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Chapter 1

Introduction

Humanity faces pressing issues. Which climate change mitigation and adaption policies should be adopted at a local and global level? What should be done about economic inequality within and between countries? How should democratic decision procedures be designed to cope with the challenges of our time?

The present thesis takes a systematic perspective on the stated issues. All of them can be understood as aggregation problems. There is a set of alternatives (climate policies, income allocations, candidates for a political office) and a population of individuals (humanity, citizens of a country). Individuals have rankings over the set that reflect their preferences over alternatives or their well-being at alternatives. An aggregation rule translates such individual inputs into an overall social ranking of alternatives. It can in turn be used to choose an alternative (the best climate policy, income allocation, candidate).

The thesis develops new concepts and frameworks for the analysis of aggregation. It uses them to examine aggregation rules and their properties. Characterizations play a key methodological role. A characterization establishes that a specific aggregation rule (or a class of rules) is the only one satisfying a certain set of (desirable) conditions. The aim of this axiomatic analysis is to identify reasonable rules to deal with social choice problems like the ones described.

The analysis of the thesis is first and foremost a contribution to social choice theory and welfare economics (Arrow, 1951, 1963, 2012; Sen, 1970, 2017; Suzumura, 2002; Campbell and Kelly, 2002; d'Aspremont and Gevers, 2002; Blackorby et al., 2002, 2005; Cato, 2010; Weymark, 2016). Social choice theory analyzes the aggregation of individual inputs into social rankings and social choices. Welfare economics is concerned with the social welfare evaluation of alternatives. Both are closely related. Following initial formal explorations of Arrow (1951, 1963) and Sen (1970), the 1970s were a particularly fruitful period where many foundational contributions on social aggregation were published (Gärdenfors, 1973; Young, 1974; Fine and Fine, 1974; Hammond, 1976, 1979; d'Aspremont and Gevers, 1977; Sen, 1977; Deschamps and Gevers, 1978; Maskin, 1978; Gevers, 1979; Roberts, 1980a,b,c).

In mainstream economics, there has been persistent skepticism surrounding the inquiry of social choice and welfare (see Suzumura, 2002; Sen, 2017: New Introduction). On the one hand, social choice theory has identified several impossibility results, the most famous one being Arrow's Impossibility Theorem. They seem to question the possibility of reasonable social evaluation and choice. On the other hand, there have been methodological concerns as social aggregation is tied to (contestable) value judgments.

The thesis takes up the challenge. The described choice problems and many other ones call for a systematic analysis of social evaluation. Impossibility results are based on assumptions. They invite for a reflection which of these assumptions should be weakened to generate possibility results. The analysis of value judgments and their implications is complex, but possible. In line with this, the thesis joins recent contributions on social choice and welfare (Fleurbaey and Maniquet, 2011, 2017, 2019; Asheim and Zuber, 2014; Fleurbaey and Tadenuma, 2014; Mongin and Pivato, 2015; Pivato, 2015, 2020; Fleurbaey and Mongin, 2016; Morreau and Weymark, 2016; Piacquadio, 2017, Bosmans et al., 2018; Brandl and Brandt, 2020; McCarthy et al., 2020; Bartholdi et al., 2021; Terzopoulou and Endriss, 2021; Bossert et al., 2023; Gustafsson et al., 2023; Karni and Weymark, 2023; Maskin, 2023; Nebel, 2023; Pivato and Tchouante, 2023; Spears and Zuber, 2023).

The thesis consists of three self-contained contributions (chapters 2-4). In the remainder of this introduction, these contributions are summarized. Afterwards, overall themes of their investigations are discussed.

Chapter 2 develops the first systematic relational theory of welfarist social evaluation. It is based on a single difference relation capturing qualitative intra- and interpersonal well-being difference and level comparisons. As the relation can be incomplete, it is consistent with all kinds of assumptions regarding the interpersonal comparability of differences and levels. The relational theory overcomes issues of the classical and still prevalent theory of social welfare in terms of Social Welfare Functionals. In contrast to the classical theory, the relational theory does not presuppose utility profile representations but derives them. It also does not employ problematic inter-profile conditions from the classical theory.

The relational theory is based on two fundamental normative conditions. Fundamental Equity is a new general condition to deal with conflicts of interest between two individuals. The property offers a unified rationale for different equity conditions. Differences between these conditions are due to differences in informational inputs. Given different assumptions on interpersonal well-being level and difference comparability, Strong Pareto and Fundamental Equity characterize Simple Majority Relation, Leximin, Utilitarianism, and a novel rich class of Additive Welfarist Relations. The results show that major aggregation approaches have a common normative basis and are solely distinguished on informational grounds. Given a weak incomparability assumption on well-being, no social ordering satisfies the two fundamental conditions Strong Pareto and Fundamental Equity.

The relational theory applies to the social evaluation of states of the world and of uncertain prospects. It is also extended to a variable population setting. This extension yields a systematic analysis of variable population social evaluation given different informational assumptions. Another extension is concerned with preference foundations of the well-being difference relation. Under relatively weak assumptions, a preference relation of an ethical observer and a profile of individual preference relations is sufficient to derive a complete difference relation. Given complete interpersonal well-being difference and level comparisons, fairly weak assumptions imply that interpersonal well-being difference ratios are well-defined. In that informational environment, many social orderings satisfy Strong Pareto and Fundamental Equity.

Chapter 3 takes the insights of Chapter 2 as a starting point to provide a systematic analysis of Generalized Utilitarianism. It investigates and connects different classes of Generalized Utilitarian Orderings. These social orderings take an additively separable form and allow to give priority to worse off individuals. The chapter clarifies discussions from the literature concerning informational requirements of Generalized Utilitarianism. It discusses different procedures to specify interpersonal well-being comparisons for social evaluation. Given meaningful well-being difference ratios, all Generalized Utilitarian Orderings are well-defined.

The chapter establishes an axiomatic analysis of different Generalized Utilitarian classes. In contrast to existing contributions, the analysis is based on a single profile of individual utility functions and uses weak and intuitive separability and compensation conditions. Compensation properties are closely related to continuity. Overall, the analysis strengthens the case for Generalized Utilitarianism.

The chapter also offers an axiomatic analysis of Utilitarianism. A single-profile version of the canonical characterization from the literature implicitly uses a strong substantive invariance requirement on social evaluation. In contrast, a new characterization is based on the novel property that social evaluation is stable under certain small well-being changes. Remarkably, the result does not employ anonymity or separability conditions. Both characterizations derive the property that equal well-being gains are socially indifferent. The offered analysis is extended to the social evaluation of uncertain prospects and yields characterizations of Expected Generalized Utilitarianism and Expected Utilitarianism. The distinguishing point of the analysis is the perfect analogy between the certainty and the uncertainty case.

Chapter 4 develops a systematic account of aggregating incomplete individual ranking information. It is based on a new concept. The Pull Ahead Index measures how many new favorable rankings need to be added to a given profile of rankings for an alternative to take the social lead over another alternative under an aggregation rule. It provides a novel perspective on the structure and quality of aggregation rules for incompletely known individual rankings. The chapter analyzes conditions on Pull Ahead Indices of aggregation rules. Some conditions have counterparts that are directly formulated in terms of social and individual rankings. This is the case for central independence, anonymity, and neutrality properties. New positive and equal responsiveness conditions do not have such counterparts.

Conditions on the Pull Ahead Index are employed for an axiomatic analysis of different classes of majority rules for incompletely known individual rankings. Relative Majority Rules determine differences in favorable rankings that are necessary for a favorable social ranking. They are characterized by the property that new opposing rankings do not change levelled Pull Ahead Indices. Quorum Majority Rules specify quorums of favorable rankings that are needed for a favorable social ranking. Their critical property is that new opposing rankings reduce levelled Pull Ahead Indices. The Simple Majority Rule forms the intersection of Relative and Quorum Majority Rules. It is characterized by the property that alternatives do not have an implicit lead in terms of the Pull Ahead Index.

The analysis of the chapter is relevant to compare the merit of different majority rules in voting and policymaking contexts. If the profile of known individual rankings is fixed, the no implicit lead condition is reasonable and provides an argument for the Simple Majority Rule. If the profile can be augmented, one can make a case for Relative or Quorum Majority Rules. A distinguishing feature of majority rules is to let the social evaluation of a pair of alternatives only depend on individual rankings over the pair. To analyze aggregation procedures with richer informational bases, extended Pull Ahead Indices in terms of favorable rankings over third alternatives and in terms of interpersonally comparable utility gains are formulated.

While chapters 2-4 are independent contributions, they are united by several **overall themes**.

The informational basis of aggregation: All chapters analyze the structure and merit of aggregation rules given different informational inputs. The difference relation of chapter 2 allows for all kinds of more or less complete interpersonal well-being level and difference comparisons. The same normative conditions characterize different aggregation rules under different informational assumptions. Chapter 3 takes a closer look at the informational environment where interpersonal well-being difference ratios are well-defined. Chapter 4 is concerned with profiles of incomplete individual rankings, corresponding to the environment where there are only intrapersonal level comparisons. However, the chapter also discusses how to extend its approach to richer informational inputs.

Single- and multi-profile approaches: In contrast to many other contributions in social choice theory and welfare economics, chapters 2-3 do not employ (contestable) conditions that relate different profiles of individual rankings or utility functions. These chapters present the first systematic single-profile analysis of welfarist social evaluation. Fundamental Equity provides a solution to the quest for a general and convincing equity condition in the single-profile literature (see Blackorby et al., 2006). Chapter 4 instead takes a multi-profile approach. In its framework, profiles of known individual rankings can be augmented. It is thus natural to formulate requirements on social rankings and Pull Ahead Indices as inter-profile conditions. The chapters show that both single- and multi-profile approaches have their merits. Taken together, they provide a deeper understanding of aggregation.

Two-person conditions: Both voting and social welfare evaluation can involve millions or even billions of individuals. It is difficult to assess trade-offs between so many people. It is easier to intuitively grasp situations where only two people have conflicting rankings over two alternatives, while all other individuals are indifferent. In line with that, conditions for such two-person situations play a key role in all three chapters. Fundamental Equity from chapter 2 implies different two-person properties under different informational assumptions. In chapter 3, new two-person compensation and stability conditions are key to derive Generalized Utilitarianism and Utilitarianism, respectively. In chapter 4, novel properties concerned with opposing rankings of two individuals are critical to distinguish Relative from Quorum Majority Rules.

Unity of analysis: The social evaluation of alternatives is a complex topic. To cope with this complexity, all chapters aim for an analysis which is based on a limited number of core ideas. Fundamental Equity from chapter 2 provides a unified information-dependent justification of different normative properties. It characterizes and connects a wide range of aggregation rules, including voting type approaches like Simple Majority and classical welfarist approaches like Utilitarianism and Leximin. Chapter 3 adds separability and closely related compensation and continuity conditions to the picture. The analysis of chapters 2-3 smoothly extends to the social evaluation of uncertain prospects and variable populations. The Pull Ahead Index from chapter 4 is a versatile tool in voting and policymaking contexts. It highlights relations between different classes of qualified majority rules.

Connecting literatures: Aggregation is a central topic in different literatures. All chapters of the thesis contribute to establish connections between these literatures. The unified analysis of welfarist social evaluation in chapters 2-3 brings together the theory of Social Welfare Functionals, voting theory, theories of social prospect evaluation and variable populations, theories of well-being, as well as decision and measurement theory. It connects discussions on social evaluation in economics and philosophy. The possible reduction of the well-being difference relation to a profile of preference relations in chapter 2 brings welfare economic analysis closer to the core of economic theory. Chapter 4 provides links between classical social choice theory concerned with the aggregation of individual orderings as well as more recent contributions on the aggregation of incomplete preferences and judgments and on dynamic information acquisition in collective decision-making.

Impossibilities and Possibilities: The chapters identify both impossibility and possibility results. On the one hand, Chapter 2 establishes the inconsistency of Strong Pareto and Fundamental Equity given weak well-being incomparability. On the other hand, chapters 2-3 show that there are attractive aggregation procedures given sufficiently rich complete interpersonal well-being level and difference comparisons. Chapter 3 derives an impossibility result of a different kind. Two seemingly weak regularity conditions are inconsistent with the idea to give priority to worse off individuals. The qualified majority rules characterized in chapter 4 face the problem of cyclic social rankings.

Social Rankings and Social Choice: In all three chapters, the output of aggregation is a social ranking of alternatives. As discussed, the motivation for the derivation of social rankings is to inform social choice. Chapters 2-3 are concerned with fairly rich sets of alternatives, containing conceivable states of the world or conceivable prospects. The social ranking over conceivable alternatives induces a social ranking of feasible alternatives. The best feasible alternative according to the ranking is the recommendation for social choice. Chapter 4 discusses that the relation between social rankings and social choice is more subtle in a dynamic context.

Chapter 2

A Single Relation Theory of Welfarist Social Evaluation

The prevalent theory of social welfare in terms of Social Welfare Functionals is based on profiles of utility functions and uses problematic inter-profile conditions. To overcome this issue, the chapter offers a systematic relational theory of social evaluation over states of the world. It is formulated in terms of a single difference relation capturing basic well-being comparisons between states. Utility representations are derived and not presupposed. A new Fundamental Equity condition deals with trade-offs between individuals. It provides a unified rationale for existing two-person conditions and identifies deep connections between seemingly distinct approaches to social evaluation. Dependent on the extent of interpersonal comparability, Strong Pareto and Fundamental Equity characterize Simple Majority Relation, Leximin, Utilitarianism, a level-based Borda Relation, and a new class of Additive Welfarist Relations. They yield a general impossibility given weak incomparability. The analysis extends to social evaluation under uncertainty and to variable populations. Well-being comparisons can be reduced to preferences of individuals and of an ethical observer. Overall, the theory connects welfare economics, social choice and voting theory, decision and measurement theory, as well as theories of social prospect evaluation and variable populations.

2.1 Introduction

Governments around the world must choose between different courses of action everyday. Among the most pressing issues are effective ways to fight climate change and reduce global poverty. Economics is meant to assist these efforts by examining the social welfare evaluation of different states (or histories) of the world. The classical and still prevalent theory of social welfare is due to Sen (1970, 2017). A Social Welfare Functional translates profiles of individual utility functions over states into social orderings of states. Numerical utility levels represent the underlying well-being of individuals. Despite offering a systematic analysis of aggregate evaluation, Social Welfare Functional theory faces deep conceptual problems. Its central condition Binary Independence conflates meaningful comparisons of well-being with meaningless comparisons of utility levels between different scales.[1] This scale problem is already noted by Sen (1977: pp. 1542-1543) and is more thoroughly discussed in Morreau and Weymark (2016), but has remained unsolved to the present day.

To overcome the scale problem and other issues of the classical theory, the chapter offers a systematic relational theory of welfarist social evaluation over states of the world. A single difference relation captures primitive well-being difference and level comparisons. It can contain judgments like "if 100 US Dollars are redistributed from a rich to a poor individual, the well-being gain of the latter is greater than the well-being loss of the former" and thereby provides a direct link between welfare economic analysis and public discourse. The relational theory is very flexible, allowing for all kinds of assumptions on the intra- and interpersonal comparability of well-being differences and levels.

The relational theory is concerned with the aggregation of the well-being difference relation into a social ordering over states of the world. All informational and normative conditions are formulated in terms of these relations. The theory is based on just two fundamental normative properties. Strong Pareto is one of the central conditions of economics and secures that social evaluation positively respond to individual gains.

Fundamental Equity is a new condition to deal with the basic problem of economics to assess trade-offs between individuals. It is concerned with a situation where individual i gains from state y to state x, individual j gains from x to y, while all other individuals are indifferent between x and y. Due to the symmetry of this two-person situation, Fundamental Equity states that a social ranking in favor of x (or y) requires a justification. It must either be the case that i's gain is greater than j's gain, or i must be worse off than j. Fundamental Equity provides a unified rationale for existing two-person conditions under different informational assumptions (including Pigou-Dalton

[1] Binary Independence requires that, if two profiles of utility functions coincide on a pair of states, the social ordering of the pair should coincide between the two profiles. But if utility functions measure well-being in different scales, comparisons of utility levels between them are as meaningless as temperature comparisons between the Celsius and the Fahrenheit scale.

and Hammond Equity).

Taken together, Strong Pareto and Fundamental Equity have remarkably far-reaching consequences. Dependent on comparability assumptions concerning well-being levels and gains, they characterize Simple Majority Relation, Leximin Relation, Utilitarian Relation, a level-based Borda Relation, and a fairly rich new class of Additive Welfarist Relations. For a general incomplete difference relation, Strong Pareto and Fundamental Equity yield an impossibility result.

The single relation theory improves, generalizes, and unifies the classical theory of social welfare in terms of Social Welfare Functionals and its canonical results (see Blackorby et al., 2002, 2005; d'Aspremont and Gevers, 2002; Bossert and Weymark, 2004; Morreau and Weymark, 2016; Weymark, 2016; Sen, 2017; Nebel, 2023). In contrast to the classical theory, the relational theory is formulated in terms of primitive well-being comparisons and does not presuppose utility representations. Being based on a single relation, it also gives up all problematic inter-profile conditions like Binary Independence and solves the scale problem. A systematic axiomatic analysis without inter-profile properties has so far been an open issue in welfare economics (see Blackorby et al., 2006).

The relational theory identifies deep normative connections between major approaches to social evaluation. It connects classical welfarist approaches like Leximin and Utilitarianism with voting type approaches like Simple Majority and Borda. Specifically, the analysis gives an explanation why the Simple Majority Relation is generally intransitive. This provides a link between welfare economics and voting theory (May, 1952; Young, 1974; Pattanaik, 2002; Barberà and Gerber, 2017; Alcantud, 2019; Horan et al., 2019; Bartholdi et al., 2021; Terzopoulou and Endriss, 2021; Maskin, 2023).

Three extensions of the basic theory for states of the world with a fixed population contribute to major debates in normative economics. First, the offered analysis for different informational assumptions also applies if social alternatives are abstract prospects and well-being comparisons are ex ante. It does not presuppose expected utility theory. This contributes to the literature on social evaluation under uncertainty (Harsanyi, 1955, 1982; Weymark, 1991; Blackorby et al., 2005; Fleurbaey, 2010; Pivato, 2013b; Mongin and Pivato, 2015; Fleurbaey and Mongin, 2016; Sprumont, 2019; Brandl and Brandt, 2020; McCarthy et al., 2020; Gustafsson et al., 2023; Karni and Weymark, 2023; Pivato and Tchouante, 2023; Spears and Zuber, 2023).

Second, the analysis for different informational environments extends to variable populations. It covers total and critical level approaches. This contributes to the variable population literature (Blackorby et al., 2005; Asheim and Zuber, 2014; Fleurbaey and Tadenuma, 2014; Pivato, 2020; Bossert et al., 2023; Gustafsson et al., 2023; Spears and Zuber, 2023).

The third extension offers a preference foundation of the well-being difference relation. Intrapersonal level and difference comparisons are reduced to individual preferences and interpersonal level comparisons are reduced to an ethical observer's preferences over one-person states. If preferences are complete, the difference relation is also complete and many attractive aggregation procedures are available. This third extension and the general theory of the chapter relate to decision and measurement theory (Krantz et al., 1971; Wakker, 1989; Köbberling, 2006; Mongin and Pivato, 2015; Li et al., 2023). In line with the latter, the theory is based on a single relation capturing primitive judgments/preferences.[2]

Like the literature on difference measurement (Krantz et al., 1971; Wakker, 1988; Köbberling, 2006; Pivato, 2013c), the theory of the chapter derives utility difference representations. Like a small literature on the aggregation of individual difference relations into social difference relations (Harvey, 1999; Pivato, 2015), it analyzes the qualitative foundations of utilitarianism. However, these contributions operate in different frameworks and use different conditions than the present chapter. Their analysis is concerned with utilitarian-type social difference relations.

The chapter takes the following structure. Section 2.2 introduces the well-being difference relation (2.2.1) and the two fundamental normative conditions (2.2.2). Section 2.3 examines different informational assumptions. It characterizes Simple Majority Relation (2.3.1), Leximin Relation (2.3.2), Utilitarian and Level Borda Relation (2.3.3), and the class of Additive Welfarist Relations (2.3.4). Section 2.3.5 states a general impossibility result. Section 2.4 deals with extensions, including uncertain prospects (2.4.1), variable populations (2.4.2), and preference foundations of the difference relation (2.4.3). Section 2.5 elaborates on the relation of the theory to the literature, covering Social Welfare Functions (2.5.1), Social Welfare Functionals (2.5.2), and other fields (2.5.3). Section 2.6 concludes. Appendix 2.A contains all proofs.

[2]The preference foundation of well-being comparisons also relates to the literature on social evaluation in economic environments (Fleurbaey and Maniquet, 2011, 2017, 2019; Fleurbaey and Tadenuma, 2014; Piacquadio, 2017, Bosmans et al., 2018).

2.2 Framework

2.2.1 Well-Being Differences and Levels

Let N be a population of individuals and X be a set of alternatives, where $|N| \geq 2$ and $|X| \geq 2$. In sections 2.2 and 2.3, N is assumed to be finite and X is interpreted as the set of all conceivable states (or histories) of the world with population N. Let $i_x = (i, x)$ denote individual $i \in N$ at state of the world $x \in X$.

The whole analysis of the chapter is based on qualitative comparisons of qualitative well-being differences. A **well-being difference relation** $\succsim$ is a relation over $(N \times X)^2$.[3] Consider individuals $i, j, k, l \in N$ and states $x, y, z, w \in X$. Figure 2.1 illustrates qualitative well-being differences. The red line represents the well-being difference between i at x and j at y, while the blue line represents the well-being difference between k at z and l at w. Now, $(i_x, j_y) \succsim (k_z, l_w)$ means that the former is at least as great as the latter.

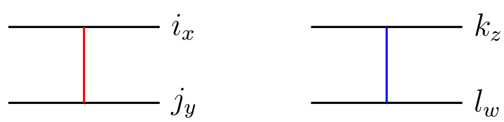

Figure 2.1: Comparisons of Well-Being Differences

Let $\sim$ and $\succ$ be the symmetric and asymmetric part of $\succsim$, respectively.[4] Then, $(i_x, j_y) \sim (k_z, l_w)$ means that the well-being difference between i at x and j at y is as great as the well-being difference between k at z and l at w. Likewise, $(i_x, j_y) \succ (k_z, l_w)$ means that the former is greater than the latter. Finally, $(i_x, j_y) \not\succsim (k_z, l_w)$ and $(k_z, l_w) \not\succsim (i_x, j_y)$ means that the well-being difference between i at x and j at y is incomparable to the well-being difference between k at z and l at w.

Difference relation $\succsim$ can represent the judgments of an ethical observer.[5] For example,

[3] A relation R over a set A is a subset of $A \times A$. For $a, b \in A$, write aRb instead of $(a, b) \in R$. Relation R is (1) reflexive if, for all $a \in A$, aRa, (2) transitive if, for all $a, b, c \in A$, aRb and bRc imply aRc, (3) complete if, for all $a, b \in A$, aRb or bRa, and (4) symmetric if, for all $a, b \in A$, aRb implies bRa. An ordering is a complete and transitive relation.

[4] For relation R over set A, the symmetric part of R is the relation I over A such that, for $a, b \in A$, aIb if and only if aRb and bRa. The asymmetric part of R is the relation P over A such that, for $a, b \in A$, aPb if and only if aRb and not bRa.

[5] More generally, $\succsim$ can also represent consensus judgments of a group of ethical observers.

suppose that $i = j$ and $k = l$ in Figure 2.1. Suppose that i is a poor individual who realizes some income gain from state y to state x, while k is a rich individual who realizes the same income gain from w to z. In that case, $(i_x, i_y) \succ (k_z, k_w)$ means that the observer judges the well-being gain of i due to the income gain to be greater than the according well-being gain of k. This judgment is a qualitative version of diminishing marginal utility of income. A preference foundation of $\succsim$ will be discussed in section 2.4.3.[6]

The difference relation approach is compatible with different well-being conceptions.[7] Under an experientialist conception, a well-being difference (say, the one between i at x and j at y in Figure 2.1) is a difference in the quality of individuals' experiences at states. Under a desire satisfaction approach, it is a difference in individuals' desire satisfaction at states. Under an objective goods conception, it is a value difference between individuals' bundles of objective goods at states. Importantly, the assumption that $\succsim$ reflects the judgments of an ethical observer does not commit to a specific well-being conception. The judgments can be in line with an experientialist, desire satisfaction, or objective goods conception.

The following analysis neither presupposes a specific procedure to determine $\succsim$ nor a specific content of the comparisons in $\succsim$. It just assumes that $\succsim$ is fixed. The following consistency conditions are standard in difference measurement.

C1: $\succsim$ is reflexive and transitive.
C2: For all $i, j, k, i', j', k' \in N$ and $x, y, z, x', y', z' \in X$, $(i_x, j_y) \succsim (i'_{x'}, j'_{y'})$ and $(j_y, k_z) \succsim (j'_{y'}, k'_{z'})$ imply $(i_x, k_z) \succsim (i'_{x'}, k'_{z'})$.
C3: For all $i, j, k, l \in N$ and $x, y, z, w \in X$, $(i_x, j_y) \succsim (k_z, l_w)$ implies $(l_w, k_z) \succsim (j_y, i_x)$.
C4: For all $i, j \in N$ and $x, y \in X$, $(i_x, i_x) \sim (j_y, j_y)$.

While well-being difference comparisons are to be transitive by C1, they do not have to be complete. This will allow for great flexibility regarding the extent of intra- and interpersonal well-being comparisons. C2 is a consistency condition on comparisons of wholes and parts. It is illustrated in Figure 2.2. Suppose that two well-being differences are split into two parts, respectively. If the first part of the first difference (red line)

[6] The chapter does not take a position whether well-being difference comparisons are matters of fact that are captured by a "true" well-being difference relation. If that is the case, this true relation should arguably be the input of aggregation. If there is no true difference relation, the judgments of the ethical observer construct well-being comparisons in the first place. See Maniquet (2016: 240-241).

[7] For a discussion of these different conceptions, see Bradley (2014), Heathwood (2014), Bykvist (2016), Haybron (2016), and Hurka (2016).

is at least as great as the first part of the second difference (blue line), and the second part of the first difference (yellow line) is at least as great as the second part of the second difference (green line), then the whole first difference (red plus yellow line) is at least as great as the whole second difference (blue plus green line). According to C3, if a first difference is at least as great as a second difference, then the reversed second difference is at least as great as the reversed first difference.

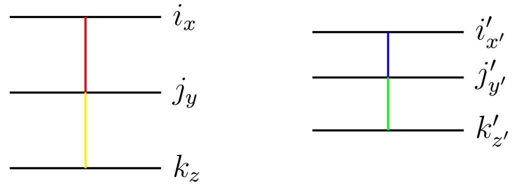

Figure 2.2: Consistent Comparisons of Wholes and Parts

Intuitively, the well-being difference between i at x and i at x is zero. C4 secures that this zero difference is unique. Indeed, the consistency conditions imply that well-being differences have signs in the following sense. If $(i_x, j_y) \succ (j_y, j_y)$, the well-being difference between i at x and j at y is a well-being gain as it is greater than the zero difference. If $(j_y, j_y) \succ (i_x, j_y)$, it is a well-being loss. If $(i_x, j_y) \sim (j_y, j_y)$, the well-being difference is zero. If $(i_x, j_y) \not\succsim (j_y, j_y)$ and $(j_y, j_y) \not\succsim (i_x, j_y)$, it is indefinite.

Given C1-C4, well-being level comparisons can be conceived as special well-being difference comparisons. Difference relation $\succsim$ gives rise to the following level relation.

Well-being level relation $\succsim^*$ over $N \times X$ exhibits, for $i, j \in N$ and $x, y \in X$, $i_x \succsim^* j_y \Leftrightarrow (i_x, j_y) \succsim (j_y, j_y)$.

In view of the previous discussion, it makes sense to interpret $i_x \succsim^* j_y$ such that i at x is at least as well off as j at y. Let $\sim^*$ and $\succ^*$ be the symmetric and asymmetric part of $\succsim^*$, respectively. That is, $i_x \sim^* j_y$ means that i at x is as well off as j at y and $i_x \succ^* j_y$ means that i at x is better off than j at y. Moreover, $i_x \not\succsim^* j_y$ and $j_y \not\succsim^* i_x$ means that i at x and j at y are incomparable.

C1-C4 imply that $\succsim^*$ is reflexive and transitive (Appendix 2.A.1, Lemma A.1.1). Since $\sim^*$ is an equivalence relation (reflexive, transitive, and symmetric), it partitions $N \times X$ into non-empty pairwise disjoint subsets (equivalence classes) whose union is $N \times X$. The equivalence class of i_x is $[i_x] = \{j_y \in N \times X \mid i_x \sim^* j_y\}$. It can be identified with the **well-being level** of i at x. This formal understanding of a well-being level will turn out to be convenient.[8] In line with it, $i_x \succsim^* j_y$ also means that the well-being level of i at x is at least as great as the well-being level of j at y. Analogous interpretations apply to $\sim^*$ and $\succ^*$. The well-being level relation is assumed to satisfy a richness assumption.

R1: For all $i, j \in N$ and $x, y, z \in X$, there exists $w \in X$ such that $i_w \sim^* i_x$, $j_w \sim^* j_y$, and, for all $k \in N \setminus \{i, j\}$, $k_w \sim^* k_z$.

Since X is the set of all conceivable states of the world with population N, R1 is defendable under most well-being conceptions. For instance, suppose that an individual's well-being level at a state solely depends on the individual's bundle of goods at the state. In that case, R1 is implied by the assumption that there is a conceivable state w where i has the same bundle as at x, j has the same bundle as at y, and all other individuals have the same bundles as at z, respectively.

Property R1 is a richness assumption on the set of conceivable states of the world X. The set of feasible states of the world $Y \subseteq X$ in some choice situation does not have to satisfy any richness assumptions. Specifically, state w in the sense of R1 does not have to be feasible. Analogous comments apply to the other richness properties to come. The following assumption will be made throughout the chapter.

Assumption: C1-C4 and R1 hold.

2.2.2 Two Fundamental Conditions for Social Orderings

A **Social Ordering (SO)** $\succsim_S$ is a complete and transitive relation over X. It represents the overall social evaluation of states of the world. For $x, y \in X$, $x \succsim_S y$ means that x is socially at least as good as y. Letting $\sim_S$ and $\succ_S$ be the symmetric and asymmetric part of $\succsim_S$, $x \sim_S y$ means that x is socially as good as y and $x \succ_S y$ means that x is socially better than y. Transitivity of $\succsim_S$ is a rationality requirement on social evaluation. Completeness reflects that $\succsim_S$ is supposed to inform (social) choice. SO $\succsim_S$ orders the set of feasible states $Y \subseteq X$ in a choice situation. A feasible state that is socially at least as good as any other feasible state is recommended for choice.

Like well-being difference relation $\succsim$, SO $\succsim_S$ can capture the judgments of an ethical observer. Suppose that the observer's preferences over states are in line with her judgments. Then, $x \succsim_S y$ holds if and only if the observer weakly prefers x over y, which is the case if and only if the observer judges that x is socially at least as good as y.[9]

[8] It leaves open whether there is something like a "real" well-being level of an individual at a state.

[9] A subject weakly prefers x over y if she prefers x over y or is indifferent between x and y. SO

The whole analysis of the chapter is based on just two fundamental normative conditions. Both are concerned with the relation between SO $\succsim_S$ and well-being difference relation $\succsim$. Since $\succsim$ represents comparisons of qualitative well-being differences and levels, the conditions are also entirely qualitative. They are defined no matter which level and difference comparisons are possible under $\succsim$.

SO $\succsim_S$ satisfies **Strong Pareto (SP)** if, for all $x, y \in X$, the following holds: If, for all $i \in N$, $i_x \succsim^* i_y$, then $x \succsim_S y$. If, in addition, there exists $i \in N$ with $i_x \succ^* i_y$, then $x \succ_S y$.

SP embodies two fundamental ideas. The first one is the welfarist view that only individuals' well-being matters for the overall evaluation of states. Specifically, SP implies that, for all $x, y \in X$ with $i_x \sim^* i_y$ for all $i \in N$, $x \sim_S y$.[10] The second idea of SP is to promote individual well-being. Ceteris paribus, an individual improvement constitutes a social improvement.

SP does not apply if individual i is better off at x than at y, individual j is better off at y than at x, while all other individuals are equally well off at x and y. This basic two-person situation is illustrated in Figure 2.3. A fundamental task of economics is to deal with such trade-offs. The following condition is meant to accomplish the task.

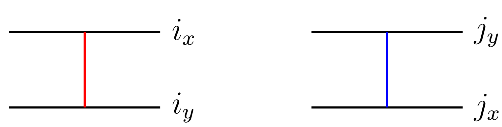

Figure 2.3: The Basic Two-Person Situation

SO $\succsim_S$ satisfies **Fundamental Equity (FE)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, the following holds: If $x \succ_S y$, then $(i_x, i_y) \succ (j_y, j_x)$ or $j_x \succ^* i_y$.

FE states necessary (but not sufficient) conditions for a strict social ranking in the basic two-person situation. It is based on the fundamental idea that all individuals are

$\succsim_S$ can also represent the consensus judgments/preferences of a group of ethical observers. In that case, the ordering assumption reflects the requirement that group deliberation should be decisive and rational.

[10]There has been some debate whether this Pareto Indifference (PI) condition is equivalent to welfarism (Kaplow and Shavell, 2001, 2004; Fleurbaey et al., 2003). This is a matter of definition. PI is weaker than a form of welfarism in Social Welfare Functional theory which entails the condition of Binary Independence (see section 2.5.2).

equally important in social evaluation. Since one individual gains from y to x and the other one loses, the default evaluation should be that x is socially as good as y. A deviation from this evaluation requires a justification.

There are arguably just two welfarist justifications for $x \succ_S y$ that are in line with equity.[11] To begin with, i's well-being gain from y to x (represented by the red line in Figure 2.3) might be greater than j's well-being gain from x to y (represented by the blue line). That is a possible reason to socially favor x over y. Suppose that this is not the case, that is, i's gain from y to x is not greater than j's gain from x to y.

There is still a possible egalitarian justification for $x \succ_S y$. It might be the case that i at y is worse off than j at x. That is, i's gain has a smaller starting level than j's gain. This is a possible reason to prioritize the former over the latter. Suppose that this case also does not apply, so that i at y is not worse off than j at x. Then, there is arguably no equitable welfarist justification for $x \succ_S y$. Consistent with the presumption in favor of equality, x should not be socially favored over y. That is precisely what FE states.

If the ethical observer endorses welfarism and equity in social evaluation, he or she has a compelling argument for FE. As will be shown in the next sections, the combination of SP and FE has remarkably far-reaching and general implications.

2.3 Characterizations

This section examines the implications of Strong Pareto and Fundamental Equity for different informational environments. The environments in sections 2.3.1-2.3.4 correspond to major assumptions from the classical theory of social welfare. But they are formulated in terms of well-being difference relation $\succsim$ and do not presuppose numerical representations. Section 2.3.5 states an impossibility result for a general incomplete difference relation.

2.3.1 No Level and Gain Comparisons

This subsection is concerned with the case where $\succsim$ exhibits intrapersonal level comparisons, but neither interpersonal level nor gain comparisons.

[11]For a justification to be welfarist, it must solely be formulated in terms of individuals' well-being gains.

Personal Level Comparability (PLC): For all $i \in N$ and $x, y \in X$, $i_x \succsim^* i_y$ or $i_y \succsim^* i_x$.

No Interpersonal Level Comparability (NILC): For all $i, j \in N$ and $x, y \in X$, $i_x \succsim^* j_y$ implies $i = j$.

Minimal Interpersonal Gain Comparability (MIGC): For all distinct $i, j \in N$ and $x, y, z, w \in X$ with $i_x \succ^* i_y$ and $j_z \succ^* j_w$, $(i_x, i_y) \succsim (j_z, j_w)$ implies $i_x \succsim^* j_z$ and $j_w \succsim^* i_y$.

NILC and MIGC together imply that there are no interpersonal gain comparisons at all.[12] Figure 2.4 illustrates intrapersonal level comparisons. Individual i is better off at x than at z and better off at z than at y. Individual j is better off at y than at x and equally well off at x and z. Certain intrapersonal gain comparisons follow from the consistency conditions on $\succsim$ (Appendix 2.A.1, Lemma A.1.1). For instance, i's gain from y to x (red line) is greater than i's gain from y to z (yellow line) as the latter is a part of the former. Assumptions PLC, NILC, and MIGC leave open whether there are intrapersonal gain comparisons in addition to these part-whole comparisons.

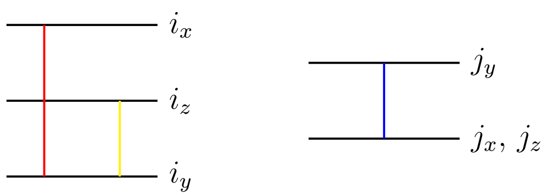

Figure 2.4: Intrapersonal Level Comparisons

Given NILC and MIGC, social evaluation cannot be based on interpersonal well-being comparisons. Under the following aggregative relation, the comparative assessment of states x and y indeed only depends on intrapersonal level comparisons over x and y.

The **Simple Majority Relation (SMR)** $\succsim_M$ over X specifies, for all $x, y \in X$, $x \succsim_M y \Leftrightarrow |\{i \in N \mid i_x \succ^* i_y\}| \geq |\{i \in N \mid i_y \succ^* i_x\}|$.

According to the SMR, state x is socially at least as good as state y if and only if the number of individuals who are better off at x than at y is at least as great as the number of individuals who are better off at y than at x. In the following, suppose that all individuals other than i and j are equally well off at x, y, and z in Figure 2.4. Under

[12] Given NILC, MIGC is equivalent to the property that, for all $i, j \in N$ and $x, y, z, w \in X$ with $i_x \succ^* i_y$ and $j_z \succ^* j_w$, $(i_x, i_y) \succsim (j_z, j_w)$ implies $i = j$.

the SMR, x is socially as good as y since i is better off at x than at y, while it is the other way around for j. This assessment is in line with the following condition.

SO $\succsim_S$ satisfies **Strong Equity (SE)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, it is the case that $x \sim_S y$.

SE embodies a strong claim. Whenever a basic two-person situation arises, the involved states should be equally good from the social point of view. The basic symmetry of the situation (one person gains from y to x, the other one loses) is taken to be sufficient for an equal social ranking.

FE offers a conditional justification of SE. It allows for two possible reasons for $x \succ_S y$ in the basic two-person situation. First, i's well-being gain from y to x might be greater than j's well-being gain from x to y. Second, i might start from a lower well-being level than j. Both justifications are based on interpersonal well-being comparisons. If such comparisons do not exist, there is no justification for $x \succ_S y$. Only an equal social ranking is acceptable. That is why FE implies SE in the absence of interpersonal comparisons.

Lemma 1: Suppose NILC and MIGC hold. SO $\succsim_S$ satisfies FE if and only if $\succsim_S$ satisfies SE.[13]

SE faces a severe problem. The only SO satisfying SP and SE is the SMR. The latter is only transitive under highly restrictive assumptions given R1. There must not be an individual with at least three well-being levels and another individual with at least two well-being levels. Since X contains conceivable states of the world, this special case hardly obtains. The following characterization of the SMR must thus be regarded as an impossibility result.

Proposition 1: Suppose PLC holds. There exists SO $\succsim_S$ satisfying SP and SE if and only if (a) there exists $k \in N$ such that, for all $i \in N \setminus \{k\}$, $|\{[i_x] \mid x \in X\}| = 1$ or (b) for all $i \in N$, $|\{[i_x] \mid x \in X\}| \leq 2$. If such $\succsim_S$ exists, then $\succsim_S = \succsim_M$.

Proposition 1 uncovers a fundamental inconsistency in social evaluation based on SE given SP. Consider the consequences of the two conditions in Figure 2.4. By SE, $z \sim_S y$. SP implies $x \succ_S z$. Social transitivity would demand $x \succ_S y$. Instead, SE induces $x \sim_S y$. The social value of j's gain from x (z) to y (blue line) is simultaneously equal to the social value of i's gain from y to x (red line) and the social value of i's smaller gain of from y to z (yellow line). This is the explanation of Proposition 1 in a nutshell.

[13] SE implies FE no matter which informational assumptions obtain.

It is particularly relevant to understand the general intransitivity of $\succsim_M$. Being based on SE, the SMR embodies the fundamental inconsistency just described.

Taken together, Lemma 1 and Proposition 1 lead to an impossibility result on combining SP and FE if there are no interpersonal well-being level and gain comparisons.

Theorem 1: Suppose PLC, NILC, and MIGC hold. There exists SO $\succsim_S$ satisfying SP and FE if and only if (a) there exists $k \in N$ such that, for all $i \in N \setminus \{k\}$, $|\{[i_x] \mid x \in X\}| = 1$ or (b) for all $i \in N$, $|\{[i_x] \mid x \in X\}| \leq 2$. If such $\succsim_S$ exists, then $\succsim_S = \succsim_M$.

2.3.2 Level Without Gain Comparisons

In the second main informational environment, interpersonal well-being level comparisons are possible. But there are no interpersonal gain comparisons except for part-whole comparisons.[14] That is, MIGC is combined with the following condition.

Interpersonal Level Comparability (ILC): $\succsim^*$ is complete.

Figure 2.5 illustrates intra- and interpersonal level comparisons. For instance, j at y is better off than i at z. But given MIGC, it is not possible to compare the magnitudes of j's gain from z to y (blue line) and i's gain from y to z (red line).

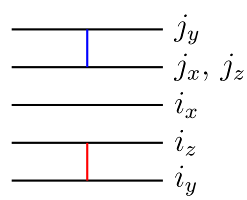

Figure 2.5: Interpersonal Level Comparisons

Under the Leximin SO, the ranking of states x and y solely depends on intra- and interpersonal well-being level comparisons regarding x and y. For $x, y \in X$, let $N_{s,xy} = \{i \in N \mid i_x \not\sim^* i_y\}$. Given PLC (which is implied by ILC), it is the set of individuals who are either better off at x than at y or the other way around. For non-empty $M \subseteq N$, let $R(M) = \{1, ..., |M|\}$.

[14] If $i_x \succsim^* j_z \succ^* j_w \succsim^* i_y$, the consistency conditions on $\succsim$ imply that $(i_x, i_y) \succsim (j_z, j_w)$.

Suppose ILC holds. The **Leximin Relation (LR)** $\succsim_L$ over X specifies, for all $x, y \in X$, $x \succsim_L y$ if and only if

(1) $N_{s,xy} = \emptyset$, or

(2) $N_{s,xy} \neq \emptyset$ and there exist bijections $\pi, \mu : R(N_{s,xy}) \to N_{s,xy}$ such that, for all $r, t \in R(N_{s,xy})$ with $r \geq t$, $\pi(r)_x \succsim^* \pi(t)_x$ and $\mu(r)_y \succsim^* \mu(t)_y$, and either

(a) $\pi(r)_x \sim^* \mu(r)_y$ for all $r \in R(N_{s,xy})$, or

(b) there exists $t \in R(N_{s,xy})$ such that, for all $r \in R(N_{s,xy})$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t)_x \succ^* \mu(t)_y$.[15]

If all individuals are equally well off at x and y, they are socially equally good according to the LR. Suppose there are individuals whose well-being levels at x and y differ. Of them, $\pi(1)$ is the worst-off individual at x, $\pi(2)$ is the second-worst-off individual at x, ..., and $\pi(|N_{s,xy}|)$ is the best-off individual at x. Likewise, $\mu(1)$ is the worst-off individual at y, ..., and $\mu(|N_{s,xy}|)$ is the best-off individual at y. Ties can be broken arbitrarily. According to the LR, x is socially better than y if the worst-off individual at x is better off than the worst-off individual at y. If the worst-off individuals are equally well off, the second-worst-off individuals are compared and so on. If individuals of all ranks are equally well off, then x is socially as good as y.

Suppose that all individuals other than i and j are equally well off at x, y, and z in Figure 2.5. Of the two concerned persons, i is the worst-off individual at both x and y, while j is the best-off individual at x and y. Since i is better off at x than at y, x is socially better than y according to the LR.

The following two conditions deal with basic two-person situations where well-being levels of the concerned individuals can be compared. The second one is due to Hammond (1976: p. 795).

SO $\succsim_S$ satisfies **Anonymity (A)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \sim^* j_y \succ^* j_x \sim^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, it is the case that $x \sim_S y$.

SO $\succsim_S$ satisfies **Hammond Equity (HE)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y \succ^* j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, it is the case that $y \succsim_S x$.

If i and j switch their well-being levels from x to y, A requires that x is socially as good as y (left side of Figure 2.6). HE is concerned with a basic two-person situation where i is better off than j (right side of Figure 2.6). The starting level of i's gain (red line) is greater than the end level of j's gain (yellow line). According to HE, y is socially at

[15]The stated definition of the LR will also cover the variable population extension in section 2.4.2.

least as good as x. The gain of the worse off individual is socially at least as valuable as the gain of the better off individual.

Figure 2.6: Anonymity and Hammond Equity

Together with SP, A and HE characterize the LR. The result employs two richness assumptions. One can always find a state where j is as well of as i at x. One can also always find a well-being level in between two well-being levels of an individual.

R2: If ILC obtains, then, for all $i, j \in N$ and $x \in X$, there exists $y \in X$ with $j_y \sim^* i_x$.

R3: For all $i \in N$ and $x, y \in X$ with $i_x \succ^* i_y$, there exist $j \in N$ and $z \in X$ with $i_x \succ^* j_z \succ^* i_y$.[16]

Proposition 2: Suppose R2-R3 and ILC hold. SO $\succsim_S$ satisfies SP, A, and HE if and only if $\succsim_S = \succsim_L$.

The proof of Proposition 2 can be illustrated with Figure 2.5. Due to HE, z is socially at least as good as y. By SP, x is socially better than z. Social transitivity yields that x is socially better than y. That is precisely what the LR states.

FE offers an unconditional justification of A. The gain of i (blue line in Figure 2.6) and the gain of j (green line) have the same starting level. Due to the consistency conditions on $\succsim$, they also have the same magnitude. There is thus no reason for x to be socially better than y or the other way around. Indeed, FE implies A no matter which informational assumptions obtain.

Lemma 2: If SO $\succsim_S$ satisfies FE, then $\succsim_S$ satisfies A.

HE can be criticized if the gain of the better off individual i (red line in Figure 2.6) is greater than the gain of the worse off individual j (yellow line). But this presupposes that interpersonal gain comparisons are meaningful. If they are not, FE provides the basis for HE. Since i's gain neither has a smaller starting level nor a greater magnitude

[16]Together, ILC, R2, and R3 have a significant structural implication. If there exists an individual with at least two well-being levels, then all individuals have infinitely many well-being levels.

than j's gain (due to the absence of gain comparisons), there is no reason to socially favor x over y.

Lemma 3: Suppose MIGC holds. If SO $\succsim_S$ satisfies FE, then $\succsim_S$ satisfies HE.

The combination of Proposition 2 and Lemmas 2-3 leads to a characterization and justification of the LR in terms of SP and FE for the informational environment where there are interpersonal level but no gain comparisons.

Theorem 2: Suppose R2-R3, ILC, and MIGC hold. SO $\succsim_S$ satisfies SP and FE if and only if $\succsim_S = \succsim_L$.

2.3.3 Gain Without Level Comparisons

In the third informational environment, interpersonal comparisons of intrapersonal well-being gains are possible, but interpersonal well-being level comparisons are not. Property NILC is combined with the next condition.

Interpersonal Difference Comparability (IDC): For all $i, j \in N$ and $x, y, z, w \in X$, $(i_x, i_y) \succsim (j_z, j_w)$ or $(j_z, j_w) \succsim (i_x, i_y)$.[17]

Figure 2.7 illustrates interpersonal comparisons of intrapersonal gains. Suppose that j's gain from x (z) to y (blue line) is as great as i's gain from y to z (yellow line). Suppose that it is also as great as i's gain from z to x (red line). It is possible to measure i's gains in units of j's gain. The magnitude of i's gains from y to z and from z to x is one unit gain, respectively. The magnitude of i's overall gain from y to x (yellow plus red line) is two unit gains.

Two additional conditions imply the measurability of intrapersonal well-being gains in terms of a unit gain. The first one is a richness assumption.

R4: For all $i, j \in N$ and $x, y, z, w \in X$ with $i_x \succ^* i_y$, $j_z \succ^* j_w$, and $(i_x, i_y) \succ (j_z, j_w)$, there exist $x', y' \in X$ such that $(i_{x'}, i_y) \sim (j_z, j_w)$ and $(i_x, i_{y'}) \sim (j_z, j_w)$.

Suppose that i's gain is greater than j's gain. According to R4, there exists another gain of i that is as great as j's gain and has the same starting level as i's original gain.

[17] IDC states that all intrapersonal well-being differences are intra- and interpersonally comparable. Due to the consistency conditions on $\succsim$, this is the case if intrapersonal level comparisons and interpersonal comparisons of intrapersonal gains are possible. Formally, IDC is equivalent to PLC and the property that, for all $i, j \in N$ and $x, y, z, w \in X$ with $i_x \succ^* i_y$ and $j_z \succ^* j_w$, $(i_x, i_y) \succsim (j_z, j_w)$ or $(j_z, j_w) \succsim (i_x, i_y)$.

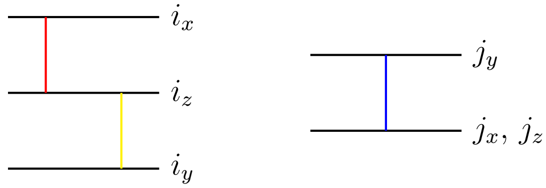

Figure 2.7: Interpersonal Gain Comparisons

There also exists a gain of i that is as great as j's gain and has the same end level as i's original gain.

The second condition makes sure that no well-being gain (loss) is infinitely larger than another well-being gain (loss). Consider $k \in N$. A sequence $x_1, x_2, ... \in X$ is a **standard sequence** of k if $(k_{x_2}, k_{x_1}) \not\sim (k_{x_1}, k_{x_1})$ and, for all x_m, x_{m+1} in the sequence, $(k_{x_{m+1}}, k_{x_m}) \sim (k_{x_2}, k_{x_1})$. It is **bounded** if there exist $y, z \in X$ such that, for all x_m in the sequence, $(k_y, k_z) \succ (k_{x_m}, k_{x_1}) \succ (k_z, k_y)$. The well-being differences of k are **Archimedean** if every bounded standard sequence of k is finite.[18]

Reduction to Archimedean Differences (RAD): If IDC obtains, then there exists $k \in N$ with Archimedean well-being differences such that, for all $i \in N$ and $x, y \in X$, there exist $z, w \in X$ with $(i_x, i_y) \sim (k_z, k_w)$.

RAD has the effect that each intrapersonal well-being difference can be reduced to an Archimedean difference of individual k with the same magnitude. Together with R4 and IDC, the condition secures that difference comparisons have a utility representation.

Lemma 4: Suppose R4, RAD, and IDC hold. There exists $u : N \times X \to \mathbb{R}$ such that, for all $i, j \in N$ and $x, y, z, w \in X$, $(i_x, i_y) \succsim (j_z, j_w) \Leftrightarrow u(i_x) - u(i_y) \geq u(j_z) - u(j_w)$. For every $v : N \times X \to \mathbb{R}$ that represents difference comparisons in this way, there exist $(\lambda_i)_{i \in N} \in \mathbb{R}^N$ and $\mu \in \mathbb{R}_+$ such that, for all $i \in N$ and $x \in X$, $v(i_x) = \lambda_i + \mu \cdot u(i_x)$.[19]

Lemma 4 implies that interpersonal comparisons of intrapersonal well-being differences can be represented by corresponding comparisons of utility differences under utility profile u. It moreover implies that there exist unique ratios between well-being differences which measure how many times one difference "fits into" another difference.[20] Once a

[18] For example, suppose that $(k_{x_2}, k_{x_1}) \succ (k_{x_1}, k_{x_1})$, so that (k_{x_2}, k_{x_1}) is a gain. Difference (k_{x_m}, k_{x_1}) is the $(m-1)$-fold concatenation of that gain. In this sense, gain (k_y, k_z) is infinitely larger than gain (k_{x_2}, k_{x_1}) if the sequence is not finite.

[19] Here, $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda > 0\}$.

[20] The ratio between i's well-being difference between x and y and j's non-zero well-being difference

unit well-being gain is fixed, each well-being difference can be numerically measured in terms of its ratio to the unit gain. In Figure 2.7, the ratio between i's gain from y to x and j's gain from x to y is 2 because the former consists of two parts that are as great as the latter, respectively.

Because intrapersonal well-being differences can be measured in terms of a unit gain, it becomes possible to add them up in a meaningful way. The Utilitarian SO ranks states x and y according to the sum of individuals' well-being differences between x and y.

Take $u : N \times X \to \mathbb{R}$ that represents interpersonal comparisons of intrapersonal differences in the sense of Lemma 4. The **Utilitarian Relation (UR)** $\succsim_U$ over X specifies, for all $x, y \in X$, $x \succsim_U y \Leftrightarrow \sum_{i \in N} \big(u(i_x) - u(i_y)\big) \geq 0$.[21]

In Figure 2.7, suppose that j's gain is the unit gain and that all individuals other than i and j are equally well off at x, y, and z. The well-being difference of i between x and y is two unit gains, while the well-being difference of j between x and y is minus one unit gain. This yields a net difference of one unit gain, meaning that x is socially better than y according to the UR. In contrast, the net difference between z and y is zero, so that they rank equally under the UR. This is in line with the next condition.

SO $\succsim_S$ satisfies **Gain Equity (GE)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \sim (j_y, j_x)$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, it is the case that $x \sim_S y$.

GE is illustrated in Figure 2.8. It is concerned with a basic two-person situation where i's gain (red line) is as great as j's gain (blue line). According to GE, x is socially as good as y, reflecting an equal social value of the two gains. With or without interpersonal level comparisons, SP and GE characterize the UR given the conditions of Lemma 4.[22]

Proposition 3: Suppose R4, RAD, and IDC hold. SO $\succsim_S$ satisfies SP and GE if and only if $\succsim_S = \succsim_U$.

The derivation of the UR can be illustrated with Figure 2.7. Individual i's gain from

between z and w is indeed a representation-invariant property. For all utility profiles u and v in the sense of Lemma 4, there exists $\mu \in \mathbb{R}_+$ such that $\frac{v(i_x)-v(i_y)}{v(j_z)-v(j_w)} = \frac{\mu \cdot (u(i_x)-u(i_y))}{\mu \cdot (u(j_z)-u(j_w))} = \frac{u(i_x)-u(i_y)}{u(j_z)-u(j_w)}$.

[21]In line with the discussion, the UR is well-defined given the uniqueness property of u from Lemma 4. Every $v \in \mathbb{R}^{N \times X}$ that represents interpersonal comparisons of intrapersonal differences in the sense of Lemma 4 has an associated $\mu \in \mathbb{R}_+$ such that $\sum_{i \in N} \big(v(i_x) - v(i_y)\big) \geq 0 \Leftrightarrow \sum_{i \in N} \mu \cdot \big(u(i_x) - u(i_y)\big) \geq 0 \Leftrightarrow \sum_{i \in N} \big(u(i_x) - u(i_y)\big) \geq 0$.

[22]Interestingly, R4 plays a double role for Proposition 3. It is used to derive the utility representation and, in turn, to derive the UR.

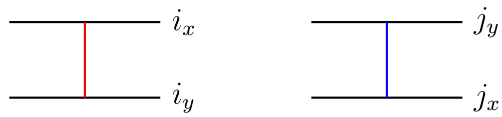

Figure 2.8: Gain Equity

y to z is as great as individual j's gain from z to y. Due to GE, this implies that z is socially as good as y. By SP, x is socially better than z. By social transitivity, that leads to x being socially better than y. This coincides with the utilitarian assessment of x and y.

FE provides a conditional justification of GE for the case where there are no interpersonal level comparisons. In Figure 2.8, one can argue against the equal social value of the two gains if one of the individuals is worse off, meaning that her gain has a smaller starting level. But without level comparisons, there is no reason to socially favor x over y or the other way around.

Lemma 5: Suppose NILC holds. If SO $\succsim_S$ satisfies FE, then $\succsim_S$ satisfies GE.

Taken together, Proposition 3 and Lemma 5 yield a characterization and justification of the UR with SP and FE in the informational environment of gain without level comparisons.

Theorem 3: Suppose R4, RAD, NILC, and IDC hold. SO $\succsim_S$ satisfies SP and FE if and only if $\succsim_S = \succsim_U$.

Given R4 and IDC, all intrapersonal well-being gains between neighboring well-being levels have the same magnitude within and between individuals (Appendix 2.A.4, Lemma A.4.2). Supposing that they connect neighboring levels, the three colored gains in Figure 2.7 must thus indeed be equal. This leads to an interesting connection to the following version of the Borda ranking.

Suppose $|\{[i_x] \mid x \in X\}| < \infty$ for each $i \in N$. The **Level Borda Relation (LBR)** $\succsim_B$ over X specifies, for all $x, y \in X$, $x \succsim_B y \Leftrightarrow \sum_{i \in N} \big(|\{[i_z] \mid i_x \succ^* i_z\}| - |\{[i_z] \mid i_y \succ^* i_z\}|\big) \geq 0$.

The LBR is only defined if all individuals have finitely many well-being levels. For individual i, the score of state x is the number of i's well-being levels below x. States x and y are socially ranked according to the sum of individual score differences between x and y. In Figure 2.7, i's score difference between x and y is 2 and j's score difference between x and y is -1. Accordingly, x is socially better than y under the LBR.

Given R4, IDC, and finite individual well-being levels, the UR coincides with the LBR. That is so because the LBR assigns a score difference of 1 to the unit well-being gain between neighboring levels. This leads to the following joint characterization.

Corollary 1: Suppose R4, RAD, NILC, and IDC hold. Suppose that $|\{[i_x] \mid x \in X\}| < \infty$ for each $i \in N$. SO $\succsim_S$ satisfies SP and FE if and only if $\succsim_S = \succsim_U = \succsim_B$.[23]

2.3.4 Level and Gain Comparisons

This subsection examines a general approach to social evaluation and characterizes it in the environment where interpersonal well-being level and gain comparisons are possible. The social evaluation of states x and y can be split into two stages. First, an individual evaluation difference between x and y is assigned to each person. Second, x and y are ranked according to the sum of individual evaluation differences. The UR is a prime example for this procedure, equating the individual evaluation difference with the individual's utility difference. The following class is made up of SOs following the two-stage approach. To the knowledge of the author, it has not been introduced before. Let $[x] = ([k_x])_{k \in N}$ be the tuple of individuals' well-being levels at state x.

Suppose PLC holds. SO $\succsim_S$ is an **Additive Welfarist Relation (AWR)** if there exists $d : N \times \{[x] \mid x \in X\}^2 \to \mathbb{R}$ such that
(a) for all $i \in N$ and $x, y \in X$,
$d(i, [x], [y]) \geq 0 \Leftrightarrow i_x \succsim^* i_y$, and
$d(i, [x], [y]) = -d(i, [y], [x])$.
(b) for all $i, j \in N$ and $x, y, z, w \in X$ with $\{x, y\} = \{z, w\}$, $i_x \succ^* i_y$, $j_z \succ^* j_w$, $(j_z, j_w) \succsim (i_x, i_y)$, and $i_y \succsim^* j_w$, it is true that $d(j, [z], [w]) \geq d(i, [x], [y])$.
(c) for all $x, y \in X$, $x \succsim_S y \Leftrightarrow \sum_{i \in N} d(i, [x], [y]) \geq 0$.

An AWR ranks states by summing individual evaluation differences that satisfy specific properties. Individual i's evaluation difference between x and y is $d(i, [x], [y])$. It only depends on tuples of well-being levels at x and y, and not on any non-well-being properties of states. By (a), the sign of individual evaluation differences is in line with intrapersonal level comparisons.

[23]In the statement of Corollary 1, RAD can be weakened. It does not have to be assumed that k's well-being differences are Archimedean. This is anyway the case if k has finitely many well-being levels.

The definition of an AWR does not presuppose an informational assumption on $\succsim$ in addition to PLC. However, property (b) is automatically satisfied unless according interpersonal level and gain comparisons are possible. Consider states x and y such that i realizes a gain from y to x, while j realizes a gain from y to x or from x to y. Individual j's gain is at least as great as individual i's gain and has an equal or smaller starting level. By (b), the positive evaluation difference assigned to j's gain is at least as great the positive evaluation difference assigned to i's gain. This is in line with the following version of the famous Pigou-Dalton condition.

SO $\succsim_S$ satisfies **Strong Pigou-Dalton (SPD)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, the following holds: If $(j_y, j_x) \succsim (i_x, i_y)$ and $i_y \succsim^* j_x$, then $y \succsim_S x$.[24]

Property SPD is illustrated in Figure 2.9. Take a basic two-person situation where j's gain (blue line) is at least as great as i's gain (red line) and has an equal or smaller starting level. According to SPD, y is socially at least as good as x. Together with SP, SPD characterizes the class of AWRs.

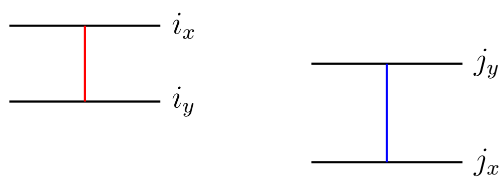

Figure 2.9: Strong Pigou-Dalton

Proposition 4: Suppose PLC holds. SO $\succsim_S$ satisfies SP and SPD if and only if $\succsim_S$ is an AWR.

Property (b) of an AWR reflects SPD by letting the social value of j's gain be at least as great as the social value of i's gain in Figure 2.9. However, (b) is much more general than SPD. It applies no matter whether all other individuals are equally well off at the considered states or not. That is why the proof of Proposition 4 is quite complicated.

FE provides an unconditional justification for SPD. Because i's gain in Figure 2.9 neither has a smaller starting level nor a greater magnitude than j's gain, there is no reason to socially favor x over y. Accordingly, FE implies SPD no matter which informational

[24]SPD implies A. The standard version of Pigou-Dalton is weaker (see d'Aspremont and Gevers, 2002: p. 506). Under it, the second sentence becomes "If $(j_y, j_x) \sim (i_x, i_y)$ and $i_y \succsim^* j_y$, then $y \succsim_S x$".

assumptions apply. If level and gain comparisons are complete, the reverse implication holds as well.

Lemma 6: Suppose ILC and IDC hold. SO $\succsim_S$ satisfies FE if and only if $\succsim_S$ satisfies SPD.

In general, not all AWRs satisfy FE. Given ILC and IDC, all of them do so. This leads to a characterization of the AWR class with SP and FE in the informational environment of interpersonal level and gain comparability.

Theorem 4: Suppose ILC and IDC hold. SO $\succsim_S$ satisfies SP and FE if and only if $\succsim_S$ is an AWR.

Neither the definition of an AWR nor the characterizations of the AWR class require difference relation $\succsim$ to be representable by a utility profile. However, if ILC and R2 are added to the properties of Lemma 4, a utility representation will exist.

Lemma 7: Suppose R2, R4, RAD, ILC, and IDC hold. There exists $u : N \times X \to \mathbb{R}$ such that, for all $i, j, k, l \in N$ and $x, y, z, w \in X$, $(i_x, j_y) \succsim (k_z, l_w) \Leftrightarrow u(i_x) - u(j_y) \geq u(k_z) - u(l_w)$. For every $v : N \times X \to \mathbb{R}$ that represents difference comparisons in this way, there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that, for all $i \in N$ and $x \in X$, $v(i_x) = \lambda + \mu \cdot u(i_x)$.

Given the assumptions of Lemma 7, utility profile u represents interpersonal comparisons of interpersonal well-being differences and, specifically, interpersonal well-being level comparisons. It also represents the uniquely determined ratios between interpersonal well-being differences.[25] Each tuple of well-being levels $[x] = ([k_x])_{k \in N}$ is represented by the vector of utility levels $(u(k_x))_{k \in N}$. Individual i's evaluation difference between x and y under an AWR can thus likewise be expressed as a function of $(u(k_x))_{k \in N}$ and $(u(k_y))_{k \in N}$.

Individual evaluation differences between x and y can depend on the whole tuples of well-being levels at x and y. Moreover, there are no consistency conditions between evaluation differences of distinct pairs of tuples. For these reasons, the class of AWRs is quite rich no matter which informational assumptions obtain. The UR and the LR are both AWRs. They satisfy SP and FE (and thus SPD) whenever they are defined. The UR sets $d(i, [x], [y]) = u(i_x) - u(i_y)$. A d-function in the sense of the LR assigns much larger evaluation differences between smaller well-being levels than between greater

[25] For every profile v in the sense of Lemma 7, this ratio between two well-being differences is $\frac{v(i_x)-v(j_y)}{v(k_z)-v(l_w)} = \frac{u(i_x)-u(j_y)}{u(k_z)-u(l_w)}$.

well-being levels, so that the former always outweigh the latter. The additive welfarist properties of an AWR are common ground of both Utilitarianism and Leximin.

The UR and the LR only come into conflict if both are well-defined. That is the case given the assumptions of Lemma 7. However, in this informational environment, many other AWRs are also candidates for social evaluation. For instance, a Generalized Utilitarian Relation $\succsim_S$ is characterized by an increasing and concave function $g : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in X$, $x \succsim_S y \Leftrightarrow \sum_{i \in N} \big(g(u(i_x)) - g(u(i_y))\big) \geq 0$. That is, $d(i, [x], [y]) = g(u(i_x)) - g(u(i_y))$. Under the UR, g is the identity. Generalized Utilitarian Relations and Leximin have in common that they do not let the social evaluation of states depend on well-being levels of unconcerned individuals. However, not all AWRs satisfy this Separability condition. SP and FE imply the property if either level or gain comparisons are possible (as the LR and the UR satisfy it), but not if both are possible.[26]

2.3.5 A General Impossibility

Sections 2.3.1-2.3.4 have dealt with four informational environments that correspond to major comparability assumptions from the classical theory of social welfare. In these environments, interpersonal well-being level comparisons are either impossible or complete. The same is true for interpersonal well-being gain comparisons. This leads to the question what implications SP and FE have for a general incomplete well-being difference relation $\succsim$ with partial level and gain comparability. Unfortunately, the answer is negative given the following mild incomparability property.

Weak Two-Person Incomparability (WTPI): There exist $i, j \in N$ and $x, y, z \in X$ with $i_x \succ^* i_z$, $i_x \succ^* i_y$, $i_z \succ^* i_y$, $j_y \succ^* j_x$, $j_y \succ^* j_z$, $j_x \sim^* j_z$, $(i_x, i_y) \not\succsim (j_y, j_x)$, $(j_y, j_x) \not\succsim (i_x, i_y)$, $i_y \not\succsim^* j_x$, $j_x \not\succsim^* i_y$, $(i_z, i_y) \not\succsim (j_y, j_z)$, $(j_y, j_z) \not\succsim (i_z, i_y)$, $i_y \not\succsim^* j_z$, $j_z \not\succsim^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, $k_y \sim^* k_z$, and $k_x \sim^* k_z$.

A two-person, three-state situation in the sense of WTPI is illustrated in Figure 2.10.

[26] SO $\succsim_S$ satisfies Separability (S) if, for all $x, y, z, w \in X$ and $M \subseteq N$ such that, for all $i \in M$, $i_x \sim^* i_z$ and $i_y \sim^* i_w$, and, for all $i \in N \setminus M$, $i_x \sim^* i_y$ and $i_z \sim^* i_w$, it is the case that $x \succsim_S y \Leftrightarrow z \succsim_S w$. As AWR evaluation differences depend on tuples of well-being levels, they can depend on well-being levels of unconcerned individuals. An example of an AWR that violates S is the Gini SO. Take u in the sense of Lemma 7. For $x, y \in X$, take bijections $\pi, \mu : R(N) \to N$ such that, for all $r, t \in R(N)$ with $r \geq t$, $u(\pi(t)_x) \geq u(\pi(r)_x)$ and $u(\mu(t)_y) \geq u(\mu(r)_y)$. According to Gini, $x \succsim_S y \Leftrightarrow \sum_{r \in R(N)} (2r - 1)\big(u(\pi(r)_x) - u(\mu(r)_y)\big) \geq 0$.

All individuals but i and j are equally well off at x, y, and z. While i is better off at x than at z and better off at z than at y, j is better off at y than at x and equally well off at x and z. Both the magnitude and the starting level of i's gain from y to x (red line) are incomparable to the magnitude and the starting level of j's gain from x to y (blue line). Likewise, the magnitude and the starting level of i's gain from y to z (yellow line) are incomparable to the magnitude and the starting level of j's gain from z to y (also blue line). Given WTPI, SP and FE are incompatible.

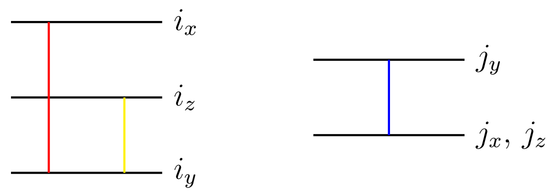

Figure 2.10: Weak Two-Person Incomparability

Theorem 5: Suppose WTPI holds. There exists no SO satisfying SP and FE.

The reasoning behind Theorem 5 is simple. Since i's gain from y to x and j's gain from x to y are incomparable, FE precludes that x is socially better than y or the other way around. By social completeness, this implies that x is socially as good as y. Analogously, FE and social completeness yield that y is socially as good as z. By social transitivity, it follows that x is socially as good as z. But by SP, x is socially better than z.

The argument is a generalization of the problem from section 2.3.1. In contrast to NILC and MIGC, WTPI does not state that all interpersonal comparisons are impossible. It just postulates that there exists one situation like in Figure 2.10. Theorem 5 holds given all kinds of more or less complete intra- and interpersonal level and gain comparisons in $\succsim$. Moreover, it employs none of the richness and consistency assumptions from previous sections. It even holds given intransitive intra- and interpersonal level and gain comparisons.[27] Accordingly, Theorem 5 is very general.

To avoid the negative conclusion of Theorem 5, one of its assumptions must be weakened. One could blame FE for the impossibility. Like SE, it induces that the social value of j's blue gain in Figure 2.10 is simultaneously equal to the social value of i's red gain and i's smaller yellow gain. This problem is avoided under the following equity

[27]The formulation of WTPI indeed allows for intransitivity of $\succsim$ and $\succsim^*$.

condition that only applies in basic two-person situations where both i's and j's gain are intrapersonally minimal.

SO $\succsim_S$ satisfies **Positional Equity (PE)** if, for all $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, the following holds: If, for all $z, w \in X$ with $i_z \succ^* i_w$, $(i_x, i_y) \not\succ (i_z, i_w)$, and, for all $z, w \in X$ with $j_z \succ^* j_w$, $(j_y, j_x) \not\succ (j_z, j_w)$, then $x \sim_S y$.

If there are intrapersonal level but no gain comparisons except for part-whole comparisons, SP and PE characterize the LBR. In that case, intrapersonally minimal gains are precisely those between neighboring well-being levels of an individual (Appendix 2.A.6, Lemma A.6.1).

Minimal Personal Gain Comparability (MPGC): For all $i \in N$ and $x, y, z, w \in X$ with $i_x \succ^* i_y$ and $i_z \succ^* i_w$, $(i_x, i_y) \succsim (i_z, i_w)$ implies $i_x \succsim^* i_z$ and $i_w \succsim^* i_y$.

Proposition 5: Suppose PLC and MPGC hold. Suppose that $|\{[i_x] \mid x \in X\}| < \infty$ for each $i \in N$. SO $\succsim_S$ satisfies SP and PE if and only if $\succsim_S = \succsim_B$.[28]

If there are no interpersonal well-being comparisons, PE is a reasonable weakening of FE. Once again, the two possible justifications for an unequal social ranking in a basic two-person situation (greater gain or smaller starting level for one individual) do not apply without interpersonal comparisons. The antecedent of PE even excludes a possible asymmetry with respect to the intrapersonal minimality of well-being gains. In contrast, the equal ranking conclusion of PE can be challenged in a basic two-person situation where magnitudes of gains and/or starting levels are comparable. FE only implies PE given NILC and MIGC.

Lemma 8: Suppose NILC and MIGC hold. If SO $\succsim_S$ satisfies FE, then $\succsim_S$ satisfies PE.

Overall, it is quite difficult to come up with intuitive and general alternatives to FE. As argued, the condition has strong normative appeal. The same is true for SP. To keep both properties, one might weaken the assumption that the social ranking is an ordering. But ideally, social evaluation should both be complete (to inform social choice) and transitive (to be in line with standards of rationality). Accordingly, section 2.4.3 will discuss a solution to the challenge of Theorem 5 which attacks WTPI.

[28]The LBR can likewise be characterized with SP and PE if intrapersonal gain comparisons are complete and R4 holds (Appendix 2.A.6, Proposition A.6.1).

2.4 Extensions

2.4.1 Uncertain Prospects

This subsection extends the previous analysis to social evaluation under uncertainty. Let X be a set of conceivable uncertain prospects for a fixed finite population N. Given each uncertain prospect $x \in X$, each individual $i \in N$ is certain to exist and faces some uncertain individual prospect.

Difference relation $\succsim$ over $(N \times X)^2$ now captures qualitative comparisons of qualitative ex ante well-being differences. Accordingly, $(i_x, j_y) \succsim (k_z, l_w)$ means that the ex ante well-being difference between i's prospect given x and j's prospect given y is at least as great as the ex ante well-being difference between k's prospect given z and l's prospect given w. Moreover, $i_x \succsim^* j_y$ means that i's prospect given x is at least as valuable as j's prospect given y.

For example, take the basic two-person situation for prospects that is illustrated in Figure 2.11. Suppose that i is certain to be a poor individual and j is certain to be a rich individual given both prospects x and y. Suppose that, compared to prospect y, individual i faces a certain income gain (say, \$100) under all circumstances given prospect x. Individual j faces the same certain income gain under all circumstances from x to y. An ethical observer might well judge that the ex ante well-being gain of poor i from y to x (red line) is greater than the ex ante well-being gain of rich j from x to y (blue line), meaning $(i_x, i_y) \succ (j_y, j_x)$. A preference foundation of ex ante comparisons will be offered in section 2.4.3.

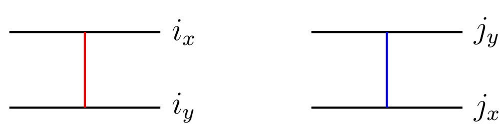

Figure 2.11: The Basic Two-Person Situation for Prospects

All conditions from sections 2.2-2.3 also apply to the prospect interpretation. SO $\succsim_S$ represents the social evaluation of prospects. According to SP, if each individual prospect is at least as valuable given prospect x as given prospect y, x is socially at least as good as y. If at least one individual prospect is more valuable, x is socially better than y. The reasoning of FE also translates to the prospect case. In Figure 2.11, x can only be socially better than y if i's ex ante gain from y to x is greater than j's ex ante gain from x to y or i's prospect given y is less valuable than j's prospect given x. FE is also normatively appealing in this context.

