific quadratic differentials as building blocks, out of which we can construct all other elements of $\Omega_x^{\otimes 2}$ and $\Omega_{2x}^{\otimes 2}$ in a way such that we can control their Laurent tails w.r.t. z.

To this end, first note that since $h^0\left(K_X^2 \otimes \mathcal{O}_X(x)\right) = 3g-2$, an element in $\Omega_x^{\otimes 2}$ is fixed up to scaling upon fixing $3g-3$ out of its $4g-3$ zeroes in total. Similarly, an element in $\Omega_{2x}^{\otimes 2}$ is fixed up to scaling upon fixing $3g-2$ out of its $4g-2$ zeroes. To characterize the meromorphic quadratic differentials that have non-trivial simple and double poles at x, let us first recall the notion of Q-generic divisors. For an effective divisor $\mathbf{x}$, let $Q_{\mathbf{x}}$ be the space of quadratic differentials whose zero divisors are bounded below by $\mathbf{x}$. We will say $\mathbf{x}$ is Q-generic if the dimension of $Q_{\mathbf{x}}$ has the minimal, expected value, namely

$$\dim Q_{\mathbf{x}} = \begin{cases} 3g-3-\deg(\mathbf{x}) & \text{for } \deg(\mathbf{x}) < 3g-3, \\ 0 & \text{for } \deg(\mathbf{x}) \geq 3g-3. \end{cases}$$

Then an element of $\Omega_x^{\otimes 2}$ ($\Omega_{2x}^{\otimes 2}$) that has a non-trivial simple (respectively, double) pole at x is characterized by the fact that if an effective divisor of degree $3g-3$ (respectively, $3g-2$) is contained in its zero divisor (and hence determines it up to scaling), then this divisor is Q-generic. It is almost obvious that by varying the pole x in its neighborhood we can construct a family of quadratic differentials that is holomorphically parameterized by this neighborhood, while keeping intact the freedom to choose the zeroes. Proposition 6.1 formalizes this intuition.

PROPOSITION 6.1. *Let $U \subset X$ be an open subset, $i \in \{1,2\}$, $d = 3g-4+i$, and $\mathbf{x}^d : U \to$*

$X^{[d]}$, $x \mapsto \mathbf{x}^d(x) = x_1(x) + ... + x_d(x)$, be a holomorphic map such that, for all $x \in U$ and $r \in \{1, ..., d\}$, $x_r(x) \neq x$ and $\mathbf{x}^d(x)$ is Q-generic. Then there exists a family of meromorphic quadratic differentials $\left\{q^{(i)}_{x,\mathbf{x}^d}\right\}_{x \in U}$ holomorphically parameterized by U such that

(a) $q^{(i)}_{x,\mathbf{x}^d}$ has a simple (double) pole at x if $i = 1$ (respectively, $i = 2$) and is holomorphic on $X \setminus \{x\}$;

(b) $\mathbf{x}^d(x)$ is contained in the support of the zero divisor of $q^{(i)}_{x,\mathbf{x}^d}$.

Proof. We show the proof for the case $i = 1$ and $d = 3g - 3$; the case $i = 2$ is similar. For each fixed $x \in U$, due to the hypotheses that $x_r(x) \neq x$ and $\mathbf{x}^d(x)$ is not contained in the zero divisor of a holomorphic quadratic differential, the above observation already guarantees the unique up to scaling existence of a meromorphic quadratic differential satisfying conditions (a) and (b). Let $-x + \mathbf{x}^d(x) + x_{3g-2} + ... + x_{4g-3}$ be the divisor associated to such a quadratic differential. This divisor is unique, hence $x_{3g-2}(x) + ... + x_{4g-3}(x)$ is determined by x and $\mathbf{x}^d(x)$. More concretely, let q be a holomorphic quadratic differential with zero divisor $\text{div}(q) = x'_1 + ... + x'_{4g-4}$ and $A : X \to \text{Jac}(X)$ an Abel map. Then $x_{3g-2} + ... + x_{4g-3}$ is the only point in $A^{-1} \circ A\left(x - \sum_{k=1}^{3g-3} x_k(x) + \text{div}(q)\right)$; in particular, $h^0\left(x_{3g-2} + ... + x_{4g-3}\right) = 1$. Since the restriction of A to the set of effective divisors that are not special, i.e. $\left\{D \in X^{[g]} \mid h^0(D) = 1\right\}$ is a biholomorphism, we have $x_{3g-2}(x), ..., x_{4g-3}(x)$ as holomorphic functions of x. Now, for each $x \in U$, note that the ratio of a quadratic differential with divisor $-x + \mathbf{x}^d(x) + x_{3g-2} + ... + x_{4g-3}$ over q is a meromorphic function defined on all of X and proportional to

$$f_x(z) = \frac{\prod_{k=1}^{3g-3} E(z, x_k(x))}{E(z, x)} \frac{\prod_{h=3g-2}^{4g-3} E(z, x_h(x))}{\prod_{j=1}^{4g-4} E(z, x'_j)}, \tag{6.5}$$

where $E(x, y)$ is a prime form of X. Since $E(x, y)$ is holomorphic in both of its variables, $f_x(z)$ is holomorphic on U. Hence $q^{(i)}_{x,\mathbf{x}^d} := f_x(z)q(z)$ varies holomorphically with respect to x. $\square$

The families parameterized by U that satisfies conditions (a) and (b) in proposition 6.1 are far from unique: scaling one such family by a function holomorphic on U produces another. Given a local coordinate z on U, however, we can look to control the Laurent tails of these quadratic differentials w.r.t. z. The resulting “normalized” families would be unique, but we would be able to impose less zeroes.

PROPOSITION 6.2. *Let (U,z) be an open coordinated subset of X.*

1. Let $\mathbf{x}^{3g-3}: U \to X^{[3g-3]}$, $x \mapsto \mathbf{x}^{3g-3}(x) = x_1(x) + ... + x_{3g-3}$, be a holomorphic map such that for all $x \in U$ and $r \in \{1,...,3g-3\}$, $x_r(x) \neq x$ and $\mathbf{x}^{3g-3}(x)$ is Q-generic. Then there exists a unique family of meromorphic quadratic differentials $\left\{q^{(i)}_{z,x,\mathbf{x}}\right\}_{x\in U}$ holomorphically parameterized by U such that, for each $x \in U$, $q^{(i)}_{z,x,\mathbf{x}^{3g-3}}$ satisfies conditions (a), (b) in Proposition 6.1, and furthermore takes the local form

$$\frac{dz^2}{(z-z(x))^i} + R^{(i)}_{z,x,\mathbf{x}^{3g-3}}(z)dz^2, \tag{6.6}$$

where $R^{(i)}_{z,x,\mathbf{x}^{3g-3}}(z)$ is the restriction of a holomorphic function $R^{(i)}(z_1,z_2)$ defined on $(U,z)\times(U,z)$ to the slice $\{z_1 = z(x), z_2 \equiv z\}$.

2. Let $\mathbf{x}^{3g-4}: U \to X^{[3g-4]}$, $x \mapsto \mathbf{x}^{3g-4}(x) = x_1(x) + ... + x_{3g-4}$, be a holomorphic map such that for all $x \in U$ and $r \in \{1,...,3g-4\}$, $x_r(x) \neq x$ and $x + \mathbf{x}^{3g-4}(x)$ is Q-generic. Then there exists a unique family of meromorphic quadratic differentials $\left\{q^{(i)}_{0,z,x,\mathbf{x}^{3g-4}}\right\}_{x\in U}$ holomorphically parameterized by U such that, for each $x \in U$, $q^{(i)}_{0,z,x,\mathbf{x}^{3g-4}}$ satisfies conditions (a), (b) in Proposition 6.1, and furthermore takes the local form

$$\frac{dz^2}{(z-z(x))^i} + R^{(i)}_{0,z,x,\mathbf{x}^{3g-4}}(z)dz^2, \tag{6.7}$$

where $R^{(i)}_{0,z,x,\mathbf{x}^{3g-4}}(z)$ is the restriction of a function $R^{(i)}_0(z_1,z_2)$, which is defined on $(U,z_1)\times(U,z_2)$ and vanishes at the diagonal $\{z_1 = z_2\}$, to the slice $\{z_1 = z(x), z_2 \equiv z\}$.

Proof. 1. For $i = 1$, consider a family $\left\{q^{(1)}_{x,\mathbf{x}^{3g-3}}\right\}_{x\in U}$ from Proposition 6.1. For each fixed $x \in U$, we can scale $q^{(1)}_{x,\mathbf{x}^{3g-3}}$ into taking the local form (6.6) to construct $q^{(1)}_{z,x,\mathbf{u}}$. For $i = 2$, choose $x_{3g-2} \in X \setminus U$ such that $x_{3g-2} + \mathbf{x}^{3g-3}$ is Q-generic, and consider two families $\left\{q^{(i)}_{x,\mathbf{x}^{3g-3}+x_{3g-2}}\right\}_{x\in U}$ for $i = 1,2$ from Proposition 6.1. For each fixed $x \in U$, combining scaling $q^{(2)}_{x,\mathbf{x}^{3g-2}+x_{3g-2}}$ and adding a scaling of $q^{(1)}_{z,x,\mathbf{x}^{3g-3}}$, we can construct $q^{(2)}_{z,x,\mathbf{x}^{3g-3}}$ that takes the local form (6.6). Since $q^{(1)}_{z,x,\mathbf{x}^{3g-3}}$ in general does not vanish at $x_{3g-2}(x)$, we can only in general guarantee $q^{(2)}_{z,x,\mathbf{x}^{3g-3}}(z)$ to vanish at $\mathbf{x}^{3g-3}(x)$. The uniqueness statement follows from the uniqueness up to scaling of $q^{(i)}_{x,\mathbf{x}^d}$.

2. By our assumption, for all $x \in U$, there exists a unique up to scaling a holomorphic quadratic

differential that vanishes at $\mathbf{x}^{3g-4}(x)$ and does not vanish at x. By adding proper scalings of this quadratic differential with the members of the family $\left\{q^{(i)}_{z,x,\mathbf{x}^{3g-4}+x_{3g-3}}\right\}_{x\in U}$, where x_{3g-3} is a point in $X\setminus U$, we can cancel the evaluation of $R^{(i)}_{z,x,\mathbf{x}^{3g-4}+x_{3g-3}}(z)$ at $z(x)$. $\square$

REMARK 6.1. To emphasize the fact that the "normalized" forms (6.6) and (6.7) are only manifest w.r.t. a choice of local coordinate, we have included the local coordinate in the notation of these families of quadratic differentials.

Some technical results. In the coming chapter we will be interested in families of quadratic differentials of certain forms. The idea is to work with families of "building blocks" of quadratic differentials and analyze the limit of these families as the poles collide.

Let (U,z) be a coordinated neighborhood of $x_0\in X$ and $\mathbf{x}'=x_3+...+x_d$ for $d\leq 3g-3$ a reduced effective divisor on X such that, for all $x_1,x_2\in U$, $x_1+x_2+\mathbf{x}'$ is Q-generic. For each $r\in\{3,...,d\}$, let $\mathbf{x}'_{\mathbf{r}}:=\mathbf{x}'-x_r$. Then, refining U if necessary, we can choose some effective divisors $\mathbf{w}=w_0+w_{d+1}+...+w_{3g-3}$ on X with support distinct from $U\cup\{x_3,...,x_d\}$ such that, for all $x_1,x_2\in U$ and $r\in\{3,...,d\}$, $x_1+\mathbf{x}'+\mathbf{w}$ and $x_1+x_2+\mathbf{x}'_{\mathbf{r}}+\mathbf{w}$ are Q-generic.

Now, for each $u\in z(U)$, let $x_\pm(u)$ be points in U having coordinates $\pm u$ respectively. We use the following short-hand notations for the unique families of quadratic differentials from proposition 6.2,

$$q^{(i)}_{u,\pm}:=q^{(i)}_{0,z,x_\pm,\mathbf{w}+\mathbf{x}'}\overset{\text{on }(U,z)}{\equiv}\frac{dz^2}{(z\mp u)^i}+R^{(i)}_{u,\pm}(z)dz^2,\qquad i\in\{1,2\},\tag{6.8}$$

where $R^{(i)}_{u,\pm}(z)$ is the restriction to the slice $\{(u,z)\}$ of a function $R^{(i)}_0(z_1,z)$ which is holomorphic in both variables and vanishes at the diagonal $\{z=u\}$.

LEMMA 6.3. *Fix $i\in\{1,2\}$.*
1. The functions $\frac{1}{u}R^{(i)}_{u,+}(-u)$ and $-\frac{1}{u}R^{(i)}_{u,-}(u)$ are holomorphic on U with variable $u=z(x_+)$. Furthermore, their evaluations at $u=0$ are equal.
2. For each fixed z, the limit

$$\check{R}^{(i)}(z):=\lim_{u\to 0}\frac{1}{u}\left(R^{(i)}_{u,+}(z)-R^{(i)}_{u,-}(z)\right)\tag{6.9}$$

exists. These limits define a holomorphic function on U with variable z.

Proof. 1. We need to show that the $u \to 0$ limit of $\frac{1}{u} R^{(i)}_{u,+}(-u)$ and $-\frac{1}{u} R^{(i)}_{u,-}(u)$ exist and are equal. Consider the Taylor expansion of $R^{(i)}_0(z_1, z) = \sum_{m,n \geq 0} R^{(i)}_{m,n} z_1^m z^n$ at its zero $(z_1, z) = (0, 0)$. Since $R^{(i)}_{0,0} = 0$,

$$R^{(i)}_{u,-}(u) \coloneqq R^{(i)}_0(-u, u) = -\left(R_{1,0} - R_{0,1}\right) u + \mathcal{O}(u^2),$$
$$R^{(i)}_{u,+}(-u) \coloneqq R^{(i)}_0(u, -u) = \left(R_{1,0} - R_{0,1}\right) u + \mathcal{O}(u^2).$$

It follows that

$$\lim_{u \to 0} \frac{1}{u} R^{(i)}_{u,+}(-u) = -\lim_{u \to 0} \frac{1}{u} R^{(i)}_{u,-}(u). \tag{6.10}$$

2. For each fixed z, it follows from the Taylor expansion $R^{(i)}_0(z_1, z) = \sum_{m,n \geq 0} R^{(i)}_{m,n} z_1^m z^n$ that,

$$R^{(i)}_{u,+}(z) - R^{(i)}_{u,-}(z) = 2u \sum_{n \geq 0} R^{(i)}_{1,n} z^n + 2u^3 \sum_{n \geq 0} R^{(i)}_{3,n} z^n + 2u^5 \sum_{n \geq 0} R^{(i)}_{5,n} z^n + ...$$

is a holomorphic function on (S, u) that vanishes at $u = 0$; we can calculate

$$\lim_{u \to 0} \frac{1}{u} \left(R^{(i)}_{u}(z) - R^{(i)}_{u,-}(z) \right) = 2 \sum_{n \geq 0} R^{(i)}_{1,n} z^n. \tag{6.11}$$

□

6.2 Parameterize SL-operators

PROPOSITION 6.4. *Let $x'_1 + ... + x'_{3g-3}$ be a reduced Q-generic divisor on X. Then there exists coordinate neighborhoods (U_r, z_r) of each x'_r, $r \in \{1, ..., 3g-3\}$, and an injective map of sets*

$$\begin{aligned} U_1 \times ... \times U_r \times \mathbb{C}^{3g-3} &\longrightarrow \{SL\text{-operators}\} \\ (\vec{x}, \vec{\nu}) = (x_1, ..., x_{3g-3}, \nu_1, ..., \nu_{3g-3}) &\longmapsto \mathcal{D}_{(\vec{x}, \vec{\nu})} \end{aligned}$$

such that $\mathcal{D}_{(\vec{x}, \vec{\nu})}$ has simple apparent singularities at each x_r with respective accessory parameters ν_r w.r.t. the local coordinates z_r.

