Modal logics and intermediate logics motivated by an open problem on c.c.c. forcing

Dissertation

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Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Eigenanteilserklärung.

I declare that the results presented in this dissertation are entirely my own research work unless stated otherwise. Critical ideas proposed by others will be marked or acknowledged. To the best of my knowledge, results attributed to others in the literature will be attributed to the primary sources or standard references; if the primary source is unknown to me, they will be attributed to "folklore".

The results in Chapter 3 are mine, based on the ideas of Inamdar from [Ina20]. My supervisor assisted with the final text of §3.5 and §3.6. The content of Chapters 4 and 5 and some related material is the result of collaborations as outlined below:

1. The main results of Chapters 4 and 5 were obtained during my visit to the ILLC at the Universiteit van Amsterdam in collaboration with Nick Bezhanishvili and Gaëlle Fontaine. In Chapter 4, theorem 4.1.4, Propositions 4.1.6 and 4.1.7 are conjectures by Bezhanishvili, and the Lemma 4.2.10 is a joint work with Fontaine. In Chapter 5, one of the proofs of the main result was obtained together with Bezhanishvili and Fontaine. However, as was pointed out to me by Bezhanishvili, the result itself is due to Vincenzo Marra, who announced it at ToLo 2016 (Topological Methods in Logic 2016) in Tbilisi, Georgia.

2. The definition of "spiking" (Definition 3.2.1) is due to Löwe.

The results in Chapter 6 and 7 are mine.

Two joint publications are planned: one with Gaëlle Fontaine containing some of the material of Chapter 4, and another with Nick Bezhanishvili and Gaëlle Fontaine including results from Chapter 5. A paper entitled "Modal and intermediate logics of spiked Boolean algebras" [LX24] contains parts of Chapter 6 and additional results by Löwe on spikings that are not included in this thesis. This paper was submitted to a conference proceedings volume.

Contents

1. Acknowledgements

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2. Introduction

2.1. General motivation

Forcing is a powerful tool for constructing models of the set theory, introduced by Paul Cohen around 1962 (see [Coh63, Coh64] for the original work of Cohen). Forcing was first used to prove that the Continuum Hypothesis CH does not follow from the axioms of ZFC. Since then, forcing has become a core technology in set theory and other research areas in mathematical logic. Hamkins and Löwe have proposed to study the modal logic of the class of models of set theory with the forcing relation, called "the modal logic of forcing" [HL08].

This proposal was inspired by provability logic, going back to Gödel's paper [Göd33] in which he attempted to respond to issues related to Brouwer's intuitionistic logic. Provability logic is the modal logic where the $\Box$ operator is interpreted as provability; Solovay identified this modal logic in [Sol76, Theorem 4.6] (cf. also [AB05, Boo95, BV06, JdJ97]). Hamkins and Löwe aimed to do for forcing what Solovay had done for provability [HL08, § 1] and gave the following forcing interpretation for the modal operator respectively:

$$\Diamond\varphi \text{ if } \text{ there exists some forcing extension such that } \varphi \text{ holds}$$

and

$$\Box\varphi \text{ if } \text{ in every forcing extensions, we have } \varphi \text{ holds.}$$

They proved that the *modal logic of forcing* is exactly $\mathsf{S4.2}$ and then continued to study modal logics of forcing restricted to natural classes of forcings. If Γ is a class of forcing notions, they defined

$$\Diamond_\Gamma\varphi \text{ if there exists some forcing extension in } \Gamma \text{ such that } \varphi \text{ holds}$$

and

$$\Box_\Gamma \varphi \text{ if in every forcing extensions in } \Gamma, \text{ we have } \varphi \text{ holds,}$$

with each forcing class Γ corresponding to a modal logic. Hamkins, Leibman and Löwe proved that a modal logic called S4.tBa is an upper bound of the modal logic of ω_1-preserving forcing [HLL15, Theorem 36]. The logic S4.tBA is the least modal companion of a logic known as Medvedev logic Med and Hamkins, Leibman and Löwe conjectured that it is also an upper bound of the modal logic of c.c.c. forcing, one of the most important classes of forcing notions.

Medvedev logic had been studied in entirely different contexts before: it was introduced in [Med62] by Medvedev for the logic of finite problems to respond to Kolmogorov's [Kol32] informal interpretation of sentences of intuitionistic logic as a logic of problems. One of the most important results about Med is that Maksimova, Skvortsov and Shehtman proved that it cannot be finitely axiomatisable.

In his Master's thesis [Ina13], Inamdar aimed to show the mentioned conjecture by Hamkins, Leibman and Löwe, but instead of proving that S4.tBA is an upper bound, he identified a different, but closely related class of structures that he called *spiked Boolean algebras* and proved that their modal logic S4.sBA is an upper bound for the modal logic of c.c.c. forcing [Ina13, § 5]. Inamdar's result remains the best-known upper bound for that modal logic.

In this thesis, we aimed to improve on Inamdar's upper bound: Van Benthem, Guram Bezhanishvili and Gehrke introduced another intermediate logic called Cheq in [vBBG03]; it plays an important role in the modal logics of different topological spaces because of its interesting spatial logic properties, and whether it can be finitely axiomatised remains a famous open problem. The modal logic S4.FPFA is the least modal companion of Cheq and is contained in both S4.tBA and S4.sBA. If one could show that S4.FPFA was an upper bound for the modal logic of c.c.c. forcing, this would, in particular, prove the conjecture of Hamkins, Leibman and Löwe. Towards that goal, we developed an idea by Inamdar [Ina20], but were only able to show that S4.FPFA is an upper bound of the modal logic of c.c.c. forcing under an additional assumption (of which we do not know whether it is true).

The story about the modal logic of c.c.c. forcing recounted above contained three modal logics, S4.tBA, S4.sBA, and S4.FPFA as well as their corresponding intermediate logics, Med, LS, and Cheq. These six logics will be the protagonists of this thesis.

2.2. Overview of the dissertation structure

The rest of Chapter 2 introduces the background for the results of this dissertation: basic notions of modal logic and simple facts of c.c.c. forcing in set theory, the concepts, as well as results of the modal logic of forcing, introduced in [HL08]. It is worth noting that not all background and notations are summarized necessary for reading this thesis in this chapter; some from combinatorial mathematics (geometric combinatorics and poset topology, graph theory, etc.) that are needed in Chapter 5 and 7 will be reviewed in the relevant chapters.

In Chapter 3, we first recall what is known about the upper bounds of the modal logic of c.c.c forcing so far. Hamkins, Leibman and Löwe conjectured that S4.tBa is an upper bound in [HLL15, § 5.6] and Inamdar proved that S4.sBa is an upper bound in [Ina13, § 5]. Inamdar never published his result since he had an idea to improve on it using gaps instead of Suslin trees. He wrote up some notes about his ideas in 2020 [Ina20]; based on these notes, S4.FPFA for finite pre-partial function algebra was conjectured to be an upper bound in Theorem 3.6.3. We have three different frames **tBa**, **sBa** and **FPFA** and corresponding logics Med, LS and Cheq, then we give a comparison of those logics in Corollary 3.4.3, Theorem 3.4.4 and 3.4.6. We will then give results on Med in Chapter 4 and Chapter 5, arrange results on the logic of **sBa** in Chapter 6, and discuss the results of Cheq in Chapter 7.

Chapter 4 gives positive answers to conjectures by Nick Bezhanishvili for generalized Medvedev logics in Corollary 4.2.2 and Theorem 4.2.12. Bezhanishvili noticed that the result by Maksimova, Skvortsov and Shehtman in [MSS79] that Med is not finitely axiomatisable is about product of 2-chains and asked whether the result can be generalized to any chain or any finite rooted frame with a top. In our proof of the conjecture, Lemma 4.2.10 is joint work with Fontaine. Besides this, Theorem 4.1.4 also shows that the logic of the product of any finite rooted frame with a top is exactly the logic KC (also inspired by Nick Bezhanishvili). At the end of this chapter, it is proved that there are at least countable many different generalized Medvedev logics in Theorem 4.3.1 and there is no least such logic in Theorem 4.3.2.

Chapter 5 connects the well-known concept of nerve and Medvedev logic Med in Theorem 5.2.1. One of the proofs was obtained together with Bezhanishvili and Fontaine. In this chapter, the main result is proved by two different methods. One of them reveals certain geometric aspects of nerves and simplicial complexes.

In Chapter 6, we consider the class of finite spiked Boolean algebra and show that the modal and intermediate logics of it are not finitely axiomatisable in Theorem 6.1.5 and Corollary 6.1.7. We prove a lemma called the bow-tie lemma and use it to show that the logic of finite spiked Boolean algebra is not finitely axiomatisable over Cheq in Theorem 6.3.1. In the proof of the results in this chapter, we obtained Proposition 6.2.5, which provides a result concerning the relationship between the generalized Medvedev logic and Cheq.

Chapter 7 summarizes that the finite partial function algebra **FPFA** can be regarded as three different algebraic structures in Theorem 7.2.5, containing geometrical aspects as the face poset of the n-cube. It also computes the maximal size of its antichain in Theorems 7.1.12. Following Kuznetsov's idea of applying Offner's edge-coloring result [Off08] to the finite axiomatisation of Cheq from his unpublished note, we discuss the two strategies of Fontaine and Shatrov, as well as Kuznetsov's alternative solution, for proving that Cheq is not finitely axiomatisable. Finally, we provide a construction showing that Cheq is not finitely axiomatisable with five or six variables in Theorem 7.5.1.

2.3. Intuitionistic logic and intermediate logics

We first recall the definitions of IPC and intermediate logics. Let $\mathcal{L}$ be a language with

1. countably many propositional variables $p_0, p_1, \ldots$, which form a set **Prop**,
2. logical connectives $\vee$, $\wedge$ and $\rightarrow$,
3. a propositional constant $\bot$.

Definition 2.3.1. *Intuitionistic propositional calculus* IPC is the least set of formulas together with the following axioms:

1. $p_0 \rightarrow (p_1 \rightarrow p_0)$
2. $(p_0 \rightarrow (p_1 \rightarrow p_2)) \rightarrow ((p_0 \rightarrow p_1) \rightarrow (p_0 \rightarrow p_2))$

3. $p_0 \wedge p_1 \to p_0$

4. $p_0 \wedge p_1 \to p_1$

5. $p_0 \to p_0 \vee p_1$

6. $p_1 \to p_0 \vee p_1$

7. $(p_0 \to p_2) \to ((p_1 \to p_2) \to ((p_0 \vee p_1) \to p_2))$

8. $\bot \to p_0$

and is closed under the inference rules *Modus Ponens*

$$\text{given } \varphi \text{ and } \varphi \to \psi, \text{ we obtain } \psi \quad \textbf{(MP)}$$

and *Substitution*

$$\text{given } \varphi(p_1, \ldots, p_n), \text{ we obtain } \varphi(\psi_1, \ldots, \psi_n). \quad \textbf{(Subst)}$$

Definition 2.3.2. *Classical propositional calculus* CPC is the smallest logic containing both IPC and $p \vee \neg p$. An *intermediate logic* L is a set of formulas that is closed under the above inference rules, contains IPC and is contained in CPC.

An intermediate logic L is called *finitely axiomatisable* if there exists a finite set of formulas C such that L is the smallest intermediate logic that contains C. $\mathsf{L} + \varphi$ is denoted as the smallest intermediate logic containing $\mathsf{L} \cup \{\varphi\}$.

2.3.1. Posets as Kripke frame

Definition 2.3.3. Given a non-empty set S, if R is a reflexive, transitive and antisymmetric binary relation on S, then R is called a *partial order* on S. We usually write R as $\leq$, and the set together with $\leq$ is a *partially ordered set*, or simply, a *poset*.

Definition 2.3.4. An *intuitionistic Kripke frame* is a partially ordered set $\mathcal{F} = \langle W, \leq \rangle$, and any $x \in W$ is called a *point* or *node*. If $x \leq y$, then y is *accessible* from x.

Definition 2.3.5. An *intuitionistic valuation* is a map V form **Prop** to the power set of W and for each $x \in V(p)$, if $x \le y$, then $y \in V(p)$. An *intuitionistic Kripke model* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is a pair of an intuitionistic frame $\mathcal{F}$ and an intuitionistic valuation V.

The relation $\vDash$ between frames (models) and formulas is defined as usual; see, e.g., [CZ97]. We define a relation $\mathcal{M}, x \vDash \varphi$, which is read as φ is *true* at x in $\mathcal{M}$, as follows:

$$\begin{aligned}
\mathcal{M}, x \vDash p &\quad \text{iff} \quad x \in V(p);\\
\mathcal{M}, x \vDash \varphi \wedge \psi &\quad \text{iff} \quad \mathcal{M}, x \vDash \varphi \text{ and } \mathcal{M}, x \vDash \psi;\\
\mathcal{M}, x \vDash \varphi \vee \psi &\quad \text{iff} \quad \mathcal{M}, x \vDash \varphi \text{ or } \mathcal{M}, x \vDash \psi;\\
\mathcal{M}, x \vDash \varphi \to \psi &\quad \text{iff} \quad \forall y \in W, (x \le y \text{ and } \mathcal{M}, y \vDash \varphi) \text{ implies } \mathcal{M}, y \vDash \psi;\\
\mathcal{M}, x \nvDash \bot.&
\end{aligned}$$

Definition 2.3.6. A formula φ is said to be *satisfied* in a model $\mathcal{M}$ if there exists some point x such that $\mathcal{M}, x \vDash \varphi$ and φ is said to be *true* in $\mathcal{M}$ if for any point x, we have $\mathcal{M}, x \vDash \varphi$, in this case, it is denoted by $\mathcal{M} \vDash \varphi$.

A formula φ is said to be *satisfied* in frame $\mathcal{F}$ if there exists some model $\mathcal{M}$ on $\mathcal{F}$ such that φ is satisfied in the model $\mathcal{M}$. A formula φ is said to be *true at x* in $\mathcal{F}$ if for all models $\mathcal{M}$ on $\mathcal{F}$, it is true at x, in this case, it is denoted by $\mathcal{F}, x \vDash \varphi$. The notation $\mathcal{F} \vDash \varphi$ denotes that φ is *valid* in $\mathcal{F}$, if for all model $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \vDash \varphi$.

Definition 2.3.7. If $\mathbf{C}$ is a class of finite frames, let $\mathsf{Log}(\mathbf{C})$ be the logic consisting of all valid formulas on every frame of $\mathbf{C}$. We call it the *logic of* $\mathbf{C}$. In this case, we say that $\mathbf{C}$ *characterizes* or *determines* the logic. Given a logic L, a frame $\mathcal{F}$ is an L-*frame* if each formula in L is valid in $\mathcal{F}$.

We introduce some general terminology that will be used to describe finite frames $\mathcal{F} = \langle W, \le \rangle$.

Given a point x, call y is a *successor* (or *predecessor*) of x if $x \le y$ (or $y \le x$) and y is *proper* if $x \ne y$ and we denote it as $x < y$ (or $y < x$). Additionally, y is *immediate* if y is proper and there is no $z \notin \{x, y\}$, such that z located in the middle between x and y. The notation $x\!\uparrow$ (or $x\!\downarrow$) denotes the *principal upset* $\{y \in W : x \le y\}$ (respectively, *principal downset* $\{y \in W : y \le x\}$).

If $\mathcal{F}$ has a least element $\bar{0}$, an element a of $\mathcal{F}$ is an *atom* if, $\bar{0} < a$ and for every $x \in \mathcal{F}$, $x \le a \to x = a$ or $x = \bar{0}$. The point x is called *maximal* if there are no $y \in \mathcal{F}$ such that $x < y$. An element b is called a *coatom* if it is not maximal and all of its immediate successors are maximal. A *top* is a point t such that for all point $x \in W$, we have $x \leq t$. We say that $\mathcal{F}$ *has a top* if there exists a top in $\mathcal{F}$ and it is obvious that the top is unique.

If $\mathcal{F}$ is transitive, we call a subset $C \subseteq W$ a *chain* if for any distinct points $x, y \in C$, $x \leq y$ or $y \leq x$ holds. The *depth* of $\mathcal{F}$ is the largest number n with the following property: one can find a n-sized chain in $\mathcal{F}$. We call a subset $A \subseteq W$ an *antichain* if for any distinct points $x, y \in A$, neither $x \leq y$ nor $y \leq x$ holds. The *width* of $\mathcal{F}$ is the largest number n with the property: one can find a n-sized antichain in $\mathcal{F}$. The *branching degree* $b(x)$ of x is the number of immediate successors of x.

2.3.2. Truth preserving operations

We recall the main operations that preserve truth between frames in this part. One may find more details in [CZ97, BRV01].

Definition 2.3.8 (Generated subframe)**.** A frame $\mathcal{S} = \langle W', \leq' \rangle$ is a *subframe* of $\mathcal{F} = \langle W, \leq \rangle$ if $W' \subseteq W$ and $\leq' = \leq \restriction_{W'}$. Furthermore, the subframe $\mathcal{S}$ is *generated* if W' is an upward closed subset of W.

Definition 2.3.9. We say that $\mathcal{S}$ is *generated by the set* X if $\mathcal{S}$ is a generated subframe and W' is the smallest upward closed subset containing X. If $\mathcal{F}$ is generated by $\{x\}$, then it is called *rooted* and x the *root of* $\mathcal{F}$. For any point x, the depth of the subframe generated by it is the *depth of* x (notation: $d(x)$).

Theorem 2.3.10. For any frame $\mathcal{F}$ and formula φ, the following are equivalent:

1. $\mathcal{F} \vDash \varphi$;
2. $\mathcal{F}' \vDash \varphi$ for every generated subframe $\mathcal{F}'$ of $\mathcal{F}$;
3. $\mathcal{F}' \vDash \varphi$ for every rooted generated subframe $\mathcal{F}'$ of $\mathcal{F}$.

Definition 2.3.11 (p-morphisms)**.** Given two frames $\mathcal{F}$ and $\mathcal{F}'$ and a map $f : W \to W'$, we call f a *p-morphism* from $\mathcal{F}$ to $\mathcal{F}'$ if:

1. for all $x, y \in W$, if $x \leq y$, then $f(x) \leq' f(y)$,
2. if $f(x) \leq' t$, then there exists $y \in W$, such that $x \leq y$ and $f(y) = t$.

If f is onto, then we call $\mathcal{F}'$ a *p-morphic image* of $\mathcal{F}$.

Theorem 2.3.12 (Folklore)**.** If $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$, then for any formula φ,

$$\mathcal{F} \vDash \varphi \text{ implies } \mathcal{F}' \vDash \varphi.$$

A class $\mathbf{C}$ of frames is *closed under rooted generated subframes* if each rooted generated subframe of element in $\mathbf{C}$ is isomorphic to an element of $\mathbf{C}$.

Theorem 2.3.13 (Jankov-de Jongh theorem)**.** Given a finite rooted frame $\mathcal{F}$, there exists a formula $\chi(\mathcal{F})$ such that for every frame $\mathcal{F}'$,

$$\mathcal{F}' \nvDash \chi(\mathcal{F}) \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathcal{F}'.$$

Corollary 2.3.14 (Folklore)**.** Given a class of finite frames $\mathbf{C}$ that is closed under rooted generated subframes, for any finite rooted frame $\mathcal{F}$,

$$\mathcal{F} \vDash \mathsf{Log}(\mathbf{C}) \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of element of } \mathbf{C}.$$

2.4. Modal logic

In this part, we are going to recall the basic concepts of modal logic. We turn to the modal language $\mathcal{ML}$ as an extension of $\mathcal{L}$ with the extra operators $\Box$ and $\Diamond$.

Definition 2.4.1. The modal logic K is the least set containing classical propositional calculus together with the following axioms:

1. $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$,
2. $\Box p \leftrightarrow \neg \Diamond \neg p$.

and is closed under the inference rules modus ponens **(MP)**, substitution **(Subst)** and *Necessitation*

$$\text{Given } \varphi, \text{ we infer } \Box \varphi \quad \textbf{(N)}.$$

Definition 2.4.2. A *modal Kripke frame* is $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set consisting of *points* and R is a binary relation on W. A *modal Kripke model* is $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a modal Kripke frame and V is a *valuation* that maps any propositional variable to some subsets of W.

We recursively define $\mathcal{M}, x \vDash \varphi$, which means φ is *true* at x in the model $\mathcal{M}$, as follows:

$$
\begin{array}{rll}
\mathcal{M}, x \vDash p & \text{iff} & x \in V(p); \\
\mathcal{M}, x \vDash \varphi \wedge \psi & \text{iff} & \mathcal{M}, x \vDash \varphi \text{ and } \mathcal{M}, x \vDash \psi; \\
\mathcal{M}, x \vDash \varphi \vee \psi & \text{iff} & \mathcal{M}, x \vDash \varphi \text{ or } \mathcal{M}, x \vDash \psi; \\
\mathcal{M}, x \vDash \varphi \rightarrow \psi & \text{iff} & \mathcal{M}, x \vDash \varphi \text{ implies } \mathcal{M}, x \vDash \psi; \\
\mathcal{M}, x \nvDash \bot; & & \\
\mathcal{M}, x \vDash \Box\varphi & \text{iff} & \text{for all } y \in W \text{ such that } xRy \text{ we have } \mathcal{M}, y \vDash \varphi.
\end{array}
$$

and so

$$
\begin{array}{rll}
\mathcal{M}, x \vDash \neg\varphi & \text{iff} & \mathcal{M}, x \nvDash \varphi; \\
\mathcal{M}, x \vDash \Diamond\varphi & \text{iff} & \text{there exists } y \in W \text{ such that } xRy \text{ and } \mathcal{M}, y \vDash \varphi.
\end{array}
$$

The definitions of truth and validity in modal frames (models), as well as truth-preserving operations, are the same as in Definition 2.3.6 and Subsection 2.3.2.

Definition 2.4.3. If $\mathbf{C}$ is a class of finite modal frames, let $\mathsf{ML}(\mathbf{C})$ be the modal logic consisting of all valid modal formulas on every frame of $\mathbf{C}$. We call it the *modal logic of* $\mathbf{C}$. In this case, we say that $\mathbf{C}$ *characterizes* or *determines* this modal logic. Given a modal logic ML, a frame $\mathcal{F}$ is a ML-*frame* if each modal formula in ML is valid in $\mathcal{F}$.

We give some examples, and more details can be found in [CZ97, BRV01].

1. $\mathsf{K4} = \mathsf{K} + (\Box p \rightarrow \Box\Box p)$, is characterized by the class of transitive frames.

2. $\mathsf{S4} = \mathsf{K4} + (\Box p \rightarrow p)$, is characterized by the class of pre-orders.

3. $\mathsf{Grz} = \mathsf{K} + \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$, is characterized by the class of finite posets.

4. $\mathsf{S4.2} = \mathsf{S4} + (\Diamond \Box p \rightarrow \Box \Diamond p)$, is characterized by the class of directed pre-orders.

In the 1930s, Gödel suggested a translation T which can embed IPC into the modal logic $\mathsf{S4}$ in [Göd33].

Definition 2.4.4. The *Gödel-translation* T is the map defined as follows:

1. $\mathrm{T}(\bot) = \bot$;
2. $\mathrm{T}(p) = \Box p$, where $p \in \mathbf{Prop}$;
3. $\mathrm{T}(\varphi_0 \wedge \varphi_1) = \mathrm{T}(\varphi_0) \wedge \mathrm{T}(\varphi_1)$;
4. $\mathrm{T}(\varphi_0 \vee \varphi_1) = \mathrm{T}(\varphi_0) \vee \mathrm{T}(\varphi_1)$;
5. $\mathrm{T}(\varphi_0 \to \varphi_1) = \Box(\mathrm{T}(\varphi_0) \to \mathrm{T}(\varphi_1))$.

In the 1940s, McKinsey and Tarski proved that the Gödel translation embeds IPC to $\mathsf{S4}$ in [MT48, § 5]. In the 1950s, Dummett and Lemmon extend this result to intermediate logics and extensions of $\mathsf{S4}$ in [DL59, Theorem 1]. In the 1970s, Esakia [Esa79b, Esa79a] developed the theory of Heyting algebras; this was independently done by Maksimova and Rybakov [MR74] and also by Blok [Blo76]. These research works lead to a theory of modal companions, which will be introduced later.

Definition 2.4.5. A modal logic $\mathsf{M} \supseteq \mathsf{S4}$ is called a *modal companion* of an intermediate logic L, if

$$\varphi \in \mathsf{L} \text{ iff } \mathrm{T}(\varphi) \in \mathsf{M}, \text{ for each intuitionistic formula } \varphi.$$

If $\mathcal{F}_0 = \langle W_0, R_0 \rangle$ is a finite poset, we say that $\mathcal{F} = \langle W, R \rangle$ is a *thickening of* $\mathcal{F}_0$ if $\mathcal{F}$ is a finite partial pre-order (i.e., R is reflexive and transitive, but not necessarily anti-symmetric), $\sim$ is the induced equivalence relation (i.e., $x \sim y \Leftrightarrow xRy$ and yRx) and $\mathcal{F}_0 = \mathcal{F}/\sim$. The *cluster* of x, denoted by $C(x)$, is the $\sim$ equivalence class containing x, i.e., $C(x) = \{y : y \sim x \wedge y \in W\}$. Given a class $\mathbf{C}$ of finite partial orders, let $\mathbf{C}^\bullet$ be the class of thickenings of elements of it; its elements are usually referred to with the prefix "pre-".

Theorem 2.4.6 (Folklore; cf. [She90, Proposition 7])**.** For any intermediate logic L, both the least and the greatest modal companions exist. Let $\tau(\mathsf{L})$ be the least modal companion and

$$\tau(\mathsf{L}) = \mathsf{S4} + \{\mathrm{T}(\varphi) : \varphi \in \mathsf{L}\},$$

let $\sigma(\mathsf{L})$ be the greatest modal companion and

$$\sigma(\mathsf{L}) = \mathsf{Grz} + \{\mathrm{T}(\varphi) : \varphi \in \mathsf{L}\}.$$

Theorem 2.4.7.

1. The map τ is an isomorphism of the lattice of intermediate logics into the lattice of extensions of $\mathsf{S4}$.
2. (The Blok-Esakia theorem) The map σ is an isomorphism from the lattice of intermediate logics onto the lattice of extensions of Grz.

Theorem 2.4.8 (Esakia; cf. [She90, Proposition 9])**.** For any class $\mathbf{C}$ of finite posets, the modal logic of $\mathbf{C}$ is the greatest modal companion of its logic, that is, $\mathsf{ML}(\mathbf{C}) = \sigma(\mathsf{Log}(\mathbf{C}))$.

Theorem 2.4.9 (Zakharyaschev; cf. [She90, Proposition 10])**.** For any class $\mathbf{C}$ of finite posets, the modal logic of $\mathbf{C}^{\bullet}$ is the least modal companion of its logic, that is, $\mathsf{ML}(\mathbf{C}^{\bullet}) = \tau(\mathsf{Log}(\mathbf{C}))$.

2.5. Forcing background

In this section, we briefly recall the definitions and some facts of forcing in the set theory. For further results and details of forcing, one can consult [Jec03, Kun14].

Assume that M is a countable transitive model of ZFC, the *ground model*.

Definition 2.5.1. A *forcing poset* is a triple $\mathbb{P} = (P, \leq, \mathbf{1})$, where $\leq$ is a partial order with separative condition and for every $p \in P$, $p \leq \mathbf{1}$. An element $p \in P$ is called *forcing condition*. We say that p is *stronger* than q if $p \leq q$.

Definition 2.5.2. For a forcing poset $\mathbb{P}$, the conditions p and q are *compatible* if there is s such that $s \leq p$ and $s \leq q$; the conditions p and q are *incompatible* if they are not compatible. A set $A \subseteq \mathbb{P}$ is an *antichain* if its elements are pairwise incompatible.

Definition 2.5.3. A set $F \subseteq P$ is a *filter* on $\mathbb{P}$ if

1. $\mathbf{1} \in F$;
2. $p \leq q$ and $p \in F$ imply $q \in F$;
3. if $p, q \in F$, then there exists $s \in F$ such that $s \leq p$ and $s \leq q$.

A set $G \subseteq P$ is $\mathbb{P}$-*generic* over M or M-*generic* for $\mathbb{P}$ or simply, *generic*, if G is a filter and for any dense $D \subseteq P$ such that $D \in M$, we have $G \cap D \neq \varnothing$.

We focus on the structure itself in the forcing poset in this work and do not intend to discuss all facts about the forcing theory. Specifically, we omit the definitions of names and the semantic and syntactic forcing relations. The main class of forcings we study here are the c.c.c. forcings.

Definition 2.5.4. A forcing poset $\mathbb{P}$ has the *countable chain condition* or to have c.c.c., if each antichain is countable.

The Knaster property is stronger than the countable chain condition.

Definition 2.5.5. A forcing poset $\mathbb{P}$ is *Knaster* if every uncountable set $A \subseteq P$ contains an uncountable subset $B \subseteq A$ of pairwise compatible conditions.

Recall the following forcing poset Coh, the set consisting of the finite partial functions from ω to $\{0,1\}$ and ordered by the reverse inclusion, it is the so-called *Cohen forcing* that adds a *Cohen real*:

1. for any $p \in \mathrm{Coh}$, $\mathrm{dom}(p)$ is a finite subset in ω;
2. p is stronger than q if $p \supseteq q$.

Let G be generic in the Cohen poset and $c = \bigcup\{p \in \mathrm{Coh} : p \in G\}$, then c is a function $c : \omega \to \{0,1\}$ and called a *Cohen real*.

Theorem 2.5.6 (Folklore; cf. [Ina13, Theorem 20])**.** Given two transitive models of set theory $V \subseteq V'$, then $\mathrm{Coh}^V = \mathrm{Coh}^{V'}$ and a Cohen real over V' is also Cohen over V.

Definition 2.5.7. In the model of set theory M, given two forcing posets $\mathbb{P}, \mathbb{Q}$, the *product*, $\mathbb{P} \times \mathbb{Q}$, is the forcing poset whose elements are (p,q), where $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ and its order is: $(p_0, q_0) \leq (p_1, q_1)$ iff $p_0 \leq p_1$ and $q_0 \leq q_1$.

Definition 2.5.8. Let $\{\mathbb{P}_i : i \in I\}$ be a collection of forcing posets, each having the greatest element $\mathbf{1}$. The *product* $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ defined as follows:

1. Every elements p of $\mathbb{P}$ is a function such that for any $i \in I$, $p(i) \in \mathbb{P}_i$ and the set $\{i \in I : p(i) \neq \mathbf{1}\}$ is finite.
2. $p \leq q$ iff $p(i) \leq q(i)$ for all $i \in I$.

For any p, we say the finite set $\{i \in I : p(i) \neq \mathbf{1}\}$ the *support* of p. For any subset $U \subseteq I$, $p \restriction U$ is a function whose support is the intersection of the support of p and U and for any $i \in U$, $p \restriction U(i) = p(i)$. Then the *projection* of $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ to U (notation: $(\prod_{i \in I} \mathbb{P}_i) \restriction U$) is the sub-poset of $\mathbb{P}$ consisting $\{p \restriction U : p \in \mathbb{P}\}$. It follows that $(\prod_{i \in I} \mathbb{P}_i) \restriction U \cong \prod_{i \in U} \mathbb{P}_i$.

Lemma 2.5.9 (Folklore; cf. [Kun14, p. 328])**.** If $\mathbb{P}$ and $\mathbb{Q}$ has the c.c.c, then the following are equivalent:

1. $\mathbb{P} \times \mathbb{Q}$ has the c.c.c.;
2. $\mathbf{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ has the c.c.c.;
3. $\mathbf{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \dot{\mathbb{P}}$ has the c.c.c.

2.6. Modal logic of forcing

Hamkins and Löwe introduced and developed the area of using modal logic to describe forcing in set theory in their original paper [HL08]. We will follow their idea and interpret the modal operator $\Box$ as "in every forcing extensions" while $\Diamond$ is interpreted as "in some forcing extension".

Definition 2.6.1. For a statement φ in set theory, $\Box\varphi$ means that for each poset $\mathbb{P}$ and condition p in it, $p \Vdash_{\mathbb{P}} \varphi$ and $\Diamond\varphi$ means that there exists some poset $\mathbb{P}$ and condition p in it, $p \Vdash_{\mathbb{P}} \varphi$.

Definition 2.6.2. Given a modal assertion $\phi(p_1, \ldots, p_n)$, if $\phi(\varphi_1, \ldots, \varphi_n)$ holds whenever φ_i is an arbitrary sentence in set theory and $\Box$ and $\Diamond$ is interpreted by the above forcing interpretation, then is said to be a *valid principle of forcing.* It is a ZFC-*provable principle of forcing* if ZFC proves all substitution instances $\phi(\varphi_1, \ldots, \varphi_n)$. This naturally generalizes to larger theories with the notion of a T-provable principle of forcing. For any model M of set theory, the modal assertion $\phi(p_1, \ldots, p_n)$ is a *valid principle of forcing in* M if all substitution instances $\phi(\varphi_1, \ldots, \varphi_n)$ are true in M.

For example, $\neg \Diamond p \leftrightarrow \Box\neg p$ is a valid principle of forcing, since φ is not forceable iff in every forcing extension, we have $\neg\varphi$. Hamkins and Löwe's provided the following theorem (see [HL08, Main Result 6]).

Theorem 2.6.3. If ZFC is consistent, then the ZFC-provable principles of forcing are exactly S4.2.

Restricting the class of forcing notions can change the interpretation of the modal operators, then the modal logic of this new class of forcing raises different questions. If Γ is a class of forcing notions, then $\Diamond_\Gamma\varphi$ means that there exists some forcing extension by forcing in Γ such that φ holds and $\Box_\Gamma \varphi$ means that φ holds in every forcing extensions by forcing in Γ. A modal assertion $\phi(p_1, \ldots, p_n)$ is said to be a *valid principle of* Γ*-forcing* if, for all substitution instances $\phi(\varphi_1, \ldots, \varphi_n)$ with arbitrary set-theoretic assertion φ_i, this modal assertion holds under the Γ-forcing interpretation for modal operators.

Given a forcing class Γ, each assignment of propositional variables p_i to set-theoretical assertions φ_i can be extended to a Γ *forcing translation.* This is a function H from the modal language $\mathcal{ML}$ to the first-order language of set theory such that $H(p_i) = \varphi_i$ and follows the rules: $H(\phi \wedge \psi) = H(\phi) \wedge H(\psi)$, $H(\neg\phi) = \neg H(\phi)$ and $H(\Box\phi) = \Box_\Gamma H(\phi)$. The last case asserts in set theory that $H(\phi)$ has Boolean value one for every forcing notion of Γ. The *modal logic of* Γ *forcing* over a model of set theory M is the set: $\{\phi \in \mathcal{ML} : M \vDash H(\phi)$ for every Γ forcing translations $H\}$. When we restrict the classes of forcing to be c.c.c. forcing, then the modal operator $\Box$ is interpreted as "in every c.c.c. forcing extensions" and $\Diamond$ is interpreted as "in some c.c.c. forcing extension". The modal logic of c.c.c forcing is denoted by $\mathsf{ML}_{\text{c.c.c.}}$. According to [HL08], the most interesting open problems are posed as follows:

Question (Hamkins and Löwe)**.** *What is the modal logic of c.c.c. forcing?*

In general, like the original method used in [HL08], in the forcing case, one may need to find or build the upper bound and lower bound for it. In the case of lower bounds, one may need to study the property of the modal frame formed by the forcing or c.c.c. forcing.

Theorem 2.6.4. All theorems of $\mathsf{S4}$ are valid principles of $\mathsf{ML}_{\text{c.c.c.}}$.

Proof. To verify K, suppose that both p_0 and $p_0 \to p_1$ hold in every c.c.c. extensions, thus in every c.c.c. extensions, we have p_1 holds and it is easy to see that $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$ holds. We have already shown that $\neg \Diamond p \leftrightarrow \Box \neg p$ as an example. S is obvious. For $\mathsf{4}$, according to the property about iteration of c.c.c. forcing, $\Box p \to \Box\Box p$ holds.

Finally, $\mathsf{ML}_{\text{c.c.c.}}$ is closed under those inference rules, moreover, for necessitation, when we already have p in every model, it is obvious that $\Box p$ holds in each model. □

Before discussing the results about the upper bounds of $\mathsf{ML}_{\text{c.c.c.}}$, we may recall the method of Hamkins, Leibman and Löwe to determine the upper bounds of the modal logic of Γ-forcing when we assume that the forcing class is Γ-forcing. The follows come from [HLL15, Definition 8] and [HLL15, Lemma 9].

Definition 2.6.5 (Hamkins, Leibman and Löwe)**.** A Γ-*labeling* of a frame $\mathcal{F}$ for a model M of set theory is an assignment $m \mapsto \phi_m$, from points in $\mathcal{F}$ to o set-theoretic statements with the following property:

1. The statements ϕ_m form a mutually exclusive partition of truth in the Γ-forcing extensions of M, i.e., given a Γ-extension, it satisfies exactly one ϕ_m.

2. Any Γ-forcing extension with ϕ_m true satisfies $\Diamond\phi_u$ iff $m \leq u$.

3. $M \vDash \phi_{m_0}$, if m_0 is initial in $\mathcal{F}$.

Theorem 2.6.6 (Hamkins, Leibman and Löwe)**.** Let $m \mapsto \phi_m$ be a Γ-labelling of finite frame $\mathcal{F}$ for model M and x_0 be initial in $\mathcal{F}$. Then for any Kripke model $\mathcal{M}$ whose frame is $\mathcal{F}$, there exists an assignment $p \mapsto \psi_p$ from **Prop** to set-theoretic assertions, and for any $\varphi(p_1, \ldots, p_n)$,

$$\mathcal{M}, x_0 \vDash \varphi(p_1, \ldots, p_n) \text{ iff } M \vDash \varphi(\psi_{p_1}, \ldots, \psi_{p_n}).$$

In conclusion, the modal logic of Γ-forcing over M is contained in the modal logic of assertions valid in $\mathcal{F}$.

Proof. A proof of this theorem can be found in [Ina13, Theorem 57]. □

The existence of a class of Γ-labellings can be imaginatively or even visually understood by the construction of statements called buttons and switches introduced in [HL08, § 2]. A *switch* in W satisfies that $W \vDash \Box(\Diamond s \wedge \Diamond\neg s)$. We call a switch is *on* in the Γ forcing extension $W[G]$ if $W[G] \vDash s$ is true and call a switch is *off* if $W[G] \vDash \neg s$. A *button* in W satisfies that $W \vDash \Box\Diamond\Box b$. It is *pushed* if $\Box b$ holds and *pure* if $W \vDash \Box(b \to \Box b)$. A finite set consisting of switches and buttons is said to be *independent* in W if each one of them can be operated without affecting the others.