All results from section 2.3 go through under the prospect interpretation. The characterized SOs rank prospects. The analysis does not require that prospects have a specific structure. However, to illustrate social evaluation under uncertainty, one concrete model of prospects will be considered in the remainder of the subsection.

Suppose that $X = S^m$, where S is a set of conceivable states of the world with fixed finite population N and $m \in \mathbb{N}$. Prospect $x \in X$ is a tuple $x = (x_1, ..., x_m)$. Each entry represents a different circumstance and it is uncertain which circumstance obtains. If the first circumstance obtains, state of the world x_1 will be realized given x, and so on. State of the world $s \in S$ can be identified with $(s, ..., s) \in X$, that is, the prospect which realizes s given all circumstances.[29]

The theory of the chapter is compatible with different approaches how ex ante well-being levels of individuals given prospects relate to their ex post well-being levels at states of the world.[30] In the following, two examples will be discussed.

Expected Utility: Consider the informational environment where interpersonal ex ante level and gain comparisons are possible. Take $u : N \times X \to \mathbb{R}$ in the sense of Lemma 7. The evaluation of individual prospects takes the expected utility form if there exist probabilities $p_1, ..., p_m \in [0, 1]$ with $\sum_{r=1}^{m} p_r = 1$ such that, for all $i \in N$ and $x \in X$, $u(i_x) = \sum_{r=1}^{m} p_r \cdot u(i_{x_r})$, where i_{x_r} is a shorthand for $i_{(x_r,...,x_r)}$. Individual i's utility level given prospect x is the sum of i's utility levels at the states that x realizes under different circumstances, weighted by probabilities of circumstances.[31]

By Theorem 4, SP and FE imply that an AWR must be employed to aggregate individuals' expected utility levels into a social evaluation of prospects. For the ranking of the LR, ex ante level comparisons are critical. They reflect comparisons of individuals' expected utility levels, that is, $i_x \succsim^* j_y \Leftrightarrow \sum_{r=1}^{m} p_r \cdot u(i_{x_r}) \geq \sum_{r=1}^{m} p_r \cdot u(j_{y_r})$.

Ex ante gain comparisons reflect comparisons of expected utility differences. Under

[29] The model presupposes that states of the world are compatible with all circumstances.

[30] As before, the ex ante well-being level of individual i given prospect x can be identified with the class of individual-prospect tuples that are as valuable as i_x according to $\succsim^*$. The ex post well-being level of i at state of the world s can be identified with i's ex ante well-being level given prospect $(s, ..., s)$.

[31] The evaluation can be based on objective probabilities or subjective probabilities of the ethical observer.

the UR, $x \succsim_U y \Leftrightarrow \sum_{i \in N} \sum_{r=1}^{m} p_r \cdot \big(u(i_{x_r}) - u(i_{y_r})\big) \geq 0$. This Expected Utility Utilitarianism ranks prospects by summing individual expected utility differences. In the informational environment of ex ante gain without level comparisons, Theorem 3 leads to a characterization of Expected Utility Utilitarianism with SP and FE.

Maximin Utility: Again, consider the environment where interpersonal ex ante level and gain comparisons are possible and take u in the sense of Lemma 7. The evaluation of individual prospects takes the maximin utility form if, for all $i \in N$ and $x \in X$, $u(i_x) = min\{u(i_{x_1}), ..., u(i_{x_m})\}$.[32] Under it, the ex ante utility level of an individual given a prospect is simply the smallest ex post utility level of the individual that can be realized by the prospect. Due to Theorem 4, a SO satisfying SP and FE is an AWR that aggregates these maximin utility levels.

On the one hand, interpersonal ex ante level comparisons are reduced to ex post level comparisons given the maximin utility assumption, meaning $i_x \succsim^* j_y \Leftrightarrow min\{u(i_{x_1}), ..., u(i_{x_m})\} \geq min\{u(j_{y_1}), ..., u(j_{y_m})\}$. They are the input of the LR. In the environment of interpersonal level without gain comparability, SP and FE characterize the LR applied to maximin utility levels by Theorem 2.[33]

On the other hand, ex ante gain comparisons are reduced to ex post gain comparisons. The UR ranks prospects by summing individual maximin utility differences, that is, $x \succsim_U y \Leftrightarrow \sum_{i \in N} \big(min\{u(i_{x_1}), ..., u(i_{x_m})\} - min\{u(i_{y_1}), ..., u(i_{y_m})\}\big) \geq 0$. Due to Theorem 3, this Maximin Utility Utilitarianism is characterized by SP and FE if interpersonal gain comparisons are possible but interpersonal level comparisons are not.

2.4.2 Variable Populations

This subsection extends the previous analysis to variable populations. It focuses on social evaluation under certainty. Section 2.4.3 deals with the uncertainty case. Let N be a potential population of individuals. It can be infinite. Let X be a set of conceivable states of the world. Each state $x \in X$ has a finite population $N(x) \subseteq N$. Difference relation $\succsim$ is still defined over $(N \times X)^2$. It now also captures comparisons between existence and non-existence. The following property is concerned with these comparisons.

V1: For all $i \in N$ and $x, y \in X$ with $i \notin N(x) \cup N(y)$, there exists $z \in X$ with $i \in N(z)$

[32]Maximin utility is examined in Segal and Sobel (2002).

[33]This result can be stated without a utility representation of $\succsim^*$.

and $i_x \sim^* i_y \sim^* i_z$.

The fact that $i_x \sim^* i_y \sim^* i_z$ is taken to mean that i's existence at z is as valuable as i's non-existence (at x and y). One possible interpretation of this statement is that the quality of i's existence at z is "neutral" in some way to be defined. In that case, call i's well-being level at z the neutral level of i. Given an experientialist well-being conception, the neutral level is the one of being unconscious. Figure 2.12 illustrates V1. Suppose that i exists at w, z, and z', but not at x and y. Individual i's existence at z is as valuable as i's non-existence.

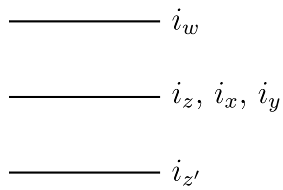

Figure 2.12: Comparisons between Existence and Non-Existence

Given V1, intra- and interpersonal level and difference comparisons of $\succsim$ involving non-existent individuals at states reduce to comparisons involving only existent individuals. For instance, take a basic two-person situation where i exists at x and y, while j exists at y but not at x. Take state x' such that j exists at x' and $j_{x'} \sim^* j_x$. Given the consistency assumptions on $\succsim$, it is the case that $(i_x, i_y) \succsim (j_y, j_x) \Leftrightarrow (i_x, i_y) \succsim (j_y, j_{x'})$ and $i_y \succsim^* j_x \Leftrightarrow i_y \succsim^* j_{x'}$. As populations of states are finite, V1 also implies the following property.

V2: For all $x, y \in X$, $|\{i \in N \mid i_x \succ^* i_y\}| < \infty$.

Given V2, all definitions of sections 2.2-2.3 go through for infinite N. The set $N_{s,xy}$ of individuals for whom x is better than y or the other way around is finite (given PLC), so that the LR is well-defined. The sum $\sum_{i \in N} \big(u(i_x) - u(i_y)\big)$ of the UR has only finitely many non-zero summands. Due to property (a) of an AWR, the same is true for its sum of individual evaluation differences.

Given V1, SP reduces the social evaluation of states with variable populations to an evaluation of states for the potential population. The social ranking of states x and y comes down to a social ranking of hypothetical states x' and y' with population N, where, for each i from N, i's existence at x' is as valuable as i's (non-)existence at x

and i's existence at y' is as valuable as i's (non-)existence at y.[34] In line with this, FE essentially applies to basic two-person situations where i and j exist at both x and y. As explained, the non-existence case reduces to this existence case.

All results from section 2.3 go through given V2 (and thus given V1). SP and FE lead to according (im)possibility results for different informational environments. Which variable population version of Utilitarianism and Leximin is characterized depends on the interpretation of the well-being level corresponding to an individual's non-existence. Under the neutral level interpretation, it represents a neutral quality of the individual's existence. If interpersonal level comparisons are possible, it is natural to assume that the neutral level is the same across individuals.[35]

Supposing that each individual's neutral level is represented by a utility level of 0 under u, the UR specifies $x \succsim_U y \Leftrightarrow \sum_{i \in N(x)} u(i_x) \geq \sum_{i \in N(y)} u(i_y)$. This is known as Total Utilitarianism. In line with that term, the other characterized SOs can likewise be called Total LR, Total SMR, Total LBR, and Total AWR under the neutral level interpretation.

Given interpersonal level and gain comparability and the neutral level interpretation, SP and FE imply what is known as the Repugnant Conclusion: A state with a large population of individuals with high well-being levels can be socially worse than a state with a larger population of individuals who are only slightly above the neutral level. This property can be avoided with an alternative interpretation.

According to the critical level interpretation, the well-being level corresponding to individual i's non-existence is the level that makes i's existence socially as valuable as i's non-existence. Under this interpretation, i's critical level can be different from i's neutral level. In Figure 2.12, i's critical level is i's level at z, but i's neutral level could be i's level at z'. If interpersonal level comparisons are possible, it is reasonable to assume that the critical level is the same across individuals (like the neutral level).

Supposing that each individual's critical level is represented by a utility level of 0 under u, the UR has the form stated above. In view of the different meaning, this version is known as Critical Level Utilitarianism. Under the critical level interpretation, the other SOs can in turn be called Critical Level LR, Critical Level SMR, Critical Level LBR, and Critical Level AWR. They are characterized by SP and FE given the according informational environments. Specifically, the Repugnant Conclusion is avoided given

[34] Given an infinite N, x' and y' are not elements of X.

[35] That is, for all $i, j \in N$ and $x, y \in X$ with $i \notin N(x)$ and $j \notin N(y)$, $i_x \sim^* j_y$.

interpersonal level and gain comparability if the critical level is greater than the neutral level.

Independently of the interpretation, the separability implications of SP and FE depend on the informational environment. The condition of Existence Independence requires that the social evaluation of states is independent of the existence of individuals who are equally well off at the states. SP and FE imply Existence Independence if there are either level or gain comparisons (as Total/Critical Level Leximin and Utilitarianism satisfy the property), but not if there are both.[36]

2.4.3 Preference Foundations of Well-Being Comparisons

This subsection covers the general variable population prospect case and discusses how well-being comparisons embodied in $\succsim$ can be reduced to preferences of individuals and of an ethical observer. Let N be a potential population of individuals and X a set of conceivable prospects. For prospect $x \in X$, let $N(x) \subseteq N$ be the finite set of individuals who have a chance to exist given x (that is, who are not certain not to exist given x). For each $i \in N$, let $S_i \subseteq X$ be the set of conceivable states of the world (certain prospects) where only i exists. That is, for $x \in S_i$, $N(x) = \{i\}$.

Difference relation $\succsim$ captures ex ante well-being difference and level comparisons. They are illustrated in Figure 2.13. In line with previous sections, $(i_x, j_y) \succsim (h_z, l_w)$ means that the ex ante well-being difference between i's prospect given x and j's prospect given y (red line) is at least as great as the ex ante well-being difference between h's prospect given z and l's prospect given w (blue line). Moreover, $i_x \succ^* j_y$ means that i's prospect given x is more valuable than j's prospect given y.

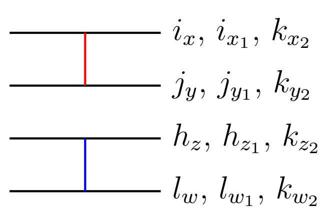

Figure 2.13: Preference Foundations of Well-Being Comparisons

[36]In the Lemma 7 environment, some possibilist rank-additive SOs from Pivato (2020) satisfy SP and FE (SPD), but not Existence Independence.

Together with V1/V2, all definitions and results from sections 2.2-2.3 go through and have a variable population prospect interpretation. The critical question is how to determine the comparisons in $\succsim$. In the following, a preference foundation will be given for them.

Let $(\succsim_i)_{i \in N}$ be a profile of individual preference relations over X. Each $\succsim_i$ captures the preferences over prospects that individual i has in some reference situation. It is assumed that i's preferences capture the intrapersonal evaluation of i's prospects.

P1: For all $i \in N$ and $x, y \in X$, $i_x \succsim^* i_y \Leftrightarrow x \succsim_i y$.

According to P1, i's prospect given x is at least as valuable as i's prospect given y if and only if i weakly prefers x over y.[37] In Figure 2.13, i's prospect given x is as valuable as i's prospect given x_1. This corresponds to the fact that i is indifferent between x and x_1.

Let $\succsim_S$ be a SO that captures the preferences of an ethical observer over X. That is, x is socially at least as good as y ($x \succsim_S y$) if and only if the observer weakly prefers x over y. In welfarist social evaluation, the comparison of states of the world x and y comes down to a comparison of individuals' well-being levels at x and y. Specifically, if both x and y are one-person states, the social evaluation of x and y reduces to comparing the well-being level of the individual who exists at x to the well-being level of the individual who exists at y. The following assumption is in line with this.

P2: For all $i, j \in N$, $x \in S_i$, and $y \in S_j$, $i_x \succsim^* j_y \Leftrightarrow x \succsim_S y$.

Because the observer's preferences over one-person states are in line with well-being level comparisons, the latter can be inferred from the former. In Figure 2.13, suppose that x_1 is a one-person state where only i exists and y_1 is a one-person state where only j exists. Individual i is better off at x_1 than individual j at y_1 (i's certain prospect given x_1 is more valuable than j's certain prospect given y_1). This corresponds to the observer's preference for x_1 over y_1.

Intrapersonal well-being difference comparisons of an individual can be reduced to preferences of the individual over concatenated alternatives. The next property captures this idea.

[37] One issue is that i might not be certain to exist given x and/or y. However, i can nevertheless form preferences over x and y in some (idealized) reference situation. Of course, i must exist in the reference situation. Moreover, i's preferences over states can be in line with different well-being conceptions. They might be based on i's experiences or i's objective goods at these states. To be in line with consistency conditions on $\succsim$, individuals' preference relations must be transitive.

P3: There exist $k \in N$ and $c : S_k \times S_k \to X$ such that, for all $x, y, z, w \in S_k$, $(k_x, k_y) \succsim (k_z, k_w) \Leftrightarrow c(x, w) \succsim_k c(y, z)$.

P3 is concerned with intrapersonal well-being difference comparisons of individual k over states of the world where only k exists. Mapping c is a concatenation operation. Prospect $c(x, w)$ is the concatenation of states x and w. One possibility is that $c(x, w)$ is a prospect where x is realized with probability 0.5 and w is realized with probability 0.5. According to P3, k's well-being difference between x and y is at least as great as k's well-being difference between z and w if and only if k weakly prefers the concatenation of x and w over the concatenation of y and z.

In Figure 2.13, suppose that $x_2, y_2, z_2, w_2 \in S_k$. Individual k prefers x_2 over y_2 and z_2 over w_2. P3 presupposes that k forms her preference over $c(x_2, w_2)$ and $c(y_2, z_2)$ by comparing the well-being gain in the first component from $c(y_2, z_2)$ to $c(x_2, w_2)$ (red line) to the well-being gain in the second component from $c(x_2, w_2)$ to $c(y_2, z_2)$ (blue line). Individual k weakly prefers $c(x_2, w_2)$ over $c(y_2, z_2)$ if and only if the former is at least as great as the latter. This identification procedure for difference comparisons assumes a component-wise assessment of prospects.[38] However, P3 is fairly weak as it only states that there is one individual whose well-being difference comparisons can be inferred from preferences over concatenated one-person states.

Taken together, P1-P3 imply that the interpersonal comparison of interpersonal ex ante well-being differences can be reduced to k's preference over concatenated one-person states. Analogously to x_1 and y_1, suppose that z_1 is a state where only h exists and w_1 is a state where only l exists. The fact that $i_x \sim^* i_{x_1}$ and $i_{x_1} \sim^* k_{x_2}$ reflect that i is indifferent between x and x_1 and the ethical observer is indifferent between one-person states x_1 and x_2. Likewise, $j_y \sim^* k_{y_2}$, $h_z \sim^* k_{z_2}$, and $l_w \sim^* k_{w_2}$ have analogous preference foundations.

Because the respective starting and end levels coincide, the consistency assumptions on $\succsim$ imply that the well-being difference between i's prospect given x and j's prospect given y (red line) is at least as great as the well-being difference between h's prospect given z and l's prospect given w (blue line) if and only if k's gain from y_2 to x_2 (again, red line) is at least as great as k's gain from w_2 to z_2 (again, blue line). The latter holds if and only if k weakly prefers prospect $c(x_2, w_2)$ over prospect $c(y_2, z_2)$.

[38]The lottery interpretation also assumes that k is risk-neutral in well-being. Individual k might be equally well off at y and z, but not at x and w. But if the gain from y to x is as great as the loss from z to w, k is indifferent between $c(x, w)$ and $c(y, z)$.

The discussed procedure gives a preference foundation to all comparisons in $\succsim$. The only assumption still needed is that indifferent states in the sense of Figure 2.13 always exist.

P4: Take $k \in N$ in the sense of P3. For all $i \in N$ and $x \in X$, there exist $y \in S_i$ and $z \in S_k$ with $x \sim_i y$ and $y \sim_S z$.

For $i \in N$ and $x \in X$, let $s(i_x) \in S_k$ be a state such that there exists $y \in S_i$ with $x \sim_i y$ and $y \sim_S s(i_x)$. By P4, such $s(i_x)$ exists. In Figure 2.13, $s(i_x) = x_2$, $s(j_y) = y_2$, $s(h_z) = z_2$, and $s(l_w) = w_2$.

Proposition 6: Suppose P1-P4 hold. Take $k \in N$ in the sense of P3-P4. For all $i, j, h, l \in N$ and $x, y, z, w \in X$, $(i_x, j_y) \succsim (h_z, l_w) \Leftrightarrow c(s(i_x), s(l_w)) \succsim_k c(s(j_y), s(h_z))$. Moreover, $\succsim$ is complete.

In view of the general impossibility result of Theorem 5, it is significant that difference relation $\succsim$ in the sense of Proposition 6 is complete. The critical assumption for this result is that preferences of the ethical observer are complete ($\succsim_S$ is an ordering). It implies that interpersonal level comparisons in the sense of P2 are complete. To be in line with the other assumptions, individual preference relations (specifically, $\succsim_k$) must also be complete. Due to Theorem 4, SP and FE characterize the AWR class for complete $\succsim$. Given the additional assumptions of Lemma 7, $\succsim$ also has a utility representation.[39] In any case, a rich number of SOs is available for aggregation.

2.5 Relation to the Literature

This section discusses the relation of the established theory to the literature. Sections 2.5.1-2.5.2 are concerned with the classical theory of social welfare in terms of Social Welfare Functions and Social Welfare Functionals. Section 2.5.3 discusses relations to other fields. The classical theory is concerned with the base interpretation from sections 2.2-2.3 where X is the set of all conceivable states of the world with fixed finite population N.

[39] Specifically, the reduction of interpersonal difference comparisons to k's intrapersonal difference comparisons in the sense of Proposition 6 is in line with RAD and the according method to construct a utility representation.

2.5.1 Social Welfare Functions

The starting point of modern social choice theory is Arrow (1951, 1963). An Arrovian **Social Welfare Function (SWF)** translates profiles of individual preference orderings over X into SOs over X. A preference profile can be conceived as a special case of a difference relation $\succsim$. If intrapersonal well-being level comparisons coincide with individuals' preferences over X (P1), there are only minimal intrapersonal gain comparisons (MPGC), and there are no interpersonal comparisons (NILC and MIGC), the information contained in $\succsim$ corresponds to the information contained in a preference profile.[40]

Arrow's Impossibility Theorem shows that there is no SWF with an unrestricted domain of profiles (UD) that ranks a pair of states by only considering preferences over the pair (Independence of Irrelevant Alternatives, IIA), respects unanimous strict preferences (Weak Pareto, WP), and is not dictatorial (ND).[41] The result and related SWF theory face the **multi-profile problem** (see Blackorby et al., 2006). In terms of the present chapter, the UD assumption means that multiple difference relations are considered. The associated social rankings are connected via the inter-profile (inter-relation) condition IIA. Given one profile, i is better off at x than at y, while it is the other way around given another one. There is a problem with variable individual well-being levels under some well-being conceptions. If x and y are comprehensive states of the world, they contain fixed experiences for individual i, respectively. Under an experientialist conception, the ranking of x and y for i must thus be fixed.

The present chapter solves the multi-profile problem. There are several differences between Theorem 1 (as well as Proposition 1) and Arrow's Impossibility Theorem. The substantive content of FE is stronger than the one of ND. However, Theorem 1 uses R1 instead of UD and does not employ an analog of the contested IIA condition.[42] More generally, the present chapter solves the **interpersonal comparison problem**

[40] Even in that case, $\succsim$ contains part-whole gain comparisons due to the consistency conditions.

[41] For reviews of SWF theory and related impossibility results, see Campbell and Kelly (2002), Cato (2010), and Sen (2017: chapters A1*, A2*). A recent contribution on Arrovian aggregation is Brandl and Brandt (2020).

[42] There is a literature formulating versions of Arrow's Impossibility Theorem for a single profile of preference relations (Kemp and Ng, 1976; Parks, 1976; Pollak, 1979; Blackorby et al., 1990). They employ the following condition: For all $x, y, z, w \in X$ such that, for all $i \in N$, $i_x \succsim^* i_y \Leftrightarrow i_z \succsim^* i_w$ and $i_y \succsim^* i_x \Leftrightarrow i_w \succsim^* i_z$, it is the case that $x \succsim_S y \Leftrightarrow z \succsim_S w$. See Fleurbaey and Mongin (2005) for a general discussion. The property replaces IIA and avoids the multi-profile problem. But it is strong and controversial even without interpersonal comparisons. None of the results of this chapter use it.

of SWF theory: The SWF framework is not rich enough to model possible interpersonal comparisons.

There is a literature solving the level part of the interpersonal comparison problem (Sen, 1970: chapter 9*, 1974; Hammond, 1976; Roberts, 1980a; Pivato, 2013a). It considers generalized SWFs whose input are relations over $N \times X$ reflecting intra- and interpersonal well-being level comparisons. A central characterization of Leximin in this framework is due to Hammond (1976: Theorem 7.2). It employs an unrestricted domain of level relations over $N \times X$ (UD), an interpersonal version of IIA, SP, HE, and a multi-person version of A (Suppes Indifference).[43] Instead, Proposition 2 of the present chapter offers a single relation characterization of Leximin.

2.5.2 Social Welfare Functionals

Both Arrovian and generalized Social Welfare Functions are not rich enough to model difference comparisons. Up to the present day, the common theory for the analysis of different assumptions on level and gain comparability is the one of Social Welfare Functionals (due to Sen, 1970: chapter 8*). Reviews of that theory are offered in Blackorby et al. (1984), Bossert (1991), Blackorby et al. (2002, 2005), d'Aspremont and Gevers (2002), Bossert and Weymark (2004), Weymark (2016), and Sen (2017: chapter A3*).

In the theory of Social Welfare Functionals, utility profiles form the input of aggregation. Profile u has an associated class $V(u) \subseteq \mathbb{R}^{N\times X}$ of utility profiles that are informationally equivalent to u. An intra- or interpersonal well-being level or difference comparison is meaningful if and only if it holds under all profiles from $V(u)$. A **Social Welfare Functional (SWFL)** on domain $D \subseteq \mathbb{R}^{N\times X}$ maps utility profile $u \in D$ into SO $\succsim_u$ over X. It represents the social evaluation of states given u. Informationally equivalent profiles must be mapped into the same SO.[44] While attacking the interpersonal comparison problem of SWF theory, SWFL theory still faces the multi-profile problem. Core results presuppose $D = \mathbb{R}^{N\times X}$, the adapted version of UD. Moreover, SWFL theory faces two new problems.

[43] To be precise, even Two-Person Suppes Indifference (Hammond, 1979: p. 1132) is a stronger version of A as it also applies if all individuals are indifferent between two alternatives. Blackorby et al. (2006) also state a multi-person version of A (p. 283). Their second anonymity condition (p. 286) is a multi-person version of SE.

[44] That is, for all $u, v \in D$, $v \in V(u)$ implies $\succsim_v = \succsim_u$.

First, SWFL theory formulates all informational and normative properties in terms of intermediate utility profiles and not in terms of the underlying primitive well-being comparisons. This is a general methodological **foundation problem** as a convincing theory should be formulated in terms of primitive objects. In line with this, the difference relation framework allows to reconstruct the SWFL approach to comparability in terms of **multiutility representations**. Difference relation $\succsim$ is represented by $V(u)$ if, for all $i, j, k, l \in N$ and $x, y, z, w \in X$, $\big((i_x, j_y) \succsim (k_z, l_w)\big) \Leftrightarrow \big(\forall v \in V(u) : v(i_x) - v(j_y) \geq v(k_z) - v(l_w)\big)$. A well-being difference comparison holds if and only if the corresponding utility difference comparison holds under all profiles from $V(u)$. The existence of a multiutility representation implies that the consistency conditions on $\succsim$ hold, but not the other way around.

Second, SWFL theory does not explicitly model scales in which well-being is measured (see Morreau and Weymark, 2016). Due to that **scale problem**, one of its most fundamental conditions mixes up meaningful utility comparisons within a given scale and meaningless utility comparisons between different scales.

Binary Independence (BI): For all $x, y \in X$ and $u, v \in D$ such that, for all $i \in N$, $u(i_x) = v(i_x)$ and $u(i_y) = v(i_y)$, it is the case that $x \succsim_u y \Leftrightarrow x \succsim_v y$.

Consider individual i and suppose, for sake of illustration, that i's well-being at a state is identified with i's body temperature at the state. If u and v measure i's well-being in the same scale (say, Celsius), the antecedent of BI with respect to i is meaningful. The individual's well-being level at x (y) coincides between u and v. If that is the case for all individuals, it is in line with welfarism that the social ranking of x and y should coincide between u and v. But if u and v measure i's well-being in different scales (say, Celsius and Fahrenheit), the equality of i's utility levels at x (y) between u and v does not have any substantive meaning. If that is the case for some or all individuals, it is not at all clear why the social ranking of x and y is supposed to coincide between the two profiles.[45]

The theory of the present chapter is formulated in terms of a single relation capturing basic well-being comparisons and thereby solves the multi-profile, foundation, and scale problems. Compared to the following four major informational SWFL assumptions and

[45] Due to the scale problem, BI fails to distinguish between real well-being changes and purely representational changes. This issue was already noted by Sen (1977: pp. 1542-1543), but has only recently received a more thorough discussion in Morreau and Weymark (2016) and Nebel (2023). Morreau and Weymark generalize SWFLs and explicitly model measurement scales. Nebel employs an aggregation framework based on abstract well-being levels.

corresponding canonical SWFL results, the informational environments and results of the chapter are more general as they do not presuppose utility representations. The results employ single relation conditions instead of multi-profile properties.[46]

Cardinal Non-Comparability (CNC)[47] corresponds to the environment of no level and gain comparability. Sen (1970: Theorem 8*2) establishes that, given CNC and UD, there is no SWFL satisfying BI, WP, and ND. Like this SWFL version of Arrow's Impossibility Theorem, Theorem 1 also applies if intrapersonal gain comparisons are possible. Theorem 5 is much more general. It waives all consistency, richness, and informational assumptions except for WTPI.

Ordinal Level Comparability (OLC)[48] corresponds to the environment of level without gain comparability. The canonical characterization of the Leximin SWFL given OLC employs UD, BI, SP, a multi-profile anonymity condition (MPA), a minimal equity condition, and a multi-profile separability condition.[49] Instead, Theorem 2 of the present chapter states a single relation characterization of Leximin. ILC and MIGC replace OLC, richness assumptions replace UD, and FE replaces MPA and minimal equity. Theorem 2 employs no analog of BI and multi-profile separability.

Cardinal Unit Comparability (CUC)[50] corresponds to the environment of gain

[46]It is possible to replace UD and BI by the so called Strong Neutrality (SN) condition. SN can be defined for fixed difference relation $\succsim$ with representation $V(u)$: For all $x, y, z, w \in X$ and $v, v' \in V(u)$ such that, for all $i \in N$, $v(i_x) = v'(i_z)$ and $v(i_y) = v'(i_w)$, it is the case that $x \succsim_S y \Leftrightarrow z \succsim_S w$. The following SWFL results go through with SN given a richness assumption on $V(u)$. See Blackorby et al. (1990: Proposition 6) and d'Aspremont and Gevers (2002: Theorem 3.5). For related analysis, see Roberts (1980c). While the SN approach avoids the multi-profile problem, it still faces the foundation and scale problems. Like BI, SN has no clear relational basis.

[47]For all $u, v \in D$, $v \in V(u)$ holds if and only if there exist $(\lambda_i)_{i \in N} \in \mathbb{R}^N$ and $(\mu_i)_{i \in N} \in \mathbb{R}^N_+$ such that, for all $i \in N$ and $x \in X$, $v(i_x) = \lambda_i + \mu_i \cdot u(i_x)$. If $V(u)$ represents $\succsim$, CNC implies PLC, NILC, and MIGC. But the latter are well-defined without the existence of a multiutility representation. Analogous comments apply to the other environments. BI is equivalent to IIA if there are no interpersonal comparisons, but not in general.

[48]For all $u, v \in D$, $v \in V(u)$ holds if and only if there exists a positive monotonic transformation $\phi : \{a \in \mathbb{R} \mid \exists i_x \in N \times X : u(i_x) = a\} \to \mathbb{R}$ such that, for all $i \in N$ and $x \in X$, $v(i_x) = \phi(u(i_x))$. A function $\phi : A \to \mathbb{R}$ on $A \subseteq \mathbb{R}$ is a positive monotonic transformation if, for all $\lambda, \mu \in A$, $\lambda > \mu$ implies $\phi(\lambda) > \phi(\mu)$. It is a positive affine transformation if there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$, such that, for all $\nu \in A$, $\phi(\nu) = \lambda + \mu \cdot \nu$.

[49]This is Theorem 7 of d'Aspremont and Gevers (1977). Their Theorem 5 is a SWFL version of Hammond's (1976) result. For related analysis of Leximin, see Sen (1977), Deschamps and Gevers (1978), Gevers (1979), Hammond (1979), Roberts (1980a), Pivato (2013a), and Ou-Yang (2018).

[50]For all $u, v \in D$, $v \in V(u)$ holds if and only if there exist $(\lambda_i)_{i \in N} \in \mathbb{R}^N$ and $\mu \in \mathbb{R}_+$ such that,

without level comparability. The canonical characterization of the Utilitarian SWFL given CUC uses UD, BI, SP, and MPA.[51] In contrast, the single relation characterization of Utilitarianism in Theorem 3 derives a utility difference representation from qualitative assumptions. UD and BI are not used and FE replaces MPA.

Blackorby et al. (2002: Theorem 10) characterize the Utilitarian SWFL with UD, BI, PI, WP, and a condition to the effect that it is socially indifferent to which individual a certain amount of utility is assigned (Incremental Equity, IE). Instead, Proposition 3 derives a utility representation and characterizes the Utilitarian Relation with SP and the qualitative GE. Blackorby et al. (2002) show that MPA and CUC imply IE (Theorem 11) and use this fact to state a version of the canonical characterization of the Utilitarian SWFL (Theorem 12). This move is analogous to the one from Proposition 3 to Theorem 3 via Lemma 5 in the present chapter.

Cardinal Full Comparability (CFC)[52] corresponds to the environment of level and gain comparability. The canonical SWFL result for CFC is due to Deschamps and Gevers (1978: Theorem 2). They establish a joint characterization of the Leximin SWFL and Utilitarian-Type SWFLs given CFC with the same conditions as in the canonical characterization of the Leximin SWFL given OLC. But while SP and FE likewise yield Leximin in the ILC-MIGC environment (Theorem 2), they characterize a much broader class in the ILC-IDC environment (Theorem 4) than the SWFL conditions given CFC.[53]

2.5.3 Other Fields

This subsection discusses how the theory of the chapter relates to different fields from the literature and connects these fields.

for all $i \in N$ and $x \in X$, $v(i_x) = \lambda_i + \mu \cdot u(i_x)$.

[51]This is Theorem 3 of d'Aspremont and Gevers (1977). For related analysis of Utilitarianism, see Deschamps and Gevers (1978), Maskin (1978), Gevers (1979), Roberts (1980b), Balasubramanian (2015), Yamamura (2017), and Bossert and Kamaga (2020).

[52]For all $u, v \in \mathbb{R}^{N \times X}$, $v \in V(u)$ holds if and only if there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that, for all $i \in N$ and $x \in X$, $v(i_x) = \lambda + \mu \cdot u(i_x)$.

[53]Utilitarian-Type Relations of Deschamps and Gevers (1978) are quite restrictive as their ranking of two states coincides with the one of the UR if utility sums differ between these states. Even if single relation Separability S is added to SP and FE in the ILC-IDC environment, the resulting class of SOs is broader than the one of the SWFL CFC result. The former contains the class of Generalized Utilitarian Relations, while the latter does not.

Voting theory is concerned with the analysis of voting rules (May, 1952; Young, 1974; Pattanaik, 2002; Barberà and Gerber, 2017; Alcantud, 2019; Horan et al., 2019; Bartholdi et al., 2021; Terzopoulou and Endriss, 2021; Maskin, 2023). Overall, the present chapter shows that there is a close axiomatic connection between classical welfarist social evaluation procedures like Utilitarianism and Leximin on the one hand, and central voting rules like Simple Majority and Borda on the other hand.

The canonical characterization of the Simple Majority SWF for two alternatives is due to May (1952). May's Theorem employs UD and inter-profile (inter-relation) conditions to the effect that individuals and alternatives are treated impartially and social evaluation positively responds to individual preferences.[54] Instead, Theorem 1 and Proposition 1 give a single relation characterization of the Simple Majority Relation for the case where all individuals have at most two well-being levels. To the knowledge of the author, Corollary 1 and Proposition 5 (as well as Proposition A.6.1 in Appendix 2.A.6) are the first characterizations of the Level Borda Relation. They contribute to the analysis of Borda scoring that has generally been set in a voting context.[55]

The **social evaluation of variable populations** is examined in Parfit (1984), Broome (2004), Blackorby et al. (2005), Asheim and Zuber (2014), Fleurbaey and Tadenuma (2014), Pivato (2020), Bossert et al. (2023), Gustafsson et al. (2023), and Spears and Zuber (2023). The present chapter contributes to this literature by offering a systematic relational analysis of variable population social evaluation under different informational assumptions.[56]

Existing analysis usually takes place in a variable population version of the SWFL framework and presupposes utility representations. But there are only few characterizations employing informational conditions. The reason for this is that the combination of a variable population version of BI and informational assumptions is highly restrictive (see Blackorby et al., 2005: pp. 176-179, 206-208). The present chapter solves

[54]Numerous characterizations have been established since. For an overview and analysis, see Alcantud (2019), Horan et al. (2019), and Bartholdi et al. (2021). See also Maskin (1995).

[55]Important early contributions are Gärdenfors (1973), Young (1974), Fine and Fine (1974), and Nitzan and Rubinstein (1981). For different versions and characterizations of the Borda Relation, see Pattanaik (2002), Cullinan et al. (2014), Terzopoulou and Endriss (2021), Maskin (2023), and Barberà and Bossert (2023).

[56]It also applies to histories of the world. In this setting, the analysis presupposes that populations of histories are finite. FE makes sure that there is no discrimination between individuals based on the time of their existence. Asheim and Tungodden (2004) and Kamaga (2016) analyze the case of infinitely many generations.

this problem by dropping any version of BI. Its analysis sheds light on discussions in the literature about total and critical level evaluation, the Repugnant Conclusion, and Existence Independence.[57]

Social evaluation under uncertainty is investigated in Harsanyi (1955, 1982), Weymark (1991), Blackorby et al. (2005), Fleurbaey (2010), Pivato (2013b), Mongin and Pivato (2015), Fleurbaey and Mongin (2016), Sprumont (2019), Brandl and Brandt (2020), McCarthy et al. (2020), Gustafsson et al. (2023), Karni and Weymark (2023), Pivato and Tchouante (2023), and Spears and Zuber (2023). The present chapter contributes to these works by establishing a systematic relational analysis of the social evaluation of abstract prospects. It sheds light on ex ante well-being comparisons, their preference foundations, and implications of different ex ante comparability assumptions.

The characterizations of prospect SOs (including the UR) are conceptually independent of expected utility theory. One of the main insights from the literature is that the social and individual evaluation of prospects must take a joint expected utility form under fairly weak conditions.[58] Corresponding results can also be formulated in the present framework by adding according assumptions.

The theory of the chapter also has methodological and substantive connections to **decision and measurement theory**. In line with the latter, it is based on a single relation capturing primitive judgments and preferences, and does not presuppose numerical representations. Cardinal (multi)utility representations for a difference relation are analyzed in Krantz et al. (1971), Wakker (1988), Köbberling (2006), and Pivato (2013c).[59] The utility representations derived in Lemmas 4 and 7 relate to this analysis.

There are a few contributions dealing with difference relations in a social evaluation context, but in a different way than the present chapter. Harvey (1999) considers the aggregation of individual difference relations into a social difference relation. Pivato (2015) is concerned with the aggregation of a difference relation over personal states into a difference relation over social states. Both derive utilitarian-type social difference

[57]The derived utility functions assign a utility level to states where the individual does not exist, reflecting the unique neutral/critical level of the individual. That is not so in Blackorby et al. (2005) and other contributions that consider variable critical levels.

[58]It entails that social evaluation and individuals' evaluation must be based on the same subjective probabilities. In the present context, they can be identified with the subjective probabilities of the ethical observer. On this issue, see Mongin and Pivato (2015).

[59]On multiutility representations for preference relations, see Ok (2002) and Nishimura and Ok (2016).

relations.[60]

Most contributions in decision theory take subjects' preferences as the sole primitive of their models (Wakker, 1989; Mongin and Pivato, 2015; Li et al., 2023). In line with this, the present chapter offers a preference foundation of the difference relation via P1-P4.[61] The approach of the chapter also relates to the literature on **social evaluation in economic environments** (Fleurbaey and Maniquet, 2011, 2017, 2019; Fleurbaey and Tadenuma, 2014; Piacquadio, 2017, Bosmans et al., 2018). In line with P1, the contributions construct well-being measures consistent with individuals' preferences over consumption bundles.[62] For interpersonal comparisons, they study multi-commodity generalizations of the idea that the well-being gain from an income gain decreases with the original income.

2.6 Conclusion

The chapter has developed a theory of welfarist social evaluation. It is based on a single difference relation capturing primitive well-being comparisons. Social orderings and conditions of all types (normative, informational, consistency, richness) are formulated in terms of this difference relation. Utility representations are not presupposed, but derived in environments with gain comparisons (Lemmas 4, 7).

Strong Pareto and Fundamental Equity have far-reaching implications for different informational assumptions. They are summarized in Table 2.1. Theorems 1-5 use different richness assumptions, but it is possible to put them all together. In that case, normative conditions as well as consistency and richness assumptions (C1-C4, R1-R4, RAD) are fixed. The characterizations only differ with respect to informational assumptions. Major approaches to social evaluation (Simple Majority, Leximin, Utilitarianism) have Strong Pareto and Fundamental Equity as their common normative basis and are solely distinguished on informational grounds.

[60]A critical condition for the results is a difference Pareto condition. This condition is quite strong as it effectively induces that the social value of an individual gain does not depend on its starting level. No according condition is employed in the present chapter.

[61]Specifically, difference measurement in terms of risk preferences has a long history in decision theory. Recently, Karni and Weymark (2023) take this approach to construct intra- and interpersonal well-being difference comparisons. They then employ a condition similar to GE to characterize Relative Utilitarianism.

[62]Fleurbaey and Tadenuma (2014) also analyze versions of P2.

Theorem	Interpersonal Comparability	Characterized Relation(s)	Via Condition(s)
1	No	Impossibility (Simple Majority Relation)	Strong Equity
2	Only Levels	Leximin Relation	Anonymity, Hammond Equity
3	Only Gains	Utilitarian Relation	Gain Equity
4	Levels and Gains	Additive Welfarist Relations	Strong Pigou-Dalton
5	Weak Incomparability	Impossibility	

Table 2.1: Implications of Strong Pareto and Fundamental Equity

Fundamental Equity unifies seemingly distinct normative arguments and conditions. It provides unconditional justifications of Anonymity and Strong Pigou-Dalton as well as conditional justifications of Strong Equity, Hammond Equity, and Gain Equity. These properties yield characterizations of Simple Majority Relation, Leximin Relation, Utilitarian Relation, and Additive Welfarist Relations via Propositions 1-4 (see Table 2.1). While Theorems 1-4 hold under more restrictive informational assumptions, they have a unified rationale in terms of Fundamental Equity.

Due to Theorem 5, Strong Pareto and Fundamental Equity yield an impossibility result for a general well-being difference relation given Weak Two-Person Incomparability. If there are no interpersonal comparisons, Positional Equity is a reasonable weakening of Fundamental Equity and characterizes the Level Borda Relation (Proposition 5). But overall, it is desirable to keep Strong Pareto, Fundamental Equity, and the assumption that the social ranking is an ordering as they are all normatively appealing.

Theorems 1-5 apply if alternatives are abstract prospects and the well-being difference relation captures ex ante comparisons. They also go through for variable populations. The characterized social orderings are either of the total or of the critical level type. Overall, this establishes a systematic relational analysis of social prospect evaluation and variable population evaluation.

The well-being difference relation can be determined via the preferences of individuals and of an ethical observer. Importantly, the difference relation inherits the completeness of preference relations (Proposition 6). This avoids the impossibility from Theorem 5 and leads to the informational environment of Theorem 4.

In the environment of complete level and gain comparability, there are several candidate orderings for social evaluation, including Leximin, Utilitarianism, Generalized Utilitarian Relations, and rank-additive approaches like Gini. They all satisfy Strong Pareto and Fundamental Equity. To restrict the class of candidate orderings, further normative single relation conditions need to be added. This is an interesting avenue for future research.

Overall, the present chapter has aimed to provide an intuitive and qualitative analysis of social evaluation. Such analysis is likewise needed to deal with open issues. It is meant to close the gap between economic theory, public discourse, and practical decision-making. Welfarist evaluation should inform welfarist choice.

2.A Appendix

2.A.1 Consistency Conditions

The following sections present proofs of the results from the main text. In view of the variable population interpretation, they allow for infinite N and assume V2. The proofs employ the following consistency conditions that are implied by C1-C4.

C5: $\succsim^*$ is reflexive and transitive.
C6: For all $i, j, k, l \in N$ and $x, y, z, w \in X$, $(i_x, k_z) \sim (j_y, k_z) \Rightarrow (i_x, l_w) \sim (j_y, l_w)$, $(k_z, i_x) \sim (k_z, j_y) \Rightarrow (l_w, i_x) \sim (l_w, j_y)$, $(i_x, k_z) \succ (j_y, k_z) \Rightarrow (i_x, l_w) \succ (j_y, l_w)$, and $(k_z, i_x) \succ (k_z, j_y) \Rightarrow (l_w, i_x) \succ (l_w, j_y)$.
C7: For all $i, j \in N$ and $x, y \in X$, $i_x \sim^* j_y \Leftrightarrow (i_x, j_y) \sim (j_y, j_y)$ and $i_x \succ^* j_y \Leftrightarrow (i_x, j_y) \succ (j_y, j_y)$.
C8: For all $i, j, k, l \in N$ and $x, y, z, w \in X$, $i_x \sim^* k_z$ and $j_y \sim^* l_w$ imply $(i_x, j_y) \sim (k_z, l_w)$.
C9: For all $i, j, k \in N$ and $x, y, z \in X$, $i_x \succ^* j_y \succ^* k_z$ implies $(i_x, k_z) \succ (i_x, j_y)$ and $(i_x, k_z) \succ (j_y, k_z)$.

Lemma A.1.1: If $\succsim$ satisfies C1-C4, then $\succsim$ satisfies C5-C9.
Proof:
(C5) Take $i \in N$ and $x \in X$. By C1, $(i_x, i_x) \succsim (i_x, i_x)$. It follows that $i_x \succsim^* i_x$. Thus, $\succsim^*$ is reflexive. Consider $i, j, k \in N$ and $x, y, z \in X$ with $i_x \succsim^* j_y$ and $j_y \succsim^* k_z$. That is, $(i_x, j_y) \succsim (j_y, j_y)$ and $(j_y, k_z) \succsim (k_z, k_z)$. By C4, $(j_y, j_y) \sim (k_z, k_z)$. Due to C1, that implies $(i_x, j_y) \succsim (k_z, k_z)$. Taken together, C2 yields $(i_x, k_z) \succsim (k_z, k_z)$. That is, $i_x \succsim^* k_z$, so that $\succsim^*$ is transitive.

(C6) To begin with, it is true that, for all $i, j, k, l \in N$ and $x, y, z, w \in X$, $(i_x, k_z) \succsim (j_y, k_z) \Rightarrow (i_x, l_w) \succsim (j_y, l_w)$ and $(k_z, i_x) \succsim (k_z, j_y) \Rightarrow (l_w, i_x) \succsim (l_w, j_y)$. To see that, suppose $(i_x, k_z) \succsim (j_y, k_z)$. By C1, $(k_z, l_w) \succsim (k_z, l_w)$. Due to C2, that implies $(i_x, l_w) \succsim (j_y, l_w)$. Suppose $(k_z, i_x) \succsim (k_z, j_y)$. Due to C1, $(l_w, k_z) \succsim (l_w, k_z)$. By C2, it follows that $(l_w, i_x) \succsim (l_w, j_y)$.

Consider $i, j, k, l \in N$ and $x, y, z, w \in X$. Suppose $(i_x, k_z) \sim (j_y, k_z)$, that is, $(i_x, k_z) \succsim (j_y, k_z)$ and $(j_y, k_z) \succsim (i_x, k_z)$. Due to the last paragraph, that implies $(i_x, l_w) \succsim (j_y, l_w)$ and $(j_y, l_w) \succsim (i_x, l_w)$. That means $(i_x, l_w) \sim (j_y, l_w)$. Suppose $(k_z, i_x) \sim (k_z, j_y)$. Since $(k_z, i_x) \succsim (k_z, j_y)$ and $(k_z, j_y) \succsim (k_z, i_x)$, the last paragraph implies $(l_w, i_x) \succsim (l_w, j_y)$ and $(l_w, j_y) \succsim (l_w, i_x)$. That is, $(l_w, i_x) \sim (l_w, j_y)$. Suppose $(i_x, k_z) \succ (j_y, k_z)$. Since

$(i_x, k_z) \succsim (j_y, k_z)$ and $(j_y, k_z) \not\succsim (i_x, k_z)$, the last paragraph induces $(i_x, l_w) \succsim (j_y, l_w)$ and $(j_y, l_w) \not\succsim (i_x, l_w)$, meaning $(i_x, l_w) \succ (j_y, l_w)$. Suppose $(k_z, i_x) \succ (k_z, j_y)$, so that $(k_z, i_x) \succsim (k_z, j_y)$ and $(k_z, j_y) \not\succsim (k_z, i_x)$. Again, the last paragraph implies $(l_w, i_x) \succsim (l_w, j_y)$ and $(l_w, j_y) \not\succsim (l_w, i_x)$, yielding $(l_w, i_x) \succ (l_w, j_y)$.

(C7) Consider $i, j \in N$ and $x, y \in X$. Suppose $i_x \sim^* j_y$. That is, $(i_x, j_y) \succsim (j_y, j_y)$ and $(j_y, i_x) \succsim (i_x, i_x)$. Due to C4 and C1, $(j_y, i_x) \succsim (j_y, j_y)$. C3 leads to $(j_y, j_y) \succsim (i_x, j_y)$, implying $(i_x, j_y) \sim (j_y, j_y)$. Suppose $i_x \succ^* j_y$. That is, $(i_x, j_y) \succsim (j_y, j_y)$ and $(j_y, i_x) \not\succsim (i_x, i_x)$. Assume $(j_y, j_y) \succsim (i_x, j_y)$. C3 implies $(j_y, i_x) \succsim (j_y, j_y)$. In turn, C4 and C1 yield the contradiction $(j_y, i_x) \succsim (i_x, i_x)$. Assumption $(j_y, j_y) \succsim (i_x, j_y)$ must be wrong, so that $(i_x, j_y) \succ (j_y, j_y)$. Suppose $(i_x, j_y) \sim (j_y, j_y)$. That is, $i_x \succsim^* j_y$. As shown, $i_x \succ^* j_y$ would imply the contradiction $(i_x, j_y) \succ (j_y, j_y)$. Hence, $i_x \sim^* j_y$. Suppose $(i_x, j_y) \succ (j_y, j_y)$. Again, $i_x \succsim^* j_y$. As argued, $i_x \sim^* j_y$ would yield the contradiction $(i_x, j_y) \sim (j_y, j_y)$, so that $i_x \succ^* j_y$ must hold.

(C8) Take $i, j, k, l \in N$ and $x, y, z, w \in X$ with $i_x \sim^* k_z$ and $j_y \sim^* l_w$. By C7, it follows that $(i_x, k_z) \sim (k_z, k_z)$. Due to C6, that implies $(i_x, l_w) \sim (k_z, l_w)$. C7 also induces $(j_y, l_w) \sim (l_w, l_w)$. By C4 and C1, that leads to $(j_y, l_w) \sim (j_y, j_y)$. In turn, C6 implies $(i_x, l_w) \sim (i_x, j_y)$. Taken together, C1 yields $(i_x, j_y) \sim (k_z, l_w)$.

(C9) Consider $i, j, k \in N$ and $x, y, z \in X$ with $i_x \succ^* j_y \succ^* k_z$. C7 implies $(j_y, k_z) \succ (k_z, k_z)$. Due to C4 and C1, it follows that $(j_y, k_z) \succ (j_y, j_y)$. By C6, that yields $(i_x, k_z) \succ (i_x, j_y)$. C7 also implies $(i_x, j_y) \succ (j_y, j_y)$. Due to C6, that results in $(i_x, k_z) \succ (j_y, k_z)$.■

2.A.2 Proofs for Section 2.3.1

Proof of Lemma 1:
For "⇒", suppose $\succsim_S$ satisfies FE. Consider $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. By NILC and MIGC, $(i_x, i_y) \not\succ (j_y, j_x)$ and $(j_y, j_x) \not\succ (i_x, i_y)$. By NILC, $j_x \not\succ^* i_y$ and $i_y \not\succ^* j_x$. Therefore, FE induces $x \not\succ_S y$ and $y \not\succ_S x$. Since $\succsim_S$ is complete, that yields $x \sim_S y$. Accordingly, $\succsim_S$ satisfies SE. For "⇐", suppose that $\succsim_S$ satisfies SE. For $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$, SE implies $x \sim_S y$. Since the antecedent of FE is never satisfied, $\succsim_S$ satisfies the condition.■

For $x, y \in X$, denote $N_{xy} = \{i \in N \mid i_x \succ^* i_y\}$ and $N_{e,xy} = \{i \in N \mid i_x \sim^* i_y\}$. Whenever considering fixed $\{x, y\}$, the shorter symbols $N_x = N_{xy}$, $N_y = N_{yx}$, $N_e =$

$N_{e,xy}$, and $N_s = N_{s,xy}$ will be used.

Proof of Proposition 1:
The third sentence of Proposition 1 will be shown first. For that purpose, suppose SO $\succsim_S$ satisfies SP and SE. Take $x, y \in X$. Let $n = |N_x|$ and $m = |N_y|$. By PLC, $N_x \cup N_y \cup N_e = N$.

Suppose $n \geq m$. Rename individuals in N_x and N_y such that $N_x = \{i^1, ..., i^m, ..., i^n\}$ as well as $N_y = \{j^1, ..., j^m\}$. Suppose $m > 0$. Inductively define a sequence $x_0, x_1, ..., x_m \in X$ with $x_0 = y$: For each $r \in \{1, ..., m\}$, $i^r_{x_r} \sim^* i^r_x$, $j^r_{x_r} \sim^* j^r_x$, and, for all $k \in N \setminus \{i^r, j^r\}$, $k_{x_r} \sim^* k_{x_{r-1}}$. Such a sequence exists by R1.

Take $r \in \{1, ..., m\}$. By individual transitivity (the intrapersonal part of C5), $i^r_y \sim^* i^r_{x_{r-1}}$. Together with $i^r_x \succ^* i^r_y$, that implies $i^r_{x_r} \succ^* i^r_{x_{r-1}}$. It follows analogously that $j^r_{x_{r-1}} \succ^* j^r_{x_r}$. Since its antecedent is satisfied, SE yields $x_r \sim_S x_{r-1}$. The transitivity of $\succsim_S$ leads to $x_m \sim_S y$. If $m = 0$, then $x_m = x_0 = y$.

Suppose $x \sim_M y$, that is, $n = m$. In that case, individual transitivity implies $k_{x_m} \sim^* k_x$ for all $k \in N$. By SP, $x_m \sim_S x$. Social transitivity yields $x \sim_S y$. Suppose $x \succ_M y$, that is, $n > m$. In that case, individual transitivity implies $k_{x_m} \sim^* k_y$ for all $k \in \{i^{m+1}, ..., i^n\}$ and $k_{x_m} \sim^* k_x$ for all $k \in N \setminus \{i^{m+1}, ..., i^n\}$. Together with $k_x \succ^* k_y$, it also implies $k_x \succ^* k_{x_m}$ for $k \in \{i^{m+1}, ..., i^n\}$. By SP, that yields $x \succ_S x_m$. Social transitivity implies $x \succ_S y$.

It has been established that $x \sim_M y \Rightarrow x \sim_S y$ and $x \succ_M y \Rightarrow x \succ_S y$ holds for all $x, y \in X$. Since $\succsim_M$ is complete, that yields $x \succsim_M y \Leftrightarrow x \succsim_S y$ for all $x, y \in X$. Hence, $\succsim_S = \succsim_M$.

For "$\Rightarrow$" of the second sentence of Proposition 1, suppose there exists SO $\succsim_S$ satisfying SP and SE. By the third sentence, $\succsim_S = \succsim_M$. Assume that neither (a) nor (b) holds. It follows that there exist $i_{x_1}, i_{x_2}, i_{x_3}, j_{y_1}, j_{y_2} \in N \times X$ with $i \neq j$, $i_{x_1} \succ^* i_{x_2} \succ^* i_{x_3}$, and $j_{y_1} \succ^* j_{y_2}$. Due to R1, there exist $x, y, z \in X$ with the following properties. First, $i_x \sim^* i_{x_1}$, $j_x \sim^* j_{y_2}$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_{x_1}$. Second, $i_y \sim^* i_{x_3}$, $j_y \sim^* j_{y_1}$, and, for all $k \in N \setminus \{i, j\}$, $k_y \sim^* k_{x_1}$. Third, $i_z \sim^* i_{x_2}$, $j_z \sim^* j_{y_2}$, and, for all $k \in N \setminus \{i, j\}$, $k_z \sim^* k_{x_1}$. It follows that $x \sim_M y$ and $y \sim_M z$, but $x \succ_M z$. This contradicts the fact that $\succsim_S$ is transitive. Accordingly, the initial assumption must be wrong. Either (a) or (b) must hold.

For "$\Leftarrow$" of the second sentence of Proposition 1, show that the SMR is the desired SO. To begin with, consider $x, y \in X$ such that, for all $i \in N$, $i_x \succsim^* i_y$. That implies

$N_{yx} = \emptyset$ and thus $x \succsim_M y$. Suppose there exists $i \in N$ with $i_x \succ^* i_y$. Then, $N_{xy} \neq \emptyset$ and hence $x \succ_M y$. Consequently, $\succsim_M$ satisfies SP. Take $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. It follows that $N_{xy} = \{i\}$ and $N_{yx} = \{j\}$, implying $x \sim_M y$. Therefore, $\succsim_M$ satisfies SE. Consider $x, y \in X$. If $|N_{xy}| \geq |N_{yx}|$, then $x \succsim_M y$. If $|N_{xy}| \leq |N_{yx}|$, then $y \succsim_M x$. Accordingly, $\succsim_M$ is complete.

Suppose there exists $k \in N$ in the sense of (a). Take $x, y, z \in X$ with $x \succsim_M y$ and $y \succsim_M z$. That implies $k_x \succsim^* k_y$ and $k_y \succsim^* k_z$. By individual transitivity, it follows that $k_x \succsim^* k_z$. That yields $x \succsim_M z$, so that $\succsim_M$ is transitive. Suppose (b) holds. In that case, $\succsim_M = \succsim_B$. For $x, y \in X$, $x \succsim_M y \Leftrightarrow |N_{xy}| \geq |N_{yx}| \Leftrightarrow \sum_{i \in N_{xy}} \big(|\{[i_z] \mid i_x \succ^* i_z\}| - |\{[i_z] \mid i_y \succ^* i_z\}|\big) \geq \sum_{i \in N_{yx}} \big(|\{[i_z] \mid i_y \succ^* i_z\}| - |\{[i_z] \mid i_x \succ^* i_z\}|\big) \Leftrightarrow \sum_{i \in N} \big(|\{[i_z] \mid i_x \succ^* i_z\}| - |\{[i_z] \mid i_y \succ^* i_z\}|\big) \geq 0 \Leftrightarrow x \succsim_B y$. Since $\succsim_B$ is transitive by Proposition 5, so is $\succsim_M$.■

Proof of Theorem 1:
By Lemma 1, there exists a SO satisfying SP and FE if and only if there exists a SO satisfying SP and SE. Via the equivalence from the second sentence of Proposition 1, that yields the equivalence from the second sentence of Theorem 1. For the third sentence of Theorem 1, suppose SO $\succsim_S$ satisfies SP and FE. By Lemma 1, $\succsim_S$ satisfies SE. Due to Proposition 1, it follows that $\succsim_S = \succsim_M$.■

2.A.3 Proofs for Section 2.3.2

For Lemmas A.3.1-A.3.4, suppose ILC holds. For finite non-empty $M \subseteq N$ and $x \in X$, bijection $\pi : R(M) \to M$ is an x-order if, for all $r, t \in R(M)$ with $r \geq t$, $\pi(r)_x \succsim^* \pi(t)_x$.

Lemma A.3.1: Take finite non-empty $M \subseteq N$ and $x \in X$. Consider x-orders $\pi, \pi' : R(M) \to M$. For all $r \in R(M)$, $\pi(r)_x \sim^* \pi'(r)_x$.
Proof:
By definition, both π and π' must assign a smaller rank to an individual with a lower well-being level at x than to an individual with a greater well-being level at x. That is, $\pi'(t)_x \succ^* \pi(r)_x$ implies $r < t$ and $\pi(r)_x \succ^* \pi'(t)_x$ implies $r > t$.■

Lemma A.3.2: Take finite non-empty $M \subseteq N$ and $x, y \in X$. Consider x-order $\pi : R(M) \to M$ and y-order $\mu : R(M) \to M$. Take $t \in R(M)$. The following conditions are equivalent:
(a) for all $r \leq t$, $\pi(r)_x \sim^* \mu(r)_y$.

(b) there exists a bijection $\sigma : \{\pi(1), ..., \pi(t)\} \to \{\mu(1), ..., \mu(t)\}$ such that, for all $i \in \{\pi(1), ..., \pi(t)\}$, $i_x \sim^* \sigma(i)_y$.

Proof:

"(a)⇒(b)": Suppose that, for all $r \leq t$, $\pi(r)_x \sim^* \mu(r)_y$. For each $r \leq t$, define $\sigma : \pi(r) \mapsto \mu(r)$. It is bijective because π and μ are. By construction, it is the case that $\pi(r)_x \sim^* \sigma(\pi(r))_y$ for each $r \leq t$.

"(b)⇒(a)": Suppose that there exists bijection $\sigma : \{\pi(1), ..., \pi(t)\} \to \{\mu(1), ..., \mu(t)\}$ such that, for all $i \in \{\pi(1), ..., \pi(t)\}$, $i_x \sim^* \sigma(i)_y$. Define $\mu' : R(M) \to M$ such that, for $r \leq t$, $\mu'(r) = \sigma(\pi(r))$ and, for $r > t$, $\mu'(r) = \mu(r)$. It is bijective because π, μ, and σ are.

Take $1 \leq r \leq r' \leq |M|$. First, suppose $r' \leq t$. Then, $\mu'(r')_y \sim^* \sigma(\pi(r'))_y \sim^* \pi(r')_x \succsim^* \pi(r)_x \sim^* \sigma(\pi(r))_y \sim^* \mu'(r)_y$. Second, suppose $r \leq t < r'$. Then, $\mu'(r) \in \{\mu(1), ..., \mu(t)\}$ and $\mu'(r') = \mu(r')$, implying $\mu'(r')_y \succsim^* \mu'(r)_y$. Third, suppose $t < r$. Then, $\mu'(r')_y \sim^* \mu(r')_y \succsim^* \mu(r)_y \sim^* \mu'(r)_y$. Taken together, that means that μ' is a y-order. Due to Lemma A.3.1, that implies $\mu(r)_y \sim^* \mu'(r)_y$ for all $r \in R(M)$. It follows that, for $r \leq t$, $\pi(r)_x \sim^* \sigma(\pi(r))_y \sim^* \mu'(r)_y \sim^* \mu(r)_y$.■

Lemma A.3.3: Take $x, y \in X$ with $N_{s,xy} \neq \emptyset$ and finite $M \subseteq N$ with $N_{s,xy} \subseteq M$. Conditions (1) and (2) are equivalent:

(1) There exist x-order $\pi : R(N_{s,xy}) \to N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \to N_{s,xy}$ such that, for all $r \in R(N_{s,xy})$, $\pi(r)_x \sim^* \mu(r)_y$.

(2) There exist x-order $\pi' : R(M) \to M$ and y-order $\mu' : R(M) \to M$ such that, for all $r \in R(M)$, $\pi'(r)_x \sim^* \mu'(r)_y$.

Proof:

"(1)⇒(2)": Suppose (1) holds. Each individual from $M \setminus N_{s,xy}$ has the same well-being level at x and y. That is, for each well-being level, the same number of individuals from M realize this level at x and at y, respectively. It is thus possible to construct π' and μ' in the sense of (2). It is indeed possible to construct π' and μ' such that, for all $r \in R(M)$ with $\pi'(r) \in M \cap N_{e,xy}$, $\pi'(r) = \mu'(r)$, and, for all $r \in R(N_{s,xy})$, there exists $r' \in R(M)$ with $\pi'(r') = \pi(r)$ and $\mu'(r') = \mu(r)$.

"(2)⇒(1)": Suppose (2) holds. It means that, for each well-being level, the same number of individuals from M realize this level at x and at y, respectively. Again, each individual from $M \setminus N_{s,xy}$ has the same well-being level at x and y. It follows that, for each well-being level, the same number of individuals from $N_{s,xy}$ realize this level at x and at y, respectively. This fact allows to construct π and μ in the sense of (1).■

Lemma A.3.4: Take $x, y \in X$ with $N_{s,xy} \neq \emptyset$ and finite $M \subseteq N$ with $N_{s,xy} \subseteq M$. Conditions (3) and (4) are equivalent:
(3) There exist x-order $\pi : R(N_{s,xy}) \to N_{s,xy}$, y-order $\mu : R(N_{s,xy}) \to N_{s,xy}$, and $t \in R(N_{s,xy})$ such that, for all $r \in R(N_{s,xy})$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t)_x \succ^* \mu(t)_y$.
(4) There exist x-order $\pi' : R(M) \to M$, y-order $\mu' : R(M) \to M$, and $t \in R(M)$ such that, for all $r \in R(M)$ with $r < t$, $\pi'(r)_x \sim^* \mu'(r)_y$, and $\pi'(t)_x \succ^* \mu'(t)_y$.
Proof:
"(3)⇒(4)": Suppose (3) holds. Define $M' = \{i \in M \cap N_{e,xy} \mid \mu(t)_y \succsim^* i_x\}$. There exist x-order bijection $\pi' : R(M) \to M$ and y-order bijection $\mu' : R(M) \to M$ such that, for all $r \in R(M)$ with $\pi'(r) \in M'$, $\pi'(r) = \mu'(r)$, and, for all $r \in R(N_{s,xy})$ with $r < t$, there is $r' \in R(M)$ with $\pi'(r') = \pi(r)$ and $\mu'(r') = \mu(r)$. They exhibit $\pi'(r)_x \sim^* \mu'(r)_y$ for all $r \in R(M)$ with $r < t + |M'|$. After rank $t - 1 + |M'|$, there is no individual in M with a smaller well-being level than $[\mu(t)_y]$ at y left (neither in $M \cap N_{e,xy}$ nor in $N_{s,xy}$). It is thus possible to set $\mu'(t + |M'|) = \mu(t)$. There is also no individual in M left whose well-being level at x is not greater than $[\mu(t)_y]$. That means $\pi'(t + |M'|)_x \succ^* \mu(t)_y \sim^* \mu'(t + |M'|)_y$.[63] Hence, (4) is satisfied with π' and μ'.

"(4)⇒(3)": Suppose (4) holds. It implies that, for each well-being level below $[\mu'(t)_y]$, the same number of individuals in M have that well-being level at x and y, respectively. In contrast, there are more individuals in M that realize well-being level $[\mu'(t)_y]$ at y than at x. Accordingly, it is possible to construct π and μ in the sense of (3) with $t^* \in R(N_{s,xy})$ such that, for all $r \in R(N_{s,xy})$ with $r < t^*$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t^*)_x \succ^* \mu'(t)_y \sim^* \mu(t^*)_y$.■

Proof of Proposition 2:
"⇒": Suppose SO $\succsim_S$ satisfies SP, A, and HE.

Consider $x, y \in X$ with $x \sim_L y$. There are two possibilities. First, $N_s = \emptyset$. In that case, SP induces $x \sim_S y$. Second, $N_s \neq \emptyset$ and there exist x-order $\pi : R(N_s) \to N_s$ and y-order $\mu : R(N_s) \to N_s$ such that, for all $r \in R(N_s)$, $\pi(r)_x \sim^* \mu(r)_y$.

Due to Lemma A.3.2, there exists bijection $\sigma : N_s \to N_s$ such that, for all $i \in N_s$, $i_x \sim^* \sigma(i)_y$. It is well-known that there are bijections $\sigma^1, ..., \sigma^n : N_s \to N_s$ with $i^1, ..., i^n, j^1, ..., j^n \in N_s$ such that $\sigma = \sigma^n \circ ... \circ \sigma^1$ and, for each σ^t, $\sigma^t(i^t) = j^t$, $\sigma^t(j^t) = i^t$, and, for all $k \in N_s \setminus \{i^t, j^t\}$, $\sigma^t(k) = k$.

Inductively define a sequence $x_0, x_1, ..., x_n \in X$ with $x_0 = x$: For $t \in \{1, ..., n\}$, take x_t

[63] Individual $\pi'(t+|M'|)$ does not have to be $\pi(t)$. There can be an individual from $M \cap N_{e,xy}$ whose well-being level at x is smaller than the well-being level of $\pi(t)$ at x.

such that $i^t_{x_t} \sim^* j^t_{x_{t-1}}$, $j^t_{x_t} \sim^* i^t_{x_{t-1}}$, and, for all $k \in N \setminus \{i^t, j^t\}$, $k_{x_t} \sim^* k_{x_{t-1}}$. That is, for all $k \in N_s$, $\sigma^t(k)_{x_t} \sim^* k_{x_{t-1}}$. Such x_t exists due to the combination of R2 and R1.

By ILC, there are three possibilities. If $i^t_{x_{t-1}} \sim^* j^t_{x_{t-1}}$, then SP induces $x_t \sim_S x_{t-1}$. If $i^t_{x_{t-1}} \succ^* j^t_{x_{t-1}}$, then A implies $x_t \sim_S x_{t-1}$. If $j^t_{x_{t-1}} \succ^* i^t_{x_{t-1}}$, then A also yields $x_t \sim_S x_{t-1}$. Because all neighboring pairs of the sequence are socially equally good, the transitivity of $\succsim_S$ leads to $x_n \sim_S x$.

For all $j \in N_e$, $j_{x_n} \sim^* j_x \sim^* j_y$. Consider $j \in N_s$. Since σ is surjective, there exists $i \in N_s$ with $j = \sigma(i) = (\sigma^n \circ ... \circ \sigma^1)(i)$. It is true that $i_{x_0} \sim^* \sigma^1(i)_{x_1} \sim^* (\sigma^2 \circ \sigma^1)(i)_{x_2} \sim^* ... \sim^* (\sigma^n \circ ... \circ \sigma^1)(i)_{x_n}$. That results in $j_{x_n} \sim^* i_x \sim^* j_y$. Taken together, SP implies $x_n \sim_S y$. Social transitivity yields $x \sim_S y$.

Consider $x, y \in X$ with $x \succ_L y$. That is, $N_s \neq \emptyset$ and there exist x-order $\pi : R(N_s) \to N_s$, y-order $\mu : R(N_s) \to N_s$, and $t \in R(N_s)$ such that, for all $r \in R(N_s)$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t)_x \succ^* \mu(t)_y$. Due to R2, there exists $z_r \in X$ with $\mu(r)_{z_r} \sim^* \pi(r)_x$ for each $r \in R(N_s)$ with $\mu(r)_y \not\succ^* \pi(r)_x$. Denote $R = \{r \in R(N_s) \mid \mu(r)_y \succ^* \pi(r)_x\}$. Clearly, $r \in R$ implies $r > t$. Suppose $R \neq \emptyset$. Define $R^* = \{1, ..., |R|\}$ and take any bijection $\gamma : R^* \to R$.

Inductively define sequence $x_0, x_1, ..., x_{|R|} \in X$ with $x_0 = y$. Take $s \in R^*$ with $\mu(t)_{z_t} \succ^* \mu(t)_{x_{s-1}}$. By R3, there exists $j_z \in N \times X$ with $\mu(t)_{z_t} \succ^* j_z \succ^* \mu(t)_{x_{s-1}}$. By R2, there exists $z' \in X$ with $\mu(t)_{z'} \sim^* j_z$. By R1, there exists $x_s \in X$ with $\mu(t)_{x_s} \sim^* \mu(t)_{z'}$, $\mu(\gamma(s))_{x_s} \sim^* \mu(\gamma(s))_{z_{\gamma(s)}}$, and, for all $i \in N \setminus \{\mu(t), \mu(\gamma(s))\}$, $i_{x_s} \sim^* i_{x_{s-1}}$.

By construction, $\mu(\gamma(s))_{x_{s-1}} \succ^* \mu(\gamma(s))_{x_s} \sim^* \pi(\gamma(s))_x \succsim^* \pi(t)_x \sim^* \mu(t)_{z_t} \succ^* \mu(t)_{x_s} \succ^* \mu(t)_{x_{s-1}}$ and, for all $i \in N \setminus \{\mu(t), \mu(\gamma(s))\}$, $i_{x_s} \sim^* i_{x_{s-1}}$. Due to HE, that yields $x_s \succsim_S x_{s-1}$. Taken together, social transitivity leads to $x_{|R|} \succsim_S y$. By R1, there exists $w_0 \in X$ with $\mu(t)_{w_0} \sim^* \mu(t)_{z_t}$ and, for all $i \in N \setminus \{\mu(t)\}$, $i_{w_0} \sim^* i_{x_{|R|}}$. SP induces $w_0 \succ_S x_{|R|}$. By social transitivity, that implies $w_0 \succ_S y$. If $R = \emptyset$, let $w_0 \in X$ be such that $\mu(t)_{w_0} \sim^* \mu(t)_{z_t}$ and, for all $i \in N \setminus \{\mu(t)\}$, $i_{w_0} \sim^* i_y$. Again, R1 makes sure that such a state exists. SP yields $w_0 \succ_S y$.

Denote $T = \{r \in R(N_s) \mid r > t \wedge \pi(r)_x \succ^* \mu(r)_y\}$. If $T = \emptyset$, set $w = w_0$. Suppose $T \neq \emptyset$. Define $T^* = \{1, ..., |T|\}$ and take any bijection $\delta : T^* \to T$. Inductively define sequence $w_1, ..., w_{|T|} \in X$ and set $w = w_{|T|}$. For $s \in T^*$, let $w_s \in X$ be such that $\mu(\delta(s))_{w_s} \sim^* \mu(\delta(s))_{z_{\delta(s)}} \sim^* \pi(\delta(s))_x$ and, for all $i \in N \setminus \{\mu(\delta(s))\}$, $i_{w_s} \sim^* i_{w_{s-1}}$. It exists due to R1. By SP, $w_s \succ_S w_{s-1}$. Taken together, social transitivity yields $w_{|T|} \succ_S y$. In any case, $w \succ_S y$.

By construction, $\mu(r)_w \sim^* \pi(r)_x$ holds for all $r \in R(N_{s,xy})$. For all $i \in N_{e,xy}$, $i_w \sim^* i_y \sim^* i_x$. It follows that $N_{s,xw} \subseteq N_{s,xy}$. If $N_{s,xw} = \emptyset$, then $x \sim_L w$. If $N_{s,xw} \neq \emptyset$, Lemma A.3.3 implies $x \sim_L w$.[64] As shown above, that yields $x \sim_S w$. Social transitivity leads to $x \succ_S y$.

It has been demonstrated that, for all $x, y \in X$, $x \sim_L y \Rightarrow x \sim_S y$ and $x \succ_L y \Rightarrow x \succ_S y$. Since $\succsim_L$ is complete, that yields $x \succsim_S y \Leftrightarrow x \succsim_L y$ for all $x, y \in X$. Therefore, $\succsim_S = \succsim_L$.

"$\Leftarrow$": Take LR $\succsim_L$. First of all, it is a SO. To see that, consider $x, y \in X$. If $N_{s,xy} = \emptyset$, then $x \sim_L y$. Suppose $N_{s,xy} \neq \emptyset$. Then, there exist x-order $\pi : R(N_{s,xy}) \to N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \to N_{s,xy}$. If $\pi(r)_x \sim^* \mu(r)_y$ for all $r \in R(N_{s,xy})$, then $x \sim_L y$. Suppose that there exists $t \in R(N_{s,xy})$ such that, for all $r \in R(N_{s,xy})$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t)_x \not\sim^* \mu(t)_y$. If $\pi(t)_x \succ^* \mu(t)_y$, then $x \succ_L y$. If $\mu(t)_y \succ^* \pi(t)_x$, then $y \succ_L x$. Thus, $\succsim_L$ is complete.

Consider $x, y, z \in X$ with $x \succsim_L y$ and $y \succsim_L z$. Define $M = N_{s,xy} \cup N_{s,yz} \cup N_{s,xz}$. Distinguish the following cases.

1. Suppose case 1 of the Leximin definition applies to $x \succsim_L y$, that is, $N_{s,xy} = \emptyset$.

1.1 Suppose case 1 of the Leximin definition applies to $y \succsim_L z$, that is, $N_{s,yz} = \emptyset$. For all $i \in N$, $i_x \sim^* i_y \sim^* i_z$, so that $N_{s,xz} = \emptyset$. The latter means $x \succsim_L z$. In the following cases, suppose $N_{s,xz} \neq \emptyset$.