Proof. Since $x'_1 + ... + x'_{3g-3}$ is Q-generic, there exists a neighborhood U_r of x'_r for each $r \in \{1, ..., 3g-3\}$ such that, for all $(x_1, ..., x_{3g-3}) \in U_1 \times ... \times U_{3g-3}$, the divisor $x_1 + ... + x_{3g-3}$ is Q-generic. Let z_1, ..., z_{3g-3} be coordinates on U_1, ..., U_{3g-3} respectively that are subordinate to a fixed holomorphic projective structure. Then for each $(\vec{x}, \vec{\nu}) = (x_1, ..., x_{3g-3}, \nu_1, ..., \nu_{3g-3}) \in U_1 \times ... \times U_r \times \mathbb{C}^{3g-3}$, we define a non-degenerate linear system of rank $3g-3$ in $H^0(K^2)$ by

$$q^{(0)}(z_r) \mid_{z_r(x_r)} + \nu_r^2 = 0, \qquad\qquad r = 1, ..., 3g-3,$$

which admits a unique solution $q^{(0)}_{(\vec{x},\vec{\nu})}$. The meromorphic quadratic differential

$$q_{(\vec{x},\vec{\nu})} = -\frac{3}{4}\sum_{r=1}^{3g-3} q^{(2)}_{0,z_r,x_r,\widehat{\mathbf{x}}_r} + \sum_{r=1}^{3g-3} \nu_r q^{(1)}_{0,z_r,x_r,\widehat{\mathbf{x}}_r} + q^{(0)}_{(\vec{x},\vec{\nu})},$$

where $\widehat{\mathbf{x}}_r := x_1 + ... + x_{3g-3} - x_r$, defines an SL-operator $\mathcal{D}_{(\vec{x},\vec{\nu})}$ by its local form $\partial^2_{z_r} + q_{(\vec{x},\vec{\nu})}(z_r)$ on (U_r, z_r). Clearly the assignment $(\vec{x}, \vec{\nu}) \mapsto \mathcal{D}_{(\vec{x},\vec{\nu})}$ is injective. $\square$

In the following we prove proposition 1.3.

PROPOSITION 6.5 (PROPOSITION 1.3). *Suppose* $\deg(\Lambda) - g$ *is odd. Let* q_0 *be a non-degenerate holomorphic quadratic differential and* $x'_1 + ... + x'_{3g-3}$ *be a reduced Q-generic divisor. If in addition there is no exceptional divisor on the spectral curve* S_{q_0} *projecting to* $x'_1 + ... + x'_{3g-3}$*, then there exist open neighborhoods* $V \subset H^0(K^2)$ *of* q_0*,* $U_r \subset X$ *of* x'_r *and an embedding*

$$\begin{aligned} U_1 \times ... \times U_{3g-3} \times V &\longrightarrow \mathcal{M}_H(\Lambda), \\ (\vec{x}, q) = (x_1, ..., x_{3g-3}, q) &\longmapsto [E_{(\vec{x},q)}, \phi_{(\vec{x},q)}] \end{aligned}$$

where $\det(\phi_{(\vec{x},q)}) = q$ *and* $E_{(\vec{x},q)}$ *admits a subbundle* $L_{\vec{x}}$ *with the zero divisor of* $c_{L_{\vec{x}}}(\phi_{(\vec{x},q)})$ *being* $x_1 + ... + x_{3g-3}$*. Furthermore, there exist coordinate* z_r *on* U_r *and an injective map of sets*

$$\begin{aligned} U_1 \times ... \times U_{3g-3} \times V &\longrightarrow \{SL\text{-operators }\}, \\ (\vec{x}, q) = (x_1, ..., x_{3g-3}, q) &\longmapsto \mathcal{D}_{(\vec{x},q)} \end{aligned}$$

where $\mathcal{D}_{(\vec{x},q)}$ has simple apparent singularities $x_1, ..., x_{3g-3}$ with respective accessory parameters $\nu_1, ..., \nu_{3g-3}$ satisfying $\nu_r^2 + q(z_r(x_r)) = 0$ for $r = 1, ..., 3g-3$.

Proof. For $r \in \{1, ..., 3g-3\}$, let U_r be a neighborhood of x'_r that contains no zero of q_0 and such that $U_r \cap U_{r'} = \emptyset$ for $r \neq r'$. For each $\vec{x} \in U_1 \times ... \times U_{3g-3}$, choose a line bundle $L_{\vec{x}}$ satisfying $KL_{\vec{x}}^{-2}\Lambda \cong \mathcal{O}_X(x_1 + ...x_{3g-3})$ in such a way that $\{L_{\vec{x}}\}_{\vec{x}\in U_1\times ...\times U_{3g-3}}$ is a family holomorphically parameterized by $U_1 \times ... \times U_{3g-3}$. Let V be a neighborhood of q_0 that contains no degenerate holomorphic quadratic differential and such that, for each $r \in \{1, ..., 3g-3\}$, the subset $\bigcup_{q\in V} S_q \mid_{\pi^{-1}(U_r)}$ of $T^*X \mid_{U_r}$, where S_q is the spectral curve defined by $q \in V$, has two distinct components [17]. As a particular consequence, if $q \in V$ then q has no zeroes located in any of the neighborhood U_r. Since V by construction allows us to choose a distinct component of $\bigcup_{q\in V} S_q \mid_{\pi^{-1}(U_r)}$, for each $r \in \{1, ..., 3g-3\}$, one can assign to $(x_r, q) \in U_r \times V$ a point $\tilde{x}_r$ on S_q that projects to x_r in a way such that $(x_r, q) \mapsto \tilde{x}_r(x_r, q)$ defines a holomorphic function $U_r \times V \to T^*X \mid_{U_r}$. Given $(\vec{x}, q) \in U_1 \times ... \times U_{3g-3} \times V$, let $D(\vec{x}, q)$ be the effective divisor on S_q defined by $\sum_{r=1}^{3g-3} \tilde{x}_r(x_r, q)$. By proposition 3.6, there exists a unique Higgs bundle $(E_{(\vec{x},q)}, \phi_{(\vec{x},q)})$ such that $\det(\phi_{(\vec{x},q)}) = q$ and $E_{(\vec{x},q)}$ admits $L_{\vec{x}}$ as a subbundle inducing $D(\vec{x}, q)$ as the corresponding Baker-Akhiezer divisor. One can, if necessary, further refine U_r and V so that $D(\vec{x}, q)$ is not an exceptional divisor on S_q for all $(\vec{x}, q) \in U_1 \times ... \times U_{3g-3} \times V$. Then the assignment $(\vec{x}, q) \mapsto [E_{(\vec{x},q)}, \phi_{(\vec{x},q)}]$ is injective. By appealing to the fact that the complex structure of each smooth Hitchin fiber is compatible with the variation of effective divisors on the corresponding spectral curve and that the Hitchin fibration is holomorphic, one sees that $(\vec{x}, q) \mapsto [E_{(\vec{x},q)}, \phi_{(\vec{x},q)}]$ defines an embedding.

Now, let $z_1, ..., z_{3g-3}$ be coordinates on $U_1, ..., U_{3g-3}$ respectively that are subordinate to a fixed holomorphic projective structure. For each $(x_r, q) \in U_r \times V$, let $\nu_r(x_r, q) \in \mathbb{C}$ be such that $\tilde{x}_r(x_r, q)$ has coordinate $(z_r(x_r), \nu_r(x_r, q))$ in the local frame of $T^*X \mid_{U_r}$ defined by dz_r. The second part of the proposition now follows from proposition 6.4. $\square$

[17] One can define such V by making sufficiently small perturbation of $3g-4$ simple zeroes of q_0 together with scaling.

Chapter 7

Collision of apparent singularities

In this chapter, we analyze the limit when two simple apparent singularities collide. As two simple apparent singularities approach each other, each contributes and blows-up the coefficient $q_{z,0}$ in the Laurent expansion in (6.1a) at the other one. The constraint (6.1b) implies that their accessory parameters will also blow-up. We will show that, if these respective accessory parameters blow-up in a particular way, the limit upon collision will be either an SL-operator with a double apparent singularity at the collision site, or an SL-operator with two less apparent singularities.

7.1 Setup

7.1.1 Conditions on the collision site.

For $2 \leq d \leq 3g-3$, let x_0 and $\mathbf{x}' = x_3 + ... + x_d$ be reduced effective divisors on X such that $x_0 \nleq \mathbf{x}'$ and

$$2x_0 + \mathbf{x}' \text{ is } Q\text{-generic.} \tag{7.1}$$

A consequence if this condition is that for all pairs of distinct points x_1 and x_2 in a sufficiently small neighborhood U of x_0, we have

$$x_1 + x_2 + \mathbf{x}' \text{ is } Q\text{-generic.} \tag{7.2}$$

Condition (7.2) is equivalent to requiring that, given a basis $\mathbf{q} = (q_1, ..., q_{3g-3})$ of $H^0\left(K_X^2\right)$, the $d \times (3g-3)$ matrix $\mathbf{q}_{\mathbf{r},\mathbf{k}} := q_k(z_r(x_r))_{1\leq r\leq d}^{1\leq k\leq 3g-3}$, defined by evaluating q_k at x_r using some local coordinate z_r, is of maximal rank d. This condition is satisfied by a generic choices of x_0 and $\mathbf{x}'$.

Determinant with simple zero at x_0. Consider the case where $d = 3g-3$, and x_0, $\mathbf{x}'$ satisfy condition (7.1). Then upon choosing local coordinates and a basis $\mathbf{q} = (q_k)$ of $H^0(K^2)$, any x_1, $x_2 \in U$ define a $(3g-3) \times (3g-3)$ non-degenerate matrix $\mathbf{q}_{\mathbf{r},\mathbf{k}}(x_1, x_2)$.

We will in the following evaluate q_k at x_1 and x_2 using the same local coordinate z on U. Furthermore, we will make the following choices of x_1 and x_2. W.l.o.g., suppose $z(x_0) = 0$. Then for each $u \in z(U)$, let $x_1(u)$ and $x_2(u)$ be points in U that have coordinates $\pm u$ respectively; in particular, $x_0 = x_1(0) = x_2(0)$. Then a choice of a basis $\mathbf{q} = (q_k)$ of $H^0(K^2)$ and local coordinates z_3, ..., z_{3g-3} for x_3, ..., x_{3g-3} together define a function $\det(\mathbf{q}_{\mathbf{r},\mathbf{k}})(u) : U \cong z(U) \to \mathbb{C}$. This function is holomorphic in u and vanishes only at $u = 0$, since at $u = 0$ the matrix $\mathbf{q}_{\mathbf{r},\mathbf{k}}$ is degenerate. The following proposition shows that if we use a certain family $(q_{k,u})_{k=1}^{3g-3}$ of basis of $H^0(K^2)$ that also varies holomorphically in u, the zero at $u = 0$ of the corresponding determinant is simple.

LEMMA 7.1. *Let x_0 and $\mathbf{x}' = x_1 + ... x_{3g-3}$ satisfy (7.1), and (U, z), $x_1(u)$, $x_2(u)$ be constructed as above. Then there exists a family $\mathbf{q}_u := \{(q_{k,u})_{k=1}^{3g-3}\}_{u \in z(U)}$ of basis of $H^0(K^2)$ that is parameterized by U and is such that the function $\det(\mathbf{q}_{\mathbf{r},\mathbf{k}})(u)$, where $\mathbf{q}_{\mathbf{r},\mathbf{k}} = q_{k,u}(z_r(x_r))$, has a simple zero at $u = 0$.*

Proof. For each k in the range $3 \leq k \leq 3g-3$ and each $u \in z(U)$, there exists a unique up to scaling quadratic differential that vanishes at $x_1(u)$, $x_2(u)$ and all x_r with $r \neq k$. By our assumption on x_0 and $\mathbf{x}'$, this quadratic differential does not vanish at x_k. By scaling, for $3 \leq k \leq 3g-3$, we define a unique quadratic differential $q_{k,u}$ with $q_{k,u}(z_r(x_r)) = \delta_{kr}$. We now define $q_1 \equiv q_{1,u}$ and $q_2 \equiv q_{2,u}$ independent of u as follows. Let q_1 be a (unique up to scaling) holomorphic quadratic differential that vanishes at $x_0 + \mathbf{x}'$. By our assumption on x_0 and $\mathbf{x}'$, its zero at x_0 is simple. For some $x_0' \in U$ with $x_0' \neq x_0$, let q_2 be a (unique up to scaling) holomorphic quadratic differential that vanishes at $x_0' + \mathbf{x}'$. By our assumption on x_0 and $\mathbf{x}'$, q_2 does not vanish at x_0 and hence is linearly independent from q_1. Then for each $u \in z(U)$, let $(q_{k,u}) := (q_1, q_2, q_{3,u}, ..., q_{3g-3,u})$: this forms a basis of $H^0(K^2)$ for each $u \in z(U)$. It follows that the function

$$
\begin{aligned}
\det(\mathbf{q}_{\mathbf{r},\mathbf{k}})(u) &= \begin{vmatrix} q_1(z(x_1)) & q_2(z(x_1)) & 0 \\ q_1(z(x_2)) & q_2(z(x_2)) & 0 \\ 0 & 0 & I_{(3g-5)\times(3g-5)} \end{vmatrix} \\
&= q_1(z(x_1))q_2(z(x_2)) - q_1(z(x_2))q_2(z(x_1)) \\
&= 2q_1'(z(x_0)))q_2(z(x_0))u + \mathcal{O}(u^2) \qquad (7.3)
\end{aligned}
$$

has a simple zero at $u = 0$. $\square$

7.1.2 Families of meromorphic quadratic differentials.

Given x_0 and $\mathbf{x}'$ satisfying condition (7.1), let (U, z), (U_3, z_3), ..., (U_d, z_d) be some respective coordinate neighborhoods of x_0, x_3, ..., x_d, which do not intersect each other and are subordinate to a fixed holomorphic projective structure. W.l.o.g., let $z(x_0) = 0$; for each $u \in z(U)$, let $x_\pm(u) \in U$ be defined by $z(x_\pm) = \pm u$. Let $\mathbf{x}'_\mathbf{r} := \mathbf{x}' - x_r$.

Let $\nu_\pm(u)$, $\nu_3(u)$, ..., $\nu_d(u)$ be holomorphic functions from $U \setminus \{x_0\}$ to $\mathbb{C}$. We want to characterize a family of SL-operators $\{\mathcal{D}_u\}_{u\in z(U), u\neq 0}$ holomorphically parameterized by $U \setminus \{x_0\}$, where $\mathcal{D}_u$ has simple apparent singularities at $x_\pm(u)$, x_3, ..., x_d and respective accessory parameters $\nu_\pm(u)$, $\nu_3(u)$, ..., $\nu_d(u)$ w.r.t. local coordinates z, z_3, ..., z_d. We do this by writing $\mathcal{D}_u = \mathcal{D}_{pr} + q_u$, where $\mathcal{D}_{pr}$ is an SL-operator defined by the chosen holomorphic projective structure, and q_u is a meromorphic quadratic differential built in terms of the "building blocks". One set of the "building blocks" comes from (6.8),

$$
q^{(i)}_{u,\pm} := q^{(i)}_{0,z,x_\pm,\mathbf{w}+\mathbf{x}'} \overset{\text{on } (U,z)}{\equiv} \frac{dz^2}{(z \mp u)^i} + R^{(i)}_{u,\pm}(z)dz^2, \qquad i \in \{1, 2\}, \qquad (7.4)
$$

some $u \to 0$ limits of which are discussed in lemma 6.3. We also use the following short-hand notations

$$
q^{(i)}_{u,x_r} := q^{(i)}_{0,z_r,x_r,x_++x_-+\mathbf{x}'_\mathbf{r}+\mathbf{w}-w_0}, \qquad 3 \le r \le d,
$$

for "normalized" quadratic differentials that have simple/double pole at x_r and vanishes at

$$x_+ + x_+ + x_3 + ...x_{r-1} + x_{r+1} + ... + x_d + w_{d+1} + ...w_{3g-3}.$$

We also denote the $u \to 0$ limits of these families by

$$q_0^{(i)} := q_{0,z,x_0,\mathbf{w}+\mathbf{x}'}^{(i)} \overset{\text{on } (U,z)}{\equiv} \frac{dz^2}{z^i} + R_0^{(i)}(z)dz^2, \qquad q_{0,x_r}^{(i)} := q_{0,z_r,x_r,2x_0+\mathbf{x}_\mathbf{r}'+\mathbf{w}-w_0}^{(i)}. \tag{7.5}$$

REMARK 7.1. We have defined $q_{u,\pm}^{(i)}$ upon choosing a divisor $\mathbf{w} = w_0 + w_{d+1} + ...w_{3g-3}$. Choosing a different choice of $\mathbf{w}$ amounts to translating the definitions of $q_{u,\pm}^{(i)}$ and $q_{u,x_r}^{(i)}$ by a holomorphic quadratic differential that vanishes at $\mathbf{x}'$ and $x_+ + x_- + \mathbf{x}_\mathbf{r}'$ respectively.

