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Bornological Algebras in Exotic Derived Categories and Condensed Mathematics

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AZ

To my late grandfather

Affidavit

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare upon oath that I have written the present thesis independently and have not used further resources and aids than those stated in the dissertation.

Hamburg, 11.11.2024

Date

Signature

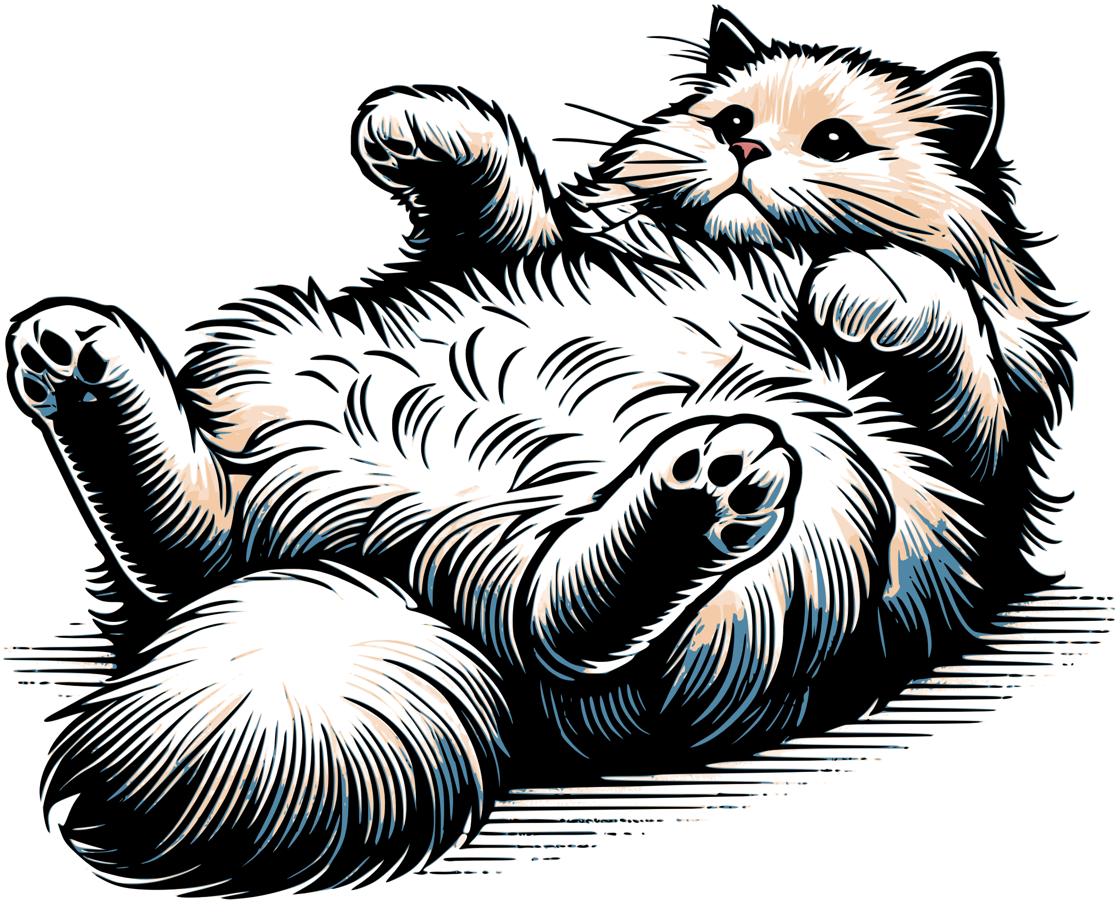
(Thomas Stempfhuber)

Declaration of original work

This thesis contains joint work with Claudia Scheimbauer that appeared as the preprint [SS23]:

> C. Scheimbauer and T. Stempfhuber. Relative field theories via relative dualizability. arXiv:2312.05051, 2023.

Drawn from [SS23] is the last section of the introduction, all of Part III and Appendix D. Almost all of the results have been obtained or generalized after submission of the author's master thesis together with the coauthor, whose results, contributions and ideas are fully acknowledged by the author of this thesis.



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Abstract

This thesis is divided into three parts.

In the first part, we examine the quasi-abelian category of complete bornological spaces and its contraderived categories. We classify all projective complete bornological spaces and show that the category has infinite global dimension. Furthermore, we demonstrate that nuclear Fréchet spaces have finite projective dimension, provided a specific cardinality condition holds. Assuming the continuum hypothesis, we establish the existence of a symmetric monoidal quasi-abelian subcategory of all complete bornological spaces, which includes Fréchet spaces, for which the contraderived category can be defined and identified with the homotopy category of projectives. This result is extended to complete bornological modules over a nuclear Fréchet algebra, with applications to smooth functions and the de Rham algebra on real smooth manifolds.

In the second part, we construct categories of $\mathcal{M}$-complete and liquid condensed vector spaces, drawing on niche notions from classical functional analysis. We demonstrate that Waelbroeck's compactological sets form a category equivalent to quasi-separated condensed sets. Building on this, we extend the idea to vector spaces, utilizing the theory of Smith spaces and compactologies to construct the category of compactological spaces and demonstrate that it is equivalent to $\mathcal{M}$-complete condensed vector spaces. Additionally, we introduce the concept of p-lensed spaces, defined via non-locally convex Smith spaces, and show that their category is equivalent to that of quasi-separated p-liquid spaces. Finally, we prove that the left heart of the quasi-abelian category of p-lensed spaces coincides with the category of p-liquid vector spaces.

In the third part, we investigate relative versions of dualizability designed for relative versions of topological field theories (TFTs), also called twisted TFTs, or quiche TFTs in the context of symmetries. In even dimensions we show an equivalence between lax and oplax fully extended framed relative topological field theories valued in an (∞, N)-category in terms of adjunctibility. Motivated by this, we systematically investigate higher adjunctibility conditions and their implications for relative TFTs. Finally, for fun we explore a tree version of adjunctibility and compute the number of equivalence classes thereof.

Zusammenfassung

Diese Arbeit ist in drei Teile gegliedert.

Im ersten Teil untersuchen wir die quasi-abelsche Kategorie der vollständigen bornologischen Räume und ihre kontraderivierten Kategorien. Wir klassifizieren alle projektiven vollständigen bornologischen Räume und beweisen, dass die Kategorie unendliche globale Dimension hat. Außerdem zeigen wir, dass nukleare Fréchet-Räume eine endliche projektive Dimension haben, sofern eine bestimmte Kardinalitätsbedingung erfüllt ist. Unter der Annahme der Kontinuumshypothese, beweisen wir die Existenz einer symmetrischen monoidalen quasi-abelschen Unterkategorie aller vollständigen bornologischen Räume, die Fréchet-Räume einschließt, für die die kontraderivierte Kategorie definiert und mit der Homotopiekategorie der Projektiven identifiziert werden kann. Dieses Ergebnis wird auf vollständige bornologische Module über einer nuklearen Fréchet-Algebra erweitert, mit Anwendungen auf glatte Funktionen und die de Rham-Algebra auf reellen glatten Mannigfaltigkeiten.

Im zweiten Teil konstruieren wir Kategorien von $\mathcal{M}$-vollständigen und flüssigen kondensierten Vektorräumen, und greifen dabei auf Nischenbegriffe aus der klassischen Funktionalanalysis zurück. Wir zeigen, dass Waelbroecks kompaktologische Mengen eine Kategorie bilden, die äquivalent zu quasiseparierten kondensierten Mengen ist. Aufbauend darauf erweitern wir die Idee auf Vektorräume, indem wir die Theorie der Smith-Räume und Kompaktologien nutzen, um die Kategorie der kompaktologischen Räume zu konstruieren und zu beweisen, dass diese äquivalent zu $\mathcal{M}$-vollständigen kondensierten Vektorräumen ist. Zusätzlich führen wir das Konzept der p-linsierten Räume ein, definiert über nicht lokal-konvexe Smith-Räume, und zeigen, dass ihre Kategorie äquivalent ist zu der der quasiseparierten p-flüssigen Räume. Abschließend beweisen wir, dass das linke Herz der quasi-abelschen Kategorie der p-linsierten Räume mit der Kategorie der p-flüssigen Vektorräume übereinstimmt.

Im dritten Teil untersuchen wir relative Versionen der Dualisierbarkeit, konzipiert für relative Versionen topologischer Feldtheorien (TFTs), auch als twisted TFTs oder quiche TFTs im Kontext von Symmetrien bezeichnet. In geraden Dimensionen zeigen wir eine Äquivalenz zwischen in einer ∞-Kategorie wertigen laxen und oplaxen voll erweiterten gerahmten relativen TFTs in Bezug auf Adjungierbarkeit. Motiviert dadurch untersuchen wir systematisch höhere Adjungierbarkeitsbedingungen und ihre Implikationen für relative TFTs. Schließlich erforschen wir zum Spaß eine Baumversion der Adjungierbarkeit und berechnen die Anzahl der Äquivalenzklassen.

Contents

Introduction

Analytic Geometry

In the 1950s and 60s, algebraic geometry advanced by drawing on ideas and techniques from complex analytic geometry, evolving into a theory of sheaves and schemes. In the following decades, however, the focus shifted predominantly towards algebraic geometry, with foundational progress in the analytic domain appearing to fall behind. A cornerstone of algebraic geometry's success lies in the use of a contravariant functor

$$\mathcal{O}(\cdot): \ \mathsf{Ringed\ Spaces} \ \to \mathsf{Rings}, \quad (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X),$$

which can be restricted to an anti-equivalence with the subcategory of affine schemes, forming essential building blocks for the theory. A challenge in extending this framework to the analytic realm is the need to cover complex, analytic, and smooth geometry in one theory. Additionally, natural analogues for Rings within Functional Analysis, such as Banach or Fréchet rings, do not have the same favorable abelian theory as Rings.

Recently, Ben-Bassat, Kremnitzer and Kelly proposed in [BBKK24] a comparable framework for a generalized approach to analytic geometry:

$$\mathcal{O}(\cdot): \ \mathsf{Analytic\ Spaces} \ \to \mathsf{complete\ Bornological\ Rings}.$$

Moreover, Clausen and Scholze, in [Analytic, Complex], have presented an alternative approach for defining analytic spaces and general analytic rings within the theory of condensed mathematics:

$$\mathcal{O}(\cdot): \ \mathsf{Analytic\ Spaces} \ \to \mathsf{Analytic\ Rings}.$$

Both frameworks support different flavors of analytic geometry, such as smooth geometry over $\mathbb{R}$, complex geometry over $\mathbb{C}$ or rigid geometry over non-Archimedean Banach rings.

Our focus will be on the smooth theory and the Archimedean field $\mathbb{R}$. In this context, both the bornological and condensed setup seek to extend the theory of Fréchet spaces, as smooth functions $\mathcal{C}^\infty(X)$ on manifolds X naturally carry a Fréchet topology. This approach leads to **complete bornological spaces of convex type** in the bornological framework and to the so-called **liquid** analytic ring structure in the condensed setup.

We will explore aspects of both of these theories. Our main contributions are

(a) calculating specific homological dimensions within the category of complete bornological spaces and analyzing the structure of exotic derived categories, and

(b) establishing a comparison between the liquid and bornological theory.

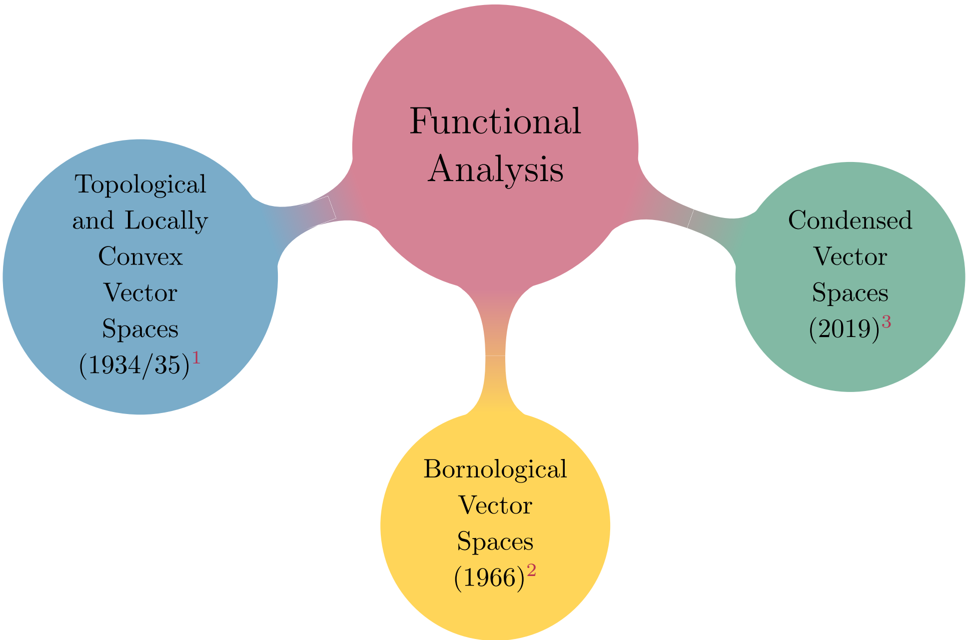


Figure 1.: Three approaches to (modern) Functional Analysis

This thesis consists of three parts. In Part I we will study contraderived categories of bornological spaces and algebras. In Part II we will construct compactological and liquid vector spaces by employing compactologies, Smith spaces and formal quotients. Part III is quite different in flavor and discusses generalized adjunctions in higher categories and applications to topological field theories.

We approach Part I and Part II from the perspective of someone familiar with category theory and homological algebra, but not necessarily the language of exact and quasi-abelian categories. Additionally, all relevant concepts from functional analysis and condensed mathematics that we use are introduced in Chapter 3 and Chapter 6. Part III requires a basic familiarity with the language of ∞-categories, and knowledge of topological field theories is also helpful.

[1] Topological Vector Spaces are primarily attributed to Kolmogoroff [Kol34], while Locally Convex Spaces were studied shortly after by von Neumann [VN35].

[2] The basic idea to focus on bounded sets can already be found in Mackey's 1942 thesis [Mac42], but bornological Spaces were systematically developed starting with Hogbe-Nlend [HN66] and Waelbroeck [Wae67].

[3] Condensed Objects were introduced by Clausen and Scholze [Condensed]. Independently Barwick and Haine [BH19] defined the closely related Pyknotic Objects.

Part I. Exotic Derived Categories of Bornological Algebras

Derived Categories as Invariants of Manifolds

The Rise of Algebraic Geometry was accompanied by huge advances in Commutative and Homological Algebra. Part of this revolution were **derived categories** thought up by Grothendieck and introduced by Verdier [Ver96]. The key idea is that, when constructing cohomology groups of a scheme or manifold, taking homology loses a lot of information, which was still present on the level of chain complexes. Two spaces can have the same cohomology groups, while not being homotopic. However, just not taking homology and purely considering a chain complex given by a triangulation, differential forms or a sheaf is not a well-defined invariant of a manifold. To overcome this problem, one considers all possible complexes with maps up to chain homotopy at once and then identifies them up to quasi-isomorphism. This construction is rather technical, but yields a good invariant: the **derived category**.

Given a complex manifold or a scheme X the specific variant of interest is the **bounded derived category of coherent sheaves** $\mathsf{D}^{\mathrm{b}}(X)$ on X. The strength of this invariant is highlighted by the well-known Reconstruction theorem of Bondal and Orlov [BO01], which states that, if X is a smooth projective variety with an ample or anti-ample canonical bundle, the triangulated category $\mathsf{D}^{\mathrm{b}}(X)$ determines X up to isomorphism. The broader question of which other schemes and manifolds this holds for remains an active area of research.

Our interest lies not in the ordinary derived category, but in its lesser-known relatives, known as exotic derived categories. The so-called **absolute, co- and contraderived categories** were introduced by Positselski [Pos11]. We will focus on the contraderived category, which is just like the ordinary derived category defined as a quotient. While for the latter we divide by all acyclic complexes, we quotient out the considerably smaller subcategory of contraacyclic complexes for the former. This leads to a finer invariant, which in particular can distinguish some complexes which are quasi-isomorphic. In applying the theory to derived categories defined through the de Rham algebra of a smooth manifold, this property is particularly advantageous, as quasi-isomorphic DG algebras may yield significantly different categories of complexes of smooth vector bundles. This is also demonstrated by the fact that the closely related category of cohesive modules, defined over the de Rham algebra of a manifold X, is equivalent to the category of infinity-local systems on X. This result, proven by Block and Smith in [BS09], can be viewed as a generalized form of the Riemann-Hilbert correspondence.

Now, let us turn to our concrete problem. Let X be a real smooth manifold. Our goal is to define the contraderived category $\mathsf{D}^{\mathrm{ctr}}(X)$ by starting with DG modules over the de Rham algebra of X. The central building block of this DG algebra is the space of smooth functions $\mathcal{C}^\infty(X)$ on X (or an open chart). Since we do not intend to restrict our attention to finitely generated modules, equipping the ring of smooth functions with a topology is necessary, allowing us to avoid dealing primarily with non-constructive and non-geometric objects. Fortunately, $\mathcal{C}^\infty(X)$ has a natural Fréchet topology, defined via bounding all derivatives of smooth functions restricted to compact subsets of X. Additionally, being smooth and not just k-differentiable for some $k \in \mathbb{N}$ makes the space $\mathcal{C}^\infty(X)$ nuclear. Nuclear spaces were defined by Grothendieck [Gro55] as topological vector spaces, whose different tensor products agree. We will see that this makes the symmetric monoidal category of nuclear Fréchet spaces very well-behaved. Thus, our framework should contain all nuclear Fréchet spaces and modules.

Now that we have decided to use topological algebras, we have introduced a new problem. The category of Fréchet spaces and linear continuous maps is not abelian. In fact, nearly none of the categories of topological vector spaces, that one considers in classical functional analysis are abelian. However, most are still **quasi-abelian**, which is a special case of **Quillen exact categories**. The latter axiomatize short exact sequences by equipping a category C with a collection of composable pairs of morphisms $\mathscr{E}$, called an **exact structure**. The quasi-abelian categories are a special case of exact categories with very large exact structures. It turns out, that both the category of Fréchet spaces and the full subcategory of nuclear Fréchet spaces are quasi-abelian. We want to incorporate this rich exact structure of nuclear Fréchet spaces into our theory. The main reason is that both modern proposals for analytic geometry, that we saw at the beginning, do include the full quasi-abelian structure of nuclear Fréchet spaces. This is in contrast to the relative theory, where one only considers sequences that split as topological vector spaces, which is used in [Ogn14] and [Blo06].

To define the contraderived category $\mathsf{D}^{\mathrm{ctr}}(X)$ we also need countable products, which is something that can fail in the topological setting. Furthermore, the contraderived category should be well-behaved, in the sense that $\mathsf{D}^{\mathrm{ctr}}(X)$ should be equivalent to the homotopy category of projective objects. Determining for which contraderived categories this holds is a central question in the theory of exotic derived categories. Currently, the most general result requires a finiteness condition on homological dimensions, which we state below in (iv).

This leads us to the following desiderata for a suitable category C:

(i) The category C should have an exact structure $\mathscr{E}$ and a tensor product making it symmetric monoidal,

(ii) The exact category $(\mathsf{C}, \mathscr{E})$ should include nuclear Fréchet spaces as a full subcategory and the exact sequences from their quasi-abelian exact structure should be part of $\mathscr{E}$,

(iii) The category C should have countable products, so that we can define the contraderived category of C,

(iv) We want a simple description of the contraderived categories as complexes of projectives, for which we need the following:

The exact category C has enough projectives and any countable product of projectives has finite projective dimension.	(ΠP)

The obvious choice is to take either Fréchet spaces $\mathsf{Fré}_{\mathbb{R}}$ or their subcategory of nuclear Fréchet spaces $\mathsf{NF}_{\mathbb{R}}$ themselves. Both $\mathsf{Fré}_{\mathbb{R}}$ and $\mathsf{NF}_{\mathbb{R}}$ are quasi-abelian and symmetric monoidal with the completed projective tensor product. They also have countable products.

Unfortunately, Fréchet spaces as a quasi-abelian category do not have enough projectives by a result from Geiler [Gei78]. The category of all locally convex spaces also has this problem [Gei72].

Fortunately, there is one theory that includes nuclear Fréchet spaces and has good homological properties. Well, there are two, but more on that in Part II. Here we will use the category of (complete) bornological vector spaces. While a topological vector space consists of a vector space and a system of open sets, such that addition and scalar multiplication are continuous, a bornological space is a vector space and a system of bounded sets, such that addition and scalar multiplication are bounded.

Bornological Spaces and Disks

For topological vector space the topological information is given by a basis of neighborhoods of the origin. Most of the time one studies locally convex spaces, where these can be chosen to consist of **disks** - convex and balanced subsets. Here balanced means that a subset D is "symmetric" at the origin, that is for all $x \in D$ we have $\lambda x \in D$ for $\|\lambda\| \leq 1$.

In the bornological setting we also want local convexity and we achieve this by requiring, that all bounded subsets are contained in some **bounded disk**. A special case of the latter will be **Banach disks** that can be turned (up to closure) into the unit ball of Banach space.

Definition. A **complete bornological space** is a vector space E equipped with a system of subsets $\mathcal{B}$, called **bounded sets**, such that

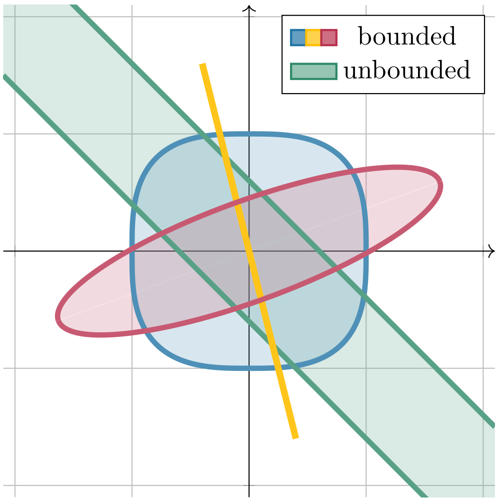


Figure 2.: Some disks in $\mathbb{R}^2$. As with topological vector spaces and neighborhoods, most interesting phenomena only appear in infinite dimensions.

(i) $\mathcal{B}$ contains all singletons and is closed under taking finite unions and subsets,

(ii) for all $B \in \mathcal{B}$ and $\lambda \in \mathbb{K}$ we have $\lambda B \in \mathcal{B}$,

(iii) for all $B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ such that C is a disk and

$$\|x\|_C := \inf \{ \lambda \in \mathbb{R}_{>0} \mid x \in \lambda C \}$$

turns $\operatorname{span}(C)$ into a Banach space.

The category $\mathsf{CBorn}_{\mathbb{K}}$ contains all complete bornological spaces and linear maps that are **bounded**.

With this definition we get a quasi-abelian category $\mathsf{CBorn}_{\mathbb{K}}$ which is complete and cocomplete. After defining the so called **completed projective tensor product** $\hat{\otimes}_\pi$ we also get a closed symmetric monoidal category $(\mathsf{CBorn}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$. We will see that there is a natural functor from nuclear Fréchet spaces to $\mathsf{CBorn}_{\mathbb{K}}$ and that it is fully faithful, exact and strong monoidal.

Homological Dimensions

Evidently, Banach spaces play an important role in the theory of complete bornological spaces. In fact, it was shown in [PS00] that $\mathsf{CBorn}_{\mathbb{K}}$ is a full subcategory of the Ind-completion of $\mathsf{Ban}_{\mathbb{K}}$. They also show that $\mathsf{CBorn}_{\mathbb{K}}$ is quasi-abelian and has enough projective objects. Furthermore, a Banach space in $\mathsf{CBorn}_{\mathbb{K}}$ is projective if and only if it is projective in the quasi-abelian category of Banach spaces and bounded maps $\mathsf{Ban}_{\mathbb{K}}$. The latter were shown by Köthe [Köt66] to be exactly the

Banach spaces of the form $\ell^1(Y)$ for some set Y. The proof that $\mathsf{CBorn}_{\mathbb{K}}$ has enough projectives [PS00] uses coproducts of $\ell^1(Y)$. We prove that these spaces are in fact all projective objects of $\mathsf{CBorn}_{\mathbb{K}}$.

Theorem 4.3.11. *Every projective object in $\mathsf{CBorn}_{\mathbb{K}}$ is isomorphic to*

$$\coprod_{i\in I} \ell^1(Y_i)$$

for some family of sets Y_i, $i \in I$.

Another consequence of the prominent role of Banach spaces is the following.

Theorem 4.3.11. *The global dimension of $\mathsf{CBorn}_{\mathbb{K}}$ is ∞.*

Clearly, the homological properties of $\mathsf{Ban}_{\mathbb{K}}$ and $\mathsf{CBorn}_{\mathbb{K}}$ are closely related, and the infinite global dimension also means that in order to address (iv) we need to study projective dimensions of countable products themselves. This is already a very difficult problem in the purely algebraic situation. Additionally, homological dimensions of Banach spaces and topological vector spaces in general are notoriously difficult.

> *"The words still resonate in our minds: a long, large, possibly endless complex to represent a Banach space X. Yes, why not. With the only added problem that (almost) nothing, zero, zip, zilch, nada is known about even the simplest question: is actually infinite the projective presentation?" (J.M.F. Castillo [Cas21])*

There is very little literature on projective dimensions of Banach spaces [Wod94, SCG21b] and the problem seems unapproachable. In order to overcome this difficulty we use a trick known as Mitchell's Theorem.

Theorem. *[Mitchell [Mit73]] Let C be a quasi-abelian category with exact products. Let I be a directed poset of cardinality $\aleph_d$ with $d \in \mathbb{N}$. Then Rlim^n vanishes for all $n > d + 1$.*

We will show that every Fréchet space in $\mathsf{CBorn}_{\mathbb{K}}$ has a basis of bounded sets with cardinality $\mathfrak{d}$. The latter is called **dominating number** and can, just like the cardinality of the continuum, be quite small or very large only assuming ZFC. After overcoming some technical difficulties concerning derived and homotopy limits in $\mathsf{CBorn}_{\mathbb{K}}$ we can use this fact and Mitchell's Theorem to show the following.

Theorem 4.3.17. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a nuclear Fréchet space and assume that the dominating number satisfies $\mathfrak{d} \leq \aleph_d$ for some $d \in \mathbb{N}$. Then the projective dimension of F is $d + 1$ or smaller.*

Small Complete Bornological Spaces

While Theorem 4.3.17 is a nice result it does not address whether $\mathsf{CBorn}_{\mathbb{K}}$ satisfies (ΠP) stated in (iv). In fact, we will not answer that question in this thesis and can only wonder whether there is a singular true answer purely inside ZFC.

Instead, we restrict our attention to a full subcategory of $\mathsf{CBorn}_{\mathbb{K}}$ of spaces, that do not have too many bounded sets.

Definition. Let κ be a regular cardinal. A complete bornological space $(E, \mathcal{B})$ is $\leq\kappa$**-small**, if there is a subset $\mathcal{C} \subset \mathcal{B}$ with cardinality $\mathrm{card}(\mathcal{C}) \leq \kappa$, such that for every bounded set $B \in \mathcal{B}$ there is a bounded set $C \in \mathcal{C}$ with $B \subset C$. This defines the full subcategory of $\mathsf{CBorn}_{\mathbb{K}}$ of $\leq\kappa$-small spaces denoted by $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$.

We discuss the properties of $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ in Section 4.4 and with another argument involving Mitchell's Theorem we can show that $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ satisfies (iv). However, we do need a cardinality assumption such as the Continuum Hypothesis or a slightly weaker variant.

Theorem (see Section 4.4)**.** *The category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ is quasi-abelian and has enough projectives. It is symmetric monoidal, when equipped with the completed projective tensor product. Assume that $2^{\aleph_0} \leq \aleph_d$. Then there is an exact, fully faithful and strong monoidal functor $\mathsf{NF}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$. Furthermore, $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ has countable products, and countable products of projective objects have finite projective dimension.*

We can define the contraderived category of complete bornological DG vector spaces $\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ and use the finiteness of products of projectives to prove the following.

Theorem 4.4.15. *Assume that $2^{\aleph_0} \leq \aleph_d$. Consider $\aleph_d$-small complete bornological DG vector spaces. The composition of functors*

$$\mathsf{Ho}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}{}_{\mathrm{proj}}) \to \mathsf{Ho}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}) \to \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d})$$

is an equivalence of categories. Here $\mathsf{Ho}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}{}_{\mathrm{proj}})$ is the homotopy category of DG vector spaces in $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$, whose underlying complexes consist of projective objects in every degree. In particular, every $\aleph_d$-small complete bornological DG vector spaces is contraequivalent to a DG vector space M with

$$M^\sharp \cong \coprod_{i\in I} \ell^1(X_i)[s_i],$$

where $\mathrm{card}(I) \leq \aleph_d$, $\{X_i\}_{i\in I}$ is some family of sets and $s_i \in \mathbb{Z}, i \in I$ are some integers denoting shifts.

Bornological Algebras

Since $\mathsf{CBorn}_{\mathbb{K}}$ is a symmetric monoidal quasi-abelian category, we can easily define complete bornological (DG) algebras and complete bornological (DG) modules. Additionally, the exact structure of $\mathsf{CBorn}_{\mathbb{K}}$ lifts to the module categories.

In Chapter 5 we will consider two different exact structures on the category of modules A-Mod over a $\mathsf{CBorn}_{\mathbb{K}}$-algebra A. The quasi-abelian exact structure on $\mathsf{CBorn}_{\mathbb{K}}$ extends to the collection of short exact sequences

$$\mathfrak{max} = \left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \;\middle|\; f, g \text{ morphisms in } A\text{-}\mathsf{Mod},\ g = \ker f, f = \operatorname{coker} g \right\}.$$

We also consider the smaller exact structure of split exact sequences

$$\mathfrak{split} = \left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \;\middle|\; (f, g) \in \mathfrak{max}, (f, g) \text{ splits in } \mathsf{CBorn} \right\}.$$

The latter is what is studied in the context of Fréchet algebras in the works of Helemskii [Hel90, Hel89] and used by Block in [Blo06]. The advantage is that it does not inherit any difficulties of the quasi-abelian structure on $\mathsf{CBorn}_{\mathbb{K}}$ or $\mathsf{Ban}_{\mathbb{K}}$. In fact, for $A = \mathcal{C}^\infty(X)$ for a real smooth d-dimensional manifold we even get finite global dimension.

Theorem (Ogneva [Ogn86, Ogn14])**.** *The category of Fréchet modules over the Fréchet algebra $\mathcal{C}^\infty(X)$ with the $\mathfrak{split}$-exact structure has global dimension $d = \dim(X)$.*

This allows us to prove the analogous result for complete bornological $\mathcal{C}^\infty(X)$-modules with the $\mathfrak{split}$-exact structure and then extend it to the de Rham algebra $\mathbf{\Omega}(X)$.

Theorem 5.2.8. *Let X be a real smooth n-dimensional manifold. Consider complete bornological DG modules over the de Rham algebra $\mathbf{\Omega}(X)$ with the $\mathfrak{split}$-exact structure. The composition of functors*

$$\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}_{\text{proj}}) \to \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)) \to \mathsf{D}^{\text{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$$

is an equivalence of categories.

In particular, every DG module over the de Rham algebra on X is contraequivalent to a DG module, whose underlying complex consists of $\mathfrak{split}$-projective objects. This also shows, that $\mathsf{D}^{\text{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$ is equivalent to Blocks category of unbounded quasi perfect complexes $q\mathcal{P}^u_{\mathbf{\Omega}(X)}$ [Blo06].

One of the defining features of this $\mathfrak{split}$-framework is that the homological theory of the underlying category of vector spaces is trivial. This allows one to obtain Ogneva's theorem and Theorem 5.2.8. As we are interested in the quasi-abelian setup, used in the works of Ben-Bassat, Kremnitzer, Kelly [BBKK24] and Clausen, Scholze [Analytic, Complex], we now turn to the $\mathfrak{max}$-exact structure.

Theorem 5.3.2. *Let X be a real smooth n-dimensional manifold. Let $M \in \mathcal{C}^\infty(X)\text{-}\mathsf{Mod}$ be a $\mathsf{CBorn}_{\mathbb{K}}$-module, that is nuclear and Fréchet . Assume $\mathfrak{d} = \aleph_d$. Then*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(M) \leq n + d + 1.$$

The statement of Theorem 5.3.2 is not strong enough to proceed as in the $\mathfrak{split}$-case. In fact, the infinite global dimension of $\mathsf{CBorn}_{\mathbb{K}}$ from Theorem 4.3.11 shows that the same approach as in the $\mathfrak{split}$-setting can not work.

However, we have shown that $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ satisfies (ΠP) and we can directly lift the condition to $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ module categories over nuclear Fréchet algebras.

Theorem 5.3.6. *Let A be a nuclear Fréchet $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$-algebra. Assume that $2^{\aleph_0} = \aleph_d$. Then $(\mathsf{A\text{-}Mod}^{\leq\aleph_d}, \mathfrak{max})$ satisfies (ΠP).*

The nuclearity of A implies flatness, which is required. More interestingly A being metrizable is what allows us to commute countable products with tensor products, which we use in the proof of Theorem 5.3.6. The analogous algebraic result requires finite presentation for this property.

Finally, we prove our main theorem.

Theorem 5.3.10. *Assume that $2^{\aleph_0} \leq \aleph_d$. Consider $\leq\aleph_d$-small complete bornological DG modules over the de Rham algebra $\mathbf{\Omega}(X)$. The composition of functors*

$$\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max}_{\mathrm{proj}}) \to \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}) \to \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max})$$

is an equivalence of categories.

Overview of Part I

Chapter 1, Chapter 2 and Chapter 3 provide mostly background needed in Chapter 4 and Chapter 5. In Chapter 1 we will review Quillen exact structures and quasi-abelian categories and their homological algebra. In Chapter 2 we introduce Positselski's exotic derived categories in the setting of exact structures and discuss the central conditions (ΣI) and (ΠP) that one needs to check for the semiorthogonal decomposition Theorem 2.2.12. Chapter 3 is a review of all the material from classical Functional Analysis necessary for this thesis.

We first recall the basic theory of complete bornological spaces in Chapter 4. We classify all projective objects in Theorem 4.3.10 and show in Theorem 4.3.17, that nuclear Fréchet spaces have finite projective dimension under a cardinality assumption. In Section 8.2 we construct a subcategory of complete bornological spaces, that under the continuum hypothesis contains all nuclear Fréchet spaces and satisfies (ΠP). This allows us to get a semiorthogonal decomposition for the contraderived category in Theorem 4.4.15. Finally, in Chapter 5 we use the results from Chapter 4 to study contraderived categories over smooth functions and the de Rham algebra with two different exact structures. We prove semiorthogonal decomposition for the split-exact and the quasi-abelian case.

In Appendix A we recall cardinal numbers and in particular the dominating number $\mathfrak{d}$, that is used in Chapter 4 and Chapter 5. In Appendix B we review the definition of Ind-categories and some variants used in this thesis.

Part II. A Comparison of Bornological and Condensed Spaces

One of the major challenges in Part I was identifying a suitable and well-behaved category that includes nuclear Fréchet spaces and possesses a robust theory of homological algebra. We addressed this issue by moving beyond the classical framework of topological and locally convex vector spaces, opting instead for bornological spaces. However, this is just one possible solution. A more modern approach, which aims to reformulate functional analysis for a better interaction with homological algebra, is called *Condensed Mathematics*. It was recently introduced by Clausen and Scholze in [Condensed]. The lecture notes [Analytic] and [Complex] expand the framework and demonstrate applications in Analytic and Complex Geometry.

The core idea of condensed mathematics is to define **condensed sets** as functors

$$\begin{Bmatrix} \text{suitable site} \\ \text{of test spaces} \end{Bmatrix}^{\mathrm{op}} \to \mathsf{Set}$$

subject to some sheaf conditions. By replacing Set with the category of vector spaces Vect one can also define **condensed vector spaces**.

In Part II we will show how one can construct condensed sets and complete condensed vector spaces with the tools from topological and bornological vector spaces. The key ingredient is that of a **compactology**. While a bornology encodes bounded sets in a space, a compactology is a compatible system of compact sets.

Definition (Waelbroeck [Wae67], Buchwalter [Buc68])**.** A compactological set is a set X equipped with a system of subsets $\mathcal{B}$, called a **compactology**, such that

(i) $\mathcal{B}$ is a bornology,

(ii) each $B \in \mathcal{B}$ has a topology and all of these are compatible,

(iii) for each $B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ with $B \subset C$ and the topology on C is compact Hausdorff.

Given a compactological set $(X, \mathcal{B})$ we can equip X itself with a topology. It is given as the final topology of all $B \in \mathcal{B}$ and we will see in Section 7.1 that it is compactly generated and weak Hausdorff. These two properties of topologies are well known from Steenrods "convenient category of topological spaces" $\mathsf{cgWHaus}$. The morphisms of compactological sets are defined to be bounded and continuous maps. We get a category $\mathsf{CompSet}$, which is complete and cocomplete as we will see in Section 7.3. While, every space in $\mathsf{cgWHaus}$ arises as the final topology of a compactological set, there can be many different compactological sets with the same final topology. The extra objects in $\mathsf{CompSet}$ play an important role in the theory, especially in improving the behavior of colimits. Instead of just remembering the colimit topology, $\mathsf{CompSet}$ encodes all the information of the underlying diagram. This leads to Theorem 7.4.4, where we only get the so called **quasi-separated** condensed sets.

All of the following comparisons with condensed sets involve some cardinality restrictions, that we will ignore in this introduction.

Theorem 7.4.4. *There is an equivalence of categories*

$$\mathsf{CompSet} \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}) \quad \cong \quad \mathsf{qsCond}(\mathsf{Set}).$$

of

- *compactological sets* $\mathsf{CompSet}$*,*
- *formal filtered colimits of compact Hausdorff spaces with injections as transition maps* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus})$*, and*
- *quasi-separated condensed sets* $\mathsf{qsCond}(\mathsf{Set})$*.*

Smith Spaces as Building Blocks

Our goal is to get a comparison as in Theorem 7.4.4 for categories of bornological and condensed vector spaces. We also add the requirement, that the vector spaces should be complete in an appropriate way. The first hurdle is to find a suitable replacement for compact Hausdorff spaces. The answer was already stated in lecture 3 of [Analytic]: **Smith spaces**. The idea is to give a vector space S a topology that is purely determined by a **universal compact** disk K. We take $K, 2K, 3K, \ldots$ as in Fig. 3 and define a subset $A \subset S$ to be closed if and only if $A \cap nK$ is closed for all $n \in \mathbb{N}$.

The category of Smith spaces is anti-equivalent to the category of Banach spaces via **Stereotype duality**. The general notion of Stereotype duals is due to Akbarov [Akb03], but in the special case of Smith and Banach spaces the dual topologies are given by the compact-open topologies. We get an anti-equivalence

$$\mathsf{Ban}_{\mathbb{K}} \to \mathsf{Smi}_{\mathbb{K}}, \quad B \mapsto B^{\vee}; \qquad \mathsf{Smi}_{\mathbb{K}} \to \mathsf{Ban}_{\mathbb{K}}, \quad S \mapsto S^{\vee},$$

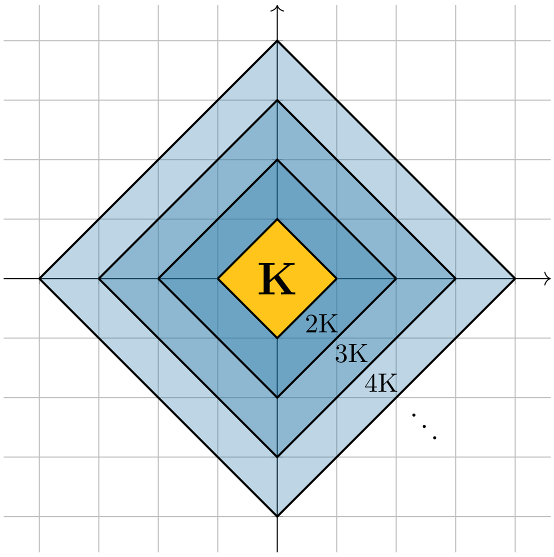


Figure 3.: Colimit diagram for a Smith space; K is the universal compact set.

that already appeared in the work of Freundlich Smith [Smi52].

While not strictly necessary for the comparison with condensed vector spaces we are interested in how Smith and Banach spaces interact with bornologies. Given a topological vector space V there are two natural choices for extracting a bornology. The first one is the so called **von Neumann bornology** $\mathrm{vN}(V)$ that consists of all bounded sets in V. The second is the **precompact bornology** $\mathrm{Cpt}(v)$, which is given by all totally bounded sets. For every Smith space S there is a Banach space B such that $\mathrm{vN}(B) = \mathrm{Cpt}(S)$, but the converse is only true if B has a predual. We will elaborate on this point and discuss how Banach and Smith spaces are related in Section 8.1.2 and Section 8.2. In particular, Theorem 8.1.17 gives a characterization of Smith spaces as k-ifications of weak*-topologies.

Compactological Vector Spaces

Given a bounded disk K in a bornological space, we say that K is a **Smith disk** if there is a **Smith topology** on K that turns $\operatorname{span} K$ into a Smith space with universal compact K.

Definition. A **compactological space** is a vector space E with a compactology $\mathcal{B}$, such that

(i) $(E, \mathcal{B})$ is a bornological vector space,

(ii) for every $B \in \mathcal{B}$ there is a $K \in \mathcal{B}$, such that $B \subset K$ and K is the universal compact set of a Smith space.

A morphism of compactological spaces is a linear bounded map, that is continuous with respect to the final topologies. We denote the category of compactological spaces and morphisms with $\mathsf{Comp}_{\mathbb{K}}$.

After discussing the quasi-abelian structure of compactological spaces we arrive at the desired comparison with the condensed world, where the counterpart is given by **$\mathcal{M}$-complete** vector spaces. As before, we set aside the details of the cardinality restrictions.

Theorem 8.3.6. *There is an equivalence of categories*

$$\mathsf{Comp}_{\mathbb{R}} \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{R}}) \quad \cong \quad \mathcal{M}\mathsf{Cond}(\mathsf{Vect})$$

of

- *compactological* $\mathbb{R}$*-vector spaces* $\mathsf{Comp}_{\mathbb{R}}$*,*
- *formal filtered colimits of Smith* $\mathbb{R}$*-vector spaces with injections as transition maps* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{R}})$*, and*
- *condensed* $\mathcal{M}$*-complete* $\mathbb{R}$*-vector spaces* $\mathcal{M}\mathsf{Cond}(\mathsf{Vect})$*.*

Non-Convexity and quasi-separated Liquid Vector Spaces

While $\mathcal{M}$-complete vector spaces do provide a nice setting for locally convex functional analysis, they are more a curiosity in [Analytic], that shows up as a first failed attempt on the way to the desired category of p-liquid vector spaces. The latter form a nice abelian category and, more importantly, constitute an analytic ring structure on $\mathbb{R}$. Our goal now is to construct p-liquid vector spaces using bornologies, starting with the quasi-separated ones.

The big difference of $\mathcal{M}$-complete vector spaces and quasi-separated liquid vector spaces is that the latter include non-locally convex spaces. The explanation, why their inclusion is necessary goes back to the late 1970s, where Roberts [Rob77a], Kalton [Kal78], and Ribe [Rib79] showed that Banach spaces are not closed under extensions in all topological vector spaces. However, all of the extensions are captured by the following generalization.

Definition. Let V be a vector space and $0 < p \leq 1$. A **p-norm** on V is a map $\langle\!\langle \cdot \rangle\!\rangle_p : V \to \mathbb{R}_{\geq 0}$, such that

(i) $\langle\!\langle \lambda v \rangle\!\rangle_p = |\lambda|^p \langle\!\langle v \rangle\!\rangle_p$, for all $\lambda \in \mathbb{K}$ and $v \in W$,

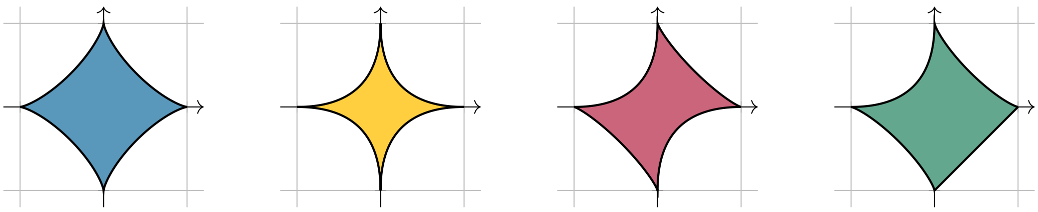
(ii) $\langle\!\langle v + w \rangle\!\rangle_p \leq \langle\!\langle v \rangle\!\rangle_p + \langle\!\langle w \rangle\!\rangle_p$ for all $v, w \in W$.

(iii) $\langle\!\langle v \rangle\!\rangle_p = 0$ if and only if $v = 0$, for all $v \in W$.

A complete topological vector space Q with its topology induced by a p-norm $\langle\!\langle \cdot \rangle\!\rangle_p$ is called a **p-Banach space**.

Kalton showed [Kal81] that any extension of p-Banach spaces is still a q-Banach space for all $0 < q < p$, with $p = 1$ being the special case of extensions of Banach spaces. Consequently, both for p-liquid vector spaces and for our definition the spaces will not be fully non-convex, but rather q-convex for all $0 < q < p$ in some sense. The role of disks, which are convex and balanced subsets, gets now taken by what we call **q-lenses**, defined as q-convex and balanced subsets.

Definition. Let $0 < p \leq 1$. A **p-lens** is a p-convex and balanced subset. A **p-Smith space** S is a Hausdorff topological vector space, whose topology is the colimit topology of $K, 2K, 3K, \ldots$ of a given **universal compact** p-lens K.



$\frac{3}{4}$-lens $\frac{1}{2}$-lens $\frac{1}{2}$-lens $\frac{1}{2}$-convex, but not balanced

Figure 4.: p-convex subsets of $\mathbb{R}^2$

From now on we fix a convexity parameter $0 < p \leq 1$. Similar to before we use bornological spaces as base to build upon, but with the difference that the bornology does not need to be of convex type. Instead of one basis of disks, we require the spaces to have a basis of q-lenses for all q smaller than p. After defining **p-Smith lenses** as p-lenses that can be made into p-Smith spaces when equipped with a **p-Smith topology** we can define non-locally convex compactological spaces.

Definition. A **compactological p-lensed space** is a vector space E with a compactology $\mathcal{B}$, such that

(i) $(E, \mathcal{B})$ is a (non-convex) bornological vector space,

(ii) for every $B \in \mathcal{B}$ and $0 < q < p$, there is a $K \in \mathcal{B}$, such that $B \subset K$ and K is a q-Smith lens.

A morphism of compactological p-lensed spaces is a linear bounded map, that is continuous with respect to the final topologies. We denote the category of compactological p-lensed spaces and morphisms with $\mathsf{CLens}_{\mathbb{K}}^{<p}$.

This gives us the quasi-separated p-liquid vector spaces. Once again, we omit any of the cardinality restrictions.

Theorem 9.3.3. *There is an equivalence of categories*

$$\mathsf{CLens}_{\mathbb{K}}^{<p} \quad \cong \quad \mathsf{qsLiq}_p$$

of

- *compactological p-lensed spaces* $\mathsf{CLens}_{\mathbb{K}}^{<p}$ *and*
- *condensed quasi-separated p-liquid* $\mathbb{R}$*-vector spaces* qsLiq_p.

All Liquid Vector Spaces as Formal Quotients

An important part of the appeal of liquid vector spaces is that they have non-separated quotients. Even if $V \hookrightarrow W$ is a dense embedding of topological or condensed vector spaces, the object W/V has an interesting condensed structure. The category of compactological p-lensed spaces does not share this feature and only quotients by closed subspaces are well-defined. To rectify this shortcoming of $\mathsf{CLens}_{\mathbb{K}}^{<p}$ we will use another idea of Waelbroeck [Wae72] , [Wae86] – quotient spaces. We will call them **formal quotients** and in their modern form they are defined in [Weg17]. Formal quotients of quasi-abelian categories agree with Schneiders [Sch99] **Left Heart** construction. The latter is defined as the heart of one of the two canonical t-structures on the derived category of a quasi-abelian category C and denoted by $\mathcal{LH}(\mathsf{C})$. In [Sch99] they use it to define $\mathcal{W}$ as the left heart of Ind-seminormed spaces $\mathsf{Ind}(\mathsf{SemiN}_{\mathbb{K}})$.

> *"We feel that $\mathcal{W}$-sheaves provides a convenient notion of topological sheaves which is suitable for applications in algebraic analysis. [...] Note that, in a private discussion some time ago, C. Houzel, conjectured that [$\mathcal{W}$][4] should be a good candidate to replace the category of locally convex topological vector spaces in problems dealing with sheaves and cohomology. He also suggested the name $\mathcal{W}$ since he expected this category to be related to the category of quotient bornological spaces introduced by Waelbroeck." (JP.Schneiders [Sch99])*

We will show that this insight of Houzel and Schneiders was well-founded. Formal quotients or equivalently the left Heart of $\mathsf{CLens}_{\mathbb{K}}^{<p}$ allow us to move from quasi-separated to all liquid vector spaces. As before, cardinality restrictions are left entirely unmentioned.

Theorem (Theorem 9.2.16 and Theorem 9.3.9)**.** *The category of compactological p-lensed spaces* $\mathsf{CLens}_{\mathbb{K}}^{<p}$ *is quasi-abelian and its left Heart is equivalent to the abelian category of p-liquid vector spaces* Liq_p.

[4] originally "a category defined through the formula in Corollary 3.2.22". The corollary gives the formula $\mathcal{W} \cong \mathcal{A}dd(\mathcal{P}, \mathcal{A}b_V)$. See Proposition 2.1.14 in [Sch99].

It is worth noting that, to achieve this comparison, we have synthesized several older and quite niche ideas from functional analysis, dating back decades. These include Smith spaces, introduced by Freundlich Smith [Smi52] in 1952, and bornologies, compactologies, and quotient spaces, developed by Waelbroeck [Wae67, Wae71, Wae72, Wae86] and Buchwalter [Buc68, Buc69] in the 1960s. Additionally, we incorporate the approach of using p-convexity to address the extension problem, as explored by Kalton [Kal78], Roberts [Rob77b], and Ribe [Rib79] in the late 1970s. When combined, these concepts lead to the definition of our quotient p-lensed spaces, which align with the modern framework of p-liquid vector spaces.

Overview of Part II

In Chapter 6 we give a short introduction to condensed mathematics and collect foundational definitions and results.

In Chapter 7 we recall the notion of compactologies and prove basic results about the category of compactological sets. We show that compactological sets are equivalently essentially monomorphic Ind-compact Hausdorff spaces or quasi-separated condensed sets. In Chapter 8 we recall the definition of Smith spaces and give a characterization of Smith topologies via k-ifications of weak*-topologies. We introduce compactological vector spaces and show that they form a quasi-abelian category. We prove our first main theorem of this chapter (Theorem 8.3.6), which is the equivalence of compactological spaces, Ind-Smith spaces and $\mathcal{M}$-complete vector spaces. We move from convexity to p-convexity and generalize Smith spaces and compactological spaces to p-Smith and p-lensed spaces in Chapter 9. We show that p-lensed spaces are equivalent to quasi-separated p-liquid spaces. Finally, we use the construction of formal quotients/the left Heart to prove the second main theorem (Theorem 9.3.9), which establishes an equivalence of quotient p-lensed spaces and p-liquid vector spaces.

In Appendix C we provide background for compact generation and weak Hausdorff spaces from classical topology. We also give proofs and counterexamples how weak Hausdorff fits in the separation hierarchy and how the different definitions of compactly generated spaces in the literature are related. For convenience there is also Appendix A on cardinals and Appendix B on Ind-categories, both of which are used throughout this chapter.

Part III. Relative Field Theories via Relative Dualizability

The notion of a functor having an adjoint is a property describing many phenomena in all areas of mathematics. Replacing the 2-category of categories, functors, and natural transformations by an arbitrary bicategory, or, even more generally, an (∞, n)-category, we obtain a notion which generalizes – simultaneously – finiteness conditions for a module (finitely generated and projective), k-handles of a manifold and handle cancellation, duals of vector spaces and perfect complexes, and smooth and proper DG-algebras or DG-categories.

In fact, we have a hierarchy of properties, which we will investigate in this article. We start with a 1-morphism in our higher category $\mathcal{C}$ and ask for this 1-morphism to have a left adjoint (or to have a right adjoint). This adjoint comes together with a unit and counit 2-morphism exhibiting adjunctibility. Now we can ask for these 2-morphisms themselves to have a left adjoint (or to have a right adjoint). In an (∞, N)-category, we can repeat this procedure n times, arriving at **n-times left (or right) adjunctibility**.

In this article, our first goal was to demonstrate that for even n, opting for always either left or right adjoints at every level leads to the same notion. This result is established through an interchange lemma generalizing such an interchange property stated in Lurie's seminal article on the Cobordism Hypothesis [Lur09].

While this result is a purely higher categorical result, our motivation for proving this comes from the classification of a **relative** version of fully extended topological field theories [FT14, ST11]. Such a relative notion of field theory was first introduced by Stolz and Teichner in [ST11] as **twisted field theories** to capture the behavior of Segal's conformal field theories [Seg04] and anomalies: rather than assigning a number to a closed top dimensional manifold viewed as an endomorphism of the ground field k, a twisted field theory should capture choosing elements in a vector space or line, viewed as a morphism $k \to V$. Here the vector space may vary depending on the choice of manifold, hence is a once-categorified field theory itself. Freed and Teleman's **relative field theories** [FT14] are meant to capture the same idea, namely, they should be a "homomorphism" of field theories $\mathbb{1} \to \alpha$, where α often (but not always) is the truncation of a field theory of one dimension higher.

The mathematical framework suggested in [ST11] to capture this behavior is that of a **symmetric monoidal lax or oplax natural transformation**

$$\mathsf{Bord}_n \underset{T}{\overset{\mathbb{1}}{\rightrightarrows}} \mathsf{C}, \qquad \Downarrow \tag{0.1}$$

between symmetric monoidal functors. Here C is a bicategory, $\mathbb{1}$ is the trivial functor, and T is called twist. In [JFS17] this notion was extended to the higher categorical setting. Moreover, using the Cobordism Hypothesis, a classification of twisted fully extended topological field theories was given precisely in terms of left (or right) n-times adjunctibility. Our main result shows that in even dimensions, these flavors occur simultaneously; while in odd dimensions, they differ.

Finally, our interchange result can be generalized: so far we have chosen either left at every stage, or right at every stage. Instead, we may choose to alternate in a chosen pattern. Our interchange lemma shows that in any dimension there are exactly two equivalence classes of such notions of adjunctibility. In even dimensions left and right n-times-adjunctibility happen to capture just one of the classes, whereas in odd dimensions, they lie in different classes. One may ask for field theoretic interpretations of all of these notions, and we discuss this in detail. In particular, we answer the question which notion of adjunctibility is the "correct" one for relative field theories: we should ask for an **oplax** natural transformation, and this is the only choice.

Before delving into the details of our categorical results, let us mention that these relative versions of (T)FTs have recently been ubiquitous: they appear as "quiche" field theories [5] in the context of (possibly non-invertible) symmetries in [FMT22]. Using the framework developed in [JFS17] which is central here, various such examples have recently been worked out, e.g. in [Dyk23a, Dyk23b, Haï23].

Overview of our higher categorical Results

We now give a more detailed overview of our higher categorical results.

Our first main result, which was the motivation for this article, is that in an (∞, N)-category C and for even $n > 0$, n-times left adjunctibility and n-times

[5] Freed–Moore–Teleman introduced the term **quiche** as being half of a **sandwich**. We would like to point out here that a quiche is not half of a sandwich as was already observed by the authors themselves. In German and Russian the word **Butterbrot** `https://en.wikipedia.org/wiki/Butterbrot` is precisely half of a sandwich, namely a piece of bread with a topping of butter (and in some regions some extra stuff). We refrain from introducing yet another different name.

right adjunctibility of a k-morphism coincide. This follows from an interchange result, Lemma 11.1.3.

Theorem 11.2.9. *Let $n \geq 2$ be even. Then a k-morphism f is n-times left adjunctible if and only if f is n-times right adjunctible.*

In the odd case, this is not true. In fact, if a k-morphism is both n-times left and n-times right adjunctible, it already is n-adjunctible (Theorem 11.2.10).

One may now ask whether in even dimensions there is a different notion of adjunctibility which is not equivalent to either purely right or purely left adjunctibility. Indeed, rather than choosing the existence of right adjoints of (co)units at each level (or always left), we can use a general "mixed" sequence of "right" and "left" to define a notion of "mixed" adjunctibility. We call such a sequence a **dexterity function** a^n and adjunctibility based thereupon a^n-adjunctibility. Given two such functions $a^n, b^n : \{1, 2, \dots, n\} \rightrightarrows \{L, R\}$, their common adjunctibility depends on the parity of

$$p(a^n, b^n) \;\coloneqq\; |(a^n)^{-1}(L)| - |(b^n)^{-1}(L)|\,.$$

The Theorem above then generalizes to

Theorem (Theorem 11.2.6)**.** *If $p(a^n, b^n)$ is even, then f is a^n-adjunctible if and only if f is b^n-adjunctible.*

In fact, the 2^n a priori different notions of mixed adjunctibility reduce to just two: for every n there are **exactly** two equivalence classes of "mixed" adjunctibility, where we call two dexterity functions a^n, b^n and their notions of mixed adjunctibility equivalent if the following holds: a morphism is a^n-adjunctible if and only if it is b^n-adjunctible. We prove that this is the case if and only their parity $p(a^n, b^n)$ is even. This is Corollary 11.2.7 and Remark 11.2.8.

If their parity is odd, we obtain a stronger condition.

Theorem (Theorem 11.2.6 and Theorem 11.2.10)**.** *If $p(a^n, b^n)$ is odd and f is both a^n-adjunctible and b^n-adjunctible, then f is n-adjunctible.*

Along the way we prove some cute results useful to proving various adjunctibility results.

For a bicategory, if we reverse only 1-morphisms, left adjoints become right adjoints and vice versa. In Section 11.3 we consider various opposite categories, reversing only some of the morphisms, in relation to mixed adjunctibility. More precisely, for two dexterity functions a^n, b^n we can precisely say for which j we have to reverse j-morphisms in C so that a^n-adjunctibility translates to b^n-adjunctibility.

Our main result suggests that for n odd, there should be a morphism f which is n-times left but not right adjunctible. In Chapter 12, Example 12.2.8 we construct an example illustrating this for $n = 3$ in a higher Morita category.

Application to relative and twisted Topological Field Theories

Relative versions of field theories should satisfy two desiderata, which we elaborate on in Chapter 13. The first, already mentioned above, is that for top-dimensional closed manifolds, we should choose an element in a vector space (for instance a line),

$$\mathbb{1} \to T(M)\,. \tag{0.2}$$

The second is that if $T \equiv \mathbb{1}$, we should recover the usual notion of "absolute" field theory valued in the looping $\Omega\mathcal{C}$,

$$\mathscr{Z}\colon \mathsf{Bord}_n^{fr} \to \Omega\mathcal{C}\,. \tag{0.3}$$

The framework suggested in [ST11] using symmetric monoidal lax or oplax natural transformations unfortunately does not satisfy both at the same time: As was shown in [JFS17], oplax field theories satisfy the former, but not the latter; whereas lax field theories satisfy the latter, but not the former.

The main goal of our categorical results was to compare the oplax and lax variant of relative field theories, and to see whether we can modify the framework to obtain both (0.2) and (0.3). Moreover, Freed–Teleman in [FT14] did not make precise which framework they would like to use, and for instance in [FT21] suggest a stronger condition in terms of adjunctibility. In this article we study the various options for the framework and compare them to this latter stronger condition.

Recall that an oplax (respectively lax) natural transformation is a functor

$$\mathsf{B} \longrightarrow \mathsf{C}^{\rightarrow} \quad \text{respectively} \quad \mathsf{B} \longrightarrow \mathsf{C}^{\downarrow}\,,$$

where $\mathsf{C}^{\rightarrow}$ and $\mathsf{C}^{\downarrow}$ are certain arrow higher categories governing the desired diagrammatics. Hence an oplax (respectively lax) relative field theory is a symmetric monoidal functor

$$\mathsf{Bord}_n \longrightarrow \mathsf{C}^{\rightarrow} \quad \text{respectively} \quad \mathsf{Bord}_n \longrightarrow \mathsf{C}^{\downarrow}\,.$$

In the fully extended framed case (using Bord_n^{fr}) and under the assumption of the **Cobordism Hypothesis** from [Lur09], lax and oplax twisted topological field theories are classified by n-dualizable objects in the target categories $\mathsf{C}^{\downarrow}$ and $\mathsf{C}^{\rightarrow}$. These were in turn computed by Johnson-Freyd and Scheimbauer.

Theorem 13.1.2 (Theorem 7.6 in [JFS17])**.** *An object $f : X \to Y$ in $\mathsf{C}^{\downarrow}$ is n-dualizable if and only if X, Y are n-dualizable and f as a morphism in C is n-times left adjunctible. An object $f : X \to Y$ in $\mathsf{C}^{\rightarrow}$ is n-dualizable if and only if X, Y are n-dualizable and f as a morphism in C is n-times right adjunctible.*

Together with the Cobordism Hypothesis, this Theorem reduces the comparison of lax and oplax fully extended framed twisted topological field theories to the categorical comparison of n-times left and right adjunctibility, which is our main categorical result Corollary 11.2.9.

This leads to the question whether, for any given dexterity function a^n, we can define arrow categories C^{a^n} whose n-dualizable objects are characterized by a^n-adjunctible morphisms as in Theorem 13.1.2. More importantly, can we choose a dexterity function a^n so that we resolve our desiderata (0.2) and (0.3)?

We answer this question negatively. To see this, we first prove that for desideratum (0.2), choosing **oplax** is the only possible choice. This follows from

Theorem 13.1.10. *Let C be a symmetric monoidal (∞, N)-category, $f : X \to Y$ a 1-morphism in C, and a^n a dexterity function. The following are equivalent:*

1. *When viewed as an object in C^{a^n}, f is n-dualizable.*
2. *The objects X and Y are n-dualizable and f is n-times right adjunctible in $\mathsf{C}^{\mathrm{op}_{a^n}}$.*
3. *The objects X and Y are n-dualizable and f is a^n-adjunctible in C.*

Here $\mathsf{C}^{\mathrm{op}_{a^n}}$ is a a certain opposite category of C, reversing j-morphisms for certain j depending on a^n.

Our conclusion now is the following: we **could** define an a^n-lax twisted field theory to be a symmetric monoidal "a^n-lax natural transformation", i.e. a symmetric monoidal functor

$$\mathsf{Bord}_n \longrightarrow \mathsf{C}^{a^n},$$

and Corollary 13.1.10 characterizes the framed fully extended framed ones in terms of a^n-adjunctibility. However, only choosing **lax** satisfies desideratum (0.3), so this does not resolve the dichotomy between (13.1) and (13.2):

Finally, we investigate the above mentioned stronger adjunctibility condition appearing in [FT21]. Teleman suggested to us that this condition might appear when looking at dualizable field theories. We confirm this in Proposition 13.2.1, but the converse direction is not true. Instead, we give a more general characterization in terms of a^n-adjunctibility in Proposition 13.2.2.

A fun Variant: Trees of Adjunctibility

In the last section Chapter 14, we play with the higher adjunctibility conditions. So far, for fixed j, we required for each (co)unit j-morphism the existence of the same sided adjoint. We generalize this to asking for each unit and counit individually whether it should have a left or right adjoint. Which sided adjoint we require is now recorded in a **dexterity tree** (rather than a dexterity function). For instance, for a morphism f the tree

$$\begin{array}{ccccccc} & & & R & & & \\ & & / & & \backslash & & \\ & L & & & & R & \\ / & & \backslash & & / & & \backslash \\ R & & R & & L & & L \end{array}$$

records the existence of

$$\begin{array}{cccc} & (f \dashv f^R, u, c) & & \\ (u^L \dashv u, u_u, c_u) & & (c \dashv c^R, u_c, c_c) & \\ (u_u \dashv u_u^R, u_{u_u}, c_{u_u}) \quad (c_u \dashv u_u^R, u_{c_u}, c_{c_u}) & & (u_c^L \dashv u_c, u_{u_c}, c_{u_c}) \quad (c_c^L \dashv c_c, u_{c_c}, c_{c_c}) & . \end{array}$$

The interchange result Lemma 11.1.3 results in a equivalence relation $t^n \sim s^n$ of such trees, which amounts to a morphism being t^n-adjunctible if and only if it is s^n-adjunctible, see Theorem 14.2.1.

Let $\mathscr{T}_n := \mathfrak{T}_n / \sim$ be the set of equivalence classes of dexterity trees.

Theorem 14.2.5. *We have*

$$|\mathscr{T}_1| = 2 \quad \textit{and} \quad |\mathscr{T}_n| = |\mathscr{T}_{n-1}|^2 + 2^{2^{n-1}-1} \quad \textit{for} \quad n \geq 2.$$

which can be found in the OEIS as A332757 [SI23]. It also describes the number of involutions in the n-fold iterated wreath product of $\mathbb{Z}/2\mathbb{Z}$ or the number of involutory automorphisms of T_{n+1}. The first five terms are $2, 6, 44, 2064$ and 4292864.

Overview of Part III

In Chapter 10 we recall the notions of duals and adjoints in higher categories and n-dualizability.

In Chapter 11 we first recall the notions of higher left and right adjunctibility and reprove the Interchange Lemma (Lemma 11.1.3). Then we generalize these notions to "mixed" versions of higher adjunctibility and prove our main theorem (Theorem 11.2.6), which together with Remark 11.2.8 shows that we have precisely two independent notions of mixed adjunctibility. In even dimensions, higher left and right adjunctibility reduce to the same case (Corollary 11.2.9). We continue with useful tools for proving adjunctibility and discuss various opposite categories and mixed adjunctibility therein.

We construct various examples of mixed adjunctible morphisms in higher Morita categories in Chapter 12.

In Chapter 13 we discuss applications of our results to relative field theories, which was the motivation for this article.

Finally, in Chapter 14 we consider even more general notions of higher adjunctibility, depending on binary trees and compute the number of different notions that appear.

In the Appendix we include some basic lemmas about adjunctions in Appendix D.1, recall and reprove a reduction of n-dualizability conditions in Appendix D.2, and include full proofs of dualizability statements in the Morita bicategory in Appendix D.3.

Notation

We will use $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive integers and write $\subset$ to mean subset or equal. Given a category C we denote by $X \in \mathsf{C}$, that X is an object of C. We use $\coprod$ for coproducts. Nearly all colimits in this thesis are filtered and we will just write colim. All graded objects are $\mathbb{Z}$-graded and unless stated otherwise we use the cohomological convention.

A list of symbols with their first appearances can be found before the bibliography.

Part I.