3. The modal logic of c.c.c. forcing and finite partial function algebras

This chapter introduces three algebraic structures: **tBa**, **sBa**, and **FPFA**, and their corresponding intermediate and modal logics. In [HLL15, Theorem 36], it was proved that S4.tBa is an upper bound of the modal logic of ω_1-preserving forcing, and Hamkins, Leibman and Löwe conjectured it to be an upper bound of $\mathsf{ML}_{\text{c.c.c.}}$. Hamkins and Löwe proved that $\mathsf{ML}_{\text{c.c.c.}}$ does not include S4.2 by applying a c.c.c.-labeling to a concrete frame, 2-fork[1], in [HL08, Theorem 34] and this frame is a topless Boolean algebra. In [Ina13, Theorem 149], Inamdar proved that the modal logic of c.c.c. forcing is contained in S4.sBa. Since

$$\mathsf{S4} \subsetneq \mathsf{S4.tBA} \subsetneq \mathsf{S4.2} \subsetneq \mathsf{S4.3}$$

and

$$\mathsf{S4} \subsetneq \mathsf{S4.sBA} \subsetneq \mathsf{S4.2} \subsetneq \mathsf{S4.3},$$

S4.tBA and S4.sBA occupy similar positions within the landscape of modal logics, and **sBa** is also a generalized form of the above 2-fork frame. Finally, the modal logic S4.FPFA is the least modal companion of the well-known logic Cheq. By developing an idea from Inamdar [Ina20], we shall show that S4.FPFA is an upper bound of the modal logic of c.c.c. forcing, under an additional assumption (of which we do not know whether it is true) in Theorem 3.6.3. The finite partial function algebra $\mathcal{R}_n \cong \mathcal{C}_n$, being the product of the 2-fork frame and also a generalized form of 2-fork, provides a suitable motivation for comparing these three structures and their related logics.

[1] We define the *2-fork* as the frame $\mathcal{C}_1 := \langle W, R\rangle$, where $W = \{0, 1, 2\}$, all nodes are R-reflexive, and both 1 and 2 are R-accessible from 0.

3.1. The logic of finite topless Boolean algebras

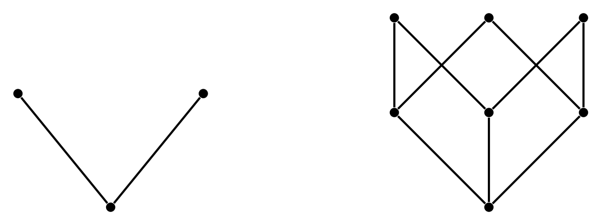


Figure 3.1-1: Medvedev frames

Definition 3.1.1 (Finite Boolean algebra)**.** For a non-empty set with n elements $[n] = \{1, 2, \ldots, n\}$, let $\mathcal{P}(n)$ denote the Kripke frame[2]:

$$\mathcal{P}(n) = \langle \{X : X \subseteq [n]\}, \supseteq \rangle.$$

We call $\mathcal{P}(n)$ a *finite Boolean algebra on n elements*.

In 1932, Kolmogorov [Kol32] suggested a constructive interpretation of intuitionistic logic as a calculus of problems. To make precise the description proposed by Kolmogorov, Medvedev established the foundational framework for the logic of finite problems in [Med62]. Inheriting such a tradition on the frame for logic of finite problems, the order in our Boolean algebra here is reverse inclusion $\supseteq$. A finite topless Boolean algebra is obtained by removing the top of a finite Boolean algebra.

Definition 3.1.2 (Finite topless Boolean algebra)**.** For a non-empty set with n elements $[n] = \{1, 2, \ldots, n\}$, let $\mathcal{P}_0(n)$ denote the Kripke frame:

$$\mathcal{P}_0(n) = \langle \{X : X \text{ is a non-empty subset of } [n]\}, \supseteq \rangle.$$

We call $\mathcal{P}_0(n)$ a *finite topless Boolean algebra on n elements* or a *Medvedev frame* (on n elements).

[2] In this definition, we denote by $\mathcal{P}(n)$ the n-th algebra structure as a poset on the set $[n] = \{1, 2, \ldots, n\}$, not on the set $n = \{0, 1, \ldots, n-1\}$.

Definition 3.1.3. The logic Med is the intermediate logic of all Medvedev frames, in other words,

$$\mathsf{Med} = \mathsf{Log}(\{\mathcal{P}_0(n) : n \in \omega\}).$$

The modal logic tBa is the greatest modal companion of Med, that is

$$\mathsf{tBa} = \sigma(\mathsf{Med}).$$

The modal logic $\mathsf{S4.tBa}$ is the least modal companion of Med, that is

$$\mathsf{S4.tBa} = \tau(\mathsf{Med}).$$

Note that $\mathcal{P}(n)$ is obtained by the product of the 2-chain, and the Kripke frame $\mathcal{P}_0(n)$ is obtained by removing the top, we will study the product of finite rooted frame with a top and more importantly, the generalized Medvedev logic in Chapter 4.

3.2. The logic of finite spiked Boolean algebras

Löwe introduced the following notion and proved results that are included in [LX24, § 5].

Definition 3.2.1 (Spiking)**.** If $\mathcal{P} = \{P, \leq_P\}$ is a partial order with a top, we call $\mathcal{S} = \{S, \leq_S\}$ a *spiking* of $\mathcal{P}$ if $\mathcal{P}$ is a suborder of $\mathcal{S}$, all elements of $\mathcal{P}$ have the same predecessors in $\mathcal{S}$ as they do in $\mathcal{P}$, $S \setminus P$ is a finite set of maximal elements of $\mathcal{S}$ (called *spikes*) that all have only coatoms of $\mathcal{P}$ as immediate predecessors, and each coatom of $\mathcal{P}$ has at most one element of $S \setminus P$ as immediate successor. A spike is called *pure* if it has exactly one coatom as immediate predecessor. A poset is said to be a *pure spiking* if each of the spikes is pure.

Consider a spiking $\mathcal{S}$ as adding spikes to a partial order $\mathcal{P}$, wherein an additional maximal elements sit as a successor of the coatoms of $\mathcal{P}$. This arrangement stipulates that each coatom has at most one spike (and coatoms can share a spike). See Figure 3.2-2 for an illustrative example.

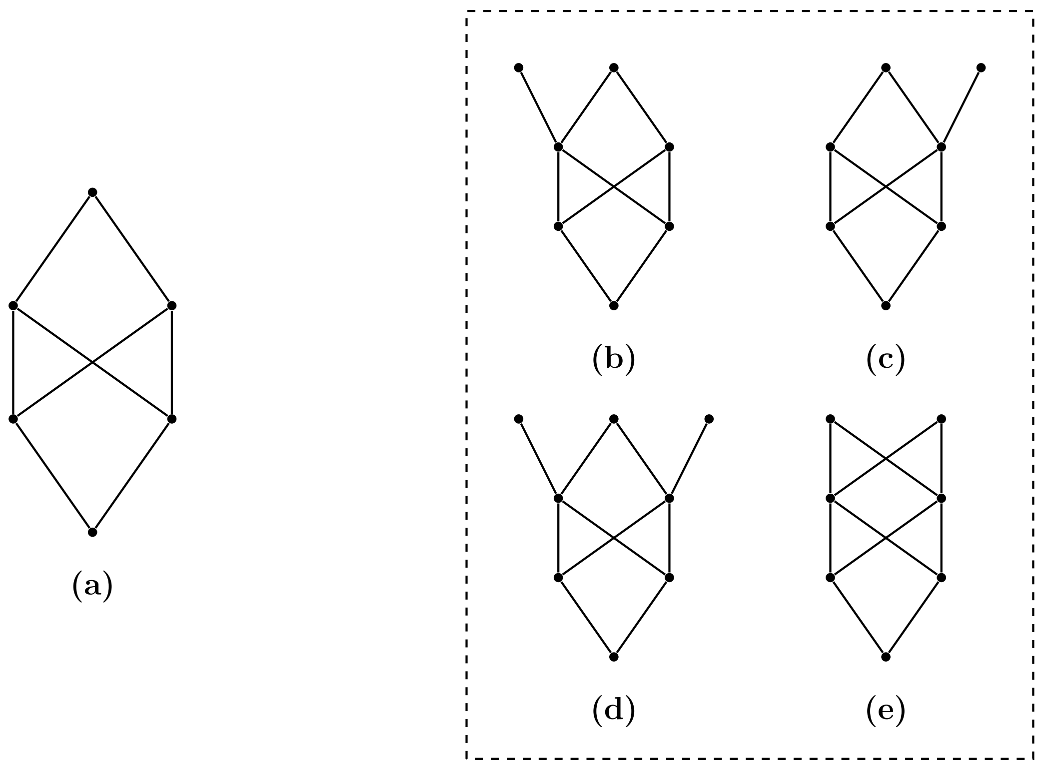


Figure 3.2-2: A partial order (a) with its five spikings (a) to (e), where (a) to (d) are pure spikings.

Definition 3.2.2 (Finite spiked Boolean Algebra)**.** A partial order $\mathcal{S}_n = \{S_n, \leq_{S_n}\}$ is called a *spiked Boolean algebra* on n elements if it is obtained from a Boolean algebra on n elements $\mathcal{P}(n) = \langle \{X : X \subseteq [n]\}, \supseteq \rangle$ by adding precisely one pure spike $\{js\}$ to each coatom $\{j\}$. For an illustrative example, see Figure 3.2-3.

In other words, for any $n \in \omega$, $\mathcal{S}_n = \langle S_n, \leq_{S_n} \rangle$ denotes the spiked Boolean algebra on n elements, where

$$S_n := \{X : X \subseteq [n]\} \cup \{\{1s\}, \{2s\}, \dots, \{ns\}\}$$

and

$$x \leq_{S_n} y :\Leftrightarrow x \supseteq y \text{ in } \mathcal{P}(n) \text{ or } \{j\} \subseteq x \subseteq [n], y = \{js\} \text{ or } x = y = \{js\}.$$

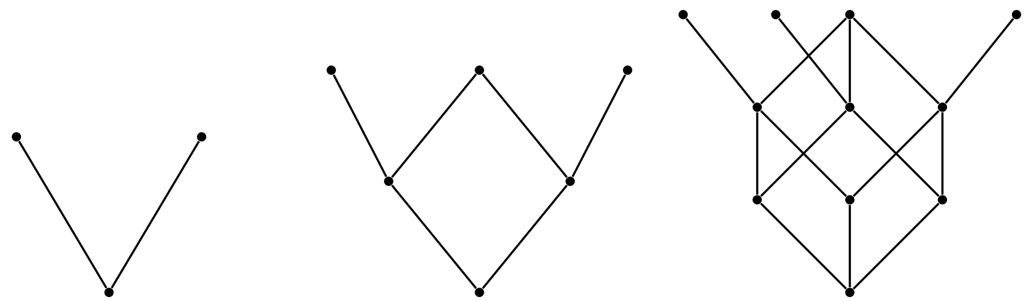


Figure 3.2-3: spiked Boolean algebras

Definition 3.2.3. Let LS denote the intermediate logic[3] of all finite spiked Boolean algebra $\{\mathcal{S}_n\}_{n\geq 1}$.

The modal logic sBa is the greatest modal companion of LS, that is

$$\mathsf{sBa} = \sigma(\mathsf{LS}).$$

The modal logic $\mathsf{S4.sBa}$ is the least modal companion of LS, that is

$$\mathsf{S4.sBa} = \tau(\mathsf{LS}).$$

3.3. The logic of finite partial function algebras

Definition 3.3.1 (Finite partial function algebra)**.** A poset $\mathcal{R}_n$ is called the *finite partial function algebra* on n elements if it consists of partial functions from $[n]$ to $\{1, 2\}$. For $a, b \in \mathcal{R}_n$, we define $a < b \in \mathcal{R}_n$ if and only if $a = b\restriction\mathrm{dom}(a)$. For any point $a \in \mathcal{R}_n$, we associate the pair $(1_a, 2_a)$ where $1_a = a^{-1}(1)$ and $2_a = a^{-1}(2)$.

Recall the 2-fork frame $\mathcal{C}_1$, let $\mathcal{C}_n$ be the Cartesian product of $\mathcal{C}_1$ taken n times. Then every point a of $\mathcal{C}_n$ can be associated with an n-tuple $(x_0, x_1, \ldots, x_{n-1})$, where $x_i \in \{0, 1, 2\}$. We will see that $\mathcal{C}_n \cong \mathcal{R}_n$ by Lemma 7.1.6.

Definition 3.3.2. Let Cheq denote the intermediate logic of all frames in $\{\mathcal{C}_n\}_{n\geq 1}$.

[3] We refer to this logic as *Inamdar logic* Inam in [LX24].

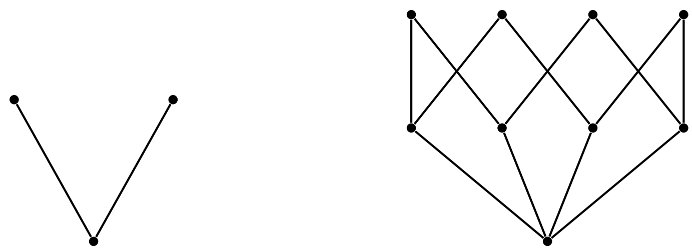


Figure 3.3-4: $\mathcal{C}_1$ and $\mathcal{C}_2$

In fact, Cheq is the intermediate analogue of the well-known modal logic of chequered subsets of $\mathbb{R}^\infty$ which was introduced in [vBBG03].

Definition 3.3.3. The modal logic FPFA is the greatest modal companion of Cheq, that is

$$\mathsf{FPFA} = \sigma(\mathsf{Cheq}).$$

The modal logic S4.FPFA is the least modal companion of Cheq, that is

$$\mathsf{S4.FPFA} = \tau(\mathsf{Cheq}).$$

We summarise in Table 3.1 the three classes of structures and related intermediate (modal) logics.

Name	**Frame**	**Intermediate Logic**	**Modal Logic**	
			τ	σ
Finite topless Boolean algebra	$\mathcal{P}_0(n)$	Med	S4.tBa	tBa
Finite spiked Boolean algebra	$\mathcal{S}_n$	LS	S4.sBa	sBa
Finite partial function algebra	$\mathcal{R}_n$	Cheq	S4.FPFA	FPFA

Table 3.1: Comparison of three classes of structures

3.4. Comparison of the relevant logics

For any node a in a spiked Boolean algebra, if there are two nodes above a, then a is located in the corresponding Boolean algebra. Let p_0 and p_1 be distinct atomic propositions, then define $\varphi_0 = (p_0 \wedge \neg p_1) \wedge \Box(p_0 \wedge \neg p_1)$ and $\varphi_1 = (p_1 \wedge \neg p_0) \wedge \Box(p_1 \wedge \neg p_0)$. Let q_0 and q_1 be another two different atomic propositions. Consider the following formula:

$$\Psi_s := \big(\Diamond (\Box q_0 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1) \wedge \Diamond(\Box q_1 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1)\big) \to \Diamond \Box (q_0 \wedge q_1).$$

Lemma 3.4.1. $\Psi_s \in \mathsf{S4.sBa}$, $\Psi_s \notin \mathsf{S4.FPFA}$, $\Psi_s \notin \mathsf{S4.tBa}$.

Proof. Let $\mathcal{S}$ be any finite spiked pre-Boolean algebra and V be a valuation on $\mathcal{S}$. Assume that a point $a \in \mathcal{S}$, $a \vDash \Diamond(\Box q_0 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1) \wedge \Diamond(\Box q_1 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1)$, then there are two nodes $x, y \geq a$ and $x \vDash \Box q_0 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$, $y \vDash \Box q_1 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$, so there are two points $u, v \geq x$, $u \vDash \varphi_0$ and $v \vDash \varphi_1$. By the definition of φ_0 and φ_1, we have $u \neq v$ and they belong to two different clusters. In fact, they don't have a join. It follows that x is not the spiked one and so does y. Thus x and y belong to the corresponding Boolean algebra, and then we have $z \in \mathcal{S}$ and $z \geq x, y$. Since $x \vDash \Box q_0$ and $y \vDash \Box q_1$, we have $z \vDash \Box(q_0 \wedge q_1)$. Since $a \leq z$, then $a \vDash \Diamond \Box (q_0 \wedge q_1)$. It follows that $\Psi_s \in \mathsf{S4.sBa}$.

Let $\mathcal{R}_2$ be the finite partial function algebra on two elements. V' be a valuation on $\mathcal{R}_2$ such that

1. $(\{1,2\}, \varnothing) \vDash \varphi_0 \wedge q_0 \wedge \neg q_1$;
2. $(\{2\}, \{1\}) \vDash \varphi_1 \wedge q_0 \wedge \neg q_1$;
3. $(\varnothing, \{1,2\}) \vDash \varphi_1 \wedge \neg q_0 \wedge q_1$;
4. $(\{1\}, \{2\}) \vDash \varphi_0 \wedge \neg q_0 \wedge q_1$.

and

1. $(\{2\}, \varnothing) \vDash q_0$;
2. $(\varnothing, \{2\}) \vDash q_1$.

Then $(\{2\}, \varnothing) \vDash \Box q_0 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$ and $(\varnothing, \{2\}) \vDash \Box q_1 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$. But $(\varnothing, \varnothing) \vDash \Box \Diamond \neg(q_0 \wedge q_1)$, thus $\Psi_s \notin \mathsf{S4.FPFA}$.

Let $\mathcal{P}_0(4)$ be the topless Boolean algebra on four elements. V'' be a valuation on $\mathcal{P}_0(4)$ such that

1. $\{1\} \vDash \varphi_0 \wedge q_0 \wedge \neg q_1$;
2. $\{2\} \vDash \varphi_1 \wedge q_0 \wedge \neg q_1$;
3. $\{3\} \vDash \varphi_1 \wedge \neg q_0 \wedge q_1$;
4. $\{4\} \vDash \varphi_0 \wedge \neg q_0 \wedge q_1$.

and

1. $\{1,2\} \vDash q_0$;

2. $\{3,4\} \vDash q_1$.

Then $\{1,2\} \vDash \Box q_0 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$ and $\{3,4\} \vDash \Box q_1 \wedge \Diamond\varphi_0 \wedge \Diamond\varphi_1$. But $\{1,2,3,4\} \vDash \Box\Diamond\neg(q_0 \wedge q_1)$, thus $\Psi_s \notin \mathsf{S4.tBa}$. □

Let p_0, p_1 and p_2 be distinct atomic propositions, then for $0 \le i \le 2$, define $\phi_i = (p_i \bigwedge_{j\neq i}(\neg p_j)) \wedge \Box(p_i \bigwedge_{j\neq i}(\neg p_j))$. Consider the following formulas which are attributed to [HLL15, § 2]:

$$\Psi_t \coloneqq (\textstyle\bigwedge_{i=0,1,2} \Diamond\phi_i) \to \Diamond(\Diamond\phi_0 \wedge \Diamond\phi_1 \wedge \Box\neg\phi_2).$$

Lemma 3.4.2. $\Psi_t \in \mathsf{S4.tBa}$, $\Psi_t \notin \mathsf{S4.FPFA}$, $\Psi_t \notin \mathsf{S4.sBa}$.

Proof. Let $\mathcal{P}_0$ be any finite topless pre-Boolean algebra corresponding to proper subsets of a finite set D and V be a valuation on $\mathcal{P}_0$. Assume that a point $a \in \mathcal{P}_0$ and $a \vDash \bigwedge_{i=0,1,2}\Diamond\phi_i$, then there are three nodes $c_0, c_1, c_2 \ge a$ such that $c_j \vDash \phi_j$ and no two of c_j have a join in $\mathcal{P}_0$. Thus $c_0 \cap c_2 = c_1 \cap c_2 = \varnothing$, it follows that $a < c_0 \cup c_1 \nleq c_2$. Let $c = c_0 \cup c_1$, then $c \vDash \Diamond\phi_0 \wedge \Diamond\phi_1$. If $c \vDash \Diamond\phi_2$, then $c \vDash \bigwedge_{i=0,1,2}\Diamond\phi_i$, and by the above argument, we can find a new $c' > c$, and $c' \vDash \Diamond\phi_0 \wedge \Diamond\phi_1$. Because $\mathcal{P}_0$ is finite, we will finally found a node $c_0 > a$ such that $c_0 \vDash \Diamond\phi_0 \wedge \Diamond\phi_1 \wedge \Box\neg\phi_2$. It follows that $\Psi_t \in \mathsf{S4.tBa}$.

Let $\mathcal{R}_2$ be a finite partial function algebra on two elements. V' be a valuation on $\mathcal{R}_2$ such that

1. $(\{1,2\},\varnothing) \vDash \phi_0$;

2. $(\varnothing,\{1,2\}) \vDash \phi_1$;

3. $(\{2\},\{1\}) \vDash \phi_2$;

4. $(\{1\},\{2\}) \vDash \phi_2$.

Then $(\varnothing,\varnothing) \vDash \bigwedge_{i=0,1,2}\Diamond\phi_i$. For any node $a \notin \{(\{1,2\},\varnothing),(\varnothing,\{1,2\})\}$, we have $a \vDash \Diamond\phi_2$. For any node $a \in \{(\{1,2\},\varnothing),(\varnothing,\{1,2\})\}$, we have $a \vDash \neg(\Diamond\phi_0 \wedge \Diamond\phi_1)$. Therefore, any point dose not have $\Diamond\phi_0 \wedge \Diamond\phi_1 \wedge \Box\neg\phi_2$ and so $\Psi_t \notin \mathsf{S4.FPFA}$.

Let $\mathcal{S}_3$ be the spiked Boolean algebra on three elements. V'' be a valuation on $\mathcal{S}_3$ such that

1. $\{1s\} \vDash p_0 \wedge \neg p_1 \wedge \neg p_2$;

2. $\{2s\} \vDash \neg p_0 \wedge p_1 \wedge \neg p_2$;

3. $\{3s\} \vDash \neg p_0 \wedge \neg p_1 \wedge p_2$;

4. $\varnothing \vDash \neg p_0 \wedge \neg p_1 \wedge p_2$.

Then $\{1,2,3\} \vDash \bigwedge_{i=0,1,2} \Diamond\phi_i$. For any node $a \notin \{\{1s\},\{2s\}\}$, we have $a \vDash \Diamond\phi_2$. For any node $a \in \{\{1s\},\{2s\}\}$, we have $a \vDash \neg(\Diamond\phi_0 \wedge \Diamond\phi_1)$. Therefore, any point dose not have $\Diamond\phi_0 \wedge \Diamond\phi_1 \wedge \Box\neg\phi_2$ and so $\Psi_t \notin$ S4.tBa. □

Corollary 3.4.3. S4.sBa is not contained in S4.FPFA, and neither is S4.tBa. S4.tBa and S4.sBa do not contain each other.

Theorem 3.4.4. S4.FPFA is strictly included in S4.tBA.

Proof. Since $\Psi_t \in$ S4.tBa and $\Psi_t \notin$ S4.FPFA, it suffices to prove that for any $n \in \omega$, there exists a p-morphism f_n from $\mathcal{R}_n$ onto $\mathcal{P}_0(n+1)$. Define a map f_n from $\mathcal{R}_n$ to $\mathcal{P}_0(n+1)$ for any $a = (1_a, 2_a)$ as follows:

$$f_n(a) = \begin{cases} [v_a] \smallsetminus 1_a, & \text{if } 2_a \neq \varnothing, \\ [n+1] \smallsetminus 1_a, & \text{if } 2_a = \varnothing, \end{cases}$$

where $v_a = \min\{v : v \in 2_a\}$.

We observe the following results:

1. $f_n((\varnothing,\varnothing)) = [n+1]$.
2. If $a = (\{u\},\varnothing)$, then $f_n(a) = [n+1] \smallsetminus \{u\}$, where $1 \leq u \leq n$.
3. If $a = (\varnothing,\{v\})$, then $f_n(a) = [v]$, where $1 \leq v \leq n$.

For any non-empty subset S of $\{1,2,\ldots,n+1\}$, if $n+1 \in S$, then we have $f_n(([n+1] \smallsetminus S,\varnothing)) = S$. If $n+1 \notin S$ but $n \in S$, then $f_n(([n] \smallsetminus S,\{n\})) = S$. More generally, let $n_S = \max S$, then $f_n(([n_S] \smallsetminus S,\{n_S\})) = S$. Thus, for each non-empty subset of $[n+1]$, it is the image of some element of $\mathcal{R}_n$, so f_n is onto.

If $a \leq b$ in $\mathcal{R}_n$ and $a = (\varnothing,\varnothing)$, b is an atom of $\mathcal{R}_n$, then $f_n(a) = [n+1] \supseteq f_n(b)$, so $f_n(a) \leq f_n(b)$ in $\mathcal{P}_0(n+1)$.

According to our construction, $f_n(a) = \bigcap f_n(x_a)$, where $x_a \leq a$ and x_a is an atom in $\mathcal{R}_n$. Thus if $a \leq b$ in $\mathcal{R}_n$, then $f_n(a) = \bigcap f_n(x_a) \supseteq \bigcap f_n(x_b) = f_n(b)$ and so $f_n(a) \leq f_n(b)$ in $\mathcal{P}_0(n+1)$.

On the other hand, if $f_n(a) \leq s$ in $\mathcal{P}_0(n+1)$, then $s \subseteq f_n(a)$ as a non-empty subset of $[n+1]$. We aim to find b in $\mathcal{R}_n$ such that $a \leq b$ and $f_n(b) = s$, there are three cases:

Case 1. If $n+1 \in f_n(a)$ and $n+1 \in s$, then $2_a = \varnothing$ and $a = (1_a, \varnothing)$, $f_n(a) = [n+1] \smallsetminus 1_a$. Let $1_b = \{u \in [n] : u \notin s\}$ and $b = (1_b, \varnothing)$. According to our construction, $f_n(b) = [n+1] \smallsetminus 1_b = s$. Since $[n+1] \smallsetminus 1_b = s \subseteq f_n(a) = [n+1] \smallsetminus 1_a$, then $1_a \subseteq 1_b$ and so $a = (1_a, \varnothing) \leq (1_b, \varnothing) = b$.

Case 2. If $n+1 \in f_n(a)$ and $n+1 \notin s$, then $a = (1_a, \varnothing)$ and $f_n(a) = [n+1] \smallsetminus 1_a$. Let $1_b = \{u \in [n] : u \notin s\}$ and $n_s = \max s$. Then $1_b \cap \{n_s\} = \varnothing$ and let $b = (1_b, \{n_s\})$, so $f_n(b) = [n_s] \smallsetminus 1_b = s$. Since $[n_s] \smallsetminus 1_b = s \subseteq f_n(a) = [n+1] \smallsetminus 1_a$, then $u \notin 1_b \rightarrow u \notin 1_a$ when $u \leq n_s$ and $u \in 1_b$ when $n_s < u \leq n$, thus $1_a \subseteq 1_b$. Therefore, $a = (1_a, \varnothing) \leq (1_b, \{n_s\}) = b$.

Case 3. If $n+1 \notin f_n(a)$ and $n+1 \notin s$, then $2_a \neq \varnothing$ and $a = (1_a, 2_a)$, $f_n(a) = [v_a] \smallsetminus 1_a$. Let $n_s = \max s$ and $1_b = \{u \in [n_s] : u \notin s\} \cup \{u : n_s < u \leq n \text{ and } u \in 1_a\}$. Since $[n_s] \smallsetminus 1_b = s \subseteq f_n(a) = [v_a] \smallsetminus 1_a$, then $n_s = \max s \leq \min\{v : v \in 2_a\} = v_a$ and $u \notin 1_b \rightarrow u \notin 1_a$ when $u \in [n_s]$, thus $1_a \subseteq 1_b$. It is easy to see that $1_b \cap (2_a \cup \{n_s\}) = (1_b \cap 2_a) \cup (1_b \cap \{n_s\})$ where $1_b \cap 2_a = (\{u \in [n_s] : u \notin s\} \cap 2_a) \cup (\{u : n_s < u \leq n \text{ and } u \in 1_a\} \cap 2_a)$ and $1_b \cap \{n_s\} = \varnothing$. Since $n_s \leq \min\{v : v \in 2_a\}$ and $n_s \in u$, then $\{u \in [n_s] : u \notin s\} \cap 2_a = \varnothing$. Because $\{u : n_s < u \leq n \text{ and } u \in 1_a\} \cap 2_a = \varnothing$, then $1_b \cap 2_a = \varnothing$. Thus $1_b \cap (2_a \cup \{n_s\}) = \varnothing$ and then we can let $b = (1_b, 2_a \cup \{n_s\})$. According to the above argument, $a = (1_a, 2_a) \leq (1_b, 2_a \cup \{n_s\}) = b$ and $f_n(b) = [\min(2_a \cup \{n_s\})] \smallsetminus 1_b = [n_s] \smallsetminus 1_b = s$.

Thus, f_n is a p-morphism from $\mathcal{R}_n$ onto $\mathcal{P}_0(n+1)$. □

Remark 3.4.5. A different construction and proof for a p-morphism between FPFA and **tBa** can be found in [Lit04, Theorem 4].

Theorem 3.4.6. S4.FPFA is strictly included in S4.sBA.

Proof. Since $\Psi_s \in$ S4.sBa and $\Psi_s \notin$ S4.FPFA, it is sufficient to get the result by proving that for any $n \in \omega$, there exists a p-morphism g_n from $\mathcal{R}_n$ onto $\mathcal{S}_n$, where $\mathcal{S}_n = \langle S_n, \leq_{S_n} \rangle$ denotes the spiked Boolean algebra on n elements.

We define a map g_n from $\mathcal{R}_n$ to $\mathcal{S}_n$ for any $a = (1_a, 2_a)$ as follows:

$$g_n(a) = \begin{cases} [n] \smallsetminus 1_a, & \text{if } 2_a = \varnothing \\ \{js\}, & \text{if } a = ([n] \smallsetminus \{j\}, \{j\}) \\ \bigcap_{j \in 2_a} \{j\}, & \text{otherwise} \end{cases}$$

It is not difficult to calculate the following cases:

1. $g_n((\varnothing, \varnothing)) = [n]$.
2. If $a = (\{u\}, \varnothing)$, then $g_n(a) = [n] \setminus \{u\}$, where $1 \le u \le n$.
3. If $a = (\varnothing, \{v\})$, $g_n(a) = \{v\}$, where $1 \le v \le n$.
4. If $a = (\{1, 2, ..., n\} \setminus \{j\}, \{j\})$, $g_n(a) = \{js\}$, where $1 \le j \le n$.

For any subset $S \subseteq [n]$, let $1_a = \{u \in [n] : u \notin S\}$, then $g_n((1_a, \varnothing)) = [n] \setminus 1_a = S$. Together with the fact that $g_n(([n] \setminus \{j\}, \{j\})) = \{js\}$, thus g_n is an onto from $\mathcal{R}_n$ to $\mathcal{S}_n$.

Case 1. If $a \le b$ in $\mathcal{R}_n$ and $2_b = \varnothing$, then $1_a \subseteq 1_b$ and so $g_n(a) = [n] \setminus 1_a \supseteq [n] \setminus 1_b = g_n(b)$, that is, $g_n(a) \le g_n(b)$.

Case 2. If $a \le b$ in $\mathcal{R}_n$ and $b = (\{1, 2, ..., n\} \setminus \{j\}, \{j\})$, then $a = (1_a, \varnothing)$ or $a = (1_a, \{j\})$ where $1_a \subseteq 1_b$. Thus $j \notin 1_a$ and so $j \in [n] \setminus 1_a$. Therefore, $g_n(a) = [n] \setminus 1_a \le \{j\} < \{js\} = g_n(b)$ if $a = (1_a, \varnothing)$ or $g_n(a) = \{j\} < \{js\} = g_n(b)$ if $a = (1_a, \{j\})$. So $g_n(a) \le g_n(b)$ in $\mathcal{S}_n$.

Case 3. If $a \le b$ in $\mathcal{R}_n$ and $b \ne (\{1, 2, ..., n\} \setminus \{j\}, \{j\})$ for any $1 \le j \le n$ and $2_b \ne \varnothing$. So there exists at least a j such that $j \in 2_b$. Then $j \notin 1_b$ and so $j \notin 1_a$. Thus $g_n(a) = [n] \setminus 1_a \le \{j\} \le \bigcap_{i \in 2_b} \{i\} = g_n(b)$ if $2_a = \varnothing$ or $g_n(a) = \bigcap_{i \in 2_a} \{i\} \le \bigcap_{i \in 2_b} \{i\} = g_n(b)$ if $2_a \ne \varnothing$. So $g_n(a) \le g_n(b)$ in $\mathcal{S}_n$.

On the other hand, if $g_n(a) \le s$ in $\mathcal{S}_n$. We aim to find b in $\mathcal{R}_n$, such that $a \le b$ and $g_n(b) = s$.

If $2_a = \varnothing$, then $g_n(a) = [n] \setminus 1_a$ and $a = (1_a, \varnothing)$. There are four cases for s as follows:

Case 1. If s is a non-empty subset of $[n]$ and $s \ne \{j\}$ for any $1 \le j \le n$. Let $b = ([n] \setminus s, \varnothing)$, then $g_n(b) = s$, and since $g_n(a) = [n] \setminus 1_a \supseteq s$, then $[n] \setminus s \supseteq 1_a$, therefore, $a = (1_a, \varnothing) \le ([n] \setminus s, \varnothing) = b$.

Case 2. If $s = \varnothing$, then let $b = ([n], \varnothing)$. It is not difficult to see $g_n(b) = \varnothing = s$ and $a = (1_a, \varnothing) \le ([n], \varnothing) = b$.

Case 3. If $s = \{j\}$, since $[n] \setminus 1_a \le \{j\}$, then $j \notin 1_a$. Let $b = (1_a, \{j\})$, then $g_n(b) = \{j\} = s$ and $a = (1_a, \varnothing) \le (1_a, \{j\}) = b$.

Case 4. If $s = \{js\}$, let $b = ([n] \setminus \{j\}, \{j\})$. Since $[n] \setminus 1_a \le \{js\}$, then $j \notin 1_a$ and so $1_a \subseteq [n] \setminus \{j\}$. Therefore, $a = (1_a, \varnothing) \le ([n] \setminus \{j\}, \{j\}) = b$ and $g_n(b) = \{js\} = s$.

When $2_a \ne \varnothing$, we are looking at two obvious cases first. If $a = ([n] \setminus \{j\}, \{j\})$, then $g_n(a) = \{js\}$ and so $s = \{js\}$ and we choose $b = a$. Another case is that if $|2_a| \ge 2$, then $g_n(a) = \varnothing$ and so $s = \varnothing$, then we choose $b = a$.

The remaining case is that if $2_a = \{j\}$ and $a \ne ([n] \setminus \{i\}, \{i\})$ for any $1 \le i \le n$, then $g_n(a) = \{j\}$. Let $a = (1_a, \{j\})$.

Case 1. If $s = \{js\}$, since $1_a \ne [n] \setminus \{j\}$, then $1_a \subseteq [n] \setminus \{j\}$ and let $b = ([n] \setminus \{j\}, \{j\})$, then $g_n(b) = \{js\} = s$ and $a = (1_a, \{j\}) \le ([n] \setminus \{j\}, \{j\}) = b$.

Case 2. If $s = \varnothing$, since $1_a \neq [n] \setminus \{j\}$, then there exists $u \neq j$ such that $u \in [n] \setminus 1_a$, thus $1_a \cap \{u, j\} = \varnothing$. Let $b = (1_a, \{u, j\})$, then $g_n(b) = \varnothing = s$ and $a = (1_a, \{j\}) \leq (1_a, \{u, j\}) = b$.

Thus, g_n is a p-morphism from $\mathcal{R}_n$ onto $\mathcal{S}_n$. □

Corollary 3.4.7.

1. $\mathsf{FPFA} \subseteq \mathsf{tBA} \cap \mathsf{sBA}$.
2. $\mathsf{Cheq} \subseteq \mathsf{Med} \cap \mathsf{LS}$.
3. $\mathsf{S4.FPFA} \subseteq \mathsf{S4.tBA} \cap \mathsf{S4.sBA}$.

We already know that Med is not finitely axiomatisable over Cheq in [Fon06, Theorem 9], we will prove that LS is not finitely axiomatisable over Cheq in Chapter 6.

3.5. Destructible gaps

In this section, we will give an overview of the basic theory of destructible gaps following [Yor07, Git11, Bag24].

We write 2^ω for the power set of ω. If $a \in 2^\omega$, we write $a(n)$ for the nth element of a in increasing enumeration. Given $a, b \in 2^\omega$, we define that a is *eventually dominated* by b, denoted as $a <^* b$, if there exist only finitely many n such that $a(n) \geq b(n)$. Let $A = \langle a_\alpha : \alpha < \omega_1 \rangle$ and $B = \langle b_\beta : \beta < \omega_1 \rangle$ be sequences in 2^ω. The pair (A, B) is called an *pregap* if for any $\alpha < \alpha' < \omega_1$ and $\beta < \beta' < \omega_1$ we have $a_\alpha <^* a_{\alpha'} <^* b_{\beta'} <^* b_\beta$.[4]

For each pregap (A, B), we define

$$\mathbb{F}(A, B) := \{\sigma \in [\omega_1]^{<\omega} : \text{ if } \alpha \neq \beta \in \sigma, \text{ we have } (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \varnothing\},$$
$$\mathbb{S}(A, B) := \{\sigma \in [\omega_1]^{<\omega} : (\bigcup_{\alpha \in \sigma} a_\alpha) \cap (\bigcup_{\beta \in \sigma} b_\beta) = \varnothing\}.$$

Both of these sets can be ordered by reverse inclusion to obtain a poset.

If $c \in 2^\omega$ is such that $a_\alpha <^* c <^* b_\alpha$ for all $\alpha < \omega_1$, then we say c *separates* the pregap (A, B). If no such c exists, the pregap (A, B) is called a *gap*. A gap is called *destructible* if there exists an ω_1-preserving forcing that adds a real separating it. The gap is called *indestructible* if it is not destructible.

[4] Note that this would usually be called an (ω_1, ω_1)-pregap or (ω_1, ω_1^*)-pregap, depending on the author, but since we only care about this case and not its generalisations, we drop the prefix in our notation for the sake of simplicity.

Theorem 3.5.1. Let (A, B) be a pregap. The following are equivalent:

1. (A, B) is a gap;

2. for all uncountable $X \subseteq \omega_1$, there are $\alpha \neq \beta \in X$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \varnothing$;

3. $\mathbb{F}(A, B)$ has the c.c.c.

Proof. This result is claimed without proof in [Yor07, Theorem 1.2.1.]. The equivalence of 1. and 2. is proved in [TF95, Lemma 9.1] or [Git11, Lemma 1.6].

($\neg 2. \Rightarrow \neg 3.$): Suppose X is an uncountable subset of ω_1 such that for any $\alpha, \beta \in X$, $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing$. Then X forms an uncountable antichain in $\mathbb{F}(A, B)$, so $\mathbb{F}(A, B)$ does not have the c.c.c.

($\neg 3. \Rightarrow \neg 1.$): If $\mathbb{F}(A, B)$ does not have the c.c.c., then we can find an uncountable antichain $\{\sigma_\alpha : \alpha < \omega_1\}$ which is a Δ-system with $\max(\sigma_{\alpha_1}) < \min(\sigma_{\alpha_2})$ when $\alpha_1 < \alpha_2$.

Furthermore, we may assume that there exists an $N < \omega$, such that for any $\alpha < \omega_1$, let $\sigma_\alpha = \{\delta_0^\alpha, \delta_1^\alpha, \dots, \delta_n^\alpha\}$, then $a_{\delta_0^\alpha} \setminus N \subseteq a_{\delta_1^\alpha} \setminus N \subseteq \dots \subseteq a_{\delta_n^\alpha} \setminus N$ and $b_{\delta_0^\alpha} \setminus N \subseteq b_{\delta_1^\alpha} \setminus N \subseteq \dots \subseteq b_{\delta_n^\alpha} \setminus N$.