Now, to construct q_u from the "building blocks", for each nonzero $u \in z(U)$, let

$$q_u = q_u^{(0)} + \sum_{s\in\{\pm\}} \nu_s(u) q_{u,s}^{(1)} - \frac{3}{4}\sum_{s\in\{\pm\}} q_{u,s}^{(2)} + \sum_{r=3}^{d} \nu_r(u) q_{u,x_r}^{(1)} - \frac{3}{4}\sum_{r=3}^{d} q_{u,x_r}^{(2)}, \tag{7.6}$$

where $q_u^{(0)}$ is the holomorphic quadratic differential that makes q_u satisfy condition (6.1b), which now due to the construction of $q_{u,\pm}^{(i)}$ and $q_{u,x_r}^{(i)}$ takes the form

$$q_u^{(0)}(z)\mid_{z=\pm u} = -\nu_\pm(u)^2 - \nu_\mp(u) q_{u,\mp}^{(1)}(z)\mid_{z=\pm u} + \frac{3}{4} q_{u,\mp}^{(2)}(z)\mid_{z=\pm u} =: C_\pm(u), \tag{7.7a}$$

$$q_u^{(0)}(z_r)\mid_{z_r=z_r(x_r)} = -\nu_r(u)^2, \qquad \text{for } 3 \le r \le d. \tag{7.7b}$$

The functions $\nu_\pm(u)$, $\nu_3(u)$, ..., $\nu_d(u)$ define via these constraints (7.7) for each $u \neq 0$ a non-homogeneous linear system in $H^0(K_X^2)$. Due to condition (7.1), the homogeneous linear system is of maximal rank d. It follows that, for each $u \neq 0$, the non-homogeneous linear system (7.7) defines via its solutions a $(3g-3-d)$-dimensional subspace in $H^0(K^2)$, and hence a $(3g-3-d)$-dimensional family of meromorphic quadratic differentials q_u with the appropriate Laurent tails at $x_\pm$, x_3, ..., x_d defined by $\nu_\pm(u)$, $\nu_3(u)$, ..., $\nu_d(u)$.

We can represent the linear system (7.7) more explicitly by using the family $\mathbf{q}_u = \{(q_{k,u})_{k=1}^{3g-3}\}_{u\in z(U)}$ of basis of $H^0(K^2)$ constructed in lemma 7.1. For $u \neq 0$, let $q_u^{(0)} = \sum_{k=1}^{3g-3} E_k(u) q_{k,u}$. Then

(7.7) decouples into a 2×2 linear system representing (7.7a),

$$\begin{pmatrix} q_1(z)\mid_{z=u} & q_2(z)\mid_{z=u} \\ q_1(z)\mid_{z=-u} & q_2(z)\mid_{z=-u} \end{pmatrix} \begin{pmatrix} E_1(u) \\ E_2(u) \end{pmatrix} = \begin{pmatrix} C_+(u) \\ C_-(u) \end{pmatrix}, \tag{7.8a}$$

and a trivial $(d-2) \times (d-2)$ one representing (7.7b),

$$E_r(u) = -\nu_r(u)^2 \text{ for } 3 \leq r \leq d. \tag{7.8b}$$

For $d < 3g-3$, the expansion coefficients $E_r(u)$ with r in the range $d < r \leq 3g-3$ are not constrained and parameterize the $(3g-3-d)$-dimensional subspace in $H^0(K^2)$ consisting of solutions to (7.7).

Limits of the solutions to the linear systems. We will be interested in families of meromorphic quadratic differentials $\{q_u\}_{u \in z(U), u \neq 0}$ that have well-defined limits $q_0 := \lim_{u \to 0} q_u$ with the Laurent tails at the poles having the appropriate forms to make these poles apparent singularities. In other words, we are interested in families $\{\mathcal{D}_u\}_{u \in z(U)}$ of SL-operators parameterized by U, with $\mathcal{D}_u$ for $u \neq 0$ having simple apparent singularities at $x_\pm$, x_3, ..., x_d and $\mathcal{D}_0$ being the limit as x_+ and x_- collide.

More specifically, we will study families determined by $\nu_3(u)$, ..., $\nu_d(u)$ being holomorphic for all $u \in z(U)$, and $\nu_\pm(u)$ holomorphic at $u \neq 0$ and having the Laurent expansions at $u = 0$ of the form

$$\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2) \tag{7.9}$$

or

$$\nu_\pm(u) = \mp\frac{1}{4u} + \nu_0 \pm \nu' u + \mathcal{O}(u^2). \tag{7.10}$$

LEMMA 7.2. *Suppose $\nu_3(u)$, ..., $\nu_d(u) : U \cong z(U) \to \mathbb{C}$ are holomorphic functions, and $\nu_\pm(u) : U \cong z(U) \to \mathbb{C}$ are meromorphic functions with simple poles and Laurent expansions at $u = 0$ of either the form (7.9) or (7.10). Then the induced non-homogeneous linear system (7.7) limits to* *a degenerate $d \times (3g-3)$ linear system of rank $d-1$.*

Proof. This can be checked explicitly by plugging (7.9) and (7.10) in (7.7a). Specifically, the coefficients at the orders u^{-1} and u^0 of $\nu_\pm(u)$ ensure that $C_\pm(u)$ are regular at $u=0$, while the coefficients at the order u^1 of $\nu_\pm(u)$ together with part 1 of Lemma 6.3 ensure that $C_+(0) = C_-(0) = C_0$, where

$$C_0 \coloneqq \begin{cases} -\nu' - \frac{3}{4}\lim\limits_{u\to 0}\frac{1}{u}R^{(1)}_{u,-}(u) & \text{for } \nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2), \\ -\nu_0^2 + \nu' - \frac{1}{4}\lim\limits_{u\to 0}\frac{1}{u}R_{u,-}(u) & \text{for } \nu_\pm(u) = \mp\frac{1}{4u} + \nu_0 \pm \nu' u + \mathcal{O}(u^2). \end{cases} \tag{7.11}$$

Then the linear system (7.7) limits to

$$q_u^{(0)}(0) = C_0, \qquad\qquad q_u^{(0)}(z_r(x_r)) = -\nu_r(0)^2, \tag{7.12}$$

for $3 \le r \le d$. Note that the dimension of $Q_{x_0+\mathbf{x}'}$ must have the minimal value $3g-2-d$ since otherwise condition (7.1) will be violated. This yields the rank d_1 of the system (7.12). $\square$

LEMMA 7.3. *Let x_0, x_3, ..., x_d satisfy condition (7.1), $\nu_0 \in \mathbb{C}$, and $q \in H^0(K^2)$ a holomorphic quadratic differential. Then there exist holomorphic functions $\nu_3(u)$, ..., $\nu_d(u) : z(U) \to \mathbb{C}$ and $\nu_\pm(u) : z(U \setminus \{x_0\}) \to \mathbb{C}$ with Laurent expansions at $u=0$ of the forms (7.9) or (7.10) (with ν_0 as the coefficient of order u^0), and a family of holomorphic quadratic differentials $\{q_u^{(0)}\}_{u\in z(U)}$ such that*

(i) $q_u^{(0)}$ solves the linear system (7.7) defined by $\nu_\pm(u)$, $\nu_3(u)$, ..., $\nu_d(u)$ for $u \neq 0$;

(ii) $q_0^{(0)} = q$ solves the corresponding limit linear system (7.12).

Proof. We will make use of the representation (7.8) of (7.7) using a family $\mathbf{q}_u = \{(q_{k,u})_{k=1}^{3g-3}\}_{u\in z(U)}$ of basis of $H^0(K^2)$ constructed in lemma 7.1. Let us use the basis corresponding to $u=0$ and expand $q = \sum_{k=1}^{3g-3} E_k^0 q_{k,0}$. We need to show that there exist $\nu_\pm(u)$ of the prescribed forms such that the solutions to (7.8a),

$$E_1(u) = \frac{C_+(u)q_2(-u) - C_-(u)q_2(u)}{q_1(u)q_2(-u) - q_1(-u)q_2(u)}, \quad E_2(u) = \frac{q_1(u)C_-(u) - q_1(-u)C_+(u)}{q_1(u)q_2(-u) - q_1(-u)q_2(u)}, \tag{7.13}$$

where $C_\pm(u)$ are defined as in (7.7), have well-defined $u \to 0$ limits which are equal to E_1^0 and E_2^0 respectively. To this end, denote by $N_1(u)$ and $N_2(u)$ the respective numerators in (7.13). Since $\lim_{u\to 0} C_\pm(u) = C_0$ (cf. (7.11)), both $N_1(u)$ and $N_2(u)$ are holomorphic and vanish at $u = 0$. Since the denominators of (7.13) have simple zero at $u = 0$ by lemma 7.1, the $u \to 0$ limits of $E_1(u)$ and $E_2(u)$ are well-defined by L'Hôpital's rule. These limits are determined by the coefficient at order u^1 of $N_1(u)$ and $N_2(u)$. With $R_{u,\pm}^{(i)}(z) = \sum_{m,n} R_{m,n}^{(i)}(\pm u)^m z^n$, one can compute explicitly these coefficients to be

$$\begin{aligned}
N_1(u) &= \frac{q_2(0)}{2}\left[-4(\nu_{+,2}+\nu_{-,2}) + 3\left(R_{0,1}^{(2)} - R_{1,0}^{(2)} + R_{2,0}^{(1)} - R_{1,1}^{(1)} + R_{0,2}^{(1)}\right)\right] u \\
&\quad + \frac{q_2'(0)}{2}\left(4\nu_1 + 3R_{1,0}^{(1)} - 3R_{0,1}^{(1)}\right) u + \mathcal{O}(u^2), \\
N_2(u) &= -\frac{q_1'(0)}{2}\left[4\nu_1 + 3R_{1,0}^{(1)} - 3R_{0,1}^{(1)}\right] u + \mathcal{O}(u^2)
\end{aligned}$$

for the ansatz $\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu_1 u + \nu_{\pm,2}u^2 + \mathcal{O}(u^3)$, and

$$\begin{aligned}
N_1(u) &= -\frac{q_2(0)}{2}\left[8\nu_0\nu_1 + 4\nu_0\left(R_{0,1}^{(1)} - R_{1,0}^{(1)}\right) + R_{2,0}^{(1)} - R_{1,1}^{(1)} + R_{2,0}^{(1)} + 3R_{1,0}^{(2)} - 3R_{0,1}^{(2)}\right] u \\
&\quad + \frac{q_2'(0)}{2}\left[4\nu_0^2 - 4\nu_1 + R_{0,1}^{(1)} - R_{1,0}^{(1)}\right] u + \mathcal{O}(u^2) \\
N_2(u) &= \frac{q_1'(0)}{2}\left(-4\nu_0^2 + 4\nu_1 - R_{0,1}^{(1)} + R_{1,0}^{(1)}\right) u + \mathcal{O}(u^2)
\end{aligned}$$

for the ansatz $\nu_\pm(u) = \mp\frac{1}{4u} + \nu_0 \pm \nu_1 u + \nu_{\pm,u}u^2 + \mathcal{O}(u^3)$.

Since $q_2(0)$ and $q_1'(0)$ are non-zero, we can tune $(\nu_1, \nu_{\pm,2})$ in the first case and (ν_0, ν_1) in the second case to tune the coefficients of order u^1 of $N_1(u)$ and $N_2(u)$ as we want. Hence there exist holomorphic functions $\nu_\pm(u) : U \setminus \{x_0\} \to \mathbb{C}$ with the Laurent expansion at $u = 0$ of the form (7.9) or (7.10), so that $\lim_{u\to 0} E_1(u) = E_1^0$ and $\lim_{u\to 0} E_2(u) = E_2^0$. For $3 \le r \le d$, let $\nu_r \in \mathbb{C}$ be such that $E_r^0 = -\nu_r^2$, and define $\nu_r(u) = \nu_r$ for all $u \in z(U)$. The tuple $(E_1(u), E_2(u), E_3(u), ..., E_{3g-3}(u)) = (E_1(u), E_2(u), E_3^0, ..., E_{3g-3}^0)$ then defines for each $u \neq 0$ a holomorphic quadratic differential $q_u^{(0)} := \sum_{k=1}^{3g-3} E_k(u) q_{k,u}$ that solves the linear system (7.8) defined by $(\nu_\pm(u), \nu_3(u), ..., \nu_d(u))$, with $\lim_{u\to 0} q_u^{(0)} = q$ a solution to the corresponding limit linear system (7.12). $\square$

7.2 Double apparent singularity as the limit

In this subchapter we will analyze the limit $\lim_{u \to 0} q_u$ if $\nu_{\pm}(u)$ take the form (7.10).

LEMMA 7.4. *Let x_0, x_3, ..., x_d be distinct points on X, (U, z) a coordinate neighborhood of x_0 with $z(x_0) = 0$, $\nu_{\pm}(u) = \mp \frac{1}{4u} + \nu_0 \pm \nu' u + \mathcal{O}(u^2)$ holomorphic functions on $U \setminus \{z_0\}$, and $q^{(i)}_{u,\pm}$ defined as in (7.4) for $u \in z(U)$, $u \neq 0$. Then the family of meromorphic quadratic differentials*

$$\sum_{s \in \{\pm\}} \nu_s(u) q^{(1)}_{u,s} - \frac{3}{4} \sum_{s \in \{\pm\}} q^{(2)}_{u,s}, \tag{7.14}$$

which is parameterized by $U \setminus \{x_0\}$, extends to a family parameterized by U. The meromorphic quadratic differential $q^{(2)}_{x_0}$ corresponding to the extension to x_0 is holomorphic on $X \setminus \{x_0\}$, vanishes at x_3, ..., x_d, and on U takes the form

$$q^{(2)}_{x_0} \equiv q^{(2)}_{x_0}(\nu_0) = \left[-\frac{2}{z^2} + \frac{2\nu_0}{z} + 2\nu_0 R^{(1)}_0(z) - \frac{1}{4} \check{R}^{(1)}(z) - \frac{3}{2} R^{(2)}_0(z) \right] dz^2, \tag{7.15}$$

where $\check{R}^{(1)}(z)$ is defined in part 2 of lemma 6.3 and $R^{(i)}_0(z)$ defined in (7.5).

Proof. We will show explicitly that, as a multi-variable function with variables u and z, (7.14) limits to (7.15) as $u \to 0$. To this end, it suffices to show that given any $z \neq 0$, the $u \to 0$ limit of the evaluation of (7.14) at z is the evaluation of (7.15) at z. Plugging $\nu_{\pm}(u) = \mp \frac{1}{4u} + \nu_0 \pm \nu' u + \mathcal{O}(u^2)$ into (7.14) yields

$$\begin{aligned} &\frac{-2z^2 - u^2}{(z-u)^2 (z+u)^2} + \frac{2\nu_0 z}{(z-u)(z+u)} - \frac{R^{(1)}_{u,+}(z) - R^{(1)}_{u,-}(z)}{4u} + \nu_0 \left(R^{(1)}_{u,+}(z) + R^{(1)}_{u,-}(z) \right) \\ &\qquad - \frac{3}{4} \left(R^{(2)}_{u,+}(z) + R^{(2)}_{u,-}(z) \right) + \mathcal{O}(u)(z). \end{aligned} \tag{7.16}$$

Here we have ignored the factor dz^2 and denoted by $\mathcal{O}(u)(z)$ a function with variables u and z such that, given any $z' \neq 0$, the function $F_{z'}(u) \coloneqq \mathcal{O}(u)(z) \mid_{z=z'}$ is holomorphic on $\{u \mid 0 \leq |u| \leq |z'|/2\}$ with $F_{z'}(u) \sim \mathcal{O}(u)$. It follows that $\lim_{u \to 0} \mathcal{O}(u)(z) = \mathcal{O}(u)(z) \mid_{u=0}$, which a priori defines a function in z, is the zero function. By a similar argument, we can take the $u \to 0$ limit of the other terms in (7.16) explicitly and obtain $q^{(2)}_{x_0}$ as in (7.15) as the limit.