1.2 Suppose case 2(a) of the Leximin definition applies to $y \succsim_L z$. By Lemma A.3.3, there exist y-order $\pi : R(M) \to M$ and z-order $\mu : R(M) \to M$ such that $\pi(r)_y \sim^* \mu(r)_z$ for all $r \in R(M)$. Since $N_{s,xy} = \emptyset$, it follows that, for all $r \in R(M)$, $\pi(r)_x \sim^* \pi(r)_y \sim^* \mu(r)_z$. Specifically, π is an x-order. Due to Lemma A.3.3, that implies $x \succsim_L z$.

1.3 Suppose case 2(b) of the Leximin definition applies to $y \succsim_L z$. By Lemma A.3.4, there exist y-order $\pi : R(M) \to M$ and z-order $\mu : R(M) \to M$ with $t \in R(M)$ such that, for all $r \in R(M)$ with $r < t$, $\pi(r)_y \sim^* \mu(r)_z$, and $\pi(t)_y \succ^* \mu(t)_z$. Since $N_{s,xy} = \emptyset$, it follows that, for all $r \in R(M)$ with $r < t$, $\pi(r)_x \sim^* \pi(r)_y \sim^* \mu(r)_z$, and $\pi(t)_x \sim^* \pi(t)_y \succ^* \mu(t)_z$. Again, π is an x-order. By Lemma A.3.4, it follows that $x \succsim_L z$.

2. Suppose case 2(a) of the Leximin definition applies to $x \succsim_L y$. Due to Lemma A.3.3, there exist x-order $\pi : R(M) \to M$ and y-order $\mu : R(M) \to M$ such that

[64] Set $M = N_{s,xy}$ and recognize that μ is a w-order bijection.

$\pi(r)_x \sim^* \mu(r)_y$ for all $r \in R(M)$.

2.1 Suppose case 1 of the Leximin definition applies to $y \succsim_L z$, so that $N_{s,yz} = \emptyset$. It follows that $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu(r)_z$ for all $r \in R(M)$. Since μ is also a z-order, Lemma A.3.3 implies $x \succsim_L z$.

2.2 Suppose case 2(a) of the Leximin definition applies to $y \succsim_L z$. By Lemma A.3.3, there exist y-order $\mu' : R(M) \to M$ and z-order $\pi' : R(M) \to M$ such that $\mu'(r)_y \sim^* \pi'(r)_z$ for all $r \in R(M)$. Lemma A.3.1 implies that, for all $r \in R(M)$, $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu'(r)_y \sim^* \pi'(r)_z$. By Lemma A.3.3, that implies $x \succsim_L z$.

2.3 Suppose case 2(b) of the Leximin definition applies to $y \succsim_L z$. Due to Lemma A.3.4, there exist y-order $\mu' : R(M) \to M$ and z-order $\pi' : R(M) \to M$ with $t \in R(M)$ such that, for all $r \in R(M)$ with $r < t$, $\mu'(r)_y \sim^* \pi'(r)_z$, and $\mu'(t)_y \succ^* \pi'(t)_z$. Lemma A.3.1 induces that, for all $r \in R(M)$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu'(r)_y \sim^* \pi'(r)_z$, and $\pi(t)_x \sim^* \mu(t)_y \sim^* \mu'(t)_y \succ^* \pi'(t)_z$. In turn, Lemma A.3.4 implies $x \succsim_L z$.

3. Suppose case 2(b) of the Leximin definition applies to $x \succsim_L y$. By Lemma A.3.4, there exist x-order $\pi : R(M) \to M$ and y-order $\mu : R(M) \to M$ with $t \in R(M)$ such that, for all $r \in R(M)$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y$, and $\pi(t)_x \succ^* \mu(t)_y$.

3.1 Suppose case 1 of the Leximin definition applies to $y \succsim_L z$, meaning $N_{s,yz} = \emptyset$. For all $r \in R(M)$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu(r)_z$, and $\pi(t)_x \succ^* \mu(t)_y \sim^* \mu(t)_z$. Again, μ is also a z-order. Due to Lemma A.3.4, $x \succsim_L z$.

3.2 Suppose case 2(a) of the Leximin definition applies to $y \succsim_L z$. Due to Lemma A.3.3, there exist y-order $\mu' : R(M) \to M$ and z-order $\pi' : R(M) \to M$ such that, for all $r \in R(M)$, $\mu'(r)_y \sim^* \pi'(r)_z$. Lemma A.3.1 implies that, for all $r \in R(M)$ with $r < t$, $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu'(r)_y \sim^* \pi'(r)_z$, and $\pi(t)_x \succ^* \mu(t)_y \sim^* \mu'(t)_y \sim^* \pi'(t)_z$. By Lemma A.3.4, it follows that $x \succsim_L z$.

3.3 Suppose case 2(b) of the Leximin definition applies to $y \succsim_L z$. By Lemma A.3.4, there exist y-order $\mu' : R(M) \to M$ and z-order $\pi' : R(M) \to M$ with $t' \in R(M)$ such that, for all $r \in R(M)$ with $r < t'$, $\mu'(r)_y \sim^* \pi'(r)_z$, and $\mu'(t')_y \succ^* \pi'(t')_z$. Define $t'' = min\{t, t'\}$. For all $r \in R(M)$ with $r < t''$, Lemma A.3.1 implies $\pi(r)_x \sim^* \mu(r)_y \sim^* \mu'(r)_y \sim^* \pi'(r)_z$. If $t'' = t \leq t'$, then $\pi(t'')_x \succ^* \mu(t'')_y$ and $\mu'(t'')_y \succsim^* \pi'(t'')_z$. If $t'' = t' \leq t$, then $\pi(t'')_x \succsim^* \mu(t'')_y$ and $\mu'(t'')_y \succ^* \pi'(t'')_z$. Together with Lemma A.3.1, it follows that $\pi(t'')_x \succsim^* \mu(t'')_y \sim^* \mu'(t'')_y \succsim^* \pi'(t'')_z$, with one comparison being strict. Hence, $\pi(t'')_x \succ^* \pi'(t'')_z$. Due to Lemma A.3.4, that yields $x \succsim_L z$. Taken together, it has been established that $\succsim_L$ is transitive.

Consider $x, y \in X$ such that, for all $i \in N$, $i_x \succsim^* i_y$. That is, $N_{yx} = \emptyset$. If $N_{xy} = \emptyset$, then $x \succsim_L y$. Suppose $N_{xy} \neq \emptyset$, meaning that there exists $i \in N$ with $i_x \succ^* i_y$. For x-order $\pi : R(N_{s,xy}) \rightarrow N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \rightarrow N_{s,xy}$, that implies $\pi(1)_x \succ^* \pi(1)_y \succsim^* \mu(1)_y$. It follows that $x \succ_L y$. Hence, $\succsim_L$ satisfies SP.

Take $x, y \in X$ and $i, j \in N$ such that $i_x \sim^* j_y \succ^* j_x \sim^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. In this (and any other) basic two-person situation, $N_{s,xy} = \{i, j\}$ and $R(N_{s,xy}) = \{1, 2\}$. For x-order $\pi : R(N_{s,xy}) \rightarrow N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \rightarrow N_{s,xy}$, $\pi(1) = j$, $\pi(2) = i$, $\mu(1) = i$, and $\mu(2) = j$. Specifically, $\pi(1)_x \sim^* \mu(1)_y$ and $\pi(2)_x \sim^* \mu(2)_y$. That means $x \sim_L y$. Accordingly, $\succsim_L$ satisfies A.

Consider $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y \succ^* j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. Now, x-order $\pi : R(N_{s,xy}) \rightarrow N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \rightarrow N_{s,xy}$ exhibit $\pi(1) = j$, $\pi(2) = i$, $\mu(1) = j$, and $\mu(2) = i$. It follows that $\mu(1)_y \succ^* \pi(1)_x$, implying $y \succ_L x$. Since $y \succsim_L x$, $\succsim_L$ satisfies HE.■

Proof of Lemma 2:
Suppose $\succsim_S$ satisfies FE. Take $x, y \in X$ and $i, j \in N$ such that $i_x \sim^* j_y \succ^* j_x \sim^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. Due to C8, that implies $(i_x, i_y) \sim (j_y, j_x)$. By FE and the completeness of $\succsim_S$, it follows that $x \sim_S y$. Thus, $\succsim_S$ satisfies A. Note that the same argument shows SPD⇒A.■

Proof of Lemma 3:
Suppose $\succsim_S$ satisfies FE. Consider $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y \succ^* j_y \succ^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. Due to MIGC, $(i_x, i_y) \succ (j_y, j_x)$ would imply the contradiction $j_x \succsim^* i_y$. Thus, $(i_x, i_y) \not\succ (j_y, j_x)$ must hold. Taken together, FE yields $x \not\succ_S y$. Since $\succsim_S$ is complete, that results in $y \succsim_S x$. Hence, $\succsim_S$ satisfies HE.■

Proof of Theorem 2:
For "⇒", suppose SO $\succsim_S$ satisfies SP and FE. By Lemmas 2-3, $\succsim_S$ satisfies A and HE. Due to Proposition 2, that yields $\succsim_S = \succsim_L$. For "⇐", take LR $\succsim_L$. By Proposition 2, $\succsim_L$ satisfies SP. Consider $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \not\succ (j_y, j_x)$, $j_x \not\succ^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. By ILC, $i_y \succsim^* j_x$. Take x-order $\pi : R(N_{s,xy}) \rightarrow N_{s,xy}$ and y-order $\mu : R(N_{s,xy}) \rightarrow N_{s,xy}$. Suppose $i_y \sim^* j_x$. Together with $j_y \sim^* j_y$, C8 implies $(j_y, i_y) \sim (j_y, j_x)$. Assume $i_x \succ^* j_y$. By C9, it follows that $(i_x, i_y) \succ (j_y, i_y)$. Due to C1, that yields the contradiction $(i_x, i_y) \succ (j_y, j_x)$. It must thus instead be the case that $j_y \succsim^* i_x$. It follows that $\mu(1)_y \sim^* i_y \sim^* j_x \sim^* \pi(1)_x$ and $\mu(2)_y \sim^* j_y \succsim^* i_x \sim^* \pi(2)_x$. Thus, $y \succsim_L x$. If $i_y \succ^* j_x$, then $\mu(1)_y \succ^* j_x \sim^* \pi(1)_x$ and $y \succ_L x$. In any case, $x \not\succ_L y$. Therefore, $\succsim_L$ satisfies FE.■

2.A.4 Proofs for Section 2.3.3

Lemma A.4.1 states some useful implications.

Lemma A.4.1:
(a) If C2 holds, then, for all $i, j, k, i', j', k' \in N$ and $x, y, z, x', y', z' \in X$, $(i_x, j_y) \sim (i'_{x'}, j'_{y'})$ and $(j_y, k_z) \sim (j'_{y'}, k'_{z'})$ imply $(i_x, k_z) \sim (i'_{x'}, k'_{z'})$.
(b) If C3 holds, then, for all $i, j, k, l \in N$ and $x, y, z, w \in X$, $(i_x, j_y) \sim (k_z, l_w)$ implies $(l_w, k_z) \sim (j_y, i_x)$.
(c) If C3 and IDC hold, then C4 holds.
(d) If C1, C3, and IDC hold, then PLC holds.
Proof:
(a) Consider $i, j, k, i', j', k' \in N$ and $x, y, z, x', y', z' \in X$ with $(i_x, j_y) \sim (i'_{x'}, j'_{y'})$ and $(j_y, k_z) \sim (j'_{y'}, k'_{z'})$. Since $(i_x, j_y) \succsim (i'_{x'}, j'_{y'})$ and $(j_y, k_z) \succsim (j'_{y'}, k'_{z'})$, C2 implies $(i_x, k_z) \succsim (i'_{x'}, k'_{z'})$. Since $(i'_{x'}, j'_{y'}) \succsim (i_x, j_y)$ and $(j'_{y'}, k'_{z'}) \succsim (j_y, k_z)$, C2 also implies $(i'_{x'}, k'_{z'}) \succsim (i_x, k_z)$.

(b) Take $i, j, k, l \in N$ and $x, y, z, w \in X$ with $(i_x, j_y) \sim (k_z, l_w)$. Since $(i_x, j_y) \succsim (k_z, l_w)$, C3 implies $(l_w, k_z) \succsim (j_y, i_x)$. Since $(k_z, l_w) \succsim (i_x, j_y)$, C3 likewise implies $(j_y, i_x) \succsim (l_w, k_z)$.

(c) Consider $i, j \in N$ and $x, y \in X$. By IDC, $(i_x, i_x) \succsim (j_y, j_y)$ or $(j_y, j_y) \succsim (i_x, i_x)$. By C3, the former implies $(j_y, j_y) \succsim (i_x, i_x)$, while the latter implies $(i_x, i_x) \succsim (j_y, j_y)$. In any case, $(i_x, i_x) \sim (j_y, j_y)$.

(d) Take $i \in N$ and $x, y \in X$. By IDC, $(i_x, i_y) \succsim (i_y, i_y)$ or $(i_y, i_y) \succsim (i_x, i_y)$. The former implies $i_x \succsim^* i_y$. Suppose the latter holds. By C3, it implies $(i_y, i_x) \succsim (i_y, i_y)$. By C4 and C1, it follows that $(i_y, i_x) \succsim (i_x, i_x)$, meaning $i_y \succsim^* i_x$.■

Proof of Lemma 4:
Take $k \in N$ in the sense of RAD. Define relation $\succsim_k$ over $(\{k\} \times X)^2$ such that, for $x, y, z, w \in X$, $(k_x, k_y) \succsim_k (k_z, k_w) \Leftrightarrow (k_x, k_y) \succsim (k_z, k_w)$. The relation has the following properties:

(1) $\succsim_k$ is complete and transitive (by IDC and C1).
(2) For all $x, y, z, w \in X$, $(k_x, k_y) \succsim_k (k_z, k_w)$ implies $(k_w, k_z) \succsim_k (k_y, k_x)$ (by C3).
(3) For all $x, y, z, x', y', z' \in X$, $(k_x, k_y) \succsim_k (k_{x'}, k_{y'})$ and $(k_y, k_z) \succsim_k (k_{y'}, k_{z'})$ imply $(k_x, k_z) \succsim_k (k_{x'}, k_{z'})$ (by C2).
(4) For all $x, y, z, w \in X$ with $(k_x, k_y) \succsim_k (k_z, k_w) \succsim_k (k_x, k_x)$, there exist $x', y' \in X$ with $(k_{x'}, k_y) \sim_k (k_z, k_w)$ and $(k_x, k_{y'}) \sim_k (k_z, k_w)$ (by C1, C4, C7, and R4).

(5) The well-being differences of k are Archimedean (by RAD).

To prove (4), consider $x, y, z, w \in X$ with $(k_x, k_y) \succsim_k (k_z, k_w) \succsim_k (k_x, k_x)$. Suppose $(k_z, k_w) \sim_k (k_x, k_x)$. In that case, set $x' = y$ and $y' = x$. By C4 and C1, that leads to $(k_{x'}, k_y) \sim_k (k_z, k_w)$ and $(k_x, k_{y'}) \sim_k (k_z, k_w)$. Suppose $(k_x, k_y) \sim_k (k_z, k_w)$. Now, $x' = x$ and $y' = y$ yield $(k_{x'}, k_y) \sim_k (k_z, k_w)$ and $(k_x, k_{y'}) \sim_k (k_z, k_w)$. Suppose $(k_x, k_y) \succ_k (k_z, k_w) \succ_k (k_x, k_x)$. By C4 and C1, that implies $(k_x, k_y) \succ_k (k_y, k_y)$ and $(k_z, k_w) \succ_k (k_w, k_w)$. Due to C7, it follows that $k_x \succ^* k_y$ and $k_z \succ^* k_w$. Taken together, R4 implies that there exist $x', y' \in X$ with $(k_{x'}, k_y) \sim_k (k_z, k_w)$ and $(k_x, k_{y'}) \sim_k (k_z, k_w)$.

Due to Theorem 4.2 of Krantz et al. (1971), the five properties imply that there exists $u_k : \{k\} \times X \to \mathbb{R}$ such that, for all $x, y, z, w \in X$, $(k_x, k_y) \succsim_k (k_z, k_w) \Leftrightarrow u_k(k_x) - u_k(k_y) \geq u_k(k_z) - u_k(k_w)$. For every other $v_k : \{k\} \times X \to \mathbb{R}$ that represents comparisons of k's differences in this way, there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that, for all $x \in X$, $v_k(k_x) = \lambda + \mu \cdot u_k(k_x)$.

Define $u : N \times X \to \mathbb{R}$ like this. For $x \in X$, set $u(k_x) = u_k(k_x)$. Fix $x_r \in X$. Take $i \in N \setminus \{k\}$ and $x \in X$. By RAD, there exist $z, w \in X$ with $(i_x, i_{x_r}) \sim (k_z, k_w)$. Set $u(i_x) = u_k(k_z) - u_k(k_w)$. The value is well-defined. For other $z', w' \in X$ with $(i_x, i_{x_r}) \sim (k_{z'}, k_{w'})$, it is true that $u_k(k_{z'}) - u_k(k_{w'}) = u_k(k_z) - u_k(k_w)$.

Claim 1: For all $i \in N$ and $x, y \in X$ with $z, w \in X$ such that $(i_x, i_y) \sim (k_z, k_w)$, it is true that $u(i_x) - u(i_y) = u(k_z) - u(k_w)$.

Together with C1, Claim 1 yields that, for all $i, j \in N$ and $x, y, z, w \in X$ with $x', y', z', w' \in X$ such that $(i_x, i_y) \sim (k_{x'}, k_{y'})$ and $(j_z, j_w) \sim (k_{z'}, k_{w'})$, it is the case that $(i_x, i_y) \succsim (j_z, j_w) \Leftrightarrow (k_{x'}, k_{y'}) \succsim (k_{z'}, k_{w'}) \Leftrightarrow u(k_{x'}) - u(k_{y'}) \geq u(k_{z'}) - u(k_{w'}) \Leftrightarrow u(i_x) - u(i_y) \geq u(j_z) - u(j_w)$. Claim 1 can be reduced to Claim 2.

Claim 2: For all $i \in N$ and $x, y \in X$ with $i_x \succsim^* i_y$ and $z, w \in X$ such that $(i_x, i_y) \sim (k_z, k_w)$, it is true that $u(i_x) - u(i_y) = u(k_z) - u(k_w)$.

To prove Claim 2, consider according $i \in N$ and $x, y \in X$. Distinguish three cases.

1. Suppose $i_x \succsim^* i_y \succsim^* i_{x_r}$. Since $(i_x, i_y) \succsim (i_y, i_y)$ and $(i_y, i_{x_r}) \succsim (i_y, i_{x_r})$, it is true that $(i_x, i_{x_r}) \succsim (i_y, i_{x_r})$. Take $x_1, y_1, z_1, w_1 \in X$ with $(i_x, i_{x_r}) \sim (k_{x_1}, k_{z_1})$ and $(i_y, i_{x_r}) \sim (k_{y_1}, k_{w_1})$. It follows that $(k_{x_1}, k_{z_1}) \succsim (k_{y_1}, k_{w_1}) \succsim (k_{x_1}, k_{x_1})$. Due to (4), there exists $y_2 \in X$ with $(k_{y_2}, k_{z_1}) \sim (k_{y_1}, k_{w_1})$. In turn, $(k_{y_2}, k_{z_1}) \sim (i_y, i_{x_r})$. It follows that $(i_{x_r}, i_y) \sim (k_{z_1}, k_{y_2})$. Combined with $(i_x, i_{x_r}) \sim (k_{x_1}, k_{z_1})$, that implies $(i_x, i_y) \sim (k_{x_1}, k_{y_2})$. Taken together, it follows that $u(k_z) - u(k_w) = u(k_{x_1}) - u(k_{y_2}) = (u(k_{x_1}) - u(k_{z_1})) - (u(k_{y_2}) - u(k_{z_1})) = u(i_x) - u(i_y)$.

2. Suppose $i_x \succsim^* i_{x_r} \succsim^* i_y$. Due to $(i_x, i_{x_r}) \succsim (i_x, i_{x_r})$ and $(i_{x_r}, i_y) \succsim (i_{x_r}, i_{x_r})$, it is the case that $(i_x, i_y) \succsim (i_x, i_{x_r})$. Take $x_1, z_1 \in X$ with $(i_x, i_{x_r}) \sim (k_{x_1}, k_{z_1})$. Since $(k_z, k_w) \succsim (k_{x_1}, k_{z_1}) \succsim (k_z, k_z)$, there exists $w_1 \in X$ with $(k_z, k_{w_1}) \sim (k_{x_1}, k_{z_1})$. Due to $(k_w, k_z) \sim (i_y, i_x)$ and $(k_z, k_{w_1}) \sim (i_x, i_{x_r})$, it follows that $(k_w, k_{w_1}) \sim (i_y, i_{x_r})$. That yields $u(k_z) - u(k_w) = (u(k_z) - u(k_{w_1})) - (u(k_w) - u(k_{w_1})) = u(i_x) - u(i_y)$.

3. Suppose $i_{x_r} \succsim^* i_x \succsim^* i_y$. Now, $(i_{x_r}, i_x) \succsim (i_{x_r}, i_x)$ and $(i_x, i_y) \succsim (i_x, i_x)$ imply $(i_{x_r}, i_y) \succsim (i_{x_r}, i_x)$. Consider $x_1, y_1, z_1, w_1 \in X$ with $(i_{x_r}, i_x) \sim (k_{z_1}, k_{x_1})$ and $(i_{x_r}, i_y) \sim (k_{w_1}, k_{y_1})$. Due to $(k_{w_1}, k_{y_1}) \succsim (k_{z_1}, k_{x_1}) \succsim (k_{w_1}, k_{w_1})$, there exists $x_2 \in X$ with $(k_{w_1}, k_{x_2}) \sim (k_{z_1}, k_{x_1})$. That implies $(k_{w_1}, k_{x_2}) \sim (i_{x_r}, i_x)$ and in turn $(i_x, i_{x_r}) \sim (k_{x_2}, k_{w_1})$. Together with $(i_{x_r}, i_y) \sim (k_{w_1}, k_{y_1})$, it follows that $(i_x, i_y) \sim (k_{x_2}, k_{y_1})$. Moreover, $(k_{y_1}, k_{w_1}) \sim (i_y, i_{x_r})$. That results in $u(k_z) - u(k_w) = u(k_{x_2}) - u(k_{y_1}) = (u(k_{x_2}) - u(k_{w_1})) - (u(k_{y_1}) - u(k_{w_1})) = u(i_x) - u(i_y)$. Taken together, this establishes Claim 2.

Consider $i \in N$ and $x, y \in X$ with $z, w \in X$ such that $(i_x, i_y) \sim (k_z, k_w)$. If $i_x \succsim^* i_y$, Claim 2 implies $u(i_x) - u(i_y) = u(k_z) - u(k_w)$. Suppose $i_y \succsim^* i_x$. Since $(k_w, k_z) \sim (i_y, i_x)$, Claim 2 induces that $u(i_y) - u(i_x) = u(k_w) - u(k_z)$. That is, $u(i_x) - u(i_y) = u(k_z) - u(k_w)$. This establishes Claim 1. Consequently, u represents difference comparisons of $\succsim$.

Take $v : N \times X \to \mathbb{R}$ that represents difference comparisons of $\succsim$. Define $v_k : \{k\} \times X \to \mathbb{R}$ such that, for $x \in X$, $v_k(k_x) = v(k_x)$. It follows that, for all $x, y, z, w \in X$, $(k_x, k_y) \succsim_k (k_z, k_w) \Leftrightarrow v_k(k_x) - v_k(k_y) \geq v_k(k_z) - v_k(k_w)$. As discussed above, there exist $\lambda_k \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that, for all $x \in X$, $v(k_x) = v_k(k_x) = \lambda_k + \mu \cdot u_k(k_x) = \lambda_k + \mu \cdot u(k_x)$. For $i \in N \setminus \{k\}$, define $\lambda_i = v(i_{x_r})$.

Consider $i \in N$ and $x \in X$. If $i = k$, then, by construction, $v(i_x) = \lambda_i + \mu \cdot u(i_x)$. Suppose $i \neq k$. Take $z, w \in X$ such that $(i_x, i_{x_r}) \sim (k_z, k_w)$. It follows that $v(i_x) = \lambda_i + v(i_x) - v(i_{x_r}) = \lambda_i + v(k_z) - v(k_w) = \lambda_i + \mu \cdot (u(k_z) - u(k_w)) = \lambda_i + \mu \cdot u(i_x)$. This establishes the uniqueness claim about u.■

Proof of Proposition 3:
"⇒": Suppose SO $\succsim_S$ satisfies SP and GE. Take $u \in \mathbb{R}^{N \times X}$ in the sense of Lemma 4.

Consider $x, y \in X$ with $x \sim_U y$. That is, $\sum_{k \in N}(u(k_x) - u(k_y)) = \sum_{k \in N_s}(u(k_x) - u(k_y)) = 0$. If $N_s = \emptyset$, then SP induces $x \sim_S y$. Suppose $N_s \neq \emptyset$. Since the sum of utility differences is zero, that implies $N_x \neq \emptyset$ and $N_y \neq \emptyset$. Rename individuals from N_x and N_y such that $N_x = \{i(1), ..., i(n)\}$ and $N_y = \{j(1), ..., j(m)\}$.

Inductively define a sequence of states $z(0), z(1), ..., z(\tilde{s}) \in X$ with $z(0) = y$. Take

$s \in \mathbb{N}$. If $u(i(r)_{z(s-1)}) = u(i(r)_x)$ for all $r \in \{1, ..., n\}$, set $s - 1 = \tilde{s}$. If $u(j(t)_{z(s-1)}) = u(j(t)_x)$ for all $t \in \{1, ..., m\}$, likewise set $s - 1 = \tilde{s}$. Otherwise, let $r \in \{1, ..., n\}$ be the smallest number such that $u(i(r)_x) > u(i(r)_{z(s-1)})$ and $t \in \{1, ..., m\}$ be the smallest number such that $u(j(t)_{z(s-1)}) > u(j(t)_x)$. There are three possibilities.

First, suppose $u(i(r)_x) - u(i(r)_{z(s-1)}) = u(j(t)_{z(s-1)}) - u(j(t)_x)$. In that case, select $z(s)$ such that $u(i(r)_{z(s)}) = u(i(r)_x)$, $u(j(t)_{z(s)}) = u(j(t)_x)$, and, for all $k \in N \setminus \{i(r), j(t)\}$, $u(k_{z(s)}) = u(k_{z(s-1)})$. Such a state exists by R1.

Second, suppose $u(i(r)_x) - u(i(r)_{z(s-1)}) > u(j(t)_{z(s-1)}) - u(j(t)_x)$. Since u represents difference comparisons, R4 implies that there exists $z \in X$ such that $u(i(r)_z) - u(i(r)_{z(s-1)}) = u(j(t)_{z(s-1)}) - u(j(t)_x)$. In that case, select $z(s)$ such that $u(i(r)_{z(s)}) = u(i(r)_z)$, $u(j(t)_{z(s)}) = u(j(t)_x)$, and, for all $k \in N \setminus \{i(r), j(t)\}$, $u(k_{z(s)}) = u(k_{z(s-1)})$. Again, such a state exists by R1.

Third, suppose $u(j(t)_{z(s-1)}) - u(j(t)_x) > u(i(r)_x) - u(i(r)_{z(s-1)})$. Due to R4, there exists $w \in X$ such that $u(j(t)_{z(s-1)}) - u(j(t)_w) = u(i(r)_x) - u(i(r)_{z(s-1)})$. Now select $z(s)$ such that $u(i(r)_{z(s)}) = u(i(r)_x)$, $u(j(t)_{z(s)}) = u(j(t)_w)$, and, for all $k \in N \setminus \{i(r), j(t)\}$, $u(k_{z(s)}) = u(k_{z(s-1)})$. Once again, such a state exists by R1.

In any case, it is true that $u(i(r)_{z(s)}) > u(i(r)_{z(s-1)})$, $u(j(t)_{z(s-1)}) > u(j(t)_{z(s)})$, $u(i(r)_{z(s)}) - u(i(r)_{z(s-1)}) = u(j(t)_{z(s-1)}) - u(j(t)_{z(s)})$, and, for all $k \in N \setminus \{i(r), j(t)\}$, $u(k_{z(s)}) = u(k_{z(s-1)})$. Specifically, $\sum_{k \in N} u(k_{z(s)}) - u(k_{z(s-1)}) = \sum_{k \in N_s} u(k_{z(s)}) - u(k_{z(s-1)}) = u(i(r)_{z(s)}) - u(i(r)_{z(s-1)}) + u(j(t)_{z(s)}) - u(j(t)_{z(s-1)}) = 0$, so that $\sum_{k \in N_s} u(k_{z(s)}) = \sum_{k \in N_s} u(k_{z(s-1)})$. By construction, the antecedent of GE is satisfied. The condition yields $z(s) \sim_S z(s-1)$. Taken together, social transitivity leads to $z(\tilde{s}) \sim_S y$. Moreover, $\sum_{k \in N_s} u(k_{z(\tilde{s})}) = \sum_{k \in N_s} u(k_y) = \sum_{k \in N_s} u(k_x)$.

For all $r \in \{1, ..., n\}$ and $t \in \{1, ..., m\}$, $u(i(r)_x) \geq u(i(r)_{z(\tilde{s})})$ and $u(j(t)_{z(\tilde{s})}) \geq u(j(t)_x)$. Suppose that there exists $r \in \{1, ..., n\}$ with $u(i(r)_x) > u(i(r)_{z(\tilde{s})})$. By construction of the sequence, that implies $u(j(t)_{z(\tilde{s})}) = u(j(t)_x)$ for all $t \in \{1, ..., m\}$. It follows that $\sum_{k \in N_s} u(k_x) > \sum_{k \in N_s} u(k_{z(\tilde{s})})$. This is a contradiction. Hence, $u(i(r)_x) = u(i(r)_{z(\tilde{s})})$ must hold for all $r \in \{1, ..., n\}$. The existence of $t \in \{1, ..., m\}$ with $u(j(t)_{z(\tilde{s})}) > u(j(t)_x)$ would lead to the contradiction $\sum_{k \in N_s} u(k_{z(\tilde{s})}) > \sum_{k \in N_s} u(k_x)$. Thus, $u(j(t)_{z(\tilde{s})}) = u(j(t)_x)$ must hold for all $t \in \{1, ..., m\}$. For all $k \in N_e$, $u(k_{z(\tilde{s})}) = u(k_y) = u(k_x)$. Taken together, SP implies $x \sim_S z(\tilde{s})$. Social transitivity yields $x \sim_S y$.

Consider $x, y \in X$ with $x \succ_U y$. That is, $\sum_{k \in N} u(k_x) - u(k_y) = \sum_{k \in N_s} u(k_x) - u(k_y) > 0$. It follows that $N_x \neq \emptyset$. If $N_y = \emptyset$, then SP induces $x \succ_S y$. Suppose $N_y \neq \emptyset$. Again denoting $N_x = \{i(1), ..., i(n)\}$ and $N_y = \{j(1), ..., j(m)\}$, reconsider sequence

$z(0), z(1), ..., z(\tilde{s}) \in X$ from above. As before, $z(\tilde{s}) \sim_S y$, $\sum_{k \in N_s} u(k_{z(\tilde{s})}) = \sum_{k \in N_s} u(k_y)$ and, for all $r \in \{1, ..., n\}$ and $t \in \{1, ..., m\}$, $u(i(r)_x) \geq u(i(r)_{z(\tilde{s})})$ and $u(j(t)_{z(\tilde{s})}) \geq u(j(t)_x)$. If $u(i(r)_x) = u(i(r)_{z(\tilde{s})})$ held for all $r \in \{1, ..., n\}$, that would yield the contradiction $\sum_{k \in N_s} u(k_x) > \sum_{k \in N_s} u(k_y) = \sum_{k \in N_s} u(k_{z(\tilde{s})}) \geq \sum_{k \in N_s} u(k_x)$. Hence, there exists $r \in \{1, ..., n\}$ with $u(i(r)_x) > u(i(r)_{z(\tilde{s})})$. By construction, that implies $u(j(t)_{z(\tilde{s})}) = u(j(t)_x)$ for all $t \in \{1, ..., m\}$. Again, all $k \in N_e$ exhibit $u(k_{z(\tilde{s})}) = u(k_y) = u(k_x)$. Taken together, SP induces $x \succ_S z(\tilde{s})$. Social transitivity leads to $x \succ_S y$.

It has been shown that, for all $x, y \in X$, $x \sim_U y \Rightarrow x \sim_S y$ and $x \succ_U y \Rightarrow x \succ_S y$. Together with the completeness of $\succsim_U$, that yields $x \succsim_S y \Leftrightarrow x \succsim_U y$ for all $x, y \in X$. Consequently, $\succsim_S = \succsim_U$.

"$\Leftarrow$": Consider UR $\succsim_U$. To begin with, it is a SO. Take $x, y \in X$. If $\sum_{i \in N} u(i_x) - u(i_y) \geq 0$, then $x \succsim_U y$. If $\sum_{i \in N} u(i_x) - u(i_y) < 0$, then $\sum_{i \in N} u(i_y) - u(i_x) = \sum_{i \in N_{s,xy}} u(i_y) - u(i_x) = -\sum_{i \in N_{s,xy}} u(i_x) - u(i_y) = -\sum_{i \in N} u(i_x) - u(i_y) > 0$, so that $y \succ_U x$. Thus, $\succsim_U$ is complete.

Take $x, y, z \in X$ with $x \succsim_U y$ and $y \succsim_U z$. That is, $\sum_{i \in N} u(i_x) - u(i_y) \geq 0$ and $\sum_{i \in N} u(i_y) - u(i_z) \geq 0$. Define $M = N_{s,xy} \cup N_{s,yz} \cup N_{s,xz}$. It is the case that $\sum_{i \in N} u(i_x) - u(i_z) = \sum_{i \in M} u(i_x) - u(i_z) = \sum_{i \in M} u(i_x) - u(i_y) + u(i_y) - u(i_z) = \sum_{i \in M} u(i_x) - u(i_y) + \sum_{i \in M} u(i_y) - u(i_z) = \sum_{i \in N} u(i_x) - u(i_y) + \sum_{i \in N} u(i_y) - u(i_z) \geq 0$. Hence, $x \succsim_U z$, so that $\succsim_U$ is transitive.

Take $x, y \in X$ such that, for all $i \in N$, $i_x \succsim^* i_y$. That implies $u(i_x) \geq u(i_y)$ for all $i \in N$ and in turn $\sum_{i \in N} u(i_x) - u(i_y) \geq 0$. Thus, $x \succsim_U y$. Suppose that, in addition, there exists $j \in N$ with $j_x \succ^* j_y$. Since $u(j_x) > u(j_y)$, that yields $\sum_{i \in N} u(i_x) - u(i_y) > 0$, implying $x \succ_U y$. Hence, $\succsim_U$ satisfies SP.

Take $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \sim (j_y, j_x)$, and, for all $k \in N \backslash \{i, j\}$, $k_x \sim^* k_y$. It follows that $u(i_x) - u(i_y) = u(j_y) - u(j_x)$ and, for all $k \in N \backslash \{i, j\}$, $u(k_x) - u(k_y) = 0$. That implies $\sum_{k \in N} u(k_x) - u(k_y) = u(i_x) - u(i_y) + u(j_x) - u(j_y) = 0$ and in turn $x \sim_U y$. Therefore, $\succsim_U$ satisfies GE.■

Proof of Lemma 5:

Suppose SO $\succsim_S$ satisfies FE. Take $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \sim (j_y, j_x)$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. By NILC, $j_x \not\succ^* i_y$ and $i_y \not\succ^* j_x$. It follows by FE that $x \not\succ_S y$ and $y \not\succ_S x$. Social completeness yields $x \sim_S y$. Hence, $\succsim_S$ satisfies GE.■

Proof of Theorem 3:

For "⇒", suppose that SO $\succsim_S$ satisfies SP and FE. By Lemma 5, $\succsim_S$ also satisfies GE. By Proposition 3, that leads to $\succsim_S = \succsim_U$. For "⇐", consider UR $\succsim_U$. Due to Proposition 3, $\succsim_U$ satisfies SP. Take $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \not\succ (j_y, j_x)$, $j_x \not\succ^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. By IDC, that implies $(j_y, j_x) \succsim (i_x, i_y)$ and thus $u(j_y) - u(j_x) \geq u(i_x) - u(i_y)$. Since $u(k_x) = u(k_y)$ for all $k \in N \setminus \{i, j\}$, it follows that $\sum_{k \in N} u(k_y) - u(k_x) = u(j_y) - u(j_x) + u(i_y) - u(i_x) \geq 0$. That means $y \succsim_U x$, so that $x \not\succ_U y$. Hence, $\succsim_U$ satisfies FE.■

Lemma A.4.2: Suppose R4 and IDC hold. Consider $i_x, i_y \in N \times X$ with $i_x \succ^* i_y$. There is no $\tilde{z} \in X$ such that $i_x \succ^* i_{\tilde{z}} \succ^* i_y$ if and only if, for all $j_z, j_w \in N \times X$ with $j_z \succ^* j_w$, $(j_z, j_w) \succsim (i_x, i_y)$.

Proof:

For "⇒", suppose that there exist $j_z, j_w \in N \times X$ with $j_z \succ^* j_w$ and $(j_z, j_w) \not\succsim (i_x, i_y)$. IDC implies $(i_x, i_y) \succ (j_z, j_w)$. By R4, there exists $x' \in X$ with $(i_{x'}, i_y) \sim (j_z, j_w)$. By C7, $(j_z, j_w) \succ (j_w, j_w)$. Due to C4 and C1, that implies $(i_{x'}, i_y) \succ (i_y, i_y)$. In turn, C7 leads to $i_{x'} \succ^* i_y$. Since C1 gives $(i_x, i_y) \succ (i_{x'}, i_y)$, C6 induces $(i_x, i_{x'}) \succ (i_{x'}, i_{x'})$. By C7, it follows that $i_x \succ^* i_{x'}$. Taken together, $i_x \succ^* i_{x'} \succ^* i_y$. For "⇐", suppose that there exists $z \in X$ such that $i_x \succ^* i_z \succ^* i_y$. Due to C9, that yields $(i_x, i_y) \succ (i_x, i_z)$.■

Proof of Corollary 1:

For "⇐", note that UR $\succsim_U$ satisfies SP and FE by Theorem 3. This is so in particular if $\succsim_U = \succsim_B$. For "⇒", suppose that SO $\succsim_S$ satisfies SP and FE. Due to Theorem 3, that implies $\succsim_S = \succsim_U$.

As noted, the fact that k in the sense of RAD has finitely many well-being levels implies that k's well-being differences are Archimedean.[65] To see that, take standard sequence $x_1, x_2, ... \in X$ of k. By IDC, there are two possibilities. First, suppose $(k_{x_2}, k_{x_1}) \succ (k_{x_1}, k_{x_1})$. It follows that $k_{x_{m+1}} \succ^* k_{x_m}$ for all x_m, x_{m+1} in the sequence. From one element to the next, k improves by at least one well-being level. Since k only has finitely many well-being levels, this implies that the sequence is finite. Second, suppose $(k_{x_1}, k_{x_1}) \succ (k_{x_2}, k_{x_1})$. That implies $(k_{x_1}, k_{x_2}) \succ (k_{x_2}, k_{x_2})$. For all x_m, x_{m+1} in the sequence, $(k_{x_1}, k_{x_2}) \sim (k_{x_m}, k_{x_{m+1}})$. Taken together, that yields $k_{x_m} \succ^* k_{x_{m+1}}$. Now, k moves down at least one well-being level from one element to the next. Again, it follows that the sequence is finite because k has finitely many well-being levels.

It remains to be shown that $\succsim_U = \succsim_B$. For $i_x, i_y \in N \times X$ such that $u(i_x) > u(i_y)$ and there is no $\tilde{z} \in X$ with $u(i_x) > u(i_{\tilde{z}}) > u(i_y)$, define $\delta = u(i_x) - u(i_y)$. To see

[65]The same is true for all other individuals.

that δ is well-defined, consider any other $j_z, j_w \in N \times X$ such that $u(j_z) > u(j_w)$ and there is no $\tilde{w} \in X$ with $u(j_z) > u(j_{\tilde{w}}) > u(j_w)$. By Lemma A.4.2, it follows that $u(j_z) - u(j_w) = u(i_x) - u(i_y)$.

Consider $x, y \in X$. For each $i \in N$, it is the case that $u(i_x) - u(i_y) = \delta \cdot \big(|\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}|\big)$. Suppose $x \sim_B y$. That is, $\sum_{i \in N} |\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}| = 0$. It follows that $\sum_{i \in N} u(i_x) - u(i_y) = \sum_{i \in N_s} u(i_x) - u(i_y) = \sum_{i \in N_s} \delta \cdot \big(|\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}|\big) = \delta \cdot \sum_{i \in N_s} |\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}| = \delta \cdot \sum_{i \in N} |\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}| = 0$. Hence, $x \sim_U y$.

Suppose $x \succ_B y$. That means $\sum_{i \in N} |\{u(i_z) \mid u(i_x) > u(i_z)\}| - |\{u(i_z) \mid u(i_y) > u(i_z)\}| > 0$. It follows analogously that $\sum_{i \in N} u(i_x) - u(i_y) > 0$, implying $x \succ_U y$. The established implications and the completeness of $\succsim_B$ induce $x \succsim_U y \Leftrightarrow x \succsim_B y$ for all $x, y \in X$. Accordingly, $\succsim_U = \succsim_B$.■

2.A.5 Proofs for Section 2.3.4

For $x, y \in X$ and $i \in N_{s,xy}$, let $x_i, y_i \in X$ be such that $\{x_i, y_i\} = \{x, y\}$ and $i_{x_i} \succ^* i_{y_i}$. Define $D(i, \{x, y\}) \subseteq N_{s,xy}$ in the following way. For $j \in N_{s,xy}$, $j \in D(i, \{x, y\})$ if and only if $(j_{x_j}, j_{y_j}) \succsim (i_{x_i}, i_{y_i})$, and $i_{y_i} \succsim^* j_{y_j}$. Set $D(i, \{x, y\})$ contains those individuals from $N_{s,xy}$ whose well-being differences between x and y weakly dominate i's well-being difference between x and y in the sense of SPD. In particular, reflexivity of $\succsim$ and $\succsim^*$ imply $i \in D(i, \{x, y\})$. For $M \subseteq N_{s,xy}$, denote $D(M, \{x, y\}) = \bigcup_{i \in M} D(i, \{x, y\})$. For j from $N_{s,xy}$ to be in $D(M, \{x, y\})$, there must be i from M whose well-being difference between x and y is weakly dominated by j's well-being difference between x and y.

For Lemmas A.5.1-A.5.6, let $\{x, y\}$ be fixed. For $i \in N_s$ and $M \subseteq N_s$, denote $D(i) = D(i, \{x, y\})$ and $D(M) = D(M, \{x, y\})$. For $S \subseteq N_s$, denote $S_x = S \cap N_x$ and $S_y = S \cap N_y$. That is, $S = S_x \cup S_y$.

Lemma A.5.1: For all $i, j, k \in N$ with $i \in D(j)$ and $j \in D(k)$, it is true that $i \in D(k)$.
Proof:
Take $i, j, k \in N$ with $i \in D(j)$ and $j \in D(k)$. It follows that $(i_{x_i}, i_{y_i}) \succsim (j_{x_j}, j_{y_j})$, $j_{y_j} \succsim^* i_{y_i}$, $(j_{x_j}, j_{y_j}) \succsim (k_{x_k}, k_{y_k})$, and $k_{y_k} \succsim^* j_{y_j}$. Transitivity of $\succsim$ and $\succsim^*$ yields $(i_{x_i}, i_{y_i}) \succsim (k_{x_k}, k_{y_k})$, and $k_{y_k} \succsim^* i_{y_i}$. Hence, $i \in D(k)$.■

Lemma A.5.2: For all $M, M' \subseteq N_s$,
(a) $M \subseteq D(M)$.
(b) $M \subseteq M' \Rightarrow D(M) \subseteq D(M')$.
(c) $D(M \cup M') = D(M) \cup D(M')$.
(d) $D(D(M) \cap D(M')) = D(M) \cap D(M')$.
Proof:
For (a), take $i \in M$. Due to $i \in D(i)$, that implies $i \in D(M)$. For (b), suppose $M \subseteq M'$. Take $i \in D(M)$. There exists $k \in M$ with $i \in D(k)$. Since $k \in M'$, that implies $i \in D(M')$. For "$\subseteq$" of (c), suppose $i \in D(M \cup M')$. There exists $k \in M \cup M'$ with $i \in D(k)$. If $k \in M$, then $i \in D(M)$. If $k \in M'$, then $i \in D(M')$. Thus, $i \in D(M) \cup D(M')$. For "$\supseteq$" of (c), suppose $i \in D(M) \cup D(M')$. If $i \in D(M)$, there exists $k \in M$ with $i \in D(k)$. If $i \in D(M')$, there exists $k \in M'$ with $i \in D(k)$. In any case, $i \in D(M \cup M')$.

For "$\subseteq$" of (d), consider $i \in D(D(M) \cap D(M'))$. There exists $j \in D(M) \cap D(M')$ with $i \in D(j)$. In turn, there exist $k \in M$ and $l \in M'$ with $j \in D(k)$ and $j \in D(l)$. By A.5.1, it follows that $i \in D(k)$ and $i \in D(l)$. That yields $i \in D(M) \cap D(M')$. On the other hand, "$\supseteq$" of (d) follows by (a).■

Lemma A.5.3: Consider $S \subseteq N_s$ such that, for all $M \subseteq N_s$, $|D(M) \cap S_x| \leq |D(M) \cap S_y|$. For all $M, M' \subseteq N_s$ with $|D(M) \cap S_x| = |D(M) \cap S_y|$ and $|D(M') \cap S_x| = |D(M') \cap S_y|$, $|D(M \cup M') \cap S_x| = |D(M \cup M') \cap S_y|$.
Proof:
Take $M, M' \subseteq N_s$ with $|D(M) \cap S_x| = |D(M) \cap S_y|$ and $|D(M') \cap S_x| = |D(M') \cap S_y|$. By assumption, $|D(M \cup M') \cap S_x| \leq |D(M \cup M') \cap S_y|$. It remains to be shown that $|D(M \cup M') \cap S_x| \geq |D(M \cup M') \cap S_y|$. By A.5.2, (c), $|D(M \cup M') \cap S_x| = |(D(M) \cup D(M')) \cap S_x|$ and $|D(M \cup M') \cap S_y| = |(D(M) \cup D(M')) \cap S_y|$. By A.5.2, (d), $|D(D(M) \cap D(M')) \cap S_x| = |D(M) \cap D(M') \cap S_x|$ and $|D(D(M) \cap D(M')) \cap S_y| = |D(M) \cap D(M') \cap S_y|$. By assumption, $|D(D(M) \cap D(M')) \cap S_x| \leq |D(D(M) \cap D(M')) \cap S_y|$.

Taken together, that yields $|D(M \cup M') \cap S_x| = |(D(M) \cup D(M')) \cap S_x| = |(D(M) \cap S_x) \cup (D(M') \cap S_x)| = |D(M) \cap S_x| + |D(M') \cap S_x| - |(D(M) \cap S_x) \cap (D(M') \cap S_x)| = |D(M) \cap S_x| + |D(M') \cap S_x| - |D(M) \cap D(M') \cap S_x| = |D(M) \cap S_x| + |D(M') \cap S_x| - |D(D(M) \cap D(M')) \cap S_x| \geq |D(M) \cap S_x| + |D(M') \cap S_x| - |D(D(M) \cap D(M')) \cap S_y| = |D(M) \cap S_y| + |D(M') \cap S_y| - |D(M) \cap D(M') \cap S_y| = |D(M) \cap S_y| + |D(M') \cap S_y| - |(D(M) \cap S_y) \cap (D(M') \cap S_y)| = |(D(M) \cap S_y) \cup (D(M') \cap S_y)| = |(D(M) \cup D(M')) \cap S_y| = |D(M \cup M') \cap S_y|$.■

Lemma A.5.4: Consider $S \subseteq N_s$ such that, for all $M \subseteq N_s$, $|D(M) \cap S_x| \leq |D(M) \cap S_y|$. For every $i \in S_x$, there exists $j \in D(i) \cap S_y$ such that, for all $M \subseteq N_s$, $|D(M) \cap S_x \setminus \{i\}| \leq |D(M) \cap S_y \setminus \{j\}|$.
Proof:
Consider $i \in S_x$. Define $\Omega = \{M \subseteq N_s \mid i \notin D(M) \wedge |D(M) \cap S_x| = |D(M) \cap S_y|\}$. Moreover, define $M^* = \bigcup_{M \in \Omega} M$. The repeated application of A.5.3 yields $|D(M^*) \cap S_x| = |D(M^*) \cap S_y|$. Suppose $D(i) \cap S_y \subseteq D(M^*) \cap S_y$. That implies $|D(M^* \cup \{i\}) \cap S_y| = |(D(M^*) \cup D(i)) \cap S_y| = |(D(M^*) \cap S_y) \cup (D(i) \cap S_y)| = |D(M^*) \cap S_y|$. On the other hand, the construction of M^* entails $i \notin D(M^*)$. Because of $i \in D(i)$, that leads to $|D(M^* \cup \{i\}) \cap S_x| = |(D(M^*) \cup D(i)) \cap S_x| = |(D(M^*) \cap S_x) \cup (D(i) \cap S_x)| > |D(M^*) \cap S_x|$. By assumption, $|D(M^* \cup \{i\}) \cap S_x| \leq |D(M^* \cup \{i\}) \cap S_y|$. Together with $|D(M^*) \cap S_x| = |D(M^*) \cap S_y|$, that yields the contradiction $|D(M^* \cup \{i\}) \cap S_x| > |D(M^* \cup \{i\}) \cap S_x|$. Consequently, it must be the case that $D(i) \cap S_y \not\subseteq D(M^*) \cap S_y$.

Take $j \in (D(i) \cap S_y) \setminus (D(M^*) \cap S_y)$. Consider $M \subseteq N_s$. If $|D(M) \cap S_x| < |D(M) \cap S_y|$, then $|D(M) \cap S_x \setminus \{i\}| \leq |D(M) \cap S_y \setminus \{j\}|$ holds anyway. Suppose $|D(M) \cap S_x| = |D(M) \cap S_y|$. If $i \in D(M)$, that implies $|D(M) \cap S_x \setminus \{i\}| \leq |D(M) \cap S_y \setminus \{j\}|$. Suppose $i \notin D(M)$. That means $M \in \Omega$ and $M \subseteq M^*$. By A.5.2, (b), it follows that $D(M) \subseteq D(M^*)$. Since $j \notin D(M^*)$, it is true that $j \notin D(M)$. Taken together, that yields $|D(M) \cap S_x \setminus \{i\}| = |D(M) \cap S_x| = |D(M) \cap S_y| = |D(M) \cap S_y \setminus \{j\}|$. In any case, $|D(M) \cap S_x \setminus \{i\}| \leq |D(M) \cap S_y \setminus \{j\}|$.■

Lemma A.5.5: Suppose that $N_x \neq \emptyset$ and that, for all $M \subseteq N_s$, $|D(M) \cap N_x| \leq |D(M) \cap N_y|$. There exists injective $\pi : N_x \to N_y$ such that, for all $i \in N_x$, $\pi(i) \in D(i)$.
Proof:
Since $D(N_s) = N_s$, it must be the case that $|N_x| \leq |N_y|$. Let $N_x = \{i^1, ..., i^n\}$. Construct π in the following way. Take $S^1 = N_s$, so that $S^1_x = N_x$ and $S^1_y = N_y$. Lemma A.5.4 implies that there exists $j^1 \in D(i^1) \cap S^1_y$ such that, for all $M \subseteq N_s$, $|D(M) \cap S^1_x \setminus \{i^1\}| \leq |D(M) \cap S^1_y \setminus \{j^1\}|$. Set $\pi(i^1) = j^1$. Specify $S^2 = S^1 \setminus \{i^1, j^1\}$, so that $S^2_x = S^1_x \setminus \{i^1\}$ and $S^2_y = S^1_y \setminus \{j^1\}$. Again, A.5.4 implies that there exists $j^2 \in D(i^2) \cap S^2_y$ such that, for all $M \subseteq N_s$, $|D(M) \cap S^2_x \setminus \{i^2\}| \leq |D(M) \cap S^2_y \setminus \{j^2\}|$. Set $\pi(i^2) = j^2$. Continuing this way, one constructs a sequence $S^1, S^2, ..., S^n \subseteq N_s$ with $S^n_x = \{i^n\}$ and a corresponding mapping $\pi : N_x \to N_y$. It exhibits $\pi(i^r) \in D(i^r)$ for all $r \in \{1, ..., n\}$. Take $r, t \in \{1, ..., n\}$ with $r < t$. By construction, that implies $\pi(i^r) \notin S^t_y$ and $\pi(i^t) \in S^t_y$, so that $\pi(i^r) \neq \pi(i^t)$. Hence, π is injective.■

Lemma A.5.6: Take SO $\succsim_S$ satisfying SP and SPD. If, for all $M \subseteq N_s$, $|D(M) \cap N_y| \leq |D(M) \cap N_x|$, then $x \succsim_S y$. If, in addition, $|N_x| > |N_y|$, then $x \succ_S y$.

Proof:
Suppose that, for all $M \subseteq N_s$, $|D(M) \cap N_y| \leq |D(M) \cap N_x|$. That implies $|N_y| \leq |N_x|$. If $N_y = \emptyset$, then $x \succsim_S y$ holds by SP. Suppose $N_y \neq \emptyset$. By Lemma A.5.5, there exists injective $\pi : N_y \to N_x$ such that, for all $i \in N_y$, $\pi(i) \in D(i)$. Rename individuals such that $N_y = \{j^1, ..., j^m\}$, $N_x = \{i^1, ..., i^m, ..., i^n\}$, and, for all $1 \leq r \leq m$, $\pi(j^r) = i^r$. Reconsider sequence $x_0, x_1, ..., x_m \in X$ from the proof of Proposition 1.

Take $1 \leq r \leq m$. To begin with, $i^r_{x_r} \sim^* i^r_x \succ^* i^r_y \sim^* i^r_{x_{r-1}}$ and $j^r_{x_{r-1}} \sim^* j^r_y \succ^* j^r_x \sim^* j^r_{x_r}$. Together with $i^r \in D(j^r)$, C8 implies $(i^r_{x_r}, i^r_{x_{r-1}}) \sim (i^r_x, i^r_y) \succsim (j^r_y, j^r_x) \sim (j^r_{x_{r-1}}, j^r_{x_r})$ and $j^r_{x_r} \sim^* j^r_x \succsim^* i^r_y \sim^* i^r_{x_{r-1}}$. Moreover, $k_{x_r} \sim^* k_{x_{r-1}}$ holds for all $k \in N \setminus \{i^r, j^r\}$. Because its antecedent is satisfied, SPD yields $x_r \succsim_S x_{r-1}$. Taken together, social transitivity implies $x_m \succsim_S y$. Following the argument from the proof of Proposition 1, one shows that $n = m$ implies $x \sim_S x_m$ and that $n > m$ implies $x \succ_S x_m$. Social transitivity yields $x \succsim_S y$ and $x \succ_S y$ in these cases, respectively.■

Lemma A.5.7: Take $x, y, z, w \in X$ with $[x] = [z]$ and $[y] = [w]$. Then, $N_{xy} = N_{zw}$, $N_{yx} = N_{wz}$, and $N_{e,xy} = N_{e,zw}$. Moreover, $D(i, \{x, y\}) = D(i, \{z, w\})$ holds for all $i \in N_{s,xy} = N_{s,zw}$.
Proof:
For each $i \in N$, $i_x \sim^* i_z$ and $i_y \sim^* i_w$. That implies $i \in N_{xy} \Leftrightarrow i_x \succ^* i_y \Leftrightarrow i_z \succ^* i_w \Leftrightarrow i \in N_{zw}$. Analogously, $i \in N_{yx} \Leftrightarrow i \in N_{wz}$ and $i \in N_{e,xy} \Leftrightarrow i \in N_{e,zw}$. For $i \in N_{s,zw}$, let $z_i, w_i \in X$ be such that $\{z_i, w_i\} = \{z, w\}$ and $i_{z_i} \succ^* i_{w_i}$. It follows that, for all $i \in N_{s,xy} = N_{s,zw}$, $i_{x_i} \sim^* i_{z_i}$ and $i_{y_i} \sim^* i_{w_i}$. Consider $i \in N_{s,xy}$. For $j \in N_{s,xy}$, it is the case that $j \in D(i, \{x, y\}) \Leftrightarrow (j_{x_j}, j_{y_j}) \succsim (i_{x_i}, i_{y_i}) \wedge i_{y_i} \succsim^* j_{y_j} \Leftrightarrow (j_{z_j}, j_{w_j}) \succsim (i_{z_i}, i_{w_i}) \wedge i_{w_i} \succsim^* j_{w_j} \Leftrightarrow j \in D(i, \{z, w\})$.■

Proof of Proposition 4:
"⇒": Suppose SO $\succsim_S$ satisfies SP and SPD. In the following, $d : N \times \{[x] \mid x \in X\}^2 \to \mathbb{R}$ will be constructed such that $\succsim_S$ is an AWR with d. Consider $x, y \in X$. Values of d for $[x]$ and $[y]$ will depend on the specification of (1) N_{xy}, N_{yx}, and $N_{e,xy}$, (2) $D(M, \{x, y\})$ for some distinguished $M \subseteq N_{xy} \cup N_{yx}$, and (3) the social ranking of x and y. They will be well-defined for the following reasons.

Take any other $z, w \in X$ with $[x] = [z]$ and $[y] = [w]$. Lemma A.5.7 implies $N_{xy} = N_{zw}$, $N_{yx} = N_{wz}$, and $N_{e,zw} = N_{e,xy}$. It also implies $D(M, \{x, y\}) = D(M, \{z, w\})$ for all $M \subseteq N_{s,xy} = N_{s,zw}$. That is so because $D(M, \{x, y\}) = \bigcup_{i \in M} D(i, \{x, y\}) = \bigcup_{i \in M} D(i, \{z, w\}) = D(M, \{z, w\})$. Moreover, SP induces $x \sim_S z$ and $y \sim_S w$. It follows by social transitivity that $x \succ_S y \Leftrightarrow z \succ_S w$, $y \succ_S x \Leftrightarrow w \succ_S z$, and $x \sim_S y \Leftrightarrow$

$z \sim_S w$.

For $i \in N_{e,xy}$, set $d(i,[x],[y]) = d(i,[y],[x]) = 0$. Abbreviate $D(i) = D(i,\{x,y\})$ and $D(M) = D(M,\{x,y\})$ for $i \in N_s$ and $M \subseteq N_s$. Distinguish the following cases.

1. Suppose $x \succ_S y$ and $|N_x| > |N_y|$. For $i \in N_x$, set $d(i,[x],[y]) = -d(i,[y],[x]) = 1$. For $i \in N_y$, set $d(i,[x],[y]) = -d(i,[y],[x]) = -1$. By construction, d satisfies property (a) for x and y. For all $i,j \in N_s$, $d(j,[x_j],[y_j]) = 1 = d(i,[x_i],[y_i])$, so that d satisfies (b) for x and y. Since $\sum_{i \in N} d(i,[x],[y]) = -\sum_{i \in N} d(i,[y],[x]) = |N_x| - |N_y| > 0$, d also satisfies (c) for x and y.

2. Suppose $x \succ_S y$ and $|N_x| \leq |N_y|$. If, for all $M \subseteq N_s$, $|D(M) \cap N_x| \leq |D(M) \cap N_y|$, Lemma A.5.6 would imply the contradiction $y \succsim_S x$. There must thus exist $M \subseteq N_x \cup N_y$ with $|D(M) \cap N_x| > |D(M) \cap N_y|$.[66] Define $D(M)^c = N_s \setminus D(M)$. For $i \in D(M)^c \cap N_x$, set $d(i,[x],[y]) = -d(i,[y],[x]) = 1$. For $i \in D(M)^c \cap N_y$, set $d(i,[x],[y]) = -d(i,[y],[x]) = -1$. For $i \in D(M) \cap N_x$, set $d(i,[x],[y]) = -d(i,[y],[x]) = \lambda > |D(M)^c \cap N_y| - |D(M)^c \cap N_x| \geq 1$. For $i \in D(M) \cap N_y$, set $d(i,[x],[y]) = -d(i,[y],[x]) = -\lambda$. By construction, this specification satisfies (a) for x and y.

Take $i,j \in N_s$ with $j \in D(i)$. Suppose $i \in D(M)$. Then, there exists $k \in M$ with $i \in D(k)$. By A.5.1, that implies $j \in D(k)$ and thus $j \in D(M)$. It follows that $d(j,[x_j],[y_j]) = \lambda = d(i,[x_i],[y_i])$. Suppose $i \in D(M)^c$. Since $d(j,[x_j],[y_j]) \in \{1,\lambda\}$, that yields $d(j,[x_j],[y_j]) \geq 1 = d(i,[x_i],[y_i])$. Hence, d satisfies (b) for x and y.

It is true that $\sum_{i \in N} d(i,[x],[y]) = -\sum_{i \in N} d(i,[y],[x]) = \sum_{i \in D(M)} d(i,[x],[y]) + \sum_{i \in D(M)^c} d(i,[x],[y]) = (|D(M) \cap N_x| - |D(M) \cap N_y|) \cdot \lambda + |D(M)^c \cap N_x| - |D(M)^c \cap N_y| \geq \lambda + |D(M)^c \cap N_x| - |D(M)^c \cap N_y| > 0$. Therefore, d satisfies (c) for x and y.

3. Suppose $y \succ_S x$ and $|N_y| > |N_x|$. Set $d(i,[x],[y]) = -d(i,[y],[x]) = 1$ for $i \in N_x$ and $d(i,[x],[y]) = -d(i,[y],[x]) = -1$ for $i \in N_y$. With an analogous argument to the first case, one shows that d satisfies (a)-(c) for x and y.

4. Suppose $y \succ_S x$ and $|N_y| \leq |N_x|$. Due to Lemma A.5.6, there must exist $M \subseteq N_s$ with $|D(M) \cap N_y| > |D(M) \cap N_x|$. Construct d for x and y analogously to the second case. It then satisfies (a)-(c) for x and y.

5. Suppose $x \sim_S y$ and $|N_x| = |N_y|$.[67] Specify $d(i,[x],[y]) = -d(i,[y],[x]) = 1$ for $i \in N_x$ and $d(i,[x],[y]) = -d(i,[y],[x]) = -1$ for $i \in N_y$. Analogous to cases 1 and 3, d satisfies (a)-(b) for x and y. Moreover, $\sum_{i \in N} d(i,[x],[y]) = -\sum_{i \in N} d(i,[y],[x]) =$

[66] In particular, $N_x \neq \emptyset$.

[67] This includes the possibility $|N_x| = |N_y| = 0$.

$|N_x| - |N_y| = 0$. Thus, d also satisfies (c) for x and y.

6. Suppose $x \sim_S y$ and $|N_x| > |N_y|$. If, for all $M \subseteq N_s$, $|D(M) \cap N_y| \leq |D(M) \cap N_x|$, Lemma A.5.6 would imply the contradiction $x \succ_S y$. Accordingly, there must exist $M \subseteq N_s$ with $|D(M) \cap N_y| > |D(M) \cap N_x|$.[68] Again, denote $D(M)^c = N_s \setminus D(M)$. Define $\lambda = \frac{|D(M)^c \cap N_x| - |D(M)^c \cap N_y|}{|D(M) \cap N_y| - |D(M) \cap N_x|} = \frac{|N_x| - |D(M) \cap N_x| - |N_y| + |D(M) \cap N_y|}{|D(M) \cap N_y| - |D(M) \cap N_x|} > 1$. Set $d(i,[x],[y]) = -d(i,[y],[x]) = 1$ for $i \in D(M)^c \cap N_x$ and $d(i,[x],[y]) = -d(i,[y],[x]) = -1$ for $i \in D(M)^c \cap N_y$. Set $d(i,[x],[y]) = -d(i,[y],[x]) = \lambda$ for $i \in D(M) \cap N_x$ and $d(i,[x],[y]) = -d(i,[y],[x]) = -\lambda$ for $i \in D(M) \cap N_y$. The specification satisfies (a) for x and y.

Consider $i, j \in N_s$ with $j \in D(i)$. Suppose $i \in D(M)$, so that there exists $k \in M$ with $i \in D(k)$. Again, A.5.1 implies $j \in D(k)$. Since $j \in D(M)$, $d(j,[x_j],[y_j]) = \lambda = d(i,[x_i],[y_i])$. Suppose $i \in D(M)^c$. Due to $d(j,[x_j],[y_j]) \in \{1, \lambda\}$, it follows that $d(j,[x_j],[y_j]) \geq 1 = d(i,[x_i],[y_i])$. Therefore, d satisfies (b) for x and y.

By construction, $\sum_{i \in N} d(i,[x],[y]) = -\sum_{i \in N} d(i,[y],[x]) = \sum_{i \in D(M)} d(i,[x],[y]) + \sum_{i \in D(M)^c} d(i,[x],[y]) = (|D(M) \cap N_x| - |D(M) \cap N_y|) \cdot \lambda + |D(M)^c \cap N_x| - |D(M)^c \cap N_y| = 0$. Consequently, d satisfies (c) for x and y.

7. Suppose $x \sim_S y$ and $|N_x| < |N_y|$. Again, Lemma A.5.6 implies that there exists $M \subseteq N_s$ with $|D(M) \cap N_y| < |D(M) \cap N_x|$. Following the idea of case 6, one constructs values of d for x and y such that d satisfies (a)-(c) for x and y.

Taken together, it has been established how to construct d such that (a)-(c) are satisfied for all $x, y \in X$. Accordingly, $\succsim_S$ is an AWR with d.

"$\Leftarrow$": Suppose SO $\succsim_S$ is an AWR with d. Take $x, y \in X$. Suppose that, for all $i \in N$, $i_x \succsim^* i_y$. That implies $d(i,[x],[y]) \geq 0$ for every $i \in N$. It follows that $\sum_{i \in N} d(i,[x],[y]) \geq 0$, implying $x \succsim_S y$. Suppose that, in addition, there exists $i \in N$ with $i_x \succ^* i_y$. It follows that $d(i,[x],[y]) > 0$. Since $\sum_{k \in N} d(k,[x],[y]) > 0$, $x \succ_S y$. Therefore, $\succsim_S$ satisfies SP.

Consider $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(j_y, j_x) \succsim (i_x, i_y)$, $i_y \succsim^* j_x$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. It follows that $d(j,[y],[x]) \geq d(i,[x],[y])$. Taken together, $\sum_{k \in N} d(k,[y],[x]) = d(j,[y],[x]) + d(i,[y],[x]) = d(j,[y],[x]) - d(i,[x],[y]) \geq 0$, implying $y \succsim_S x$. Hence, $\succsim_S$ satisfies SPD.■

Proof of Lemma 6:
For "$\Rightarrow$", suppose that SO $\succsim_S$ satisfies FE. Suppose that the antecedent of SPD holds.

[68]In particular, $N_y \neq \emptyset$.

It implies $(i_x, i_y) \not\succ (j_y, j_x)$ and $j_x \not\succ^* i_y$. FE and social completeness yield $y \succsim_S x$, so that $\succsim_S$ satisfies SPD. For "$\Leftarrow$", suppose that $\succsim_S$ satisfies SPD. Take $x, y \in X$ and $i, j \in N$ such that $i_x \succ^* i_y$, $j_y \succ^* j_x$, $(i_x, i_y) \not\succ (j_y, j_x)$, $j_x \not\succ^* i_y$, and, for all $k \in N \setminus \{i, j\}$, $k_x \sim^* k_y$. By IDC and ILC, that implies $(j_y, j_x) \succsim (i_x, i_y)$ and $i_y \succsim^* j_x$. Due to SPD, that leads to $y \succsim_S x$. Since $x \not\succ_S y$, $\succsim_S$ satisfies FE.■

Proof of Theorem 4:
For "$\Rightarrow$", suppose SO $\succsim_S$ satisfies SP and FE. By Lemma 6, $\succsim_S$ also satisfies SPD. By Proposition 4, $\succsim_S$ is an AWR. For "$\Leftarrow$", suppose SO $\succsim_S$ is an AWR. By Proposition 4, $\succsim_S$ satisfies SP and SPD. By Lemma 6, $\succsim_S$ satisfies FE.■

Proof of Lemma 7:
Take $k \in N$. By Lemma 4, there exists $u' : N \times X \to \mathbb{R}$ such that, for all $x, y, z, w \in X$, $(k_x, k_y) \succsim (k_z, k_w) \Leftrightarrow u'(k_x) - u'(k_y) \geq u'(k_z) - u'(k_w)$. Define $u : N \times X \to \mathbb{R}$ like this. Take $i \in N$ and $x \in X$. By R2, there exists $y \in X$ with $k_y \sim^* i_x$. Set $u(i_x) = u'(k_y)$. The value is well-defined. For any $z \in X$ with $k_z \sim^* i_x$, C5 and C7 imply $(k_y, k_z) \sim (k_z, k_z)$, which in turn yields $u'(k_y) - u'(k_z) = 0$.

Consider $i, j, i', j' \in N$ and $x, y, z, w \in X$. By R2, there exist $x', y', z', w' \in X$ with $k_{x'} \sim^* i_x$, $k_{y'} \sim^* j_y$, $k_{z'} \sim^* i'_z$, and $k_{w'} \sim^* j'_w$. Due to C8, that implies $(i_x, j_y) \sim (k_{x'}, k_{y'})$ and $(i'_z, j'_w) \sim (k_{z'}, k_{w'})$. Together with C1, it follows that $(i_x, j_y) \succsim (i'_z, j'_w) \Leftrightarrow (k_{x'}, k_{y'}) \succsim (k_{z'}, k_{w'}) \Leftrightarrow u'(k_{x'}) - u'(k_{y'}) \geq u'(k_{z'}) - u'(k_{w'}) \Leftrightarrow u(i_x) - u(j_y) \geq u(i'_z) - u(j'_w)$.

Take $v : N \times X \to \mathbb{R}$ that represents difference comparisons like u. Specifically, for all $i, j \in N$ and $x, y, z, w \in X$, $(i_x, i_y) \succsim (j_z, j_w) \Leftrightarrow u(i_x) - u(i_y) \geq u(j_z) - u(j_w)$ and $(i_x, i_y) \succsim (j_z, j_w) \Leftrightarrow v(i_x) - v(i_y) \geq v(j_z) - v(j_w)$. By Lemma 4, there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that, for all $x \in X$, $v(k_x) = \lambda + \mu \cdot u(k_x)$. Take $i \in N$ and $x \in X$. By R2, there exists $y \in X$ with $k_y \sim^* i_x$. Again, C7 implies $(i_x, k_y) \sim (k_y, k_y)$, so that $u(i_x) - u(k_y) = 0$ and $v(i_x) - v(k_y) = 0$. Taken together, that yields $v(i_x) = v(k_y) = \lambda + \mu \cdot u(k_y) = \lambda + \mu \cdot u(i_x)$.■

2.A.6 Proofs for Section 2.3.5

Proof of Theorem 5:
Suppose there exists SO $\succsim_S$ satisfying SP and FE. Consider $i, j \in N$ and $x, y, z \in X$ in the sense of WTPI. Due to FE, $x \not\succ_S y$, $y \not\succ_S x$, $z \not\succ_S y$, and $y \not\succ_S z$. Social completeness implies $x \sim_S y$ and $y \sim_S z$. By social transitivity, that leads to $x \sim_S z$.

But by SP, it follows that $x \succ_S z$. This is a contradiction.■

Lemma A.6.1: Suppose MPGC holds. Consider $i_x, i_y \in N \times X$ with $i_x \succ^* i_y$. There is no $\tilde{z} \in X$ such that $i_x \succ^* i_{\tilde{z}} \succ^* i_y$ if and only if, for all $i_z, i_w \in N \times X$ with $i_z \succ^* i_w$, $(i_x, i_y) \not\succ (i_z, i_w)$.

Proof:

For "⇒", suppose there is no $\tilde{z} \in X$ such that $i_x \succ^* i_{\tilde{z}} \succ^* i_y$. Take $i_z, i_w \in N \times X$ with $i_z \succ^* i_w$. Suppose $(i_x, i_y) \succ (i_z, i_w)$. By MPGC, it follows that $i_x \succsim^* i_z$ and $i_w \succsim^* i_y$. That implies $i_x \succ^* i_w$ and $i_z \succ^* i_y$. Since i_x and i_y are neighboring, it must thus be the case that $i_x \sim^* i_z$ and $i_y \sim^* i_w$. By C8, that yields $(i_x, i_y) \sim (i_z, i_w)$. This is a contradiction. Hence, $(i_x, i_y) \not\succ (i_z, i_w)$ must hold. For "⇐", suppose there exists $z \in X$ such that $i_x \succ^* i_z \succ^* i_y$. By C9, that implies $(i_x, i_y) \succ (i_x, i_z)$.■

Proof of Proposition 5:

"⇒": Suppose SO $\succsim_S$ satisfies SP and PE. Consider $x, y \in X$. Let $n = |N_x|$ and $m = |N_y|$. Rename individuals in N_x and N_y such that $N_x = \{i(1), ..., i(n)\}$ and $N_y = \{j(1), ..., j(m)\}$. For $i \in N_x$, define $b_i = |\{[i_z] \mid i_x \succ^* i_z \succsim^* i_y\}|$. For $i \in N_y$, define $b_i = |\{[i_z] \mid i_y \succ^* i_z \succsim^* i_x\}|$. It is true that $x \succsim_B y \Leftrightarrow \sum_{i \in N_s} |\{[i_z] \mid i_x \succ^* i_z\}| - |\{[i_z] \mid i_y \succ^* i_z\}| \geq 0 \Leftrightarrow \sum_{i \in N_x} |\{[i_z] \mid i_x \succ^* i_z\}| - |\{[i_z] \mid i_y \succ^* i_z\}| \geq \sum_{i \in N_y} |\{[i_z] \mid i_y \succ^* i_z\}| - |\{[i_z] \mid i_x \succ^* i_z\}| \Leftrightarrow \sum_{p=1}^{n} b_{i(p)} \geq \sum_{p=1}^{m} b_{j(p)}$. Define $\lambda = \sum_{p=1}^{n} b_{i(p)}$ and $\mu = \sum_{p=1}^{m} b_{j(p)}$.