Let $c_\alpha = a_{\delta_0^\alpha} \setminus N$ and $d_\alpha = b_{\delta_0^\alpha} \setminus N$ for any α. If there are $\alpha \neq \beta < \omega_1$, $c_\alpha \cap d_\beta \neq \varnothing$, then for any $\delta \in \sigma_\alpha$ and $\delta' \in \sigma_\beta$, $a_\delta \cap b_{\delta'} \neq \varnothing$, hence σ_α and σ_β is not incompatible. So $(c_\alpha \cap d_\beta) \cup (c_\beta \cap d_\alpha) = \varnothing$ for any $\alpha \neq \beta < \omega_1$. Now let $c := \bigcup_{\alpha < \omega_1} c_\alpha$. For any a_α, there is β such that $\alpha \leq \min(\sigma_\beta)$, so $a_\alpha \leq^* c_\beta$ and then $a_\alpha \leq^* c$. For any b_α, there is β such that $\alpha \leq \min(\sigma_\beta)$, then there is n such that $b_\alpha \setminus n \subseteq d_\beta \setminus n$. Note that $d_\beta \cap c = d_\beta \cap (\cup_{\alpha < \omega_1} c_\alpha) = \varnothing$ since $(c_\alpha \cap d_\beta) \cup (c_\beta \cap d_\alpha) = \varnothing$ for any $\alpha \neq \beta < \omega_1$, then $b_\alpha \cap c$ is finite.

Therefore, c separates (A, B) and so (A, B) is not a gap. □

Theorem 3.5.2 (Kunen)**.** Let (A, B) be a gap. The following are equivalent:

1. (A, B) is destructible;

2. for all uncountable $X \subseteq \omega_1$, there are $\alpha \neq \beta \in X$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing$;

3. $\mathbb{S}(A, B)$ has the c.c.c.

Proof. This result is claimed without proof in [Yor07, Theorem 1.2.2]. The equivalence of 1. and 2. is proved in [Git11, Theorem 3.2].

(3. $\Rightarrow$ 2.): Let $p_\alpha = \{\alpha\}$ be a condition of $\mathbb{S}(A,B)$ since $a_\alpha \cap b_\alpha = \varnothing$. Fix an uncountable $X \subseteq \omega_1$, consider $Y = \{p_\alpha : \alpha \in X\}$. Because $\mathbb{S}(A,B)$ has the c.c.c., there are p_α and p_β in Y and p_α and p_β are compatible. Now assume $q \in [\omega_1]^{<\omega} \in \mathbb{S}(A,B)$ be the condition that $q \leq p_\alpha$ and $q \leq p_\beta$, then $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing$.

($\neg$3. $\Rightarrow$ $\neg$2.): Suppose that $\mathbb{S}(A,B)$ does not have the c.c.c., fix an uncountable antichain $\{\sigma_\alpha : \alpha < \omega_1\}$ in $\mathbb{S}(A,B)$. Let $\{\gamma_\alpha\}_{\alpha<\omega_1}$ be an increasing sequence where $\gamma_\alpha > \max(\sigma_\alpha)$. Furthermore, due to the pigeonhole principle, we may assume there exists $n < \omega$ and $N < \omega$, such that

1. $|\sigma_\alpha| = n$ for any $\alpha < \omega_1$,
2. all $\{a_\delta \cap N : \delta \in \sigma_\alpha\}$ are the same for any $\alpha < \omega_1$,
3. all $\{b_\delta \cap N : \delta \in \sigma_\alpha\}$ are the same for any $\alpha < \omega_1$, and
4. $a_\delta \setminus N \subseteq a_{\gamma_\alpha}$ and $b_\delta \setminus N \subseteq b_{\gamma_\alpha}$ for any $\delta \in \sigma_\alpha$.

For any $\alpha \neq \beta < \omega_1$, σ_α and σ_β are incompatible, then there are $\delta \in \sigma_\alpha$ and $\delta' \in \sigma_\beta$ such that we can find $N' < \omega$ and $N' \in (a_\delta \cap b_{\delta'}) \cup (a_{\delta'} \cap b_\delta)$. If $N' < N$ and $N' \in a_\delta \cap b_{\delta'}$, since all $\{a_\delta \cap N : \delta \in \sigma_\alpha\}$ are the same, then there is $\delta'' \in \sigma_\beta$, $N' \in a_{\delta''} \cap b_{\delta'}$, but $\delta'', \delta' \in \sigma_\beta$ in $\mathbb{S}(A,B)$, a contradiction. $N' < N$ and $N' \in a_{\delta'} \cap b_\delta$ also lead to s contradiction, thus $N' \geq N$. Because $a_\delta \setminus N \subseteq a_{\gamma_\alpha}$ and $b_\delta \setminus N \subseteq b_{\gamma_\alpha}$ for any $\delta \in \sigma_\alpha$, $N' \in (a_{\gamma_\alpha} \cap b_{\gamma_\beta}) \cup (a_{\gamma_\beta} \cap b_{\gamma_\alpha})$. It is obvious that $\{\gamma_\alpha\}_{\alpha<\omega_1}$ is uncountable. ☐

The following crucial properties are claimed in [Yor07, p. 130] without proof.

Theorem 3.5.3.
1. If (A,B) is a destructible gap, then forcing with $\mathbb{S}(\mathcal{A},\mathcal{B})$ separates (A,B).
2. If (A,B) forms a gap, then forcing with $\mathbb{F}(A,B)$ makes (A,B) indestructible.

Clearly, if (A,B) is a destructible gap, both $\mathbb{F}(A,B)$ and $\mathbb{S}(A,B)$ are c.c.c., but their product cannot be, so these forcing notions are not *productively c.c.c.* (for definitions, cf. [Bag24, Definition 3.1]). The question of what happens with products of forcings for different gaps leads to the next definitions.

Let I be an index set and $\mathcal{G} := \{(A_i, B_i) : i \in I\}$ be a family of destructible gaps. We call a forcing notion $\mathbb{P}$ a $\mathcal{G}$-*Yorioka product* (or just *Yorioka product* if $\mathcal{G}$ is clear from the context) if there is a subset $J \subseteq I$ and a sequence $\{\mathbb{X}_j : j \in J\}$ such that $\mathbb{X}_j$ is either $\mathbb{F}(A_j, B_j)$ or $\mathbb{S}(A_j, B_j)$ and $\mathbb{P} = \prod_{j \in J} \mathbb{X}_j$. We call a family $\mathcal{G}$ *independent* if every $\mathcal{G}$-Yorioka product is c.c.c. By Theorem 3.5.3, forcing with a Yorioka product will destroy those gaps (A_j, B_j) for which $\mathbb{S}(A_j, B_j)$ occurs in the product and make those gaps (A_j, B_j) indestructible for which $\mathbb{F}(A_j, B_j)$ occurs in the product.

Theorem 3.5.4 (Yorioka)**.** The axiom $\diamond$ implies that there is an infinite independent family of destructible gaps. In particular, there is one in **L**.

Proof. Cf. [Yor07, Theorem 2.1]. □

If $\mathcal{G} := \{(A_i, B_i) : i \in I\}$ is a family of destructible gaps, and we go to a c.c.c. forcing extension, some of the gaps may not be destructible anymore. We call $\mathcal{G}$ *stably independent* if it is independent and the following holds: in any c.c.c. generic extension N, define $I_N := \{i \in I : (A_i, B_i)$ is still destructible in $N\}$; then $\mathcal{G}_N := \{(A_i, B_i) ; i \in I_N\}$ is still an independent family of destructible gaps in N.

We do not know whether Yorioka's independent families from Theorem 3.5.4 are stably independent; it could be that families of destructible gaps produced by Cohen forcing as in [TF95, Theorem 9.3] are stably independent. In particular, we do not know whether the following statement is consistent:

> there is an absolutely definable (without parameters) infinite stably independent family of destructible gaps. $(\diamondsuit)$

3.6. The c.c.c.-labelling

In this section, we shall assume that $\mathsf{ZFC} + \diamondsuit$ is consistent and prove that $\mathsf{S4.FPFA}$ is an upper bound for the modal logic of c.c.c. forcing. As mentioned at the end of § 3.5, we do not know whether $\mathsf{ZFC} + \diamondsuit$ is consistent; in the (unfortunate) case that it is not, we discuss in Remark 3.6.4 that our proof still gives an upper bound for a modal logic of a rather unnatural forcing class Γ.

Working over a model M of $\mathsf{ZFC} + \diamondsuit$, we pick the absolutely definable stably independent family of destructible gaps $\mathcal{G} = \{(A_i, B_i) : i \in \omega\}$ that exists in M by $\diamondsuit$.

Recall the notion of a Γ-labelling defined by Hamkins, Leibman and Löwe in 2.6.5. In order to show that $\mathsf{S4.FPFA}$ is an upper bound for the modal logic of c.c.c. forcing, we need to provide c.c.c.-labellings for every thickening of every finite partial function algebra $\mathcal{R}_n$.

Given one of these finite partial function algebras $\mathcal{R}_n$, we consider the subfamily $\mathcal{G}_n := \{(A_i, B_i) : i \in [n]\}$ which is an independent family of destructible gaps. The absolute definability means that statements such as "the gap (A_i, B_i) is separated" or "the gap (A_i, B_i) is indestructible" are sentences in the language of set theory and can be used as control statements.

If N is any c.c.c. extension of M, we can define its *poset label*: if $X, Y \subseteq [n]$ and $X \cap Y = \varnothing$, then (X, Y) is the poset label of N if

(a) $i \notin X \cup Y$ iff (A_i, B_i) is destructible gap in N;

(b) $i \in X$ iff (A_i, B_i) is separated in N;

(c) $i \in Y$ iff (A_i, B_i) is indestructible gap in N.

and

$$(X, Y) \leq (X', Y') \text{ iff } X \subseteq X' \text{ and } Y \subseteq Y'.$$

Remember that the elements $a \in \mathcal{R}_n$ are partial functions from $[n]$ to $\{1, 2\}$. The map $a \mapsto (a^{-1}(1), a^{-1}(2))$ is a bijection between $\mathcal{R}_n$ and the set of poset labels preserving the orders $\leq$. To each poset label (X, Y), we assign the sentence $\varphi_{(X,Y)}$ of the language of set theory that expressed the conjunction of the n statements that the label expresses via (a) to (c).

Lemma 3.6.1. If N is a c.c.c. extension of $M \models \mathsf{ZFC} + \diamondsuit$ with poset label (X, Y) and (X', Y') is any poset label, then there is a c.c.c. extension of N with poset label (X', Y') if and only if $X \subseteq X'$ and $Y \subseteq Y'$.

Proof. $(\Rightarrow)$: Assume $\mathbb{P}$ is a c.c.c. forcing and N' is the extension of N by forcing with $\mathbb{P}$. If a gap is separated in N, it will be separated in any c.c.c. extension; similarly, if a gap is indestructible in N, then it is indestructible in any c.c.c. extension since $\mathbb{P}$ is ω_1-preserving. Thus, it is clear that $X \subseteq X'$ and $Y \subseteq Y'$.

$(\Leftarrow)$: We are working over N where all gaps with index in X are separated and all gaps with index in Y are indestructible. Since (X', Y') is a poset label, we have $Y' \cap X' = \varnothing$, so the conditions $X \subseteq X'$ and $Y \subseteq Y'$ imply that $X \cap Y' = X' \cap Y = \varnothing$. Thus, for all indices $i \in X' \cup Y'$ that are not in $X \cup Y$, the gap (A_i, B_i) is destructible, so $\mathbb{S}(A_i, B_i)$ separates (A_i, B_i) by Theorem 3.5.3 and is c.c.c. by Theorem 3.5.2; and $\mathbb{F}(A_i, B_i)$ makes (A_i, B_i) indestructible by Theorem 3.5.3 and is c.c.c. by Theorem 3.5.1.

It is clear that $(X' \smallsetminus X) \cap (Y' \smallsetminus Y) = \varnothing$. By $\diamondsuit$ in M which implies that $\mathcal{G}_n$ is stably independent, we know that $\mathcal{G}_n$ is still independent in the c.c.c. extension N. Consider the following Yorioka product

$$\mathbb{P} = \prod_{i \in X' \smallsetminus X} \mathbb{S}(A_i, B_i) \times \prod_{j \in Y' \smallsetminus Y} \mathbb{F}(A_j, B_j)$$

which is c.c.c. by independence of $\mathcal{G}_n$. Therefore, forcing with $\mathbb{P}$ over N to obtain a generic extension N' achieves what we want: the gaps with index in X' are separated and the gaps with index in Y' are indestructible.

We still need to show that for any $k \notin X' \cup Y'$, the gap (A_k, B_k) remains a destructible gap in N'. Both $\mathbb{P} \times \mathbb{F}(A_k, B_k)$ and $\mathbb{P} \times \mathbb{S}(A_k, B_k)$ are Yorioka products, so by $\diamondsuit$ in M, they are c.c.c. in N and therefore, by Lemma 2.5.9, $\mathbb{F}(A_k, B_k)$ and $\mathbb{S}(A_k, B_k)$ remain c.c.c. in N'. By Theorems 3.5.1 & 3.5.2, this implies that (A_k, B_k) is a destructible gap in N' and, thus, the poset label of the extension is (X', Y'). □

Corollary 3.6.2. If $M \models \mathsf{ZFC}+\diamondsuit$, then for any n there exists a c.c.c.-labelling of the finite partial function algebra $\mathcal{R}_n$.

Proof. If $a \in \mathcal{R}_n$, we define $(X, Y) := (a^{-1}(1), a^{-1}(2))$ and assign the sentence $\varphi_a := \varphi_{(X,Y)}$ to a. Theorem 3.6.1 implies that this is a c.c.c.-labeling. □

Corollary 3.6.2 gives us a c.c.c.-labeling for the skeletons for the relevant frames. We now need to extend this to thickenings by adding arbitrarily large finite clusters at each point; we do this by Cohen forcing.

Theorem 3.6.3. If $\mathsf{ZFC} + \diamondsuit$ is consistent, then $\mathsf{ML}_{\text{c.c.c.}} \subseteq \mathsf{S4.FPFA}$.

Proof. We are working over $M \models \diamondsuit$. For any n, we need to provide a c.c.c.-labelling for a thickening $\mathcal{C}$ of $\mathcal{R}_n$. Corollary 3.6.2 provides us with a c.c.c.-labeling of the skeleton. Fix m such that the size of the clusters in $\mathcal{C}$ is bounded by 2^m. Without loss of generality, we can assume that they all have precisely size 2^m.

For each ordinal β, we let m_β be its 2^m-modulus, i.e., the unique number $\ell < 2^m$ such that there is a natural number k and a limit ordinal λ with $\beta = \lambda + k$ and $k \equiv \ell \mod 2^m$.

For any $0 \leq j < m$, let s_j be the set-theoretic statement "if the number of Cohen reals over M is $\aleph_\beta$, then the jth digit of the binary expansion of m_β is 1". Since we can force the value of m_β to be anything by adding Cohen reals, we can flip each s_j on or off without affecting the other statements s_i independently. For any regular cardinal κ, the forcing poset $\mathbb{Q}$ adding κ-many Cohen reals is productively c.c.c. In particular, for any Yorioka product $\mathbb{P}$, the product $\mathbb{P} \times \mathbb{Q}$ is c.c.c.

Thus, if $a \in \mathcal{R}_n$ and $k = \sum_{i<m} b_i < 2^m$ be a binary expansion for $k < 2^m$, let (a, k) represent the kth cluster point in the thickening of $\mathcal{R}_n$ and assign the statement $\psi_{a,k} := \varphi_a \wedge \bigwedge_{b_i=1} s_i$ to the pair (a, k). We have to show that this is a c.c.c.-labelling of $\mathcal{C}$.

If $a \nleq b$, and $N \vDash \psi_{a,k}$, then by Corollary 3.6.2, there cannot be a c.c.c. extension that satisfies φ_b. Otherwise, $a \leq b$, so again by (the proof of) Corollary 3.6.2, there is a Yorioka product $\mathbb{P}$ which is c.c.c. and forcing with $\mathbb{P}$ makes φ_b true. Now to obtain an arbitrary $\psi_{b,\ell}$ find the right product of Cohen forcing $\mathbb{Q}$ that makes $\psi_{b,\ell}$ true. By the above remark, $\mathbb{P} \times \mathbb{Q}$ is c.c.c. □

As mentioned, we do not know whether the assumption of Theorem 3.6.3 is true. In particular, we do not know whether the **L**-least independent family of destructible gaps produced by $\diamond$ is stably independent. In case it is not, the proof still yields some insight.

Remark 3.6.4. Let $\mathcal{G} = \{(A_i, B_i) : i \in \omega\}$ be the **L**-least independent infinite family of destructible gaps. For any model M, we define $I_M := \{i \in \omega : (A_i, B_i)$ is a destructible gap in $M\}$. A model M is called *$\mathcal{G}$-stable* if $\{(A_i, B_i) : i \in I_M\}$ is an independent family of destructible gaps in M. A forcing $\mathbb{P}$ is called *$\mathcal{G}$-stabilising* if it is ω_1-preserving and preserves the property of being $\mathcal{G}$-stable; we write $\Gamma_\mathcal{G}$ for the class of $\mathcal{G}$-stabilising forcings. Then our proof shows that the modal logic of $\Gamma_\mathcal{G}$-forcing is contained in $\mathsf{S4.FPFA}$.

4. Generalized Medvedev logics

This chapter introduces a new generalized form of Medvedev's logic, derived by removing the maximal element from the product of finite rooted frames with a top element. We find that the logic corresponding to the product of such frames, even before any modification, is essentially the KC logic. Maksimova, Skvortsov and Shehtman [MSS79] proved the impossibility of a finite axiomatisation for Medvedev logic. In response, Nick Bezhanishvili proposed two stronger conjectures. The main result of this chapter is proving the non-finite axiomatisability of every generalized Medvedev logic, thus giving positive answers to Nick's conjectures. Additionally, we examine whether there are infinitely many generalized Medvedev logics and whether there is a smallest one.

4.1. Medvedev Logic generalized via products

Definition 4.1.1. Let $\mathcal{P}_0 = \langle P_0, \leq_0 \rangle$ and $\mathcal{P}_1 = \langle P_1, \leq_1 \rangle$ be posets. Let $P = P_0 \times P_1$ be the Cartesian product of P_0 and P_1. A binary relation $\leq$ on P is defined as follows:

$$(x, y) \leq (x', y') \text{ iff } x \leq_0 x' \text{ and } y \leq_1 y'.$$

We call the poset $\mathcal{P} = \langle P, \leq \rangle$ the *product* of $\mathcal{P}_0$ and $\mathcal{P}_1$. Given a poset $\mathcal{P}$, $\mathcal{P}^2$ denotes the product $\mathcal{P} \times \mathcal{P}$. Furthermore, $\mathcal{P}^n$ denotes the product of $\mathcal{P}^{n-1}$ and $\mathcal{P}$.

In Section 3.1, we introduce the definitions of Medvedev frames and its logic Med in Definitions 3.1.2 and 3.1.3, respectively. Since the Medvedev logic can be regarded as a logic of the topless products of the 2-chains, Nick Bezhanishvili proposed two stronger conjectures to extend the notion of Medvedev logic to arbitrary n-chain and even arbitrary non-singleton, finite, rooted frame that has a top. At the same time, this raises the question of whether generalized Medvedev logic is finitely axiomatisable compared to the non-finite axiomatisability of Med.

Assume $\mathcal{F}$ is a finite rooted frame with a top, let $\mathcal{F}^n$ be the product of $\mathcal{F}$ on n-times. The simplest example for such $\mathcal{F}$ is precisely the *n-chain*, i.e.,

Definition 4.1.2. For a natural number $n \geq 2$, the frame $\mathcal{H}_n = \langle [n], \leq \rangle$ is called the *n-chain* or *finite chain with n points*, where $[n] = \{1, 2, \ldots, n\}$ and $\leq$ is the standard order on the natural numbers.

It is obvious that $\mathcal{F}^n$ has a top since $\mathcal{F}$ has one. For each $\mathcal{F}^n$, let $\mathcal{F}^n_{\mathrm{t}}$ be the frame which is obtained from the frame $\mathcal{F}^n$ by removing its top element. It is not difficult to see that every Medvedev frame $\mathcal{P}_0(n)$ is exactly $\mathcal{F}^n_{\mathrm{t}}$ when $\mathcal{F}$ is the 2-chain $\mathcal{H}_2$.

Definition 4.1.3. Let $\mathsf{P}_{\mathcal{F}}$ denote the logic characterized by all $\mathcal{F}^n$, where $n \geq 1$, i.e., the logic $\mathsf{Log}(\{\mathcal{F}^n\}_{n\geq 1})$.

Let $\mathsf{TLP}_{\mathcal{F}}$ denote the logic characterized by all $\mathcal{F}^n_{\mathrm{t}}$, where $n \geq 1$, i.e., the logic $\mathsf{Log}(\{\mathcal{F}^n_{\mathrm{t}}\}_{n\geq 1})$. We call $\mathsf{TLP}_{\mathcal{F}}$ a *generalized Medvedev logic*. A frame is called a $\mathsf{TLP}_{\mathcal{F}}$-frame if all the theorems of the logic $\mathsf{TLP}_{\mathcal{F}}$ are valid in it.

The following two theorems will give us an approximate picture of $\mathsf{P}_{\mathcal{F}}$ and $\mathsf{TLP}_{\mathcal{F}}$.

First, the logic characterized by the product of $\mathcal{F}$ is the logic of the weak excluded middle $\mathsf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$. Indeed, since KC is the logic of directed intuitionistic frames and is the logic of the product of 2-chains $\mathcal{H}_2$ at the same time, the following result is relatively natural. The following result was conjectured by Nick Bezhanishvili, who suggested it for our study, and I provided the subsequent proof.

Theorem 4.1.4. $\mathsf{P}_{\mathcal{F}} = \mathsf{KC}$.

Proof. Because KC is the intermediate logic of all directed frames which include $\{\mathcal{F}^n\}_{n\geq 1}$, it is immediate to obtain that $\mathsf{P}_{\mathcal{F}} \supseteq \mathsf{KC}$.

Let t be the top of the frame $\mathcal{F}$ and $\mathcal{H}_2$ be the 2-chain, that is, $\mathcal{H}_2 = \langle \{1, 2\}, \leq \rangle$. We define a map f_1 from $\mathcal{F}$ to the 2-chain $\mathcal{H}_2$ as follows:

$$f_1(x) = \begin{cases} 2, & \text{if } x = t \text{ is the top of } \mathcal{F} \\ 1, & \text{otherwise} \end{cases}$$

It is easy to check that f_1 is a p-morphism from $\mathcal{F}$ to the 2-chain and then define a map f_n from $\mathcal{F}^n$ to the frame $\mathcal{P}(n)$, which is the n product of the 2-chain, as follows:

$$f_n((x_1, x_2, \ldots, x_n)) = (f_1(x_1), f_1(x_2), \ldots, f_1(x_n)).$$

So f_n is a p-morphism from $\mathcal{F}^n$ to the frame $\mathcal{P}(n)$ and thus $\mathsf{Log}(\mathcal{F}^n) \subseteq \mathsf{Log}(\mathcal{P}(n))$. Because $\mathsf{KC} = \bigcap_{n\in\omega} \mathsf{Log}(\mathcal{P}(n))$ (see, e.g., [MSS79, Corollary 1]), in conclusion, $\mathsf{P}_{\mathcal{F}} = \bigcap_{n\in\omega} \mathsf{Log}(\mathcal{F}^n) \subseteq \bigcap_{n\in\omega} \mathsf{Log}(\mathcal{P}(n)) = \mathsf{KC}$.

Putting everything together, we have $\mathsf{P}_{\mathcal{F}} = \mathsf{KC}$. □

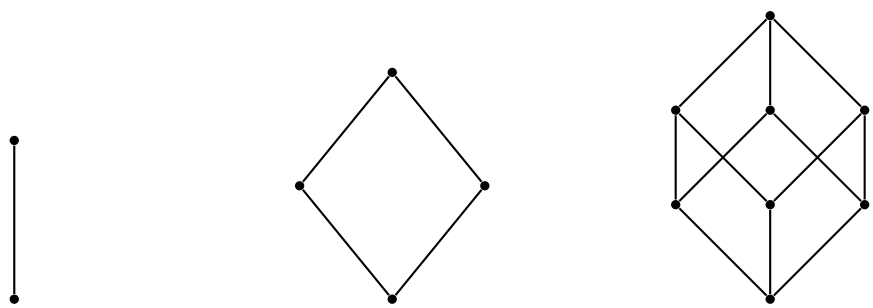


Figure 4.1-1: Product of the 2-chain

When we remove the top of $\mathsf{P}_{\mathcal{F}}$, the well-known logic Med is actually the greatest generalized Medvedev logic.

Theorem 4.1.5. $\mathsf{TLP}_{\mathcal{F}} \subseteq \mathsf{Med}$.

Proof. We use the construction of f_n from the proof of Theorem 4.1.4 and then $f_n^0 = f_n \smallsetminus \{\underbrace{(t, t, \ldots, t)}_{n}, \underbrace{(1, 1, \ldots, 1)}_{n}\}$ is a p-morphism from $\mathcal{F}_{\mathrm{t}}^n$ to $\mathcal{P}_0(n)$, so $\mathsf{TLP}_{\mathcal{F}} = \bigcap_{n\in\omega} \mathsf{Log}(\mathcal{F}_{\mathrm{t}}^n) \subseteq \bigcap_{n\in\omega} \mathsf{Log}(\mathcal{P}_0(n)) = \mathsf{Med}$. □

The primary focus of this chapter is to answer the two conjectures made by Nick Bezhanishvili. We will then develop a characterization of generalized Medvedev logics.

Proposition 4.1.6 (Bezhanishvili's first conjecture)**.** $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable if $\mathcal{F}$ is an n-chain $\mathcal{H}_n$.

Proposition 4.1.7 (Bezhanishvili's second conjecture)**.** $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable for any non-singleton finite rooted frame $\mathcal{F}$ with a top.

In fact, since the Medvedev logic Med is exactly $\mathsf{TLP}_{\mathcal{F}}$ when $\mathcal{F}$ is chosen to be the 2-chain, the above two conjectures generalize the theorem that Medvedev logic Med is not finitely axiomatisable; the latter was proved by Maksimova, Skvortsov and Shehtman in [MSS79, Corollary 5].

4.2. Non-finite axiomatisation of generalized Medvedev logic

4.2.1. The chain case

We first proceed to deal with the first conjecture, the simpler case.

Theorem 4.2.1. $\mathsf{TLP}_{\mathcal{H}_n} = \mathsf{Med}$ for every finite chain $\mathcal{H}_n$.

Proof. For a given number n, let $\mathcal{H}_n$ be the n-chain, in other words, $\mathcal{H}_n = \langle\{1, 2, \ldots, n\}, \leq\rangle$, and let $\mathcal{P}(n)$ be n product of 2-chain. According to the definition, $\mathsf{TLP}_{\mathcal{H}_2} = \mathsf{Med}$. We consider the n-chain for $n \geq 3$.

We build a map $\hat{f}_1$ from $\mathcal{P}(n-1)$ to $\mathcal{H}_n$ as follows:

$$\hat{f}_1(x) = n + 1 - d(x), \text{ where } 1 \leq d(x) \leq n \text{ is the depth of } x \text{ in } \mathcal{P}(n-1).$$

If $x \leq y$ in $\mathcal{P}(n-1)$, then $d(y) \leq d(x)$, and so $\hat{f}_1(x) = n + 1 - d(x) \leq n + 1 - d(y) = \hat{f}_1(y)$ in $\mathcal{H}_n$. If $\hat{f}_1(x) \leq s$ in $\mathcal{H}_n$, then it is easy to find a y such that $x \leq y$ in $\mathcal{P}(n-1)$ and $d(y) = n + 1 - s$, thus $\hat{f}_1(y) = s$. Therefore $\hat{f}_1$ is a p-morphism from $\mathcal{P}(n-1)$ to $\mathcal{H}_n$. In addition, only the top element of $\mathcal{P}(n-1)$ has depth 1 and is sent to n in $\mathcal{H}_n$, which is the top element of $\mathcal{H}_n$.

Then $\hat{f}_1$ can be extended to a p-morphism $\hat{f}_m$ from $\underbrace{\mathcal{P}(n-1) \times \ldots \mathcal{P}(n-1)}_{m}$ to $\underbrace{\mathcal{H}_n \times \ldots \mathcal{H}_n}_{m}$ in the following way:

$$\hat{f}_m((x_1, \ldots, x_m)) = (\hat{f}_1(x_1), \ldots, \hat{f}_1(x_m)).$$

Note that only the top element of $\mathcal{P}(n-1) \times \ldots \mathcal{P}(n-1)$ can be sent to the top element of $\mathcal{H}_n \times \ldots \mathcal{H}_n$, so there exists a p-morphism from $(\mathcal{P}(n-1))^m_{\mathrm{t}}$ to $(\mathcal{H}_n)^m_{\mathrm{t}}$ for every natural number m. In conclusion, $\mathsf{TLP}_{\mathcal{P}(n-1)} \subseteq \mathsf{TLP}_{\mathcal{H}_n}$.

Since $\mathcal{P}(n-1) = \underbrace{\mathcal{H}_2 \times \ldots \times \mathcal{H}_2}_{n-1}$ and so $\mathsf{Med} = \mathsf{TLP}_{\mathcal{H}_2} \subseteq \mathsf{TLP}_{\mathcal{P}(n-1)}$, therefore $\mathsf{Med} \subseteq \mathsf{TLP}_{\mathcal{H}_n}$. But according to Theorem 4.1.5, Med is the greatest generalized Medvedev logic, so $\mathsf{TLP}_{\mathcal{H}_n} = \mathsf{Med}$. □

Corollary 4.2.2. $\mathsf{TLP}_{\mathcal{H}_n}$ is not finitely axiomatisable.

4.2.2. Chinese lanterns

In this section, we provide an overview of results by Maksimova, Skvortsov and Shehtman from [MSS79] and apply them to build a connection between $\mathsf{TLP}_{\mathcal{F}}$-frames and their Chinese lantern frames.

Definition 4.2.3 (Chinese lantern $\Phi(s,n)$)**.** Given integers s and n, the *Chinese lantern* $\Phi(s,n)$ is the left frame in Figure 4.2-2

$$\begin{aligned}\{(i,j) \in \omega \times \omega :\\ &(0 \le i \le s \text{ and } 0 \le j \le 1)\\ &\vee(i = s+1 \text{ and } 1 \le j \le n)\\ &\vee(i = s+2 \text{ and } j = 0)\}\end{aligned}$$

equipped with an accessibility relation that is defined as an ordering:

$$(i_0, j_0) \le (i_1, j_1) \text{ iff } i_0 > i_1 \text{ or } (i_0, j_0) = (i_1, j_1).$$

Definition 4.2.4 (Chinese lantern $\Phi'(s,n,m)$)**.** Given $m \le s$, let $\Phi'(s,n,m)$ be the right frame in Figure 4.2-2. It is formed by $\Phi'(s,n,m) = \Phi(s,n) \smallsetminus \{(m,1)\}$.

Lemma 4.2.5 (Maksimova, Skvortsov and Shehtman)**.** If $\mathcal{F}$ is a finite rooted frame with a top, then $\mathcal{F}$ is a Med-frame.

Proof. A proof of this lemma can be found in [Fon07, Claim 9]. □

Definition 4.2.6 (Suspension)**.** The *suspension* $\mathcal{F}^{(1)} = \langle F^{(1)}, \le_{(1)} \rangle$ of a frame $\mathcal{F} = \langle F, \le \rangle$ is defined as follows:

$$F^{(1)} = F \times \{\overline{0}\} \cup \{\overline{0}, \overline{1}\} \times \{\overline{1}\}$$

and

$$(x,a) \le_{(1)} (y,b) \text{ iff } (a = b = \overline{0} \wedge x \le y) \text{ or } (a = \overline{0} \wedge b = \overline{1}) \text{ or } (a = b \wedge x = y).$$

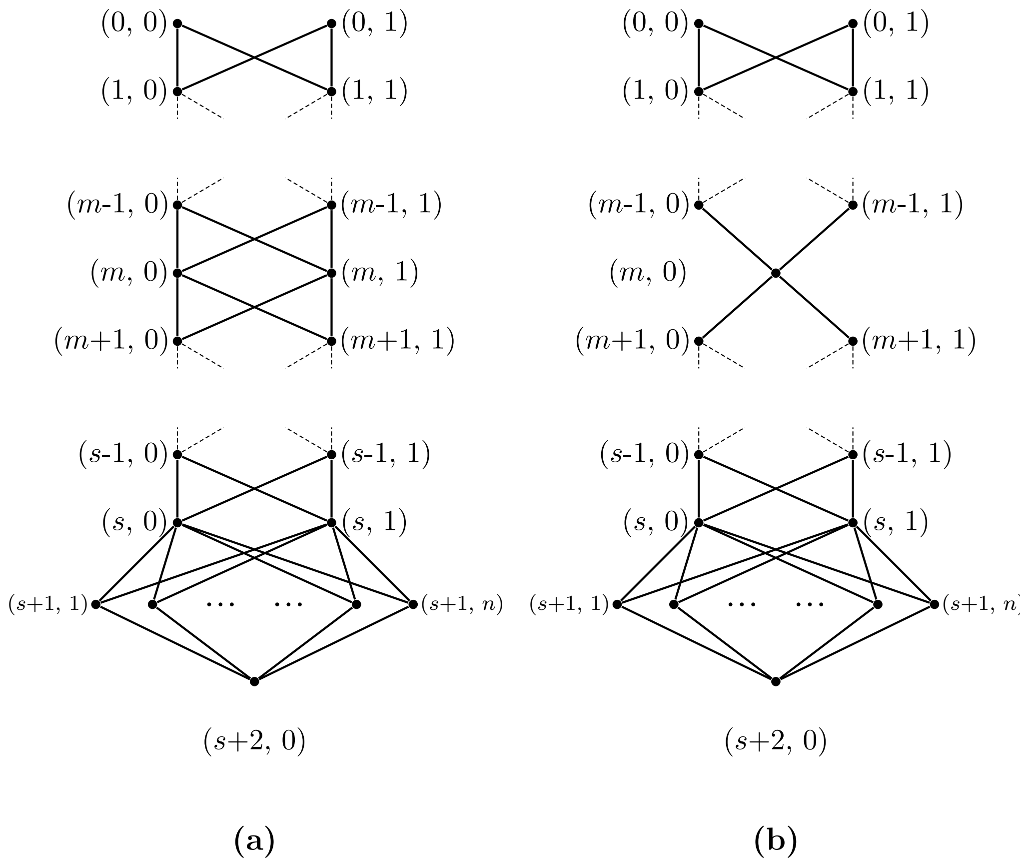


Figure 4.2-2: The frames $\Phi(s,n)$ and $\Phi'(s,n,m)$

The *n-th suspension* $\mathcal{F}^{(n)} = \langle F^{(n)}, \leq_{(n)} \rangle$ of $\mathcal{F}$ is the suspension of $\mathcal{F}^{(n-1)}$, thus we have

$$F^{(n)} = F^{(n-1)} \times \{\overline{0}\} \cup \{\overline{0}, \overline{1}\} \times \{\overline{1}\}$$

and

$$(x,a) \leq_{(n)} (y,b) \text{ iff } (a = b = \overline{0} \wedge x \leq_{(n-1)} y) \text{ or } (a = \overline{0} \wedge b = \overline{1}) \text{ or } (a = b \wedge x = y).$$

Lemma 4.2.7 (Maksimova, Skvortsov and Shehtman)**.** If $\mathcal{F}$ is a finite rooted frame with a top, then for any $n \geq 0$, the n-th suspension $\mathcal{F}^{(n)}$ of $\mathcal{F}$ is a p-morphic image of $\mathcal{P}_0(j)$ for some j.

Proof. If $n = 0$, then $\mathcal{F}^{(n)} = \mathcal{F}$ is a finite rooted frame with a top. So by

Lemma 4.2.5, $\mathcal{F}$ is a p-morphic image of $\mathcal{P}_0(j)$ for some j. We prove the proposition by induction for n.

Now suppose for $n \geq 1$, there exists a p-morphism f from $\mathcal{P}_0(j)$ to $\mathcal{F}^{(n)}$. We define a map f' from $\mathcal{P}_0(j+1)$ to $\mathcal{F}^{(n+1)}$ as follows:

$$f'(x) = \begin{cases} (\overline{0}, \overline{1}), & \text{if } x = \{j+1\} \\ (\overline{1}, \overline{1}), & \text{if } x \subseteq [j] \\ (f(x \smallsetminus \{j+1\}), \overline{0}), & \text{otherwise} \end{cases}$$

If $x \leq y$ in $\mathcal{P}_0(j+1)$, then $y \subseteq x \subseteq [j+1]$. Consider the following cases:

Case 1. If $y = \{j+1\}$, then $j+1 \in x$, so $f'(x) \leq (\overline{0}, \overline{1})$ and $f'(y) = (\overline{0}, \overline{1})$. Thus, $f'(x) \leq f'(y)$ in $\mathcal{F}^{(n+1)}$.

Case 2. If $\{j+1\} \subsetneq y$, then $\{j+1\} \subsetneq x$. Since f is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}^{(n)}$, we have $f(x \smallsetminus \{j+1\}) \leq f(y \smallsetminus \{j+1\})$. Therefore, $f'(x) = (f(x \smallsetminus \{j+1\}), \overline{0}) \leq (f(y \smallsetminus \{j+1\}), \overline{0}) = f'(y)$ in $\mathcal{F}^{(n+1)}$.

Case 3. If $j+1 \notin y$ and $j+1 \notin x$, then $y \subseteq x \subseteq [j]$, leading to $f'(x) = f'(y) = (\overline{1}, \overline{1})$.

Case 4. if $j+1 \notin y$ and $j+1 \in x$, then $y \subseteq [j]$ and $\{j+1\} \subsetneq x$. Thus, $f'(x) \leq (\overline{1}, \overline{1})$ and $f'(y) = (\overline{1}, \overline{1})$. Therefore, $f'(x) \leq f'(y)$ in $\mathcal{F}^{(n+1)}$.

If $f'(x) < t$ in $\mathcal{F}^{(n+1)}$ for some $x \in \mathcal{P}_0(j+1)$, consider the following cases:

Case 1. If $t = (\overline{0}, \overline{1})$, then $\{j+1\} \subsetneq x$. Let $y = \{j+1\}$, so $f'(x) < (\overline{0}, \overline{1}) = f'(y)$ and $y \subsetneq x$, which implies $x < y$.

Case 2. If $t = (\overline{1}, \overline{1})$, then $\{j+1\} \subsetneq x$. Let $y = x \smallsetminus \{j+1\} \subseteq [j]$, so $f'(x) < (\overline{1}, \overline{1}) = f'(y)$, and $y \subsetneq x$, which implies $x < y$.

Case 3. If $t \notin \{(\overline{0}, \overline{1}), (\overline{1}, \overline{1})\}$, since f is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}^{(n)}$, there exists $y_1 \in \mathcal{P}_0(j)$ such that $x \smallsetminus \{j+1\} < y_1$ and $f'(y_1 \cup \{j+1\}) = t$. Let $y = y_1 \cup \{j+1\}$, then $x < y$.

In conclusion, f' is a p-morphism from $\mathcal{P}_0(j+1)$ to $\mathcal{F}^{(n+1)}$. By induction for n, the statement holds for any $n \geq 0$, that is, for any $n \geq 0$, $\mathcal{F}^{(n)}$ is a p-morphic image of some $\mathcal{P}_0(j)$. $\square$

Lemma 4.2.8. Let $\mathcal{F}$ be a finite rooted frame with a top, then each $\Phi'(s, n, m)$ is a $\mathsf{TLP}_{\mathcal{F}}$-frame.