To show that $q^{(2)}_{x_0}$ is holomorphic on $X \setminus \{x_0\}$, it suffices to show holomorphicity on $X \setminus U$. Given $p \in X \setminus U$ and a coordinate neighborhood (V, w) of p, let $q^{(i)}_{u,\pm}(w)dw^2$ be the local form of $q^{(i)}_{u,\pm}$ on V. We want to show that the evaluation of (7.14) at p, which defines the function

$$\sum_{s\in\{\pm\}} \nu_s(u) q^{(1)}_{u,s}(w(p)) - \frac{3}{4} \sum_{s\in\{\pm\}} q^{(2)}_{u,s}(w(p)) = \frac{1}{4u} \left(-q^{(1)}_{u,+}(w(p)) + q^{(1)}_{u,-}(w(p)) \right) + \mathcal{O}(1) \tag{7.17}$$

in u which is holomorphic at $u \neq 0$ has a well-defined $u \to 0$ limit. Since $q^{(i)}_{u,\pm} \overset{u\to 0}{\to} q^{(i)}_0$, which is holomorphic on $X \setminus \{x_0\}$, it suffices to show that the leading term $\frac{1}{4u}\left(-q^{(1)}_{u,+}(w(p)) + q^{(1)}_{u,-}(w(p)) \right)$ of (7.17) has a well-defined $u \to 0$ limit. This is achieved by an argument similar to the proof of part 2 of lemma 6.3. Furthermore, for $p \in \{u_3, ..., u_d\}$, since (7.17) vanishes for $u \neq 0$, it is identically zero for all $u \in z(U)$. $\square$

In the following, we fix a holomorphic projective structure and a corresponding SL-operator $\mathcal{D}_{pr}$ (which has no apparent singularity). Recall again that any other SL-operator can be written as $\mathcal{D}_{pr} + q$ where q is a meromorphic quadratic differential having double poles with appropriate Laurent tails in coordinates subordinate to the chosen holomorphic projective structure.

PROPOSITION 7.5. *Let x_0, x_3, ..., x_d be distinct points on X such that $2x_0 + x_3 + ...x_d$ is Q-generic. Let z be a coordinate on U subordinate to the chosen holomorphic projective structure, with $z(x_0) = 0$. Let $\nu_3(u)$, ..., $\nu_d(u)$ be holomorphic functions on U, $\nu_\pm(u) = \mp\frac{1}{4u} + \nu_0 \pm \nu' u + \mathcal{O}(u^2)$ holomorphic functions on $U \setminus \{z_0\}$, and $\{\mathcal{D}_u = \mathcal{D}_{pr} + q_u\}_{u\in z(U), u\neq 0}$ the corresponding family of SL-operators parameterized by $U \setminus \{x_0\}$ where q_u is defined as in (7.6), with $\{q^{(0)}_u\}_{u\in z(U), u\neq 0}$ the corresponding family of holomorphic quadratic differentials.*

If there exists a holomorphic quadratic differential $q^{(0)}_0$ with $q^{(0)}_0 = \lim_{u\to 0} q^{(0)}_u$, then $\{\mathcal{D}_u\}_{u\in z(U), u\neq 0}$ extends to a family of SL-operators parameterized by U. The SL-operator corresponding to the extension to x_0 is $\mathcal{D}_0 = \mathcal{D}_{pr} + q_0$, where

$$q_0 \overset{on\ (U,z)}{=} q^{(0)}_0 + q^{(2)}_{x_0}(\nu_0) + \sum_{r=3}^{d} \nu_r(0) q^{(1)}_{0,x_r} - \frac{3}{4} \sum_{r=3}^{d} q^{(2)}_{0,x_r}. \tag{7.18}$$

(Recall the definition of $q_{0,x_r}^{(2)}$ in (7.15).) In particular, the apparent singularities of $\mathcal{D}_0$ consists of a double apparent singularity at x_0 with accessory parameter ν_0, and simple apparent singularities at x_3, ..., x_d with respective accessory parameters $\nu_3(0)$, ..., $\nu_d(0)$.

Proof. The existence of $\mathcal{D}_0 = \mathcal{D}_{pr} + q_0$ with q_0 of the form (7.18) follows from the definition (7.6) of q_u and lemma 7.4. Let $q_0^{reg}(z)$ be the regular part on U of $q_0(z)$, i.e.

$$\begin{aligned} q_0^{reg}(z) &:= q_0(z) - \left(-\frac{2}{z^2} + \frac{2\nu_0}{z}\right) dz^2 \\ &= q_0^{(0)}(z) + \left[2\nu_0 R_0^{(1)}(z) - \frac{1}{4}\check{R}^{(1)}(z) - \frac{3}{2}R_0^{(2)}(z)\right] dz^2 + \sum_{r=3}^{d} \nu_r(0) q_{0,x_r}^{(1)}(z) - \frac{3}{4}\sum_{r=3}^{d} q_{0,x_r}^{(2)}(z). \end{aligned}$$

To prove the proposition, it remains to show that the coefficients of the Laurent tail of $q_0(z)$ satisfy the condition making x_0 a double apparent singularity, i.e. for $q_0^{reg}(z) = \sum_{k\geq 0} q_{0,k} z^k$ we have

$$\nu_0^3 + q_{0,0}\nu_0 + \frac{q_{0,1}}{2} = 0. \tag{7.19}$$

To this end, for each $u \neq 0$, let $q_u^{reg}(z)$ be the regular part on U of $q_u(z)$, i.e.

$$\begin{aligned} q_u^{reg}(z) &:= q_u(z) - \left(-\frac{3}{4(z-u)^2} - \frac{3}{4(z+u)^2} + \frac{\nu_+(u)}{z-u} + \frac{\nu_-(u)}{z+u}\right) \\ &= q_u^{(0)}(z) + \sum_{s\in\{\pm\}} \nu_s(u) R_{u,s}^{(1)}(z) - \frac{3}{4}\sum_{s\in\{\pm\}} R_{u,s}^{(2)}(z) + \sum_{r=3}^{d} \nu_r(u) q_{u,x_r}^{(1)}(z) - \frac{3}{4}\sum_{r=3}^{d} q_{u,x_r}^{(2)}(z). \end{aligned}$$

As a function of both u and z, the leading term in u of $q_u^{reg}(z)$ comes from the leading terms in $\nu_\pm(u) R_{u,\pm}^{(1)}(z)$ and is equal to $-(R_{u,+}^{(1)}(z) - R_{u,-}^{(1)}(z))/(4u)$. By part 2 of lemma 6.3, this has a well-defined $u \to 0$ limit. Hence $q_u^{reg}(z)$ has a well-defined $u \to 0$ limit, and in fact $q_u^{reg}(z) \xrightarrow{u\to 0} q_0^{reg}(z)$.

Now, this enables us to expand $q_u^{reg}(z) = \sum_{m,n\geq 0} q_{m,n} u^m z^n$, where letting $m = 0$ gives the coefficients $q_{0,n}$ of the Taylor expansion $q_0^{reg}(z) = \sum_{n\geq 0} q_{0,n} z^n$. Plugging this expansions of $q_0^{reg}(z)$ and and $\nu_\pm(u)$ into condition (6.1b) for $z = \pm u$,

$$\nu_\pm(u)^2 - \frac{3}{4}\frac{1}{(2u)^2} + \frac{\nu_\mp(u)}{(\pm 2u)} + q_u^{reg}(\pm u) = 0, \qquad u \neq 0,$$

we obtain a series of constraints by order of u starting from the order u^{-2}. The constraints up to order u^{-1} are automatically satisfied by the leading terms of $\nu_{\pm}(u)$. Solving the constraints at order u^0 and u^1, we obtain (7.19). $\square$

PROPOSITION 7.6. *(Proposition 1.7) Let $\mathcal{D}$ be an SL-operator with div$(\mathcal{D}) = 2x_0 + x_3 + ... + x_d$ being Q-generic and $d \leq 3g-3$. Then there exists a coordinate neighborhood (U, z) of x_0, where $U \subset U'$ and $z(x_0) = 0$, and a family of SL-operators $\{\mathcal{D}_u\}_{u \in z(U)}$ parameterized by U such that*

(i) $\mathcal{D}_0 = \mathcal{D}$;

(ii) for $u \neq 0$, $\mathcal{D}_u$ has simple apparent singularities at x_3, ..., x_d and $x_{\pm} \in U$ with $z(x_{\pm}) = \pm u$;

(iii) for $u \neq 0$, the accessory parameters $\nu_{\pm}(u)$ of $x_{\pm}$ w.r.t. the local coordinate z, as functions of u, have simple poles at $u = 0$ and Laurent expansions $\nu_{\pm}(u) = \mp\frac{1}{4u} + \nu_0^{\mathcal{D}} \pm \nu' u...$, where $2\nu_0^{\mathcal{D}}$ is the accessory parameter of the double apparent singularity x_0 of $\mathcal{D}$.

Furthermore, this family defines via taking monodromy a holomorphic map $U \to Hom(\pi_1, PSL_2(\mathbb{C}))$, which is injective for $d < 2g-2$.

Proof. Fix a holomorphic projective structure with corresponding SL-operator $\mathcal{D}_{pr}$, and choose coordinate neighborhoods (U_r, z_r) of x_r for $3 \leq r \leq d$ and (U, z) of x subordinate to this holomorphic projective structure, where U satisfies condition (7.1). We can write $\mathcal{D} = \mathcal{D}_{pr} + q_{\mathcal{D}}$, where $q_{\mathcal{D}}$ is a meromorphic quadratic differential with local expression $-\frac{3}{4(z-z_r(x_r))^2}dz_r^2 + \frac{\nu_r^{\mathcal{D}}}{z-z_r(x_r)}dz_r^2 + ...$ on U_r, $3 \leq r \leq d$, and $-\frac{2}{z^2}dz^2 + \frac{2\nu_0^{\mathcal{D}}}{z}dz^2 + ...$ on U. Then there exists a unique holomorphic quadratic differential $q_0^{(0)}(\mathcal{D})$ such that $q_{\mathcal{D}}$ can be written in the form (7.18), i.e.

$$q_{\mathcal{D}} = q_0^{(0)}(\mathcal{D}) + q_{x_0}^{(2)}(\nu_0^{\mathcal{D}}) + \sum_{r=3}^{d} \nu_r^{\mathcal{D}} q_{0,x_r}^{(1)} - \frac{3}{4}\sum_{r=3}^{d} q_{0,x_r}^{(2)}.$$

By lemma 7.3, there exist holomorphic functions $\nu_3(u)$, ..., $\nu_d(u) : z(U) \to \mathbb{C}$, $\nu_{\pm}(u) : z(U \setminus \{x_0\}) \to \mathbb{C}$ with Laurent expansions at $u = 0$ of the forms (7.10) (in particular, with $\nu_0^{\mathcal{D}}$ as the coefficient of order u^0), and a family $\{q_u^{(0)}\}_{u \in z(U)}$ of holomorphic quadratic differentials with

$\lim_{u\to 0} q_u^{(0)} = q_0^{(0)}(\mathcal{D})$ such that

$$q_u = q_u^{(0)} + \sum_{s\in\{\pm\}} \nu_s(u) q_{u,s}^{(1)} - \frac{3}{4}\sum_{s\in\{\pm\}} q_{u,s}^{(2)} + \sum_{r=3}^{3g-3} \nu_r(u) q_{u,x_r}^{(1)} - \frac{3}{4}\sum_{r=3}^{3g-3} q_{u,x_r}^{(2)}, \qquad u \neq 0,$$

is a meromorphic quadratic differential with simple apparent singularities at $x_3, ..., x_d$ and $x_\pm \in U$ with $z(x_\pm) = \pm u$. By proposition 7.5, $q_0 \coloneqq \lim_{u\to 0} q_u = q_{\mathcal{D}}$, and hence $\{\mathcal{D}_u \coloneqq \mathcal{D}_{pr} + q_u\}_{u\in z(U)}$ defines a family we seek.

To see that taking the projective monodromy representation of $\mathcal{D}_u$ defines a holomorphic map $U \to \mathrm{Hom}(\pi_1, PSL_2(\mathbb{C}))$, consider a coordinate neighborhood (V, w), where V is distinct from $U \cup \{x_3, ..., x_d\}$ and w is subordinate to the chosen holomorphic projective structure. On (V, w), $\mathcal{D}_u$ takes the form $\partial_w^2 + q_u(w)$ where $q_u(w)$ is a function holomorphic on $U \times V$. By standard results on differential equations that vary holomorphically with respect to deformation parameters [38], the local solutions to $\partial_w^2 + q_u(w)$ are holomorphic functions on $U \times V$. Analytically continuing the ratio of two such linearly independent solutions defines the projective monodromy representation of $\mathcal{D}_u$, which is now holomorphic in u. By corollary 5.4, if the number of apparent singularities is less than $2g - 2$ to start with, this holomorphic map is injective. $\square$

EXAMPLE 7.2. Let $d \in \mathbb{Z}_+$ be even, and x_0, x_3, ..., x_d, (U, z) and $x_\pm$ as in proposition 7.6. Suppose $\{(F_u, \nabla_u)\}_{u\in z(U)}$ is a family of irreducible $SL_2(\mathbb{C})$-holomorphic connections where F_u admits a subbundle L_u such that the zero divisor of $c_{L_u}(\nabla_u)$ is $x_\pm(u) + x_3 + ... + x_d$ for $u \neq 0$ and $2x_0 + x_3 + ... + x_d$ for $u = 0$. In other words, at the limit $u \to 0$, $c_{L_u}(\nabla_u)$ forms a double zero at x_0.

We claim that the accessory parameters $\nu_\pm(u)$ of the apparent singularities $x_\pm(u)$ from the induced family $\{\mathcal{D}_u \coloneqq \mathcal{D}_{(L_u\hookrightarrow F_u, \nabla_u)}\}_{u\neq 0}$ are of the form (7.10). Suppose on U and in certain local frame adapted to L_u, ∇_u takes the form $\partial_z + \begin{pmatrix} a_u(z) & b_u(z) \\ c_u(z) & -a_u(z) \end{pmatrix}$. Then $a_u(z)$, $b_u(z)$ and $c_u(z)$ are holomorphic on both u and z. In particular, we can write $c_u(z) = (z - u)(z + u) f_u(z)$ where $f_u(z)$ is a function holomorphic on both u and z such that for all $u \in z(U)$, $f_u(z)$ is nonzero everywhere on U. Expanding $a_u(z) = \sum_{m,n\geq 0} a_{m,n} u^m z^n$ and $f_u(z) = \sum_{m,n\geq 0} f_{m,n} u^m z^n$, one can observe that the accessory parameters of the apparent singularities $x_\pm$ of $\mathcal{D}_u$ (cf. (5.17)) have Laurent expansions

$$a_u(\pm u) - \frac{c_u''(\pm u)}{4c_u'(\pm u)} = \mp\frac{1}{4u} + \left(a_{0,0} - \frac{f_{0,1}}{2f_{0,0}}\right) \pm \left(a_{0,1} + a_{1,0} \right. \\ \left. + \frac{f_{0,1}^2 + f_{0,1}f_{1,0} - f_{0,0}(2f_{0,2} + f_{1,1})}{2f_{0,0}^2}\right)u + \mathcal{O}(u^2)$$

which satisfy the form (7.10).

7.3 Reduction of the number of apparent singularities as the limit

In this subchapter we will analyze the limit $\lim_{u\to 0} q_u$ if $\nu_\pm(u)$ take the form (7.9). We skip the proofs of the following results as they are similar to the proofs in the previous subchapter.