Exotic Derived Categories of Bornological Algebras

1. Quasi-abelian Categories and Exact Structures

Contents

Most of the categories appearing in this thesis will not be abelian. However, there is a generalization of the concept known as Quillen exact categories, which extends the notion of exactness beyond the abelian context. This chapter provides background for quasi-abelian categories and their exact structures. We mainly follow [Qui73], [Bue10] and [FS10].

1.1. Quillen Exact Categories

Notation 1.1.1. Given a map $f : X \to Y$ in an additive category we use the notation $\operatorname{Ker} f$ for the object and $\ker f : \operatorname{Ker} f \to X$ for the morphism. Similarly, we have $\operatorname{coker} f : Y \to \operatorname{Coker} f$.

Definition 1.1.2. Let C be an additive category. A **kernel-cokernel pair** (f, g) in C is a pair of composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

such that f is a kernel of g and g is a cokernel of f.

If a class $\mathscr{E}$ of kernel-cokernel pairs on C is fixed, an **$\mathscr{E}$-admissible monomorphism** is a morphism f such that there exists a morphism g with $(f,g) \in \mathscr{E}$. Dually, an **$\mathscr{E}$-admissible epimorphism** is a morphism g such that there exists a morphism f with $(f,g) \in \mathscr{E}$. We use the arrows

$$X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$$

to denote that f is an $\mathscr{E}$-admissible monomorphism and g is an $\mathscr{E}$-admissible epimorphism.

Notation 1.1.3. If the class of kernel-cokernel pairs $\mathscr{E}$ is clear from the context we shorten the name to admissible monomorphism and admissible epimorphism.

Definition 1.1.4. Let C be an additive category. An **exact structure** on C is a class $\mathscr{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms.

- [E0] For all objects $X \in \mathsf{C}$, the identity morphism id_X is an admissible monomorphism.
- [E0$^{\mathrm{op}}$] For all objects $X \in \mathsf{C}$, the identity morphism id_X is an admissible epimorphism.
- [E1] The class of admissible monomorphisms is closed under composition.
- [E1$^{\mathrm{op}}$] The class of admissible epimorphisms is closed under composition.
- [E2] The pushout of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism.
- [E2$^{\mathrm{op}}$] The pullback of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

The elements of $\mathscr{E}$ are called **short exact sequences**.

An **exact category** is a pair $(\mathsf{C}, \mathscr{E})$ consisting of an additive category C and an exact structure $\mathscr{E}$.

A basic but important observation is that we always get a short exact sequence from an admissible mono- or epimorphism by taking the cokernel or kernel.

Proposition 1.1.5. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category. If $f : W \to X$ is an admissible monomorphism, then f has a cokernel and*

$$W \overset{f}{\rightarrowtail} X \twoheadrightarrow \operatorname{Coker} f$$

is a short exact sequence. Similarly, if $g : Y \to Z$ is an admissible epimorphism, then g has a kernel and

$$\operatorname{Ker} g \rightarrowtail Y \xrightarrow{g} \twoheadrightarrow Z$$

is a short exact sequence.

Proposition 1.1.6. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category. Isomorphisms in C are admissible mono- and epimorpisms. Conversely, an admissible epimorphism, that is also a monomorphism, is an isomorphism and an admissible monomorphism, that is also an epimorphism, is an isomorphism.*

Proof. The first statement is Remark 2.3 in [Bue10]. Let f be an admissible epimorphism. Then f is the coequalizer of $\ker f, 0 : \operatorname{Ker} f \rightrightarrows X$. If f is also monic then $\ker f = 0$ and their coequalizer is an isomorphism. Dually, the last statement follows from epic equalizers being isomorphisms. □

Recall, that in any category C a morphism $r : Y \to Z$ is a **retraction** if it admits a **section** $s : Z \to Y$ with $r \circ s = \mathrm{id}_Z$. Dually, a morphism $c : X \to Y$ is a **coretraction** if there is a section $s : Y \to X$ with $s \circ c = \mathrm{id}_Z$.

Lemma 1.1.7 (Lemma 3.8 in [FS10])**.** *Let C be an additive category and (f, g) a kernel-cokernel pair. The following are equivalent*

(i) g is a retraction,

(ii) f is a coretraction,

(iii) There is a commutative diagram

$$\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow\!\!\!\!\!\twoheadrightarrow & Z \\
\Big\| {\scriptstyle f} & & \Big\downarrow {\scriptstyle \beta} & & \Big\| {\scriptstyle g} \\
X & \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} & X \coprod Z & \overset{[0\ 1]}{\twoheadrightarrow} & Z,
\end{array}$$

where β is an isomorphism.

Definition 1.1.8. Let C be an additive category and (f, g) a kernel-cokernel pair. If (f, g) satisfies the conditions in Lemma 1.1.7, we say that (f, g) is **split exact**.

It follows from the axioms of exact structures, that they always contain all split-exact sequences. Additionally, there is always a minimal exact structure on any additive category.

Proposition 1.1.9 (Proposition 3.10 in [FS10]). *Let* C *be an additive category. The class*

$$\mathfrak{min} := \{\, (f,g) \;\mid\; (f,g) \textit{ is a split exact kernel-cokernel pair} \,\}$$

is an exact structure and any other possible exact structure $\mathscr{E}$ *on* C *contains* $\mathfrak{min}$.

Definition 1.1.10. Let C be an additive category. The class $\mathfrak{min}$ from Proposition 1.1.9 is called **minimal exact structure**. If there is an exact structure $\mathfrak{max}$ on C, such that every other exact structure $\mathscr{E}$ on C satisfies $\mathscr{E} \subset classEmax$, then $\mathfrak{max}$ is called the **maximal exact structure**.

Remark 1.1.11. A class of kernel-cokernel pairs $\mathscr{E}$ is an exact structure on an additive category C if and only if $\mathscr{E}^{\mathrm{op}}$ is an exact structure on C^{op}. If $\mathscr{E}$ is the minimal/maximal exact structure then so is $\mathscr{E}^{\mathrm{op}}$. ◇

Definition 1.1.12. Let $(\mathsf{C}, \mathscr{E})$ be an exact category and D a full additive subcategory of C. We say that D is **closed under extensions** in C if for all exact sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathscr{E}$ with $X, Z \in \mathsf{D}$ then Y is isomorphic to an object in D.

Remark 1.1.13. Most of the examples of exact categories in this thesis will be subcategories of topological vector spaces. In this context being closed under extension is known as three-space property and a lot of research has been done. See Example 3.19 in [FS10] for a short list and pointers to the literature. ◇

Proposition 1.1.14 (Lemma 10.20 in [Bue10]). *Let* $(\mathsf{C}, \mathscr{E})$ *be an exact category and* D *a full additive subcategory of* C*, that is closed under extensions. Then*

$$\mathscr{E}' := \{\, (f,g) \in \mathscr{E} \;\mid\; (f,g) \textit{ is a sequence in } \mathsf{D} \,\}$$

is an exact structure on D.

In this case we say that $(\mathsf{D}, \mathscr{E}')$ *is a* ***fully exact subcategory*** *of* $(\mathsf{C}, \mathscr{E})$.

As in the abelian setting we have the usual definition of exact functors.

Definition 1.1.15. Let $(\mathsf{C}, \mathscr{E}_{\mathsf{C}})$ and $(\mathsf{D}, \mathscr{E}_{\mathsf{D}})$ be exact categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is **exact** if $F(\mathscr{E}_{\mathsf{C}}) \subset \mathscr{E}_{\mathsf{D}}$.

1.2. Quasi-abelian Categories

We have seen, that we always have minimal exact structures on an additive category C. If we impose more conditions on C we also get a maximal exact structure.

Definition 1.2.1. An additive category C is **pre-abelian** if it has all kernels and cokernels.

Remark 1.2.2. A category admits finite limits if and only if it has all equalizers and finite products. But an equalizer of f and g in an additive category is the kernel of $f - g$. Therefore, a pre-abelian category has all finite limits. Similarly, it follows that a pre-abelian category admits finite colimits. $\diamond$

Proposition 1.2.3 (Theorem 3.3 in [SW11])**.** *Let C be a pre-abelian category. Then C has a maximal exact structure $\mathfrak{max}$. Every other exact structure $\mathscr{E}$ on C satisfies $\mathscr{E} \subset \mathfrak{max}$.*

In [SW11] or chapter 4 of [FS10] one can find a description of the kernel-cokernel pairs that are part of the maximal exact structure. However, we will only work with the special case of quasi-abelian categories, where the maximal exact structure contains all kernel-cokernel pairs.

Definition 1.2.4. A pre-abelian category C is **quasi-abelian** if

(i) The class of kernels is stable under pushout along arbitrary morphisms,

(ii) The class of cokernels is stable under pullback along arbitrary morphisms,

The notion of quasi-abelian categories is older than exact structures and goes back to Yoneda [Yon60].

Proposition 1.2.5 (Proposition 1.1.7 in [Sch99])**.** *Let A be a quasi-abelian category. Then the class $\mathfrak{max}$ of all kernel-cokernel pairs is the maximal exact structure on A. Conversely, a pre-abelian category, that has all kernel-cokernel pairs as its maximal exact structure, is quasi-abelian.*

The terminology that one often encounters in the theory of quasi-abelian category is that of strict morphisms. The definition uses the image and coimage.

Proposition 1.2.6. *Let C be an additive category and $f : X \to Y$ a morphism admitting a kernel and a cokernel. There is a canonical morphism $\tilde{f}$ from the image $\operatorname{Im} f = \ker(\operatorname{coker} f)$ to the coimage $\operatorname{Coim} f = \operatorname{coker}(\ker f)$ of f. The map is given by the commutative diagram:*

$$
\begin{array}{ccccccc}
\operatorname{Ker} f & \longrightarrow & X & \xrightarrow{\;f\;} & Y & \longrightarrow & \operatorname{Coker} f \\
 & & \downarrow & & \uparrow & & \\
 & & \operatorname{Coim} f & \overset{\tilde{f}}{\dashrightarrow} & \operatorname{Im} f & &
\end{array}
$$

Definition 1.2.7. Let C be an additive category and $f : X \to Y$ a morphism admitting a kernel and a cokernel. We say that f is **strict** if the canonical map $\tilde{f} : \operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Proposition 1.2.8 (Proposition 2.27 in [FS10])**.** *Let C be an additive category. A morphism f is*

(i) a strict monomorphism if and only if it is the kernel of its cokernel,

(ii) a strict epimorphism if and only if it is the cokernel of its kernel.

Therefore, the maximal exact structure in a quasi-abelian category is that of strict monomorphism-epimorphism pairs.

Remark 1.2.9. Another weakening of abelian categories is that of **semi-abelian categories**. These are preabelian categories, where for every morphism $f : X \to Y$ the canonical morphism $\tilde{f} : \operatorname{Coim} f \to \operatorname{Im} f$ from Proposition 1.2.6 is both a mono- and an epimorphism. Abelian categories can be defined similarly as preabelian categories, where for every morphism $f : X \to Y$ the canonical morphism $\tilde{f} : \operatorname{Coim} f \to \operatorname{Im} f$ from Proposition 1.2.6 is an isomorphism, i.e. strict. We have the following hierarchy

$$\left\{\begin{matrix}\text{additive}\\ \text{categories}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{preabelian}\\ \text{categories}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{semi-abelian}\\ \text{categories}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{quasi-abelian}\\ \text{categories}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{abelian}\\ \text{categories}\end{matrix}\right\}.$$

Note, that being (pre-/semi-/quasi-)abelian is a property and not extra structure. This is in contrast to general exact categories, where we specify a class of short exact sequences. All inclusions in the hierarchy above are strict. The most difficult one to see is that not every semi-abelian category is quasi-abelian. This is the opposite of what is known as Raĭkov's conjecture. Counterexamples were constructed by Bonet and Dierolf [BD06] as well as Rump [Rum08, Rum11]. We will briefly revisit the original counterexample from [BD06] in Chapter 4. ◇

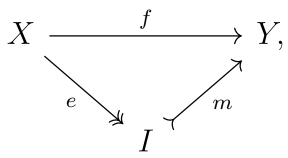
We have seen in Proposition 1.1.14 that full extension-closed subcategories of exact categories are exact. For quasi-abelian categories we have a simple criterion.

Proposition 1.2.10 (Proposition 4.20 in [FS10])**.** *Let C be a quasi-abelian category and D a full additive subcategory of C, that reflects kernels and cokernels. Then D is also quasi-abelian.*

1.3. Homological Algebra in Exact Categories

1.3.1. Long Exact Sequences

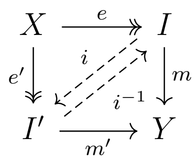
Definition 1.3.1. Let C be an exact category. A morphism $f : X \to Y$ is **admissible** if it factors as



where e is an admissible epimorphism and m is an admissible monomorphism.

Remark 1.3.2. The terminology admissible is consistent in the sense that a map is an admissible monomorphism/epimorphism if and only if it is admissible and a monomorphism/epimorphism by Proposition 1.1.6. ◇

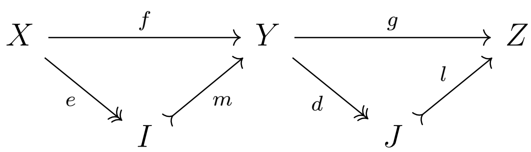
Proposition 1.3.3 (Lemma 8.4 in [Bue10])**.** *Let C be an exact category and $f : X \to Y$ an admissible morphism. Then the factorization is unique up to unique isomorphism. That is in the commutative diagram of solid arrows*



there is a isomorphism i with inverse i^{-1} making the diagram commute.

Proposition 1.3.4. *Proposition 3.1 in [Fre65] Let C be an exact category, such that every morphism is admissible Then C is abelian.*

Definition 1.3.5. Let C be an exact category. A sequence of admissible morphisms



is **exact** (at Y) if $I \rightarrowtail Y \twoheadrightarrow J$ is a short exact sequence. Long sequence of admissible morphisms are **exact** or **acyclic** if every sequence of two composable morphisms is exact.

Lemma 1.3.6 (Lemma 7.1 in [Bue10])**.** *Let C be an additive category. The following are equivalent.*

1. *Every coretraction has a cokernel,*
2. *Every retraction has a kernel.*

If C *is an exact category we can extend the list by*

3. *Every coretraction is an admissible monomorphism,*
4. *Every retraction is an admissible epimorphism.*

Definition 1.3.7. Let C be an additive category. If the equivalent conditions in Lemma 1.3.6 hold, then C is **weakly idempotent complete**.

We will mostly work with quasi-abelian categories, which always satisfy this.

Corollary 1.3.8. *Let* C *be a pre-abelian category. Then* C *is weakly idempotent complete.*

Proof. By definition all morphisms have kernels and cokernels. □

Remark 1.3.9. For weakly idempotent exact categories we have the usual tools to work with exact sequences in the form of diagram lemmas. The Four- and Five-Lemma, the Snake Lemma and the 3×3 Lemma are all proven in [Bue10]. ◇

1.3.2. Resolutions

Definition 1.3.10. Let C be an additive category. A **(cochain) complex** is a diagram $(A^\bullet, d^\bullet)$

$$\dots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \dots$$

in C, such that $d^n \circ d^{n+1} = 0$ for all $n \in \mathbb{Z}$. Dually we define chain complexes.

We denote the category of chain complexes and chain maps with $\mathsf{Ch}(\mathsf{C})$.

Proposition 1.3.11. *Let* $(\mathsf{C}, \mathscr{E})$ *be an exact category. Then* $\mathsf{Ch}(\mathsf{C})$ *is an exact category with respect to the class* $\mathsf{Ch}(\mathsf{C})$ *of short sequences of chain maps, which are exact in each degree. If* $(\mathsf{C}, \mathscr{E})$ *is (quasi-)abelian, then the category* $(\mathsf{Ch}(\mathsf{C}), \mathsf{Ch}(\mathscr{E}))$ *is (quasi-)abelian.*

Proof. The proof is as in Lemma 9.1 in [Bue10], where the quasi-abelian case follows from the characterization in Proposition 1.2.5. □

Definition 1.3.12. Let C be an exact category and $(A^\bullet, d_A^\bullet), (B^\bullet, d_B^\bullet) \in \mathsf{Ch}(\mathsf{C})$. The **mapping cone** of a chain map $f^\bullet : A^\bullet \to B^\bullet$ is the complex

$$\operatorname{cone}(f)^n = A^{n+1} \amalg B^n \qquad \text{differential} \qquad d_f^n = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}$$

The **translation functor** on $\mathsf{Ch}(\mathsf{C})$ is defined by

$$\Sigma : \mathsf{Ch}(\mathsf{C}) \to \mathsf{Ch}(\mathsf{C}), \qquad A^\bullet \mapsto \operatorname{cone}(A^\bullet \to 0).$$

As in the abelian case we get a triangulated category, but in order to avoid naming confusion the exact triangles are called **strict triangles** in [Bue10].

Definition 1.3.13. Let C be an exact category and $(A^\bullet, d_A^\bullet), (B^\bullet, d_B^\bullet) \in \mathsf{Ch}(\mathsf{C})$. A chain map $f^\bullet : A^\bullet \to B^\bullet$ is **chain homotopic to zero** if there are morphisms $h^n : A^n \to B^{n-1}$, such that $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$. We define the set

$$N(A, B) := \{ \text{chain maps } f^\bullet : A^\bullet \to B^\bullet \mid f \text{ is chain homotopic to zero} \}.$$

Definition 1.3.14. Let C be an exact category. The **homotopy category** $\mathsf{Ho}(\mathsf{C})$ is the category of chain complexes $\mathsf{Ch}(\mathsf{C})$ as objects and

$$\mathrm{Hom}_{\mathsf{Ho}(\mathsf{C})}(A, B) := \mathrm{Hom}_{\mathsf{Ch}(\mathsf{C})}(A, B) \Big/ N(A, B).$$

The category $\mathsf{Ho}(\mathsf{C})$ is a triangulated category, when equipped with the translation functor and strict triangles in $\mathsf{Ch}(\mathsf{C})$.

1.3.3. Verdier Quotients and Derived Categories

Definition 1.3.15. Let C be an exact category. A chain complex $(A^\bullet, d^\bullet)$ is **acyclic** if each differential d^n factors $A^n \twoheadrightarrow Z^{n+1}A \rightarrowtail A^{n+1}$, such that all $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1}$ are exact.

With this definition an acyclic complex is a complex with admissible differentials, which is exact. From here on the same machinery as in the abelian case leads to the following definition. See section 10 in [Bue10] for details.

Proposition 1.3.16. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category. The homotopy category of acyclic complexes, denoted by $\mathsf{Ac}(\mathsf{C}, \mathscr{E})$, is a triangulated subcategory of $\mathsf{Ho}(\mathsf{C})$.*

Definition 1.3.17. Let $(\mathsf{C}, \mathscr{E})$ be an exact category. The **derived category** of C is the Verdier quotient

$$\mathsf{D}(\mathsf{C}, \mathscr{E}) := \mathsf{Ho}(\mathsf{C}) \Big/ \mathsf{Ac}(\mathsf{C}, \mathscr{E}).$$

1.3.4. Injectives and Projectives

The theory of projective and injective objects translates from the abelian to the exact world.

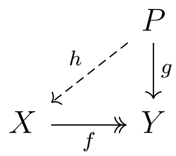
Definition 1.3.18. Let $(\mathsf{C}, \mathscr{E})$ be an exact category. We say that an object $P \in \mathsf{C}$ is **$\mathscr{E}$-projective** if the represented functor $\mathrm{Hom}_{\mathsf{C}}(P, \cdot) : \mathsf{C} \to \mathsf{Ab}$ is exact. We say that an object $Q \in \mathsf{C}$ is **$\mathscr{E}$-injective** if the corepresented functor $\mathrm{Hom}_{\mathsf{C}}(\cdot, Q) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

If the exact structure is clear from the context we drop the $\mathscr{E}$- prefix.

We also have the familiar lifting properties.

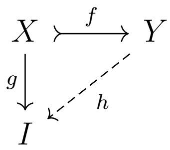
Proposition 1.3.19 (Proposition 11.3 in [Bue10])**.** *Let $(\mathsf{C}, \mathscr{E})$ be an exact category.*

An object $P \in \mathsf{C}$ is $\mathscr{E}$-projective if and only if for all admissible epimorphisms $f : X \twoheadrightarrow Y$ and all morphisms $g : P \to Y$ there is a morphism $h : P \to X$ making the diagram



commute.

An object $I \in \mathsf{C}$ is $\mathscr{E}$-injective if and only if for all admissible monomorphisms $f : X \rightarrowtail Y$ and all morphisms $g : X \to I$ there is a morphism $h : Y \to I$ making the diagram



commute.

Definition 1.3.20. Let $(\mathsf{C}, \mathscr{E})$ be an exact category.

We say that C has **enough $\mathscr{E}$-injectives** if for all objects $X \in \mathsf{C}$ there is an $\mathscr{E}$-injective object I and an $\mathscr{E}$-admissible monomorphism $q : X \to I$.

We say that C has **enough $\mathscr{E}$-projectives** if for all objects $Y \in \mathsf{C}$ there is a $\mathscr{E}$-projective object P and an $\mathscr{E}$-admissible epimorphism $p : P \to Y$.

Definition 1.3.21. Let $(\mathsf{C}, \mathscr{E})$ be an exact category and $X \in \mathsf{C}$ an object.

A **left resolution** of X is a chain complex $L_\bullet$ with $L_i = 0$ for $i < 0$, such that the **augmented complex**

$$\cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow X \longrightarrow 0$$

is exact. The number $\inf\{\, d \in \mathbb{N}_0 \,|\, L_k = 0 \text{ for all } k > d \,\} \in \mathbb{N}_0 \cup \{\infty\}$ is the **length** of $L_\bullet$. An **$\mathscr{E}$-projective resolution** of X is a left resolution consisting of $\mathscr{E}$-projective objects.

A **right resolution** of X is a cochain complex $R^\bullet$ with $R^j = 0$ for $j < 0$, such that the **augmented complex**

$$0 \longrightarrow X \longrightarrow R^0 \longrightarrow R^1 \longrightarrow R^2 \longrightarrow \cdots$$

is exact. The number $\inf\{\, d \in \mathbb{N}_0 \,|\, R^l = 0 \text{ for all } l > d\,\} \in \mathbb{N}_0 \cup \{\infty\}$ is the **length** of $R^\bullet$. An $\mathscr{E}$**-injective resolution** of X is a right resolution consisting of $\mathscr{E}$-injective objects.

Proposition 1.3.22. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives. Then every object $X \in \mathsf{C}$ has a $\mathscr{E}$-projective resolution.*

Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-injectives. Then every object $X \in \mathsf{C}$ has an $\mathscr{E}$-injective resolution.

Lemma 1.3.23. *Let*

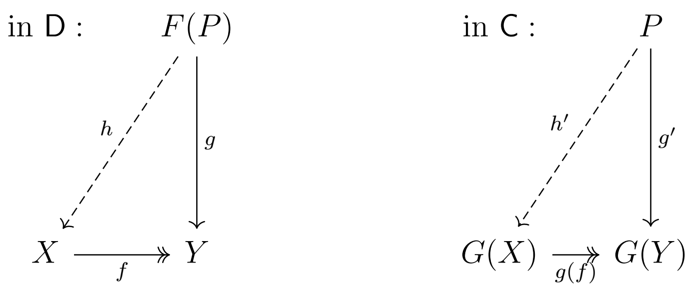
$$F : (\mathsf{C}, \mathscr{E}_\mathsf{C}) \leftrightarrows F : (\mathsf{D}, \mathscr{E}_\mathsf{D}) : G$$

be an adjunction $F \dashv G$ of additive functors between exact categories. We have that

(i) if G preserves $\mathscr{E}_\mathsf{D}$-admissible epimorphisms, then F preserves $\mathscr{E}_\mathsf{C}$-projective objects,

(ii) if F preserves $\mathscr{E}_\mathsf{C}$-admissible monomorphisms, then G preserves $\mathscr{E}_\mathsf{D}$-injective objects.

Proof. (i) Let P be an $\mathscr{E}_\mathsf{C}$-projective object in C. Let $f : X \twoheadrightarrow Y$ be an $\mathscr{E}_\mathsf{D}$-admissible epimorphism and $g : F(P) \to Y$ a morphism in D. By assumption $G(f) : G(F) \to G(Y)$ is an $\mathscr{E}_\mathsf{D}$-admissible epimorphism and by adjointness we get a morphism $g' : P \to G(Y)$. Using that P is $\mathscr{E}_\mathsf{C}$-projective we get a morphism $h' : P \to G(X)$ with $g(f) \circ h' = g'$. By adjointness we have a morphism $g : F(P) \to X$ with $f \circ h = g$.



(ii) Dual to (i).

□

Definition 1.3.24. Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives. Let $A \in \mathsf{C}$ be an object.

The **projective dimension** of A with respect to $(\mathsf{C}, \mathscr{E})$ is

$$\mathrm{pd}_{\mathsf{C},\mathscr{E}}(A) := \min\{ d \in \mathbb{N}_0 \cup \{\infty\} \mid \text{A has a } \mathscr{E}\text{-projective resolution of length d} \}.$$

The **injective dimension** of A with respect to $(\mathsf{C}, \mathscr{E})$ is

$$\mathrm{id}_{\mathsf{C},\mathscr{E}}(A) := \min\{ d \in \mathbb{N}_0 \cup \{\infty\} \mid \text{A has a } \mathscr{E}\text{-injective resolution of length d} \}$$

The **global dimension** of $(\mathsf{C}, \mathscr{E})$ is

$$\mathrm{g}\ell\,(\mathsf{C}, \mathscr{E}) := \sup\{ \mathrm{pd}_{\mathsf{C},\mathscr{E}}(A) \mid A \in \mathsf{C} \text{ object} \}$$

An exact version of Schanuel's Lemma holds here, with the same proof as in the abelian case (see, for example, Lemma 5.1 in [Lam99]). Instead of stating the lemma itself, we highlight its most significant consequence.

Proposition 1.3.25. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives. Let $A \in \mathsf{C}$ be an object and $n \in \mathbb{N}_0$. The following are equivalent.*

(i) $\mathrm{pd}_{\mathsf{C},\mathscr{E}}(A) \leq n$

(ii) for any truncated $\mathscr{E}$-projective resolution

$$P_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A$$

the kernel $\operatorname{Ker} d_{n-1}$ *is $\mathscr{E}$-projective.*

1.3.5. Derived Functors

Similar to the abelian case, we can define left and right derived functors. See section 1.3 in [Sch99].

Corollary 1.3.26 (section 1.3 in [Sch99])**.** *Let C, D be quasi-abelian categories and $F : \mathsf{C} \to \mathsf{D}$ an additive Functor.*

(i) If C has enough $\mathfrak{max}$-projectives, then F has a left derived functor.

(ii) If C has enough $\mathfrak{max}$-injectives, then F has a right derived functor.

1.3.6. Ext Functor

One of the applications of derived functor is defining Ext. With the same proof as in the abelian case we get the following two results.

Proposition 1.3.27. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives, $X \in \mathsf{C}$ an object and $n \in \mathbb{N}_0$. Then $\operatorname{pd}_{\mathsf{C},\mathscr{E}}(A) \leq n$ if and only if*

$$\operatorname{Ext}^i_{\mathsf{C},\mathscr{E}}(X, L) = 0 \quad \text{for all} \quad i > d \quad \text{and all objects} \quad L \in \mathsf{C}.$$

Proposition 1.3.28. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives and $X \rightarrowtail Y \twoheadrightarrow Z$ a short exact sequence. Then there is a long exact sequence*

$$\cdots \longrightarrow \operatorname{Ext}^i(Z, L) \longrightarrow \operatorname{Ext}^{i+1}(Y, L) \longrightarrow \operatorname{Ext}^{i+1}(X, L) \longrightarrow \cdots.$$

for every $L \in \mathsf{C}$.

Proposition 1.3.29. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives. Let $n, d \in \mathbb{N}_0$ and*

$$Q_n \rightarrowtail \cdots \longrightarrow Q_1 \longrightarrow Q_0 \twoheadrightarrow A$$

be an exact complex with $\operatorname{pd}(Q_i) \leq d$ for $i = 0, 1, \ldots, n$. Then

$$\operatorname{pd}(A) \leq n + d.$$

Proof. If $n = 0$ we have $A \cong Q_0$ and therefore $\operatorname{pd}(A) = \operatorname{pd}(Q_0) \leq d$. In the case $n \geq 1$ we use induction over n.

For $n = 1$ we have a a short exact sequence

$$Q_1 \rightarrowtail Q_0 \twoheadrightarrow A.$$

Let L be on object in C. We use the long exact sequence from Proposition 1.3.28, that is

$$\cdots \longrightarrow \operatorname{Ext}^i(Q_1, L) \longrightarrow \operatorname{Ext}^{i+1}(A, L) \longrightarrow \operatorname{Ext}^{i+1}(Q_0, L) \longrightarrow \cdots.$$

For $i > d$ we have $\operatorname{Ext}^i(Q_1, L) = 0 = \operatorname{Ext}^{i+1}(Q_0, L)$ by Proposition 1.3.27 and therefore by exactness also $\operatorname{Ext}^{i+1}(A, L) = 0$. From Proposition 1.3.27 it follows that $\operatorname{pd}(A) \leq d + 1$, which is what we wanted to show.

For $n \geq 2$ we have an exact sequence

$$Q_n \overset{d}{\rightarrowtail} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \twoheadrightarrow A,$$

that we can split up into a one degree shorter exact sequence

$$Q_{n-1}/\operatorname{Im} d \rightarrowtail Q_{n-2} \longrightarrow \cdots \longrightarrow Q_0 \twoheadrightarrow A \tag{1.1}$$

and a short exact sequence

$$Q_n \rightarrowtail Q_{n-1} \twoheadrightarrow Q_{n-1}/\operatorname{Im} d. \tag{1.2}$$

The induction hypothesis applied to (1.2) gives us $\operatorname{pd}\left(Q_{n-1}/\operatorname{Im} d\right) \leq 1+d$. With this we can apply the induction hypothesis to (1.1) and get

$$\operatorname{pd}(A) \leq (n-1)+(d+1) = n+d.$$

□

We also get the following characterization of projective dimensions. The proof is as in the abelian case.

Proposition 1.3.30. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with enough $\mathscr{E}$-projectives. Let $\mathsf{D}(\mathsf{C}, \mathscr{E})$ be the derived category of C and $X \in \mathsf{C}$ an object. Then*

$$\operatorname{pd}_{\mathsf{C},\mathscr{E}}(X) = n \quad \Leftrightarrow \quad \operatorname{Hom}_{\mathsf{D}(\mathsf{C},\mathscr{E})}(F, L[i]) = 0 \quad \textit{for all } L \in \mathsf{C}, i > n.$$

1.3.7. Algebras and Modules

Proposition 1.3.31. *Let $(\mathsf{C}, \mathscr{E})$ be a symmetric monoidal exact category with tensor product $\otimes$ and countable coproducts. The category of chain complexes $\mathsf{Ch}(\mathsf{C})$ is symmetric monoidal with tensor product defined by*

$$(A^\bullet \otimes B^\bullet)^n = \coprod_{i+j=n} A^i \otimes B^j$$

for $A^\bullet, B^\bullet \in \mathsf{Ch}(\mathsf{C})$.

Proposition 1.3.32 (Lemma 2.2.2 in [HSS00])**.** *Let C be a cocomplete, symmetric monoidal category with tensor product $\otimes$ and unit $\mathbb{1}$. Let A be a commutative monoid in $(\mathsf{C}, \otimes, \mathbb{1})$, such that $A \otimes (\cdot) : \mathsf{C} \to \mathsf{C}$ preserves coequalizers. Then the category A-Mod of A-modules is symmetric monoidal with tensor product*

$$M \otimes_A N \; := \operatorname{colim}(M \otimes A \otimes N \rightrightarrows M \otimes N)$$

and unit A.

Proposition 1.3.33 (Lemma 2.2.8 in [HSS00])**.** *Let* C *be a bicomplete, closed symmetric monoidal category with tensor product* $\otimes$ *and unit* $\mathbb{1}$*. Let* A *be a commutative monoid in* $(\mathsf{C}, \otimes, \mathbb{1})$*. Then there is an* A*-module* $\mathrm{Hom}_A(M, N)$*, natural for* $M, N \in \mathsf{C}$*, such that the functor* $(\cdot) \otimes_A M$ *is left adjoint to the functor* $\mathrm{Hom}_A(M, (\cdot))$*.*

The following is the generalization of Proposition 1.5.1 from [Sch99] to general exact categories.

Proposition 1.3.34. *Let* $(\mathsf{C}, \mathscr{E})$ *be a symmetric monoidal exact category and* A *a (commutative) monoid object in* C*. Let*

$$u : A\text{-}\mathsf{Mod} \to \mathsf{C}$$

be the forgetful functor. Then the category A-Mod *of* A*-modules in* C *is exact with the exact structure*

$$\left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \;\middle|\; f, g \text{ morphisms in } A\text{-}\mathsf{Mod},\; u\left(X \xrightarrow{f} Y \xrightarrow{g} Z\right) \in \mathscr{E} \right\}.$$

Moreover, the forgetful functor u *preserves limits and colimits. A morphism* f *in* A-Mod *is admissible if and only if* $u(f)$ *is admissible in* C*.*

2. Derived Categories of the Second Kind

Contents

Derived categories of the second kind, also known as exotic derived categories, are extensions of the classical derived categories introduced by Grothendieck and Verdier in the 1960s. The definitions and much of the theory presented in this chapter are due to Leonid Positselski, who developed these categories as part of his work on nonhomogeneous Koszul duality in the 1990s. A detailed account of the historical development and terminology can be found in Remark 9.2 of [PŠ22] and Section 7.2 of [Pos23]. Besides Positselski's memoir [Pos11], a more detailed introduction to exotic derived categories is available in [Pos23].

We review the basic definitions and state them in the context of exact categories. Note that our setup is not as general as the exact DG categories defined in [Pos21].

2.1. Coderived and Contraderived Categories

2.1.1. DG Algebras and DG Modules

We will briefly define DG Algebras and DG modules in the exact setting. See section 3 in [Pos11] or section 4 in [Pos23] for more details.

Let $(\mathsf{C}, \mathscr{E})$ be an exact category. By Proposition 1.3.11 we have an exact structure $\mathscr{E}_{\mathsf{Ch}}$ on $\mathsf{Ch}(\mathsf{C})$, which is given by

$$\mathscr{E}_{\mathsf{Ch}} = \left\{ L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet \;\middle|\; (f^n, g^n) \in \mathscr{E} \text{ for all } n \in \mathbb{Z} \right\}.$$

We assume that C has countable coproducts, and is (symmetric) monoidal with tensor product $\otimes$ and unit $\mathbb{1}$. By Proposition 1.3.31 the category $\mathsf{Ch}(\mathsf{C})$ is (symmetric) monoidal with tensor product

$$(M^\bullet \otimes N^\bullet)^n = \coprod_{i+j=n} M^i \otimes N^j$$

and unit $\mathbb{K}$ seen as a complex in degree 0.

Definition 2.1.1. A **DG algebra** A in C is a monoid in $(\mathsf{Ch}(\mathsf{C}), \otimes, \mathbb{K})$. We denote the category of left DG A-modules with $\mathsf{DG}\text{-}A$.

Notation 2.1.2. Given a DG algebra A, or DG module M. We denote their underlying graded objects with $A^\bullet$ and $M^\bullet$.

Remark 2.1.3. We will use this theory in the case where C is given by complete bornological spaces. In this case DG algebras and DG modules can be characterized with their usual definitions given for discrete algebras and modules, with the additional condition that all involved morphisms are bounded.

In particular, using the explicit definitions it is easy to add curvature to the theory and work over complete bornological CDG algebras and modules. ◇

2.1.2. The Contraderived and Coderived Categories

Let $(\mathsf{C}, \mathscr{E})$ be as above. Let A be a DG algebra in C. We have seen that the category of left DG A-modules has an exact structure.

Definition 2.1.4. A **closed morphism** $f : M \to N$ in $\mathsf{DG}\text{-}A$ is an element of $\mathrm{Hom}^0_{\mathsf{DG}\text{-}A}(M, N)$, such that $d(f) = 0$. The category of left DG A-modules and closed morphisms is denoted by $\mathsf{Z}^0(\mathsf{DG}\text{-}A)$.

We have an exact structure on $\mathsf{Z}^0(\mathsf{DG}\text{-}A)$ inherited from $\mathscr{E}$. A triple in $\mathsf{Z}^0(\mathsf{DG}\text{-}A)$ is a short exact sequence if the triple of graded modules is exact. See the Remark in section 3.5 in [Pos11] or Example 5.3.(1) in [Pos21] for more details. If we consider short exact sequences of $\mathsf{Z}^0(\mathsf{DG}\text{-}A)$ as a finite complex of DG modules we can totalize. This leads to three exotic derived categories.

Definition 2.1.5. A left DG A-module is **absolute acyclic** if it belongs to the minimal thick subcategory $\mathsf{Ac}^{\mathrm{abs}}(\mathsf{DG}\text{-}A, \mathscr{E})$ of the homotopy category $\mathsf{Ho}(\mathsf{DG}\text{-}A)$ containing the total DG modules of exact triples of left DG A-modules. The Verdier quotient

$$\mathsf{D}^{\mathrm{abs}}(\mathsf{DG}\text{-}A, \mathscr{E}) := \mathsf{Ho}(\mathsf{DG}\text{-}A) \big/ \mathsf{Ac}^{\mathrm{abs}}(\mathsf{DG}\text{-}A, \mathscr{E}).$$

is the **absolute derived category** of left DG A-modules.

Definition 2.1.6. A left DG A-module is **contraacyclic** if it belongs to the minimal thick subcategory $\mathsf{Ac}^{\mathrm{ctr}}(\mathsf{DG}\text{-}A, \mathscr{E})$ of the homotopy category $\mathsf{Ho}(\mathsf{DG}\text{-}A)$ containing the total DG modules of exact triples of left DG A-modules and closed under countable products. The Verdier quotient

$$\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}A, \mathscr{E}) := \mathsf{Ho}(\mathsf{DG}\text{-}A) \big/ \mathsf{Ac}^{\mathrm{ctr}}(\mathsf{DG}\text{-}A, \mathscr{E})$$

is the **contraderived category** of left DG A-modules.

Definition 2.1.7. A left DG A-module is **coacyclic** if it belongs to the minimal thick subcategory $\mathsf{Ac}^{\mathrm{co}}(\mathsf{DG}\text{-}A, \mathscr{E})$ of the homotopy category $\mathsf{Ho}(\mathsf{DG}\text{-}A)$ containing the total DG modules of exact triples of left DG A-modules and closed under countable coproducts. The Verdier quotient

$$\mathsf{D}^{\mathrm{co}}(\mathsf{DG}\text{-}A, \mathscr{E}) := \mathsf{Ho}(\mathsf{DG}\text{-}A) \big/ \mathsf{Ac}^{\mathrm{co}}(\mathsf{DG}\text{-}A, \mathscr{E})$$

is the **coderived category** of left DG A-modules.

Remark 2.1.8. Note, that we slightly change the definition from [Pos11] and only require countable products for the contraderived and countable coproducts for the coderived category. ◇

2.2. Semiorthogonal Decompositions

2.2.1. Positselski's Homological Dimension Conditions

Let $(\mathsf{C}, \mathscr{E})$ be an exact category. We consider the conditions

The category C has enough $\mathscr{E}$-injectives and any countable coproduct of $\mathscr{E}$-injective objects has finite $\mathscr{E}$-injective dimension in C.	(ΣI)

The exact category C has enough $\mathscr{E}$-projectives and any countable direct product of $\mathscr{E}$-projectives objects has finite $\mathscr{E}$-projective dimension in C.	(ΠP)

They originally appeared in section 3 of [Pos11] and were stated for the abelian category of (graded) modules over (graded) rings. In practice both of these conditions are very hard to check, but Positselski in [Pos11] already identifies cases where they hold for modules over a ring R.

Example 2.2.1. Let R be a ring. We say that R satisfies (ΣI) or (ΠP) if the module category R-Mod does so.

(i) If R has finite global dimension it satisfies both (ΣI) and (ΠP).

(ii) If R is Noetherian it satisfies (ΣI). More generally, one can use Baer's criterion for injectivity to show that a ring S is right Noetherian if and only if countable coproducts of injective right S-modules are injective. This is known as the Bass-Papp Theorem [Bas59]. For a proof see (3.46) in [Lam99].

(iii) If R is Artinian it satisfies (ΠP). More generally, a ring S is right Artinian if and only if all products of projective right S-modules are projective. This is Chase's Theorem [Cha60]. For a proof see (4.47) in [Lam99].

◇

The condition on global dimension immediately generalizes to exact categories.

Proposition 2.2.2. $(\mathsf{C}, \mathscr{E})$ *be an exact category with enough $\mathscr{E}$-projectives and $\mathscr{E}$-injectives. If C has finite global dimension it satisfies* (ΣI) *and* (ΠP)*.*

One might ask whether there exist rings R that do not satisfy (ΣI) or (ΠP). Here is a way to get a counterexample for (ΠP).

Theorem 2.2.3 (Kaplansky, [Bas60])**.** *The following are equivalent for a commutative ring R.*

(i) Every R-module has projective dimension 0 or ∞.

(ii) There is an isomorphism

$$R \cong \bigoplus_{i=1}^{n} R_i,$$

where (R_i, m_i) are local left perfect rings.

Note, that Theorem 2.2.3 is reformulation of the theroem stated in [Bas60] using Bass' Theorem P form the same article.

Corollary 2.2.4. *Let R be a commutative non-Artinian ring. Then there is a countable product of projective left R-modules that is not projective as an R-module.*

Proposition 2.2.5. *Let R be a commutative local perfect ring that is non-Artinian. Then R does not satisfy* (ΠP)*.*

Proof. By Chase's Theorem ((4.47) in [Lam99]) we get a projective left R-module M, such that $\prod_{\mathbb{N}} M$ is not projective. It follows from Kaplansky's Theorem 2.2.3 that the projective dimension of the product is ∞. □

An example for a ring with the properties in Proposition 2.2.5 is the trivial extension $\mathbb{Q} \rtimes V$ of an infinite dimensional $\mathbb{Q}$-vector space V. See [DAR24].

Example 2.2.6. In Part II we will study condensed vector spaces and their Archimedean analytic ring structure of liquid vector spaces. The non-Archimedean analogue is the category of **solid abelian groups** Solid. For this example the only relevant information we need is that Solid is a complete and cocomplete abelian category, which admits compact projective generators which are exactly the condensed abelian groups $\prod_I \mathbb{Z}$. for some set I. This was shown in Theorem 5.8 (i) and Theorem 6.2 (i) in [Condensed]. It follows from the description of the generators that all their countable products are again projective. It follows that this is true for all projective objects in Solid, which shows that Solid satisfies (ΠP). ◇

2.2.2. The Graded Conditions

Let $(\mathsf{C}, \mathscr{E})$ be an exact category, that is symmetric monoidal with tensor product $\otimes$. Let A be a monoid in C. Again we consider the conditions (ΣI) and (ΠP) for the exact categories of left A-modules. If A is a graded algebra we also define graded variants.

The category $A^\bullet$-Mod has enough $\mathscr{E}$-injectives and any countable sum of $\mathscr{E}$-injective graded left $A^\bullet$-modules has finite $\mathscr{E}$-injective dimension as a graded left $A^\bullet$-module	(#ΣI)
The category $A^\bullet$-Mod has enough $\mathscr{E}$-projectives and any countable product of $\mathscr{E}$-projective graded left $A^\bullet$-modules has finite $\mathscr{E}$-projective dimension as a graded left $A^\bullet$-module	(#ΠP)

Since we will only use projectives and the contraderived category in Chapter 4 and Chapter 5 we focus on (ΠP) and (#ΠP).

Proposition 2.2.7. *A graded algebra $A^\bullet$ satisfies (#ΠP) if and only if A satisfies (ΠP).*

To prove this, we need to transfer some well-known facts from the abelian theory to our setting.

Lemma 2.2.8. *Let $A^\bullet$ be a graded algebra. A graded left $A^\bullet$-module $P^\bullet$ is graded projective if and only if it is ungraded projective as a left A-module.*

For one direction we will use the following Lemma.

Lemma 2.2.9. *Let $A^\bullet$ be a graded algebra and $M^\bullet, N^\bullet, P^\bullet$ be graded left $A^\bullet$-modules. Consider the diagram*

$$\begin{array}{ccc} & & P^\bullet \\ & & \downarrow{\scriptstyle f} \\ M^\bullet & \xrightarrow[g]{} & N^\bullet \end{array}$$

together with a homomorphism $h : P \to M$ such that $f = g \circ h$ in A-Mod. Then there is a graded homomorphism $\bar{h} : P^\bullet \to M^\bullet$ with $f = g \circ \bar{h}$ in A-GrMod.

Proof. Let $n \in \mathbb{Z}$ be an integer. Using the assumption $f = g \circ h$ and the fact that f and g are graded we get $g^n \circ \mathrm{pr}_{M^n} h|_{P^n} = f^n$, where pr_{M^n} is the projection $M^\bullet \to M^n$. Defining $\bar{h}^n := \mathrm{pr}_{M^n} h|_{P^n}$ for all $n \in \mathbb{Z}$, gives us the desired graded homomorphism. □

proof of Lemma 2.2.8. Let $P^\bullet$ be a graded left $A^\bullet$-module. Assume that P is projective as a left A-module. Then for a diagram as in Lemma 2.2.9 with g being an epimorphism we get a map $h : P \to M$ in A-Mod from projectivity. Applying Lemma 2.2.9 gives us the desired graded homomorphism.

Conversely, assume that $P^\bullet$ is graded projective. Then $P^\bullet$ is a coproduct of projective $A^\bullet$-modules. Since coproducts of projective are projective, $P^\bullet$ is also projective in A-Mod. □

In particular, graded projective resolutions give us projective resolutions and vice versa.

Corollary 2.2.10. *Let $A^\bullet$ be a graded ring and $M^\bullet$ a graded left $A^\bullet$-module. Then*

$$\mathrm{pd}_{A^\bullet,\mathscr{E}^\bullet}(M^\bullet) = \mathrm{pd}_{A,\mathscr{E}}(M).$$

Proof. It follows from Lemma 2.2.8 that every graded projective resolution of length n is also a length n projective resolution in A-Mod.

For the other direction consider a truncated graded projective resolution of $M^\bullet$

$$P^\bullet_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow P^\bullet_1 \longrightarrow P^\bullet_0 \longrightarrow M^\bullet.$$

By Lemma 2.2.8 this is also a projective resolution of M in A-Mod. If $\mathrm{pd}_{A,\mathscr{E}}(M) \leq n$ the kernel $\operatorname{Ker} d_{n-1}$ is projective in A-Mod by Proposition 1.3.25. Using Lemma 2.2.8 again, it follows that $\operatorname{Ker} d_{n-1}$ is also graded projective. □

proof of Proposition 2.2.7. First note, that the projective graded modules over $A^\bullet$ are given by degreewise projective A-modules. Thus A-Mod has enough projectives if and only if $\mathsf{A}^\bullet$-GrMod has enough projectives. Furthermore, A-Mod has countable products if and only if $\mathsf{A}^\bullet$-GrMod has countable products. The equivalence of the projective dimension conditions follows from Corollary 2.2.10. □

Another simplification is that it suffices to check (ΠP) products, where every factor is isomorphic.

Proposition 2.2.11. *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with countable products and enough $\mathscr{E}$-projectives. If for every $\mathscr{E}$-projective object P the $\mathscr{E}$-projective dimension of $\prod_{\mathbb{N}} P$ is finite, then $(\mathsf{C}, \mathscr{E})$ satisfies (ΠP).*

Proof. Let $P_i \in \mathsf{C}$ be $\mathscr{E}$-projective objects for $i \in \mathbb{N}$. Then $P := \coprod_{i=1}^{\infty} P_i$ is also $\mathscr{E}$-projective. The product $\prod_{i=1}^{\infty} P_i$ is a direct summand of $\prod_{\mathbb{N}} P$ and the latter has finite $\mathscr{E}$-projective dimension by the assumption. It follows that $\prod_{i=1}^{\infty} P_i$ also has finite $\mathscr{E}$-projective dimension. □

2.2.3. Positselski's Semiorthogonal Decomposition Theorem

Here is why (ΠP) is so important for the theory of contraderived categories.

Theorem 2.2.12 (Positselski). *Let $(\mathsf{C}, \mathscr{E})$ be an exact category with countable products and enough $\mathscr{E}$-projectives. Let A be a C DG algebra. Assume that the underlying graded algebra $A^\sharp$ satisfies (♯ΠP). Then The composition of functors*

$$\mathsf{Ho}(\mathsf{DG}\text{-}A, \mathscr{E}_{\mathrm{proj}}) \to \mathsf{Ho}(\mathsf{DG}\text{-}A) \to \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}A, \mathscr{E})$$

is an equivalence of triangulated categories.

The statement is a slight generalization of the Theorem in section 3.8 in [Pos11], but the proof applies unchanged. It also follows from Theorem 5.10 in [Pos21] with the exact DG category defined as in Example 5.3.(1) and 4.39 from [Pos21].

3. Topological Vector Spaces

Contents

In this chapter, we introduce the fundamental concepts of Functional Analysis and review essential aspects of the theory of topological vector spaces, aiming to provide a useful compilation of reference material. The resources referenced throughout include [SW99], [Con19], [Tre67] [MV97], [Jar81] and [NB10]. The last two, in particular, are especially useful for readers who wish to look up additional details.

3.1. Fundamental Definitions and Properties

3.1.1. Topological Fields and Vector Spaces

Definition 3.1.1. A **topological ring** pair (R, υ) is a pair given by a ring R equipped with a topology υ such that the addition

$$R \times R \to R, \quad (s,t) \mapsto s+t$$

and multiplication

$$R \times R \to R, \quad (s,t) \mapsto st$$

are continuous with respect to υ on R and the product topology of υ and υ on $R \times R$.

A **topological field** is a topological ring $(\mathbb{K}, \upsilon)$, such that $\mathbb{K}$ is a field and multiplicative inversion

$$\mathbb{K} \setminus \{0\} \to \mathbb{K} \setminus \{0\}, \quad s \mapsto s^{-1}$$

is continuous with respect to the subspace topology of υ on $\mathbb{K} \setminus \{0\}$.

Definition 3.1.2. Let $(\mathbb{K}, \upsilon)$ be a topological field. A **topological vector space** over $\mathbb{K}$ is a pair (V, τ) given by a vector space V equipped with a topology τ, such that addition and multiplication are jointly continuous:

(i) addition

$$V \times V \to V, \quad (x,y) \mapsto x+y$$

is continuous with respect to τ on V and the product topology of τ and τ on $V \times V$,

(ii) scalar multiplication

$$\mathbb{K} \times V \to V, \quad (\lambda, x) \mapsto \lambda x$$

is continuous with respect to τ on V and the product topology of υ and τ on $K \times V$.

We denote the category of topological vector spaces and continuous $\mathbb{K}$-linear maps with $\mathsf{TVS}_{\mathbb{K}}$.

While the definition above applies to any topological field, one almost always works with a field whose topology is defined by an absolute value.

Definition 3.1.3. Let $\mathbb{K}$ be a field. An **absolute value** on $\mathbb{K}$ is a function

$$\|\cdot\| :\to \mathbb{R}_{\geq 0}$$

such that for all $a, b \in \mathbb{K}$

(i) $\|a\| = 0$ if and only if $a = 0$,

(ii) $\|a \cdot b\| = \|a\| \cdot \|b\|$,

(iii) $\|a + b\| \leq \|a\| + \|b\|$ (triangle inequality).

If the triangle inequality is strengthened to

$$\|a + b\| \leq \max(\|a\|, \|b\|)$$

we say that $(\mathbb{K}, \|\cdot\|)$ is **non-Archimedean**. If this is not the case we say that $(\mathbb{K}, \|\cdot\|)$ is **Archimedean**.

Remark 3.1.4. A field $\mathbb{K}$ with absolute value $\|\cdot\|$ is non-Archimedean if and only if there is a $C \in \mathbb{R}_{\geq 0}$ with $\|n\| \leq C$ for all $n = 1 + \ldots + 1 \in \mathbb{K}$. ◇

A field $\mathbb{K}$ with absolute value $\|\cdot\|$ has a topology with a basis given by sets of the form

$$\{ a \in \mathbb{K} \mid a < \epsilon \}$$

for $\epsilon \in \mathbb{R}_{>0}$.

Example 3.1.5. The real and complex numbers $(\mathbb{R}, \|\cdot\|)$ and $(\mathbb{C}, \|\cdot\|)$ with their usual Euclidean norms are Archimedean fields.

Any field $\mathbb{K}$ with the trivial absolute value

$$\|a\|_0 := \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{else.} \end{cases}$$

is a non-Archimedean field. The p-adic numbers $(\mathbb{Q}_p, \|\cdot\|_p)$ with their p-adic absolute value are non-Archimedean. ◇

Both complete bornological spaces and condensed vector spaces provide a good framework for non-Archimedean functional analysis. However, we will focus on the Archimedean case. From now on, unless stated otherwise, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, with their Euclidean absolute value.

3.1.2. Subsets of Topological Vector Spaces

Other than open, closed and (covering) compact sets there are other important subsets of topological vector spaces.

Definition 3.1.6. Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a topological vector space.

For two subsets $A, B \subset V$ we say that A **absorbs** B if

$$\bigcup_{0<|t|\leq r} tB \subset A \qquad \text{for some } r > 0.$$

A subset X of V is called

(i) **bounded** if for each neighborhood U of 0 there is a $\lambda \in \mathbb{K}$ such that $X \subset \lambda U$,

(ii) **totally bounded** or **precompact** if for each neighborhood U of 0 there is a finite subset $X_0 \subset X$ such that $X \subset X_0 + U$,

(iii) **absorbing** if X absorbs every finite subset of V,

(iv) **bornivorous** if X absorbs every bounded subset of V,

(v) **convex** if for all $x, y \in X$ and all real $t \in (0, 1)$ we have $tx + (1 - t)y \in X$,

(vi) **balanced** or **circled** if for all $|t| \leq 1$ we have $tX \subset X$,

(vii) a **disk** or **absolutely convex** if X is convex and balanced,

(viii) a **barrel** if X is a closed absorbing disk.

3.1.3. Local Convexity

Throughout this thesis, we consider several full subcategories of $\mathsf{TVS}_{\mathbb{K}}$, most of which consist exclusively of locally convex topological vector spaces.

Definition 3.1.7. A topological vector space is **locally convex** if there is a neighborhood basis of convex sets at 0. We denote the full subcategory of all locally convex spaces in $\mathsf{TVS}_{\mathbb{K}}$ with $\mathsf{LCS}_{\mathbb{K}}$.

Remark 3.1.8. Some authors such as Schaefer in [SW99] require locally convex spaces to be Hausdorff. We denote the Hausdorff locally convex spaces with $\mathsf{SLCS}_{\mathbb{K}}$. ◇

There are other equivalent definitions of locally convex spaces.

Proposition 3.1.9 (Theorem 4.5.1 in [NB10])**.** *Let X be a topological vector space. The following are equivalent*

(i) X is locally convex,

(ii) the origin in X has a neighborhood basis consisting of barrels,

(iii) the origin in X has a neighborhood basis consisting of open disks,

(iv) every neighborhood basis of a point $x \in X$ contains, a convex neighborhood basis.

All the characterizations above are concerned with neighborhood bases. A different way to define locally convex spaces is to use seminorms.

Definition 3.1.10. Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a topological vector space. A **seminorm** on V is a map $p : V \to \mathbb{R}$ such that

(i) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in V$,

(ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Remark 3.1.11. It follows from the axioms of a seminorm $p : V \to \mathbb{R}$ that

$$p(0) = |0| \cdot p(0) = 0$$

and

$$p(x) = \frac{1}{2}\left(p(x) + |-1|p(x)\right) = \frac{1}{2}\left(p(x) + p(-x)\right) \geq 0$$

for all $x \in V$. ◇

Definition 3.1.12. Let $\mathcal{P}$ be a family of seminorms on a $\mathbb{K}$-vector space V. The topology τ **generated** by $\mathcal{P}$ is the **initial topology** with respect to $\mathcal{P}$, i.e. the coarsest topology such that all $p : (V, \tau) \to \mathbb{R}, p \in \mathcal{P}$ are continuous.

Proposition 3.1.13 (section 4.4 in [SW99]). *A topological vector space (V, τ) is locally convex if and only if there is a family of seminorms $\mathcal{P}$ on V generating τ. In this case $\mathcal{P}$ is a **fundamental system of seminorms** for V.*

Throughout Chapter 4 bounded sets will play a crucial role, and the following characterization will be useful.

Proposition 3.1.14 (Theorem 6.1.5 in [NB10]). *Let V be a locally convex space with a fundamental system $\mathcal{P}$ of seminorms. A subset $B \subset V$ is bounded if and only if $p(B)$ is bounded for all $p \in \mathcal{P}$.*

Remark 3.1.15. Most topological vector spaces that can be found in mathematics and physics are locally convex. Some interesting examples of non-locally convex spaces are given by ℓ^p-spaces with $0 < p < 1$. We will encounter them in Chapter 9. ◇

One of the defining features of locally convex spaces is that the Hahn-Banach theorem holds. A topological vector space $V \in \mathsf{TVS}_{\mathbb{K}}$ has the **Hahn-Banach extension property** if any continuous linear functional f on a linear subspace $U \subset V$ can be extended to a continuous linear functional on X.

Theorem 3.1.16 (Hahn-Banach, see chapter 7 [NB10]). *Let $V \in \mathsf{Vect}_{\mathbb{K}}$ be $\mathbb{K}$-vector space, $D \subset V$ an absorbing disk and $U \subset V$ a linear subspace of V. If f is a linear functional on U, such that $|f| \leq 1$ on $D \cap U$, then there exists a linear functional F on V extending f, such that $|F| \leq 1$ on D.*

In particular, every locally convex vector space has the Hahn-Banach extension property.

A partial converse is also true.

Theorem 3.1.17 (Theorem 4.8 in [KPR84]). *Every complete metrizable topological vector space with the Hahn-Banach extension property is locally convex.*

3.1.4. Completeness

Whether a topological space is (Cauchy) complete makes sense for metric spaces and more generally for uniform spaces. The latter are defined via uniformities and one can show, that every topological vector space has a uniformity. We can also use Cauchy nets.

Definition 3.1.18. A topological vector space $V \in \mathsf{TVS}_{\mathbb{K}}$ is **complete** if every Cauchy net converges. A topological vector space $V \in \mathsf{TVS}_{\mathbb{K}}$ is **sequentially-complete** if all Cauchy sequences converge.

Definition 3.1.19. A **pseudometric** d on a topological space X is a function

$$d : X \times X \to \mathbb{R}_{\geq 0},$$

such that for all $x, y, z \in X$,

(i) $d(x, x) = 0$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Note that, unlike a metric, a pseudometric allows $d(x, y) = 0$ for $x \neq y$.
For pseudometrizable spaces, which include Banach and Fréchet spaces, it suffices to check Cauchy sequences.

Proposition 3.1.20 (Proposition 3.2.2 in [Jar81])**.** *Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a topological vector space with topology induced by a pseudometric d. Then V is complete if and only if only V is sequentially-complete.*

Notation 3.1.21. We denote the full subcategory of $\mathsf{LCS}_{\mathbb{K}}$ of complete locally convex spaces by $\mathsf{CLCS}_{\mathbb{K}}$.

Proposition 3.1.22 (Corollary 3.5.3 in [Jar81])**.** *Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a Hausdorff topological vector space. A subset $A \subset V$ is compact if and only if A is totally bounded and complete.*

Definition 3.1.23. Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a topological vector space. We say that V is **quasi-complete** if every closed and bounded subset is complete and V is **pseudo-complete** if every closed and totally bounded subset is complete.

We have the following hierarchy

$$\left\{\begin{matrix}\text{Complete}\\ \text{Vector Spaces}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Quasi-complete}\\ \text{Vector Spaces}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Pseudo-complete}\\ \text{Vector Spaces}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{All Topological}\\ \text{Vector Spaces}\end{matrix}\right\}.$$

3.1.5. Limits and Colimits

The standard terminology in the context of topological vector spaces is to refer to **projective limits** of **projective systems** and **inductive limits** of **inductive systems**. In categorical terms, these correspond to codirected limits and directed colimits. We will use the slightly more general notions of cofiltered limits and filtered colimits, although most instances in this thesis will involve diagrams given by posets.

Consider a (co)filtered diagram of locally convex vector spaces

$$I \to \mathsf{LCS}_{\mathbb{K}}, \quad i \mapsto (V_i, \tau_i)$$

with topologies $\tau_i, i \in I$. For cofiltered limits it is irrelevant if we take them in topological spaces Top, topological vector spaces $\mathsf{TVS}_{\mathbb{K}}$ or locally convex spaces $\mathsf{LCS}_{\mathbb{K}}$. In any case, the limit topology is linear and locally convex. In particular, products in $\mathsf{LCS}_{\mathbb{K}}$ are given by the usual product topology. See section 3. and 6.6 in [Jar81] for details and proofs. In contrast, filtered colimits are way worse behaved. A good overview is provided by [Bie88]. Interesting examples of badly behaved colimits are given in [Kōm64].

A special case of colimits are coproducts/direct sums, which we denote with $\amalg$.

Definition 3.1.24. Let V be a topological vector space and $W \subset V$ a subspace. Then W is a **complemented subspace** of V, if there is another subspace $U \subset V$, such that $W \amalg U \cong V$.

3.1.6. Normability and Metrizability

Definition 3.1.25. A topological vector space (V, τ) is **normable/ seminormable/ metrizable/ pseudometrizable** if its topology τ can be obtained from a norm/ seminorm/ metric/ pseudometric on V.

We denote the full subcategory of $\mathsf{TVS}_{\mathbb{K}}$ of normable spaces with $\mathsf{Norm}_{\mathbb{K}}$ and the full subcategory of $\mathsf{TVS}_{\mathbb{K}}$ of seminormable spaces with $\mathsf{SemiN}_{\mathbb{K}}$.

Remark 3.1.26. The metric in the definition of a metrizable topological vector space need not be translation-invariant, i.e.

$$d(x, y) = d(x + z, y + z) \quad \text{for all } x, y, z \in V$$

is not a requirement. In particular the induced uniformity might differ from the canonical uniformity associated to a topological vector space. ◇

Definition 3.1.27. Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a seminormable vector space with a chosen seminorm p. Then

$$B_{\leq r}(V) = B_{\leq r}(V, p) := \{ v \in V \mid p(v) \leq r \}$$

is the **(closed) ball of radius** $r \in \mathbb{R}_{\geq 0}$ of V (with respect to p) and

$$B_{\leq 1}(V) = B_{\leq 1}(V, p) := \{ v \in V \mid p(v) \leq 1 \}$$

is the **(closed) unit ball** of V (with respect to p).

There is a very useful normability criterion due to Kolmogoroff.

Theorem 3.1.28 (Kolmogoroff [Kol34], see 2.1 in [SW99])**.** *A topological vector space is normable if and only if it is Hausdorff and admits a bounded, convex neighborhood of* 0*.*

For metrizability there is a result due to Birkhoff and Kakutani. Originally stated for topological groups we can reformulate it to get the following for topological vector spaces.

Theorem 3.1.29 (Birkhoff-Kakutani, see Theorem 6.1 in [SW99])**.** *Let* (V, τ) *be a topological vector space. Then the following conditions are equivalent:*

(i) V *is Hausdorff and there is a countable basis of neighborhoods of 0.*

(ii) There is a translation-invariant metric on V *that induces the topology* τ *on* V*,*

(iii) (V, τ) *is metrizable as a topological space,*

(iv) (V, τ) *is metrizable as a topological vector space.*

3.1.7. Separation Hierarchy

There is a big zoo of separation axioms for topological spaces. We recall some of them, mainly for the purpose to see that they all agree in the case of topological vector spaces.

Let X be a topological space. Some of the separation axioms are

[T_0] (Kolmogoroff) Any two distinct points are topologically distinguishable, i.e.

For $x, y \in X$ with $x \neq y$ there is an open set $U \subset X$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$.

[T_1] Any two distinct points are separated, i.e.

For $x, y \in X$ with $x \neq y$ there is a neighborhood U of x, such that $y \notin U$,

[T_2] (Hausdorff) Any two distinct points are separated by neighborhoods, i.e.

For $x, y \in X$ with $x \neq y$ there are neighborhoods U of x and V of y, such that $U \cap V = \emptyset$,

[T_3] (regular Hausdorff) T_0 and **regular**, that is any closed set and point not contained in the set are separated by neighborhoods, i.e.

For a closed set $F \in X$ and a point $x \in X$ with $x \notin F$ there is a neighborhood U of x and V of F, such that $V \cap U = \emptyset$.

[$T_{3\frac{1}{2}}$] (Tychonoff) T_0 and **completely regular**, that is any closed set and point not contained in the set are separated via a continuous function, i.e.

For a closed set $F \in X$ and a point $x \in X$ with $x \notin F$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f_F = 0$.

These 5 axioms have hierarchy indicated by their subscripts:

$$T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1 \implies T_0$$

But for topological vector spaces we have the following.

Proposition 3.1.30 (Theorem 8.4 in [HR79])**.** *Let (V, τ) be a topological vector space such that the underlying topological space is T_0. Then the space is also $T_{3\frac{1}{2}}$ (Tychonoff).*

In particular, every T_0 topological vector spaces is also Hausdorff, and Hausdorff will be the one separation property that we use in statements if needed.

For example, only with the Hausdorff assumption we can recover the uniqueness of finite dimensional topological vector spaces with given dimension, that we have in the algebraic setting. See Theorem 3.1.31.

3.1.8. Finite Dimensional Vector Spaces

Theorem 3.1.31 (Theorem 3.5.6 in [Jar81])**.** *Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be Hausdorff. The following are equivalent*

1. *V is finite dimensional,*
2. *$V \cong \mathbb{K}^n$, where $n = \dim V$,*
3. *V has a precompact neighborhood of the origin.*

Corollary 3.1.32. *Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be Hausdorff and let W be a finite dimensional subspace. Then W is closed in V.*

Example 3.1.33. Consider the $\mathbb{R}$-vector space $\mathbb{R}^2$, where the topology κ is given by the seminorm

$$p(x, y) = |x|.$$

Then $(\mathbb{R}^2, \kappa)$ is a non Hausdorff topological vector space. ◇

3.1.9. Dual Spaces

Given a $\mathbb{K}$-vector space W its **algebraic dual** is the vector space

$$W^* := \mathrm{Hom}(W, \mathbb{K}) = \{ f : W \to \mathbb{K} \mid f \text{ is } \mathbb{K}\text{-linear} \}$$

equipped with pointwise addition and scalar multiplication. Given a topological vector space V we restrict V^* to the vector space containing only the continuous linear maps $V \to \mathbb{K}$. There are different topologies that we can put on this dual vector space.

Definition 3.1.34. Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a topological vector space and $\mathcal{A}$ be a collection of bounded subsets of V. The **continuous dual** of V is the vector space

$$\{ f : V \to \mathbb{K} \mid f \text{ is } \mathbb{K}\text{-linear and continuous} \} .$$

The **topology of uniform convergence on** $\mathcal{A}$ on the continuous dual is defined by the collection seminorms

$$\|f\|_A = \sup_{x \in A} |f(x)| \qquad \text{for all } A \in \mathcal{A}.$$

Some important special cases are the following.