Proof. The downset $(m, 0)\downarrow$ in $\Phi'(s, n, m)$ is a finite rooted frame with a top $(m, 0)$. Then $\Phi'(s, n, m)$ can be obtained by taking the m-th suspension of $(m, 0)\downarrow$. In other words, $\Phi'(s, n, m) = ((m, 0)\downarrow)^{(m)}$ and by applying Lemma 4.2.7, it is obvious that $\Phi'(s, n, m)$ is a p-morphic image of some $\mathcal{P}_0(j)$.

By Theorem 4.1.5, every generalized Medvedev logic $\mathsf{TLP}_{\mathcal{F}} \subseteq \mathsf{Med}$. Since every $\Phi'(s,n,m)$ is a Med-frame, then each $\Phi'(s,n,m)$ is a $\mathsf{TLP}_{\mathcal{F}}$-frame. □

Proposition 4.2.9 (Maksimova, Skvortsov and Shehtman)**.** For any formula ϕ with s variables, there is an $m \le s$ such that

$$\Phi(s,n) \vDash \phi \text{ iff } \Phi'(s,n,m) \vDash \phi.$$

Proof. ($\Rightarrow$) The nodes $(m,0)$ and $(m,1)$ have the same immediate successors in $\Phi(s,n)$, there is a p-morphism g_m from $\Phi(s,n)$ to $\Phi'(s,n,m)$.

$$g_m(x) = \begin{cases} (m,0), & \text{if } x \in \{(m,0),(m,1)\} \\ x, & \text{otherwise} \end{cases}$$

If $\Phi(s,n) \vDash \phi$, then $\Phi'(s,n,m) \vDash \phi$ for any $m \le s$ due to the property of p-morphism.

($\Leftarrow$) Now we suppose that $\Phi(s,n) \nvDash \phi$, then there exists a valuation V such that $(\Phi(s,n),V) \nvDash \phi$. Suppose ϕ is a formula with s proposition variables $p_1,\dots,p_s$. According to the definition, the valuation $V(p_i)$ of a given proposition variable p_i is an upset in $\Phi(s,n)$, thus if p_i is true in x of $\Phi(s,n)$ under the valuation V, then p_i is true in all $y \ge x$ of $\Phi(s,n)$ under the valuation V. So for any p_i, it is impossible to have $a \ne b$ such that

$$(a,0) \text{ and } (a,1) \text{ do not agree on } p_i,$$

and at the same time,

$$(b,0) \text{ and } (b,1) \text{ do not agree on } p_i.$$

In other words, there exists at most one a such that $(a,0)$ and $(a,1)$ do not agree on p_i. We have s proposition variables $\{p_i\}_{1\le i\le s}$ and so there exists at least one m_0 such that $(m_0,0)$ and $(m_0,1)$ agree on every variable p_i. Then we define a valuation V_1 on $\Phi'(s,n,m_0)$ as follows:

$$V_1(p_i) = V(p_i) \smallsetminus \{(m_0,1)\}, \text{ for any proposition variable } p_i$$

Based on this, the above defined g_{m_0} is a map from $(\Phi(s,n),V)$ to $(\Phi'(s,n,m_0),V_1)$. For every $x \ne (m_0,1)$, $g_{m_0}(x) = x$ and then $x \in V(p_i)$ iff $g_{m_0}(x) \in V_1(p_i)$. For $x = (m_0,1)$, $x \in V(p_i)$ iff $g_{m_0}(x) = (m_0,0) \in V_1(p_i)$ since $(m_0,0)$ and $(m_0,1)$ agree on every p_i, so $x \in V(p_i)$ iff $g_{m_0}(x) \in V_1(p_i)$ for every point x.

Therefore g_{m_0} is a p-morphism from $(\Phi(s,n),V)$ to $(\Phi'(s,n,m_0),V_1)$. Because $\Phi(s,n) \nvDash \phi$, we also obtain a m_0 such that $\Phi'(s,n,m_0) \nvDash \phi$. □

Therefore, Chinese lanterns or similar structures can serve as an effective tool for measuring the "gap" between certain logics, i.e., whether they differ in a finite axiomatisation.

4.2.3. The general case

To prove Conjecture 4.1.7, the key point is to prove that for any finite rooted frame $\mathcal{F}$ with a top, we can find a Chinese lantern that is not a $\mathsf{TLP}_{\mathcal{F}}$-frame when $\mathcal{F}$ is an arbitrary finite rooted frame with a top. The following lemma is joint work with Fontaine.

Lemma 4.2.10. Let $\mathcal{F}$ be a finite rooted frame and no point in $\mathcal{F}$ has a single immediate successor, $\mathcal{A}$ be an arbitrary non-singleton finite rooted frame with a top and $b = \max\{b(u) : u \in \mathcal{A}\}$. If $\mathcal{F}$ is a p-morphic image of a generated subframe of some $\mathcal{A}^m_{\mathrm{t}}$, then $b(x) < b \times 2^{d(x)}$ for any x in $\mathcal{F}$, where $b(x)$ is the branching degree of x and $d(x)$ is the depth of x.

Proof. Let f be a p-morphism from $\mathcal{S}$ to frame $\mathcal{F}$ where $\mathcal{S}$ is a generated subframe of $\mathcal{A}^m_{\mathrm{t}}$. For any point $u = (u_1, u_2, \dots, u_m)$ in $\mathcal{S}$, let t be the top of $\mathcal{A}$, then let $\#(u)$ be the cardinality of set $\{u_i : u_i \neq t\}$ and $1 \leq \#(u) \leq m$.

We begin with proving that for any point x in $\mathcal{F}$, there is a point u_x in $\mathcal{S}$ such that

$$f(u_x) = x \text{ and } \#(u_x) < 2^{d(x)}$$

We prove this by induction on the depth of x in $\mathcal{F}$.

If $d(x) = 1$, then x is a maximal point in $\mathcal{F}$. According to the property of p-morphism, there is a point u in $\mathcal{S}$ such that $f(u) = x$, then the maximal point $u_x \geq u$ will imply $f(u_x) \geq f(u)$. So $f(u_x) = x$ and $\#(u_x) = 1 < 2 = 2^{d(x)}$.

Assume the above proposition holds for $d(x) = d$. We turn to the case $d(x) = d + 1$. Let u be a point in $\mathcal{S}$ such that $f(u) = x$. The frame $u\uparrow$ in $\mathcal{S}$ is also a generated subframe of $\mathcal{A}^m_{\mathrm{t}}$, and f is a p-morphism from $u\uparrow$ to the finite rooted subframe $x\uparrow$ of $\mathcal{F}$. According to the induction hypothesis, there are two distinct points y and z such that y and z are immediate successors of x, there are points u_y and u_z such that $f(u_y) = y$ and $f(u_z) = z$, $\#(u_y) < 2^{d(y)}$ and $\#(u_z) < 2^{d(z)}$.

Let $u_y = (y_1, y_2, \dots, y_m)$ and $u_z = (z_1, z_2, \dots, z_m)$ where $y_i, z_i \in \mathcal{A}$, we then define $u_x = (x_1, x_2, \dots, x_m)$ as follows:

$$x_i = \min\{y_i, z_i\} \text{ for any } 1 \leq i \leq m$$

So $u_x \geq u$ by its definition and $u_x = (x_1, x_2, \dots, x_m) < u_y$, $u_x < u_z$. Because f is a p-morphism, $f(u_x) \geq f(u) = x$ and $f(u_x) \leq f(u_y) = y$, $f(u_x) \leq f(u_z) = z$. But y and z are both immediate successors of x, thus $f(u_x) = x$.

Then we calculate the number of elements that is not equal to t in $u_x = (x_1, x_2, \dots, x_m)$. If x_i is not the top t of $\mathcal{A}$, then at least one of y_i and z_i is not the top t of $\mathcal{A}$ since $x_i = \min\{y_i, z_i\}$. Vice versa, once one of y_i and z_i is not the top t, then x_i is not t. In conclusion, $\{i : x_i \neq t\} = \{i : y_i \neq t\} \cup \{i : z_i \neq t\}$, so $\#(u_x) = |\{i : x_i \neq t\}| = |\{i : y_i \neq t\} \cup \{i : z_i \neq t\}| = |\{i : y_i \neq t\}| + |\{i : z_i \neq t\}| - |\{i : y_i \neq t\} \cap \{i : z_i \neq t\}| \leq |\{i : y_i \neq t\}| + |\{i : z_i \neq t\}| = \#(u_y) + \#(u_z) < 2^{d(y)} + 2^{d(z)}$. By the hypothesis, $d(y) = d(z) = d$, so $\#(u_x) < 2^{d+1} = 2^{d(x)}$.

This indicates that for a point x in $\mathcal{F}$, there is at least a point u_x in $\mathcal{S}$ such that $f(u_x) = x$ and $\#(u_x) < 2^{d(x)}$. Recall that if $u_x = (x_1, x_2, \dots, x_m)$, then the branching degree $b(u_x) = b(x_1) + b(x_2) + \dots + b(x_m)$. Furthermore, due to $b = \max\{b(u) : u \in \mathcal{A}\}$, then if x_i is not the top element t, its branching degree $b(x_i)$ is bounded by b, $b(x_i) \leq b$. If x_i is the top element t, $b(x_i) = 0$. Therefore there are at most $b \times \#(u_x)$ immediate successors of u_x and then the branching degree $b(u_x) \leq b \times \#(u_x) < b \times 2^{d(x)}$.

In order to conclude the proof, we still have to show that $b(x) \leq b(u_x)$ and we will do so by proving that for any point x in $\mathcal{F}$, since $\mathcal{A}_t^m$ is finite, we can always choose a maximal element u_x such that $f(u_x) = x$ and $\#(u_x) < 2^{d(x)}$. Then there exists no $u'_x > u_x$ that meets this requirement. Let y be an arbitrary immediate successor of x, because $f(u_x) = x \leq y$ and f is a p-morphism, then there is a u_y in $\mathcal{S}$ such that $f(u_y) = y$ and $u_x \leq u_y$. So there exists at least one immediate successor u' that is located between u_x and u_y, $u_x \leq u' \leq u_y$ implies $x = f(u_x) \leq f(u') \leq f(u_y) = y$ because of the property of p-morphism. Suppose $u_x = (x_1, x_2, \dots, x_m)$ and $u' = (x'_1, x'_2, \dots, x'_m)$, if x'_i is not the top t, then x_i is not the top t since $x_i \leq x'_i$. So $\#(u') \leq \#(u_x) < 2^{d(x)}$. If $f(u') = x$, then it would contradict the maximality of the choice of u_x. Therefore $f(u') = y$ is an immediate successor of x.

So for any immediate successor y of x in $\mathcal{F}$, we can always find an immediate successor u' of u_x in $\mathcal{S}$ such that $f(u') = y$, thus the branching degree of x is no more than the branching degree of u_x. Together with $b(u_x) < b \times 2^{d(x)}$, we conclude that $b(x) \leq b(u_x) < b \times 2^{d(x)}$. □

Corollary 4.2.11. $\Phi(s, b \times 2^{s+3})$ is not a $\mathsf{TLP}_{\mathcal{F}}$-frame if $\mathcal{F}$ is a non-singleton finite rooted frame with a top and $b = \max\{b(u) : u \in \mathcal{F}\}$.

Proof. Let x be the root of $\Phi(s, b \times 2^{s+3})$, then its depth $d(x) = s + 3$ and its branching degree $b(x) = b \times 2^{s+3}$. By Lemma 4.2.10, $\Phi(s, b \times 2^{s+3})$ is not a p-morphic image of any generated subframe of $\mathcal{F}_{\mathrm{t}}^m$.

Let χ be the Jankov-de Jongh formula of $\Phi(s, b \times 2^{s+3})$, then according to Jankov-de Jongh theorem, for any integer m,

$$\mathcal{F}_{\mathrm{t}}^m \nvDash \chi \text{ iff } \Phi(s, b \times 2^{s+3}) \text{ is } p\text{-morphic image of a generated subframe of } \mathcal{F}_{\mathrm{t}}^m.$$

Thus $\mathcal{F}_{\mathrm{t}}^m \vDash \chi$ for any m and therefore $\chi \in \bigcap_{m \in \omega} \mathsf{Log}(\mathcal{F}_{\mathrm{t}}^m) = \mathsf{TLP}_{\mathcal{F}}$. But it is clear that $\Phi(s, b \times 2^{s+3}) \nvDash \chi$, so $\Phi(s, b \times 2^{s+3})$ is not a $\mathsf{TLP}_{\mathcal{F}}$-frame. □

Theorem 4.2.12. $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable for any non-singleton finite rooted frame $\mathcal{F}$ with a top.

Proof. Suppose $\mathsf{TLP}_{\mathcal{F}}$ is finitely axiomatisable with s variables, we may assume that $\mathsf{TLP}_{\mathcal{F}}$ is axiomatised by a single formula $\phi(p_1, \ldots, p_s)$.

By Proposition 4.2.9, there is an $m \leq s$ such that

$$\Phi(s, b \times 2^{s+3}) \vDash \phi \text{ iff } \Phi'(s, b \times 2^{s+3}, m) \vDash \phi.$$

According to Corollary 4.2.11, $\Phi(s, b \times 2^{s+3}) \nvDash \phi$, but by Lemma 4.2.8, $\Phi'(s, b \times 2^{s+3}, m) \vDash \phi$ for any $m \leq s$, a contradiction.

Therefore $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable with s variables. Furthermore, any generalized Medvedev logic $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable. □

4.3. Further results on generalized Medvedev logics

Since any generalized Medvedev logic is not finitely axiomatisable, then many well-known logics are not generalized Medvedev logics. Together with the fact that Med is the greatest generalized Medvedev logic and $\mathsf{Med} \subseteq \mathsf{KC}$, we give an approximate description of the "geographic coordinates" of generalized Medvedev logics, i.e., the generalized Medvedev logics from an island within these logics.

Theorem 4.3.1. There are at least countably many different generalized Medvedev logic $\mathsf{TLP}_{\mathcal{F}}$.

Proof. We will construct a family of frames $\{\mathcal{D}(i)\}_{i \geq 1}$, where every frame $\mathcal{D}(i)$ is a finite rooted frame with a top. We begin with setting $\mathcal{D}(1) = \mathcal{H}_3$, where $\mathcal{H}_3$ is the 3-chain, then $\mathsf{TLP}_{\mathcal{D}(1)} = \mathsf{Med}$ by Theorem 4.2.1.

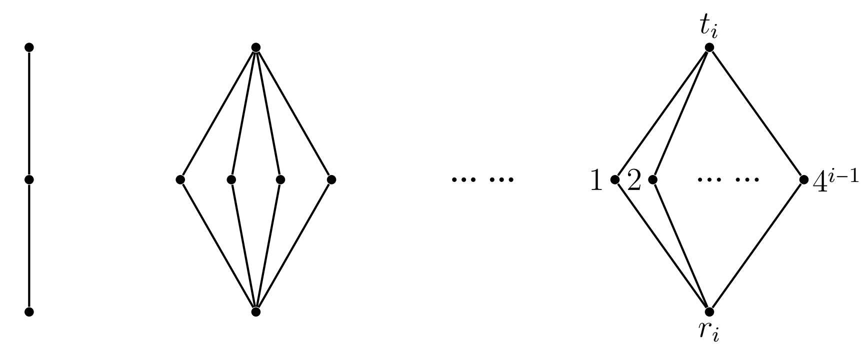


Figure 4.3-3: $\{\mathcal{D}(i)\}_{i\geq 1}$

As a next step, we construct $\mathcal{D}(2) = \langle D_2, R_2\rangle$ as follows:

$$D_2 = \{r_2, 1, 2, 3, 4, t_2\}$$

and

$$R_2 = \{(r_2, x) : x \in D_2\} \cup \{(x, t_2) : x \in D_2\} \cup \{(x, x) : x \in D_2\}.$$

Thus r_2 is the root of frame $\mathcal{D}(2)_{\mathrm{t}}$, we then consider some parameters related to the root in $\mathcal{D}(2)_{\mathrm{t}}$. It is obvious that the depth of r_2 is 2, and its branching degree is 4. According to Lemma 4.2.10, $\mathcal{D}(2)_{\mathrm{t}}$ is not a p-morphic image of a generated subframe of $\mathcal{D}(1)_{\mathrm{t}}^m$ for any natural number m, so the logic of $\mathcal{D}(2)_{\mathrm{t}}$ is strictly contained in $\mathsf{TLP}_{\mathcal{D}(1)} = \mathsf{Med}$. Since $\mathsf{TLP}_{\mathcal{D}(2)} \subseteq \mathsf{Log}(\mathcal{D}(2)_{\mathrm{t}})$, it is immediate to obtain that $\mathsf{TLP}_{\mathcal{D}(2)} \neq \mathsf{TLP}_{\mathcal{D}(1)}$.

Following the above approach, we then give the general construction scheme for $\mathcal{D}(i) = \langle D_i, R_i\rangle$:

$$D_i = \{r_i, 1, 2, \ldots, 4^{i-1}, t_i\}$$

and

$$R_i = \{(r_i, x) : x \in D_i\} \cup \{(x, t_i) : x \in D_i\} \cup \{(x, x) : x \in D_i\}.$$

It is easy to build a p-morphism f_{ji} from $\mathcal{D}(j)$ to $\mathcal{D}(i)$ when $1 \leq i < j$:

$$f_{ji}(x) = \begin{cases} 4^{i-1}, & \text{if } x \in \{4^{i-1}+1, \ldots, 4^{j-1}\} \\ x, & \text{otherwise} \end{cases}.$$

In addition, only the top t_i of $\mathcal{D}(i)$ can be sent to the top t_j of $\mathcal{D}(j)$ via f_{ji}, so when $1 \leq i < j$, we have the relation $\mathsf{TLP}_{\mathcal{D}(j)} \subseteq \mathsf{TLP}_{\mathcal{D}(i)}$.

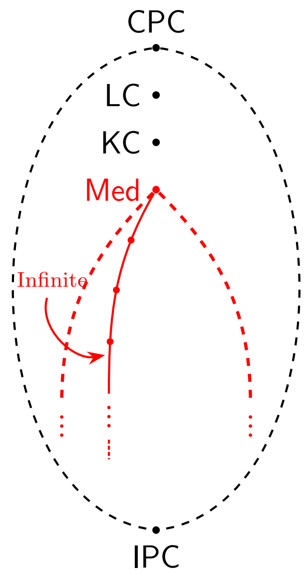


Figure 4.3-4: The red parts from Med down represent the current landscape of generalized Medvedev logics, all of which are not finitely axiomatisable.

The maximal branching degree of nodes in $\mathcal{D}(i)$ is the number of immediate successors of the root r_i, which is exactly 4^{i-1}. By applying Lemma 4.2.10, $\mathcal{D}(j)_t$ can not be a $\mathsf{TLP}_{\mathcal{D}(i)}$-frame, since $d(r_j) = 2$ in $\mathcal{D}(j)_\mathrm{t}$, but $b(r_j) = 4^{j-1} \geq 2^{d(r_j)} \times 4^{i-1}$. Therefore, $\mathsf{TLP}_{\mathcal{D}(j)}$ is strictly contained in $\mathsf{TLP}_{\mathcal{D}(i)}$ when $1 \leq i < j$, we then obtained countably many different generalized Medvedev logic $\{\mathsf{TLP}_{\mathcal{D}(i)}\}_{i\geq 1}$. □

Theorem 4.3.2. There is no least generalized Medvedev logic, i.e., the intersection of all generalized Medvedev logics is no longer a generalized Medvedev logic.

Proof. If not, then there is a finite rooted frame $\mathcal{F}_0$ with a top, such that $\mathsf{B} = \mathsf{TLP}_{\mathcal{F}_0}$ and $\mathsf{B} \subseteq \mathsf{TLP}_{\mathcal{F}}$ for every finite rooted frame $\mathcal{F}$ that has a top. Let $b_0 = \max\{b(u) : u \in \mathcal{F}_0\}$ be the maximal branching degree of points in the frame $\mathcal{F}_0$. Based on b_0, we construct a finite frame $\mathcal{D} = \langle D, R\rangle$:

$$D = \{r, 1, 2, \ldots, 4b_0, t'\}$$

and

$$R = \{(r, x) : x \in D\} \cup \{(x, t') : x \in D\} \cup \{(x, x) : x \in D\}.$$

In the frame $\mathcal{D}_\mathrm{t}$, no point has a single immediate successor and r is the root with its depth $d(r) = 2$. Since the branching degree of the root is $4b_0$, thus $\mathsf{B} = \mathsf{TLP}_{\mathcal{F}_0}$ is not contained in the logic of $\mathcal{D}_\mathrm{t}$ due to Lemma 4.2.10.

But $\mathsf{B} \subseteq \mathsf{TLP}_{\mathcal{D}} = \bigcap_{m\in\omega} \mathsf{Log}(\mathcal{D}_\mathrm{t}^m) \subseteq \mathsf{Log}(\mathcal{D}_\mathrm{t})$, a contradiction. □

It should be noted that, since subsequent conclusions are needed for the proof, we will compare the generalized Medvedev logic with Cheq and provide Proposition 6.2.5, placing this result in Chapter 6.

5. Nerves and Medvedev frames

This chapter builds a bridge between the construction of *nerves* of posets and Medvedev frames. The main result states that the logic of any *dual nerve* is the logic of some Medvedev frame. The presented proof was obtained together with Nick Bezhanishvili and Gaëlle Fontaine. However, as was pointed out by Nick Bezhanishvili, the result itself is due to Vincenzo Marra, who announced it at ToLo 2016 (Topological Methods in Logic 2016) in Tbilisi, Georgia.

To prove this result, we use two different methods. One of them indicates some geometric aspects of nerves and simplicial complexes.

5.1. Nerves and dual nerves

We give a brief introduction to Alexandrov's notion of the nerve from [Ale28] for a poset. For a poset $\mathcal{F}$, its *nerve*, $\mathcal{N}(\mathcal{F})$, is defined as the collection of finite non-empty chains in $\mathcal{F}$, ordered by inclusion.

Definition 5.1.1. Let $\mathcal{F} = \langle W, \leq_W \rangle$ be a poset. The *dual nerve* of $\mathcal{F}$, denoted by $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$, is the set of all finite, non-empty chains in $\mathcal{F}$ ordered by the reverse inclusion relation, that is, for two chains $\mathcal{H}$ and $\mathcal{H}'$ in $\mathcal{F}$, we say $\mathcal{H} \leq \mathcal{H}'$ in $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ iff $\mathcal{H}'$ is a subchain of $\mathcal{H}$.

Lemma 5.1.2. Let $\mathcal{F}$ be a finite poset. Then there exists a p-morphism from $\mathcal{N}(\mathcal{F})$ to $\mathcal{F}$.

Proof. Let us consider the map $f_{\max} : \mathcal{N}(\mathcal{F}) \to \mathcal{F}$, which sends a chain to its maximum element.

Assume $\mathcal{H}$ and $\mathcal{H}'$ be two chains of $\mathcal{F}$. Let c and c' be the maximal element of the chains $\mathcal{H}$ and $\mathcal{H}'$, respectively. If $\mathcal{H} \leq \mathcal{H}'$ in $\mathcal{N}(\mathcal{F})$, then $\mathcal{H} \subseteq \mathcal{H}'$ and then $f_{\max}(\mathcal{H}) = c \leq c' = f_{\max}(\mathcal{H}')$.

If $f_{\max}(\mathcal{H}) \le c'$, let $\mathcal{H}' = \mathcal{H} \cup \{c'\}$, then $\mathcal{H}'$ is a chain in $\mathcal{F}$ and $\mathcal{H} \le \mathcal{H}'$ in $\mathcal{N}(\mathcal{F})$. Since c' is greater than the maximal element of $\mathcal{H}$, then $f_{\max}(\mathcal{H}') = f_{\max}(\mathcal{H} \cup \{c'\}) = c'$.

Therefore the max map $f_{\max}$ is a p-morphism from $\mathcal{N}(\mathcal{F})$ to $\mathcal{F}$. □

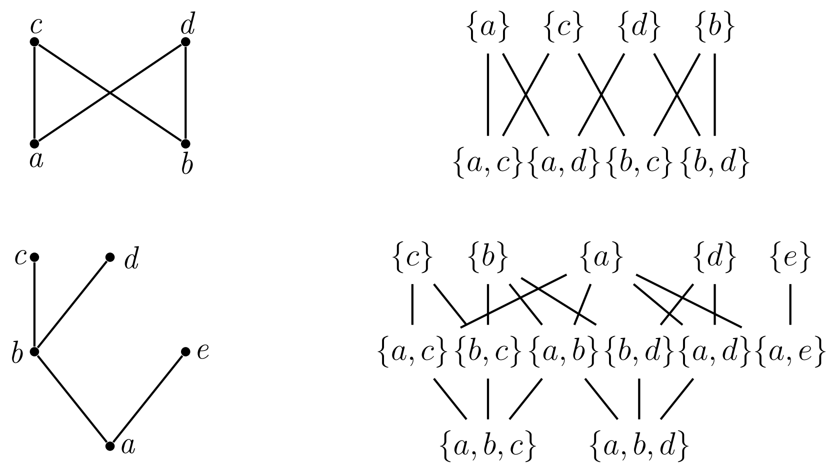


Figure 5.1-1: Two frames and their dual nerves

Lemma 5.1.3. $\mathsf{Log}(\mathcal{P}_0(m)) \subsetneq \mathsf{Log}(\mathcal{P}_0(n))$ when $n < m$.

Proof. For $n < m$, let $E = \{n, n+1, \ldots, m\}$. We construct a map f from $\mathcal{P}_0(m)$ to $\mathcal{P}_0(n)$ as follows:

$$f(X) = \begin{cases} X, & X \cap E = \varnothing \\ (X \smallsetminus E) \cup \{n\}, & X \cap E \neq \varnothing \end{cases}$$

For any non-empty subset $X \subseteq [m]$, we regard the above set E as a new element n. We can then study the behavior of $\mathcal{P}_0(m)$ like $\mathcal{P}_0(n)$.

If $X \le Y$ in $\mathcal{P}_0(m)$, then $Y \subseteq X$.

Case 1. When $X \cap E = \varnothing$, we have $Y \cap E = \varnothing$, thus $f(Y) = Y \subseteq X = f(X)$, $f(X) \le f(Y)$.

Case 2. If $X \cap E \neq \varnothing$ and $Y \cap E \neq \varnothing$, then $f(Y) = (Y \smallsetminus E) \cup \{n\} \subseteq (X \smallsetminus E) \cup \{n\} = f(X)$, $f(X) \le f(Y)$.

Case 3. If $X \cap E \neq \varnothing$ and $Y \cap E = \varnothing$, then $Y \subseteq X \smallsetminus E$ and so $f(Y) = Y \subseteq (X \smallsetminus E) \cup \{n\} = f(X)$, $f(X) \le f(Y)$.

If $f(X) \leq U$ in $\mathcal{P}_0(n)$ and $X \cap E = \varnothing$, then $U \subseteq f(X) = X$ and $U \cap E = \varnothing$, thus $f(U) = U$ and $X \leq U$ in $\mathcal{P}_0(m)$.

If $f(X) \leq U$ in $\mathcal{P}_0(n)$ and $X \cap E \neq \varnothing$, then $U \subseteq (X \smallsetminus E) \cup \{n\}$.

Case 1. When $n \in U$, it is easy to see that $U \cap E = \{n\}$. Since $U \smallsetminus \{n\} \subseteq X \smallsetminus E$, then $(U \smallsetminus \{n\}) \cup (X \cap E) \subseteq X$. Therefore $f((U \smallsetminus \{n\}) \cup (X \cap E)) = (U \smallsetminus \{n\}) \cup \{n\} = U$ and $X \leq (U \smallsetminus \{n\}) \cup (X \cap E)$ in $\mathcal{P}_0(m)$.

Case 2. When $n \notin U$, we have $U \subseteq X \smallsetminus E$ and then $U \cap E = \varnothing$, so $f(U) = U$ and $X \leq U$ in $\mathcal{P}_0(m)$.

In conclusion, f is a p-morphism from $\mathcal{P}_0(m)$ to $\mathcal{P}_0(n)$, so $\mathsf{Log}(\mathcal{P}_0(m)) \subseteq \mathsf{Log}(\mathcal{P}_0(n))$. Recall that we can bound the width of every rooted subframe by using the following formula:

$$\mathsf{bw_n} = \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j), \text{ for } n \geq 1.$$

Since Sperner's theorem from [Spe28] indicates that the maximal number of subsets of $[n]$ such that no one contains another is $\binom{n}{\lfloor n/2 \rfloor}$, then we have $\mathcal{P}_0(n) \vDash \mathsf{bw}_{\binom{\mathsf{n}}{\lfloor \mathsf{n}/2 \rfloor}}$ and $\mathcal{P}_0(m) \nvDash \mathsf{bw}_{\binom{\mathsf{n}}{\lfloor \mathsf{n}/2 \rfloor}}$.

Putting everything together, $\mathsf{Log}(\mathcal{P}_0(m)) \subsetneq \mathsf{log}(\mathcal{P}_0(n))$. □

5.2. The main result

The following theorem is the main result in this chapter and it builds a bridge between the (dual) nerves and Medvedev logic.

Theorem 5.2.1. $\mathcal{N}^{\mathrm{d}}(\mathcal{F}) \vDash \mathsf{Med}$ for any finite frame $\mathcal{F}$.

To prove this theorem, we begin by studying the simplest case: the chain case.

Lemma 5.2.2. For any integer $n \geq 2$, the dual nerve of an n-chain is precisely the Medvedev frame on n elements, that is, $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n) = \mathcal{P}_0(n)$ up to the isomorphism of posets.

Proof. Let $\mathcal{H}_n = \langle \{x_1, x_2, \dots, x_n\}, \leq \rangle$, then every non-empty finite chain $\mathcal{H}$ of the n-chain $\mathcal{H}_n$ is, in fact, a subset of $\{1, 2, \dots, n\}$ that ordered by $\leq$. Thus it is natural to have the following map f from the non-empty chain $\mathcal{H}$ of $\mathcal{H}_n$ to the non-empty subset S of $[n]$:

$$f(\mathcal{H}) = S, \text{ if } S = \{i : x_i \in \mathcal{H}\}.$$

The map f is a one-to-one such that sends a subchain to its index set. Furthermore,

$$\begin{aligned} &\mathcal{H} \le \mathcal{H}' \text{ in } \mathcal{N}^{\mathrm{d}}(\mathcal{H}_n) \\ &\quad \iff \mathcal{H} \supseteq \mathcal{H}' \\ &\quad \iff S = f(\mathcal{H}) \supseteq f(\mathcal{H}') = S' \\ &\quad \iff S \le S' \text{ in } \mathcal{P}_0(n). \end{aligned}$$

Therefore, the map f is a poset isomorphism from $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n)$ to $\mathcal{P}_0(n)$, thus $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n) \cong \mathcal{P}_0(n)$. □

Proof of Theorem 5.2.1. For an arbitrary finite frame $\mathcal{F}$, $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ is the dual nerve of $\mathcal{F}$. Let u be any point in $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$, then according to the definition, u is a finite non-empty chain of $\mathcal{F}$, assume $u = \mathcal{H}_n$, so the rooted generated subframe of $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ which is generated by u is exactly all finite non-empty chains of $\mathcal{H}_n$, that is, the dual nerve of an n-chain, $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n)$. By Lemma 5.2.2, $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n) \cong \mathcal{P}_0(n)$, thus the logic $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n)) = \mathsf{Log}(\mathcal{P}_0(n))$.

Since the logic of every rooted generated subframe of $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ is equal to the logic of some Medvedev frame, thus the logic Med is valid in any rooted generated subframe of $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$, therefore $\mathcal{N}^{\mathrm{d}}(\mathcal{F}) \vDash \mathsf{Med}$. □

Corollary 5.2.3. For any finite frame $\mathcal{F}$, there exists $m \in \omega$ such that the logic $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{F})) = \mathsf{Log}(\mathcal{P}_0(m))$.

Proof. Let the maximal chain of $\mathcal{F}$ be $\mathcal{H}_m$, that is, the size of the greatest chain of $\mathcal{F}$ is m.

According to the proof of Theorem 5.2.1, each rooted generated subframe of $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ is a dual nerve of some finite chain, that is, $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_n)$, which is isomorphic to a Medvedev frame $\mathcal{P}_0(n)$, where $n \le m$.

Thus the logic of $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ is the intersection of the logics of some $\mathcal{P}_0(i)$, $i \le m$. By Lemma 5.1.3, $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{F})) = \mathsf{Log}(\mathcal{P}_0(m))$. □

5.3. Another proof of the main result

In this section, we consider a given finite frame $\mathcal{F}$ and fix the size of its largest chain to be m. The following definition and lemma are due to Bezhanishvili [Bez06, Lemma 3.1.6].

Assume a and b have the same immediate successors in the frame $\mathcal{F} = \langle W, \leq \rangle$. Let E be the smallest equivalence relation identifying a and b, with this equivalence relation, the *quotient frame* $\mathcal{F}/E = (W/E, \leq')$ is a frame such that

$$W/E = \{E(x) : x \in W\}, \text{ where } E(x) = \{y \in W : xEy\},$$

and

$$E(x) \leq' E(y) \text{ iff } x' \leq y' \text{ for some } x' \in E(x) \text{ and } y' \in E(y).$$

The *β-reduction* is a map $f_E : W \to W/E$ which is defined as follows:

$$f_E(x) = E(x).$$

Lemma 5.3.1. The map f_E is a p-morphism.

Proof. Since a and b have the same immediate successors, then a and b are incomparable.

If $x \leq a$ in W, then $E(x) \leq' E(a)$, thus $f_E(x) \leq' f_E(a)$.

If $a \leq x$ in W, following the same argument, $f_E(a) \leq' f_E(x)$.

If $f_E(x) <' E(a)$ in W/E, then $E(x) <' E(a)$, so either $x < a$ or $x < b$, and $f_E(a) = f_E(b) = E(a)$.

If $E(a) <' f_E(x)$ in W/E, then either $a < x$ or $b < x$. Because the immediate successors of a and b are the same, no matter whether a or b is sent to $E(a)$ by f_E, we always have $a < x$ and $b < x$. □

For the given finite frame $\mathcal{F}$, it is certain that we can sequentially disentangle each of its maximal (finite) chains from $\mathcal{F}$ through a finite number of steps. Upon collecting these maximal chains, we align their diagrams adjacently and regard them as one big diagram to obtain a new poset, which we call the *bunch* (denoted by $\mathbf{B}(\mathcal{F})$). It is not difficult to have the following lemma.

Lemma 5.3.2. $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))) = \mathsf{Log}(\mathcal{P}_0(m))$.

Proof. From the above construction of $\mathbf{B}(\mathcal{F})$, it follows that the bunch is the disjoint union of a family of maximal chains of the frame $\mathcal{F}$. Let this finite family of maximal chains be $\{\mathcal{H}^i : i \in I\}$ and there exists some $\mathcal{H}^j = \mathcal{H}_m$ since the size of the largest chain of $\mathcal{F}$ is m.

By Lemma 5.2.2, after taking the dual nerve, $\mathcal{N}^{\mathrm{d}}(\mathcal{H}^i) \cong \mathcal{P}_0(n_i)$. As the disjoint union, $\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F})) = \sum_{i\in I} \mathcal{N}^{\mathrm{d}}(\mathcal{H}^i) \cong \sum_{i\in I} \mathcal{P}_0(n_i)$, thus its logic $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))) = \bigcap_{i\in I} \mathsf{Log}(\mathcal{P}_0(n_i))$. By Lemma 5.1.3, $\bigcap_{i\in I} \mathsf{Log}(\mathcal{P}_0(n_i)) = \mathsf{Log}(\mathcal{P}_0(m))$. Hence, it can be concluded that the logic corresponding to the bunch is equivalent to the logic of the largest chain of $\mathcal{F}$. □

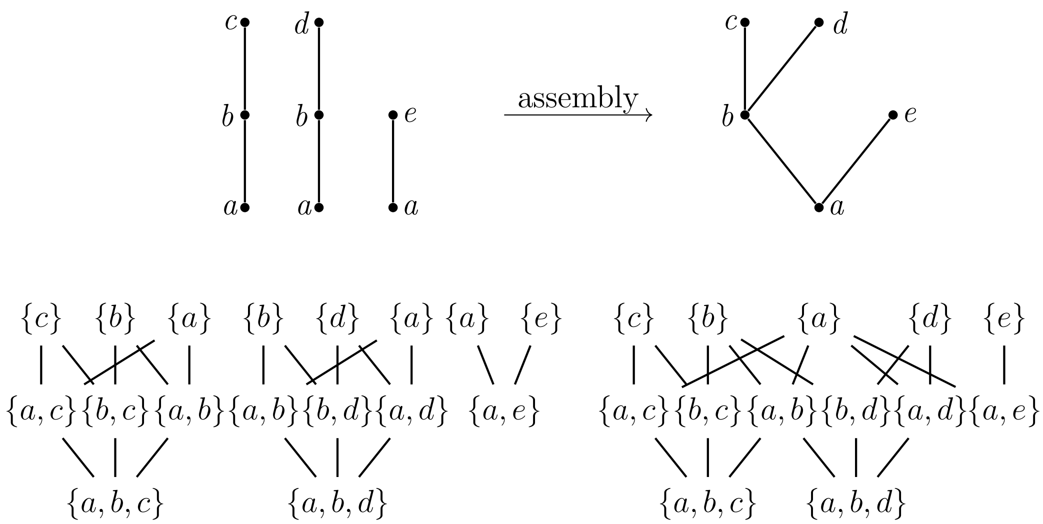


Figure 5.3-2: Dual nerve through assembly

Conversely, given the collection of the maximal chains in the bunch $\mathbf{B}(\mathcal{F})$ as components, we engage in a process to reconstruct $\mathcal{F}$. This reconstruction leverages the points of the bunch as instructions. Specifically, when a common point is identified across multiple maximal chains within $\mathbf{B}(\mathcal{F})$, these chains are glued together at the shared point. The reconstruction is deemed complete once there remains no point that needs to be glued. This process, from the bunch $\mathbf{B}(\mathcal{F})$ to $\mathcal{F}$, is called an *assembly*.

It is noteworthy that, in the process of reconstructing the frame $\mathcal{F}$ through assembly, the same gluing instructions at those share points transform the bunch's dual nerve into the dual nerve of $\mathcal{F}$, in fact, constitute β-reductions from $\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))$ to $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$.

Lemma 5.3.3. $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))) \subseteq \mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{F}))$.

Proof. During the process of reconstructing $\mathcal{F}$ from its bunch $\mathbf{B}(\mathcal{F})$, if two distinct non-empty finite chains X and Y are mapped to the same point under the above assembly program instructions, then X and Y were originally the same chain in the frame $\mathcal{F}$, thus the corresponding subchains of X and Y will likewise be mapped to the same point during the assembly process. This indicates that the execution of the assembly directs some points in $\mathcal{B}(\mathcal{F})$ with the same immediate successors to the same point in $\mathcal{F}$. By Lemma 5.3.1, there is a p-morphism from $\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))$ to $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$, therefore $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathbf{B}(\mathcal{F}))) \subseteq \mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{F}))$. □

As a next step, we attempt to horizontally compress $\mathcal{F}$ into a chain, considering $\mathcal{H}_m$ as its largest chain with the aim of compressing $\mathcal{F}$ into $\mathcal{H}_m$. A natural approach is gluing points that share the same depth. This process, from the frame $\mathcal{F}$ to its largest chain $\mathcal{H}_m$, where all points with a depth equal to d are compressed into a new point of the chain, whose depth remains d. We call this process a *consolidation*.