LEMMA 7.7. *Let x_0, x_3, ..., x_d be distinct points on X, (U, z) a coordinate neighborhood of x_0 with $z(x_0) = 0$, $\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2)$ holomorphic functions on $U \setminus \{x_0\}$, and $q_{u,\pm}^{(i)}$ defined as in (7.4) for $u \in z\,(U)$, $u \neq 0$. Then the family of meromorphic quadratic differentials*

$$\sum_{s\in\{\pm\}} \nu_s(u) q_{u,s}^{(1)} - \frac{3}{4}\sum_{s\in\{\pm\}} q_{u,s}^{(2)}, \tag{7.20}$$

which is parameterized by $U \setminus \{x_0\}$, extends to a family parameterized by U. The quadratic differential $\Delta q_{x_0}^{(0)}$ corresponding to the extension to x_0 is holomorphic on X, vanishes at x_3, ..., x_d, and on U takes the form

$$\Delta q_{x_0}^{(0)} = \left[\frac{3}{4}\check{R}^{(1)}(z) - \frac{3}{2}R_0^{(2)}(z)\right] dz^2. \tag{7.21}$$

Proof. Similar to the proof of lemma 7.4. □

Fix a holomorphic projective structure, and denote by $\mathcal{D}_{pr}$ the corresponding SL-operator that has no apparent singularity. In the following we use coordinates subordinate to the chosen holomorphic projective structure.

PROPOSITION 7.8. *Let x_0, x_3, ..., x_d be distinct points on X such that $2x_0 + x_3 + \ldots x_d$ is Q-generic. Let z be a coordinate on U subordinate to the chosen holomorphic projective structure,* *with $z(x_0) = 0$. Let $\nu_3(u)$, ..., $\nu_d(u)$ be holomorphic functions on U, $\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2)$ holomorphic functions on $U \setminus \{z_0\}$, and $\{\mathcal{D}_u = \mathcal{D}_{pr} + q_u\}_{u\in z(U), u\neq 0}$ the corresponding family of SL-operators parameterized by $U \setminus \{x_0\}$ where q_u is defined as in (7.6), with $\{q_u^{(0)}\}_{u\in z(U), u\neq 0}$ the corresponding family of holomorphic quadratic differentials.*

If there exists a holomorphic quadratic differential $q_0^{(0)}$ with $q_0^{(0)} = \lim_{u\to 0} q_u^{(0)}$, then $\{\mathcal{D}_u\}_{u\in z(U), u\neq 0}$ extends to a family of SL-operators parameterized by U. The SL-operator corresponding to the extension to x_0 is $\mathcal{D}_0 = \mathcal{D}_{pr} + q_0$, where

$$q_0 \overset{on\ (U,z)}{=} q_0^{(0)} + \Delta q_{x_0}^{(0)} + \sum_{r=3}^{d} \nu_r(0) q_{0,x_r}^{(1)} - \frac{3}{4}\sum_{r=3}^{d} q_{0,x_r}^{(2)}, \tag{7.22}$$

where $\Delta q_{x_0}^{(0)}$ is defined as in (7.21). In particular, the apparent singularities of $\mathcal{D}_0$ are simple and located at x_3, ..., x_d with accessory parameters $\nu_3(0)$, ..., $\nu_d(0)$.

Proof. The proof follows directly from lemma 7.7. □

PROPOSITION 7.9. *Let $\mathcal{D}$ be an SL-operator with div($\mathcal{D}$) $= x_3 + ... + x_d$ for $d \leq 3g - 3$, and x_0 be a point on X such that $2x_0 + x_3 + ...x_d$ is Q-generic. Then there exists a coordinate neighborhood (U, z) of x_0 and a family of SL-operators $\{\mathcal{D}_u\}_{u\in z(U)}$ parameterized by U such that*

(i) $\mathcal{D}_0 = \mathcal{D}$;

(ii) for $u \neq 0$, $\mathcal{D}_u$ has simple apparent singularities at x_3, ..., x_d and $x_\pm \in U$ with $z(x_\pm) = \pm u$;

(iii) for $u \neq 0$, the accessory parameters $\nu_\pm(u)$ of $x_\pm$ w.r.t. the local coordinate z, as functions of u, have simple poles at $u = 0$ and Laurent expansions $\nu_\pm(u) = \pm\frac{3}{4u} \pm \nu' u + \mathcal{O}(u^2)$.

Furthermore, this family defines via taking monodromy a holomorphic map $U \to Hom(\pi_1, PSL_2(\mathbb{C}))$, which is injective for $d < 2g - 2$.

Proof. Similar to the proof of proposition 7.6. □

Relation to isomonodromic operation. Bubbling is an isomonodromic operation that takes as input a complex projective structure subordinate to a Riemann surface X' and a path γ on the underlying surface S, such that the restriction to γ of the developing map of the projective structure is injective. By cutting open X' along γ and gluing in an entire copy of $\mathbf{P}^1$ along the image of γ, one obtains another complex projective structure. After bubbling, the two end points of γ on S have an angle excess of 2π under the developing map, and hence are apparent singularities of the output projective coordinate. It is clear that the output projective structure realizes the same projective monodromy representation as the input projective structure, but has two more apparent singularities and is subordinate to a different complex structure X'' of the underlying surface S.

It is natural to ask if we can identify a given projective structure subordinate to X as the output of a bubbling. The answer is it is sufficient to find two paths γ_1 and γ_2 which (i) start and end at the same points, (ii) have the same image under the developing map, and (iii) bound a simply connected subset of the surface [8]. The input projective structure and Riemann surface of the bubbling can be recovered by "debubbling", i.e. collapsing the subset bounded by γ_1 and γ_2.

We suggest that in the setup of proposition 7.8 and 7.9, $x_\pm$ are the apparent singularities that appear as the result of a bubbling. In the region $|z| \sim |u| << 1$, the leading orders of $\mathcal{D}_u$ takes the form $\partial_z^2 - \frac{3u^2}{(z-u)^2(z+u)^2}$. One can check that $\chi(z) = \left(z^2 - u^2\right)^{-1/2}(z-u)^2$ and $\chi(z) = \left(z^2 - u^2\right)^{-1/2}(z+u)^2$ are solutions to this approximation of $\mathcal{D}_u$, and hence $w = \frac{(z-u)^2}{(z+u)^2}$ approximates the developing map. Since

$$w\left(ue^{i\theta}\right) = -\tan\left(\theta/2\right)^2 = w\left(ue^{-i\theta}\right),$$

the paths $\gamma_1 = ue^{i\theta}$ and $\gamma_2 = ue^{-i\theta}$, for $\theta \in [0, \pi]$, have the same image under $w(z)$. Hence, for each $u \neq 0$, we have identified an "approximate bubbling": the copy of $\mathbb{P}^1$ glued in is the disc defined by the boundary $\left\{|u|e^{i\theta} \mid \theta \in [0, 2\pi]\right\}$. The suggested image is that the bubble glued in "shrinks" as $u \to 0$ and completely disappears at the limit.

Appendix A

Rank-2 bundles as extensions of line bundles

Let X be a compact Riemann surface of genus $g \geq 2$. Given a line bundle L on X, all rank-2 bundles on X of determinant Λ that admits L as a subbundle can be realized as an extension of the form

$$0 \to L \to E \to L^{-1}\Lambda \to 0. \tag{A.1}$$

This is an example of an extension of $L^{-1}\Lambda$ by L.

We say two extensions of $L^{-1}\Lambda$ by L that realize E and E' are equivalent if there exists an isomorphism $E \xrightarrow{\sim} E'$ that commutes with the embeddings of L into E and E' (and hence also commutes with the projections to $L^{-1}\Lambda$). The moduli space $\operatorname{Ext}\left(L^{-1}\Lambda, L\right)$ of extensions of $L^{-1}\Lambda$ by L is the set of all such extensions modulo these equivalences. It is well-known that $\operatorname{Ext}\left(L^{-1}\Lambda, L\right)$ is canonically isomorphic to $H^1\left(L^2\Lambda^{-1}\right)$. The isomorphism is given by tensoring (A.1) with $L\Lambda^{-1}$ (or equivalently applying the functor $\operatorname{Hom}\left(L^{-1}\Lambda, \_\right)$) and taking the image of 1 via the coboundary map $H^0\left(\mathcal{O}\right) = H^0\left(L^{-1}\Lambda \otimes L\Lambda^{-1}\right) \to H^1\left(L^2\Lambda^{-1}\right)$.

We will use this invariant formulation of the isomorphism $\operatorname{Ext}\left(L^{-1}\Lambda, L\right) \cong H^1\left(L^2\Lambda^{-1}\right)$ in the proof of Lange-Narasimhan's result below [42]. On the other hand, we can understand how (A.1) can be regarded as an element of $H^1\left(L^2\Lambda^{-1}\right)$ more concretely as follows. The data equivalent to this s.e.s is E together with an embedding $L \overset{i}{\hookrightarrow} E$. Concretely, with respect to the local decomposition $E|_{U_\alpha} \cong L|_{U_\alpha} \oplus (L^{-1}\Lambda)|_{U_\alpha}$ over each open subset U_α indexed by $\alpha \in \mathcal{I}$, the transition functions $(E)_{\alpha\beta}$ of E are of the form

$$(E)_{\alpha\beta} = \begin{pmatrix} l_{\alpha\beta} & l_{\alpha\beta}\epsilon_{\alpha\beta} \\ 0 & l_{\alpha\beta}^{-1}\lambda_{\alpha\beta} \end{pmatrix} \tag{A.2}$$

where $l_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ are transition functions of L and Λ respectively. The cocycle conditions

$(E)_{\alpha\beta} = (E)^{-1}_{\alpha\beta}$ and $(E)_{\alpha\gamma} = (E)_{\alpha\beta}(E)_{\beta\gamma}$ are respectively equivalent to the constraints

$$\epsilon_{\alpha\beta} = -l^2_{\beta\alpha}\lambda^{-1}_{\beta\alpha}\epsilon_{\beta\alpha}, \qquad -\epsilon_{\beta\gamma} + \epsilon_{\alpha\gamma} - l^2_{\gamma\beta}\lambda^{-1}_{\gamma\beta}\epsilon_{\alpha\beta} = 0. \tag{A.3}$$

on the local functions $\epsilon_{\alpha\beta}$. These two conditions are precisely the definition of a 1-cocyle of the line bundle $L^2\Lambda^{-1}$, regarded as a locally free $\mathcal{O}_c$-module. The first condition means that the local functions $\epsilon_{\beta\alpha}$ and $\epsilon_{\alpha\beta}$, up to a sign, are representatives of the same section of $L^2\Lambda^{-1}$ over $U_{\alpha\beta}$, but in the representations defined by the restrictions $\left(L^2\Lambda^{-1}\right)(U_\alpha)|_{U_{\alpha\beta}}$ and $\left(L^2\Lambda^{-1}\right)(U_\beta)|_{U_{\alpha\beta}}$ respectively, with the isomorphism from the former to the latter given by multiplying with $l^2_{\beta\alpha}\lambda^{-1}_{\beta\alpha}$ [18]. The second condition can be read as the cocycle condition of the data $(\epsilon_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ written in the restriction $\left(L^2\Lambda^{-1}\right)(U_\gamma)|_{U_{\alpha\beta\gamma}}$. Therefore any extension of the form (A.2) defines a 1-cocycle of $L^2\Lambda^{-1}$ and vice versa. One can show that the equivalences of 1-cocycle of $L^1\Lambda^{-1}$ and extensions of the form (A.1) are compatible with this correspondence, hence the isomorphism $\mathrm{Ext}\left(L^{-1}\Lambda, L\right) \cong H^1\left(L^2\Lambda^{-1}\right)$. The scaling of an embedding L into E corresponds to scaling the extension (A.1), hence the moduli space we are interested in is $\mathbb{P} := \mathbb{P}H^1\left(L^2\Lambda^{-1}\right)$.

Secants and secant varieties of X in $\mathbb{P}$. We first recall that an element of the projectivization $\mathbb{P}V$ of a vector space $V \cong \mathbb{C}^{n+1}$, by definition representing a line in V, is equivalent to a hyperplane, i.e. a codimension-1 linear subspace, of the dual space V^*. This hyperplane in V^* is defined as the kernel of the line in V. In coordinates, if $(v_i) \in \mathbb{C}^n$ is the coordinate of a representative element of a line in V, then in the dual coordinates $\check{v}_i$ the corresponding hyperplane in V^* is defined by the equation $\sum v_i\check{v}_i = 0$. Hence in our case where V is taken to be $H^1\left(L^2\Lambda^{-1}\right) \cong \mathbb{C}^{n+1}$, the moduli space $\mathbb{P} \cong \mathbb{P}^n$ equivalently characterizes the hyperplanes in $H^0\left(KL^{-2}\Lambda\right)$ via Serre duality.

We will be mostly interested in the situation where $\deg\left(\Lambda L^{-2}\right) \geq 2$. In this case, due to degree reason no point on X is a common zero of all sections of $KL^{-2}\Lambda$, and hence imposing the

[18] If $L^2\Lambda^{-1} \cong \mathcal{O}(Q)$ for a divisor D then the corresponding 1-cocyle relation in $\mathcal{O}(Q)$, the sheaf of local meromorphic functions with poles bounded below by $-Q$, is $-\epsilon_{\beta\gamma} + \epsilon_{\alpha\gamma} - \epsilon_{\alpha\beta} = 0$. The factor $l^2_{\gamma\beta}\lambda^{-1}_{\gamma\beta}$ is needed when we transit from the descriptions in terms of local functions to local sections of line bundles.

vanishing condition at a given point p defines a hyperplane in $H^0\left(KL^{-2}\Lambda\right)$. We can define a map

$$\begin{aligned}\text{Span} : X &\longrightarrow \mathbb{P} \\ p &\mapsto \left[\left\{s \in H^0\left(KL^{-2}\Lambda\right) \mid s(p) = 0\right\}\right].\end{aligned} \tag{A.4}$$

One can describe this map in the homogeneous coordinates of $\mathbb{P}$ as follows. Let s_0, ..., s_n be a basis of $H^0\left(KL^{-2}\Lambda\right) \cong \mathbb{C}^{n+1}$ and $(\check{v}_i)_{i=0}^n$ be the coordinates with respect to this basis. A section $s = (\check{v}_i(s)) = \sum \check{v}_i(s)s_i$ vanishes at p if and only if $\sum \check{v}_i(s)s_i(p) = 0$. By definition of Span$(p)$ as the kernel of the space of all sections s satisfying $\sum \check{v}_i(s)s_i(p) = 0$, we can write Span$(p) = [s_0(p), ..., s_n(p)]$ in the dual coordinates on $H^1\left(L^2\Lambda^{-1}\right)$ and upon choosing a local trivialization on a neighborhood of p (this coordination is independent of the choice of the local trivialization since different choices differ by a locally nowhere-vanishing holomorphic function). Furthermore, when $\deg\left(\Lambda L^{-2}\right) \geq 3$ then $X \overset{\text{Span}}{\to} \mathbb{P}$ is an embedding

Given an effective divisor $D = p_1 + ... + p_d$, define Span$\,(D)$ to be the linear subspace of $\mathbb{P}$ spanned by Span$\,(p_1)$, ..., Span$\,(p_d)$. By definition Span(D) is the projectivization of the linear subspace in $H^1\left(L^2\Lambda^{-1}\right)$ which is the kernel of the subspace in $H^0\left(KL^{-2}\Lambda\right)$ consisting of sections vanishing at D. For a generic divisor $D = p_1 + ... + p_d$ of degree $d \leq n + 1$, the space of such sections is of codimension d, or equivalently the matrix $(s_i\,(p_j))$ formed by homogeneous coordinates of Span$\,(p_j)$ is of maximal rank d. For such a generic D, the points Span(p_1), ..., Span(p_d) are linearly independent and hence Span(D) is of dimension $d-1$. Hence Span$(D) = \mathbb{P}$ for a generic effective divisor D of degree $d \geq n + 1 = h^0\left(KL^{-2}\Lambda\right)$. This reflects the fact the only section of $KL^{-2}\Lambda$ that vanishes at such a generic divisor D is the zero section (since we only have $h^0\left(KL^{-2}\Lambda\right) - 1$ degrees of freedom to move the zeroes of sections of $KL^{-2}\Lambda$), and hence by definition any nonzero element of $H^1\left(L^2\Lambda^{-1}\right)$ can represent an element in Span(D).

Given an effective divisor D of degree d we say Span(D) is a d-secant of X in $\mathbb{P}$. We say the closure of the union of all such d-secants the d-secant variety of X in $\mathbb{P}$ and denote it by Sec$_d(X)$. It can be shown that Sec$_d(X)$ is an irreducible variety of dimension $2d - 1$ if it is not already the whole $\mathbb{P}$. In particular, Sec$_1\,(X)$ is the embedding of X in $\mathbb{P}$.