(i) The **strong topology** is the topology of uniform convergence on all bounded subsets. We denote the **strong dual** of V with V'.

(ii) The **stereotype topology** is the topology of uniform convergence on all totally bounded subsets. We denote the **stereotype dual** of V with $V^\vee$.

(iii) The **weak topology** is the topology of uniform convergence on all finite subsets.

Note, that by definition all of the above topologies are locally convex. If one wants to work with duals of non-locally convex vector spaces the question whether the dual **separates points**, is very important. See [AEK06] for the general theory. We will stick to the locally convex setting.

Definition 3.1.35. Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex vector space. If the (strong dual) **evaluation map**

$$\iota'_V : V \to V'', \qquad \iota_V(x)(f) = f(x), \quad x \in V, f \in V',$$

is an isomorphism of topological vector spaces, we say that V is **reflexive**. If V is Hausdorff and the (stereotype dual) **evaluation map**

$$\iota^\vee_V : V \to V^{\vee\vee}, \qquad \iota_V(x)(f) = f(x), \quad x \in V, f \in V^\vee,$$

is an isomorphism of topological vector spaces, we say that V is **stereotype**.

Remark 3.1.36. We will use some elements of the theory of Stereotype spaces in Chapter 8. A good reference is the book [Akb22], which also contains most of the material from [Akb03]. ◇

Remark 3.1.37. The category of reflexive spaces is rather small. For example, not every Banach space is reflexive. In contrast, the category of stereotype spaces is quite large. Every quasi-complete barreled space is stereotype. Even more generally a Hausdorff locally convex space is stereotype if and only if it is pseudo-complete and **pseudo-saturated** by Theorem 4.1.1 in [Akb22]. We say that a Hausdorff locally convex space V is pseudo-saturated if every closed **capacious** disk $D \subset V$ is a neighborhood of the origin, where capacious means that for all totally bounded subsets $B \subset V$ there is a finite set $J \subset V$ with $S \subset J + D$. ◇

3.1.10. Tensor Products

There are multiple ways to endow the algebraic tensor product of two topological vector spaces with a suitable topology. Grothendieck systematically investigated these topologies in [Gro53] and [Gro55]. Among the notable examples of topological tensor products are the projective, injective, and inductive topologies. In this work, we concentrate on projective tensor products, as they are the only ones we use explicitly.

Definition 3.1.38. Let $V, W \in \mathsf{LCS}_{\mathbb{K}}$ be locally convex spaces. The **π-topology** on $V \otimes W$ is the finest topology, which makes the canonical map

$$(\cdot) \otimes (\cdot) : V \times Y \to V \otimes W, \qquad (v, w) \mapsto v \otimes w$$

continuous. The **projective tensor product** $V \otimes_\pi W$ of V and W is the algebraic tensor product $V \otimes W$ equipped with the π-topology

Remark 3.1.39. The *pi*-topology from Definition 3.1.38 can be defined as the finest topology, since it is always locally convex for $V, W \in \mathsf{LCS}_{\mathbb{K}}$ by Proposition 6.6.2 in [Jar81]. ◇

We can also describe the seminorms, which define the π-topology. We state this here, but use some definitions from Chapter 4.

Proposition 3.1.40 (section 15.1 in [Jar81])**.** *Let $V, W \in \mathsf{LCS}_{\mathbb{K}}$ be locally convex spaces with fundamental systems of seminorms $\mathcal{P}_V$ and $\mathcal{P}_W$. For two seminorms $\mathcal{P}_V, q \in \mathcal{P}_W$, we define $p \otimes q$ to be the Minkowski functional of the disked hull of the set $B_{\leq 1}(V, p) \otimes B_{\leq 1}(W, q)$ of all elementary tensors formed from elements of the unit balls of p and q. A fundamental system of seminorms of $V \otimes_\pi W$ is given by*

$$\{\, p \otimes q \mid p \in \mathcal{P}_V, q \in \mathcal{P}_W \,\}.$$

Proposition 3.1.41 (Proposition 15.1.3 in [Jar81])**.** *If V, W are Hausdorff, then so is $V \otimes_\pi W$.*

While separation carries over to the projective tensor product of two separated spaces, completeness does not. The solution is to simply add completion to the definition, when one considers complete locally vector spaces or a subcategory.

Definition 3.1.42. Let $V, W \in \mathsf{CLCS}_{\mathbb{K}}$ be complete locally convex spaces. The **completed projective tensor product** of V, W is defined by

$$V \hat{\otimes}_\pi W := \widehat{V \otimes_\pi W}.$$

The inductive tensor product $\otimes_\iota$ differs from the projective tensor product, by only requiring, that the canonical map from Definition 3.1.38 is separately continuous.

The definition of the injective tensor product $\otimes_\epsilon$ is slightly more involved and can be found, for example, in chapter 16 in [Jar81]. Its completion is denoted by $\hat{\otimes}_\epsilon$.

3.2. Categories of Topological Vector Spaces

Topological vector spaces come in many types, each with specific topological constraints to suit various applications. We review the types relevant to this thesis and compile some of their categorical and homological properties.

3.2.1. Baire spaces

The Baire category theorem is a fundamental result from topology, but plays an invaluable role in functional analysis. It is used in the proofs of the open mapping theorem, the closed graph theorem, and the uniform boundedness principle.

Definition 3.2.1. Let X be a topological space. We say that X is a **Baire space** if every countable intersection of dense open sets is dense in X.

Theorem 3.2.2 (Baire category theorem, see Theorem 34 in [Kel75])**.** *We have the following classes of Baire spaces.*

(**BCT1**) *Every complete pseudometric spaces is a Baire space.*

(**BCT2**) *Every locally compact regular space is a Baire space.*

In particular, all completely metrizable and all locally compact Hausdorff spaces are Baire spaces.

3.2.2. Locally Convex Spaces

	Locally convex spaces $\mathsf{LCS}_{\mathbb{K}}$		
Complete	Yes	Proposition 3.2.3	
Cocomplete	Yes	Proposition 3.2.3	
Quasi-Abelian	Yes	Proposition 3.2.4	
	$\mathsf{LCS}_{\mathbb{K}}$ with exact structure $\mathfrak{max}$		
Enough Projectives	No, only $\coprod \mathbb{K}$	Proposition 3.2.5	

Proposition 3.2.3. *The categories* $\mathsf{LCS}_{\mathbb{K}}$*,* $\mathsf{SLCS}_{\mathbb{K}}$ *and* $\mathsf{CLCS}_{\mathbb{K}}$ *are complete and cocomplete.*

Proposition 3.2.4 (Propositions 2.1.11 and 3.1.8 in [Pro00])**.** *The categories* $\mathsf{LCS}_{\mathbb{K}}$ *and* $\mathsf{SLCS}_{\mathbb{K}}$ *are quasi-abelian.*

However, the category $\mathsf{CLCS}_{\mathbb{K}}$ is not quasi-abelian as shown in Proposition 4.1.14 in [Pro00].

Proposition 3.2.5 (section 6 in [Köt66] and [Gei72])**.** *The categories* $\mathsf{LCS}_{\mathbb{K}}$ *and* $\mathsf{SLCS}_{\mathbb{K}}$ *do not have enough* $\mathfrak{max}$*-projective objects. All* $\mathfrak{max}$*-projectives objects in* $\mathsf{LCS}_{\mathbb{K}}$ *and* $\mathsf{SLCS}_{\mathbb{K}}$ *are coproducts of* $\mathbb{K}$*.*

3.2.3. Banach Spaces

When considering Banach spaces, we do not treat the norm as part of the structure. Instead, this role is taken by the topology, leading to the following definition.

Definition 3.2.6. A **Banach space** is a topological vector space that is normable and complete.

We denote the full subcategory of all Banach spaces in $\mathsf{TVS}_{\mathbb{K}}$ with $\mathsf{Ban}_{\mathbb{K}}$.

Example 3.2.7. Let Y be a set and $1 \leq p < \infty$. The vector space

$$\ell^p(Y) := \left\{ f : Y \to \mathbb{K} \;\middle|\; (|f(y)|^p)_{y\in Y} \text{ is summable} \right\}$$

with topology given by the norm

$$\|f\|_p := (\Sigma_{y\in Y} |f(y)|^p)^{\frac{1}{p}}$$

is the space of **absolutely p-summable functions** on Y. If $Y = \mathbb{N}$ we write $\ell^p := \ell^p(\mathbb{N})$. ◇

Example 3.2.8. Let Y be a set. The vector space

$$\ell^\infty(Y) := \left\{ f : Y \to \mathbb{K} \;\middle|\; \sup_{y \in Y} |f(y)| < \infty \right\}$$

with topology given by the norm

$$\|f\|_\infty := \sup_{y \in Y} |f(y)|$$

is the space of **bounded functions** on Y. If $Y = \mathbb{N}$ we write $\ell^\infty := \ell^\infty(\mathbb{N})$. ◇

	Banach spaces $\mathsf{Ban}_{\mathbb{K}}$	
Countable Products	No	Proposition 3.2.9
Countable Coproducts	No	Proposition 3.2.9
Quasi-Abelian	Yes	Proposition 3.2.10
Symmetric Mondoidal	Yes, $\mathsf{Ban}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$	Proposition 3.2.16
	$\mathsf{Ban}_{\mathbb{K}}$ with exact structure $\mathfrak{max}$	
Enough Projectives	Yes, $\ell^1(X)$	Proposition 3.2.12
Enough Injectives	Yes, $\ell^\infty(X)$ and summands	Proposition 3.2.11
Global Dimension	∞	Proposition 3.2.13

Banach spaces are closely related to normable spaces and seminormable spaces the forgetful functor

$$\mathsf{Ban}_{\mathbb{K}} \to \mathsf{Norm}_{\mathbb{K}} \to \mathsf{SemiN}_{\mathbb{K}}$$

have left adjoints

$$\widehat{(\cdot)} : \mathsf{Norm}_{\mathbb{K}} \to \mathsf{Ban}_{\mathbb{K}}, \qquad V \mapsto \widehat{V}$$

and

$$\mathrm{sep} : \mathsf{SemiN}_{\mathbb{K}} \to \mathsf{Norm}_{\mathbb{K}}, \qquad V \mapsto V/\{\bar{0}\},$$

called the **completion functor** $\widehat{(\cdot)}$ and **separation functor** sep.

If we fix a norm or a seminorm on all Banach/ normable/ seminormable spaces, we can consider their subcategories with non-expanding maps $\mathsf{Ban}_{\mathbb{K}}^{\leq 1}$, $\mathsf{Norm}_{\mathbb{K}}^{\leq 1}$ and $\mathsf{SemiN}_{\mathbb{K}}^{\leq 1}$. These categories play an important role in the theory of Banach/ normable/ seminormable spaces, since they are complete and cocomplete and one can transfer properties to the categories with bounded maps, as long as no diagrams with maps of increasing norms are involved. See section 3.2 in [Sch99].

The theory of $\mathsf{Ban}_{\mathbb{K}}$, $\mathsf{Norm}_{\mathbb{K}}$ and $\mathsf{SemiN}_{\mathbb{K}}$ is very similar and we focus only on $\mathsf{Ban}_{\mathbb{K}}$.

Proposition 3.2.9. *The category* $\mathsf{Ban}_{\mathbb{K}}$ *has no non-trivial infinite products or coproducts.*

Proposition 3.2.10 (Section 4.3 in [Pro00])**.** *The category* $\mathsf{Ban}_{\mathbb{K}}$ *is quasi-abelian.*

Proposition 3.2.11. *The category* $\mathsf{Ban}_{\mathbb{K}}$ *has enough* $\mathfrak{max}$*-injectives. All* $\mathfrak{max}$*-injectives are* $\ell^\infty(Y)$ *for sets* Y*, and complemented subspaces of these.*

Proposition 3.2.12 (section 3 in [Köt66])**.** *The category* $\mathsf{Ban}_{\mathbb{K}}$ *has enough* $\mathfrak{max}$*-projectives. All* $\mathfrak{max}$*-projectives* $\ell^1(Y)$ *for sets* Y*.*

Proposition 3.2.13 (Proposition 5.1 in [SCG21a], [Wod94])**.** *We have for the global dimension*

$$\mathrm{g}\ell\left(\mathsf{Ban}_{\mathbb{K}}, \mathfrak{max}\right) = \infty.$$

Example 3.2.14. The closed subspace of null-sequence c_0 in ℓ^∞ is not complemented. Thus,

$$c_0 \rightarrowtail \ell^\infty \twoheadrightarrow \ell^\infty / c_0$$

does not split. ◇

More generally, we have the following.

Theorem 3.2.15 (Lindenstrauss & Tzafriri, Theorem 1 in [LT71])**.** *Let* $E \in \mathsf{Ban}_{\mathbb{K}}$ *be a Banach space. All* $\mathfrak{max}$*-short exact sequences of the form*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

split, if and only if E *is isomorphic to a Hilbert space.*

Proposition 3.2.16. *The category* $\mathsf{Ban}_{\mathbb{K}}$ *is closed symmetric monoidal with the completed projective tensor products* $\hat{\otimes}_\pi$ *and unit* $\mathbb{K}$*.*

In [LP68] Lindenstrauss and Pelczyński introduced so called $\mathcal{L}_p$**-spaces**, which are locally similar to L_p. We do not provide a definition here, but we want to remark, that it follows from Theorem 2.20 in [Rya02] they are flat if $p = 1$.

Proposition 3.2.17. *Let* $V \in \mathsf{Ban}_{\mathbb{K}}$ *be a* $\mathcal{L}_1$*-space. Then* V *is flat with respect to* $\hat{\otimes}_\pi$*.*

Definition 3.2.18. Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. If there is another Banach space $A \in \mathsf{Ban}_{\mathbb{K}}$, such that B is the strong dual of A, we say that B is a **dual Banach space** and A is a **predual** of B.

We will use the Banach-Alaoglu theorem in Chapter 8, but we will only need the following special case of Banach spaces. Refer to section 8.2 in [NB10] for the definition of the weak*-topology.

Theorem 3.2.19 (Banach-Alaoglu, see Theorem 8.4.1 in [NB10])**.** *Let* $V \in \mathsf{Ban}_{\mathbb{K}}$ *be a Banach space. The unit ball of its strong dual* V' *is compact in the weak***-topology.*

3.2.4. Fréchet Spaces

Definition 3.2.20. A **Fréchet space** F is a topological vector space that is

(i) locally convex,

(ii) metrizable by a translation invariant metric, i.e. a metric

$$d : F \times F \to F, \quad \text{such that } d(x, y) = d(x+z, y+z) \text{ holds for all } x, y, z \in F,$$

(iii) complete.

We denote the full subcategory of all Fréchet spaces in $\mathsf{TVS}_{\mathbb{K}}$ with $\mathsf{Fré}_{\mathbb{K}}$.

Another characterization is given by seminorms, which is a corollary of the Birkhoff-Kakutani Theorem 3.1.29.

Proposition 3.2.21. *Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex space. Then V is a Fréchet space if and only if E has a conuntable system of seminorms. In this case, the seminorms can be chosen to be $\mathcal{P} = \{ \|\cdot\|_i \mid i \in \mathbb{N} \}$ with*

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3 \leq \dots$$

	Fréchet spaces $\mathsf{Fré}_{\mathbb{K}}$	
Countable Products	Yes	Proposition 3.2.22
Countable Coproducts	No	Proposition 3.2.22
Quasi-Abelian	Yes	Proposition 3.2.23
Symmetric Mondoidal	Yes, $\mathsf{Fré}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$	Proposition 3.2.25
	$\mathsf{Fré}_{\mathbb{K}}$ with exact structure $\mathfrak{max}$	
Enough Projectives	No	Proposition 3.2.24
Enough Injectives	Yes	Proposition 3.2.24

Proposition 3.2.22. *The category $\mathsf{Fré}_{\mathbb{K}}$ has countable products. It has no non-trivial uncountable products or non-trivial infinite coproducts*

Proposition 3.2.23 (Proposition 4.4.5 in [Pro00])**.** *The category $\mathsf{Fré}_{\mathbb{K}}$ is quasi-abelian.*

Proposition 3.2.24 (Proposition 4.4.6 in [Pro00], [Gei78])**.** *The category $\mathsf{Fré}_{\mathbb{K}}$ has enough $\mathfrak{max}$-injective objects. It does not have enough $\mathfrak{max}$-projective objects.*

Proposition 3.2.25. *The category $\mathsf{Fré}_{\mathbb{K}}$ is closed symmetric monoidal with the completed projective tensor product $\hat{\otimes}_\pi$ and unit $\mathbb{K}$.*

Example 3.2.26. Let X be a real smooth n-dimensional manifold. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable covering of X by charts U_i. Since each U_i is homeomorphic to $\mathbb{R}^n$ we can choose compact subsets $K_{i,j} \subset U_i, j \in N$, such that every other compact subset of is contained in one $K_{i,j}$. Let $\{\rho_i\}_{i\in\mathbb{N}}$ be a partition of unity subordinate to $\{U_i\}_{i\in\mathbb{N}}$. For all $i, j \in \mathbb{N}$ and multi-indices $\alpha \in {\mathbb{N}_0}^n$ define the seminorms

$$\|f\|_{i,j,\alpha} := \sup_{x\in K_{i,j}} \|\partial^\alpha(\rho_i f(x))\|| \qquad \text{for } f \in \mathcal{C}^\infty(X).$$

Since the collection of seminorms

$$\left\{ \|\cdot\|_{i,j,\alpha} : \mathcal{C}^\infty(X) \to \mathbb{R}_{\geq 0} \;\middle|\; i, j \in \mathbb{N}, \alpha \in {\mathbb{N}_0}^n \right\}$$

is countable we get a Fréchet topology on the $\mathbb{R}$-vector space $\mathcal{C}^\infty(X)$ by Theorem 3.1.29. ◇

In the rest of this chapter we collect some more classes of spaces, which will appear once or twice in this thesis.

3.2.5. Barreled Spaces

Definition 3.2.27. A topological vector space V is a **barrelled space** if every barrel in V is a neighborhood of 0.

Proposition 3.2.28 (see [SW99] chapter II 7.1)**.** *Let V be a locally convex Baire space. Then V is barreled.*

In particular, all Fréchet spaces are barreled.

3.2.6. Montel Spaces

Definition 3.2.29. Let V be a Hausdorff locally convex space. We say that V has the **Heine-Borel property** if a subset $K \subset V$ is compact if and only if K is closed and bounded in V.

Definition 3.2.30. A topological vector space V is a **Montel space** if V is barreled and has the Heine-Borel property.

Remark 3.2.31. A Banach space is Montel if and only if it is finite dimensional. ◇

Proposition 3.2.32 (see 19 in [SW99])**.** *Montel spaces are closed under limits and countable colimits.*

3.2.7. Nuclear Spaces

We have seen that various tensor products can be defined for topological vector spaces. While these tensor products generally differ, there exists a particular class of spaces for which they coincide. This class, known as nuclear spaces, was introduced and studied by Grothendieck in [Gro55].

Definition 3.2.33. A locally convex space $V in \mathsf{LCS}_{\mathbb{K}}$ is nuclear if the canonical map $V\hat{\otimes}_\pi W \to V\hat{\otimes}_\epsilon W$ is an isomorphism for all locally convex spaces W. We denote the full subcategory of $\mathsf{LCS}_{\mathbb{K}}$ of nuclear spaces with $\mathsf{Nuc}_{\mathbb{K}}$.

Proposition 3.2.34 (chapter II theoreme 9 in [Gro55])**.** *The category $\mathsf{Nuc}_{\mathbb{K}}$ of nuclear spaces is complete and has countable colimits.*

Proposition 3.2.35 (Theorem 7.5 in [SW99])**.** *Let V, W be nuclear spaces. Then the (projective) tensor product $V \otimes W$ as well as its completion $V\hat{\otimes}W$ are nuclear.*

Proposition 3.2.36 (Proposition 50.6 in [Tre67])**.** *A Fréchet space $F \in \mathsf{Fré}_{\mathbb{K}}$ is nuclear if and only if its strong dual F' is nuclear*

Proposition 3.2.37 (See statement 50.12 in [Tre67])**.** *Every nuclear Fréchet space is separable and Montel.*

Proposition 3.2.38 (see chapter 51 in [Tre67])**.** *Let X be a real smooth manifold. Then the Fréchet space $\mathcal{C}^\infty(X)$ from Example 3.2.26 is nuclear.*

3.2.8. LB-spaces

Definition 3.2.39. Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex space. Then V is a LB-space if it is a countable filtered colimit of Banach spaces in LCS. If all transition maps of the colimit are embeddings, V is a **strict LB-space**.

3.2.9. DF spaces

Definition 3.2.40. Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex space. If V is isomorphic to the strong dual of a Fréchet space, then V is a **DF-space**. If V is isomorphic to the strong dual of a nuclear Fréchet space, then V is a **DNF-space**.

Proposition 3.2.41 (Corollary 2.6 in [Vog00])**.** *Let V be a DF space. Then V has a countable fundamental system of bounded sets. That is, there is a countable set $\mathcal{B}$ of bounded sets, such that for every bounded set $C \subset V$ there is a $B \in \mathcal{B}$ and an $r \in \mathbb{R}_{>0}$ with $C \subset rB$.*

4. Bornological Spaces

Contents

Bornologies and bornological spaces were never a mainstay in the field of functional analysis. The idea of emphasizing bounded sets dates back to Mackey's thesis on duality theory in in 1942 [Mac42]. During the 1950s and 1960s, many authors recognized that some major problems in the theory of locally convex spaces could be addressed more effectively by considering spaces of bounded sets [HN70]. Much of the early theory was developed by Henri Hogbe-Nlend [HN66, HN71, HN73, HN77] and Lucien Waelbroeck [Wae67, Wae71]. Other important early references are [Fer72] and [Hou73]. In 2000 Fabienne Prosmans and Jean-Pierre Schneiders proved that bornological spaces and some interesting subcategories are quasi-abelian and compared them to Ind-completions [PS00]. In the early 2000's Ralf Meyer used bornologies for analytic cyclic homology [Mey99, Mey04, Mey07]. Recently, there is renewed interest in using the quasi-abelian structure of bornological spaces in both the archimedean and non-archimedean case as a foundation for (derived) analytic geometry [BBB16, BBBK18, BBK23, BBKK24].

After reviewing the basic definitions and results in Section 4.1 and Section 4.2 we proceed to study homological dimensions in Section 4.3 and Section 4.4.

4.1. Bornological Sets

4.1.1. Bornologies

Definition 4.1.1. Let X be a Set. A **bornology** on X is a family $\mathcal{B} \neq \emptyset$ of subsets of X such that

(i) $\mathcal{B}$ covers X:

$$\bigcup_{B \in \mathcal{B}} B = X,$$

(ii) $\mathcal{B}$ is stable under inclusions:

$$\text{if } B \in \mathcal{B}, \text{ then every subset of } B \text{ is in } \mathcal{B},$$

(iii) $\mathcal{B}$ is stable under finite unions:

$$\text{if } \; B_1, \dots B_n \in \mathcal{B}, \quad \text{then } \; \bigcup_{i=1}^{n} B_i \; \in \mathcal{B}.$$

We say that $(X, \mathcal{B})$ is a **bornological set** and the elements of $\mathcal{B}$ are **bounded sets** (with respect to $\mathcal{B}$).

Given two bornological sets $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ a map of sets $f : X \to Y$ is **bounded** if $f(B) \in \mathcal{B}_Y$ holds for all $B \in \mathcal{B}_X$. A **morphism of bornological sets** $g : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is given by a bounded map of sets $g : X \to Y$.

We denote the category of bornological sets and bounded maps by BornSet.

Remark 4.1.2. Bornological sets are often called bornological spaces. We reseverve the latter for bornological vector spaces. ◇

Similar to topological spaces we have notions of base and subbase.

Definition 4.1.3. Let $(X, \mathcal{B}) \in \mathsf{BornSet}$ be a bornological set. A family of sets $\mathcal{A}$ is a **base** of $\mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$ and for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ with $B \subset A$. A family of sets $\mathcal{S}$ is a **subbase** of $\mathcal{B}$ if $\mathcal{S} \subset \mathcal{B}$ and

$$\left\{ \bigcup_{i=1}^{n} S_i \;\middle|\; n \in \mathbb{N} \text{ and } S_1, \dots S_n \in \mathcal{S} \right\}.$$

is a base of $\mathcal{B}$.

4.1.2. Constructions with Bornological Sets

Example 4.1.4. Let X be a set. The finest bornology on X is the **discrete bornology** $\mathcal{P}(X) = \{B \subset X\}$. The coarsest bornology on X is the **indiscrete bornology** $\{B \subset X \,|\, B \text{ finite}\}$. ◇

Example 4.1.5. Let $(X, \mathcal{B}_X)$ be a bornological set and $Y \subset X$. The collection

$$\mathcal{B}_Y \;:=\; \{\, B \cap Y \mid B \in \mathcal{B}_X \,\}$$

ist called the **subspace bornology** on Y. ◇

Example 4.1.6. Given a family of bornological sets $(X_i, \mathcal{B}_i)_{i \in I}$ indexed by I we can define a **product bornology** $\mathcal{B}$ on $X = \prod_{i \in I} X_i$ via

$$\mathcal{B} \;=\; \left\{ \prod_{i \in I} B_i \;\middle|\; B_i \in \mathcal{B}_i \text{ for all } i \in I \right\}$$

and get a bornological set $(X, \mathcal{B})$. ◇

4.1.3. Bornologies from Topology

Example 4.1.7. Let X be a T_1 topological space. Then

$$\mathcal{C} \;=\; \{\, C \subset X \mid C \text{ is relatively compact in } X \,\}$$

defines the **relatively compact bornology** on X. The assumption T_1 guarentees that singleton sets are relatively compact. ◇

Example 4.1.8. Let (Y, d) be a metric space. Then

$$\mathcal{B} \;=\; \left\{ B \subset Y \;\middle|\; \sup_{a, b \in B} d(a, b) < \infty \right\}$$

defines the **metric bornology** on Y. ◇

4.2. Bornological Vector Spaces

4.2.1. Vector Bornologies

We now turn to bornologies on vector spaces. As before we will only consider the Archimedean case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The Euclidean norm $|\cdot|$ on $\mathbb{K}$ defines a bornology on $\mathbb{K}$ given by the family of bounded sets with respect to $|\cdot|$.

Definition 4.2.1. Consider the field $\mathbb{K}$ with its bornology $\mathcal{B}_{\mathbb{K}}$. A **bornological ($\mathbb{K}$-vector) space** $(E, \mathcal{B})$ is given by a $\mathbb{K}$-vector space E and a bornology $\mathcal{B}$ on the underlying set of E, such that

(i) addition $E \times E \to E,\ (x, y) \mapsto x + y$ is bounded with respect to $\mathcal{B}$ on E and the product bornology of $\mathcal{B}$ and $\mathcal{B}$ on $E \times V$,

(ii) scalar multiplication $\mathbb{K} \times E \to E,\ (\lambda, x) \mapsto \lambda x$ is bounded with respect to $\mathcal{B}$ on V and the product bornology of $\mathcal{B}_{\mathbb{K}}$ and $\mathcal{B}$ on $K \times E$.

If $\mathcal{B}$ has a basis given by disks, we say that $(E, \mathcal{B})$ is of **convex type**.

We denote by $\mathsf{Born}_{\mathbb{K}}$ the category of bornological $\mathbb{K}$-vector spaces of convex type with bounded maps.

Remark 4.2.2. Every seminormed space (E, p) can be given a bornology of convex type, that is induced by all bounded subsets with respect to p. In fact, the category of seminormed $\mathbb{K}$-vector spaces with continuous maps is a full subcategory of $\mathsf{Born}_{\mathbb{K}}$. ◇

We will focus on the category $\mathsf{Born}_{\mathbb{K}}$ and omit the "of convex type" from now on. Only later in Chapter 9 we will also consider bornological spaces that are not of convex type.

4.2.2. Diskworld

As we will see, the central concept in the theory of bornological spaces is that of a bounded disk, along with various forms and variants of bounded disks. Recall that a disk is a balance and convex subset of a vector space.

Definition 4.2.3. Let $(E, \mathcal{B}) \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space and $B \in \mathcal{B}$ a bounded subset. If B is a disk, that is a convex and balanced subset of B, we define a seminormed space E_B given by

$$(\operatorname{span}_{\mathbb{K}}(B)\ ,\ p_B) \quad \text{with} \quad p_B : E_B \to \mathbb{R}_{\geq 0}, \quad x \mapsto \inf\{\, \lambda > 0 \,|\, x \in \lambda B \,\}.$$

Here p_B is called the **gauge seminorm** or the **Minkowski functional** of B.

If p_B is a norm we say that B is **norming**. If E_B is complete we say that B is **complete** or a **Banach disk**. We call E_B its **associated seminormed/normed/Banach space** of B.

We have

$$\mathcal{B}_{\mathrm{B}}(E) \subset \mathcal{B}_{\mathrm{N}}(E) \subset \mathcal{B}_{\mathrm{D}}(E) \subset \mathcal{B}$$

for the sets of Banach disks, norming disks, bounded disks and bounded subsets in E.

On all the above sets of disks we can define a partial order $\subset$ given by set inclusion. From now on we will use the notion of cofinality. See Appendix A.4 for a definition.

Proposition 4.2.4 (Lemma 1.8 in [Mey07])**.** *Let $(E, \mathcal{B}) \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space. The sets $\mathcal{B}_{\mathrm{B}}(E), \mathcal{B}_{\mathrm{N}}(E), \mathcal{B}_{\mathrm{D}}(E), \mathcal{B}$ ordered by inclusion $\subset$ are filtered. $\mathcal{B}_{\mathrm{D}}(E) \subset \mathcal{B}$ is cofinal.*

Remark 4.2.5. A sufficient condition for a bounded disk B in a vector space V to be a Banach disk is that there exists a locally convex topology τ on V, such that B is τ-sequentially complete and closed. ◇

Definition 4.2.6. A bornological space $E \in \mathsf{Born}_{\mathbb{K}}$ **separated** if $\mathcal{B}_{\mathrm{N}}(E) = \mathcal{B}_{\mathrm{D}}(E)$ and **complete** if $\mathcal{B}_{\mathrm{B}}(E) \subset \mathcal{B}_{\mathrm{D}}(E)$ is cofinal.

We denote the full subcategory of $\mathsf{Born}_{\mathbb{K}}$ of separated bornological spaces by $\mathsf{SBorn}_{\mathbb{K}}$ and the full subcategory of $\mathsf{Born}_{\mathbb{K}}$ of complete bornological spaces by $\mathsf{CBorn}_{\mathbb{K}}$.

In the case of separated bornological spaces the condition of cofinality actually implies that every bounded disk is already norming. We also have the following characterization.

Proposition 4.2.7 (Section 4 in [PS00])**.** *A bornological space $E \in \mathsf{Born}_{\mathbb{K}}$ is separated if and only if $\{0\}$ is the only bounded linear subspace of E.*

The sets $\mathcal{B}_{\mathrm{D}}(E), \mathcal{B}_{\mathrm{B}}(E)$ are closed under intersections. Thus, as long as there is a bounded/Banach disk containing a bounded set $B \subset E$ there is a minimal one with respect to $\subset$. Existence is guaranteed by cofinality of the sets $\mathcal{B}_{\mathrm{B}}(E)$ and $\mathcal{B}_{\mathrm{D}}(E)$ in $\mathcal{B}$.

Definition 4.2.8. Let $(E, \mathcal{B}) \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space and $B \in \mathcal{B}$ a bounded subset. The smallest bounded disk containing B is called **disked hull** of B and denoted by $B^{\diamond}$. If E is complete the smallest Banach disk containing B is called **complete disked hull** of B and denoted by $B^{\diamond\!\!\!\diamond}$.

Proposition 4.2.9. *Let $(E, \mathcal{B}) \in \mathsf{CBorn}_{\mathbb{K}}$. If there is a cofinal subset $\mathcal{D} \subset \mathcal{B}_{\mathrm{D}}(E)$ of cardinality κ. Then there is also a cofinal subset $\mathcal{D}' \subset \mathcal{B}_{\mathrm{B}}(E)$ of Banach disks and cardinality κ.*

Proof. Take the complete disked hull of all elements of $\mathcal{D}$. □

The connection of these subsets of disks and their surrounding bornological spaces is given by the following result.

Proposition 4.2.10 (Proposition §2.2.2 in [Fer72] or Proposition 2.9 in [PS00])**.** *Let $E \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space. Then there is a canonical isomorphism*

$$\operatorname*{colim}_{B \in \mathcal{B}_{\mathrm{D}}(E)} E_B \to E.$$

The seminormed spaces E_B are considered as bornological spaces by Remark 4.2.2. The diagram is filtered and consists of monomorphisms.

If E is separated there is a canonical isomorphism

$$\operatorname*{colim}_{B \in \mathcal{B}_{\mathrm{N}}(E)} E_B \to E.$$

and if E is complete there is a canonical isomorphism

$$\operatorname*{colim}_{B \in \mathcal{B}_{\mathrm{B}}(E)} E_B \to E.$$

We have defined three categories with by definition fully faithful inclusions

$$\mathsf{CBorn}_{\mathbb{K}} \hookrightarrow \mathsf{SBorn}_{\mathbb{K}} \hookrightarrow \mathsf{Born}_{\mathbb{K}}.$$

Both of the inclusions admit left adjoints. The first one is the **completion functor**

$$\widehat{(\cdot)} : \mathsf{SBorn}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}, \qquad E \mapsto \widehat{E} = \operatorname*{colim}_{B \in \mathcal{B}_{\mathrm{D}}(E)} \widehat{E_B}, \tag{4.1}$$

where $\widehat{E_B}$ is the completion of the normed space E_B. The second left adjoint is the **Separation functor**

$$\operatorname{sep}(\cdot) : \mathsf{Born}_{\mathbb{K}} \to \mathsf{SBorn}_{\mathbb{K}}, \qquad E \mapsto E/\overline{\{0\}}, \tag{4.2}$$

where $\overline{\{0\}}$ is the closure of $\{0\}$ given by the span of all closed bounded subspaces of E. For $E \in \mathsf{Born}_{\mathbb{K}}$ we write $\widehat{E} := \widehat{\operatorname{sep}(E)} \in \mathsf{CBorn}_{\mathbb{K}}$. For more details and proofs see section 3.3 in [BBB16].

Next, we will establish the basic characteristics of the category of complete bornological spaces. Here is a short overview:

	complete bornological spaces $\mathsf{CBorn}_{\mathbb{K}}$	
Complete	Yes	Corollary 4.2.14
Cocomplete	Yes	Corollary 4.2.14
Quasi-Abelian	Yes	Proposition 4.2.34
Symmetric Monoidal	Yes, $(\mathsf{CBorn}_{\mathbb{K}}, \hat{\otimes}_{\pi}, \mathbb{K})$	Theorem 4.2.27
	$\mathsf{CBorn}_{\mathbb{K}}$ with exact structure $\mathfrak{max}$	
Enough Projectives	Yes, $\ell^1(X)$ and coproducts thereof	Proposition 4.3.2, Theorem 4.3.10
Global Dimension	∞	Theorem 4.3.11

4.2.3. Limits and Colimits

Proposition 4.2.11 (Propositions 1.2, 4.5, 5.5 in [PS00]). *Let I be a set and $\{(E_i, \mathcal{B}_i)\}_{i\in I}$ be a collection of bornological spaces. Their coproduct in $\mathsf{Born}_{\mathbb{K}}, \mathsf{SBorn}_{\mathbb{K}}$ and $\mathsf{CBorn}_{\mathbb{K}}$ is given by the vector space $\coprod_{i\in I} E_i$ with bornology generated by the basis*

$$\left\{ \coprod_{i\in I} B_i \subset \coprod_{i\in I} E_i \;\middle|\; B_i \in \mathcal{B}_i, \operatorname{card}\{\, i \in I \mid B_i \neq \{0\} \,\} < \infty \right\}.$$

Their product in $\mathsf{Born}_{\mathbb{K}}, \mathsf{SBorn}_{\mathbb{K}}$ and $\mathsf{CBorn}_{\mathbb{K}}$ is given by the vector space $\prod_{i\in I} E_i$ with bornology generated by the basis

$$\left\{ \prod_{i\in I} B_i \subset \prod_{i\in I} E_i \;\middle|\; B_i \in \mathcal{B}_i \right\}.$$

Example 4.2.12. Let us consider the countable infinite coproduct and product

$$\coprod_{i\in\mathbb{N}} \mathbb{K}, \qquad \prod_{i\in\mathbb{N}} \mathbb{K}.$$

A basis for the standard bornology on $\mathbb{K}$ is given by all closed balls centered at 0. We write $B_{\leq r}(\mathbb{K}) = \{\, x \in \mathbb{K} \mid \|x\| \leq r \,\}$ for $r \in \mathbb{R}_{>0}$. In particular, in the case $\mathbb{K} = \mathbb{R}$ the bornology is generated by all intervals $[-r, r]$.

For the bornology of $\coprod_{i\in\mathbb{N}} \mathbb{K}$ we have the basis

$$\left\{ \coprod_{i\in\mathbb{N}} B_{\leq r_i}(\mathbb{K}) \;\middle|\; r_i \in \mathbb{R}_{\geq 0},\ r_i \neq 0 \text{ for only finitely many } i \in \mathbb{N} \right\}.$$

and for the bornology of $\prod_{i\in\mathbb{N}} \mathbb{K}$ we have the basis

$$\left\{ \coprod_{i\in\mathbb{N}} B_{\leq r_i}(\mathbb{K}) \;\middle|\; r_i \in \mathbb{R}_{\geq 0} \right\}.$$

by Proposition 4.2.11.

Later in this chapter we will be interested in the cofinality of bounded sets in order to determine homological dimensions. In our example here the cofinality of bounded sets in $\coprod_{i\in\mathbb{N}}\mathbb{K}$ is $\mathbb{N}$ and a basis of that cardinality is given by

$$\left\{ \coprod_{i=1}^{N} B_{\leq N}(\mathbb{K}) \;\middle|\; N\in\mathbb{N} \right\}.$$

To determine the cofinality of $\prod_{i\in\mathbb{N}}\mathbb{K}$ we first note that

$$\mathcal{D} := \left\{ \coprod_{i\in\mathbb{N}} B_{\leq n_i}(\mathbb{K}) \;\middle|\; n_i\in\mathbb{N} \right\}$$

is cofinal in the bounded sets of the product. The preorder $(\mathcal{D},\subset)$ is isomorphic to the preorder $(\prod_{\mathbb{N}}\mathbb{N},\leq)$ described in Appendix A.4. The cofinality is $\mathrm{cf}(\mathcal{D})=\mathfrak{d}$ and by Theorem A.4.3 we have

$$\aleph_1 \;\leq\; \mathfrak{d} \;\leq\; \beth_1 = 2^{\aleph_0},$$

but cannot prove better bounds on $\mathfrak{d}$ inside ZFC. By Proposition A.4.1 it follows that the bounded sets of $\prod_{i\in\mathbb{N}}\mathbb{K}$ also have cofinality $\mathfrak{d}$ and a corresponding basis is given by $\mathcal{D}$. ◇

Lemma 4.2.13 (Propositions 1.5, 4.6 in [PS00])**.** *The categories* $\mathsf{Born}_{\mathbb{K}}, \mathsf{SBorn}_{\mathbb{K}}$ *and* $\mathsf{CBorn}_{\mathbb{K}}$ *are preabelian. Let $f: E\to F$ be a bounded map between bornological spaces E, F. Then in* $\mathsf{Born}_{\mathbb{K}}$

(i) $\mathrm{Ker}\, f$ *is given by* $f^{-1}(0)$ *with the subspace bornology,*
(ii) $\mathrm{Im}\, f$ *is given by* $f(E)$ *with the subspace bornology,*
(iii) $\mathrm{Coker}\, f$ *is given by the quotient space* $F/f(E)$*,*
(iv) $\mathrm{Coim}\, f$ *is given by the quotient space* $E/f^{-1}(0)$*.*

If E, F are separated/complete then in $\mathsf{SBorn}_{\mathbb{K}}/\mathsf{CBorn}_{\mathbb{K}}$*,*

(i) $\mathrm{Ker}\, f$ *is given by* $f^{-1}(0)$ *with the subspace bornology,*
(ii) $\mathrm{Im}\, f$ *is given by* $\overline{f(E)}$ *with the subspace bornology,*
(iii) $\mathrm{Coker}\, f$ *is given by the quotient space* $F/\overline{f(E)}$*,*
(iv) $\mathrm{Coim}\, f$ *is given by the quotient space* $E/f^{-1}(0)$*.*

Corollary 4.2.14. *The categories* $\mathsf{Born}_{\mathbb{K}}, \mathsf{SBorn}_{\mathbb{K}}$ *and* $\mathsf{CBorn}_{\mathbb{K}}$ *are complete and cocomplete.*

Proof. This follows from Proposition 4.2.11 and Lemma 4.2.13. □

4.2.4. Duality of Bornologies and Topologies

Theorem 4.2.15 (See chapter 1 in [Fer72]). *There is an adjunction*

$$\mathsf{Born}_{\mathbb{K}} \underset{\mathrm{vN}}{\overset{\mathrm{t}}{\rightleftarrows}} \mathsf{LCS}_{\mathbb{K}} \;(\bot).$$

The functor t *equips a space* $E \in \mathsf{Born}_{\mathbb{K}}$ *with the topology whose basis is defined to consist of all bornivorous sets. The functor* vN *endows a space* $F \in \mathsf{LCS}_{\mathbb{K}}$ *with its* ***von Neumann bornology*** *that is defined to consist of all subsets of* F *that are absorbed by all neighborhoods of the origin.*

Proposition 4.2.16. *The von Neumann bornology functor commutes with small limits and small coproducts.*

Proof. As a right adjoint, vN commutes with small limits. Let $\{V_i\}_{i\in I}$ be a family of locally convex spaces. Then the coproduct $\coprod_{i\in I} V_i$ in $\mathsf{LCS}_{\mathbb{K}}$ has the algebraic coproduct $\coprod_{i\in I} V_i$ as underlying vector space. Denote the canonical inclusion by $\iota_i : V_i \to \coprod_{i\in I} V_i$. A neighborhood basis of 0 is given by convex hulls

$$W_{\{U_i\}_{i\in I}} \;=\; \text{convex}\left\{ \sum_{j\subset J} u_j \;\middle|\; J\subset I \text{ finite, } u_j \in \iota_j(U_j) \right\}$$

for all $\{U_i\}_{i\in I}, U_i \subset V_i$ open.

A subset $B\subset V$ is bounded if and only if for all $W_{\{U_i\}_i}$ there is a $\lambda \in \mathbb{R}_{>0}$ with $B\subset \lambda W_{\{U_i\}_i}$. Thus, B is bounded if and only if there is a finite set $J\subset I$ and bounded subsets $B_j \subset V_j$ with $B \subset \coprod_{j\in J} B_j$. Comparing this to the coproduct in $\mathsf{Born}_{\mathbb{K}}$ from Proposition 4.2.11 we see that vN commutes with all small coproducts. □

We could have defined "bornological" to be a property of topological vector spaces similar to "normable", "Banach" or "nuclear". A **bornological topological vector space** E is a locally convex vector space that is isomorphic to its **bornologification** $\mathrm{t}(\mathrm{vN}(E))$. We denote the full subcategory of $\mathsf{LCS}_{\mathbb{K}}$ of bornological topological vector spaces by $\mathsf{BTVS}_{\mathbb{K}}$.

Proposition 4.2.17 (section 13.2 in [NB10]). *Let* $E\in \mathsf{LCS}_{\mathbb{K}}$ *be a locally convex space. Then* E *is contained in* $\mathsf{BTVS}_{\mathbb{K}}$ *if and only if every bornivorous set in* E *is a neighborhood of* 0*.*

However, $\mathsf{BTVS}_{\mathbb{K}}$ is not quasi-abelian and therefore the wrong category to look at from our perspective.

Proposition 4.2.18 ([BD06] and [Rum11])**.** *The category* $\mathsf{BTVS}_{\mathbb{K}}$ *is semi-abelian but not quasi-abelian.*

The following Proposition shows that seminormed and in particular Banach and Fréchet spaces are bornological topological vector spaces.

Proposition 4.2.19 (Example 13.2.8.(b) in [NB10])**.** *Let* $V \in \mathsf{LCS}_{\mathbb{K}}$ *be a locally convex space. If* V *is pseudometrizable, then* V *is a bornological topological vector space.*

Using the adjunction in Theorem 4.2.15 we can transfer topological notions, such as metrizability.

Definition 4.2.20. A bornological space $E \in \mathsf{Born}_{\mathbb{K}}$ is metrizable if there is a metrizable locally convex space with V with $\mathrm{vN}(V) \cong E$.

Remark 4.2.21. For (sequential) topological vector spaces the property of being complete can be expressed with Cauchy sequences. There is an analogous concept called Mackey sequneces for bornological spaces. See [Bam16] or [BBBK18] for more information. ◇

Proposition 4.2.22 (Corollary 1.18 in [Bam16])**.** *Let* $V \in \mathsf{LCS}_{\mathbb{K}}$ *be a metrizable locally convex space. Then* V *is complete (topologically) if and only if* $\mathrm{vN}(V)$ *is complete (bornologically).*

Proposition 4.2.23 (Section 1.1.6 in [Mey07])**.** *Taking von Neumann bornology of Fréchet spaces is a fully faithful functor.*

Definition 4.2.24. A complete bornological space $F \in \mathsf{CBorn}_{\mathbb{K}}$ is **Fréchet** , if there is a topological Fréchet vector space V with $\mathrm{vN}(V) \cong F$.

4.2.5. Ind-categories and dissection functors

In Proposition 4.2.10 we have seen that bornological spaces can always be written as filtered colimits over objects from certain subcategories. This suggests, that they are closely related to so called Ind-categories consisting of formal filtered colimits. See Appendix B for a definition and some details. Homological methods in Ind-categories of seminormed, normed and Banach spaces were already used in [Sch99] and [PS99]. The connection to different categories of bornological spaces was established in [PS00].

Given a complete bornological vector space $E \in \mathsf{CBorn}_{\mathbb{K}}$ there is an isomorphism

$$E \;\cong\; \operatorname*{colim}_{B \in \mathfrak{B}_{\mathrm{B}}(E)} E_B$$

by Proposition 4.2.10. Replacing E with by the diagram of this colimit we define a **dissection functor**

$$\mathrm{diss}: \mathsf{CBorn}_{\mathbb{K}} \to \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}), \qquad E \mapsto \text{“}\operatorname*{colim}_{B \in \mathcal{B}_{\mathrm{B}}(E)}\text{”} E_B. \tag{4.3}$$

Note, that $\mathsf{Ban}_{\mathbb{K}}$ is not essentially small, but we define $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ to consist of small formal filtered colimits. We discuss some consequences of this fact to Local Presentability in Section 4.2.9.

Proposition 4.2.25 (Proposition 3.51 in [BBB16] and Theorem 1.139 in [Mey07])**.** *The functor (4.3) defines an adunction*

$$\mathsf{CBorn}_{\mathbb{K}} \underset{\text{colim}}{\overset{\text{diss}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})\,.$$

It commutes with small limits and coproducts, but not with cokernels in general. The dissection functor induces an equivalence of categories

$$\mathsf{CBorn}_{\mathbb{K}} \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{Ban}_{\mathbb{K}})$$

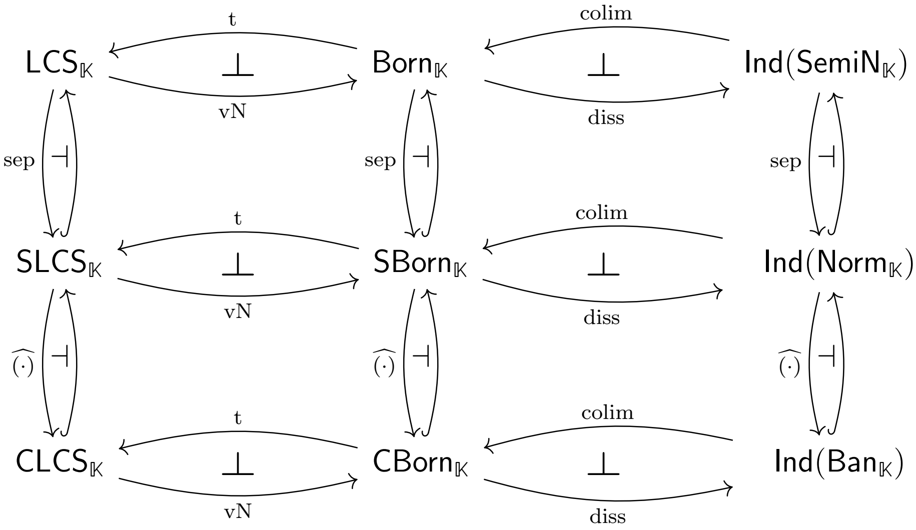
of

- *complete bornological spaces* $\mathsf{CBorn}_{\mathbb{K}}$*,*
- *the full reflective subcategory of* $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ *given by formal filtered colimits of Banach spaces with monomorphisms as transition maps* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Ban}_{\mathbb{K}})$*.*

Analogous to Proposition 4.2.25 we have

$$\mathsf{Born}_{\mathbb{K}} \cong \mathsf{Ind}_{\rightarrowtail}(\mathsf{SemiN}_{\mathbb{K}}) \qquad \text{and} \qquad \mathsf{SBorn}_{\mathbb{K}} \cong \mathsf{Ind}(\mathsf{Norm}_{\mathbb{K}}).$$

We can illustrate the situation in a diagram of adjunctions.



4.2.6. Tensor Product

Tensor products of bornological spaces were already studied in 1972 by Houzel in chapter 4 of [Fer72]. See also [Bam14].

We define the (completed) projective tensor product in $\mathsf{Born}_\mathbb{K}$ and $\mathsf{CBorn}_\mathbb{K}$.

Definition 4.2.26. Let $E, F \in \mathsf{Born}_\mathbb{K}$ be bornological spaces. Then equipping the algebraic tensor product $E \otimes_\mathbb{K} F$ with a bornology given by the basis

$$\left\{ (C \otimes D)^{\diamond} \;\middle|\; C \in \mathcal{B}_\mathrm{D}(E), D \in \mathcal{B}_\mathrm{D}(F) \right\}, \quad \text{where}$$
$$C \otimes D = \{ c \otimes d \mid c \in C,\, d \in D \} \quad \text{for} \quad C \in \mathcal{B}_\mathrm{D}(E), D \in \mathcal{B}_\mathrm{D}(F),$$

defines the **projective tensor product** $E \otimes_\pi F$.

For $E, F \in \mathsf{CBorn}_\mathbb{K}$ we define the **completed projective tensor product** as the completion

$$E \hat{\otimes}_\pi F := \widehat{E \otimes_\pi F}.$$

Theorem 4.2.27 (Section 3.3 in [BBB16])**.** *The category* $\mathsf{CBorn}_\mathbb{K}$ *with tensor product* $\hat{\otimes}_\pi$ *and unit* $\mathbb{K}$ *is closed, symmetric monoidal.*

Remark 4.2.28. The bornological completed projective tensor product has inherited the name of the topological completed projective tensor product since it agrees with the latter on Banach spaces. For general locally conves spaces V, W the relationship between $\mathrm{vN}(V)\hat{\otimes}_\pi \mathrm{vN}(W)$ and $\mathrm{vN}(V\hat{\otimes}_\pi W)$ is complicated. See section 1.3.6 in [Mey07]. ◇

Proposition 4.2.29. *The von Neumann bornology functor restricted to nuclear Fréchet spaces* $\mathrm{vN} : (\mathsf{NF}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$ *is strongly monoidal.*

Proof. This follows from Theorem 1.87 in [Mey07] and the fact, that nuclear Fréchet spaces are Montel from Proposition 3.2.37. ☐

Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological vector space. The functor

$$\mathsf{CBorn}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}, \qquad F \mapsto E \hat{\otimes}_\pi F$$

is right exact, but not always left exact with respect to $\mathfrak{max}$.

Definition 4.2.30. A complete bornological space $E \in \mathsf{CBorn}_{\mathbb{K}}$ is **flat** if $\cdot \hat{\otimes}_\pi E : \mathsf{CBorn}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ is $\mathfrak{max}$-exact.

Proposition 4.2.31. *Let* $E \in \mathsf{CBorn}_{\mathbb{K}}$ *be a complete bornological space. If there is a cofinal subset* $\mathcal{D} \subset \mathcal{B}_{\mathrm{B}}(E)$*, such that all associated Banach spaces* $E_B, B \in \mathcal{D}$ *are* $\mathcal{L}_1$*, the space* E *is flat in* $(\mathsf{CBorn}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$*.*

Proof. Every $\mathcal{L}_1$-Banach space is flat in $\mathsf{CBorn}_{\mathbb{K}}$ by Proposition 3.2.17. Since $\mathsf{CBorn}_{\mathbb{K}}$ is closed by Theorem 4.2.27, colimits commute with $\hat{\otimes}_\pi$. By Lemma 2.3, Lemma 3.8 and Proposition 4.8 in [PS00] monomorphic filtered colimits are $\mathfrak{max}$-exact. Thus, the monomorphic filtered colimit $E \cong \operatorname{colim}_{B \in \mathcal{D}} E_B$ of flat spaces is flat. ☐

Question 4.2.32. Is every flat object in $\mathsf{CBorn}_{\mathbb{K}}$ of the form described in Proposition 4.2.31?

One can also define the completed projective tensor product $\hat{\otimes}_\pi$ in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$.

Proposition 4.2.33 (section 3 in [BBK23])**.** *The category* $(\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}), \hat{\otimes}_\pi, \mathbb{K})$ *is a closed symmetric monoidal category.*

4.2.7. Exact Structure

Proposition 4.2.34 (Proposition 5.6 and Corollary 5.7 in [PS00])**.** *The category* $\mathsf{CBorn}_{\mathbb{K}}$ *is quasi-abelian. A morphism* $f : (E, \mathcal{B}_E) \to (E, \mathcal{B}_F)$ *is*

(i) an admissible monomorphism if f *is injective and for all* $C \in \mathcal{B}_F$ *we have* $f^{-1}(C) \in \mathcal{B}_E$*,*

(ii) an admissible epimorphism if f *is surjective and for all* $C \in \mathcal{B}_F$ *there is a* $B \in \mathcal{B}_E$ *with* $f(B) = C$*.*

The description of the admissible monomorphisms and epimorphisms is that of kernels and cokernels from Lemma 4.2.13, which by Proposition 1.2.5 form the maximal exact structure $\mathfrak{max}$ on $\mathsf{CBorn}_{\mathbb{K}}$.

In Proposition 4.2.23 we have seen that $\mathrm{vN} : \mathsf{Fr\acute{e}}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ is fully faithful.

Lemma 4.2.35 (Lemma 3.71 in [BBBK18])**.** *Given a short* $\mathfrak{max}$*-exact sequence*

$$E \rightarrowtail F \twoheadrightarrow G$$

of Fréchet spaces in $\mathsf{CBorn}_{\mathbb{K}}$ *the sequence*

$$\mathrm{t}(E) \rightarrowtail \mathrm{t}(F) \twoheadrightarrow \mathrm{t}(G)$$

is $\mathfrak{max}$*-exact in* $\mathsf{LCS}_{\mathbb{K}}$*.*

For the converse statement we need nuclearity in general. See Lemma 4.2.40.

Proposition 4.2.36 (section 1.4.2 in [Sch99])**.** *The category* $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ *is quasi-abelian.*

4.2.8. Nuclear Spaces

We can define nuclearity for complete bornological spaces via their description as filtered colimits of Banach spaces. For more details see section 3 in [BBBK18] and section 4 in [BBK23].

Definition 4.2.37. A complete bornological space $E \in \mathsf{CBorn}_{\mathbb{K}}$ is said to be **nuclear** if there is an isomorphism

$$E \cong \operatorname*{colim}_{i \in I} B_i$$

in $\mathsf{CBorn}_{\mathbb{K}}$, where $I \to \mathsf{Ban}_{\mathbb{K}}, i \mapsto B_i$ is a diagram of Banach spaces with nuclear monomorphisms $B_i \to B_j$ as transition maps for all $i < j$.

We denote the full subcategory of nuclear objects in $\mathsf{CBorn}_{\mathbb{K}}$ by $\mathsf{NCBorn}_{\mathbb{K}}$.

Remark 4.2.38. Unlike nuclear locally convex spaces, every nuclear bornological space is by definition complete. ◇

Proposition 4.2.39 (Lemma 3.60 in [BBBK18])**.** *Let* $F \in \mathsf{NF}_{\mathbb{K}}$ *be nuclear Fréchet space. Then* $\mathrm{vN}(F) \in \mathsf{CBorn}_{\mathbb{K}}$ *is nuclear in the sense of Definition 4.2.45.*

Thus, by Proposition 4.2.23 von Neumann bornology defines a fully faithful functor $\mathsf{NF}_{\mathbb{K}} \to \mathsf{NCBorn}_{\mathbb{K}}$. Additionally, the exact structures match.

Lemma 4.2.40 (Lemma 3.71 in [BBBK18])**.** *The sequence*

$$U \longrightarrow V \longrightarrow W$$

of nuclear Fréchet spaces in $\mathsf{NF}_{\mathbb{K}}$ *is short* $\mathfrak{max}$*-exact if and only if*

$$\mathrm{vN}(U) \longrightarrow \mathrm{vN}(V) \longrightarrow \mathrm{vN}(W)$$

is a short $\mathfrak{max}$*-exact sequence in* $\mathsf{CBorn}_{\mathbb{K}}$*.*

Lemma 4.2.41 (Theorem 3.50 in [BBBK18])**.** *Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be nuclear. Then E is flat.*

Nuclear bornological spaces have similar stability properties as nuclear locally convex spaces.

Proposition 4.2.42 (Propositions 3.51 and 3.52 in [BBBK18])**.** *Small colimits and countable limits of nuclear objects in $\mathsf{CBorn}_{\mathbb{K}}$ are nuclear.*

Proposition 4.2.43. *The category $\mathsf{NCBorn}_{\mathbb{K}}$ is quasi-abelian.*

Proof. By Proposition 1.2.10 it suffices to check that $\mathsf{NCBorn}_{\mathbb{K}}$ reflects kernels and cokernels. This follows from Proposition 4.2.42. □

Remark 4.2.44. In [RD81] Dierolf and Roelcke proved that the subcategory of nuclear locally convex vector spaces are closed under extensions in all locally convex vector spaces. Proposition 4.2.43 is the analogous result in the bornological setting. ◇

We can also define nuclearity for all Ind-Banach spaces.

Definition 4.2.45 (Definition 2.15 in [PS99])**.** A formal filtered colimit of Banach spaces "$\mathrm{colim}_{i\in I}$"$V_i \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ is said to be **nuclear** if it is isomorphic to an object "$\mathrm{colim}_{j\in J}$"$W_j \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$, where all transition maps $W_j \to W_k$ with $j, k \in J$ are nuclear.

Proposition 4.2.46 (Lemma 2.17 in [PS99])**.** *Let* "$\mathrm{colim}_{i\in I}$"$V_i \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ *be a nuclear Ind-Banach space. Then there is a diagram of Banach spaces*

$$J \to \mathsf{Ban}_{\mathbb{K}}, \qquad j \mapsto \ell^1$$

with nuclear transition maps and $\mathrm{card}(I) = \mathrm{card}(J)$*, such that*

$$E \;\cong\; \operatorname*{colim}_{j\in J} \ell^1.$$

A generalization of this result to Modules over Banach rings can be found as Lemmas 4.19 and 4.20 in [BBK23].

4.2.9. Local Presentability

At a first glance $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ and $\mathsf{CBorn}_{\mathbb{K}} \cong \mathsf{Ind}_{\rightarrowtail}(\mathsf{Ban}_{\mathbb{K}})$ might look like locally presentable and in particular $\aleph_0$-accessible categories. However, the fact that we consider small formal filtered colimits in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$, while $\mathsf{Ban}_{\mathbb{K}}$ is not essentially small, complicates things. Additionally, $\mathsf{CBorn}_{\mathbb{K}}$ only corresponding to essentially monomorphic objects introduces additional difficulties.

Definition 4.2.47. Let κ be a regular cardinal and C a cocomplete category with a full subcategory S. The **$<\kappa$-small colimit completion** $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ is the smallest full subcategory $\mathsf{S} \subset \mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S}) \subset \mathsf{C}$, which is closed under the formation of $<\kappa$-small colimits.

We can construct $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ via a transfinite induction. We set $\mathsf{S}_0 := \mathsf{S}$ and define a successor $\mathsf{S}_{\alpha+1}$ to be the full subcategory of C consisting of $<\kappa$-small colimits of objects in S_α. In the limit case we take unions. Let γ be the smallest ordinal with $\mathrm{card}(\gamma) = \kappa$. We have $\mathsf{S}_\gamma = \mathsf{S}_{\gamma+1}$, since any $<\kappa$-small collection of objects in S_γ is contained in some S_β with $\beta < \gamma$. Thus, we have constructed

$$\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S}) \;=\; \mathsf{S}_\gamma. \tag{4.4}$$

Lemma 4.2.48. *Let κ be a regular cardinal and C a cocomplete category with a full subcategory S. Let I be a set and $V = \mathrm{colim}_{i\in I} V_i$ a colimit in C with $V_i \in \mathsf{S}$ for all $i \in I$. Then we can write V as a colimit over some κ-filtered diagram K:*

$$V \cong \operatorname*{colim}_{j\in K} W_j, \qquad \text{with} \quad W_j \in \mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S}) \quad \text{for all} \quad j \in K.$$

Proof. Take the diagram $F_0 : I \to \mathsf{C}$ and add all colimits of $J \subset I, \mathrm{card}(J) < \kappa$ to get $F_1 : I_1 \to \mathsf{C}$. Continuing via transfinite induction, where we take the union of the I_α in the limit case, we get $F_\gamma : I_\gamma \to \mathsf{C}, j \mapsto W_j$ with $\mathrm{card}(\gamma) = \kappa$. By construction all I_α and I_γ are small, all W_j are contained in $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$, $K := I_\gamma$ is κ-filtered and $V = \mathrm{colim}\, F_\gamma$. □

Proposition 4.2.49. *Let κ be a regular cardinal. Let C be locally small, cocomplete category, that has a small full subcategory S of κ-compact objects, that generate C under arbitrary colimits. Then C is κ-accessible and a set of generators under κ-filtered colimits is given by $\mathrm{Colim}^{\kappa}_{\mathsf{C}}(\mathsf{S})$.*

Proof. A $<\kappa$-small colimit of κ-compact objects is κ-compact by Proposition 1.16 in [AR94]. Thus, in every step S_α of the transfinite construction (4.4) of $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ the subcategory S_α consists of κ-compact objects. Furthermore, all S_α and therefore also $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ are small categories.

Consider the colimit functor

$$L_\kappa : \mathsf{Ind}_\kappa(\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})) \to \mathsf{C}, \qquad \text{“}\operatorname*{colim}_{i\in I}\text{”} X_i \mapsto \operatorname*{colim}_{i\in I} X_i$$

from formal κ-filtered colimits in $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ to C. It is fully faithful, since all objects of $\mathrm{Colim}^{<\kappa}_{\mathsf{C}}(\mathsf{S})$ are κ-compact in C and therefore

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(L_\kappa(\text{“}\operatorname*{colim}_{i\in I}\text{”} X_i), L_\kappa(\text{“}\operatorname*{colim}_{j\in J}\text{”} Y_j)) &= \mathrm{Hom}_{\mathsf{C}}(\operatorname*{colim}_{i\in I} X_i, \operatorname*{colim}_{j\in J} Y_j) \\
&\cong \lim_{i\in I} \mathrm{Hom}_{\mathsf{C}}(X_i, \operatorname*{colim}_{j\in J} Y_j) \\
&\cong \lim_{i\in I} \operatorname*{colim}_{j\in J} \mathrm{Hom}_{\mathsf{C}}(X_i, Y_j),
\end{aligned}$$

which matches the Hom-Sets in $\mathsf{Ind}_\kappa(\mathrm{Colim}_{\mathsf{C}}^{\lhd\kappa}(\mathsf{S}))$. The functor L_κ is also essentially surjective. Indeed, every object of C is a colimit of objects of S and we can rewrite any such colimit as a κ-filtered colimit of objects in $\mathrm{Colim}_{\mathsf{C}}^{\lhd\kappa}(\mathsf{S})$ by Lemma 4.2.48. Thus, L_κ is an equivalence of categories and C is κ-accessible. □

To apply Proposition 4.2.49 to the situation we are interested in, we first need some information about compactness in $\mathsf{CBorn}_{\mathbb{K}}$ and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$.

Lemma 4.2.50 (Example 7, Corollary 1 in [BK17])**.** *Let $B \in \mathsf{CBorn}_{\mathbb{K}}$ be a Banach space. The space B is $\aleph_0$-compact relative to the subcategory of monomorphisms in $\mathsf{CBorn}_{\mathbb{K}}$. If $\kappa \geq \aleph_1$ is a regular cardinal, such that B has a dense subset D with $\mathrm{card}(D) < \kappa$, then B is κ-compact.*

In contrast, all Banach spaces in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ are $\aleph_0$-compact as stated in Remark B.1.5.

Every Banach space is a quotient of $\ell^1(X)$ for some set X by Proposition 3.2.12. Furthermore we can write $\ell^1(X)$ as a $\aleph_1$-filtered colimit

$$\ell^1(X) = \operatorname*{colim}_{\substack{J \subset X \\ J \text{ countable}}} \ell^1. \tag{4.5}$$

However, this does not imply that every object of $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ or $\mathsf{CBorn}_{\mathbb{K}}$ is a colimit of ℓ^1 as we will see. The problem with taking the generating class $\{\, \ell^1(X) \mid X \text{ a set} \,\}$ is that it is not a set. The only way to resolve this is to restrict the category $\mathsf{CBorn}_{\mathbb{K}}$.

Definition 4.2.51 (Definition 10 in [BK17])**.** Let κ be a regular cardinal. A complete bornological space E is **$<\kappa$-dense** if there is an admissible epimorphism

$$\coprod_{i \in I} \ell^1(X_i) \twoheadrightarrow E \tag{4.6}$$

with $\mathrm{card}(X_i) \leq \kappa$ for all $i \in I$. We denote the full subcategory of $<\kappa$-dense complete bornological spaces by $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$.

We will see in Proposition 4.3.2 that there is a an admissible epimorphism as in (4.6) into every complete bornological space. Thus, $\mathsf{CBorn}_{\mathbb{K}}$ is a large union over all $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ for regular cardinals κ.

Example 4.2.52. (i) For $\kappa = \aleph_0$ the category $\mathsf{CBorn}_{\lhd\aleph_0,\mathbb{K}}$ contains only **fine bornological spaces**. These are spaces, whose bounded sets are contained and bounded in finite dimensional subspaces. In fact, $\mathsf{CBorn}_{\lhd\aleph_0,\mathbb{K}}$ is equivalent to the category Vect. See section 2.2 in [Mey99] or section 1.3 in [Mey07] for more details.