Suppose $x \sim_B y$, so that $\lambda = \mu$. Set $x_0 = y$. Suppose $\mu > 0$. Inductively construct a sequence $x_1, ..., x_\mu \in X$. Take $1 \leq t \leq \mu$. There exist $r \in \{1, ..., n\}$ and $s \in \{1, ..., m\}$ such that $\sum_{p=1}^{r-1} b_{i(p)} < t \leq \sum_{p=1}^{r} b_{i(p)}$ and $\sum_{p=1}^{s-1} b_{j(p)} < t \leq \sum_{p=1}^{s} b_{j(p)}$. State x_t is to exhibit the following properties. First, $i(r)_{x_t} \succ^* i(r)_{x_{t-1}}$ and there exists no $z \in X$ such that $i(r)_{x_t} \succ^* i(r)_z \succ^* i(r)_{x_{t-1}}$. Second, $j(s)_{x_{t-1}} \succ^* j(s)_{x_t}$ and there exists no $z \in X$ such that $j(s)_{x_{t-1}} \succ^* j(s)_z \succ^* j(s)_{x_t}$. Third, $k_{x_t} \sim^* k_{x_{t-1}}$ for all $k \in N \setminus \{i(r), j(s)\}$. Such x_t exists due to R1 and because every $k \in N$ only has finitely many well-being levels.

By Lemma A.6.1, $(i(r)_{x_t}, i(r)_{x_{t-1}})$ is minimal in the sense of PE. The same is true for $(j(s)_{x_{t-1}}, j(s)_{x_t})$. Accordingly, PE implies $x_t \sim_S x_{t-1}$. Since that is true for all $1 \leq t \leq \mu$, the transitivity of $\succsim_S$ yields $x_\mu \sim_S y$. If $\mu = 0$, the latter holds because $\succsim_S$ is reflexive.

Starting from her well-being level at y, each $k \in N_x$ moves up b_k well-being levels from x_0 to x_μ. By construction, her well-being levels at x_μ and x are the same. Likewise, each $k \in N_y$ moves down b_k well-being levels from her well-being level at y to her well-being level at x (= her well-being level at x_μ). That is, $k_{x_\mu} \sim^* k_x$ holds for all $k \in N_s$. The transitivity of $\succsim^*$ implies $k_{x_\mu} \sim^* k_x$ for all $k \in N_e$.[69] Hence, SP implies $x_\mu \sim_S x$. Taken together, the transitivity of $\succsim_S$ yields $x \sim_S y$.

Suppose $x \succ_B y$, implying $\lambda > \mu$. If $\mu > 0$, reconsider the above sequence. Exactly the same argument yields $x_\mu \sim_S y$. If $\mu = 0$, this holds because of $x_0 = y$. From y to x_μ, all individuals from N_y move down to their well-being level at x. All individuals from N_e have the same well-being level at x_μ as at x and y. That is, for all $k \in N_y \cup N_e$, $k_x \sim^* k_{x_\mu}$. However, not all individuals from N_x have moved up to their well-being level at x. Some are still below it at x_μ. For all $k \in N_x$, $k_x \succsim^* k_{x_\mu}$, and there exists $k \in N_x$ with $k_x \succ^* k_{x_\mu}$. SP implies $x \succ_S x_\mu$. Due to the transitivity of $\succsim_S$, that results in $x \succ_S y$.

Taken together, $x \sim_B y \Rightarrow x \sim_S y$ and $x \succ_B y \Rightarrow x \succ_S y$ holds for all $x, y \in X$. Due to the completeness of $\succsim_B$, that yields $x \succsim_B y \Leftrightarrow x \succsim_S y$ for all $x, y \in X$. Therefore, $\succsim_S = \succsim_B$.

"$\Leftarrow$": Suppose $\succsim_S = \succsim_B$. For $i_x \in N \times X$, denote $\beta(i_x) = |\{[i_z] \mid i_x \succ^* i_z\}|$. That is, $x \succsim_B y \Leftrightarrow \sum_{i \in N} \beta(i_x) - \beta(i_y) \geq 0$. First of all, $\succsim_B$ is a SO. Take $x, y \in X$. Since $\sum_{i \in N} \beta(i_x) - \beta(i_y) \geq 0$ or $\sum_{i \in N} \beta(i_y) - \beta(i_x) \geq 0$, it follows that $x \succsim_B y$ or $y \succsim_B x$. Thus, $\succsim_B$ is complete. Take $x, y, z \in X$ with $x \succsim_B y$ and $y \succsim_B z$. That is, $\sum_{i \in N} \beta(i_x) - \beta(i_y) \geq 0$ and $\sum_{i \in N} \beta(i_y) - \beta(i_z) \geq 0$. Denote $N' = N_{s,xy} \cup N_{s,xz} \cup N_{s,yz}$. It follows that $\sum_{i \in N} \beta(i_x) - \beta(i_z) = \sum_{i \in N'} \beta(i_x) - \beta(i_z) = \sum_{i \in N'} \beta(i_x) - \beta(i_y) + \beta(i_y) - \beta(i_z) = \sum_{i \in N'} \beta(i_x) - \beta(i_y) + \sum_{i \in N'} \beta(i_y) - \beta(i_z) = \sum_{i \in N} \beta(i_x) - \beta(i_y) + \sum_{i \in N} \beta(i_y) - \beta(i_z) \geq 0$. Hence, $x \succsim_B z$, so that $\succsim_B$ is transitive.

Consider $x, y \in X$. Suppose that, for all $i \in N$, $i_x \succsim^* i_y$. That implies $\beta(i_x) \geq \beta(i_y)$ for all $i \in N$. In turn, $\sum_{i \in N} \beta(i_x) - \beta(i_y) \geq 0$, so that $x \succsim_B y$. Suppose that, in addition, there exists $i \in N$ with $i_x \succ^* i_y$. That yields $\beta(i_x) > \beta(i_y)$ and $\sum_{i \in N} \beta(i_x) - \beta(i_y) > 0$, leading to $x \succ_B y$. Therefore, $\succsim_B$ satisfies SP.

Suppose the antecedent of PE is satisfied for $x, y \in X$ and $i, j \in N$. By Lemma A.6.1, that implies $\beta(i_x) - \beta(i_y) = 1$, $\beta(j_x) - \beta(j_y) = -1$, and, for all $k \in N \setminus \{i, j\}$, $\beta(k_x) - \beta(k_y) = 0$. It follows that $\sum_{k \in N} \beta(k_x) - \beta(k_y) = 0$, implying $x \sim_B y$. Accordingly, $\succsim_B$ satisfies PE.■

Proof of Lemma 8:
Suppose $\succsim_S$ satisfies FE. By Lemma 1, $\succsim_S$ satisfies SE. Suppose the antecedent of PE

[69] If $\mu > 0$, it is in fact possible to set $x_\mu = x$.

obtains for $x, y \in X$ and $i, j \in N$. Due to SE, $x \sim_S y$. Hence, $\succsim_S$ satisfies PE.■

As noted in the main text, it is also possible to characterize the LBR if intrapersonal difference comparisons are complete.

Personal Difference Comparability (PDC): For all $i \in N$ and $x, y, z, w \in X$, $(i_x, i_y) \succsim (i_z, i_w)$ or $(i_z, i_w) \succsim (i_x, i_y)$.

Lemma A.6.2: Suppose R4 and PDC hold. Consider $i_x, i_y \in N \times X$ with $i_x \succ^* i_y$. There is no $\tilde{z} \in X$ such that $i_x \succ^* i_{\tilde{z}} \succ^* i_y$ if and only if, for all $i_z, i_w \in N \times X$ with $i_z \succ^* i_w$, $(i_z, i_w) \succsim (i_x, i_y)$.
Proof:
For "⇒", suppose that there exist $i_z, i_w \in N \times X$ with $i_z \succ^* i_w$ and $(i_z, i_w) \not\succsim (i_x, i_y)$. By PDC, that implies $(i_x, i_y) \succ (i_z, i_w)$. Due to R4, there exists $x' \in X$ with $(i_{x'}, i_y) \sim (i_z, i_w)$. An analogous argument to the one in the proof of Lemma A.4.2 yields $i_x \succ^* i_{x'} \succ^* i_y$. For "⇐", suppose that there exists $z \in X$ such that $i_x \succ^* i_z \succ^* i_y$. Like in the proof of Lemma A.4.2, C9 implies $(i_x, i_y) \succ (i_x, i_z)$.■

Proposition A.6.1: Suppose R4 and PDC hold. Suppose that $|\{[i_x] \mid x \in X\}| < \infty$ for each $i \in N$. SO $\succsim_S$ satisfies SP and PE if and only if $\succsim_S = \succsim_B$.
Proof:
For "⇒", the proof of Proposition 5 can be replicated. The only difference is that Lemma A.6.2 instead of Lemma A.6.1 guarantees the applicability of PE. For "⇐", it is shown like in the proof of Proposition 5 that $\succsim_B$ satisfies SP. Suppose the antecedent of PE is satisfied for $x, y \in X$ and $i, j \in N$. By PDC, it implies that, for all $i_z, i_w \in N \times X$ with $i_z \succ^* i_w$, $(i_z, i_w) \succsim (i_x, i_y)$, and, for all $j_z, j_w \in N \times X$ with $j_z \succ^* j_w$, $(j_z, j_w) \succsim (j_y, j_x)$. Due to Lemma A.6.2, it follows that $\beta(i_x) - \beta(i_y) = 1$, $\beta(j_x) - \beta(j_y) = -1$, and, for all $k \in N \setminus \{i, j\}$, $\beta(k_x) - \beta(k_y) = 0$. That yields $\sum_{k \in N} \beta(k_x) - \beta(k_y) = 0$ and $x \sim_B y$. Consequently, $\succsim_B$ also satisfies PE under PDC.■

2.A.7 Proofs for Section 2.4.3

Proof of Proposition 6:
For all $i \in N$ and $x \in X$, it is the case that $i_x \sim^* k_{s(i_x)}$. To see that, consider $i \in N$ and $x \in X$. Take $y \in S_i$ with $x \sim_i y$ and $y \sim_S s(i_x)$. On the one hand, P1 implies $i_x \sim^* i_y$. On the other hand, P2 induces $i_y \sim^* k_{s(i_x)}$. By C5, that yields $i_x \sim^* k_{s(i_x)}$.

Consider $i, j, h, l \in N$ and $x, y, z, w \in X$. Due to the last paragraph, $i_x \sim^* k_{s(i_x)}$,

$j_y \sim^* k_{s(j_y)}$, $h_z \sim^* k_{s(h_z)}$, and $l_w \sim^* k_{s(l_w)}$. It follows by C8 that $(i_x, j_y) \sim (k_{s(i_x)}, k_{s(j_y)})$ and $(h_z, l_w) \sim (k_{s(h_z)}, k_{s(l_w)})$. By C1, that implies $(i_x, j_y) \succsim (h_z, l_w) \Leftrightarrow (k_{s(i_x)}, k_{s(j_y)}) \succsim (k_{s(h_z)}, k_{s(l_w)})$. By P3, $(k_{s(i_x)}, k_{s(j_y)}) \succsim (k_{s(h_z)}, k_{s(l_w)}) \Leftrightarrow c(s(i_x), s(l_w)) \succsim_k c(s(j_y), s(h_z))$. Taken together, that yields $(i_x, j_y) \succsim (h_z, l_w) \Leftrightarrow c(s(i_x), s(l_w)) \succsim_k c(s(j_y), s(h_z))$.

For each $i \in N$, $\succsim_i$ is complete. To see that, take $i \in N$ and $x, y \in X$. By P4, there exist $z, w \in S_i$ with $x \sim_i z$ and $y \sim_i w$. It follows by P1 that $i_x \sim^* i_z$ and $i_y \sim^* i_w$. Since $\succsim_S$ is complete, $z \succsim_S w$ or $w \succsim_S z$ holds. Due to P2, that implies $i_z \succsim^* i_w$ or $i_w \succsim^* i_z$. Taken together, C5 yields $i_x \succsim^* i_y$ or $i_y \succsim^* i_x$.

Take $i, j, h, l \in N$ and $x, y, z, w \in X$. Since $\succsim_k$ is complete, it must either be true that $c(s(i_x), s(l_w)) \succsim_k c(s(j_y), s(h_z))$ or that $c(s(j_y), s(h_z)) \succsim_k c(s(i_x), s(l_w))$. By the established equivalence, that implies $(i_x, j_y) \succsim (h_z, l_w)$ or $(j_y, i_x) \succsim (l_w, h_z)$. By C3, the latter induces $(h_z, l_w) \succsim (i_x, j_y)$. Hence, $\succsim$ is complete.■

Chapter 3

Generalized Utilitarianism and Well-Being Comparability

The chapter offers a unified analysis of Generalized Utilitarianism with weak informational and normative assumptions. It connects different literatures. First, the chapter clarifies informational requirements. Contrary to other discussions, Generalized Utilitarian Orderings (including Prioritarian Orderings) are well-defined given sufficiently rich qualitative well-being difference and level comparisons. Second, the chapter provides characterizations of Generalized Utilitarian Orderings with weak separability and compensation conditions. Existing and new compensation properties are closely related to continuity. Third, the chapter examines attempts to justify Utilitarianism. Maskin's influential characterization implicitly employs a strong substantive invariance requirement on social evaluation. A different characterization of Utilitarianism is based on a new stability condition and neither needs anonymity nor separability assumptions. Fourth, the chapter extends the analysis to social evaluation under uncertainty, characterizing Expected Generalized Utilitarianism and Expected Utilitarianism. It establishes a perfect analogy between the certainty and the uncertainty case.

3.1 Introduction

Should a society give priority to worse off individuals? To the present day, this is a fundamental open question. Two major theories of social evaluation give radically different answers. According to Utilitarianism, the social value of a well-being gain should be independent of the initial well-being level. According to Leximin, a well-being gain of a worse off individual should get absolute priority over a (possibly much larger) well-being gain of a better off individual. Generalized Utilitarianism offers a compromise by allowing the social value of a well-being gain to decrease with the initial well-being level.

The purpose of the present chapter is to provide a unified analysis of Generalized Utilitarianism with weak and intuitive informational and normative assumptions. Specifically, the chapter makes four contributions.

First, the chapter clarifies the informational basis of Generalized Utilitarianism. The critical input are qualitative comparisons of the form "individual i at state of the world x is better off than individual j at state of the world y" and "the well-being gain of i from y to x is greater than the well-being gain of j from w to z". If such comparisons are complete and transitive and some additional assumptions hold, it is possible to measure well-being difference ratios. Generalized Utilitarian Orderings are defined in terms of these ratios. No concatenation operation on well-being is required.

Second, the chapter provides an axiomatic analysis of Generalized Utilitarianism in terms of a single utility profile that represents well-being comparisons. Generalized Utilitarian Orderings are characterized with Strong Pareto, a Pigou-Dalton condition, a weak separability property, and a restricted compensation condition. Restricted compensation is shown to be equivalent to continuity. A new intuitive compensation condition for small well-being changes leads to a characterization of all surjective Generalized Utilitarian Orderings.

Third, the chapter investigates two axiomatic approaches to Utilitarianism. Each one derives the property that equal well-being gains are socially indifferent. It is shown that a single-profile version of Maskin's (1978) famous characterization implicitly employs a strong substantive invariance requirement on social evaluation. An alternative characterization employs Strong Pareto, restricted compensation, and a new condition to the effect that the social ordering should be stable under small differences in well-being differences. Remarkably, the result neither uses an anonymity nor a separability property.

Fourth, the chapter extends the analysis to the social evaluation of uncertain prospects. It works out a close formal analogy between the certainty and the uncertainty case. Qualitative ex post well-being comparisons form the informational basis in both cases. The uncertainty counterparts of the certainty conditions lead to characterizations of Expected Generalized Utilitarianism and Expected Utilitarianism. They imply that social inequality and risk aversion coincide.

Generalized Utilitarianism is a central subject in different literatures. The chapter aims to contribute to and connect these separate literatures. First of all, it is a contribution to welfare economics and the theory of Social Welfare Functionals (see d'Aspremont and Gevers, 2002; Bossert and Weymark, 2004; Weymark, 2016; Sen, 2017: chapter A3*). Specifically, the chapter contributes to the single-profile literature (Roberts, 1980c; Blackorby et al., 1990; Fleurbaey and Mongin, 2005; Blackorby et al., 2006; chapter 2 of this thesis).

The chapter does not employ the inter-profile conditions of Binary Independence and Strong Neutrality (critically discussed in Morreau and Weymark, 2016). This is a key difference to existing analysis of Utilitarianism and Generalized Utilitarianism in terms of Social Welfare Functionals (d'Aspremont and Gevers, 1977; Maskin, 1978; Deschamps and Gevers, 1978; Gevers, 1979; Roberts, 1980b; Blackorby et al., 2002; Balasubramanian, 2015; Yamamura, 2017; Bossert and Kamaga, 2020). That analysis is concerned with a more restrictive notion of Generalized Utilitarianism which requires a richer informational basis than well-being difference ratios.

The chapter also contributes to the literature on utilitarianism, prioritarianism, and egalitarianism at the intersection of economics and philosophy (Parfit, 1991; Broome, 1991; McCarthy, 2006, 2008, 2017; Fleurbaey, 2015; Holtug, 2015; Adler, 2019; Buchak, 2023). Despite the same subject, there are just few cross citations between this literature and the one on Social Welfare Functionals. The characterized Generalized Utilitarian class contains Prioritarian Orderings (assigning greater social value to a gain from a lower starting level) and the Utilitarian Ordering. The offered axiomatic analysis improves the discussion on the merit of prioritarianism and utilitarianism.

Methodologically, the chapter works out the connection between Generalized Utilitarianism and two different approaches to additive conjoint measurement (Wakker, 1989, 1991; Bouyssou and Pirlot, 2005; Ghosh et al., 2023). While the better known topological approach (due to Debreu, 1960) is based on continuity, the algebraic approach (due to Krantz et al., 1971) is based on solvability (compensation). The results of the chapter stress the close relation between the two approaches. They also incorporate insights to derive full separability from weak separability assumptions due to Gorman (1968) and Blackorby et al. (1998).

The chapter contributes to work on additively separable aggregation in the context of risk and uncertainty (Harsanyi, 1953, 1955, 1982; Weymark, 1991; McCarthy, 2006, 2008, 2017; Grant et al., 2010; Mongin and Pivato, 2015; Fleurbaey and Mongin, 2016;

Eden, 2020; McCarthy et al., 2020; Gustafsson et al., 2023; Li et al., 2023; Pivato and Tchouante, 2023). A distinguishing feature of the present chapter is that it provides an analogous analysis of the certainty and the uncertainty case.

The chapter also relates to contributions on well-being measurement. Economists tend to be skeptical about the possibility of making interpersonal well-being comparisons. However, the social evaluation of alternatives requires to assess trade-offs between different individuals. It is thus a reasonable approach to construct interpersonal well-being comparisons and then employ them for social evaluation. There are several promising procedures, including happiness comparisons (Hirschauer et al., 2015; Perez-Truglia, 2020), comparisons based on extended preferences (Harsanyi, 1953, 1955; Adler, 2014), and comparisons of resources (Fleurbaey and Maniquet, 2008, 2011, 2017; Fleurbaey and Tadenuma, 2014; Piacquadio, 2017; Bosmans et al., 2018). The latter literature also analyzes generalized utilitarian aggregation of resource-based well-being measures. The present chapter is consistent with any kind of well-being conception.

The chapter takes the following structure. Section 3.2 discusses well-being comparisons and utility representations (3.2.1) as well as the definition of Generalized Utilitarianism (3.2.2). Section 3.3 establishes characterizations of Generalized Utilitarian classes (3.3.1) and Utilitarianism (3.3.2). Section 3.4 extends the analysis to social prospect evaluation. Section 3.5 concludes. Appendix 3.A contains all proofs (3.A.1 to 3.A.3) and independence checks (3.A.4).

3.2 Framework

3.2.1 Well-Being and Utility

Let $N = \{1, ..., n\}$ be a population of individuals and X be a set of conceivable states of the world, where $n \geq 3$. It is assumed that each individual from N exists at all states from X. A variable population extension is discussed in section 3.5.

In the entire chapter, qualitative well-being level and difference comparisons between individuals at states of the world form the input of aggregation. They can be captured by a difference relation $\succsim_d$ over $(N \times X)^2$.[1] For $i, j, k, l \in N$ and $x, y, z, w \in X$,

[1]A relation R over a set B is a subset of $B \times B$. For $a, b \in B$, write aRb instead of $(a, b) \in R$. Relation R is complete if, for all $a, b \in B$, aRb or bRa holds. It is transitive if, for all $a, b, c \in B$, aRb and bRc imply aRc.

$\big((i,x),(j,y)\big) \succsim_d \big((k,z),(l,w)\big)$ means that the well-being difference between i at x and j at y is at least as great as the well-being difference between k at z and l at w. Let $\succ_d$ and $\sim_d$ be the asymmetric and the symmetric part of $\succsim_d$.[2] That is, $\big((i,x),(j,y)\big) \succ_d \big((k,z),(l,w)\big)$ means that the well-being difference between i at x and j at y is greater than the well-being difference between k at z and l at w. Likewise, $\big((i,x),(j,y)\big) \sim_d \big((k,z),(l,w)\big)$ means that the former is as great as the latter.

Well-being level comparisons can be conceived as special well-being difference comparisons. For $i,j \in N$ and $x,y \in X$, $\big((i,x),(j,y)\big) \succsim_d \big((j,y),(j,y)\big)$ means that the well-being difference between i at x and j at y is at least as great as the zero difference between j at y and j at y. In other words, i at x is at least as well off as j at y.

It is assumed throughout the chapter that qualitative well-being difference and level comparisons can be represented by a utility profile $u : N \times X \to \mathbb{R}$. For $i \in N$ and $x \in X$, denote $u_i(x) = u(i,x)$ and $u(x) = (u_1(x), ..., u_n(x))$.

A1: There exists $u : N \times X \to \mathbb{R}$ such that, for all $i,j,k,l \in N$ and $x,y,z,w \in X$, $\big((i,x),(j,y)\big) \succsim_d \big((k,z),(l,w)\big) \Leftrightarrow u_i(x) - u_j(y) \geq u_k(z) - u_l(w)$. For every other $v : N \times X \to \mathbb{R}$ that represents $\succsim_d$ in this way, there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_{++}$ such that, for all $i \in N$ and $x \in X$, $v_i(x) = \lambda + \mu \cdot u_i(x)$.[3]

Given A1, profile u represents all intra- and interpersonal well-being level and difference comparisons from $\succsim_d$. Individual i at state of the world x is at least as well off as individual j at state of the world y if and only if $u_i(x) \geq u_j(y)$. Likewise, the well-being difference between i at x and j at y is at least as great as the well-being difference between k at z and l at w if and only if $u_i(x) - u_j(y) \geq u_k(z) - u_l(w)$. Moreover, well-being difference ratios are well-defined. The value $\frac{u_i(x)-u_j(y)}{u_k(z)-u_l(w)}$ is invariant under all utility profiles that represent $\succsim_d$ in the sense of A1.[4] For instance, $u_i(x) - u_j(y) = 6$ and $u_k(z) - u_l(w) = 3$ means that the positive well-being difference between i at x and j at y is twice as large as the positive well-being difference between k at z and l at w.

Krantz et al. (1971), Köbberling (2006), Pivato (2013c), and chapter 2 of this thesis examine conditions on $\succsim_d$ that are sufficient for it to have a utility representation in

[2] The asymmetric part of relation R over B is relation P over B such that, for all $a,b \in B$, aPb if and only if aRb and not bRa. The symmetric part of R is relation I over B such that, for all $a,b \in B$, aIb if and only if aRb and bRa.

[3] Denote $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$ and $\mathbb{R}_{++} = \{\lambda \in \mathbb{R} \mid \lambda > 0\}$.

[4] For every utility profile v in the sense of A1, there exists $\mu \in \mathbb{R}_{++}$ such that $\frac{v_i(x)-v_j(y)}{v_k(z)-v_l(w)} = \frac{\mu\cdot(u_i(x)-u_j(y))}{\mu\cdot(u_k(z)-u_l(w))} = \frac{u_i(x)-u_j(y)}{u_k(z)-u_l(w)}$.

the sense of A1. Lemma 7 of chapter 2 states such conditions. Critically, intra- and interpersonal well-being level and difference comparisons in $\succsim_d$ must be complete and transitive.

Several promising approaches to well-being comparisons have been developed. They can serve as a basis for systematic social evaluation and, specifically, to assess unavoidable trade-offs between different individuals. The remainder of the subsection gives three examples.

Example 1: According to experience-based well-being conceptions, well-being differences between individuals at states of the world are quality differences between the experiences that individuals have at these states (see Heathwood, 2014). Happiness research provides an empirical basis for this approach (Hirschauer et al., 2015; Perez-Truglia, 2020). In happiness surveys, individuals are asked a question of the form "how happy are you on a scale from 0 to 10?". Let $u_i(x) \in \{0, ..., 10\}$ be the response of individual i at state x to this question.

Two assumptions are needed for u to be a utility profile in the sense of A1. First, well-being must be identified with happiness. That is, $\big((i,x),(j,y)\big) \succsim_d \big((k,z),(l,w)\big)$ holds if and only if the happiness difference between i at x and j at y is at least as great as the happiness difference between k at z and l at w. Second, comparisons of score differences must indeed represent comparisons of happiness differences. Specifically, higher reported scores reflect higher levels of happiness within and between individuals. The unit score difference represents a unit difference in happiness.

The assumptions can be challenged. However, if one is able to defend them, one will have a utility profile in the sense of A1.[5] Specifically, the approach gives rise to well-being difference ratios in a natural way. If $u_i(x) = 9$, $u_i(y) = 5$, $u_j(z) = 6$, and $u_j(w) = 4$, i's happiness difference between x and y is twice as large as j's happiness difference between z and w because the former is four times as large as the unit happiness difference, while the latter is twice as large as that unit difference.

Example 2: In line with economic theory, it is possible to construct well-being comparisons based on individual preferences over consumption bundles (Fleurbaey and Maniquet, 2008, 2011, 2017; Fleurbaey and Tadenuma, 2014; Piacquadio, 2017; Bosmans et

[5]Profile u will indeed be unique up to a positive affine transformation. Every other utility profile representing the underlying happiness comparisons can only differ from u by assigning a different number to the smallest happiness level and a different number to the unit happiness difference. These are just two degrees of freedom.

al., 2018). An according example is the following.

Let $B = \mathbb{R}^m_+$ be a set of bundles of m goods. Suppose that each individual has a preference relation over B. Suppose that the set of conceivable states is the set of conceivable allocations, that is, $X = B^n$. Allocation $x = (x_1, ..., x_n) \in X$ contains bundle $x_i \in B$ for individual $i \in N$. Under standard assumptions, there exists unique $\lambda(x_i) \in \mathbb{R}_+$ such that i is indifferent between x_i and $\lambda(x_i) \cdot (1, ..., 1) \in B$. This $\lambda(x_i)$ is the equivalent value of x_i relative to reference bundle $(1, ..., 1)$. It is possible to construct utility profile u in the sense of A1 by setting $u_i(x) = \lambda(x_i)$ for all $i \in N$ and $x \in X$. Such a resource-based approach rests on the assumption that equivalent value differences represent well-being differences. That is, $\big((i, x), (j, y)\big) \succsim_d \big((k, z), (l, w)\big)$ holds if and only if $\lambda(x_i) - \lambda(y_j) \geq \lambda(z_k) - \lambda(w_l)$.

Example 3: The previous example is based on individual preferences. It is also possible to reduce interpersonal well-being level and difference comparisons to preferences of an ethical observer over different lives (Harsanyi, 1953, 1955; Adler, 2014). An according procedure is the following. Let tuple $\big((i, x), (j, y)\big) \in (N \times X)^2$ denote the alternative where the ethical observer first lives the life of i at x and then the life of j at y. Suppose that the observer has a preference relation $\succsim_e$ over such alternatives.

It is possible to set $\big((i, x), (j, y)\big) \succsim_d \big((k, z), (l, w)\big) \Leftrightarrow \big((i, x), (l, w)\big) \succsim_e \big((j, y), (k, z)\big)$. This equivalence rests on the assumption that the observer's weak preference for $\big((i, x),$ $(l, w)\big)$ over $\big((j, y), (k, z)\big)$ reflects that the well-being difference between i's life at x and j's life at y is at least as great as the well-being difference between k's life at z and l's life at w. Additional assumptions secure that there exists a utility profile u representing $\succsim_d$ (and $\succsim_e$) in the sense of A1 (see Krantz et al., 1971: chapter 6). Clearly, there are also issues with this third approach. One of the most important ones is to identify who the relevant ethical observer is. While the approach is elegant, it is also less determinate than the previous two.

In the following, it is assumed that one utility profile u in the sense of A1 is fixed. The following analysis is formulated in terms of u. It employs an idealized richness assumption which allows to focus on substantive issues. There is no smallest well-being gain and individuals can independently realize well-being levels.

A2: There exists an interval $A \subseteq \mathbb{R}$ such that utility profile u exhibits $u(X) = \{a \in \mathbb{R}^n \mid \exists x \in X : u(x) = a\} = A^n$.[6]

[6] The interval can take the form (a) $A = [\lambda, \mu]$ with $\lambda, \mu \in \mathbb{R}$ and $\lambda < \mu$, (b) $A = [\lambda, \infty)$ with $\lambda \in \mathbb{R}$, (c) $A = (-\infty, \mu]$ with $\mu \in \mathbb{R}$, or (d) $A = \mathbb{R}$.

Assumptions on images of utility profiles are standard in the theory of Social Welfare Functionals. Condition A2 differs from a richness assumption on the joint image of utility profiles called domain attainability (see d'Aspremont and Gevers, 2002: p. 467).[7] The rectangular field assumption in McCarthy (2008: pp. 5-6) is a version of A2.

Assumption A2 is in line with economic theory and the resource-based approach to well-being measurement. The utility profile constructed in Example 2 exhibits $u(X) = A^n$ with $A = \mathbb{R}_+$. Assumption A2 can also be defended for the other approaches. For individuals to be able to indicate all possible well-being levels in Example 1, an "idealized" happiness survey will need to allow them to report any number from $A = [0, 10]$.

Importantly, it is possible that the set of feasible states of the world Y in some choice situation is a (finite) subset of X which does not exhibit the richness properties of X embodied in A2. That is, there can be a smallest well-being gain with respect to all feasible states. Moreover, different well-being levels can be feasible for different individuals.

3.2.2 Generalized Utilitarian Social Orderings

The fixed utility profile u allows to compare the relative merit of states of the world for each respective individual. To reach an overall evaluation of states, it is necessary to aggregate this information. That is done by means of an aggregate or social ranking.

A **Social Ordering (SO)** $\succsim$ is a complete and transitive relation over X. For $x, y \in X$, $x \succsim y$ means that state of the world x is socially at least as good as state of the world y. Let $\succ$ and $\sim$ be the asymmetric and the symmetric part of $\succsim$. That is, $x \succ y$ ($x \sim y$) means that state x is socially better than (as good as) state y. These evaluations can reflect the judgments of an ethical observer. Transitivity is a well-established consistency requirement on social evaluation. SO $\succsim$ is also assumed to be complete as it is meant to inform (social) choice. Suppose that only states x and y are feasible in a decision situation. If x is socially better than y, then x should be chosen, and the other way around. If x is socially as good as y, then any of the two states can be selected.

The remainder of the section introduces Generalized Utilitarian approaches to social evaluation. It will be explained that these approaches are well-defined given A1. Their

[7] The latter is almost always combined with the problematic Strong Neutrality condition in characterizations. This condition is employed in none of the following results except for Theorem 3.

merit will be examined in the next section. To begin with, Utilitarianism is one of the most common social evaluation procedures.

SO $\succsim$ is the **Utilitarian Ordering (UO)** if, for all $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{i\in N} u_i(x) \geq \sum_{i\in N} u_i(y)$. Comparisons of utilitarian sums are invariant under all utility profiles v in the sense of A1.[8] However, the absolute value of a state's utilitarian sum is clearly profile-dependent.

There is a way to describe the UO as being based on a profile-independent social value of each state of the world. Supposing $[0, 1] \subseteq A$, take $j \in N$ and $z, w \in X$ with $u_j(z) = 1$ and $u_j(w) = 0$. Under u, j's well-being level at w is the reference zero level, and j's well-being difference between z and w is the reference unit difference. Consider $x \in X$. For $i \in N$, $\frac{u_i(x)-u_j(w)}{u_j(z)-u_j(w)}$ is the unique ratio of the difference between i's well-being level at x and the reference level to the reference difference. It is a profile-independent individual value of x for i. In turn, $\sum_{i\in N} \frac{u_i(x)-u_j(w)}{u_j(z)-u_j(w)}$ is a profile-independent social value of x. According to the UO, $x \succsim y \Leftrightarrow \sum_{i\in N} \frac{u_i(x)-u_j(w)}{u_j(z)-u_j(w)} \geq \sum_{i\in N} \frac{u_i(y)-u_j(w)}{u_j(z)-u_j(w)}$. States are socially ranked according to their social values. This is a profile-independent formulation of Utilitarianism.

On the one hand, Utilitarianism is often criticized for not taking the distribution of well-being into account. Ceteris paribus, it is indifferent between a distribution where one individual has a well-being level of 2 and a second individual has a well-being level of 0 and a distribution where both individuals have a level of 1. On the other hand, its approach to rank states according to the sum of their individual values has several advantages. It is thus reasonable to look for aggregation procedures that both take an additive form and are distribution-sensitive. Generalized Utilitarianism provides a solution.

SO $\succsim$ is a **Generalized Utilitarian Ordering (GUO)** if there exists increasing, continuous, and concave $g : A \to \mathbb{R}$ such that, for all $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{i\in N} g(u_i(x)) \geq \sum_{i\in N} g(u_i(y))$.[9] The UO is the GUO with $g : \lambda \mapsto \lambda$. One can state a profile-independent formulation of a GUO that is analogous to the one for the UO. Once

[8]It is indeed true that $\sum_{i\in N} v_i(x) \geq \sum_{i\in N} v_i(y) \Leftrightarrow \sum_{i\in N}(\lambda + \mu \cdot u_i(x)) \geq \sum_{i\in N}(\lambda + \mu \cdot u_i(y)) \Leftrightarrow \sum_{i\in N} u_i(x) \geq \sum_{i\in N} u_i(y)$.

[9]Function $g : A \to \mathbb{R}$ is (a) increasing if, for all $\lambda, \mu \in A$ with $\lambda > \mu$, $g(\lambda) > g(\mu)$, (b) continuous if, for all $\lambda \in A$ and $\epsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$ such that, for all $\mu \in A$ with $|\mu - \lambda| < \delta$, $|g(\mu) - g(\lambda)| < \epsilon$, (c) concave if, for all $\lambda, \mu \in A$ and $t \in [0, 1]$, $g(t\cdot\lambda + (1-t)\cdot\mu) \geq t\cdot g(\lambda) + (1-t)\cdot g(\mu)$, (d) strictly concave if $\geq$ is replaced by $>$ for $\lambda \neq \mu$ and $t \in (0, 1)$ in (c), and (e) surjective if, for all $\rho \in \mathbb{R}$, there exists $\lambda \in A$ with $\rho = g(\lambda)$.

again, suppose $[0,1] \subseteq A$ and take $j \in N$ and $z, w \in X$ with $u_j(z) = 1$ and $u_j(w) = 0$. Consider GUO $\succsim$ with associate g. Now, $g\big(\frac{u_i(x)-u_j(w)}{u_j(z)-u_j(w)}\big)$ is a profile-independent value of $x \in X$ for $i \in N$. It is a transformation of the ratio of the difference between i's well-being level at x and the reference level to the reference difference. Summing up individual values, $\sum_{i \in N} g\big(\frac{u_i(x)-u_j(w)}{u_j(z)-u_j(w)}\big)$ is a profile-independent social value of x. GUO $\succsim$ ranks states of the world according to these social values. For example, suppose $A = \mathbb{R}$, $g(\lambda) = 2 \cdot \lambda$ for $\lambda < 0$, and $g(\lambda) = \lambda$ for $\lambda \geq 0$. The zero level under u might be a poverty line (applying to well-being and not to income). According to $\succsim$, a unit well-being loss below the line can be socially compensated by two unit gains above the line.

The introduced Generalized Utilitarian class has two subclasses that are discussed in different fields. In the theory of Social Welfare Functionals, a more restrictive notion of Generalized Utilitarianism is considered. Suppose $A = \mathbb{R}$. GUO $\succsim$ with associate g is **Restricted Generalized Utilitarian** if, for all utility profiles v in the sense of A1 and $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{i \in N} g(v_i(x)) \geq \sum_{i \in N} g(v_i(y))$.[10] Function g must not only represent the social ranking when being applied to u, but also when being applied to any other profile representing well-being comparisons. At first view, this seems to capture a form of profile independence. However, as shown, every GUO can be stated in a profile-independent form. Restricted Generalized Utilitarianism really embodies a strong invariance of social evaluation under certain well-being changes (see section 3.3.2).

Another subclass of GUOs is much discussed at the intersection of economics and philosophy. Prioritarianism is the view that priority should be given to worse off individuals. Formally, a **Prioritarian** SO is a GUO whose associate g is strictly concave. Under every GUO, the social value of a well-being gain does not increase with the initial well-being level. Under a Prioritarian GUO, this social value decreases. While the UO is a GUO, it is not Prioritarian. The same is true for the GUO from the above example. However, both orderings belong to a different subclass of GUOs. For $A = \mathbb{R}$, GUO $\succsim$ with associate g is a **Surjective Generalized Utilitarian Ordering (SGUO)** if g is surjective. The classes of GUOs and SGUOs will be characterized in the next section.

[10]See Blackorby et al. (2002). However, they do not impose that g is concave.

3.3 Characterizations

3.3.1 Generalized Utilitarianism

This section provides an axiomatic analysis of Generalized Utilitarianism. It discusses normative conditions that together characterize different Generalized Utilitarian classes. To begin with, the following condition is one of the most fundamental to welfare economics.

SO $\succsim$ satisfies **Strong Pareto (SP)** if, for all $x, y \in X$, the following holds: If, for all $i \in N$, $u_i(x) \geq u_i(y)$, then $x \succsim y$. If, in addition, there exists $i \in N$ with $u_i(x) > u_i(y)$, then $x \succ y$.

SP embodies two central ideas. First, only individual well-being is relevant for the social evaluation of states of the world. This welfarist paradigm is presupposed in the entire chapter. SP indeed implies that, for all $x, y \in X$ with $u_i(x) = u_i(y)$ for all $i \in N$, it is the case that $x \sim y$ (**Pareto Indifference (PI)**). The second idea of SP is that aggregate evaluation should positively respond to well-being. Ceteris paribus, an individual well-being gain constitutes a social improvement.

SO $\succsim$ satisfies **Pigou-Dalton (PD)** if, for all $x, y \in X$ and $i, j \in N$ such that $u_i(x) - u_i(y) = u_j(y) - u_j(x) > 0$, $u_j(x) \geq u_i(y)$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$, it is the case that $x \succsim y$.

PD is a second fundamental normative condition. The dispute over prioritarianism and egalitarianism (Parfit, 1991; Broome, 1991; Holtug, 2015; Fleurbaey, 2015; Buchak, 2023) is best reconstructed in terms of this condition.

The prioritarian justification of PD emphasizes that evaluating states x and y comes down to comparing the social value of i's gain from y to x to the social value of j's gain from x to y. According to it, the fact that i's gain is as great as j's gain and has an equal or lower starting point is sufficient for i's gain to be socially at least as valuable as j's gain. For that reason, x is socially at least as good as y. The egalitarian justification of PD points to the fact that the sum total of well-being is the same at x and y, but that inequality does not increase from y to x since $u_j(y) \geq u_i(x) > u_i(y)$ and $u_j(y) > u_j(x) \geq u_i(y)$. For that reason, x is socially at least as good as y.

Some philosophers emphasize that it is important to distinguish between these two justifications. They mark the divide between a prioritarian and an egalitarian approach to social evaluation. However, it is necessary to distinguish between social rankings and their justification (Fleurbaey, 2015). Both views lead to the same PD condition. For the rest of the chapter, prioritarianism is entirely understood in terms of social rankings. As defined in the last section, a Prioritarian SO is a GUO with strictly concave g.

PD must be distinguished from two other formulations of the Pigou-Dalton principle. A weaker version replaces $u_j(x) \geq u_i(y)$ by $u_j(x) \geq u_i(x)$. PD is equivalent to the combination of this version with the condition of Anonymity (Appendix 3.A.1, Lemma A.1.4). Anonymity states that permutations of individual well-being levels do not affect the social ranking. Via this property, PD secures that all individuals have equal importance in social evaluation. On the other hand, PD does not imply a stronger version of Pigou-Dalton to the effect that, for $u_j(x) \geq u_i(x)$, $x \succ y$. The condition only demands that the gain of a worse off individual is socially at least as valuable as an equal gain of a better off individual, but not that it is more valuable. That is why the UO satisfies PD.

SO $\succsim$ satisfies **Weak Binary Separability (WBS)** if there exists $M \subseteq N$ with $|M| = 2$ so that, for all $x, y, z, w \in X$ such that, for all $i \in M$, $u_i(x) = u_i(z)$ and $u_i(y) = u_i(w)$, and, for all $i \in N \setminus M$, $u_i(x) = u_i(y)$ and $u_i(z) = u_i(w)$, it is the case that $x \succsim y \Leftrightarrow z \succsim w$.

For sake of concreteness, suppose $M = \{1, 2\}$. According to WBS, the social evaluation of states where all individuals except for 1 and 2 are equally well off only depends on well-being levels of 1 and 2 at these states. It does not depend on well-being levels of unconcerned individuals outside M. The fact that these well-being levels might differ between x (y) and z (w) does not lead to a different assessment of x vis-à-vis y and z vis-à-vis w. This separability clearly simplifies social evaluation.

Full Separability requires that the statement of WBS holds for all $M \subseteq N$ (see Appendix 3.A.1, Lemma A.1.6). WBS is much weaker as it only demands that social evaluation is separable for one binary set of individuals. It might be that 1 and 2 live on a different planet than individuals from $N \setminus M$ and there is no relation whatsoever between the two groups at x, y, z, and w (see Parfit, 1991: pp. 6-8). For these states, the argument that social evaluation should not depend on well-being levels of unconcerned individuals from $N \setminus M$ is particularly strong. Now, consider states x', y', z', and w' where all individuals from N together live on one planet, but where, for all $i \in N$, $u_i(x') = u_i(x)$, $u_i(y') = u_i(y)$, $u_i(z') = u_i(z)$, and $u_i(w') = u_i(w)$. Due to PI, $x' \sim x$, $y' \sim y$, $z' \sim z$, and $w' \sim w$. So if one accepts the conclusion of WBS for x, y, z, and w, one must also do so for x', y', z', and w'.

SO $\succsim$ satisfies **Continuity (C)** if, for all $x \in X$, the sets $\{a \in A^n \mid \exists y \in X : a = u(y) \wedge y \succsim x\}$ and $\{a \in A^n \mid \exists y \in X : a = u(y) \wedge x \succsim y\}$ are closed in A^n.

Continuity conditions are prevalent in economics. C is a property to the effect that "small" well-being changes do not lead to "large" changes in the social ranking of states. In contrast to C, the following condition does not refer to a topological concept.

SO $\succsim$ satisfies **Restricted Compensation (RC)** if, for all $x, y, z \in X$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $u_k(y) = u_k(z)$, $y \succsim x$, and $x \succsim z$, there exists $w \in X$ such that, for all $k \in N \setminus \{i\}$, $u_k(w) = u_k(z)$, and $w \sim x$.

The critical case is the one where $y \succ x$ and $x \succ z$. All individuals but i are equally well off at y and z. Given SP, i must realize a well-being gain from z to y which is sufficient to turn the social ranking vis-à-vis x around. RC demands that it is possible to find a (smaller) gain for i such that the resulting state w is socially as good as x. In other words, the superiority of the well-being distribution at x over the one at z is compensated by i's gain from z to w. It is intuitive and reasonable to demand such a form of restricted compensation.

RC is a version of the well-known Restricted Solvability condition in the theory of additive conjoint measurement (see references from section 3.1). While C is a topological property, RC is an algebraic condition on solvability for $\sim$. Given SP, it is in fact the case that C and RC are equivalent. This insight might be of independent interest.

Proposition 1: Suppose SO $\succsim$ satisfies SP. Then, $\succsim$ satisfies C if and only if $\succsim$ satisfies RC.[11]

Taken together, the discussed conditions lead to a characterization of the GUO class. In view of the equivalence of RC and C, Theorem 1 connects an algebraic and a topological approach to Generalized Utilitarianism.

Theorem 1: SO $\succsim$ satisfies SP, PD, WBS, and RC (or C) if and only if $\succsim$ is a GUO.

Theorem 1 unifies related characterizations of generalized utilitarian orderings from the literature. Blackorby et al. (2002: Theorem 5) essentially characterize the Restricted

[11]Proposition 1 allows to dispense with an Archimedean condition that is generally needed in the algebraic approach to additive conjoint measurement. Ghosh et al. (2023) examine the logical relation between different continuity and solvability properties, including versions of C and RC. Their Proposition 3 could be used for the proof of "RC⇒C". Instead, a self-contained proof is given in Appendix 3.A.1. It nicely illustrates the interplay of SP and RC in the derivation of C.

Generalized Utilitarian class with anonymity, full separability, and continuity. They implicitly use the condition of Strong Neutrality. Blackorby et al. (1998: pp. 70-73) sketch how to derive full separability from one separable set of two individuals given anonymity and continuity assumptions. As discussed in the proof of Theorem 1, a critical step for such a derivation is due to Gorman (1968).

If the stronger version of Pigou-Dalton to the effect that the social value of a well-being gain decreases with the starting level replaces PD in Theorem 1, this leads to a characterization of the Prioritarian class. McCarthy (2008: Theorem 1) employs such a stronger version together with full separability and continuity to characterize prioritarian orderings. He also assumes a utility profile that is unique up to a positive affine transformation.

If there is neither a smallest nor a greatest well-being level, it is possible to formulate the following new two-person compensation condition.

Suppose $A = \mathbb{R}$. SO $\succsim$ satisfies **Compensation for Small Well-Being Changes (CSWC)** if there exists $\epsilon \in \mathbb{R}_{++}$ such that, for all $\lambda \in \mathbb{R}$, $x \in X$, and distinct $i, j \in N$ with $|u_i(x) - \lambda| \leq \epsilon$, there exists $y \in X$ with $u_i(y) = \lambda$, for all $k \in N \setminus \{i, j\}$, $u_k(y) = u_k(x)$, and $y \sim x$.

According to CSWC, an individual's well-being change whose magnitude does not exceed ϵ can be socially compensated by a well-being change of another individual. Since there is no smallest well-being gain by A2, ϵ can be very small. For instance, it might be the well-being change of winning or losing one US Dollar for a millionaire.

Let $u_i(x) > \lambda$ in the definition, so that i faces a small well-being loss from $u_i(x)$ to λ. By CSWC, it can be socially compensated by a well-being gain of individual j, leaving all other individuals equally well off. If i is worse off than j, the required gain for j can be much larger than the loss for i. The requirement that it is at all possible to find a compensating gain for the tiny loss is thus quite mild.

The condition also states for $u_i(x) < \lambda$ that a small gain of i is worth some loss of j. However, if i is better off than j, the acceptable loss can be very small, even much smaller than i's gain. Overall, CSWC is thus an intuitive and convincing condition. Together with SP, PD, and WBS, it leads to a characterization of Surjective Generalized Utilitarianism.

Theorem 2: Suppose $A = \mathbb{R}$. SO $\succsim$ satisfies SP, PD, WBS, and CSWC if and only if $\succsim$ is a SGUO.

Theorem 2 characterizes the class of GUOs with surjective g. In contrast, a non-surjective g has an upper bound. If i is sufficiently bad off and j is sufficiently well off, it is not possible to compensate a small loss for i by any large gain for j under the corresponding GUO. Under every SGUO, a well-being loss of any size can be socially compensated by some well-being gain. That is implied by the intuitively mild CSWC condition.[12] However, even a small loss of a bad off individual might only be compensated by a very large gain of a better off individual. From a practical point of view, a SGUO can thus still effectively give almost all weight to the improvement of worse off individuals.[13]

Overall, the analysis supports (Surjective) Generalized Utilitarianism as an attractive approach to social evaluation. However, the class of (S)GUOs is rich. Justifying a single element of it seems to be more difficult than justifying the class as a whole. The next subsection and section 3.4 are concerned with this second step.

3.3.2 Utilitarianism

The central characterization of Utilitarianism in the context of A1 is due to Maskin (1978).[14] The present subsection reconstructs and critically discusses Maskin's Theorem. It then offers a different justification of Utilitarianism. For both approaches, the following two-person condition is important.

SO $\succsim$ satisfies **Gain Equity (GE)** if, for all $x, y \in X$ and $i, j \in N$ such that $u_i(x) - u_i(y) = u_j(y) - u_j(x) > 0$ and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$, it is the case that $x \sim y$.

GE embodies the idea that well-being gains of equal magnitude are socially indifferent. Since i's gain from y to x is as great as j's gain from x to y, x is socially as good as y. Together with SP, the condition leads to a characterization of Utilitarianism.

[12] CSWC turns out to be equivalent to a version of the Unrestricted Solvability condition from additive conjoint measurement (Appendix 3.A.1, Lemma A.1.7).

[13] Moreover, while a non-surjective GUO does not allow every loss of a bad off individual to be compensated by a gain of a single better off individual, it does so if sufficiently many better off individuals gain. This point is discussed in Fleurbaey and Tungodden (2010). The additive nature of Generalized Utilitarianism thus anyway commits to a form of unconditional compensation.

[14] For discussions, see Balasubramanian (2015), Weymark (2016), and Sen (2017: chapter A3*). Maskin's Theorem can be obtained from a joint characterization of Leximin and utilitarian-type procedures due to Deschamps and Gevers (1978).

Proposition 2: SO $\succsim$ satisfies SP and GE if and only if $\succsim$ is the UO.

If well-being gain comparisons are possible, but well-being level comparisons are not, GE is implied by the Fundamental Equity condition introduced in chapter 2 of this thesis.[15] However, the present chapter assumes that level comparisons are possible. In this setting, GE is clearly controversial.[16] It directly contradicts the main idea of prioritarianism that a gain of a worse off individual is socially more valuable than an equal gain of a better off individual. Instead, GE embodies the main idea of Utilitarianism in two-person situations. While Proposition 2 fails to offer an independent justification of Utilitarianism, it is methodologically important. Each of the following characterizations will be reduced to it.

Maskin's Theorem is stated in a multi-profile setting where individual well-being levels at states of the world are variable. In contrast, well-being levels at states are fixed in the present chapter. However, it is possible to formulate a version of Maskin's Theorem in this setting. The original result presupposes an unrestricted domain of utility profiles and the condition of Binary Independence. It is well-known that, together with PI, these conditions imply Strong Neutrality. If well-being levels at states are fixed, it is possible to directly assume that property.[17]

SO $\succsim$ satisfies **Strong Neutrality (SN)** if, for all $x, y, z, w \in X$ and utility profiles v, v' in the sense of A1 with $v(x) = v'(z)$ and $v(y) = v'(w)$, it is the case that $x \succsim y \Leftrightarrow z \succsim w$.

Despite its central importance in the literature on Social Welfare Functionals, SN is a puzzling condition. This is most clearly seen in the present setting where individual well-being levels at states are fixed. Different utility profiles in the sense of A1 correspond to different measurement scales. While it makes sense to compare numerical utility levels for a given profile (scale), it makes no sense to conduct such comparisons between

[15]Proposition 3 from chapter 2 is a version of Proposition 2 which employs weaker richness assumptions and derives utility representations. Without level comparisons, it leads to an informational justification of Utilitarianism (chapter 2, Theorem 3). Blackorby et al. (2002: Theorems 10-12) develop a related argument based on the condition of Incremental Equity. In contrast to the present chapter and chapter 2, they employ an unrestricted domain of utility profiles and the property of Binary Independence. While the results in chapter 2 require SP, both Proposition 2 and the results in Blackorby et al. (2002) go through with Weak Pareto and PI.

[16]Given A1 and SP, the Fundamental Equity condition from chapter 2 does not imply GE, but is equivalent to PD.

[17]In particular, this approach has lead to single-profile versions of Arrow's Impossibility Theorem. See Kemp and Ng (1976), Parks (1976), Pollak (1979), Roberts (1980c), Blackorby et al. (1990), and Fleurbaey and Mongin (2005).

different profiles (scales). Accordingly, it is not clear why the identity of utility levels between v and v' is supposed to restrict the social evaluation of states in the sense of SN.[18] The property turns out to be equivalent to the following condition which can be stated in terms of the single profile (scale) u.

SO $\succsim$ satisfies **Strong Invariance (SI)** if, for all $x, y, z, w \in X$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}_{++}$ such that, for all $i \in N$, $u_i(x) = \alpha + \beta \cdot u_i(z)$ and $u_i(y) = \alpha + \beta \cdot u_i(w)$, it is the case that $x \succsim y \Leftrightarrow z \succsim w$.

Proposition 3: SO $\succsim$ satisfies SN if and only if $\succsim$ satisfies SI.

The good news is that SI is a clear condition and captures the substantive implications of SN. It requires the social ranking of states to be invariant under certain changes of well-being levels and differences. For instance, a common well-being level change from z (w) to x (y) for each individual should not lead to a different social ranking of x vis-à-vis y and z vis-à-vis w.[19] Together with the properties from the last subsection, SI (SN) characterizes Utilitarianism.

Theorem 3: SO $\succsim$ satisfies SP, PD, WBS, RC (or C), and SI (or SN) if and only if $\succsim$ is the UO.[20]

Theorem 3 is a version of Maskin's Theorem for fixed well-being levels at states. The original formulation uses anonymity, separability, and continuity. As shown in the proof of Theorem 3, it is insightful to consider the consequences of SI for a GUO with concave associate g. The proof derives GE and thereby reduces Theorem 3 to Proposition 2. In contrast, continuity of g plays the central role in Maskin's (1978) proof.

The bad news is that SI is a controversial condition. It is not clear how to justify the strong substantive invariance requirements it imposes on the social evaluation of states.[21] Indeed, SI runs counter to the main idea of prioritarianism that the social

[18] A similar critique applies to Binary Independence given an unrestricted domain of utility profiles. The condition is unable to distinguish meaningful comparisons of utility levels for a given scale from meaningless comparisons of utility levels between different scales. See Morreau and Weymark (2016) for a discussion. Without Binary Independence, Maskin's Theorem does not go through.

[19] In contrast to information invariance conditions for Social Welfare Functionals that are meant to secure invariance of the social ranking under merely representational changes, SI is a strong substantive condition requiring invariance of the social ranking under the stated real well-being changes. For a discussion of this distinction, see Nebel (2023).

[20] A Restricted Generalized Utilitarian SO satisfies SN. Theorem 3 thus shows that the UO is the only element of that class in the context of A1.

[21] For a related critique of Maskin's Theorem, see Nebel (2023: section 5.1).

value of a well-being gain decreases with its starting level. To see that, consider states x, y, and z, and individuals i and j such that $u_i(x) = 4$, $u_i(z) = 2$, $u_i(y) = u_j(y) = 0$, $u_j(z) = -1$, $u_j(x) = -2$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y) = u_k(z) = 0$.

Suppose $z \sim y$, that is, two unit gains from starting level 0 are socially as valuable as one unit gain from starting level -1. According to the prioritarian idea, two unit gains from a level greater than 0 should in turn be socially less valuable than a unit gain from a level smaller than -1. But due to SI, it follows that $x \sim y$ (set $w = y$, $\alpha = 0$, and $\beta = 2$), implying $x \sim z$. The condition thus induces that two unit gains from starting level 2 are socially as valuable as one unit gain from starting level -2. For that reason, a prioritarian can and should reject SI. Theorem 3 is not a convincing justification of Utilitarianism as long as there is no convincing argument for SI.

The remainder of the section develops a different justification of Utilitarianism. Again, Proposition 2 is part of the argument. The following new two-person condition is key. It captures the idea that very small differences in well-being differences should not lead to a complete reversal of the social ranking.

SO $\succsim$ satisfies **Stability (STB)** if, for all $x, y \in X$ and $i, j \in N$ with $u_j(y) > u_j(x) = u_i(x) > u_i(y)$, $u_j(y) - u_j(x) \neq u_i(x) - u_i(y)$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$, there exists $\epsilon \in \mathbb{R}_{++}$ such that, for all $y', y'' \in X$ with $|u_j(y') - u_j(y)| + |u_j(y'') - u_j(y)| < \epsilon$ and, for all $k \in N \setminus \{j\}$, $u_k(y') = u_k(y'') = u_k(y)$, it is the case that $x \succ y' \Rightarrow x \succsim y''$.

For the sake of concreteness, suppose that j's gain from x to y is greater than i's gain from y to x. States y' and y'' have "almost" the same well-being distribution as y. All individuals but j are equally well off at the three states, and j's well-being differences between y, y', and y'' are very small, respectively. As long as ϵ is sufficiently small, it is indeed the case that j does not only gain from x to y' (y''), but that j's gain from x to y' (y'') is greater than i's gain from i's level at y to i's level at x. Because all individuals except i and j are equally well off at x and y' (y''), the social evaluation of these states comes down to a comparative assessment of i's gain and j's gain from x to y' (y'').

Suppose that i's gain is deemed socially more valuable than j's greater gain from x to y', that is, x is socially better than y'. Qualitatively, the evaluation situation of x vis-à-vis y' is the same as the evaluation situation of x vis-à-vis y''. Individual i realizes the same gain from y' to x as from y'' to x. Individual j realizes a greater gain from x to y' and y'', respectively. There is only a very small difference between j's gain from x to y' and j's gain from x to y''. According to STB, this very small difference should not lead to a complete reversal of social evaluation. Considering the fact that i's gain is socially more valuable than j's gain from x to y', it should still be socially at least as valuable as j's gain from x to y''. That is, x should be socially at least as good as y''.

If STB is violated, social evaluation embodies highly subtle value judgments. Qualitatively equal situations (one person realizes a loss, the other person realizes a greater gain) are treated completely differently due to a very small difference in well-being differences. It is difficult to form such complicated value judgments (particularly in public deliberation). The idea of STB is somewhat similar to the one of C. Nevertheless, the two conditions are logically independent.[22] Indeed, the combination of STB and C (or RC) with SP yields a characterization of Utilitarianism.

Theorem 4: SO $\succsim$ satisfies SP, RC (or C), and STB if and only if $\succsim$ is the UO.

The proof of Theorem 4 shows that SP, RC (C), and STB together imply GE, so that Proposition 2 can be applied. To illustrate, compare a gain ϵ of individual i and an equal gain of individual j whose starting level coincides with the end level of i's gain. According to every Prioritarian SO, i's gain is socially more valuable than j's gain. However, i's gain is socially as valuable as some greater gain δ of j. Take a gain δ^+ that is slightly greater than δ and a gain δ^- that is slightly smaller than δ. Supposing that differences in differences are sufficiently small, δ^+ and δ^- are both greater than ϵ. Despite this equal qualitative relation, δ^+ (from j's starting level) is socially more valuable than ϵ (from i's starting level), while it is the other way around for δ^-. These differences contradict STB.

Every Prioritarian SO embodies highly specific quantitative value judgments. For a given gain from some starting level, it specifies a unique fraction of that gain that is equally socially valuable when realized from a lower starting level. As discussed, it is difficult to form and agree on such quantitative specifications. In contrast, the Utilitarian Ordering is solely based on qualitative value judgments to the effect that a greater gain is socially better than a smaller gain and that two equal gains are socially indifferent. This is captured by Theorem 4.

The justification of Utilitarianism via Theorem 4 has some advantages over the one via Theorem 3. In contrast to SI, STB has a clear rationale. Moreover, Theorem 4 does not need WBS. What is even more remarkable is that Theorem 4 does not employ any anonymity condition (neither PD nor some weakening of it). It solely uses SP and the

[22] As discussed in the following, a Prioritarian SO violates STB but satisfies C (RC). A Special-Level Utilitarian SO is discussed in Appendix 3.A.5. It satisfies SP and STB, but violates C (RC) and SI (SN). This shows that, even given SP, STB neither implies C (RC) nor SI (SN).

regularity conditions RC (C) and STB. The reasoning that has been discussed for Prioritarian SOs also excludes non-anonymous approaches like Weighted Utilitarianism.[23] Someone who finds the insensitivity of Utilitarianism to the distribution of well-being problematic might interpret Theorem 4 as an impossibility result.

3.4 Uncertainty

This section extends the analysis to an uncertainty framework. Let $N = \{1, ..., n\}$ be a population of individuals, $C = \{1, ..., m\}$ a set of circumstances, and X a set of social prospects, where $n \geq 2$ and $m \geq 2$. The state of the world that is realized by prospect $x \in X$ depends on the circumstance $c \in C$ that obtains. Accordingly, the well-being level of $i \in N$ given x is contingent on the realized circumstance. Which circumstance will obtain is not known. It is assumed that, for all circumstances $c, d \in C$, c is neither more likely than d nor the other way around.

Well-being level and difference comparisons are now captured by a difference relation $\succsim_d$ over $(N \times C \times X)^2$. For $i, j, k, l \in N$, $c, d, e, f \in C$, and $x, y, z, w \in X$, $\big((i, c, x), (j, d, y)\big) \succsim_d \big((k, e, z), (l, f, w)\big)$ means that the well-being difference between i at c given x and j at d given y is at least as great as the well-being difference between k at e given z and l at f given w. These well-being difference (and level) comparisons are ex post. They apply to individuals at states of the world that are realized by prospects for given circumstances. The discussion and examples of section 3.2.1 still apply.

The reformulation of A1 states that there exists $u : N \times C \times X \to \mathbb{R}$, unique up to a positive affine transformation, that represents $\succsim_d$. Fix such a utility profile u. For $i \in N$, $c \in C$, and $x \in X$, denote $u_{i,c}(x) = u(i, c, x)$ and $u(x) = (u_{i,c}(x))_{(i,c) \in N \times C}$. The new version of A2 states that there exists an interval $A \subseteq \mathbb{R}$ such that $u(X) = \{a \in \mathbb{R}^{N \times C} \mid \exists x \in X : u(x) = a\} = A^{N \times C}$.

SO $\succsim$ is a complete and transitive relation over X and captures the relative social evaluation of prospects. It is an **Expected Generalized Utilitarian Ordering (EGUO)** if there exists increasing, continuous, and concave $g : A \to \mathbb{R}$ such that, for all $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} g(u_{i,c}(x)) \geq \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} g(u_{i,c}(y))$.

In line with the assumption that no circumstance is more likely than another and the

[23] Under Weighted Utilitarianism, different individuals get different weights. It is discussed in Appendix 3.A.4.

principle of insufficient reason, an EGUO assigns a (subjective) probability of $\frac{1}{m}$ to each circumstance. The social value of prospect x for a given circumstance c is the sum of transformed utilities given c and x. The overall social value of x is the expected value of these circumstance values.[24] The **Expected Utilitarian Ordering (EUO)** is the EGUO with $g : \lambda \mapsto \lambda$. EGUOs can be characterized with uncertainty versions of the conditions from section 3.3.

SO $\succsim$ satisfies **Strong Pareto (SP)** if, for all $x, y \in X$, the following holds: If, for all $(i, c) \in N \times C$, $u_{i,c}(x) \geq u_{i,c}(y)$, then $x \succsim y$. If, in addition, there exists $(i, c) \in N \times C$ with $u_{i,c}(x) > u_{i,c}(y)$, then $x \succ y$.

SO $\succsim$ satisfies **Pigou-Dalton (PD)** if, for all $x, y \in X$, $(i, c), (j, d) \in N \times C$ such that $u_{i,c}(x) - u_{i,c}(y) = u_{j,d}(y) - u_{j,d}(x) > 0$, $u_{j,d}(x) \geq u_{i,c}(y)$, and, for all $(k, e) \in (N \times C) \setminus \{(i, c), (j, d)\}$, $u_{k,e}(x) = u_{k,e}(y)$, it is the case that $x \succsim y$.

SO $\succsim$ satisfies **Weak Binary Separability (WBS)** if there exists $M \subseteq N \times C$ with $|M| = 2$ so that, for all $x, y, z, w \in X$ such that, for all $(i, c) \in M$, $u_{i,c}(x) = u_{i,c}(z)$ and $u_{i,c}(y) = u_{i,c}(w)$, and, for all $(i, c) \in (N \times C) \setminus M$, $u_{i,c}(x) = u_{i,c}(y)$ and $u_{i,c}(z) = u_{i,c}(w)$, it is the case that $x \succsim y \Leftrightarrow z \succsim w$.

SO $\succsim$ satisfies **Restricted Compensation (RC)** if, for all $x, y, z \in X$ and $(i, c) \in N \times C$ such that, for all $(k, e) \in N \times C \setminus \{(i, c)\}$, $u_{k,e}(y) = u_{k,e}(z)$, $y \succsim x$, and $x \succsim z$, there exists $w \in X$ such that, for all $(k, e) \in N \times C \setminus \{(i, c)\}$, $u_{k,e}(w) = u_{k,e}(z)$, and $w \sim x$.

It is important to understand that the uncertainty versions are perfectly analogous to their certainty counterparts. The only formal difference is that prospects replace states of the world and individual-circumstance tuples replace individuals. SP becomes a reasonable dominance condition to the effect that, if each individual given each circumstance is at least as well off given prospect x as given prospect y, then x is socially at least as good as y. If an individual is better off given a circumstance, then x is socially better than y.

PD is concerned with prospects x and y that only differ with respect to two individual-circumstance tuples. Individual i at circumstance c faces a gain from y to x, while individual j at circumstance d faces a gain of equal magnitude from x to y. In view of the fact that the former gain does not have a greater starting level than the latter, it is

[24] Like in the certainty case, one can state a profile-independent formulation of an EGUO in terms of well-being difference ratios.

socially at least as valuable as the latter.[25] Compared to its certainty version, the weak inequality aversion of PD does not only have an interpersonal component, but also an intrapersonal one. If $i = j$, the individual gains from y to x given c, but gains from x to y given d. That the former gain is socially at least as valuable as the latter gain reflects weak risk aversion in the social evaluation of individual well-being prospects.

WBS captures the weak assumption that there exists a separable binary set of individual-circumstance tuples $M = \{(i, c), (j, d)\}$. For $i \neq j$, the discussion for the certainty version of WBS applies. But it is also possible that social evaluation is separable for two circumstances $c \neq d$ with respect to a single individual $i = j$. Finally, RC also has an analogous interpretation to the certainty case. If a gain at individual-circumstance tuple (i, c) from prospect z to prospect y can turn the social ranking vis-à-vis prospect x around, there is a (smaller) gain at (i, c) from z to prospect w that leads to social indifference between w and x.

Taken together, the uncertainty conditions characterize the class of EGUOs. The entire formal argument to establish Theorem 5 is perfectly analogous to the argument for Theorem 1. Index set $N \times C$ replaces index set N.

Theorem 5: SO $\succsim$ satisfies SP, PD, WBS, and RC if and only if $\succsim$ is an EGUO.

Under an EGUO, g simultaneously captures the degree of social inequality aversion and the degree of risk aversion in the social evaluation of individual prospects. That is so because $\sum_{i \in N} g(u_{i,c}(x))$ is the social value of prospect x for a given circumstance c and $\sum_{c \in C} \frac{1}{m} \cdot g(u_{i,c}(x))$ is the social value of x for a given individual i. The conditions of Theorem 5 imply this equality of social inequality and risk aversion. If one of them is determined, the other will be determined as well. The approach to determine social inequality aversion in terms of risk aversion goes back to Harsanyi (1953, 1955) and is examined in Grant et al. (2010) and Eden (2020).

In contrast to the setup of this section, Harsanyi's (1955) Aggregation Theorem operates in a lottery framework where probability distributions are variable. Using social and individual expected utility assumptions and an ex ante version of Pareto, it shows that social evaluation takes an additive form. One problem is that interpersonal well-being gain comparisons are not specified, so that the definition of utilitarianism is ambiguous.[26] This problem is avoided in the present chapter. Moreover, Theorem 5

[25] The assumption that d is not more likely than c is relevant here. If d was more likely than c, that would be a reason for $y \succ x$.

[26] On this issue, see Harsanyi (1982), Weymark (1991), Fleurbaey and Mongin (2016), and Sen (2017:

does not presuppose the expected utility form of aggregation, but derives it.

As for Theorem 1, a stronger Pigou-Dalton condition requiring the social value of a well-being gain to decrease with the starting level leads to a characterization of Expected Prioritarianism, that is, Expected Generalized Utilitarianism with strictly concave g. McCarthy (2008: Theorem 3) characterizes expected prioritarianism using expected utility and variable population separability assumptions. McCarthy et al. (2020) provide a general analysis of utilitarian-type aggregation in a lottery framework.[27]

Theorem 5 also relates to recent contributions that characterize fully additively separable social evaluation in the circumstance framework of this section (Mongin and Pivato, 2015; Gustafsson et al., 2023; Li et al., 2023; Pivato and Tchouante, 2023).[28] In line with Gorman (1968), they employ social and individual dominance conditions as well as continuity and compensation assumptions to derive full separability. The critical point is to exploit the two-dimensional structure of individuals and circumstances. Instead, Theorem 5 does not make use of this structure, but establishes a perfect analogy between the uncertainty case and the one-dimensional certainty case. Given continuity assumptions, WBS is implied by social and individual dominance. In contrast to the other contributions, Theorem 5 makes critical use of PD to derive full separability and the specific form $g(u_{i,c}(x))$ of the social value of prospect x with respect to individual i and circumstance c.

The characterization of the EUO via the uncertainty version of STB in Theorem 6 is likewise formally analogous to the characterization of the UO in Theorem 4.

SO $\succsim$ satisfies **Stability (STB)** if, for all $x, y \in X$, $(i,c),(j,d) \in N \times C$ with $u_{j,d}(y) > u_{j,d}(x) = u_{i,c}(x) > u_{i,c}(y)$, $u_{j,d}(y) - u_{j,d}(x) \neq u_{i,c}(x) - u_{i,c}(y)$, and, for all $(k,e) \in (N \times C) \setminus \{(i,c),(j,d)\}$, $u_{k,e}(x) = u_{k,e}(y)$, there exists $\epsilon \in \mathbb{R}_{++}$ such that, for all $y', y'' \in X$ with $|u_{j,d}(y') - u_{j,d}(y)| + |u_{j,d}(y'') - u_{j,d}(y)| < \epsilon$ and, for all $(k,e) \in (N \times C) \setminus \{(j,d)\}$, $u_{k,e}(y') = u_{k,e}(y'') = u_{k,e}(y)$, it is the case that $x \succ y' \Rightarrow x \succsim y''$.

Theorem 6: SO $\succsim$ satisfies SP, RC, and STB if and only if $\succsim$ is the EUO.

Again, for sake of concreteness, suppose that j's gain at d from x to y' (y'') is greater than i's gain at c from y' (y'') to x. According to STB, the small difference in j's gain

chapter A3*).

[27]McCarthy (2006, 2008, 2017) and McCarthy et al. (2020) discuss different definitions of utilitarianism and prioritarianism in the context of risk.

[28]These contributions are also concerned with other applications than social evaluation under uncertainty.

at d from x to y' and from x to y'' should not completely reverse the social ranking.[29] The uncertainty version of STB also has an intrapersonal component. By adapting the argument why a Prioritarian SO violates STB in the certainty case, one shows that the uncertainty versions of SP, RC, and STB exclude the possibility of risk aversion in the social evaluation of individual prospects. A gain of individual i at circumstance c which is socially indifferent to a greater gain of i at circumstance d whose starting level coincides with the end level of the former gain leads to a violation of STB.

3.5 Conclusion

The chapter has offered a systematic analysis of Generalized Utilitarianism given qualitative well-being difference and level comparisons. There are several promising approaches to construct interpersonal well-being comparisons in terms of happiness, resources, and extended preferences. Generalized Utilitarian Orderings are defined in terms of well-being difference ratios. The Generalized Utilitarian class contains all Prioritarian Orderings as well as the Utilitarian Ordering.

All characterizations of the chapter are formulated in terms of a single utility profile which represents well-being difference and level comparisons. Strong Pareto, Pigou-Dalton, Weak Binary Separability, and Restricted Compensation characterize the class of Generalized Utilitarian Orderings. Given Strong Pareto and the framework assumptions of the chapter, Restricted Compensation and Continuity are equivalent. The new Compensation for Small Well-Being Changes condition turns out to be stronger and characterizes the subclass of Surjective Generalized Utilitarian Orderings. The results provide a good justification for (Surjective) Generalized Utilitarianism. Their distinguishing features are weak and intuitive separability and compensation conditions.

The Utilitarian Ordering is characterized by Strong Pareto and Gain Equity. This insight is valuable to derive characterizations. A single-profile version of Maskin's Theorem shows that the Utilitarian Ordering is the only Generalized Utilitarian Ordering satisfying Strong Invariance. However, it is difficult to defend this property. The Utilitarian Ordering is also characterized by Strong Pareto, Restricted Compensation (Continuity), and the new Stability condition. The result provides a more promising argument for Utilitarianism than Maskin's Theorem. It does not need anonymity

[29]If (subjective) probabilities of circumstances are unequal, STB becomes too strong. In that case, even the reformulation of Expected Utilitarianism violates the condition.

or separability conditions. Stability reduces the sensitivity of social evaluation with respect to subtle well-being changes.

The analysis extends to social evaluation under uncertainty. Aggregation is still based on qualitative ex post well-being comparisons. Uncertainty versions of Strong Pareto, Pigou-Dalton, Weak Binary Separability, and Restricted Compensation characterize the class of Expected Generalized Utilitarian Orderings. Uncertainty versions of Strong Pareto, Restricted Compensation, and Stability characterize the Expected Utilitarian Ordering. The distinguishing feature of these results is the perfect analogy to the certainty case. If one endorses the conditions for Expected Generalized Utilitarianism, one faces the task to jointly determine social inequality and risk aversion.

Two extensions of the offered analysis are of particular interest.

1. In the fixed population model of the chapter, social evaluation under certainty comes down to ranking utility vectors of length n. It is possible to extend the analysis to variable finite populations of states of the world. Suppose that there is a unique well-being level of existence which is socially as valuable as non-existence. If this critical level is represented by a utility level of 0, social evaluation comes down to ranking utility vectors of infinite length with finitely many non-zero entries. With appropriate reformulations, the analysis of the chapter yields characterizations of critical-level versions of the respective Generalized Utilitarian Orderings. An according extension is also possible for the uncertainty case.

2. The analysis shows that weak conditions yield a fully separable additive social ordering. There are some issues with full separability in the uncertainty case (see Grant et al., 2010; Mongin and Pivato, 2015; Li et al., 2023). For instance, a permutation of well-being levels between individuals i and j at circumstance c is socially indifferent under every Expected Generalized Utilitarian Ordering. But one can argue that the social assessment of the permutation should depend on how well off i is relative to j at other circumstances. This challenges the uncertainty version of Pigou-Dalton. One way to deal with the issue is to weaken that property.