Explicit constructions of secants of X in $\mathbb{P}$. Before discussing an explicit construction of the extensions that define $\mathrm{Span}(D)$, we recall a formulation of the Serre duality. Given a divisor Q on X, the space $H^0(K_{-Q})$ of global meromorphic differentials whose poles are bounded below by Q is canonically isomorphic to the dual of the space $H^1(\mathcal{O}(Q))$ of equivalence classes of 1-cocyles whose poles are bounded below by $-Q$. This canonical isomorphism is defined via

$$\langle .,. \rangle : H^0(K_{-Q}) \times H^1(\mathcal{O}(Q)) \longrightarrow H^1(K) \overset{\mathrm{Res}}{\cong} \mathbb{C},$$
$$(\omega, [\epsilon_{\alpha\beta}]) \mapsto [\omega\epsilon_{\alpha\beta}]. \tag{A.5}$$

By construction, for $\omega \in H^0(K_{-Q})$ and $\epsilon_{\alpha\beta} \in H^1(\mathcal{O}(Q))$, the product $\omega\epsilon_{\alpha\beta}$ is regular and is a 1-cocyle of the sheaf K of holomorphic differentials. The isomorphism $H^1(K) \overset{\mathrm{Res}}{\cong} \mathbb{C}$ can be described in terms of the Mittag-Leffler distributions of the 1-cocyles of K. Recall that a Mittag-Leffler distribution of a 1-cocyle $\omega_{\alpha\beta}$ of K is a collection (ω_α) of local meromorphic differentials on each U_α, such that $\omega_{\alpha\beta} = \omega_\alpha - \omega_\beta$: by construction $\underset{p}{\mathrm{Res}}(\omega_\alpha) = \underset{p}{\mathrm{Res}}(\omega_\beta)$ for all $p \in U_\alpha \cap U_\beta$. If (ω'_α) is another Mittag-Leffler distribution of $\omega_{\alpha\beta}$, then $(\omega_\alpha - \omega'_\alpha)$ defines a global meromorphic differential on X, and hence $\sum_{p\in X} \underset{p}{\mathrm{Res}}(\omega_\alpha - \omega'_\alpha) = 0$. It follows that the assignment $[\omega_{\alpha\beta}] \mapsto \mathrm{Res}([\omega_{\alpha\beta}]) := \sum_{p\in X} \underset{p}{\mathrm{Res}}(\omega_\alpha)$ defines a well-defined morphism, and it can be shown to be an isomorphism by dimension count and checking that the image is nonzero.

To make use of this formulation of the Serre duality in terms of meromorphic differentials and functions, it is at first convenient to characterize $L^2\Lambda^{-1}$ as a divisor Q, and match the holomorphic sections of $KL^{-2}\Lambda$ and 1-cocyles of $L^2\Lambda^{-1}$ with the corresponding meromorphic objects in K_{-Q} and $\mathcal{O}(Q)$ respectively. To be even more concrete, let us characterize L and Λ respectively in terms of some divisors $D_L = \sum l_i p_{L,i}$ and $D_\Lambda = \sum \lambda_i p_{\Lambda,i}$, which we can assume to be distinct. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{I}}$ is a covering of X with unique indices $\alpha(L,i)$ and $\alpha(\Lambda,i) \in \mathcal{I}$ satisfying $p_{L,i} \in U_{\alpha(L,i)}$ and $p_{\Lambda,i} \in U_{\alpha(\Lambda,i)}$. We can obtain a new covering by refining each $U_{\alpha(L,i)}$ into $U'_{(L,i)} \cup D_{L,i}$, where $U'_{(L,i)} = U_{\alpha(L,i)} \setminus \{p_{L,i}\}$ and $D_{L,i}$ is a small neighborhood with local coordinate $z_{L,i}$ centered at $p_{L,i}$ not intersecting any other elements of $\mathcal{U}$, and similarly refining $U_{\alpha(\Lambda,i)}$ into $U'_{(\Lambda,i)} \cup D_{\Lambda,i}$. The line bundle L is defined over $\mathcal{U}'$ with trivial transition functions everywhere except $(L)_{U'_{(L,i)}D_{L,i}} := z_{L,i}^{-l_i}$. The transition functions of Λ and $L^2\Lambda^{-1}$, which can be characterized by $Q := 2D_L - D_\Lambda = \sum_i 2l_i p_{L,i} - \sum_j \lambda_j p_{\Lambda,j}$, are defined similarly over $\mathcal{U}'$. With the line bundles defined this way, there is a canonical isomorphism from $L^2\Lambda^{-1}$ to $\mathcal{O}(Q)$, defined by multiplication by $z_{L,i}^{-2l_i}$ on $D_{L,i}$, by $z_{\Lambda,i}^{\lambda_i}$ on $D_{\Lambda,i}$, and by 1 elsewhere. Similarly, the sheaf of holomorphic sections of $KL^{-2}\Lambda$ is canonically isomorphic to the sheaf K_{-Q} of meromorphic differentials with poles bounded below by Q, via multiplication by $z_{L,i}^{2l_i}$ on $D_{L,i}$, by $z_{\Lambda,i}^{-\lambda_i}$ on $D_{\Lambda,i}$, and by 1 elsewhere.

We now give an explicit construction of a representative of Span$(p) \in \mathbb{P}$, which is unique up to scaling. This representative extension is a modification around p of the split extension $L \oplus L^{-1}\Lambda$: we add a upper triangular transition functions near p while keeping elsewhere the diagonal transition functions with diagonal elements being transition functions of L and $L^{-1}\Lambda$. W.l.o.g. suppose p is contained in a unique element $U_{\alpha_p} \equiv U_p$ of the covering $\mathcal{U}'$ defined above. Refining U_p into $U_p' = U_p \setminus \{p\}$ and a small neighborhood D_p with local coordinate z_p centered at p, we define the extension $E(\epsilon_p)$ of $L^{-1}\Lambda$ by L via the transition function

$$\left(E\left(\epsilon_p\right)\right)_{D_p U_p'} = \begin{pmatrix} 1 & \frac{\epsilon_p}{z_p} \\ 0 & 1 \end{pmatrix}, \tag{A.6}$$

while defining its transition functions elsewhere to be the same as those of $L \oplus L^{-1}\Lambda$. This extension is characterized by the 1-cocyle of $L^2\Lambda^{-1}$ that takes the value $z_p^{-1}\epsilon_p$ on $D_p \cap U_p'$ and is zero elsewhere. The corresponding 1-cocyle $\vec{\epsilon_p}'$ in $\mathcal{O}(Q)$, which is also zero everywhere except on $D_p \cap U_p'$, would take the same value if $p \notin Q$ since then w.l.o.g. we can assume U_p contains no point in Q and hence the isomorphism between the two sheaves on U_p is just multiplication by 1. If $p = p_{L,i}$ then U_p coincides with $D_{L,i}$, in which cases $\left(\vec{\epsilon_p}'\right)_{D_p U_p'} = z_p^{-1-2l_i}\epsilon_p$. Similarly, if $p = p_{\Lambda,i}$ then $\left(\vec{\epsilon_p}'\right)_{D_p U_p'} = z_p^{-1+\lambda_i}\epsilon_p$.

PROPOSITION A.1. *For any $p \in X$ and any $\epsilon_p \neq 0$, the extension $E\left(\epsilon_p\right)$ represents Span(p).*

Proof. It suffices to show that the Serre duality pairing of $E\left(\epsilon_p\right)$ with any section $s \in H^0\left(KL^{-2}\Lambda\right)$ is $\langle E\left(\epsilon_p\right), s\rangle = \epsilon_p s(p)$. We do this by evaluating the Serre duality pairing of the corresponding objects in $H^1\left(\mathcal{O}(Q)\right)$ and $H^0\left(K_{-Q}\right)$, namely $\left[\vec{\epsilon_p}'\right]$ and the meromorphic differential $\omega_s \in H^0\left(K_{-Q}\right)$ corresponding to s. For $p \notin Q$ this follows from the fact that the isomorphisms $H^1\left(L^2\Lambda^{-1}\right) \xrightarrow{\sim} H^1\left(\mathcal{O}(Q)\right)$ and $H^0\left(KL^{-2}\Lambda\right) \xrightarrow{\sim} H^0\left(K_{-Q}\right)$ are locally just the identities. For p coinciding with $p_{L,i}$ or $p_{\Lambda,i}$, this follows from how the isomorphisms cancel each other. $\square$

Let $E\left(p\right) := E\left(\epsilon_p = 1\right)$. By definition an element of Span(D) for $D = p_1 + ... + p_d$ can be represented by a linear combination of $E\left(p_1\right)$, ..., $E\left(p_d\right)$, which we shall write as $\sum_{i=1}^{d} \epsilon_i E\left(p_i\right)$. In terms of transition functions, such an extension can be obtained by repeating the procedure of constructing $E\left(p\right)$ at each point in D (these procedures are commutative).

REMARK A.1. Suppose $D = p_1 + ... + p_d$ is a divisor of degree $d \leq n+1$ such that Span(D) is of dimension $d-1$, i.e. given a basis s_0', ..., s_n' of $H^0\left(KL^{-2}\Lambda\right)$ the square matrix $\left(s_i'\left(p_j\right)\right)$ is of maximal rank d.Then there exists a different basis s_1, ..., s_n of $H^0\left(KL^{-2}\Lambda\right)$ satisfying $s_i(p_j) = \delta_{ij}$. The dual basis in $H^1\left(L^2\Lambda^{-1}\right)$ coincides with the basis provided by $E(p_1)$, ..., $E(p_d)$. In particular if $d = n+1$ then Span$(D) = \mathbb{P}$, and $E(p_1)$, ..., $E(p_{n+1}$ provide a basis for $H^1\left(L^2\Lambda^{-1}\right)$.

Secant varieties and Segre stratification. From the above discussion one can see that the higher degree of D, the more modifications we make to the split extension $L \oplus L^{-1}\Lambda$ to obtain an element in Span(D). In this process we obstruct an embedding of $L^{-1}\Lambda$ into the rank-2 bundle, and in a sense we go "further away" from the split extension which is the unique one that admits both L and $L^{-1}\Lambda$ as subbundles. The following proposition, which is an adaptation of the results in [42] [43], makes precise this statement.

With $\deg\left(\Lambda L^{-2}\right) \geq 2$, let E be a bundle arising as an extension of the form (A.1), $[E] \in \mathbb{P}^n$ the equivalence class of the extension up to scaling, and D an effective divisor. Let $L' := L^{-1}\Lambda(-D)$ and denote by s_D the canonical injection $L' \to L^{-1}\Lambda$ which introduces zeroes at D to sections of L'.

PROPOSITION A.2. *Suppose D is of degree $d \leq n = g-2+\deg\left(\Lambda L^{-2}\right)$. Then $[E] \in$ Span(D) if and only if there exists an injection (which is not necessarily an embedding) $L' \to E$ such that the composition $L' \to E \twoheadrightarrow L^{-1}\Lambda$ vanishes at D, i.e. it is s_D up to scaling.*

Proof. Abusing the notation we also denote the induced injection $H^0\left(KL^{-2}\Lambda(-D)\right) \to H^0\left(KL^{-2}\Lambda\right)$ as s_D, and by s_D^* we mean the dual map $H^1\left(L^2\Lambda^{-1}\right) \to H^1\left(L^2\Lambda^{-1}(D)\right)$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}\left(L', E\right) & \xrightarrow{j} & \mathrm{Hom}\left(L', L^{-1}\Lambda\right) & \xrightarrow{\delta} & H^1\left(L^2\Lambda^{-1}(D)\right) \\ & & {\scriptstyle \circ s_D}\uparrow & & {\scriptstyle s_D^*}\uparrow \\ & & \mathrm{Hom}\left(L^{-1}\Lambda, L^{-1}\Lambda\right) & \xrightarrow{\gamma} & H^1\left(L^2\Lambda^{-1}\right), \end{array} \tag{A.7}$$

where the upper and lower horizontal exact rows are induced by respectively applying the functor $\mathrm{Hom}\left(L', \_\right)$ and $\mathrm{Hom}\left(L^{-1}\Lambda, \_\right)$ to (A.1). Note that $[E] = \gamma(1)$, and $\circ s_D(1) = s_D$. By commutativity and exactness, observe that $s_D^*\left([E]\right) = 0$ if and only if $s_D \in \mathrm{im}\,(j)$, i.e. s_D is equal to a composition of the form $L' \to E \twoheadrightarrow L^{-1}\Lambda$.

Suppose $[E] \in \mathrm{Span}(D)$. Then by definition its representatives evaluate to zero all sections of $KL^{-2}\Lambda$ that vanish at D. Since the image of s_D contains only such sections, $s_D^*\left([E]\right) = 0$. It follows that there exists some injection $L' \to E$ such that s_D is the composition $L' \to E \twoheadrightarrow L^{-1}\Lambda$.

Suppose $L' \to E$ is an injection such that s_D is the composition $L' \to E \twoheadrightarrow L^{-1}\Lambda$. It follows that $0 = \left\langle s_D^*\left([E]\right), s'\right\rangle = \left\langle [E], s_D(s')\right\rangle$ for any section $s' \in H^0\left(KL^{-2}\Lambda(-D)\right)$. A priori $\mathrm{im}\,(s_D)$ is contained in the space V_D of all sections of $KL^{-2}\Lambda$ that vanish at D. But since $\deg\,(D) \leq n$, the degrees of freedom to move the zeroes of sections of $KL^{-2}\Lambda$, we can construct an inverse of s_D from V_D to $H^0\left(KL^{-2}\Lambda(-D)\right)$, i.e. $V_D = \mathrm{im}\,(s_D)$. It follows that $[E]$ evaluates all of V_D to zero, i.e. $[E] \in \mathrm{Span}(D)$. □

COROLLARY A.3. *Suppose* $\deg\,(D) \leq \deg\left(\Lambda L^{-2}\right)$ *and if equality occurs then* $\mathcal{O}(D) \ncong \Lambda L^{-2}$. *Then* $[E] \in$ *Span*(D') *for some effective divisor* D' *belonging to the linear equivalence class* $[D]$ *if and only if there exists a nonzero injection* $L' \to E$.

Proof. It suffices to show that with D being such an effective divisor then any nonzero injection $L' \to E$ does not factor through $L \hookrightarrow E$, i.e. the composition $L' \to E \twoheadrightarrow L^{-1}\Lambda \in H^0\left(\mathcal{O}(D)\right)$ is nonzero. If $\deg\,(D) < \deg\left(\Lambda L^{-2}\right)$ then $\deg\,(L') > \deg\,(L)$, so this follows. In the other case, since $L' \ncong L$, this also follows. □

REMARK A.2. For $\deg\,(D) \leq \deg\left(\Lambda L^{-2}\right)$, the open dense subset of $\mathrm{Span}(D)$ defined as the complement of $\bigcup_{D' < D} \mathrm{Span}(D)'$ consists precisely of the extensions corresponding to rank-2 bundles that admit both L and L' as subbundles.

Appendix B

Higgs bundles in terms of extension classes

The lower-left components of the Higgs fields. Given a Higgs field ϕ on a rank-2 bundle E realized as an extension of the form (A.1), consider the composition

$$c_L(\phi) : L \hookrightarrow E \xrightarrow{\phi} EK \twoheadrightarrow L^{-1}\Lambda K, \tag{B.1}$$

which is a section of the line bundle $KL^{-2}\Lambda$. Concretely, this section can be realized as follows. Suppose over each open set U_α and with respect to certain local frames of $E|_{U_\alpha}$ that are adapted to the embedding of L, the Higgs field takes the form $\phi_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \otimes 1_{K,\alpha}$. These local expressions of the Higgs field transforms as

$$\begin{aligned} \phi_\alpha &= (E)_{\alpha\beta}\phi_\beta(E)^{-1}_{\alpha\beta} \\ &= \begin{pmatrix} a_\beta + \epsilon_{\alpha\beta}c_\beta & l^2_{\alpha\beta}\lambda^{-1}_{\alpha\beta}\left(b_\beta - \epsilon_{\alpha\beta}(a_\beta - d_\beta) - \epsilon^2_{\alpha\beta}c_\beta\right) \\ l^{-2}_{\alpha\beta}\lambda_{\alpha\beta}c_\beta & d_\beta - \epsilon_{\alpha\beta}c_\beta \end{pmatrix} \otimes \left(k_{\alpha\beta}1_{K,\alpha}\right), \end{aligned} \tag{B.2}$$

where $1_{K,\alpha}$ and $k_{\alpha\beta}$ are the local generators and transition functions of K. The local functions $\{c_\alpha\}_{\alpha\in\mathcal{I}}$ in particular glue into a section $c_L(\phi) \in H^0(KL^{-2}\Lambda)$. In particular, if $c_L(\phi)$ vanishes at $x \in X$, the subspace $L\,|_x \subset E\,|_x$ is an eigen-space of ϕ.