In other words, the consolidation is a map $\hat{f}$ from $\mathcal{F}$ to $\mathcal{H}_m$, such that

$$\hat{f}(x) = d(x), \text{ where } d(x) \text{ is the depth of } x.$$

The consolidation $\hat{f}$ naturally induces a correspondence f_{con} from the non-empty finite chains of $\mathcal{F}$ to the non-empty finite chains of $\mathcal{H}_m$:

$$f_{\mathrm{con}}(\{x_1, x_2, \ldots, x_s\}) = \{\hat{f}(x_1), \hat{f}(x_2), \ldots, \hat{f}(x_s)\}$$

It will soon be proved that f_{con} is a map from $\mathcal{N}^{\mathrm{d}}(\mathcal{F})$ to $\mathcal{N}^{\mathrm{d}}(\mathcal{H}_m)$, reflecting the β-reductions from the former to the latter. When $x_i \neq x_j$ in a chain $X = \{x_1, x_2, \ldots, x_s\}$, then $d(x_i) \neq d(x_j)$, so $\hat{f}(x_i) \neq \hat{f}(x_j)$. Hence, f_{con} does not change the size of the chain.

Lemma 5.3.4. $\mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{F})) \subseteq \mathsf{Log}(\mathcal{N}^{\mathrm{d}}(\mathcal{H}_m))$.

Proof. For two different nonempty finite chains X and Y of $\mathcal{F}$, we investigate what will happen when $f_{\mathrm{con}}(X) = f_{\mathrm{con}}(Y)$. According to the above argument, the cardinality of X is equal to the cardinality of Y. Let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$, by the definition of f_{con}, $\{\hat{f}(x_1), \hat{f}(x_2), \ldots, \hat{f}(x_s)\} = f_{\mathrm{con}}(X) = f_{\mathrm{con}}(Y) = \{\hat{f}(y_1), \hat{f}(y_2), \ldots, \hat{f}(y_s)\}$. For any subchain X' of X, $f_{\mathrm{con}}(X')$ is a non-empty subset of $f_{\mathrm{con}}(X)$, so $f_{\mathrm{con}}(X')$ is also a non-empty subset of $f_{\mathrm{con}}(Y)$, e.g., $f_{\mathrm{con}}(X') = f_{\mathrm{con}}(Y')$ where Y' is a subchain of Y. Furthermore, different subchains of X will correspond to different subchains

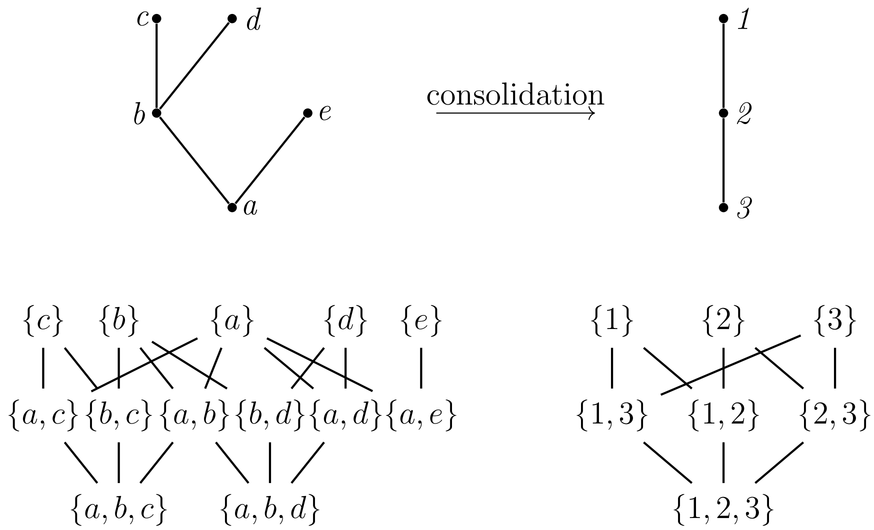


Figure 5.3-3: Dual nerve through consolidation

of Y and vice versa. In particular, $f_{\text{con}}(X)$ and $f_{\text{con}}(Y)$ have the same immediate successors. According to Lemma 5.3.1, f_{con} is a p-morphism from $\mathcal{N}^{\text{d}}(\mathcal{F})$ to $\mathcal{N}^{\text{d}}(\mathcal{H}_m)$ and so $\mathsf{Log}(\mathcal{N}^{\text{d}}(\mathcal{F})) \subseteq \mathsf{Log}(\mathcal{N}^{\text{d}}(\mathcal{H}_m))$. □

Another Proof of Theorem 5.2.1. By Lemma 5.3.2 and Lemma 5.3.3, the logic $\mathsf{Log}(\mathcal{P}_0(m))$ is contained in $\mathsf{Log}(\mathcal{N}^{\text{d}}(\mathcal{F}))$. According to Lemma 5.2.2 and Lemma 5.3.4, the logic $\mathsf{Log}(\mathcal{N}^{\text{d}}(\mathcal{F}))$ is contained $\mathsf{Log}(\mathcal{P}_0(m))$. Therefore, $\mathsf{Log}(\mathcal{N}^{\text{d}}(\mathcal{F})) = \mathsf{Log}(\mathcal{P}_0(m))$ and $\mathcal{N}^{\text{d}}(\mathcal{F}) \vDash \mathsf{Med}$. □

5.4. Geometric aspects

5.4.1. Simplices and simplicial complexes

This section will discuss geometric aspects of Theorems 5.2.1, Lemma 5.2.2 and Corollary 5.2.3, as well as the relationship between nerves and barycentric subdivision. To this end, a brief review of the (algebraic and geometric) content of simplices and simplicial complexes is in order.

Let $S = \{p_i\}_{i\in I}$ be a finite set in the w-dimensional Euclidean space $\mathbb{R}^w$. For given points of S, their *affine combination* is a point $p = \sum_{i\in I} \alpha_i p_i$ where $\sum_{i\in I} \alpha_i = 1$. We say an affine combination $p = \sum_{i\in I} \alpha_i p_i$ is a *convex combination* if $\alpha_i \geq 0$ for all $i \in I$. The *convex hull* of S, (denote by $\text{conv}(S)$), is the set of all convex combinations of points in S.

Furthermore, points in S are *affinely independent* if no point is an affine combination of other points. We give a definition of the *simplex*.

Definition 5.4.1. A *simplex* X is a convex hull of affinely independent points.

If $S = \{p_1, \ldots, p_{m+1}\}$ is an affinely independent set in the w-dimensional Euclidean space $\mathbb{R}^w$, then $X = \text{conv}(S)$ is an *m-simplex*, its dimension $\dim(X) = m$. An m-simplex X_m is an m-dimensional polytope or the simplest kind of m-dimensional polyhedron.

Suppose the $m+1$ points $p_1, \ldots, p_{m+1}$ are affinely independent, which means the m vectors $p_2 - p_1, \ldots, p_{m+1} - p_1$ are linearly independent, and then

$$X = \{\sum_{1 \leq i \leq m+1} \alpha_i p_i : \sum_{1 \leq i \leq m+1} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for } 1 \leq i \leq m+1\}.$$

When $T \subseteq S$ is a subset of those affinely independent points, the points of T are obviously affinely independent. Then $Y = \text{conv}(T)$, the convex hull of T, is a simplex. In fact, it is not difficult to see

$$Y = \left\{ \sum_{1 \leq i \leq m+1} \alpha_i p_i \ : \ \begin{array}{l} \sum_{1 \leq i \leq m+1} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for } 1 \leq i \leq m+1 \\ \text{and } \alpha_i = 0 \text{ for } p_i \notin T \end{array} \right\}.$$

We say the simplex Y is a *face* of X, denoted as $Y \preceq X$. Write $Y \prec X$ if $Y \preceq X$ and $Y \neq X$. More specifically, $Y = \varnothing$ and $Y = X$ are called the *improper faces* and the other faces are called *proper faces*. When $\dim(Y) = m' \leq m$, we call it an m'-face of X.

The non-empty faces of a simplex X form a poset ordered by the set inclusion. We call this poset the *face poset* $\mathcal{Q}(X)$ of the simplex X. The *dual* of this face poset, denoted as $\mathcal{Q}^{\text{d}}(X)$, is obtained by reversing the order in the $\mathcal{Q}(X)$.

Definition 5.4.2. A *simplicial complex* Σ is a finite set of simplices that satisfies the following conditions:

1. Every face of a simplex from Σ is in Σ.

2. The non-empty intersection of any two simplices X and X' from Σ is a face of both X and X'.

The *support* of Σ is the set $|\Sigma| = \bigcup \Sigma$. We say that Σ is a *triangulation* of the polyhedron $|\Sigma|$.

Under the above order $\leq$, Σ forms a poset $\mathcal{Q}(\Sigma)$, we also call this poset a *face poset* of the simplicial complex. $\mathcal{Q}^{\mathrm{d}}(\Sigma)$ denotes the *dual* face poset of the simplicial complex Σ, which is defined to be the poset of non-empty faces ordered by reverse inclusion.

For each simplicial complex Σ, its *vertex set* is the set of all its 0-faces, that is, $V = \{Y : Y \in \Sigma \text{ and } \dim(Y) = 0\}$. If we only focus on its vertex set, we are ignoring the geometric property of the simplex and studying its combinatorial structure. This leads to an important concept in combinatorics, *abstract simplicial complex*. If V is a finite set of points and $A(\Sigma)$ is a collection of non-empty finite subsets of V, we say that $A(\Sigma)$ is an *abstract simplicial complex* on the finite vertex set V if

1. $\{v\} \in A(\Sigma)$ for all $v \in V$.
2. $X \in A(\Sigma)$ and $Y \subseteq X$ imply $Y \in A(\Sigma)$.

Related concepts can be extended from geometric to abstract simplicial complex. The elements of $A(\Sigma)$ are called the *faces* and the maximal faces are called *facets*. The elements of the vertex set are called the *vertices* and each face is a finite subset of the vertex set. We say that a face X has dimension m, that is $\dim(X) = m$, if $m = |X| - 1$. The *dimension* $\dim(A(\Sigma))$ of $A(\Sigma)$ is defined to be $\max\{\dim(X) : X \in A(\Sigma)\}$.

Let Σ be a simplicial complex. We can obtain an abstract simplicial complex $A(\Sigma)$ by regarding the faces of $A(\Sigma)$ be the vertex sets of the simplices of Σ. Each abstract simplicial complex $A(\Sigma)$ can be constructed in this way. Although the choice of Σ is not unique, the underlying topological space, which is formed by taking the union of the simplices in Σ endowed with the standard topology on $\mathbb{R}^w$, is unique up to homeomorphism. We call this space the *geometric realization* of $A(\Sigma)$.

An abstract simplicial complex $A(\Sigma)$ together with the set inclusion relation will form a poset, called the *abstract simplicial complex poset* $\mathcal{A} = \langle A(\Sigma), \subseteq \rangle$, then $\mathcal{A} \cong \mathcal{Q}(\Sigma)$.

5.4.2. Face posets and order complexes

To build a bridge between face posets and the complexes, we need the following definitions and facts, which are classical results from the textbook of poset topology [Wac07, § 1.1].

For any poset $\mathcal{F}$, one can associate an abstract simplicial complex with it, denoted by $\Delta(\mathcal{F})$, called the *order complex* of $\mathcal{F}$. The vertices of $\Delta(\mathcal{F})$ are those points of $\mathcal{F}$ and the faces of $\Delta(\mathcal{F})$ are those chains of $\mathcal{F}$. Consider the face poset of $\Delta(\mathcal{F})$, it will be the collection of all non-empty faces of the order complex together with the set inclusion. Thus a non-empty chain in $\mathcal{F}$ is a non-empty face of $\Delta(\mathcal{F})$. According to the definition of nerve, the face poset $\mathcal{Q}(\Delta(\mathcal{F}))$ is obtained by taking the nerve of the original poset, i.e., $\mathcal{N}(\mathcal{F}) = \mathcal{Q}(\Delta(\mathcal{F}))$.

If we start with a simplicial complex Σ, we can then obtain the face poset $\mathcal{Q}(\Sigma)$ and then take the order complex to get the $\Delta(\mathcal{Q}(\Sigma))$, the final simplicial complex is known as the *barycentric subdivision* of Σ, write as $\mathbf{Sd}(\Sigma)$, that is, $\mathbf{Sd}(\Sigma) = \Delta(\mathcal{Q}(\Sigma))$.

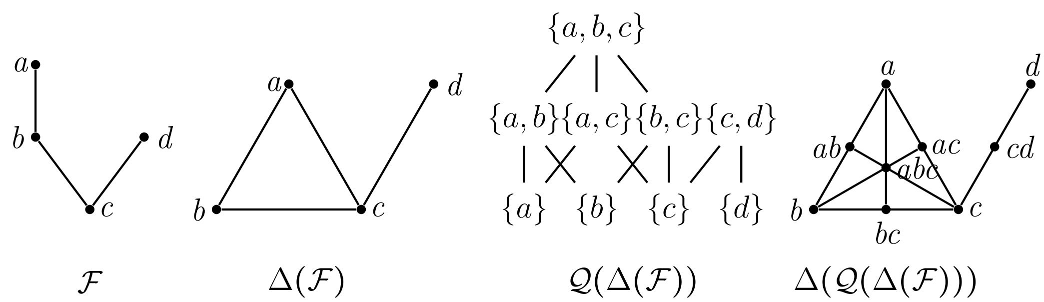


Figure 5.4-4: Nervers and barycentric subdivision

Thus, if we consider either face poset, order complex, or nerve, barycentric subdivision as operators, then based on the argument above, we get the following relation:

$$\mathcal{N} = \mathcal{Q} \cdot \Delta$$

and

$$\mathbf{Sd} = \Delta \cdot \mathcal{Q}$$

Assume we fix a simplicial complex Σ, let its abstract simplicial complex poset be $\mathcal{A}$, then $\mathcal{A} = \mathcal{Q}(\Sigma)$ up to the isomorphism. So

$$\mathcal{N}(\mathcal{A}) = \mathcal{Q} \cdot \Delta(\mathcal{A}) = \mathcal{Q} \cdot \Delta \cdot \mathcal{Q}(\Sigma) = \mathcal{Q}(\mathbf{Sd}(\Sigma))$$

and thus $\mathcal{N}(\mathcal{A})$ is the abstract simplicial complex poset of $\mathbf{Sd}(\Sigma)$.

Furthermore, if we write $\mathcal{N}^{(m)} = \mathcal{N} \cdot \mathcal{N}^{(m-1)}$ and $\mathbf{Sd}^{(m)} = \mathbf{Sd} \cdot \mathbf{Sd}^{(m-1)}$, then

$$\begin{aligned} \mathcal{N}^{(m)}(\mathcal{A}) &= \underbrace{(\mathcal{Q} \cdot \Delta) \cdots (\mathcal{Q} \cdot \Delta)}_{m}(\mathcal{Q}(\Sigma)) \\ &= \mathcal{Q}\underbrace{(\Delta \cdot \mathcal{Q}) \cdots (\Delta \cdot \mathcal{Q})}_{m}\Sigma \\ &= \mathcal{Q}(\mathbf{Sd}^{(m)}(\Sigma)) \end{aligned}$$

and thus $\mathcal{N}^{(m)}(\mathcal{A})$ is the abstract simplicial complex poset of $\mathbf{Sd}^{(m)}(\Sigma)$.

Therefore the dual $\mathcal{Q}^{\mathrm{d}}(\mathbf{Sd}^{(m)}(\Sigma)) = \mathcal{N}^{\mathrm{d}}(\mathcal{N}^{(m-1)}(\mathcal{A}))$ for any integer m, together with Corollary 5.2.3, we get the following proposition:

Proposition 5.4.3. For any n-dimensional simplicial complex Σ, the logic of $\mathcal{Q}^{\mathrm{d}}(\mathbf{Sd}^{(m)}(\Sigma))$ includes Medvedev logic Med.

Finally, we describe the geometric version of Theorems 5.2.1 and Lemma 5.2.2, i.e., the version with respect to the face poset of simplex and face poset of simplicial complex.

Theorem 5.4.4. Let X_n be an n-dimensional simplex, then the dual of its face poset $\mathcal{Q}^{\mathrm{d}}(X_n) \cong \mathcal{P}_0(n+1)$.

Proof. Let the vertex set of X_n be $\{x_1, x_2, \ldots, x_{n+1}\}$, then each X_n's non-empty face Y is a non-empty subset of the vertex set. The following map f sends a face to its vertex set:

$$f(Y) = S, \text{ if } S = \{i : x_i \in Y\}.$$

So f is a one-to-one map from the subset of vertex set to the subset of $[n+1]$, and

$$\begin{aligned} Y_0 \leq Y_1 \text{ in } \mathcal{Q}^{\mathrm{d}}(X_n) & \\ &\iff Y_1 \leq Y_0 \\ &\iff S_1 = f(Y_1) \subseteq f(Y_0) = S_0 \\ &\iff S_0 \leq S_1 \text{ in } \mathcal{P}_0(n+1). \end{aligned}$$

Therefore f is a poset isomorphism and $\mathcal{Q}^{\mathrm{d}}(X_n) \cong \mathcal{P}_0(n+1)$. ☐

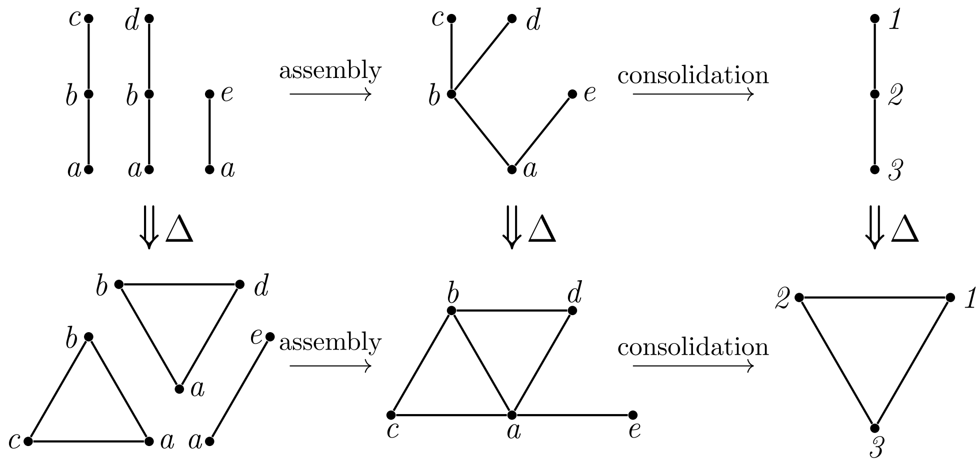


Figure 5.4-5: assembly and consolidation of Σ

Theorem 5.4.5. Let Σ be an n-dimensional simplicial complex, then

$$\mathsf{Log}(\mathcal{Q}^{\mathrm{d}}(\Sigma)) = \mathsf{Log}(\mathcal{P}_0(n+1)).$$

Proof. This proof will be organized by following the proof in Section 5.3 and using the result of dual nerve. For a given poset $\mathcal{F}$, its bunch $\mathbf{B}(\mathcal{F})$ reconstructs it via assembly and it will form a chain with $n+1$ elements by consolidation. From a geometric view, the above procedure is gluing the simplices to reconstruct the simplicial complex and collapse into a simplex, see Fig. 5.4-5.

Suppose Σ is a set of simplices $\{X^1, X^2, \ldots, X^s\}$, each m-dimensional simplex X^i is the order complex of a chain $\mathcal{H}^i$ of $m+1$ elements. In the procedure of simplices $\{X^1, X^2, \ldots, X^s\}$ forming the simplicial complex Σ, if we glue vertices belong to X^i and X^j together, then we will glue the corresponding elements of chains $\mathcal{H}^i$ and $\mathcal{H}^j$ together. Finally, we will obtain the poset $\mathcal{F}$ via the assembly and Σ is the order complex of $\mathcal{F}$. Recall the consolidation described in Section 5.3, every vertex of Σ will become a vertex of a n-dimensional simplex since the dimension of Σ is n, thus consolidation indicates how to collapse Σ to a n-dimensional simplex and every maximal chain of $\mathcal{F}$ becomes a face of the n-dimensional simplex, at the same time, $\mathcal{F}$ becomes a chain $\mathcal{H}_{n+1}$.

In the above argument, each simplicial complex in every step is the order complex of the poset, therefore the face posets of those simplicial complexes are the nerves of those poset since $\mathcal{N} = \mathcal{Q}{\cdot}\Delta$. So the discussion about the dual face poset is about the dual nerves of posets. Because the maximal dimension of simplices of Σ is n, by Corollary 5.2.3, $\mathsf{Log}(\mathcal{Q}^{\mathrm{d}}(\Sigma)) = \mathsf{Log}(\mathcal{P}_0(n+1))$. $\square$

6. The logic of spiked Boolean algebras

In this chapter, we consider the intermediate logic and modal logic of finite spiked Boolean algebras and prove that the modal and intermediate logics associated to it are not finitely axiomatisable. Furthermore, we prove that LS is not finitely axiomatisable over Cheq.

6.1. Non-finite axiomatisability of LS, sBa and S4.sBa

The following lemma is a direct consequence of the suspension Lemma 4.2.7 by Maksimova, Skvortsov and Shehtman.

Lemma 6.1.1. Assume $\mathcal{F}$ is a finite rooted frame with a top, then for any $n \geq 0$, the n-th suspension $\mathcal{F}^{(n)}$ of $\mathcal{F}$ is a p-morphic image of some spiked Boolean algebra $\mathcal{S}_j$.

Proof. If $n = 0$, then $\mathcal{F}^{(n)} = \mathcal{F}$ is a finite rooted frame with a top 1_F. So by Lemma 4.2.5, this frame $\mathcal{F}$ is a p-morphic image of $\mathcal{P}_0(j)$ for some j, say the p-morphism f_0. Since $\mathcal{S}_j$ is obtained by adding a top $\varnothing$ and spikes $\{\{is\} : 1 \leq i \leq j\}$ to $\mathcal{P}_0(j)$, we then define a map f from $\mathcal{S}_j = \langle S_j, \leq_{S_j} \rangle$ to $\mathcal{F} = \langle F, \leq \rangle$ as follows:

$$f(x) = \begin{cases} 1_F, & \text{if } x \in \{\varnothing\} \cup \{\{is\} : 1 \leq i \leq j\} \\ f_0(x), & \text{otherwise} \end{cases}.$$

If $x \leq_{S_j} y$ in $\mathcal{S}_j$, then there are three different cases:
Case A1. If $x \in \{\varnothing\} \cup \{\{is\} : 1 \leq i \leq j\}$ and $y \in \{\varnothing\} \cup \{\{is\} 1 \leq i \leq j\}$, thus $x = y$ and $f(x) = f(y) = 1_F$.

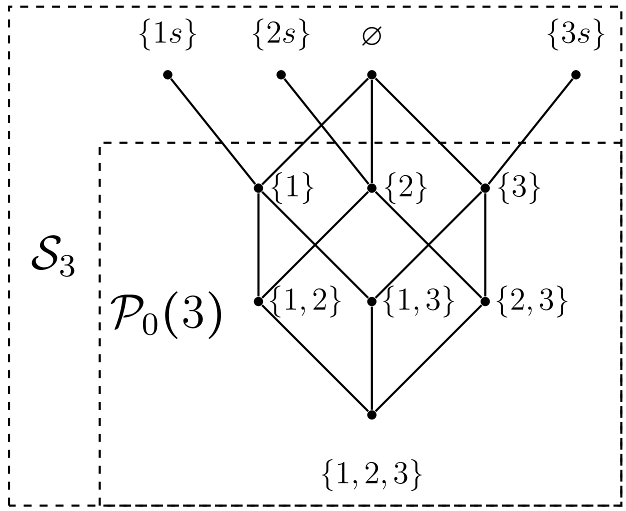


Figure 6.1-1: spiked Boolean algebra $\mathcal{S}_3$ and $\mathcal{P}_0(3)$

Case A2. If $x \notin \{\varnothing\} \cup \{\{is\} : 1 \le i \le j\}$ and $y \in \{\varnothing\} \cup \{\{is\} : 1 \le i \le j\}$, thus $f(x) = f_0(x)$ and $f(y) = 1_F$, then $f(x) \le f(y)$.

Case A3. If $x \notin \{\varnothing\} \cup \{\{is\} : 1 \le i \le j\}$ and $y \notin \{\varnothing\} \cup \{\{is\} : 1 \le i \le j\}$, thus $f(x) = f_0(x)$ and $f(y) = f_0(y)$. Because f_0 is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}$ and $x \le_{S_j} y$ in $\mathcal{S}_j$, we obtain $f(x) \le f(y)$.

If $f(x) \le t$ in $\mathcal{F}$ for some $x \in \mathcal{S}_j$, then there are two different cases:

Case B1. If $f(x) \le t$ and $t = 1_F$, then $f(\varnothing) = 1_F$ and $x \le_{S_j} \varnothing$.

Case B2. If $f(x) \le t$ and $t \ne 1_F$, then there exists a y such that $x \le_{S_j} y$ and $f(y) = f_0(y) = t$ since f_0 is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}$.

In conclusion, f is a p-morphism from $\mathcal{S}_j$ to $\mathcal{F}$.

If $n = 1$, then $\mathcal{F}^{(n)} = \mathcal{F}^{(1)}$ is a suspension of $\mathcal{F}$, where $\mathcal{F}$ is a finite rooted frame with a top. By Lemma 4.2.7, $\mathcal{F}$ is a p-morphic image of $\mathcal{P}_0(j)$ for some j, say the map f_1. We then define a map f' from $\mathcal{S}_j = \langle S_j, \le_{S_j} \rangle$ to $\mathcal{F}^{(1)} = \langle F^{(1)}, \le_1 \rangle$ as follows:

$$f'(x) = \begin{cases} (\overline{0}, \overline{1}), & \text{if } x = \varnothing \\ (\overline{1}, \overline{1}), & \text{if } x \in \{\{is\} : 1 \le i \le j\} \\ (f_1(x), \overline{0}), & \text{otherwise} \end{cases}$$

If $x \le_{S_j} y$ in $\mathcal{S}_j$, then there are four different cases:

Case C1. If $x \in \{\varnothing\} \cup \{\{is\} : 1 \le i \le j\}$, then $x = y$ and $f'(x) = f'(y)$.

Case C2. If $y = \varnothing$, then $f'(x) = (f_1(x), \overline{0}) \in \mathcal{F} \times \{\overline{0}\}$ and $f'(y) = (\overline{0}, \overline{1})$, thus $f'(x) \leq f'(y)$.

Case C3. If $y \in \{\{is\} : 1 \leq i \leq j\}$, then $f'(x) = (f_1(x), \overline{0}) \in \mathcal{F} \times \{\overline{0}\}$ and $f'(y) = (\overline{1}, \overline{1})$, thus $f'(x) \leq f'(y)$.

Case C4. If $x \notin \{\varnothing\} \cup \{\{is\} : 1 \leq i \leq j\}$ and $y \notin \{\varnothing\} \cup \{\{is\} : 1 \leq i \leq j\}$, then $f'(x) = (f_1(x), \overline{0})$ and $f'(y) = (f_1(y), \overline{0})$. Because f_1 is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}$ and $x \leq_{S_j} y$ in $\mathcal{S}_j$, we obtain $f'(x) \leq f'(y)$.

If $f'(x) \leq t$ in $\mathcal{F}^{(1)}$ for some $x \in \mathcal{S}_j$, then there are three different cases:

Case D1. If $f'(x) \leq t$ and $t = (\overline{0}, \overline{1})$, then $f(\varnothing) = (\overline{0}, \overline{1})$ and $x \leq_{S_j} \varnothing$.

Case D2. If $f'(x) \leq t$ and $t = (\overline{1}, \overline{1})$, then there exists $1 \leq i \leq j$ such that $i \in x$, then $x \leq_{S_j} \{is\}$ and $f'(\{is\}) = (\overline{1}, \overline{1})$.

Case D3. If $f'(x) \leq t$ and $t \in \mathcal{F} \times \{\overline{0}\}$, then there exists a y such that $x \leq_{S_j} y$ and $f'(y) = (f_1(y), 0) = t$ since f_1 is a p-morphism from $\mathcal{P}_0(j)$ to $\mathcal{F}$.

In conclusion, f' is a p-morphism from $\mathcal{S}_j$ to $\mathcal{F}^{(1)}$.

If $n > 1$, then $\mathcal{F}^{(n)}$ is a suspension of $\mathcal{F}^{(n-1)}$, where $\mathcal{F}^{(n-1)}$ the $n-1$th suspension of a finite rooted frame with a top. By Lemma 4.2.7, $\mathcal{F}^{n-1}$ is a p-morphic image of $\mathcal{P}_0(j)$ for some j, say the map f_2. We then define a map f'' from $\mathcal{S}_j = \langle S_j, \leq_{S_j} \rangle$ to $\mathcal{F}^{(n)} = \langle F^{(n)}, \leq_n \rangle$ as follows:

$$f''(x) = \begin{cases} (\overline{0}, \overline{1}), & \text{if } x = \varnothing \\ (\overline{1}, \overline{1}), & \text{if } x \in \{\{is\} : 1 \leq i \leq j\} \\ (f_2(x), \overline{0}), & \text{otherwise} \end{cases}$$

By the above argument, we have f'' is a p-morphism from $\mathcal{S}_j$ to $\mathcal{F}^{(n)}$ for $n > 1$. So for any $n \geq 0$, we have $\mathcal{F}^{(n)}$ is the p-morphic image of some $\mathcal{S}_j$.

□

Corollary 6.1.2. Each Chinese lantern $\Phi'(s, n, m)$ is a LS-frame.

Proof. It is obvious that the downset $(m, 0)\downarrow$ in $\Phi'(s, n, m)$ is a finite rooted frame with a top $(m, 0)$. Then we can obtain $\Phi'(s, n, m)$ by taking the m-th suspension of $(m, 0)\downarrow$.

In other words, $\Phi'(s, n, m) = ((m, 0)\downarrow)^{(m)}$ and applying Lemma 6.1.1, it is obvious that $\Phi'(s, n, m)$ is a p-morphic image of some spiked Boolean algebra. □

Lemma 6.1.3. Let $\Phi(s, n)$ be a Chinese lantern. If it is a p-morphic image of some spiked Boolean algebra $\mathcal{S}_m$, then $n < 2^{s+2}$, that is, the branching degree of the root is less than 2^{s+2}.

Proof. Let f be the p-morphism from a finite spiked Boolean algebra $\mathcal{S}_m$ to $\Phi_{s,n}$. Let C be the set of coatoms of $\mathcal{S}_m$: we can think of $\mathcal{S}_m$ as the power set of C with additional spikes, i.e., $\varnothing$ is the top of the Boolean algebra corresponding to $\mathcal{S}_m$, the singletons $\{c\}$ are the coatoms, and C is the root of $\mathcal{S}_m$. Let $\mathcal{P}_0(m)$ be the corresponding topless Boolean algebra, i.e., the set of non-empty subsets of C.

Clearly, both the top of $\mathcal{S}_m$ and its spikes have to be sent by f to maximal elements in $\Phi(s,n)$, i.e., either $(0,0)$ or $(0,1)$. Without loss of generality, the top is mapped to $(0,0)$. This means that at least one spike is mapped to $(0,1)$. Note furthermore that any element of $\mathcal{S}_m$ that is not a spike cannot be mapped to $(0,1)$ (since it is below $\varnothing$). We consider two cases.

Case 1. All spikes are mapped by f to $(0,1)$. Note that the part of $\Phi(s,n)$ that is strictly below $(0,1)$ is just $\Phi(s-1,n)$. This means that every coatom of $\mathcal{S}_m$ has a successor mapped to $(0,0)$ and a successor mapped to $(0,1)$, so it cannot be mapped to either and therefore has to be mapped to an element of $\Phi(s-1,n)$. Thus, $f{\restriction}\mathcal{P}_0(m)$ is a p-morphism from $\mathcal{P}_0(m)$ onto $\Phi(s-1,n)$, so by [MSS79, Lemma 6] or Lemma 4.2.10, we have $n < 2^{(s-1)+3} = 2^{s+2}$.

Case 2. Some spikes are mapped by f to $(0,0)$ and some to $(0,1)$. In this case, let C_0 and C_1 be the sets of coatoms whose corresponding spikes are mapped to $(0,0)$ and $(0,1)$, respectively. Both sets are non-empty in this case. All coatoms in C_0 have no successor mapped to $(0,1)$, so they must be mapped by f to $(0,0)$ as well; similarly, all subsets of C_0 must be mapped to $(0,0)$.

For any $X \subseteq C$, let $X_0 := X \cap C_0$ and $X_1 := X \cap C_1$. Note that if $X_1 \neq \varnothing$, then there is some $c \in X_1 = C_1 \cap X$, so X_1 lies below the spike associated with c which is mapped to $(0,1)$. Thus $f(X_1) \neq (0,0)$. Note that we observed before that $f(X_1) \neq (0,1)$ for any $X \subseteq C$.

Claim. For each $X \subseteq C$, we have $f(X) = f(X_1)$.

[We can prove the claim by induction on the size of X. The case of $|X| = 0$, i.e., $X = X_1 = \varnothing$ is trivial. We deal with two special cases first:

1. If $f(X_1) = (s+1,0)$, i.e., the bottom element of the Chinese lantern, then $f(X) \leq f(X_1)$ and thus $f(X) = (s+1,0) = f(X_1)$.

2. If $X_1 = \varnothing$, then $X \subseteq C_0$, so $f(X) = f(X_1) = f(\varnothing) = (0,0)$ by the above remark.

Therefore, without loss of generality, we can assume from now on that $f(X_1)$ is neither $(0,0)$, $(0,1)$, nor $(s+1,0)$. Suppose that we have $f(X) < f(X_1)$

for some X. Because of our assumption, we know $f(X_1) \neq (s+1,0)$. Thus, there is some x in the Chinese lantern that is incomparable with $f(X_1)$ and $f(X) \leq x$. Since the Chinese lantern is the p-morphic image of f, there is some element of the spiked Boolean algebra mapping to x. Note that by our assumption $(0,0) \neq x \neq (0,1)$ and therefore the preimage of x can neither be the top nor any spike. So, let $Z \subsetneq X$ such that $x = f(Z)$. Since it is a proper subset, the induction hypothesis applies to Z, so $f(Z_1) = f(Z)$. But $Z_1 \subseteq X_1$, and so $x = f(Z) = f(Z_1) \geq f(X_1)$ in contradiction to the choice of x.]

Let $\mathcal{P}_0(m_1)$ be the topless Boolean algebra corresponding to the power set Boolean algebra of C_1. Our claim implies that $f\restriction\mathcal{P}_0(m_1)$ is a p-morphism from $\mathcal{P}_0(m_1)$ onto $\Phi(s-1,n)$, so by [MSS79, Lemma 6] or Lemma 4.2.10, we have $n < 2^{(s-1)+3} = 2^{s+2}$. □

Corollary 6.1.4. For each natural number $s \geq 1$, the frame $\Phi(s, 2^{s+2})$ is **not** a LS-frame.

Proof. By Lemma 6.1.3, $\Phi(s, 2^{s+2})$ is not a p-morphic image of any spiked Boolean algebra $\mathcal{S}_m$.

It is not difficult to see that the spiked Boolean algebra is closed under rooted generated subframes. Thus, according to Jankov-de Jongh Theorem

$$\Phi(s, 2^{s+2}) \vDash \mathsf{LS} \text{ iff } \Phi(s, 2^{s+2}) \text{ is a } p\text{-morphic image of some } \mathcal{S}_m.$$

Thus $\Phi(s, 2^{s+2})$ is not a LS-frame. □

Theorem 6.1.5. The intermediate logic of spiked Boolean algebra LS is **not** finitely axiomatisable.

Proof. If not, assume $\mathsf{LS} = \phi(p_1, \dots, p_s)$ and ϕ is a formula with s variables. Due to Proposition 4.2.9, there is a $m \leq s$ such that

$$\Phi(s,n) \vDash \phi \text{ iff } \Phi'(s,n,m) \vDash \phi.$$

In particular, there is a $m \leq s$ such that

$$\Phi(s, 2^{s+2}) \vDash \phi \text{ iff } \Phi'(s, 2^{s+2}, m) \vDash \phi.$$

According to Corollary 6.1.4, $\Phi(s, 2^{s+2}) \nvDash \mathsf{LS}$. By Corollary 6.1.2, we have $\Phi'(s, 2^{s+2}, m)$ is LS-frame. The contradiction means that ϕ is not axiomatisable with s variables for any $s \in \omega$ and so LS is not finitely axiomatisable. □

Lemma 6.1.6. Let L be an intermediate logic, then L is finitely axiomatisable iff $\sigma(\mathsf{L})$ is finitely axiomatisable.

Proof. This lemma is based on [She90, Corollary 8], which the author contribute it to an observation made by Maksimova. If L is finitely axiomatisable, then $\sigma(\mathsf{L})$ is finitely axiomatisable since $\sigma(\mathsf{L}) = \mathsf{Grz} + \{\mathrm{T}(\varphi) : \varphi \in \mathsf{L}\}$.

If L is not finitely axiomatisable, then it could be the union of an ascending chain of logics,

$$\mathsf{L} = \bigcup \mathsf{L}_i$$

and

$$\mathsf{L}_0 \subsetneqq \mathsf{L}_1 \subsetneqq \dots$$

But according to Blok-Esakia isomorphism theorem,

$$\sigma(\mathsf{L}_0) \subsetneqq \sigma(\mathsf{L}_1) \subsetneqq \dots$$

and $\sigma(\mathsf{L})$ is the union of this ascending chain, so $\sigma(\mathsf{L})$ is not finitely axiomatisable. □

Thus, if we already have $\tau(\mathsf{L})$ is finitely axiomatisable, then $\sigma(\mathsf{L}) = \mathsf{Grz} + \tau(\mathsf{L})$ is and so L is finitely axiomatisable. As a consequence, it is enough to show that the intermediate logic L is not finitely axiomatisable in order to have all three logics L, $\sigma(\mathsf{L})$, and $\tau(\mathsf{L})$ are not finitely axiomatisable.

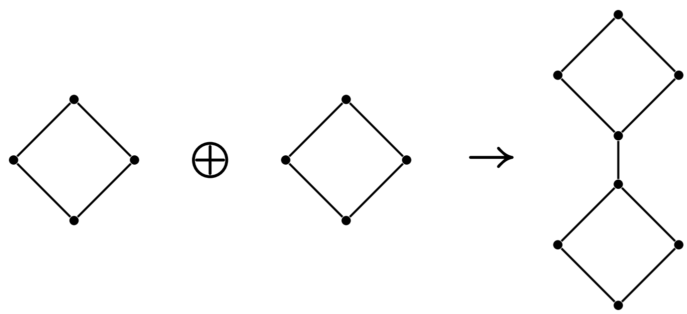
Corollary 6.1.7. The modal logic of spiked Boolean algebra sBa and the modal logic of spiked pre-Boolean algebra $\mathsf{S4.sBa}$ are **not** finitely axiomatisable.