The social evaluation of alternatives is not just a theoretical exercise. It is meant to inform practical decision-making. In line with that, the chapter has provided an investigation of well-being aggregation. This is a first step to effectively promote individual well-being.

3.A Appendix

3.A.1 Proofs for Section 3.3.1

In line with the literature on Social Welfare Functionals (see references from section 3.1), it will be convenient to consider orderings over utility space A^n that are associate to SOs. A **Social Welfare Ordering (SWO)** is a complete and transitive relation over A^n. SWO $\succsim^*$ is an **associate** of SO $\succsim$ if, for all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \succsim^* u(y)$. SO conditions from the main text have corresponding SWO conditions.

SWO $\succsim^*$ satisfies **Strong Pareto (SP*)** if, for all $a, b \in A^n$ such that, for all $i \in N$, $a_i \geq b_i$, and there exists $i \in N$ with $a_i > b_i$, it is the case that $a \succ^* b$.

SWO $\succsim^*$ satisfies **Pigou-Dalton (PD*)** if, for all $a, b \in A^n$ and $i, j \in N$ such that $a_i - b_i = b_j - a_j > 0$, $a_j \geq b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$, it is the case that $a \succsim^* b$.

SWO $\succsim^*$ satisfies **Weak Binary Separability (WBS*)** if there exists $M \subseteq N$ with $|M| = 2$ so that, for all $a, b, c, d \in A^n$ such that, for all $i \in M$, $a_i = c_i$ and $b_i = d_i$, and, for all $i \in N \setminus M$, $a_i = b_i$ and $c_i = d_i$, it is the case that $a \succsim^* b \Leftrightarrow c \succsim^* d$.

SWO $\succsim^*$ satisfies **Continuity (C*)** if, for all $b \in A^n$, the sets $\{a \in A^n \mid a \succsim^* b\}$ and $\{a \in A^n \mid b \succsim^* a\}$ are closed in A^n.

SWO $\succsim^*$ satisfies **Restricted Compensation (RC*)** if, for all $a, b, c \in A^n$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $b_k = c_k$, $b \succsim^* a$, and $a \succsim^* c$, there exists $d \in A^n$ such that, for all $k \in N \setminus \{i\}$, $d_k = c_k$, and $d \sim^* a$.

Suppose $A = \mathbb{R}$. SWO $\succsim^*$ satisfies **Compensation for Small Well-Being Changes (CSWC*)** if there exists $\epsilon \in \mathbb{R}_{++}$ such that, for all $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^n$, and distinct $i, j \in N$ with $|a_i - \lambda| \leq \epsilon$, there exists $b \in \mathbb{R}^n$ with $b_i = \lambda$, for all $k \in N \setminus \{i, j\}$, $b_k = a_k$, and $b \sim^* a$.

Lemma A.1.1: SO $\succsim$ satisfies SP if and only if $\succsim$ has an associate SWO $\succsim^*$ satisfying SP*.
Proof:
"$\Rightarrow$": Suppose SO $\succsim$ satisfies SP. Define SWO $\succsim^*$ such that, for all $a, b \in A^n$, $a \succsim^* b$ if and only if there exist $x, y \in X$ with $u(x) = a$, $u(y) = b$, and $x \succsim y$. To see that $\succsim^*$ is indeed a SWO, take $a, b \in A^n$. By A2, there exist $x, y \in X$ with $u(x) = a$ and $u(y) = b$. Because $\succsim$ is complete, $x \succsim y$ or $y \succsim x$ holds. By definition, that implies

$a \succsim^* b$ or $b \succsim^* a$. Thus, $\succsim^*$ is complete. Consider $a, b, c \in A^n$ with $a \succsim^* b$ and $b \succsim^* c$. By definition, there exist $x, y, z, w \in X$ with $u(x) = a$, $u(y) = b$, and $x \succsim y$, as well as $u(z) = b$, $u(w) = c$, and $z \succsim w$. Due to SP, $y \sim z$, so that the transitivity of $\succsim$ yields $x \succsim w$. It follows by definition that $a \succsim^* c$. Hence, $\succsim^*$ is transitive.

Consider $x, y \in X$. Suppose $x \succsim y$. By definition, that implies $u(x) \succsim^* u(y)$. Suppose $u(x) \succsim^* u(y)$. By definition, there exist $z, w \in X$ with $u(z) = u(x)$, $u(w) = u(y)$, and $z \succsim w$. SP and social transitivity imply $x \succsim y$. Taken together, this establishes that $\succsim^*$ is associate to $\succsim$. Take $a, b \in A^n$ such that, for all $i \in N$, $a_i \geq b_i$, and there exists $i \in N$ with $a_i > b_i$. There exist $x, y \in X$ with $u(x) = a$ and $u(y) = b$. By SP of $\succsim$, it is true that $x \succ y$. Since $\succsim^*$ is associate to $\succsim$, it follows that $a \succ^* b$. Therefore, $\succsim^*$ satisfies SP*.

"$\Leftarrow$": Suppose SO $\succsim$ has associate SWO $\succsim^*$ satisfying SP*. Take $x, y \in X$ such that, for all $i \in N$, $u_i(x) \geq u_i(y)$. If $u(x) = u(y)$, the reflexivity of $\succsim^*$ implies $u(x) \sim^* u(y)$ and in turn $x \sim y$. If $u(x) \neq u(y)$, SP* of $\succsim^*$ yields $u(x) \succ^* u(y)$ and thus $x \succ y$. Accordingly, $\succsim$ satisfies SP.■

Lemma A.1.2: Suppose SWO $\succsim^*$ is associate to SO $\succsim$. Then,
(a) $\succsim$ satisfies PD if and only if $\succsim^*$ satisfies PD*.
(b) $\succsim$ satisfies WBS if and only if $\succsim^*$ satisfies WBS*.
(c) $\succsim$ satisfies C if and only if $\succsim^*$ satisfies C*.
(d) $\succsim$ satisfies RC if and only if $\succsim^*$ satisfies RC*.
(e) given $A = \mathbb{R}$, $\succsim$ satisfies CSWC if and only if $\succsim^*$ satisfies CSWC*.
Proof:
(a) For "$\Rightarrow$", suppose $\succsim$ satisfies PD. Take $a, b \in A^n$ and $i, j \in N$ such that $a_i - b_i = b_j - a_j > 0$, $a_j \geq b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. There exist $x, y \in X$ with $u(x) = a$ and $u(y) = b$. By PD, $x \succsim y$. Since $\succsim^*$ is associate to $\succsim$, that yields $a \succsim^* b$. Consequently, $\succsim^*$ satisfies PD*. For "$\Leftarrow$", suppose $\succsim^*$ satisfies PD*. Take $x, y \in X$ and $i, j \in N$ such that $u_i(x) - u_i(y) = u_j(y) - u_j(x) > 0$, $u_j(x) \geq u_i(y)$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$. PD* yields $u(x) \succsim^* u(y)$ and in turn $x \succsim y$. Thus, $\succsim$ satisfies PD.

(b) For "$\Rightarrow$", suppose $\succsim$ satisfies WBS with $M \subseteq N$. Consider $a, b, c, d \in A^n$ and such that, for all $i \in M$, $a_i = c_i$ and $b_i = d_i$, and, for all $i \in N \setminus M$, $a_i = b_i$ and $c_i = d_i$. There exist $x, y, z, w \in X$ with $u(x) = a$, $u(y) = b$, $u(z) = c$, and $u(w) = d$. Due to WBS, it follows that $x \succsim y \Leftrightarrow z \succsim w$. This yields $a \succsim^* b \Leftrightarrow c \succsim^* d$, so that $\succsim^*$ satisfies WBS* with M. For "$\Leftarrow$", suppose $\succsim^*$ satisfies WBS* with with $M \subseteq N$. Consider

$x, y, z, w \in X$ such that, for all $i \in M$, $u_i(x) = u_i(z)$ and $u_i(y) = u_i(w)$, and, for all $i \in N \setminus M$, $u_i(x) = u_i(y)$ and $u_i(z) = u_i(w)$. By WBS*, $x \succsim y \Leftrightarrow u(x) \succsim^* u(y) \Leftrightarrow u(z) \succsim^* u(w) \Leftrightarrow z \succsim w$. Hence, $\succsim$ satisfies WBS with M.

(c) For "$\Rightarrow$", suppose $\succsim$ satisfies C. Take $b \in A^n$. There exists $x \in X$ with $u(x) = b$. For $c \in A^n$, it follows that $c \in \{a \in A^n \mid a \succsim^* b\} \Leftrightarrow c \in \{a \in A^n \mid \exists y \in X : a = u(y) \wedge y \succsim x\}$. That is, $\{a \in A^n \mid a \succsim^* b\} = \{a \in A^n \mid \exists y \in X : a = u(y) \wedge y \succsim x\}$. Since the latter set is closed in A^n due to C, so is the former set. Analogously, $\{a \in A^n \mid b \succsim^* a\} = \{a \in A^n \mid \exists y \in X : a = u(y) \wedge x \succsim y\}$ is closed in A^n. Therefore, $\succsim^*$ satisfies C*. For "$\Leftarrow$", suppose $\succsim^*$ satisfies C*. Take $x \in X$. For $c \in A^n$, it is true that $c \in \{a \in A^n \mid \exists y \in X : a = u(y) \wedge y \succsim x\} \Leftrightarrow c \in \{a \in A^n \mid a \succsim^* u(x)\}$ and thus $\{a \in A^n \mid \exists y \in X : a = u(y) \wedge y \succsim x\} = \{a \in A^n \mid a \succsim^* u(x)\}$. As the latter set is closed in A^n by C*, so is the former. Likewise, $\{a \in A^n \mid \exists y \in X : a = u(y) \wedge x \succsim y\} = \{a \in A^n \mid u(x) \succsim^* a\}$ is closed in A^n. Accordingly, $\succsim$ satisfies C.

(d) For "$\Rightarrow$", suppose $\succsim$ satisfies RC. Consider $a, b, c \in A^n$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $b_k = c_k$, $b \succsim^* a$, and $a \succsim^* c$. There are $x, y, z \in X$ with $u(x) = a$, $u(y) = b$, and $u(z) = c$. It follows that $y \succsim x$ and $x \succsim z$. Due to RC, there exists $w \in X$ such that, for all $k \in N \setminus \{i\}$, $u_k(w) = u_k(z)$, and $w \sim x$. Since $u(w) \sim^* u(x)$, $\succsim^*$ satisfies RC* with $d = u(w)$. For "$\Leftarrow$", suppose $\succsim^*$ satisfies RC*. Consider $x, y, z \in X$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $u_k(y) = u_k(z)$, $y \succsim x$, and $x \succsim z$. That is, $u(y) \succsim^* u(x)$ and $u(x) \succsim^* u(z)$. By RC*, there exists $d \in A^n$ such that, for all $k \in N \setminus \{i\}$, $d_k = u_k(z)$, and $d \sim^* u(x)$. Take $w \in X$ with $u(w) = d$. Because $w \sim x$, $\succsim$ satisfies RC with w.

(e) For "$\Rightarrow$", suppose $\succsim$ satisfies CSWC with $\epsilon \in \mathbb{R}_{++}$. Consider $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^n$, and distinct $i, j \in N$ with $|a_i - \lambda| \leq \epsilon$. Take $x \in X$ with $u(x) = a$. By CSWC, there exists $y \in X$ such that $u_i(y) = \lambda$, for all $k \in N \setminus \{i, j\}$, $u_k(y) = u_k(x)$, and $y \sim x$. Because $u(y) \sim^* u(x)$, $\succsim^*$ satisfies CSWC* with ϵ. For "$\Leftarrow$", suppose $\succsim^*$ satisfies CSWC* with $\epsilon \in \mathbb{R}_{++}$. Take $\lambda \in \mathbb{R}$, $x \in X$, and distinct $i, j \in N$ with $|u_i(x) - \lambda| \leq \epsilon$. Due to CSWC*, there exists $b \in \mathbb{R}^n$ such that $b_i = \lambda$, for all $k \in N \setminus \{i, j\}$, $b_k = u_k(x)$, and $b \sim^* u(x)$. There exists $y \in X$ with $u(y) = b$. It follows that $y \sim x$, so that $\succsim$ satisfies CSWC with ϵ.■

For the proof of Proposition 1, the following notion of continuity is relevant.

SWO $\succsim^*$ satisfies m-**Continuity** **(m-C*)** for $m \in N$ if, for all $b, c \in A^n$, the sets $\{a \in A^n \mid (\forall k \in \{m+1, ..., n\} : a_k = c_k) \wedge (a \succsim^* b)\}$ and $\{a \in A^n \mid (\forall k \in \{m+1, ..., n\} : a_k = c_k) \wedge (b \succsim^* a)\}$ are closed in A^n.

Lemma A.1.3: If SWO $\succsim^*$ satisfies SP* and RC*, then, for all $m \in N$, $\succsim^*$ satisfies m-C*.

Proof:

Suppose SWO $\succsim^*$ satisfies SP* and RC*. The stated claim will be proved by induction.

$m = 1$: Consider $b, c \in A^n$ and a sequence $(a^t)_{t\in\mathbb{N}} \in (A^n)^{\mathbb{N}}$ that converges to $a \in A^n$, and exhibits, for all $t \in \mathbb{N}$, $a^t_k = c_k$ for all $k \in \{2, ..., n\}$, and $a^t \succsim^* b$. It follows that $a_k = c_k$ for all $k \in \{2, ..., n\}$. Suppose $b \succ^* a$. By RC*, there exists $d \in A^n$ such that, for all $k \in \{2, ..., n\}$, $d_k = c_k$, and $d \sim^* b$. Since $d \succ^* a$, SP* implies $d_1 > a_1$. Define $\epsilon = d_1 - a_1 > 0$. Because the sequence converges to a, there exists $s \in \mathbb{N}$ such that, for all $t \geq s$, $|a^t_1 - a_1| < \epsilon$. For $t \geq s$, that implies $d_1 > a^t_1$. By SP*, that yields $d \succ^* a^t$ and in turn $b \succ^* a^t$. This is a contradiction. The initial assumption $b \succ^* a$ must be wrong. Instead, $a \succsim^* b$ must hold. This argument establishes that the set $\{a \in A^n \mid (\forall k \in \{2, ..., n\} : a_k = c_k) \wedge (a \succsim^* b)\}$ is closed in A^n. One shows analogously that the set $\{a \in A^n \mid (\forall k \in \{2, ..., n\} : a_k = c_k) \wedge (b \succsim^* a)\}$ is closed in A^n. Hence, $\succsim^*$ satisfies 1-C*.

$(m - 1) \Rightarrow m$: Suppose $\succsim^*$ satisfies $(m - 1)$-C* for some $m \in \{2, ..., n\}$. Consider $b, c \in A^n$ and a sequence $(a^t)_{t\in\mathbb{N}} \in (A^n)^{\mathbb{N}}$ that converges to $a \in A^n$, and exhibits, for all $t \in \mathbb{N}$, $a^t_k = c_k$ for all $k \in \{m + 1, ..., n\}$, and $a^t \succsim^* b$. That implies $a_k = c_k$ for all $k \in \{m+1, ..., n\}$. Suppose there exists $s \in \mathbb{N}$ such that, for all $t \geq s$, $a_m \geq a^t_m$. Define sequence $(d^t)_{t\in\mathbb{N}} \in (A^n)^{\mathbb{N}}$ such that, for all $t \in \mathbb{N}$, $d^t_m = a_m$, and, for all $k \in N \setminus \{m\}$, $d^t_k = a^t_k$. By SP*, $d^t \succsim^* a^t$ holds for all $t \geq s$. Taken together, the sequence converges to a, is constant for all $k \in \{m, ..., n\}$, and exhibits, for all $t \geq s$, $d^t \succsim^* b$. Due to $(m - 1)$-C*, that implies $a \succsim^* b$.

Suppose that, for all $s \in \mathbb{N}$, there exists $t \geq s$ with $a^t_m > a_m$. Suppose $b \succ^* a$. Take $p \in \mathbb{N}$ with $a^p_m > a_m$. Define sequence $(d^t)_{t\in\mathbb{N}} \in (A^n)^{\mathbb{N}}$ such that, for all $t \in \mathbb{N}$, $d^t_m = a^p_m$, and, for all $k \in N \setminus \{m\}$, $d^t_k = a^t_k$. Since $(a^t)_{t\in\mathbb{N}}$ converges to a, there exists $p' \in \mathbb{N}$ such that, for all $t \geq p'$, $d^t_m > a^t_m$. By SP*, that implies $d^t \succ^* a^t$ for all $t \geq p'$. Taken together, $(d^t)_{t\in\mathbb{N}}$ converges to $d = (a_1, ..., a_{m-1}, a^p_m, a_{m+1}, ..., a_n)$, is constant for all $k \in \{m, ..., n\}$, and exhibits, for all $t \geq p'$, $d^t \succ^* b$. Due to $(m - 1)$-C*, that implies $d \succsim^* b$. By RC*, there exists $\lambda \in A$ with $f = (a_1, ..., a_{m-1}, \lambda, a_{m+1}, ..., a_n) \sim^* b$. Due to SP*, $\lambda > a_m$.

By assumption, there exists $s \in \mathbb{N}$ with $\lambda > a^s_m > a_m$. Define sequence $(e^t)_{t\in\mathbb{N}} \in (A^n)^{\mathbb{N}}$ such that, for all $t \in \mathbb{N}$, $e^t_m = a^s_m$, and, for all $k \in N \setminus \{m\}$, $e^t_k = a^t_k$. There exists $s' \in \mathbb{N}$ such that, for all $t \geq s'$, $e^t_m > a^t_m$. Due to SP*, that leads to $e^t \succ^* a^t$ for all $t \geq s'$.

Taken together, $(e^t)_{t\in\mathbb{N}}$ converges to $e = (a_1, ..., a_{m-1}, a_m^s, a_{m+1}, ..., a_n)$, is constant for all $k \in \{m, ..., n\}$, and exhibits, for all $t \geq s'$, $e^t \succ^* b$. By $(m-1)$-C*, it follows that $e \succsim^* b$.

Due to RC*, there exists $\mu \in A$ with $g = (a_1, ..., a_{m-1}, \mu, a_{m+1}, ..., a_n) \sim^* b$. By SP*, $a_m^s \geq \mu$. Since $\lambda > \mu$, SP* also yields $f \succ^* g$. But social transitivity implies $f \sim^* g$. This is a contradiction. Accordingly, the initial assumption $b \succ^* a$ must be wrong. It must instead be true that $a \succsim^* b$. Therefore, in all cases, $a \succsim^* b$. This demonstrates that the set $\{a \in A^n \mid (\forall k \in \{m+1, ..., n\} : a_k = c_k) \wedge (a \succsim^* b)\}$ is closed in A^n. An analogous argument shows that the set $\{a \in A^n \mid (\forall k \in \{m+1, ..., n\} : a_k = c_k) \wedge (b \succsim^* a)\}$ is closed in A^n. Consequently, $\succsim^*$ satisfies m-C*.■

Proof of Proposition 1:
Suppose SO $\succsim$ satisfies SP. By Lemma A.1.1, $\succsim$ has an associate SWO $\succsim^*$ satisfying SP*.

"⇐": Suppose $\succsim$ satisfies RC. By Lemma A.1.2, $\succsim^*$ satisfies RC*. Due to Lemma A.1.3, $\succsim^*$ satisfies n-C*. This condition is identical to C*. Accordingly, Lemma A.1.2 implies that $\succsim$ satisfies C.

"⇒": Suppose $\succsim$ satisfies C. By Lemma A.1.2, $\succsim^*$ satisfies C*. Take $a, b, c \in A^n$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $b_k = c_k$, $b \succsim^* a$, and $a \succsim^* c$. If $b \sim^* a$, $\succsim^*$ satisfies RC* with $d = b$. If $c \sim^* a$, $\succsim^*$ satisfies RC* with $d = c$. Suppose $b \succ^* a \succ^* c$. By SP*, that implies $b_i > c_i$. For $\lambda \in [c_i, b_i] \subseteq A$, define $d(\lambda) = (c_1, ..., c_{i-1}, \lambda, c_{i+1}, ..., c_n) \in A^n$. Define $L = \{\lambda \in [c_i, b_i] \mid a \succsim^* d(\lambda)\}$ and $U = \{\lambda \in [c_i, b_i] \mid d(\lambda) \succsim^* a\}$. Since $c_i \in L$ and $b_i \in U$, both sets are non-empty. Moreover, SP* implies that, for $\lambda \in L$ and $\mu \in U$, $\lambda \leq \mu$. Accordingly, L has a smallest upper bound $\lambda_L \in [c_i, b_i]$ and U has a greatest lower bound $\lambda_U \in [c_i, b_i]$.

Suppose $\lambda_L > \lambda_U$. Define $\rho = \frac{\lambda_L - \lambda_U}{2} > 0$. There exists $\lambda \in L$ with $\lambda > \lambda_L - \rho$. Otherwise, $\lambda_L - \rho$ would be a smaller upper bound of L than λ_L. Likewise, there exists $\mu \in U$ with $\mu < \lambda_U + \rho$. Otherwise, $\lambda_U + \rho$ would be a greater lower bound of U than λ_U. Taken together, $\lambda \leq \mu < \lambda_U + \rho = \lambda_U + 2 \cdot \rho - \rho = \lambda_L - \rho < \lambda$. This is a contradiction. Hence, $\lambda_L \leq \lambda_U$. Suppose $\lambda_L < \lambda_U$. Take $\lambda \in [c_i, b_i]$ with $\lambda_L < \lambda < \lambda_U$. It follows that $\lambda \notin L$ and $\lambda \notin U$. This contradicts the completeness of $\succsim^*$. Consequently, $\lambda_L = \lambda_U$. Define $\lambda^* = \lambda_L = \lambda_U$.

Consider $\lambda \in [c_i, b_i]$ with $\lambda < \lambda^*$. There exists $\mu \in L$ with $\lambda < \mu$. Otherwise λ would be a smaller upper bound of L than λ^*. By SP*, $d(\mu) \succ^* d(\lambda)$. Since $a \succsim^* d(\mu)$, that implies $a \succ^* d(\lambda)$. Because that is the case for all $\lambda \in [c_i, b_i]$ with $\lambda < \lambda^*$, C* induces

$a \succsim^* d(\lambda^*)$. Take $\lambda \in [c_i, b_i]$ with $\lambda > \lambda^*$. Now, there exists $\mu \in U$ with $\lambda > \mu$. Otherwise λ would be a greater lower bound of U than λ^*. Due to SP*, $d(\lambda) \succ^* d(\mu)$. Together with $d(\mu) \succsim^* a$, it follows that $d(\lambda) \succ^* a$. Again, this is so for all $\lambda \in [c_i, b_i]$ with $\lambda > \lambda^*$, so that C* yields $d(\lambda^*) \succsim^* a$. Taken together, $d(\lambda^*) \sim^* a$. Hence, $\succsim^*$ satisfies RC* with $d = d(\lambda^*)$. Lemma A.1.2 implies that $\succsim$ satisfies RC.■

Condition PD* is equivalent to the following two SWO properties. It is likewise possible to formulate corresponding SO versions.

SWO $\succsim^*$ satisfies **Classical Pigou-Dalton (CPD*)** if, for all $a, b \in A^n$ and $i, j \in N$ such that $a_i - b_i = b_j - a_j > 0$, $a_j \geq a_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$, it is the case that $a \succsim^* b$.

SWO $\succsim^*$ satisfies **Anonymity (A*)** if, for all $a \in A^n$ and all bijections $\pi : N \to N$, it is the case that $a \sim^* (a_{\pi(1)}, ..., a_{\pi(n)})$.

Lemma A.1.4: SWO $\succsim^*$ satisfies PD* if and only if $\succsim^*$ satisfies CPD* and A*.
Proof:
"$\Rightarrow$": Suppose SWO $\succsim^*$ satisfies PD*. Take $a, b \in A^n$ and $i, j \in N$ in the sense of CPD*. Since $a_j > b_i$, PD* implies $a \succsim^* b$. Hence, $\succsim^*$ satisfies CPD*.

Consider $a \in A^n$ and bijection $\pi : N \to N$. It is well-known that there exist bijections $\pi_1, ..., \pi_s : N \to N$ such that, for all $t \in \{1, ..., s\}$, there exist $i, j \in N$ with $\pi_t(i) = j$, $\pi_t(j) = i$, and $\pi_t(k) = k$ for all $k \in N \setminus \{i, j\}$, and $\pi = \pi_s \circ ... \circ \pi_1$. For each $t \in \{0, 1, ..., s\}$, define $a^t = (a_{(\pi_t \circ ... \circ \pi_0)(1)}, ..., a_{(\pi_t \circ ... \circ \pi_0)(n)})$, where $\pi_0 : N \to N, i \mapsto i$.

Take $t \in \{1, ..., s\}$. By construction, there exist $i, j \in N$ with $(\pi_t \circ ... \circ \pi_0)(i) = (\pi_{t-1} \circ ... \circ \pi_0)(j)$, $(\pi_t \circ ... \circ \pi_0)(j) = (\pi_{t-1} \circ ... \circ \pi_0)(i)$, and $(\pi_t \circ ... \circ \pi_0)(k) = (\pi_{t-1} \circ ... \circ \pi_0)(k)$ for all $k \in N \setminus \{i, j\}$. That implies $a_i^t = a_{(\pi_t \circ ... \circ \pi_0)(i)} = a_{(\pi_{t-1} \circ ... \circ \pi_0)(j)} = a_j^{t-1}$, $a_j^t = a_{(\pi_t \circ ... \circ \pi_0)(j)} = a_{(\pi_{t-1} \circ ... \circ \pi_0)(i)} = a_i^{t-1}$, and, for all $k \in N \setminus \{i, j\}$, $a_k^t = a_{(\pi_t \circ ... \circ \pi_0)(k)} = a_{(\pi_{t-1} \circ ... \circ \pi_0)(k)} = a_k^{t-1}$.

Suppose $a_i^{t-1} = a_j^{t-1}$. In that case, $a^t = a^{t-1}$, implying $a^{t-1} \sim^* a^t$. Suppose $a_i^{t-1} > a_j^{t-1}$. That implies $a_i^{t-1} - a_i^t = a_j^t - a_j^{t-1} > 0$. Due to PD*, it follows that $a^{t-1} \sim^* a^t$. Suppose $a_i^{t-1} < a_j^{t-1}$. That analogously implies $a_i^t - a_i^{t-1} = a_j^{t-1} - a_j^t > 0$. Again, PD* induces $a^{t-1} \sim^* a^t$. In any case, $a^{t-1} \sim^* a^t$ holds for each $t \in \{1, ..., s\}$. By construction $a^0 = a$ and $a^s = (a_{\pi(1)}, ..., a_{\pi(n)})$. Taken together, that yields $a \sim^* (a_{\pi(1)}, ..., a_{\pi(n)})$. Consequently, $\succsim^*$ satisfies A*.

"$\Leftarrow$": Suppose $\succsim^*$ satisfies CPD* and A*. Consider $a, b \in A^n$ and $i, j \in N$ such that $a_i - b_i = b_j - a_j > 0$, $a_j \geq b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Suppose $a_j \geq a_i$. By CPD*, that implies $a \succsim^* b$.

Suppose $a_i > a_j$. That is, $b_j \geq a_i > a_j \geq b_i$. Suppose $a_j = b_i$. Since $a_i = b_j$, A* implies $a \sim^* b$ in that case. Suppose $a_j > b_i$. That is, $b_j > a_i > a_j > b_i$. Take $c \in A^n$ with $c_i = b_j$, $c_j = b_i$, and, for all $k \in N \setminus \{i, j\}$, $c_k = b_k$. Due to A*, $c \sim^* b$. It is the case that $a_j - c_j = c_i - a_i > 0$, $a_i > a_j$, and, for all $k \in N \setminus \{i, j\}$, $a_k = c_k$. Due to CPD*, that implies $a \succsim^* c$. Taken together, $a \succsim^* b$. Since it is true in all cases that $a \succsim^* b$, $\succsim^*$ satisfies PD*.■

Set $M \subseteq N$ is **separable** (with respect to $\succsim^*$) if, for all $a, b, c, d \in A^n$ such that, for all $i \in M$, $a_i = c_i$ and $b_i = d_i$, and, for all $i \in N \setminus M$, $a_i = b_i$ and $c_i = d_i$, it is the case that $a \succsim^* b \Leftrightarrow c \succsim^* d$. WBS* states that there exists one binary separable M. As such, it is much weaker than the following full separability condition: SWO $\succsim^*$ satisfies **Separability (S*)** if every $M \subseteq N$ is separable. Together with the other SWO conditions in the sense of Theorem 1, WBS* implies S*. The proof of this implication employs the following result due to Gorman (1968: Theorem 1).

Lemma A.1.5: Suppose SWO $\succsim^*$ satisfies SP* and C*. If the sets $M, M' \subseteq N$ are separable and $M \cap M' \neq \emptyset$, then $M \cap M'$ and $M \cup M'$ are separable.
Proof:
Due to SP*, every non-empty $M \subseteq N$ is strictly essential in the sense that, for all $a \in A^n$, there exist $b, c \in A^n$ with $b_i = c_i = a_i$ for all $i \in N \setminus M$ and $b \succ^* c$. To see that, take $\lambda, \mu \in A$ with $\lambda > \mu$ and set $b_i = \lambda$ and $c_i = \mu$ for all $i \in M$. Consider separable $M, M' \subseteq N$ with $M \cap M' \neq \emptyset$. If $M \subseteq M'$ or $M' \subseteq M$, then $M \cap M'$ and $M \cup M'$ are clearly separable. Suppose that $M \nsubseteq M'$ and $M' \nsubseteq M$. Since all of its assumptions are satisfied, Theorem 1 of Gorman (1968) implies that $M \cap M'$ and $M \cup M'$ are separable.■

Lemma A.1.6: If SWO $\succsim^*$ satisfies SP*, A*, WBS*, and C*, then $\succsim^*$ satisfies S*.
Proof:
To begin with, the reflexivity of $\succsim^*$ implies that the empty set is separable. Let M^* be the separable set in the sense of WBS*. Consider $M \subseteq N$ with $|M| = 2$. Take $a, b, c, d \in A^n$ such that, for all $l \in M$, $a_l = c_l$ and $b_l = d_l$, and, for all $l \in N \setminus M$, $a_l = b_l$ and $c_l = d_l$.

First, suppose that M and M^* have one individual in common. That is, there exist distinct $i, j, k \in N$ with $M = \{i, j\}$ and $M^* = \{j, k\}$. Define bijection $\pi : N \to N$ such that $\pi(i) = k$, $\pi(k) = i$, and, for all $l \in N \setminus \{i, k\}$, $\pi(l) = l$. Define $a' = (a_{\pi(1)}, ..., a_{\pi(n)})$, $b' = (b_{\pi(1)}, ..., b_{\pi(n)})$, $c' = (c_{\pi(1)}, ..., c_{\pi(n)})$, and $d' = (d_{\pi(1)}, ..., d_{\pi(n)})$. Due to A*, $a \sim^* a'$, $b \sim^* b'$, $c \sim^* c'$, and $d \sim^* d'$. By construction, it is the case that $a'_j = c'_j$ and $b'_j = d'_j$ as well as $a'_k = c'_k$ and $b'_k = d'_k$. Moreover, $a'_i = b'_i$ and $c'_i = d'_i$, and, for all $l \in N \setminus \{i, j, k\}$, $a'_l = b'_l$ and $c'_l = d'_l$. Since M^* is separable, it follows that $a' \succsim^* b' \Leftrightarrow c' \succsim^* d'$. This yields $a \succsim^* b \Leftrightarrow c \succsim^* d$, so that M is separable.

Second, suppose that M and M^* have no individual in common. That is, there exist distinct $i, j, k, h \in N$ with $M = \{i, j\}$ and $M^* = \{k, h\}$. Define bijection $\pi : N \rightarrow N$ such that $\pi(i) = k$, $\pi(k) = i$, $\pi(j) = h$, $\pi(h) = j$, and, for all $l \in N \setminus \{i, j, k, h\}$, $\pi(l) = l$. Again, define $a' = (a_{\pi(1)}, ..., a_{\pi(n)})$, $b' = (b_{\pi(1)}, ..., b_{\pi(n)})$, $c' = (c_{\pi(1)}, ..., c_{\pi(n)})$, and $d' = (d_{\pi(1)}, ..., d_{\pi(n)})$. As before, A* implies $a \sim^* a'$, $b \sim^* b'$, $c \sim^* c'$, and $d \sim^* d'$. It is the case that $a'_k = c'_k$, $b'_k = d'_k$, $a'_h = c'_h$, and $b'_h = d'_h$. Furthermore, $a'_i = b'_i$, $c'_i = d'_i$, $a'_j = b'_j$, $c'_j = d'_j$, and, for all $l \in N \setminus \{i, j, k, h\}$, $a'_l = b'_l$ and $c'_l = d'_l$. The separability of M^* implies $a \succsim^* b \Leftrightarrow a' \succsim^* b' \Leftrightarrow c' \succsim^* d' \Leftrightarrow c \succsim^* d$. Taken together, the argument establishes that every $M \subseteq N$ with $|M| = 2$ is separable.

In view of $n \geq 3$, consider distinct $i, j, k \in N$. As shown, $\{i, j\}$ and $\{j, k\}$ are separable. By Lemma A.1.5, $\{j\} = \{i, j\} \cap \{j, k\}$ is separable. This establishes that every $M \subseteq N$ with $|M| = 1$ is separable.

It is proved by induction that, for all $m \in \{3, ..., n\}$, every $M \subseteq N$ with $|M| = m$ is separable. Take $m \in \{3, ..., n\}$ and suppose that every $M \subseteq N$ with $|M| = m - 1$ is separable. Consider $M \subseteq N$ with $|M| = m$. Take distinct $i, j \in M$. As shown, $\{i, j\}$ is separable. By assumption, $M \setminus \{j\}$ is also separable. Due to Lemma A.1.5, it follows that $M = \{i, j\} \cup (M \setminus \{j\})$ is separable. Taken together, it has been demonstrated that every $M \subseteq N$ is separable. Hence, $\succsim^*$ satisfies S*.■

Proof of Theorem 1:
"⇐": Take GUO $\succsim$ with associate g. Define its associate SWO $\succsim^*$ such that, for $a, b \in A^n$, $a \succsim^* b$ if and only if there exist $x, y \in X$ with $u(x) = a$, $u(y) = b$, and $x \succsim y$. Take $a, b \in A^n$. Suppose $a \succsim^* b$. Since there exist $x, y \in X$ with $u(x) = a$, $u(y) = b$, and $x \succsim y$, it follows that $\sum_{i \in N} g(a_i) \geq \sum_{i \in N} g(b_i)$. Suppose $\sum_{i \in N} g(a_i) \geq \sum_{i \in N} g(b_i)$. For $x, y \in X$ with $u(x) = a$ and $u(y) = b$, that implies $x \succsim y$ and in turn $a \succsim^* b$. Taken together, $a \succsim^* b \Leftrightarrow \sum_{i \in N} g(a_i) \geq \sum_{i \in N} g(b_i)$. For $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{i \in N} g(u_i(x)) \geq \sum_{i \in N} g(u_i(y)) \Leftrightarrow u(x) \succsim^* u(y)$, so that $\succsim^*$ is indeed associate to $\succsim$. It will be shown that $\succsim^*$ satisfies SP*, PD*, WBS*, and C*. By Lemmas A.1.1-A.1.2 and Proposition 1, it will then follow that $\succsim$ satisfies SP, PD, WBS, C, and RC.

Consider $a, b \in A^n$ such that, for all $i \in N$, $a_i \geq b_i$, and there exists $j \in N$ with $a_j > b_j$. Since g is increasing, it follows that $\sum_{i \in N} g(a_i) > \sum_{i \in N} g(b_i)$. That implies $a \succ^* b$.

Accordingly, $\succsim^*$ satisfies SP*.

It is true that, for all $\lambda, \mu \in A$ with $\lambda > \mu$ and $\epsilon \in \mathbb{R}_{++}$ with $(\lambda+\epsilon) \in A$, $g(\lambda+\epsilon)-g(\lambda) \leq g(\mu+\epsilon)-g(\mu)$. To see that, take $\lambda, \mu \in A$ with $\lambda > \mu$ and $\epsilon \in \mathbb{R}_{++}$ with $(\lambda+\epsilon) \in A$. Define $t = \frac{\lambda-\mu}{\lambda-\mu+\epsilon} \in (0,1)$ and $s = \frac{\epsilon}{\lambda-\mu+\epsilon} \in (0,1)$. Using the fact that g is concave, it follows that $g(\lambda)+g(\mu+\epsilon) = g(t\cdot(\lambda+\epsilon)+(1-t)\cdot\mu)+g(s\cdot(\lambda+\epsilon)+(1-s)\cdot\mu) \geq t\cdot g(\lambda+\epsilon)+(1-t)\cdot g(\mu)+s\cdot g(\lambda+\epsilon)+(1-s)\cdot g(\mu) = (t+s)\cdot g(\lambda+\epsilon)+(2-t-s)\cdot g(\mu) = g(\lambda+\epsilon)+g(\mu)$. That is, $g(\lambda+\epsilon)-g(\lambda) \leq g(\mu+\epsilon)-g(\mu)$.

Take $a, b \in A^n$ and $i, j \in N$ such that $a_i - b_i = b_j - a_j > 0$, $a_j \geq b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Define $\epsilon = b_j - a_j$. Due to the last paragraph, it follows that $g(a_i) - g(b_i) = g(b_i+\epsilon) - g(b_i) \geq g(a_j+\epsilon) - g(a_j) = g(b_j) - g(a_j)$. That yields $\sum_{k \in N} g(a_k) - \sum_{k \in N} g(b_k) \geq 0$ and in turn $a \succsim^* b$. Consequently, $\succsim^*$ satisfies PD*.

Consider $a, b, c, d \in A^n$ and $M \subseteq N$ such that, for all $i \in M$, $a_i = c_i$ and $b_i = d_i$, and, for all $i \in N \setminus M$, $a_i = b_i$ and $c_i = d_i$. It follows that $\sum_{i \in N} g(a_i) - \sum_{i \in N} g(b_i) = \sum_{i \in M} g(a_i) - \sum_{i \in M} g(b_i) = \sum_{i \in M} g(c_i) - \sum_{i \in M} g(d_i) = \sum_{i \in N} g(c_i) - \sum_{i \in N} g(d_i)$. That yields $a \succsim^* b \Leftrightarrow \sum_{i \in N} g(a_i) - \sum_{i \in N} g(b_i) \geq 0 \Leftrightarrow \sum_{i \in N} g(c_i) - \sum_{i \in N} g(d_i) \geq 0 \Leftrightarrow c \succsim^* d$. Hence, $\succsim^*$ satisfies S* and, in particular, WBS*.

Take $b \in A^n$ and a sequence $(a^t)_{t \in \mathbb{N}} \in (A^n)^{\mathbb{N}}$ that converges to $a \in A^n$, and exhibits, for all $t \in \mathbb{N}$, $a^t \succsim^* b$. Suppose $b \succ^* a$. That is, $\sum_{i \in N} g(b_i) > \sum_{i \in N} g(a_i)$. Define $\epsilon = \sum_{i \in N} g(b_i) - \sum_{i \in N} g(a_i) > 0$. Since g is continuous, there exists $\delta_i > 0$ for each $i \in N$ such that, for all $\mu \in A$ with $|\mu - a_i| < \delta_i$, $|g(\mu) - g(a_i)| < \frac{\epsilon}{n}$. Because $(a^t)_{t \in \mathbb{N}}$ converges to a, there exists $s \in \mathbb{N}$ such that, for all $i \in N$, $|a_i^s - a_i| < \delta_i$. Taken together, it follows that $\sum_{i \in N} g(a_i^s) = \sum_{i \in N}(g(a_i^s) - g(a_i) + g(a_i)) \leq \sum_{i \in N}(|g(a_i^s) - g(a_i)| + g(a_i)) < \sum_{i \in N}(\frac{\epsilon}{n} + g(a_i)) = \epsilon + \sum_{i \in N} g(a_i) = \sum_{i \in N} g(b_i)$. That yields the contradiction $b \succ^* a^s$. Accordingly, it must instead be the case that $a \succsim^* b$. The argument establishes that the set $\{a \in A^n \mid a \succsim^* b\}$ is closed in A^n. One demonstrates analogously that the set $\{a \in A^n \mid b \succsim^* a\}$ is closed in A^n. Consequently, $\succsim^*$ satisfies C*.

"$\Rightarrow$": Suppose SO $\succsim$ satisfies SP, PD, WBS, and RC (or C). Due to Lemmas A.1.1-A.1.2 and Proposition 1, $\succsim$ has an associate SWO $\succsim^*$ satisfying SP*, PD*, WBS*, and C*. By Lemma A.1.4, $\succsim^*$ satisfies CPD* and A*. Due to Lemma A.1.6, it also satisfies S*.

Because $\succsim^*$ is complete and transitive, and satisfies SP*, S*, and C*, all conditions of Theorem 6.14 in Krantz et al. (1971: p. 302) are satisfied. Due to that Theorem, there exists $h_i : A \to \mathbb{R}$ for each $i \in N$ such that, for all $a, b \in A^n$, $a \succsim^* b \Leftrightarrow \sum_{i \in N} h_i(a_i) \geq$

$\sum_{i\in N} h_i(b_i)$. Take $\mu \in A$. For each $i \in N$, define $g_i : A \to \mathbb{R}, \lambda \mapsto h_i(\lambda) - h_i(\mu)$. For $a, b \in A^n$, $\sum_{i\in N} g_i(a_i) \geq \sum_{i\in N} g_i(b_i) \Leftrightarrow \sum_{i\in N}(h_i(a_i)-h_i(\mu)) \geq \sum_{i\in N}(h_i(b_i)-h_i(\mu)) \Leftrightarrow \sum_{i\in N} h_i(a_i) \geq \sum_{i\in N} h_i(b_i) \Leftrightarrow a \succsim^* b$. For each $i \in N$, $g_i(\mu) = h_i(\mu) - h_i(\mu) = 0$.

Consider $i, j \in N$ and $\lambda \in A$. Take $a, b \in A^n$ such that $a_i = \lambda = b_j$, $a_j = \mu = b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k = \mu$. By A*, $a \sim^* b$. That is, $0 = \sum_{k\in N} g_k(a_k) - \sum_{k\in N} g_k(b_k) = g_i(\lambda) + g_j(\mu) - g_i(\mu) - g_j(\lambda) = g_i(\lambda) - g_j(\lambda)$. It follows that, for all $i, j \in N$, $g_i = g_j$. Define $g = g_i$ for any $i \in N$. Taken together, it is the case that, for all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \succsim^* u(y) \Leftrightarrow \sum_{i\in N} g(u_i(x)) \geq \sum_{i\in N} g(u_i(y))$. It remains to be shown that g is increasing, continuous, and concave.

Consider $\lambda, \mu \in A$ with $\lambda > \mu$. Take $a = (\lambda, \mu, ..., \mu)$ and $b = (\mu, \mu, ..., \mu)$ from A^n. By SP*, $a \succ^* b$. That implies $0 < \sum_{i\in N} g(a_i) - \sum_{i\in N} g(b_i) = g(\lambda) - g(\mu)$. Hence, g is increasing.

Take $\mu \in A$. Suppose that μ is not the greatest well-being level, that is, there exists $\lambda \in A$ with $\lambda > \mu$. Define $U = \{g(\lambda) \in g(A) \mid \lambda > \mu\}$. By assumption, this set is non-empty. Since g is increasing, $g(\lambda) > g(\mu)$ holds for all $g(\lambda) \in U$. As $g(\mu)$ is a lower bound of U, U has a greatest lower bound $\rho \in \mathbb{R}$ with $\rho \geq g(\mu)$. Suppose $\rho > g(\mu)$. Define $\epsilon = \rho - g(\mu) > 0$. There exists $g(\nu) \in U$ with $g(\nu) < \rho + \epsilon$. Otherwise, $\rho + \epsilon$ would be a greater lower bound of U than ρ. Recognizing that $\nu > \mu$, take $\lambda \in (\mu, \nu) \subseteq A$. Since $g(\lambda) \in U$ and g is increasing, $g(\nu) > g(\lambda) \geq \rho$. Define $\delta = g(\nu) - \rho > 0$.

There exists $g(\tilde{\nu}) \in U$ with $g(\tilde{\nu}) < \rho + \frac{\delta}{2}$. Otherwise, $\rho + \frac{\delta}{2}$ would be a greater lower bound of U than ρ. Recognizing that $\tilde{\nu} > \mu$, define $a^\lambda = (\lambda, \nu, \mu, ..., \mu) \in A^n$ for $\lambda \in [\mu, \tilde{\nu}]$. Define $b = (\tilde{\nu}, \tilde{\nu}, \mu, ..., \mu) \in A^n$. For $\lambda \in (\mu, \tilde{\nu}]$, $g(\lambda) \in U$, so that $g(\lambda) \geq \rho$. It follows that $\sum_{i\in N} g(a_i^\lambda) - \sum_{i\in N} g(b_i) = g(\lambda) + g(\nu) - 2 \cdot g(\tilde{\nu}) > g(\lambda) + g(\nu) - 2 \cdot \rho - \delta = g(\lambda) - \rho \geq 0$, so that $a^\lambda \succ^* b$. Due to C*, that implies $a^\mu \succsim^* b$. But, in fact, $\sum_{i\in N} g(a_i^\mu) - \sum_{i\in N} g(b_i) = g(\mu) + g(\nu) - 2 \cdot g(\tilde{\nu}) \leq g(\mu) + g(\nu) - 2 \cdot \rho < g(\mu) + \rho + \epsilon - 2 \cdot \rho = 0$. That yields the contradiction $b \succ^* a^\mu$. Accordingly, assumption $\rho > g(\mu)$ must be wrong. It must instead be the case that $\rho = g(\mu)$. That is, $g(\mu)$ is the greatest lower bound of U. If μ is not the smallest well-being level, one shows analogously that $g(\mu)$ is the smallest upper bound of $L = \{g(\lambda) \in g(A) \mid \lambda < \mu\}$.

Consider $\mu \in A$ that is neither the smallest nor the greatest well-being level and $\epsilon \in \mathbb{R}_{++}$. Since $g(\mu)$ is the greatest lower bound of U, there exists $g(\nu) \in U$ with $g(\mu) + \epsilon > g(\nu)$. Since $g(\mu)$ is the smallest upper bound of L, there exists $g(\tilde{\nu}) \in L$ with $g(\mu) - \epsilon < g(\tilde{\nu})$. Define $\delta = min\{\nu - \mu, \mu - \tilde{\nu}\} > 0$. Take $\lambda \in A$ with $|\lambda - \mu| < \delta$. Suppose $\lambda \geq \mu$. Then, $\lambda - \mu = |\lambda - \mu| < \nu - \mu$, so that $\lambda < \nu$. That implies

$|g(\lambda) - g(\mu)| = g(\lambda) - g(\mu) < g(\nu) - g(\mu) < \epsilon$. Suppose $\lambda < \mu$. In that case, $\mu - \lambda = |\lambda - \mu| < \mu - \tilde{\nu}$, so that $\lambda > \tilde{\nu}$. That yields $|g(\lambda) - g(\mu)| = g(\mu) - g(\lambda) < g(\mu) - g(\tilde{\nu}) < \epsilon$. Accordingly, g is continuous at μ.

Suppose that μ is the smallest well-being level from A and take $\epsilon \in \mathbb{R}_{++}$. As $g(\mu)$ is the greatest lower bound of U, there exists $g(\nu) \in U$ with $g(\mu) + \epsilon > g(\nu)$. Define $\delta = \nu - \mu > 0$. For $\lambda \in A$ with $|\lambda - \mu| < \delta$, it follows that $\lambda < \nu$ and thus $|g(\lambda) - g(\mu)| = g(\lambda) - g(\mu) < g(\nu) - g(\mu) < \epsilon$. That is, g is continuous at μ. One shows analogously that g is continuous at the greatest well-being level from A (supposing it exists). Therefore, g is continuous.

It is true that, for all $\lambda, \mu \in A$ with $\lambda > \mu$ and $\epsilon \in \mathbb{R}_{++}$ with $\lambda + \epsilon \in A$, $g(\lambda + \epsilon) - g(\lambda) \leq g(\mu + \epsilon) - g(\mu)$. To see that, take $\lambda, \mu \in A$ with $\lambda > \mu$ and $\epsilon \in \mathbb{R}_{++}$ with $\lambda + \epsilon \in A$. Consider $a = (\lambda, \mu + \epsilon, \mu, ..., \mu)$ and $b = (\lambda + \epsilon, \mu, \mu, ..., \mu)$. Due to PD*, $a \succsim^* b$. That implies $\sum_{i \in N} g(a_i) \geq \sum_{i \in N} g(b_i)$ and in turn $g(\mu + \epsilon) - g(\mu) \geq g(\lambda + \epsilon) - g(\lambda)$.

Suppose there exist $\lambda, \mu \in A$ with $\lambda > \mu$ and $t \in [0, 1]$ with $g(t \cdot \lambda + (1 - t) \cdot \mu) < t \cdot g(\lambda) + (1 - t) \cdot g(\mu)$. Defining $\nu = t \cdot \lambda + (1 - t) \cdot \mu$, that implies $t \in (0, 1)$ and $g(\lambda) - g(\nu) > g(\lambda) - t \cdot g(\lambda) - (1 - t) \cdot g(\mu) = (1 - t) \cdot (g(\lambda) - g(\mu))$.

Suppose $t \in \mathbb{Q}$. That is, there exist $s, m \in \mathbb{N}$ with $s < m$ and $t = \frac{s}{m}$. For $p \in \{0, 1, ..., m\}$, define $\nu_p = \frac{p}{m} \cdot \lambda + (1 - \frac{p}{m}) \cdot \mu$. In particular, $\nu_0 = \mu$, $\nu_s = \nu$, and $\nu_m = \lambda$. For each $p \in \{1, ..., m\}$, it is true that $\nu_p - \nu_{p-1} = \frac{p}{m} \cdot \lambda + (1 - \frac{p}{m}) \cdot \mu - \frac{p-1}{m} \cdot \lambda - (1 - \frac{p-1}{m}) \cdot \mu = \frac{\lambda - \mu}{m} > 0$. That implies $g(\nu_m) - g(\nu_s) = \sum_{p=s+1}^{m} (g(\nu_p) - g(\nu_{p-1})) \leq (m - s) \cdot (g(\nu_{s+1}) - g(\nu_s))$. It follows that $(m - s) \cdot (g(\nu_{s+1}) - g(\nu_s)) \geq g(\nu_m) - g(\nu_s) > (1 - \frac{s}{m}) \cdot (g(\lambda) - g(\mu)) = \frac{m-s}{m} \cdot (g(\lambda) - g(\mu))$. It is in turn true that, for all $p \in \{1, ..., s\}$, $g(\nu_p) - g(\nu_{p-1}) \geq g(\nu_{s+1}) - g(\nu_s) > \frac{1}{m} \cdot (g(\lambda) - g(\mu))$. Taken together, that yields $g(\lambda) - g(\mu) = g(\nu_m) - g(\nu_s) + \sum_{p=1}^{s} (g(\nu_p) - g(\nu_{p-1})) > (1 - \frac{s}{m}) \cdot (g(\lambda) - g(\mu)) + s \cdot \frac{1}{m} \cdot (g(\lambda) - g(\mu)) = g(\lambda) - g(\mu)$. This is a contradiction.

Suppose $t \notin \mathbb{Q}$. Define $\epsilon = t \cdot g(\lambda) + (1 - t) \cdot g(\mu) - g(\nu) > 0$. Because g is continuous, there exists $\delta \in \mathbb{R}_{++}$ such that, for all $\nu' \in A$ with $|\nu' - \nu| < \delta$, $|g(\nu') - g(\nu)| < \epsilon$. Take $t' \in (0, 1) \cap \mathbb{Q}$ with $t < t' < t + \frac{\delta}{\lambda - \mu}$. Define $\nu' = t' \cdot \lambda + (1 - t') \cdot \mu$. It is true that $\nu' > \nu$ and $t' \cdot g(\lambda) + (1 - t') \cdot g(\mu) > t \cdot g(\lambda) + (1 - t) \cdot g(\mu)$.[30] Moreover, $t' - t < \frac{\delta}{\lambda - \mu}$ implies $|\nu' - \nu| = \nu' - \nu = (t' - t) \cdot (\lambda - \mu) < \delta$. Continuity of g induces $g(\nu') - g(\nu) = |g(\nu') - g(\nu)| < \epsilon = t \cdot g(\lambda) + (1 - t) \cdot g(\mu) - g(\nu)$. Taken together, it follows

[30]Indeed, $t' > t \Leftrightarrow t' \cdot (\lambda - \mu) > t \cdot (\lambda - \mu) \Leftrightarrow t' \cdot \lambda + (1 - t') \cdot \mu - \mu > t \cdot \lambda + (1 - t) \cdot \mu - \mu \Leftrightarrow \nu' > \nu$. Likewise, $t' > t \Leftrightarrow t' \cdot (g(\lambda) - g(\mu)) > t \cdot (g(\lambda) - g(\mu)) \Leftrightarrow t' \cdot g(\lambda) + (1 - t') \cdot g(\mu) - g(\mu) > t \cdot g(\lambda) + (1 - t) \cdot g(\mu) - g(\mu) \Leftrightarrow t' \cdot g(\lambda) + (1 - t') \cdot g(\mu) > t \cdot g(\lambda) + (1 - t) \cdot g(\mu)$.

that $g(t' \cdot \lambda + (1-t') \cdot \mu) = g(\nu') < t \cdot g(\lambda) + (1-t) \cdot g(\mu) < t' \cdot g(\lambda) + (1-t') \cdot g(\mu)$. From here, the same argument as before derives a contradiction. Hence, the initial assumption must be wrong. It is instead true that, for all $\lambda, \mu \in A$ with $\lambda > \mu$ and $t \in [0,1]$, $g(t \cdot \lambda + (1-t) \cdot \mu) \geq t \cdot g(\lambda) + (1-t) \cdot g(\mu)$.

Consider $\lambda, \mu \in A$ and $t \in [0,1]$. If $\lambda = \mu$, $g(t \cdot \lambda + (1-t) \cdot \mu) \geq t \cdot g(\lambda) + (1-t) \cdot g(\mu)$ is automatically true. For $\lambda > \mu$, it has been shown that $g(t \cdot \lambda + (1-t) \cdot \mu) \geq t \cdot g(\lambda) + (1-t) \cdot g(\mu)$. Suppose $\lambda < \mu$. Define $s = 1-t$. Since $s \in [0,1]$, the stated argument yields $g(t \cdot \lambda + (1-t) \cdot \mu) = g(s \cdot \mu + (1-s) \cdot \lambda) \geq s \cdot g(\mu) + (1-s) \cdot g(\lambda) = t \cdot g(\lambda) + (1-t) \cdot g(\mu)$. Consequently, g is concave. Taken together, this establishes that $\succsim$ is a GUO.■

CSWC* is equivalent to the following two-person condition.

Suppose $A = \mathbb{R}$. SWO $\succsim^*$ satisfies **Unrestricted Compensation (UC*)** if, for all $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^n$, and distinct $i, j \in N$, there exists $b \in \mathbb{R}^n$ with $b_i = \lambda$, for all $k \in N \setminus \{i, j\}$, $b_k = a_k$, and $b \sim^* a$.

Lemma A.1.7: Suppose $A = \mathbb{R}$. SWO $\succsim^*$ satisfies CSWC* if and only if $\succsim^*$ satisfies UC*.
Proof:
For "$\Leftarrow$", suppose SWO $\succsim^*$ satisfies UC*. It immediately follows that $\succsim^*$ satisfies CSWC* with every $\epsilon \in \mathbb{R}_{++}$. For "$\Rightarrow$", suppose SWO $\succsim^*$ satisfies CSWC* with $\epsilon \in \mathbb{R}_{++}$. Consider $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^n$, and distinct $i, j \in N$. For $\succsim^*$ to satisfy UC*, there must be $b \in \mathbb{R}^n$ with $b_i = \lambda$, for all $k \in N \setminus \{i, j\}$, $b_k = a_k$, and $b \sim^* a$. If $a_i = \lambda$, the conclusion of UC* is satisfied with $b = a$.

Suppose $a_i < \lambda$. Inductively define a sequence $b^0, b^1, ... \in \mathbb{R}^n$ with $b^0 = a$. For $t \in \mathbb{N}$, let b^t be such that $b^t_i = min\{b^{t-1}_i + \epsilon, \lambda\}$, for all $k \in N \setminus \{i, j\}$, $b^t_k = b^{t-1}_k$, and $b^t \sim^* b^{t-1}$. Since $|b^{t-1}_i - min\{b^{t-1}_i + \epsilon, \lambda\}| \leq \epsilon$, such b^t exists by CSWC*. By construction, one finds $b^r \in \mathbb{R}^n$ with $b^r_i = \lambda$ and, for all $k \in N \setminus \{i, j\}$, $b^r_k = a_k$. Social transitivity yields $b^r \sim^* a$. That is, the conclusion of UC* is satisfied with $b = b^r$.

Suppose $a_i > \lambda$. Again, inductively define a sequence $b^0, b^1, ... \in \mathbb{R}^n$ with $b^0 = a$. For $t \in \mathbb{N}$, let b^t be such that $b^t_i = max\{b^{t-1}_i - \epsilon, \lambda\}$, for all $k \in N \setminus \{i, j\}$, $b^t_k = b^{t-1}_k$, and $b^t \sim^* b^{t-1}$. Due to $|b^{t-1}_i - max\{b^{t-1}_i - \epsilon, \lambda\}| \leq \epsilon$, such b^t exists by CSWC*. Once again, there is $b^r \in \mathbb{R}^n$ with $b^r_i = \lambda$, for all $k \in N \setminus \{i, j\}$, $b^r_k = a_k$, and $b^r \sim^* a$. The conclusion of UC* is satisfied with $b = b^r$.■

Lemma A.1.8: Suppose $A = \mathbb{R}$. If SWO $\succsim^*$ satisfies CSWC*, then $\succsim^*$ satisfies RC*.
Proof:

Suppose SWO $\succsim^*$ satisfies CSWC*. By Lemma A.1.7, $\succsim^*$ satisfies UC*. Consider $a, b, c \in A^n$ and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $b_k = c_k$, $b \succsim^* a$, and $a \succsim^* c$. Inductively define a sequence $d^0, d^1, ..., d^n \in A^n$ with $d^0 = a$. Take $j \in N$. If $j = i$, set $d^j = d^{j-1}$. If $j \neq i$, select d^j such that $d^j_j = c_j$, for all $k \in N \setminus \{i, j\}$, $d^j_k = d^{j-1}_k$, and $d^j \sim^* d^{j-1}$. Such d^j exists by UC*. By construction, it is the case that, for all $k \in N \setminus \{i\}$, $d^n_k = c_k$. Social transitivity implies $d^n \sim^* a$. Hence, $\succsim^*$ satisfies RC* with $d = d^n$.■

Proof of Theorem 2:
"⇐": Take SGUO $\succsim$ with associate g and associate SWO $\succsim^*$. As a GUO, $\succsim$ satisfies SP, PD, and WBS by Theorem 1. Take $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^n$, and distinct $i, j \in N$. Since g is surjective, there exists $\mu \in \mathbb{R}$ with $g(\mu) = g(a_i) + g(a_j) - g(\lambda)$. Consider $b \in \mathbb{R}^n$ with $b_i = \lambda$, $b_j = \mu$, and, for all $k \in N \setminus \{i, j\}$, $b_k = a_k$. By construction, it follows that $\sum_{k \in N} g(b_k) - \sum_{k \in N} g(a_k) = g(b_i) - g(a_i) + g(b_j) - g(a_j) = 0$. Therefore, $b \sim^* a$, so that $\succsim^*$ satisfies UC*. By Lemma A.1.7, $\succsim^*$ satisfies CSWC*. In turn, Lemma A.1.2 implies that $\succsim$ satisfies CSWC.

"⇒": Suppose SO $\succsim$ satisfies SP, PD, WBS, and CSWC. By Lemmas A.1.1-A.1.2, $\succsim$ has an associate SWO $\succsim^*$ satisfying SP*, PD*, WBS*, and CSWC*. Due to Lemmas A.1.7-A.1.8, $\succsim^*$ also satisfies UC* and RC*. By Lemma A.1.2, $\succsim$ satisfies RC. Theorem 1 implies that $\succsim$ is a GUO. It remains to be shown that its associate g is surjective.

Take $\rho \in \mathbb{R}$. There exists $m \in \mathbb{Z}$ with $m < \frac{\rho - g(0)}{g(1) - g(0)}$ and $m < 0$. For all $p \in \mathbb{Z}$ with $p < 0$, $g(p+1) - g(p) \geq g(1) - g(0)$. That implies $g(0) - g(m) = \sum_{p=m}^{-1} g(p+1) - g(p) \geq |m| \cdot (g(1) - g(0)) = -m \cdot (g(1) - g(0)) > g(0) - \rho$. Hence, $g(m) < \rho$. This shows that g does not have a lower bound.

Suppose g has smallest upper bound $\rho \in \mathbb{R}$, that is, for all $\lambda \in \mathbb{R}$, $g(\lambda) \leq \rho$. Define $\epsilon = \rho - g(0) > 0$. There exists $\lambda \in \mathbb{R}$ with $g(\lambda) > \rho - \frac{\epsilon}{2}$. Otherwise, $\rho - \frac{\epsilon}{2}$ would be a smaller upper bound of g than ρ. Take $a = (\lambda, \lambda, 0, ..., 0)$. By UC*, there exists $\mu \in \mathbb{R}$ with $b = (\mu, 0, 0, ..., 0) \sim^* a$. That implies $\sum_{i \in N} g(a_i) - \sum_{i \in N} g(b_i) = 0$ and thus $g(\lambda) + g(\lambda) - g(\mu) - g(0) = 0$. It follows that $g(\mu) = 2 \cdot g(\lambda) - g(0) > 2 \cdot \rho - \epsilon - g(0) = \rho$. This contradicts the fact that ρ is an upper bound of g. The initial assumption that g has a smallest upper bound must be wrong. Accordingly, g has no upper bound.

Consider $\rho \in \mathbb{R}$. Define $L = \{\lambda \in \mathbb{R} \mid g(\lambda) < \rho\}$ and $U = \{\lambda \in \mathbb{R} \mid g(\lambda) > \rho\}$. Because g is unbounded, both sets are non-empty. Each element of U is an upper bound of L and each element of L is a lower bound of U. Thus, L has a smallest upper bound $\lambda_L \in \mathbb{R}$ and U has a greatest lower bound $\lambda_U \in \mathbb{R}$.

Suppose $\lambda_L > \lambda_U$. Define $\epsilon = \lambda_L - \lambda_U > 0$. As λ_L is the smallest upper bound of L, there exists $\lambda \in L$ with $\lambda > \lambda_L - \frac{\epsilon}{2}$. Because λ_U is the greatest lower bound of U, there exists $\mu \in U$ with $\mu < \lambda_U + \frac{\epsilon}{2}$. That implies $\lambda - \mu > \lambda_L - \frac{\epsilon}{2} - \lambda_U - \frac{\epsilon}{2} = 0$. Taken together, $\rho > g(\lambda) > g(\mu) > \rho$. This is a contradiction. It must thus be the case that $\lambda_L \leq \lambda_U$.

Suppose $\lambda_L < \lambda_U$. Take $\lambda, \mu \in \mathbb{R}$ with $\lambda_L < \lambda < \mu < \lambda_U$. Since $\lambda > \lambda_L$, $\lambda \notin L$, so $g(\lambda) \geq \rho$. As $\lambda < \lambda_U$, $\lambda \notin U$, so $g(\lambda) \leq \rho$ and in turn $g(\lambda) = \rho$. Analogously, $\lambda_L < \mu < \lambda_U$ implies $\mu \notin L$ and $\mu \notin U$, so that $g(\mu) = \rho$. Taken together, $\rho = g(\mu) > g(\lambda) = \rho$. As this is a contradiction, it must be the case that $\lambda_L = \lambda_U$.

Define $\lambda = \lambda_L = \lambda_U$. Suppose $\rho > g(\lambda)$. Define $\epsilon = \rho - g(\lambda) > 0$. Because g is continuous, there exists $\delta \in \mathbb{R}_{++}$ such that, for all $\mu \in \mathbb{R}$ with $|\mu - \lambda| < \delta$, $|g(\mu) - g(\lambda)| < \epsilon$. Take $\mu \in \mathbb{R}$ with $\lambda < \mu < \lambda + \delta$. Since $\mu - \lambda < \delta$, continuity of g induces $g(\mu) - g(\lambda) < \epsilon = \rho - g(\lambda)$. Due to $g(\mu) < \rho$, $\mu \in L$. That implies $\lambda = \lambda_L \geq \mu > \lambda$. Since this is a contradiction, it must be true that $\rho \leq g(\lambda)$.

Suppose $\rho < g(\lambda)$. Define $\epsilon = g(\lambda) - \rho > 0$. Again, continuity of g implies that there exists $\delta \in \mathbb{R}_{++}$ such that, for all $\mu \in \mathbb{R}$ with $|\mu - \lambda| < \delta$, $|g(\mu) - g(\lambda)| < \epsilon$. Consider $\mu \in \mathbb{R}$ with $\lambda > \mu > \lambda - \delta$. Continuity of g induces $g(\lambda) - g(\mu) < \epsilon = g(\lambda) - \rho$ and thus $g(\mu) > \rho$. Since $\mu \in U$, it follows that $\lambda > \mu \geq \lambda_U = \lambda$. This is another contradiction. The only remaining possibility is $\rho = g(\lambda)$. Consequently, g is surjective. Taken together, this demonstrates that $\succsim$ is a SGUO.■

3.A.2 Proofs for Section 3.3.2

Theorems 3 and 4 will be reduced to Proposition 2. As before, it will be convenient to operate in utility space.

SWO $\succsim^*$ satisfies **Gain Equity (GE*)** if, for all $a, b \in A^n$, $\epsilon \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$, it is the case that $a \sim^* b$.

Lemma A.2.1: Suppose SWO $\succsim^*$ is associate to SO $\succsim$. Then, $\succsim$ satisfies GE if and only if $\succsim^*$ satisfies GE*.
Proof:
"$\Rightarrow$": Suppose $\succsim$ satisfies GE. Take $a, b \in A^n$, $\epsilon \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. There exist $x, y \in X$ with

$u(x) = a$ and $u(y) = b$. It is true that $u_i(x) - u_i(y) = \epsilon = u_j(y) - u_j(x) > 0$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$. By GE, $x \sim y$, so that $u(x) \sim^* u(y)$. Hence, $\succsim^*$ satisfies GE*.

"$\Leftarrow$": Suppose $\succsim^*$ satisfies GE*. Consider $x, y \in X$ and $i, j \in N$ such that $u_i(x) - u_i(y) = u_j(y) - u_j(x) > 0$ and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$. Define $\epsilon = u_i(x) - u_i(y)$. That is, $u_i(y) = u_i(x) - \epsilon$ and $u_j(y) = u_j(x) + \epsilon$. Due to GE*, that implies $u(x) \sim^* u(y)$ and in turn $x \sim y$. Therefore, $\succsim$ satisfies GE.■

Proof of Proposition 2:
"$\Leftarrow$": Suppose $\succsim$ is the UO with associate $\succsim^*$ as defined in the proof of Theorem 1. As a GUO, $\succsim$ satisfies SP by Theorem 1. Consider $a, b \in A^n$, $\epsilon \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. It follows that $\sum_{k \in N}(a_k - b_k) = 0$. That implies $a \sim^* b$. Hence, $\succsim^*$ satisfies GE*. By Lemma A.2.1, $\succsim$ satisfies GE.

"$\Rightarrow$": Suppose $\succsim$ satisfies SP and GE. By Lemmas A.1.1 and A.2.1, $\succsim$ has an associate SWO $\succsim^*$ satisfying SP* and GE*. For $a \in A^n$, define $\mu(a) = \frac{1}{n} \cdot \sum_{i \in N} a_i$, $N_+(a) = \{i \in N \mid a_i > \mu(a)\}$, $N_-(a) = \{i \in N \mid a_i < \mu(a)\}$, and $N_=(a) = \{i \in N \mid a_i = \mu(a)\}$. If $N_=(a) \neq N$, then both $N_+(a)$ and $N_-(a)$ are non-empty.[31]

Consider $a \in A^n$. Inductively define a sequence $a^0, a^1, ..., a^s \in A^n$ with $a^0 = a$. Take $t \in \mathbb{N}$ such that a^{t-1} is defined. If $N_=(a^{t-1}) = N$, set $s = t-1$. Suppose $N_=(a^{t-1}) \neq N$. Take $i \in N_+(a^{t-1})$ and $j \in N_-(a^{t-1})$. Define $\epsilon = min\{a_i^{t-1} - \mu(a^{t-1}), \mu(a^{t-1}) - a_j^{t-1}\} > 0$. Specify $a^t \in A^n$ such that $a_i^t = a_i^{t-1} - \epsilon$, $a_j^t = a_j^{t-1} + \epsilon$, and, for all $k \in N \setminus \{i, j\}$, $a_k^t = a_k^{t-1}$. It follows that $\sum_{i \in N} a_i^t = \sum_{i \in N} a_i$ and thus $\mu(a^t) = \mu(a)$ for all a^t of the sequence. Accordingly, $N_=(a^t)$ contains at least one more individual than $N_=(a^{t-1})$ for all a^t. For that reason, there indeed exists $s \in \mathbb{N}_0$ with $a^s = (\mu(a^s), ..., \mu(a^s)) = (\mu(a), ..., \mu(a))$. If $s \geq 1$, GE* implies that, for all $t \in \{1, ..., s\}$, $a^t \sim^* a^{t-1}$. Social transitivity yields $a^s \sim^* a$.

Consider $a, b \in A^n$. Due to the last paragraph, it is the case that $a \sim^* (\mu(a), ..., \mu(a))$ and $b \sim^* (\mu(b), ..., \mu(b))$. Together with SP* (or the SWO version of Weak Pareto), that implies $a \succsim^* b \Leftrightarrow (\mu(a), ..., \mu(a)) \succsim^* (\mu(b), ..., \mu(b)) \Leftrightarrow \mu(a) \geq \mu(b) \Leftrightarrow \sum_{i \in N} a_i \geq \sum_{i \in N} b_i$. For $x, y \in X$, that yields $x \succsim y \Leftrightarrow u(x) \succsim^* u(y) \Leftrightarrow \sum_{i \in N} u_i(x) \geq \sum_{i \in N} u_i(y)$. Consequently, $\succsim$ is the UO.■

[31] That is so because $N_+(a) \neq \emptyset$ and $N_-(a) = \emptyset$ would imply the contradiction $\sum_{i \in N} a_i > n \cdot \mu(a) = \sum_{i \in N} a_i$. Likewise, $N_+(a) = \emptyset$ and $N_-(a) \neq \emptyset$ would imply $\sum_{i \in N} a_i < n \cdot \mu(a) = \sum_{i \in N} a_i$.

SWO $\succsim^*$ satisfies **Strong Invariance (SI*)** if, for all $a, b, c, d \in A^n$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}_{++}$ such that, for all $i \in N$, $a_i = \alpha + \beta \cdot c_i$ and $b_i = \alpha + \beta \cdot d_i$, it is the case that $a \succsim^* b \Leftrightarrow c \succsim^* d$.

Lemma A.2.2: Consider SO $\succsim$. The following statements are pairwise equivalent:
(a) $\succsim$ satisfies SN.
(b) $\succsim$ satisfies SI.
(c) $\succsim$ has an associate SWO $\succsim^*$ satisfying SI*.
Proof:
"(a)⇒(b)": Suppose $\succsim$ satisfies SN. Take $x, y, z, w \in X$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}_{++}$ such that, for all $i \in N$, $u_i(x) = \alpha + \beta \cdot u_i(z)$ and $u_i(y) = \alpha + \beta \cdot u_i(w)$. Take $v = \alpha + \beta \cdot u$. By construction, it is true for all $i \in N$ that $u_i(x) = \alpha + \beta \cdot u_i(z) = v_i(z)$ and $u_i(y) = \alpha + \beta \cdot u_i(w) = v_i(w)$. Due to SN, that implies $x \succsim y \Leftrightarrow z \succsim w$. Thus, $\succsim$ satisfies SI.

"(b)⇒(c)": Suppose $\succsim$ satisfies SI. Take $x, y \in X$ with $u(x) = u(y)$. Since, for all $i \in N$, $u_i(x) = 0 + 1 \cdot u_i(y)$ and $u_i(y) = 0 + 1 \cdot u_i(x)$, SI induces $x \succsim y \Leftrightarrow y \succsim x$. Social completeness yields $x \sim y$. Accordingly, $\succsim$ satisfies PI. Reconsidering the proof of Lemma A.1.1, one recognizes that PI is sufficient to establish the existence of SWO $\succsim^*$ that is associate to $\succsim$.

Consider $a, b, c, d \in A^n$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}_{++}$ such that, for all $i \in N$, $a_i = \alpha + \beta \cdot c_i$ and $b_i = \alpha + \beta \cdot d_i$. There exist $x, y, z, w \in X$ with $u(x) = a$, $u(y) = b$, $u(z) = c$, and $u(w) = d$. That is, for all $i \in N$, $u_i(x) = \alpha + \beta \cdot u_i(z)$ and $u_i(y) = \alpha + \beta \cdot u_i(w)$. By SI, it follows that $x \succsim y \Leftrightarrow z \succsim w$. Taken together, that yields $a \succsim^* b \Leftrightarrow c \succsim^* d$. Hence, $\succsim^*$ satisfies SI*.