Note that, given a subbundle L of E, we have defined a map $c_L : \mathrm{End}(E) \otimes K \to KL^{-2}\Lambda$ in (B.1). The kernel of c_L consists of Higgs fields that preserve L, i.e. $\{\phi \in \mathrm{End}(E) \otimes K \mid \phi(L) \subset LK\}$. These Higgs fields are of upper-triangular form in local frames adapted to L. We can assign a morphism $E \to LK$ to such a Higgs field as follows. Over each open set U_α and in local frames adapted to $L \hookrightarrow E$, an upper-triangular Higgs field of the form $\begin{pmatrix} a_\alpha & b_\alpha \\ 0 & d_\alpha \end{pmatrix}$ can be regarded as a local morphism $E\,|_{U_\alpha} \to (LK)\,|_{U_\alpha}$ defined by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\alpha \mapsto a_\alpha - d_\alpha$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_\alpha \mapsto b_\alpha$. If we transit from local frames of $E\mid_{U_\alpha}$ to $E\mid_{U_\beta}$, both adapted to the embedding $L\hookrightarrow E$, using the transition function (A.2), the local sections $\begin{pmatrix}1\\0\end{pmatrix}_\alpha$ and $\begin{pmatrix}0\\1\end{pmatrix}_\alpha$ now take the from $\begin{pmatrix}l_{\beta\alpha}\\0\end{pmatrix}_\beta$ and $\begin{pmatrix}l_{\beta\alpha}\epsilon_{\beta\alpha}\\l_{\beta\alpha}^{-1}\lambda_{\beta\alpha}\end{pmatrix}_\beta$ respectively. It follows from (B.2) that the local morphism $E\mid_{U_\beta}\to(LK)\mid_{U_\beta}$ sends

$$\begin{aligned}\begin{pmatrix}l_{\beta\alpha}\\0\end{pmatrix}_\beta &\mapsto l_{\beta\alpha}(a_\beta-d_\beta)=l_{\beta\alpha}k_{\beta\alpha}(a_\alpha-d_\alpha),\\ \begin{pmatrix}l_{\beta\alpha}\epsilon_{\beta\alpha}\\l_{\beta\alpha}^{-1}\lambda_{\beta\alpha}\end{pmatrix}_\beta &\mapsto l_{\beta\alpha}\epsilon_{\beta\alpha}(a_\beta-d_\beta)+l_{\beta\alpha}^{-1}\lambda_{\beta\alpha}b_\beta=l_{\beta\alpha}k_{\beta\alpha}b_\alpha.\end{aligned}\tag{B.3}$$

Hence these local morphisms glue into a morphism $E\to LK$.

If we restrict to trace-zero Higgs fields, then this assignment furthermore is clearly unique. In other words, the subbundle of $\mathrm{End}_0(E)\otimes K$ consisting of trace-zero Higgs fields preserving L is isomorphic to $\mathrm{Hom}\,(E,LK)\cong E^*LK$, and so we have the s.e.s.

$$0\to E^*LK\to \mathrm{End}_0(E)\otimes K\xrightarrow{c_L} KL^{-2}\Lambda\to 0.\tag{B.4}$$

together with its induced l.e.s.

$$0\to H^0\left(E^*LK\right)\to H^0\left(\mathrm{End}_0(E)\otimes K\right)\xrightarrow{c_L} H^0\left(KL^{-2}\Lambda\right)\to H^1\left(E^*LK\right)\to ...\tag{B.5}$$

The image of c_L consists of all lower-left components (B.1) picked out from all trace-zero Higgs fields on E using local frames adapted to $L\hookrightarrow E$. We can compute its dimension by computing the dimension of $\ker\left(c_L\right)\cong H^0\left(E^*LK\right)$. The Riemann-Roch theorem and Serre duality give

$$h^0\left(E^*LK\right)=2g-2+h^0\left(L^{-1}E\right)-\deg\left(L^{-2}\Lambda\right).\tag{B.6}$$

If E is stable and hence $h^0(\mathrm{End}_0(E)\otimes K)=3g-3$, we have

$$\dim\mathrm{im}\left(c_L\right)=g-1-h^0\left(L^{-1}E\right)+\deg\left(L^{-2}\Lambda\right).\tag{B.7}$$

In case $\deg\left(L^{-2}\Lambda\right) > 0$, we have $h^0\left(KL^{-2}\Lambda\right) = g-1+\deg\left(L^{-2}\Lambda\right)$, and hence im (c_L) is a subspace of codimension $h^0\left(L^{-1}E\right)$ in $H^0\left(KL^{-2}\Lambda\right)$. If E is not "overcounted" as an extension of the form (A.1), i.e. it admits a unique embedding from L up to scaling, then im (c_L) is a hyperplane in $H^0\left(KL^{-2}\Lambda\right)$. As shown in the discussion that follows, im (c_L) is always contained in the hyperplane defined as the kernel of the extension representing E. Hence when E is not "overcounted", a section of $KL^{-2}\Lambda$ forms the lower-left component of a Higgs field on E if and only if it lies in the kernel of the extension representing E.

Serre duality constraint. We claim that if $L \hookrightarrow E$ is a subbundle of E and $\phi \in H^0\left(\mathrm{End}_0(E)\otimes K\right)$ a Higgs field on it, the section $c_L(\phi) \in H^0\left(KL^{-2}\Lambda\right)$ defined by (B.1) satisfies

$$\langle c_L(\phi), [E]\rangle = 0, \tag{B.8}$$

where $[E] \in H^1\left(L^2\Lambda^{-2}\right)$ is the equivalence class of an extension realizing E of the form (A.1), and the pairing is via Serre duality. Equivalently, the image of c_L is contained in $\ker([E]) = \{s \in H^0(KL^{-2}\Lambda) \mid \langle s, [E]\rangle = 0\}$.

To see this, observe that by choosing N sufficiently high, we can choose $p_1, ..., p_N \in X$ such that $c_L(\phi)(p_i) \neq 0$ for all $i = 1, ..., N$ and $[E]$ corresponds to a point in Span$(p_1 + ... + p_N) \subset \mathbb{P}H^1\left(L^2\Lambda^{-2}\right)$. Then we can define E in terms of its transition functions w.r.t. local frames adapted to L that are of the form (A.6) for p_i, ..., p_N and are diagonal otherwise. The regularity at each p_i, $i = 1, ..., N$ of the diagonal components of the local form (B.2) of ϕ implies that $-\epsilon_{p_i} c_L(\phi)(z_{p_i}(p_i))$ is the residue at p_i of a meromorphic differential that has a simple pole at each p_i and is holomorphic elsewhere. The sum of residues of such a differential must vanish. It then remains to observe that $\langle c_L(\phi), [E]\rangle = \sum_{i=1}^N \epsilon_{p_i} c(z_{p_i}(p_i))$.

Two special cases. There are two situations in which the image of c_L can be described more explicitly. One situation is when c_L is injective, i.e. a Higgs field on E can be uniquely represented by its lower-left component defined via (B.1). It follows from (B.6) that c_L is injective when E is not "overcounted", i.e. $h^0\left(L^{-1}E\right) = 1$, and the degree of L is such that $\deg\left(L^{-2}E\right) = 2g-1$. In this case, the hyperplane $\ker([E])$, which is of dimension $3g-3$, is in 1-1 correspondence with trace-less Higgs fields on E.

Another situation is when $h^0\left(L^{-2}\Lambda\right)=0$. For example, this is generically true when E is not a maximally stable bundle with L being its maximal subbundle, since then $\deg\left(L^{-2}\Lambda\right)<g-1$. In this case, there is an alternative way to (B.6) to compute $h^0\left(E^*LK\right)$ by considering the s.e.s.

$$0\to KL^2\Lambda^{-1}\to E^*LK\to K\to 0. \tag{B.9}$$

Here $KL^2\Lambda^{-1}$ is the bundle of nilpotent Higgs fields admitting L as the kernel, i.e. strictly upper-triangular Higgs fields in the local frames adapted to $L\hookrightarrow E$, while the quotient bundle K represents the diagonal elements of upper-triangular Higgs fields, according to the transformation rules (B.2). It follows from the l.e.s.

$$0\to H^0\left(KL^2\Lambda^{-1}\right)\to H^0\left(E^*LK\right)\to H^0\left(K\right)\to H^1\left(KL^2\Lambda^{-1}\right)\to ... \tag{B.10}$$

and Serre duality $H^1\left(KL^2\Lambda^{-1}\right)\cong H^0\left(L^{-2}\Lambda\right)^*$, which is zero by our assumption, that

$$h^0\left(E^*LK\right)=h^0(K)+h^0\left(KL^2\Lambda^{-1}\right)=g+h^0\left(KL^2\Lambda^{-1}\right). \tag{B.11}$$

Hence

$$\dim\operatorname{im}(c_L)=h^0\left(\mathrm{End}_0(E)\otimes K\right)-h^0\left(E^*LK\right)=2g-3-h^0\left(KL^2\Lambda^{-1}\right). \tag{B.12}$$

REMARK B.1. By comparing (B.6) and (B.11), we see that if $h^0\left(L^{-2}\Lambda\right)=0$, then $h^0\left(L^{-1}E\right)=1$ (no "overcount").

The upper-right and diagonal components when $\deg\left(L^{-2}\Lambda\right)\leq g-1$. First, consider the case where $\deg\left(L^{-2}\Lambda\right)\leq g-1$. Then $L^2\Lambda^{-1}$ is isomorphic to $\mathcal{O}_X(Q)$ with $Q=Q_0-\sum_{i=1}^{g}q_i$, where Q_0 is an effective divisor $Q_0=q_{0,1}+...+q_{0,m}$ for $m\geq 1$. The Riemann-Roch theorem

$$h^0\left(K_Q\right)-h^0\left(-Q\right)=m-1, \qquad m\geq 1. \tag{B.13}$$

implies that $h^0\left(-Q\right)=h^0\left(L^{-2}\Lambda\right)>0$ if and only if there exists a holomorphic differential vanishing at q_1, ..., q_g. Indeed, recall that $H^0\left(K_Q\right)$ is the space of meromorphic differentials that have zeroes at $q_1, ..., q_g$ and might have simple poles at $q_{0,1}, ..., q_{0,m}$. Since the sum of the residues at $q_{0,1}, ..., q_{0,m}$ of such differentials vanishes, the total degrees of freedom to adjust these residues is at most $m-1$. Hence $h^0\left(K_Q\right) > m-1$ if and only if there exists at least two such differentials that have the same residue at each $q_{0,j}$, $j = 1, ..., m$. Their difference is a holomorphic differential that vanishes at $q_1, ..., q_g$. Note that the existence of such a holomorphic differential is equivalent to the fact that the matrix $\omega_i(q_j)$, where $\omega_1, ..., \omega_g$ is a basis of $H^0\left(K\right)$, is degenerate. Hence $h^0\left(L^{-2}\Lambda\right) > 0$ if and only if $\omega_i(q_j)$ is degenerate.

Suppose the extension realizing E represents an element in $\mathrm{Span}(D) \subset \mathbb{P}\mathrm{Ext}\left(L^{-1}\Lambda, L\right)$ for some effective divisor $D = p_1 + ... + p_{\deg(D)}$, i.e. according to (A.6) it differs from the split bundle $L \oplus L^{-1}\Lambda$ by transition functions of the form $\begin{pmatrix} 1 & \epsilon_i/z_i \\ 0 & 1 \end{pmatrix}$ around p_i. It follows from (B.2) together with the explicit transition function (A.6) that, in local frames adapted to L, the diagonal and upper-right components of a Higgs field preserving L can be represented respectively by

(a) a holomorphic differential $A \in H^0\left(K\right)$,

(b) a meromorphic differential $B \in H^0\left(K_{D+Q}\right)$ (i.e. B vanishes at $q_1, ..., q_g$ and is allowed to have simple poles at each point p_i of D and $q_{0,j}$ of Q_0), the residue of which at each p_i is $\underset{p_i}{\mathrm{Res}}\left(B\right) = 2\underset{p_i}{\mathrm{Res}}\left(A\epsilon_i/z_i\right) = 2A(p_i)\epsilon_i$.

Given a fixed $A \in H^0\left(K\right)$, there exists some meromorphic differential $B_0 \in H^0\left(K_{D+Q_0}\right)$, which is allowed to have simple pole at each point of D and Q_0, with the residue at p_i being $2A(p_i)\epsilon_i$. Such a meromorphic differential B_0 exists if and only if the sum of its residues vanishes, i.e. the sum of residues at $q_{0,1}, ..., q_{0,m}$ is equal to $-2\sum_{i=1}^{\deg(D)} A(p_i)\epsilon_i$. If ω is a holomorphic differential satisfying $\omega(q_i) + B_0(q_i) = 0$ for all $i = 1, ..., g$, then $B := B_0 + \omega$ is a meromorphic differential satisfying condition (2) above. Such a holomorphic differential ω is a solution to a non-homogeneous linear system associated to $\omega_i(q_j)$, and exists if and only if $\omega_i(q_j)$ is non-degenerate. Hence if and only if $h^0\left(L^{-2}\Lambda\right) = 0$, given any $A \in H^0\left(K\right)$ there exists some Higgs field that is L-invariant with the diagonal elements represented by A.

When $h^0\left(L^{-2}\Lambda\right) = 0$ and given a fixed holomorphic differential A, the space of meromorphic differentials B satisfying condition (2) above is isomorphic to $H^0\left(KL^2\Lambda^{-1}\right)$. Indeed, let B be such a meromorphic differential. We can keep A fixed and deform the residues $\underset{q_{0,i}\in Q_0}{\mathrm{Res}} B$ by adding a meromorphic differential that vanishes at q_1, ..., q_g and has simple pole only at $q_{0,1}$, ..., $q_{0,m}$, i.e. by adding an element of $H^0\left(K_Q\right) \cong H^0\left(KL^2\Lambda^{-1}\right)$. Hence the first term in (B.11) corresponds to the freedom to choose the diagonal component, while the second term corresponds to the freedom to choose the upper-right component of the Higgs fields once the diagonal components have been fixed. In particular, if $h^0\left(KL^2\Lambda^{-1}\right) = 0$ (which is the generic case when $\deg\left(L^{-2}\Lambda\right) = g-1$), the morphism $H^0\left(E^*LK\right) \to H^0\left(K\right)$ in (B.10) is an isomorphism.

When $h^0\left(L^{-2}\Lambda\right) = h^0\left(-Q\right) > 0$, not all holomorphic differential make up the diagonal components of an L-invariant Higgs field. In this case, there exists some effective divisor $q_1' + q_2' + ... + q_{g-m}'$, $m \geq 1$, that is linearly equivalent to $-Q$. Given a fixed $A \in H^0\left(K\right)$, the requirement of the vanishing sum of residues of $B \in H^0\left(K_{D+Q}\right)$ now translates to a constraint on A: $\sum_{i=1}^{\deg(D)} A(p_i)\epsilon_i = 0$. This reflects the fact that the morphism $H^0\left(E^*LK\right) \to H^0\left(K\right)$ in (B.10) when $h^0\left(L^{-2}\Lambda\right) > 0$ is in general not surjective, and can even be zero. Once such a meromorphic differential A satisfying $\sum_{i=1}^{\deg(D)} A(p_i)\epsilon_i = 0$ exists, a meromorphic differential B satisfying condition (2) above is guaranteed to exist by a similar construction in the case $h^0\left(L^{-2}\Lambda\right) = 0$.