Proof. Because LS is not finitely axiomatisable, together with $\mathsf{sBa} = \sigma(\mathsf{LS})$ and $\mathsf{S4.sBa} = \tau(\mathsf{LS})$, then sBa and $\mathsf{S4.sBa}$ are not finitely axiomatisable. □

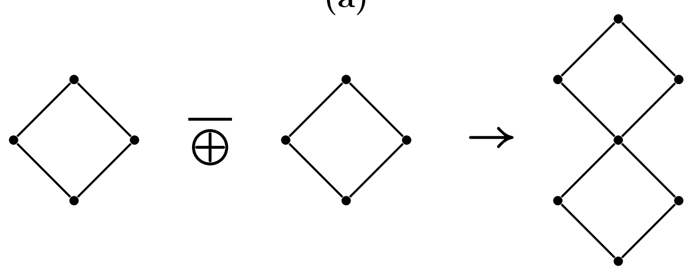
6.2. The bow-tie lemma

Definition 6.2.1 (Linear sum and vertical sum)**.** Let $\mathcal{E}_0 = (E_0, \leq_0)$ and $\mathcal{E}_1 = (E_1, \leq_1)$ be two disjoint partial order sets. The *linear sum* of $\mathcal{E}_0$ and $\mathcal{E}_1$, denoted by $\mathcal{E}_0 \oplus \mathcal{E}_1$, is the union $E_0 \cup E_1$ equipped with a new order $\leq$: for any x, y, we have $x \leq y$ if and only if $(x, y \in E_0 \wedge x \leq_0 y)$ or $(x, y \in E_1 \wedge x \leq_1 y)$ or $(x \in E_0 \wedge y \in E_1)$.

If $\mathcal{E}_0$ has a greatest element $1_{\mathcal{E}_0}$ and $\mathcal{E}_1$ has a least element $0_{\mathcal{E}_1}$, the *vertical sum* of $\mathcal{E}_0$ and $\mathcal{E}_1$, denoted by $\mathcal{E}_0 \overline{\oplus} \mathcal{E}_1$, is obtained from $\mathcal{E}_0 \oplus \mathcal{E}_1$ by identifying $1_{\mathcal{E}_0}$ with $0_{\mathcal{E}_1}$.



(a)



(b)

Figure 6.2-2: (a) is an example of a linear sum, while (b) is an example of a vertical sum.

Lemma 6.2.2. Suppose $\mathcal{E}_0 = (E_0, \leq_0)$ be a frame with a greatest element, and $\mathcal{E}_1 = (E_1, \leq_1)$ forms a frame with a least element. If $\mathcal{E}_0$ is a p-morphic image of $\mathcal{C}_n$ and $\mathcal{E}_1$ is a p-morphic image of $\mathcal{C}_m$, it follows that the vertical sum $\mathcal{E}_0 \overline{\oplus} \mathcal{E}_1$ is a p-morphic image of $\mathcal{C}_{n+m}$.

Proof. Let f_0 be the p-morphism from $\mathcal{C}_n$ onto $\mathcal{E}_0 = (E_0, \leq_0)$ and f_1 be the p-morphism from $\mathcal{C}_m$ onto $\mathcal{E}_1 = (E_1, \leq_1)$. For any $x \in \mathcal{C}_{n+m}$, assume $x = (x_0, x_1, \ldots, x_{n-1}, x_n, \ldots, x_{n+m-1})$, we then define a map f from $\mathcal{C}_{n+m}$ onto $\mathcal{E}_0 \overline{\oplus} \mathcal{E}_1$ as follows:

$$f(x) = \begin{cases} f_0(x_0, \ldots, x_{n-1}) & \text{if } x_n = x_{n+1} = \ldots = x_{n+m-1} = 0 \\ f_1(x_n, \ldots, x_{n+m-1}) & \text{otherwise} \end{cases}$$

It is tedious but not difficult to check that f is a p-morphism from the frame $\mathcal{C}_{n+m}$ onto the frame $\mathcal{E}_0 \overline{\oplus} \mathcal{E}_1$. □

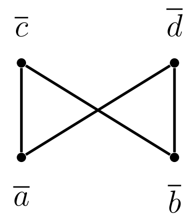


Figure 6.2-3: Bow-tie

We denote by $\mathcal{B} = \langle B, R\rangle$ the *bow-lie*, i.e., the partial order with four elements $B = \{\overline{a}, \overline{b}, \overline{c}, \overline{d}\}$, two incomparable minimal elements $\{\overline{a}, \overline{b}\}$ and two incomparable maximal elements $\{\overline{c}, \overline{d}\}$ such that each maximal element is above each minimal element; cf. Figure 6.2-3.

Lemma 6.2.3 (Bow-tie lemma)**.** Let $\mathcal{A}$ be a finite rooted frame and $\mathcal{B}$ be a bow tie. Then the linear sum $\mathcal{A} \oplus \mathcal{B}$ is a p-morphic image of some $\mathcal{C}_n$.

Proof. It is obvious that $\mathcal{A} \oplus \overline{a}$ is a finite rooted frame with the greatest element $\overline{a}$, then $\mathcal{A} \oplus \overline{a}$ is a p-morphic image of some Medvedev frame $\mathcal{P}_0(n)$ and we assume this p-morphism is f. Note that $\mathcal{P}_0(n)$ is exactly the frame

$$\mathcal{P}_0(n) = \langle \{(x_0, x_1, \ldots, x_{n-1}) : (x_i = 0 \vee x_i = 1) \wedge \sum_i x_i \neq n\}, \leq\rangle$$

where

$$(x_0, x_1, \ldots, x_{n-1}) \leq (x'_0, x'_1, \ldots, x'_{n-1}) \text{ iff } x_i \leq x'_i \text{ for any } 0 \leq i \leq n-1.$$

So f is a p-morphism from $\mathcal{P}_0(n)$ onto $\overline{a}{\downarrow}$ in $\mathcal{A} \oplus \mathcal{B}$.

For the given n, consider the frame $\mathcal{C}_n$. Every node x of $\mathcal{C}_n$ could be associate with an n-tuple $(x_0, x_1, \ldots, x_{n-1})$ where $x_i \in \{0, 1, 2\}$, every node of $\mathcal{P}_0(n)$ could be ssociate with an n-tuple $(x_0, x_1, \ldots, x_{n-1})$ where $x_i \in \{0, 1\}$ and $\sum_i x_i \neq n$, thus $\mathcal{P}_0(n)$ is a subframe (not a generated subframe) of $\mathcal{C}_n$.

Now we let M be the set of all maximal (not necessarily greatest) elements of $\mathcal{A}$, in other words, $M = \{x$ is a node of $\mathcal{A}$: the depth of x in $\mathcal{A}$ is equal to $1\}$. We already know that f is a p-morphism from $\mathcal{P}_0(n)$ to $\mathcal{A} \oplus \overline{a}$, then let V denote $f^{-1}(M)$, that is, $V = \{v$ is a node of $\mathcal{P}_0(n) : f(v) \in M\}$.

In this configuration, for any $v = (v_0, v_1, \ldots, v_{n-1}) \in V$, we construct a $w_v = (w_0, w_1, \ldots, w_{n-1})$ as follows:

$$w_i = \begin{cases} 0, & \text{if } v_i = 1 \\ 2, & \text{if } v_i = 0 \end{cases}$$

For any such w_v, let $w_v\uparrow$ be the upset of w_v in $\mathcal{C}_n$ and we obtain a set of nodes in $\mathcal{C}_n$ by collecting all $w_v\uparrow$ and a node $(1,1,\ldots,1)$. So we assume $W = \bigcup_{v\in V} w_v\uparrow \cup \{(1,1,\ldots,1)\}$.

As a next step, we will construct some set of nodes in $\mathcal{C}_n$ via turning to consider $f^{-1}(\overline{a})$. Recall f is a p-morphism from $\mathcal{P}_0(n)$ to $\mathcal{A}\oplus\overline{a}$, now let $X = \{x \text{ is a node of } \mathcal{P}_0(n) : f(x) = \overline{a}\}$.

For any $x \in X$, where $x = (x_0, x_1, \ldots, x_{n-1})$, let $I_0 = \{i : 0 \leq i \leq n-1 \text{ and } x_i = 0\}$ and $I_1 = \{i : 0 \leq i \leq n-1 \text{ and } x_i = 1\}$, we construct some $y = (y_0, y_1, \ldots, y_{n-1})$ as follows:

If $i \in I_0$, then $y_i \neq 1$ and there is at least one $i \in I_0$ such that $y_i \neq 0$. If $i \in I_1$, then $y_i = 1$.

For all $x \in X$, let Y be the set of all y which is built by those x via the above procedure. For those nodes that do not occur in $Y \cup W$ nor $\mathcal{P}_0(n)$, we collect them and then form a set Z.

We define a map f_1 from $\mathcal{C}_n$ onto $\mathcal{A}\oplus\mathcal{B}$ as follows:

$$f_1(u) = \begin{cases} \overline{c}, & \text{if } u \in Y \\ \overline{d}, & \text{if } u \in W \\ f(u), & \text{if } u \text{ is a node in } \mathcal{P}_0(n) \\ \overline{b}, & \text{if } u \in Z \end{cases}$$

Now we start to prove that f_1 is a p-morphism from $\mathcal{C}_n$ onto $\mathcal{A}\oplus\mathcal{B}$.

First, we need to prove Y, W, Z and the set of nodes in $\mathcal{P}_0(n)$ are mutually disjoint partitions of nodes in $\mathcal{C}_n$.

According to the procedure to produce Y, for any $y = (y_0, y_1, \ldots, y_{n-1}) \in Y$, together with some I_0 which is not empty and there is a $y_i = 2$, so y is not a node in $\mathcal{P}_0(n)$. It is obvious that for any $w = (w_0, w_1, \ldots, w_{n-1}) \in W$, there is a $w_i = 2$ or $w = (1,1,\ldots,1)$, therefore w is not a node in $\mathcal{P}_0(n)$.

Since $\overline{a} \geq m$ for any $m \in M$ in $\mathcal{A}$ and f is a p-morphism from $\mathcal{P}_0(n)$ to $\mathcal{A}\oplus\overline{a}$, it is not possible to find a $v = (v_0, v_1, \ldots, v_{n-1}) \in V$ in $\mathcal{P}_0(n)$ and $x = (x_0, x_1, \ldots, x_{n-1}) \in X$ in $\mathcal{P}_0(n)$ such that $v_i = 0 \to x_i = 0$ for every $0 \leq i \leq n-1$. If not, then we can find $v' \geq x'$ in $\mathcal{P}_0(n)$ and then $f(v') \geq f(x')$ in $\mathcal{A}\oplus\overline{a}$. But according to the definition, $f(v') \in M$ and $f(x') = \overline{a}$, a contradiction. Therefore, for every $v = (v_0, v_1, \ldots, v_{n-1}) \in V$ and $x = (x_0, x_1, \ldots, x_{n-1}) \in X$, there is a $0 \leq i \leq n-1$ such that $v_i = 0$ and $x_i = 1$. According to our procedure, for any $w = (w_0, w_1, \ldots, w_{n-1}) \in w_v\uparrow$, $w_i = 2$ and x will produce some $y = (y_0, y_1, \ldots, y_{n-1})$ with $y_i = 1$. Together with the fact that $(1,1,\ldots,1) \in W$

and $(1,1,\ldots,1) \notin Y$, the above argument indicates that for any $w \in W$ and $y \in Y$, there is at least one $0 \le i \le n-1$, $w_i \ne y_i$ and then $w \ne y$.

By the definition of Z and putting everything together, Y, W, Z and nodes in $\mathcal{P}_0(n)$ are mutually disjoint. Our map f_1 is well-defined.

The next step is to prove that $u \le x$ implies $f_1(u) \le f_1(x)$.

If $u = (u_0, u_1, \ldots, u_{n-1})$ is a node in $\mathcal{P}_0(n)$ and $u \le x$, then we consider the following cases:

Case 1. If x is also a node in $\mathcal{P}_0(n)$. It is obvious that $f_1(u) = f(u) \le f(x) = f_1(x)$ since f is a p-morphism.

Case 2. If $f_1(u) = f(u) \ne \overline{a}$ and x is not a node in $\mathcal{P}_0(n)$, then $f_1(x) \in \{\overline{b}, \overline{c}, \overline{d}\}$ and then $f_1(u) \le f_1(x)$.

Case 3. If $f_1(u) = f(u) = \overline{a}$ and x is not a node in $\mathcal{P}_0(n)$, then $u \in X$. If $x = (1,1,\ldots,1)$, then $f_1(u) = \overline{a} \le \overline{d} = f_1(x)$. Otherwise, there is at least one $0 \le i \le n-1$ such that $x_i = 2$ since x is not in $\mathcal{P}(n)$. In $u\!\uparrow$ in $\mathcal{C}_n$, those elements which are not in $\mathcal{P}(n)$ will be sent to $\overline{c}$ via f_1. Therefore $f_1(u) = \overline{a} \le \overline{c} = f_1(x)$.

If $u = (u_0, u_1, \ldots, u_{n-1})$ is a node in Z and $u \le x$, then there is at least one $u_i = 2$ and so $x_i = 2$. Thus x is in W, Y or Z and $f_1(x) \in \{\overline{b}, \overline{c}, \overline{d}\}$, we always have $f_1(u) = \overline{b} \le f_1(x)$.

If $u = (u_0, u_1, \ldots, u_{n-1})$ is a node in Y and $u \le x$, then we consider the following cases:

Case 1. If there is no $u_i = 0$, then $x = u$ and so $f_1(u) = f_1(x)$.

Case 2. Because $u \in Y$, assume $t = (t_0, t_1, \ldots, t_{n-1}) \in X$ produce u by the procedure, then $\{i : 0 \le i \le n-1 \text{ and } u_i \ne 1\} = \{i : 0 \le i \le n-1 \text{ and } t_i = 0\}$. Since $u \le x$, it is obvious that $u_i \le x_i$, thus $\{i : 0 \le i \le n-1 \text{ and } x_i \ne 1\} \subseteq \{i : 0 \le i \le n-1 \text{ and } u_i \ne 1\}$. Suppose $I = \{i : 0 \le i \le n-1 \text{ and } x_i \ne 1\}$, we then construct a element $t' = (t'_0, t'_1, \ldots, t'_{n-1})$ such that:

$$t'_i = \begin{cases} 0, & \text{if } i \in I \\ 1, & \text{otherwise} \end{cases}$$

Since there at least one $u_i = 2$, so $x_i = 2$ and then $t'_i = 0$, $t \ne (1,1,\ldots,1)$. Therefore t' is a node in $\mathcal{P}_0(n)$. Since $\{i : 0 \le i \le n-1 \text{ and } t'_i = 0\} = I \subseteq \{i : 0 \le i \le n-1 \text{ and } u_i \ne 1\} = \{i : 0 \le i \le n-1 \text{ and } t_i = 0\}$, then it is obvious that $t_i \le t'_i$ for any $0 \le i \le n-1$. In conclusion, $t \le t'$ and $\overline{a} = f(t) \le f(t')$, thus $f(t') = \overline{a}$ and $t' \in X$. Due to our construction, we have t' produces x by the procedure from X to Y, so $x \in Y$. Putting everything together, $f_1(u) = \overline{c} = f_1(x)$.

If $u = (u_0, u_1, \dots, u_{n-1})$ is a node in W and $u \leq x$, then we consider the following cases:

Case 1. If $u = (1, 1, \dots, 1)$, then $x = (1, 1, \dots, 1)$ and so $f_1(u) = f_1(x)$.

Case 2. If $u \in w_v{\uparrow}$ for some $v \in V$, then $x \geq u$ will also be an element in $w_v{\uparrow}$, thus $x \in W$ and $f_1(u) = \overline{d} = f_1(x)$.

The final step is to finish the proof of p-morphism. Since $f_1(X) = \overline{a}$, $f_1(Z) = \overline{b}$, $f_1(Y) = \overline{c}$, $f_1(W) = \overline{d}$ and $f_1(V) = M$, what we need to figure out are the following non-trivial cases:

Case 1. If $f_1(u) \leq s$ in $\mathcal{A} \oplus \overline{a}$, then there is a x in $\mathcal{P}_0(n)$ such that $f_1(x) = f(x) = s$ and $u \leq x$ since f is a p-morphism from $\mathcal{P}_0(n)$ to $\mathcal{A} \oplus \overline{a}$.

Case 2. If $f_1(u) \leq \overline{b}$ and $f_1(u) \in M$, because every node with only one digit is 0 in $\mathcal{P}_0(n)$ will be sent to $\overline{a}$ via f, then u has at least two digits $u_i = u_j = 0$, where $i \leq j$. Let $x = (x_0, x_1, \dots, x_{n-1})$ be a node defined as follows:

$$x_m = \begin{cases} u_m, & \text{if } m \notin \{i, j\} \\ 0, & \text{if } m = i \\ 2, & \text{if } m = j \end{cases}$$

It is obvious that $u_m \leq x_m$ for any $0 \leq m \leq n-1$ and then $u \leq x$. Since $f(u) \in M$, u is not an element of X, thus x is not an element of Y. In the meantime, every node of V will have at least two digits that are 0, so every node of W except $(1, 1, \dots, 1)$ will have at least two digits that are 2. Since x has exactly one digit x_j which is 2, therefore $x \notin w_v{\uparrow}$ for any $v \in V$ and $x \neq (1, 1, \dots, 1)$, then x is not an element of W. It is not difficult to see $x \in Z$. In conclusion, $u \leq x$ and $f_1(x) = \overline{b}$.

Case 3. $f_1(u) = \overline{a} \leq \overline{c}$, then $u \in X$, $x = (x_0, x_1, \dots, x_{n-1})$ be a node defined as follows:

$$x_i = \begin{cases} 1, & \text{if } u_i = 1 \\ 2, & \text{if } u_i = 0 \end{cases}$$

Then x is produced from u and we have $u \leq x$ and $x \in Y$, $f_1(x) = \overline{c}$.

Case 4. In this case, $f_1(u) = \overline{a} \leq \overline{d}$, then let $x = (1, 1, \dots, 1)$, then $f_1(x) = \overline{d}$ and $u \leq x$.

Case 5. $f_1(u) = \overline{b} \leq \overline{c}$, we need to find a x such that $f_1(x) = \overline{c}$, that is, $x \in Y$, and $u \leq x$.

Because $u \in Z$, there is at least one $u_i = 2$ and at least one $u_j \neq 2$. We build a $t = (t_0, t_1, \dots, t_{n-1})$ as follows:

$$t_m = \begin{cases} 0, & \text{if } u_m = 2 \\ 1, & \text{if } u_m \neq 2 \end{cases}$$

If $t \notin X \cup V$, then there is a $s \in M$ such that $f(t) \leq s$. Since f is a p-morphism, there is a $t' \in V$ such that $f(t') = s \in M$ and $t \leq t'$. We construct $w = (w_0, x_1, \dots, w_{n-1})$ as follows:

$$w_m = \begin{cases} 0, & \text{if } t'_m = 1 \\ 2, & \text{if } t'_m = 0 \end{cases}$$

Then $w_m = 2$ implies $t'_m = 0$, and then $t_m = 0$ since $t \leq t'$, and so $u_m = 2$ by the definition. Therefore $w_m \leq u_m$ for any $0 \leq m \leq n-1$ and so $w \leq u$. Since $t' \in V$, we have $w \in W$ due to our construction. In particular, $u \in W$, a contradiction.

If $t \in V$, then $u \in w_t\uparrow$ and so $u \in W$, a contradiction.

If $t \in X$, then we consider $x = (x_0, x_1, \dots, x_{n-1})$ as follows:

$$x_m = \begin{cases} 2, & \text{if } t_m = 0 \\ 1, & \text{if } t_m = 1 \end{cases}$$

Then x is produced from t and so $x \in Y$, it is obvious that $u_m \leq x_m$ and so $u \leq x$.

In conclusion, $t \in X$ and we have $x \in Y$ and $u \leq x$.

Case 6. In this case, $f_1(u) = \overline{b} \leq \overline{d}$, we need to find a x such that $f_1(x) = \overline{d}$, that is, $x \in W$ and $u \leq x$. Because $u \in Z$, there is at least one $u_i = 2$ and at least one $u_j \neq 2$. We build a $t = (t_0, t_1, \dots, t_{n-1})$ as follows:

$$t_m = \begin{cases} 0, & \text{if } u_m = 2 \\ 1, & \text{if } u_m = 1 \\ 0, & \text{if } u_m = 0 \end{cases}$$

If $t \notin X \cup V$, then there is a $s \in M$ such that $f(t) \leq s$. Since f is a p-morphism, there is a $t' \in V$ such that $f(t') = s \in M$ and $t \leq t'$.

We construct $w = (w_0, x_1, \ldots, w_{n-1})$ and $x = (x_0, x_1, \ldots, x_{n-1})$ as follows:

$$w_m = \begin{cases} 0, & \text{if } t'_m = 1 \\ 2, & \text{if } t'_m = 0 \end{cases}$$

and

$$x_m = \begin{cases} 1, & \text{if } t'_m = 1 \text{ and } t_m = 1 \\ 2, & \text{otherwise} \end{cases}$$

Then $u_m = 1$ indicates $t_m = 1$, since $t \le t'$, $t'_m = 1$, so $x_m = 1$. If $u_m \neq 1$, then $t_m = 0$ and then $x_m = 2$. Therefore $u_m \le x_m$ for $0 \le m \le n-1$. In conclusion, $u \le x$.

Note that $w_m = 2$ implies $t'_m = 0$, and then $x_m = 2$. Thus $x \in w\uparrow$. Since $t' \in V$, according to the definition of w, $w \in W$ and thus $x \in W$.

If $t \in X$, then $t_m = 1$ implies $u_m = 1$ and $t_m = 0$ implies $t_m \le u_m$. Thus u is produced from t by the procedure from X to Y, and then $u \in Y$, a contradiction.

If $t \in V$, then consider the following $x = (x_0, x_1, \ldots, x_{n-1})$:

$$x_m = \begin{cases} 1, & \text{if } t_m = 1 \\ 2, & \text{if } t_m = 0 \end{cases}$$

then $u_m \le x_m$ due to our construction and then $u \le x$ and $x \in W$ since $x \in w_t\uparrow$.

In conclusion, $t \notin X$ and we always have $x \in W$ and $u \le x$.

So far we have proved that f_1 is a p-morphism from $\mathcal{C}_n$ onto $\mathcal{A} \oplus \mathcal{B}$. □

We will use the bow-tie lemma to prove that LS is not finitely axiomatisable over Cheq. Recall the Chinese lanterns $\Phi(s,n)$ and $\Phi'(s,n,m)$ from Chapter 4. We will see that:

Corollary 6.2.4. $\Phi(s,n)$ and $\Phi'(s,n,m)$ are Cheq-frames.

Proof. Each $\Phi(s,n)$ is a linear sum of a finite rooted frame and a bow-tie. Thus it is a Cheq-frame according to the bow-tie Lemma 6.2.3. For $m \ge 0$, $(m,0)\uparrow$ in $\Phi'(s,n,m)$ depends on m. When $m \ge 2$, it is a linear sum of a finite rooted frame and the bow-tie. When $m = 1$, it is $\mathcal{C}_1$ itself. When $m = 0$, it is a singleton point. In conclusion, the frame $(m,0)\uparrow$ is a Cheq-frame by the bow-tie Lemma 6.2.3.

Because $(m,0)\downarrow$ is a finite rooted frame with a top, it is a Med-frame. Since $\mathsf{Cheq} \subseteq \mathsf{Med}$ and so $(m,0)\downarrow$ is a Cheq-frame.

Because $\Phi'(s,n,m)$ is the vertical sum of $(m,0)\downarrow$ and $(m,0)\uparrow$, that is,

$$\Phi'(s,n,m) = (m,0)\downarrow\overline{\oplus}(m,0)\uparrow.$$

By Lemma 6.2.2 and putting everything together, $\Phi'(s,n,m)$ is always a Cheq-frame. □

The bow-tie lemma enriches the properties of Cheq-frames and proves that a particular class of frames are all Cheq-frames.

It is useful for addressing and resolving Cheq-frame problems. One application is that, given $\mathsf{Cheq} \subseteq \mathsf{TLP}_{\mathcal{F}}$, to demonstrate that $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable over Cheq, we must ensure that both the Chinese lanterns $\Phi(s,n)$ and $\Phi'(s,n,m)$ are Cheq-frames, as established by Corollary 6.2.4. Following the method in Section 4.2, we arrive at a generalization of Fontaine's result:

Proposition 6.2.5. If $\mathsf{TLP}_{\mathcal{F}}$ includes Cheq, then $\mathsf{TLP}_{\mathcal{F}}$ is not finitely axiomatisable over Cheq.

Since $\mathsf{Med} = \mathsf{TLP}_{\mathcal{H}_2}$ and $\mathsf{Cheq} \subseteq \mathsf{Med}$, it follows that Med is not finitely axiomatisable over Cheq, as shown in [Fon06, Theorem 9]. Proposition 6.2.5 generalizes this theorem.

6.3. LS is not finitely axiomatisable over Cheq

Theorem 6.3.1. The intermediate logic of spiked Boolean algebra LS is **not** finitely axiomatisable over Cheq.

Proof. Since $\mathsf{Cheq} \subseteq \mathsf{LS}$, if LS is finitely axiomatisable over Cheq, then let us assume $\mathsf{LS} = \mathsf{Cheq} + \phi(p_1,\ldots,p_s)$ and ϕ is a formula with s variables. Due to proposition 4.2.9, there is a $m \leq s$ such that

$$\Phi(s,n) \vDash \phi \text{ iff } \Phi'(s,n,m) \vDash \phi.$$

In particular, there is a $m \leq s$ such that

$$\Phi(s,2^{s+2}) \vDash \phi \text{ iff } \Phi'(s,2^{s+2},m) \vDash \phi.$$

By Corollary 6.2.4, $\Phi(s,n)$ and $\Phi'(s,n,m)$ are Cheq-frames. So it is impossible that one of $\Phi(s,2^{s+2})$ and $\Phi'(s,2^{s+2},m)$ is a LS-frame and the other is not.

According to Corollary 6.1.2, $\Phi'(s,2^{s+2},m) \models \mathsf{LS}$ while $\Phi(s,2^{s+2}) \not\models \mathsf{LS}$ by Corollary 6.1.4. The contradiction means that ϕ is not axiomatisable with s variables for any $s \in \omega$ and so LS is not finitely axiomatisable over Cheq. $\square$

7. The axiomatisation of Cheq and related modal logics

This chapter first investigates the modal logic $\mathsf{ML}(\mathcal{C}_n)$, the modal logic of the Cartesian product of the 2-fork frame $\mathcal{C}_1$. We prove that the frame $\mathcal{C}_n$ is isomorphic to:

1. The finite partial function algebra $\mathcal{R}_n$ on n elements;
2. The dual face poset of n-cube.

Additionally, we prove that $\mathcal{C}_n \vDash \mathsf{Grz} + \mathsf{bw}_{\mathsf{s(n)}} + \mathsf{bd}_{\mathsf{n+1}}$, since the largest antichain in $\mathcal{C}_n$ has a size of $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$.

Kuznetsov [Kuz19] suggested using the edge-coloring result of Offner [Off08, Theorem 2] to discuss the strategies of Fontaine and Shatrov for proving that Cheq is not finitely axiomatisable. We introduce Kuznetsov's work and then discuss his alternative solution by bow-tie lemma.

Finally, we prove that Cheq is not finitely axiomatisable with five or six variables.

7.1. The logic of cartesian products of 2-fork frame

Recall the 2-fork frame $\mathcal{C}_1$ and its Cartesian product $\mathcal{C}_n$. Each point x of $\mathcal{C}_n$ can be associated with an n-tuple $(x_0, x_1, \dots, x_{n-1})$, where $x_i \in \{0, 1, 2\}$. For ease of notation, we may sometimes write $(x_0, x_1, \dots, x_{n-1})$ directly as the string $x_0 x_1 \dots x_{n-1}$.

Definition 7.1.1. Let $\mathsf{ML}(\mathcal{C}_1)$ be the modal logic characterized by the 2-fork frame $\mathcal{C}_1$, and let $\mathsf{ML}(\mathcal{C}_n)$ be the modal logic characterized by the frame $\mathcal{C}_n$.

Theorem 7.1.2. The logics $\{\mathsf{ML}(\mathcal{C}_i)\}_{i \geq 1}$ form a descending chain, that is, $\mathsf{ML}(\mathcal{C}_1) \supseteq \mathsf{ML}(\mathcal{C}_2) \supseteq \dots \supseteq \mathsf{ML}(\mathcal{C}_n) \supseteq \dots$

Proof. We begin with proving $\mathsf{ML}(\mathcal{C}_1) \supseteq \mathsf{ML}(\mathcal{C}_2)$ by providing a p-morphism. Assuming $\mathcal{C}_1 = \langle W_1, R_1\rangle$ and $\mathcal{C}_2 = \langle W_2, R_2\rangle$, where $W_1 = \{0,1,2\}$ and $W_2 = \{00,01,02,10,20,11,12,21,22\}$. We define a map f_1 from $\mathcal{C}_2$ onto $\mathcal{C}_1$ as follows:

$$f_1(x) = \begin{cases} 0, & x \in \{00,01,02,10,20\} \\ 1, & x \in \{11,22\} \\ 2, & x \in \{12,21\} \end{cases}.$$

To check f_1 is a p-morphism, if $x_1 < x_2$, then $f_1(x_1) = 0$ and it is obvious that $f_1(x_1) < f_1(x_2)$. For any $x_1 \in W_2$ and $f_1(x_1) < y$, it means that $f_1(x_1) = 0$ and we can choose $x_2 = 11$ or 22 when $y = 1$, choose $x_2 = 12$ or 21 when $y = 2$. It guarantees that we always have a $x_2 \in W_2$ such that $f_1(x_2) = y$ and $x_1 < x_2$. So f_1 is a p-morphism from $\mathcal{C}_2$ onto $\mathcal{C}_1$ and then $\mathsf{ML}(\mathcal{C}_2) \subseteq \mathsf{ML}(\mathcal{C}_1)$.

Now we begin to prove that $\mathsf{ML}(\mathcal{C}_n) \supseteq \mathsf{ML}(\mathcal{C}_{n+1})$ for $n \geq 2$ by constructing a p-morphism f_n from $\mathcal{C}_{n+1}$ onto $\mathcal{C}_n$. Assuming $\mathcal{C}_n = \langle W_n, R_n\rangle$ and $\mathcal{C}_{n+1} = \langle W_{n+1}, R_{n+1}\rangle$, for any $x \in W_{n+1}$, x is associated with an $n+1$-tuple $(x_0, x_1, ..., x_n)$ where $x_i \in \{0,1,2\}$. We can regard (x_0, x_1) as an element of $\mathcal{C}_2$, and let

$$x_{01} \coloneqq f_1\big((x_0, x_1)\big).$$

Then $y = (x_{01}, x_2, ..., x_n)$ is an n-tuple of $\{0,1,2\}$, thus an element of W_n. According to the above discussion, we define a map from $\mathcal{C}_{n+1}$ onto $\mathcal{C}_n$ as follows:

$$f_n(x) = y, \text{ where } x = (x_0, x_1, ..., x_n) \text{ and } y = \big(x_{01}, x_2, ..., x_n\big).$$

If $s \leq t$ in $\mathcal{C}_{n+1}$ where $s = (s_0, s_1, ..., s_n)$ and $t = (t_0, t_1, ..., t_n)$, then $(s_0, s_1) \leq (t_0, t_1)$ in $\mathcal{C}_2$ and $(s_2, ..., s_n) \leq (t_2, ..., t_n)$ in $\mathcal{C}_{n-1}$. We have $s_{01} = f_1\big((s_0, s_1)\big) \leq f_1\big((t_0, t_1)\big) = t_{01}$ since f_1 is a p-morphism from $\mathcal{C}_2$ to $\mathcal{C}_1$. Thus $f_n(s) = (s_{01}, s_2, ..., s_n) \leq (t_{01}, t_2 ..., t_n) = f_n(t)$.

If for any $x \in \mathcal{C}_{n+1}$, $f_n(x) \leq y$, assuming $x = (x_0, x_1, ..., x_n)$, $f_n(x) = \big(x_{01}, x_2, ..., x_n\big)$ and $y = (y_1, y_2, ..., y_n)$, then $f_1\big((x_0, x_1)\big) = x_{01} \leq y_1$ and $(x_2, ..., x_n) \leq (y_2, ..., y_n)$. Since f_1 is a p-morphism from $\mathcal{C}_2$ to $\mathcal{C}_1$, there is a $(y'_0, y'_1) \in \mathcal{C}_2$ such that $(x_0, x_1) \leq (y'_0, y'_1)$ and $f_1\big((y'_0, y'_1)\big) = y_1$. So $y' = (y'_0, y'_1, y_2, ..., y_n) \geq x$ in $\mathcal{C}_{n+1}$ and $f_n(y') = (y_1, y_2, ..., y_n) = y$ according to the construction of f_n. Therefore f_n is a p-morphism from $\mathcal{C}_{n+1}$ onto $\mathcal{C}_n$ and then $\mathsf{ML}(\mathcal{C}_{n+1}) \subseteq \mathsf{ML}(\mathcal{C}_n)$.

In conclusion, $\{\mathsf{ML}(\mathcal{C}_n)\}_{n\geq 1}$ forms a descending chain. □

We will prove that $\mathsf{ML}(\mathcal{C}_{n+1}) \subsetneq \mathsf{ML}(\mathcal{C}_n)$, i.e., the containment relation is strict in Theorem 7.1.10.

We introduce the following three classes of formulas:

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p.$$

$$\mathsf{bd_{n+1}} = p_{n+1} \to \Box(\Diamond p_{n+1} \vee \mathsf{bd_n}) \text{ and } \mathsf{bd_1} = p_1 \to \Box\Diamond p_1.$$

$$\text{For } n \geq 1,\ \mathsf{bw_n} = \textstyle\bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0\leq i\neq j\leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

Regarding these formulas, we have the following three classic results describing frames:

Proposition 7.1.3 (Folklore; cf. [CZ97, Proposition 3.48]). A frame $\mathcal{F}$ validates grz iff it is a Noetherian[1] partial order.

Proposition 7.1.4 (Folklore; cf. [CZ97, Proposition 3.44]). A transitive frame $\mathcal{F}$ validates $\mathsf{bd_n}$ iff the depth of the frame $d(\mathcal{F}) \leq n$.

Proposition 7.1.5 (Folklore; cf. [CZ97, Corollary 3.43]). A transitive frame $\mathcal{F}$ validates $\mathsf{bw_n}$ iff the width of each rooted subframe of $\mathcal{F}$ is at most n.

Recall that we are defining the modal logic $\mathsf{ML}(\mathcal{C}_n)$ by the n-product of 2-fork frame $\mathcal{C}_1$. We turn to consider another structure, the finite partial function algebra $\mathcal{R}_n$. This allows us, for each concrete $\mathcal{C}_n$, to consider the relevant property or problem as a whole, through another well-defined structure.

Lemma 7.1.6. The frame $\mathcal{C}_n$ is isomorphic to the finite partial function algebra $\mathcal{R}_n$ on n elements.

Proof. The finite partial function algebra $\mathcal{R}_n$ is a poset of partial functions from $[n]$ to $\{1,2\}$ and for any $a \in \mathcal{R}_n$, we correspond $(1_a, 2_a)$ where $1_a = a^{-1}(1)$ and $2_a = a^{-1}(2)$. Definite a map f from $\mathcal{R}_n$ to $\mathcal{C}_n$ as follows:

$$f(a) = x \text{ and } x_i = \begin{cases} 0, & i \notin \mathrm{dom}(a) \\ 1, & i \in 1_a \\ 2, & i \in 2_a \end{cases}, \text{ for } 1 \leq i \leq n.$$

If $a \leq b$ in $\mathcal{R}_n$ and assume $f(a) = x = (x_1, x_2, \ldots, x_n)$, $f(b) = y = (y_1, y_2, \ldots, y_n)$, then $x_i \leq y_i$ since $a = b\restriction\mathrm{dom}(a)$, then $f(a) \leq f(b)$ in $\mathcal{C}_n$. Because f is one-to-one from $\mathcal{R}_n$ to $\mathcal{C}_n$, then f is an isomorphism from $\mathcal{R}_n$ to $\mathcal{C}_n$, thus $\mathcal{R}_n \cong \mathcal{C}_n$ for any $n \in \omega$ and it is obvious that $\mathsf{ML}(\mathcal{C}_n) = \mathsf{ML}(\mathcal{R}_n)$. ☐

[1]It means that there are no infinite ascending chains of distinct points.

The relation between points in $\mathcal{C}_n$ and elements in the finite partial function algebra $\mathcal{R}_n$ is established through the defined isomorphism. This perspective can offer deeper insights into problems related to modal logics of $\mathcal{C}_n$.

Lemma 7.1.7. The size of the largest antichain in $\mathcal{C}_n$ is $\binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$.

Proof. We begin with proving in $\mathcal{R}_n$. Define a function $w_n(d)$ as the number of points in $\mathcal{R}_n$ with depth equal to d. Note that $|\mathcal{R}_n| = 3^n = (1+2)^n = \sum_{s=0}^{n} \binom{n}{s} \times 2^s$, where 1 in $1+2$ indicates an index i that is still not included in the domain of the function, 2 in $1+2$ indicates the number of choices for the function from i to $\{1,2\}$.

Thus, the number of points of depth $n+1-s$ is $\binom{n}{s} \times 2^s$. Comparing $\binom{n}{s} \times 2^s$ and $\binom{n}{s+1} \times 2^{s+1}$, we have $\binom{n}{s} \times 2^s \leq \binom{n}{s+1} \times 2^{s+1}$ iff $s \leq \frac{2n-1}{3}$. Since $s \in \mathbb{Z}$, then $\binom{n}{s} \times 2^s$ is maximal when $s = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1$. Because $\left\lfloor \frac{2n-1}{3} \right\rfloor + 1 + \left\lfloor \frac{n}{3} \right\rfloor = n$, we obtain that $w_n(d)_{\max} = w_n\left(\left\lfloor \frac{n}{3} \right\rfloor + 1\right) = \binom{n}{\lfloor \frac{2n-1}{3} \rfloor + 1} \times 2^{\lfloor \frac{2n-1}{3} \rfloor + 1} = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$.

In conclusion, the points in $\mathcal{R}_n$ with depth $\left\lfloor \frac{n}{3} \right\rfloor + 1$ form an antichain of size $\binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$.

We then prove that the above is indeed the largest antichain by returning to study the frame $\mathcal{C}_n$, which is easier to compute. First, we prove that $\mathcal{C}_n$ has $(2n)!!$ maximal chains by induction, where $(2n)!!$ is the double factorial of $2n$.

When $n = 1$, the 2-fork frame $\mathcal{C}_1$ has 2 maximal chains.

Suppose that the proposition holds for $n = s$, then for $n = s+1$, then the atoms of the root $(0,0,\ldots,0)$ has $2 \times (s+1)$ many choices. Then for any atom x of $\mathcal{C}_{s+1}$, since the proposition holds for s and so $x\uparrow$ has $(2s)!!$ many maximal chains. Therefore, there are $(2s+2) \times (2s)!! = (2s+2)!!$ many maximal chains in $\mathcal{C}_{s+1}$. By the induction, $\mathcal{C}_n$ has $(2n)!!$ many maximal chains.

For any x as a point of $\mathcal{C}_n$, let $\#(x)$ be the number of 0s in x. Following the same argument, we obtain that $x\uparrow$ has $(2 \times \#(x))!!$ maximal chains. Since there are $(n - \#(x))!$ path choices from the root to x, then number of maximal chains of $\mathcal{C}_n$ contain x is $(2 \times \#(x))!! \times (n - \#(x))! = \#(x)! \times (n - \#(x))! \times 2^{\#(x)}$.