(ii) For $\kappa = \aleph_1$ the category $\mathsf{CBorn}_{\lessdot\aleph_1,\mathbb{K}}$ consists of all complete **locally separable bornological spaces** as defined in Definition 1.162 [Mey07]. Let V be a separable Fréchet space. Then $\mathrm{vN}(F)$ is locally separable by Theorem 1.164 in [Mey07] Since all nuclear Fréchet spaces are separable by Proposition 3.2.37, they are contained in $\mathsf{CBorn}_{\lessdot\aleph_1,\mathbb{K}}$.

◇

Notation 4.2.53. We write $\ell^1(\lambda)$, λ cardinal, for $\ell^1(X)$ with a set X with $\mathrm{card}(X) = \lambda$. In this case, $\ell^1(\lambda)$ is determined up to isomorphism.

Proposition 4.2.54 (Proposition 12, 14 in [BK17]). *Let κ be a regular cardinal. The inclusion $\mathsf{CBorn}_{<\kappa,\mathbb{K}} \hookrightarrow \mathsf{CBorn}_{\mathbb{K}}$ preserves colimits. The category $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ with $\hat{\otimes}_\pi$ is a closed symmetric monoidal, quasi-abelian, complete and cocomplete category. It has has enough $\mathfrak{max}$-projectives given by*

$$\left\{ \ell^1(\lambda) \;\middle|\; \lambda < \kappa \textit{ cardinal} \right\}.$$

In [BK17] the authors use Lemma 4.2.50 and Proposition 4.2.54 to prove that $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is locally presentable. However, the reasoning behind the existence of a set that generates $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ under κ-filtered colimits, as well as its description, remains somewhat unclear to us. With Proposition 4.2.49 we can clarify this by providing a more detailed argument.

Theorem 4.2.55 (compare to Theorem 1 in [BK17]). *Let κ be a regular cardinal. The category $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ of $<\kappa$-dense complete bornological spaces is locally presentable. Specifically, $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is κ-accessible and if $\kappa \geq \aleph_1$ a set of generators under κ-filtered colimits is given by*

$$\mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{\mathbb{K}}}(\{\ell^1\}) \;\cong\; \mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{\mathbb{K}}}(\left\{ \ell^1(\lambda) \;\middle|\; \lambda < \kappa \textit{ cardinal} \right\})$$

Proof. Let κ be a regular cardinal. By Proposition 4.2.54 the category $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is cocomplete. Since $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is also locally small it remains to show that it is accessible. If $\kappa = \aleph_0$ the category $\mathsf{CBorn}_{\lessdot\aleph_0,\mathbb{K}}$ is equivalent to Vect and therefore $\aleph_0$-accessible by Example 4.2.52. If $\kappa \geq \aleph_1$ we use that every object in $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is a quotient of $\ell^1(X)$ with $\mathrm{card}(X) < \kappa$ by Proposition 4.2.54. Thus, the set

$$\left\{ \ell^1(\lambda) \;\middle|\; \lambda < \kappa \text{ cardinal} \right\}$$

defines a small full subcategory S of $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$, that generates $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ under colimits and contain only κ-compact objects by Lemma 4.2.50. Applying Proposition 4.2.49 we see that $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is κ-accessible and generated under κ-filtered colimits by $\mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{<\kappa,\mathbb{K}}}(\mathsf{S})$. By (4.5) every $\ell^1(X)$ is $<\kappa$-small colimit of ℓ^1. By (4.2.54) we can calculate all colimits in $\mathsf{CBorn}_{\mathbb{K}}$. Therefore,

$$\mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{<\kappa,\mathbb{K}}}(\mathsf{S}) \;\cong\; \mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{\mathbb{K}}}(\mathsf{S}) \;\cong\; \mathrm{Colim}^{\lessdot\kappa}_{\mathsf{CBorn}_{\mathbb{K}}}(\{\ell^1\}).$$

□

Theorem 4.2.55 shows that $\mathsf{CBorn}_{\mathbb{K}}$ is a union of locally presentable categories. We can show the analogous result for $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$, where the compactness of all Banach spaces makes the situation less complicated.

Definition 4.2.56. Let κ be a regular cardinal. A Banach space B is **$<\kappa$-dense** if there is an admissible epimorphism

$$\ell^1(X) \twoheadrightarrow E$$

with $\mathrm{card}(X) < \kappa$. We denote the full subcategory of $<\kappa$-dense Banach spaces by $\mathsf{Ban}_{<\kappa,\mathbb{K}}$. We define **$<\kappa$-dense Ind-Banach spaces** to be the category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$.

With the same proof as Proposition 4.2.54 we get the analogous statement.

Proposition 4.2.57. *Let κ be a regular cardinal. The inclusion $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}}) \hookrightarrow \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ preserves colimits. The category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ with $\hat{\otimes}_\pi$ is a closed symmetric monoidal, quasi-abelian, complete and cocomplete category and has enough $\mathfrak{max}$-projectives given by $\ell^1(X)$ for sets X with $\mathrm{card}(X) < \kappa$.*

For $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ we can immediately conclude that the category is $\aleph_0$-accessible, since $\mathsf{Ban}_{<\kappa,\mathbb{K}}$ is a essentially small category.

Proposition 4.2.58. *The category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ is locally presentable. It is $\aleph_0$-accessible.*

Remark 4.2.59. The category of Banach spaces and linear bounded maps $\mathsf{Ban}_{\mathbb{K}}$ and the category of Banach spaces and linear contractions $\mathsf{Ban}_{\mathbb{K}}^{\leq 1}$ are both $\aleph_1$-accessible. The latter is Example 1.48 in [AR94] and for the former proof is sketched in [MSE]. ◇

We get, that $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ is a union of locally presentable categories, but is it locally presentable itself?

Proposition 4.2.60. *The categories $\mathsf{CBorn}_{\mathbb{K}}$ and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ are neither locally presentable nor accessible.*

Proof. Assume that $\mathsf{CBorn}_{\mathbb{K}}$ is accessible. Then there is a small set S that generates $\mathsf{CBorn}_{\mathbb{K}}$ under colimits. For every complete bornological space E there is a cardinal κ_E with $E \in \mathsf{CBorn}_{\kappa_E,\mathbb{K}}$. For the set of cardinals $\{ \kappa_E \mid E \in S \}$ we can choose a κ that is larger then all its elements and we get $S \subset \mathsf{CBorn}_{<\kappa,\mathbb{K}}$. Since $\mathsf{CBorn}_{<\kappa,\mathbb{K}}$ is cocomplete and colimits are computed in $\mathsf{CBorn}_{\mathbb{K}}$ by Proposition 4.2.54 the set S cannot generate all of $\mathsf{CBorn}_{\mathbb{K}}$ under colimits and we get a contradiction. The same argument can be applied to $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$. □

4.2.10. Derived and Homotopy Limits and Colimits

In Section 4.3 it will be quite useful to know which colimits are homotopy colimits. Let us first talk about deriving the colimit and limit functor. We will see in Proposition 4.3.2 and Corollary 4.3.3 that the categories $\mathsf{CBorn}_{\mathbb{K}}$, $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ have enough projectives. By Corollary 1.3.26 we can derive right exact functors. In particular, we get a derived colimit Lcolim. Conditions for deriving the limit functor are given in section 3 of [Pro99]. In our case of $\mathsf{CBorn}_{\mathbb{K}}$ and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ the derived limit functor Rlim exists and is given by the Roos complex. See Vista 3.5.12 in [Wei94] for a definition of the Roos complex and Rlim^n in the abelian case and section 3.6 in [BBBK18] and section 3 in [BBK23] for details on the Roos complex for $\mathsf{CBorn}_{\mathbb{K}}$ and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$.

In order to construct homotopy limits and colimits the weak equivalences will be part of a model structure. Specifically, we want to use the combinatorial projective model structure from [Wal15], that is defined on any monoidal Grothendieck quasi-abelian category with enough projectives. Recall that a quasi-abelian category C is said to be **Grothendieck** if it is locally presentable and its collection of admissible monomorphisms is closed under small filtered colimits. In Proposition 1.9 of [Wal15], it is claimed that the category $\mathsf{CBorn}_{\mathbb{K}}$ is Grothendieck quasi-abelian, which is deduced from Proposition 1.7. The proof of the latter includes the assertion that "any cocomplete category of the form $\mathsf{Ind}(\mathsf{C})$ is locally presentable by definition." However, this statement is generally false for non-small categories C. If, instead, we consider two Grothendieck universes $\mathbb{U} \in \mathbb{V}$, as in [Wal15], where $\mathsf{Ban}_{\mathbb{K}}$ is $\mathbb{V}$-small and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ is formed within $\mathbb{V}$, then $\mathsf{CBorn}_{\mathbb{K}}$ is no longer a full reflective subcategory of $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$. As a result, the argument in Proposition 1.9 of [Wal15] fails in this case. To fix this issue while still employing the combinatorial model structure from section 2 of [Wal15], we instead rely on the fact that $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ is a union of locally presentable categories, which we discussed in Section 4.2.9.

Proposition 4.2.61. *Let κ be a regular cardinal. The category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ is Grothendieck quasi-abelian. There is a left proper combinatorial model structure on the category of cochain complexes in $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. A set of generating cofibrations is given by admissible monomorphisms*

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{K} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle m} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{K} & \xrightarrow{\mathrm{id}} & \mathbb{K} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

in degree n for all $n \in \mathbb{Z}$.

Proof. By Proposition 4.2.57 the category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ is symmetric monoidal and quasi-abelian with enough projectives. By Proposition 4.2.58 it is locally presentable. As an Ind-completion of a quasi-abelian category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ has exact filtered colimits. Therefore $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ is Grothendieck quasi-abelian. The rest of the statement now follows from Proposition 2.12 in [Wal15], where the set of generating cofibrations is defined in the proof. □

Let κ be a regular cardinal. The homotopy limit and colimit in the model category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ exist. They are defined as

$$\operatorname*{holim}_{\mathsf{D}} F := \operatorname*{Rlim}_{\mathsf{D}} F, \qquad \operatorname*{hocolim}_{\mathsf{D}} F := \operatorname*{Lcolim}_{\mathsf{D}} F,$$

for a functor $F : \mathsf{D} \to \mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. Note that, they agree with the derived colimit and limit in the exact category $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. In particular, the homotopy limit is given up to quasi-isomorphism by the Roos-complex.

Lemma 4.2.62. *Let κ be a regular cardinal. In $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ all filtered colimits are homotopy colimits.*

For any filtered colimit $\operatorname{colim}_{i\in I} V_i \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ and $L \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ we have

$$\mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(\operatorname*{colim}_{i\in I} V_i, L) = \operatorname*{Rlim}_{i\in I} \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(V_i, L).$$

Proof. Fix a regular cardinal κ. We consider the combinatorial model structure from Proposition 4.2.61 on $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. The generating set of cofibrations has complexes of zeros and $\mathbb{K}$ as domains and codomains. These are $\aleph_0$-compact in $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. It is shown in Proposition A.1.2.5 in [HTT] that if the domains and codomains of the generating cofibrations in a combinatorial model category are λ-compact, then the functorial factorizations automatically preserve λ-filtered colimits. It follows, that we have functorial factorizations preserving $\aleph_0$-filtered colimits. Now the proof of Proposition 7.3 in [Dug01] shows that $\aleph_0$-filtered colimits of weak equivalences are again weak equivalences. Thus, filtered colimits are already homotopy colimits in $\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. This allows us to calculate

$$\begin{aligned}\mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})}(\operatorname*{colim}_{i\in I} W_i, M) &= \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})}(\operatorname*{hocolim}_{i\in I} W_i, M)\\ &= \operatorname*{holim}_{i\in I} \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})}(W_i, M)\\ &= \operatorname*{Rlim}_{i\in I} \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})}(W_i, M).\end{aligned} \tag{4.7}$$

for any filtered colimit $\operatorname{colim}_{i\in I} W_i \in \mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ and $M \in \mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$.

To get the statement in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ we need the following observation. For any object $F = $ "$\operatorname{colim}_{i\in I}$"$B_i \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ there is a cardinal κ_F with $B_i \in \mathsf{Ban}_{\kappa_F,\mathbb{K}}$ for all $i \in I$. We get $F \in \mathsf{Ind}(\mathsf{Ban}_{\kappa_F,\mathbb{K}})$. Thus, for all $L \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ and all filtered colimits

$$\operatorname*{colim}_{i\in I} F_i$$

with $F_i \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}), i \in I$, we can choose a regular cardinal κ, such that both with $F_i \in \mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$ for all $i \in I$ and $L \in \mathsf{Ind}(\mathsf{Ban}_{<\kappa,\mathbb{K}})$. Now, in this cardinality restricted situation we can use (4.7). □

4.3. Homological Dimensions in $\mathsf{CBorn}_{\mathbb{K}}$

4.3.1. Projective Objects

Proposition 4.3.1. *Let $B \in \mathsf{CBorn}_{\mathbb{K}}$ be a Banach space. Then B is $\mathfrak{max}$-projective in $\mathsf{CBorn}_{\mathbb{K}}$ if and only if B is $\mathfrak{max}$-projective in $\mathsf{Ban}_{\mathbb{K}}$.*

Proof. The proof of Lemma 2.12 in [PS00] with seminormed spaces replaced by Banach spaces and $\mathsf{Born}_{\mathbb{K}}$ replaced by $\mathsf{CBorn}_{\mathbb{K}}$ works. This is also remarked in chapter 5 [PS00]. □

The following was proven in Proposition 5.8 in [PS00]. Let us add some details to the statement.

Proposition 4.3.2. *The category $\mathsf{CBorn}_{\mathbb{K}}$ has enough $\mathfrak{max}$-projectives. In particular, for all $E \in \mathsf{CBorn}_{\mathbb{K}}$ there is an admissible epimorphism*

$$e: \coprod_{i\in I} \ell^1(X_i) \twoheadrightarrow E$$

for some family of sets $X_i, i \in I$. The set I can be chosen to have cardinality $\mathrm{cf}(\mathcal{B}_{\mathrm{B}}(E))$.

proof from Proposition 2.13 [PS00]. For $E \in \mathsf{CBorn}_{\mathbb{K}}$ choose a cofinal subset $\{B_i\}_{i\in I}$ of $\mathcal{B}_{\mathrm{B}}(E)$. We have an admissible epimorphism

$$\coprod_{i\in I} E_{B_i} \twoheadrightarrow \mathrm{colim}_{i\in I} E_{B_i} \cong E.$$

Since $\mathsf{Ban}_{\mathbb{K}}$ has enough projectives of the form $\ell^1(X)$ by Proposition 3.2.12 there is an admissible epiomorphism $\ell^1(X_i) \twoheadrightarrow E_{B_i}$ for all $i \in I$. By composition we get an admissible epimorphism

$$\coprod_{i\in I} \ell^1(X_i) \twoheadrightarrow \coprod_{i\in I} E_{B_i} \twoheadrightarrow E.$$

Since all $\ell^1(X_i), i \in I$ are $\mathfrak{max}$-projective in $\mathsf{CBorn}_{\mathbb{K}}$ by Proposition 4.3.1 the space $\coprod_{i\in I} \ell^1(X_i)$ is also $\mathfrak{max}$-projective. Note that $\mathrm{card}(I) = \mathrm{cf}(\mathcal{B}_{\mathrm{B}}(E))$ by construction. □

The same argument works for $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$.

Corollary 4.3.3. *The category* $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ *has enough projectives.*

In the following we will show that the spaces from Proposition 4.3.2 are in fact all of the $\mathfrak{max}$-projectives. Every such object in $\mathsf{CBorn}_{\mathbb{K}}$ is a coproduct of spaces of the form $\ell^1(X)$. We do this by combining the analogous result for LB-spaces from Domanski [Dom92] with the proof idea from a classical result due to Kaplansky [Kap58].

Definition 4.3.4. A complete bornological space $E \in \mathsf{CBorn}_{\mathbb{K}}$ is a **bornological LB-space** if there is an isomorphism

$$E \cong \operatorname*{colim}_{n \in \mathbb{N}} E_n,$$

for countably many Banach spaces E_n, where all transition maps are monomorphisms.

Remark 4.3.5. Definition 4.3.4 does not correspond to all LB-spaces in $\mathsf{LCS}_{\mathbb{K}}$, but rather only to **regular LB-spaces** via the von Neumann bornology. See section 3 in [BBBK18]. ◇

Proposition 4.3.6. *Let* $E \in \mathsf{CBorn}_{\mathbb{K}}$ *be a complete bornological space. Then the following are equivalent.*

(i) *E is a bornological LB-space,*

(ii) *E is a quotient of a countable coproduct of Banach spaces,*

(iii) $\mathcal{B}_{\mathrm{B}}(E)$ *ordered by inclusion has cofinality* $\leq \aleph_0$.

Proof. Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space.

- *(i)* ⇒ *(ii)* : Given an isomorphism $E \cong \operatorname{colim}_{n \in \mathbb{N}} E_n$ we have the exact sequence

$$\operatorname{Ker} e \rightarrowtail \coprod_{n \in \mathbb{N}} E_n \xrightarrow{\ e\ } E\,.$$

- *(ii)* ⇒ *(iii)* : Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space and let $B_n \in \mathsf{CBorn}_{\mathbb{K}}, n \in \mathbb{N}$ be Banach spaces. A basis for the quotient bornology of $\left(\coprod_{n \in \mathbb{N}} B_n\right)/F$ with canonical projection q is given by

$$\left\{ q\left(\prod_{n=1}^{m} m \cdot B_{\leq 1}(B_n) \right) \,\middle|\, m \in \mathbb{N} \right\},$$

which is countable. The statement now follows from Proposition 4.2.9.

- *(iii)* ⇒ *(i)* : If $\{ D_n \mid n \in \mathbb{N} \} \subset \mathcal{B}_{\mathrm{B}}(E)$ is cofinal, we have

$$E \cong \operatorname*{colim}_{n \in \mathbb{N}} E_{B_n}.$$

□

Lemma 4.3.7. *Let $L \in \mathsf{CBorn}_{\mathbb{K}}$ be $\mathfrak{max}$-projective and a bornological LB-space. Then there is an isomorphism*

$$L \cong \coprod_{i\in\mathbb{N}} \ell^1(Y_i)$$

for some sets $Y_n, n \in \mathbb{N}$.

Proof. By Proposition 4.3.2 there is an admissible epimorphism $e : \coprod_{n\in\mathbb{N}} \ell^1(X_n) \to P$. Since P is $\mathfrak{max}$-projective, e splits and P is a complemented subspace of $\coprod_{n\in\mathbb{N}} \ell^1(X_n)$.

By Theorem 4.2.15 the functor $\mathrm{t} : \mathsf{Born}_{\mathbb{K}} \to \mathsf{LCS}_{\mathbb{K}}$ commutes with coproducts. Furthermore, for all Banach spaces $B \in \mathsf{Ban}_{\mathbb{K}}$ considered as a bornological space via the von Neumann bornology $\mathrm{vN}(B) \in \mathsf{Born}_{\mathbb{K}}$, we have $\mathrm{t}(\mathrm{vN}(B)) \cong B$ by Proposition 4.2.19 Thus,

$$\mathrm{t}\left(\coprod_{n\in\mathbb{N}} \ell^1(X_n)\right) \cong \coprod_{n\in\mathbb{N}} \mathrm{t}\left(\ell^1(X_n)\right) \cong \coprod_{n\in\mathbb{N}} \ell^1(X_n) \ \in \mathsf{LCS}_{\mathbb{K}}$$

and $\mathrm{t}(P) \in \mathsf{LCS}_{\mathbb{K}}$ is an LB-space complemented in $\coprod_{n\in\mathbb{N}} \ell^1(X_n)$. The main Theorem of [Dom92] now states that $\mathrm{t}(P)$ is isomorphic to $\coprod_{i\in\mathbb{N}} \ell^1(Y_i)$ for some sets $Y_i, i \in \mathbb{N}$. Thus, P in $\mathsf{CBorn}_{\mathbb{K}}$ is isomorphic to $\coprod_{i\in\mathbb{N}} \ell^1(Y_i) \in \mathsf{CBorn}_{\mathbb{K}}$ by Proposition 4.2.16. □

For the proof of Lemma 4.3.9 we will use the following standard fact regarding direct summands.

Lemma 4.3.8. *Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space with complemented subspaces $F, G \subset E$, such that $F \subset G$. Then F is complemented in G.*

Proof. There are $F', G' \subset E$ with $E \cong F \amalg F' \cong G \amalg G'$. We will show that $F' \cap G$ is a complement of $F \amalg G'$ in E.

We have $F \cap F' = \{0\}$ and $G \cap G' = \{0\}$. Since $F \subset G'$, we also have $F \cap G' = \{0\}$ and it makes sense to consider $F \amalg G'$ as a subspace of E. Let $f \in F$ and $g' \in G'$. If $f + g' \in F' \cap G$ then with $f \in F \subset G$ we get $g' \in G$ and therefore $g' = 0$. Then we have $finF' \cap F$ and $f = 0$. Thus, $(F \amalg G') \cap (F' \cap G) = \{0\}$.

For all $e \in E \cong G \amalg G'$ there are $g \in G, g' \in G'$ with $e = g + g'$. Using $E \cong F \amalg F'$ we also get $f \in F, f' \in F'$ with $g = f + f'$. As $F \subset G$, we have $f' = g - f \in G$. Thus,

$$e = g + g' = (f + g') + f' \in (F \amalg G') + (F' \cap G),$$

which completes the proof that $F' \cap G$ is complemented in E. It follows that $G \cong E/G' \cong F \amalg (F' \cap G)$. □

Lemma 4.3.9. *Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a coproduct*

$$E \cong \coprod_{i\in I} B_i,$$

where the B_i are Banach spaces. Every complemented subspace of E is isomorphic to a coproduct of bornological LB-spaces.

The following proof is based on Kaplansky's approach to Theorem 1 in [Kap58]. The primary distinction, aside from the setting, lies in the construction of the S_α.

Proof. Let $P \in \mathsf{CBorn}_{\mathbb{K}}$ be a complemented subspace of E. Then there is a $Q \in \mathsf{CBorn}_{\mathbb{K}}$, such that $E \cong P \amalg Q$. We have the canonical projections $\mathrm{pr}_P, \mathrm{pr}_Q$, that is bounded linear maps with $\mathrm{pr}_P|_P = \mathrm{id}_P$, $\mathrm{pr}_Q\big|_Q = \mathrm{id}_Q$, such that pr_P vanishes on Q and pr_Q vanishes on P.

By transfinite induction over an ordinal α we construct

- base case: Set $S_0 = \{0\}$.
- successor: Given S_α choose a B_j not contained in S_α. If there is no such B_j left, the construction is done. We have

  $$B_j \subset \mathrm{pr}_P(B_j) + \mathrm{pr}_Q(B_j),$$

  since for every $b \in B_j$ there are $p \in P, q \in Q$ with $b = p + q = \mathrm{pr}_P(b) + \mathrm{pr}_Q(b)$. The images $\mathrm{pr}_P(B_{\leq 1}(B_j)), \mathrm{pr}_Q(B_{\leq 1}(B_j))$ are bounded. Taking their complete disked hulls and then their associated Banach spaces, we see that $\mathrm{pr}_P(B_j)$, $\mathrm{pr}_Q(B_j)$ are contained in Banach spaces $P_{B_j} \subset P, Q_{B_j} \subset Q$. By definition of the coproduct bornology there are finite sets $J_P, J_Q \subset I$ with

  $$P_{B_j} \subset \coprod_{i\in J_P} B_i \qquad \text{and} \qquad Q_{B_j} \subset \coprod_{i\in J_Q} B_i.$$

  Set $J_1 = J_P \cup J_Q$. We repeat the procedure above for every B_i with $i \in J_1$ to get a new finite index set for each of the finitely many B_i's in J_1. Let J_2 be the union of those sets, which is again a finite subset of I. Iteratively, construct J_n for all $n \in \mathbb{N}$. Finally, set $J = \cup_{n\in\mathbb{N}} J_n$ and note that J is countable since all J_n are finite. Define

  $$S_{\alpha+1} = S_\alpha + \coprod_{i\in J} B_i$$

- limit: For a limit ordinal α we define

  $$S_\alpha := \bigcup_{\beta<\alpha} S_\beta$$

We have the following:

(i)

$$E = \bigcup_{\alpha} S_{\alpha},$$

(ii) if α is a limit ordinal, S_α is the union of all S_β with $\beta < \alpha$,

(iii) Each S_α is a coproduct of a subset of the B_i's,

(iv) $S_{\alpha+1}/S_\alpha$ is a countable coproduct of Banach spaces,

(v)

$$S_\alpha = P_\alpha + Q_\alpha\,, \quad \text{where} \quad P_\alpha := S_\alpha \cap P, \quad Q_\alpha := S_\alpha \cap Q.$$

The first three statements follow easily from the construction. For (iii) note that both S_α and $S_{\alpha+1}$ consist of coproducts of the B_i's. Thus, the quotient is also a coproduct of the B_i's. The countability follows from the construction. For the assertion in (v) we first note that $P_\alpha + Q_\alpha \subset S_\alpha$. For the other inclusion we consider an element $s \in S_\alpha$. By (iii) we can assume without loss of generality that $s \in B_j$ for some B_j that is part of the coproduct for S_α. We have $s = \mathrm{pr}_P(s) + \mathrm{pr}_Q(s)$. With $E \cong \coprod_{i \in I} B_i$ we know that $\mathrm{pr}_P(s)$ is contained in $\coprod_{i \in L} B_i$ for some finite set $L \subset I$. By construction each of the B_i for $i \in L$ was added as a summand to S_α. Therefore, $\mathrm{pr}_P(s) \in P_\alpha$. The same argument for $\mathrm{pr}_Q(s)$ shows $\mathrm{pr}_Q(s) \in Q_\alpha$ and thus $s \in P_\alpha + Q_\alpha$.

Now, let us complete the proof. We have that P_α is a direct summand of S_α by (v) and $E \cong P \amalg Q$. Furthermore, S_α is a direct summand of E by (iii) and $E \cong \coprod_{i \in I} B_i$. Thus, P_α is a direct summand of E, from which it follows that P_α is also a direct summand of $P_{\alpha+1}$ by Lemma 4.3.8. Since,

$$S_{\alpha+1}/S_\alpha \;=\; P_{\alpha+1}/P_\alpha \amalg Q_{\alpha+1}/Q_\alpha$$

we see that $P_{\alpha+1}/P_\alpha$ is a bornological LB-space by (iv) and Proposition 4.3.6. It follows from (ii), that P_α for a limit ordinal α is the union of all P_β with $\beta < \alpha$. Alltogether, this shows that P is a coproduct of bornological LB-spaces. □

Theorem 4.3.10. *Let $P \in \mathsf{CBorn}_{\mathbb{K}}$ be a $\mathfrak{max}$-projective object. Then there is an isomorphism*

$$P \cong \coprod_{i \in I} \ell^1(Y_i)$$

for some family of sets $Y_i, i \in I$.

Proof. By Proposition 4.3.2 there is an admissible epimorphism $e : \coprod_{i \in I} \ell^1(X_i) \to P$. Since P is $\mathfrak{max}$-projective, e splits and P is a complemented subspace of $\coprod_{i \in I} \ell^1(X_i)$. By Lemma 4.3.9 we have an isomorphism

$$P \cong \coprod_{j \in J} L_j \tag{4.8}$$

where L_j are bornological LB-spaces. For all $j \in J$ the spaces L_j are complemented in P and therefore $\mathfrak{max}$-projective. By Lemma 4.3.7 we have isomorphisms

$$L_j \cong \coprod_{l \in L_j} \ell^1(Y_{j,l}) \tag{4.9}$$

for some sets $Y_{j,1}, Y_{j,2}, \dots$, for all $j \in J$. The statement now follows from (4.8) together with (4.9). ☐

We can give a nice overview of the relations of flat, projective and nuclear objects in $\mathsf{CBorn}_{\mathbb{K}}$. By combining Proposition 4.2.31, Lemma 4.2.41, Proposition 4.2.46, Theorem 4.3.10 and the fact that $\ell^1(X)$ is nuclear if and only if X is finite, we get the following.

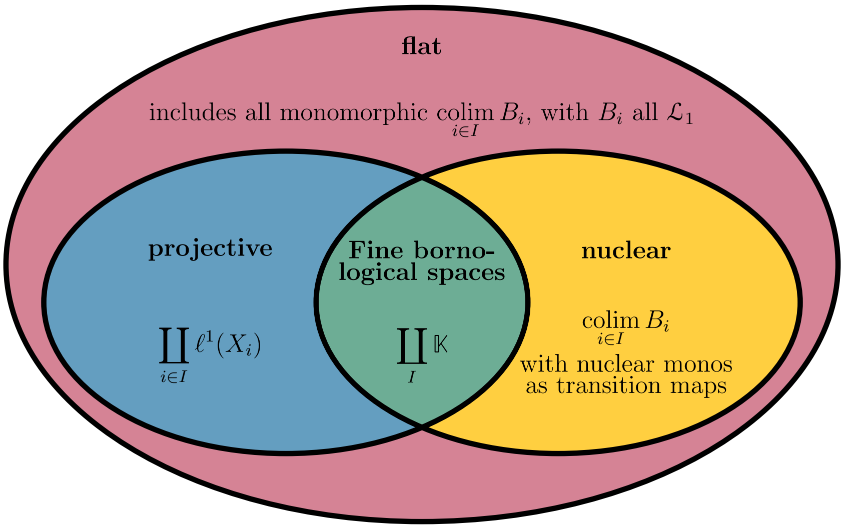


Figure 4.1.: Some special classes of objects in $\mathsf{CBorn}_{\mathbb{K}}$

4.3.2. Global Dimension

One of the conditions that allows us to apply (IIP) is having finite global dimension. Unfortunately, this is not the case for $(\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max})$.

Theorem 4.3.11. *We have for the global dimension*

$$\mathrm{g}\ell\left(\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}\right) = \infty.$$

Proof. By Proposition 3.2.13 there is a Banach space B with $\mathfrak{max}$-projective dimension ∞ in $\mathsf{Ban}_{\mathbb{K}}$. Thus, B has a $\mathfrak{max}$-projective resolution $P_\bullet \to B$ of length ∞ in $\mathsf{Ban}_{\mathbb{K}}$ that does not split. By Proposition 4.3.1 $(\mathrm{vN}(P))_\bullet \to \mathrm{vN}(B)$ is a $\mathfrak{max}$-projective resolution in $\mathsf{CBorn}_{\mathbb{K}}$. Since $\mathsf{Ban}_{\mathbb{K}}$ is a full subcategory of $\mathsf{CBorn}_{\mathbb{K}}$ via vN we have

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}}(\mathrm{vN}(B)) = \infty.$$

□

4.3.3. Projective Dimensions

Our goal for this section is to use Mitchell's Theorem 4.3.12 to get a bound on projective dimensions dependent on certain cardinal numbers.

In [Mit73] Mitchell proved the following theorem for R-modules using the Roos complex to define Rlim^n. This is possible in a general exact category, which has exact products (AB4*). The proof in this general setup is completely analogous to the original one.

Theorem 4.3.12 (Mitchell [Mit73])**.** *Let* C *be an exact category satisfying* (AB4*)*. Let* I *be a directed poset of cardinality* $\aleph_d$ *with* $d \in \mathbb{N}$*. Then* Rlim^n *vanishes for all* $n > d+1$*.*

In combination with Mitchell's theorem we want to use Lemma 4.2.62 that says that filterd colimits in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ are homotopy colimits. In order to then make statements about $\mathsf{CBorn}_{\mathbb{K}}$ and not just $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ we need the following.

Proposition 4.3.13. *Let* $F \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ *be a formal filtered colimit. For the* $\mathfrak{max}$*-projective dimension we have*

$$\mathrm{pd}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}), \mathfrak{max}}(F) \;=\; \mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}}(\operatorname{colim} F)\,.$$

Similarly, for a complete bornological space $E \in \mathsf{CBorn}_{\mathbb{K}}$*.*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}}(E) \;=\; \mathrm{pd}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}), \mathfrak{max}}(\operatorname{diss} E)$$

Proof. By Proposition 5.16 in [PS00] the derived categories of $\mathsf{CBorn}_{\mathbb{K}}$ and $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ are equivalent and the equivalence is induced by

$$\mathsf{CBorn}_{\mathbb{K}} \underset{\text{colim}}{\overset{\text{diss}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})\,.$$

Therefore,

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}}),\mathfrak{max})}(F, L[i]) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max})}(\operatorname{colim} F, \operatorname{colim} L[i])$$

for all $L \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ and $i \in \mathbb{Z}$. The first statement now follows from Proposition 1.3.30. The second assertion is shown in an analogous manner. □

Theorem 4.3.14. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space, such that there is an isomorphism*

$$F \cong \operatorname*{colim}_{i\in I} \ell^1(X_i)$$

for sets $X_i, i \in I$ and $\operatorname{card}(I) \leq \aleph_d$. Then

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(F) \leq d+1.$$

Proof. Let F be isomorphic to $\operatorname{colim}_{i\in I} \ell^1(X_i)$ with $\operatorname{card}(I) \leq \aleph_d$. By Proposition 4.3.13 we can calulate the projective dimension of F as the projective dimension of $\operatorname{diss} F$ in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$. Let $L \in \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ be another Ind-Banach space. We need to show that $\mathrm{Ext}^i(F, L) = 0$ for $i > d+1$. For that we use that $\ell^1(X_i)$ is $\mathfrak{max}$-projective and compute with Lemma 4.2.62

$$\begin{aligned}
\mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(\text{``}\operatorname*{colim}_{i\in I}\text{''}\ell^1(X_i), L) &= \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(\operatorname*{colim}_{i\in I} \ell^1(X_i), L) \\
&= \operatorname*{holim}_{i\in I} \mathrm{RHom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(\ell^1(X_i), L) \\
&= \operatorname*{holim}_{i\in I} \mathrm{Hom}_{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})}(\ell^1(X_i), L).
\end{aligned}$$

Note that $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ satisifes (AB4*) by Corollary 4.3.3 and Proposition 1.4.5 of [Sch99]. This allows us to apply Theorem 4.3.12 and it follows from $\operatorname{card}(I) \leq \aleph_d$ and the computation above, that $\mathrm{Ext}^i(F, L) = 0$ for all $i > d+1$. □

Nuclear Fréchet spaces We want to apply Theorem 4.3.14 to nucelar Fréchet spaces. One specific cardinal number we will use is the bounding number $\mathfrak{d}$. It is defined as the cofinality of the poset $(\prod_{\mathbb{N}} \mathbb{N}, \leq)$, where $\leq$ is pointwise domination. See Appendix A for more details.

Proposition 4.3.15. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a Fréchet space. Then the cofinality of $(\mathcal{B}_{\mathrm{B}}(F), \subset)$ is $\mathfrak{d}$ or smaller. If F is not normable it is exactly $\mathfrak{d}$.*

Proof. Let $E \in \mathsf{Fré}_{\mathbb{K}}$ be a Fréchet space with $\mathrm{vN}(E) = F$. By Proposition 3.2.21 there is a fundamental system of seminorms

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots \tag{4.10}$$

for E. Let $B_i := \{ x \in E \mid \|x\|_i \leq 1 \}$ be the unit ball for the seminorm $\|\cdot\|_i$ for all $i \in \mathbb{N}$. It follows from (4.10), that $B_i \subset B_j$ for $i \geq j$. By Proposition 3.1.14 a subset $B \subset E$ is bounded if and only if $\|B\|_i$ is bounded for all $i \in \mathbb{N}$. Thus, the set

$$\mathcal{A} := \left\{ \bigcap_{i \in \mathbb{N}} n_i \cdot B_i \;\middle|\; (n_i)_{i \in \mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{N} \right\}$$

is a basis for the von Neumann bornology of E. The partially ordered set $(\mathcal{A}, \subset)$ has cofinality equal to or smaller than the cofinality of $(\prod_{\mathbb{N}} \mathbb{N}, \leq)$ described in Appendix A.4. Thus, the cofinality of $\mathcal{B}_{\mathrm{D}}(F)$ and therefore also of $\mathcal{B}_{\mathrm{B}}(F)$ is at most $\mathfrak{d}$. If F is not normable, the fundamental system (4.10) has infinitely many strict inclusions. In this case, $(\mathcal{A}, \subset)$ is isomorphic to the poset $(\prod_{\mathbb{N}} \mathbb{N}, \leq)$ and has cofinality $\mathfrak{d}$. □

Corollary 4.3.16. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a nuclear Fréchet space and infinite-dimensional. There is an isomorphism*

$$\operatorname*{colim}_{i \in I} \ell^1 \cong \operatorname{diss} F$$

in $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$, for some set I with cardinality $\operatorname{card}(I) = \mathfrak{d}$.

Proof. Combine Proposition 4.3.15 and Proposition 4.2.46. □

Theorem 4.3.17. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a nuclear Fréchet and assume that the dominating number satisfies $\mathfrak{d} \leq \aleph_d$ for some $d \in \mathbb{N}$. Then*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}}(F) \leq d + 1.$$

Proof. If F is nuclear Fréchet and $\mathfrak{d} \leq \aleph_d$ we can apply Corollary 4.3.16 to get an isomorphism $\operatorname{diss} F \cong \operatorname{colim}_{i \in I} \ell^1$, where I has cardinality $\mathfrak{d} \leq \aleph_d$. The statement now follows from Theorem 4.3.14. □

Corollary 4.3.18. *Let X be a real smooth manifold. Assuming CH we have*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}}(\mathcal{C}^\infty(X)) \leq 2.$$

Dual Nuclear Fréchet spaces

Definition 4.3.19. A bornological space E is a **dual nuclear Fréchet space (DNF)** if there is a $V \in \mathsf{LCS}_{\mathbb{K}}$ that is the strong dual of a nuclear Fréchet space and $\mathrm{vN}(V) \cong E$.

Recall from Proposition 3.2.36 that all DNF spaces are nuclear.

Corollary 4.3.20. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a DNF space. Then*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(F) \leq 1.$$

If F is infinite dimensional, then

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(F) = 1.$$

Proof. Let $E \in \mathsf{LCS}_{\mathbb{K}}$ be a DNF space with $\mathrm{vN}(E) \cong F$. By Proposition 3.2.41 the space E has a countable fundamental system of bounded sets. Thus, the von Neumann bornology of E has a countable basis of bounded disks and also of Banach disks by Proposition 4.2.9. By Proposition 4.2.46 there is a countable diagram $I \to \mathsf{Ban}_{\mathbb{K}}, i \mapsto \ell^1(X_i)$ such that $F \cong \mathrm{colim}_{i\in\mathbb{N}} \ell^1(X_i)$ for some sets $X_i, i \in \mathbb{N}$. Proceeding as in Theorem 4.3.14 we get

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(F) \leq 1.$$

It remains to show that F is not $\mathfrak{max}$-projective, for infinite dimensional F. From the classification Theorem 4.3.10 we see that a projective object is nuclear if and only if it is a coproduct of $\mathbb{K}$. In this case the only option for a pre-dual of F is again a coproduct of $\mathbb{K}$. The latter cannot be infinite, since F is Fréchet . Thus, F is projective if and only if F is finite dimensional. □

Example 4.3.21. Let $n \in \mathbb{N}$. The locally convex space of **Distributions with compact support** $\mathcal{D}_c(\mathbb{R}^n)$ is defined as the strong dual of the nuclear Fréchet space $\mathcal{C}^\infty(\mathbb{R}^n)$ from Example 3.2.26. Taking the von Neumann bornology and using Corollary 4.3.20 we have

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(\mathrm{vN}(\mathcal{D}_c(\mathbb{R}^n))) = 1.$$

◇

Question 4.3.22. Does the category $(\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max})$ satisfy $(\Pi\mathrm{P})$. Specifically, do countable products of $\mathfrak{max}$-projective have finite $\mathfrak{max}$-projective dimension? Is this question decidable within ZFC?

The simplest infinite product to consider is the space $\prod_{\mathbb{N}} \mathbb{K}$. This space is nuclear Fréchet , so by Theorem 4.3.17 its $\mathfrak{max}$-projective dimension is finite if $\mathfrak{d} = \aleph_d$ for some $d \in \mathbb{N}$. Can methods similar to those used by Osofsky in [Oso68] be applied to show that the $\mathfrak{max}$-projective dimension of $\prod_{\mathbb{N}} \mathbb{K}$ is infinite if $\mathfrak{d} \geq \aleph_\omega$?

4.4. Small Bornological Spaces and small Ind-categories

In this section we generalize the notion of bornological LB-spaces from Definition 4.3.4. The goal is to find a category, that under some cardinality assumption and using Theorem 4.3.14 satisfies $(\Pi\mathrm{P})$.

4.4.1. Definition and Dissection

Definition 4.4.1. Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space and κ a regular cardinal number. We say that E is **$\leq\kappa$-small** if there is an isomorphism

$$E \cong \operatorname*{colim}_{i \in I} E_i,$$

for Banach spaces $E_i, i \in I$ and $\operatorname{card}(I) \leq \kappa$.

Denote the full subcategory of $\mathsf{CBorn}_{\mathbb{K}}$ of $\leq\kappa$-small complete bornological spaces by $\mathsf{CBorn}_{\mathbb{K}}^{\leq\kappa}$.

Example 4.4.2.

(i) Every bornological LB-space is $\leq\aleph_0$-small.

(ii) Every DNF-space is $\leq\aleph_0$-small by Proposition 3.2.41.

(iii) Every Banach space is $\leq\aleph_0$-small.

It seems that intuitively a Banach space should be 1-small, since one disk suffices to define the topology. This can be fixed by changing the preorder on Banach disks from inclusion to absorption. See section 1.1.2 in [Mey07]. For "true" infinite colimits, such as LB-spaces that are not Banach, the preorder makes no difference.

◇

Remark 4.4.3. For any regular cardinal κ the category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\kappa}$ is not essentially small. Indeed, it always contains $\ell^1(X)$ for all sets X. Since the isomorphism class of $\ell^1(X)$ in $\mathsf{Ban}_{\mathbb{K}}$ and also in $\mathsf{CBorn}_{\mathbb{K}}$ is determined by $\operatorname{card}(X)$ the class $\operatorname{ob}(\mathsf{CBorn}_{\mathbb{K}}^{\leq\kappa})$ up to isomorphism is not a set. ◇

Proposition 4.4.4. *Let $E \in \mathsf{CBorn}_{\mathbb{K}}$ be a complete bornological space and κ a regular cardinal number. Then the following are equivalent.*

(i) E is $\leq\kappa$-small,

(ii) E is a quotient of a coproduct of $\leq \kappa$ Banach spaces,

(iii) $\mathcal{B}_{\mathrm{B}}(E)$ ordered by inclusion has cofinality $\leq \kappa$,

(iv) there is a cofinal subset of bounded sets in E with cardinality $\leq \kappa$.

Proof. The proof is the same as Proposition 4.3.6 with the appropriate cardinalities replaced. The last equivalence follows from Proposition A.4.1. □

For the rest of this section fix $d \in \mathbb{N}$. We will consider $\leq\aleph_d$-small spaces.

Proposition 4.4.5. *There is an equivalence of categories*

$$\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d} \;\cong\; \mathsf{Ind}_{\rightarrowtail}^{\aleph_d}(\mathsf{Ban}_{\mathbb{K}})$$

of

- $\leq\aleph_d$*-small complete bornological spaces* $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$*, and*
- *the category* $\mathsf{Ind}_{\rightarrowtail}^{\aleph_d}(\mathsf{Ban}_{\mathbb{K}})$ *of formal filtered colimits over diagrams of size at most* $\aleph_d$*, where all transition maps are monomorphisms.*

Proof. Given an $\leq\aleph_d$-small complete bornological space E, write it as a filtered colimit over an $\leq\aleph_d$-small set $\mathcal{D}$ of Banach disks For the dissection functor from Proposition 4.2.25 we have

$$\operatorname{diss}(E) \cong \text{“}\operatorname*{colim}_{D\in\mathcal{D}}\text{”}\, E_D.$$

Thus, the essential image of $\operatorname{diss} : \mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d} \to \mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ is $\mathsf{Ind}_{\rightarrowtail}^{\aleph_d}(\mathsf{Ban}_{\mathbb{K}})$. □

4.4.2. Exact Structure, Limits, Colimits and Tensor Product

Lemma 4.4.6. *The category* $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$ *is a full additive subcategory of* $\mathsf{CBorn}_{\mathbb{K}}$ *that reflects kernels and cokernels.*

Proof. Let $f : E \to F$ be a morphism in $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$. It suffices to check that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are $\leq\aleph_d$-small.

The kernel $\operatorname{Ker} f$ is $f^{-1}(0)$ equipped with the subspace bornology. Clearly, if $\mathcal{D}_E$ is a $\leq\aleph_d$-small cofinal subset of $\mathcal{B}_{\mathrm{B}}(E)$, then $\{\, D \cap f^{-1}(0) \mid D \in \mathcal{D}_E \,\}$ is $\leq\aleph_d$-small and cofinal in all bounded sets of $\operatorname{Ker} F$. By Proposition 4.4.4 this suffices to show $\operatorname{Ker}(f) \in \mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$.

The cokernel $\operatorname{Coker} f$ is the quotient space $F/\overline{f(E)}$ with the quotient bornology. Write $\pi : F \to F/\overline{f(E)}$ for the canonical map. Then

$$\mathcal{B}_{\operatorname{Coker} \{} = \{\, \pi(B) \mid B \text{ bounded in } F \,\}$$

is the bornology of $\operatorname{Coker} f$. If $\mathcal{D}_F$ has cardinality $\leq \aleph_d$ and is a cofinal subset of $\mathcal{B}_{\mathrm{B}}(F)$ it follows that $\{\, \pi(D) \mid D \in \mathcal{D}_F \,\}$ has cardinality $\leq \aleph_d$ and is a cofinal in $\mathcal{B}_{\operatorname{Coker}\{}$. By Proposition 4.4.4 this suffices to show $\operatorname{Ker}(f) \in \mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$. □

Proposition 4.4.7. *The category* $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$ *is quasi-abelian and has* $\leq\aleph_d$*-small colimits.*

Proof. The property quasi-abelian follows from Lemma 4.4.6 and combined with Proposition 1.2.10.

It suffices to show that $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ has cokernels and $\leq\aleph_d$-small coproducts. We have already seen, that we have cokernels since the category is quasi-abelian. For $\leq\aleph_d$-small coproducts consider $\leq\aleph_d$-small $E_i \in \mathsf{CBorn}_{\mathbb{K}}$ for $i \in I$ with $\operatorname{card} I \leq \aleph_d$. Let $\mathcal{D}_i$ be a cofinal subset of $\mathcal{B}_{\mathrm{B}}(E_i)$ for all $i \in I$. Then

$$\mathcal{D} := \left\{ \coprod_{\substack{J\subset I\\ J \text{ finite}}} D_j[j] \subset \coprod_{i\in I} E_i[i] \;\middle|\; D_j \in \mathcal{D}_j \right\}$$

is a cofinal subset of $\mathcal{B}_{\mathrm{B}}\left(\coprod_{i\in I} E_i\right)$. For any fixed finite set $J \subset I$ the set

$$\left\{ \coprod_{j\in J} D_j[j] \;\middle|\; D_j \in \mathcal{D}_j \right\}$$

has cardinality $\aleph_d^{\operatorname{card} J} = \aleph_d$. Furthermore, the set $\{\, J \subset I \mid J \text{ finite} \}$ also has cardinality $\aleph_d$. Thus, $\operatorname{card}(\mathcal{D}) = \aleph_d \cdot \aleph_d = \aleph_d$ and therefore $\coprod_{i\in I} E_i \in \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$. □

Proposition 4.4.8. *The category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ has enough $\mathfrak{max}$-projectives. All $\mathfrak{max}$-projectives are isomorphic to a space of the form*

$$\coprod_{i\in I} \ell^1(X_i)$$

with X_i sets and $\operatorname{card}(I) \leq \aleph_d$.

Proof. Note that $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ is a full subcategory of $\mathsf{CBorn}_{\mathbb{K}}$, where a pair of morphisms is a kernel-cokernel pair if and only if it is a kernel-cokernel pair in $\mathsf{CBorn}_{\mathbb{K}}$. Thus, the second statement follows directly from Theorem 4.3.10.

Let $E = \operatorname{colim}_{D\in\mathcal{D}} E_D$ be an $\leq\aleph_d$-small complete bornological space and D a $\leq\aleph_d$-small cofinal set of Banach disks in E. The proof of Proposition 4.3.2 shows that there is an admissible epimorphism

$$\textstyle\coprod_{D\in\mathcal{D}} \ell^1(B_{\leq 1}(E_D)) \longrightarrow\!\!\!\!\!\rightarrow \coprod_{D\in\mathcal{D}} E_D \longrightarrow\!\!\!\!\!\rightarrow E\,.$$

□

Proposition 4.4.9. *The category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$, equipped with the completed projective tensor product, is symmetric monoidal.*

Proof. We will use that $(\mathsf{CBorn}_{\mathbb{K}}, \hat{\otimes}_\pi, \mathbb{K})$ is symmetric monoidal by Theorem 4.2.27 and show that the full subcategory $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ is closed under $\hat{\otimes}_\pi$. Let $E, F \in \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ be $\leq\aleph_d$-small complete bornological vector spaces. Let $\mathcal{D}_E \subset \mathcal{B}_{\mathrm{B}}(E)$, $\mathcal{D}_F \subset \mathcal{B}_{\mathrm{B}}(F)$ be cofinal and $\leq\aleph_d$-small subsets. By Definition 4.2.26 the projective tensor product has basis

$$\left\{ (C \otimes D)^{\diamond} \;\middle|\; C \in \mathcal{D}_E, D \in \mathcal{D}_F \right\},$$

which has cardinality $\aleph_d \cdot \aleph_d = \aleph_d$. Completing it, we still have a basis of cardinality $\aleph_d$. We have shown that $E \hat{\otimes}_\pi F$ is $\leq\aleph_d$-small. Since we also have the unit $\mathbb{K} \in \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$, the statement follows. □

Remark 4.4.10. Unlike $\mathsf{CBorn}_{\mathbb{K}}$ the category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ with tensor product $\hat{\otimes}_\pi$ is not closed. The problem is that the internal Hom

$$\underline{\mathrm{Hom}}_{\mathsf{CBorn}_{\mathbb{K}}}\left(\prod_{i\in\aleph_d} \mathbb{K}, \prod_{i\in\aleph_d} \mathbb{K} \right)$$

is not $\leq\aleph_d$-small. ◇

4.4.3. Smallness of Nuclear Fréchet Spaces

One of the requirements on a suitable theory for studying smooth functions and the de Rham algebra is that it has to contain nuclear Fréchet spaces.

Proposition 4.4.11. *Assume that $\mathfrak{d} \leq \aleph_d$. Then there is a fully faithful functor* $\mathsf{Fré}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$.

Proof. By Proposition 4.2.23 the functor $\mathrm{vN} : \mathsf{Fré}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ is fully faithful. By Proposition 4.3.15 the a Fréchet space in $\mathsf{CBorn}_{\mathbb{K}}$ is $\mathfrak{d}$-small. The assertion now follows from the assumption $\mathfrak{d} \leq \aleph_d$. □

To define the contraderived category we also need countable products.

Proposition 4.4.12. *Assume that $2^{\aleph_0} \leq \aleph_d$. Then the category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ is closed under countable products.*

Proof. Let $E_i \in \mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ be $\leq\aleph_d$-small complete bornological spaces for $i \in \mathbb{N}$. Let $\mathcal{D}_i$ be a cofinal subset of $\mathcal{B}_{\mathrm{B}}(E_i)$ for all $i \in \mathbb{N}$. The product $\prod_{i\in\mathbb{N}} E_i$ has a basis for its bornology given by

$$\mathcal{D} = \left\{ \prod_{i\in\mathbb{N}} D_i[i] \subset \prod_{i\in\mathbb{N}} E_i[i] \;\middle|\; D_i \in \mathcal{D}_i \right\}.$$

We calculate with Corollary A.2.3

$$\operatorname{card}\mathcal{D} = \prod_{i\in\mathbb{N}} \operatorname{card}\mathcal{D}_i \leq \prod_{i\in\mathbb{N}} \aleph_d = \aleph_d^{\aleph_0} = 2^{\aleph_0}\cdot\aleph_d\,,$$

which by assumption is at most $\aleph_d$. By Proposition 4.2.9 we get a cofinal subset of $\mathcal{B}_{\mathrm{B}}\left(\prod_{i\in\mathbb{N}} E_i\right)$ of cardinality $\aleph_d$. Thus, $\prod_{i\in\mathbb{N}} E_i$ is $\leq\aleph_d$-small. □

Remark 4.4.13. We can not replace the assumption $2^{\aleph_0}\leq\aleph_d$ in Proposition 4.4.12 with something strictly weaker. Indeed, the cofinality of $\mathcal{B}_{\mathrm{B}}\left(\prod_{\aleph_0}\left(\coprod_{\aleph_d}\mathbb{K}\right)\right)$ is $2^{\aleph_0}\cdot\aleph_d$. ◇

4.4.4. Contraderived Category

Assume that $\mathfrak{d}\leq\aleph_d$. We have seen that $(\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}, \hat{\otimes}_\pi, \mathbb{K})$ is a symmetric monoidal quasi-abelian category, which has countable products. We can form the category of chain complexes consisting of graded $\leq\aleph_d$-small complete bornological spaces. By considering $\mathbb{K}$ as a in degree 0 concentrated DG algebra we define the category of DG vector spaces $\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}$. As discussed in Chapter 2 and using that we have countable products we can define the contraderived category $\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}, \mathfrak{max})$. Note, that the differentials of a DG vector space need not be admissible. However, the short exact sequences used for defining contraacyclic objects consist of $\mathfrak{max}$-admissible maps.

Theorem 4.4.14. *Assume that* $2^{\aleph_0}\leq\aleph_d$*. Then the exact category* $(\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}, \mathfrak{max})$ *satisfies* (ΠP).

Proof. Using the cardinality assumption, the category $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ has countable products by Proposition 4.4.12. Let P be a $\mathfrak{max}$-projective object in $\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d}$, which by Proposition 4.4.8 we can write as

$$P \cong \coprod_{i\in I}\ell^1(X_i)$$

with X_i sets and $\operatorname{card}(I)\leq\aleph_d$. Since we always have an admissible epimorphism $\ell^1(Y')\to\ell^1(Y)$ for sets Y, Y' with $\operatorname{card}(Y)\leq\operatorname{card}(Y')$, we can assume without loss of generality that $X_i = X$ for some set X and all $i\in I$.

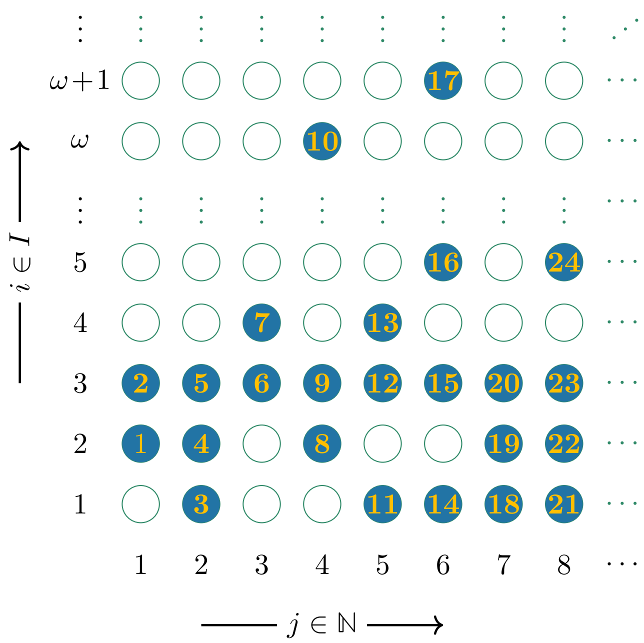
Our goal is to get the bound on the projective dimension from Theorem 4.3.14. Let us first construct a diagram given by a small enough poset. Let S be any cofinal subset of $(\prod_{\mathbb{N}}\mathbb{N}, \leq)$. So any $s\in S$ consists of a sequence of natural numbers $(s_l)_{l\in\mathbb{N}}$ and we order them by pointwise domination in $(\mathbb{N}, \leq)$. That is $s\leq s'$ if $s_l\leq s'_l$ for all $l\in\mathbb{N}$.

We write the elements of $\mathbb{N} \times I$ as $[j,i]$ and define

$$\mathfrak{T} := \{\, T \subset \mathbb{N} \times I \mid \text{for all } j \in \mathbb{N},\ T \text{ contains only finitely many [j,i] with } i \in I \,\}\,.$$

For every $T \in \mathfrak{T}$ we choose an enumeration

$$\varphi_T : \mathbb{N} \to T, \qquad l \mapsto \varphi_T(l).$$



An Element $T \in \mathfrak{T}$ illustrated as a subset of $\mathbb{N} \times I$ by the filled **blue circles**. The set T contains only finitely many elements in each column.

For T we have the enumeration $\varphi_T : \mathbb{N} \to T$ represented by the **yellow numbers**.

For every $s \in S$ this defines an element of $\mathfrak{T} \times S$. We imagine that (T,s) has value $s_l \in \mathbb{N}$ at the **circle** with number $\varphi_T(l)$ and zero at all the unfilled **circles**. The preorder on $\mathfrak{T} \times S$ is then given by pointwise domination.

Figure 4.2.: Grid representing an element of $\mathfrak{T}$

We have a map

$$\sigma : \mathfrak{T} \times S \;\to \prod_{[j,i] \in \mathbb{N} \times I} \mathbb{N}_0[j,i], \qquad (T,s) \mapsto \sigma(T,s) = \prod_{l \in \mathbb{N}} s_l\, \varphi_T(l), \tag{4.11}$$

where all the other entries are zero. That is, at $[j,i] \in \mathbb{N} \times I$ the number in $\sigma(T,s)$ is

$$= \begin{cases} s_l & \text{if there is a } l \in \mathbb{N} \text{ with } [j,i] = \varphi_T(l), \\ 0 & \text{else.} \end{cases}$$

We order $\prod_{[j,i] \in \mathbb{N} \times I} \mathbb{N}_0[j,i]$ by pointwise domination as we did with S. Then σ with (4.11) defines a preorder on $\mathfrak{T} \times S$, where $(T,s) \le (T',s')$ if $\sigma(T,s) \le \sigma(T',s')$. It might be helpful to visualize the elements of $\mathfrak{T}, \mathfrak{T} \times S$ and the preorder in a grid as in Fig. 4.2.

Now we move to defining bornological spaces for all elements in $\mathfrak{T} \times S$. For $T \in \mathfrak{T}, s \in S$ we set $E_{T,s} := \ell^1(\mathbb{N} \times X)$ and write an element of $E_{T,s}$ as $a = (a^1, a^2, \dots)$ with $a^l \in \ell^1(X), l \in \mathbb{N}$. Note that since $a \in \ell^1(\mathbb{N} \times X)$, the a^l are also 1-summable when considered together. We will use the entries s_l of s to scale the a^l. Next, we define monomorphisms

$$\rho_{T,s} : \; E_{T,s} \to \prod_{j \in \mathbb{N}} \coprod_{i \in I} \ell^1(X)[j,i] \quad (a^1, a^2, \dots) \mapsto \sum_{l \in \mathbb{N}} (a^l \cdot s_l) \varphi_T(l) \tag{4.12}$$

Note that the monomorphisms are regular and by construction

$$E_{T,s} \cong \mathrm{Im}(\rho_{T,s}) \subset \mathrm{Im}(\rho_{T',s'}) \cong E_{T',s'}$$

if and only if $(T, s) \leq (T', s')$. Thus, we have a diagram

$$L \to \mathsf{CBorn}_{\mathbb{K}}, \qquad (T, s) \;\mapsto\; E_{T,s}$$

with transition maps given by inclusions. We can take the colimit and automatically get a monomorphism

$$\varPsi : \operatorname*{colim}_{(T,s) \in \mathfrak{T} \times S} E_{T,s} \to \prod_{j \in \mathbb{N}} \coprod_{i \in I} \ell^1(X)$$

from the universal property and the inclusions from (4.12). To see that $\varPsi$ is an isomorphism it suffices to show that it is an admissible epimorphism by Proposition 1.1.6. We will do this by, proving that the bounded sets of all $\mathrm{Im}(\rho_{T,s})$ are cofinal in the bornology of $\prod_{j \in \mathbb{N}} \coprod_{i \in I} \ell^1(X)$. Let B be a bounded set in $\prod_{j \in \mathbb{N}} \coprod_{i \in I} \ell^1(X)[j,i]$. By the description of the product and coproduct bornologies from Proposition 4.2.11, there are bounded disks $(B_{j,i})_{j \in \mathbb{N}, i \in I}$, where $B \subset \prod_{j \in \mathbb{N}, i \in I} B_{j,i}$, $B_{j,i} \subset \ell^1(X)[j,i]$ bounded and for each fixed $j \in \mathbb{N}$ only finitely many $B_{j,i}$ are non-zero. Thus, we can choose a $T \in \mathfrak{T}$, that contains all indices for non-zero $B_{j,i}$.

Each $B_{j,i}$ is norm-bounded in $\ell^1(X)[j,i]$. So there is a $s_{j,i} \in \mathbb{N}$ such that $B_{j,i} \subset s_{j,i} \cdot B_{\leq 1}(\ell^1(X)[j,i])$. Thus, with the cofinality of S we can choose $s \in S$, such that

$$B \subset \prod_{j \in \mathbb{N}, i \in I} B_{j,i} \subset \rho_{T,s}\left(B_{\leq 1}(\ell^1(X \times \mathbb{N})[s])\right),$$

which is bounded in $\mathrm{Im}(\rho_{T,s})$. This concludes the proof, that $\varPsi$ is an isomorphism.

Next, let us determine the size of the poset $\mathfrak{T} \times S$. Since I is infinite, the number of finite subsets of I is $\mathrm{card}\, I \leq \aleph_d$. Thus, the cardinality of $\mathfrak{T}$ is $\mathrm{card}(T) \leq {\aleph_d}^{\aleph_0}$. For $S \subset \prod_{\mathbb{N}} \mathbb{N}$ we have $\mathrm{card}(S) \leq {\aleph_0}^{\aleph_0} = 2^{\aleph_0}$. With the cardinality assumption and (A.2.3) we get

$$\mathrm{card}(\mathfrak{T} \times S) = \leq \aleph_d^{\aleph_0} \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \cdot \aleph_d \cdot 2^{\aleph_0} = \aleph_d.$$

Thus, $\prod_{j\in\mathbb{N}}\coprod_{i\in I}\ell^1(X)$ is a monomorphic filtered colimit of $\leq\aleph_d$ spaces, that are $\mathfrak{max}$-projective. By Theorem 4.3.14 we get

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}}^{\leq\aleph_d},\mathfrak{max}}\left(\prod_{j\in\mathbb{N}}\coprod_{i\in I}\ell^1(X)\right)\leq d+1.$$

The assertion now follows from Proposition 2.2.11. □

Theorem 4.4.15. *Assume that $2^{\aleph_0}\leq\aleph_d$. Consider $\leq\aleph_d$-small complete bornological DG-vector spaces. The composition of functors*

$$\mathsf{Ho}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d},\mathfrak{max}_{\mathrm{proj}})\to\mathsf{Ho}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d})\to\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d},\mathfrak{max})$$

is an equivalence of categories. In particular, every $\leq\aleph_d$-small complete bornological DG vector spaces is contraequivalent to a DG vector space M with

$$M^\sharp\cong\coprod_{i\in I}\ell^1(X_i)[s_i],$$

where $\mathrm{card}(I)\leq\aleph_d$*, $\{X_i\}_{i\in I}$ is some family of sets and $s_i\in\mathbb{Z},i\in I$ are some integers denoting shifts.*

Proof. The category $\mathsf{DG\text{-}CBorn}_{\mathbb{K}}^{\leq\aleph_d}$ satisfies (ΠP) by Theorem 4.4.14 and Proposition 2.2.7. We get the semiorthogonal decomposition from Theorem 2.2.12. The second assertion follows from the description of all $\mathfrak{max}$-projectives in Proposition 4.4.8. □

5. Bornological Algebras

Contents

In this chapter, we will build upon the results concerning homological dimensions from Chapter 4, extending them to categories of bornological (graded) modules over bornological (graded) algebras. Our primary focus will be on two key examples: smooth functions and the de Rham algebra on real smooth manifolds. Transitioning from vector spaces to modules introduces two significant exact structures, that we will explore. The first is an extension of the maximal exact structure on $\mathsf{CBorn}_{\mathbb{K}}$, which reflects the fact that the module categories are quasi-abelian. The second, a simpler exact structure, extends the minimal trivial structure on $\mathsf{CBorn}_{\mathbb{K}}$ and will contain short exact sequences that split if one forgets the module action.

The primary objective of this chapter is to leverage the results on homological dimensions from Chapter 4 to apply Theorem 2.2.12, leading to semiorthogonal decompositions of contraderived categories.

5.1. Bornological Algebras and Exact Structures

5.1.1. Bornological Algebras and Modules

We have seen in Theorem 4.2.27, that $\mathsf{CBorn}_{\mathbb{K}}$ with the tensor product $\hat{\otimes}_\pi$ from Section 4.2.6 and unit $\mathbb{K}$ is symmetric monoidal. We will denote $\hat{\otimes}_\pi$ by $\otimes_{\mathbb{K}}$ in this chapter. The category of chain complexes in $\mathsf{CBorn}_{\mathbb{K}}$ is denoted by $\mathsf{Ch}(\mathsf{CBorn}_{\mathbb{K}})$ and is also symmetric monoidal by Proposition 1.3.31.

Definition 5.1.1. A **complete bornological algebra** or a **$\mathsf{CBorn}_{\mathbb{K}}$-algebra** A is a monoid in $(\mathsf{CBorn}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. The left modules over A are called **complete bornological left A-modules** or a **$\mathsf{CBorn}_{\mathbb{K}}$ left A-modules** and their category of modules and modules homomorphisms is denoted by $A\text{-}\mathsf{Mod}$. Similarly, we define right modules and denote them by $\mathsf{Mod}\text{-}A$.

A $\mathsf{CBorn}_{\mathbb{K}}$ **DG algebra** $A^\bullet$ is a monoid in $(\mathsf{Ch}(\mathsf{CBorn}_{\mathbb{K}}), \otimes_{\mathbb{K}}, \mathbb{K})$. A $\mathsf{CBorn}_{\mathbb{K}}$ **left DG module** is a left module over $A^\bullet$.

Remark 5.1.2. We can reformulate Definition 5.1.1 to be more explicit. A $\mathsf{CBorn}_{\mathbb{K}}$-algebra is an associative $\mathbb{K}$-algebra, which is a complete bornological space, such that the multiplication is bounded. A $\mathsf{CBorn}_{\mathbb{K}}$ left A-module is a left A-modules, which is a complete bornological space, such that the module action is bounded. ◇

We will focus on the commutative case. Given a complete bornological commutative algebra A the category $A\text{-}\mathsf{Mod}$ of complete bornological A-modules is monoidal with the induced tensor product $\otimes_A$ and unit A by Proposition 1.3.32.

Recall that $\otimes_A$ is defined via

$$M \otimes_A N = \operatorname{colim}\left(M \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} N \rightrightarrows M \otimes_{\mathbb{K}} N\right)$$

for $\mathsf{CBorn}_{\mathbb{K}}$-modules M, N.