"(c)⇒(a)": Suppose $\succsim$ has an associate SWO $\succsim^*$ satisfying SI*. Consider $x, y, z, w \in X$ and utility profiles v, v' in the sense of A1 with $v(x) = v'(z)$ and $v(y) = v'(w)$. There exist $\alpha, \alpha' \in \mathbb{R}$ and $\beta, \beta' \in \mathbb{R}_{++}$ such that $u = \alpha + \beta \cdot v$ and $v' = \alpha' + \beta' \cdot u$. Define $\alpha^* = \alpha + \beta \cdot \alpha'$ and $\beta^* = \beta \cdot \beta' > 0$. It follows that, for all $i \in N$, $u_i(x) = \alpha + \beta \cdot v_i(x) = \alpha + \beta \cdot v'_i(z) = \alpha + \beta \cdot (\alpha' + \beta' \cdot u_i(z)) = \alpha^* + \beta^* \cdot u_i(z)$. Likewise, for all $i \in N$, $u_i(y) = \alpha + \beta \cdot v_i(y) = \alpha + \beta \cdot v'_i(w) = \alpha + \beta \cdot (\alpha' + \beta' \cdot u_i(w)) = \alpha^* + \beta^* \cdot u_i(w)$. Due to SI*, that implies $x \succsim y \Leftrightarrow u(x) \succsim^* u(y) \Leftrightarrow u(z) \succsim^* u(w) \Leftrightarrow z \succsim w$. Therefore, $\succsim$ satisfies SN.■

Proof of Proposition 3:
The proposition is equivalence "(a)⇔(b)" of Lemma A.2.2.■

Proof of Theorem 3:

"$\Leftarrow$": Suppose $\succsim$ is the UO with associate $\succsim^*$. As a GUO, $\succsim$ satisfies SP, PD, WBS, RC, and C (Theorem 1). Consider $a, b, c, d \in A^n$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}_{++}$ such that, for all $i \in N$, $a_i = \alpha + \beta \cdot c_i$ and $b_i = \alpha + \beta \cdot d_i$. It follows that $a \succsim^* b \Leftrightarrow \sum_{i \in N} a_i \geq \sum_{i \in N} b_i \Leftrightarrow \sum_{i \in N}(\alpha + \beta \cdot c_i) \geq \sum_{i \in N}(\alpha + \beta \cdot d_i) \Leftrightarrow \sum_{i \in N} c_i \geq \sum_{i \in N} d_i \Leftrightarrow c \succsim^* d$. Thus, $\succsim^*$ satisfies SI*. By Lemma A.2.2, $\succsim$ satisfies SI and SN.

"$\Rightarrow$": Suppose SO $\succsim$ satisfies SP, PD, WBS, RC (or C), and SI (or SN). Due to Theorem 1, $\succsim$ is a GUO with associate $\succsim^*$ and associate g. Specifically, $\succsim^*$ satisfies SP*, PD*, WBS*, RC*, and C* (Lemmas A.1.1-A.1.2, Proposition 1). By Lemma A.2.2, $\succsim^*$ satisfies SI*. Since g is concave, it is the case that, for all $\lambda, \mu \in A$ with $\lambda > \mu$ and $\epsilon \in \mathbb{R}_{++}$ with $(\lambda + \epsilon) \in A$, $g(\lambda + \epsilon) - g(\lambda) \leq g(\mu + \epsilon) - g(\mu)$ (see proof of Theorem 1).

Consider $a, b \in A^n$, $\epsilon \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, $a_i = a_j$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Define $\lambda_2 = a_i + \epsilon$, $\lambda_1 = a_i + \frac{\epsilon}{2}$, $\lambda_0 = a_i$, $\lambda_{-1} = a_i - \frac{\epsilon}{2}$, and $\lambda_{-2} = a_i - \epsilon$. By SP* and PD*, it is true that $(\lambda_0, \lambda_1, \lambda_0, ..., \lambda_0) \succ^* (\lambda_0, \lambda_0, \lambda_0, ..., \lambda_0)$ and $(\lambda_0, \lambda_0, \lambda_0, ..., \lambda_0) \succsim^* (\lambda_{-1}, \lambda_1, \lambda_0, ..., \lambda_0)$. Due to RC*, there exists $\mu_{-1} \in A$ with $(\mu_{-1}, \lambda_1, \lambda_0, ..., \lambda_0) \sim^* (\lambda_0, \lambda_0, \lambda_0, ..., \lambda_0)$. SP* implies $\lambda_0 > \mu_{-1} \geq \lambda_{-1}$. Define $\delta = 2 \cdot (\lambda_0 - \mu_{-1}) > 0$. That is, $\delta \leq \epsilon$ and $\mu_{-1} = \lambda_0 - \frac{\delta}{2}$. Define $\mu_{-2} = \lambda_0 - \delta \geq \lambda_{-2}$.

Set $\alpha = -\lambda_0$ and $\beta = 2$. Then, $(\mu_{-2}, \lambda_2, \lambda_0, ..., \lambda_0) = \alpha \cdot (1, ..., 1) + \beta \cdot (\mu_{-1}, \lambda_1, \lambda_0, ..., \lambda_0)$ and $(\lambda_0, ..., \lambda_0) = \alpha \cdot (1, ..., 1) + \beta \cdot (\lambda_0, ..., \lambda_0)$. Due to SI*, that implies $(\mu_{-2}, \lambda_2, \lambda_0, ..., \lambda_0) \sim^* (\lambda_0, \lambda_0, \lambda_0, ..., \lambda_0)$. It follows that $g(\mu_{-2}) + g(\lambda_2) = g(\lambda_0) + g(\lambda_0) = g(\mu_{-1}) + g(\lambda_1)$. On the one hand, $g(\lambda_2) - g(\lambda_1) \leq g(\lambda_1) - g(\lambda_0)$. On the other hand, $g(\lambda_2) - g(\lambda_1) = g(\mu_{-1}) - g(\mu_{-2}) \geq g(\lambda_0) - g(\mu_{-1}) = g(\lambda_1) - g(\lambda_0)$. Taken together, $g(\lambda_2) - g(\lambda_1) = g(\lambda_1) - g(\lambda_0)$. That is, $(\lambda_0, \lambda_2, \lambda_1, ..., \lambda_1) \sim^* (\lambda_1, \lambda_1, \lambda_1, ..., \lambda_1)$.

Set $\alpha = -\frac{\epsilon}{2}$ and $\beta = 1$. Then, $(\lambda_{-1}, \lambda_1, \lambda_0, ..., \lambda_0) = \alpha \cdot (1, ..., 1) + \beta \cdot (\lambda_0, \lambda_2, \lambda_1, ..., \lambda_1)$ and $(\lambda_0, ..., \lambda_0) = \alpha \cdot (1, ..., 1) + \beta \cdot (\lambda_1, ..., \lambda_1)$. By SI*, it follows that $(\lambda_{-1}, \lambda_1, \lambda_0, ..., \lambda_0) \sim^* (\lambda_0, ..., \lambda_0)$. That is, $g(\lambda_{-1}) + g(\lambda_1) = g(\lambda_0) + g(\lambda_0) = g(\mu_{-1}) + g(\lambda_1)$. Since g is injective, $g(\lambda_{-1}) = g(\mu_{-1})$ implies $\lambda_{-1} = \mu_{-1}$. In turn, $\delta = \epsilon$ and $\lambda_{-2} = \mu_{-2}$. Taken together, $\sum_{k \in N}(g(a_k) - g(b_k)) = g(\lambda_0) + g(\lambda_0) - g(\lambda_{-2}) - g(\lambda_2) = 0$, so that $a \sim^* b$.

Consider $a, b \in A^n$, $\epsilon \in \mathbb{R}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, $a_i = a_j$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. For $\epsilon > 0$, it has been shown that $a \sim^* b$. If $\epsilon = 0$, then $a = b$, so that $a \sim^* b$. If $\epsilon < 0$, define $\epsilon' = -\epsilon > 0$. Since $b_j = a_j - \epsilon'$, $b_i = a_i + \epsilon'$, the stated argument also yields $a \sim^* b$. That is, for all $a, b \in A^n$, $\epsilon \in \mathbb{R}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, $a_i = a_j$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$, it is true that $a \sim^* b$.

Consider $a, b \in A^n$, $\epsilon \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \epsilon$, $b_j = a_j + \epsilon$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Define $c \in A^n$ such that $c_i = c_j = \frac{a_i + a_j}{2} = \frac{b_i + b_j}{2}$ and, for all $k \in N \setminus \{i, j\}$, $c_k = a_k$. Let $\delta = c_i - a_i$ and $\delta' = c_i - b_i$. Since $a_i = c_i - \delta$ and $a_j = c_j + \delta$, it follows that $c \sim^* a$. Since $b_i = c_i - \delta'$ and $b_j = c_j + \delta'$, it likewise follows that $c \sim^* b$. Hence, $a \sim^* b$, so that $\succsim^*$ satisfies GE*. Due to Lemma A.2.1 and Proposition 2, it follows that $\succsim$ is the UO.■

SWO $\succsim^*$ satisfies **Stability (STB*)** if, for all $a, b \in A^n$ and $i, j \in N$ with $b_j > a_j = a_i > b_i$, $b_j - a_j \neq a_i - b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$, there exists $\epsilon \in \mathbb{R}_{++}$ such that, for all $b', b'' \in A^n$ with $|b'_j - b_j| + |b''_j - b_j| < \epsilon$ and, for all $k \in N \setminus \{j\}$, $b'_k = b''_k = b_k$, it is the case that $a \succ^* b' \Rightarrow a \succsim^* b''$.

Lemma A.2.3: Suppose SWO $\succsim^*$ is associate to SO $\succsim$. Then, $\succsim$ satisfies STB if and only if $\succsim^*$ satisfies STB*.

Proof:

"$\Rightarrow$": Suppose $\succsim$ satisfies STB. Consider $a, b \in A^n$ and $i, j \in N$ with $b_j > a_j = a_i > b_i$, $b_j - a_j \neq a_i - b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. There exist $x, y \in X$ with $u(x) = a$ and $u(y) = b$. That is, $u_j(y) > u_j(x) = u_i(x) > u_i(y)$, $u_j(y) - u_j(x) \neq u_i(x) - u_i(y)$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$. Take $\epsilon \in \mathbb{R}_{++}$ in the sense of STB for x, y, i, and j. Consider $b', b'' \in A^n$ with $|b'_j - b_j| + |b''_j - b_j| < \epsilon$, for all $k \in N \setminus \{j\}$, $b'_k = b''_k = b_k$, and $a \succ^* b'$. Again, there exist $y', y'' \in X$ with $u(y') = b'$ and $u(y'') = b''$. It follows that $|u_j(y') - u_j(y)| + |u_j(y'') - u_j(y)| < \epsilon$, for all $k \in N \setminus \{j\}$, $u_k(y') = u_k(y'') = u_k(y)$, and $u(x) \succ^* u(y')$. In turn, $x \succ y'$. Due to STB, that implies $x \succsim y''$. It follows that $u(x) \succsim^* u(y'')$, so that $a \succsim^* b''$. Thus, $\succsim^*$ satisfies STB*.

"$\Leftarrow$": Suppose $\succsim^*$ satisfies STB*. Consider $x, y \in X$ and $i, j \in N$ with $u_j(y) > u_j(x) = u_i(x) > u_i(y)$, $u_j(y) - u_j(x) \neq u_i(x) - u_i(y)$, and, for all $k \in N \setminus \{i, j\}$, $u_k(x) = u_k(y)$. Take $\epsilon \in \mathbb{R}_{++}$ in the sense of STB* for $u(x)$, $u(y)$, i, and j. Consider $y', y'' \in X$ with $|u_j(y') - u_j(y)| + |u_j(y'') - u_j(y)| < \epsilon$, for all $k \in N \setminus \{j\}$, $u_k(y') = u_k(y'') = u_k(y)$, and $x \succ y'$. Since $u(x) \succ^* u(y')$, STB* induces $u(x) \succsim^* u(y'')$ and in turn $x \succsim y''$. Hence, $\succsim$ satisfies STB.■

Proof of Theorem 4:

"$\Leftarrow$": Suppose $\succsim$ is the UO with associate $\succsim^*$. Once again, $\succsim$ satisfies SP, RC, and C (Theorem 3). Take $a, b \in A^n$ and $i, j \in N$ with $b_j > a_j = a_i > b_i$, $b_j - a_j \neq a_i - b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Define $\epsilon = min\{b_j - a_j, |(b_j - a_j) - (a_i - b_i)|\} > 0$. Consider $b', b'' \in A^n$ with $|b'_j - b_j| + |b''_j - b_j| < \epsilon$ and, for all $k \in N \setminus \{j\}$, $b'_k = b''_k = b_k$. Suppose $b_j - a_j > a_i - b_i$. It follows that $(b'_j - a_j) - (a_i - b'_i) = (b'_j - b_j) + (b_j - a_j) - (a_i - b_i) \geq$

$(b'_j - b_j) + \epsilon \geq -|b'_j - b_j| + \epsilon > -\epsilon + \epsilon = 0$. That implies $b' \succ^* a$, so that the implication of STB* is satisfied. Suppose $b_j - a_j < a_i - b_i$. Now, $(a_i - b''_i) - (b''_j - a_j) = (a_i - b_i) - (b_j - a_j) - (b''_j - b_j) \geq \epsilon - (b''_j - b_j) \geq \epsilon - |b''_j - b_j| > \epsilon - \epsilon = 0$. That means $a \succ^* b''$, so that the implication of STB* is again satisfied. Hence, $\succsim^*$ satisfies the condition. By Lemma A.2.3, $\succsim$ satisfies STB.

"$\Rightarrow$": Suppose SO $\succsim$ satisfies SP, RC (or C), and STB. Due to Lemmas A.1.1-A.1.2 and A.2.3 (and Proposition 1), $\succsim$ has an associate SWO $\succsim^*$ satisfying SP*, RC*, and STB*. Take $a, b \in A^n$, $\tilde{\epsilon} \in \mathbb{R}_{++}$, and $i, j \in N$ such that $b_i = a_i - \tilde{\epsilon}$, $b_j = a_j + \tilde{\epsilon}$, $a_i = a_j$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$.

Suppose $a \succ^* b$. By SP*, $(b_1, ..., b_{i-1}, a_i, b_{i+1}, ..., b_n) \succ^* a$. Due to RC*, there exists $b^1 \in A^n$ such that $b^1 = (b_1, ..., b_{i-1}, b^1_i, b_{i+1}, ..., b_n) \sim^* a$. By social transitivity and SP*, $a_i > b^1_i > b_i$. Define $\delta = a_i - b^1_i > 0$ and $b^2 = (b^1_1, ..., b^1_{j-1}, a_j + \delta, b^1_{j+1}, ..., b^1_n) \in A^n$. Since $b^1_j - a_j = b_j - a_j = a_i - b_i > a_i - b^1_i = \delta$, it is the case that $b^1_j > b^2_j$. On the one hand, SP* implies $b^1 \succ^* b^2$ and thus $a \succ^* b^2$. On the other hand, it implies $(b^2_1, ..., b^2_{i-1}, a_i, b^2_{i+1}, ..., b^2_n) \succ^* a$. Due to RC*, there exists $b^3 \in A^n$ such that $b^3 = (b^2_1, ..., b^2_{i-1}, b^3_i, b^2_{i+1}, ..., b^2_n) \sim^* a$. SP* secures $a_i > b^3_i > b^2_i = b^1_i$. Taken together, it follows that $b_j > b^3_j > a_j = a_i > b^3_i$, $b^3_j - a_j = \delta = a_i - b^1_i > a_i - b^3_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b^3_k$.

Consider $\epsilon \in \mathbb{R}_{++}$. Define $c, d \in A^n$ such that $b^3_j + \frac{\epsilon}{2} > d_j > b^3_j > c_j > b^3_j - \frac{\epsilon}{2}$, and, for all $k \in N \setminus \{j\}$, $c_k = d_k = b^3_k$. Specifically, $|c_j - b^3_j| + |d_j - b^3_j| < \epsilon$. Due to SP*, $b^3 \succ^* c$ and $d \succ^* b^3$. Social transitivity yields $a \succ^* c$ and $d \succ^* a$. Hence, there exists no ϵ in the sense of STB* for a, b^3, i, and j. This is a contradiction. Accordingly, assumption $a \succ^* b$ must be wrong.

Suppose $b \succ^* a$. By SP*, $a \succ^* (b_1, ..., b_{j-1}, a_j, b_{j+1}, ..., b_n)$. By RC*, there exists $b' \in A^n$ such that $b' = (b_1, ..., b_{j-1}, b'_j, b_{j+1}, ..., b_n) \sim^* a$. Social transitivity and SP* imply $b_j > b'_j > a_j$, so that $b'_j - a_j < a_i - b'_i$. Analogously to the last paragraph, one establishes that there is no ϵ in the sense of STB* for a, b', i, and j. Since this is another contradiction, assumption $b \succ^* a$ must also be wrong. Social completeness yields $a \sim^* b$. As shown in the proof of Theorem 3, this property induces that $\succsim^*$ satisfies GE*. Again, Lemma A.2.1 and Proposition 2 imply that $\succsim$ is the UO.■

3.A.3 Proofs for Section 3.4

Formally, the uncertainty framework is analogous to the certainty framework. Rename individual-circumstance tuples from $N \times C$ such that each tuple is represented by a number from $R = \{1, ..., n \cdot m\}$.[32] Accordingly, $\succsim_d$ is a difference relation over $(R \times X)^2$. For $u : R \times X \to \mathbb{R}$, $r \in R$, and $x \in X$, denote $u_r(x) = u(r, x)$. Denote $u(x) = (u_1(x), ..., u_{n \cdot m}(x)) \in \mathbb{R}^{n \cdot m}$.

The reformulation of A1 states that there exists $u : R \times X \to \mathbb{R}$, unique up to a positive affine transformation, such that, for all $r, s, t, q \in R$ and $x, y, z, w \in X$, $\big((r, x), (s, y)\big) \succsim_d \big((t, z), (q, w)\big) \Leftrightarrow u_r(x) - u_s(y) \geq u_t(z) - u_q(w)$. Fix such a utility profile u. The reformulation of A2 states that there exists an interval $A \subseteq \mathbb{R}$ such that $u(X) = \{a \in \mathbb{R}^{n \cdot m} \mid \exists x \in X : u(x) = a\} = A^{n \cdot m}$.

For every condition from the certainty framework, it is possible to define the corresponding uncertainty condition by replacing N with R, $i, j, k, ...$ with $r, s, t, ...$, and n with $n \cdot m$. For instance, SO $\succsim$ satisfies the uncertainty version of Strong Pareto (SP) if, for all $x, y \in X$, the following holds: If, for all $r \in R$, $u_r(x) \geq u_r(y)$, then $x \succsim y$. If, in addition, there exists $r \in R$ with $u_r(x) > u_r(y)$, then $x \succ y$.

The uncertainty version of a SWO is a complete and transitive relation over $A^{n \cdot m}$. Again, all definitions go through with the according replacements. For instance, SWO $\succsim^*$ satisfies the uncertainty version of Pigou-Dalton (PD*) if, for all $a, b \in A^{n \cdot m}$ and $r, s \in R$ such that $a_r - b_r = b_s - a_s > 0$, $a_s \geq b_r$, and, for all $t \in R \setminus \{r, s\}$, $a_t = b_t$, it is the case that $a \succsim^* b$.

Proof of Theorem 5:
Following the argument for Theorem 1 with the according replacements, one establishes that SO $\succsim$ satisfies the uncertainty versions of SP, PD, WBS, and RC if and only if there exists increasing, continuous, and concave $g : A \to \mathbb{R}$ such that, for all $x, y \in X$, $x \succsim y \Leftrightarrow \sum_{r \in R} g(u_r(x)) \geq \sum_{r \in R} g(u_r(y)) \Leftrightarrow \sum_{c \in C} \sum_{i \in N} g(u_{i,c}(x)) \geq \sum_{c \in C} \sum_{i \in N} g(u_{i,c}(y)) \Leftrightarrow \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} g(u_{i,c}(x)) \geq \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} g(u_{i,c}(y))$.■

Proof of Theorem 6:
For "$\Leftarrow$", one notes that the EUO is an EGUO and thus satisfies the uncertainty versions of SP and RC. One adapts the argument from Theorem 4 to show that the EUO satisfies STB. For "$\Rightarrow$", one adapts the argument from Theorem 4 to establish that SO $\succsim$ satisfying the uncertainty versions of SP, RC, and STB exhibits, for all $x, y \in X$,

[32] Formally, this can be done by fixing a bijection $\sigma : N \times C \to R$.

$x \succsim y \Leftrightarrow \sum_{r \in R} u_r(x) \geq \sum_{r \in R} u_r(y) \Leftrightarrow \sum_{c \in C} \sum_{i \in N} u_{i,c}(x) \geq \sum_{c \in C} \sum_{i \in N} u_{i,c}(y) \Leftrightarrow \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} u_{i,c}(x) \geq \sum_{c \in C} \frac{1}{m} \cdot \sum_{i \in N} u_{i,c}(y)$.■

3.A.4 Independence of Conditions

The Universal Indifference SO $\succsim$ specifies, for all $x, y \in X$, $x \sim y$. It satisfies PD, WBS, RC, C, CSWC, GE, SI, SN, and STB, but violates SP.

A Weighted Utilitarian SO $\succsim$ is characterized by $\lambda_1, ..., \lambda_n \in \mathbb{R}_{++}$ with $\lambda_i \neq \lambda_j$ for some $i, j \in N$ such that $x \succsim y \Leftrightarrow \sum_{i \in N} \lambda_i \cdot u_i(x) \geq \sum_{i \in N} \lambda_i \cdot u_i(y)$. It satisfies SP, WBS, RC, C, CSWC, SI, and SN, but violates PD, GE, and STB.

For $x, y \in X$, take bijections $\pi, \mu : N \to N$ such that, for all $r, t \in N$ with $r > t$, $u_{\pi(t)}(x) \geq u_{\pi(r)}(x)$ and $u_{\mu(t)}(y) \geq u_{\mu(r)}(y)$. The Gini SO specifies $x \succsim y \Leftrightarrow \sum_{r \in N}(2r - 1) \cdot u_{\pi(r)}(x) \geq \sum_{r \in N}(2r - 1) \cdot u_{\mu(r)}(y)$. It satisfies SP, PD, RC, C, CSWC, SI, and SN, but violates WBS, GE, and STB.

The Leximin SO $\succsim$ specifies $x \succsim y$ if and only if (a) $u_{\pi(r)}(x) = u_{\mu(r)}(y)$ for all $r \in N$, or (b) there exists $t \in N$ such that, for all $r > t$, $u_{\pi(r)}(x) = u_{\mu(r)}(y)$ and $u_{\pi(t)}(x) > u_{\mu(t)}(y)$. It satisfies SP, PD, WBS, SI, SN, and STB, but not RC, C, CSWC, and GE.

A Prioritarian SGUO satisfies SP, PD, WBS, RC, C, and CSWC, but violates GE, SI, SN, and STB.

Overall, these SOs establish that the conditions of Theorems 1-4 and Proposition 2 are logically independent. The uncertainty versions of the SOs satisfy and violate the corresponding uncertainty conditions like their certainty counterparts. This shows that the conditions of Theorems 5-6 are logically independent.

3.A.5 Special-Level Utilitarianism

Suppose $A = \mathbb{R}$ and consider SO $\succsim$ with associate SWO $\succsim^*$ such that, for all $a, b \in \mathbb{R}^n$, the following holds. If $a \in \mathbb{R}^n_+$ and $b \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, then $a \succ^* b$. If $a, b \in \mathbb{R}^n_+$ or $a, b \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, then $a \succsim^* b \Leftrightarrow \sum_{i \in N} a_i \geq \sum_{i \in N} b_i$. The SO gives top priority for everyone to reach the special level 0. Within the group of states where everyone reaches that special level (where someone does not reach that special level), states are ranked according to their utilitarian sums.

By construction, $\succsim^*$ is complete. Take $a, b, c \in \mathbb{R}^n$ with $a \succsim^* b$ and $b \succsim^* c$. Suppose

$c \in \mathbb{R}^n_+$. Since $b \succsim^* c$, that implies $b \in \mathbb{R}^n_+$. Together with $a \succsim^* b$, that in turn implies $a \in \mathbb{R}^n_+$. By construction, $\sum_{i \in N} a_i \geq \sum_{i \in N} b_i \geq \sum_{i \in N} c_i$, so that $a \succsim^* c$. Suppose $c \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. If $a \in \mathbb{R}^n_+$, then $a \succsim^* c$ follows immediately. Suppose $a \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Since $a \succsim^* b$, that implies $b \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Again, by construction, $\sum_{i \in N} a_i \geq \sum_{i \in N} b_i \geq \sum_{i \in N} c_i$, implying $a \succsim^* c$. Hence, $\succsim^*$ is transitive, so that it is indeed a SWO.[33]

Consider $a, b \in \mathbb{R}^n$ such that, for all $i \in N$, $a_i \geq b_i$, and there exists $i \in N$ with $a_i > b_i$. That is, $\sum_{i \in N} a_i > \sum_{i \in N} b_i$. Suppose $b \in \mathbb{R}^n_+$. That implies $a \in \mathbb{R}^n_+$ and in turn $a \succ^* b$. Suppose $b \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. If $a \in \mathbb{R}^n_+$, then $a \succ^* b$ follows immediately. If $a \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, the comparison of utilitarian sums yields $a \succ^* b$. Thus, $\succsim^*$ satisfies SP*.

Take $a, b \in \mathbb{R}^n$ and $i, j \in N$ with $b_j > a_j = a_i > b_i$, $b_j - a_j \neq a_i - b_i$, and, for all $k \in N \setminus \{i, j\}$, $a_k = b_k$. Define $\epsilon = min\{b_j - a_j, |(b_j - a_j) - (a_i - b_i)|\} > 0$. Consider $b', b'' \in \mathbb{R}^n$ with $|b'_j - b_j| + |b''_j - b_j| < \epsilon$ and, for all $k \in N \setminus \{j\}$, $b'_k = b''_k = b_k$. By construction, $b'_j > a_j$ and $b''_j > a_j$. Suppose $b'' \in \mathbb{R}^n_+$. That implies $a, b' \in \mathbb{R}^n_+$. The three utility vectors are thus ranked according to their utilitarian sums under $\succsim^*$. Following the argument from the proof of Theorem 4, one shows that the implication of STB* is satisfied in that case. Suppose $b'' \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. If $a \in \mathbb{R}^n_+$, then $a \succ^* b''$, so that the implication of STB* is satisfied. Suppose $a \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. That implies $b' \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Again, a, b', and b'' are ranked according to their utilitarian sums. Once more, the argument from the proof of Theorem 4 shows that the implication of STB* is satisfied in that case. Taken together, this establishes that $\succsim^*$ satisfies the condition.

It is the case that $(0, ..., 0) \succ^* (1, \lambda, 0, ..., 0)$ for all $\lambda \in \mathbb{R} \setminus \mathbb{R}_+$. C* would thus demand $(0, ..., 0) \succsim^* (1, 0, 0, ..., 0)$. But instead, $(1, 0, 0, ..., 0) \succ^* (0, ..., 0)$. Hence, $\succsim^*$ violates C*. It is also the case that $(2, 0, 1, ..., 1) \sim^* (1, 1, 1, ..., 1)$. SI* would demand $(1, -1, 0, ..., 0) \sim^* (0, 0, 0, ..., 0)$. But in fact, $(0, 0, 0, ..., 0) \succ^* (1, -1, 0, ..., 0)$. Therefore, $\succsim^*$ violates SI*. Due to Lemmas A.1.1-A.1.2, A.1.8, and A.2.2-A.2.3, as well as Proposition 1, SO $\succsim$ satisfies SP and STB, but violates C, RC, CSWC, SI, and SN. One can likewise show that $\succsim$ satisfies PD, but violates WBS and GE.

[33] That also implies that $\succsim$ is indeed a SO. For $x, y \in X$, $u(x) \succsim^* u(y)$ or $u(y) \succsim^* u(x)$, yielding $x \succsim y$ or $y \succsim x$. Consider $x, y, z \in X$ with $x \succsim y$ and $y \succsim z$. Since $u(x) \succsim^* u(y)$ and $u(y) \succsim^* u(z)$, transitivity of $\succsim^*$ implies $u(x) \succsim^* u(z)$ and in turn $x \succsim z$.

Chapter 4

How Much Do You Need to Take the Lead? Majority Rules for Incomplete Rankings

While votes are counted, the lead can switch between candidates. Based on this observation, the chapter offers a new concept to study the aggregation of incompletely known individual rankings. The Pull Ahead Index measures the number of new favorable rankings an alternative needs to take the social lead over another alternative. By studying conditions on the Index, novel insights into the structure of aggregation are identified. New and classical versions of Independence of Irrelevant Alternatives, Anonymity, and Neutrality are generally equivalent. Other properties do not have classical equivalents. Relative and Quorum Majority Rules for incomplete inputs are characterized with conditions on the Index. While rules from both classes satisfy a Positive Responsiveness condition, they are distinguished by the response of the Index to opposing information. The Simple Majority Rule is characterized with the condition that alternatives should not have an implicit lead in terms of the Index. Its merit vis-à-vis qualified majority rules depends on whether ranking information is static or dynamic. This is both true in voting and policy evaluation contexts. One can also define Pull Ahead Indices for richer informational inputs.

4.1 Introduction

An election night can be very exciting. At the beginning, one candidate possibly looks like the clear winner. But as more results from polling places come in, another candidate might catch up and eventually pull ahead. In the 2020 United States Presidential Election, Donald Trump lead in several key states like Georgia, Michigan, and Pennsylvania. However, Joe Biden won the states after all votes were counted. The present chapter takes these dynamic properties of aggregation as a starting point.

The chapter offers a systematic account of aggregating incompletely known individual preference rankings. It is based on the new concept of a Pull Ahead Index. An aggregation rule translates a profile of incomplete individual rankings over alternatives into a social ranking of alternatives. The Pull Ahead Index of alternative x over alternative y at a profile is the minimum number of new strict individual rankings of x over y that need to be added for x to socially rank higher than y under the rule. That is, x will pull ahead of y if these rankings are reported. The Index sheds light on the relation between social and individual rankings.

The chapter examines the structure and quality of aggregation rules by analyzing conditions on their Pull Ahead Indices. New versions of Binary Independence of Irrelevant Alternatives, Anonymity, and Neutrality are formulated. They are generally equivalent to their counterparts which are directly formulated in terms of individual and social rankings. Several other conditions do not have according equivalents. In particular, a reasonable Positive Responsiveness condition on the Pull Ahead Index is introduced.

The chapter employs conditions on the Pull Ahead Index to establish characterizations of two classes of qualified majority rules for incomplete profiles. According to Relative Majority Rules, individual rankings favoring x over y must surpass individual rankings favoring y over x by some threshold for x to socially rank higher than y. According to Quorum Majority Rules, individual rankings favoring x over y must exceed individual rankings favoring y over x and a quorum for such a strict social ranking. Two characterizations of each class are offered, respectively. They present new insights into the structure of majority rules. On the one hand, rules from both classes satisfy Positive Responsiveness. On the other hand, the specification of levelled Pull Ahead Indices is critical to distinguish Relative from Quorum Majority Rules.

The Simple Majority Rule forms the intersection of Relative and Quorum Majority Rules. The Pull Ahead Index uncovers that the Simple Majority Rule avoids the implicit lead of an alternative. This leads to new characterizations that do not presuppose May's (1952) Positive Responsiveness. They derive the condition as an implication of No Implicit Lead and the less controversial new Positive Responsiveness property.

The analysis of the chapter is of interest in all situations where incompletely known individual rankings are aggregated. A first application is voting and democratic decision-making. In elections, individuals generally do not submit their full ordering over the set of candidates. Individuals who do not show up report no preferences at all. A second application is policymaking based on social welfare considerations. Here, information acquisition can have several stages. Incomplete well-being ranking information is augmented in the process over time.

The analysis offers new arguments to identify reasonable aggregation rules for incomplete individual rankings. If social rankings represent evaluations of alternatives and are to yield immediate decisions, the Simple Majority Rule will be the most convincing majority rule under plausible assumptions. This is the case in voting and in policymaking contexts. If new preference information can be acquired, one can make a case for qualified majority rules. One way to avoid problems of majority rules is to let the social ranking of two alternatives not only depend on individual rankings over the pair. In line with that, it is possible to extend the developed approach by considering Pull Ahead Indices in terms of favorable individual rankings over third alternatives or favorable interpersonally comparable well-being gains.

The chapter connects and contributes to different literatures. Classical social choice theory investigates the aggregation of complete and transitive individual rankings into social rankings (Arrow, 1951, 1963, 2012; Sen, 1970, 2017; Campbell and Kelly, 2002; Weymark, 2016). Recent contributions on democratic decision procedures are Bartholdi et al. (2021) and Holliday and Pacuit (2021). The aggregation of incomplete preferences has received less attention so far. Existing analysis is set in a voting context (Ackerman et al., 2013; Cullinan et al., 2014; Terzopoulou and Endriss, 2021). Related investigations are concerned with the aggregation of incomplete judgments (Dietrich and List, 2010; Terzopoulou, 2020; Barberà and Bossert, 2023). Dynamic information acquisition and information aggregation in collective decision-making are studied in Chan et al. (2018) and Bouton et al. (2018).

The chapter's results contribute new insights to the axiomatic literature on majority rules. That literature focuses on the relationship of social rankings and individual orderings. Following the characterization of May (1952), the Simple Majority Rule has been studied most extensively. Characterizations for two alternatives and a fixed society are stated in Fishburn (1983), Tideman (1986), Goodin and List (2006), Llamazares (2006), Quesada (2013a), Alcantud (2020), and Freixas and Pons (2021). Characterizations for two alternatives and variable societies are found in Aşan and Sanver (2002), Woeginger (2003, 2005), Miroiu (2004), Quesada (2010, 2011, 2012, 2013b), Xu and Zhong (2010), Alcantud (2019, 2020), and McMorris et al. (2021). For the case of more than two alternatives, Campbell (1988), Maskin (1995), Cantillon and Rangel (2002), Dasgupta and Maskin (2008, 2020), Cato (2011), and Horan et al. (2019) provide characterizations of the Simple Majority Rule. Ackerman et al. (2013) discuss the definition of the rule for incomplete preferences.

Majority rules based on differences in strict individual rankings have come into focus more recently in a voting context. Characterizations of these rules in a fixed population two alternative framework are established in Llamazares (2006), Houy (2007a), Sanver (2009), and King and Powers (2018). Due to framework differences, both the definition and characterizations of Relative Majority Rules in the sense of the present chapter differ from these contributions. Houy (2007a) and Sanver (2009) also examine generalized relative majority rules with quorums. King and Powers (2018) characterize a class of generalized relative majority rules where the threshold for a strict social ranking can depend on the considered alternative.

Pauly (2013) provides a characterization of majority rules based on quorums of favorable rankings in a referendum context. The prevalent absolute majority rules instead require that a (qualified) absolute majority of a fixed population ranks x higher than y for x to socially rank higher than y. Aşan and Sanver (2006) and Quesada (2010) state characterizations. Sanver (2009) conducts a unified examination of relative and absolute majority rules. Terzopoulou (2020) analyzes relative and absolute quota rules for incomplete judgments over multiple issues.

Houy (2009) and Jeong and Ju (2017) consider majority rules with a quorum of overall strict individual rankings over a pair that are necessary for a strict social ranking of the pair. Houy (2007b) studies non-neutral qualified majority rules concerned with the ratio of opposing individual rankings. The question of tiebreaking is analyzed in Campbell and Kelly (2000), Goodin and List (2006), Jeong and Ju (2017), Dasgupta and Maskin (2020), and McMorris et al. (2021). A graphical investigation of different majority rules is pursued in Cantillon and Rangel (2002). Goodin and List (2006) discuss the rationale for qualified majorities. Barberà and Gerber (2017) examine the manipulability of the agenda under different qualified majority rules in sequential decision-making.

The chapter takes the following structure. Section 4.2 introduces Social Ranking Rules and the Pull Ahead Index (4.2.1), formulates conditions for the Pull Ahead Index and investigates their relations to classical conditions (4.2.2). Section 4.3 establishes characterizations of Relative Majority Rules (4.3.1), Quorum Majority Rules (4.3.2), and

Simple Majority Rule (4.3.3). Section 4.4 discusses implications of the results for voting and policymaking (4.4.1) and considers extended Pull Ahead Indices (4.4.2). Section 4.5 concludes. Appendix 4.A contains proofs of results (4.A.1 to 4.A.4), independence checks (4.A.5), and further logical relations (4.A.6).

4.2 Framework

4.2.1 Social Rankings and the Pull Ahead Index

Let X be a set of alternatives faced by a society, where $|X| \geq 2$. In an election, X is the set of (conceivable or actual) candidates. In policymaking, X is a set of (conceivable or feasible) policies. Let N be an infinite set of potential individuals of society. The actual population N_a of society is assumed to be a finite subset of N.

Let $S(X)$ be the set of irreflexive relations over X.[1] For an individual $i \in N$, relation r_i from $S(X)$ represents a possible specification of i's known rankings over distinct alternatives from X. It neither has to be complete nor transitive. Let s_i and p_i denote the symmetric and asymmetric part of r_i.[2] For distinct $x, y \in X$, $(x, y) \in r_i$ means that i is known to rank x at least as high as y. Likewise, $(x, y) \in s_i$ means that i is known to rank x as high as y, and $(x, y) \in p_i$ means that i is known to rank x higher than y. The latter is a strict ranking. Finally, $(x, y), (y, x) \notin r_i$ means that i's ranking of x and y is unknown. In an election, rankings from r_i are preferences over candidates submitted by i. In policymaking, they represent empirically identified well-being comparisons over alternatives for i.

Let $D \subseteq S(X)^N$ be a non-empty domain of profiles of known individual rankings. Profile $r \in D$ assigns a known ranking $r_i = r(i)$ to each individual $i \in N$. Domain D can contain hypothetical non-actual profiles.

It is assumed that an actual profile r^a from D only contains known rankings of individ-

[1]For a set A, a relation q over A is a subset of $A \times A$. Relation q is irreflexive if, for all $a \in A$, $(a, a) \notin q$. It is complete if, for all distinct $a, b \in A$, $(a, b) \in q$ or $(b, a) \in q$. It is transitive if, for all distinct $a, b, c \in A$, $(a, b) \in q$ and $(b, c) \in q$ imply $(a, c) \in q$. It is asymmetric if, for all distinct $a, b \in A$, $(a, b) \in q$ implies $(b, a) \notin q$. If q is irreflexive, complete, and transitive, it is an ordering. If it is also asymmetric, it is a linear ordering.

[2]The symmetric part q_s of relation q over A is defined such that, for $a, b \in A$, $(a, b) \in q_s \Leftrightarrow (a, b) \in q \wedge (b, a) \in q$. The asymmetric part q_p of q is defined such that, for $a, b \in A$, $(a, b) \in q_p \Leftrightarrow (a, b) \in q \wedge (b, a) \notin q$.

uals from actual population N_a. That is, for all $i \in N \setminus N_a$, $r_i^a = \emptyset$. The empty profile without any known individual rankings over distinct alternatives is denoted by $r^{\emptyset}$. For all $i \in N$, $r_i^{\emptyset} = \emptyset$.

The aim of aggregation is to translate a given profile of known individual rankings into a social ranking of the set of alternatives. This is formalized by the following concept.

A **Social Ranking Rule (SRR)** on non-empty domain $D \subseteq S(X)^N$ is a mapping $\phi : D \to S(X)$ such that $\phi(r)$ is complete for all $r \in D$.

A SRR assigns a complete social ranking $\phi(r)$ of distinct alternatives from X to each profile r from D. Let $\phi_s(r)$ and $\phi_p(r)$ be the symmetric and asymmetric part of $\phi(r)$, respectively. For distinct $x, y \in X$, $(x, y) \in \phi(r)$ means that x socially ranks at least as high as y under r. Moreover, $(x, y) \in \phi_s(r)$ and $(x, y) \in \phi_p(r)$ mean that x socially ranks as high as y under r and that x socially ranks higher than y under r, respectively. For now, it is left open what a social ranking means exactly. The question is taken up in section 4.4.1.

In a voting context, it is natural to assume that the input of aggregation is a profile of incompletely known preferences. In social welfare evaluation, this assumption means that only intrapersonal well-being level comparisons can be identified, but no interpersonal well-being comparisons. Richer informational inputs are discussed in section 4.4.2.

How should a convincing SRR be selected? In the literature, the prevalent procedure is to impose conditions on how to translate individual rankings into social rankings. The present chapter proposes a new approach to assess SRRs. It examines what is required for an alternative to take the social lead over another alternative under a SRR.

For SRR ϕ on D, $r \in D$, and distinct $x, y \in X$, the **Pull Ahead Index (PAI)** $d_{xy}(r)$ of x over y at r under ϕ is the smallest $n^* \in \mathbb{N}_0$ such that there exist $N^* \subseteq N$ and $r^* \in D$ with
(a) $|N^*| = n^*$,
(b) for all $i \in N^*$, $(x, y), (y, x) \notin r_i$ and $r_i^* = r_i \cup \{(x, y)\}$,
(c) for all $i \in N \setminus N^*$, $r_i^* = r_i$,
(d) $(x, y) \in \phi_p(r^*)$.
If such n^* does not exist, then $d_{xy}(r) = \infty$.

The Pull Ahead Index is the central concept of the present chapter. The PAI of x over y at r under ϕ is the smallest number of strict individual rankings favoring x

over y that induce a strict social ranking of x over y under ϕ when being added to r. The corresponding set of individuals and augmented profile are denoted by N^* and r^*, respectively. All individuals outside N^* keep their initial ranking. The ranking of x and y by an individual i from N^* is not known at r. In contrast, i is known to rank x higher than y at r^*. All other rankings of i remain the same. An additional strict individual ranking of x over y in the sense of Pull Ahead Indices will also be called individual improvement of x over y. For example, $d_{xy}(r) = 3$ means that x needs three individual improvements over y to take the social lead over y. It is possible that $d_{yx}(r) = 5$, so that y needs more individual improvements to pull ahead of x.

The following sections aim to show that the analysis of Pull Ahead Indices leads to a deeper understanding of the structure of Social Ranking Rules. That is the case because PAIs contain more information than social rankings. If one knows the PAIs of x over y and of y over x, one will also know the social ranking of x and y. The reverse is not true. Specifically, x socially ranks higher than y if and only if the PAI of x over y is zero. The latter means that no additional individual rankings in favor of x are required for x to socially rank higher than y. Formally, $(x, y) \in \phi_p(r)$ is equivalent to $d_{xy}(r) = 0$ for all $r \in D$ and distinct $x, y \in X$.[3] Accordingly, either the PAI of x over y or the PAI of y over x must be positive at each profile. The two alternatives socially rank equally if and only if both PAIs are positive.

The concept of a PAI can be applied to any SRR. It does not impose a condition. However, take domain D and distinct alternatives $x, y \in X$ such that $(x, y) \notin p_i$ holds for all $r \in D$ and all $i \in N$. For each $r \in D$, $d_{xy}(r)$ is equal to infinity unless $(x, y) \in \phi_p(r)$ holds. PAIs are more informative given the following domain condition.

Domain D is **additive** if, for all $r, r' \in S(X)^N$, distinct $x, y \in X$, and finite $N' \subseteq N$ such that, for all $i \in N'$, $(x, y), (y, x) \notin r_i$ and $r'_i = r_i \cup \{(x, y)\}$, and, for all $i \in N \setminus N'$, $r'_i = r_i$, it is the case that $r \in D \Rightarrow r' \in D$. That is, if finitely many strict individual rankings for an alternative over another are added to a profile from an additive domain, the augmented profile will also belong to the domain. On additive domains, PAIs are informative. They reflect the reaction of the considered SRR to every possible finite addition of strict individual rankings.

The characterizations of the chapter deal with the following additive domain. For $r \in S(X)^N$, $Y \subseteq X$, and $i \in N$, let $r_i|Y$ be the restriction of r_i to Y.[4] Let $r|Y \in S(X)^N$

[3] Assume $(x, y) \in \phi_p(r)$. Then, $N^* = \emptyset$, $n^* = |N^*|$, and $r^* = r$ satisfy (a)-(d). That is, $d_{xy}(r) = n^* = 0$. Assume $d_{xy}(r) = n^* = 0$. That implies $N^* = \emptyset$, $r^* = r$, and in turn $(x, y) \in \phi_p(r)$.

[4] The restriction of relation q over A to $B \subseteq A$ is $q|B = q \cap (B \times B)$.

be the restriction of r to Y in the sense that $(r|Y)(i) = r_i|Y$ for each $i \in N$. Define

$$D_F = \Big\{r \in S(X)^N \mid \forall x, y \in X : |\{i \in N \mid (x, y) \in p_i\}| < \infty \wedge |\{i \in N \mid r_i|\{x, y\} = \emptyset\}| = \infty\Big\}.$$

Domain D_F contains profiles such that, for all alternatives x and y, the number of people who are known to strictly rank x over y is finite, and there are infinitely many potential individuals whose ranking over $\{x, y\}$ is not known.

How do PAIs relate to the distinction between infinite potential population N and finite actual population N_a? As assumed, an actual profile only contains non-empty rankings for individuals from N_a. There are infinitely many potential individuals whose ranking of a pair $\{x, y\}$ is unknown. However, only finitely many of them are in N_a. The PAI of x over y at r under SRR ϕ can be finite because it considers possible new strict rankings of x over y by all potential individuals. But if strict rankings of x over y by non-actual individuals are needed for x to pull ahead of y, then x cannot realize the social lead over y. For instance, the PAI of x over y might be 10, while there are only 5 actual individuals left whose ranking of the pair is unknown. The examples in section 4.3 will illustrate this point.

4.2.2 Conditions for the Pull Ahead Index

This subsection introduces conditions for Pull Ahead Indices of Social Ranking Rules. It also studies how these properties relate to classical conditions which are directly concerned with the relation between social and individual rankings. The focus will be on additive domains.

Since Arrow's (1951, 1963, 2012) foundational work on social choice theory, it has been a central question whether the social ranking of a pair of alternatives should be allowed to depend on individual rankings over other alternatives. Arrow's (Binary) Independence of Irrelevant Alternatives precludes that. The social ranking of the pair should only depend on individual orderings over the pair. This is a strong and controversial condition. Arguments for and against it are discussed in Patty and Penn (2019). The following definitions state a refined version of the classical condition for incomplete individual rankings and an analogous condition for PAIs.

SRR ϕ on D satisfies **Classical Independence of Irrelevant Alternatives (CIIA)** if, for all $r, r' \in D$ and distinct $x, y \in X$ with $r|\{x, y\} = r'|\{x, y\}$, it is the case that

$(x, y) \in \phi(r) \Leftrightarrow (x, y) \in \phi(r')$.

SRR ϕ on D satisfies **Independence of Irrelevant Alternatives (IIA)** if, for all $r, r' \in D$ and distinct $x, y \in X$ with $r|\{x, y\} = r'|\{x, y\}$, it is the case that $d_{xy}(r) = d_{xy}(r')$.

According to CIIA, the social ranking of a pair of alternatives should only depend on known individual rankings over the pair. According to IIA, the PAIs of the pair should only depend on known individual rankings over the pair. On an additive domain, IIA's antecedent secures that potential improvements of x over y are symmetric between r and r'. Any set of new strict individual rankings of x over y can be added to r if and only if it can be added to r'. On a non-additive domain, IIA and other of the following PAI conditions can be too strong as they fail to recognize an asymmetry in potential improvements of alternatives between different profiles.[5] In general, IIA is stronger than CIIA. However, the two conditions are equivalent on additive domains.

Lemma 1:
(a) If SRR ϕ satisfies IIA, then ϕ satisfies CIIA.
(b) If SRR ϕ on additive D satisfies CIIA, then ϕ satisfies IIA.

Anonymity and neutrality conditions are prevalent in the literature and go back to May's (1952) initial characterization of the Simple Majority Rule. The present chapter formulates versions of these conditions applying to incomplete individual rankings and introduces analogous conditions for PAIs.

SRR ϕ on D satisfies **Classical Anonymity (CA)** if, for all $r, r' \in D$ with a bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$, it is the case that, for all distinct $x, y \in X$, $(x, y) \in \phi(r) \Leftrightarrow (x, y) \in \phi(r')$.

SRR ϕ on D satisfies **Anonymity (A)** if, for all $r, r' \in D$ with a bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$, it is the case that, for all distinct $x, y \in X$, $d_{xy}(r) = d_{xy}(r')$.

CA and A aim for an equal treatment of individuals in aggregation. Assume that two profiles exhibit the same collection of rankings. They differ as these rankings are assigned to different individuals, respectively. Condition CA demands that this should

[5] For instance, consider $X = \{x, y, z\}$, $i \in N$, and $D = \{r^{\emptyset}, r, r'\}$ such that $r_i = \{(x, z)\}$, $r'_i = \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r_k = r'_k = \emptyset$. Take SRR ϕ on D with $(x, y) \in \phi_s(r)$. By construction, $d_{xy}(r) = \infty$, so that IIA demands $d_{xy}(r^{\emptyset}) = \infty$. But in contrast to r, a strict individual ranking of x over y can be added to $r^{\emptyset}$.

not affect the social ranking of X. Condition A requires that PAIs for all tuples of alternatives from X should coincide between the profiles. On an additive domain, the antecedent secures that potential individual improvements of x over y are symmetric between r and r' except for renaming of individuals. A difference in PAIs of x over y between the profiles would then be due to an intrinsic discrimination between people. Analogous to IIA and CIIA, it turns out that A is generally stronger than CA, but equivalent to it on additive domains.

Lemma 2:
(a) If SRR ϕ satisfies A, then ϕ satisfies CA.
(b) If SRR ϕ on additive D satisfies CA, then ϕ satisfies A.

SRR ϕ on D satisfies **Classical Neutrality (CN)** if, for all $r, r' \in D$ with a bijection $\sigma : X \to X$ such that, for all $i \in N$ and $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$, it is the case that, for all distinct $x, y \in X$, $(x, y) \in \phi(r) \Leftrightarrow (\sigma(x), \sigma(y)) \in \phi(r')$.[6]

SRR ϕ on D satisfies **Neutrality (N)** if, for all $r, r' \in D$ with a bijection $\sigma : X \to X$ such that, for all $i \in N$ and $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$, it is the case that, for all distinct $x, y \in X$, $d_{xy}(r) = d_{\sigma(x)\sigma(y)}(r')$.

CN and N are concerned with an equal treatment of alternatives in aggregation. Assume that known individual rankings coincide between two profiles except that alternatives have different names at the latter. According to CN, the social ranking at the first profile should coincide with the social ranking of renamed alternatives at the second profile. According to N, the PAI of each tuple from $X \times X$ at the former should coincide with the PAI of the renamed tuple at the latter. Given the antecedent and additivity, one can add new favorable rankings of x over y for some individuals to r if and only if one can add new favorable rankings of $\sigma(x)$ over $\sigma(y)$ for these individuals to r'. Due to this symmetry, a difference in PAIs of x over y at r and $\sigma(x)$ over $\sigma(y)$ at r' would reflect a discrimination between alternatives. Once again, the PAI condition is generally stronger than the classical condition, but equivalent to it on additive domains. Taken together, Lemmas 1-3 shed light on the structure of three central conditions in social choice theory.

Lemma 3:
(a) If SRR ϕ satisfies N, then ϕ satisfies CN.
(b) If SRR ϕ on additive D satisfies CN, then ϕ satisfies N.

[6] Since σ is injective, $x \neq y$ implies $\sigma(x) \neq \sigma(y)$.

It is an important question how the individual and social evaluation of alternatives should relate. The following condition is a PAI version of May's (1952) Positive Responsiveness. Different classical variants of May's condition are discussed in Cantillon and Rangel (2002), Miroiu (2004), Woeginger (2005), and Cato (2011).

SRR ϕ on D satisfies **May Positive Responsiveness (MPR)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i \in N$ such that $(x, y), (y, x) \notin r_i$, $r'_i = r_i \cup \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$, it is the case that $d_{yx}(r) > 0 \Rightarrow d_{xy}(r') = 0$.

Profile r' is generated by adding one strict individual ranking of x over y to r. If the PAI of y over x is positive (x socially ranks at least as high as y) at r, MPR requires that the PAI of x over y is zero (x socially ranks higher than y) at r'. MPR is weaker than a formulation of May's condition which applies to a wider range of individual improvements on D_F (Appendix 4.A.6, Proposition A.6.1).

MPR's idea has been criticized in the literature (Campbell and Kelly, 2000: p. 699; Aşan and Sanver, 2002: p. 409). The fact that x improves relative to y for a single individual turns an equal social ranking of x and y into a strict ranking in favor of x. One might argue that the change at the individual level is "too small" to alter the social ranking like this. For those who agree with the critique but nevertheless want the social assessment of alternatives to positively respond to individual rankings, the following new PAI condition can be attractive.

SRR ϕ on D satisfies **Positive Responsiveness (PR)** if, for all $r \in D$ and distinct $x, y \in X$, it is the case that
(a) $|\{i \in N \mid r_i|\{x, y\} = \emptyset\}| = \infty \Rightarrow d_{xy}(r) < \infty$,
and for all $r' \in D$ and $i \in N$ such that $(x, y), (y, x) \notin r_i$, $r'_i = r_i \cup \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$, it is the case that
(b) $0 < d_{xy}(r) < \infty \Rightarrow d_{xy}(r') < d_{xy}(r)$, and
(c) $d_{xy}(r) = 0 \Rightarrow d_{xy}(r') = 0$, and
(d) $d_{yx}(r') \geq d_{yx}(r)$.

All subconditions of PR are reasonable on additive domains. Given infinitely many unknown rankings over x and y, x can take the social lead if enough favorable rankings for it are added to the profile (a). While the number of necessary improvements must be finite, it can be arbitrarily large. For a profile from D_F, the antecedent of (a) is satisfied for each pair of alternatives, so that all PAIs are finite given PR.

Conditions (b)-(d) are concerned with the case where a strict individual ranking of x over y is added to a profile. If the initial PAI of x over y is positive but finite, it should be reduced by the augmentation (b). That is reasonable. While the single strict ranking might not be enough for x to take the social lead over y, it should bring x closer to that lead. If x does not need any individual improvements to pull ahead of y (x socially ranks higher than y), the same is true after the addition (c). Finally, the PAI of y over x should not get smaller due to the change (d). That is, y is not supposed to benefit from the strict individual ranking in favor of x.

MPR and PR are logically independent on D_F (see Appendix 4.A.6). In an important sense, PR is weaker than MPR. Assume that x and y socially rank equally at profile r, that is, $d_{yx}(r) > 0$. Add a strict individual ranking of x over y to r. MPR requires that x socially ranks higher than y at the new profile r'. It induces $d_{xy}(r) = 1$. Instead, PR only demands that the PAI of x over y is reduced by the addition, while the PAI of y over x is not. It is consistent with, say, $d_{xy}(r) = 4$, $d_{yx}(r) = 5$, $d_{xy}(r') = 3$, and $d_{yx}(r') = 6$. In that case, an equal social ranking prevails.

PR is a good example to illustrate the merit of considering conditions for PAIs. It imposes a certain change in PAIs which does not have to translate into a change in social rankings. In the remainder of the section, two further conditions on PAIs are introduced. Like PR, they do not have classical counterparts.

SRR ϕ on D satisfies **Independence of Indifference Information (III)** if, for all $r, r' \in D$ such that, for all $x, y \in X$ and all $i \in N$, $(x, y) \in r_i \Rightarrow (x, y) \in s_i$ and $(x, y) \in r'_i \Rightarrow (x, y) \in s'_i$, it is the case that, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, $d_{xy}(r) = d_{x'y'}(r')$.

At profiles r and r', there are no strict individual rankings. Condition III requires that all PAIs at the two profiles coincide. The number of necessary strict rankings for the social lead should be independent of individual indifferences. It is not allowed that, say, $d_{xy}(r) = 5$ and $d_{x'y'}(r') = 9$.

Condition III is not innocuous. It excludes the possibility that the strict social ranking of x over y depends on a quorum of people for whom the ranking of x and y is known. Individuals who are indifferent between x and y could be known at r. Given some quorum, that might bring x closer to the social lead over y at r than x' to a lead over y' at r'. However, if one aims for an equal treatment of alternatives and does not wish to impose such quorums, III will be a reasonable condition on D_F.[7] Given absent strict

[7]III can be too strong on some other additive domains. For example, take SRR ϕ on $D = S(X)^N$ and $r^{\emptyset}, r \in D$ such that $(x, y) \in s_i$ holds for all $i \in N$ and all distinct $x, y \in X$. For distinct $x, y \in X$ with $(x, y) \in \phi_s(r)$, $d_{xy}(r) = \infty$. Accordingly, III would induce $d_{xy}(r^{\emptyset}) = \infty$. Some of the conditions

individual rankings, the condition precludes an intrinsic distinction between alternatives via different PAIs. For a profile r which only consists of individual indifferences, III implies $d_{xy}(r) = d_{yx}(r)$ for all distinct alternatives x and y. In particular, x and y socially rank equally.

SRR ϕ on D satisfies **Equal Responsiveness (ER)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k$, it is the case that $d_{xy}(r) = d_{yx}(r) \Rightarrow d_{xy}(r') = d_{yx}(r')$.

Condition ER is concerned with the case where PAIs of two alternatives x and y are levelled at profile r and two opposing strict rankings of individuals i and j are added from r to r'. According to ER, the two PAIs should still coincide after this augmentation. For instance, given $d_{xy}(r) = d_{yx}(r) = 5$, $d_{xy}(r') = d_{yx}(r') = 10$ is consistent with ER. Note that the condition does not apply to cases where two alternatives socially rank equally but have unlevelled PAIs.

ER embodies the idea to treat alternatives and individuals equally.[8] There are two explanations for $d_{xy}(r') < d_{yx}(r')$ on an additive domain. First, x might have an intrinsic advantage over y, so that the favorable ranking for x is more significant than the favorable ranking for y. Second, individual i might receive a greater weight in aggregation than individual j, so that i's ranking has a larger effect than j's ranking. ER excludes such discrimination between alternatives and individuals.

4.3 Characterizations

4.3.1 Relative Majority Rules

The present subsection introduces and characterizes a class of Relative Majority Rules in the considered infinite potential population framework where individuals have incompletely known rankings over alternatives.

SRR ϕ on $D \subseteq D_F$ is the **δ-Relative Majority Rule (δ-RMR)** with $\delta \in \mathbb{N}$ if, for all $r \in D$ and distinct $x, y \in X$,
$(x, y) \in \phi(r) \Leftrightarrow |\{i \in N \mid (x, y) \in p_i\}| + \delta - 1 \geq |\{i \in N \mid (y, x) \in p_i\}|$.

in the next sections face similar problems on certain (additive) domains where the number of unknown individual rankings over a pair can be finite.

[8]Nevertheless, the combination of IIA, A, N, and III does not imply ER on D_F (see Appendix 4.A.6).

Consider natural number δ. Alternative x has a relative majority of at least δ over y if and only if the number of individuals who are known to strictly rank x over y exceeds the number of individuals who are known to strictly rank y over x by at least δ. A relative majority of at least 1 is a simple majority. The δ-RMR does the following. If x has a relative majority of at least δ over y, then x socially ranks higher than y. Likewise, if y has a relative majority of at least δ over x, then y socially ranks higher than x. If neither of the two alternatives has such a qualified relative majority over the other, then x and y socially rank equally.

Example: Consider a vote between two alternatives from $X = \{x, y\}$. Assume that the actual population $N_a \subseteq N$ of society consists of 20 individuals. Take domain $D = D_F$. At profile $r \in D$, 6 people are known to strictly rank x over y (they voted for x), 2 people are known to strictly rank y over x (they voted for y), and 4 people are known to rank x as high as y (they reported indifference). In addition, there are 8 actual individuals whose rankings of x and y are not known. They have not cast their ballot (yet). Rankings of non-actual individuals from $N \setminus N_a$ are not known as they do not vote at an actual profile.

Alternative x has a relative majority of 4 over y at r. According to the 3-RMR, x socially ranks higher than y at that profile. Moreover, the PAIs are $d_{xy}(r) = 0$ and $d_{yx}(r) = 7$. While x needs no additional votes to take the social lead over y, y needs 7 additional votes to turn the social ranking. They consist of 4 votes to catch up with x and 3 more to acquire the qualified relative majority over x. Alternative y can still realize this social lead if 7 of the remaining 8 actual individuals turn out and vote for y. Things are different under the 6-RMR. Here, x socially ranks as high as y. The PAIs are $d_{xy}(r) = 2$ and $d_{yx}(r) = 10$. That is, x (y) needs 2 (10) additional votes to socially rank higher than y (x). While that is factually possible for x, y can only realize the social lead if two more individuals join N_a.

The first characterization of the class of Relative Majority Rules on D_F is based on a strengthening of ER.

SRR ϕ on D satisfies **Independent Equal Responsiveness (IER)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k$, it is the case that $d_{xy}(r) = d_{yx}(r) \Rightarrow d_{xy}(r') = d_{yx}(r') = d_{xy}(r)$.

Like ER, IER is concerned with the case where PAIs over x and y are levelled at r and two opposing strict rankings are added from r to r'. But IER does not only demand that the two PAIs still coincide at r'. It also requires that they are equal to their initial value at r. Given $d_{xy}(r) = d_{yx}(r) = 5$, IER induces $d_{xy}(r') = d_{yx}(r') = 5$.

There is a clear reasoning behind IER on D_F. A strict individual ranking of one alternative over another is relative positive evidence for the former over the latter. Given the antecedent of IER, both alternatives need the same amount of relative positive evidence to take the social lead over the other. The idea of the condition is that new opposing evidence cancels out. It keeps the amount of relative positive evidence required for a strict social ranking.

Combined with IIA, III, and PR, IER characterizes the class of Relative Majority Rules. Due to Lemma 1 and the fact that D_F is additive, it is possible to substitute CIIA for IIA in Theorem 1.

Theorem 1:
SRR ϕ on $D = D_F$ satisfies IIA, III, PR, and IER if and only if ϕ is a RMR.[9]

The next condition also deals with levelled PAIs.

SRR ϕ on D satisfies **Independence of Balancing Information (IBI)** if, for all $r, r' \in D$ and distinct $x, y \in X$, $(d_{xy}(r) = d_{yx}(r) < \infty \wedge d_{xy}(r') = d_{yx}(r') < \infty) \Rightarrow d_{xy}(r) = d_{xy}(r')$.

If PAIs over x and y are levelled and finite at r and r', IBI requires that they coincide between the profiles. For example, assume that 5 individuals strictly rank x over y and 5 other individuals strictly rank y over x at r. At r', 10 individuals strictly rank x over y and 10 individuals strictly rank y over x. Assume that, due to the equal number of opposing individual rankings, PAIs over x and y are levelled and finite at both profiles. According to IBI, they have to coincide.

IBI has a similar idea as IER. Levelled PAIs of x and y mean that evidence in favor of x over y and evidence in favor of y over x cancel out. The amount of relative positive evidence x needs to take the social lead over y should not depend on the specification of that cancelling balancing information.

Although their reasoning is similar, IBI and IER are logically independent on D_F (see Appendix 4.A.6). Together with IIA, A, N, and PR, the former characterizes the class of Relative Majority Rules. In view of Lemmas 1-3, one can also employ the classical versions of IIA, A, and N in Theorem 2.

[9]Implication "$\Rightarrow$" holds on every domain $D_2 \subseteq D \subseteq D_F$, where $D_2 = \{r \in D_F \mid \forall i \in N : |r_i| \leq 2\}$. The same is true for the following characterizations. This is shown in Appendices 4.A.2 to 4.A.4.

Theorem 2:
SRR ϕ on $D = D_F$ satisfies IIA, A, N, PR, and IBI if and only if ϕ is a RMR.

Llamazares (2006), Houy (2007a), Sanver (2009), and King and Powers (2018) characterize relative majority rules based on differences in strict individual rankings in a framework with a fixed finite population and two alternatives.[10] They employ anonymity, neutrality, monotonicity, and cancellation conditions. A version of monotonicity for incomplete individual rankings is weaker than PR.[11] A version of cancellation requires that the social ranking of a pair of alternatives should not change if two opposing individual rankings of the pair are newly identified. It turns out to be stronger than IER and independent of IBI on D_F (Appendix 4.A.6, Proposition A.6.2).

4.3.2 Quorum Majority Rules

This subsection introduces and examines the following class of majority rules with quorums for incomplete individual rankings.

SRR ϕ on $D \subseteq D_F$ is the **λ-Quorum Majority Rule (λ-QMR)** with $\lambda \in \mathbb{N}$ if, for all $r \in D$ and distinct $x, y \in X$,
$(x, y) \in \phi(r) \Leftrightarrow max\{|\{i \in N \mid (x, y) \in p_i\}|, \lambda - 1\} \geq |\{i \in N \mid (y, x) \in p_i\}|$.

Under the λ-QMR, λ is a quorum of favorable individual rankings required for a strict social ranking. If x has a simple majority over y and the number of individuals known to strictly rank x over y is at least λ, then x socially ranks higher than y. If y has a simple majority over x and the number of individuals known to strictly rank y over x is at least λ, then y socially ranks higher than x. If neither of the two alternatives has such a qualified majority over the other, they socially rank equally. The prevalent absolute majority rules are a subclass of QMRs in the following sense. An absolute majority rule with respect to the actual finite population $N_a \subseteq N$ of society is a λ-QMR with $\frac{|N_a|}{2} < \lambda \leq |N_a|$.

Example: Reconsider the voting example from the last section where 6 individuals voted for x, 2 individuals voted for y, 4 individuals reported indifference, and 8 actual

[10]That framework does not distinguish between indifference and abstention. Both are represented by the number 0.

[11]Take an improvement of x over y from r to r' in the sense of PR. According to monotonicity, $(x, y) \in \phi(r) \Rightarrow (x, y) \in \phi(r')$ and $(x, y) \in \phi_p(r) \Rightarrow (x, y) \in \phi_p(r')$. These two implications are entailed by (d) and (c) of PR, respectively. Only PR excludes the rule assigning social indifference between all alternatives at all profiles from D_F.

individuals have not (yet) turned out at r. Under the 5-QMR, $d_{xy}(r) = 0$ and $d_{yx}(r) = 5$. Alternative x socially ranks higher than y because it has a qualified majority over y given $\lambda = 5$. In contrast, y needs 5 additional votes to gain the simple majority over x. Since it also satisfies the quorum with these additions, they would secure the social lead for y.

Under the 11-QMR, an absolute majority of votes is needed for a strict social ranking given N_a. It leads to $d_{xy}(r) = 5$ and $d_{yx}(r) = 9$. Both alternatives need the respective number of additional votes to reach the quorum. But since only 8 actual individuals can still cast their ballot, it is factually impossible for y to improve accordingly. If one new individual joins N_a and the absolute majority quorum is upheld, y can realize the social lead.

IER leads to a characterization of RMRs. The following condition is another strengthening of ER and embodies a different idea.

SRR ϕ on D satisfies **Accumulative Equal Responsiveness (AER)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k$, it is the case that
(a) $d_{xy}(r) = d_{yx}(r) > 1 \Rightarrow d_{xy}(r') = d_{yx}(r') = d_{xy}(r) - 1$.
(b) $d_{xy}(r) = d_{yx}(r) = 1 \Rightarrow d_{xy}(r') = d_{yx}(r') = 1$.

Once again, PAIs over x and y are levelled at r and two opposing rankings are added from r to r'. As ER and IER, AER demands that the two PAIs remain levelled at r'. As under IER, the PAIs should not change if they are equal to one. However, if the PAIs are greater than one, they should be reduced by one, respectively. For example, if $d_{xy}(r) = d_{yx}(r) = 4$, then $d_{xy}(r') = d_{yx}(r') = 3$ according to AER.

One possible justification for AER on D_F is the following. An alternative needs to accumulate positive evidence over another alternative to socially rank higher than the latter. A strict individual ranking of x over y (y over x) will add to this positive evidence for x (y), even if it is accompanied by a strict ranking of y over x (x over y). That is why the new opposing rankings reduce the required additional amount of positive evidence for the social lead by one. The idea connects to Llamazares' (2006: pp. 315-316) criterion that an alternative should have "wide support" to get elected.

Theorem 3:
SRR ϕ on $D = D_F$ satisfies IIA, III, PR, and AER if and only if ϕ is a QMR.

Theorem 3 complements Theorem 1. Given (C)IIA, III, and PR, AER as opposed to IER distinguishes the class of Quorum Majority Rules from the class of Relative Majority Rules. The following condition is also concerned with the structure of levelled PAIs.

SRR ϕ on D with $r^{\emptyset} \in D$ satisfies **Accumulation of Balancing Information (ABI)** if, for all $r \in D$ and distinct $x, y \in X$, $d_{xy}(r) = d_{yx}(r) \Rightarrow d_{xy}(r) = max\{d_{xy}(r^{\emptyset}) - |\{i \in N \mid (x,y) \in p_i\}|, 1\}$.

PAIs over x and y are levelled at r. According to ABI, the PAI of x over y should be equal to the PAI of x over y when no information is available minus the number of strict individual rankings of x over y at r (as long as that difference is positive). That is, the number of favorable rankings necessary to take the social lead should decrease with the number of favorable rankings already known. For example, assume $d_{xy}(r) = d_{yx}(r)$ and $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset}) = 10$. Assume as well that there are three individuals ranking x higher than y and another three individuals ranking y higher than x at r. ABI requires that $d_{xy}(r) = d_{yx}(r) = 7$.

ABI and AER are logically independent on D_F (see Appendix 4.A.6). However, the justification of AER can also be applied to ABI. Some positive evidence for an alternative over another one is taken to be necessary for the former to reach the social lead over the latter. Favorable individual rankings contribute to this positive evidence. So even if PAIs of x and y are balanced, more known strict individual rankings of x over y should ceteris paribus reduce the additional positive evidence that x needs to take the social lead over y.

Theorem 4:
SRR ϕ on $D = D_F$ satisfies IIA, A, N, PR, and ABI if and only if ϕ is a QMR.

Given (C)IIA, (C)A, (C)N, and PR, the choice between IBI and ABI marks the divide between Relative Majority Rules and Quorum Majority Rules. This is due to Theorems 2 and 4. Pauly (2013: Theorem 1) characterizes majority rules with quorums of favorable rankings in the fixed population two alternative framework that is used for characterizations of relative majority rules from the literature. The result employs anonymity, neutrality, weak Pareto, quorum, and strategy proofness conditions.

4.3.3 The Simple Majority Rule

The Simple Majority Rule is among the most influential aggregation rules. This subsection formulates the rule for incomplete individual rankings and offers multiple characterizations of it.

The **Simple Majority Rule (SMR)** on $D \subseteq D_F$ is the 1-Relative Majority Rule and the 1-Quorum Majority Rule on D. That is, for all $r \in D$ and distinct $x, y \in X$, SMR ϕ exhibits $(x, y) \in \phi(r) \Leftrightarrow |\{i \in N \mid (x, y) \in p_i\}| \geq |\{i \in N \mid (y, x) \in p_i\}|$.

According to the SMR, alternative x socially ranks at least as high as alternative y if and only if the number of individuals who are known to rank x higher than y is at least as great as the number of individuals who are known to rank y higher than x. Since the SMR constitutes the intersection of the class of RMRs and the class of QMRs, there are two immediate corollaries to the analysis of the previous sections.

Corollary 1:
SRR ϕ on $D = D_F$ satisfies IIA, III, PR, IER, and AER if and only if ϕ is the SMR.

Corollary 2:
SRR ϕ on $D = D_F$ satisfies IIA, A, N, PR, IBI, and ABI if and only if ϕ is the SMR.

Given the other conditions of Corollary 1 (2), combining IER and AER (IBI and ABI) means imposing a strengthened form of positive responsiveness. If $d_{xy}(r) = d_{yx}(r)$ holds, then $d_{xy}(r) = d_{yx}(r) = 1$. That is, if PAIs of x and y are levelled, adding a single strict individual ranking of x over y (y over x) will lead to a strict social ranking of x over y (y over x). This property characterizes the SMR.

The property is implied by MPR's requirement that an additional strict individual ranking of x over y should turn an initial equal social ranking of the pair into a strict social ranking in favor of x. Indeed, two characterizations of the SMR with MPR are established in the remainder of this section. But as discussed, one might find it difficult to justify the condition. There is a way to characterize the SMR which does not presuppose MPR, but derives it as an implication. The next condition will be central.

SRR ϕ on D satisfies **No Implicit Lead (NIL)** if, for all $r \in D$ and distinct $x, y \in X$, $d_{xy}(r) < d_{yx}(r) \Rightarrow d_{xy}(r) = 0$.

So far, it has been allowed that PAIs of a pair of alternatives deviate despite an equal social ranking of the pair. For example, $d_{xy}(r) = 3$ and $d_{yx}(r) = 5$ might obtain. Here, x has a lead over y in the sense that it needs fewer individual improvements to reach a strict social ranking over its competitor. The lead is implicit because the current social ranking is equal. Condition NIL precludes such an implicit lead. If the PAI of x over y is smaller than the PAI of y over x, the former PAI should be zero (x should socially rank higher than y). Arguments for NIL are discussed in the next section.

NIL turns out to be weaker than MPR on a range of domains. However, given PR, NIL implies MPR on every subdomain of D_F. Domain D is one-additive if, for all $r \in D$ and distinct $x, y \in X$, there exist $r' \in D$ and $i \in N$ such that $(x, y), (y, x) \notin r_i$, $r'_i = r_i \cup \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$. In particular, D_F is one-additive.

Lemma 4:
(a) If SRR ϕ on one-additive D satisfies MPR, then ϕ satisfies NIL.
(b) If SRR ϕ on $D \subseteq D_F$ satisfies PR and NIL, then ϕ satisfies MPR.

Lemma 4 will turn out to be powerful. If one agrees with PR and NIL, one will be committed to MPR. In characterizations, the former two conditions can replace the latter. Given (C)IIA, III, and ER, a SRR satisfies PR and NIL (MPR) if and only if it is the Simple Majority Rule.

Proposition 1:
SRR ϕ on $D = D_F$ satisfies IIA, III, ER, and MPR if and only if ϕ is the SMR.

Theorem 5:
SRR ϕ on $D = D_F$ satisfies IIA, III, PR, ER, and NIL if and only if ϕ is the SMR.

Given (C)IIA, (C)A, and (C)N, a SRR satisfies MPR if and only if it is the Simple Majority Rule. This is stated in Proposition 2. In view of the fact that the property is weaker than a classical formulation of May's condition (Appendix 4.A.6, Proposition A.6.1), the result is a generalized and somewhat tightened version of May's Theorem (May, 1952) for incomplete individual rankings. It leads to a characterization of the Simple Majority Rule with PR and NIL.

Proposition 2:
SRR ϕ on $D = D_F$ satisfies IIA, A, N, and MPR if and only if ϕ is the SMR.

Theorem 6:
SRR ϕ on $D = D_F$ satisfies IIA, A, N, PR, and NIL if and only if ϕ is the SMR.

4.4 Discussion

4.4.1 Social Rankings and Social Choice

The social ranking is a central concept of social choice theory. Nevertheless, its exact meaning is not always clarified. The present subsection discusses three possible interpretations. For each of them, it assesses the merit of the conditions used to characterize different majority rules. Table 4.1 summarizes which rules satisfy which properties on D_F. A Proper Relative Majority Rule (PRMR) is a RMR with $\delta > 1$. A Proper Quorum Majority Rule (PQMR) is a QMR with $\lambda > 1$. A "✓" means that the condition is satisfied.