Constraints on upper-right and diagonal components when $\deg\left(L^{-2}\Lambda\right) \geq g$. As the degree of $L^{-2}\Lambda$ increases there will be more constraints on the diagonal components. Consider the case where $L^{-2}\Lambda \cong \mathcal{O}_X(-Q)$ where $-Q = q_1 + ... + q_{g+m}$. Generically Q satisfies $h^0\left(-Q\right) = m+1$, or equivalently $h^0\left(K_Q\right) = 0$. Let E be realized by an extension representing an element in Span$(D) \subset \mathbb{P}$ for some $D = p_1 + ... + p_{\deg(D)}$. Note that for a generic bundle E realized as an extension of $L^{-1}\Lambda$ by L, we need $\deg(D) = h^0\left(K_{-Q}\right) = 2g-1+m$. In the following we suppose E is such a generic bundle and we set $\deg(D) = 2g-1+m$.

The diagonal and upper-right components of an upper-triangular Higgs field $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ are respectively represented by a holomorphic differential A and a meromorphic differential $B \in H^0\left(K_{D+Q}\right)$, i.e. with zeroes at q_1, ..., q_{g+m} and simple poles at p_1, ..., p_{2g-1+m} satisfying

$$\operatorname*{Res}_{p_i}(B) - 2\epsilon_i A(p_i) = 0, \qquad i = 1, ..., 2g-1+m. \tag{B.14}$$

When Q is in generic position, $h^0\left(K_{D+Q}\right) = 2g-2$. This implies that the residues at p_i, $i =$

$1, ..., 2g-1+m$ of an element of $H^0\left(K_{D+Q}\right)$ determine it, and furthermore only $2g-2$ of them are free variables. Let $r_i(B) \coloneqq \operatorname*{Res}_{p_i}(B)$ for $1 \leq i \leq 2g-2$ be the coordinates in $H^0\left(K_{D+Q}\right)$. Then the rest of the residues $\operatorname*{Res}_{2g-1 \leq j \leq 2g-1+m}(B)$ are linear functions in $r_1(B)$, ..., $r_{2g-2}(B)$. Equations for $1 \leq i \leq 2g-2$ then allows us to write $\operatorname*{Res}_{2g-1 \leq j \leq 2g-1+m}(B)$ as linear combinations of $A(p_i)$. Plugging them in for $2g-1 \leq i \leq 2g-1+m$, we get a homogeneous linear system of $m+1$ equations for $A \in H^0(K) \cong \mathbb{C}^g$. If $m+1 < g$ and if the linear system is of maximal rank, there exists a $(g-1-m)$-dimensional family of solutions for A, and given each such A, there exists a unique upper-triangular Higgs field with A representing the diagonal components since (B) determines the residues of B and hence B itself. Comparing with (B.6), we see that the (generic) assumption that the linear system is of maximal rank is equivalent to $h^0\left(L^{-1}E\right) = 1$ (no “overcount”).

The cases $m = g-1$ and $m = 0$ are the two extreme cases. In the former case, as followed from (B.6), there is generically no L-invariant Higgs field (and hence the lower-left components are in 1-1 correspondence with Higgs fields). In the latter case, the only constraint on A comes from the vanishing residue condition of B.

Appendix C

Baker-Akhiezer divisors for $GL_2(\mathbb{C})$-Higgs bundles

Consider the moduli space $\mathcal{M}_H(GL_2(\mathbb{C}))$ of rank-2 Higgs bundles where the determinant line bundle of the underlying bundles is of odd degree. Via tensoring with a line bundle, this situation is equivalent to one where the determinant line bundle $\det(E)$ of each underlying bundle E is of degree $2g-1$. By Riemann-Roch, $h^0(E) \geq \deg(\det(E)) - 2(g-1) = 1$, and so any such E admits a morphism $\mathcal{O} \to E$.

Denote by $\mathcal{M}_H^s \subset \mathcal{M}_H(GL_2(\mathbb{C}))$ the loci of Higgs bundles with non-degenerate spectral curves. The notion of Baker-Akhiezer divisors we have defined (cf. definition 3.1) for trace-less Higgs fields generalize to general rank-2 Higgs fields. The eigen-line bundle Φ of any Higgs bundle $(E, \phi) \in \mathcal{M}_H^s$ is isomorphic to $\pi_\phi^*(K^{-1})(D)$, where $S_\phi \xrightarrow{\pi_\phi} X$ is the corresponding spectral curve and D is the Baker-Akhiezer divisor associated to the data $\left(\mathcal{O} \xrightarrow{i} E, \phi\right)$. Note that $\deg(D) = \deg(K \det(E)) = 4g-3$, the genus of the spectral curves. The advantage of working with Baker-Akhiezer divisors of this degree is that generically we will be able to determine the spectral curve once we know the divisor, and hence knowing a point D in the symmetric product $(T^*X)^{[4g-3]}$ completely determines the data $(\mathcal{O} \to E, \phi)$. Within this appendix, by Baker-Akhiezer divisors we will mean those of this type.

Undercount. If a bundle E with $\det(E) \cong \Lambda$ of degree $2g-1$ has a nowhere-vanishing section $i \in H^0(E)$ then it arises, up to scaling of i, as an extension of the form

$$0 \to \mathcal{O} \xrightarrow{i} E \to \Lambda \to 0. \tag{C.1}$$

Extensions of this form are elements of $\mathrm{Ext}(\det(E), \mathcal{O}) \cong H^1\left(\det(E)^{-1}\right) \cong \mathbb{C}^{3g-2}$. Not all of the bundles that make up Higgs bundles in $\mathcal{M}_H^s$ has a nowhere-vanishing section so that it could fit in (C.1) though. These bundles are "undercounted" if we want to use (C.1) to model the moduli of the bundles that make up Higgs bundles in $\mathcal{M}_H^s$.

As an example, a generic Higgs bundle (E, ϕ) with E unstable has no embedding from $\mathcal{O}$ and so is "undercounted" in this sense. Indeed, let $M \hookrightarrow E$ be the unique destabilizing subbundle of degree $\deg(M) \geq \frac{\deg(E)}{2} = g - \frac{1}{2}$, and suppose there exists a nowhere-vanishing morphism $\mathcal{O} \xrightarrow{i} E$. Then i cannot factor through $M \hookrightarrow E$ as $\deg(M) > 0$. Hence the composition $\mathcal{O} \to E \to M^{-1}\Lambda$ is nonzero. But $\deg\left(M^{-1}\Lambda\right) \leq g - \frac{1}{2}$, and so the condition $h^0(M^{-1}\Lambda) > 0$ can only be satisfied non-generically.

Nevertheless, the underlying bundle E of a generic point $[E, \phi] \in \mathcal{M}_H^s$ fits in (C.1). Indeed, consider a generic, non-degenerate spectral curve S. It follows from theorem 1 in [6] that a generic bundle E can be recovered as the direct image of a line bundle $\Phi(\mathcal{R})$ on S , or equivalently it has a Higgs field ϕ with eigen-line bundle Φ on S. Now, if such E has no nowhere-vanishing section, then the Baker-Akhiezer divisor on S associated to $(\mathcal{O} \xrightarrow{s'} E, \phi)$ for any nonzero $s' \in H^0(E)$ will contain the pull-back from the zero divisor on X of s'. In other words, this Baker-Akhiezer divisor lies in the set

$$\left\{D \in S^{[4g-3]} \mid \mathcal{O}_X\left(\pi(D)\right) \cong K\Lambda,\ D \text{ has some pull-back of a divisor from } X\right\}. \tag{C.2}$$

Since this is a positive codimension subset of $\left\{D \in S^{[4g-3]} \mid \mathcal{O}_X\left(\pi(D)\right) \cong K\Lambda\right\}$, a generic point $[E, \phi] \in \mathcal{M}_H^s$ will not produce a Baker-Akhiezer divisor lying in this set and so will not be "undercounted".

We note that even when E cannot be represented as an element of $H^1\left(\det(E)^{-1}\right)$, we can still mark their occurrences by including the set (C.2) in our consideration. For example, although a generic unstable bundle E with its destabilizing subbundle $M$is "undercounted" in (C.1), it always admits sections of the form $\mathcal{O} \xrightarrow{s_{\mathbf{m}}} M \hookrightarrow E$ vanishing at an effective divisor $\mathbf{m}$ representing M since $\deg(M) \geq g$. Hence while generically E would not be counted in (C.1), the Baker-Akhiezer divisors associated to $(M \hookrightarrow E, \phi)$, which contains $\pi^{-1}(\mathbf{m})$, are contained in (C.2). In other words, Baker-Akhiezer divisors associated to sections that vanish somewhere [19] of these "undercounted" Higgs bundles behave predictably.

[19]Note that these bundles will then fit in the s.e.s of the form $0 \to \mathcal{O}(\text{div}(s')) \to E \to \mathcal{O}(-\text{div}(s'))\Lambda \to 0$, where $\text{div}(s')$ is the zero divisor of a nonzero section $s' \in H^0(E)$, with Baker-Akhiezer divisor being the non-σ-invariant part of the Baker-Akhiezer divisor associated to $\mathcal{O} \xrightarrow{s'} E$ in discussion.

Overcount and exceptional divisors. If a bundle E has two linearly independent sections, then for any two Higgs field ϕ the two corresponding Baker-Akhiezer divisors would be different but linearly equivalent. They are examples of exceptional divisors, i.e. in our case effective divisors of degree $4g-3$ on a spectral curve that has a non-trivial family of linearly equivalent divisors. The converse direction is also clear: two different but linearly equivalent effective divisors of degree $4g-3$ are Baker-Akhiezer divisors defining one same Higgs bundles (E, ϕ) but associated to two linearly independent sections of E. We say a bundle E is "overcounted" if $h^0(E) \geq 2$. It follows that E is over-counted if and only if the Baker-Akhiezer divisor associated to one/any data $(\mathcal{O} \to E, \phi)$ is exceptional.

Note that our notions of "undercount" and "overcount" are not mutually exclusive, since a bundle can have many linearly independent sections (overcounted) but none of them is nowhere-vanishing (undercounted). For example, consider an unstable bundle E with destabilizing subbundle M of degree $\deg(M) \geq g+1$; these make up Higgs bundles in the strata lower than the highest unstable stratum where $\deg(M) = g$. Besides generically being "undercounted" as discussed above, it also admits at least two linearly independent sections factoring through M and so is "overcounted".

We note that, though, a generic Higgs bundle (E, ϕ) with E unstable is "undercounted" but not "overcounted". Indeed, in a generic situation, the destabilizing subbundle $M \hookrightarrow E$ is of degree g and satisfies $h^0(M) = 1$, $h^0(M^{-1} \det(E)) = 0$. All sections of E then factor through M, which vanishes at g points on X, and so E is not "overcounted". On the other hand, in a non-generic situation where E is not "undercounted", it would be "overcounted" due to having nowhere-vanishing sections besides the sections induced by sections of M. Hence if a Higgs bundle (E, ϕ) induces a Baker-Akhiezer divisor that is both non-exceptional and has no σ-invariant contribution, E must be stable.

Similarly to the "undercount" situation, an "overcount" situation is non-generic since an exceptional divisor of degree $4g-3$ is non-generic on the spectral curve.

REMARK C.1. (a) There are two basic ways for a bundle E to be "overcounted". First, E might have a section s whose zero divisor div(s) is an exceptional divisor or has $\deg(\text{div}(s)) > g$ (as in the above case where s is the composition $\mathcal{O} \to M \hookrightarrow E$ and $\deg(M) \geq g+1$).

The Baker-Akhiezer divisor associated to s then inherits the degrees of freedom to move the $\pi^{-1}(\mathrm{div}(s))$ part around while staying in its linear equivalence class. Second, E might have two or more linearly independent sections with the same zero divisor (which might be trivial, as in the case of a nowhere-vanishing section). In this case the corresponding exceptional Baker-Akhiezer divisors have some degrees of freedom to move the non-σ-invariant part around.

(b) Suppose E is overcounted and, in addition, is not split. Then either E has a section that vanishes somewhere, or all of its sections are nowhere-vanishing and there are two linearly independent, nowhere-vanishing sections i_1 and i_2 which must be be parallel at some [20] points. In the latter case, a linear combination of i_1 and i_2 induces a section of E that vanishes at some of these points, bringing us to the former case. The Baker-Akhiezer divisor in the former case is contained in the set (C.2). In short, if E is overcounted and not split, then the family of corresponding exceptional Baker-Akhiezer divisors contains some divisors having some σ-invariant part.

(c) If E is split, then $E = M \oplus M^{-1}\Lambda$ with $\deg(M) \geq g$ also admits a section of the form $\mathcal{O} \to M \hookrightarrow E$ vanishing at an effective divisor $\mathbf{m}$ representing M. The Baker-Akhiezer divisor associated to any data $(\mathcal{O} \to M \hookrightarrow E, \phi)$ contains $\pi^{-1}(\mathbf{m})$.

The overall picture. We document in table 1 some examples of how the properties of the Baker-Akhiezer divisors associated to certain data $(\mathcal{O} \to E, \phi)$ depends on E. The list of examples is not exhaustive. Note that in the non-exceptional Baker-Akhiezer divisor row, a Higgs bundle (E, ϕ) could only be categorized into one of the two columns, while in the exceptional Baker-Akhiezer divisor row, it can fit in both columns.

Let $\mathcal{N}_{GL_2}$ be the moduli space of stable rank-2 bundle of degree $2g-1$. Let $\mathcal{N}'_{GL_2}$ be the loci consisting of stable bundles that has exactly one section up to scaling, and this section is nowhere-vanishing; it is an open dense subspace of $\mathcal{N}_{GL_2}$. Let $T^*_s\mathcal{N}'_{GL_2}$ be the set of equivalence classes of Higgs bundles made from these bundles such that the spectral curves of the Higgs fields are non-degenerate. The construction of Baker-Akhiezer divisors using the unique up to scaling

[20] i_1 and i_2 are parallel at $2g-1$ points (counted with multiplicity), which is the zero divisor of the composition $i_1 \hookrightarrow E \to \Lambda$, where the last arrow is the quotient map of $i_2 \hookrightarrow E$.

	contains no pull-back of divisors from X (induced by an embedding $\mathcal{O} \hookrightarrow E$)	contains pull-back of divisors from X (induced by $\mathcal{O} \to E$ with zeroes)
non-exceptional ($h^0(E) = 1$)	a generic stable bundle; NO unstable bundles;	a generic unstable bundle (with destabilizing subbundle of degree g);
exceptional ($h^0(E) > 1$)	some bundles, both stable and unstable, with maximal subbundle M satisfying $h^0(KM^{-2}\det(E))$ and $h^0(M^{-1}\det(E)) > 0$, e.g. $E = M \oplus M^{-1}\det(E)$ with M as such;	a bundle admitting a subbundle that has > 1 linearly independent section, e.g. unstable bundle with destabilizing subbundle of degree $> g$;

Table 1: Properties and examples of Baker-Akhiezer divisors associated to $(\mathcal{O} \to E, \phi)$.

section of the underlying bundles defines a map $BA : T^*\mathcal{N}^s_{GL_2} \to (T^*X)^{[4g-3]}$. The produced Baker-Akhiezer divisors are non-exceptional and contain no pull-back of divisors from X.

$$\begin{tikzcd} & (T^*X)^{[4g-3]}_s \arrow[dr, "A^s"] & \\ T^*_s\mathcal{N}^s_{GL_2} \arrow[ur, "BA"] \arrow[rr, hook] & & \mathcal{M}^s_{H,GL_2} \end{tikzcd} \tag{C.3}$$

Given a generic point $\mathbf{P}$ in the image of BA, we can recover the unique spectral curve S passing through the $4g-4$ points in T^*X defined by $\mathbf{P}$. For each such spectral curve S, the image of BA defines an open dense subset of the set of equivalence classes of divisors of degree $4g-3$. By taking into account effective divisors of degree $4g-3$ of the other three types in table 1, we can cover the rest of the equivalence classes and corresponding Higgs bundles. For example, by including divisors that contains pull-back of divisors on X as summand, we would include all Higgs bundles with unstable underlying bundles.

Completing the set of divisors for all non-degenerate spectral curve, we complete the image of BA to an open dense subspace $(T^*X)^{[4g-3]}_s$ of $(T^*X)^{[4g-3]}$ defined by all effective divisors of degree $4g-3$ on all non-degenerate spectral curves. The Abel map on each non-degenerate spectral curve together define a map $A^s : (T^*X)^{[4g-3]}_s \to \mathcal{M}^s_{H,GL_2}$. The map A^s is surjective by the completion we did.

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