Throughout this chapter, we will denote the forgetful functor by

$$\mathrm{vs} : A\text{-}\mathsf{Mod} \to \mathsf{CBorn}_{\mathbb{K}}, \quad E \mapsto \mathrm{vs}(E). \tag{5.1}$$

It assigns to a bornological A-module its underlying bornological vector space.

5.1.2. The Maximal and Split Exact Structures

We have seen in Proposition 1.3.34 that any exact structure on $\mathsf{CBorn}_{\mathbb{K}}$ extends to an exact structure on $A\text{-}\mathsf{Mod}$. So far we have mostly worked with the maximal exact structure on $\mathsf{CBorn}_{\mathbb{K}}$. It extends to

$$\mathfrak{max} = \left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \;\middle|\; f, g \text{ morphisms in } A\text{-}\mathsf{Mod},\ g = \ker f, f = \operatorname{coker} g \right\},$$

On the other hand the minimal exact structure of all split short exact sequences in $\mathsf{CBorn}_{\mathbb{K}}$ extends to

$$\mathfrak{split} = \left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \;\middle|\; (f,g) \in \mathfrak{max}, (f,g) \text{ splits in } \mathsf{CBorn} \right\}.$$

The last condition means that given a short exact sequence

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z$$

the underlying short exact sequence

$$\mathrm{vs}(X) \xrightarrow{\;f\;} \mathrm{vs}(Y) \xrightarrow{\;g\;} \mathrm{vs}(Z)$$

in $\mathsf{CBorn}_{\mathbb{K}}$ splits. Thus, we have $\mathrm{vs}(Y) \cong \mathrm{vs}(X) \coprod \mathrm{vs}(Z)$. However, the A-module structure might not be compatible with the splitting. In particular, $\mathfrak{split}$ is not the minimal exact structure on A-Mod. The latter only consists of exact sequence that split as A-modules.

Example 5.1.3. Let $\mathcal{C}^\infty(\mathbb{R}) \in \mathsf{CBorn}_{\mathbb{R}}$ be the $\mathsf{CBorn}_{\mathbb{R}}$-algebra of smooth functions on $\mathbb{R}$. The underlying complete bornological spaces is given by the Fréchet structure from Example 3.2.26. Let $\mathbb{R}_0$ be the 1-dimensional $\mathbb{R}$-vector space with module action

$$\mathcal{C}^\infty(\mathbb{R}) \times \mathbb{R}_0 \to \mathbb{R}_0, \qquad f \cdot r \mapsto f(0) \cdot r.$$

The bounded linear map

$$e : \mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}_0, \qquad f \mapsto f(0)$$

is a morphism in $\mathcal{C}^\infty(\mathbb{R})$-$\mathsf{Mod}$. There is no $\mathcal{C}^\infty(\mathbb{R})$-linear map $s : \mathbb{R}_0 \to \mathcal{C}^\infty(\mathbb{R})$ and consequently no section for e. Thus, by Lemma 1.1.7 the kernel-cokernel pair

$$\{ f \in \mathcal{C}^\infty(\mathbb{R}) \mid f(0) = 0 \} \rightarrowtail \mathcal{C}^\infty(\mathbb{R}) \xrightarrow{\;e\;} \mathbb{R}_0 \tag{5.2}$$

does not split in $\mathcal{C}^\infty(\mathbb{R})$-$\mathsf{Mod}$. The underlying sequence of complete bornolgical spaces does split, since $\mathrm{vs}(\mathbb{R}_0) \cong \mathbb{R} \in \mathsf{CBorn}_{\mathbb{R}}$ is finite dimensional. This shows, that (5.2) is a short exact sequence in the $\mathfrak{split}$-exact structure. ◇

5.1.3. Projective and Free Modules

Definition 5.1.4. Let A be a $\mathsf{CBorn}_{\mathbb{K}}$-algebra. A left $\mathsf{CBorn}_{\mathbb{K}}$ A-module M is **free** if there is an isomorphism

$$M \cong A \otimes_{\mathbb{K}} E$$

for some complete bornological space $E \in \mathsf{CBorn}_{\mathbb{K}}$.

Remark 5.1.5. The "free" construction is a functor $\mathsf{CBorn}_{\mathbb{K}} \to A\text{-}\mathsf{Mod}$ that is left adjoint to the forgetful functor from (5.1). We have

$$\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(A \otimes_{\mathbb{K}} E, N) \cong \mathrm{Hom}_{\mathsf{CBorn}_{\mathbb{K}}}(E, \mathrm{vs}(N))$$

for all $E \in \mathsf{CBorn}_{\mathbb{K}}$, $N \in A\text{-}\mathsf{Mod}$. With the different exact structures this induces two adjunctions of additive functors

$$(\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{max}) \underset{\mathrm{vs}}{\overset{A \otimes_{\mathbb{K}} \cdot}{\rightleftarrows}} (A\text{-}\mathsf{Mod}, \mathfrak{max})$$

$$(\mathsf{CBorn}_{\mathbb{K}}, \mathfrak{min}) \underset{\mathrm{vs}}{\overset{A \otimes_{\mathbb{K}} \cdot}{\rightleftarrows}} (A\text{-}\mathsf{Mod}, \mathfrak{split}).$$

In both cases the functor vs preserves admissible epimorphisms. ◇

Proposition 5.1.6. *Let A be a $\mathsf{CBorn}_{\mathbb{K}}$-algebra and $A \otimes_{\mathbb{K}} E$ a free left $\mathsf{CBorn}_{\mathbb{K}}$ A-module with $E \in \mathsf{CBorn}_{\mathbb{K}}$. Then $A \otimes_{\mathbb{K}} E$ is $\mathfrak{split}$-projective. If E is $\mathfrak{max}$-projective in $\mathsf{CBorn}_{\mathbb{K}}$, then $A \otimes_{\mathbb{K}} E$ is $\mathfrak{max}$-projective in A-Mod.*

Proof. Every object in $\mathsf{CBorn}_{\mathbb{K}}$ is $\mathfrak{split}$-projective. Both assertions now follow from the adjunction in Remark 5.1.5 and Lemma 1.3.23. □

Corollary 5.1.7. *Let A be a $\mathsf{CBorn}_{\mathbb{K}}$-algebra. Then A-Mod has enough $\mathfrak{split}$- and enough $\mathfrak{max}$-projectives.*

Proof. Let M be a left $\mathsf{CBorn}_{\mathbb{K}}$ A-module. The epimorphism

$$A \otimes_{\mathbb{K}} M \twoheadrightarrow M$$

is $\mathfrak{split}$-admissible and $A \otimes_{\mathbb{K}} M$ is $\mathfrak{split}$-projective by Proposition 5.1.6. For the $\mathfrak{max}$-case, consider $\mathrm{vs}(M) \in \mathsf{CBorn}_{\mathbb{K}}$. Since $\mathsf{CBorn}_{\mathbb{K}}$ has enough projectives by Proposition 4.3.2, there is $\mathfrak{max}$-projective $P \in \mathsf{CBorn}_{\mathbb{K}}$ and an admissible epimorphism

$$P \twoheadrightarrow \mathrm{vs}(M) \qquad \text{in } \mathsf{CBorn}_{\mathbb{K}}.$$

Extension of Scalars gives us a $\mathfrak{max}$-admissible epimorphism, that we can compose as in

$$A \otimes_{\mathbb{K}} P \twoheadrightarrow A \otimes_{\mathbb{K}} \mathrm{vs}(M) \twoheadrightarrow M \qquad \text{in } A\text{-}\mathsf{Mod}$$

to get the desired statement. □

By Theorem 4.3.10 all $\mathfrak{max}$-projectives in $\mathsf{CBorn}_{\mathbb{K}}$ are coproducts of $\ell^1(Y)$ for sets Y. The proof of Corollary 5.1.7 shows that enough $\mathfrak{max}$-projectives are given by

$$\left\{ A \otimes_{\mathbb{K}} \left(\coprod_{i \in I} \ell^1(Y_i) \right) \;\middle|\; \{Y_i\}_{i \in I} \text{ a family of sets} \right\}.$$

We can describe the underlying bornological space of the projective objects more precisely.

Proposition 5.1.8 (Corollary 15.6.8 in [Jar81] or Theorem 6.15 in [Vog00])**.** *Let $V \in \mathsf{CLCS}_{\mathbb{K}}$ be a complete locally convex vector space with a fundamental system of seminorms $\mathcal{P}$. Then we have a canonical isomorphism*

$$\begin{aligned} V\hat{\otimes}_\pi \ell^1(Y) &\xrightarrow{\sim} \left\{ (v_y)_{y\in Y} \,\middle|\, v_y \in V, q_p(v) := \sum_{y\in Y} p(v_y) < \infty, p \in \mathcal{P} \right\}, \\ ((x_y)_y, v) &\mapsto ((x_y \cdot v)_y), \end{aligned}$$

equipped with the seminorms $\{ q_p \mid p \in \mathcal{P} \}$.

Notation 5.1.9. The space from Proposition 5.1.8 is denoted by $\ell^1\{Y, V\}$.

Proposition 5.1.10. *Let $F \in \mathsf{CBorn}_{\mathbb{K}}$ be a nuclear Fréchet space with* $\mathrm{vN}(V) = F, V \in \mathsf{NF}_{\mathbb{K}}$*. Then*

$$F \otimes_{\mathbb{K}} \coprod_{i\in I} \ell^1(Y_i) \cong \coprod_{i\in I} \mathrm{vN}\left(\ell^1\{Y_i, V\}\right)$$

Proof. The statement follows from Proposition 4.2.29. □

Similar to ℓ^1 we simply write $\ell^1\{Y, F\} := \mathrm{vN}\left(\ell^1\{Y, V\}\right)$ for a set Y and $\mathrm{vN}(V) = F \in \mathsf{CBorn}_{\mathbb{K}}$.

5.1.4. Continuous Functions and the de Rham Algebra

Here is our main example. Let X be a real smooth n-dimensional manifold. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable covering of X by charts U_i. Since each U_i is homeomorphic to $\mathbb{R}^n$ we can choose compact subsets $K_{i,j} \subset U_i, j \in N$, such that every other compact subset of is contained in one $K_{i,j}$. Let $\{\rho_i\}_{i\in\mathbb{N}}$ be a partition of unity subordinate to $\{U_i\}_{i\in\mathbb{N}}$. For all $i, j \in \mathbb{N}$ and multi-indices $\alpha \in {\mathbb{N}_0}^n$ define the seminorms

$$\|f\|_{i,j,\alpha} := \sup_{x\in K_{i,j}} \|\partial^\alpha(\rho_i f(x))\|| \qquad \text{for } f \in \mathcal{C}^\infty(X).$$

Since the collection of seminorms

$$\left\{ \|\cdot\|_{i,j,\alpha} : \mathcal{C}^\infty(X) \to \mathbb{R}_{\geq 0} \,\middle|\, i, j \in \mathbb{N}, \alpha \in {\mathbb{N}_0}^n \right\}$$

is countable we get a Fréchet topology on the $\mathbb{R}$-vector space $\mathcal{C}^\infty(X)$ by Theorem 3.1.29. The topology is nuclear by Proposition 3.2.38.

On every open subset U_i the space of $\mathcal{C}^\infty$-differential k-forms on U_i is given by

$$\boldsymbol{\Omega}^k(U_i) \;=\; \mathcal{C}^\infty(U_i)\hat{\otimes}_\pi \boldsymbol{\Omega}^k,$$

where $\boldsymbol{\Omega}^k$ is the finite dimensional vector space of k-forms. Note, that $\boldsymbol{\Omega}^k$ being finite dimensional also means that the projective tensor product with $\mathcal{C}^\infty(U_i)$ is already complete. With the partition of unity $\{\rho_i\}_{i\in\mathbb{N}}$ from above we can define $\boldsymbol{\Omega}^k(X)$ and get the usual der Rham complex

$$0 \longrightarrow \boldsymbol{\Omega}^0(X) \xrightarrow{d^0} \boldsymbol{\Omega}^1(X) \xrightarrow{d^1} \boldsymbol{\Omega}^2(X) \xrightarrow{d^2} \boldsymbol{\Omega}^3(X) \longrightarrow \dots$$

By construction all $\boldsymbol{\Omega}^k(X)$ carry a nuclear Fréchet topology.

Lemma 5.1.11. *Let X be a real smooth n-dimensional manifold. The de Rham algebra $\boldsymbol{\Omega}(X)$ is a $\mathsf{CBorn}_{\mathbb{K}}$ DG algebra. The underlying complete bornological space of $\boldsymbol{\Omega}(X)^\sharp$ is nuclear and Fréchet .*

Example 5.1.12. Consider the special case of a zero-dimensional manifold X. If we also assume that X is connected it only consists of one point $* \cong X$. Smooth functions on X are $\mathcal{C}^\infty(*) \cong \mathbb{R}$ with its usual Euclidean bornology. Thus, $\mathcal{C}^\infty(*)\text{-}\mathsf{Mod} \cong \mathsf{CBorn}_{\mathbb{K}}$ with the usual exact structures and tensor product. We get

$$\mathrm{g}\ell_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(\mathcal{C}^\infty(*)) = \infty$$

by Theorem 4.3.11. ◇

5.2. Contraderived Categories with the $\mathfrak{split}$-exact structure

The split-exact structure is similar to the purely algebraic situation, where the abelian category of vector spaces also does not have any non-split exact sequences.

5.2.1. Ogneva's Results on split-exact Modules over $\mathcal{C}^\infty(M)$

In 1986 Ogneva published a result [Ogn86], showing that in the split-exact structure the dimensions of the manifold and the Fréchet algebra of smooth functions coincide. They also gave a more detailed proof later in [Ogn14].

Theorem 5.2.1 (Theorem 1 in [Ogn14])**.** *Let X be a real smooth n-dimensional manifold. Consider the Fréchet algebra of smooth functions $\mathcal{C}^\infty(X)$ on X and the category $\mathcal{C}^\infty(X)\text{-}\mathsf{Mod}$ of Fréchet modules over $\mathcal{C}^\infty(X)$ with the $\mathbb{K}$-split exact* *structure. The enveloping algebra of* $\mathcal{C}^\infty(X)$ *is* $\mathcal{C}^\infty(X)^e := \mathcal{C}^\infty(X)\hat{\otimes}_\pi\mathcal{C}^\infty(X)$*. We have*

$$\mathrm{pd}_{\mathcal{C}^\infty(X)^e\text{-}\mathsf{Mod},\mathfrak{split}}\left(\mathcal{C}^\infty(X)\right) = n$$

and the global dimension of $\mathcal{C}^\infty(X)$ *is*

$$\mathrm{g}\ell_{(\mathsf{Fr\acute{e}}_{\mathbb{K}},\mathfrak{split})}\left(\mathcal{C}^\infty(X)\right) = n.$$

Part of the proof are the following results.

Proposition 5.2.2 (Theorem 2 in [Ogn14])**.** *Let* X *be a real smooth* n*-dimensional manifold and* $U \subset X$ *open and contained in a chart. Then* $\mathcal{C}^\infty(U)$ *is a* $\mathfrak{split}$*-projective* $\mathcal{C}^\infty(X)$*-module.*

Proposition 5.2.3 (Theorem 4 in [Ogn14])**.** *Let* X *be a real smooth* n*-dimensional manifold. Let* $x \in X$ *be an arbitrary point and* C_0 *the* 1*-dimensional* $\mathcal{C}^\infty(X)$*-module with action* $f \cdot \lambda = f(x)\lambda$ *for* $f \in \mathcal{C}^\infty(X)$ *and* $\lambda \in \mathbb{K}$*. Then*

$$\mathrm{pd}_{\mathcal{C}^\infty(X)\text{-}\mathsf{Mod},\mathfrak{split}}\left(C_0\right) = n.$$

Remark 5.2.4. One of the reasons we focus on projective objects and the contraderived category is that in the $\mathfrak{split}$-setting we do not necessarily have enough injective objects. For Fréchet spaces this was shown by Pirkovskii in [Pir98]. ◇

5.2.2. Contraderived Category of the de Rham Algebra I

To extend Ogneva's result to $\mathsf{CBorn}_{\mathbb{K}}$ we need the following lemma.

Lemma 5.2.5. *Let* A *be a commutative* $\mathsf{CBorn}_{\mathbb{K}}$*-algebra with enveloping algebra* $A^e = A \otimes A$*. For the induced* $\mathfrak{split}$*-exact structures on* A-Mod *and* A-Mod^e *we get*

$$\mathrm{g}\ell\left(A\text{-}\mathsf{Mod}, \mathfrak{split}\right) \leq \mathrm{pd}_{A^e\text{-}\mathsf{Mod},\mathfrak{split}}\left(A\right).$$

Proof. This follows from Lemma 9.1.9 in [Wei94] and the fact that the relative Ext-functor in the split-exact structure is the same as the regular Ext-functor. □

Corollary 5.2.6. *Let* X *be a real smooth* n*-dimensional manifold. We consider the* $\mathsf{CBorn}_{\mathbb{K}}$ *smooth functions* $\mathcal{C}^\infty(X)$ *and the de Rham algebra* $\mathbf{\Omega}(X)$ *both with the* $\mathfrak{split}$*-exact structure. Then, for the underlying graded* $\mathsf{CBorn}_{\mathbb{K}}$*-algebra* $\mathcal{C}^\infty(X)$*, concentrated in degree* 0*,*

$$\mathrm{g}\ell_{(\mathrm{Gr}(\mathsf{CBorn}_{\mathbb{K}}),\mathfrak{split})}\left(\mathcal{C}^\infty(X)\right) = n.$$

The underlying graded algebra of $\mathbf{\Omega}(X)$ *with the* $\mathfrak{split}$*-exact structure satisfies* (#ΠP)*.*

Proof. The proof of Theorem 5.2.1 gives us a free resolution of $\mathcal{C}^\infty(X)$ as a $\mathcal{C}^\infty(X)^e$-module of length n. This is still a free resolution when considered as $\mathsf{CBorn}_{\mathbb{K}}$-modules over the $\mathsf{CBorn}_{\mathbb{K}}$-algebra $\mathcal{C}^\infty(X)^e$, using that $\mathcal{C}^\infty(X)^e$ is nuclear Fréchet and Proposition 4.2.29. The first statement now follows from Lemma 5.2.5 and Proposition 2.2.7.

The de Rham algebra is finitely generated as a $\mathcal{C}^\infty(X)$ module. The second statement now follows by arguing as in the Gorenstein case in section 3.10 in [Pos11]. It is also possible to use that $\mathbf{\Omega}(X)$ is nuclear Fréchet and use the same argument as in Section 5.3.2 to lift (#ΠP) from $\mathcal{C}^\infty(X)$ to $\mathbf{\Omega}(X)$. □

Remark 5.2.7. The de Rham algebra $\mathbf{\Omega}(X)$ with the $\mathfrak{split}$-exact structure does not have finite global dimension. See Example 5.2.14. ◇

Now we are ready to prove our desired result for the contraderived category with $\mathfrak{split}$-contraacyclics. Recall from Chapter 2 that we have

$$\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}) := \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)) \Big/ \mathsf{Ac}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}),$$

where $\mathsf{Ac}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$ is the minimal triangulated subcategory of $\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X))$ containing the total DG modules of $\mathfrak{split}$-exact triples of $\mathsf{CBorn}_{\mathbb{K}}$ DG modules over $\mathbf{\Omega}(X)$ and closed under countable products.

Theorem 5.2.8. *Let X be a real smooth n-dimensional manifold. Consider complete bornological DG modules over the de Rham algebra $\mathbf{\Omega}(X)$ with the $\mathfrak{split}$-exact structure. The composition of functors*

$$\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}_{\mathrm{proj}}) \to \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)) \to \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$$

is an equivalence of categories. Here $\mathsf{DG}\text{-}\mathbf{\Omega}(X)_{\mathrm{proj}}$ denotes the complete bornological DG modules M over $\mathbf{\Omega}(X)$, such that $M^\sharp$ is degreewise $\mathfrak{split}$-projective.

Proof. The statement follows from Corollary 5.2.6 and Theorem 2.2.12. □

Remark 5.2.9. Enough $\mathfrak{split}$-projectives are given by free modules as seen in the proof of Corollary 5.1.7. Thus, we also have

$$\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)_{\mathrm{proj}}) \cong \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)_{\mathrm{free}}).$$

◇

Next, we want to compare $\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X))$ to cohesive modules over $\mathbf{\Omega}(X)$, introduced by Block in [Blo05]. The original definition is for CDG algebras with no topology, but in [Blo06] Block defines a version for Fréchet algebras, that immediately generalizes to the bornological setting. Under this generalization the Definition reads as follows.

Definition 5.2.10 (Definition 2.11 in [Blo06]). Let $(A^\bullet, d, c)$ be a complete bornological curved DG algebra. The DG category $q\mathcal{P}_{A^\bullet}$ of **quasi-perfect twisted complexes**, whose objects are $E = (E^\bullet, \mathbb{E})$ where $E^\bullet$ is a bounded $\mathbb{Z}$-graded right complete bornological A-module, which is $\mathfrak{split}$-projective and

$$\mathbb{E} : E^\bullet \to E^\bullet \otimes_A A^\bullet$$

is a $\mathbb{Z}$-connection, which is bounded and satisfies

$$\mathbb{E} \circ \mathbb{E}(e) = -ec.$$

The morphisms between $E_1 = (E_1^\bullet, \mathbb{E}_1)$ and $E_2 = (E_2^\bullet, \mathbb{E}_2)$ of degree k are

$$\mathrm{Hom}_{q\mathcal{P}_{A^\bullet}}(E_1, E, 2) = \left\{ \Phi : E_1^\bullet \otimes_A A^\bullet \to E_1^\bullet \otimes_A A^\bullet \;\middle|\; \Phi(ea) = (-1)^{k|a|}\Phi(e)a \right\},$$

where all morphisms are required to be bounded. The differential on morphisms is given by

$$d(\Phi)(e) = \mathbb{E}_2(\Phi(e)) - (-1)^{|\Phi|}\Phi(\mathbb{E}_1(e)).$$

Comnparing quasi-prefect twisted complexes with over the de Rham algebra $\mathbf{\Omega}(X)$ of a real smooth manifold X with $\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}_{\mathrm{proj}}))$ we see that the only difference lies in the boundedness of the complexes in $q\mathcal{P}^u_{\mathbf{\Omega}(X)}$. Denote by $q\mathcal{P}^u_{\mathbf{\Omega}(X)}$ the unbounded quasi-perfect twisted complexes over $\mathbf{\Omega}(X)$. We have the following.

Corollary 5.2.11. *Let X be a smooth real manifold. Consider complete bornological DG modules over the de Rham algebra $\mathbf{\Omega}(X)$ with the $\mathfrak{split}$-exact structure. We have the following equivalences of triangulated categories*

$$q\mathcal{P}^u_{\mathbf{\Omega}(X)} \cong \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split}_{\mathrm{proj}}) \cong \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$$

Remark 5.2.12. The triangulated equivalence of $q\mathcal{P}^u_{\mathbf{\Omega}(X)}$ and $\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X), \mathfrak{split})$ comes from a composition of functors with a natural DG enhancement, it induces an equivalence of DG categories. ◇

Remark 5.2.13. In [CHL21] Chuang, Holstein and Lazarev define and study **twisted A-module** over a DG algebra A. They are defined as DG A-modules M, such that $M^\sharp$ is free as an $A^\sharp$-module. Section 3 of [CHL21] also contains a comparison of twisted modules with both cohesive modules over A as well as the contraderived category of A. The theory is inherently topological, but restricted to finitely generated modules (over Arens-Michael algebras) or pseudo-compact modules over pseudo-compact algebras. Here pseudo-compact, can be understood as given by projective limits of finitely generated free $\mathbb{K}$-modules. In contrast, our setup allows for any topology/bornology in $\mathsf{CBorn}_{\mathbb{K}}$. It is worth noting that the Arens-Michael algebras mentioned above are specific examples of m-convex algebras. The condition of m-convexity becomes essential when studying deformations, and thus, this condition should be incorporated into our framework if this is the goal. ◇

Example 5.2.14. Let X be a real smooth n-dimensional manifold. Let $x \in X$ be an arbitrary point and C_0 the 1-dimensional $\mathcal{C}^\infty(X)$-module with action $f \cdot \lambda = f(x)\lambda$ for $f \in \mathcal{C}^\infty(X)$ and $\lambda \in \mathbb{K}$. We have seen in Corollary 5.3.3 that

$$\mathrm{pd}_{\mathcal{C}^\infty(X)\text{-}\mathsf{Mod},\mathfrak{split}}(C_0) \;=\; n.$$

This is no longer true for the de Rham algebra. By Corollary 2.2.10 we can forget the grading to show this. Consider $\mathcal{C}^\infty(\mathbb{R})dx$ as a $\boldsymbol{\Omega}(\mathbb{R})$ module. The sequence

$$\mathrm{Ker}(dx) \rightarrowtail \boldsymbol{\Omega}(\mathbb{R}) \xrightarrow{dx} \mathcal{C}^\infty(\mathbb{R})dx$$

is $\mathfrak{split}$-exact, but does not split over $\boldsymbol{\Omega}(\mathbb{R})$. Since $\boldsymbol{\Omega}(\mathbb{R})$ is free it is $\mathfrak{split}$-projective and

$$\ldots \longrightarrow \boldsymbol{\Omega}(\mathbb{R}) \longrightarrow \boldsymbol{\Omega}(\mathbb{R}) \longrightarrow \boldsymbol{\Omega}(\mathbb{R}) \twoheadrightarrow \mathcal{C}^\infty(\mathbb{R})dx$$

is a non-split $\mathfrak{split}$-projective resolution of $\mathcal{C}^\infty(\mathbb{R})$, which shows that

$$\mathrm{pd}_{\boldsymbol{\Omega}(\mathbb{R}),\mathfrak{split}}(\boldsymbol{\Omega}(\mathbb{R})) = \infty.$$

◇

Remark 5.2.15. Let X be a smooth real manifold. Consider the DG category of cohesive module $\mathcal{P}_{\boldsymbol{\Omega}(X)}$, which differs from $q\mathcal{P}_{A^\bullet}$ in that the complexes are finitely generated projective. In this situation a bornology or a topology is irrelevant. In [BS09] Block and Smith prove that $\mathcal{P}_{\boldsymbol{\Omega}(X)}$ is equivalent to the DG category of infinity local systems on X. Another proof of this fact can be found in [CHL21], where the infinity locally systems are understood to be cohomologically constant complexes of sheaves.

The category $\mathcal{P}_{\boldsymbol{\Omega}(X)}$ and therefore the category of infinity local systems is a full subcategory of $q\mathcal{P}^u_{\boldsymbol{\Omega}(X)} \cong \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\boldsymbol{\Omega}(X), \mathfrak{split})$. ◇

5.3. Contraderived Categories with the $\mathfrak{max}$-exact structure

5.3.1. From $\mathfrak{split}$-Projective to $\mathfrak{max}$-Projective Dimensions

Lemma 5.3.1. *Let A be a flat $\mathsf{CBorn}_{\mathbb{K}}$ algebra and let M be a complete bornological left A-module. If we have*

$$\mathrm{pd}_{A\text{-}\mathsf{Mod},\mathfrak{split}}(M) = l < \infty \quad \textit{and}$$
$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}\left(\mathrm{vs}(A^{\otimes l+1} \otimes M)\right) = k < \infty,$$

then

$$\mathrm{pd}_{A\text{-}\mathsf{Mod},\mathfrak{max}}(M) \leq k + l.$$

Proof. With the assumption $\mathrm{pd}_{A\text{-Mod},\mathfrak{split}}(M) = l$ and Corollary 5.1.7 the Bar resolution gives us a $\mathfrak{split}$-projective resolution

$$\operatorname{Ker} d_l \rightarrowtail A^{\otimes l} \otimes M \xrightarrow{d_l} \cdots \longrightarrow A^{\otimes 2} \otimes M \longrightarrow A \otimes M \twoheadrightarrow M\,. \tag{5.3}$$

We also consider the longer $\mathfrak{split}$-projective resolution

$$\operatorname{Ker} d_{l+1} \rightarrowtail A^{\otimes l+1} \otimes M \xrightarrow{d_{l+1}} \cdots \longrightarrow A^{\otimes 2} \otimes M \longrightarrow A \otimes M \twoheadrightarrow M\,.$$

We can apply the generalized Schanuel's Lemma as stated in Corollary 5.5 in [Lam99]. A proof of the generalized Schanuel's Lemma for exact categories can be found in [MR23]. We get

$$A^{\otimes l+1} \otimes M \;\cong\; \operatorname{Ker} d_l \amalg \operatorname{Ker} d_{l+1}$$

in A-Mod. Thus, we also have for the maximal exact structure

$$\mathrm{pd}_{A\text{-Mod},\mathfrak{max}}(\operatorname{Ker} d_l) \;\le\; \mathrm{pd}_{A\text{-Mod},\mathfrak{max}}\left(A^{\otimes l+1} \otimes M\right). \tag{5.4}$$

Combining the second assumption and (5.4) all of the modules in the left resolution (5.3) of M have $\mathfrak{max}$-projective dimension k or smaller. The statement now follows from Proposition 1.3.29. □

Theorem 5.3.2. *Let X be a real smooth n-dimensional manifold. Let $M \in \mathcal{C}^\infty(X)$-Mod be a $\mathsf{CBorn}_{\mathbb{K}}$-module, that is nuclear and Fréchet . Assume $\mathfrak{d} = \aleph_d$ for some $d \in \mathbb{N}$. Then*

$$\mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{max}}(M) \le \mathrm{pd}_{\mathsf{CBorn}_{\mathbb{K}},\mathfrak{split}}(M) + d + 1 \le n + d + 1$$

Proof. We have seen in Proposition 3.2.38 that $\mathcal{C}^\infty(X)$ is nuclear Fréchet . By Theorem 4.3.14 we can choose $k = n + 1$ in Lemma 5.3.1. The second inequality follows from Theorem 5.2.1. □

Corollary 5.3.3. *Let X be a real smooth n-dimensional manifold. Assume that the dominating number satisfies $\mathfrak{d} = \aleph_d$ with $d \in \mathbb{N}$. Let $U \subset X$ be an open subset contained in a chart. Then*

$$\mathrm{pd}_{\mathcal{C}^\infty(X)\text{-Mod},\mathfrak{max}}(\mathcal{C}^\infty(U)) \le n + 1.$$

We also have

$$\mathrm{pd}_{\mathcal{C}^\infty(X)^e\text{-Mod},\mathfrak{max}}(\mathcal{C}^\infty(X)) \le n + d + 1.$$

Proof. The first statement follows from Theorem 5.3.2 and Proposition 5.2.2.

For the second one we can use Lemma 5.3.1 with $l = n$ and $k = d + 1$ by Theorem 5.2.1 and Theorem 4.3.14. □

5.3.2. Lifting (ΠP) to nuclear Fréchet algebras

The assertion about the global dimension in Ogneva's Theorem 5.2.1 is a consequence of the calculating the bidimension and Lemma 5.2.5. We do have a similar bound on the bidimension by Corollary 5.3.3. However, even the simplest case of Example 5.1.12 has infinite global dimension. Instead, we show something weaker. We have seen in Section 4.4 that $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$ satisfies (ΠP) under the assumption $\mathfrak{d} \leq \aleph_d$. We will see how we can lift this to categories of modules of nuclear Fréchet algebras in $\mathsf{CBorn}_{\mathbb{K}}$ and their $\mathfrak{max}$-exact structure.

Corollary 5.3.4. *Let A be a nuclear $\mathsf{CBorn}_{\mathbb{K}}$-algebra. Then the extension of scalars functor*

$$A \otimes (-) : (\mathsf{CBorn}_{\mathfrak{K}}, \mathfrak{max}) \to (A\text{-}\mathsf{Mod}, \mathfrak{max})$$

is exact.

Proof. This follows from A being flat by Lemma 4.2.41. □

Lemma 5.3.5 (Lemma 5.20 in [BBK23])**.** *Let $E \in \mathsf{CBorn}_{\mathbb{K}}$. Then $E \otimes (-)$ commutes with countable products if and only if E is metrizable.*

Note that, in the algebraic setting, the analogue statement requires E to be finitely presented.

Theorem 5.3.6. *Let A be a nuclear Fréchet $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$-algebra. Assume that $2^{\aleph_0} = \aleph_d$ for some $d \in \mathbb{N}$. Then the exact category $(\mathsf{A}\text{-}\mathsf{Mod}^{\leq \aleph_d}, \mathfrak{max})$ satisfies (ΠP).*

Proof. First we note that $(\mathsf{A}\text{-}\mathsf{Mod}^{\leq \aleph_d}, \mathfrak{max})$ has enough $\mathfrak{max}$-projectives by Proposition 4.4.8.

Let E be a $\mathfrak{max}$-projective object in $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$. By Theorem 4.4.14 the space $\prod_{\mathbb{N}} E$ has finite $\mathfrak{max}$-projective dimensions in $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$. There is a $\mathfrak{max}$-projective resolution

$$Q_l \rightarrowtail \dots \longrightarrow Q_0 \twoheadrightarrow \prod_{\mathbb{N}} E$$

for some $l \in N$. Using that A is nuclear by Corollary 5.3.4, extension of scalars gives us an exact sequence

$$A \otimes Q_l \rightarrowtail \dots \longrightarrow A \otimes Q_0 \twoheadrightarrow A \otimes \left(\prod_{\mathbb{N}} E\right).$$

in $(\mathsf{A}\text{-}\mathsf{Mod}^{\leq \aleph_d}, \mathfrak{max})$. Since A is metrizable we can apply Lemma 5.3.5 and get the resolution

$$A \otimes Q_l \rightarrowtail \dots \longrightarrow A \otimes Q_0 \twoheadrightarrow \prod_{\mathbb{N}} (A \otimes E) \tag{5.5}$$

By Proposition 5.1.6 the exact sequence (5.5) is a $\mathfrak{max}$-projective resolution of $\prod_{\mathbb{N}}(A \otimes E)$ in $\mathsf{A\text{-}Mod}^{\leq \aleph_d}$.

Now, let $P \in A\text{-}\mathsf{Mod}$ be any $\mathfrak{max}$-projective module. Since $\mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$ has enough $\mathfrak{max}$-projectives there is a $\mathfrak{max}$-projective space $L \in \mathsf{CBorn}_{\mathbb{K}}^{\leq \aleph_d}$ and an admissible epimorphism

$$L \longrightarrow\!\!\!\!\rightarrow \mathrm{vs}(P).$$

We have already seen that extension of scalars is exact and applying it here gives us an admissible epimorphism

$$A \otimes L \longrightarrow\!\!\!\!\rightarrow A \otimes \mathrm{vs}(P).$$

Postcomposing it with the action map $A \otimes \mathrm{vs}(P) \to P$ we get an admissible epimorphism $A \otimes L \twoheadrightarrow P$. By $\mathfrak{max}$-projectivity of P we get a split and there exists a $S \in \mathsf{A\text{-}Mod}^{\leq \aleph_d}$ with $P \coprod S \cong A \otimes L$. Taking the countable product we see that $\prod_{\mathbb{N}} P$ is a direct summand of $\prod_{\mathbb{N}}(P \coprod S)$ and therefore also of $\prod_{\mathbb{N}}(A \otimes L)$. But the latter has finite $\mathfrak{max}$-projective dimension by the argument above. Thus, the module $\prod_{\mathbb{N}} P$ has finite $\mathfrak{max}$-projective dimension.

The result now follows from Proposition 2.2.11. □

5.3.3. Contraderived Category of the de Rham Algebra II

We fix $d \in \mathbb{N}$ for this section.

Proposition 5.3.7. *Let A be a complete bornological algebra that is nuclear Fréchet and $\leq\aleph_d$-small. We consider the category of complete bornological and $\leq\aleph_d$-small modules over A and denote it by $\mathsf{A\text{-}Mod}^{\leq \aleph_d}$. Assume that $2^{\aleph_0} \leq \aleph_d$.*

Then the exact category $\left(\mathsf{A\text{-}Mod}^{\leq \aleph_d}, \mathfrak{max}\right)$ satisfies (ΠP).

Proof. This follows from Theorem 4.4.14 and Theorem 5.3.6. □

Corollary 5.3.8. *Let X be a smooth real manifold. Assume $2^{\aleph_0} \leq \aleph_d$. Then the exact category $\left(\mathcal{C}^\infty(\mathsf{X})\text{-}\mathsf{Mod}^{\leq \aleph_d}, \mathfrak{max}\right)$ satisfies (ΠP).*

Proof. By Proposition 3.2.38 the algebra $\mathcal{C}^\infty(X)$ is nuclear Fréchet and by Proposition 4.4.11 it is $\leq\aleph_d$-small. The statement now follows from Proposition 5.3.7. □

Lemma 5.3.9. *Assume that $2^{\aleph_0} \leq \aleph_d$. Then the category of graded complete bornological $\leq\aleph_d$-small Modules over $\mathbf{\Omega}(X)^\sharp$ satisfies (ΠP) with respect to its $\mathfrak{max}$-exact structure.*

Proof. By Theorem 5.3.6 and Lemma 5.1.11 the statement is true if we consider ungraded module over $\mathbf{\Omega}(X)^\sharp$, where we forget the grading. By Proposition 2.2.7 it follows that the statement for graded modules is also true. □

Theorem 5.3.10. *Assume that* $2^{\aleph_0} \leq \aleph_d$. *Consider* $\leq\aleph_d$*-small complete bornological DG modules over the de Rham algebra* $\mathbf{\Omega}(X)$. *The composition of functors*

$$\mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max}_{\mathrm{proj}}) \to \mathsf{Ho}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}) \to \mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max})$$

is an equivalence of categories. In particular, every $\leq\aleph_d$*-small complete bornological DG module over* $\mathbf{\Omega}(X)$ *is contraequivalent to a DG module* M *with*

$$M^\sharp \cong \coprod_{i\in I} \ell^1\{Y_i, \mathbf{\Omega}(X)^{\leq\aleph_d}\}[s_i],$$

where $\{Y_i\}_{i\in I}$ *is some family of sets with* $\mathrm{card}(Y_i) \leq \aleph_d$ *and* $s_i \in \mathbb{Z}, i \in I$ *are some integers denoting shifts.*

Proof. The statement follows from Lemma 5.3.9, Proposition 2.2.7 and Theorem 2.2.12 □

Remark 5.3.11. The homotopy category of an exact category depends solely on its additive structure. Thus, whether we consider a $\mathsf{CBorn}_{\mathbb{K}}$ (DG) algebra and its modules with the $\mathfrak{split}$- or $\mathfrak{max}$-exact structure makes no difference. However, (absolute/co-/contra-)acyclic objects are all determined by short exact sequences. Larger exact structures have more acyclic objects. Thus, the contraderived category with respect to the $\mathfrak{max}$-exact structure $\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max})$ is a Verdier quotient of the contraderived category with respect to the $\mathfrak{split}$-exact structure $\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{split})$. The advantage of the former is that all of the complicated exact sequences, that $\mathsf{CBorn}_{\mathbb{K}}$ has, are quotiented out and we are left with what is ideally an easier to handle invariant of the manifold X. ◇

Remark 5.3.12. In Remark 5.2.15 we discussed infinity local systems and cohesive modules. We have seen that they are a subcategory of the $\mathfrak{split}$-exact contraderived category. To compare the cohesive modules with the $\mathfrak{max}$-exact contraderived category first note, that $\mathbf{\Omega}(X)$ is $\mathfrak{max}$-projective as a module over itself by Proposition 5.1.6. Thus, coproducts of $\mathbf{\Omega}(X)$ and their direct summands are $\mathfrak{max}$-projective. This includes cohesive module, since they are finitely generated and (discrete) projective.

Therefore, cohesive modules are also a subcategory of $\mathsf{D}^{\mathrm{ctr}}(\mathsf{DG}\text{-}\mathbf{\Omega}(X)^{\leq\aleph_d}, \mathfrak{max})$. ◇

Example 5.3.13. Let us return to the example, where $x \in X$ is an arbitrary point and C_0 is the 1-dimensional $\mathcal{C}^\infty(X)$-module with action $f \cdot \lambda = f(x)\lambda$ for $f \in \mathcal{C}^\infty(X)$ and $\lambda \in \mathbb{K}$. It follows from Theorem 5.3.10 that we get a possible infinitely long complex, that in each degree is of the form $\coprod_{i\in I} \ell^1\{Y_i, \mathbf{\Omega}(X)^{\leq\aleph_d}\}$ to resolve C_0.

In fact, it has to be infinitely long or of infinite rank. This follows from the result of Block and Smith discussed in Remark 5.3.12, since otherwise it would correspond to a locally constant sheaf. ◇

Part II.

A Comparison of Bornological and Condensed Spaces

6. The Basics of Condensed Mathematics

Contents

We gather some basic definitions and results about condensed sets, introduced by Dustin Clausen and Peter Scholze 2019 in [Condensed]. In addition to the original lecture notes and the follow up courses [Analytic, Complex], Dagur Ásgeirsson master thesis [Ás21] is a good reference.

Although we will not be using them, it is worth mentioning the concept of pyknotic objects, developed by Barwick and Haine [BH19], independently of condensed mathematics. Pyknotic and condensed objects are essentially equivalent, with the primary distinction lying in how they address issues related to set theory.

The core idea of condensed math is to define a notion of topology with sheaf theory, that works well when paired with algebraic structures. A condensed set T will be a functor

$$T : \begin{Bmatrix} \text{suitable site} \\ \text{of test spaces} \end{Bmatrix}^{\mathrm{op}} \to \mathsf{Set} \tag{6.1}$$

subject to some sheaf conditions. Replacing the category Set with for example $\mathsf{Vect}_{\mathbb{K}}$ allows one to define condensed vector spaces. Unlike classical topological vector space categories such as $\mathsf{TVS}_{\mathbb{K}}$, $\mathsf{LCS}_{\mathbb{K}}$, etc. , that are at best quasi-abelian, $\mathsf{Cond}(\mathsf{Vect}_{\mathbb{K}})$ is a nice abelian category.

6.1. Different Sites

One suitable category of test spaces are compact Hausdorff spaces, but we are also interested in two subcategories.

Definition 6.1.1. A topological space X is **totally disconnected** if its only connected subsets are singletons. A topological space X is a **profinite set** or a **Stone space** if it is a compact totally disconnected Hausdorff space.

We denote the category of profinite sets and continuous maps by Prof.

Proposition 6.1.2 (Lemma 08ZY in [Sta24])**.** *A topological space is a profinite set if and only if it is a cofiltered limit of finite discrete spaces in the category* Top*. The category of profinite sets* Prof *is equivalent to the pro-category of finite sets* $\mathsf{Pro}(\mathsf{Fin})$*.*

Definition 6.1.3. A topological space X is **extremally disconnected** if the closure of every open set is open.

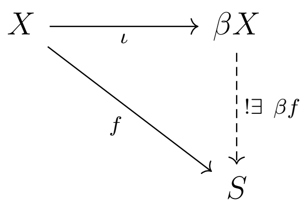
We denote the category of extremally disconnected spaces and continuous maps by Extr.

Unlike profinite sets, extremally disconnected spaces do not need to be Hausdorff or compact. Indeed, every discrete and every indiscrete space is extremally disconnected. For compact Hausdorff spaces we have the following alternative characterization by Gleason [Gle58].

Proposition 6.1.4 (Theorem 2.5 in [Gle58])**.** *The projective objects in* cHaus *are exactly the extremally disconnected compact Hausdorff spaces. I.e. a compact Hausdorff space S is extremally disconnected if and only if every surjection $T \to S$ from a compact Hausdorff space T splits.*

The question now is whether we have enough projectives in cHaus. The answer is yes, relying on the following well-known construction, originally by Tychonoff [Tyc30], Stone [Sto37] and Čech [Čec37].

Definition 6.1.5. Let X be a topological space. Its **Stone-Čech compactification** is a compact Hausdorff space βX together with a continuous map $\iota : X \to \beta X$ that has the following universal property: Every continuous map $f : X \to S$ to a compact Hausdorff space S extends uniquely to a continuous map $\beta f : \beta X \to S$.



Remark 6.1.6. The Stone-Čech compactification is only an actual compactification, when X is Tychonoff. In this case the map ι is an embedding. See chapter 1 in [Wal74] for more details. ◇

Using the axiom of choice one can prove existence. See chapter 1 in [Wal74], but note that the complete regularity assumption is only needed to show that ι is an embedding.

Proposition 6.1.7. *For every topological space X its Stone-Čech compactification βX as defined in Definition 6.1.5 exists. It defines a functor $\beta : \mathsf{Top} \to \mathsf{cHaus}$ that is left adjoint to the inclusion and exhibits cHaus as a reflective subcategory of Top.*

Proposition 6.1.8 (Proposition 10.47 and Theorem 3.2 in [Wal74])**.** *Let X be an infinite discrete topological space with cardinality κ. Its Stone-Čech compactification βX is the space of ultrafilters on X, which is extremally disconnected and has cardinality 2^{2^κ}.*

Proposition 6.1.9 (Proposition 1.2.5 in [Ás21])**.** *An extremally disconnected space is totally disconnected.*

Thus, we have the following inclusions of full subcategories

$$\mathsf{Extr} \subset \mathsf{Prof} \subset \mathsf{cHaus}\,. \tag{6.2}$$

We will provide an ad hoc definition of condensed sets in Definition 6.2.3, but let us state the basic idea how it fits in the framework of abstract sheaf-theory. Informally, a **site** is a category C, that allows us to define sheaves on C. In particular, C has to allow gluing. For this one defines a collection of **coverings** given by families of morphisms with a fixed target, that satisfy certain axioms. See chapter 1.1 in [Ás21] for an actual definition.

For any category C in (6.2) coverings will be given by **jointly surjective** finite families. That is for $S \in \mathsf{C}$ and $\{S_i \to S\}_{i \in I}$ continuous, the induced map

$$\coprod_{i \in I} S_i \to S$$

has to be surjective. The finite families of jointly surjective continuous maps do form what is called a **pre-coverage**. The site is then given by C together with the pre-coverage. One can take the pre-coverage and extend it to a **Grothendieck topology**. See chapter 1.1 in [Ás21] for more details.

6.2. Cardinals and Size Issues

The categories $\mathsf{cHaus}, \mathsf{Prof}$ and Extr are all large categories. So using them as sketched in (6.1) and considering functor categories has some set-theoretic issues. To address these, we will fix a cardinal number κ and restrict our attention to spaces of cardinality at most κ.

To ensure the existence of projective objects, we require Stone-Čech compactifications by Proposition 6.1.4. From Proposition 6.1.8 it follows that κ must be sufficiently large to not be reached by power set operations. Consequently, we fix a strong limit cardinal κ in the sense of Definition A.2.1. Additionally, since $\aleph_0$ is a strong limit cardinal, we impose that κ be uncountable. For instance, κ could be $\beth_\alpha$ for any limit ordinal α. We say that a space $S \in \mathsf{cHaus}$ is κ-small if $\mathrm{card}(S) < \kappa$.

Notation 6.2.1. We denote the site of κ-small extremally disconnected spaces/ profinite sets/ compact Hausdorff spaces and jointly surjective finite families of maps by

$$\mathsf{Extr}_\kappa\,, \qquad \mathsf{Prof}_\kappa \qquad \text{and} \qquad \mathsf{cHaus}_\kappa.$$

We denote the category of sheaves on a site C by $\mathsf{Sh}(\mathsf{C})$.

By (6.2) we also have

$$\mathsf{Extr}_\kappa \subset \mathsf{Prof}_\kappa \subset \mathsf{cHaus}_\kappa\,.$$

Since they all have coverings defined in the same way, any sheaf $\mathcal{F} : \mathsf{cHaus}_\kappa^{\mathrm{op}} \to \mathsf{Set}$ is also a sheaf on Prof_κ and on Extr_κ when restricted to those sites. It turns out that, although extremally disconnected spaces are very sparse in cHaus, all three sites define equivalent category of sheaves.

Theorem 6.2.2 (2.3 and 2.7 in [Condensed] or 1.2.6 in [Ás21])**.** *The restriction functors*

$$\mathsf{Sh}(\mathsf{cHaus}_\kappa) \;\to\; \mathsf{Sh}(\mathsf{Prof}_\kappa) \;\to\; \mathsf{Sh}(\mathsf{Extr}_\kappa)$$

are equivalences of categories.

Definition 6.2.3. Let κ be an uncountable strong limit cardinal. A **κ-condensed set** T is a sheaf on one of the equivalent sites Extr_κ, Prof_κ or cHaus_κ. Explicitly, for C being one of these sites, T is given by a functor

$$T : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$$

satisfying $T(\emptyset) = *$ and the following two (sheaf) conditions:

(i) For any $S_1, S_2 \in \mathsf{C}$ the natural map

$$T(S_1 \sqcup S_2) \to T(S_1) \times T(S_2) \tag{6.3}$$

is a bijection.

(ii) For any surjection $S' \to S$ of objects in C, let p_1 and p_2 denote the projections $S' \times_S S' \to S'$. Then the map

$$T(S) \to \{\, x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S') \,\} \tag{6.4}$$

is a bijection.

A morphism of κ-condensed sets is a morphism of sheaves. Denote the category of κ-condensed sets by $\mathsf{Cond}_\kappa(\mathsf{Set})$.

All three possible sites have advantages in certain situations. In particular, extremally disconnected spaces are not very approachable, but the sheaf conditions simplify significantly in their case.

Proposition 6.2.4 (Theorem 1.2.18 in [Ás21]). *A presheaf $T : \mathsf{Extr}_\kappa^{\mathrm{op}} \to \mathsf{Set}$ is a κ-condensed set if and only if it satisfies (6.3).*

One of the big advantages of the formalism is that we can now easily define condensed objects in categories other than Set.

Definition 6.2.5. Let C be a category. A **κ-condensed object** T is a sheaf of C on one of the equivalent sites Extr_κ, Prof_κ or cHaus_κ. We denote the category of κ-condensed objects in C with $\mathsf{Cond}_\kappa(\mathsf{C})$

This way we get κ-condensed vector spaces T that are given by presheaves $T : \mathsf{cHaus}_\kappa \to \mathsf{Vect}_{\mathbb{K}}$ subject to (6.3) and (6.4).

There is a way, to get rid of the fixed cardinal κ. Note that strong limit cardinals are ordered by size and cofinal in all cardinal numbers.

Proposition 6.2.6 (Proposition 2.9 in [Condensed]). *For $\kappa' > \kappa$ uncountable strong limit cardinals there is a functor from the category of κ-condensed to κ'-condensed sets. This functor is fully faithful commutes with all colimits and $\mathrm{cf}(\kappa)$-small limits.*

Definition 6.2.7. The category of **condensed sets** is the (large) filtered colimit of $\mathsf{Cond}_\kappa(\mathsf{Set})$ along the poset of strong limit cardinals.

We denote the category of condensed sets by $\mathsf{Cond}(\mathsf{Set})$.

This construction works for all categories with small colimits. In particular, we get condensed abelian groups $\mathsf{Cond}(\mathsf{Ab})$, condensed vector spaces $\mathsf{Cond}(\mathsf{Vect}_{\mathbb{K}})$ and condensed rings $\mathsf{Cond}(\mathsf{Ring})$. See Appendix to lecture 2 in [Condensed] for more details.

6.3. Topological Spaces and Condensation

The translation from topological spaces to condensed set is given by the following. For $X \in \mathsf{Top}$ define

$$\underline{X} : \mathsf{cHaus}_\kappa \to \mathsf{Set}, \qquad S \mapsto \mathrm{Cont}(S, T),$$

where $\mathrm{Cont}(S, T)$ is the set of continuous maps $S \to T$. This defines a functor

$$\underline{(\,\cdot\,)} : \mathsf{Top} \to \mathsf{Cond}_\kappa(\mathsf{Set})$$

called **condensation functor**. For any space $X \in \mathsf{Top}$ we have

$$\underline{X}(*) = \mathrm{Cont}(*, X) \cong X.$$

For this reason we call $T(*)$ the **underlying set** of T for any $T \in \mathsf{Cond}_\kappa(\mathsf{Set})$.

Given a κ-condensed set T we can define a topology on its underlying set $T(*)$. From Yoneda we get the correspondence

$$\mathrm{Hom}(\underline{S}, T) \cong T(S),$$

which shows that we can identify every $f \in T(S)$ with a map $\underline{S} \to T$. At the point $*$ we get $\tilde{f} : \underline{S}(*) = S \to T(*)$. Now define $T(*)$ to have the finest topology such that for all $S \in \mathsf{cHaus}_\kappa$ and $f \in T(S)$, the map $\tilde{f}$ is continuous. We get a functor

$$(\cdot)(*) : \mathsf{Cond}_\kappa(\mathsf{Set}) \to \mathsf{Top}\,.$$

The construction of $T(*)$ is closely related to k-ification from Definition C.3.10 as the following proposition shows.

Proposition 6.3.1 (proposition 1.7 in [Condensed])**.** *We have an adjunction*

$$\mathsf{Top} \underset{\underline{(\cdot)}}{\overset{(\cdot)(*)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Cond}_\kappa(\mathsf{Set})\,.$$

For $X \in \mathsf{Top}$ the counit produces its k-ification $\underline{X}(*) \cong kX$.

Thus, if X is compactly generated (CG 2 Definition C.3.4) we have $\underline{X}(*) \cong X$.

Remark 6.3.2. We can define a condensation functor with non-size restricted compact Hausdorff spaces. However, for $X \in \mathsf{Top}$ not T_1

$$\underline{X} : \mathsf{cHaus} \to \mathsf{Set}, \qquad S \mapsto \mathrm{Cont}(S, T),$$

is not an element of $\mathsf{Cond}(\mathsf{Set})$. See Warning 2.14 in [Condensed]. If X is T_1 we do get a condensed set. Conversely, for a condensed set T such that all maps from points are quasi-compact, we can define the space $T(*)$ via the quotient construction above. It is T_1 and compactly generated (CG 2). See proposition 2.15 in [Condensed]. ◇

6.4. Quasi-compact and Quasi-separated Condensed Sets

There are two important sheaf-theoretic notions that are important in condensed math. Quasi-compact describes that given a cover we can choose a finite subcover.

Definition 6.4.1. A sheaf F on a site C is **quasi-compact** or **qc** if for any collection of maps of sheaves $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$, such that the induced map

$$\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$$

is an epimorphism, there exists a finite subset $J \subset I$, such that

$$\coprod_{i \in J} \mathcal{F}_i \to \mathcal{F}$$

is an epimorphism.

Quasi-separated says that quasi-compactness is stable under forming pullbacks over it.

Definition 6.4.2. A sheaf F on a site C is **quasi-separated** or **qs** if for any morphisms of sheaves $\mathcal{G} \to \mathcal{F}, \mathcal{H} \to \mathcal{F}$ with $\mathcal{G}, \mathcal{H}$ qc, their pullback $\mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ is also qc.

Notation 6.4.3. We abbreviate the property quasi-compact, quasi-separated with qcqs.

We use the notation $\mathsf{qcCond}(\mathsf{C}), \mathsf{qsCond}(\mathsf{C}), \mathsf{qcqsCond}(\mathsf{C})$ and their κ-variants for the qc, qs and qcqs (κ)-condensed subobjects of a category C.

The importance of these subcategories will be made clear by the following theorem.

Theorem 6.4.4 (Proposition 1.2 in [Analytic]). *The condensation functor*

$$\mathsf{Top} \to \mathsf{Cond}_\kappa(\mathsf{Set}), \qquad X \mapsto \underline{X}$$

*(i) is fully faithful, when restricted to compactly generated spaces (*CG 2 *Definition C.3.4),*

(ii) induces an equivalence of compact Hausdorff spaces and $\mathsf{qcqsCond}_\kappa(\mathsf{Set})$*,*

(iii) induces a fully faithful functor from κ-small compactly generated weak Hausdorff spaces to qc κ-condensed sets. The category $\mathsf{qcCond}_\kappa(\mathsf{Set})$ *is equivalent to* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}_\kappa)$*. The latter is the full subcategory of* $\mathsf{Ind}(\mathsf{cHaus}_\kappa)$ *consisting of formal colimits of spaces in* cHaus_κ *with injective transition maps.*

6.5. Condensed Vector Spaces

We will focus on $\mathbb{R}$-vector spaces and denote their category by $\mathsf{Vect}_{\mathbb{R}}$.

Theorem 2.2 in [Condensed] describes κ-condensed abelian groups as a very nice abelian category. The same proof works for $\mathsf{Cond}_\kappa(\mathsf{Vect}_{\mathbb{R}})$ and we get the following.

Theorem 6.5.1 (see to Theorem 2.2 in [Condensed]). *The category $\mathsf{Cond}_\kappa(\mathsf{Vect}_{\mathbb{R}})$ is abelian and satisfies the Grothendieck axioms* (AB3), (AB4), (AB5*), (AB6), (AB3*) *and* (AB4*). *I.e. it is complete and cocomplete, also products, coproducts and filtered colimits are exact and for any index set J and filtered categories $I_j, j \in J$ with functors $i \mapsto M_i$ from I_j to κ-condensed vector spaces, the natural map*

$$\operatorname*{colim}_{(i_j\in I_j)_j} \prod_{j\in J} M_{i_j} \;\to\; \prod_{j\in J} \operatorname*{colim}_{i_j\in I_j} M_{i_j}$$

is an isomorphism.

Moreover, the categories are generated by compact projectives, given by $\mathbb{R}[S]$ for all $S \in \mathsf{Extr}_\kappa$.

Let us explain the notation $\mathbb{R}[S]$ above. The forgetful functor $\mathsf{Cond}_\kappa(\mathsf{Vect}_{\mathbb{R}}) \to \mathsf{Cond}_\kappa(\mathsf{Set})$ has a left adjoint $T \mapsto \mathbb{R}[T]$, where $\mathbb{R}[T]$ is the sheafification of sending an extremally disconnected S to the free vector space on $T(S)$:

$$\mathsf{Extr}_\kappa^{\mathrm{op}} \to \mathsf{Cond}_\kappa(\mathsf{Vect}_{\mathbb{R}}), \qquad S \mapsto \coprod_{T(S)} \mathbb{R}\,.$$

We write $\mathbb{R}[S]$ for $\mathbb{R}[\underline{S}]$.

From Theorem 6.4.4 we find that the condensation functor

$$\mathsf{TVS}_{\mathbb{R}} \to \mathsf{Cond}_\kappa(\mathsf{Vect}_{\mathbb{R}}), \qquad V \mapsto \underline{V}$$

is fully faithful on compactly generated topological vector spaces. This class encompasses all F-spaces by Corollary C.3.17, and thus includes Banach and Fréchet spaces.

We want a robust notion of completeness for condensed vector spaces that aligns with the traditional topological definition. In condensed mathematics, this is achieved through analytic ring structures; see Lecture 7 in [Condensed] for the general framework. A primary example of this is the completion of condensed abelian groups to solid abelian groups, which serves as the non-Archimedean counterpart to the desired Archimedean theory for condensed $\mathbb{R}$-vector spaces. However, the latter proves to be more complex. Before Clausen and Scholze introduced the liquid analytic ring structure in Lecture 6 of [Analytic], their initial approach of $\mathcal{M}$-complete vector spaces is also quite interesting from our perspective. We will revisit these notions in Chapter 8 and Chapter 9, where we construct them using compactologies, introduced in the next chapter.

7. Compactologies and Compactological Sets

Contents

In lecture 2 of [Complex] Clausen and Scholze mention that the notion of quasi-separated condensed sets aligns with that of compactological spaces. The aim of this chapter is to expand upon this observation in detail. This provides a good starting point for the analogous comparison in presence of vector space structure in Chapter 8 and Chapter 9.

The original definition of compactological spaces is due to Waelbrock in [Wae67]; see also chapter 3 in [Wae71]. Note, that Buchwalter proposed a slightly different definition in [Buc68] and [Buc69]. Unlike them, we will refer to these spaces as compactological to align with the terminology used for bornological sets and to distinguish them from the compactological vector spaces defined later.

7.1. Compactological Sets and the Colimit Topology

Definition 7.1.1 (Definition 1, Chapter 3 [Wae71])**.** Let X be a set. A **compactology** $(\mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ on X is a bornology $\mathcal{B}$ with topologies τ_B for all $B \in \mathcal{B}$, such that the following hold.

(i) If $B \subset B'$, then τ_B is the restriction of $\tau_{B'}$ to B,

(ii) for all $B \in \mathcal{B}$ there is a $K \in \mathcal{B}$ with $B \subset K$ and (K, τ_K) compact Hausdorff.

A **compactological set** is a triple $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ consisting of a set X equipped with a compactology $(\mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$.

A morphism of compactological sets

$$u : (X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}}) \;\to\; (Y, \mathcal{C}, (\tau_C)_{C\in\mathcal{C}})$$

is a map $u : X \to Y$, such that

(i) u is a morphism of bornological sets, i.e. $u(B) \in \mathcal{C}$ for all $B \in \mathcal{B}$.

(ii) for all $B \in \mathcal{B}$, the restriction $u_B : (B, \tau_B) \to (u(B), \tau_{u(B)})$ is continuous.

We denote the category of compactological sets with $\mathsf{CompSet}$.

Recall, that we say that a subset $B \subset X$ is bounded if it is an element of the bornology $\mathcal{B}$.

Remark 7.1.2. A priori, we do not have a topology on X itself. However, given a compactological set $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ we can take the colimit (i.e. final) topology of the $(\tau_B)_{B\in\mathcal{B}}$ and get a topology τ on X. A subset C in (X, τ) is closed if and only if $C \cap U$ is closed in (B, τ_B) for all $B \in \mathcal{B}$. In fact, instead of the system of topologies on all bounded sets we could have only required one topology on X that is the final topology coherent with all bounded subspaces. This is the definition found in [Com23]. ◇

Notation 7.1.3. Let $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ be a compactological set. We write τ_X for the final topology on X described in Remark 7.1.2.

Next, we prove some basic results about the topology τ_X. Most of these results can already be found in [Wae71] or in [Com23].

Lemma 7.1.4. *Let $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ be a compactological set and $B \in \mathcal{B}$ a bounded subset. Its closure $(\overline{B}, \tau_{\overline{B}})$ with respect to the colimit topology τ_X is bounded and a compact Hausdorff space.*

Proof. Let $B \in \mathcal{B}$ be bounded. By definition there is a set K such that (K, τ_K) is compact Hausdorff and $B \subset K$. The topology on B is the subspace topology of the topology on K, which is again the subspace topology of the colimit topology τ_X. Since τ_K is compact and Hausdorff, it is closed. Thus, the closure $\overline{B}$ is also contained in K and by the definition of a bornology also bounded. The corresponding topology $\tau_{\overline{B}}$ as a subspace topology of τ_K is Hausdorff. Furthermore, being closed in a compact space it is also compact. □

Proposition 7.1.5. *Given a compactological set $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$, the poset $(\mathcal{B}, \subset)$ is filtered and the subset of compact Hausdorff spaces in $\mathcal{B}$ is cofinal. The corresponding colimit topology τ_X on X from Remark 7.1.2 is compactly generated and weak Hausdorff.*

Proof. The diagram $(\mathcal{B}, \subset)$ is a preorder. As a bornology $\mathcal{B}$ is stable under finite unions. Thus, $(\mathcal{B}, \subset)$ is filtered. The subset of compact Hausdorff spaces $\mathcal{B}$ is cofinal by Lemma 7.1.4.

For the second claim recall Remark 7.1.2. We will show that (X, τ_X) is CG 3 as defined in Definition C.3.5. Let $A \subset X$ be a closed subset and $S \subset X$ be a compact Hausdorff subspace. Then $A \cap S$ is closed in S by definition. Conversely, let $A \subset X$ be a subset such that $A \cap S$ is closed in S for all compact Hausdorff subspaces $S \subset X$. Let $B \in \mathcal{B}$ be a bounded set. Then $A \cap \overline{B}$ is closed in $\overline{B}$ by Lemma 7.1.4. Thus, $A \cap B$ is closed in B and we have shown that X is CG 3. It follows from Theorem C.3.8 that X is compactly generated.

The last step is to show that X is weak Hausdorff. Let S be a compact Hausdorff space and $f : S \to X$ be continuous. Let $B \in \mathcal{B}$ be a closed bounded subset of X. The set $f(S) \cap B$ is closed in $f(S)$ and therefore compact. Thus, $f(S) \cap B$ is a compact subspace of the Hausdorff space B and hence closed in B. Let $B' \in \mathcal{B}$ be a not necessarily closed bounded subset of X. Then $\overline{B'}$ is closed by Lemma 7.1.4. By the argument above $f(S) \cap \overline{B'}$ is closed in $\overline{B'}$. Hence, $f(S) \cap \overline{B'} \cap B' = f(S) \cap B'$ is closed in B'. Thus, $f(S)$ is closed with respect to τ_X. □

Corollary 7.1.6. *Let $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$, $(Y, \mathcal{C}, (\tau_C)_{C \in \mathcal{C}})$ be compactological sets and $u : X \to Y$ a bounded map. Then u is a morphism in* CompSet *if and only if $u : (X, \tau_X) \to (Y, \tau_Y)$ is continuous.*

Proof. Combine Proposition 7.1.5 and Lemma C.3.13 □

Before we discuss more properties let us take a look at some examples to get a feeling for the colimit topology of a compactology.

Example 7.1.7 (Example 1.4.7 in [Com23]). Let $X = [0, 1]$ be a closed interval. Consider the bornology $\mathcal{B}$ consisting of countable unions of points in X. Every $B \in \mathcal{B}$ is equipped with its subspace topology τ_B from $\mathbb{R}$. Then $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ is a compactological set.

The colimit topology τ_X on X coincides with the Euclidean subspace topology $\tau_{\mathbb{R}}$.

Proof. Let $U \in \tau_{\mathbb{R}}$ be a Euclidean open subset and $B \in \mathcal{B}$. Then $U \cap B$ is open in B since B has the subspace topology with respect to $\tau_{\mathbb{R}}$. Thus, U is open in τ_X and $\tau_{\mathbb{R}} \subset \tau_X$.

Conversely, let $U \in \tau_X$ be open. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X \setminus U$ with Euclidean limit point $x \in X$. The set $B := \{x, x_1, x_2, \ldots\}$ is bounded. Thus, $(X \setminus U) \cap B$ is closed in B. Hence, $x \in (X \setminus U)$ shows that $X \setminus U$ is closed with respect to $\tau_{\mathbb{R}}$. □

In particular, not all compact Hausdorff subsets of X are bounded, i.e. elements of $\mathcal{B}$.

If we replace the bornology $\mathcal{B}$ of countable subsets with the bornology of finite subsets, the colimit topology τ_X is the discrete topology on X. ◇

The statement in Example 7.1.7 and its proof generalize in the following way.

Proposition 7.1.8. *Let $(X, \tau) \in \mathsf{Top}$ be a topological space. Consider the bornology $\mathcal{B}_{\aleph_1}$ consisting of countable unions of points in X. Every $B \in \mathcal{B}_{\aleph_1}$ is equipped with its subspace topology τ_B from X. The colimit topology τ_X of the compactological set $(X, \mathcal{B}_{\aleph_1}, (\tau_B)_B)$ coincides with τ if and only if (X, τ) is a sequential space.*

Proof. See Example 7.1.7 for one direction. Conversely, assume τ is not sequential. The colimit τ_X is by definition sequential and cannot agree with τ. □

Similarly, the Compactology $(\mathcal{B}_{\aleph_0}, (\tau_B)_B)$ of finite points characterizes discrete topological spaces.