	IIA	A	N	III	PR	MPR	NIL	ER	IER	IBI	AER	ABI
SMR	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
PRMR	✓	✓	✓	✓	✓			✓	✓	✓		
PQMR	✓	✓	✓	✓	✓			✓			✓	✓

Table 4.1: Majority Rules and PAI Conditions

1. Social Rankings as Relative Evaluations: Social rankings can represent a comparative social evaluation of alternatives given available information. This is both relevant in a voting and in a policymaking context. One might like to compare candidates for a political office or policies. A SRR ϕ generating such relative social evaluations will be called Evaluation Ranking Rule (ERR). Under ERR ϕ, $(x, y) \in \phi_p(r)$ means that available information at r favors x over y, and $(x, y) \in \phi_s(r)$ means that available information at r neither favors x over y nor y over x.[12]

It will be assumed that Evaluation Ranking Rule ϕ on D_F satisfies A, N, ER, III, and PR. Social evaluation should treat individuals and alternatives equally, be independent of indifference information, and respond positively to individual evaluation. It will also be assumed that ϕ satisfies IIA. Given the assumptions, the critical question is whether to impose NIL on ϕ.

Assume that the PAI of x over y is smaller than the PAI of y over x at profile r under ϕ. By assumption, social evaluation should treat alternatives and individuals equally, so

[12]The interpretation is in line with the assumption that social rankings are complete. However, it can lead to intransitive social rankings.

that the asymmetry cannot be due to discrimination between alternatives or individuals. It also cannot be due to an asymmetry in potential improvements on D_F. For each set of individuals whose rankings of x and y are unknown at r, r can be augmented so that either all these individuals rank x higher than y or the other way around. It thus makes sense to infer from the asymmetry in PAIs that available information at r favors x over y under ϕ. Consequently, the very idea of an ERR requires that x socially ranks higher than y and the PAI of x over y is zero. Hence, NIL is reasonable for an ERR.

The argument together with Theorems 5 and 6 provide support to employ the Simple Majority Rule as Evaluation Ranking Rule given the assumptions. According to the SMR as ERR, available information at r favors x over y if and only if x has a simple majority over y. Under any RMR or QMR other than the SMR, the social ranking does not always reflect favorable available information.

To illustrate, reconsider the example from section 4.3 where 6 individuals are known to rank x higher than y, while 2 individuals are known to have the opposite ranking at profile r. All RMRs and QMRs agree in specifying $d_{xy}(r) < d_{yx}(r)$. But while the SMR sets $d_{xy}(r) = 0$ and $d_{yx}(r) = 5$, other rules exhibit an equal social ranking. For instance, the 5-RMR specifies $d_{xy}(r) = 1 < 9 = d_{yx}(r)$ and the 10-QMR sets $d_{xy}(r) = 4 < 8 = d_{yx}(r)$. While their PAIs apparently acknowledge that available information at r favors x over y, their social ranking does not.

2. Social Rankings for Fixed Decision-Making: Social rankings can also be characterized by their relationship to social choice. Consider the case where society must choose an alternative from $X = \{x, y\}$ given profile r of known individual rankings. SRR ϕ is a Fixed Decision Ranking Rule (FDRR) if its social rankings relate to social choice in the following way. If $(x, y) \in \phi_p(r)$, then x will be chosen. If $(y, x) \in \phi_p(r)$, then y will be chosen. If $(x, y) \in \phi_s(r)$, then x and y will be chosen with probability 0.5, respectively. In any case, social rankings of a FDRR lead to a social choice given profile r.[13] The selection of a FDRR will be discussed for two different cases.

Voting: Suppose x and y are candidates for a political office. Profile r contains the final result of all submitted ballots by individuals from the actual population N_a. In the example from section 4.3, the vote count is 6 to 2 in favor of x. Moreover, 4 people reported indifference and 8 actual individuals did not cast their ballot. Assume that a

[13]One can also define the concept of a FDRR for more than two alternatives. In that case, the social choice is to be selected randomly from the set of best alternatives under a FDRR (if this set is non-empty). Dasgupta and Maskin (2020) examine social choice functions with such a tiebreaking mechanism.

candidate must be selected given r. There is no possibility of delay.

It is a reasonable normative assumption that only submitted ballots matter. That is, r contains all relevant information to decide who is elected. The winner should arguably be determined according to the relative social evaluation of the two candidates at r. In other words, a FDRR in a voting context should be an ERR. The presented argument thus not only supports the Simple Majority Rule as an ERR, but also as a FDRR. The example illustrates that point. According to the SMR as ERR, available information at r favors x over y. It is thus sensible to elect x. That is what the SMR as FDRR does. In contrast, the 5-RMR and the 10-QMR specify an equal social ranking. As FDRRs, they would induce a coin toss to decide between x and y. Considering the favorable evaluation for x, it is unclear how to justify such action.

Welfarist policymaking: Suppose x and y are different policy options. Profile r consists of known individual well-being rankings over x and y. The policy with greater social welfare for N_a is to be chosen. Consider an according reinterpretation of the voting example. For 12 out of 20 individuals from N_a, well-being information has been empirically identified at r. There are 6 individuals whose well-being at x is known to be greater than at y, 2 individuals whose well-being at y is known to be greater than at x, and 4 individuals whose well-being at x is known to be as great as at y. The well-being ranking of the other 8 actual individuals is unknown. Assume that a deadline or cost considerations force to make a decision with the given available information r.

The social welfare evaluation of x and y for N_a is the social evaluation of x and y at profile r' where all well-being rankings over N_a are known. Call this the N_a-complete profile. Assume that the SMR is employed as ERR. Given that assumption, social welfare at x is greater than at y if and only if x has a simple majority over y at the N_a-complete profile r'. Specifically, the social welfare difference between x and y is the difference between strict well-being rankings of x over y and of y over x. In the example, assume that 9 individuals have a greater well-being level at x than at y, 7 individuals have a greater well-being level at y than at x, and 4 individuals have an equal well-being level at x and y given r'. That is, 3 of the remaining actual individuals favor x, while 5 of them favor y. The social welfare difference between x and y is $9 - 7 = 2$.

Social choice must be made given r and not given r'. A welfarist policymaker faces the statistical problem to predict the social welfare difference given r. The sample welfare difference is the difference in strict well-being rankings of x over y and of y over x at r. In the example, this sample welfare difference is $6 - 2 = 4$. Consider any statistical assumption such that a positive sample welfare difference is equivalent to a positive expected social welfare difference.[14] Given such an assumption, the SMR is the appropriate FDRR for welfarist policymaking. It always chooses the alternative with a favorable expected social welfare difference.

3. Social Rankings for Open Decision-Making: It is still assumed that society aims to choose an alternative from $X = \{x, y\}$. However, it is now possible to either decide for an alternative given incomplete profile r or for acquiring new information. SRR ϕ is an Open Decision Ranking Rule (ODRR) if its social rankings relate to social choice in the following way. As under a FDRR, $(x, y) \in \phi_p(r)$ induces that x is chosen and $(y, x) \in \phi_p(r)$ implies that y is chosen. However, $(x, y) \in \phi_s(r)$ now means that the decision between x and y is postponed and new information on individual rankings will be acquired. That is, r will be augmented.[15] An ODRR will be needed if welfarist policymakers can either choose an alternative given available well-being rankings or decide to acquire more information.[16]

Assume that the SMR is still employed as ERR. It yields a social welfare difference at N_a-complete profile r' and a sample welfare difference at incomplete profile r as before. A positive sample welfare difference between x and y is arguably still necessary for a strict social ranking of x over y under an ODRR. That is, a simple majority of x over y at r is necessary for society to settle on x. But it might not be sufficient anymore. One can argue that a "significant" sample welfare difference in favor of x is needed for a strict social ranking of x over y. An ODRR might accordingly violate NIL.

Given the other conditions of Theorems 1-4, a decision for IER or IBI (AER or ABI) means that a RMR (QMR) should be employed as ODRR. A RMR as ODRR specifies a fixed sample welfare difference that an alternative needs to become the social choice. A QMR as ODRR demands a quorum of favorable individual well-being rankings as well as a positive sample welfare difference for that. In each case, a higher δ (λ) implies that an alternative must meet a stricter requirement to become the social choice.[17]

[14] A simplistic assumption would be that each individual whose ranking over x and y is unknown at r is equally likely to have a greater well-being level at x than at y and to have a greater well-being level at y than at x.

[15] The definition can be generalized for more than two alternatives. One stops acquiring information and chooses an alternative if and only if it is the unique best alternative under an ODRR.

[16] Elections with runoffs pose a different problem. If a runoff takes place, profile r consisting of first-round ballots will not be augmented but replaced by a new profile of rankings over remaining candidates.

[17] That is, decisiveness decreases. See Llamazares (2006) for a discussion of this property.

4.4.2 Pull Ahead Indices for Extended Informational Inputs

This subsection discusses how the concept of a Pull Ahead Index can be extended for richer informational inputs of aggregation. It gives two examples that are both motivated by limitations of majority rules.

1. It is well-known that majority rules can lead to cyclic social rankings if there are more than two alternatives.[18] Specifically, the set of best alternatives is empty in that case. Both in a voting and in a policymaking context, this poses a problem for majority rules as FDRRs and ODRRs.[19] One way to deal with this issue is to weaken IIA and let the Pull Ahead Index of x over y depend on individual rankings involving third alternatives. One might argue that strict individual rankings of x over a third alternative z and of z over y provide positive evidence for x over y. Call such rankings favorable external rankings for x over y. It is possible to define an External Pull Ahead Index that measures the smallest number of favorable external rankings for x over y that x needs to take the social lead over y.

For example, $|\{(i,z) \in N \times X \mid (x,z) \in p_i\}| - |\{(i,z) \in N \times X \mid (z,x) \in p_i\}|$ is the Borda score of x at r. According to the γ-Relative Borda Rule with natural number γ, x socially ranks higher than y if and only if the difference in Borda scores of x and y is at least γ.[20] The critical properties of the γ-Relative Borda Rule are that its External Pull Ahead Index of x over y given no information is γ and that this index in terms of favorable rankings vis-à-vis third alternatives is twice as large as the Pull Ahead Index of x over y in terms of favorable rankings of x over y. One strict individual ranking of x over y gets twice the weight of an external favorable ranking.

2. In a voting context, individuals usually only submit incomplete preferences over alternatives. But it is reasonable to consider interpersonal well-being level and difference comparisons in social welfare evaluation (Sen, 1970, 2017; d'Aspremont and Gevers, 1977, 2002; Maskin, 1978; Weymark, 2016; chapters 2-3 of this thesis). SRRs do not incorporate such extended informational inputs. Majority rules only depend on intrapersonal well-being level comparisons and do not account for differences in well-

[18] A cyclic ranking is one of the form $(x,y),(y,z),(z,x) \in \phi_p(r)$.

[19] The discussed specification of a FDRR for more than two alternatives presupposes that the set of best alternatives is non-empty at the fixed profile. Under the discussed specification of the ODRR, information acquisition continues if there is no single best alternative.

[20] Different versions of the Borda rule are analyzed in Gärdenfors (1973), Young (1974), Nitzan and Rubinstein (1981), Pattanaik (2002), Cullinan et al. (2014), Terzopoulou and Endriss (2021), Barberà and Bossert (2023), and chapter 2 of this thesis.

being levels or well-being differences between individuals.[21]

One way to augment the input of aggregation is to consider information profiles which capture interpersonally comparable utility differences between alternatives for individuals. An Extended Social Ranking Rule translates a profile of identified individual utility differences between alternatives into a social ranking of alternatives.[22] It is possible to define an Extended Pull Ahead Index for the Extended Social Ranking Rule which measures the sum of newly identified utility differences in favor of x over y that needs to be surpassed for x to take the social lead over y.

According to the c-Relative Utilitarian Rule with non-negative real number c, x socially ranks higher than y if and only if the sum of identified individual utility differences between x and y is greater than c. The critical properties of this rule are that the Extended Pull Ahead Index given no utility difference information is c and that a newly identified utility difference between x and y of 1 generally reduces the Extended Pull Ahead Index of x over y by 1.

4.5 Conclusion

The present chapter has offered a new approach to aggregation. Pull Ahead Indices measure necessary new favorable individual rankings for an alternative to take the social lead over another alternative. Their richer informational content as compared to social rankings allows to formulate novel conditions. Independence of Irrelevant Alternatives, Anonymity, and Neutrality are equivalent to their classical counterparts for social rankings on additive domains (Lemmas 1-3). The other introduced conditions for Pull Ahead Indices do not have classical equivalents.

New class characterizations have been identified. Relative Majority Rules are distinguished by the fact that a levelled Pull Ahead Index is independent of (new) opposing information (Theorems 1-2). The critical property of Quorum Majority Rules is that favorable individual rankings accumulate and decrease a levelled Pull Ahead Index

[21]Theorem 1 of chapter 2 characterizes the Simple Majority Relation as a social welfare evaluation ordering for the case where there are no interpersonal well-being level and difference comparisons. The employed normative conditions of Strong Pareto and Fundamental Equity yield other social welfare evaluation orderings under different informational assumptions.

[22]Formally, such an extended profile can be defined as a collection of real numbers $u(i, x, y)$, one for each tuple of distinct alternatives x and y and each individual i for whom the utility difference between x and y is identified.

(Theorems 3-4).

Positive Responsiveness is an attractive condition that is satisfied by all considered majority rules. Together with the requirement that alternatives should not have an implicit lead in terms of the Pull Ahead Index, it implies May's Positive Responsiveness (Lemma 4). This insight has been employed to establish new characterizations of the Simple Majority Rule (Theorems 5-6, Propositions 1-2).

No Implicit Lead is a reasonable condition for social evaluation, voting, and policymaking given a fixed informational basis. It provides an argument to employ the Simple Majority Rule rather than qualified majority rules in these contexts. If it is possible to acquire new information, Proper Relative or Quorum Majority Rules can be plausible.

The presented analysis suggests several relevant extensions.

1. One can also characterize other classes of majority rules with conditions on the Pull Ahead Index. Call a quorum of favorable individual rankings in the sense of Quorum Majority Rules a quorum of type (1). Majority rules with a quorum of type (2) specify a number of overall strict individual rankings over a pair (x over y plus y over x) that are necessary for a strict social ranking over the pair (see Houy, 2009; Jeong and Ju, 2017). Compared to Relative and Quorum Majority Rules, their critical property is that two new opposing individual rankings generally reduce a levelled Pull Ahead Index by 2. Majority rules with a quorum of type (3) set up a number of overall known individual rankings over a pair (strict plus indifferent) which are necessary for a strict social ranking of the pair. In contrast to the other classes, a newly identified indifference over x and y generally reduces the Pull Ahead Index of x over y by 1.

2. In the context of policymaking with information acquisition, two extensions are interesting. First, it is relevant to consider rules where the required relative majority (quorum) for a strict social ranking depends on sample and population size. If most individual rankings are known, a small positive sample welfare difference can suffice to establish a positive social welfare difference. Second, it is possible to allow for incomplete social rankings that have a different meaning than equal social rankings.

3. For a comprehensive understanding of aggregation, it is relevant to examine the discussed extensions of the Pull Ahead Index in more detail. The External Pull Ahead Index in terms of new favorable rankings over third alternatives is particularly helpful to analyze the introduced class of Relative Borda Rules. Likewise, the Extended Pull Ahead Index in terms of new utility differences can be used to study the discussed class of Relative Utilitarian Rules.

As the identified results and potential extensions show, the Pull Ahead Index presents a new and fruitful perspective on aggregation. This perspective is forward-looking. The Index is not only concerned with the current social ranking and current majorities. It is about pathways to new majorities.

4.A Appendix

The Appendix has several parts. Results from the main text are proved in sections 4.A.1 to 4.A.4. The logical independence of the respective conditions from the characterizations is demonstrated in section 4.A.5. Noted logical relations between conditions from the main text are verified in section 4.A.6.

4.A.1 Proofs for Section 4.2.2

Lemma 1:
(a) Assume SRR ϕ on D satisfies IIA. Consider $r, r' \in D$ and distinct $x, y \in X$ such that $r|\{x,y\} = r'|\{x,y\}$. By IIA, that implies $d_{yx}(r) = d_{yx}(r')$. It follows that $(x,y) \in \phi(r) \Leftrightarrow d_{yx}(r) > 0 \Leftrightarrow d_{yx}(r') > 0 \Leftrightarrow (x,y) \in \phi(r')$. Hence, ϕ satisfies CIIA.

(b) Assume SRR ϕ on additive D satisfies CIIA. Take $r, r' \in D$ and distinct $x, y \in X$ such that $r|\{x,y\} = r'|\{x,y\}$. Assume $d_{xy}(r) \neq d_{xy}(r')$. Without loss of generality, let $d_{xy}(r) < d_{xy}(r')$. In particular, $d_{xy}(r) < \infty$, so that there exists $n^* \in \mathbb{N}_0$ with $d_{xy}(r) = n^*$. Consider the corresponding $N^* \subseteq N$ and $r^* \in D$ from the definition of the PAI. For $i \in N^*$, it is the case that $r'_i|\{x,y\} = r_i|\{x,y\} = \emptyset$. Define $r'' \in S(X)^N$ such that, for all $i \in N^*$, $r''_i = r'_i \cup \{(x,y)\}$, and, for all $i \in N \setminus N^*$, $r''_i = r'_i$. Since D is additive, it follows that $r'' \in D$.

For each $i \in N^*$, it is true that $r^*_i|\{x,y\} = \{(x,y)\} = r''_i|\{x,y\}$. For each $i \in N \setminus N^*$, $r^*_i|\{x,y\} = r_i|\{x,y\} = r'_i|\{x,y\} = r''_i|\{x,y\}$ holds. Taken together, that implies $r^*|\{x,y\} = r''|\{x,y\}$. It is true by definition that $(x,y) \in \phi_p(r^*)$. By CIIA, that yields $(x,y) \in \phi_p(r'')$. In words, adding strict rankings of x over y for the n^* individuals from N^* to r' leads to a strict social ranking of x over y. That is, $d_{xy}(r') \leq n^* = d_{xy}(r)$. This contradicts the initial assumption $d_{xy}(r) < d_{xy}(r')$. Consequently, $d_{xy}(r) = d_{xy}(r')$ must hold, so that ϕ satisfies IIA.■

Lemma 2:
(a) Assume SRR ϕ on D satisfies A. Take $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Consider distinct $x, y \in X$. Due to A, it follows that $d_{yx}(r) = d_{yx}(r')$. That yields $(x,y) \in \phi(r) \Leftrightarrow d_{yx}(r) > 0 \Leftrightarrow d_{yx}(r') > 0 \Leftrightarrow (x,y) \in \phi(r')$. Thus, ϕ satisfies CA.

(b) Assume SRR ϕ on additive D satisfies CA. Consider $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Take distinct $x, y \in X$. Assume $d_{xy}(r) \neq$

$d_{xy}(r')$. Again, $d_{xy}(r) < d_{xy}(r')$ can be supposed without loss of generality. There exists $n^* \in \mathbb{N}_0$ with $n^* = d_{xy}(r) < \infty$. Take the corresponding N^* and r^*. For all $i \in N^*$, $(x, y), (y, x) \notin r_i$ and thus $(x, y), (y, x) \notin r'_{\pi(i)}$ holds. Define $r'' \in S(X)^N$ such that, for each $i \in N^*$, $r''_{\pi(i)} = r'_{\pi(i)} \cup \{(x, y)\}$, and, for each $i \in N \setminus N^*$, $r''_{\pi(i)} = r'_{\pi(i)}$. It follows that $r'' \in D$ as D is additive.

For $i \in N^*$, it is true that $r^*_i = r_i \cup \{(x, y)\} = r'_{\pi(i)} \cup \{(x, y)\} = r''_{\pi(i)}$. For $i \in N \setminus N^*$, it is true that $r^*_i = r_i = r'_{\pi(i)} = r''_{\pi(i)}$. By definition, $(x, y) \in \phi_p(r^*)$. As its antecedent is satisfied with respect to r^* and r'', CA induces $(x, y) \in \phi_p(r'')$. Adding strict rankings of x over y for n^* individuals to r' generates a strict social ranking of x over y. In contrast to the proof of Lemma 1, the relevant individuals are not the ones from N^*, but their counterparts under π. Still, $d_{xy}(r') \leq n^* = d_{xy}(r)$ holds, contradicting the initial assumption $d_{xy}(r) < d_{xy}(r')$. Hence, $d_{xy}(r) = d_{xy}(r')$ must hold, implying that ϕ satisfies A.■

Lemma 3:
(a) Assume SRR ϕ on D satisfies N. Consider $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$. Take distinct $x, y \in X$. N implies $d_{yx}(r) = d_{\sigma(y)\sigma(x)}(r')$. That yields $(x, y) \in \phi(r) \Leftrightarrow d_{yx}(r) > 0 \Leftrightarrow d_{\sigma(y)\sigma(x)}(r') > 0 \Leftrightarrow (\sigma(x), \sigma(y)) \in \phi(r')$. Therefore, ϕ satisfies CN.

(b) Assume SRR ϕ on additive D satisfies CN. Take $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$. Consider distinct $x, y \in X$ and define $x' = \sigma(x)$ and $y' = \sigma(y)$. Assume $d_{xy}(r) \neq d_{x'y'}(r')$. Without loss of generality, suppose $d_{xy}(r) < d_{x'y'}(r')$. Take $n^* \in \mathbb{N}_0$ with $n^* = d_{xy}(r) < \infty$ and the corresponding N^* and r^*. For $i \in N^*$, $(x, y), (y, x) \notin r_i$ and hence $(x', y'), (y', x') \notin r'_i$ holds. Define $r'' \in S(X)^N$ such that, for each $i \in N^*$, $r''_i = r'_i \cup \{(x', y')\}$, and, for each $i \in N \setminus N^*$, $r''_i = r'_i$. The additivity of D secures $r'' \in D$.

Consider $i \in N$. Take $a, b \in X$ with $\{a, b\} \neq \{x, y\}$. It is true that $(a, b) \in r^*_i \Leftrightarrow (a, b) \in r_i \Leftrightarrow (\sigma(a), \sigma(b)) \in r'_i \Leftrightarrow (\sigma(a), \sigma(b)) \in r''_i$. It is also true that $(y, x) \in r^*_i \Leftrightarrow (y, x) \in r_i \Leftrightarrow (y', x') \in r'_i \Leftrightarrow (y', x') \in r''_i$. Assume $i \in N \setminus N^*$. Then, $(x, y) \in r^*_i \Leftrightarrow (x, y) \in r_i \Leftrightarrow (x', y') \in r'_i \Leftrightarrow (x', y') \in r''_i$. Assume $i \in N^*$. Then, $(x, y) \in r^*_i$ and $(x', y') \in r''_i$. By definition, $(x, y) \in \phi_p(r^*)$. Because its antecedent holds with respect to r^* and r'', CN implies $(x', y') \in \phi_p(r'')$. Adding n^* strict rankings of x' over y' to r' induces that x' socially ranks higher than y'. But then $d_{x'y'}(r') \leq n^* = d_{xy}(r)$ holds, contradicting the assumption $d_{xy}(r) < d_{x'y'}(r')$. Accordingly, $d_{xy}(r) = d_{x'y'}(r')$ must be true. That is, ϕ satisfies N.■

4.A.2 Proofs for Section 4.3.1

For profile $r \in D$ and distinct $x, y \in X$, denote $N_{xy}(r) = \{i \in N \mid (x,y) \in p_i\}$, $N_{yx}(r) = \{i \in N \mid (y,x) \in p_i\}$, $N_{d,xy}(r) = \{i \in N \mid (x,y) \in s_i\}$, $m_{xy}(r) = |N_{xy}(r)|$, and $m_{yx}(r) = |N_{yx}(r)|$. If r and $\{x,y\}$ are clear from the context, denote $N_{xy} = N_{xy}(r)$, $N_{yx} = N_{yx}(r)$, $N_d = N_{d,xy}(r)$, $m_{xy} = m_{xy}(r)$, $m_{yx} = m_{yx}(r)$, and $m = min\{m_{xy}, m_{yx}\}$.

Theorem 1:
"⇒": Assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, III, PR, and IER.

Define $\delta = d_{xy}(r)$, where $r \in D$ is any profile containing no strict individual rankings and $x, y \in X$ are any distinct alternatives. In view of $r^{\emptyset} \in D$, such a profile exists. To see that δ is well-defined, consider any other profile $r' \in D$ containing no strict individual rankings and any other distinct $x', y' \in X$. By III, it follows that $d_{xy}(r) = d_{x'y'}(r')$. In particular, $d_{xy}(r) = d_{yx}(r)$ holds. This can only be true if $d_{xy}(r) > 0$, implying $\delta > 0$. Due to PR(a) of ϕ and the fact that $r \in D_F$, δ is finite.

It is to be established that ϕ is the δ-RMR. To do so, consider $r \in D$ and distinct $x, y \in X$. Since $r \in D_F$, $m_{xy} < \infty$ and $m_{yx} < \infty$. Relabel individuals from N_{xy} and N_{yx} such that $N_{xy} = \{i_1, ..., i_{m_{xy}}\}$ and $N_{yx} = \{j_1, ..., j_{m_{yx}}\}$.

To show that ϕ is the δ-RMR, the following equivalence must be verified:

$$(x,y) \in \phi(r) \Leftrightarrow m_{xy} + \delta - 1 \geq m_{yx} \tag{4.1}$$

Step 1: Define $r^0 \in D_2$ in the following way. For $i \in N_d$, set $r_i^0 = \{(x,y),(y,x)\}$. For $i \in N \setminus N_d$, set $r_i^0 = \emptyset$. Profile r^0 contains the indifferent individual rankings of $\{x,y\}$ from r, and nothing else. By assumption, $r^0 \in D$. As there are no strict individual rankings under r^0, III implies $d_{xy}(r^0) = d_{yx}(r^0) = \delta$.

Step 2: If $m = 0$, continue with Step 3. If $m > 0$, inductively define a sequence of profiles $r^0, ..., r^m$ in the following way. For each $t \in \{1, ..., m\}$, set $r_{i_t}^t = \{(x,y)\}$, $r_{j_t}^t = \{(y,x)\}$, and, for all $k \in N \setminus \{i_t, j_t\}$, $r_k^t = r_k^{t-1}$. Profile r^0 is subsequently augmented by pairs of rankings from r. For each pair, one individual strictly ranks x over y and the other ranks y over x. For all $t \in \{1, ..., m\}$, $r^t \in D_2$, so that $r^t \in D$ holds by assumption. By IER, it is true that, for all $t \in \{1, ..., m\}$, $d_{xy}(r^{t-1}) = d_{yx}(r^{t-1}) = \delta \Rightarrow d_{xy}(r^t) = d_{yx}(r^t) = \delta$. Together with $d_{xy}(r^0) = d_{yx}(r^0) = \delta$, it follows that $d_{xy}(r^m) = d_{yx}(r^m) = \delta$.

Step 3: Define $r^h = r|\{x,y\} \in D_2$. It contains all individual rankings over $\{x,y\}$ from r, and nothing else. By assumption, $r^h \in D$. Show "⇒" of equivalence (4.2) by demonstrating $m_{yx} > m_{xy} + \delta - 1 \Rightarrow (x, y) \notin \phi(r^h)$.

$$(x, y) \in \phi(r^h) \Leftrightarrow m_{xy} + \delta - 1 \geq m_{yx} \tag{4.2}$$

Assume $m_{yx} > m_{xy} + \delta - 1$. That is, $m_{yx} \geq m_{xy} + \delta$ and $m_{xy} = m$. Inductively define a sequence of profiles $r^m, ..., r^{m_{yx}} \in D$ in the following way. For each $t \in \{m+1, ..., m_{yx}\}$, set $r^t_{j_t} = \{(y, x)\}$, and, for all $k \in N \setminus \{j_t\}$, $r^t_k = r^{t-1}_k$. By construction, $r^h = r^{m_{yx}}$ is generated by augmenting r^m with the rankings of individuals $j_{m+1}, ..., j_{m_{yx}}$ who favor y over x. As shown, $d_{yx}(r^m) = \delta > 0$. Due to PR(b), the PAI for y is reduced by at least one with each element of the sequence, respectively. Because $m_{yx} - m \geq \delta$, there exists $t^* \in \{m+1, ..., m_{yx}\}$ such that $d_{yx}(r^{t^*}) = 0$. The repeated application of PR(c) to the sequence after t^* yields $d_{yx}(r^h) = 0$. It follows that $(y, x) \in \phi_p(r^h)$. That is, $(x, y) \notin \phi(r^h)$.

Step 4: Establish "$\Leftarrow$" of equivalence (4.2). Assume $m_{xy} + \delta - 1 \geq m_{yx}$. Again, $d_{yx}(r^m) = \delta > 0$. Distinguish three cases.

First, assume $m_{xy} = m_{yx} = m$. In that case, $r^m = r^h$. Since $d_{yx}(r^h) > 0$, it follows that $(y, x) \notin \phi_p(r^h)$ and in turn $(x, y) \in \phi(r^h)$.

Second, assume $m_{xy} > m_{yx} = m$. Analogous to Step 3, inductively define a sequence of profiles $r^m, ..., r^{m_{xy}} \in D$ such that, for each $t \in \{m+1, ..., m_{xy}\}$, $r^t_{i_t} = \{(x, y)\}$, and, for all $k \in N \setminus \{i_t\}$, $r^t_k = r^{t-1}_k$. Profile $r^h = r^{m_{xy}}$ is constructed by augmenting r^m with the rankings of individuals $i_{m+1}, ..., i_{m_{xy}}$ who strictly rank x over y. The repeated application of PR(d) to the sequence yields $d_{yx}(r^h) \geq d_{yx}(r^m)$. Together with $d_{yx}(r^m) > 0$, it follows that $(y, x) \notin \phi_p(r^h)$. That is, $(x, y) \in \phi(r^h)$.

Third, assume $m_{yx} > m_{xy} = m$. Assume $(y, x) \in \phi_p(r^h)$. But r^h is generated by augmenting r^m with $m_{yx} - m$ individual rankings favoring y over x. Since $m_{yx} - m \leq \delta - 1 < \delta$, that implies $d_{yx}(r^m) < \delta$. This is a contradiction to $d_{yx}(r^m) = \delta$. Therefore, $(y, x) \notin \phi_p(r^h)$ and in turn $(x, y) \in \phi(r^h)$ must hold.

Step 5: Due to Lemma 1, ϕ satisfies CIIA. It is true by construction that $r|\{x, y\} = r^h|\{x, y\}$. Accordingly, CIIA implies $(x, y) \in \phi(r) \Leftrightarrow (x, y) \in \phi(r^h)$. Combined with equivalence (4.2), that yields equivalence (4.1). It has been established that ϕ is the δ-RMR.

"$\Leftarrow$": Suppose that ϕ is the δ-RMR on $D = D_F$. It is the case that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\}$. The PAI is always finite because $|\{i \in N \mid r_i|\{x, y\} = \emptyset\}| = \infty$.

Consider $r, r' \in D$ and distinct $x, y \in X$ with $r|\{x,y\} = r'|\{x,y\}$. For all $i \in N$, $i \in N_{xy}(r) \Leftrightarrow (x,y) \in p_i \Leftrightarrow (x,y) \in p'_i \Leftrightarrow i \in N_{xy}(r')$. That implies $N_{xy}(r) = N_{xy}(r')$ and hence $m_{xy}(r) = m_{xy}(r')$. Analogously, it is the case that $N_{yx}(r) = N_{yx}(r')$ and $m_{yx}(r) = m_{yx}(r')$. Taken together, it follows that $d_{xy}(r) = \max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\} = \max\{m_{yx}(r') - m_{xy}(r') + \delta, 0\} = d_{xy}(r')$. Thus, ϕ satisfies IIA.

Take $r, r' \in D$ such that, for all $x, y \in X$ and all $i \in N$, $(x,y) \in r_i \Rightarrow (x,y) \in s_i$ and $(x,y) \in r'_i \Rightarrow (x,y) \in s'_i$. Consider $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$. By assumption, it is true that $m_{xy}(r) = m_{yx}(r) = m_{x'y'}(r') = m_{y'x'}(r') = 0$. It follows that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\} = \delta = max\{m_{y'x'}(r') - m_{x'y'}(r') + \delta, 0\} = d_{x'y'}(r')$. Accordingly, ϕ satisfies III.

Consider $r \in D$ and distinct $x, y \in X$. As $d_{xy}(r) < \infty$, ϕ satisfies PR(a). Take $r' \in D$ and $i \in N$ such that $(x,y), (y,x) \notin r_i$, $r'_i = r_i \cup \{(x,y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$. Since $i \notin N_{xy}(r)$, $N_{xy}(r') = N_{xy}(r) \cup \{i\}$, and $N_{yx}(r') = N_{yx}(r)$, it follows that $m_{xy}(r') = m_{xy}(r) + 1$ and $m_{yx}(r') = m_{yx}(r)$.

For (b) of PR, assume $d_{xy}(r) > 0$. That is, $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + \delta > 0$. It follows that $m_{yx}(r') - m_{xy}(r') + \delta = m_{yx}(r) - m_{xy}(r) - 1 + \delta \geq 0$ and thus $d_{xy}(r') = m_{yx}(r') - m_{xy}(r') + \delta$. Taken together, that implies $d_{xy}(r') = m_{yx}(r') - m_{xy}(r') + \delta < m_{yx}(r) - m_{xy}(r) + \delta = d_{xy}(r)$. For (c), assume $d_{xy}(r) = 0$. That implies $m_{yx}(r') - m_{xy}(r') + \delta < m_{yx}(r) - m_{xy}(r) + \delta \leq 0$. Accordingly, $d_{xy}(r') = 0$. For (d), distinguish two cases. First, assume $d_{yx}(r) = 0$. That immediately implies $d_{yx}(r') \geq d_{yx}(r)$. Second, assume $d_{yx}(r) > 0$. It follows that $0 < d_{yx}(r) = m_{xy}(r) - m_{yx}(r) + \delta < m_{xy}(r') - m_{yx}(r') + \delta = d_{yx}(r')$. Therefore, ϕ satisfies PR.

Consider $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x,y), (y,x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x,y)\}$, $r'_j = r_j \cup \{(y,x)\}$, and, for all $k \in N \setminus \{i,j\}$, $r'_k = r_k$. Assume $d_{xy}(r) = d_{yx}(r) > 0$. It follows that $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + \delta$ and $d_{yx}(r) = m_{xy}(r) - m_{yx}(r) + \delta$. That yields $m_{xy}(r) = m_{yx}(r)$ and thus $d_{xy}(r) = \delta$. It is true that $i \notin N_{xy}(r)$, $N_{xy}(r') = N_{xy}(r) \cup \{i\}$, $j \notin N_{yx}(r)$, $N_{yx}(r') = N_{yx}(r) \cup \{j\}$. That implies $m_{xy}(r') = m_{xy}(r) + 1$, $m_{yx}(r') = m_{yx}(r) + 1$, and in turn $m_{xy}(r') = m_{yx}(r')$. It follows that $d_{xy}(r') = \max\{m_{yx}(r') - m_{xy}(r') + \delta, 0\} = \delta = \max\{m_{xy}(r') - m_{yx}(r') + \delta, 0\} = d_{yx}(r')$. That is, $d_{xy}(r') = d_{yx}(r') = d_{xy}(r)$. Hence, ϕ satisfies IER.■

Theorem 2:
"⇒": Assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, A, N, PR, and IBI.

Define $\delta = d_{xy}(r^{\emptyset})$ for $r^{\emptyset} \in D$ and any distinct $x, y \in X$. To see that δ is well-defined, consider any other distinct $x', y' \in X$. Take bijection $\sigma : X \to X$ such that $\sigma(x) = x'$

and $\sigma(y) = y'$. For all $i \in N$ and all $a, b \in X$, $(a, b), (\sigma(a), \sigma(b)) \notin r_i^{\emptyset}$ and thus $(a, b) \in r_i^{\emptyset} \Leftrightarrow (\sigma(a), \sigma(b)) \in r_i^{\emptyset}$. By N, it follows that $d_{xy}(r^{\emptyset}) = d_{x'y'}(r^{\emptyset})$. In particular, $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset})$, so that $\delta > 0$. PR(a) secures $\delta < \infty$.

It will be demonstrated that ϕ is the δ-RMR. Consider $r \in D$ and distinct $x, y \in X$. Again, relabel individuals from N_{xy} and N_{yx} such that $N_{xy} = \{i_1, ..., i_{m_{xy}}\}$ and $N_{yx} = \{j_1, ..., j_{m_{yx}}\}$. Define $N'_{xy} = \{i_1, ..., i_m\}$ and $N'_{yx} = \{j_1, ..., j_m\}$. It must be established that equivalence (4.1) from the proof of Theorem 1 holds.

Consider profile $r' \in D_2$ which looks like this:
(a) For $i \in N'_{xy}$, $r'_i = \{(x, y)\}$.
(b) For $i \in N'_{yx}$, $r'_i = \{(y, x)\}$.
(c) For $i \in N_d$, $r'_i = \{(x, y), (y, x)\}$.
(d) For $i \in N \setminus (N'_{xy} \cup N'_{yx} \cup N_d)$, $r'_i = \emptyset$.

Consider profile $r'' \in D_2$ which looks like this:
(a) For $i \in N'_{xy}$, $r''_i = \{(y, x)\}$.
(b) For $i \in N'_{yx}$, $r''_i = \{(x, y)\}$.
(c) For $i \in N_d$, $r''_i = \{(x, y), (y, x)\}$.
(d) For $i \in N \setminus (N'_{xy} \cup N'_{yx} \cup N_d)$, $r''_i = \emptyset$.

By assumption, $r', r'' \in D$. Consider bijection $\sigma : X \to X$ such that $\sigma(x) = y$, $\sigma(y) = x$, and, for all $z \in X \setminus \{x, y\}$, $\sigma(z) = z$. For $i \in N$, it is true that $(x, y) \in r'_i \Leftrightarrow (y, x) \in r''_i$, $(y, x) \in r'_i \Leftrightarrow (x, y) \in r''_i$, and, for $a, b \in X$ with $\{a, b\} \neq \{x, y\}$, $(a, b), (\sigma(a), \sigma(b)) \notin r'_i \cup r''_i$. That is, for all $i \in N$ and all $a, b \in X$, $(a, b) \in r'_i \Leftrightarrow (\sigma(a), \sigma(b)) \in r''_i$. Hence, N implies $d_{xy}(r') = d_{yx}(r'')$ and $d_{yx}(r') = d_{xy}(r'')$.

Consider bijection $\pi : N \to N$ such that, for all $t \in \{1, ..., m\}$, $\pi(i_t) = j_t$ and $\pi(j_t) = i_t$, and, for all $i \in N \setminus (N'_{xy} \cup N'_{yx})$, $\pi(i) = i$. By construction, it is true that, for all $i \in N$, $r'_i = r''_{\pi(i)}$. Due to A, it follows that $d_{xy}(r') = d_{xy}(r'')$ and $d_{yx}(r') = d_{yx}(r'')$. Together with the last paragraph, it follows that $d_{xy}(r') = d_{xy}(r'') = d_{yx}(r')$. By PR(a), $d_{xy}(r') = d_{yx}(r') < \infty$. IBI and the fact that $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset}) = \delta$ yield $d_{xy}(r') = d_{yx}(r') = d_{xy}(r^{\emptyset}) = \delta$.

Recognize that r' is identical to r^m in "$\Rightarrow$" from the proof of Theorem 1. Accordingly, one can repeat Steps 3-5 from that proof, employing PR and IIA. That establishes equivalence (4.1), so that ϕ is the δ-RMR.

"$\Leftarrow$": Suppose that ϕ is the δ-RMR on $D = D_F$. Again, it is true that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\}$. Due to Theorem 1, ϕ satisfies IIA and PR.

Consider $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Take distinct $x, y \in X$. For all $i \in N$, $i \in N_{xy}(r) \Leftrightarrow (x, y) \in p_i \Leftrightarrow (x, y) \in p'_{\pi(i)} \Leftrightarrow \pi(i) \in N_{xy}(r')$, implying $m_{xy}(r) = m_{xy}(r')$. Likewise, $m_{yx}(r) = m_{yx}(r')$. It follows that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\} = max\{m_{yx}(r') - m_{xy}(r') + \delta, 0\} = d_{xy}(r')$. Thus, ϕ satisfies A.

Take $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$. Consider distinct $x, y \in X$ and denote $x' = \sigma(x)$ as well as $y' = \sigma(y)$. For all $i \in N$, $i \in N_{xy}(r) \Leftrightarrow (x, y) \in p_i \Leftrightarrow (x', y') \in p'_i \Leftrightarrow i \in N_{x'y'}(r')$, so that $m_{xy}(r) = m_{x'y'}(r')$. Analogously, $m_{yx}(r) = m_{y'x'}(r')$. That implies $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\} = max\{m_{y'x'}(r') - m_{x'y'}(r') + \delta, 0\} = d_{x'y'}(r')$. Hence, ϕ satisfies N.

Consider $r, r' \in D$ and distinct $x, y \in X$ with $d_{xy}(r) = d_{yx}(r)$ and $d_{xy}(r') = d_{yx}(r')$. As shown in the paragraph on IER in "$\Leftarrow$" from the proof of Theorem 1, that implies $m_{xy}(r) = m_{yx}(r)$ and $m_{xy}(r') = m_{yx}(r')$. It follows that $d_{xy}(r) = \max\{m_{yx}(r) - m_{xy}(r) + \delta, 0\} = \delta = \max\{m_{yx}(r') - m_{xy}(r') + \delta, 0\} = d_{xy}(r')$. Consequently, ϕ satisfies IBI.■

4.A.3 Proofs for Section 4.3.2

Theorem 3:
"$\Rightarrow$": Assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, III, PR, and AER.

Define $\lambda = d_{xy}(r)$ for any distinct $x, y \in X$ and any $r \in D$ without strict individual rankings. As argued in the proof of Theorem 1, III and PR of ϕ secure that λ is well-defined, finite, and positive. The following argument will establish that ϕ is the λ-QMR.

Consider $r \in D$ and distinct $x, y \in X$. Again, relabel individuals from N_{xy} and N_{yx} such that $N_{xy} = \{i_1, ..., i_{m_{xy}}\}$ and $N_{yx} = \{j_1, ..., j_{m_{yx}}\}$.

To establish that ϕ is the λ-QMR, the following is to be shown:

$$(x, y) \in \phi(r) \Leftrightarrow max\{m_{xy}, \lambda - 1\} \geq m_{yx} \tag{4.3}$$

Step 1: Construct $r^0 \in D$ as in the proof of Theorem 1. As discussed in Step 1 from that proof, III implies $d_{xy}(r^0) = d_{yx}(r^0) = \lambda$.

Step 2: If $m = 0$, continue with Step 3. If $m > 0$, construct the sequence of profiles

$r^0, ..., r^m$ from the proof of Theorem 1. That is, for each $t \in \{1, ..., m\}$, $r^t_{i_t} = \{(x, y)\}$, $r^t_{j_t} = \{(y, x)\}$, and, for all $k \in N \setminus \{i_t, j_t\}$, $r^t_k = r^{t-1}_k$. Again, for all $t \in \{1, ..., m\}$, $r^t \in D$ holds. By AER of ϕ, it is the case that, for all $t \in \{1, ..., m\}$, $d_{xy}(r^{t-1}) = d_{yx}(r^{t-1}) = max\{\lambda - (t-1), 1\} \Rightarrow d_{xy}(r^t) = d_{yx}(r^t) = max\{\lambda - t, 1\}$. Together with $d_{xy}(r^d) = d_{yx}(r^d) = \lambda$, it follows that $d_{xy}(r^m) = d_{yx}(r^m) = max\{\lambda - m, 1\}$.

Step 3: Again, define $r^h = r|\{x, y\} \in D$. Demonstrate "$\Rightarrow$" of equivalence (4.4) by verifying $m_{yx} > max\{m_{xy}, \lambda - 1\} \Rightarrow (x, y) \notin \phi(r^h)$.

$$(x, y) \in \phi(r^h) \Leftrightarrow max\{m_{xy}, \lambda - 1\} \geq m_{yx} \tag{4.4}$$

Assume $m_{yx} > max\{m_{xy}, \lambda - 1\}$. That is, $m_{xy} = m$. Inductively define the sequence of profiles $r^m, ..., r^{m_{yx}} \in D$ from the proof of Theorem 1. That is, for each $t \in \{m+1, ..., m_{yx}\}$, $r^t_{j_t} = \{(y, x)\}$, and, for all $k \in N \setminus \{j_t\}$, $r^t_k = r^{t-1}_k$. In particular, $r^h = r^{m_{yx}}$. As it is true that $d_{yx}(r^m) = max\{\lambda - m, 1\}$, distinguish two cases.

First, assume $d_{yx}(r^m) = 1$. By PR(b), it follows that $d_{yx}(r^{m+1}) = 0$. The repeated application of PR(c) to the sequence then induces $d_{yx}(r^h) = 0$. That is, $(y, x) \in \phi_p(r^h)$ holds, implying $(x, y) \notin \phi(r^h)$.

Second, assume $d_{yx}(r^m) = \lambda - m > 1$. By PR(b), the PAI for y is reduced by at least one with each element of the sequence, respectively. Since $m_{yx} > \lambda - 1$, it is true that $m_{yx} - m \geq \lambda - m$. Accordingly, there exists $t^* \in \{m+1, ..., m_{yx}\}$ such that $d_{yx}(r^{t^*}) = 0$. The repeated application of PR(c) to the sequence after t^* again leads to $d_{yx}(r^h) = 0$ and thus to $(y, x) \in \phi_p(r^h)$. That implies $(x, y) \notin \phi(r^h)$.

Step 4: Show "$\Leftarrow$" of equivalence (4.4). Assume $max\{m_{xy}, \lambda - 1\} \geq m_{yx}$. Once again, $d_{yx}(r^m) = max\{\lambda - m, 1\}$. Distinguish three cases.

First, assume $m_{xy} = m_{yx} = m$, so that $r^m = r^h$. Because $d_{yx}(r^m) > 0$, it follows that $(y, x) \notin \phi_p(r^h)$ and thus $(x, y) \in \phi(r^h)$.

Second, assume $m_{xy} > m_{yx} = m$. Again, inductively define the sequence of profiles $r^m, ..., r^{m_{xy}} \in D$ such that, for each $t \in \{m+1, ..., m_{xy}\}$, $r^t_{i_t} = \{(x, y)\}$, and, for all $k \in N \setminus \{i_t\}$, $r^t_k = r^{t-1}_k$. In particular, $r^h = r^{m_{xy}}$. The repeated application of PR(d) to the sequence leads to $d_{yx}(r^h) \geq d_{yx}(r^m)$. Combined with the fact that $d_{yx}(r^m) > 0$, that implies $(y, x) \notin \phi_p(r^h)$ and thus $(x, y) \in \phi(r^h)$.

Third, assume $m_{yx} > m_{xy} = m$. By assumption, that implies $\lambda - 1 \geq m_{yx}$. It follows that $\lambda - m > m_{yx} - m > 0$. Accordingly, $d_{yx}(r^m) = \lambda - m$. Assume $(y, x) \in \phi_p(r^h)$. Now, r^h is generated by augmenting r^m with $m_{yx} - m$ strict individual rankings of y over

x. That implies $d_{yx}(r^m) \leq m_{yx} - m < \lambda - m$. This is a contradiction. Consequently, $(y, x) \notin \phi_p(r^h)$ must hold, implying $(x, y) \in \phi(r^h)$.

Step 5: Repeat Step 5 from the proof of Theorem 1 to show that IIA (CIIA) yields $(x, y) \in \phi(r) \Leftrightarrow (x, y) \in \phi(r^h)$. Together with the fact that equivalence (4.4) holds, it follows that equivalence (4.3) is true. SRR ϕ is the λ-QMR.

"$\Leftarrow$": Suppose that ϕ is the λ-QMR on $D = D_F$. The PAI $d_{xy}(r)$ looks like this. If $m_{yx}(r) \geq m_{xy}(r)$, then $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\}$. If $m_{xy}(r) > m_{yx}(r)$, then $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\}$. In any case, the PAI is finite since $|\{i \in N \mid r_i|\{x, y\} = \emptyset\}| = \infty$.

Take $r, r' \in D$ and distinct $x, y \in X$ with $r|\{x, y\} = r'|\{x, y\}$. As shown in the proof of Theorem 1, that implies $m_{xy}(r) = m_{xy}(r')$ and $m_{yx}(r) = m_{yx}(r')$. First, assume $m_{yx}(r) \geq m_{xy}(r)$. It follows that $m_{yx}(r') \geq m_{xy}(r')$ and $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\} = m_{yx}(r') - m_{xy}(r') + max\{\lambda - m_{yx}(r'), 1\} = d_{xy}(r')$. Second, assume $m_{xy}(r) > m_{yx}(r)$. That implies $m_{xy}(r') > m_{yx}(r')$ and $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\} = max\{\lambda - m_{xy}(r'), 0\} = d_{xy}(r')$. Hence, ϕ satisfies IIA.

Consider $r, r' \in D$ such that, for all $x, y \in X$ and all $i \in N$, $(x, y) \in r_i \Rightarrow (x, y) \in s_i$ and $(x, y) \in r'_i \Rightarrow (x, y) \in s'_i$. Take $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$. It follows that $m_{xy}(r) = m_{yx}(r) = m_{x'y'}(r') = m_{y'x'}(r') = 0$. That in turn implies $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\} = \lambda = m_{y'x'}(r') - m_{x'y'}(r') + max\{\lambda - m_{y'x'}(r'), 1\} = d_{x'y'}(r')$. Therefore, ϕ satisfies III.

Take $r \in D$ and distinct $x, y \in X$. As discussed, $d_{xy}(r) < \infty$ holds, so that ϕ satisfies PR(a). Consider $r' \in D$ and $i \in N$ such that $(x, y), (y, x) \notin r_i$, $r'_i = r_i \cup \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$. As discussed in the proof of Theorem 1, that implies $m_{xy}(r') = m_{xy}(r) + 1$ and $m_{yx}(r') = m_{yx}(r)$.

For PR(b), assume $d_{xy}(r) > 0$. First, assume $m_{xy}(r) = m_{yx}(r)$. That implies $m_{xy}(r') > m_{yx}(r')$. Accordingly, $d_{xy}(r) = max\{\lambda - m_{xy}(r), 1\}$ and $d_{xy}(r') = max\{\lambda - m_{xy}(r'), 0\} = max\{\lambda - m_{xy}(r) - 1, 0\}$. If $\lambda - m_{xy}(r) \leq 1$, then $\lambda - m_{xy}(r) - 1 \leq 0$ and thus $d_{xy}(r') = 0 < 1 = d_{xy}(r)$. If $\lambda - m_{xy}(r) > 1$, then $\lambda - m_{xy}(r) - 1 > 0$ and in turn $d_{xy}(r') = \lambda - m_{xy}(r) - 1 < \lambda - m_{xy}(r) = d_{xy}(r)$.

Second, assume $m_{xy}(r) > m_{yx}(r)$, implying $m_{xy}(r') > m_{yx}(r')$. Since $d_{xy}(r) > 0$, it follows that $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\} = \lambda - m_{xy}(r)$ and in turn $\lambda - m_{xy}(r') = \lambda - m_{xy}(r) - 1 \geq 0$. Taken together, that implies $d_{xy}(r') = max\{\lambda - m_{xy}(r'), 0\} = \lambda - m_{xy}(r') < \lambda - m_{xy}(r) = d_{xy}(r)$. Third, assume $m_{yx}(r) > m_{xy}(r)$, implying $m_{yx}(r') \geq$

$m_{xy}(r')$. It follows that $d_{xy}(r') = m_{yx}(r') - m_{xy}(r') + max\{\lambda - m_{yx}(r'), 1\} < m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\} = d_{xy}(r)$. In any case, $d_{xy}(r') < d_{xy}(r)$.

For PR(c), assume $d_{xy}(r) = 0$, implying $(x, y) \in \phi_p(r)$. On the one hand, it follows that $m_{xy}(r') > m_{xy}(r) > m_{yx}(r) = m_{yx}(r')$. On the other hand, $m_{xy}(r') > m_{xy}(r) \geq \lambda$. Taken together, that implies $d_{xy}(r') = max\{\lambda - m_{xy}(r'), 0\} = 0$.

For PR(d), assume $m_{xy}(r) \geq m_{yx}(r)$ first. That yields $m_{xy}(r') > m_{yx}(r')$. If $max\{\lambda - m_{xy}(r), 1\} = 1$, then $d_{yx}(r) = m_{xy}(r) - m_{yx}(r) + max\{\lambda - m_{xy}(r), 1\} = m_{xy}(r) - m_{yx}(r) + 1 \leq m_{xy}(r') - m_{yx}(r') + max\{\lambda - m_{xy}(r'), 1\} = d_{yx}(r')$. If $max\{\lambda - m_{xy}(r), 1\} = \lambda - m_{xy}(r) > 1$, then $d_{yx}(r) = m_{xy}(r) - m_{yx}(r) + max\{\lambda - m_{xy}(r), 1\} = m_{xy}(r) - m_{yx}(r) + \lambda - m_{xy}(r) = \lambda - m_{yx}(r') = m_{xy}(r') - m_{yx}(r') + \lambda - m_{xy}(r') \leq m_{xy}(r') - m_{yx}(r') + max\{\lambda - m_{xy}(r'), 1\} = d_{yx}(r')$.

Second, assume $m_{xy}(r) < m_{yx}(r)$, implying $m_{xy}(r') \leq m_{yx}(r')$. If $d_{yx}(r) = 0$, then automatically $d_{yx}(r) \leq d_{yx}(r')$. Assume $d_{yx}(r) > 0$. If $m_{xy}(r') = m_{yx}(r')$, then $d_{yx}(r) = max\{\lambda - m_{yx}(r), 0\} = \lambda - m_{yx}(r) = \lambda - m_{xy}(r') \leq m_{xy}(r') - m_{yx}(r') + max\{\lambda - m_{xy}(r'), 1\} = d_{yx}(r')$. If $m_{xy}(r') < m_{yx}(r')$, then $d_{yx}(r) = max\{\lambda - m_{yx}(r), 0\} = \lambda - m_{yx}(r') = max\{\lambda - m_{yx}(r'), 0\} = d_{yx}(r')$. Consequently, ϕ satisfies PR.

Consider $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k$. Assume $d_{xy}(r) = d_{yx}(r) > 0$. Assume $m_{xy}(r) > m_{yx}(r)$. That implies $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\} = \lambda - m_{xy}(r) < m_{xy}(r) - m_{yx}(r) + max\{\lambda - m_{xy}(r), 1\} = d_{yx}(r)$. This is a contradiction. Since $m_{xy}(r) < m_{yx}(r)$ leads to an analogous contradiction, it must be true that $m_{xy}(r) = m_{yx}(r)$.

As discussed in the proof of Theorem 1, the structure of r and r' implies $m_{xy}(r') = m_{xy}(r) + 1$, $m_{yx}(r') = m_{yx}(r) + 1$, and thus $m_{xy}(r') = m_{yx}(r')$. It is true that $d_{xy}(r) = d_{yx}(r) = max\{\lambda - m_{yx}(r), 1\}$ and $d_{xy}(r') = d_{yx}(r') = max\{\lambda - m_{yx}(r'), 1\}$. Assume $d_{xy}(r) = d_{yx}(r) > 1$, that is, $d_{xy}(r) = d_{yx}(r) = \lambda - m_{yx}(r) > 1$. It follows that $\lambda - m_{yx}(r') \geq 1$, so that $d_{xy}(r') = d_{yx}(r') = \lambda - m_{yx}(r') = \lambda - m_{yx}(r) - 1 = d_{xy}(r) - 1$. Assume $d_{xy}(r) = d_{yx}(r) = 1$. That implies $\lambda - m_{yx}(r) \leq 1$ and thus $\lambda - m_{yx}(r') \leq 1$. Hence, $d_{xy}(r') = d_{yx}(r') = 1$. That is, ϕ satisfies AER.■

Theorem 4:
"⇒": Assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, A, N, PR, and ABI. Define $\lambda = d_{xy}(r^{\emptyset})$, where $r^{\emptyset} \in D$ and $x, y \in X$ are any distinct alternatives. As shown in the proof of Theorem 2, N and PR make sure that λ is well-defined, finite, and positive.

Show that ϕ is the λ-QMR. For that purpose, consider $r \in D$ and distinct $x, y \in X$ and establish equivalence (4.3) from the proof of Theorem 3. Reconsider profile r^m from the proof of Theorem 1. As argued in the proof of Theorem 2, A and N imply $d_{xy}(r^m) = d_{yx}(r^m)$. Since $m = |\{i \in N \mid (x, y) \in p_i^m\}| = |\{i \in N \mid (y, x) \in p_i^m\}|$, ABI induces $d_{xy}(r^m) = d_{yx}(r^m) = max\{\lambda - m, 1\}$. It is accordingly possible to repeat Steps 3-5 from the proof of Theorem 3, using PR and IIA. That establishes ϕ as the λ-QMR.

"$\Leftarrow$": Suppose that ϕ is the λ-QMR on $D = D_F$. Again, PAI $d_{xy}(r)$ takes the following form. If $m_{yx}(r) \geq m_{xy}(r)$, then $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\}$. If $m_{xy}(r) > m_{yx}(r)$, then $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\}$. By Theorem 3, ϕ satisfies IIA and PR.

Take $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Consider distinct $x, y \in X$. As discussed in the proof of Theorem 2, that implies $m_{xy}(r) = m_{xy}(r')$ and $m_{yx}(r) = m_{yx}(r')$. If $m_{yx}(r) \geq m_{xy}(r)$, then $m_{yx}(r') \geq m_{xy}(r')$. It follows that $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\} = m_{yx}(r') - m_{xy}(r') + max\{\lambda - m_{yx}(r'), 1\} = d_{xy}(r')$. If $m_{xy}(r) > m_{yx}(r)$, then $m_{xy}(r') > m_{yx}(r')$. That yields $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\} = max\{\lambda - m_{xy}(r'), 0\} = d_{xy}(r')$. In any case, $d_{xy}(r) = d_{xy}(r')$ holds. Therefore, ϕ satisfies A.

Consider $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$. Take distinct $x, y \in X$ and denote $x' = \sigma(x)$ as well as $y' = \sigma(y)$. The proof of Theorem 2 yields $m_{xy}(r) = m_{x'y'}(r')$ and $m_{yx}(r) = m_{y'x'}(r')$. If $m_{yx}(r) \geq m_{xy}(r)$, then $m_{y'x'}(r') \geq m_{x'y'}(r')$. That implies $d_{xy}(r) = m_{yx}(r) - m_{xy}(r) + max\{\lambda - m_{yx}(r), 1\} = m_{y'x'}(r') - m_{x'y'}(r') + max\{\lambda - m_{y'x'}(r'), 1\} = d_{x'y'}(r')$. If $m_{xy}(r) > m_{yx}(r)$, then $m_{x'y'}(r') > m_{y'x'}(r')$. It follows that $d_{xy}(r) = max\{\lambda - m_{xy}(r), 0\} = max\{\lambda - m_{x'y'}(r'), 0\} = d_{x'y'}(r')$. Anyway, $d_{xy}(r) = d_{x'y'}(r')$ holds. Thus, ϕ satisfies N.

Take $r, r^{\emptyset} \in D$ and distinct $x, y \in X$ such that $d_{xy}(r) = d_{yx}(r)$. Due to the paragraph on AER in "$\Leftarrow$" from the proof of Theorem 3, it follows that $m_{xy}(r) = m_{yx}(r)$. This yields $d_{xy}(r) = max\{\lambda - m_{xy}(r), 1\}$. It is also true that $m_{xy}(r^{\emptyset}) = m_{yx}(r^{\emptyset}) = 0$, implying $d_{xy}(r^{\emptyset}) = \lambda$. Taken together, that induces $d_{xy}(r) = max\{d_{xy}(r^{\emptyset}) - m_{xy}(r), 1\}$. Consequently, ϕ satisfies ABI.■

4.A.4 Proofs for Section 4.3.3

Corollary 1:
For "$\Rightarrow$", assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, III, PR, IER, and AER. By

Theorems 1 and 3, there exist $\delta, \lambda \in \mathbb{N}$ such that ϕ is the δ-RMR and the λ-QMR. Consider $r \in D$ with distinct $x, y \in X$ and $N', N'' \subseteq N$, where $|N'| = |N''| = \lambda$. For all $i \in N'$, $r_i = \{(x, y)\}$, for all $i \in N''$, $r_i = \{(y, x)\}$, and, for all $i \in N \setminus (N' \cup N'')$, $r_i = \emptyset$. That is, there are λ individuals known to rank x higher than y and the same number of individuals known to rank y higher than x at $r \in D_2$. Since ϕ is the λ-QMR, it is the case that $d_{xy}(r) = d_{yx}(r) = 1$. As ϕ is the δ-RMR, it is likewise true that $d_{xy}(r) = d_{yx}(r) = \delta$. It follows that $\delta = 1$ and in turn $\lambda = 1$. SRR ϕ is the SMR. For "$\Leftarrow$", consult Theorems 1 and 3 to see that SMR ϕ satisfies IIA, III, PR, IER, and AER on $D = D_F$.■

Corollary 2:
For "$\Rightarrow$", assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, A, N, PR, IBI, and ABI. Due to Theorems 2 and 4, there exist $\delta, \lambda \in \mathbb{N}$ such that ϕ is the δ-RMR and the λ-QMR. As shown in the proof of Corollary 1, that yields $\delta = \lambda = 1$, so that ϕ is the SMR. For "$\Leftarrow$", recognize that SMR ϕ satisfies IIA, A, N, PR, IBI, and ABI on $D = D_F$ due to Theorems 1, 2, and 4.■

Lemma 4:
(a) Assume SRR ϕ on one-additive D satisfies MPR. Consider $r \in D$ and distinct $x, y \in X$ with $d_{xy}(r) < d_{yx}(r)$. Assume $d_{xy}(r) > 0$. That implies $d_{yx}(r) > 1$. By assumption, there exists $r' \in D$ with $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r'_k = r_k$, $(x, y), (y, x) \notin r_i$, and $r'_i = r_i \cup \{(y, x)\}$. By MPR of ϕ, it follows that $d_{yx}(r') = 0$. But that yields $d_{yx}(r) \leq 1$, contradicting $d_{yx}(r) > 1$. Therefore, $d_{xy}(r) = 0$ must hold, so that ϕ satisfies NIL.

(b) Assume SRR ϕ on $D \subseteq D_F$ satisfies PR and NIL. Take $r, r' \in D$, distinct $x, y \in X$, and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r'_k = r_k$, $(x, y), (y, x) \notin r_i$ and $r'_i = r_i \cup \{(x, y)\}$. Assume $d_{yx}(r) > 0$. First, suppose $d_{xy}(r) = 0$. Due to PR(c), that induces $d_{xy}(r') = 0$. Second, assume $d_{xy}(r) > 0$. By NIL, that implies $d_{xy}(r) = d_{yx}(r)$. PR(a) secures $d_{xy}(r) < \infty$. The combination of PR(b) and PR(d) yields $d_{xy}(r') < d_{xy}(r) = d_{yx}(r) \leq d_{yx}(r')$. Due to NIL, it follows that $d_{xy}(r') = 0$. Accordingly, ϕ satisfies MPR.■

Proposition 1:
"$\Rightarrow$": Assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, III, ER, and MPR. Take $r \in D$ and distinct $x, y \in X$. Once again, rename individuals such that $N_{xy} = \{i_1, ..., i_{m_{xy}}\}$ and $N_{yx} = \{j_1, ..., j_{m_{yx}}\}$. To show that ϕ is the SMR, equivalence (4.1) from the proof of Theorem 1 with $\delta = 1$ is to be demonstrated.

Reconsider $r^0, r^h \in D$ from the proof of Theorem 1. Because r^0 only contains equal individual rankings of $\{x,y\}$, III implies $d_{xy}(r^0)=d_{yx}(r^0)$.

Reconsider the sequence of profiles $r^0,...,r^m \in D$ from the proof of Theorem 1. For each $t\in\{1,...,m\}$, $r^t_{i_t}=\{(x,y)\}$, $r^t_{j_t}=\{(y,x)\}$, and, for all $k\in N\setminus\{i_t,j_t\}$, $r^t_k=r^{t-1}_k$. ER of ϕ implies that, for all $t\in\{1,...,m\}$, $d_{xy}(r^{t-1})=d_{yx}(r^{t-1})\Rightarrow d_{xy}(r^t)=d_{yx}(r^t)$. Together with $d_{xy}(r^0)=d_{yx}(r^0)$, that yields $d_{xy}(r^m)=d_{yx}(r^m)>0$.

To show "$\Rightarrow$" of equivalence (4.2) from the proof of Theorem 1 with $\delta=1$, assume $m_{yx}>m_{xy}=m$. Reconsider the sequence of profiles $r^m,...,r^{m_{yx}}\in D$ such that, for each $t\in\{m+1,...,m_{yx}\}$, $r^t_{j_t}=\{(y,x)\}$, and, for all $k\in N\setminus\{j_t\}$, $r^t_k=r^{t-1}_k$. By MPR, it follows that, for each $t\in\{m+1,...,m_{yx}\}$, $d_{xy}(r^{t-1})>0$ implies $d_{yx}(r^t)=0$ and thus $d_{xy}(r^t)>0$. Combined with $d_{xy}(r^m)>0$, this yields $d_{yx}(r^h)=d_{yx}(r^{m_{yx}})=0$ and $(x,y)\notin\phi(r^h)$.

For "$\Leftarrow$" of equivalence (4.2), assume $m_{xy}\geq m_{yx}$. If $m_{xy}=m_{yx}=m$, then $r^h=r^m$. Due to $d_{yx}(r^m)>0$, that implies $(x,y)\in\phi(r^h)$. Assume $m_{xy}>m_{yx}=m$. Construct $r^m,...,r^{m_{xy}}\in D$ such that, for each $t\in\{m+1,...,m_{xy}\}$, $r^t_{i_t}=\{(x,y)\}$, and, for all $k\in N\setminus\{i_t\}$, $r^t_k=r^{t-1}_k$. Analogously, MPR induces that, for each $t\in\{m+1,...,m_{xy}\}$, $d_{yx}(r^{t-1})>0$ implies $d_{xy}(r^t)=0$ and $d_{yx}(r^t)>0$. Due to $d_{yx}(r^m)>0$, it follows that $d_{xy}(r^h)=d_{xy}(r^{m_{xy}})=0$ and hence $(x,y)\in\phi(r^h)$.

As shown in the proof of Theorem 1, equivalence (4.2) and IIA (CIIA) imply equivalence (4.1). In the present case, $\delta=1$. Consequently, ϕ is the SMR.

"$\Leftarrow$": Assume that ϕ is the SMR on $D=D_F$. By Theorem 1, ϕ satisfies IIA, III, and IER. It thus satisfies ER in particular. Consider $r,r'\in D$, distinct $x,y\in X$, and $i\in N$ such that, for all $k\in N\setminus\{i\}$, $r'_k=r_k$, $(x,y),(y,x)\notin r_i$ as well as $r'_i=r_i\cup\{(x,y)\}$. Assume $d_{yx}(r)>0$. That implies $(x,y)\in\phi(r)$ and in turn $m_{xy}(r)\geq m_{yx}(r)$. Since $m_{xy}(r')=m_{xy}(r)+1>m_{yx}(r)=m_{yx}(r')$, it follows that $(x,y)\in\phi_p(r')$ and $d_{xy}(r')=0$. SMR ϕ satisfies MPR.■

Theorem 5:
For "$\Rightarrow$", assume SRR ϕ on $D_2\subseteq D\subseteq D_F$ satisfies IIA, III, PR, ER, and NIL. By Lemma 4(b), ϕ satisfies MPR. Accordingly, Proposition 1 implies that ϕ is the SMR. For "$\Leftarrow$", assume that ϕ is the SMR on $D=D_F$. Due to Theorem 1 and Proposition 1, ϕ satisfies IIA, III, PR, ER, and MPR. Since D is one-additive, ϕ satisfies NIL by Lemma 4(a).■

Proposition 2:
"$\Rightarrow$": Assume SRR ϕ on $D_2\subseteq D\subseteq D_F$ satisfies IIA, A, N, and MPR. Consider $r\in D$

and distinct $x, y \in X$. Reconsider $r^m \in D$ from the proof of Theorem 1. Consult the proof of Theorem 2 to show that A and N imply $d_{xy}(r^m) = d_{yx}(r^m) > 0$. From there, follow the proof of Proposition 1 to establish equivalence (4.1) from the proof of Theorem 1 with $\delta = 1$. This requires MPR and IIA. Accordingly, ϕ is the SMR.

"$\Leftarrow$": Assume that ϕ is the SMR on $D = D_F$. Due to Theorems 1 and 2, as well as Proposition 1, ϕ satisfies IIA, A, N, and MPR.■

Theorem 6:
For "$\Rightarrow$", assume SRR ϕ on $D_2 \subseteq D \subseteq D_F$ satisfies IIA, A, N, PR, and NIL. Again, ϕ satisfies MPR by Lemma 4(b). Due to Proposition 2, it follows that ϕ is the SMR. On the other hand, SMR ϕ on $D = D_F$ satisfies IIA, A, N, PR, and NIL due to Theorems 1, 2, and 5. That establishes "$\Leftarrow$".■

4.A.5 Independence of Conditions from Characterizations

For the remainder of the Appendix, assume $D = D_F$. For $r \in S(X)^N$, distinct $x, y \in X$, and $i \in N$, define $\eta_{xy}(p_i) = 1$ if $(x, y) \in p_i$ and $\eta_{xy}(p_i) = 0$ if $(x, y) \notin p_i$. Define $\tau_{xy}(r)$ in the following way. Set $\tau_{xy}(r) = 1$ if there exist $z \in X \setminus \{x, y\}$ and $k \in N$ such that $(x, z) \in p_k$. Set $\tau_{xy}(r) = 0$ otherwise. Employ the conventions $\infty + a = \infty$ and $max\{\infty, a\} = \infty$ for each $a \in \mathbb{R}$. Take the following four SRRs.

SRR ϕ on $D = D_F$ is the **Universal Social Indifference Rule (USIR)** if, for all $r \in D$ and all distinct $x, y \in X$, $(x, y) \in \phi(r)$.

SRR ϕ on $D = D_F$ is a **Tiebreaking Majority Rule (TMR)** with linear tiebreaking ordering $r_T \in S(X)$ if, for all $r \in D$ and all distinct $x, y \in X$ with $(x, y) \in p_T$,
$|\{i \in N \mid (x, y) \in p_i\}| \geq |\{i \in N \mid (y, x) \in p_i\}| \Rightarrow (x, y) \in \phi_p(r)$,
$|\{i \in N \mid (x, y) \in p_i\}| < |\{i \in N \mid (y, x) \in p_i\}| \Rightarrow (y, x) \in \phi_p(r)$.

SRR ϕ on $D = D_F$ is a **Distinguished Individual Majority Rule (DIMR)** if there exists $k \in N$ such that, for all $r \in D$ and all distinct $x, y \in X$,
$(x, y) \in \phi(r) \Leftrightarrow \sum_{i \in N \setminus \{k\}} \eta_{xy}(p_i) + \frac{1}{2} \cdot \eta_{xy}(p_k) \geq \sum_{i \in N \setminus \{k\}} \eta_{yx}(p_i) + \frac{1}{2} \cdot \eta_{yx}(p_k)$.

SRR ϕ on $D = D_F$ is the **External Majority Rule (EMR)** if, for all $r \in D$ and all distinct $x, y \in X$,
$(x, y) \in \phi(r) \Leftrightarrow |\{i \in N \mid (x, y) \in p_i\}| + \tau_{xy}(r) \geq |\{i \in N \mid (y, x) \in p_i\}| + \tau_{yx}(r)$.

Table 4.2 is an extension of Table 4.1 and summarizes which SRRs satisfy which conditions on $D = D_F$, respectively. The properties of USIR, TMR, DIMR, and EMR are

	IIA	A	N	III	PR	MPR	NIL	ER	IER	IBI	AER	ABI
SMR	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
PRMR	✓	✓	✓	✓	✓			✓	✓	✓		
PQMR	✓	✓	✓	✓	✓			✓			✓	✓
USIR	✓	✓	✓	✓			✓	✓	✓	✓	✓	✓
TMR	✓	✓			✓	✓	✓	✓	✓	✓	✓	✓
DIMR	✓		✓	✓	✓	✓	✓			✓		✓
EMR		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 4.2: SRRs and PAI Conditions

	T1	T2	T3	T4	T5	T6	C1	C2	P1	P2
PRMR			×	×	×	×	×	×		
PQMR	×	×					×	×		
USIR	×	×	×	×	×	×	×	×	×	×
TMR	×	×	×	×	×	×	×	×	×	×
DIMR		×		×	×	×		×	×	×
EMR	×	×	×	×	×	×	×	×	×	×

Table 4.3: Independence of Conditions from Characterizations

verified in Propositions A.5.1-A.5.4, respectively. Table 4.3 shows which combinations of SRRs establish the independence of the conditions from the respective results. For example, the conditions that characterize the class of RMRs in Theorem 1 are independent as a PQMR violates IER, the USIR violates PR, a TMR violates III, and the EMR violates IIA. They satisfy the remaining three conditions, respectively.

Proposition A.5.1:
USIR ϕ on $D = D_F$ satisfies IIA, A, N, III, NIL, ER, IER, IBI, AER, and ABI, but violates PR and MPR.
Proof:
For $r \in D$ and distinct $x, y \in X$, the PAI of x over y is $d_{xy}(r) = \infty$. No matter how many strict individual rankings of x over y are added, x will never socially rank higher than y under USIR ϕ.

Since $|\{i \in N \mid r_i|\{x,y\} = \emptyset\}| = \infty$ holds for $r \in D$ and distinct $x, y \in X$, PR(a) would demand $d_{xy}(r) < \infty$. Thus, ϕ violates PR. Consider $r^{\emptyset}, r \in D$, distinct $x, y \in X$, and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r_k = \emptyset$ and $r_i = \{(x, y)\}$. It is true that $d_{yx}(r^{\emptyset}) = \infty > 0$. MPR would demand $d_{xy}(r) = 0$. In contrast, ϕ violates the condition because $d_{xy}(r) = \infty$.

USIR ϕ satisfies IIA, A, N, III, IER (and thus ER), IBI, and ABI because their respective conclusions are always satisfied. That will be true in particular if their antecedents hold, respectively. For ABI, consider $r^{\emptyset}, r \in D$ and distinct $x, y \in X$. It is true that $d_{xy}(r) = \infty = max\{\infty - m_{xy}(r), 1\} = max\{d_{xy}(r^{\emptyset}) - m_{xy}(r), 1\}$. USIR ϕ satisfies AER. To see that, take $r, r' \in D$ and distinct $x, y \in X$. On the one hand, antecedent (b) never holds. On the other hand, the conclusion for antecedent (a) always holds because $d_{xy}(r') = d_{yx}(r') = \infty = \infty - 1 = d_{xy}(r) - 1$. The antecedent of NIL is never satisfied. There are no $r \in D$ and distinct $x, y \in X$ with $d_{xy}(r) < d_{yx}(r)$. Hence, ϕ satisfies the condition.■

Proposition A.5.2:
TMR ϕ with linear tiebreaking ordering $r_T \in S(X)$ on $D = D_F$ satisfies IIA, A, PR, MPR, NIL, ER, IER, IBI, AER, and ABI, but violates N and III.
Proof:
For $r \in D$ and distinct $x, y \in X$ with $(x, y) \in p_T$, it is the case that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r), 0\}$ and $d_{yx}(r) = max\{m_{xy}(r) - m_{yx}(r) + 1, 0\}$.