Let $\mathcal{A}$ be an antichain. Since any maximal chain intersects with at most one element of $\mathcal{A}$, then $\sum_{x \in \mathcal{A}} \#(x)! \times (n - \#(x))! \times 2^{\#(x)} \leq (2n)!!$ and then $\sum_{x \in \mathcal{A}} \#(x)! \times (n - \#(x))! \times 2^{\#(x)} \leq n! \times 2^n$.

Divide by $n! \times 2^n$, we have $\sum_{x \in \mathcal{A}} \frac{\#(x)! \times (n - \#(x))!}{n!} \times 2^{\#(x)-n} \leq 1$ and thus $\sum_{x \in \mathcal{A}} \frac{1}{\binom{n}{\#(x)} \times 2^{n-\#(x)}} \leq 1$.

By the above argument, $\binom{n}{\#(x)} \times 2^{n-\#(x)}$ is maximal when $\#(x) = \lfloor \frac{n}{3} \rfloor$. Therefore $\sum_{x \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}} \leq 1$ and so $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$. □

Corollary 7.1.8. For ease of writing, assuming $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$, then the width of $\mathcal{C}_n$ (or $\mathcal{R}_n$) is $s(n)$ and so $\mathcal{C}_n \vDash \mathsf{bw_{s(n)}}$ and $\mathcal{C}_n \nvDash \mathsf{bw_{s(n-1)}}$.

Proof. One can immediately conclude that the size of the largest antichain $\mathcal{C}_n$ (or $\mathcal{R}_n$) is equal to $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$ by Lemma 7.1, and then the width of $\mathcal{C}_n$ (or $\mathcal{R}_n$) is $s(n)$ according to the definition of width of a frame.

Since the width of $\mathcal{C}_n$ is $s(n)$ and $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$ is strictly increasing, thus $\mathcal{C}_n \vDash \mathsf{bw_{s(n)}}$. According to Proposition 7.1.5, $\mathcal{C}_n \nvDash \mathsf{bw_{s(n)-1}}$ and $\mathcal{C}_n \nvDash \mathsf{bw_{s(n-1)}}$. □

Proposition 7.1.9. The depth of $\mathcal{C}_n$ is $n+1$, so $\mathcal{C}_n \vDash \mathsf{bd_{n+1}}$ and $\mathcal{C}_n \nvDash \mathsf{bd_n}$.

Proof. Let $x^i = (\underbrace{1, 1, \ldots, 1}_{i}, 0, \ldots, 0)$ be a point in $\mathcal{C}_n$, then $x^0 < x^1 < \ldots < x^n$ is a chain of size $n+1$. Suppose that if there is a chain $y^0 < y^1 < \ldots < y^{n+1}$ in $\mathcal{C}_n$, then we can find distinct points y^i and y^j such that the number of 0s in y^i is equal to it in y^j, thus it is impossible to have $y^i < y^j$ or $y^j < y^i$. Therefore, the depth of $\mathcal{C}_n$ is $n+1$.

Therefore $\mathcal{C}_n \vDash \mathsf{bd_{n+1}}$ and $\mathcal{C}_n \nvDash \mathsf{bd_n}$ by Proposition 7.1.4. □

Theorem 7.1.10. The logics $\{\mathsf{ML}(\mathcal{C}_i)\}_{i \geq 1}$ form a strict descending chain, that is, $\mathsf{ML}(\mathcal{C}_1) \supsetneq \mathsf{ML}(\mathcal{C}_2) \supsetneq \ldots \supsetneq \mathsf{ML}(\mathcal{C}_n) \supsetneq \ldots$.

Proof. Since $\mathcal{C}_n \vDash \mathsf{bw_{s(n)}}$ but $\mathcal{C}_{n+1} \nvDash \mathsf{bw_{s(n)}}$, then $\mathsf{bw_{s(n)}} \in \mathsf{ML}(\mathcal{C}_n)$ and $\mathsf{bw_{s(n)}} \notin \mathsf{ML}(\mathcal{C}_{n+1})$. Together with $\mathsf{ML}(\mathcal{C}_n) \supseteq \mathsf{ML}(\mathcal{C}_{n+1})$, we conclude that the containment is strict. □

Remark 7.1.11. The above theorem is from a statement in [vBB07, p. 255].

Theorem 7.1.12. $\mathcal{C}_n \vDash \mathsf{Grz} + \mathsf{bw_{s(n)}} + \mathsf{bd_{n+1}}$, where $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$.

Proof. $\mathcal{C}_n \vDash \mathsf{grz}$ since $\mathcal{C}_n$ is a Noetherian partial order and by Proposition 7.1.3. Together with Corollary 7.1.8 and Proposition 7.1.9, $\mathcal{C}_n \vDash \mathsf{bw_{s(n)}} + \mathsf{bd_{n+1}}$. In conclusion, $\mathcal{C}_n \vDash \mathsf{Grz} + \mathsf{bw_{s(n)}} + \mathsf{bd_{n+1}}$. □

The following corollary is exact [AvBB03, Theorem 6.9], which is obtained from a topological viewpoint. We apply Theorem 7.1.12 to get it as the axiomatisation of $\mathcal{C}_1$.

Corollary 7.1.13. $\mathsf{ML}(\mathcal{C}_1) = \mathsf{Grz} + \mathsf{bw}_2 + \mathsf{bd}_2$.

Proof. Since $s(1) = \binom{1}{\lfloor \frac{1}{3} \rfloor} \times 2^{1-\lfloor \frac{1}{3} \rfloor} = 2$, by applying Theorem 7.1.12, we get $\mathcal{C}_1 \vDash \mathsf{Grz} + \mathsf{bw}_2 + \mathsf{bd}_2$. Assume that $\mathsf{L} = \mathsf{Grz} + \mathsf{bw}_2 + \mathsf{bd}_2$, then for any rooted L-frame $\mathcal{F}$, there is a p-morphism from $\mathcal{C}_1$ to $\mathcal{F}$. Therefore, the modal logic of the 2-fork frame $\mathcal{C}_1$ is $\mathsf{Grz} + \mathsf{bw}_2 + \mathsf{bd}_2$. □

For any n, what $\mathsf{ML}(\mathcal{C}_n)$ specifically is remains an open question. Due to Lemma 7.1.6, their intersection is the modal logic FPFA, which is the greatest modal companion of Cheq. Whether Cheq is finitely axiomatisable is also a well-known open question, which we will discuss in the final two sections of this chapter.

7.2. The logic of the dual face poset of n-cube

Recall that we have already given two equivalent definitions of the frame $\mathcal{C}_n$: the n-product of 2-fork frame $\mathcal{C}_1$ and the finite partial function algebra $\mathcal{R}_n$ on $[n]$ by Lemma 7.1.6. We provide an additional description of this frame, the dual face poset of the n-cube. Recall that we introduce the notion of face poset in Chapter 5. Specifically, for a simplicial complex Σ,

$$Y \leq X \text{ iff } Y \text{ is a face of } X$$

and Σ forms a poset $\mathcal{Q}(\Sigma)$ under the order $\leq$. We consider the dual of the face poset when ordered by the reverse $\leq$ and proved that if X_n is an n-dimensional simplex, then the dual of its face poset $\mathcal{Q}^{\mathrm{d}}(X_n) \cong \mathcal{P}_0(n+1)$. In this section, we restrict our discussing on the n-cube Q_n and prove that $\mathcal{Q}^{\mathrm{d}}(Q_n) \cong \mathcal{C}_n$.

We first introduce some concepts of lattice theory.

Definition 7.2.1 (Lattice)**.** Let $\mathcal{P} = (P, \leq)$ be a partially ordered set. If for any x, $y \in \mathcal{P}$, $\inf\{x,y\} \in \mathcal{P}$ and $\sup\{x,y\} \in \mathcal{P}$, then we call $\mathcal{P}$ a *lattice*. Then $x \wedge y$ denotes $\inf\{x,y\}$ and $x \vee y$ denotes $\sup\{x,y\}$. If there exists a maximal

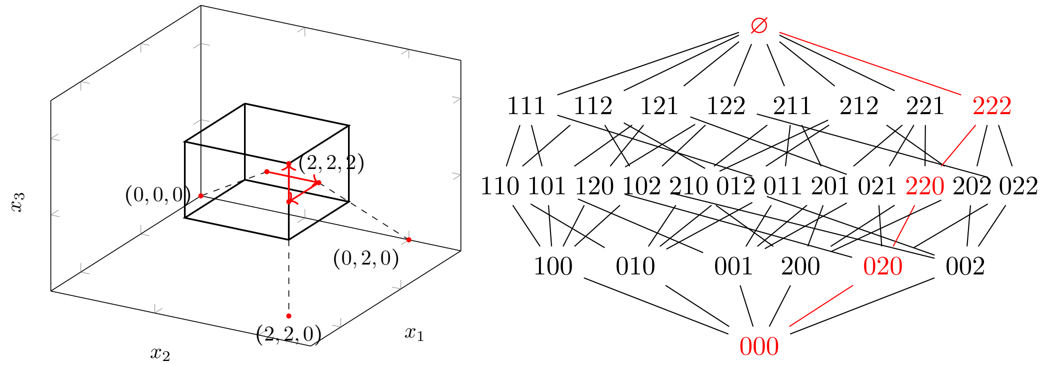


Figure 7.2-1: Cube and its face lattice

element $\bar{1}$ and a minimal element $\bar{0}$ in $\mathcal{P}$, then we call $\mathcal{P}$ a *bounded lattice*. An *atom* x in $\mathcal{P}$ if x is not $\bar{0}$ and there is no y such that $0 < y < x$. Let $|\mathbf{at}(\mathcal{P})|$ be the cardinality of the atoms in $\mathcal{P}$. A lattice is *atomistic* if every element is a join of atoms.

Definition 7.2.2 (The dual face poset)**.** Assuming X is a polytope in $\mathbb{R}^n$, a hyperplane c of $\mathbb{R}^n$ is *supporting* X if one of the two closed half-spaces of c contains X. A subset F of X is called a *face* of X if it is either $\varnothing$, X itself or the intersection of X with a supporting hyperplane c.

The empty set $\varnothing$ and the X itself are defined to be *trivial faces*. Otherwise, the face is called *proper*. The maximal proper face under the inclusion is called *facet* while the minimal proper face is called *vertex*.

The *dual face poset* $\mathcal{Q}^{\mathrm{d}}(X)$ is the poset of non-empty faces of X, ordered by reverse inclusion:

$$F_0 \leq F_1 \text{ iff } F_0 \supseteq F_1.$$

If we add the top $\varnothing$ to $\mathcal{Q}^{\mathrm{d}}(X)$, then it forms a bound lattice $\mathcal{Q}^{\mathrm{d}}_1(X)$ since $F_0 \vee F_1 = F_0 \cap F_1$ and $F_0 \wedge F_1 = \bigcap_{(F_0 \cup F_1) \subseteq F} F$. We call $\mathcal{Q}^{\mathrm{d}}_1(X)$ the *face lattice* of X.

Note that, in the lattice theory, the face lattice of X is the opposite lattice of $\mathcal{Q}^{\mathrm{d}}_1(X)$, however, we will continue to use our terminology, and this should not cause any confusion.

For the convenience of the discussion, we use the coordinate system to give a definition of the n-cube Q_n and adjust the coordinates to match our previous definition of the frame $\mathcal{C}_n$.

Definition 7.2.3. An *n-cube* Q_n is the set $\{(x_1, ..., x_n) \in \mathbb{R}^n : 1 \le x_i \le 2 \text{ for } 1 \le i \le n\}$.

[Ben82, Theorem 2.1] provided a description of the face lattice and we state the result below.

Theorem 7.2.4. A lattice $\mathcal{F}$ is the face lattice of Q_n iff $|\mathcal{F}| = 3^n + 1$, $|\mathbf{at}(\mathcal{F})| = 2n$, $\mathcal{F}$ is atomistic and for each atom a, there exists a unique atom a' with $a \vee a' = \bar{1}$.

Proof. ($\Rightarrow$) Let $\mathcal{F} = \mathcal{Q}_1^{\mathrm{d}}(Q_n)$ be the face lattice of the n-cube Q_n. For any face F in Q_n and every $x \in F$, $x = (x_1, \ldots, x_n)$ is in the coordinate system. If $(x_1, \ldots, x_n)$ satisfies that each x_i is equal to 0, 1 or 2, we then call $(x_1, \ldots, x_n)$ is *good.*

For every face F in Q_n, let $\pi_1(F)$ denote the set $\{i : \text{for all } x \in F,\ x_i = 1\}$ and $\pi_2(F)$ denote $\{i : \text{for all } x \in F,\ x_i = 2\}$ and then define $\pi(F) := (x_1, \ldots, x_n)$ as follows:

$$x_i = \begin{cases} 1, & i \in \pi_1(F) \\ 2, & i \in \pi_2(F) \\ 0, & \text{otherwise} \end{cases}, \text{ for } 1 \le i \le n.$$

Thus $\pi(F)$ is good. In fact, $\pi(F)$ is a projection of F and every face F, except the empty face, corresponds to a good $\pi(F) = (x_1, \ldots, x_n)$. Furthermore, $\pi(F_0) = \pi(F_1)$ implies $F_0 = F_1$. For any good $x = (x_1, \ldots, x_n)$, let $\pi_1^{-1}(x) := \{i : x_i = 1 \text{ in } x\}$ and $\pi_2^{-1}(x) := \{i : x_i = 2 \text{ in } x\}$, then there exists a face $F = \{(y_1, ..., y_n) : 1 \le y_i \le 2\}$ such that $y_i = 1$ iff $i \in \pi_1^{-1}(x)$, $y_i = 2$ iff $i \in \pi_2^{-1}(x)$ and $1 < y_i < 2$ iff $i \notin \pi_1^{-1}(x) \cup \pi_2^{-1}(x)$ in the coordinate system and $\pi(F) = (x_1, \ldots, x_n)$.

Therefore π is a one-to-one map from the non-empty face F of Q_n to a good $x = (x_1, \ldots, x_n)$. Since for any x_i of good x, it has three choices 0, 1 and 2, then there are 3^n good $x = (x_1, \ldots, x_n)$, together with an empty one, the cardinality of $\mathcal{F} = \mathcal{Q}_1^{\mathrm{d}}(Q_n)$ is equal to $3^n + 1$.

The atoms of $\mathcal{Q}_1^{\mathrm{d}}(Q_n)$ are exactly the facets of Q_n, then the cardinality of atoms is $2n$. The $2n$ facets are $\{x^i\}_{1 \le i \le n}$ and $\{y^i\}_{1 \le i \le n}$ where $x^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0)$ and $y^i = (\underbrace{0, \ldots, 0}_{i-1}, 2, 0, \ldots, 0)$. Because any face F which is not Q_n itself of the n-cube is the intersection of the facets containing F, we have $\mathcal{Q}_1^{\mathrm{d}}(Q_n)$ atomistic.

For any $1 \le i \le n$, $x^i \vee y^i = \bar{1}$ and for $i \neq j$. When $1 \le i \neq j \le n$, $x^i \vee y^j = (z_1, ..., z_n)$ where $z_i = 1$, $z_j = 2$ and $z_s = 0$ for $s \notin \{i, j\}$, $x^i \vee x^j = (z_1, ..., z_n)$ where $z_i = z_j = 1$ and $z_s = 0$ for $s \notin \{i, j\}$, $y^i \vee y^j = (z_1, ..., z_n)$ where $z_i = z_j = 2$ and $z_s = 0$ for $s \notin \{i, j\}$.

($\Leftarrow$) Suppose $\mathcal{F}$ has the above property in the theorem. Let the $2n$ atoms be $\{\bar{x}^i\}_{1 \le i \le n}$ and $\{\bar{y}^i\}_{1 \le i \le n}$ where $\bar{x}_i \vee \bar{y}_i = \bar{1}$. Since $\mathcal{F}$ is atomistic, every element $\bar{x}$ except the top in $\mathcal{F}$ will be of the form $(\bigvee \bar{x}^j) \vee (\bigvee \bar{y}^i)$ while $\bar{x}^i$ and $\bar{y}^i$ cannot occur at the same time for $1 \le i \le n$ because $\bar{x}_i \vee \bar{y}_i = \bar{1}$. The choice of such form is at most $\sum_{t=0}^{n} \binom{n}{t} \times 2^t = (2+1)^n = 3^n$, hence the cardinality of elements of $\mathcal{F}$ together with the top is less than or equal to $3^n + 1$. Because the cardinality of $\mathcal{F}$ is exactly $3^n + 1$, to attain equality, any element $\bar{x}$, except the top, is uniquely expressed in the form $(\bigvee \bar{x}^j) \vee (\bigvee \bar{y}^i)$. Let f_0 be a map from $\bar{x}$ to its choice function $f_0(\bar{x}) = c_{\bar{x}}$ and $c_{\bar{x}} = (c_{\bar{x}}(1), \ldots, c_{\bar{x}}(n))$ is defined as follows:

$$c_{\bar{x}}(i) = \begin{cases} 1, & \bar{x}^i \text{ occurs in the form of } \bar{x} \\ 2, & \bar{y}^i \text{ occurs in the form of } \bar{x} \\ 0, & \text{otherwise} \end{cases}, \text{ for } 1 \le i \le n.$$

Then $c_{\bar{x}}$ is a finite partial function and we have $\bar{x} \le \bar{y}$ in $\mathcal{F}$ iff $c_{\bar{y}}(i) = c_{\bar{x}}(i)$ for any i with $c_{\bar{x}}(i) \neq 0$ and we add a greatest one $c_{\bar{1}}$ among all $c_{\bar{x}}$ which is send from the top $\bar{1}$ of $\mathcal{F}$ by f_0. We then prove that the lattice $\mathcal{F}$ is isomorphic to the face lattice $\mathcal{Q}_1^{\mathrm{d}}(Q_n)$.

Let F be any non-empty face in $\mathcal{Q}_1^{\mathrm{d}}(Q_n)$ and $\pi(F)$ is defined as above. Then $\pi(F) = (x_1, \ldots, x_n) = x$ could be regarded as a finite partial function on $\{1, \ldots, n\}$. The above discussion shows that π is a one-to-one map from the non-empty face F of Q_n to a good x, and then to a $c_{\bar{x}}$ with $c_{\bar{x}}(i) = x_i$ for $1 \le i \le n$ and f_0 is a one-to-one map from any $\bar{x}$ in $\mathcal{F}$ to $c_{\bar{x}}$. Let the empty face in Q_n correspond to $\bar{1}$ in $\mathcal{F}$, then $\mathcal{F}$ is isomorphism to $\mathcal{Q}_1^{\mathrm{d}}(X)$. □

Theorem 7.2.5. $\mathcal{C}_n = \mathcal{R}_n \cong \mathcal{Q}^{\mathrm{d}}(Q_n)$, the dual face poset of n-cube.

Proof. Let $\mathcal{C}'_n$ be a frame obtained by adding a top $\bar{1}$ to $\mathcal{C}'_n$. The cardinality of $\mathcal{C}'_n$ is $3^n + 1$ and $\mathcal{C}'_n$ has $2n$ atoms $\{x^i\}_{1 \le i \le n}$ and $\{y^i\}_{1 \le i \le n}$ where $x^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0)$ and $y^i = (\underbrace{0, \ldots, 0}_{i-1}, 2, 0, \ldots, 0)$. For any $1 \le i \neq j \le n$, it is easy to see $x^i \vee y^i = \bar{1}$. Since $x^i \vee y^j$, $x^i \vee x^j$ and $y_i \vee y^j$ are all in $\mathcal{C}_n$, thus not $\bar{1}$. For any point $a \neq \bar{1}$ in $\mathcal{C}'_n$, let $a = (a_1, \ldots, a_n)$, then $a = (\bigvee_{i \in \pi_1} x_i) \vee (\bigvee_{j \in \pi_2} y_j)$, where $\pi_1 = \{i : a_i = 1\}$ and $\pi_2 = \{i : a_i = 2\}$.

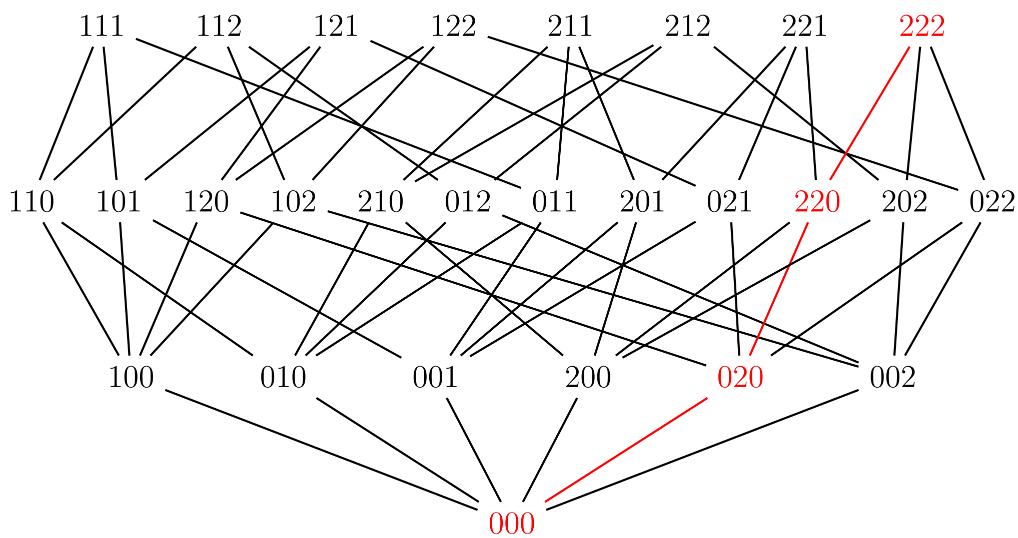


Figure 7.2-2: $\mathcal{Q}^{\mathrm{d}}(Q_3)$ or $\mathcal{C}_3$

Therefore $\mathcal{C}'_n \cong \mathcal{Q}^{\mathrm{d}}_1(Q_n)$ by Theorem 7.2.4. By removing the top $\bar{1}$, $\mathcal{C}_n$ is exactly the frame $\mathcal{Q}^{\mathrm{d}}(Q_n)$. □

In conclusion, the modal logic $\mathsf{ML}(\mathcal{C}_n)$ is the modal logic of the dual face poset of the n-cube.

In [MR78a, § 3], it is proved that the face lattice of an n-cube is Sperner[2] and the largest antichain within this lattice has a size of $s(n) = \binom{n}{\lfloor \frac{n}{3} \rfloor} \times 2^{n-\lfloor \frac{n}{3} \rfloor}$. In conclusion, we have $\mathcal{C}_n \vDash \mathsf{bw}_{\mathsf{s(n)}} \wedge \mathsf{bw}_{\mathsf{s(n)-1}}$.

7.3. Polychromatic colorings of n-cube

The n-dimensional hypercube Q_n is the graph whose vertex set is $\{1,2\}^n$. Let E_n denote the set of all edges in Q_n. The two vertices v and w of an edge e of the hypercube differ in exactly one coordinate, thus we represent the edge $e \in E_n$ by a n-vector $(e_1, e_2, \ldots, e_n)$ with

$$e_i = \begin{cases} 0, & \text{if } v \text{ and } w \text{ differ in } i\text{-th coordinate} \\ v_i, & \text{if } v_i = w_i \end{cases}$$

[2] A graded poset is said to be *Sperner* if no antichain within it has a larger size than the largest level.

Let G_0 and G_1 be two graphs. A subgraph of G_0 isomorphic to G_1 is called an *embedding* of G_1 in G_0. Furthermore, an embedding of Q_d in Q_n (as a subcube) is a n-vector with d coordinates set to 0.

Fix a set P of p colors. An *edge-coloring* μ of a graph G with p colors is a surjective function

$$\mu : E(G) \to P$$

which associate every edge of G with a color in P.

For a given hypercube Q_n and a fixed subgraph G, an edge-coloring μ of a hypercube with p colors is called a *G-polychromatic p-coloring of Q_n* if every embedding of G in Q_n contains every color. Let $p(G)$ be the maximum number of colors for which a *G-polychromatic coloring* is possible for the edges of any hypercube (containing G). $p(G)$ is called the *polychromatic number* of G.

The case $G = Q_d$, a sub-hypercube in Q_n, was introduced by Alon, Krech and Szabó in [AKS07, Theorem 4].

Theorem 7.3.1 (Alon, Krech and Szabó)**.**

$$\binom{d+1}{2} \geq p(Q_d) \geq \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$$

Offner gave the exact value of the polychromatic number of Q_d in [Off08, Theorem 2].

Theorem 7.3.2 (Offner)**.**

$$p(Q_d) = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$$

Consider an edge e of the hypercube Q_n. Let $\ell(e)$ denote the number of 1's to the left of the 0 in e, and $r(e)$ as the count of 1's to the right of the 0.

Recall that an edge-coloring μ of a hypercube Q_n is a function whose domain is the set of all edges E_n of the hypercube. We call the coloring μ is *simple* if $\mu(e)$ is determined by the value of $\ell(e)$ and $r(e)$ for every edge e.

The following lemma, which can be found in [Off08, Lemma 4] or [GLM+18, Lemma 3], indicates that it is sufficient to only consider simple colorings on the polychromatic coloring of the hypercube.

Lemma 7.3.3. Let G be a subgraph of Q_n and its polychromatic number $p(G) = s$. Then there exists a G-polychromatic s-coloring μ on Q_n, such that μ is simple.

Proof. A proof of this lemma can be found in [Off08, Lemma 4]. □

Since the above lemma allows us to only study the simple coloring, we can assign the color of each edge e with $(\ell(e), r(e))$, which accounting the number of 1's in e. For example, the edge 121021 can be identified with $(2, 1)$, and $(2, 1)$ indicates the color of the edge 121021. Since each edge e is arranged to a color class $(\ell(e), r(e))$ belonging to $\mathbb{N} \times \mathbb{N}$, we set all color classes to a triangular array, with the ith row consisting all color classes (x, y) such that $x + y = i$, and the jth diagonal containing all color classes with the form (j, y), where $i, j \geq 0$, see Figure 7.3-3.

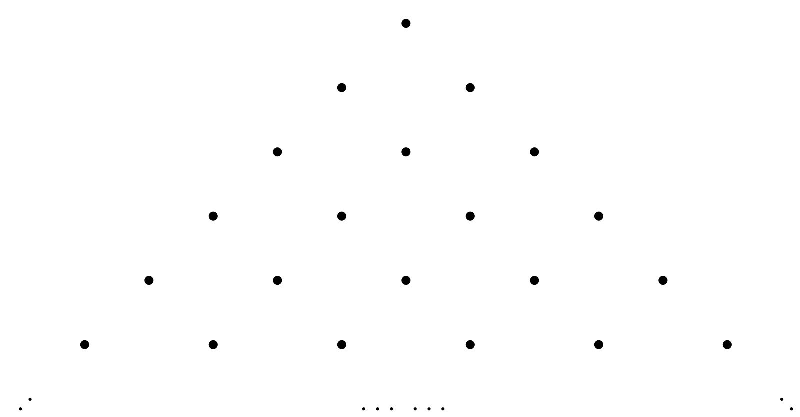


Figure 7.3-3: A triangular array of color classes

Define the *region* O as the color classes that lie in some consecutive rows and diagonals. The *width* of the region $w(O)$ is the number of the diagonals of O and the *height* of the region $h(O)$ is the number of the rows of O.

A *cover* Cov is a finite set of the form (x, y), that is, $\text{Cov} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$. We call a cover Cov is *located at d-th diagonal* if $\min_{1 \leq j \leq k}\{x_j\} = d$ and say d is the *spot number* of Cov. The *width* of the cover $w(\text{Cov}) = \max_{1 \leq i,j \leq k} |x_i - x_j| + 1$.

A *cover sequence* Seq is defined as a finite sequence of covers $\{\text{Cov}_1, \ldots, \text{Cov}_s\}$ that the spot number of Cov_i is less than or equal to the spot number of Cov_j whenever $i < j$. The *width* of the cover sequence $w(\text{Seq}) = \max|x_i - x_j| + 1$

and the *height* of the cover sequence $h(\text{Seq}) = \max|(x_i + y_i) - (x_j + y_j)| + 1$, where (x_i, y_i) and (x_j, y_j) are come from some covers of Seq. We say a cover $\text{Cov}' = \{(x'_j, y'_j) : 1 \le j \le k\}$ is a *translation copy* of a cover Cov, if there exists a $(t_x, t_y) \in \mathbb{Z} \times \mathbb{Z}$, such that $(x'_j, y'_j) = (x_j, y_j) + (t_x, t_y)$ for every $1 \le j \le k$, then call two cover sequences Seq and Seq′ are *equivariant* if $\text{Seq}' = \{\text{Cov}'_1, \text{Cov}'_2, \ldots, \text{Cov}'_s\}$ and every cover Cov'_j of Seq′ is a translation copy of the corresponding cover Cov_j of Seq.

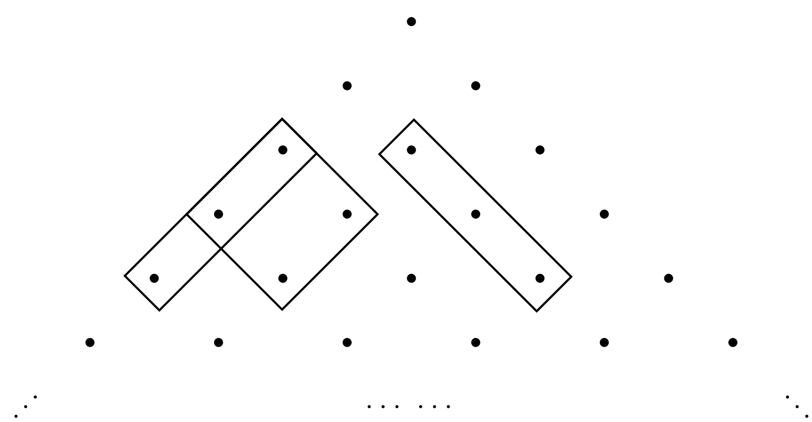


Figure 7.3-4: A cover sequence

We set some configurations for a region O, its rows are $(i+1)$-th, $(i+2)$-th, …, $(i + h(O))$-th rows of the triangular array for some i. For a given cover sequence Seq, assume the height of the region $h(O) = h(\text{Seq})$ and the width of the region $w(O) \ge w(\text{Seq})$. It guarantees that we can study the cover sequence Seq in region O.

For a cover Cov_j in Seq, let v^j_m denote the number of elements of Cov_j in the $(i+m)$-th row, then we can translate the cover Cov_j to a *cover vector* $\overrightarrow{v^j} = (v^j_1, v^j_2, \ldots, v^j_{h(O)})$. In particular, a vector called *maxiamal cover vector* $\overrightarrow{v^{\text{Seq}}} = (v^{\text{Seq}}_1, v^{\text{Seq}}_2, \ldots, v^{\text{Seq}}_{h(O)})$ if

$$v^{\text{Seq}}_m = \max_{1 \le j \le s}\{v^j_m\}.$$

Let $\left\|\overrightarrow{v^{\text{Seq}}}\right\|_1 = \sum_{m=1}^{h(O)} v^{\text{Seq}}_m$ be the 1-*norm* of the maximal cover vector.

For the given cover sequence Seq, a simple coloring μ of a triangular array with p colors is called a Seq-*polychromatic p-coloring* if every cover sequence

Seq′ contains every color when Seq′ and Seq are equivariant. Let $p(\text{Seq})$ be the *maximum* number of colors for which a Seq-*polychromatic coloring* is possible for any triangular array. This $p(\text{Seq})$ is called the *polychromatic number* of Seq.

So for a simple coloring μ, every (x, y) in the region O is corresponding to a color $t \in \{1, 2, \ldots, p(\text{Seq})\}$. For a color t of $\{1, 2, \ldots, p(\text{Seq})\}$, let u_m^t denote the number of color t in the $(i+m)$-th row, then we can translate the color t to a *color vector* $\overrightarrow{u^t} = (u_1^t, u_2^t, \ldots, u_{h(O)}^t)$ and $\sum_{t=1}^{p(\text{Seq})} \overrightarrow{u^t} = (w(O), w(O), \ldots, w(O)) = w(O)\overrightarrow{e}$, where $\overrightarrow{e} = \underbrace{(1, 1, \ldots, 1)}_{h(O)}$.

For the cover sequence Seq, the following proposition from [GLM+18, Lemma 5] provides an upper bound for its polychromatic number $p(\text{Seq})$.

Proposition 7.3.4.

$$p(\text{Seq}) \leq \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1 .$$

To prove this result, we need a lemma that provides a special region with a nice coloring property, as referenced in [Off08, Theorem 2].

Lemma 7.3.5. Given the cover sequence Seq and the region O with the Seq-polychromatic $p(\text{Seq})$-coloring, there is a region O' as a subset of O with the same height $h(O') = h(\text{Seq})$, for any given color t, there exists a cover $\text{Cov}_{j(t)}$ in the cover sequence Seq such that every horizontal translation copy of the cover $\text{Cov}_{j(t)}$ would contain the color t.

Proof. Assume the coloring assigns every (x, y) to a color of $\{1, 2, \ldots, p(\text{Seq})\}$, the the diagonals of region O are $\{1, 2, \ldots, w(O)\}$.

To start with the diagonal 1, we look for the first diagonal d_1^1 which is the first diagonal such that a horizontal translation copy of cover Cov_1 with the spot number d_1^1 does not have the color 1. After that, we start to search the smallest $d_2^1 \geq d_1^1$ such that a copy of Cov_2 with the spot number d_2^1 does not have the color 1. In general, if we already have d_m^1, what we need to do is to find the smallest $d_{m+1}^1 \geq d_m^1$ such that a copy of Cov_{m+1} with the spot number d_{m+1}^1 does not have the color 1. One thing we should pay attention to is that if we can't find such d_{m+1}^1 till diagonal $w(O)$, which means every Cov_{m+1} to the right of d_m^1 contains color 1, then we would stop the program of color 1. The program of color 1 will stop because if we already have $d_1^1, d_2^1, \ldots, d_{s-1}^1$, where the corresponding covers $\mathrm{Cov}_1, \mathrm{Cov}_2, \dots, \mathrm{Cov}_{s-1}$ with those spot numbers do not contain the color 1, then every Cov_s with spot number $\geq d^1_{s-1}$ must contain color 1. The reason is that $\mathrm{Seq} = \{\mathrm{Cov}_1, \mathrm{Cov}_2, \dots, \mathrm{Cov}_s\}$ contain every color since it is a Seq-polychromatic $p(\mathrm{Seq})$-coloring.

We execute the program for color $1, 2, \dots, p(\mathrm{Seq})$, and every program for color m returns a finite set of spot numbers I_m. We then use those spot numbers in $I_1, I_2, \dots, I_{p(\mathrm{Seq})}$ to divide the region O into finite parts and each part is a region. For any region O' and any color t, recall the program for color t, O' is located between two diagonals d^t_m and d^t_{m+1}, then all copies of Cov_{m+1} located at O' will contain color t.

Note that we can extend the width $w(O)$ as large as we want, and then there exists a region O' with its width $w(O')$ as large as we want. □

Proof of Proposition 7.3.4. According to Lemma 7.3.5, in the region O', for any given color t, there exists a cover $\mathrm{Cov}_{j(t)}$ in the cover sequence Seq such that every horizontal translation copy of the cover $\mathrm{Cov}_{j(t)}$ would contain the color t.

Assume $\overrightarrow{u^t} = (u^t_1, u^t_2, \dots, u^t_{h(O)})$ is a color vector of color t and $\overrightarrow{v^{j(t)}} = (v^{j(t)}_1, v^{j(t)}_2, \dots, v^{j(t)}_{h(O)})$ is a cover vector of the cover $\mathrm{Cov}_{j(t)}$, the *inner product* of them is $\langle \overrightarrow{u^t}, \overrightarrow{v^{j(t)}} \rangle = \sum_{i=1}^{h(O)} u^t_i v^{j(t)}_i$.

We now regard the horizontal translation copies of the cover $\mathrm{Cov}_{j(t)}$ as a horizontal movement of $\mathrm{Cov}_{j(t)}$. During the movement, we count the number when an element of $\mathrm{Cov}_{j(t)}$ passes through a point with color t in the region and let $\#(t)$ be this number of the appearance of the color t in $\mathrm{Cov}_{j(t)}$'s. So the inner product is, in fact, the upper bound of $\#(t)$, in other words, $\#(t) \leq \langle \overrightarrow{u^t}, \overrightarrow{v^{j(t)}} \rangle$.

Since each copy of $\mathrm{Cov}_{j(t)}$ contains at least one t color and there are $w(O') - w(\mathrm{Cov}_{j(t)}) + 1$ copies of $\mathrm{Cov}_{j(t)}$ in the region O', thus we obtain a lower bound of $\#(t)$, that is, $w(O') - w(\mathrm{Cov}_{j(t)}) + 1 \leq \#(t)$. Let $w_{\mathrm{Seq}} = \max_{1 \leq j \leq s}\{w(\mathrm{Cov}_j) - 1\}$ and add the number $\#(t)$ for all color, then we have $p(\mathrm{Seq})(w(O') - w_{\mathrm{Seq}}) \leq \sum_{t=1}^{p(\mathrm{Seq})} \#(t) \leq \sum_{t=1}^{p(\mathrm{Seq})} \langle \overrightarrow{u^t}, \overrightarrow{v^{j(t)}} \rangle$.

On the other hand, we already construct a maximal cover vector $\overrightarrow{v^{\mathrm{Seq}}}$, by its definition, we get that $\langle \overrightarrow{u^t}, \overrightarrow{v^{j(t)}} \rangle \leq \langle \overrightarrow{u^t}, \overrightarrow{v^{\mathrm{Seq}}} \rangle$. Consider inner product $\langle \overrightarrow{u^t}, \overrightarrow{v^{\mathrm{Seq}}} \rangle$ for all color $1 \leq t \leq p(\mathrm{Seq})$ and add up those inner products, then $\sum_{t=1}^{p(\mathrm{Seq})} \langle \overrightarrow{u^t}, \overrightarrow{v^{\mathrm{Seq}}} \rangle = \langle \sum_{t=1}^{p(\mathrm{Seq})} \overrightarrow{u^t}, \overrightarrow{v^{\mathrm{Seq}}} \rangle$. It is easy to see that $\sum_{t=1}^{p(\mathrm{Seq})} \overrightarrow{u^t} =$

$(w(O'), w(O'), \dots, w(O')) = w(O')\overrightarrow{e}$, so $\langle \sum_{t=1}^{p(\text{Seq})} \overrightarrow{u^t}, \overrightarrow{v^{\text{Seq}}} \rangle = w(O')\langle \overrightarrow{e}, \overrightarrow{v^{\text{Seq}}} \rangle = w(O') \sum_{m=1}^{h(O)} v_m^{\text{Seq}} = w(O') \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1$ and so $\sum_{t=1}^{p(\text{Seq})} \langle \overrightarrow{u^t}, \overrightarrow{v^{j(t)}} \rangle \le \sum_{t=1}^{p(\text{Seq})} \langle \overrightarrow{u^t}, \overrightarrow{v^{\text{Seq}}} \rangle = w(O') \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1$. Therefore, $p(\text{Seq})(w(O') - w_{\text{Seq}}) \le w(O') \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1$. In conclusion, $p(\text{Seq}) \le \frac{w(O')}{w(O') - w_{\text{Seq}}} \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1$.