We can also create finer topologies on a topological space by choosing a bornology and equipping it with its subspace topologies.

Example 7.1.9. Consider the closed disk $X = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \}$ in $\mathbb{R}^2$. We consider the bornology $\mathcal{B}$ with a basis given by chords in X. That is, a subset $B \subset X$ is bounded if and only if it is a subset of a finite union of chords. See Fig. 7.1.

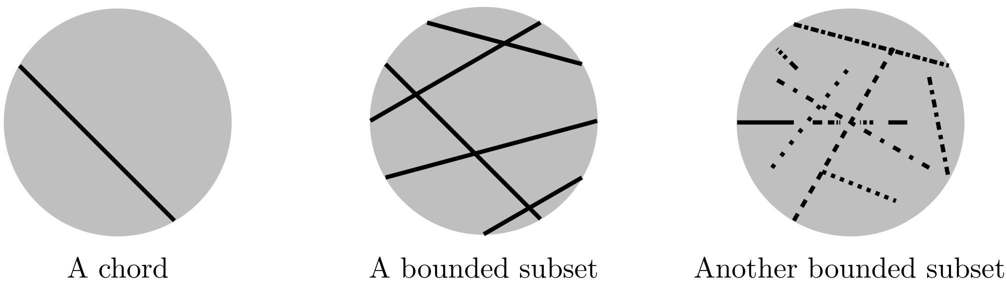


Figure 7.1.: The compactological set of chords

On each bounded set B we consider its subspace topology τ_B with respect to $\mathbb{R}^2$. By the Heine-Borel theorem a finite union of chords is compact. Thus, $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ is a compactological set.

The following are true.

- *Every bounded subset $B \in \mathcal{B}$ has empty interior $B^\circ = \emptyset$ with respect to τ_X.*
- *The colimit topology τ_X of X is strictly finer than the usual Euclidean subspace topology $\tau_{\mathbb{R}^2}$ of X.*

Proof.
- Without loss of generality let $B \in \mathcal{B}$ be a finite union of chords. Let $x \in B$ be a point. Then x locally lies on an intersection of at most finitely many and possible only one line. In any case we find another chord C through x that locally intersects B only in x. Any neighborhood of x has to have an open intersection with C. In particular, it has to contain a small interval on C that contains x and therefore cannot be a subset of B.
- Let $U \in \tau_{\mathbb{R}^2}$ be a Euclidean open subset and $B \in \mathcal{B}$. Then $U \cap B$ is open in B since B has the subspace topology with respect to $\tau_{\mathbb{R}^2}$. Thus, U is open in τ_X and $\tau_{\mathbb{R}^2} \subset \tau_X$.

  There is a Euclidean dense subset C of X which contains no three collinear points. See for example [Gow08] for a construction. The intersection of C with any closed subset of τ_X is finite and therefore closed. Thus, C itself is closed in τ_X. But C cannot be Euclidean closed, since $C \neq X$ and C is dense.

□

◇

7.2. Adjunction with Topological Spaces

In Proposition 7.1.5 we saw that every compactological set can be an assigned a topological space in cgWHaus. A morphism of compactological sets

$$u : (X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}}) \;\to\; (Y, \mathcal{C}, (\tau_C)_{C\in\mathcal{C}})$$

induces a morphism $\mathrm{Ct}(u) : (X, \tau_X) \to (Y, \tau_Y), x \mapsto u(x)$ in cgWHaus. The continuity of $\mathrm{Ct}(u)$ follows Corollary 7.1.6. This construction is functorial and part of an adjunction. Its right adjoint CSet : cgWHaus → CompSet assigns to a compactly generated weak Hausdorff space X the compactology generated by all of its compact Hausdorff subsets. The bounded sets are

$$\mathcal{B} = \{\, S \subset K \mid K \text{ a compact Hausdorff subspace of } X \,\},$$

and we equip them with their subspace topologies.

Proposition 7.2.1 (Proposition 2.2.4 in [Buc69])**.** *There is an adjunction*

$$\mathsf{CompSet} \underset{\mathrm{CSet}}{\overset{\mathrm{Ct}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{cgWHaus}$$

of the categories of compactological sets CompSet *and compactly generated weak Hausdorff spaces* cgWHaus*.*

There are more compactological sets then the ones given by images of CSet. By Proposition 7.1.8 every sequential space X, in particular every metrizable space, can be recovered from its compactology of countable sets. In fact, it can be recovered from any compactology that contains the compactology of countable sets and where the topologies are given by subspace topologies of X. Thus, there are a large amount of different compactological sets, that generate the topology of X.

Example 7.2.2. Consider $\mathbb{R}$ with its Euclidean topology. The compactology of $\mathrm{CSet}(\mathbb{R})$ consists of all pre-compact subspaces of $\mathbb{R}$. By the Heine-Borel Theorem these are exactly the spaces contained in some closed intervall. Another compactology on $\mathbb{R}$ is the compactology of countable sets $(\mathcal{B}_{\aleph_1}, (\tau_B))$, which is not isomorphic to $\mathrm{CSet}(\mathbb{R})$. Indeed, every isomorphism of compactological sets is an isomorphism of the bornologies and bijective, but $[-1, 1]$ is bounded in $\mathrm{CSet}(\mathbb{R})$ and cannot have a bounded image in $\mathcal{B}_{\aleph_1}$ under a bijection.

The same argument shows that no space $X \in \mathsf{cgWHaus}$ has an image under CSet that is isomorphic to $(\mathbb{R}, \mathcal{B}_{\aleph_1}, (\tau_B))$. The functor CSet is not essentially surjective. ◇

What about topological spaces that are not compactly generated and weak Hausdorff? The functor CSet extends to a functor $\mathsf{Top} \to \mathsf{CompSet}$, assigning to a space X the compactology of all of its compact Hausdorff subsets. Since, at least all of the singletons in X are compact Hausdorff, this is well-defined. If X is weak Hausdorff the topology $\mathrm{Ct}(\mathrm{CSet}(X))$ is the k-ification of the original topology on X by Theorem C.3.9.

Example 7.2.3. Consider the example from the proof of Theorem C.3.8, that shows that not every CG 2 space is CG 3: the Sierpinski space $X = \{0, 1\}$ with topology $\{\emptyset, \{1\}, X\}$. The compactology of $\mathrm{CSet}(X)$ is given by $\mathcal{B} = \{\emptyset, \{0\}, \{1\}\}$. Thus, $\mathrm{Ct}(\mathrm{CSet}(X))$ has the discrete topology. But the k-ification of X is X itself, since the space is already CG 2. ◇

7.3. Limits, Colimits and Quotients

We will see that the category of compactological sets $\mathsf{CompSet}$ has all small limits and colimits. This mostly follows from $\mathsf{BornSet}$ and $\mathsf{cgWHaus}$ being complete and cocomplete, but let us write down the details. We start with products and coproducts.

Lemma 7.3.1. *The category* $\mathsf{CompSet}$ *has arbitrary products and coproducts.*

Proof. Let I be a set and $(X_i, \mathcal{B}_i, (\tau_B)_{B \in \mathcal{B}_i})_{i \in I}$ a family of compactological sets.

Let us start with coproducts. The coproduct of bornological sets is given by their disjoint union $X = \bigsqcup_{i \in I} X_i$ equipped with the bornology $\mathcal{B}$ generated by the disjoint union of all their bornologies. We have

$$\mathcal{B} = \left\{ \bigsqcup_{i \in I} B_i \;\middle|\; B_i \in \mathcal{B}_i,\ B_i \neq \emptyset \text{ for only finitely many } i \in I \right\}.$$

Equipping each $B = \bigsqcup_{i \in I} B_i \in \mathcal{B}$ with its disjoint union topology τ_B induced by the τ_{B_i}, defines a system of topologies compatible with restrictions. Since the disjoint unions in question are all finite, the defined topologies are compact Hausdorff if and only if all their components are compact Hausdorff. In particular, it follows from $(\mathcal{B}_i, (\tau_B)_{B \in \mathcal{B}_i})$ being compactologies for all $i \in I$ that every B is contained in a compact Hausdorff space. Thus $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ is a compactological set. It follows from the universal properties of the coproduct of bornological sets and topological spaces that $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ has the universal property of the coproduct in CompSet.

Now we turn to products. The product of bornological sets as described in Example 4.1.6 is given by the set $X = \prod_{i \in I} X_i$ equipped with

$$\mathcal{B} := \left\{ \prod_{i \in I} B_i \;\middle|\; B_i \in \mathcal{B}_i \text{ for all } i \in I \right\}.$$

On $B = \prod_{i \in I} B_i$ we define the product topology τ_B induced by the τ_{B_i}. By Tychonoff's theorem and $(\mathcal{B}_i, (\tau_B)_{B \in \mathcal{B}_i})$ being compactologies, it follows that every B is contained in a bounded compact Hausdorff space. Thus $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ is a compactological set. It follows from the universal properties of the product of bornological sets and cgWHaus that $(X, \mathcal{B}, (\tau_B)_{B \in \mathcal{B}})$ has the universal property of the product in CompSet. □

Remark 7.3.2. Both the coproduct and product constructions in Lemma 7.3.1 are the expected adaptions from the corresponding notions in BornSet. However, note that by Proposition 7.2.1 the final topology on the product space is the corresponding product topology in cgWHaus and not in Top. ◇

Proposition 7.3.3. *The category* CompSet *is complete and cocomplete.*

Proof. By Lemma 7.3.1 it suffices to show that CompSet has equalizers and coequalizers.

Let

$$f, g : (X, \mathcal{B}_X, (\tau_B)_{B \in \mathcal{B}_X}) \rightrightarrows (Y, \mathcal{B}_Y, (\tau_B)_{B \in \mathcal{B}_Y})$$

be two parallel morphisms in CompSet.

Their equalizer is given by $E = \{ x \in X \mid f(x) = g(x) \}$ with bornology $\mathcal{B}_E = \{ B \subset E \mid B \in \mathcal{B}_X \}$ and topologies $(\tau_B)_{B \in \mathcal{B}_E}$.

For their coequalizer we consider the set $C = Y/_\sim$ with $\sim$ generated by $f(x) \sim g(x)$. The bornology $\mathcal{B}_C$ on C is the quotient bornology of $\mathcal{B}_Y$ and the topologies are the quotient topologies $(\tau_B/_\sim)_{B \in \mathcal{B}_C}$. □

Note, that the coequalizer construction only works, because the equivalence relation $\sim$ is closed, in the sense that $\sim$ is closed in $k(Y \times Y)$. By Proposition C.4.2 this is a necessary and sufficient condition for a quotient X/E of a compactological set X by an equivalence relation E to be a compactological set again. So what happens if E is not closed?

The following idea is taken from 1.4.15 in [Com23]. For a general equivalence relation $E \subset X \times X$ on a compactological set $(X, \mathcal{B}_X, (\tau_B)_{B \in \mathcal{B}_X})$ we can consider its topological quotient $X/E \in \mathsf{cTop}$. It is still compactly generated by Proposition C.3.20. With the quotient map $q : X \to X/E$ we can form a new equivalence relation $\tilde{E}$ via pullback in cTop:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ X & \xrightarrow[q]{} & X/E \end{array}$$

By definition $\tilde{E}$ is the equalizer of $q \circ \mathrm{pr}_1, q \circ \mathrm{pr}_2 : k(X \times X) \rightrightarrows X/E$, since $k(X \times X)$ is the product in cTop. We see that $\tilde{E}$ is the closure of E in $k(X \times X)$. Thus, $\tilde{E}$ is now a closed equivalence relation on X and we get a compactological set $X/\tilde{E}$. Adding the quotient X/E as a formal quotient to $\mathsf{CompSet}$ is exactly what condensed sets achieve. We will see in Theorem 7.4.4 that the compactological sets are exactly the quasi-separated condensed sets. Every non-quasi-separated condensed sets can be understood as a formal quotient of compactological sets.

7.4. Compactological Sets as formal filtered Colimits

For the comparison with condensed sets the following description of $\mathsf{CompSet}$ will be useful.

Consider the Ind-completion $\mathsf{Ind}(\mathsf{cHaus})$ of the category of compact Hausdorff spaces. Recall, that an object in $\mathsf{Ind}(\mathsf{cHaus})$ is essentially monomorphic if it can be represented by a formal filtered colimit of compact Hausdorff spaces with transition functions that are monomorphisms. We denote the full subcategory of essentially monomorphic objects of $\mathsf{Ind}(\mathsf{cHaus})$ with $\mathsf{Ind}_{\hookrightarrow}(\mathsf{cHaus})$. Recall, that the monomorphisms in cHaus are the closed embeddings. Thus, $\mathsf{Ind}_{\hookrightarrow}(\mathsf{cHaus})$ is the category of formal filtered colimits of compact Hausdorff spaces with closed embeddings as transition maps.

Consider the mapping

$$\mathrm{diss} : \mathsf{CompSet} \to \mathsf{Ind}(\mathsf{cHaus}), \qquad (X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}}) \mapsto \text{“}\operatorname*{colim}_{C\in\mathcal{C}}\text{”}\,(C, \tau_C),$$

where $\mathcal{C} = \{\, C \in \mathcal{B} \mid (C, \tau_C) \text{ compact Hausdorff} \,\}$. The mapping diss actually defines a formal filtered colimit by Proposition 7.1.5. The transition functions are given by closed embeddings.

Let

$$u : (X, \mathcal{B}_X, (\tau_B)_{B\in\mathcal{B}_X}) \to (Y, \mathcal{B}_Y, (\tau_B)_{B\in\mathcal{B}_Y})$$

be a morphism of compactological sets. By definition $u(B)$ is bounded and $u_B : (B, \tau_B) \to (u(B), \tau_{u(B)})$ is continuous for all $B \in \mathcal{B}$. If (B, τ_B) is compact Hausdorff, $(u(B), \tau_{u(B)})$ is also compact and Hausdorff by Proposition 7.1.5. Thus, u defines a morphism

$$\mathrm{diss}(f) : \text{“}\operatorname*{colim}_{\substack{B\in\mathcal{B}_X \\ \text{compact } T_2}}\text{”}\,(B, \tau_B) \mapsto \text{“}\operatorname*{colim}_{\substack{B\in\mathcal{B}_Y \\ \text{compact } T_2}}\text{”}\,(B, \tau_B). \tag{7.1}$$

We get the **dissection functor** diss, that is named after its counterpart (4.3) for bornological spaces.

For the comparison with condensed sets we will also consider compactological sets, whose size is restricted by some cardinal.

Definition 7.4.1. Let κ be a cardinal number. We say that a compactological set $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ is κ-**bounded** if every $B \in \mathcal{B}$ has cardinality $\mathrm{card}(B) < \kappa$.

We denote the full category of $\mathsf{CompSet}$ consisting of all κ-bounded compactological sets with $\mathsf{CompSet}_\kappa$.

Remark 7.4.2. Given a topological space X, its associated compactological set $\mathrm{CSet}(X)$ is κ-bounded if and only if all of its compact Hausdorff subspaces have cardinality $< \kappa$. This includes all spaces of cardinality $< \kappa$, but also some larger spaces such as $\bigsqcup_{i\in I} K$, where K is a κ-small compact Hausdorff space K and I is any set. ◇

Recall, that we denote the category of compact Hausdorff spaces of cardinality $< \kappa$ with cHaus_κ.

Proposition 7.4.3. *The dissection functor*

$$\mathrm{diss} : \mathsf{CompSet} \to \mathsf{Ind}(\mathsf{cHaus})$$

is fully faithful, commutes with limits and coproducts.

It defines an equivalence of categories of $\mathsf{CompSet}$ *and* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus})$*. The latter is a full subcategory of* $\mathsf{Ind}(\mathsf{cHaus})$*, given by all essentially monomorphic objects.*

Given an uncountable strong limit cardinal κ *this restricts to an equivalence*

$$\mathrm{diss}: \mathsf{CompSet}_\kappa \to \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}_\kappa),$$

where $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}_\kappa)$ *is the full subcategory of* $\mathsf{Ind}(\mathsf{cHaus})$*, consisting of formal colimits over diagrams of* κ*-small compact Hausdorff spaces with injective transition maps.*

Proof. By definition of compactological sets, diss factors through $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus})$. Furthermore, $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus})$ is exactly the essential image of diss, since every object "$\mathrm{colim}_{i\in I}$"$K_i \in \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus})$ defines a compactological set $(X, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$, where X is the underlying set of

$$K = \operatorname*{colim}_{i\in I} K_i,$$

$\mathcal{B}$ is the relatively compact bornology of K, and the topologies on bounded sets are given by the corresponding subspace topologies of the K_i for all $i \in I$. The functor diss is right adjoint to this colimit construction and therefore preserves limits. The coproducts in $\mathsf{CompSet}$ are given by disjoint union. Thus, diss preserves coproducts.

With the description of diss on morphisms in (7.1) it is not hard to see, that the functor is fully faithful.

The last assertion also easily follows from the description of diss. □

For the full comparison with quasi-separated condensed sets let us record both the cardinality restricted and the unrestricted equivalence.

Theorem 7.4.4. *Let* κ *be an uncountable strong limit cardinal. There is an equivalence of categories*

$$\mathsf{CompSet}_\kappa \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}_\kappa) \quad \cong \quad \mathsf{qsCond}_\kappa(\mathsf{Set}).$$

of

- κ*-bounded compactological sets* $\mathsf{CompSet}_\kappa$*,*
- *essentially monomorphic ind-compact Hausdorff spaces* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}_\kappa)$*, given by formal filtered colimits of* κ*-small compacta with closed immersions as transition maps, and*
- *quasi-separated* κ*-condensed sets* $\mathsf{qsCond}_\kappa(\mathsf{Set})$*.*

There is an equivalence of categories

$$\mathsf{CompSet} \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}) \quad \cong \quad \mathsf{qsCond}(\mathsf{Set}).$$

Proof. The first equivalence is given by the dissection functor and follows from Proposition 7.4.3. The second one is Theorem 6.4.4. □

Overall, we get the following diagram of categories with topological spaces on the left and condensed sets on the right.

$$\begin{array}{ccccccc} \mathsf{Top}_{\mathsf{T}_1} & & & \xrightarrow[\underline{(\,\cdot\,)}]{} & & & \mathsf{Cond}(\mathsf{Set}) \\ \uparrow & & & & & & \uparrow \\ \mathsf{cgWHaus} & \underset{\mathrm{CSet}}{\overset{\mathrm{Ct}}{\rightleftarrows}}\ \perp & \mathsf{CompSet} & \xrightarrow[\cong]{\mathrm{diss}} & \mathsf{Ind}_{\rightarrowtail}(\mathsf{cHaus}) & \xrightarrow[\cong]{} & \mathsf{qsCond}(\mathsf{Set}) \\ \uparrow & & & & & & \uparrow \\ \mathsf{cHaus} & & & \xrightarrow[\cong]{\underline{(\,\cdot\,)}} & & & \mathsf{qcqsCond}(\mathsf{Set}) \end{array} \tag{7.2}$$

Note, that the condensation functor $\underline{(\,\cdot\,)}$ is not fully faithful on $\mathsf{Top}_{\mathsf{T}_1}$, becomes fully faithful on $\mathsf{cgWHaus}$ and is an equivalence with $\mathsf{qcqsCond}(\mathsf{Set})$ on cHaus. The last statement is Theorem 6.4.4.

8. Locally Convex Compactological and Condensed Spaces

Contents

We saw in Theorem 7.4.4 that compact Hausdorff spaces play a key part, when comparing bornologies and quasi-separated condensed sets. For topological and bornological vector spaces a similar role is fulfilled by Smith spaces.

Our goal is to get a comparison analogous to (7.2). Furthermore, we want to understand how $\mathsf{Ind}(\mathsf{Ban}_{\mathbb{K}})$ and complete bornological spaces fit into this picture.

8.1. Smith spaces

The notion of Smith spaces is due to Marianne Freundlich Smith and was introduced 1952 in the article "The Pontrjagin duality theorem in linear spaces" [Smi52]. A good reference is chapter 4 in [Akb22].

8.1.1. Definition and Anti-equivalence with Banach spaces

Definition 8.1.1. Let $V \in \mathsf{CLCS}_{\mathbb{K}}$ be a Hausdorff complete locally convex $\mathbb{K}$-vector space and let $K \subset V$ be a compact disk. Equip the space

$$\mathbb{R}_{>0}K \; := \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda K,$$

with the compactly generated topology of the sets λK: A subset $U \subset \mathbb{R}_{>0}K$ is open if and only if for all $\lambda > 0$ the set $U \cap \lambda K$ is an open subset of λK, where the latter has its subspace topology from V. If $V \cong \mathbb{R}_{>0}K$ we say that V is a **Smith space** and K a **universal compact subset** of V.

We denote the category of Smith spaces and continuous linear maps with $\mathsf{Smi}_{\mathbb{K}}$.

Remark 8.1.2. Every compact subset S of a Smith space S with universal compact K is contained in some nK for an $n \in \mathbb{N}$. Thus, the Smith topology on S is hemicompact. ◇

The definition also provides us a recipe to create examples of Smith spaces. For any complete locally convex space $V \in \mathsf{CLCS}_{\mathbb{K}}$ all of its compact disks $K \subset V$ have an associated Smith space $\mathbb{R}_{>0}K = \operatorname{span} K$ with its compactly generated topology. For Fréchet spaces we can get all compact sets as closed subsets of disked hulls of null sequences. This is due to an old result by Grothendieck.

Proposition 8.1.3 (Lemma §4 $n°2$ Lemma 12 in [Gro55]). *Let $F \in \mathsf{Fré}_{\mathbb{K}}$ be a Fréchet space and $K \subset F$ a compact subset. Then there is a null sequence $(x_n)_{n \in \mathbb{N}}$ in F, such that K is contained in the closed disked hull of $\{ x_n \mid n \in \mathbb{N} \}$.*

We also have a converse to this statement.

Proposition 8.1.4. *Let $F \in \mathsf{LCS}_{\mathbb{K}}$ be a Fréchet space. The closed disked hull of a null sequence $(x_n)_{n \in \mathbb{N}}$ in F is compact.*

We will prove this in greater generality in Proposition 8.2.27.

Recall the notion of stereotype duality from Definition 3.1.34, where we equip the continuous dual with the topology of uniform convergence with respect to totally bounded sets. The following proposition shows that we could also have defined Smith spaces as stereotype duals of Banach spaces.

Proposition 8.1.5 (Proposition 4.1.7 in [Akb22]). *Let V be a topological vector spaces. Then V is Banach if and only if $V^\vee$ is Smith, and V is Smith if and only if $V^\vee$ is Banach. Both Banach and Smith spaces are stereotype. This defines an anti-equivalence between the categories $\mathsf{Ban}_{\mathbb{K}}$ and $\mathsf{Smi}_{\mathbb{K}}$.*

Remark 8.1.6. Throughout this chapter and the next we will solely focus on complete vector spaces. One can get non-complete spaces by replacing $\mathsf{Smi}_{\mathbb{K}}$ with the dual categories of $\mathsf{SemiN}_{\mathbb{K}}$ or $\mathsf{Norm}_{\mathbb{K}}$. ◇

In an infinite dimensional Banach or Fréchet space E we can never recover the topology via $\mathbb{R}_{>0}K$ with $K \subset E$.

Proposition 8.1.7 (section 4.1.3 in [Akb22])**.** *A Smith space $S \in \mathsf{Smi}_{\mathbb{K}}$ that is also Fréchet is finite dimensional.*

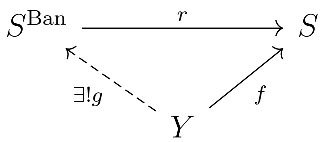
8.1.2. Banach Refinements and Smith Envelopes

Other than taking stereotype duals there is another, but closely related, way to get Smith spaces from Banach spaces and vice versa.

Definition 8.1.8 (Akbarov [Akb22])**.** Let $S \in \mathsf{Smi}_{\mathbb{K}}$ be a Smith space. Its **Banach refinement** is a Banach space S^{Ban} together with a linear and continuous map

$$r : S^{\mathrm{Ban}} \to S,$$

such that for all Banach spaces $Y \in \mathsf{Ban}_{\mathbb{K}}$ and linear continuous maps $f : Y \to S$, there is a unique linear continuous map $g : Y \to S^{\mathrm{Ban}}$ such that

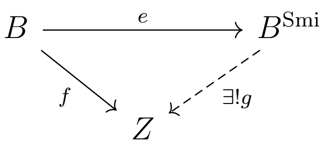


commutes.

Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. Its **Smith envelope** is a Smith space B^{Smi} together with a linear and continuous map

$$e : B \to B^{\mathrm{Smi}},$$

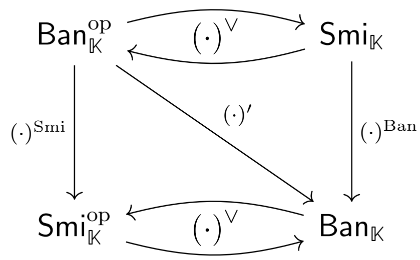
such that for all Smith spaces $Z \in \mathsf{Smi}_{\mathbb{K}}$ and linear continuous maps $f : B \to Z$, there is a unique linear continuous map $g : B^{\mathrm{Smi}} \to Z$ such that



commutes.

Proposition 8.1.9 (Theorems 4.1.13 and 4.1.16 in [Akb22])**.** *Banach refinements exist for all Smith spaces $S \in \mathsf{Smi}_{\mathbb{K}}$ and are given by $(S^{\vee})'$. Smith envelopes exist for all Banach spaces $B \in \mathsf{Ban}_{\mathbb{K}}$ and are given by $(B')^{\vee}$.*

We can describe the definitions of Banach refinements and Smith envelopes with the following commutative diagram



In particular we have the dualities,

$$\left(B^{\mathrm{Smi}}\right)^{\vee} \cong \left(B^{\vee}\right)^{\mathrm{Ban}} \qquad \text{and} \qquad \left(S^{\mathrm{Ban}}\right)^{\vee} \cong \left(S^{\vee}\right)^{\mathrm{Smi}} \tag{8.1}$$

for $B \in \mathsf{Ban}_{\mathbb{K}}$ and $S \in \mathsf{Smi}_{\mathbb{K}}$. It follows from (8.1) that for reflexive Banach or Smith spaces the Banach refinement and the Smith envelope cancel each other. Even if $S \in \mathsf{Smi}_{\mathbb{K}}$ is not reflexive, its Banach refinement has an easy description.

Proposition 8.1.10 (Proposition 4.1.14 in [Akb22])**.** *Let $S \in \mathsf{Smi}_{\mathbb{K}}$ be a Smith space with universal compact K. Its Banach refinement S^{Ban} can be identified with the underlying vector space of S equipped with the Minkowski functional defined by K. In particular, K is the unit ball of S^{Ban} and the mapping $B_S : S^{\mathrm{Ban}} \to S$ is given by the identity $s \mapsto s$.*

Remark 8.1.11. Recall the adjunction of bornologies and locally convex spaces

$$\mathsf{Born}_{\mathbb{K}} \underset{\mathrm{vN}}{\overset{\mathrm{t}}{\underset{\perp}{\rightleftarrows}}} \mathsf{LCS}_{\mathbb{K}}$$

from Theorem 4.2.15. Let $S \in \mathsf{Smi}_{\mathbb{K}}$ be a Smith space. Then its Banach refinement is

$$S^{\mathrm{Ban}} \;\cong\; \mathrm{t}\left(\mathrm{vN}(S)\right),$$

which is the set S with the finest locally convex topology having the same bounded subsets as S. Thus, the Banach refinement of a Smith space is isomorphic to its Bornologification. ◇

One consequence of Proposition 8.1.10 is, that the universal compact set in a Smith space is always a Banach disk. This raises the question if, given a Banach space B with unit ball U, can we always construct a Smith space with universal compact set U. It turns out that this property is a characterization of dual Banach spaces.

Proposition 8.1.12. *Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. Then B is a dual Banach space if and only if there is a Smith space S with the same underlying vector space as B and with the unit ball $B_{\leq 1}(B)$ as universal compact set.*

Proof.

"⇒": If B is a dual space we can equip it with the weak*-topology given by a predual. By the Banach-Alaoglu Theorem 3.2.19 the unit ball $B_{\leq 1}(B)$ is a compact subset of B in the weak*-topology. Then the space $\mathbb{R}_{>0}B_{\leq 1}(B)$ is the Smith space we are looking for.

"⇐": Assume that the underlying vector space of B has a Smith topology τ, such that $B_{\leq 1}(B)$ is τ-compact. Note that τ as a Smith topology is also locally convex. It is a theorem by Ng [Ng71] improving a statement of Dixmier [Dix48], that in this case B is a dual Banach space.

□

The prominent appearance of the Banach-Alaoglu theorem in the above proof, suggests that the weak*-topology on B might have already been Smith. This however turns out to be true if and only if B is finite-dimensional. We will prove this in Theorem 8.1.17. We need the following well-known characterization of weak*-compact sets.

Lemma 8.1.13. *Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. Consider the strong dual space B' with norm p. A subset $K \subset B'$ is weak*-compact if and only if K is weak*-closed and bounded with respect to p.*

Proof. We write $U = B_{\leq 1}(B', p)$ for the closed unit ball of p in B'.

"⇐": Let $K \subset B'$ be weak*-closed and bounded. The latter implies, that there is a $n \in \mathbb{N}$ with $K \subset n \cdot U$. By the Banach-Alaoglu theorem the set $n \cdot U$ is weak*-compact. Thus, its weak*-closed subspace K is also weak*-compact.

"⇒": Let $\emptyset \neq K \subset B'$ be a weak*-compact subset. The weak*-topology is Hausdorff and therefore K is weak*-closed. By the Banach-Alaoglu theorem U is weak*-compact and therefore also weak*-closed. Thus,

$$K = \cup_{n=1}^{\infty} (n \cdot U \cap K)$$

is a countable union of weak*-closed sets. Since K is non-empty and Baire by Theorem 3.2.2, there is a $m \in \mathbb{N}$ such that $m \cdot U \cap K$ has non-empty interior in K. There is a $z \in (m \cdot U \cap K)$ and a weak*-open neighborhood N around z, such that N is contained in $m \cdot U$, i.e. is bounded. Translating N from z to all points of K we get a weak*-open covering of K. Using that K is weak*-compact, it follows that we can cover K by finitely many bounded sets, which implies that K itself is bounded.

□

Corollary 8.1.14. *Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space and B' its strong dual with norm p. Let $\tau^*_{B'}$ be the weak*-topology on B'. Then the k-ification of $\tau^*_{B'}$ is given by*

$$\begin{aligned} k(\tau^*_{B'}) &= \{\, U \subset B' \mid U \text{ is relatively weak*-open in } \lambda \cdot B_{\leq 1}(B', p) \text{ for all } \lambda > 0 \,\} \\ &= \{\, U \subset B' \mid (B' \setminus U) \text{ is weak*-closed in } \lambda \cdot B_{\leq 1}(B', p) \text{ for all } \lambda > 0 \,\}, \end{aligned}$$

where $B_{\leq 1}(B', p)$ is the unit ball of B' with respect to p.

Proof. A set $A \subset B'$ in the k-ification of $\tau^*_{B'}$ is closed if and only if $A \cap K$ is closed for all compact subsets. Note that by Banach-Alaoglu, $B_{\leq 1}(B', p)$ is closed. By Lemma 8.1.13 every compact subset is closed and is contained in a compact set of the form $\lambda \cdot B_{\leq 1}(B', p)$ for some $\lambda > 0$. Therefore the closed sets with respect to $k(\tau^*_{B'})$ are precisely the sets $(B' \setminus U)$ described in the statement. □

Lemma 8.1.15. *Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. Then the dual B' with the weak*-topology $\tau^*_{B'}$ is not compactly generated and its k-ification defines a locally convex vector topology, with a fundamental system of neighborhoods of 0 given by*

$$(x_n)^\circ_{n \in \mathbb{N}} = \{\, f \in B' \mid |f(x_n)| \leq 1 \text{ for all } n \in \mathbb{N} \,\}, \tag{8.2}$$

where $(x_n)_{n \in \mathbb{N}}$ are sequences in B converging to 0. If B is infinite dimensional then $k(\tau^*_{B'})$ is strictly finer than $\tau^*_{B'}$.

Proof. A system of neighborhoods of zero in $\tau^*_{B'}$ is given by

$$Z^\circ = \{\, f \in B' \mid |f(x)| \leq 1 \text{ for all } x \in Z \,\}, \tag{8.3}$$

where $Z \subset B$ is finite. See section 5.1 in [Con19].

Let $C = B_{\leq 1}(B)$ be the unit ball of B. Its polar C° is the unit ball of B' with respect to its dual norm p defined by

$$p(f) = \sup_{y \in C} |f(y)| \qquad \text{for all } f \in B'.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to 0 in B. For any $\lambda \in \mathbb{R}_{>0}$ there is only a finite subset Z_λ of all x_n that are not contained in $\frac{1}{\lambda}C$. Thus,

$$\begin{aligned}(x_n)^\circ_{n\in\mathbb{N}} \cap \lambda C^\circ &= \{\, f \in B' \mid |f(x_n)| \le 1 \text{ for } n \in \mathbb{N}, |f(y)| \le \lambda \text{ for } y \in C \,\} \\ &= \left\{ f \in B' \,\middle|\, |f(x_n)| \le 1 \text{ for } x_n \in Z_\lambda, |f(y)| \le 1 \text{ for } y \in \frac{1}{\lambda}C \right\} \\ &= Z^\circ_\lambda \cap \lambda C^\circ\,,\end{aligned}$$

which shows that $(x_n)^\circ_{n\in\mathbb{N}}$ is a neighborhood of 0 in $k(\tau^*_{B'})$ by Corollary 8.1.14.

Now, let V be any neighborhood of 0 in $k(\tau^*_{B'})$. We construct a neighborhood as in (8.3) that is contained in V using induction. By Corollary 8.1.14 we have that $V \cap C^\circ$ is a $\tau^*_{B'}$-neighborhood of 0 in C°. So there is a finite set Z_1 with $Z_1^\circ \cap C^\circ \subset V$. Given Z_m with $Z_m \cap mC^\circ \subset V$, first choose a new subset $\tilde{Z}_{m+1}$, such that $\tilde{Z}^\circ_{m+1} \cap (m+1)C^\circ \subset V$. Set

$$Z_{m+1} := Z_m \cup \left(\tilde{Z}_{m+1} \cap \frac{1}{m}C\right) \tag{8.4}$$

We have

$$\begin{aligned}Z^\circ_{m+1} \cap (m+1)C^\circ &= \left(Z_m \cup \tilde{Z}_{m+1} \cap \frac{1}{m}C\right)^\circ \cap (m+1)C^\circ \\ &= Z^\circ_m \cap \left(\tilde{Z}^\circ_{m+1} \cup mC^\circ\right) \cap (m+1)C^\circ \\ &= \left(Z^\circ_m \cap \tilde{Z}^\circ_{m+1} \cap (m+1)C^\circ\right) \cup (Z^\circ_m \cap mC^\circ) \\ &\subset V.\end{aligned}$$

Note that we added only elements to Z_m of norm $\frac{1}{m}$ or smaller to construct Z_{m+1} in (8.4). Therefore, starting with 1 and putting all finite sets Z_m into one sequence $(x_n)_{n\in\mathbb{N}}$ we obtain a sequence converging to 0 in B. We calculate

$$\begin{aligned}(x_n)^\circ_{n\in\mathbb{N}} &= \left\{ f \in B' \,\middle|\, |f(x)| \le 1 \text{ for all } x \in \bigcup_{m\in\mathbb{N}} Z_m \right\} \\ &= \bigcap_{m\in\mathbb{N}} Z^\circ_m \\ &\subset V,\end{aligned}$$

where the last step follows from every element $f \in \bigcap_{m\in\mathbb{N}} Z^\circ_m$ being contained in some $Z^\circ_l \cap lC^\circ$ for an $l \in \mathbb{N}$. This completes the proof that the sets (8.3) are a neighborhood basis of 0.

If B is infinite dimensional we can choose a sequence $(x_n)_{n\in\mathbb{N}}$ converging to 0 in B, such that $\{ x_n \mid n \in \mathbb{N} \}$ is linearly independent. We saw that $(x_n)^\circ_{n\in\mathbb{N}}$ is a neighborhood of 0 in $k(\tau^*_{B'})$. But none of the neighborhoods in (8.3) are contained in $(x_n)^\circ_{n\in\mathbb{N}}$. Thus, $k(\tau^*_{B'})$ has to be strictly finer then $\tau^*_{B'}$. □

Remark 8.1.16. The sets in (8.2) are exactly the polars of sequences converging to zero. We will talk more about polars and how they connect Banach and Smith spaces in Section 8.2. ◇

Theorem 8.1.17. *Let $S \in \mathsf{Smi}_{\mathbb{K}}$ be a Smith space with topology τ_S. Then its Banach refinement S^{Ban} is a dual Banach space with a predual given by the stereotype dual $B := S^\vee$. Let $\tau^*_{B'}$ be the weak*-topology on $B' \cong S^{\mathrm{Ban}}$. We have, that*

(i) τ_S is the k-ification of $\tau^*_{B'}$,

(ii) τ_S is strictly finer than $\tau^*_{B'}$ if and only if S is infinite-dimensional. A fundamental system of convex neighborhoods of the origin in τ_S is given by polars of null sequences in B.

(iii) If $C \subset B'$ is convex, then C is closed in τ_S if and only if C is closed in $\tau^*_{B'}$.

Proof. Let S, B be as above. By definition, we have

$$B' \cong (S^\vee)' \cong S^{\mathrm{Ban}}.$$

Equip B' with the weak*-topology $\tau^*_{B'}$.

(i) We have $B^\vee \cong S$, and we can identify S with space of continuous linear functionals on B with the stereotype topology τ_S. Since B is a Banach space, all totally bounded sets are relatively compact and τ_S coincides with the compact-open topology.

By Proposition 8.1.3 all compact subsets of B are contained in closed disked hulls of null sequences in B. Conversely, every closed disked hull of null sequences in B is compact by Proposition 8.2.27. Thus, τ_S has the same neighborhood basis of the origin as $k(\tau^*_{B'})$ by Lemma 8.1.15. Therefore, $(S, \tau_S) \cong (B', k(\tau^*_{B'}))$.

(ii) This is the statement of Lemma 8.1.15.

(iii) This is a consequence of the characterization of τ_k-closed subsets in Corollary 8.1.14 and a theorem by Krein and Šmulian [KŠ40]. The latter can also be found as Theorem 12.1 in [Con19]. [1]

□

[1] See also the closely related Banach-Dieudonné theorem

Example 8.1.18. The prototypical example of a non-dual Banach space is c_0, the set of sequences converging to zero, equipped with the sup norm. By Proposition 8.1.12 the unit ball of c_0 cannot be made into the universal compact set of a Smith space c_0. But we can take the Smith envelope to get a Smith space c_0^{Smi}. We have

$$c_0^{\mathrm{Smi}} \cong (c_0')^\vee \cong \left(\ell^1\right)^\vee,$$

which by Theorem 8.1.17 we can identify with the vector space ℓ^∞ with the k-ification of the weak*-topology induced from $\ell^{1\prime} \cong \ell^\infty$.

Another way to get a Smith space is the stereotype dual $c_0{}^\vee$. Using Proposition 8.1.10 we see that $c_0{}^\vee$ is the space ℓ^1 equipped with the k-ification of the weak*-topology induced by $c_0' \cong \ell^1$. ◇

For the definition of compactological spaces later, it will be helpful to understand which linear maps of Smith spaces are continuous.

Example 8.1.19. Consider the dual space $(c_0)' \cong \ell^1$. The summation functional

$$s : \ell^1 \to \mathbb{K}, \qquad (x_i)_{i\in\mathbb{N}} \mapsto \sum_{i=1}^{\infty} x_i$$

has operator norm 1 and is therefore continuous with respect to the norm topologies. However, if we consider ℓ^1 (and $\mathbb{K} \cong \mathbb{K}'$) with the weak*-topology, we lose continuity of s. Indeed, the sequences of unit vectors $(e_n)_{n\,\in\mathbb{N}}$, where e_n has a 1 at position n and zero elsewhere, weak*-converges to 0 in ℓ^1, but $s(e_n) = 1 \neq 0$. The non-continuity remains, when we consider the Smith topologies. We have that

$$0 \notin s^{-1}(\{1\}) \cap B_{\leq 1}(\ell^1),$$

but 0 is in the weak*-closure of $s^{-1}(\{1\}) \cap B_{\leq 1}(\ell^1)$ in $B_{\leq 1}(\ell^1)$, by the argument above. Thus, the closed subset $\{1\}$ has a non-closed preimage in ℓ^1 with the Smith topology. ◇

Corollary 8.1.20. *Let $B_1, B_2 \in \mathsf{Ban}_{\mathbb{K}}$ be Banach spaces and $f : B_1' \to B_2'$ a linear map of their continuous duals. On B_1' and B_2' we have the strong, the weak*- and the Smith topologies, where the latter are given by the k-ification of the weak*-topologies. The following are equivalent.*

(i) There is a morphism of Banach spaces $g : B_2 \to B_1$ with $g' = f$,

(ii) f is continuous, when B_1' and B_2' have their weak-topology,*

(iii) f is continuous, when B_1' and B_2' have their Smith topology,

Proof. The equivalence (i) ⇔ (ii) is a standard result.

The equivalence (i) ⇔ (iii) follows from the anti-equivalence in Proposition 8.1.5 and the fact that by Theorem 8.1.17 the spaces B_1' and B_2' with their Smith topologies are isomorphic to the stereotype duals $B_1^\vee$ and $B_2^\vee$. □

8.1.3. The Smith space of Radon Measures

The projective objects in $\mathsf{Smi}_{\mathbb{K}}$ will be certain spaces of signed Radon measures.

Let us first recall what a signed Radon measure is. While they can be defined more generally we are only interested in the special case of Radon measures on (locally) compact Hausdorff spaces.

Definition 8.1.21. Let X be a locally compact Hausdorff space with $\mathcal{B}$, the Borel σ-algebra on X. Then a Borel measure μ is **inner regular** on a Borel set $B \subset \mathcal{B}$ if

$$\mu(B) = \sup \{ \mu(K) \mid K \subset B, K \text{ compact} \}$$

and **outer regular** on a Borel set $B \subset \mathcal{B}$ if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} .$$

A **Radon measure** is a Borel measure, that has finite measure on compact sets, is outer regular on all Borel sets and inner regular on open sets.

A **signed measure** ν is a measure that can take values in $\mathbb{R} \cup \{\infty, -\infty\}$. The **positive/negative variations** of ν are (non-signed) measures ν^+, ν^-, such that $\nu = \nu^+ - \nu^-$. [2]

A **signed Radon measure** is a signed Borel measure, whose positive and negative variations are Radon measures.

In our setting of locally compact Hausdorff spaces there is an important representation result due to Riesz [Rie07], Markov [Mar38] and Kakutani [Kak41]. Its modern formulation with a proof can be found as Theorem 7.2 in [Fol99]. Recall that we denote the space of continuous functions with compact support on a space X with $\mathcal{C}_c(X)$.

Theorem 8.1.22 (Riesz-Markov-Kakutani)**.** *Let X be a locally compact Hausdorff space and ψ a linear functional on $\mathcal{C}_c(X)$. Then there exists a unique Radon measure μ on X, such that*

$$\psi(f) = \int_X f \, d\mu \qquad \text{for all } f \in \mathcal{C}_c(X).$$

Corollary 8.1.23. *Let K be a compact Hausdorff space. Consider the space of continuous functions $\mathcal{C}(K)$ on K with* sup *norm*

$$\|f\|_\infty = \sup_{x \in K} |f(x)| .$$

[2]see the Hahn-Decomposition theorem, stated for example as 3.3 in [Fol99]

Its dual Banach space is the ***space of signed Radon measures*** $\mathcal{M}(K)$ *with norm*

$$\|\mu\| = \sup_{\substack{f\in\mathcal{C}(K)\\ \|f\|_\infty\leq 1}} \left|\int_K f\,d\mu\right|.$$

Corollary 8.1.24. *Let K be a compact Hausdorff space. The space $\mathcal{M}(K)$ with the k-ification of its weak*-topology induced by the predual $\mathcal{C}(K)$ is a Smith space.*

By Theorem 8.1.17 the Smith topology on $\mathcal{M}(K)$ is strictly finer than the weak*-topology as long as K is an infinite set. The latter was also an observation in Exercise 3.3 in [Analytic].

Example 8.1.25. We consider the space ℓ^∞ of all bounded sequences. For every $(x)_{i\in\mathbb{N}} \in \ell^\infty$ the boundedness implies, that the map

$$\mathbb{N} \to \mathbb{K}, \qquad i \mapsto x_i$$

factors through a compact Hausdorff space K. We get a continuous function f from the discrete space $\mathbb{N}$ to K. By the universal property of the Stone-Čech compactification of $\mathbb{N}$, there is a unique extension $\beta f : \beta\mathbb{N} \to K$. This defines and isomorphism of Banach spaces

$$\ell^\infty \;\cong\; \mathcal{C}(\beta\mathbb{N}).$$

Since $\beta\mathbb{N}$ is compact, $\mathcal{C}(\beta\mathbb{N})$ is the same as the space $\mathcal{C}_c(\beta\mathbb{N})$ of compactly supported continuous functions. To find the dual space we apply the Riesz-Markov-Kakutani representation Theorem 8.1.22. We get

$$(\ell^\infty)' \;\cong\; \mathcal{C}(\beta\mathbb{N})' \;\cong\; \mathcal{M}(\beta\mathbb{N}),$$

where $\mathcal{M}(\beta\mathbb{N})$ is the space of Radon measures on $\beta\mathbb{N}$ equipped with its dual norm.

The example generalizes to the follwing statement.

For any set X we have $(\ell^\infty(X))' \;\cong\; \mathcal{C}(\beta X)' \;\cong\; \mathcal{M}(\beta X)$. ◇

While Borel sets can be very complicated, we have a simpler description of signed Radon measures on profinite sets.

Proposition 8.1.26 (Exercise 3.3 in [Analytic] part 2)**.** *Let $S \in \mathsf{Prof}$ be a profinite set. A signed Radon measure on S is equivalent to a map ν assigning to each open and closed subset $U \subset S$ of S a real number $\nu(U) \in \mathbb{R}$, such that*

1. *$\nu(U \sqcup V) = \nu(U) + \nu(V)$ for $U, V \subset S$ disjoint and*
2. *there is some constant $C \in \mathbb{R}$, such that for all disjoint decompositions $S = U_1 \sqcup \ldots \sqcup U_n$, we have*

$$\sum_{i=1}^{n} |\nu(U_i)| \;\leq\; C.$$

We can even extend this to compact Hausdorff spaces.

Proposition 8.1.27 (Exercise 3.3 in [Analytic] part 3)**.** *Let $K \in \mathsf{cHaus}$ be a compact Hausdorff space and $S \to K$ be a surjection from a profinite set S. Then $\mathcal{M}(K)$ is given by the coequalizer*

$$\mathcal{M}(S \times_K S) \rightrightarrows \mathcal{M}(S)$$

of the projections.

8.1.4. Exact Structure on $\mathsf{Smi}_{\mathbb{K}}$

By the anti-equivalence of $\mathsf{Ban}_{\mathbb{K}}$ and $\mathsf{Smi}_{\mathbb{K}}$ from Proposition 8.1.5 there is an exact structure on $\mathsf{Smi}_{\mathbb{K}}$ in the sense of Remark 1.1.11.

Proposition 8.1.28. *The category $\mathsf{Smi}_{\mathbb{K}}$ of Smith spaces is quasi-abelian. The functor*

$$\mathsf{Ban}_{\mathbb{K}}^{\mathrm{op}} \to \mathsf{Smi}_{\mathbb{K}}, \qquad B \mapsto B^{\vee}$$

is exact with respect to their maximal exact structures.

Let us collect some basic homological facts about Smith spaces.

Proposition 8.1.29. *The following are true.*

(i) $\mathsf{Smi}_{\mathbb{K}}$ *has enough $\mathfrak{max}$-injectives. All $\mathfrak{max}$-injectives are up to isomorphism of the form $\ell^{\infty}(X)$ with its weak*-topology for some set X.*

(ii) $\mathsf{Smi}_{\mathbb{K}}$ *has enough $\mathfrak{max}$-projectives. All $\mathfrak{max}$-projectives are up to isomorphism complemented subspaces of $\mathcal{M}(\beta Y)$ with its weak*-topology for some set Y.*

(iii) The $\mathfrak{max}$-global dimension of $\mathsf{Smi}_{\mathbb{K}}$ is ∞.

Proof. This follows from the dual statements for Banach spaces. See Proposition 3.2.12, Proposition 3.2.11 and Proposition 3.2.13. □

8.2. Smith spaces and Bornologies

Recall that an epimorphism of condensed vector spaces T, T' is a morphism $T \to T'$, such that $T(S) \to T'(S)$ are surjective for all $S \in \mathsf{cHaus}_{\kappa}$. Thus, given a morphism $f : V \to W$ of compactly generated vector spaces, the condensation $\underline{f} : \underline{V} \to \underline{W}$ is an epimorphism if and only if for all compact Hausdorff spaces $L \subset W$, there is a compact Hausdorff spaces $K \subset V$ with $f(K) = L$. Compare this to Proposition 4.2.34, where we saw that a surjective bounded map $g : E \to F$ of bornological spaces is an admissible epimorphism if and only if for all bounded set $C \subset F$ there is a bounded set $B \subset V$ with $g(B) = C$. This suggests, that we want to consider precompact bornologies and not von Neumann bornologies on topological vector spaces. Additionally, Smith spaces will take the role that Banach spaces played in the theory of $\mathsf{CBorn}_{\mathbb{K}}$. But first we need to compare their bornologies.

8.2.1. Bornologies of Smith and Banach Spaces

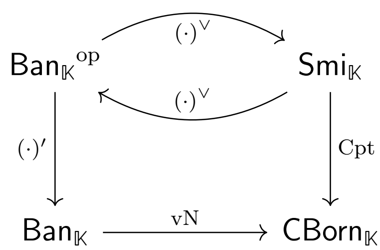
In Chapter 4 we utilized the von Neumann bornology, which consists of all bounded subsets of a topological vector space. An alternative approach often applied to assign a bornology to a locally convex space $V \in \mathsf{LCS}_{\mathbb{K}}$ is the **precompact bornology** $\mathrm{Cpt}(V)$. This bornology is given by all precompact, i.e. totally bounded, sets. Recall from Proposition 3.1.22 that a subset of a Hausdorff topological vector space is compact if and only if it is totally bounded and complete. In particular, all relatively compact sets are totally bounded. Moreover, Proposition 3.1.22 shows the following.

Proposition 8.2.1. *Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a Banach, Fréchet or Smith space. Then*

$$\mathrm{Cpt}(V) = \left\{ K \subset V \;\middle|\; \overline{K} \subset V \text{ is compact} \right\}.$$

Let us compare Cpt with the von Neumann bornology vN.

Proposition 8.2.2. *The diagram*



commutes.

Proof. First note, that two stereotype duals cancel each other and the top two arrows commute.

Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space with unit ball $C = B_{\leq 1}(B)$. The unit ball of its strong dual B' is given by the polar C°. Thus, a subset $A \subset B'$ is bounded if and only if there is a $\lambda \in \mathbb{R}_{>0}$ with $A \subset \lambda C^\circ$. The universal compact of $B^\vee$ is also the set C° and since it is universal a set $A \subset B^\vee$ is precompact if and only if there is a $\lambda \in \mathbb{R}_{>0}$ with $A \subset \lambda C^\circ$ by Proposition 8.2.1. It follows that $\mathrm{vN}(B') = \mathrm{Cpt}(B^\vee)$. Furthermore, for a morphism $f : B \to C$ in $\mathsf{Ban}_{\mathbb{K}}$ we have $\mathrm{vN}(f) = \mathrm{Cpt}(f^\vee)$ by definition. $\square$

Corollary 8.2.3. *Let $B \in \mathsf{Ban}_{\mathbb{K}}$ be a Banach space. We have*

$$\mathrm{vN}(B') \cong \mathrm{Cpt}(B^{\vee}) \cong (B', \{\, K \subset B' \mid K \textit{ is relatively weak}^{*}\textit{-compact} \,\})$$

in $\mathsf{CBorn}_{\mathbb{K}}$.

Proof. The first isomorphism is given by Proposition 8.2.2. For the second one, note that weak*-compact sets are the same as the Smith-compact sets in $B^{\vee}$. This follows, since the latter is the k-ification of the weak*-topology, as was shown in Theorem 8.1.17. □

Let us also state a more general relationship between sets and their polars of Banach and Smith spaces.

Proposition 8.2.4 (see sections 3.1 and 4.1 in [Akb22])**.** *Let $V, W \in \mathsf{LCS}_{\mathbb{K}}$ be stereotype dual to each other. Then taking polars*

$$A \mapsto A^{\circ} = \{\, f \in \mathrm{Hom}(V, \mathbb{K}) \mid |f(x)| \leq 1 \textit{ for all } x \in A \,\}$$

of sets $A \subset V$, induces bijections

$$\begin{aligned}
\{\textit{closed disks in } V\} &\leftrightarrow \{\textit{closed disks in } W\},\\
\{\textit{compact sets in } V\} &\leftrightarrow \{\textit{neighborhoods of } 0 \textit{ in } W\},\\
\{\textit{bounded sets in } V\} &\leftrightarrow \{\textit{barrels in } W\}\\
\{\textit{totally bounded sets in } V\} &\leftrightarrow \{\textit{capacious sets in } W\}.
\end{aligned}$$

Proposition 8.2.5 (Section 1.1.6 in [Mey07])**.** *The restriction* $\mathrm{Cpt} : \mathsf{Fré}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ *to Fréchet spaces is fully faithful.*

This is however false for Smith spaces.

Example 8.2.6. Consider the summation functional s from Example 8.1.19. It is continuous with respect to the norm-topologies and therefore defines a bounded map of bornological spaces

$$\mathrm{vN}(\ell^{1}, \|\cdot\|_{1}) \ \to \ \mathbb{K}.$$

By Proposition 8.2.2 we have $\mathrm{vN}(\ell^{1}, \|\cdot\|_{1}) \cong \mathrm{Cpt}(\ell^{1}, \text{Smith-topology})$. We have also seen that it is not continuous with respect to the Smith topologies. Thus, the functor Cpt is not full, when restricted to Smith spaces. ◇

8.2.2. Compactological Spaces

We defined a bornological space to be complete if its Banach disks are cofinal in all bounded disks. To make the analogous definition for Smith spaces we need a notion of Smith disks with respect to the precompact bornologies. We can use the characterization of Proposition 8.1.12 via dual Banach spaces.

Corollary 8.2.7. *Let $E \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space and $D \in \mathcal{B}_{\mathrm{D}}(E)$ a bounded disk. There is a Smith space S with universal compact set D if and only if D is a Banach disk and its associated Banach space E_D has unit ball D and is dual. In this case, $\mathrm{vN}(E'_D) = \mathrm{Cpt}(S)$.*

Proof. This follows from Proposition 8.1.12 and Proposition 8.2.2. □

Remark 8.2.8. There is some subtlety in Corollary 8.2.7, where D has to be the unit ball of E_D. In general a Banach disk D is the unit ball of E_D if and only if D is a closed subset. Otherwise, $B_{\leq 1}(E_D) = \overline{D}$ and if this space is dual, there is a Smith space with universal compact $\overline{D}$, but not with D. ◇

Definition 8.2.9. Let $E \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space. A bounded disk $D \in \mathcal{B}_{\mathrm{D}}(E)$ is a **Smith disk** if D satisfies one of the equivalent conditions in Corollary 8.2.7. We denote the set of Smith disks in E by $\mathcal{B}_{\mathrm{S}}(E)$.

We denote the smallest Smith disk containing a bounded set $B \subset E$ with $B^{\diamondsuit}$ and call it **Smith disked hull** (if it exists).

There is an important difference between Banach and Smith disks. The former have a unique Banach topology on their span, given by the Minkowski functional. The latter however can have different Smith topologies on the span with the same set as universal compact.

Example 8.2.10. We consider the space ℓ^1 with its usual norm. We saw that c_0 is a predual of ℓ^1. But the space of all converging sequences c with sup norm is also a predual of ℓ^1 and not isomotrically isomorphic to c_0. In fact, ℓ^1 has at least $\aleph_1$ non-isomorphic preduals by a result of Bessaga and Pełczyński [BP60]. All of these induce different weak*-topologies on ℓ^1, which also leads to different Smith topologies. ◇

Definition 8.2.11. Let E be a bornological space and $D \in \mathcal{B}_{\mathrm{S}}(E)$ a Smith disk. A topology τ_D on D is a **Smith topology** if

$$\operatorname{span} D = \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda D$$

with the colimit topology of all $\lambda(D, \tau_D)$, is a Smith space.

Remark 8.2.12. Every Smith disk has at least one Smith topology. By Corollary 8.2.7 and Theorem 8.1.17 the Smith topologies on a Smith disk D are the k-ifications of weak*-topologies for all possible preduals of E_D up to isometric isomorphism. ◇

Another difference between Banach and Smith spaces is that for the former not every monomorphism is regular. Indeed, ℓ^1 is dense in ℓ^2. So $\ell^1 \to \ell^2, x \mapsto x$ has image ℓ^2, which as a Hilbert space is not isomorphic to ℓ^1. However, for Smith spaces we have the following.

Lemma 8.2.13. *Let $S, T \in \mathsf{Smi}_{\mathbb{K}}$ be Smith spaces with universal compact sets K and L. Let $m : S \to T$ be an injective linear continuous map. Then m is a regular monomorphism, that is S is isomorphic to* $\operatorname{Im}(m) = \overline{m(S)}$ *with its subspace topology.*

Proof. Since m is continuous, $m(K)$ is compact in T. Using that T is Smith, there is a $\lambda \in \mathbb{R}_{>0}$, such that $m(K) \subset \lambda L$. Thus, m restricted to K is a monomorphism of compact Hausdorff spaces and therefore an embedding. We have $K \cong \overline{m(K)}$. The result follows from linear continuation. □

Now we can define the vector space analogue of compactological sets.

Definition 8.2.14. A **compactological space** is a triple $(E, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$, such that

(i) $(E, \mathcal{B})$ is a bornological space,

(ii) $(\mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ is a compactology on E,

(iii) the Smith disks $\mathcal{B}_{\mathrm{S}}(E)$ are cofinal in all bounded sets $\mathcal{B}$,

(iv) the topology τ_D is a Smith topology for all Smith disks $D \in \mathcal{B}_{\mathrm{S}}(E)$.

A morphism of compactological spaces

$$f : (E, \mathcal{B}_E, (\tau_B)_{B\in\mathcal{B}_E}) \to (F, \mathcal{B}_F, (\tau_B)_{B\in\mathcal{B}_F})$$

is a morphism of compactological sets, that is linear.

We denote the category of compactological spaces and morphisms with $\mathsf{Comp}_{\mathbb{K}}$.

Remark 8.2.15. Definition 8.2.14 amounts to having a bornological space whose bornology is fully determined by Smith disks and we choose compatible Smith topologies for every such disk. ◇

Definition 8.2.16. Let $(E, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ be a compactological space. For every Smith disk $D \in \mathcal{B}_{\mathrm{S}}(E)$ we call the Smith space induced by (D, τ_D) from Definition 8.2.11 the **associated Smith space** to D and write it as E^D.

As with bornological spaces we shorten the notation of compactological spaces to E with the bornology and the topologies left implicit.

Let $f : E \to F$ be a morphism of compactological spaces E, F. Let $C \in \mathcal{B}_{\mathrm{S}}(E)$ be a Smith disk. Then $f(C)$ is bounded and we can take the Smith disked hull $D = f(C)^{\diamondsuit}$ to get a linear map

$$f^C : E^C \to F^D. \tag{8.5}$$

By Corollary 8.2.7 we also have a linear map

$$f_C : E_C \to F_D \tag{8.6}$$

between dual Banach spaces. By Theorem 8.1.17 we can equip the spaces with the weak*-topology induced from $\left(E^C\right)^\vee$ and $\left(F^D\right)^\vee$.

Proposition 8.2.17. *Let $f : E \to F$ be a linear and bounded map between compactological spaces E, F. The following are equivalent.*

(i) f is a morphism in $\mathsf{Comp}_{\mathbb{K}}$,

(ii) for all Smith disks $C \in \mathcal{B}_{\mathrm{S}}(E)$ the maps f^C from (8.5) are continuous,

(iii) for all Smith disks $C \in \mathcal{B}_{\mathrm{S}}(E)$ the maps f_C from (8.6) are weak-weak* continuous,*

(iv) for all Smith disks $C \in \mathcal{B}_{\mathrm{S}}(E)$ the maps f_C from (8.6) are dual to a morphism of Banach spaces $F^{D\vee} \to E^{C\vee}$

Proof. The equivalence of (ii), (iii) and (iv) is Corollary 8.1.20. The equivalence of (i) and (ii) follows from the Smith disks being cofinal in all bounded sets. □

Remark 8.2.18. Since every Smith disk is a Banach disk, there is a forgetful functor $u : \mathsf{Comp}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$. By Example 8.2.6 the functor is not full. Since Smith disks in the von Neumann bornology of non-dual Banach spaces such as $\mathrm{vN}(c_0)$ are not cofinal, u is also not essentially surjective. It is however faithful. ◇

Lemma 8.2.19. *The category $\mathsf{Comp}_{\mathbb{K}}$ has all small products and coproducts.*

Proof. Let $\{E_i\}_{i\in I}$ be a family of compactological spaces. We consider the coproduct $\coprod_{i\in I} E_i$ and product $\prod_{i\in I} E_i$ compactologies.

Let $\{S_i\}_{i\in I}$ be Smith spaces with universal compact $K_i, i \in I$. Then $\prod_{i\in I} S_i$ is a complete locally convex space. Products of closed convex disks are closed convex disks and by Tychonoff $\prod_{i\in I} K_i$ is also compact. Thus $\mathbb{R}_{>0}\prod_{i\in I} K_i$ is a Smith space and $\prod_{i\in I} K_i$ is a Smith disk. Applying this argument to $\mathcal{B}_{\mathrm{S}}\left(\prod_{i\in I} E_i\right)$ shows that they are cofinal in all bounded disks of the product. Thus, $\prod_{i\in I} E_i$ is a compactological space. Similarly, $\coprod_{i\in I} E_i$ is compactological.

Next, we check the universal properties with the characterization of compactological maps from Proposition 8.2.17. Let T be a compactological space with morphisms $t_i : T \to E_i$. From the universal property of the product in $\mathsf{CBorn}_{\mathbb{K}}$ we get a unique linear bounded map $u : T \to \prod_{i\in I} E_i$ with $t_i = \mathrm{pr}_i \circ u$. Given a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(T)$ the maps ${t_i}^D$ are continuous. Since we can factor

$$ {t_i}^D = \mathrm{pr}_i \circ u^D = {\mathrm{pr}_i}^{u(D)^{\blacklozenge}} \circ u^D $$

for all $i \in I$ it follows that u^D is continuous. Thus, u is a morphism in $\mathsf{Comp}_{\mathbb{K}}$. A similar argument shows that $\coprod_{i\in I} E_i$ has the correct universal property. □

We use the usual definition for closed sets of a bornology.

Definition 8.2.20. Let E be a compactological space and $H \subset E$ be a subspace of E. Then H is **closed** if $H \cap E^L$ is closed for all Smith disks $L \in \mathcal{B}_{\mathrm{S}}(E)$. The **closure** of H is given by

$$ \overline{H} = \bigcap_{\substack{H\subset G\subset E \\ G \text{ closed in } E}} G\,. $$

Lemma 8.2.21. *Let $f : E \to F$ be a compactological map between compactological spaces E, F. Then in $\mathsf{Comp}_{\mathbb{K}}$,*

(i) $\operatorname{Ker} f$ *is given by* $f^{-1}(0)$ *with the subspace compactology,*

(ii) $\operatorname{Im} f$ *is given by* $\overline{f(E)}$ *with the subspace compactology,*

(iii) $\operatorname{Coker} f$ *is given by the quotient space* $F/\overline{f(E)}$*,*

(iv) $\operatorname{Coim} f$ *is given by the quotient space* $E/f^{-1}(0)$*.*

Proof. To check that all of these spaces are contained in $\mathsf{Comp}_{\mathbb{K}}$ it suffices to see that compactological spaces are stable under subspaces and quotients by closed subspaces.

Let $H \subset G$ be a compactological space with the subspace compactology. Given a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(G)$, the set $D \cap H$ is also a compact disk. It is the universal compact of the Smith space $\mathbb{R}_{>0}D \cap H$. Since G is compactological, these Smith disks are cofinal in the bounded sets of H, showing that H is also a compactological space.

Now we consider the quotient space G/H with H closed and q the quotient map. Given a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(G)$, the set $q(D)$ is a compact disk, since q is continuous and linear. Since H is closed, it follows that $q(\mathbb{R}_{>0}D) = \mathbb{R}_{>0}q(D)$ is a Smith space. Since G is compactological, these Smith disks are cofinal in the bounded sets of G/H, showing that G/H is also a compactological space.

This also shows, that the inclusion maps of subspaces and quotient maps are morphisms in $\mathsf{Comp}_{\mathbb{K}}$. What remains to prove is that the universal properties are satisfied. We use the characterization of compactological maps from Proposition 8.2.17.

We start with the kernel. Let T be a compactological space and $t : T \to E$ a morphism in $\mathsf{Comp}_{\mathbb{K}}$, such that $f \circ t = 0$. We get a unique linear bounded morphism $u : T \to f^{-1}(0)$ in $\mathsf{CBorn}_{\mathbb{K}}$ from the universal property of the kernel. Given a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(T)$ the map t^D is continuous. Since we can factor

$$t^D = \ker(f) \circ u^D = \ker(f)^{u(D)^{\blacklozenge}} \circ u^D$$

and $\ker(f)^{u(D)^{\blacklozenge}}$ is continuous and injective, u^D is also continuous. Therefore, $f^{-1}(0)$ has the universal property and is the kernel of f in $\mathsf{Comp}_{\mathbb{K}}$.

Next, we check the universal property of the cokernel. Let V be a compactological space and $v : F/\overline{f(E)} \to V$ a compactological map, such that $v \circ f = 0$. We get a unique linear bounded morphism $u : F/\overline{f(E)} \to V$ in $\mathsf{CBorn}_{\mathbb{K}}$ from the universal property of the cokernel. Given a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(F)$ the map v^D is continuous. Since we can factor

$$v^D = u \circ \operatorname{coker}(f)^D = u^{\operatorname{coker}(D)^{\blacklozenge}} \circ \operatorname{coker}(f)^D$$

it follows from $\operatorname{coker}(f)^D$ being a continuous quotient map, that $u^{\operatorname{coker}(D)^{\blacklozenge}}$ is also continuous. Therefore, $F/\overline{f(E)}$ has the universal property and is the cokernel of f in $\mathsf{Comp}_{\mathbb{K}}$.