Consider $r^{\emptyset} \in D$ and distinct $x, y \in X$ with $(x, y) \in p_T$. As discussed, both N and III would demand $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset})$, respectively. But in fact, $d_{xy}(r^{\emptyset}) = 0 < 1 = d_{yx}(r^{\emptyset})$. Consequently, ϕ violates N and III.

Take $r, r' \in D$ and distinct $x, y \in X$ with $(x, y) \in p_T$ such that $r|\{x, y\} = r'|\{x, y\}$. As discussed in the proof of Theorem 1, that implies $m_{xy}(r) = m_{xy}(r')$ and $m_{yx}(r) = m_{yx}(r')$. It follows that $d_{xy}(r) = max\{m_{yx}(r) - m_{xy}(r), 0\} = max\{m_{yx}(r') - m_{xy}(r'), 0\} = d_{xy}(r')$. Likewise, $d_{yx}(r) = max\{m_{xy}(r) - m_{yx}(r) + 1, 0\} = max\{m_{xy}(r') - m_{yx}(r') + 1, 0\} = d_{yx}(r')$. Accordingly, ϕ satisfies IIA.

Consider $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Take distinct $x, y \in X$ with $(x, y) \in p_T$. As discussed in the proof of Theorem 2, it is the case that $m_{xy}(r) = m_{xy}(r')$ and $m_{yx}(r) = m_{yx}(r')$. As shown in the last paragraph, that yields $d_{xy}(r) = d_{xy}(r')$ and $d_{yx}(r) = d_{yx}(r')$. Therefore, ϕ satisfies A.

Take $r \in D$ and distinct $x, y \in X$. TMR ϕ satisfies PR(a) because $d_{xy}(r) < \infty$. Consider $r' \in D$ and $i \in N$ such that $(x, y), (y, x) \notin r_i$, $r'_i = r_i \cup \{(x, y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$. It follows that $m_{xy}(r') > m_{xy}(r)$ and $m_{yx}(r') = m_{yx}(r)$.

Assume $d_{xy}(r) > 0$. If $d_{xy}(r') = 0$, then $d_{xy}(r') < d_{xy}(r)$. Assume $d_{xy}(r') > 0$. If $(x, y) \in p_T$, then $d_{xy}(r') = m_{yx}(r') - m_{xy}(r') < m_{yx}(r) - m_{xy}(r) = d_{xy}(r)$. If $(y, x) \in p_T$, then $d_{xy}(r') = m_{yx}(r') - m_{xy}(r') + 1 < m_{yx}(r) - m_{xy}(r) + 1 = d_{xy}(r)$. This verifies PR(b) of ϕ. Assume $d_{xy}(r) = 0$. That implies $m_{xy}(r') > m_{xy}(r) \geq m_{yx}(r) = m_{yx}(r')$. It follows that $(x, y) \in \phi_p(r')$ and $d_{xy}(r') = 0$. This is true no matter whether $(x, y) \in p_T$ or $(y, x) \in p_T$ holds. Hence, ϕ satisfies PR(c).

If $d_{yx}(r) = 0$, then $d_{yx}(r) \leq d_{yx}(r')$. Assume $d_{yx}(r) > 0$. If $(x, y) \in p_T$, then $0 < d_{yx}(r) = m_{xy}(r) - m_{yx}(r) + 1 < m_{xy}(r') - m_{yx}(r') + 1 = d_{yx}(r')$. If $(y, x) \in p_T$, then $0 < d_{yx}(r) = m_{xy}(r) - m_{yx}(r) < m_{xy}(r') - m_{yx}(r') = d_{yx}(r')$. Because (d) also holds, ϕ satisfies PR.

Again, assume $d_{yx}(r) > 0$. It follows that $(x, y) \in \phi_p(r)$ and $m_{xy}(r) \geq m_{yx}(r)$. That yields $m_{xy}(r') > m_{xy}(r) \geq m_{yx}(r) = m_{yx}(r')$ and thus $d_{xy}(r') = 0$. TMR ϕ satisfies MPR and, by Lemma 4(a), NIL.

Each of the antecedents of ER, IER, IBI, AER, and ABI presupposes that there exist $r \in D$ and distinct $x, y \in X$ with $d_{xy}(r) = d_{yx}(r)$, respectively. That implies $(x, y) \in \phi_s(r)$. But there are no equal social rankings under ϕ, so that the antecedents of the five conditions never hold, respectively. Hence, ϕ satisfies these conditions.■

Proposition A.5.3:
DIMR ϕ with distinguished individual $k \in N$ on $D = D_F$ satisfies IIA, N, III, PR, MPR, NIL, IBI, and ABI, but violates A, ER, IER, and AER.
Proof:
For $r \in D$ and distinct $x, y \in X$, it is the case that $d_{xy}(r) = max\{\lfloor \sum_{i \in N \setminus \{k\}} \eta_{yx}(p_i) + \frac{1}{2} \cdot \eta_{yx}(p_k) - \sum_{i \in N \setminus \{k\}} \eta_{xy}(p_i) - \frac{1}{2} \cdot \eta_{xy}(p_k) \rfloor + 1, 0\}$.[23]

Consider distinct $x, y \in X$ and $r^{\emptyset}, r \in D$ such that $r_k = \{(x, y)\}$, $r_j = \{(y, x)\}$, and, for all $i \in N \setminus \{k, j\}$, $r_i = \emptyset$. Since $\sum_{i \in N \setminus \{k\}} \eta_{xy}(p_i^{\emptyset}) + \frac{1}{2} \cdot \eta_{xy}(p_k^{\emptyset}) = 0 = \sum_{i \in N \setminus \{k\}} \eta_{yx}(p_i^{\emptyset}) + \frac{1}{2} \cdot \eta_{yx}(p_k^{\emptyset})$, it follows that $d_{xy}(r^{\emptyset}) = 1 = d_{yx}(r^{\emptyset})$. ER would hence require $d_{xy}(r) = d_{yx}(r)$. However, $\sum_{i \in N \setminus \{k\}} \eta_{xy}(p_i) + \frac{1}{2} \cdot \eta_{xy}(p_k) = \frac{1}{2} < 1 = \sum_{i \in N \setminus \{k\}} \eta_{yx}(p_i) + \frac{1}{2} \cdot \eta_{yx}(p_k)$. It is true that $(y, x) \in \phi_p(r)$ and $d_{xy}(r) = 1 > 0 = d_{yx}(r)$. Consequently, ϕ violates ER. In particular, it violates IER and AER.

In addition to r, take $r' \in D$ such that $r'_k = \{(y, x)\}$, $r'_j = \{(x, y)\}$, and, for all $i \in N \setminus \{k, j\}$, $r'_i = \emptyset$. Construct bijection $\pi : N \to N$ such that $\pi(k) = j$, $\pi(j) = k$, and, for all $i \in N \setminus \{k, j\}$, $\pi(i) = i$. For all $i \in N$, it is the case that $r_i = r'_{\pi(i)}$.

[23] For $a \in \mathbb{R}$, $\lfloor a \rfloor$ is the largest $b \in \mathbb{Z}$ with $b \leq a$.

A would impose $d_{xy}(r) = d_{xy}(r')$. But $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) + \frac{1}{2} \cdot \eta_{xy}(p'_k) = 1 > \frac{1}{2} = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2} \cdot \eta_{yx}(p'_k)$. That is, $d_{xy}(r) > 0 = d_{xy}(r')$. Therefore, ϕ violates A.

Consider $r, r' \in D$ and distinct $x, y \in X$ such that $r|\{x,y\} = r'|\{x,y\}$. For all $i \in N$, $\eta_{xy}(p_i) = \eta_{xy}(p'_i)$ and $\eta_{yx}(p_i) = \eta_{yx}(p'_i)$. That yields $d_{xy}(r) = max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot\eta_{yx}(p_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) - \frac{1}{2}\cdot\eta_{xy}(p_k)\rfloor + 1, 0\} = max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2}\cdot\eta_{yx}(p'_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) - \frac{1}{2} \cdot \eta_{xy}(p'_k)\rfloor + 1, 0\} = d_{xy}(r')$. Accordingly, ϕ satisfies IIA.

Take $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$, $(x, y) \in r_i \Leftrightarrow (\sigma(x), \sigma(y)) \in r'_i$. Consider distinct $x, y \in X$ and denote $x' = \sigma(x)$ and $y' = \sigma(y)$. For all $i \in N$, $\eta_{xy}(p_i) = \eta_{x'y'}(p'_i)$ and $\eta_{yx}(p_i) = \eta_{y'x'}(p'_i)$. That induces $d_{xy}(r) = max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2} \cdot \eta_{yx}(p_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) - \frac{1}{2} \cdot \eta_{xy}(p_k)\rfloor + 1, 0\} = max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{y'x'}(p'_i) + \frac{1}{2}\cdot\eta_{y'x'}(p'_k) - \sum_{i\in N\setminus\{k\}} \eta_{x'y'}(p'_i) - \frac{1}{2}\cdot\eta_{x'y'}(p'_k)\rfloor + 1, 0\} = d_{x'y'}(r')$. Thus, ϕ satisfies N.

Consider $r, r' \in D$ such that, for all $x, y \in X$ and all $i \in N$, $(x, y) \in r_i \Rightarrow (x, y) \in s_i$ and $(x, y) \in r'_i \Rightarrow (x, y) \in s'_i$. Take $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$. As neither r nor r' contains strict individual rankings, it is the case that $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + \frac{1}{2} \cdot \eta_{xy}(p_k) = 0 = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2} \cdot \eta_{yx}(p_k)$ and $\sum_{i\in N\setminus\{k\}} \eta_{x'y'}(p'_i) + \frac{1}{2} \cdot \eta_{x'y'}(p'_k) = 0 = \sum_{i\in N\setminus\{k\}} \eta_{y'x'}(p'_i) + \frac{1}{2} \cdot \eta_{y'x'}(p'_k)$. That is, $d_{xy}(r) = 1 = d_{x'y'}(r')$. Hence, ϕ satisfies III.

Take $r \in D$ and distinct $x, y \in X$. Due to $d_{xy}(r) < \infty$, ϕ satisfies PR(a). Consider $r' \in D$ and $j \in N$ such that $(x, y), (y, x) \notin r_j$, $r'_j = r_j \cup \{(x, y)\}$, and, for all $i \in N \setminus \{j\}$, $r'_i = r_i$. For each $i \in N$, $\eta_{xy}(p'_i) \geq \eta_{xy}(p_i)$ and $\eta_{yx}(p'_i) = \eta_{yx}(p_i)$. Moreover, $\eta_{xy}(p'_j) = 1 > 0 = \eta_{xy}(p_j)$.

Assume $d_{xy}(r) > 0$. If $d_{xy}(r') = 0$, then $d_{xy}(r') < d_{xy}(r)$. Assume $d_{xy}(r') > 0$. Suppose $j = k$, so that $\eta_{xy}(p_k) = \eta_{yx}(p_k) = \eta_{yx}(p'_k) = 0$, $\eta_{xy}(p'_k) = 1$, $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) = \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i)$, and $\sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i)$. It follows that $d_{xy}(r') = \lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2} \cdot \eta_{yx}(p'_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) - \frac{1}{2} \cdot \eta_{xy}(p'_k)\rfloor + 1 = \lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) - \frac{1}{2}\rfloor + 1 = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) < \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + 1 = d_{xy}(r)$.

Suppose $j \neq k$. That induces $\eta_{xy}(p'_k) = \eta_{xy}(p_k)$, $\eta_{yx}(p'_k) = \eta_{yx}(p_k)$, $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) = \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + 1$, and $\sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i)$. It follows that $d_{xy}(r') = \lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2}\cdot\eta_{yx}(p'_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) - \frac{1}{2}\cdot\eta_{xy}(p'_k)\rfloor + 1 = \lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot \eta_{yx}(p_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) - \frac{1}{2}\cdot\eta_{xy}(p_k)\rfloor < \lfloor\sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot\eta_{yx}(p_k) - \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) - \frac{1}{2} \cdot \eta_{xy}(p_k)\rfloor + 1 = d_{xy}(r)$. DIMR ϕ satisfies PR(b).

Assume $d_{xy}(r) = 0$. That implies $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) + \frac{1}{2} \cdot \eta_{xy}(p'_k) > \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + \frac{1}{2} \cdot$

$\eta_{xy}(p_k) > \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot\eta_{yx}(p_k) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2}\cdot\eta_{yx}(p'_k)$. It follows that $d_{xy}(r') = 0$, so that ϕ satisfies PR(c).

By construction, $d_{yx}(r') = max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) + \frac{1}{2}\cdot\eta_{xy}(p'_k) - \sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) - \frac{1}{2}\cdot \eta_{yx}(p'_k)\rfloor + 1, 0\} \geq max\{\lfloor\sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + \frac{1}{2}\cdot\eta_{xy}(p_k) - \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) - \frac{1}{2}\cdot\eta_{yx}(p_k)\rfloor +1, 0\} = d_{yx}(r)$. As this establishes (d), ϕ satisfies PR.

Consider $r, r' \in D$, distinct $x, y \in X$, and $j \in N$ such that, for all $i \in N \setminus \{j\}$, $r'_i = r_i$, $(x, y), (y, x) \notin r_j$, and $r'_j = r_j \cup \{(x, y)\}$. Assume $d_{yx}(r) > 0$. The latter implies $(x, y) \in \phi(r)$. It follows that $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) + \frac{1}{2}\cdot\eta_{xy}(p'_k) > \sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + \frac{1}{2}\cdot \eta_{xy}(p_k) \geq \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot\eta_{yx}(p_k) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2}\cdot\eta_{yx}(p'_k)$. That yields $(x, y) \in \phi_p(r')$ and $d_{xy}(r') = 0$. Accordingly, ϕ satisfies MPR. By Lemma 4(a), it also satisfies NIL.

Take $r^\emptyset, r, r' \in D$ and distinct $x, y \in X$ such that $d_{xy}(r) = d_{yx}(r)$ and $d_{xy}(r') = d_{yx}(r')$. That means $(x, y) \in \phi_s(r) \cap \phi_s(r')$. In turn, $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p_i) + \frac{1}{2}\cdot\eta_{xy}(p_k) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p_i) + \frac{1}{2}\cdot\eta_{yx}(p_k)$ and $\sum_{i\in N\setminus\{k\}} \eta_{xy}(p'_i) + \frac{1}{2}\cdot\eta_{xy}(p'_k) = \sum_{i\in N\setminus\{k\}} \eta_{yx}(p'_i) + \frac{1}{2}\cdot \eta_{yx}(p'_k)$. It follows that $d_{xy}(r) = 1 = d_{xy}(r')$. Consequently, ϕ satisfies IBI. As discussed before, $d_{xy}(r^\emptyset) = 1$, implying $d_{xy}(r) = 1 = max\{d_{xy}(r^\emptyset) - m_{xy}(r), 1\}$. Therefore, ϕ also satisfies ABI.■

Proposition A.5.4:
EMR ϕ on $D = D_F$ satisfies A, N, III, PR, MPR, NIL, ER, IER, IBI, AER, and ABI, but, for $|X| \geq 3$, violates IIA.
Proof:
For $r \in D$ and distinct $x, y \in X$, it is true that $d_{xy}(r) = max\{m_{yx}(r)+\tau_{yx}(r)-m_{xy}(r)-\tau_{xy}(r) + 1, 0\}$.

Consider distinct $x, y, z \in X$, $i \in N$, and $r^\emptyset, r \in D$ such that $r_i = \{(x, z)\}$ and, for all $k \in N \setminus \{i\}$, $r_k = \emptyset$. Since $r^\emptyset|\{x, y\} = r|\{x, y\}$, IIA would demand $d_{xy}(r^\emptyset) = d_{xy}(r)$. However, $m_{xy}(r^\emptyset) + \tau_{xy}(r^\emptyset) = 0 = m_{yx}(r^\emptyset) + \tau_{yx}(r^\emptyset)$ and $m_{xy}(r) + \tau_{xy}(r) = 1 > 0 = m_{yx}(r) + \tau_{yx}(r)$. That is, $d_{xy}(r^\emptyset) = 1 > 0 = d_{xy}(r)$. Accordingly, ϕ violates IIA.

Take $r, r' \in D$ and bijection $\pi : N \to N$ such that, for all $i \in N$, $r_i = r'_{\pi(i)}$. Consider distinct $x, y \in X$. As discussed in the proof of Theorem 2, it is true that $m_{xy}(r) = m_{xy}(r')$ and $m_{yx}(r) = m_{yx}(r')$. Moreover, $\tau_{xy}(r) = \tau_{xy}(r')$ and $\tau_{yx}(r) = \tau_{yx}(r')$. It follows that $d_{xy}(r) = max\{m_{yx}(r) + \tau_{yx}(r) - m_{xy}(r) - \tau_{xy}(r) + 1, 0\} = max\{m_{yx}(r') + \tau_{yx}(r') - m_{xy}(r') - \tau_{xy}(r') + 1, 0\} = d_{xy}(r')$. Thus, ϕ satisfies A.

Consider $r, r' \in D$ and bijection $\sigma : X \to X$ such that, for all $i \in N$ and all $x, y \in X$,

$(x,y) \in r_i \Leftrightarrow (\sigma(x),\sigma(y)) \in r'_i$. Take distinct $x,y \in X$. As verified in the proof of Theorem 2, it is the case that $m_{xy}(r) = m_{\sigma(x)\sigma(y)}(r')$ and $m_{yx}(r) = m_{\sigma(y)\sigma(x)}(r')$. Assume $\tau_{xy}(r) = 1$. There exist $z \in X \setminus \{x,y\}$ and $i \in N$ such that $(x,z) \in p_i$. It follows that $(\sigma(x),\sigma(z)) \in p'_i$. Since σ is a bijection, $\sigma(z) \in X \setminus \{\sigma(x),\sigma(y)\}$, implying $\tau_{\sigma(x)\sigma(y)}(r') = 1$. Assume $\tau_{\sigma(x)\sigma(y)}(r') = 1$. There exist $z \in X \setminus \{\sigma(x),\sigma(y)\}$ and $i \in N$ such that $(\sigma(x),z) \in p'_i$. That implies $(x,\sigma^{-1}(z)) \in p_i$. Because σ is a bijection, $\sigma^{-1}(z) \in X \setminus \{x,y\}$, so that $\tau_{xy}(r) = 1$. That induces $\tau_{xy}(r) = \tau_{\sigma(x)\sigma(y)}(r')$. Analogously, $\tau_{yx}(r) = \tau_{\sigma(y)\sigma(x)}(r')$. It follows that $d_{xy}(r) = max\{m_{yx}(r) + \tau_{yx}(r) - m_{xy}(r) - \tau_{xy}(r) + 1, 0\} = max\{m_{\sigma(y)\sigma(x)}(r') + \tau_{\sigma(y)\sigma(x)}(r') - m_{\sigma(x)\sigma(y)}(r') - \tau_{\sigma(x)\sigma(y)}(r') + 1, 0\} = d_{\sigma(x)\sigma(y)}(r')$. Hence, ϕ satisfies N.

Take $r, r' \in D$ such that, for all $x,y \in X$ and all $i \in N$, $(x,y) \in r_i \Rightarrow (x,y) \in s_i$ and $(x,y) \in r'_i \Rightarrow (x,y) \in s'_i$. Consider $x,y,x',y' \in X$ with $x \neq y$ and $x' \neq y'$. Since there are neither strict individual rankings under r nor r', it follows that $m_{xy}(r) = m_{yx}(r) = \tau_{xy}(r) = \tau_{yx}(r) = 0$ as well as $m_{x'y'}(r') = m_{y'x'}(r') = \tau_{x'y'}(r') = \tau_{y'x'}(r') = 0$. That implies $d_{xy}(r) = 1 = d_{x'y'}(r')$. Therefore, ϕ satisfies III.

Consider $r \in D$ and distinct $x,y \in X$. Since $d_{xy}(r) < \infty$, ϕ satisfies PR(a). Take $r' \in D$ and $i \in N$ such that $(x,y),(y,x) \notin r_i$, $r'_i = r_i \cup \{(x,y)\}$, and, for all $k \in N \setminus \{i\}$, $r'_k = r_k$. As discussed in the proof of Theorem 1, that implies $m_{xy}(r') = m_{xy}(r) + 1$ and $m_{yx}(r') = m_{yx}(r)$. Moreover, $\tau_{xy}(r') = \tau_{xy}(r)$ and $\tau_{yx}(r') = \tau_{yx}(r)$.

Assume $d_{xy}(r) > 0$. If $d_{xy}(r') = 0$, then $d_{xy}(r') < d_{xy}(r)$. Assume $d_{xy}(r') > 0$. It follows that $d_{xy}(r') = m_{yx}(r') + \tau_{yx}(r') - m_{xy}(r') - \tau_{xy}(r') + 1 < m_{yx}(r) + \tau_{yx}(r) - m_{xy}(r) - \tau_{xy}(r) + 1 = d_{xy}(r)$. That verifies PR(b) of ϕ. Assume $d_{xy}(r) = 0$. It follows that $m_{yx}(r') + \tau_{yx}(r') - m_{xy}(r') - \tau_{xy}(r') + 1 < m_{yx}(r) + \tau_{yx}(r) - m_{xy}(r) - \tau_{xy}(r) + 1 \leq 0$. That yields $d_{xy}(r') = 0$, so that ϕ satisfies PR(c). If $d_{yx}(r) = 0$, then $d_{yx}(r') \geq d_{yx}(r)$. Assume $d_{yx}(r) > 0$. It follows that $d_{yx}(r') \geq m_{xy}(r') + \tau_{xy}(r') - m_{yx}(r') - \tau_{yx}(r') + 1 > m_{xy}(r) + \tau_{xy}(r) - m_{yx}(r) - \tau_{yx}(r) + 1 = d_{yx}(r)$. That yields PR(d). Accordingly, ϕ satisfies PR.

Take $r, r' \in D$, distinct $x,y \in X$, and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r'_k = r_k$, $(x,y),(y,x) \notin r_i$, and $r'_i = r_i \cup \{(x,y)\}$. Assume $d_{yx}(r) > 0$. It follows that $m_{xy}(r') = m_{xy}(r) + 1$, $m_{yx}(r') = m_{yx}(r)$, $\tau_{xy}(r') = \tau_{xy}(r)$, $\tau_{yx}(r') = \tau_{yx}(r)$, and $(x,y) \in \phi(r)$. That yields $m_{xy}(r') + \tau_{xy}(r') > m_{xy}(r) + \tau_{xy}(r) \geq m_{yx}(r) + \tau_{yx}(r) = m_{yx}(r') + \tau_{yx}(r')$. Thus, $(x,y) \in \phi_p(r')$ and $d_{xy}(r') = 0$. EMR ϕ satisfies MPR. By Lemma 4(a), it also satisfies NIL.

Consider $r, r' \in D$, distinct $x,y \in X$, and $i,j \in N$ such that $(x,y),(y,x) \notin r_i \cup r_j$,

$r_i' = r_i \cup \{(x,y)\}$, $r_j' = r_j \cup \{(y,x)\}$, and, for all $k \in N \setminus \{i,j\}$, $r_k' = r_k$. Assume $d_{xy}(r) = d_{yx}(r) > 0$. That is, $(x,y) \in \phi_s(r)$, implying $m_{xy}(r) + \tau_{xy}(r) = m_{yx}(r) + \tau_{yx}(r)$. In particular, $d_{xy}(r) = max\{m_{yx}(r) + \tau_{yx}(r) - m_{xy}(r) - \tau_{xy}(r) + 1, 0\} = 1$. It is true that $m_{xy}(r') = m_{xy}(r) + 1$, $m_{yx}(r') = m_{yx}(r) + 1$, $\tau_{xy}(r') = \tau_{xy}(r)$, and $\tau_{yx}(r') = \tau_{yx}(r)$. That yields $m_{xy}(r') + \tau_{xy}(r') = m_{yx}(r') + \tau_{yx}(r')$. It follows that $d_{xy}(r') = 1 = d_{yx}(r')$. Consequently, ϕ satisfies IER (and thus ER). As shown, $d_{xy}(r) = d_{yx}(r)$ implies $d_{xy}(r) = d_{yx}(r) = 1$ and in turn $d_{xy}(r') = d_{yx}(r') = 1$. Hence, ϕ also satisfies AER.

Take $r, r' \in D$ and distinct $x, y \in X$ such that $d_{xy}(r) = d_{yx}(r)$ and $d_{xy}(r') = d_{yx}(r')$. Again, that implies $m_{xy}(r) + \tau_{xy}(r) = m_{yx}(r) + \tau_{yx}(r)$ and $m_{xy}(r') + \tau_{xy}(r') = m_{yx}(r') + \tau_{yx}(r')$. It follows that $d_{xy}(r) = 1 = d_{xy}(r')$. Therefore, ϕ satisfies IBI. Since $d_{xy}(r^{\emptyset}) = 1$, it is moreover true that $d_{xy}(r) = 1 = max\{d_{xy}(r^{\emptyset}) - m_{xy}(r), 1\}$. Accordingly, ϕ satisfies ABI.■

4.A.6 Further Logical Relations

In the remainder of the Appendix, further logical relations between conditions on $D = D_F$ are explored. To begin with, it is shown that MPR is weaker than the following version of May's Positive Responsiveness. It is concerned with improvements of an alternative in a broader sense. For instance, the property also applies if an individual indifference changes into a strict ranking.

SRR ϕ on D satisfies **Classical May Positive Responsiveness (CMPR)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r_k' = r_k$, and either (a) $(x,y) \notin r_i$, $r_i' = r_i \cup \{(x,y)\}$ or (b) $(y,x) \notin r_i'$, $r_i = r_i' \cup \{(y,x)\}$, it is the case that $(x,y) \in \phi(r) \Rightarrow (x,y) \in \phi_p(r')$.[24]

Proposition A.6.1:
(a) If SRR ϕ satisfies CMPR, then ϕ satisfies MPR.
(b) There exists SRR ϕ on $D = D_F$ that satisfies MPR, but violates CMPR.
Proof:
(a) Assume SRR ϕ on D satisfies CMPR. Consider $r, r' \in D$, distinct $x, y \in X$, and $i \in N$ such that, for all $k \in N \setminus \{i\}$, $r_k' = r_k$, and $(x,y), (y,x) \notin r_i$ as well as $r_i' = r_i \cup \{(x,y)\}$. Assume $d_{yx}(r) > 0$. That is, $(x,y) \in \phi(r)$. By CMPR of ϕ, it follows that

[24] As May's (1952) original formulation, CMPR considers a ranking improvement for a single person. Recent contributions use formulations where several individual improvements can occur at once. There is a straightforward connection, as multiple improvements are covered by applying CMPR several times.

$(x, y) \in \phi_p(r')$, so that $d_{xy}(r') = 0$. Thus, ϕ satisfies MPR.

(b) Consider SRR ϕ on $D = D_F$ which looks like this. Take a linear ordering $r_I \in S(X)$. For a distinguished pair of distinct alternatives $x, y \in X$, $(x, y) \in \phi(r) \Leftrightarrow |\{i \in N \mid (x, y) \in p_i\}| \geq |\{i \in N \mid (y, x) \in r_i\}|$ for each $r \in D$. Likewise, $(y, x) \in \phi(r) \Leftrightarrow |\{i \in N \mid (y, x) \in r_i\}| \geq |\{i \in N \mid (x, y) \in p_i\}|$ for each $r \in D$. While $|\{i \in N \mid (x, y) \in p_i\}| < \infty$, it is possible that $|\{i \in N \mid (y, x) \in r_i\}| = \infty$. For all distinct alternatives $x', y' \in X$ with $\{x', y'\} \neq \{x, y\}$, $(x', y') \in \phi(r) \Leftrightarrow (x', y') \in r_I$.

Consider $r, r' \in D$, distinct $x', y' \in X$, and $k \in N$ such that, for all $i \in N \setminus \{k\}$, $r'_i = r_i$, $(x', y'), (y', x') \notin r_k$ as well as $r'_k = r_k \cup \{(x', y')\}$. Assume $d_{y'x'}(r) > 0$. That is, $(x', y') \in \phi(r)$. First, assume $\{x', y'\} \neq \{x, y\}$. By construction, it follows that $(x', y') \in p_I$, implying $(x', y') \in \phi_p(r')$ and in turn $d_{x'y'}(r') = 0$.

Second, assume $x' = x$ and $y' = y$. That is, $d_{yx}(r) > 0$ and $(x, y) \in \phi(r)$. It follows that $|\{i \in N \mid (x, y) \in p'_i\}| > |\{i \in N \mid (x, y) \in p_i\}| \geq |\{i \in N \mid (y, x) \in r_i\}| = |\{i \in N \mid (y, x) \in r'_i\}|$. That yields $(x, y) \in \phi_p(r')$ and hence $d_{xy}(r') = d_{x'y'}(r') = 0$. Third, assume $x' = y$ and $y' = x$. That means $d_{xy}(r) > 0$ and $(y, x) \in \phi(r)$. It follows that $|\{i \in N \mid (y, x) \in r_i\}| \geq |\{i \in N \mid (x, y) \in p_i\}| = |\{i \in N \mid (x, y) \in p'_i\}|$. If $|\{i \in N \mid (y, x) \in r_i\}| = \infty$, then clearly $|\{i \in N \mid (y, x) \in r'_i\}| > |\{i \in N \mid (x, y) \in p'_i\}|$. If $|\{i \in N \mid (y, x) \in r_i\}| < \infty$, then $|\{i \in N \mid (y, x) \in r'_i\}| = |\{i \in N \mid (y, x) \in r_i\}| + 1$. Again, $|\{i \in N \mid (y, x) \in r'_i\}| > |\{i \in N \mid (x, y) \in p'_i\}|$. In any case, $(y, x) \in \phi_p(r')$, so that $d_{yx}(r') = d_{x'y'}(r') = 0$. SRR ϕ satisfies MPR.

Take $r, r' \in D$ and $i, j \in N$ such that $r_i = \{(y, x)\}$, $r'_i = \{(y, x), (x, y)\}$, $r'_j = r_j = \{(x, y)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k = \emptyset$. Since $(x, y) \in \phi_s(r)$, CMPR would demand $(x, y) \in \phi_p(r')$. But in fact, $(x, y) \in \phi_s(r')$. Therefore, ϕ violates CMPR.■

PR⇏MPR (PR⇏CMPR): A PRMR (a PQMR) on D_F satisfies PR, but violates MPR and thus CMPR.

CMPR⇏PR (MPR⇏PR): Consider SRR ϕ on $D = D_F$ which is imposed with linear ordering $r_I \in S(X)$ such that, for all $r \in D$ and all distinct $x, y \in X$, $(x, y) \in \phi(r) \Leftrightarrow (x, y) \in r_I$. For $r, r' \in D$ and distinct $x, y \in X$, $(x, y) \in \phi(r)$ implies $(x, y) \in p_I$, which in turn implies $(x, y) \in \phi_p(r')$. That will be true in particular if the antecedent of CMPR holds. SRR ϕ satisfies the condition (and thus MPR). But ϕ violates PR. For $x, y \in X$ with $(x, y) \in p_I$, it is true that $d_{yx}(r^{\emptyset}) = \infty$, violating PR(a).

Due to USIR and PRMR, it is true that **NIL⇏MPR**, **NIL⇏PR**, and **PR⇏NIL** on D_F. That is, PR and NIL are logically independent.

The following is a version of Cancellation for incomplete individual rankings. Proposition A.6.2 shows that IER is a weakening of the condition.

SRR ϕ on D satisfies **Cancellation (C)** if, for all $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x,y),(y,x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x,y)\}$, $r'_j = r_j \cup \{(y,x)\}$, and, for all $k \in N \setminus \{i,j\}$, $r'_k = r_k$, it is the case that $(x,y) \in \phi(r) \Leftrightarrow (x,y) \in \phi(r')$.

Proposition A.6.2:
(a) If SRR ϕ on $D = D_F$ satisfies C, then ϕ satisfies IER.
(b) There exists SRR ϕ on $D = D_F$ which satisfies IER, but violates C.
Proof:
(a) Assume SRR ϕ on $D = D_F$ satisfies C. Consider $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x,y),(y,x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x,y)\}$, $r'_j = r_j \cup \{(y,x)\}$, and, for all $k \in N \setminus \{i,j\}$, $r'_k = r_k$. Assume $d_{xy}(r) = d_{yx}(r) = n$.

Suppose $d_{xy}(r') = n' < n$. That is, there exists $r'' \in D$ where n' individuals are added to r' who do not rank $\{x,y\}$ at r' and rank x higher than y at r'' and $(x,y) \in \phi_p(r'')$. Consider $r''' \in D$ which is the same as r'' except that the strict individual rankings of i and j over $\{x,y\}$ are removed. By C of ϕ, it is true that $(x,y) \in \phi_p(r''')$. But r''' is constructed by adding n' strict individual rankings of x over y to r. That leads to the contradiction $d_{xy}(r) \leq n' < n$. Accordingly, $d_{xy}(r') \geq n$ must hold. The analogous argument yields $d_{yx}(r') \geq n$.

In case $n = \infty$, it follows that $d_{xy}(r') = d_{yx}(r') = d_{xy}(r) = \infty$. Assume $n < \infty$. Let r^1 be the profile where new strict rankings of x over y for n individuals are added to r, so that $(x,y) \in \phi_p(r^1)$. Denote this set of n individuals as N^1. Distinguish four cases.

First, suppose $i, j \notin N^1$, so that i and j do not rank $\{x,y\}$ at r^1. Let $r^2 \in D$ be the profile which is the same as r^1 except that i ranks x higher than y and j ranks y higher than x. By C of ϕ, it is true that $(x,y) \in \phi_p(r^2)$. But r^2 is generated from r' by adding n new strict individual rankings of x over y to r'. Thus, $d_{xy}(r') \leq n$.

Second, suppose $i \in N^1$ and $j \notin N^1$. Consider $k \in N \setminus \{j\}$ who does not rank $\{x,y\}$ at r^1. Let $r^2 \in D$ be the profile which is the same as r^1 except that k ranks x higher than y and j ranks y higher than x. Due to C of ϕ, it follows that $(x,y) \in \phi_p(r^2)$. Profile r^2 is the same as r' except that n new strict individual rankings of x over y are added to r'. Hence, $d_{xy}(r') \leq n$.

Third, suppose $i \notin N^1$ and $j \in N^1$. Consider $k, k' \in N \setminus \{i\}$ who do not rank $\{x,y\}$ at r^1. Let $r^2 \in D$ be the profile which is the same as r^1 except that k ranks x higher than y and k' ranks y higher than x. Let $r^3 \in D$ be the profile which is the same as r^2 except that j and k' do not rank $\{x, y\}$ anymore. Finally, let $r^4 \in D$ be the profile which is the same as r^3 except that i ranks x higher than y and j ranks y higher than x. By C of ϕ, the initial strict social ranking of x over y at r^1 is kept in every step, so that $(x, y) \in \phi_p(r^4)$. Profile r^4 is the same as r' except that n new strict individual rankings of x over y are added to r'. Therefore, $d_{xy}(r') \leq n$.

Fourth, suppose $i, j \in N^1$. Reconsider r^3 from the last paragraph. Take $k'' \in N \setminus \{j\}$ who does not rank $\{x, y\}$ at r^3. Let $r^4 \in D$ be the profile which is the same as r^3 except that k'' ranks x higher than y and j ranks y higher than x. Condition C still secures that the social ranking of $\{x, y\}$ remains invariant in every step, implying $(x, y) \in \phi_p(r^4)$. Again, r^4 is the same as r' except that n new strict individual rankings of x over y are added to r'. Accordingly, $d_{xy}(r') \leq n$.

Taken together, it has been established that $d_{xy}(r') \leq n$ and $d_{xy}(r') \geq n$, so that $d_{xy}(r') = n$. The analogous reasoning yields $d_{yx}(r') = n$. It follows that $d_{xy}(r') = d_{yx}(r') = d_{xy}(r)$. SRR ϕ satisfies IER.

(b) Consider SRR ϕ on $D = D_F$ which coincides with the SMR on every $r \in D$ with $r \neq r^{\emptyset}$. On $r^{\emptyset} \in D$, alternatives are socially ranked according to an external linear ordering $r_I \in S(X)$. Take $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r'_k = r_k$. Assume $d_{xy}(r) = d_{yx}(r)$. That implies $(x, y) \in \phi_s(r)$ and in turn $r, r' \neq r^{\emptyset}$. It follows that $d_{xy}(r') = d_{yx}(r') = d_{xy}(r) = 1$. Hence, ϕ satisfies IER. In contrast, consider $r \in D$, $i, j \in N$, and distinct $x, y \in X$ with $(x, y) \in p_I$ such that $r_i = \{(x, y)\}$, $r_j = \{(y, x)\}$, and, for all $k \in N \setminus \{i, j\}$, $r_k = \emptyset$. It follows that $(x, y) \in \phi_p(r^{\emptyset})$ and $(x, y) \in \phi_s(r)$. Thus, ϕ violates C.■

(AER$\wedge$ABI)$\not\Rightarrow$C: A PQMR on D_F satisfies AER and ABI, but violates C.

C$\not\Rightarrow$(AER$\vee$ABI): A PRMR on D_F satisfies C, but violates AER and ABI.

IBI$\not\Rightarrow$IER (IBI$\not\Rightarrow$C): A DIMR on D_F satisfies IBI, but violates IER (and thus C).

C$\not\Rightarrow$IBI (IER$\not\Rightarrow$IBI): Consider SRR ϕ on $D = D_F$ with distinguished individual $k \in N$. It socially ranks distinct x and y at profile $r \in D$ in the following way. If $(x, y) \in s_k$, then x and y are socially ranked as under the 2-RMR. If $(x, y) \notin s_k$, they are ranked as under the SMR. Take $r, r' \in D$, distinct $x, y \in X$, and $i, j \in N$ such that $(x, y), (y, x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x, y)\}$, $r'_j = r_j \cup \{(y, x)\}$, and, for all $l \in N \setminus \{i, j\}$, $r'_l = r_l$. If $(x, y) \in s_k$, then $(x, y) \in s'_k$. In that case, $(x, y) \in \phi(r) \Leftrightarrow$

$m_{xy}(r)+1 \geq m_{yx}(r) \Leftrightarrow m_{xy}(r')+1 \geq m_{yx}(r') \Leftrightarrow (x,y) \in \phi(r')$. If $(x,y) \notin s_k$, then $(x,y) \notin s'_k$. In that case, $(x,y) \in \phi(r) \Leftrightarrow m_{xy}(r) \geq m_{yx}(r) \Leftrightarrow m_{xy}(r') \geq m_{yx}(r') \Leftrightarrow (x,y) \in \phi(r')$. Taken together, ϕ satisfies C and thus IER. Consider $r^{\emptyset}, r \in D$ with $r_k = \{(x,y),(y,x)\}$ and, for all $i \in N \setminus \{k\}$, $r_i = \emptyset$. It is true that $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset}) = 1$ and $d_{xy}(r) = d_{yx}(r) = 2$. Hence, ϕ violates IBI.

AER$\not\Rightarrow$ABI: Reconsider SRR ϕ from the last paragraph, but now assume that, if $(x,y) \in s_k$, then x and y are socially ranked as under the 2-QMR at $r \in D$. Consider $r, r' \in D$, distinct $x,y \in X$, and $i,j \in N$ such that $(x,y),(y,x) \notin r_i \cup r_j$, $r'_i = r_i \cup \{(x,y)\}$, $r'_j = r_j \cup \{(y,x)\}$, for all $l \in N \setminus \{i,j\}$, $r'_l = r_l$. Assume $d_{xy}(r) = d_{yx}(r) > 0$. Now, $(x,y) \notin s_k$ implies $(x,y) \notin s'_k$ and $d_{xy}(r') = d_{yx}(r') = d_{xy}(r) = d_{yx}(r) = 1$. Suppose $(x,y) \in s_k$, implying $(x,y) \in s'_k$. If $m_{xy}(r) = m_{yx}(r) = 0$, then $d_{xy}(r) = d_{yx}(r) = 2$ and $d_{xy}(r') = d_{yx}(r') = 1$. If $m_{xy}(r) = m_{yx}(r) > 0$, then $d_{xy}(r') = d_{yx}(r') = d_{xy}(r) = d_{yx}(r) = 1$. Therefore, ϕ satisfies AER. Reconsider $r^{\emptyset}, r \in D$ with $r_k = \{(x,y),(y,x)\}$ and, for all $i \in N \setminus \{k\}$, $r_i = \emptyset$. Still, $d_{xy}(r^{\emptyset}) = d_{yx}(r^{\emptyset}) = 1$ and $d_{xy}(r) = d_{yx}(r) = 2$. Accordingly, ϕ violates ABI.

ABI$\not\Rightarrow$AER: A DIMR on D_F satisfies ABI and violates AER.

(IIA$\wedge$A$\wedge$N)$\not\Rightarrow$III: Consider the following simple majority rule with quorum. An alternative socially ranks higher than another alternative if the former has a simple majority over the latter and there are at least 5 strict or indifferent individual rankings over the pair. Otherwise, the pair socially ranks equally. The rule satisfies IIA, A, and N, but violates III on $D = D_F$. For the latter, consider $r^{\emptyset}, r \in D$, $i \in N$, and distinct $x,y \in X$ such that $r_i = \{(x,y),(y,x)\}$ and, for all $k \in N \setminus \{i\}$, $r_k = \emptyset$. It is true that $d_{xy}(r^{\emptyset}) = 5 \neq 4 = d_{xy}(r)$.

(IIA$\wedge$A$\wedge$N$\wedge$III)$\not\Rightarrow$ER: Take SRR ϕ on $D = D_F$ which looks like this. If there are at most 2 strict individual rankings of a pair, then the pair socially ranks equally. If there are at least 4 strict individual rankings of the pair, then the pair socially ranks as under the SMR. Assume there are three strict individual rankings of the pair. If the ranking is unanimously in favor of one alternative, then the social ranking also favors that alternative. If there is no unanimity (that is, one individual is for one alternative, the two others are for the other alternative), then the social ranking favors the alternative with only one favorable individual ranking. On the one hand, ϕ satisfies IIA, A, N, and III. The social ranking only depends on known individual rankings over the pair and it does not depend on names of individuals or alternatives. Without strict individual rankings, the PAI of each alternative over any other alternative is 3. On the other hand,

ϕ violates ER (and thus IER and AER). Consider $r, r' \in D$ and distinct $x, y \in X$ such that two individuals are known to rank x higher than y at r. The ranking of the pair by other individuals is not known. At r', two opposing individual rankings of the pair are added to r. It is true that $d_{xy}(r) = 1 = d_{yx}(r)$, but $d_{xy}(r') = 0 < 3 = d_{yx}(r')$.

(IER$\wedge$AER$\wedge$IBI$\wedge$ABI)$\not\Rightarrow$(A$\vee$N): Consider SRR ϕ on $D = D_F$ which looks like this. If dictator $i \in N$ ranks an alternative higher than another, that is reflected by the social ranking. If i's ranking of a pair is indifferent or unknown, then the pair socially ranks according to an external linear ordering $r_I \in S(X)$. By construction, the social ranking of a pair is always strict. The antecedents of IER, AER, IBI, and ABI are thus never satisfied, respectively, so that ϕ satisfies all these conditions. In contrast, ϕ violates A and N.

References

Ackerman M; Choi S; Coughlin P; Gottlieb E; Wood J (2013), Elections with Partially Ordered Preferences. Public Choice 157, 145-168

Adler M (2014), Extended Preferences and Interpersonal Comparisons: A New Account. Economics & Philosophy 30, 123-162

Adler M (2019), Measuring Social Welfare: An Introduction. New York: Oxford University Press

Alcantud J (2019), Yet Another Characterization of the Majority Rule. Economics Letters 177, 52-55

Alcantud J (2020), Simple Majorities with Voice but No Vote. Group Decision and Negotiation 29, 803-822

Arrow K (1951), Social Choice and Individual Values. New York: Wiley

Arrow K (1963), Social Choice and Individual Values, Second Edition. New York: Wiley

Arrow K (2012), Social Choice and Individual Values, Third Edition. New Haven and London: Yale University Press

Asheim G and Tungodden B (2004), Resolving Distributional Conflicts Between Generations. Economic Theory 24, 221-230

Asheim G and Zuber S (2014), Escaping the Repugnant Conclusion: Rank-Discounted Utilitarianism with Variable Population. Theoretical Economics 9, 629-650

Aşan G and Sanver M (2002), Another Characterization of the Majority Rule. Economics Letters 75, 409-413

Aşan G and Sanver M (2006), Maskin Monotonic Aggregation Rules. Economics Letters

91, 179-183

Balasubramanian A (2015), On Weighted Utilitarianism and an Application. Social Choice and Welfare 44, 745-763

Barberà S and Bossert W (2023), Opinion Aggregation: Borda and Condorcet Revisited. Journal of Economic Theory 210, Article 105654

Barberà S and Gerber A (2017), Sequential Voting and Agenda Manipulation. Theoretical Economics 12, 211-247

Bartholdi L; Hann-Caruthers W; Josyula M; Tamuz, O; Yariv L (2021), Equitable Voting Rules. Econometrica 89, 563-589

Blackorby C; Bossert W; Donaldson D (2002), Utilitarianism and the Theory of Justice, in: Arrow K; Sen A; Suzumura K (eds.), Handbook of Social Choice and Welfare, Volume 1. Amsterdam: Elsevier North Holland, 543-596

Blackorby C; Bossert W; Donaldson D (2005), Population Issues in Social Choice Theory, Welfare Economics, and Ethics. Cambridge: Cambridge University Press

Blackorby C; Bossert W; Donaldson D (2006), Anonymous Single-Profile Welfarism. Social Choice and Welfare 27, 279-287

Blackorby C; Donaldson D; Weymark J (1984), Social Choice with Interpersonal Utility Comparisons: A Diagrammatic Introduction. International Economic Review 25, 327-356

Blackorby C; Donaldson D; Weymark J (1990), A Welfarist Proof of Arrow's Theorem. Recherches Economiques de Louvain 56, 259-286

Blackorby C; Primont D; Russell R (1998), Separability: A Survey, in: Barberà S; Hammond P; Seidl C (eds.), Handbook of Utility Theory, Volume 1. Dordrecht: Kluwer, 49-92

Bosmans K; Decancq K; Ooghe E (2018), Who's Afraid of Aggregating Money Metrics? Theoretical Economics 13, 467-484

Bossert W (1991), On Intra- and Interpersonal Utility Comparisons. Social Choice and Welfare 8, 207-219

Bossert W and Kamaga K (2020), An Axiomatization of the Mixed Utilitarian-Maximin Social Welfare Orderings. Economic Theory 69, 451-473

Bossert W and Weymark J (2004), Utility in Social Choice, in: Barberà S; Hammond P; Seidl C (eds.), Handbook of Utility Theory, Volume 2. Boston: Kluwer, 1099-1177

Bossert W; Cato S; Kamaga K (2023), Revisiting Variable-Value Population Principles. Economics & Philosophy 39, 468-484

Bouton L; Llorente-Saguer A; Malherbe F (2018), Get Rid of Unanimity Rule: The Superiority of Majority Rules with Veto Power. Journal of Political Economy 126, 107-149

Bouyssou D and Pirlot M (2005), Conjoint Measurement Tools for MCDM: A Brief Introduction, in: Figueira J; Greco S; Ehrgott M (eds.), Multiple Criteria Decision Analysis: State of the Art Surveys. New York: Springer, 73-130

Bradley B (2014), Objective Theories of Well-Being, in: Eggleston B and Miller D (eds.), The Cambridge Companion to Utilitarianism. Cambridge: Cambridge University Press, 220-238

Brandl F and Brandt F (2020), Arrovian Aggregation of Convex Preferences. Econometrica 88, 799-844

Broome J (1991), Weighing Goods: Equality, Uncertainty and Time. Oxford: Blackwell Publishers

Broome J (2004), Weighing Lives. Oxford: Oxford University Press

Buchak L (2023), Relative Priority. Economics & Philosophy 39, 199-229

Bykvist K (2016), Preference-Based Views of Well-Being, in: Adler M and Fleurbaey M (eds.), The Oxford Handbook of Well-Being and Public Policy. New York: Oxford University Press, 321-346

Campbell D (1988), A Characterization of Simple Majority Rule for Restricted Domains. Economics Letters 28, 307-310

Campbell D and Kelly J (2000), A Simple Characterization of Majority Rule. Economic Theory 15, 689-700

Campbell D and Kelly J (2002), Impossibility Theorems in the Arrovian Framework, in: Arrow K; Sen A; Suzumura K (eds.), Handbook of Social Choice and Welfare, Volume 1. Amsterdam: Elsevier North Holland, 35-94

Cantillon E and Rangel A (2002), A Graphical Analysis of Some Basic Results in Social Choice. Social Choice and Welfare 19, 587-611

Cato S (2010), Brief Proofs of Arrovian Impossibility Theorems. Social Choice and Welfare 35, 267-284

Cato S (2011), Pareto Principles, Positive Responsiveness, and Majority Decisions. Theory and Decision 71, 503-518

Chan J; Lizzeri A; Suen W; Yariv L (2018), Deliberating Collective Decisions. Review of Economic Studies 85, 929-963

Cullinan J; Hsiao S; Polett D (2014), A Borda Count for Partially Ordered Ballots. Social Choice and Welfare 42, 913-926

d'Aspremont C and Gevers L (1977), Equity and the Informational Basis of Collective Choice. Review of Economic Studies 44, 199-209

d'Aspremont C and Gevers L (2002), Social Welfare Functionals and Interpersonal Comparability, in: Arrow K; Sen A; Suzumura K (eds.), Handbook of Social Choice and Welfare, Volume 1. Amsterdam: Elsevier North Holland, 459-541

Dasgupta P and Maskin E (2008), On the Robustness of Majority Rule. Journal of the European Economic Association 6, 949-973

Dasgupta P and Maskin E (2020), Strategy-Proofness, Independence of Irrelevant Alternatives, and Majority Rule. American Economic Review: Insights 2, 459-474

Debreu G (1960), Topological Methods in Cardinal Utility Theory, in: Arrow K; Karlin S; Suppes P (eds.), Mathematical Methods in the Social Sciences. Stanford: Stanford University Press, 16-26

Deschamps R and Gevers L (1978), Leximin and Utilitarian Rules: A Joint Characterization. Journal of Economic Theory 17, 143-163

Dietrich F and List C (2010), Majority Voting on Restricted Domains. Journal of Economic Theory 145, 512-543

Eden M (2020), Welfare Analysis with Heterogeneous Risk Preferences. Journal of Political Economy 128, 4574-4613

Fine B and Fine K (1974), Social Choice and Individual Rankings (II). Review of Economic Studies 41, 459-475

Fishburn P (1983), A New Characterization of Simple Majority. Economics Letters 13, 31-35

Fleurbaey M (2010), Assessing Risky Social Situations. Journal of Political Economy 118, 649-680

Fleurbaey M (2015), Equality Versus Priority: How Relevant is the Distinction? Economics & Philosophy 31, 203-217

Fleurbaey M and Maniquet F (2008), Fair Social Orderings. Economic Theory 34, 25-45

Fleurbaey M and Maniquet F (2011), A Theory of Fairness and Social Welfare. Cambridge: Cambridge University Press

Fleurbaey M and Maniquet F (2017), Fairness and Well-Being Measurement. Mathematical Social Sciences 90, 119-126

Fleurbaey M and Maniquet F (2019), Well-Being Measurement with Non-Classical Goods. Economic Theory 68, 765-786

Fleurbaey M and Mongin P (2005), The News of the Death of Welfare Economics is Greatly Exaggerated. Social Choice and Welfare 25, 381-418

Fleurbaey M and Mongin P (2016), The Utilitarian Relevance of the Aggregation Theorem. American Economic Journal: Microeconomics 8, 289-306

Fleurbaey M and Tadenuma K (2014), Universal Social Orderings: An Integrated Theory of Policy Evaluation, Inter-Society Comparisons, and Interpersonal Comparisons. Review of Economic Studies 81, 1071-1101

Fleurbaey M and Tungodden B (2010), The Tyranny of Non-Aggregation Versus the Tyranny of Aggregation in Social Choices: A Real Dilemma. Economic Theory 44, 399-414

Fleurbaey M; Tungodden B; Chang H (2003), Any Non-Welfarist Method of Policy Assessment Violates the Pareto Principle: A Comment. Journal of Political Economy 111, 1382-1385

Freixas J and Pons M (2021), An Extension and an Alternative Characterization of May's Theorem. Annals of Operations Research 302, 137-150

Gärdenfors P (1973), Positionalist Voting Functions. Theory and Decision 4, 1-24

Gevers L (1979), On Interpersonal Comparability and Social Welfare Orderings. Econometrica 47, 75-89

Ghosh A; Khan M; Uyanik M (2023), Continuity Postulates and Solvability Axioms in Economic Theory and in Mathematical Psychology: A Consolidation of the Theory of Individual Choice. Theory and Decision 94, 189-210

Goodin R and List C (2006), Special Majorities Rationalized. British Journal of Political Science 36, 213-241

Gorman W (1968), The Structure of Utility Functions. Review of Economic Studies 35, 367-390

Grant S; Kajii A; Polak B; Safra Z (2010), Generalized Utilitarianism and Harsanyi's Impartial Observer Theorem. Econometrica 78, 1939-1971

Gustafsson J; Spears D; Zuber S (2023), Utilitarianism is Implied by Social and Individual Dominance. IZA Discussion Paper No. 16561. https://ssrn.com/abstract=4619698 [Last Access: December 14, 2023]

Hammond P (1976), Equity, Arrow's Conditions, and Rawls' Difference Principle. Econometrica 44, 793-804

Hammond P (1979), Equity in Two Person Situations: Some Consequences. Econometrica 47, 1127-1135

Harsanyi J (1953), Cardinal Utility in Welfare Economics and in the Theory of Risk-Taking. Journal of Political Economy 61, 434-435

Harsanyi J (1955), Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. Journal of Political Economy 63, 309-321

Harsanyi J (1982), Morality and the Theory of Rational Behaviour, in: Sen A and Williams B (eds.), Utilitarianism and Beyond. Cambridge: Cambridge University Press, 39-62

Harvey C (1999), Aggregation of Individuals' Preference Intensities into Social Preference Intensity. Social Choice and Welfare 16, 65-79

Haybron D (2016), Mental State Approaches to Well-Being, in: Adler M and Fleurbaey M (eds.), The Oxford Handbook of Well-Being and Public Policy. New York: Oxford University Press, 347-378

Heathwood C (2014), Subjective Theories of Well-Being, in: Eggleston B and Miller D (eds.), The Cambridge Companion to Utilitarianism. Cambridge: Cambridge University Press, 199-219

Hirschauer N; Lehberger M; Musshoff O (2015), Happiness and Utility in Economic Thought – Or: What Can We Learn from Happiness Research for Public Policy Analysis and Public Policy Making? Social Indicators Research 121, 647-674

Holliday W and Pacuit E (2021), Axioms for Defeat in Democratic Elections. Journal of Theoretical Politics 33, 475-524

Holtug N (2015), Theories of Value Aggregation: Utilitarianism, Egalitarianism, Prioritarianism, in: Hirose I and Olson J (eds.), The Oxford Handbook of Value Theory. New York: Oxford University Press, 267-284

Horan S; Osborne M; Sanver R (2019), Positively Responsive Collective Choice Rules and Majority Rule: A Generalization of May's Theorem to Many Alternatives. International Economic Review 60, 1489-1504

Houy N (2007a), Some Further Characterizations for the Forgotten Voting Rules. Mathematical Social Sciences 53, 111-121

Houy N (2007b), A Characterization for Qualified Majority Voting Rules. Mathematical Social Sciences 54, 17-24

Houy N (2009), A Characterization of Majority Voting Rules with Quorums. Theory and Decision 67, 295-301

Hurka T (2016), Objective Goods, in: Adler M and Fleurbaey M (eds.), The Oxford Handbook of Well-Being and Public Policy. New York: Oxford University Press, 379-402

Jeong H and Ju B (2017), Resolute Majority Rules. Theory and Decision 82, 31-39

Kamaga K (2016), Infinite-Horizon Social Evaluation with Variable Population Size. Social Choice and Welfare 47, 207-232

Kaplow L and Shavell S (2001), Any Non-Welfarist Method of Policy Assessment Violates the Pareto Principle. Journal of Political Economy 109, 281-286

Kaplow L and Shavell S (2004), Any Non-Welfarist Method of Policy Assessment Violates the Pareto Principle: Reply. Journal of Political Economy 112, 249-251

Karni E and Weymark J (2023), Impartiality and Relative Utilitarianism. Working Paper. https://john-weymark.github.io/research/ [Last Access: December 14, 2023]

Kemp M and Ng Y (1976), On the Existence of Social Welfare Functions, Social Orderings and Social Decision Functions. Economica 43, 59-66

King S and Powers R (2018), Beyond Neutrality: Extended Difference of Votes Rules. Mathematical Social Sciences 93, 146-152

Köbberling V (2006), Strength of Preference and Cardinal Utility. Economic Theory 27, 375-391

Krantz D; Luce R; Suppes P; Tversky A (1971), Foundations of Measurement, Volume I: Additive and Polynomial Representations. New York: Academic Press

Li C; Rohde K; Wakker P (2023), The Deceptive Beauty of Monotonicity, and the Million-Dollar Question: Row-First or Column-First Aggregation? Working Paper. https://personal.eur.nl/wakker/newps.htm [Last Access: December 14, 2023]

Llamazares B (2006), The Forgotten Decision Rules: Majority Rules Based on Difference of Votes. Mathematical Social Sciences 51, 311-326

Maniquet F (2016), Social Ordering Functions, in: Adler M and Fleurbaey M (eds.), The Oxford Handbook of Well-Being and Public Policy. New York: Oxford University Press, 227-245

Maskin E (1978), A Theorem on Utilitarianism. Review of Economic Studies 45, 93-96

Maskin E (1995), Majority Rule, Social Welfare Functions, and Game Forms, in: Basu K; Pattanaik P; Suzumura K (eds.), Choice, Welfare, and Development. Oxford: Clarendon Press, 100-109

Maskin E (2023), Borda's Rule and Arrow's Independence of Irrelevant Alternatives. Working Paper. https://scholar.harvard.edu/maskin/publications/arrows-theorem-mays-axiom-bordas-rule [Last Access: December 14, 2023]

May K (1952), A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision. Econometrica 20, 680-684

McCarthy D (2006), Utilitarianism and Prioritarianism I. Economics & Philosophy 22, 335-363

McCarthy D (2008), Utilitarianism and Prioritarianism II. Economics & Philosophy 24, 1-33

McCarthy D (2017), The Priority View. Economics & Philosophy 33, 215-257

McCarthy D; Mikkola K; Thomas T (2020), Utilitarianism with and without Expected Utility. Journal of Mathematical Economics 87, 77-113

McMorris F; Mulder H; Novick B; Powers R (2021), Majority Rule for Profiles of Arbitrary Length, With an Emphasis on the Consistency Axiom. Mathematical Social Sciences 109, 164-174

Miroiu A (2004), Characterizing Majority Rule: From Profiles to Societies. Economics Letters 85, 359-363

Mongin P and Pivato M (2015), Ranking Multidimensional Alternatives and Uncertain Prospects. Journal of Economic Theory 157, 146-171

Morreau M and Weymark J (2016), Measurement Scales and Welfarist Social Choice. Journal of Mathematical Psychology 75, 127-136

Nebel J (2023), Ethics Without Numbers. Philosophy and Phenomenological Research, forthcoming

Nishimura H and Ok E (2016), Utility Representation of an Incomplete and Nontransitive Preference Relation. Journal of Economic Theory 166, 164-185

Nitzan S and Rubinstein A (1981), A Further Characterization of Borda Ranking Method. Public Choice 36, 153-158

Ok E (2002), Utility Representation of an Incomplete Preference Relation. Journal of Economic Theory 104, 429-449

Ou-Yang K (2018), Equity, Hierarchy, and Ordinal Social Choice. Mathematical Social Sciences 91, 75-84

Parfit D (1984), Reasons and Persons. Oxford: Oxford University Press

Parfit D (1991), Equality or Priority? The Lindley Lecture. University of Kansas, Department of Philosophy. https://kuscholarworks.ku.edu/handle/1808/12405 [Last Access: December 14, 2023]

Parks R (1976), An Impossibility Theorem for Fixed Preferences: A Dictatorial Bergson-Samuelson Welfare Function. Review of Economic Studies 43, 447-450

Pattanaik P (2002), Positional Rules of Collective Decision-Making, in: Arrow K; Sen A; Suzumura K (eds.), Handbook of Social Choice and Welfare, Volume 1. Amsterdam: Elsevier North Holland, 361-394

Patty J and Penn E (2019), A Defense of Arrow's Independence of Irrelevant Alternatives. Public Choice 179, 145-164

Pauly M (2013), Characterizing Referenda with Quorums via Strategy-Proofness. Theory and Decision 75, 581-597

Perez-Truglia R (2020), The Effects of Income Transparency on Well-Being. American Economic Review 110, 1019-1054

Piacquadio P (2017), A Fairness Justification of Utilitarianism. Econometrica 85, 1261-1276

Pivato M (2013a), Social Welfare with Incomplete Ordinal Interpersonal Comparisons. Journal of Mathematical Economics 49, 405-417

Pivato M (2013b), Risky Social Choice with Incomplete or Noisy Interpersonal Comparisons of Well-Being. Social Choice and Welfare 40, 123-139

Pivato M (2013c), Multiutility Representations for Incomplete Difference Preorders. Mathematical Social Sciences 66, 196-220

Pivato M (2015), Social Choice with Approximate Interpersonal Comparison of Welfare Gains. Theory and Decision 79, 181-216

Pivato M (2020), Rank-Additive Population Ethics. Economic Theory 69, 861-918

Pivato M and Tchouante E (2023), Bayesian Social Aggregation with Non-Archimedean Utilities and Probabilities. Economic Theory, forthcoming

Pollak R (1979), Bergson-Samuelson Social Welfare Functions and the Theory of Social Choice. The Quarterly Journal of Economics 93, 73-90

Quesada A (2010), Monotonicity + Efficiency + Continuity = Majority. Mathematical Social Sciences 60, 149-153

Quesada A (2011), Parallel Axiomatizations of Majority and Unanimity. Economics Letters 111, 151-154

Quesada A (2012), A Short Step Between Democracy and Dictatorship. Theory and Decision 72, 149-166

Quesada A (2013a), The Majority Rule with a Chairman. Social Choice and Welfare 40, 679-691

Quesada A (2013b), To Majority Through the Search for Unanimity. Journal of Public Economic Theory 15, 729-735

Roberts K (1980a), Possibility Theorems with Interpersonally Comparable Welfare Levels. Review of Economic Studies 47, 409-420

Roberts K (1980b), Interpersonal Comparability and Social Choice Theory. Review of Economic Studies 47, 421-439

Roberts K (1980c), Social Choice Theory: The Single-Profile and Multi-Profile Approaches. Review of Economic Studies 47, 441-450

Sanver M (2009), Characterizations of Majoritarianism: A Unified Approach. Social Choice and Welfare 33, 159-171

Segal U and Sobel J (2002), Min, Max, and Sum. Journal of Economic Theory 106, 126-150

Sen A (1970), Collective Choice and Social Welfare. San Francisco: Holden-Day

Sen A (1974), Informational Bases of Alternative Welfare Approaches: Aggregation and Income Distribution. Journal of Public Economics 3, 387-403

Sen A (1977), On Weights and Measures: Informational Constraints in Social Welfare Analysis. Econometrica 45, 1539-1572

Sen A (2017), Collective Choice and Social Welfare, Expanded Edition. London: Penguin Books

Spears D and Zuber S (2023), Foundations of Utilitarianism Under Risk and Variable Population. Social Choice and Welfare 61, 101-129

Sprumont Y (2019), Relative Utilitarianism Under Uncertainty. Social Choice and Welfare 53, 621-639

Suzumura K (2002), Introduction, in: Arrow K; Sen A; Suzumura K (eds.), Handbook of Social Choice and Welfare, Volume 1. Amsterdam: Elsevier North Holland, 1-32

Terzopoulou Z (2020), Quota Rules for Incomplete Judgments. Mathematical Social Sciences 107, 23-36

Terzopoulou Z and Endriss U (2021), The Borda Class: An Axiomatic Study of the Borda Rule on Top-Truncated Preferences. Journal of Mathematical Economics 92, 31-40

Tideman N (1986), A Majority-Rule Characterization With Multiple Extensions. Social Choice and Welfare 3, 17-30

Wakker P (1988), The Algebraic Versus the Topological Approach to Additive Representations. Journal of Mathematical Psychology 32, 421-435

Wakker P (1989), Additive Representations of Preferences: A New Foundation of Decision Analysis. Dordrecht: Springer

Wakker P (1991), Additive Representations of Preferences, A New Foundation of Decision Analysis; The Algebraic Approach, in: Doignon J and Falmagne J (eds.), Mathematical Psychology: Current Developments. New York: Springer, 71-87

Weymark J (1991), A Reconsideration of the Harsanyi-Sen Debate on Utilitarianism, in: Elster J and Roemer J (eds.), Interpersonal Comparisons of Well-Being. Cambridge: Cambridge University Press, 255-320

Weymark J (2016), Social Welfare Functions, in: Adler M and Fleurbaey M (eds.), The Oxford Handbook of Well-Being and Public Policy. New York: Oxford University Press, 126-159

Woeginger G (2003), A New Characterization of the Majority Rule. Economics Letters 81, 89-94

Woeginger G (2005), More on the Majority Rule: Profiles, Societies, and Responsiveness. Economics Letters 88, 7-11

Xu Y and Zhong Z (2010), Single Profile of Preferences With Variable Societies: A Characterization of Simple Majority Rule. Economics Letters 107, 119-121

Yamamura H (2017), Interpersonal Comparison Necessary for Arrovian Aggregation. Social Choice and Welfare 49, 37-64

Young H (1974), An Axiomatization of Borda's Rule. Journal of Economic Theory 9, 43-52

Anhang der Dissertation

Liste der Einzelarbeiten dieser kumulativen Dissertation

Die kumulative Dissertation besteht aus drei Einzelarbeiten:

1. A Single Relation Theory of Welfarist Social Evaluation (Chapter 2)
2. Generalized Utilitarianism and Well-Being Comparability (Chapter 3)
3. How Much Do You Need to Take the Lead? Majority Rules for Incomplete Rankings (Chapter 4)

Liste der aus dieser Dissertation hervorgegangenen Veröffentlichungen

Die drei Einzelarbeiten der Dissertation sind bisher unveröffentlicht.

Abstract

The thesis analyzes the aggregation of individual inputs like preferences or well-being into an overall social evaluation of alternatives. It employs new concepts and frameworks to examine the structure and quality of aggregation rules. The results are relevant for social welfare evaluation and democratic decision-making.

The prevalent theory of social welfare in terms of Social Welfare Functionals is based on profiles of utility functions and uses problematic inter-profile conditions. To overcome this issue, chapter 2 offers a systematic relational theory of social evaluation over states of the world. It is formulated in terms of a single difference relation capturing basic well-being comparisons between states. Utility representations are derived and not presupposed. A new Fundamental Equity condition deals with trade-offs between individuals. It provides a unified rationale for existing two-person conditions and identifies deep connections between seemingly distinct approaches to social evaluation.

Dependent on the extent of interpersonal comparability, Strong Pareto and Fundamental Equity characterize Simple Majority Relation, Leximin, Utilitarianism, a level-based Borda Relation, and a new class of Additive Welfarist Relations. They yield a general impossibility given weak incomparability. The analysis extends to social evaluation under uncertainty and to variable populations. Well-being comparisons can be reduced to preferences of individuals and of an ethical observer. Overall, the theory connects welfare economics, social choice and voting theory, decision and measurement theory, as well as theories of social prospect evaluation and variable populations.

Chapter 3 offers a unified analysis of Generalized Utilitarianism with weak informational and normative assumptions. It connects different literatures. First, the chapter clarifies informational requirements. Contrary to other discussions, Generalized Utilitarian Orderings (including Prioritarian Orderings) are well-defined given sufficiently rich qualitative well-being difference and level comparisons. Second, the chapter provides characterizations of Generalized Utilitarian Orderings with weak separability and compensation conditions. Existing and new compensation properties are closely related to continuity.

Third, the chapter examines attempts to justify Utilitarianism. Maskin's influential characterization implicitly employs a strong substantive invariance requirement on social evaluation. A different characterization of Utilitarianism is based on a new stability condition and neither needs anonymity nor separability assumptions. Fourth, the chap-

ter extends the analysis to social evaluation under uncertainty, characterizing Expected Generalized Utilitarianism and Expected Utilitarianism. It establishes a perfect analogy between the certainty and the uncertainty case.

While votes are counted, the lead can switch between candidates. Based on this observation, chapter 4 offers a new concept to study the aggregation of incompletely known individual rankings. The Pull Ahead Index measures the number of new favorable rankings an alternative needs to take the social lead over another alternative. By studying conditions on the Index, novel insights into the structure of aggregation are identified. New and classical versions of Independence of Irrelevant Alternatives, Anonymity, and Neutrality are generally equivalent. Other properties do not have classical equivalents.

Relative and Quorum Majority Rules for incomplete inputs are characterized with conditions on the Index. While rules from both classes satisfy a Positive Responsiveness condition, they are distinguished by the response of the Index to opposing information. The Simple Majority Rule is characterized with the condition that alternatives should not have an implicit lead in terms of the Index. Its merit vis-à-vis qualified majority rules depends on whether ranking information is static or dynamic. This is both true in voting and policy evaluation contexts. One can also define Pull Ahead Indices for richer informational inputs.

Zusammenfassung

Die Arbeit analysiert die Aggregation von individuellen Inputs wie Präferenzen oder Wohlbefinden in eine gesellschaftliche Evaluation von Alternativen. Sie verwendet neue Konzepte und Frameworks um die Struktur und Qualität von Aggregationsregeln zu untersuchen. Die Resultate sind relevant für gesellschaftliche Wohlfahrtsevaluation und demokratische Entscheidungsfindung.

Die vorherrschende Theorie gesellschaftlicher Wohlfahrt mit sozialen Wohlfahrtsfunktionalen basiert auf Profilen von Nutzenfunktionen und nutzt problematische Interprofilbedingungen. Zur Lösung dieses Problems entwickelt Kapitel 2 eine systematische relationale Theorie gesellschaftlicher Wohlfahrt über Weltzustände. Sie basiert auf einer einzelnen Differenzrelation, die Wohlbefindensvergleiche zwischen Zuständen repräsentiert. Nutzenrepräsentationen werden abgeleitet und nicht vorausgesetzt. Eine neue fundamentale Gleichheitsbedingung bewertet Konflikte zwischen Individuen. Sie führt zu einer einheitlichen Begründung von existierenden Zweipersonenbedingungen und identifiziert tiefliegende Zusammenhänge zwischen scheinbar verschiedenen Aggregationsansätzen.

Abhängig von der Möglichkeit verschiedener interpersoneller Vergleiche charakterisieren die Paretobedingung und die Gleichheitsbedingung die einfache Mehrheitsregel, Leximin, Utilitarismus, eine Bordaregel und eine neue Klasse von additiven Wohlfahrtsregeln. Mit einer schwachen Unvergleichbarkeitsannahme führen sie zu einem Unmöglichkeitsresultat. Die Analyse schließt auch die gesellschaftliche Evaluation von Unsicherheit und variablen Bevölkerungen ein. Wohlbefindensvergleiche können auf die Präferenzen von Individuen und eines ethischen Beobachtenden reduziert werden. Insgesamt verbindet die Theorie Wohlfahrtsökonomik, Social Choice Theorie, Wahltheorie, Entscheidungs- und Messtheorie sowie Theorien gesellschaftlicher Evaluation unter Unsicherheit und mit variablen Bevölkerungen.

Kapitel 3 entwickelt eine einheitliche Analyse eines verallgemeinerten Utilitarismus mit schwachen normativen und Informationsannahmen. Es verbindet unterschiedliche Literaturen. Erstens klärt das Kapitel Informationsanforderungen. Im Gegensatz zu anderen Diskussionen sind verallgemeinerte Utilitarismusregeln (die Prioritarismusregeln einschließen) wohldefiniert, wenn bestimmte qualitative Wohlbefindensvergleiche möglich sind. Zweitens charakterisiert das Kapitel verallgemeinerte Utilitarismusregeln mit schwachen Separabilitäts- und Kompensationsbedingungen. Alte und neue Kompensationsbedingungen sind eng mit einer Stetigkeitsbedingung verbunden.

Drittens untersucht das Kapitel Versuche, den Utilitarismus zu rechtfertigen. Eine einflussreiche Charakterisierung benutzt implizit eine starke Invarianzbedingung. Eine andere Charakterisierung basiert auf einer neuen Stabilitätsbedingung und benutzt weder Anonymitäts- noch Separabilitätsbedingungen. Viertens erweitert das Kapitel die Analyse für die gesellschaftliche Evaluation unter Unsicherheit. Es charakterisiert den verallgemeinerten Erwartungsutilitarismus und den Erwartungsutilitarismus. Es etabliert eine perfekte Analogie zwischen den Fällen von Sicherheit und Unsicherheit.

Wenn Stimmen ausgezählt werden, kann die Führung zwischen Kandidierenden wechseln. Basierend auf dieser Beobachtung entwickelt Kapitel 4 ein neues Konzept für die Analyse der Aggregation von unvollständig bekannten individuellen Rankings. Der Pull Ahead Index misst die Anzahl der neuen positiven Rankings, die eine Alternative benötigt, um gegenüber einer anderen Alternative in Führung zu gehen. Über die Untersuchung von Bedingungen an den Index werden neue Einsichten in die Struktur der Aggregation identifiziert. Neue und klassische Versionen von Unabhängigkeits-, Anonymitäts- und Neutralitätsbedingugen sind im Allgemeinen äquivalent. Andere Bedingungen haben keine klassischen Äquivalente.

Relative und Quorumsmehrheitsregeln für unvollständige Inputs werden mit Bedingungen an den Index charakterisiert. Während Regeln beider Klassen eine positive Responsivitätsbedingung erfüllen, unterscheiden sie sich im Hinblick auf die Reaktion des Index auf entgegengesetzte Rankings. Die einfache Mehrheitsregel wird mit der Bedingung charakterisiert, dass Alternativen keine implizite Führung im Hinblick auf den Index haben. Ihre Qualität im Vergleich zu qualifizierten Mehrheitsregeln hängt davon ab, ob die Rankinginformation statisch oder dynamisch ist. Das ist sowohl bei Abstimmungen als auch bei der Politikevaluation der Fall. Man kann auch Pull Ahead Indizes für reichere Informationsinputs definieren.