Recall in the proof of Lemma 7.3.5, we could ask the width $w(O')$ as large as we want, w_{Seq} is a constant, so $p(\text{Seq}) \le \left\| \overrightarrow{v^{\text{Seq}}} \right\|_1$. □

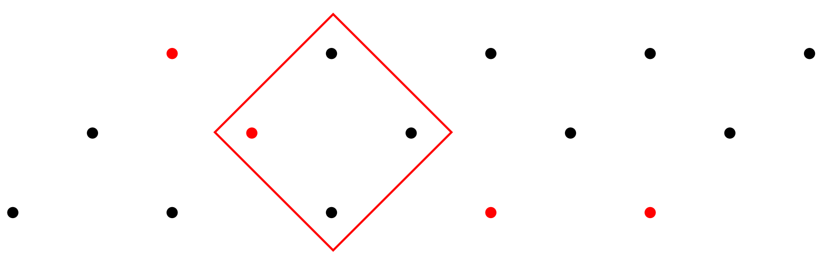


Figure 7.3-5: One copy of a cover for red color in the region O'

Proof of Theorem 7.3.2. We first build a simple edge-coloring μ, for any edge e of the hypercube Q_n, we have $\mu(e) = (\ell(e) \bmod \lfloor \frac{d+1}{2} \rfloor, r(e) \bmod \lceil \frac{d+1}{2} \rceil)$, where $\text{Cl} = \{(x, y) : 0 \le x \le \lfloor \frac{d+1}{2} \rfloor - 1 \text{ and } 0 \le y \le \lceil \frac{d+1}{2} \rceil - 1\}$ represents the set of $\lfloor \frac{(d+1)^2}{4} \rfloor$ different colors.

For any subcube Q_d of Q_n, there are exactly d coordinates $\{x_{q_1}, \dots, x_{q_d}\}$ of Q_d such that those coordinates $x_{q_i} = 0$. Let E be the subset of edges of this Q_d such that only the $q_{\lfloor \frac{d+1}{2} \rfloor}$-th coordinate is 0, that is, $E = \{(x_1, x_2, \dots, x_n) : x_{q_{\lfloor \frac{d+1}{2} \rfloor}} = 0 \text{ and } (x_1, x_2, \dots, x_n) \text{ is an edge of } Q_d\}$.

For any $e \in E$, there are at most $\lfloor \frac{d+1}{2} \rfloor - 1$ many 1's to the left of $x_{q_{\lfloor \frac{d+1}{2} \rfloor}}$ and at most $\lceil \frac{d+1}{2} \rceil - 1$ many 1's to the right, thus E as a subset of edges of Q_d contains all $\lfloor \frac{(d+1)^2}{4} \rfloor$ different colors of the set $\{(x, y) : 0 \le x \le \lfloor \frac{d+1}{2} \rfloor - 1 \text{ and } 0 \le y \le \lceil \frac{d+1}{2} \rceil - 1\}$.

For any edge e of Q_d but $e \notin E$, $\mu(e)$ is a color of Cl, so μ is a Q_d-polychromatic $\lfloor \frac{(d+1)^2}{4} \rfloor$-coloring on any hypercube Q_n containing Q_d. By the definition, $p(Q_d) \ge \lfloor \frac{(d+1)^2}{4} \rfloor$.

Then we prove $p(Q_d) \leq \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$ by building a bridge between the Q_d-polychromatic coloring of edges and Seq_{Q_d}-polychromatic coloring of triangular array.

According to Lemma 7.3.3, it suffices to consider a Q_d-polychromatic $p(Q_d)$-coloring μ on a sufficiently large hypercube and μ is a simple coloring. We now describe the cover sequence Seq_{Q_d} for every embedding of Q_d in Q_n.

Assume in Q_d, the d coordinates $\{x_{q_1}, x_{q_2}, \ldots, x_{q_d}\}$ be those coordinates $x_{q_i} = 0$. Let E^1 be the set of all edges of Q_d such that the first coordinate $x_{q_1} = 0$, then for any edge $e \in E^1$, we have $\ell(e) = m_0$ and $m_1 \leq r(e) \leq m_1 + d - 1$, where m_0 indicates how many 1's appear to the left of x_{q_1} in Q_d, and m_1 indicates how many 1's appear to the right. Thus $\mu(e) = (m_0, y),$ where $m_1 \leq y \leq m_1 + d - 1$ and $\mu(E^1) = \{(m_0, m_1), (m_0, m_1 + 1), \ldots, (m_0, m_1 + d - 1)\}$.

Thus $\mu(E^1)$ forms the cover Cov_1 of Seq_{Q_d}, geometrically speaking, Cov_1 is a $d \times 1$ rectangle whose spot number is m_0.

In general, Let E^i be the set of all edges of Q_d such that the first coordinate $x_{q_i} = 0$, then for any edge $e \in E^i$, we have $m_0^i \leq \ell(e) \leq m_0^i + i - 1$ and $m_0 + m_1 - m_0^i \leq r(e) \leq m_0 + m_1 - m_0^i + d - i$, where m_0^i indicates how many 1's appear to the left of x_{q_i} in Q_d, so $m_0^i \leq m_0^j$ when $i < j$. It is easy to see that $m_0 + m_1 - m_0^i$ indicates how many 1's appear to the right of x_{q_i} in Q_d. Thus $\mu(e) = (x, y),$ where $m_0^i \leq x \leq m_0^i + i - 1$ and $m_0 + m_1 - m_0^i \leq y \leq m_0 + m_1 - m_0^i + d - i$ and $\mu(E^i) = \{(m_0^i + \alpha, m_0 + m_1 - m_0^i + \beta) : 0 \leq \alpha \leq i - 1, 0 \leq \beta \leq d - i\}$

Thus $\mu(E^i)$ forms the cover Cov_i of Seq_{Q_d}, geometrically speaking, Cov_i is a $(d+1-i) \times i$ rectangle whose spot number is m_0^i. For any i, the height of Cov_i is d. Since $(m_0^i, m_0 + m_1 - m_0^i)$ is at the row $(m_0 + m_1)$, all Cov_i occupy the same rows in the triangular array.

Therefore, the cover sequence $\text{Seq}_{Q_d} = \{\text{Cov}_1, \text{Cov}_2, \ldots, \text{Cov}_d\}$, where Cov_i is a $(d + 1 - i) \times i$ rectangle whose spot number is m_0^i and occupies from $(m_0 + m_1)$ row to $(m_0 + m_1 + d - 1)$ row, and $m_0 = m_0^1 \leq m_0^2 \leq \ldots \leq m_0^{\text{d}}$. See Figure 7.3-4 for an example of a cover sequence for Q_3. From the viewpoint of the cover vector,

$$\overrightarrow{v^{\text{Seq}_{Q_d}}} = \begin{cases} (1, 2, \ldots, \frac{d}{2}, \frac{d}{2}, \ldots, 2, 1), & \text{if } d \text{ is even} \\ (1, 2, \ldots, \frac{d+1}{2}, \ldots, 2, 1), & \text{if } d \text{ is odd} \end{cases}$$

It is easy to see that $\left\| \overrightarrow{v^{\text{Seq}_{Q_d}}} \right\|_1 = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$, in conclusion, $p(Q_d) = p(\text{Seq}) \leq \left\| \overrightarrow{v^{\text{Seq}_{Q_d}}} \right\|_1 = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$. Putting everything together, $p(Q_d) = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$. □

In the next section, we will see an application of Theorem 7.3.2 in a completely different area.

7.4. Three strategies for proving non-finite axiomatisability

By using Offner's result 7.3.2, Kuznetsov in [Kuz19] discussed the two strategies belonging to Fontaine and Shatrov for proving that Cheq is not finitely axiomatisable, respectively. He also presented an alternative solution as an open problem for proving the non-finitely axiomatisability of Cheq.

7.4.1. Fontaine's structures and Shatrov's structures

Recall that Maksimova, Skvortsov and Shehtman provided a method to prove that Med is not finitely axiomatisable in [MSS79, Corollary 5] by showing Chinese lantern $\Phi(s, 2^{s+3})$ is not Med-frame while Chinese lantern $\Phi'(s, 2^{s+3}, m)$ is Med-frame for any $s \geq 1$.

In [Fon07, § 5], Fontaine gave two kinds of structures $\Omega(s)$ and $\Omega'(s, m)$ as frames depicted in Figure 7.4-6. As we did in § 4.2.2, using similar structures and results by Maksimova, Skvortsov and Shehtman, Fontaine proved that $\Omega(s)$ is not a Cheq-frame for any $s \geq 1$ and want to prove that $\Omega'(s, m)$ is a Cheq-frame. Lemma 6.2.2 gives us a motivation of focusing on the $(m, 0)\uparrow$ in $\Omega'(s, m)$ since $(m, 0)\downarrow$ is already a Cheq-frame. Let $\mathcal{G}_m$ be the frame $(m, 0)\uparrow$, then Fontaine provided the following proposition.

Proposition 7.4.1. If $\mathcal{G}_m$ is Cheq-frame for any m, then Cheq is not finitely axiomatisable.

Proof. Suppose Cheq is finitely axiomatisable with s variables, we may assume that Cheq is axiomatised by a single formula $\phi(p_1, \ldots, p_s)$.

As in the proof of Proposition 4.2.9, there is an $m \leq s$ such that

$$\Omega(s) \vDash \phi \text{ iff } \Omega'(s, m) \vDash \phi.$$

Since $(m, 0)\downarrow$ in $\Omega'(s, m)$ is a finite rooted frame with a top, then it is a Med-frame and so a Cheq-frame. Since $\mathcal{G}_m$ is a Cheq-frame, thus $\Omega'(s, m)$, as the vertical sum of the above frames, is a Cheq-frame by Lemma 6.2.2.

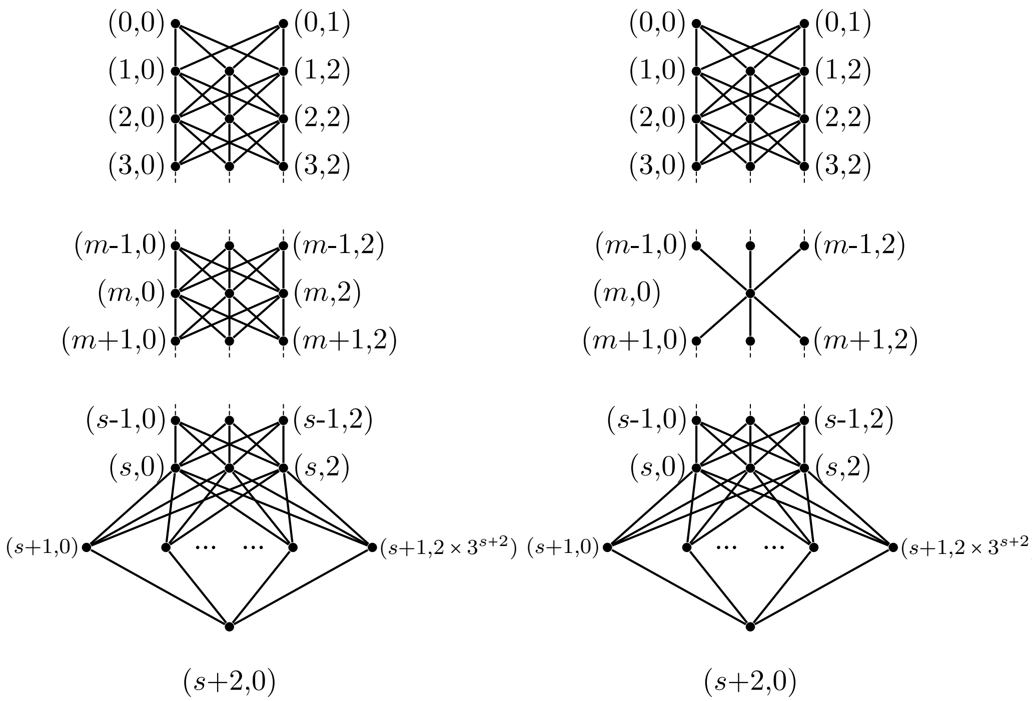


Figure 7.4-6: The frames $\Omega(s)$ and $\Omega'(s,m)$

According to [Fon07, Claim 16], the frame $\Omega(s)$ is not a Cheq-frame, a contradiction.

Therefore if $\mathcal{G}_m$ is Cheq-frame for any m, then Cheq is not finitely axiomatisable with s variables for any natural number s. □

Definition 7.4.2. A collection $\{X_0, \ldots, X_{n-1}\}$ is called a *full n-partition* of X with respect to Y if:

1. $X = \bigcup_{i=0}^{n-1} X_i$,
2. $X_i \cap X_j = \emptyset$ for $0 \leq i \neq j \leq n-1$,
3. for each i and for every $y \in Y$, there exists $x_i \in X_i$ satisfying $y \leq x_i$.

Consider the set $D(i,j)$, consisting of all x in the frame $\mathcal{C}_i$, where the depth $d(x)$ equals $j+1$ (the number of 0 that occur in x is equal to j). Fontaine showed that if, for every $i > 1$, a 3-full partition of $D(i,1)$ with

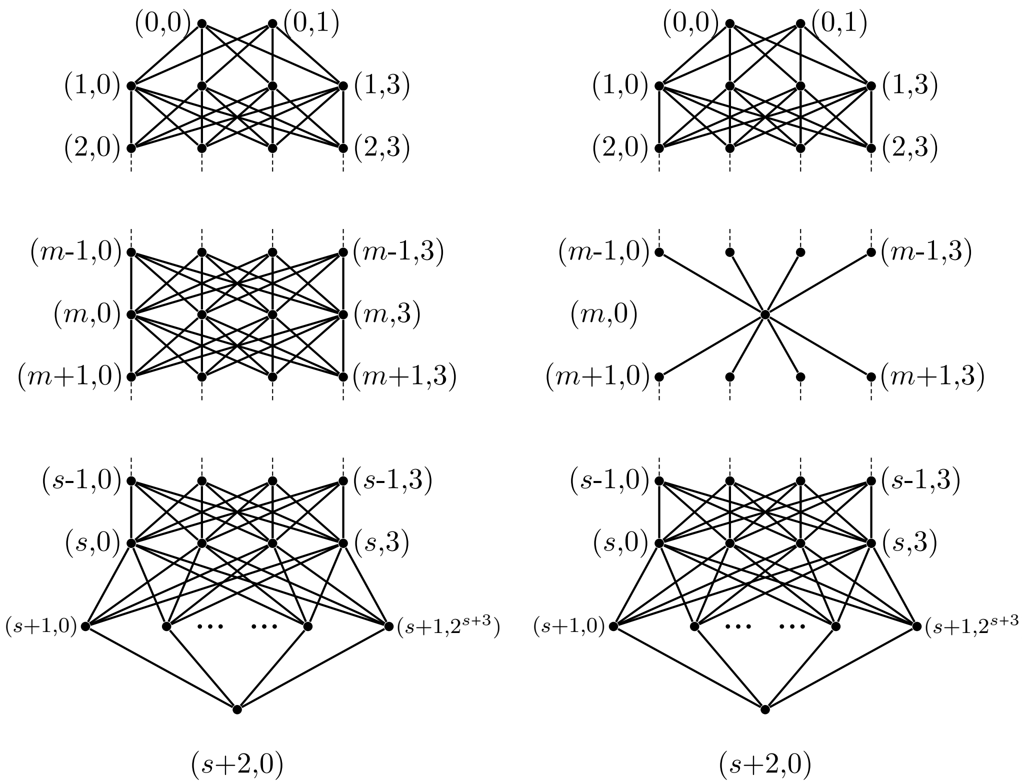


Figure 7.4-7: The frames $\Psi(s)$ and $\Psi'(s,m)$

respect to $D(i,2)$ exists, then it is possible to construct a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$ for any $j > 1$. Consequently, Fontaine suggests the following proposition in [Fon07, Proposition 22].

Proposition 7.4.3. If for each $i > 1$, a 3-full partition of $D(i,1)$ with respect to $D(i,2)$ exists, then $\mathcal{G}_m$ is a Cheq-frame for any m and so Cheq is not finitely axiomatisable.

According to Theorem 7.2.5, each $\mathcal{C}_i$ can be regarded as the dual face poset of hypercube Q_i. Then $D(i,1)$ is the set of edges of Q_i, and $D(i,2)$ is the set of faces of Q_i. The assumption of Proposition 7.4.3 is to ask to have edge-coloring μ of any hypercube Q_i with 3 colors and μ is Q_2-polychromatic 3-coloring of Q_i since every face of Q_i contains every color. But by Theorem

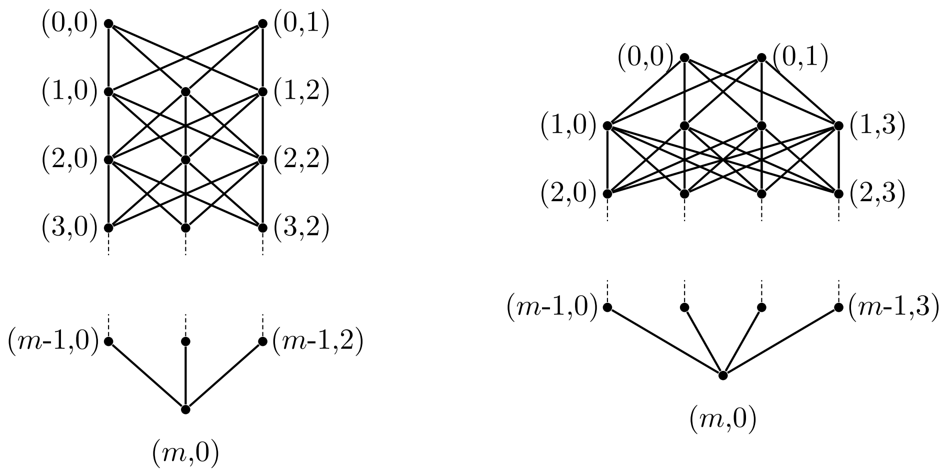


Figure 7.4-8: The frames $\mathcal{G}_m$ and $\mathcal{V}_m$

7.3.2, the polychromatic number of Q_2

$$p(Q_2) = \left\lfloor \frac{(2+1)^2}{4} \right\rfloor = 2$$

Thus for a sufficiently large integer N, the hypercube Q_N can not allow a Q_2-polychromatic 3-coloring, then the assumption of Proposition 7.4.3 fails.

Shatrov claimed to prove the non-finitely axiomatisability of Cheq at the conference Algebraic and Topological Methods in Non-Classical Logics (TANCL 07) in Oxford by showing that $\Psi'(s,m)$ is a Cheq-frame while $\Psi(s)$ is not, where $\Psi(s)$ and $\Psi'(s,m)$ are frames depicted in Figure 7.4-7.

Shatrov's strategy involves proving that each frame $\mathcal{V}_m$, which is $(m,0)\uparrow$ in $\Psi'(s,m)$, is a Cheq-frame. If we consider the possibility of a 4-full partition of $D(i,1)$ with respect to $D(i,2)$ to achieve this goal, it fails due to $p(Q_2) = 2$. However, since Shatrov has not published his proof, we do not know his concrete assumptions for proving that $\mathcal{V}_m \vDash \mathsf{Cheq}$.

7.4.2. Kuznetsov's alternative

After briefly describing the infeasibility of the above schemes to prove that Cheq is not finitely axiomatisable, Kuznetsov provided his strategy by building the following alternative structure in [Kuz19].

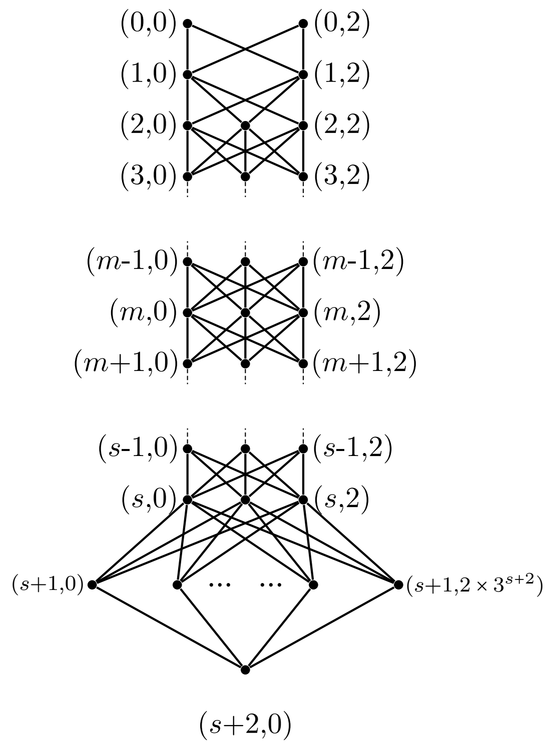


Figure 7.4-9: The frame $\mathcal{K}(s)$

Definition 7.4.4. We define the *Kuznetsov frame* $\mathcal{K}(s)$ to be the frame shown in Figure 7.4-9.

The result of the polychromatic number of Q_2 can not directly negate the Kuznetsov frame $\mathcal{K}(s)$. Kuznetsov aimed to prove that each Kuznetsov frame $\mathcal{K}(s)$ is not a Cheq-frame as the key step to follow the established method of proving that Cheq is not finitely axiomatisable. But we have the following proposition:

Proposition 7.4.5. $\mathcal{K}(s)$ is a Cheq-frame.

Proof. Each $\mathcal{K}(s)$ is a linear sum of a finite rooted frame and the bow-tie frame. By the bow-tie Lemma 6.2.3, every Kuznetsov frame $\mathcal{K}(s)$ is a Cheq-frame. □

7.5. Cheq is not axiomatisable with 5 or 6 variables

In [Fon07, Proposition 22], Fontaine proved that if one can construct a 3-full partition of $D(i,1)$ with respect to $D(i,2)$, then Cheq is not axiomatisable with i variables, and Fontaine provided a case for $i = 4$: $A(4,1)$, $B(4,1)$ and $C(4,1)$ where

$$\begin{aligned} A(4,1) =& \{(0,1,1,1),(1,2,1,0),(2,0,1,2),(0,2,2,2),(2,2,0,2),\\ &(2,1,0,1),(2,0,2,1),(1,1,2,0),(1,2,0,1),(1,0,1,2),\\ &(0,1,2,2)\}; \end{aligned}$$

$$\begin{aligned} B(4,1) =& \{(1,0,1,1),(2,1,1,0),(0,2,1,2),(1,2,2,0),(2,2,0,1),\\ &(2,0,2,2),(0,1,2,1),(1,1,0,2),(0,2,1,1),(1,0,2,1)\}; \end{aligned}$$

$$\begin{aligned} C(4,1) =& \{(2,0,1,1),(1,1,1,0),(2,2,1,0),(1,2,0,2),(2,2,2,0),\\ &(2,1,2,0),(1,1,0,1),(1,0,2,2),(2,1,0,2),(0,2,2,1),\\ &(0,1,1,2)\}. \end{aligned}$$

In conclusion, Cheq is not axiomatisable with four variables. Fontaine raised the question of whether there is a 3-full partition for $i = 5$.

In the following, I construct a new algorithm which can give a 3-full partition for $n = 5, 6$ cases:

Step 1. Construct $A(n,1)$. There are n possible positions where 0 can occur. Choose any four numbers from $\{1,2,\ldots,n\}$, denoted as n_1, n_2, n_3, n_4. For those $x = (x_1,x_2,\ldots,x_n)$ in $A(n,1)$ where x_{n_1} is 0, construct 2^{n-4} elements such that $x_{n_2} = a$, $x_{n_3} = b$, where $a,b \in \{1,2\}$, and the total number of 1s in the other x_i is odd; the other 2^{n-4} with $x_{n_2} = 3-a$, $x_{n_3} = 3-b$ and the total number of 1 in other x_i is even. There are 2^{n-3} elements x in $A(n,1)$ with $x_{n_1} = 0$;

Step 2. Construct $x = (x_1,x_2,\ldots,x_n)$ in $A(n,1)$ where $x_{n_2} = 0$. For all x constructed in Step 1, let its $x_{n_2} = 0$, yielding 2^{n-3} distinct elements, forming set A. There are 2^{n-2} elements y in total such that $y_{n_1} = y_{n_2} = 0$, forming set B. For any $x \in B$ but $x \notin A$, then $x_{n_1} = x_{n_2} = 0$. Then we obtain a new element c associate with above $x \in B$ and $x \notin A$, when $c_{n_1} = c_x$ and $c_i = x_i$ otherwise, where $c_x \in \{1,2\}$. The set of all such c is precisely those elements in $A(n,1)$ where $x_{n_2} = 0$, where c_x is to be determined for each x;

Step 3. Repeat Step 2 to construct $x = (x_1, x_2, \ldots, x_n)$ in $A(n,1)$ when set $x_{n_3} = 0$;

Step 4. Repeat Step 2 to construct $x = (x_1, x_2, \ldots, x_n)$ in $A(n,1)$ when set $x_{n_4} = 0$;

Step 5. For any elements x obtained in Step 2, let $x_{n_3} = 0$ to get an element x_{n_2,n_3}, and let $x_{n_4} = 0$ to get an element x_{n_2,n_4}. For any elements y obtained in Step 3, let $y_{n_2} = 0$ to get an element y_{n_2,n_3}, and let $y_{n_4} = 0$ to get an elemet y_{n_3,n_4}. For any elements z obtained in Step 4 and any $n \neq n_1$, let $z_n = 0$ to get an element z_{n,n_4}. Require all these elements to be distinct so that the unknown variables to be determined in Steps 2 to 4 are solvable. There are two sets of solutions: the first of which is, for c in Step 2, we have $c_{n_1} = c_x = c_{n_4}$ and for c in Step 3, we have $c_{n_1} = c_x = 3 - c_{n_4}$; another solution is, for c in Step 2, we have $c_{n_1} = c_x = 3 - c_{n_4}$ and for c in Step 3, we have $c_{n_1} = c_x = c_{n_4}$;

Step 6. Each Step of 1 to 4 yields 2^{n-3} elements respectively. For all $x = (x_1, \ldots, x_n)$ constructed in Step 1, fix $n_s \notin \{n_1, n_2, n_3, n_4\}$, let $x_{n_s} = 0$, obtaining 2^{n-3} distinct elements, forming set E. For all $y = (y_1, \ldots, y_n)$ constructed in Step 2, let $y_{n_s} = 0$, obtaining 2^{n-3} distinct elements, forming set F. Then for any $x \in E$, there exists y^- not in F and $y^-_{n_2} = y^-_{n_s} = 0$, such that there exists a unique t with $t_{n_s} = 0$, such that we can obtain x when $t_{n_1} = 0$ and obtain y^- when $t_{n_2} = 0$. For any $y \in F$, there exists $x^- \notin E$ and $x^-_{n_1} = x^-_{n_s} = 0$, such that there exists a unique t with $t_{n_s} = 0$, such that we can obtain x^- when $t_{n_1} = 0$ and obtain y when $t_{n_2} = 0$. All these 2^{n-2} ts form the set of elements in $A(n,1)$ where the n_s-th digit is 0;

Step 7. Repeat Step 6 to obtain the set of elements in $A(n,1)$ where the n_t-th digit is 0, completing the construction of $A(n,1)$;

Step 8. When $n \leq 6$, we can check that after $A(n,1)$ is constructed as above, the remaining elements of $D(n,1)$ can be evenly divided into two parts $B(n,1)$ and $C(n,1)$ that satisfy the condition.

When $n = 6$, $a = 1$, $b = 2$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$ and $n_4 = 6$, selecting the first set of solutions in Step 5, we obtain a 3-full partition for $n = 6$ by the above algorithm:

$$\begin{aligned} A(6,1) = \{&(0,1,2,2,1,2),(0,1,2,1,2,2),(0,2,1,1,1,2),(0,1,2,1,1,1),\\ &(0,2,1,1,2,1),(0,1,2,2,2,1),(0,2,1,2,1,1),(1,0,1,1,1,1),\\ &(0,2,1,2,2,2),(1,0,1,2,2,1),(1,0,2,2,1,1),(1,0,2,1,2,1),\\ &(2,0,1,1,2,2),(2,0,2,1,1,2),(2,0,2,2,2,2),(2,0,1,2,1,2),\end{aligned}$$

$$
\begin{aligned}
&(1,1,0,2,2,2),(1,2,0,1,2,2),(1,1,0,1,1,2),(1,2,0,2,1,2),\\
&(2,1,0,2,1,1),(2,2,0,1,1,1),(2,1,0,1,2,1),(1,1,1,0,1,1),\\
&(1,1,1,0,2,1),(2,2,0,2,2,1),(1,1,2,0,1,2),(1,1,2,0,2,2),\\
&(1,2,1,0,2,2),(1,2,1,0,1,2),(1,2,2,0,2,1),(1,2,2,0,1,1),\\
&(2,1,1,0,2,2),(2,1,1,0,1,2),(2,1,2,0,2,1),(2,1,2,0,1,1),\\
&(2,2,1,0,2,1),(2,2,1,0,1,1),(2,2,2,0,2,2),(2,2,2,0,1,2),\\
&(1,1,1,1,0,2),(1,1,2,1,0,1),(1,1,2,2,0,1),(1,1,1,2,0,2),\\
&(1,2,1,2,0,1),(1,2,1,1,0,1),(1,2,2,2,0,2),(1,2,2,1,0,2),\\
&(2,1,1,1,0,1),(2,1,2,1,0,2),(2,1,1,2,0,1),(2,2,1,1,0,2),\\
&(2,1,2,2,0,2),(2,2,2,1,0,1),(2,2,1,2,0,2),(2,2,2,2,0,1),\\
&(1,2,2,1,1,0),(2,2,2,2,1,0),(1,2,2,2,2,0),(2,1,1,2,2,0),\\
&(2,2,2,1,2,0),(2,1,1,1,1,0),(1,1,1,1,2,0),(1,1,1,2,1,0)\};
\end{aligned}
$$

$$
\begin{aligned}
B(6,1) =\{&(0,1,1,1,2,1),(0,1,1,1,1,2),(0,1,1,2,1,1),(0,1,2,2,1,1),\\
&(0,1,2,2,2,2),(0,1,1,2,2,2),(0,2,1,2,2,1),(0,2,1,2,1,2),\\
&(0,2,2,1,2,2),(0,2,2,1,1,1),(0,2,2,2,2,1),(1,0,1,1,2,1),\\
&(0,2,2,2,1,2),(1,0,1,2,1,2),(1,0,2,1,1,1),(1,0,1,1,2,2),\\
&(1,0,2,1,1,2),(2,0,1,1,1,1),(1,0,2,2,2,2),(2,0,1,1,1,2),\\
&(2,0,1,2,2,1),(2,0,2,1,2,2),(2,0,2,1,2,1),(2,0,2,2,1,1),\\
&(1,1,0,1,1,1),(1,1,0,2,2,1),(1,1,0,2,1,2),(1,2,0,2,1,1),\\
&(1,2,0,1,2,1),(2,1,0,1,2,2),(2,1,0,2,1,2),(1,2,0,2,2,2),\\
&(2,2,0,1,1,2),(2,1,0,2,2,1),(2,2,0,2,1,1),(1,1,1,0,2,2),\\
&(2,2,0,2,2,2),(1,2,1,0,1,1),(1,1,2,0,2,1),(2,1,1,0,1,1),\\
&(1,2,2,0,1,2),(2,1,2,0,1,2),(2,2,2,0,2,1),(2,2,1,0,2,2),\\
&(1,1,2,1,0,2),(1,2,1,1,0,2),(1,1,1,2,0,1),(1,2,2,2,0,1),\\
&(2,1,1,2,0,2),(2,2,1,1,0,1),(2,2,2,2,0,2),(2,1,2,1,0,1),\\
&(1,1,1,1,1,0),(2,2,1,2,1,0),(1,1,2,2,1,0),(1,2,1,2,2,0),\\
&(1,2,2,1,2,0),(1,2,1,1,1,0),(2,1,2,1,1,0),(2,2,2,1,1,0),\\
&(2,2,1,1,2,0),(2,1,2,2,2,0),(1,1,2,1,2,0),(2,1,1,1,2,0)\};
\end{aligned}
$$

$$
\begin{aligned}
C(6,1) =\{&(0,1,1,1,2,2),(0,1,1,1,1,1),(0,1,1,2,2,1),(0,1,1,2,1,2),\\
&(0,1,2,1,2,1),(0,1,2,1,1,2),(0,2,1,1,2,2),(0,2,1,1,1,1),\\
&(0,2,2,1,2,1),(0,2,2,1,1,2),(0,2,2,2,2,2),(0,2,2,2,1,1),
\end{aligned}
$$

$$
\begin{aligned}
&(1,0,1,1,1,2),(1,0,1,2,2,2),(1,0,1,2,1,1),(1,0,2,2,1,2),\\
&(1,0,2,1,2,2),(2,0,1,1,2,1),(1,0,2,2,2,1),(2,0,1,2,1,1),\\
&(2,0,1,2,2,2),(2,0,2,2,1,2),(2,0,2,1,1,1),(1,1,0,1,2,1),\\
&(2,0,2,2,2,1),(1,1,0,2,1,1),(1,1,0,1,2,2),(1,2,0,1,1,1),\\
&(1,2,0,2,2,1),(1,2,0,1,1,2),(2,1,0,2,2,2),(2,1,0,1,1,1),\\
&(2,1,0,1,1,2),(2,2,0,1,2,1),(2,2,0,2,1,2),(1,1,1,0,1,2),\\
&(2,2,0,1,2,2),(1,2,1,0,2,1),(1,1,2,0,1,1),(2,1,1,0,2,1),\\
&(1,2,2,0,2,2),(2,2,1,0,1,2),(2,1,2,0,2,2),(1,1,1,1,0,1),\\
&(2,2,2,0,1,1),(1,1,2,2,0,2),(1,2,2,1,0,1),(1,2,1,2,0,2),\\
&(2,1,1,1,0,2),(2,2,1,2,0,1),(2,1,2,2,0,1),(2,2,2,1,0,2),\\
&(2,2,1,2,2,0),(1,1,2,2,2,0),(2,2,2,2,2,0),(1,2,2,2,1,0),\\
&(1,2,1,1,2,0),(1,2,1,2,1,0),(2,1,2,1,2,0),(2,1,1,2,1,0),\\
&(2,2,1,1,1,0),(2,1,2,2,1,0),(1,1,1,2,2,0),(1,1,2,1,1,0)\}.
\end{aligned}
$$

When $n = 5$, $a = 1$, $b = 2$, $n_1 = 3$, $n_2 = 1$, $n_3 = 2$ and $n_4 = 4$, selecting the second set of solutions in step 5, we obtain a 3-full partition for $n = 5$ by the above algorithm:

$$
\begin{aligned}
A(5,1) =\{&(0,2,2,1,1),(0,1,1,2,1),(0,1,2,1,2),(0,2,1,2,2),\\
&(2,0,2,2,1),(2,0,1,1,2),(1,0,2,2,2),(1,0,1,1,1),\\
&(2,1,0,1,1),(1,2,0,2,1),(2,1,0,2,2),(1,2,0,1,2),\\
&(1,1,1,0,2),(1,1,2,0,1),(2,2,1,0,1),(2,2,2,0,2),\\
&(1,2,2,1,0),(2,1,2,1,0),(2,1,1,2,0),(1,2,1,2,0),\\
&(1,1,2,2,0),(2,2,1,1,0),(2,2,2,2,0),(1,1,1,1,0)\};
\end{aligned}
$$

$$
\begin{aligned}
B(5,1) =\{&(0,1,1,1,2),(0,1,2,2,2),(0,2,2,2,1),(0,1,2,1,1),\\
&(0,1,1,2,2),(0,2,1,1,1),(1,0,2,1,2),(2,0,1,1,1),\\
&(1,0,2,1,1),(1,0,1,2,1),(2,0,2,2,2),(1,0,1,2,2),\\
&(2,2,0,1,2),(1,1,0,2,1),(2,2,0,2,1),(1,1,0,1,2),\\
&(1,2,0,2,2),(2,1,0,2,1),(2,2,1,0,2),(2,1,2,0,1),\\
&(2,1,2,0,2),(1,2,2,0,1),(1,1,1,0,1),(1,2,2,0,2),\\
&(2,2,2,1,0),(1,2,1,1,0),(2,2,1,2,0),(2,1,1,1,0)\};
\end{aligned}
$$

$$
C(5,1) =\{(0,2,2,2,2),(0,2,2,1,2),(0,2,1,1,2),(0,1,1,1,1),
$$

$$(0,2,1,2,1),(0,1,2,2,1),(1,0,2,2,1),(2,0,1,2,1),$$
$$(2,0,1,2,2),(2,0,2,1,1),(1,0,1,1,2),(2,0,2,1,2),$$
$$(1,1,0,1,1),(2,2,0,1,1),(1,1,0,2,2),(2,1,0,1,2),$$
$$(2,2,0,2,2),(1,2,0,1,1),(1,2,1,0,2),(1,2,1,0,1),$$
$$(2,1,1,0,1),(2,2,2,0,1),(1,1,2,0,2),(2,1,1,0,2),$$
$$(1,1,2,1,0),(2,1,2,2,0),(1,2,2,2,0),(1,1,1,2,0)\}.$$

In conclusion, we prove the following theorem:

Theorem 7.5.1. Cheq is not axiomatisable with five or six variables.

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Appendices

A.1. Summary

We investigate three intermediate logics and their modal counterparts that play an important role in the study of modal logics of forcing classes. Hamkins, Leibman and Löwe have conjectured that S4.tBA is an upper bound of the modal logic of c.c.c. forcing; Inamdar has proved that S4.sBA is such an upper bound; in Chapter 3 of our thesis, we prove under additional assumptions that S4.FPFA, a modal logic that is contained in the intersection of the two other logics, is an upper bound. We do not know whether our additional assumptions are true; if so, our result proves the conjecture by Hamkins, Leibman and Löwe.

The remaining chapters of the thesis study the six mentioned logics in more detail. In Chapter 4, we prove two conjectures by Nick Bezhanishvili on generalized Medvedev logics; in Chapter 5, we connect the concept of nerve to Medvedev logic; in Chapter 6, we prove that S4.sBA is not finitely axiomatisable over Cheq; and finally, in Chapter 7, we prove that Cheq is not axiomatisable with five or six variables.

A.2. German Summary

Wir untersuchen drei Zwischenlogiken (Logiken, die zwischen intuitionistischer und klassischer Aussagenlogik liegen) und ihre modalen Gegenstücke, welche eine wichtige Rolle im Studium der Modallogiken von Erzwingungsklassen spielen. Hamkins, Leibman und Löwe vermuteten, daß S4.tBA eine obere Schranke für die Modallogik der Erzwingungsordnungen mit abzählbarer Kettenbedingung ist; Inamdar bewies, daß S4.sBA eine solche obere Schranke ist; in Kapitel 3 unserer Dissertation beweisen wir unter zusätzlichen Voraussetzungen, daß S4.FPFA, eine Modallogik, welche im Schnitte der beiden anderen Logiken enthalten ist, eine obere Schranke ist. Wir wissen nicht, ob unsere zusätzlichen Annahmen wahr sind. Falls ja, beweist unser Resultat die Vermutung von Hamkins, Leibman und Löwe.

Die verbleibenden Kapitel der Dissertation untersuchen die sechs erwähnten Logiken genauer. In Kapitel 4 beweisen wir zwei Vermutungen von Nick Bezhanishvili über verallgemeinerte Medvedevlogiken; in Kapitel 5 verknüpfen wir den Begriff der Nerven zu Medvedevlogik; in Kapitel 6 beweisen wir, daß S4.sBA nicht endlich über Cheq axiomatisierbar ist; und schließlich beweisen wir in Kapitel 7, daß Cheq nicht mit fünf oder sechs Variablen axiomatisierbar ist.