Lastly, note that

$$\begin{aligned}
\operatorname{Im} f &= \operatorname{Ker}(\operatorname{coker} f) \\
&= \operatorname{Ker}\left(F \twoheadrightarrow F/\overline{f(E)}\right) \\
&= \overline{f(E)}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Coim} f &= \operatorname{Coker}(\ker f) \\
&= \operatorname{Coker}\left(f^{-1}(0) \rightarrowtail E\right) \\
&= E/\overline{f^{-1}(0)} \\
&= E/f^{-1}(0).
\end{aligned}$$

□

Theorem 8.2.22. *The category* $\mathsf{Comp}_{\mathbb{K}}$ *is quasi-abelian. The forgetful functor* $u : \mathsf{Comp}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ *is exact and reflects small limits and colimits. In particular, in the maximal exact structure a morphism* $f : (E, \mathcal{B}_E) \to (F, \mathcal{B}_F)$ *is* *(i) an admissible monomorphism if and only if f is injective and $f^{-1}(B)$ is bounded for all $B \in \mathcal{B}_F$,*

(ii) an admissible epimorphism if and only if f is surjective and for all bounded sets $C \in \mathcal{B}_F$ there exists $B \in \mathcal{B}_E$ with $f(B) = C$.

Proof. We have seen in Lemma 8.2.19 that $\mathsf{Comp}_{\mathbb{K}}$ has products and coproducts. By Lemma 8.2.21 it also has kernels and cokernels. We get that $\mathsf{Comp}_{\mathbb{K}}$ is pre-abelian.

We have also seen that products, coproducts, kernels and cokernels all coincide with the constructions in $\mathsf{CBorn}_{\mathbb{K}}$. So $\mathsf{Comp}_{\mathbb{K}} \to \mathsf{CBorn}_{\mathbb{K}}$ reflects all small limits and colimits. It follows, just as in $\mathsf{CBorn}_{\mathbb{K}}$, that in $\mathsf{Comp}_{\mathbb{K}}$ kernels are stable under pushouts and cokernels are stable under pullbacks. Thus, $\mathsf{Comp}_{\mathbb{K}}$ is quasi-abelian and a morphism of $\mathsf{Comp}_{\mathbb{K}}$ is admissible if and only if it is admissible in $\mathsf{CBorn}_{\mathbb{K}}$. □

8.2.3. Comparison with Locally Convex Spaces

We have a functor

$$\mathrm{Clcs} : \mathsf{Comp}_{\mathbb{K}} \to \mathsf{LCS}_{\mathbb{K}}, \qquad E \mapsto \operatorname*{colim}_{D \in \mathcal{B}_{\mathrm{S}}(E)} E^D, \tag{8.7}$$

where the filtered colimit is taken in $\mathsf{LCS}_{\mathbb{K}}$. Since, the transition maps in the diagram are regular monomorphisms by Lemma 8.2.13, we say that the filtered colimit is **strict**. Another important feature of the diagram in (8.7) is that it is possibly uncountable. This complicates things, since nearly all positive results on strict filtered colimits require the diagram to be countable. For example, countable strict inductive limits preserve quasi-completeness. But we will see in Example 8.2.25 that in our general situation the colimit space does not even satisfy an even weaker version of completeness.

A filtered colimit of locally convex spaces is called **regular**, if all bounded subsets in the colimit are contained and bounded in some space that is part of the diagram. This is also not the case in our setting as can be seen in Example 8.2.25.

In the other direction we have the functor

$$\mathrm{CSp} : \mathsf{LCS}_{\mathbb{K}} \to \mathsf{Comp}_{\mathbb{K}}, \qquad V \mapsto \operatorname*{colim}_{\substack{K \subset V \\ \text{compact } T_2 \text{ disk}}} \mathbb{R}_{>0} K \tag{8.8}$$

where the compact Hausdorff subsets of V are ordered by inclusion and the diagram of $\mathbb{R}_{>0}K, K \subset V$ has monomorphic transition maps.

Proposition 8.2.23. *The functors* (8.7) *and* (8.8) *define an adjunction*

$$\mathsf{Comp}_{\mathbb{K}} \underset{\text{CSp}}{\overset{\text{Clcs}}{\rightleftarrows}} \mathsf{LCS}_{\mathbb{K}}\,.$$

Let us denote the essential image of Clcs with $\mathsf{UCT}_{\mathbb{K}}$.

Question 8.2.24. Which locally convex spaces are in $\mathsf{UCT}_{\mathbb{K}}$? Is there a combination of well-known adjectives for $\mathsf{LCS}_{\mathbb{K}}$ that describes $\mathsf{UCT}_{\mathbb{K}}$.

Recall, that a topological vector space is pseudo-complete if all totally bounded subsets are complete. This is a weaker notion than quasi-completeness or completeness. The following example shows that not all spaces in $\mathsf{UCT}_{\mathbb{K}}$ are pseudo-complete.

Example 8.2.25. The following example is taken from section 1 Proposition 8 in [Bie88], but is originally due to Douady [Dou63]. Consider the Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers with discrete topology. Let $x \in \beta\mathbb{N} \setminus \mathbb{N}$ and define $X := \beta\mathbb{N} \setminus \{x\}$. For any compact $K \subset X$ define the Banach space $\mathcal{C}_K := \{ f \in \mathcal{C}(X) \mid \operatorname{supp}(f) \subset K \}$ endowed with the topology of uniform convergence on K. The set of all compact subsets of X ordered by inclusion is filtered, and we define the filtered colimit

$$\mathcal{C}_c(X) \;=\; \operatorname*{colim}_{\substack{K\subset X\\ \text{compact}}} \mathcal{C}_K.$$

It is shown in section 1 Proposition 8 (d), that there is a compact subset $A \subset \mathcal{C}_c(X)$ such that the closed disked hull of A is not even sequentially complete. Thus, $\mathcal{C}_c(X)$ is not pseudo-complete. However, $\mathcal{C}_c(X)$ is a strict filtered colimit of Banach spaces and writing each Banach space as strict filtered colimit of its Smith subspaces we get

$$\mathcal{C}_c(X) = \operatorname*{colim}_{\substack{K\subset X\\ \text{compact}}} \mathcal{C}_K = \operatorname*{colim}_{\substack{K\subset X\\ \text{compact}}} \operatorname*{colim}_{L\subset \mathcal{B}_S(\mathcal{C}_K)} {\mathcal{C}_K}^L,$$

showing that $\mathcal{C}_c(X)$ is a strict filtered colimit of Smith spaces, but not pseudo-complete. ◇

Remark 8.2.26. All stereotype spaces are pseudo-complete by Remark 3.1.37. So Example 8.2.25 also shows that $\mathsf{UCT}_{\mathbb{K}}$ is not a subcategory of $\mathsf{Ste}_{\mathbb{K}}$. Although $\mathsf{Ste}_{\mathbb{K}}$ is cocomplete its colimits differs from the colimits in $\mathsf{LCS}_{\mathbb{K}}$ by Example 4.2.3 in [Akb22]. Considering only locally convex spaces in $\mathsf{Ste}_{\mathbb{K}}$ might give a comparison with $\mathsf{Comp}_{\mathbb{K}}$ with nicer properties. ◇

Proposition 8.2.27. *Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be Hausdorff, locally convex and pseudo-complete. Let $A \subset V$ be a compact subset. Then the closed disked hull of A is also compact. If V is also metrizable, then the closed disked hull of a null sequence $(x_n)_{n\in\mathbb{N}}$ in V is compact.*

Proof. Since V is locally convex the disked hull of any precompact set is precompact by Theorem II.4.3 [SW99]. Since compact sets are totally bounded and V is pseudo-complete, the closed disked hull of every compact set is complete and precompact, which implies it is compact.

If V is also metrizable, the set $\{0\} \cup \{ x_n \mid n \in \mathbb{N} \}$ is sequential compact and therefore (covering) compact in V. Since 0 is always contained in the disked hull of non-empty sets, the statement follows from the first part. □

Proposition 8.2.28. *Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex space, that is pseudo-complete and Hausdorff. Then*

$$\mathrm{Cpt}(V) \cong \operatorname*{colim}_{\substack{D \subset V \\ \textit{compact disk}}} \mathrm{Cpt}\left(\mathbb{R}_{>0} D\right)$$

Proof. Consider the posets of all compact Hausdorff subspaces and of all compact disks with subspace topology in V; both ordered by $\subset$. We have an inclusion of the finer into the coarser topology of the filtered colimits

$$\operatorname*{colim}_{\substack{D \subset \mathcal{B}_{\mathrm{D}}(V) \\ \text{compact}}} D \hookrightarrow \operatorname*{colim}_{\substack{S \subset V \\ \text{compact}}} S \tag{8.9}$$

in Top. By 8.2.27 the closed disked hulls of all compact Hausdorff subspaces are compact. Thus, the inclusion in (8.9) is a homeomorphism.

Thus, compact disks are cofinal in all compact sets of V and the result follows. □

Proposition 8.2.29. *Let $V \in \mathsf{LCS}_{\mathbb{K}}$ be a locally convex space, that is pseudo-complete, compactly generated and Hausdorff. Then the map*

$$\operatorname*{colim}_{\substack{D \subset \mathcal{B}_{\mathrm{D}}(V) \\ \textit{compact}}} \mathbb{R}_{>0} D \hookrightarrow V$$

is an isomorphism.

Proof. As in the proof of Proposition 8.2.28 we get the homemorphism from (8.9).

If V is compactly generated and Hausdorff, its topology is the colimit topology with respect to all compact subspaces of V by Theorem C.3.8. By the homeomorphism (8.9) the topology of V agrees with the colimit over all compact disks in V. It follows that

$$V \cong \operatorname*{colim}_{\substack{D \subset \mathcal{B}_{\mathrm{D}}(V) \\ \text{compact}}} \mathbb{R}_{>0} D.$$

□

Example 8.2.30. Consider the Banach space $\ell^1 \in \mathsf{Ban}_{\mathbb{K}}$. A subset $L \subset \ell^1$ is said to be **uniformly 1-summable** if

$$\lim_{N\to\infty} \sup_{(x_i)_i \in L} \sum_{i=N}^{\infty} |x_i| \ = \ 0$$

With this notion we have the following well-known characterization of the compact subsets of ℓ^1.

A subset L of ℓ^1 is compact if and only if it is closed, bounded and uniformly 1-summable.

For example,

$$K \ = \ \left\{ (x_i)_{i\in\mathbb{N}} \in \ell^1 \ \middle| \ x_i \in \left[0, \frac{1}{i}\right] \text{ for } i \in \mathbb{N} \right\}$$

is a compact subset of ℓ^1. We have

$$\mathbb{R}_{>0}K = \left\{ (x_i)_i \in \ell^1 \ \middle| \ \text{there is } \lambda > 0, \text{ s.t. for all } i \in \mathbb{N} \text{ we have } x_i \in \left[0, \frac{\lambda}{i}\right] \right\}$$

with its compactly generated topology is a Smith space. A subset $A \subset \mathbb{R}_{>0}K$ is closed if and only if for all $\lambda > 0$ the set $A \cap \lambda\mathbb{R}_{>0}K$ is closed in ℓ^1.

By Proposition 8.2.29 we have

$$\ell^1 \ \cong \ \underset{\substack{D\subset\ell^1 \\ \text{closed, bounded,} \\ \text{unif. 1-summable}}}{\operatorname{colim}} \ \mathbb{R}_{>0}D.$$

◇

One might be tempted to think that since Banach spaces are compactly generated, filtered colimits of Banach spaces, that is Hausdorff ultrabornological spaces, are compactly generated. However, this is false as the following example shows.

Example 8.2.31. Consider the ground field $\mathbb{K}$ as a topological vector space and object of $\mathsf{LCS}_{\mathbb{K}}$. The uncountable product

$$V := \prod_{\mathbb{R}} \mathbb{K} \ \in \mathsf{LCS}_{\mathbb{K}}.$$

By the Mackey-Ulam Theorem (see Theorem 13.5.4 in [Jar81]) any product V is a bornological topological vector space if $\operatorname{card}(\mathbb{R}) = 2^{\aleph_0}$ does not admit an Ulam measure. Ulam has shown in Satz (A) in [Ula30] that every cardinal admitting an Ulam measure is strongly inaccessible, which $2^{\aleph_0}$ is clearly not. Thus, V is bornological and by completeness ultrabornological. However, V is not compactly generated. See chapter 7 exercise J (b) in [Kel75].

Since V is ultrabornological we can write it as a filtered colimit of Banach spaces. If we write every Banach space in the diagram as a colimit over its Smith spaces as in Proposition 8.2.29 we get V as a filtered colimit of Smith spaces. However, the Banach spaces and therefore also the Smith spaces do not have the subspace topology of V. Thus, this construction is different then taking all compact disks in V and considering the filtered colimit of the induced Smith spaces. Indeed, the latter describes not V but the k-ification of V.

Note that this also provides an example for a non-compactly generated nuclear Montel space. ◇

8.3. Ind-Smith and $\mathcal{M}$-complete Condensed Spaces

8.3.1. Ind-Smith Spaces

From now on let κ be an uncountable regular strong limit cardinal. We say that a compactological space E is κ-bounded if every bounded subset $B \subset E$ has cardinality $< \kappa$. Note, that this implies that Smith spaces associated to Smith disks also have cardinality $< \kappa$. We denote their full subcategory in $\mathsf{Comp}_{\mathbb{K}}$ by $\mathsf{Comp}_{\mathbb{K}}^{\kappa}$.

For the comparison with $\mathsf{Ind}(\mathsf{Smi}_{\mathbb{K}})$ we need another **dissection functor**.

Consider the mapping

$$\operatorname{diss} : \mathsf{Comp}_{\mathbb{K}} \to \mathsf{Ind}(\mathsf{Smi}_{\mathbb{K}}), \qquad E \mapsto \text{“}\operatorname*{colim}_{D \in \mathcal{B}_{\mathrm{S}}(E)}\text{”}\, E^D. \tag{8.10}$$

Using Lemma 8.2.13 we see that the essential image are the diagrams with monomorphisms as transition maps.

Let $f : E \to F$ be a morphism in $\mathsf{Comp}_{\mathbb{K}}$. Let $C \in \mathcal{B}_{\mathrm{S}}(E)$ be a Smith disk. By Corollary 8.2.7 there is a Smith disk $D \in \mathcal{B}_{\mathrm{S}}(F)$ such that $f^C : E^C \to F^D$ is a morphism of Smith spaces. This collection for all Smith disks in E defines a map between the formal filtered colimits $\operatorname{diss}(f) : \operatorname{diss}(E) \to \operatorname{diss}(F)$.

Analogous to Proposition 7.4.3 we get the following.

Proposition 8.3.1. *The dissection functor (8.10) is fully faithful and commutes with limits and coproducts. It defines an equivalence of categories of* $\mathsf{Comp}_{\mathbb{K}}$ *and* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{K}})$. *The latter is a full subcategory of* $\mathsf{Ind}(\mathsf{Smi}_{\mathbb{K}})$, *given by all essentially monomorphic objects.*

Given an uncountable strong limit cardinal κ this restricts to

$$\operatorname{diss} : \mathsf{Comp}_{\mathbb{K}}^{\kappa} \to \mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{K}}^{\kappa}),$$

where $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{K}}^{\kappa})$ *is the full subcategory of* $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{K}})$ *consisting of formal colimits over diagrams of κ-bounded Smith spaces with monomorphisms as transition maps.*

Remark 8.3.2. With the anti-equivalence from Proposition 8.1.5 and Remark B.1.6 we have

$$\mathsf{Ind}(\mathsf{Smi}_{\mathbb{K}}) \cong \mathsf{Pro}({\mathsf{Smi}_{\mathbb{K}}}^{\mathrm{op}})^{\mathrm{op}} \cong \mathsf{Pro}(\mathsf{Ban}_{\mathbb{K}})^{\mathrm{op}}.$$

Under this anti-equivalence and with Remark B.2.2 we can identify $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}_{\mathbb{K}}^{\kappa})$ with essentially epimorphic κ-bounded Pro-Banach spaces. ◇

8.3.2. $\mathcal{M}$-complete Condensed Spaces

We only consider $\mathbb{K} = \mathbb{R}$ in this section. In $\mathsf{Cond}_{\kappa}(\mathsf{Vect})$ we say that V is a Smith space if $V \cong \underline{E}$ for some $E \in \mathsf{Smi}_{\mathbb{R}}$.

Recall the Smith space of signed Radon measures from Section 8.1.3. With condensation we get a condensed $\mathbb{R}$-vector space $\underline{\mathcal{M}(S)}$ for all $S \in \mathsf{Prof}$.

Definition 8.3.3 (Definition 4.1 in [Condensed])**.** Let V be a κ-condensed $\mathbb{R}$-vector space. Then V is **$\mathcal{M}$-complete** if the underlying condensed set of V is quasi-separated and for all maps $f : S \to V$ from a profinite set, there is an extension to a map

$$\tilde{f} : \mathcal{M}(S) \to V.$$

We denote the full subcategory of $\mathsf{Cond}_{\kappa}(\mathsf{Vect})$ of all κ-condensed $\mathcal{M}$-complete $\mathbb{R}$-vector spaces by $\mathcal{M}\mathsf{Cond}_{\kappa}(\mathsf{Vect})$.

It suffices to check this on extremally disconnected spaces.

Proposition 8.3.4 (Exercise 4.5 in [Analytic])**.** *A condensed $\mathbb{R}$-vector space V is $\mathcal{M}$-complete if and only if for all continuous maps $f : S \to V$ from an extremally disconnected $S \in \mathsf{Extr}$, there is an extension to a continuous map*

$$\tilde{f} : \mathcal{M}(S) \to V.$$

Proposition 8.3.5 (Proposition 4.6 in [Condensed])**.** *Let V be a condensed $\mathcal{M}$-complete vector space. Then the set of all condensed Smith subspaces of V ordered by inclusion is filtered. There is an isomorphism*

$$V \cong \operatorname*{colim}_{S \subset V} S.$$

Conversely, any filtered colimit of Smith spaces with injective transition maps is $\mathcal{M}$-complete .

We have the obvious condensation functor

$$\underline{(\cdot)} : \mathsf{Ind}(\mathsf{Smi}_{\mathbb{K}}) \to \mathsf{Cond}(\mathsf{Vect}_{\mathbb{K}}), \qquad \text{“}\operatorname*{colim}_{i \in I}\text{”} S_i \mapsto \operatorname*{colim}_{i \in I} \underline{S_i}. \tag{8.11}$$

Theorem 8.3.6. *The functors* (8.10) *and* (8.11) *define equivalences of categories*

$$\mathsf{Comp}^{\kappa}_{\mathbb{R}} \quad \cong \quad \mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}^{\kappa}_{\mathbb{R}}) \quad \cong \quad \mathcal{M}\mathsf{Cond}_{\kappa}(\mathsf{Vect})$$

of

- *κ-bounded compactological $\mathbb{R}$-vector spaces $\mathsf{Comp}^{\kappa}_{\mathbb{R}}$,*
- *essentially monomorphic Ind-Smith $\mathbb{R}$-vector spaces $\mathsf{Ind}_{\rightarrowtail}(\mathsf{Smi}^{\kappa}_{\mathbb{R}})$, given by formal filtered colimits of κ-bounded Smith spaces with injections as transition maps, and*
- *κ-condensed $\mathcal{M}$-complete $\mathbb{R}$-vector spaces $\mathcal{M}\mathsf{Cond}_{\kappa}(\mathsf{Vect})$.*

Proof. The first equivalence is Proposition 8.3.1. The second one is Proposition 8.3.5. □

Corollary 8.3.7. *The category of κ-condensed $\mathcal{M}$-complete $\mathbb{R}$-vector spaces is quasi-abelian.*

9. Non-Locally Convex Compactological and Condensed Spaces

Contents

In Chapter 8 we have seen how the locally convex $\mathcal{M}$-complete vector spaces can be built using bornologies. However, those do fall short of the aspirations of Condensed Math. They do not form an abelian category. The latter is achieved by adding more non-locally convex spaces to get liquid vector spaces.

In this chapter our goal is to first construct quasi-separated and then all liquid vector spaces using bornologies.

9.1. Non-Locally Convex Banach and Smith Spaces

In the category of Banach spaces, we saw that ℓ^1 is projective. Specifically, we have $\mathrm{Ext}^1_{\mathsf{Ban}_{\mathbb{K}}}(\ell^1, \mathbb{K}) = 0$. By the Hahn-Banach theorem, $\mathbb{K}$ is injective in $\mathsf{LCS}_{\mathbb{K}}$, and therefore $\mathrm{Ext}^1_{\mathsf{LCS}_{\mathbb{K}}}(\ell^1, \mathbb{K}) = 0$. However, in the late 1970s, Roberts [Rob77a], Kalton [Kal78], and Ribe [Rib79] all constructed non-zero elements of the group

$\mathrm{Ext}^1_{\mathsf{TVS}_{\mathbb{K}}}(\ell^1, \mathbb{K})$. For more details, see lecture V in [Analytic], where Clausen and Scholze discuss Ribe's example and use it to demonstrate that $\mathcal{M}$-complete vector spaces are not closed under extensions in all condensed $\mathbb{R}$-vector spaces. In fact, any extension-closed subcategory of $\mathsf{Cond}(\mathsf{Vect}_{\mathbb{R}})$, that contains all Banach spaces, must also include some non-locally convex spaces.

9.1.1. Non-Locally Convex Banach Spaces

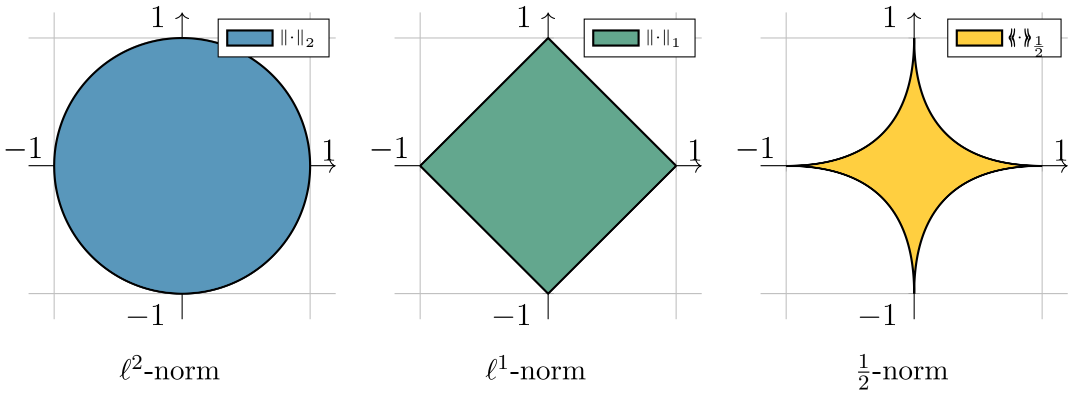


Figure 9.1.: Unit balls of two norms and a $\frac{1}{2}$-norm

Example 9.1.1. Consider the vector space $\mathbb{R}^2$ with its usual Euclidean topology. This space is normable. For example, we can consider l^p-norms defined by

$$\|(x, y)\|_p := (|x|^p + |y|^p)^{\frac{1}{p}}$$

for a real number $p \geq 1$. The analogous definition with $p < 1$ does not satisfy the triangle inequality. For example with $p = \frac{1}{2}$ we have

$$\|(1, 1)\|_{\frac{1}{2}} = (1 + 1)^2 = 4 > 1 + 1 = \|(1, 0)\|_{\frac{1}{2}} + \|(0, 1)\|_{\frac{1}{2}}.$$

We can change the definition to

$$⟪\cdot⟫_p : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, \qquad ⟪(x, y)⟫_p = |x|^p + |y|^p$$

and get a function that maps only 0 to 0 and satisfies the triangle inequality. The price we pay is that the scaling behaviour is different. For $\lambda, x, y \in \mathbb{R}$ we have

$$⟪\lambda(x, y)⟫_p = |\lambda x|^p + |\lambda y|^p = |\lambda|^p (|x|^p + |y|^p) = |\lambda|^p ⟪(x, y)⟫_p.$$

Looking at the unit balls in Fig. 9.1 we see that we lost local convexity. However, since $\mathbb{R}^2$ is finite dimensional we do not yet see the difference in the topology of spaces that are normable and ones that are only p-normable for a $0 < p < 1$. ◇

Motivated by the example above we make the following definition.

Definition 9.1.2. Let W be a $\mathbb{K}$-vector space and $0 < p \leq 1$. A **p-seminorm** on W is a map $\langle\!\langle \cdot \rangle\!\rangle_p : W \to \mathbb{R}_{\geq 0}$ such that

(i) $\langle\!\langle \lambda v \rangle\!\rangle_p = |\lambda|^p \langle\!\langle v \rangle\!\rangle_p$, for all $\lambda \in \mathbb{K}$ and $v \in W$,

(ii) $\langle\!\langle v + w \rangle\!\rangle_p \leq \langle\!\langle v \rangle\!\rangle_p + \langle\!\langle w \rangle\!\rangle_p$ for all $v, w \in W$.

If additionally,

(iii) $\langle\!\langle v \rangle\!\rangle_p = 0$ if and only if $v = 0$, for all $v \in W$,

we say that $\langle\!\langle \cdot \rangle\!\rangle_p$ is a **p-norm**.

A p-norm on W defines a topology the same way a norm does. The open balls

$$\left\{ w \in W \;\middle|\; \langle\!\langle w \rangle\!\rangle_p < \epsilon \right\}$$

for $\epsilon \in \mathbb{R}_{>0}$ are a neighborhood basis of the origin.

Definition 9.1.3. A topological $\mathbb{K}$-vector space $V \in \mathsf{TVS}_{\mathbb{K}}$ is **p-(semi)normable**, if its topology is induced by a p-(semi)norm.

If V is p-normable and complete we say that V is a **p-Banach space**.

The category of p-Banach spaces and linear continuous maps is denoted by $\mathsf{QBan}^p_{\mathbb{K}}$. The category of q-Banach spaces for any $0 < q < p$ and linear continuous maps is denoted by $\mathsf{QBan}^{<p}_{\mathbb{K}}$.

For all $0 < p \leq 1$ the category $\mathsf{QBan}^p_{\mathbb{K}}$ has the same problem extension problem as $\mathsf{Ban}_{\mathbb{K}}$. However, a classical result from Kalton shows that $\mathsf{QBan}^{<p}_{\mathbb{K}}$ is extension closed in all topological vector spaces.

Theorem 9.1.4 (Kalton [Kal81])**.** *Any extension of q-Banach spaces is a q' Banach space for all $q' < q$.*

Remark 9.1.5. All p-Banach spaces are so called quasi-Banach spaces. The term **quasi-Banach space** refers to vector spaces Q with a **quasi-norm** $(\!| \cdot |\!)$. Unlike p-norms, for a quasi-norm the scaling behaviour is that of a norm, but the triangle inequality is modified. It is replaced by the condition that there exists a $\rho \geq 1$, such that

$$(\!| x + y |\!) \leq \rho \left((\!| x |\!) + (\!| y |\!) \right) \qquad \text{for all } x, y \in Q.$$

However, any such quasi-normed space is p-convex for some $0 < p \leq 1$. This is called the Aoki-Rolewicz theorem and was proven independently by both Aoki [Aok42] and Rolewicz [Rol57]. For more details on how to find a suitable value for p see [KPR84]. ◇

Remark 9.1.6. Let $0 < p \leq 1$ and $V \in \mathsf{TVS}_{\mathbb{K}}$ be a p-Banach space with p-norm $\langle\!\langle \cdot \rangle\!\rangle_p$. For any $0 < q < p$ the map $\langle\!\langle \cdot \rangle\!\rangle_p^{\frac{q}{p}}$ defines an equivalent q-norm on V. Indeed, for all $v, w \in V$ and $\lambda \in \mathbb{K}$ we have

$$\langle\!\langle \lambda v \rangle\!\rangle_p^{\frac{q}{p}} = \left(|\lambda|^p \langle\!\langle v \rangle\!\rangle_p \right)^{\frac{q}{p}} = |\lambda|^q \langle\!\langle v \rangle\!\rangle_p^{\frac{q}{p}}$$

and using that $\frac{q}{p} \leq 1$ also

$$\langle\!\langle v + w \rangle\!\rangle_p^{\frac{q}{p}} \leq \left(\langle\!\langle v \rangle\!\rangle_p + \langle\!\langle w \rangle\!\rangle_p \right)^{\frac{q}{p}} \leq \langle\!\langle v \rangle\!\rangle_p^{\frac{q}{p}} + \langle\!\langle w \rangle\!\rangle_p^{\frac{q}{p}}$$

◇

The unit ball of a Banach space is always a disk. Given a p-Banach space we also want to have a name for the shape of their unit balls.

Definition 9.1.7. Let V be a $\mathbb{K}$-vector space and $0 < q \leq 1$. A subset $X \in V$ is called

(i) **q-convex** if for all $x, y \in X$ and $s, t \in \mathbb{R}_{\geq 0}$ with $s^p + t^p = 1$ we have $sx + ty \in X$,

(ii) a q-**lens** if it is q-convex and balanced,

(iii) a **lens** if it is a q'-lens for some $0 < q' \leq 1$.

Remark 9.1.8. Standard nomenclature for q-lenses would be q-disks as used in [Köt60]. To distinguish it from the convex setting we chose a different name. ◇

Definition 9.1.9. Let V be a vector space and $L \subset V$ a p-lens in V. The **p-Minkowski functional** of L

$$\langle\!\langle x \rangle\!\rangle_L := \inf \{ \lambda^p \mid \lambda \in \mathbb{R}_{\geq 0},\ x \in \lambda L \}$$

is a p-seminorm on $\operatorname{span}(L)$. If $(\operatorname{span}(L), \langle\!\langle \cdot \rangle\!\rangle_L)$ is p-Banach we denote it by V_{pL} and call it the **associated** p**-Banach space** of L.

Definition 9.1.10. Let A be a subset of a vector space V. We can form the p**-convex hull** and the p**-lensed hull** $A^{p\diamondsuit}$ of A

$$\bigcap_{\substack{A \subset C \\ C\ p\text{-convex}}} C\,, \qquad\qquad A^{p\diamondsuit} := \bigcap_{\substack{A \subset L \\ L\ p\text{-lens}}} L.$$

We also have the description via p-convex combinations

$$A^{p\diamondsuit} = \left\{ \sum_{i=1}^{n} t_j a_j \;\middle|\; n \in \mathbb{N},\ a_j \in A,\ \sum_{i=1}^{n} |t_j|^p \leq 1 \right\}$$

Definition 9.1.11. A topological vector space is **locally** p**-convex** if it has a neighborhood basis of the origin consisting of p-convex sets.

9.1.2. Non-Locally Convex Smith Spaces

We have the obvious generalization of Smith spaces to the p-convex setting.

Definition 9.1.12. Let $V \in \mathsf{TVS}_{\mathbb{K}}$ be a Hausdorff complete locally p-convex $\mathbb{K}$-vector space and let $K \subset V$ be a compact p-lens. Equip the space

$$\mathbb{R}_{>0}K \ := \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda K,$$

with the compactly generated topology of the sets λK: A subset $A \subset \mathbb{R}_{>0}K$ is closed if and only if for all $\lambda > 0$ the set $A \cap \lambda K$ is closed. If $V \cong \mathbb{R}_{>0}K$ we say that V is a **p-Smith space** and K a **universal compact subset** of V.

Remark 9.1.13. If $p < 1$ it is possible to remove the p-convexity condition in Definition 9.1.12 for the space V. A theorem by Kalton [Kal77] shows that every compact p-lens in a topological vector space can be linearly embedded in locally p-convex space. This is in contrast to the situation for $p = 1$. Roberts [Rob77b] showed that there is a non-empty, compact, convex subset K of a non-locally convex space, such that K has no extreme points. It follows from the Krein-Milman theorem (see Theorem 7.5.1 in [Jar81]) that K can not be linearly embedded in a locally convex space. ◇

Example 9.1.14. In Section 8.1.3 we discussed spaces of signed Radon measures as examples for Smith spaces. We also have p-Smith spaces consisting of signed Radon measures. The following construction (except the k-ification) is from [Kal77].

Let X be a compact Hausdorff space. For every point $x \in X$ the Dirac measure

$$\delta_x(A) \ := \begin{cases} 1\,, & \text{if } x \in A \\ 0\,, & \text{if } x \notin A \end{cases}$$

is an element of the space $\mathcal{M}(X)$. We denote the set of all Dirac measures on X with $\delta(X) := \{\, \delta_x \mid x \in X \,\}$. For $0 < p \leq 1$ we define a subspace

$$\mathcal{M}_p(X) := \left\{ \mu = \sum_{i=1}^{\infty} a_i \delta_{x_i} \;\middle|\; a_i \in \mathbb{R},\ \sum_{i=1}^{\infty} |a_i|^p < \infty,\ x_i \in X \text{ pairwise distinct} \right\}$$

of $\mathcal{M}(X)$ with its p-norm

$$\langle\!\langle \mu \rangle\!\rangle_p \ = \sum_{i=1}^{\infty} |a_i|^p \,.$$

We consider the unit ball

$$K := B_{\leq 1}(\mathcal{M}_p(X)) = \left\{ \mu \in \mathcal{M}_p(X) \;\middle|\; \langle\!\langle \mu \rangle\!\rangle_p \leq 1 \right\}$$

and the subspace topology on $\mathcal{M}_p(X)$ given by the k-ification of the weak*-topology on $\mathcal{M}(X)$.

Let τ_p be the colimit topology over all $\lambda \cdot K, \lambda \in \mathbb{R}_{>0}$ with their k-ifiaction of the weak*-subspace topology.

Then τ_p is a p-Smith topology on $\operatorname{span} K$. We get the p-Smith space $\mathbb{R}_{>0}K$ and denote it by $\mathcal{M}_p(X)$. ◇

Proposition 9.1.15. *Let S be a p-Smith space with universal compact K. The p-Minkowski functional $\langle\!\langle \cdot \rangle\!\rangle_K$ of K turns S into a p-Banach space.*

Proof. Completely analogous to Theorem 13.1.2 in [NB10]. □

9.2. Non-Locally Convex Compactological Spaces

Following the Literature in [Hou73] and [PS00] we have defined bornological spaces to always refer to bornological spaces of convex type. In this section we will consider a larger category, where we replace the basis of disks in the definition by a basis of p-lenses. Note that, while this will allow for some non-convex spaces, it is still stricter than just considering all vector spaces equipped with vector bornologies. We will again focus on complete spaces.

Fix a real number $0 < p \leq 1$, that will be an upper bound for the convexity that we consider.

9.2.1. Lensed Spaces

Definition 9.2.1. Let $(E, \mathcal{B}) \in \mathsf{Born}_{\mathbb{K}}$ be a bornological space in the sense of Definition 4.2.1 but not necessarily of convex type. For $0 < p \leq 1$ we denote the set of bounded p-lenses of E by

$$\mathfrak{L}^p(E) := \{ B \in \mathcal{B} \mid B \text{ is a } p\text{-lens} \}$$

and also use

$$\mathfrak{L}^{<p}(E) := \{ B \in \mathcal{B} \mid B \text{ is a } q\text{-lens for some } 0 < q < p \}$$

A bounded set $L \in \mathcal{B}$ is a ***p*-Banach lens** if $E_L := \operatorname{span}(L)$ with the Minkowski functional

$$E_L \to \mathbb{R}_{\geq 0}, \quad x \mapsto \inf \{ \lambda \in \mathbb{R}_{\geq 0} \mid x \in \lambda L \}$$

is a p-Banach space.

A bounded set $L \in \mathcal{B}$ is a ***p*-Smith lens** if there is a p-Smith space S with with L as universal compact. In this case the subspace topology of $L \subset S$ is called a ***p*-Smith topology**.

We denote the sets of bounded p- and $<p$-Banach and Smith lenses in E with

$$\mathfrak{L}^p_{\mathrm{B}}(E), \quad \mathfrak{L}^{<p}_{\mathrm{B}}(E), \quad \mathfrak{L}^p_{\mathrm{S}}(E) \quad \text{and} \quad \mathfrak{L}^{<p}_{\mathrm{S}}(E)$$

respectively.

We have the usual preoder given by inclusion on all sets of lenses.

Remark 9.2.2. For $0 < q' < q \leq 1$ every q-lens is also a q'-lens. Thus, the set of all bounded lenses of a bornological space $(E, \mathcal{B})$ is $\mathfrak{L}^1(E)$. ◇

Definition 9.2.3. A **p-lensed (bornological) space** E is a (non-convex) bornological space in the sense of Definition 4.2.1, such that for all q with $0 < q < p$ the bornology $\mathcal{B}$ has a Basis of q-Banach lenses.

We denote the category of p-lensed spaces and bounded maps with $\mathsf{Lens}^{<p}_{\mathbb{K}}$.

Example 9.2.4. For all $0 < p \leq 1$, the von Neumann bornology of a p-Banach space defines a p-lensed space. ◇

Example 9.2.5. The follwing example is due to Bourgin [Bou43]. For all $n \in \mathbb{N}$ set

$$p_n = \frac{1}{1 + \frac{1}{\sqrt{\log(n+1)}}}.$$

We define a $\mathbb{R}$-vector space

$$\ell_{(p_n)} := \left\{ (a_n)_{n \in \mathbb{N}} \;\middle|\; a_n \in \mathbb{R}, \; \sum_{n=1}^{\infty} |a_n|^{p_n} < \infty \right\}$$

with topology given by neighborhoods of 0 of the form

$$\left\{ (a_n)_{n \in \mathbb{N}} \in \ell_{(p_n)} \;\middle|\; \sum_{n=1}^{\infty} |a_n|^{p_n} < \epsilon \right\}$$

for $\epsilon \in \mathbb{R}_{>0}$. In section 7 of [Bou43] it is shown that $\ell_{(p_n)}$ is metrizable, q-convex for all $0 < q < 1$, but not locally convex. Thus, for the von Neumann bornology $\mathrm{vN}(\ell_{(p_n)}) \in \mathsf{Lens}^{<1}_{\mathbb{K}}$, although the space itself is not of convex type.

The p_n are chosen, such that they increase monotonously to 1. With a different choice for the p_n, one can construct examples with von Neumann bornology in $\mathsf{CLens}^{<p}_{\mathbb{K}}$, that are not p-convex. ◇

Remark 9.2.6. Let $(E, \mathcal{B})$ be a bornological space. The condition that $\mathfrak{L}^{<p}_{\mathrm{B}}(E)$ are cofinal is necessary for E to be p-lensed, but not sufficient. ◇

Proposition 9.2.7. *Let E be a p-lensed space. Then there is an isomorphism*

$$E \;\cong\; \operatorname*{colim}_{L \in \mathfrak{L}^{<p}_{\mathrm{B}}(E)} E_L.$$

9.2.2. Compactological Lensed Spaces

Definition 9.2.8. A **compactological p-lensed space** is a triple $(E, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$, such that

(i) $(E, \mathcal{B})$ is a (non-convex) bornological space,

(ii) $(\mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ is a compactology on E,

(iii) for all q with $0 < q < p$ the q-Smith lenses $\mathfrak{L}^q_{\mathrm{S}}(E)$ are cofinal in all bounded sets $\mathcal{B}$,

(iv) for all q with $0 < q < p$ and q-Smith lenses $L \in \mathfrak{L}^q_{\mathrm{S}}(E)$ the topology τ_D is a q-Smith topology.

A morphism of compactological p-lensed space

$$f : (E, \mathcal{B}_E, (\tau_B)_{B\in\mathcal{B}_E}) \to (F, \mathcal{B}_F, (\tau_B)_{B\in\mathcal{B}_F})$$

is a morphism of compactological sets, that is linear.

We denote the category of compactological p-lensed spaces and morphisms with $\mathsf{CLens}^{<p}_{\mathbb{K}}$.

Notation 9.2.9. For a compactological p-lensed space $(E, \mathcal{B}, (\tau_B)_{B\in\mathcal{B}})$ and a q-Smith disk L in E we denote the by τ_L induced q-Smith space with E^L.

Example 9.2.10. For all $0 < p \leq 1$, the precompact bornology with the subspace topologies of a p-Smith space defines a compactological p-lensed space. ◇

Example 9.2.11. Recall the q-Smith spaces $\mathcal{M}_q(X)$ of q-measures on a compact Hausdorff space X from Example 9.1.14. For $0 < q < p$ we eqip with their precompact bornologies. For $0 < q < q' < p$ we have an embedding $\mathrm{Cpt}(\mathcal{M}_q(X)) \to \mathrm{Cpt}(\mathcal{M}_{q'}(X))$. This gives us a diagram and we can define the space

$$\mathcal{M}_{<p}(X) := \operatorname*{colim}_{0<q<p} \mathrm{Cpt}(\mathcal{M}_q(X))$$

in (non-convex) bornological spaces. Equipping all the q-Smith disks in $\mathrm{Cpt}(\mathcal{M}_q(X))$ with with their q-Smith topology from Example 9.1.14 defines a compactology on $\mathcal{M}_{<p}(X)$. We get a compactological p-lensed space $\mathcal{M}_{<p}(X)$. ◇

Let $f : E \to F$ be a linear and bounded map between compactological p-lensed spaces and $L \in \mathfrak{L}^{<p}_{\mathrm{S}}(E)$ a q-lens with $0 < q < p$. Then $f(L)$ is bounded in L and we can form the q-Smith lensed hull M of $f(L)$ in F. We get a linear map of q-Smith spaces

$$f^L : E^L \to F^M. \tag{9.1}$$

Since $\mathfrak{L}^{<p}_{\mathrm{S}}(E)$ is cofinal in all bounded sets of E, we get the following.

Proposition 9.2.12. *Let E, F be compactological p-lensed spaces. A linear bounded map $f : E \to F$ is a morphism in $\mathsf{CLens}_{\mathbb{K}}^{<p}$ if and only if for all $L \in \mathfrak{L}_{\mathrm{S}}^{<p}(E)$ the map f^L from (9.1) is continuous.*

Lemma 9.2.13. *The category $\mathsf{CLens}_{\mathbb{K}}^{<p}$ has all small products and coproducts.*

Proof. Completely analogous to Lemma 8.2.19. □

Definition 9.2.14. Let E be a compactological p-lensed space and $H \subset E$ be a subspace of E. Then H is **closed** if $H \cap E^L$ is closed for all q-Smith lenses $L \in \mathfrak{L}_{\mathrm{S}}^{<p}(E)$. The **closure** of H is given by

$$\overline{H} = \bigcap_{\substack{H \subset G \subset E \\ G \text{ closed in } E}} G\,.$$

Lemma 9.2.15. *Let $f : E \to F$ be a compactological map between p-lensed spaces E, F. Then in $\mathsf{CLens}_{\mathbb{K}}^{<p}$,*

(i) $\operatorname{Ker} f$ *is given by* $f^{-1}(0)$ *with the subspace compactology,*

(ii) $\operatorname{Im} f$ *is given by* $\overline{f(E)}$ *with the subspace compactology,*

(iii) $\operatorname{Coker} f$ *is given by the quotient space* $F/\overline{f(E)}$*,*

(iv) $\operatorname{Coim} f$ *is given by the quotient space* $E/f^{-1}(0)$*.*

Proof. Completely analogous to Lemma 8.2.21. □

Theorem 9.2.16. *The category $\mathsf{CLens}_{\mathbb{K}}^{<p}$ is quasi-abelian. A morphism $f : (E, \mathcal{B}_E) \to (F, \mathcal{B}_F)$ in $\mathsf{CLens}_{\mathbb{K}}^{<p}$ is*

(i) an admissible monomorphism if and only if f is injective, has closed range and $f^{-1}(C) \in \mathcal{B}_E$ for all $C \in \mathcal{B}_F$,

(ii) an admissible epimorphism if and only if f is surjective and for all $C \in \mathcal{B}_F$ there is a $B \in \mathcal{B}_E$ with $f(B) = C$,

Proof. The statement follows from Lemma 9.2.13 and Lemma 9.2.15 after checking that kernels are stable under pushouts and cokernels are stable under pullback analogous to Theorem 8.2.22. □

From now on let κ be an uncountable regular strong limit cardinal. We say that a compactological p-lensed space E is κ-bounded if every bounded subset $B \subset E$ has cardinality $< \kappa$. Note, that this implies that to q-Smith disks associated q-Smith spaces also have cardinality $< \kappa$. We denote their full subcategory in $\mathsf{CLens}_{\mathbb{K}}^{<p}$ by $\mathsf{CLens}_{\kappa}^{<p}$.

Corollary 9.2.17. *The category $\mathsf{CLens}_\kappa^{<p}$ of κ-bounded compactological p-lensed spaces is quasi-abelian.*

Proof. By Theorem 9.2.16 and Proposition 1.2.10 it suffices to see that $\mathsf{CLens}_\kappa^{<p}$ reflects kernels and cokernels in $\mathsf{CLens}_\mathbb{K}^{<p}$, which follows from Lemma 9.2.15. □

9.3. From Lensed to Liquid Vector Spaces

9.3.1. Quasi-separated Liquid Vector spaces

Definition 9.3.1 (Theorem 6.5.(3) [Analytic])**.** Fix a $0 < p \leq 1$. A κ-condensed $\mathbb{R}$-vector space V is **p-liquid** if V is the cokernel of a map

$$\coprod_{i \in I} \mathcal{M}_{<p}(S_i) \;\to\; \coprod_{j \in J} \mathcal{M}_{<p}(S'_j)\,.$$

We denote the full subcategory of $\mathsf{Cond}(\mathsf{Vect}_\mathbb{R})$ of p-liquid condensed $\mathbb{R}$-vector spaces by Liq_p.

We have the following characterization of quasi-separated p-liquid spaces.

Theorem 9.3.2 (Theorem 2.24 in [Complex])**.** *Let V be a quasi-separated condensed $\mathbb{R}$-vector space and $0 < p \leq 1$. Then V is p-liquid if and only if for every $0 < q < p$, every quasi-compact subobject of V is contained in a quasi-compact q-convex subobject of V.*

Since qc subobjects of qs condensed sets are qcqs, they correspond to compact Hausdorff spaces. It follows that the functor (9.2) below actually lands in quasi-separated p-liquid spaces. We can write a compactological p-lensed space E as a filtered colimit of its Smith spaces and then condense it.

$$(\cdot)^{\lozenge}: \;\mathsf{CLens}_\kappa^{<p} \to \mathsf{Liq}_{\kappa,p}\,, \qquad E \;\mapsto\; E^{\lozenge} \;=\; \operatorname*{colim}_{L \in \mathfrak{L}_\mathrm{S}^{<p}(E)} \underline{E^L}. \tag{9.2}$$

Morphisms of compactological p-lensed spaces are mapped to the corresponding morphisms of filtered colimits using Proposition 9.2.12.

Theorem 9.3.3. *The functor (9.2) defines an equivalence of categories*

$$\mathsf{CLens}_\kappa^{<p} \quad\cong\quad \mathsf{qsLiq}_{\kappa,p}$$

of

- *κ-bounded compactological p-lensed spaces $\mathsf{CLens}_\kappa^{<p}$ and*
- *κ-condensed qs p-liquid $\mathbb{R}$-vector spaces $\mathsf{qsLiq}_{\kappa,p}$.*

Proof. Let V be a κ-condensed qs p-liquid vector space. By Theorem 9.3.2 we can write V as a colimit over all qcqs q-convex subobjects with $0 < q < p$. Each of those qcqs q-convex subobjects is a q-Smith disk in $V(*)$. Thus, there is a compactological p-lensed space E, defined as the compactology of all those q-Smith disks with q-Smith topology, such that $E^{\lozenge} \cong V$. The functor (9.2) is essentially surjective. Morphisms of qs p-liquid spaces are determined by their restrictions to qcqs subobjects. Thus, by Proposition 9.2.12 the functor is also fully faithful. □

Proposition 9.3.4. *The functor* (9.2) *commutes with products, coproducts and kernels. It also commutes with cokernels of maps with closed image.*

Proof. Products, coproducts, kernels and cokernels of maps with closed image in $\mathsf{Liq}_{\kappa,p}$ are the same as in $\mathsf{qsLiq}_{\kappa,p}$. By Theorem 9.3.3 the functor (9.2) commutes with these. □

9.3.2. Liquid Vector Spaces and Formal Quotients

In Chapter 7, we have constructed quasi-separated condensed sets as compactological sets. Additionally, we have observed that non-quasi-separated condensed sets can be viewed as quotients of the quasi-separated ones. This observation suggests that, to construct all liquid vector spaces, we should consider them as formal quotients of quasi-separated liquid vector spaces. The idea of transitioning from topological vector spaces to abstract quotients originates with Waelbroeck, who not only studied quotient Banach spaces [Wae82] but also quotient bornological spaces [Wae86]. A more modern approach to Waelbroeck's formal quotient construction, which does not require the category to be quasi-abelian, is provided by Wegner in [Weg17]. In the context of quasi-abelian categories, this formal quotient construction is equivalent to considering the left heart, a concept introduced by Beĭlinson, Bernstein, and Deligne. They, in Example 1.3.22 of [DBB83], consider the left heart of an exact category with maximal exact structure as an example of hearts of t-structures. Furthermore, in 1.3.24 of [DBB83], they note that in the case of Banach spaces, this construction is similar to Waelbroeck's notion of quotient Banach spaces, as discussed in a preprint of [Wae72]. The state-of-the-art reference for the two canonical t-structures on the derived category of a quasi-abelian category can be found in Schneider book [Sch99].

We will follow the construction in [Weg17].

Consider the category, whose objects are monomorphisms in $\mathsf{CLens}_\kappa^{<p}$. A morphism $(\alpha,\beta) : f \to g$ between two monomorphisms $f : A \rightarrowtail B, g : C \rightarrowtail D$ is given by a commutative square

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \underset{g}{\rightarrowtail} & D. \end{array} \tag{9.3}$$

We denote it by $\mathsf{Mon}(\mathsf{CLens}_\kappa^{<p})$.

Given a monomorphism $f : A \rightarrowtail B$, we can condense it via the functor from (9.2) and get a monomorphism in κ-condensed $\mathbb{R}$-vector spaces between two quasi-separated p-liquid objects. Taking the cokernel of this map in κ-condensed $\mathbb{R}$-vector spaces is a p-liquid vector space, but not necessarily quasi-separated. For a morphism (α,β) as in (9.3) we get

$$\begin{array}{ccccc} A^{\lozenge} & \overset{f^{\lozenge}}{\rightarrowtail} & B^{\lozenge} & \twoheadrightarrow & \mathrm{Coker}(f^{\lozenge}) \\ {\scriptstyle\alpha^{\lozenge}}\downarrow & & \downarrow{\scriptstyle\beta^{\lozenge}} & & \vdots\,{\scriptstyle(\alpha,\beta)^{\lozenge}} \\ C^{\lozenge} & \underset{g^{\lozenge}}{\rightarrowtail} & D^{\lozenge} & \twoheadrightarrow & \mathrm{Coker}(g^{\lozenge}) , \end{array}$$

where the map $(\alpha,\beta)^{\lozenge}$ is given by the universal property of the cokernel. This defines a functor

$$\mathsf{Mon}(\mathsf{CLens}_\kappa^{<p}) \to \mathsf{Liq}_{\kappa,p}, \qquad f : A \rightarrowtail B \;\mapsto \mathrm{Coker}(f^{\lozenge}). \tag{9.4}$$

By definition every p-liquid vector space is a cokernel of quasi-separated p-liquid vector spaces. Thus, (9.4) is essentially surjective. The question now is, which monomorphisms and maps between them in $\mathsf{Mon}(\mathsf{CLens}_\kappa^{<p})$ do we have to identify, to also make it fully faithful.

We first define $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$ to have the same objects as $\mathsf{Mon}(\mathsf{CLens}_\kappa^{<p})$ and morphisms given by

$$\mathrm{Hom}_{\mathsf{hMon}}(f,g) = \mathrm{Hom}_{\mathsf{Mon}}(f,g)/J(f,g),$$

where

$$J(f,g) = \left\{ \begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \underset{g}{\rightarrowtail} & D \end{array} \;\middle|\; \text{there is a } \rho, \text{such that } \begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle\alpha}\downarrow & \swarrow_{\rho} & \downarrow{\scriptstyle\beta} \\ C & \underset{g}{\rightarrowtail} & D \end{array} \text{ commutes} \right\}.$$

This is the homotopy category of complexes in $\mathsf{CLens}_\kappa^{<p}$ of the form

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0\,,$$

where f is a monomorphism.

A commutative square that is both a pullback and a pushout is called a pulation in [Weg17]. We will use the term bicartesian square.

Proposition 9.3.5. *Let $f : A \hookrightarrow B, g : C \hookrightarrow D$ objects of $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$. The collection*

$$\{\,(\alpha, \beta) : f \to g \mid \textit{the square } \beta \circ f = g \circ \alpha \textit{ is bicartesian}\,\} \tag{9.5}$$

is a multiplicative system in $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$. Let σ be the collection of all the multiplicative systems (9.5). The localization of $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})[\Sigma^{-1}]$ is abelian and equivalent to the left heart of $\mathsf{CLens}_\kappa^{<p}$.

Proof. Since the category is $\mathsf{CLens}_\kappa^{<p}$ quasi-abelian by Corollary 9.2.17 all statements directly follow from Corollary 1.2.20 and 1.2.21 in [Sch99]. □

Using the characterization as the left heart we write

$$\mathcal{LH}(\mathsf{CLens}_\kappa^{<p}) = \mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})[\Sigma^{-1}]$$

from now on. To show that $\mathcal{LH}(\mathsf{CLens}_\kappa^{<p})$ is equivalent to $\mathsf{Liq}_{\kappa,p}$ we need a standard fact from homological algebra.

Lemma 9.3.6 (Proposition 2.12 in [Bue10])**.** *Let C be an abelian category. Consider the commutative square*

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ C & \underset{g}{\rightarrowtail} & D, \end{array}$$

where f and g are monomorphisms. The square is bicartesian if and only if the canonical map $c : \operatorname{Coker} f \to \operatorname{Coker} g$ is an isomorphism.

For two monomorphisms with the same cokernel we get a zig-zag of bicartesian squares.

Lemma 9.3.7. *Let C be an abelian category with enough projectives and $f : A \rightarrowtail B, g : C \rightarrowtail D$ monomorphisms with isomorphic cokernel. Then there is a commutative diagram of bicartesian squares*

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{h}{\twoheadrightarrow} & \operatorname{Coker} f \\ \uparrow & & \uparrow p & & \| \\ \operatorname{Ker}(h \circ p) & \rightarrowtail & P & & \| \\ \downarrow & & \downarrow & & \wr\| \\ \operatorname{Ker}(j \circ q) & \rightarrowtail & Q & & \| \\ \downarrow & & \downarrow q & & \| \\ C & \underset{g}{\rightarrowtail} & D & \underset{j}{\twoheadrightarrow} & \operatorname{Coker} g \end{array} \tag{9.6}$$

for any P, Q projective objects with epimorphisms p, q to B and D.

Proof. Choose P, Q projective and p, g as in (9.6). We get the middle morphism $P \to Q$ from P being projective. The three morphisms on the left are all from the universal property of the Kernel. The statement now follows by applying Lemma 9.3.6 for all three squares. □

Lemma 9.3.8. *Let $f : A \hookrightarrow B, g : C \hookrightarrow D$ be two objects in $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$. Then $f \cong g$ in $\mathcal{LH}(\mathsf{CLens}_\kappa^{<p})$ if and only if $\operatorname{Coker}(f^{\lozenge}) \cong \operatorname{Coker}(g^{\lozenge})$ in $\mathsf{Liq}_{\kappa,p}$.*

Proof. The category $\mathsf{Liq}_{\kappa,p}$ is abelian and has enough quasi-separated projectives by Theorem 6.6 in [Analytic]. If $\operatorname{Coker}(f^{\lozenge}) \cong \operatorname{Coker}(g^{\lozenge})$ we have the diagram (9.6) consisting of bicartesian squares of qs p-liquid objects that are images of objects and morphisms in $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$ under $(\cdot)^{\lozenge}$. By Proposition 9.3.4 we also have a zig-zag of bicartesian squares in $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$, which shows that we identify f and g in the localization at σ.

Conversely, if $f \cong g$ in $\mathcal{LH}(\mathsf{CLens}_\kappa^{<p})$ there is a zig-zag of bicartesian squares of monomorphisms connecting f and g in $\mathsf{hMon}(\mathsf{CLens}_\kappa^{<p})$. By Theorem 9.3.3 and Lemma 9.3.6 we get $\operatorname{Coker}(f^{\lozenge}) \cong \operatorname{Coker}(g^{\lozenge})$. □

Using Lemma 9.3.8 we see that the functor (9.2) factors through $\mathcal{LH}(\mathsf{CLens}_\kappa^{<p})$. We get a functor

$$(\cdot)^{\blacklozenge} : \mathcal{LH}(\mathsf{CLens}_\kappa^{<p}) \to \mathsf{Liq}_{\kappa,p}, \qquad (f : A \rightarrowtail B) \mapsto \operatorname{Coker}(f^{\lozenge}). \tag{9.7}$$

We already have seen that (9.7) is essentially surjective. Let $f : E \to F$ be a morphism of κ-condensed p-liquid $\mathbb{R}$-vector spaces E, F. Then E and F are cokernels of monomorphisms of quasi-separated spaces $m : K \rightarrowtail P$, and $n : L \rightarrowtail Q$, where P and Q are projective. The morphism f induces $\beta : P \to Q$ and $\alpha : K \to L$ using that P is projective and the universal property of the kernel.

$$\begin{array}{ccccc} K & \overset{m}{\rightarrowtail} & P & \twoheadrightarrow & E \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle f} \\ L & \underset{n}{\rightarrowtail} & Q & \twoheadrightarrow & F \end{array}$$

The possible choices for (α, β) differ only by an element in $J(m, n)$ and are therefore unique in $\mathrm{Hom}_{\mathsf{hMon}}(m, n)$. The assignment $f \mapsto (\alpha, \beta)$ is by construction inverse to $(\cdot)^{\blacklozenge}$ and we see that (9.3.9) induces a bijection

$$\mathrm{Hom}_{\mathcal{LH}(\mathsf{CLens}_\kappa^{<p})}(m, n) \cong \mathrm{Hom}_{\mathsf{Liq}_{\kappa,p}}(E, F)$$

We obtain our main theorem.

Theorem 9.3.9. *The functor* (9.7) *defines an equivalence of abelian categories*

$$\mathcal{LH}(\mathsf{CLens}_\kappa^{<p}) \quad \cong \quad \mathsf{Liq}_{\kappa,p}$$

of

- *the left heart of* $\mathsf{CLens}_\kappa^{<p}$, *given by formal quotients of* κ*-bounded compactological* p*-lensed spaces and*
- κ*-condensed* p*-liquid* $\mathbb{R}$*-vector spaces* $\mathsf{Liq}_{\kappa,p}$.

Part III.

Relative Field Theories via Relative Dualizability

10. Higher Duals and Adjoints

Contents

In this section we review the terminology of adjoints and duals as well as their higher categorical analogues. For more details, see [Lur09] and [JFS17].

10.1. Adjoints in Bicategories

Definition 10.1.1. Let C be a bicategory and $A, B \in \mathsf{C}$. An **adjunction** between two 1-morphisms

$$l : A \to B, \qquad r : B \to A,$$

consists of a pair of 2-morphisms

$$u : \mathrm{id}_A \Rightarrow r \circ l, \qquad c : l \circ r \Rightarrow \mathrm{id}_B,$$

satisfying the **zig-zag identities**

$$\left(l \overset{\sim}{\Rightarrow} l \circ \mathrm{id}_A \xRightarrow{\mathrm{id}_l \times u} l \circ r \circ l \xRightarrow{c \times \mathrm{id}_l} \mathrm{id}_B \circ l \overset{\sim}{\Rightarrow} l \right) = \mathrm{id}_l, \tag{zig}$$

$$\left(r \overset{\sim}{\Rightarrow} \mathrm{id}_A \circ r \xRightarrow{u \times \mathrm{id}_r} r \circ l \circ r \xRightarrow{\mathrm{id}_r \times c} r \circ \mathrm{id}_B \overset{\sim}{\Rightarrow} r \right) = \mathrm{id}_r\,. \tag{zag}$$

We denote such an adjunction as $(l \dashv r, u, c)$ and call l the **left adjoint** of r and r the **right adjoint** of l. We say that u is the **unit** and c the **counit** witnessing the adjunction $l \dashv r$.

Example 10.1.2. A special case of this definition is the well known notion of adjoint functors. To see this, consider the bicategory $\mathsf{C} = \mathsf{Cat}_2$ whose objects are categories, 1-morphisms are functors, and 2-morphisms are natural transformations. Then the definition of a 1-morphism having a left/right adjoint in C is precisely that of having a left/right adjoint in the classical sense. ◇

Example 10.1.3. In the Morita bicategory Alg_1, objects are algebras, and, for two algebras A and B, a 1-morphism from A to B is an (A, B)-bimodule M. The 1-morphism M has a left adjoint if and only if M is finitely presented and projective as an A-module. We review this with details and proofs in Chapter 12. ◇

Example 10.1.4. Let $\mathbb{K}$ be a field. The bicategory LinCat has as objects finite abelian $\mathbb{K}$-linear categories, right exact linear functors as 1-morphisms and natural transformations as 2-morphisms. A 1-morphism F has a left adjoint if and only if F is left exact. While every 1-morphism G has a right adjoint linear functor H it only is a right adjoint in LinCat if G is right exact. For more details and proofs see [DSPS20]. ◇

For a bicategory C there is not just one opposite category but there are three different opposites. The 1-cell dual C^{op} reverses the direction of the 1-morphisms but not the 2-morphisms. The 2-cell dual C^{co} reverses the direction of the 2-morphisms but not the 1-morphisms. The bidual $\mathsf{C}^{\mathrm{coop}}$ reverses both 1- and 2-morphisms. By reversing the specified morphisms in Definition 10.1.1 we can see what happens to an adjunction if we pass to one of these dual categories.

Proposition 10.1.5. *Let C be a bicategory and $(l \dashv r, u, c)$ be an adjunction in C. We get the corresponding adjunctions*

$$(r \dashv l, u, c) \text{ in } \mathsf{C}^{\mathrm{op}}, \quad (r \dashv l, c, u) \text{ in } \mathsf{C}^{\mathrm{co}} \quad \text{and} \quad (l \dashv r, c, u) \text{ in } \mathsf{C}^{\mathrm{coop}}.$$

A special case of adjunctibility are so called ambijunctions or ambidextrous adjunctions, where the morphisms are simultanous left and right adjunctions. These were first considered by Morita in [Mor65]. A good reference is [Lau06].

Definition 10.1.6. Let C be a bicategory. An **ambidextrous adjunction** for a 1-morphism g are two adjunctions $f \dashv g \dashv h$ with units and counits such that f and h are 2-isomorphic.

It is well-known that left and right adjoints of functors are unique in a certain sense. This carries over to our general setting with essentially the same proof, which we make precise below.

Lemma 10.1.7. *Let C be a bicategory and $(f \dashv h, u, c)$ and $(g \dashv h, u', c')$ be adjunctions. Then the two left adjoints f and g of h are 2-isomorphic in C. Moreover, the witnessing 2-isomorphism is unique, and the category of adjunctibility data is contractible. The same holds for right adjoints.*

Hence, having an adjoint is a property, although at first sight one must specify the data of the adjoint, unit, and counit.

Definition 10.1.8. A 1-morphism f in C is **left adjunctible** if it has a left adjoint f^L, i.e. there exists an adjunction $f^L \dashv f$. A 1-morphism f in C is **right adjunctible** if it has a right adjoint f^R, i.e. there exists an adjunction $f \dashv f^R$. A 1-morphism f in C is **adjunctible** if it is **both** left and right adjunctible.

In the rest of this subsection we record some Lemmas about properties of adjoints. Although well-known to experts, we include proofs of the following three lemmas in Appendix D.1 for reference .

Lemma 10.1.9. *Let C be a bicategory. Let $(l \dashv r, u, c)$ be an adjunction in C and $\mu : l \overset{\sim}{\Rightarrow} l'$ and $\nu : r \overset{\sim}{\Rightarrow} r'$ be 2-isomorphisms. Then $(l' \dashv r', u', c')$ is an adjunction with*

$$\textit{unit} \quad u' \;=\; \Big(\quad \mathrm{id} \xRightarrow{u} r \circ l \xRightarrow{\nu \times \mu} r' \circ l' \,\Big), \tag{10.1}$$

$$\textit{counit} \quad c' \;=\; \Big(\quad l' \circ r' \xRightarrow{\mu^{-1} \times \nu^{-1}} l \circ r \xRightarrow{c} \mathrm{id} \,\Big). \tag{10.2}$$

Note that we do not have uniqueness for the unit and counit in an adjunction. Lemma 10.1.9 shows that we can always change the units and counits by composition with an automorphism of the right (or left) adjoint to get another adjunction. This is in fact the only possibility.

Lemma 10.1.10. *Let C be a bicategory. Let $(l \dashv r, u, c)$ and $(l \dashv r, u', c')$ be two adjunctions for the same 1-morphisms l, r in C. Then there is a 2-automorphism $\varphi : r \to r$, such that*

$$(\varphi^{-1} \times \mathrm{id}_l) \circ u = u' \qquad \textit{and} \qquad c \circ (\mathrm{id}_l \times \varphi) = c'.$$

Next we record the fact that we can compose adjunctions and the resulting units and counits.

Lemma 10.1.11. *Let C be a bicategory. Let $f : X \to Y$ and $g : Y \to Z$ be 1-morphisms in C with adjunctions*

$$(f^L \dashv f, u_f, c_f) \qquad \textit{and} \qquad (g^L \dashv g, u_g, c_g).$$

Then

$$(f^L \circ g^L \dashv g \circ f, \mathtt{U}, \mathtt{C}),$$

where

$$\textit{unit} \quad \mathtt{U} \;=\; \Big(\quad \mathrm{id}_Z \xRightarrow{u_g} g \circ g^L \xRightarrow{\mathrm{id} \times u_f \times \mathrm{id}} g \circ f \circ f^L \circ g^L \,\Big),$$

$$\textit{counit} \quad \mathtt{C} \;=\; \Big(\quad f^L \circ g^L \circ g \circ f \xRightarrow{\mathrm{id} \times c_g \times \mathrm{id}} f^L \circ f \xRightarrow{c_f} \mathrm{id}_X \,\Big)$$

is an adjunction.

10.2. Adjoints in higher Categories

The generalization to adjoints of higher morphisms in higher categories is done by looking at their image in the relevant homotopy bicategories.

Definition 10.2.1. Let C be an (∞, N)-category. A 1-morphism $f : X \to Y$ is **left adjunctible** if f is left adjunctible in the homotopy bicategory $\mathrm{h}_2\,\mathsf{C}$. For $k \geq 2$ fix two parallel $(k-2)$-morphisms a and b. Let V and W be two $(k-1)$-morphisms from a to b. A k-morphism $f : V \to W$ is **left adjunctible** if f is left adjunctible in the homotopy bicategory $\mathrm{h}_2\,\mathsf{C}(a, b)$, where $\mathsf{C}(a, b)$ is the Hom-$(\infty, N-k+1)$-category from a to b. The same definition holds for **right adjunctible** with every "left" changed to "right".

Remark 10.2.2. Instead of using the homotopy categories we will work within the (∞, N)-category C and check that for adjoints and duals the zig-zag identities (zig) and (zag) hold up to equivalence. This implies that they hold as equality in the corresponding homotopy bicategories. ◇

In an (∞, N)-category we can extend the notions of duals and adjoints to higher categorical versions. These do not only consist of one or a pair of adjunctions but instead also demand that there are adjunctions for the involved units and counits up to a certain level.

Definition 10.2.3. Let C be an (∞, N)-category and f a k-morphism. A **set of 1-adjunctibility data** for f is a left adjunction $(f^L \dashv f, u_1, c_1)$ and right adjunction $(f \dashv f^R, u_2, c_2)$ of f. For $n \geq 2$ a **set of n-adjunctibility data** is a set of $(n-1)$-adjunctibility data together with left and right adjunctions for all the unit and counit $(k+n-1)$-morphisms in the $(n-1)$-adjunctibility data. A k-morphism f is **n-adjunctible** if there exists n-adjunctibility data for f. If all of the pairs of adjunctions in a set of n-adjunctibility data are ambidextrous we call it a **set of ambidextrous n-adjunctibility data** und a morphism f with such data **ambidextrous n-adjunctible**.

As is the case for adjunctibility, being n-adjunctible also is a property. This can be seen by showing that the n-adjunctibility data for a k-morphism f is contractible, by using the following proposition. The first part is stated as Lemma 4.1.5 in [Ara17], which expresses the same proof in string diagrams.

Proposition 10.2.4. *Let C be an (∞, N)-category. Let $(l \dashv r, u, c)$ and $(l' \dashv r', u', c')$ be adjunctions of k-morphisms, such that $l \cong l'$ and $r \cong r'$ are isomorphic and u is left adjunctible. Then u' is also left adjunctible. The same is true for the counit and both statements also hold with "right" instead of "left".*

Hence, the choice of $(n-1)$-adjunctibility data of a morphism does not influence whether it is n-adjunctible or not.

Proof. Applying Lemma 10.1.9 to both isomorphisms $l \cong l'$ and $r \cong r'$ we get an adjunction $(l \dashv r, u'', c'')$, where u'' and c'' are compositions of isomorphisms with u and c as in (10.1) and (10.2). By Lemma 10.1.10 we see that u'' and therefore also u' are given by u composed with an isomorphism. Hence, if u is left adjunctible so is u' by Lemma 10.1.11.

The last part follows by induction. □

In n-adjunctibility data of a k-morphism f we always have "adjoint triples" or "two-step towers" of adjunctions

k-morphisms	$(f^L \dashv f, u_1, c_1)$, $(f \dashv f^R, u_2, c_2)$
$(k+1)$-morphisms	$u_1^L \dashv u_1 \dashv u_1^R$, $c_1^L \dashv c_1 \dashv c_1^R$, $u_2^L \dashv u_2 \dashv u_2^R$, $c_2^L \dashv c_2 \dashv c_2^R$
$\vdots$	$\vdots$
$(k+n-1)$-morphisms	2^{2n-1} adjunctions for all the units and counits from step $n-1$.

which leads to $2^{2(j-k)+1}$ adjunctions of j-morphisms and $\frac{2}{3} \cdot (4^n - 1)$ adjunctions overall.

10.3. Duals

Finally, we define duals of objects and higher dualizability as a special case of the notions we have introduced so far.

Given a monoidal category C, we can construct the following bicategory $\mathbf{B}\mathsf{C}$, which is a one-object delooping. It has a unique object and the category of endomorphisms of this object is C. Composition of 1-morphisms in $\mathbf{B}\mathsf{C}$ is given by the monoidal product in C. In this setting, the notion of left and right adjoints in $\mathbf{B}\mathsf{C}$ can be translated to C as the notion of left and right duals of objects, where the dual object is given by the corresponding adjoint. We will work with symmetric monoidal categories where the braiding leads to an isomorphism between left and right duals - see Lemma D.2.3 - and hence, we do not have to distinguish between left and right. Unravelling this, we arrive at the following definition.

Definition 10.3.1. Let $(\mathsf{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. An object $X \in \mathsf{C}$ is **dualizable** or 1-**dualizable** if there is a **dual** object $X^* \in \mathsf{C}$ and morphisms

$$u : \mathbb{1} \to X \otimes X^*, \qquad c : X^* \otimes X \to \mathbb{1},$$

in C called **unit** and **counit**, satisfying the following identities, called **zig-zag identities**

$$(X \cong \mathbb{1} \otimes X \xrightarrow{u \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes c} X \otimes \mathbb{1} \cong X) = \mathrm{id}_X, \quad \text{(zig)}$$
$$(X^* \cong X^* \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{X^*} \otimes u} X^* \otimes X \otimes X^* \xrightarrow{c \otimes \mathrm{id}_{X^*}} \mathbb{1} \otimes X^* \cong X^*) = \mathrm{id}_{X^*}. \quad \text{(zag)}$$

Using these notions we define higher dualizability for objects.

Definition 10.3.2. Let C be a symmetric monoidal (∞, N)-category. An object $X \in \mathsf{C}$ is 1-**dualizable** if it is so in the homotopy category $\mathrm{h}\,\mathsf{C}$. For $n \geq 2$ an object X in C is n-**dualizable** if it is 1-dualizable and the unit and counit in C are $(n-1)$-adjunctible.

Remark 10.3.3. The most well known notion of higher dualizability is that of fully dualizable objects as defined in [Lur09]. We will review it in Appendix D.2 and see that for $n = N$ it is equivalent to Definition 10.3.2. ◇

11. Variants of Adjunctibility

Contents

So far, for higher adjunctibility, we required the existence of both left and right adjoints of the morphism itself and all unit and counit morphisms. Now we vary this condition and allow for different choices. For instance, we could require that all morphisms in question have only left adjoints; or have only right adjoints; or we could even demand infinitely long towers of adjunctions. These choices give rise to a plethora of a priori different notions of higher adjunctibility. Our main result shows that this plethora in fact reduces to two cases.

Of course, we have a good motivation for investigating these choices: the former two arise naturally when studying fully extended twisted/relative topological field theories [JFS17], which we will recall in Chapter 13. This was our main motivation for this article.

As is the case for ordinary adjunctibility our definitions will focus on the data involved and define properties by demanding that such data exists.

11.1. Higher left and right Adjunctibility

We first recall the definitions of n-times left/right adjunctibility from [JFS17].

Definition 11.1.1 ([JFS17], p. 194, before Theorem 7.6)**.** Let C be an (∞, N)-category and f a k-morphism. A **set of** 1**-times left adjunctibility data** for f is a left adjoint f^L together with unit u and counit c witnessing the adjunction between f and f^L. For $n \geq 2$ a **set of** n**-times left adjunctibility data** is a set of $(n-1)$-times left adjunctibility data together with left adjunctions for all the unit and counit $(k+n-1)$-morphisms in the $(n-1)$-times left adjunctibility data.

A k-morphism f is **n-times left adjunctible** if there exists a set of n-times left adjunctibility data.

The same definition is made for **n-times right adjunctibility** with every "left" changed to "right".

Note that as for n-adjunctibility data, the category of such is contractible and hence, being n-times left adjunctible is a property.

Explicitly, a set of n-times left adjunctibility data for a k-morphism f consists of adjunctions of

k-morphisms	$(f^L \dashv f, u, c)$,
$(k+1)$-morphisms	$(u^L \dashv u, u_u, c_u),\ (c^L \dashv c, u_c, c_c)$,
$(k+2)$-morphisms	$(u_u^L \dashv u_u, u_{uu}, c_{uu}),\ (u_c^L \dashv u_c, u_{uc}, c_{uc})$, $(c_u^L \dashv c_u, u_{cu}, c_{cu}),\ (c_c^L \dashv c_c, u_{cc}, c_{cc})$.
$\vdots$	$\vdots$
$(k+n-1)$-morphisms	2^n adjunctions for all the units and counits from step $n-1$.

For $k \leq j < k+n$ the number of adjunctions of j-morphisms is 2^{j-k}. Thus n-times left adjunctibility data consists of $2^n - 1$ adjunctions overall.

For examples we ask the reader to jump straight to Chapter 12 where we take a look at 3-times left and 3-times right adjunctibility in a higher Morita category, which is a generalization of Alg_1.

Remark 11.1.2. Note that n-times left together with n-times right adjunctibility does not in general imply the stronger property of n-adjunctibility, since the data of left and right adjunctions do not need to be compatible. We will discuss this in more detail in Theorem 11.2.10. ◇

Our main goal is to discuss when n-times left and n-times right adjunctibility are equivalent. This is motivated by the following. While ordinary left and right adjunctibility are in general vastly different notions we will see that this difference vanishes for 2-times left and 2-times right adjunctibility. The categorical fact responsible for this is the following Lemma, which can be found as Remark 3.4.22 in [Lur09]. A proof is also given in [DSPS20, Lemma 1.4.4]. We will give a slightly different proof, and be very explicit in exhibiting the necessary higher morphisms.

Lemma 11.1.3. *(Interchange Lemma) Let C be an (∞, N)-category. Let f be a k-morphism that admits a left adjoint f^L with unit u and counit c. If u and c admit left adjoints u^L and c^L, then $(f \dashv f^L, c^L, u^L)$ is an adjunction. Similarly, if u and c admit right adjoints u^R and c^R, then $(f \dashv f^L, c^R, u^R)$ is an adjunction.*

Before proving the Lemma, we first look at an immediate consequence.

Corollary 11.1.4. *Let f be a 1-morphism in an $(\infty,3)$-category C. Then f is 2-times left adjunctible if and only if f is 2-times right adjunctible.*

Proof. Applying the above Lemma to a set of 2-times left adjunctibility data for f,

$$\begin{aligned} &(f^L \dashv f, u, c),\\ &(u^L \dashv u, u_u, c_u), \qquad (c^L \dashv c, u_c, c_c), \end{aligned} \tag{11.1}$$

we obtain the adjunction $(f \dashv f^L, c^L, u^L)$, which together with (11.1) forms 2-times right adjunctibility data for f.

Conversely, by Lemma 11.1.3, we can also turn 2-times right adjunctibility data into 2-times left adjunctibility data. □

Thus, 2-times left and 2-times right adjunctibility agree in contrast to the classical notions of left- and right-adjunctibility. Additionally, the adjunctions for f above also form an ambidextrous adjunction, i.e. a **2-step tower** of adjunctions $f^L \dashv f \dashv f^L$ for f with the property that the units and counits of the two adjunctions are adjoint to each other. Of course, by symmetry we could continue this tower to a tower of infinite length

$$\cdots \dashv f^L \dashv f \dashv f^L \dashv f \dashv f^L \dashv \cdots$$

Proof of Lemma 11.1.3. The adjunction $(f^L \dashv f, u, c)$ consists of the data

$$Y \underset{f}{\overset{f^L}{\rightleftarrows}} X, \qquad u : \mathrm{id}_Y \Rightarrow f\, f^L, \quad c : f^L f \Rightarrow \mathrm{id}_X,$$

with (zig) and (zag)

$$\alpha : \left(f \xRightarrow{u \times \mathrm{id}_f} f\, f^L f \xRightarrow{\mathrm{id}_f \times c} f \right) \xRightarrow{\sim} \mathrm{id}_f, \tag{11.2}$$

$$\beta : \left(f^L \xRightarrow{\mathrm{id}_{f^L} \times u} f^L f\, f^L \xRightarrow{c \times \mathrm{id}_{f^L}} f^L \right) \xRightarrow{\sim} \mathrm{id}_{f^L}, \tag{11.3}$$

where α and β are $(k+2)$-equivalences.

To show that $(f \dashv f^L, c^L, u^L)$ is an adjunction we need $(k+2)$-equivalences

$$\left(f^L \xRightarrow{c^L \times \mathrm{id}_{f^L}} f^L f\, f^L \xRightarrow{\mathrm{id}_{f^L} \times u^L} f^L \right) \xRightarrow{\sim} \mathrm{id}_{f^L}, \tag{11.4}$$

$$\left(f \xRightarrow{\mathrm{id}_f \times c^L} f\, f^L f \xRightarrow{u^L \times \mathrm{id}_f} f \right) \xRightarrow{\sim} \mathrm{id}_f. \tag{11.5}$$

We construct an equivalence for (11.4). From the adjunction data $(u^L \dashv u, \eta_u, \epsilon_u)$ for the left adjoint of u we obtain an adjunction

$$(\ \mathrm{id}_{f^L} \times u^L \ \dashv\ \mathrm{id}_{f^L} \times u\,,\ \mathrm{id}_{\mathrm{id}_{f^L}} \times \eta_u\,,\ \mathrm{id}_{\mathrm{id}_{f^L}} \times \epsilon_u\ ). \tag{11.6}$$

Similarly, from the adjunction data $(c^L \dashv c, \eta_c, \epsilon_c)$ for the left adjoint of c we obtain an adjunction

$$(\ c^L \times \mathrm{id}_{f^L} \ \dashv\ c \times \mathrm{id}_{f^L}\,,\ \eta_c \times \mathrm{id}_{\mathrm{id}_{f^L}}\,,\ \epsilon_c \times \mathrm{id}_{\mathrm{id}_{f^L}}\ ). \tag{11.7}$$

Both (11.6) and (11.7) are adjunctions in $\mathrm{Hom}_{\mathsf{C}}(X, Y)$. Applying Lemma 10.1.11 to

$$f^L \circ \mathrm{id}_Y \underset{\mathrm{id}_{f^L} \times u^L}{\overset{\mathrm{id}_{f^L} \times u}{\rightleftarrows}} f^L \circ f \circ f^L \underset{c^L \times \mathrm{id}_{f^L}}{\overset{c \times \mathrm{id}_{f^L}}{\rightleftarrows}} \mathrm{id}_X \circ f^L$$

yields an adjunction

$$\Big(\ (\mathrm{id}_{f^L} \times u^L\,) \circ (\,c^L \times \mathrm{id}_{f^L}\,)\ \dashv\ (\,c \times \mathrm{id}_{f^L}\,) \circ (\,\mathrm{id}_{f^L} \times u\,),\ \mathtt{U},\ \mathtt{C}\ \Big) \tag{11.8}$$

with $\mathtt{U}$ and $\mathtt{C}$ given by Lemma 10.1.11. The right adjoint in (11.8) is equivalent to id_{f^L} by assumption, via β given by (11.2). By Lemma 10.1.9 with $\mu = \mathrm{id}$ and $\nu = \beta$ we have an adjunction

$$\Big(\ (\,\mathrm{id}_{f^L} \times u^L\,) \circ (\,c^L \times \mathrm{id}_{f^L}\,)\ \dashv\ \mathrm{id}_{f^L},\ \mathtt{U}',\ \mathtt{C}'\ \Big), \tag{11.9}$$

where the unit and counit are given by the compositions from Lemma 10.1.9. By the uniqueness of adjoints as stated in Lemma 10.1.7 and the fact that identities are self-adjoint the left adjoint in (11.9) is equivalent to id_{f^L}. We choose such an equivalence. This is the desired $(k+2)$-equivalence for (11.4).

The construction of the equivalence for (11.5) is analogous. □

11.2. Mixed Adjunctibility and reducing Conditions

In this subsection we will prove our main Theorem, which shows that higher left and right adjunctibility agree in even dimensions as a consequence of Lemma 11.1.3. In fact, we will prove a more general version for mixed versions of higher adjunctibility. Indeed, we will have 2^n a priori different notions of mixed adjunctibility, and our main Theorem reduces these cases to just two. In even dimensions, higher left and right adjunctibility reduce to the same case, whereas in odd dimensions they do not.

We start with mixed adjunctibility. We will do this by keeping track of when we require left and when we require right adjoints in a dexterity function.

Definition 11.2.1. Let C be an (∞, N)-category. Let

$$a^n : \{1, 2, \dots, n\} \to \{L, R\}.$$

Let f be a k-morphism in C. A **set of a^n-adjunctibility data** for f with **dexterity function** a^n is defined inductively as

- step 1: the data of a $\begin{cases} \text{left adjoint,} & \text{if} \quad a^n(1) = L, \\ \text{right adjoint,} & \text{if} \quad a^n(1) = R, \end{cases}$
  together with unit and counit witnessing the adjunction.

- step $j = 2, \dots, n$: for all units and counits from step $j-1$
  the data of $\begin{cases} \text{left adjoints,} & \text{if} \quad a^n(j) = L, \\ \text{right adjoints,} & \text{if} \quad a^n(j) = R, \end{cases}$
  together with units and counits witnessing the adjunctions.

A k-morphism is **a^n-adjunctible** if there exists a set of a^n-adjunctibility data for f.

Example 11.2.2. Let f be an n-adjunctible k-morphism f and a^n any dexterity function. Then f is a^n-adjunctible. ◇

Example 11.2.3. Let

$$l^n, r^n : \{1, 2, \dots, n\} \to \{L, R\}, \qquad l^n(i) = L, \quad r^n(i) = R,$$

be the constant dexterity functions with values 'L' and 'R', respectively. Then l^n-adjunctibility is n-times left adjunctibility and r^n-adjunctibility is n-times right adjunctibility.

We note that Corollary 11.1.4 shows that a 1-morphism is l^2-adjunctible if and only if it is r^2-adjunctible. Generalizing this statement is one of our main goals. ◇

Example 11.2.4. Another natural pair of dexterity functions is given as follows. One is simply the constant 'R' dexterity function

$$\text{even}^n = r^n \tag{11.10}$$

from Example 11.2.3. The second one is given by the function

$$\begin{aligned} &\text{odd}^n : \{1, 2, \dots, n\} \to \{L, R\}, \\ &\text{odd}^n(n) = L, \qquad \text{odd}^n(i) = R \quad \text{for} \quad 1 \leq i \leq n-1. \end{aligned} \tag{11.11}$$

The names even and odd refer to the parity of the number of 'L's in the image, which will become important below. ◇

These two dexterity functions are natural in the following sense. Our main theorem proves that the 2^n a priori different notions of a^n-adjunctibility from Definition 11.2.1 reduce to just two, namely exactly the even and odd ones. The distinction for a given dexterity function a^n is given by it being a parity dexterity of even^n or odd^n in the sense of the following definition.

Definition 11.2.5. Let $a^n, b^n : \{1, 2, \ldots, n\} \rightrightarrows \{L, R\}$ be two dexterity functions. If

$$| (a^n)^{-1}(L) | \equiv | (b^n)^{-1}(L) | \mod 2,$$

we say b^n is a **parity dexterity (function) of** a^n and the pair a^n, b^n is a **parity pair** of dexterity functions. If

$$| (a^n)^{-1}(L) | \not\equiv | (b^n)^{-1}(L) | \mod 2,$$

we call b^n a **nonparity dexterity (function) of** a^n and the pair a^n, b^n a **nonparity pair** of dexterity functions.

Now we formulate our main theorem that makes the equivalences of different a^n-adjunctibilities precise.

Theorem 11.2.6. *Let* C *be an* (∞, N)*-category and* $n \geq 1$*.*
Let $a^n, b^n : \{1, 2, \ldots, n\} \rightrightarrows \{L, R\}$ *be two dexterity functions.*

1. *If* a^n, b^n *are a parity pair then a* k*-morphism* f *is* a^n*-adjunctible if and only if* f *is* b^n*-adjunctible.*
2. *If* a^n, b^n *are a nonparity pair then a* k*-morphism* f *with an* $a^n(1)$*-adjoint* g *is* a^n*-adjunctible if and only if* g *is* b^n*-adjunctible.*

We will prove this theorem later in this section. Before that we record some immediate consequences. The first follows straight from the definitions of even^n and odd^n.

Corollary 11.2.7. *Let* C *be an* (∞, N)*-category and* $n \geq 1$*.*
Let a^n *be a dexterity function such that* f *is* a^n*-adjunctible. If*

$$|(a^n)^{-1}(L)| \equiv 0 \mod 2,$$

then f *is* even^n*-adjunctible. If*

$$|(a^n)^{-1}(L)| \equiv 1 \mod 2,$$

then f *is* odd^n*-adjunctible.*

Remark 11.2.8. One can construct examples of morphisms which are even^n-adjunctible, but not odd^n-adjunctible and vice versa. We will do so in Morita categories for $n = 1$ in Example 12.1.4, $n = 2$ in Example 12.2.10, and $n = 3$ in Example 12.2.8. From these examples it is straightforward how to generalize to examples for arbitrary n in the higher Morita categories Alg_n from [Sch14]. Hence, there are precisely two independent (mixed) adjunctibility conditions we can impose on a k-morphism, which are even^n and odd^n. ◇

A special case of Theorem 11.2.6 is one of our main goals, namely, the equivalence between n-times left and right adjunctibility if n is even.

Corollary 11.2.9. *Let C be an (∞, N)-category and f a k-morphism. Let $n \geq 2$ be even. Then f is n-times left adjunctible if and only if f is n-times right adjunctible.*

Before we prove Theorem 11.2.6 let us state a theorem that describes the connection of the two classes of mixed adjunctibility to n-adjunctibility. A k-morphism f that is both left and right adjunctible is adjunctible by definition. This generalizes in the following way.

Theorem 11.2.10. *Let C be an (∞, N)-category and $n \geq 2$.*
Let $a^n : \{1, 2, \ldots, n\} \to \{L, R\}$ be a dexterity function and f an a^n-adjunctible k-morphism.

1. *Then f is ambidextrous $(n-1)$-adjunctible.*
2. *If b^n is a nonparity dexterity of a^n and f also is b^n-adjunctible, then f is n-adjunctible.*

We will prove both this theorem and Theorem 11.2.6 later in this section.

As a consequence of the second statement we see that there is a direct relation between n-adjunctibility and a^n-adjunctibility. However, although n-times left and n-times right adjunctibility represented by the dexterity functions r^n and l^n seemed like natural choices of one-sided adjunctibility, we see that even^n- and odd^n-adjunctibility is a better choice, since they represent the two adjunctibility choices and together imply n-adjunctibility.

Corollary 11.2.11. *Let C be an (∞, N)-category and $n \geq 1$. A k-morphism f is n-adjunctible if and only if it is both* even^n*- and* odd^n*-adjunctible.*

The first part of the theorem allows us to determine the adjunctibility type of the adjoint of a morphism.

Corollary 11.2.12. *Let C be an (∞, N)-category and $n \geq 2$.*
Let $a^n, b^n : \{1, 2, \ldots, n\} \rightrightarrows \{L, R\}$ be a nonparity pair of dexterity functions. Let f be an a^n-adjunctible k-morphism. Then any adjoint g of f is b^n-adjunctible.

Proof. By Theorem 11.2.10 (1) and since $n \geq 2$ the adjunction is ambidextrous, so we have that $g \dashv f \dashv g$. We can extend this data to a^n-adjunctibility data for f. If $a^n(1) = L$, then we use the adjunction $g \dashv f$ and see that the a^n-adjunctibility data for f gives c^n-adjunctibility data for g, where

$$c^n(i) = \begin{cases} R & i = 1, \\ a^n(i) & i \neq 1. \end{cases}$$

Then by Theorem 11.2.6 (1) g also is b^n-adjunctible. The case $a^n(1) = R$ is similar. □

Next, we obtain a corollary which generalizes the (quite obvious) statement that if a morphism f is left adjunctible with left adjunctible left adjoint g, then g is adjunctible.

Corollary 11.2.13. *Let C be an (∞, N)-category and $n \geq 1$. Let a^n be a dexterity function. Let $g \dashv f$ be an adjunction of k-morphisms such that both f and g are a^n-adjunctible.*

1. *If $a^n(1) = L$ then g is n-adjunctible,*
2. *If $a^n(1) = R$ then f is n-adjunctible.*

Proof. The case $n = 1$ is immediate. For $n \geq 2$ we apply Corollary 11.2.12 and obtain that g is b^n-adjunctible for aa nonparity dexterity of a^n. By Theorem 11.2.10 we have that g is n-adjunctible. The case $a^n(1) = R$ is similar. □

Corollary 11.2.14. *Let C be an (∞, N)-category and $n \geq 2$. Let f be an n-adjunctible k-morphism. Then any adjoint g of f is also n-adjunctible.*

Proof. Since f is n-adjunctible, it is a^n-adjunctible for any dexterity function, for instance for even^n and odd^n. Then by Corollary 11.2.12 g is odd^n and even^n-adjunctible, respectively. Hence, by Theorem 11.2.10 (2) g is n-adjunctible. □

Notation 11.2.15. We denote the function on $\{R, L\}$ switching direction by

$$-\colon \{R, L\} \longrightarrow \{R, L\}, \qquad -L = R, \qquad -R = L.$$

To prove our theorems we will need a generalized version of the Interchange Lemma 11.1.3. By applying the Interchange Lemma to one "layer" of a^n-adjunctibility data, we switch both the direction of the adjunctions in this and the consecutive "layer" as follows.

Lemma 11.2.16. *Let C be an (∞, N)-category and $n \geq 2$.*
Let $a^n : \{1, 2, \ldots, n\} \to \{L, R\}$ be a dexterity function. Let $0 < j < n$ and define

$$b^n : \{1, 2, \ldots, n\} \to \{L, R\}, \qquad i \mapsto \begin{cases} -a^n(i) & i = j \text{ or } i = j+1, \\ a^n(i) & i \neq j \text{ and } i \neq j+1. \end{cases}$$

Then a k-morphism f is a^n-adjunctible if and only if it is b^n-adjunctible.

In this case there is compatible a^n- and b^n-adjunctibility data in the sense that the adjunctions for $(k+j)$-morphisms together form 2-step towers

$$h \dashv g \dashv h\,,$$

such that the unit of the left (right) adjunction is adjoint to the counit of the right (left) adjunction, while all the other adjunctions are the same.

Proof. Let a^n be a dexterity function and f be a k-morphism that is a^n-adjunctible. Let $0 < j < n$. There are four different cases:

(i) $a^n(j) = L,\ a^n(j+1) = L,$

(ii) $a^n(j) = R,\ a^n(j+1) = R,$

(iii) $a^n(j) = L,\ a^n(j+1) = R,$

(iv) $a^n(j) = R,\ a^n(j+1) = L.$

We explain the situation in the first case, the others are similar. Assume we have that $a^n(j) = a^n(j+1) = L$. Let g be a $(k+j-1)$-morphism in set of a^n-adjunctibility data for f. We have adjunctions of the form

$$(g^L \dashv g, u, c), \tag{11.12}$$

$$(u^L \dashv u, u_u, c_u), \qquad (c^L \dashv c, u_c, c_c). \tag{11.13}$$

Applying Lemma 11.1.3 we can switch the adjunction to an adjunction

$$(g \dashv g^L, c^L, u^L),$$

exhibiting g as having a **right** adjoint. Then (11.13) exhibits right adjoints of the new unit c^L and counit u^L. Hence, we have exchanged the left adjoints with right adjoints.

We note that the new data fits with the rest of the data of adjunctions of i-morphisms for $i < (k+j-1)$ and $i > k+j$. This relies on the fact that the units and counits u_u, u_c, c_u, c_c of the adjunctions of $(k+j)$-morphisms are the same before and after the exchange, which proves the desired compatibility.

Doing this for all $(k+j-1)$-morphisms in the a^n-adjunctibility data creates compatible b^n-adjunctibility data, where b^n is defined as above.

For the converse direction, start with a k-morphism f that is b^n-adjunctible. If we again assume that a^n satisfies (i), then b^n satisfies (ii) and so by swapping the role of a^n and b^n in the first part we obtain that f is a^n-adjunctible. (Note that in this argument the cases (i) and (ii) interchange roles; and (iii) and (iv) interchange roles.) □

Equipped with Lemma 11.2.16 one could give a direct proof of Corollary 11.2.9 by applying it to every second "layer" and iteratively switching '$LL \leftrightarrow RR$' in the constant dexterity functions l^n and r^n from Example 11.2.3. Alternatively, we can use

$$|\,(l^n)^{-1}(L)\,| = n \;\equiv\; 0 = |\,(r^n)^{-1}(L)\,| \mod 2 \qquad \text{iff } n \text{ is even}$$

and Theorem 11.2.6 which we proof now.

Proof of Theorem 11.2.6. We start with the first claim. In the case $n = 1$ two dexterity functions that form a parity pair are the same function and the statement is immediate. Let now $n \geq 2$. By Lemma 11.2.16 we can switch consecutive entries

$$\text{`}LL\text{'} \leftrightarrow \text{`}RR\text{'} \qquad \text{and} \qquad \text{`}LR\text{'} \leftrightarrow \text{`}RL\text{'} \tag{11.14}$$

of the dexterity functions to get equivalent notions of a^n-adjunctibility. Note that both of these do not change the parity of the number of 'L's and 'R's. Let a^n, b^n be a parity pair of dexterity functions. Let $a_1^n := a^n$. For $j = 1, 2, \ldots, (n-1)$, we define a new dexterity function a_{j+1}^n as follows. If $a_j^n(j) = b^n(j)$, set $a_{j+1}^n = a_j^n$. If $a_j^n(j) \neq b^n(j)$, apply the exchange from (11.14) to entry j and $j+1$ of a_j^n to obtain a_{j+1}^n. By Lemma 11.2.16 a k-morphism f is a_{j+1}^n-adjunctible if and only if it is a_j^n-adjunctible, and hence a^n-adjunctible.

After $n-1$ steps we constructed a_n^n that describes an to a^n equivalent notion of mixed adjunctibility and has $a_n^n(j) = b^n(j)$ for $1 \leq j \leq n-1$ by construction. Since we did not change the parity of the number of 'L's in this process we have

$$|(a_n^n)^{-1}(L)| \;\equiv\; |(a^n)^{-1}(L)| \;\equiv\; |(b^n)^{-1}(L)| \mod 2$$

and therefore also $a_n^n(n) = b^n(n)$. This shows the equivalence of a^n- and b^n-adjunctibility for a parity pair.

For the second part of the theorem let a^n, b^n be a nonparity pair of dexterity functions. Let f be an a^n-adjunctible morphism with $a^n(1)$-adjoint g. The case $n = 1$ just states that left adjoints have right adjoints as well as the converse. For $n \geq 2$ it follows from the uniqueness of adjoints from Lemma 10.1.7 that a^n-adjunctibility data for f is also c^n-adjunctibility data for g, where c^n is the dexterity function with

$$c^n(j) := a^n(j) \text{ for } j = 2, 3, \ldots, n \text{ and } c^n(1) := -a^n(1).$$

Since a^n, b^n and a^n, c^n are nonparity pairs it follows that b^n, c^n are a parity pair of dexterity functions. The claim now follows from the first part of the Theorem. □

At last we prove the second main Theorem. Part of the proof ist the following Lemma.

Lemma 11.2.17. *Let* C *be an* (∞, N)*-category. Let*

$$g \dashv f \dashv g$$

be an ambidextrous adjunction given by $(g \dashv f, u_1, c_1)$ *and* $(f \dashv g, u_2, c_2)$*, such that* u_1 *and* c_1 *are* 1*-adjunctible. Then* u_2 *and* c_2 *are also* 1*-adjunctible.*

Proof. Since u_1 and c_1 are 1-adjunctible there are adjunctions

$$u_1^L \dashv u_1 \dashv u_1^R \qquad \text{and} \qquad c_1^L \dashv c_1 \dashv c_1^R.$$

Applying Lemma 11.1.3 for the left and the right adjoints of u and c yields

$$(f \dashv g, c_1^L, u_1^L) \qquad \text{and} \qquad (f \dashv g, c_1^R, u_1^R).$$

By Proposition 10.2.4 the morphisms u_2 and c_2 are left and right adjunctible. □

Proof of Theorem 11.2.10. Let f be a k-morphism and a^n a dexterity function such that f is a^n-adjunctible. Consider a set of a^n-adjunctibility data. Apply Lemma 11.2.16 with $j = 1$ to get ambidextrous adjunctions for the $(k+1)$-morphisms. In particular, we have a set of ambidextrous 1-adjunctibility data for f and its $(k+1)$-morphisms all have one adjoint $(k+1)$-morphism in the a^n-adjunctibility data by the compatibility condition in the Lemma.

Inductively, in step $l = 2, \ldots, n-1$ assume we are given a set of ambidextrous $(l-1)$-adjunctibility data such that the $(k+l)$-morphisms have adjoints or are itself part of the a^n-adjunctibility data. Then we can apply Lemma 11.2.16 with $j = l$ to a^n and get ambidextrous adjunctions for its $(k+l)$-morphisms. By the induction hypothesis this extends the ambidextrous $(l-1)$- to ambidextrous l-adjunctibility data. Furthermore by the compatibility statement of Lemma 11.2.16 the $(l+k+1)$-morphisms in this data all have adjoints in or are itself part of the $(l+k+1)$-morphisms of the a^n-adjunctibility data. After step $l = n-1$ we have constructed the desired ambidextrous $(n-1)$-adjunctibility data.

For the second claim let f also be b^n-adjunctible for a nonparity dexterity b^n of a^n. Consider another dexterity function c^n given by

$$c^n(i) = a^n(i) \;\text{ for }\; 1 \leq i \leq n-1 \qquad \text{and} \qquad c^n(n) = -a^n(n).$$

We see that c^n is a nonparity dexterity for a^n and therefore a parity dexterity for b^n. By Theorem 11.2.6 we have that f is c^n-adjunctible. Thus, we can take the corresponding adjunctions of the ambidextrous $(n-1)$-adjunctibility data and extend it to c^n-adjunctibility data. But since c^n agrees with a^n up to level $n-1$ we can take the same data and extend it to a^n-adjunctibility data. It follows from $c^n(n) = -a^n(n)$ that we have both left and right adjoints for the unit and counit $(k+n+1)$-morphisms of the a^n- and c^n-data. This gives us half of the adjunctions of $(k+n+1)$-morphisms that we need for n-adjunctibility. The missing half are adjunctions for the units and counits witnessing the adjunctions of $(k+n)$-morphisms that do not appear in the a^n- and c^n-data. We get those from Lemma 11.2.17. □

11.3. Adjoints in Opposite Categories

Similar to Proposition 10.1.5 for one adjunction let us record what happens if we pass to an opposite category. We will use this in Chapter 13 for relative field theories.

Definition 11.3.1. Let C be an (∞, N)-category. An **opposite function** is a function

$$\mathrm{op}^N : \{1, 2, \ldots, N\} \to \{\mathrm{id}, \mathrm{op}\}\,.$$

From an opposite function op^N we can construct the corresponding dual op^N-category $\mathsf{C}^{\mathrm{op}^N}$, which reverses the direction of all j-morphisms with $\mathrm{op}^N(j) = \mathrm{op}$ for all j but not any other morphisms.

Example 11.3.2. Two opposite functions we will use later are

$$\mathrm{even\,op}(j) := \begin{cases} \mathrm{op} & j \text{ even}, \\ \mathrm{id} & \text{else}, \end{cases} \qquad \text{and} \qquad \mathrm{odd\,op}(j) := \begin{cases} \mathrm{op} & j \text{ odd}, \\ \mathrm{id} & \text{else}\,. \end{cases}$$

◇

We will use the map

$$\delta : \{0, 1\} \to \{\mathrm{id}, \mathrm{op}\}, \qquad 0 \mapsto \mathrm{id}, \quad 1 \mapsto \mathrm{op},$$

to define an opposite function corresponding to a pair of dexterity functions in the following proposition.

Proposition 11.3.3. *Let C be an (∞, N)-category and f a k-morphism in C. Let a^n, b^n be two dexterity functions. Consider the opposite function $\mathrm{op}_{a^n,b^n,k} : \{1, 2, \ldots, N\} \to \{\mathrm{id}, \mathrm{op}\}$ given by*

$$\mathrm{op}_{a^n,b^n,k}(j+k) = \delta\left(|(a^j)^{-1}(L) + (b^j)^{-1}(L)| \mod 2 \right)$$

for $j = 0, 1, \ldots, n$, where all the other unassigned values are id. *Here a^j denotes the dexterity function a^n restricted to $\{1, \ldots, j\}$ with a^0 having empty domain.*

The following are equivalent:

1. *f is a^n-adjunctible in C*
2. *f is b^n-adjunctible when considered as a k-morphism in $\mathsf{C}^{\mathrm{op}_{a^n,b^n,k}}$.*
3. *f is b^n-adjunctible when considered as a k-morphism in $\mathsf{C}^{-\mathrm{op}_{a^n,b^n,k}}$, where $-\mathrm{op}_{a^n,b^n,k}$ has the opposite entries of $\mathrm{op}_{a^n,b^n,k}$.*

Proof. Since adjunctions in C are defined via the corresponding homotopy bicategory this follows from Proposition 10.1.5. □

We now unravel this complicated looking statement in some examples.

Example 11.3.4. Let C be an (∞, N)-category and let f be a 1-morphism which is n-times right adjunctible in C. Set $a^n = r^n$.

1. Setting $b^n = a^n = r^n$, we have that

$$\mathrm{op}_{a^n,b^n,1} \equiv \mathrm{id} \quad \text{and} \quad -\mathrm{op}_{a^n,b^n,1} \equiv \mathrm{op}$$

   are the constant functions valued id and op, respectively.

2. Setting $b^n = \mathrm{odd}^n$, we see that f is odd^n-adjunctible in $\mathsf{C}^{\mathrm{op}_{a^n,b^n,1}}$ and in $\mathsf{C}^{-\mathrm{op}_{a^n,b^n,1}}$ with

$$\mathrm{op}_{a^n,b^n,1}(j) = \begin{cases} \mathrm{op} & j = n+1, \\ \mathrm{id} & \text{else}. \end{cases}$$

3. Choosing $b^n = l^n$, we see that f is n-times left adjunctible in $\mathsf{C}^{\mathrm{op}_{a^n,b^n,1}}$ and in $\mathsf{C}^{-\mathrm{op}_{a^n,b^n,1}}$ with

$$\mathrm{op}_{a^n,b^n,1}(j) = \begin{cases} \mathrm{op} & j \leq n+1, \ j \text{ even}, \\ \mathrm{id} & \text{else}. \end{cases}$$

   If $n + 1 = N$ these are $\mathrm{op}_{a^n,b^n,1} = \mathrm{even\,op}$ and $-\mathrm{op}_{a^n,b^n,1} = \mathrm{odd\,op}$ from Example 11.3.2.

◇

12. Adjunctibility in Morita Categories

Contents

In this section we take a look at the Morita bicategory and similar Morita 3- and 4-categories. We construct examples of morphisms which satisfy different notions of adjunctibility introduced in the previous section.

12.1. The Morita Bicategory

In [Mor58] Morita introduced what is now called Morita equivalences between rings. In [Bén67] Bénabou recognized that these are equivalences in a bicategory, the Morita bicategory Alg_1. It can be described as follows. Let $\mathbb{K}$ be a commutative ring or field.

- objects are associative $\mathbb{K}$-algebras,
- for two $\mathbb{K}$-algebras A and B the 1-morphisms $A \to B$ are bimodules of the form ${}_AM_B$. Composition of two bimodules ${}_AM_B$ and ${}_BN_C$ is given by the relative tensor product

  $${}_BN_C \circ {}_AM_B = {}_AM_B \otimes_B {}_BN_C,$$

  which is an (A, C)-bimodule,
- 2-morphisms are given by bimodule homomorphisms.

Recall that ${}_AM_B$ means that M is a left A-module and right B-module such that $(am)b = a(mb)$ holds for all $a \in A, b \in B$ and $m \in M$. Alternatively, a bimodule

${}_AM_B$ is the same thing as a left module over the $\mathbb{K}$-algebra $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$, where B^{op} is the opposite algebra of B. A bimodule homomorphism ${}_AM_B \to {}_AN_B$ is a linear map which is both a left A-module and a right B-module homomorphism $M \to N$. Together with the tensor product of $\mathbb{K}$-algebras Alg_1 is a symmetric monoidal bicategory. Full details can be found in [Bén67].

Remark 12.1.1. **Warning:** Sometimes a different convention for the direction of 1-morphisms is chosen, namely, that ${}_AM_B$ is a 1-morphism from B to A. ◇

To characterize 2-dualizability we need to know which $\mathbb{K}$-algebras are 1-dualizable and which bimodules have left and which have right adjoints. These results are well-known, but we include proofs in Appendix D.3 for the interested reader.

Proposition 12.1.2. *Every object in Alg_1 is 1-dualizable. For a $\mathbb{K}$-algebra A its dual is the opposite algebra A^{op}, the unit morphism is the bimodule ${}_{\mathbb{K}}A_{A\otimes_{\mathbb{K}}A^{\mathrm{op}}}$ and the counit is ${}_{A\otimes_{\mathbb{K}}A^{\mathrm{op}}}A_{\mathbb{K}}$.*

Proposition 12.1.3. *A bimodule ${}_AR_B$ has a left adjoint if and only if R is finitely presented projective as a left A-module. In this case a dual is ${}_BL_A$, where $L \cong \mathrm{Hom}_A(R, A)$.*

Dually, a bimodule ${}_BL_A$ has a right adjoint if and only if L is finitely presented projective as a right A-module. In this case the dual is ${}_AR_B$, where $R \cong \mathrm{Hom}_A(L, A)$.

Example 12.1.4. With this proposition, we immediately find examples of right adjunctible but not left adjunctible 1-morphisms in Alg_1: Take any $\mathbb{K}$-algebra M which is a left A-module and as such is not finitely presented projective. Then M is an (A, M)-bimodule. Since M as a trivial M-module is finitely presented projective by Proposition 12.1.3 we have that the morphism ${}_AM_M$ is right adjunctible but not left adjunctible. Likewise, ${}_MM_{A^{\mathrm{op}}}$ is left adjunctible but not right adjunctible. Note that in addition by Proposition 12.1.2 both A and M are 1-dualizable. This is important when interpreting this as a relative TFT.

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12.2. Higher Morita Categories

There is a natural generalization of the Morita bicategory to a higher category $\mathsf{Alg}_n(\mathsf{C})$ whose objects are E_n-algebras in some suitable symmetric monoidal (possibly higher) category C. If C is a symmetric monoidal (∞, k)-category satisfying some mild conditions, then $\mathsf{Alg}_n(\mathsf{C})$ is a symmetric monoidal $(\infty, n+k)$-category, sometimes called a Morita $(n+k)$-category. See [Hau17], [Sch14], [JFS17] for the technical details. The classical Morita bicategory described above is the special case $n = 1$ and $\mathsf{C} = \mathsf{Vect}_{\mathbb{K}}$.

In the rest of this section we will look at two other special cases in which the construction reduces to familiar objects: In both cases we take $\mathsf{C} = \mathsf{Vect}_{\mathbb{K}}$. We first look at the (weak) 3-category $\mathsf{Alg}_2 = \mathsf{Alg}_2(\mathsf{Vect}_{\mathbb{K}})$ and then we consider the (weak) 4-category $\mathsf{Alg}_3 = \mathsf{Alg}_3(\mathsf{Vect}_{\mathbb{K}})$.

Objects in Alg_2 and Alg_3 are simply commutative $\mathbb{K}$-algebras. Certain morphisms are bimodules which themselves have an algebra structure. These satisfy a compatibility condition as follows:

Definition 12.2.1. Let C be a commutative algebra and M an algebra, which also has a C-module structure. Then the action factors through the unit of M,

$$C \otimes M \longrightarrow M, \qquad c \otimes m \longmapsto c \cdot m = c \cdot 1_M \cdot m,$$

so the action is determined by a homomorphism

$$C \longrightarrow M, \qquad c \longmapsto c \cdot 1_M \,.$$

We say that M is a **central C-module** if this map factors through the center $Z(M)$. An algebra which is an (A, B)-bimodule is central if it is central as an $A \otimes B^{\mathrm{op}}$-module.

Example 12.2.2. If M is commutative, any module is central.

◇

We start describing the weak 4-category Alg_3 and state some dualizability and adjunctibility results. Rather than repeating the same statements and arguments for Alg_2, we use that $\mathsf{Alg}_2 = \mathrm{Hom}_{\mathsf{Alg}_3}(\mathbb{K}, \mathbb{K})$ afterwards. The reader feeling more comfortable starting with a lower-dimensional setting can jump to Example 12.2.10 first and work their way backwards.

Informally, Alg_3 can be described as follows.

- Objects are commutative $\mathbb{K}$-algebras,
- a 1-morphism between two commutative algebras $T \to S$ is a commutative $\mathbb{K}$-algebra A which also is a bimodule ${}_T A_S$;
- a 2-morphism between two 1-morphisms $A \to B$ is an associative $\mathbb{K}$-algebra M which also is a central bimodule ${}_A M_B$;
- a 3-morphism between two 2-morphisms $N \to M$ is a bimodule ${}_M \Sigma_N$ such that for every $\sigma \in \Sigma$ and $c \in A \otimes B^{\mathrm{op}}$, we have that

$$(c \cdot 1_M) \cdot \sigma = \sigma \cdot (c \cdot 1_N) \; ; \tag{12.1}$$

- and a 4-morphism is a homomorphism, meaning it is compatible with all actions of the source and target objects, 1-morphisms, and 2-morphisms.

Compositions of morphisms which are bimodules is given by the relative tensor product, similarly to the composition of 1-morphisms in Alg_1. Composition of 4-morphisms is composition of homomorphisms.

Remark 12.2.3. The structure of a 3-morphism as defined above automatically is compatible with the action of the source and target objects, meaning that the actions on Σ also commute as in (12.1). ◇

Notation 12.2.4. We iterate the subscript notation for bimodules as a compact notation for higher morphisms. For instance, for a commutative $\mathbb{K}$-algebra T the identity 3-morphism on the identity of the identity of T is

$$\mathrm{id}_{\mathrm{id}_{\mathrm{id}_T}} = {}_{{}_{T}T_{T}\,{}_{T}T_{T}}T_{{}_{T}T_{T}\,{}_{T}T_{T}}.$$

We will need two basic facts about existence of duals and adjoints in Alg_3, described in the following two propositions. The first is a generalization of Proposition 12.1.2.

Proposition 12.2.5 (Special case of Theorem 5.1 in [GS18])**.** *The symmetric monoidal weak 4-category* Alg_3 *is fully 3-dualizable, i.e.*

1. *every object* T *in* Alg_3 *is 3-dualizable with dual* $T^{\mathrm{op}} = T$*; and*
2. *any* 1*-morphism* ${}_TA_S$ *and any* 2*-morphism* ${}_AM_B$ *in* Alg_3 *has both a left and a right adjoint whose underlying algebras are* $A^{\mathrm{op}} = A$ *and* M^{op}*.*

Proof. We spell out the data in the case of a 1-morphism $T \to S$ for later reference. The proof of the other statements is similar.

Let T and S be commutative $\mathbb{K}$-algebras, and let A be a commutative $\mathbb{K}$-algebra which is a bimodule ${}_TA_S$. By commutativity we have that $A^{\mathrm{op}} = A$ and hence A also is an ${}_SA_T$ bimodule.

A right adjoint of ${}_TA_S$ is given by ${}_SA_T$ with

$$\text{unit} \qquad (\, u : \mathrm{id}_T \to {}_TA_S \otimes_S {}_SA_T\,) \;=\; {}_{{}_TT_T}A_{({}_TA_S)\otimes_S({}_SA_T)},$$

$$\text{counit} \qquad (\, c : {}_SA_T \otimes_T {}_TA_S \to \mathrm{id}_S\,) \;=\; {}_{({}_SA_T)\otimes_T({}_TA_S)}A_{{}_SS_S}.$$

For the unit u we have an adjunction $(u \dashv u^R, u_u, c_u)$ with

$$u^R = {}_{({}_TA_S)\otimes_S({}_SA_T)}A_{{}_TT_T}, \tag{12.2}$$

$$\text{unit} \quad u_u = {}_TA_{({}_TA_{A\otimes_S A})\,\otimes_{(A\otimes_S A)}\,({}_{A\otimes_S A}A_T)}, \tag{12.3}$$

$$\text{counit} \quad c_u = {}_{({}_{A\otimes_S A}A_T)\,\otimes_T\,({}_TA_{A\otimes_S A})}A_{{}_{A\otimes_S A}(A\otimes_S A)_{A\otimes_S A}}. \tag{12.4}$$

For the counit c we have an adjunction $(c \dashv c^R, u_c, c_c)$ with

$$c^R = {}_{S}S_{S}A_{({}_SA_T)\otimes_T({}_TA_S)}, \quad (12.5)$$

$$\text{unit} \quad u_c = {}_{({}_SA_{A\otimes_TA})\otimes_{(A\otimes_TA)}({}_{A\otimes_TA}A_S)}A_{(A\otimes_SA_S)\otimes_S{}_SA_{A\otimes_TA}}, \quad (12.6)$$

$$\text{counit} \quad c_c = {}_{(A\otimes_TA A_T)\otimes_T({}_TA_{A\otimes_TA})}A_S. \quad (12.7)$$

Note that we shortened the notation of the 3-morphisms u_u, u_c, c_u, c_c and wrote A instead of ${}_TA_S$ or ${}_SA_T$ on the lowest level. Similarly, the T and S in u_u and c_c are actually

$${}_{{}_TT_T}T_{{}_TT_T} \qquad \text{and} \qquad {}_{{}_SS_S}S_{{}_SS_S}.$$

All the adjunctions above form 2-times right adjunctibility data for ${}_TA_S$. □

The second statement we need is a generalization of Proposition 12.1.3. Let T, S be objects, A, B be 1-morphisms from T to S, M, N 2-morphisms from A to B, and ${}_M\Sigma_N$ from M to N a 3-morphism in Alg_3.

Proposition 12.2.6. *The 3-morphism ${}_M\Sigma_N$ in Alg_3 has a left adjoint if and only if Σ is finitely presented and projective merely as a module over the underlying algebra of M, and similarly for the right adjoint. The adjoints are given by* $\mathrm{Hom}_M(\Sigma, M)$ *and* $\mathrm{Hom}_N(\Sigma, N)$*, respectively.*

For this we need a statement about "lifting" the adjoints from Proposition 12.1.3 to adjoints of 3-morphisms. This lifting procedure is exactly the same as the one described in [BJS21, Proposition 5.17].

Proposition 12.2.7. *The (N, M)-module* $\mathrm{Hom}_M(\Sigma, M)$ *has a canonical structure of an (A, B)-bimodule, given for $F \in \mathrm{Hom}_M(\Sigma, M)$ and $c \in A \otimes B^{\mathrm{op}}$ by*

$$c \cdot F(-) := F(- \cdot (1_N c)) = F((1_M c) \cdot -) = (1_M c) \cdot F(-) = F(-) \cdot (1_M c) =: F(-) \cdot c\,.$$

Similarly, this structure is compatible with the structure of A, B being (R, S)-modules and hence $\mathrm{Hom}_M(\Sigma, M)$ *is a 3-morphism from N to M.*

Proof. The first equality holds by (12.1). The second holds because F is an M-module. The third equality holds because the action of c on 1_M lands in the center $Z(M)$. □

Proof of Proposition 12.2.6. We discuss the case of the left adjoint; the statement for the right adjoint is similar. Let ${}_M\Sigma_N$ be a 3-morphism in Alg_3, where M and N are (A, B)-bimodules, and A and B are (T, S)-modules. There is a natural map from the (homotopy) bicategory $\mathrm{h}_2\,\mathsf{C}(A,B)$ to Alg_1, forgetting the extra bimodule structures. We can apply Proposition 12.1.3 to the image of ${}_M\Sigma_N$ and obtain a left adjoint $\mathrm{Hom}_M(\Sigma, M)$ in Alg_1 if and only if Σ is finitely presented and projective over M. Now applying Proposition 12.2.7 Σ can be given the structure of a 3-morphism from N to M. The unit and counit of the adjunction in Alg_1 are homomorphisms of the underlying bimodules, but since the actions of T, S, A, B are all central, the unit and counit respect this action as well. □

Example 12.2.8. Now we take T, S commutative $\mathbb{K}$-algebras such that $A = T$ is a right S-module which is not finitely presented and projective. We claim that in Alg_3, as a 1-morphism ${}_AA_S$ is 2-times right adjunctible, but not 3-times right adjunctible. Moreover, it is 3-times left adjunctible.

To see this, in Proposition 12.2.5 we can read off the 2-times right adjunctibility data of the adjunction

$$ {}_AA_S \dashv {}_SA_A\,. \tag{12.8} $$

First look at the counit c_c in (12.7). By Proposition 12.1.3 it does not have a right adjoint. So we cannot complete the 2-times right adjunctibility data to 3-times right adjunctibility data. By the uniqueness of adjoints up to isomorphism (see Proposition 10.2.4) it follows that ${}_AA_S$ is not 3-times right adjunctible.

To look at left adjunctibility we could take the general adjunctibility data of

$$ ({}_TA_S \dashv {}_SA_T, u, c) $$

from above and swap S and T to obtain 2-times left adjunctibility data of ${}_TA_S$.

Instead we apply Lemma 11.2.16 to the adjunction of 1-morphisms (12.8) to obtain the adjunction

$$ ({}_SA_A \dashv {}_AA_S, c^R, u^R). $$

Left adjoints for c^R and u^R are given by (12.2) and (12.5) with $T = A$. Units and counits of these adjunctions are given by u_u, u_c, c_u and c_c with $T = A$ in (12.3), (12.4), (12.6) and (12.7). These are all A, viewed as a left module for – respectively –

$$ A, \qquad A \otimes_A A, \qquad A \otimes_{(A\ \otimes_A\ A)} A \qquad \text{and} \qquad A \otimes_A A\,. $$

Since in each case it is finitely presented projective, u_u, u_c, c_u and c_c all have left adjoints by Proposition 12.2.6. This is a 3-times left adjunctibility datum. Additionally, A and S are 3-dualizable in C, since all objects of Alg_3 are 3-dualizable. ◇

Remark 12.2.9. It is possible to generalize this idea and find examples in all higher categories $\mathsf{Alg}_n(\mathsf{Vect}_{\mathbb{K}})$ for n odd. ◇

Morita categories also allow us to look at an example of mixed adjunctibility in the sense of Definition 11.2.1. For $n = 2$ we have 4 possible dexterity functions a^n given by 'LL', 'RR' 'RL' and 'LR', where the first two and the latter two define equivalent notions of adjunctibility by Theorem 11.2.6. To see that in general these two classes are different we consider 1-morphisms in the weak 3-category Alg_2. We could define $\mathsf{Alg}_2 = \mathrm{Hom}_{\mathsf{Alg}_3}(\mathbb{K}, \mathbb{K})$, but let us unravel this. Informally, Alg_2 can be described as follows.

- Objects are commutative $\mathbb{K}$-algebras,
- a 1-morphism between two 1-morphisms $A \to B$ is an associative $\mathbb{K}$-algebra M which also is a central bimodule ${}_A M_B$;
- a 2-morphism between two 2-morphisms $N \to M$ is a bimodule ${}_M \Sigma_N$ such that for every $\sigma \in \Sigma$ and $c \in A \otimes B^{\mathrm{op}}$, we have that

$$(c \cdot 1_M) \cdot \sigma = \sigma \cdot (c \cdot 1_N) \; ; \tag{12.9}$$

- and a 3-morphism is a homomorphism, meaning it is compatible with all actions of the source and target objects and 1-morphisms.

Compositions of morphisms which are bimodules is given by tensoring over the middle algebra, similarly to the composition of 1-morphisms in Alg_1. Composition of 3-morphisms is composition of homomorphisms.

Let T and S be commutative $\mathbb{K}$-algebras and ${}_T A_S$ be a 1-morphism $T \to \mathbb{K}$. Similar to Alg_3 we have an adjunction $({}_T A_S \dashv {}_S A_T^{\mathrm{op}}, u, c)$ with

$$\text{unit} \qquad (\, u : \mathrm{id}_T \to {}_T A_S \otimes_S {}_S A_T \,) \;=\; {}_{T T_T} A_{({}_T A_S) \otimes_S ({}_S A_T)}, \tag{12.10}$$

$$\text{counit} \qquad (\, c : {}_S A_T \otimes_T {}_T A_S \to \mathrm{id}_S \,) \;=\; {}_{({}_S A_T) \otimes_T ({}_T A_S)} A_{S S_S}. \tag{12.11}$$

Example 12.2.10. Consider the 1-morphism ${}_T A_S$ in Alg_2 with $A = S = \mathbb{K}$ and $T = \mathbb{K}[x]/(x^2)$ acting trivially. Recall that the dexterity function odd^2 is given by $\mathrm{odd}^2(1) = R$ and $\mathrm{odd}^2(2) = L$. We have data of a right adjoint of ${}_T A_S$ as given above. The question is whether we can extend this to odd^2-adjunctibility data. This is not the case, since A is not projective as a left T-module and by Proposition 12.1.3 the unit u in (12.10) is not left adjunctible. Thus ${}_T A_S = {}_{\mathbb{K}[x]/(x^2)} \mathbb{K}_{\mathbb{K}}$ is not odd^2-adjunctible.

On the other hand, $A = \mathbb{K}$ is trivially a finitely presented projective module over $\mathbb{K} = \mathbb{K}_{\mathbb{K}} \otimes_{\mathbb{K}} {}_{\mathbb{K}}\mathbb{K}^{\mathrm{op}}$. Thus, by Proposition 12.1.3 the unit u in (12.10) and the counit c in (12.11) with $A = S = \mathbb{K}$ have right adjoints. Since $\mathrm{even}^2(1) = \mathrm{even}^2(2) = R$, we get that ${}_T A_S = {}_{\mathbb{K}[x]/(x^2)} \mathbb{K}_{\mathbb{K}}$ is even^2-adjunctible.

◇

13. Applications to Field Theories

Contents

One of our motivations for this article was to understand the dualizability requirements for the notion of a relative version of topological field theories. In this section we explain our results in this context and prove some variations and extensions.

Relative versions of field theories were first proposed by Stolz–Teichner in [ST11] using the name **twisted field theory**, and by Freed–Teleman in [FT14] using the name **relative field theory**. More recently, Freed–Moore–Teleman [FMT22] introduced the term **quiche** as being half of a **sandwich field theory**, which has a boundary on two sides; both in the context of symmetries of field theories.

Relative theories should capture the following idea. Instead of attaching a number to top-dimensional closed manifolds, we choose an element in a vector space (for instance a line), implemented as a morphism $\mathbb{1} \to V$, and this vector space should depend on the manifold M,

$$\mathbb{1} \to T(M)\,. \tag{13.1}$$

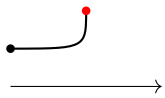
A first approximation to the notion of a relative field theory, motivated by physical bulk-boundary systems, is an **(n+1)-dimensional field theory with boundaries**, that is, a symmetric monoidal functor

$$\mathscr{Z}^\partial \colon \mathsf{Bord}^\partial_{n+1} \longrightarrow \mathcal{C},$$

where $\mathsf{Bord}^\partial_{n+1}$ is a (possibly higher) category[1] of cobordisms with free (=marked) boundaries, and $\mathcal{C}$ is a symmetric monoidal (∞, N)-category for $N > n$. In the fully

[1] Such a higher category is outlined in [FT21] and is being worked out in detail by William Stewart in his PhD thesis.

extended framed situation, an extension of the Cobordism Hypothesis, namely Lurie's Cobordism Hypothesis with singularities [Lur09] explained that these are fully determined by their value at an interval with one incoming boundary component (black) and one free boundary (red) viewed as a bordism from one point to the empty set:



This value is an n-adjunctible 1-morphism from an $(n+1)$-dualizable object to the unit in $\mathcal{C}$. In turn, according to the Cobordism Hypothesis with singularities, every such n-adjunctible 1-morphism gives rise to a framed fully extended $(n+1)$-dimensional TFT with boundaries, and in particular a framed fully extended $(n+1)$-dimensional TFT $\mathscr{Z}^{bulk}$ by restricting to bordisms with no free (red) boundary via the inclusion

$$\mathsf{Bord}_{n+1}^{fr} \hookrightarrow \mathsf{Bord}_{n+1}^{\partial}.$$

Often the theory $\mathscr{Z}^{bulk}$ is required to be invertible and is called an **anomaly**.

Note the dimension: we would like to consider such a theory as a generalization of an "absolute" theory of one dimension less valued in the looping $\Omega\mathcal{C}$, namely, the values "at the free boundary",

$$\mathscr{Z}\colon \mathsf{Bord}_n^{fr} \to \Omega\mathcal{C}\,. \tag{13.2}$$

A slight generalization is a **defect theory**, in which we allow two bulk theories separated by defect, namely a codimension 1 submanifold. In the framed fully extended case, according to the Cobordism Hypothesis with singularities these are characterized by an n-adjunctible 1-morphism between two $(n+1)$-dualizable objects in $\mathcal{C}$.

In contrast, a **relative or twisted field theory** should not require the bulk theory to be $(n+1)$-dimensional, but rather merely be n-dimensional. Geometrically, this **should** amount to only asking for (some) cylinders over the free boundary. At present, such a cobordism (higher) category has not been constructed[2].

Instead, Stolz–Teichner in [ST11] defined a twisted field theory to be a symmetric monoidal natural transformation between symmetric monoidal functors

$$\mathscr{Z}^{tw}: \qquad \mathsf{Bord}_n \underset{T}{\overset{S=\mathbb{1}}{\Downarrow}} \mathsf{C}\,. \tag{13.3}$$

[2]This will be worked out in William Stewart's PhD thesis.

Here, if Bord_n is an (∞, n)-category, then $\mathcal{C}$ should be an (∞, N)-category for $N > n$, as above. Moreover, we often require that the source $S = \mathbb{1}$ is the trivial functor valued the unit in which case T is called **twist**. However, for this definition to be meaningful, the natural transformation must be **lax** or **oplax**, as was explained in the bicategorical situation in [ST11].

For higher categories lax and oplax natural transformations were defined in [JFS17]. Unfortunately, there is a slightly unsatisfying dichotomy: symmetric monoidal **oplax** natural transformations from $\mathbb{1}$ to T enjoy the desired feature (13.1) that for any closed bordism b we have a morphism $\mathbb{1} \to T(b)$, so we are chosing an element in $T(b)$. In the **lax** situation, the arrow switches direction depending on the dimension (see [JFS17, Example 7.3]). On the other hand, **lax** symmetric monoidal natural transformations from $\mathbb{1}$ to $\mathbb{1}$ are equivalent to untwisted ("usual") field theories valued in $\Omega\mathcal{C}$ and hence satisfy (13.2), whereas **oplax** ones are equivalent to untwisted ("usual") field theories in $\Omega\mathcal{C}^{\mathrm{odd\,op}}$ ([JFS17, Theorem 7.4 & Remark 7.5]).

Furthermore, in [JFS17] a classification of framed fully extended twisted field theories was given using the Cobordism Hypothesis in terms of the value at a point. This value is required to be an a^n-adjunctible 1-morphism between n-dualizable objects, where $a^n = l^n$ or $a^n = r^n$ (see Example 11.2.3) for lax and oplax, respectively [JFS17, Theorem 1.6]. We saw in Corollary 11.2.9 that, as a consequence of our main theorem, in terms of the dualizability/adjunctibility these notions coincide in even dimensions and that in this case there is a second class of conditions we could have chosen. This suggests the question of whether we could have changed the definition of (op)lax TFTs slightly by reversing directions in a way to simultaneously satisfy (13.1) and (13.2).

In this section we prove that none of the choices resolves this dichotomy. Hence, as the first requirement of chosing elements in the theory T seems to be generally considered the more important one, we have no choice but to **define** a relative or twisted field theory to be a symmetric monoidal **oplax** natural transformation from $\mathbb{1}$ to some categorified field theory T.

Moreover, Freed–Teleman in [FT14] avoid giving a precise definition of relative field theory, and for instance in the situation in [FT21] a stronger condition than the ones we just considered appears: namely, an n-adjunctible 1-morphism between objects which are only n-dualizable.

Summarizing, to obtain a framed fully extended boundary/defect, relative/twisted TFT, or dualizable relative field theory, we require the following data for a 1-morphism $f \colon X \to Y$. The above mentioned stronger condition is indicated in the last column.

	defect	twisted lax	twisted oplax	[FT21]
objects X, Y	$(n+1)$-dualizable	n-dualizable	n-dualizable	n-dualizable
1-morphism f	left & right adjoints	left adjoint	right adjoint	left & right adjoints
(co)unit 2-morphisms	left & right adjoints	left adjoint	right adjoint	left & right adjoints
⋮	⋮	⋮	⋮	⋮
(co)unit n-morphisms	left & right adjoints	left adjoint	right adjoint	left & right adjoints

Our second goal in this section is to compare dualizable relative/twisted field theories to the final option.

13.1. Comparing lax and oplax natural Transformations and Variants

In this section we freely use the notation from [JFS17]. In brief, given an (∞, N)-category $\mathcal{C}$ two arrow categories are defined, namely $\mathsf{C}^{\to}$ and $\mathsf{C}^{\downarrow}$, which we will call the **oplax and lax arrow categories of** $\mathcal{C}$, respectively. Both have 1-morphisms in $\mathcal{C}$ as objects and squares as 2-morphisms, but the squares are filled with 2-morphisms in different directions. They are defined by

$$\mathsf{C}^{\to}_{\vec{k}} = \mathrm{map}(\Theta^{(1);\vec{k}}, \mathsf{C}) \qquad \text{and} \qquad \mathsf{C}^{\downarrow}_{\vec{k}} = \mathrm{map}(\Theta^{\vec{k};(1)}, \mathsf{C})$$

and $\Theta^{\vec{k};(1)}$ is glued from certain computads $\Theta^{(j);(1)}$, and similarly for $\Theta^{(1);\vec{k}}$, see [JFS17, Definition 5.7, 5.10].

Using these categories, we briefly recall the definition of lax and oplax natural transformations.

Definition 13.1.1. [JFS17, Definition 1.3 & Definition 6.7] Let $\mathcal{C}$ be an (∞, N)-category. A **lax natural transformation** is a functor

$$\mathcal{B} \longrightarrow \mathsf{C}^{\downarrow}$$

and an **oplax natural transformation** is a functor

$$\mathcal{B} \longrightarrow \mathsf{C}^{\to}\,.$$

Symmetric monoidal (op)lax natural transformations are symmetric monoidal such functors.

Using the Cobordism Hypothesis, the following Theorem characterizes (op)lax twisted fully extended framed topological field theories.

Theorem 13.1.2 (Theorem 7.6 in [JFS17])**.** *Let $n > 0$ and C be a symmetric monoidal (∞, N)-category. An object $f : X \to Y$ in $\mathsf{C}^{\downarrow}$ is n-dualizable if and only if X, Y are n-dualizable and f as a morphism in C is n-times left adjunctible. An object $f : X \to Y$ in $\mathsf{C}^{\to}$ is n-dualizable if and only if X, Y are n-dualizable and f as a morphism in C is n-times right adjunctible.*

Our first proposition is a connection between the lax and oplax arrow categories.

Proposition 13.1.3. *Let $\mathcal{C}$ be an (∞, N)-category. With the opposite function from Example 11.3.2 we have an equivalence of $(\infty, N-1)$-categories*

$$\left(\mathsf{C}^{\mathrm{odd\,op}\to}\right)^{\mathrm{odd\,op}} \simeq \mathsf{C}^{\downarrow},$$

which on j-morphisms is induced by the isomorphism of computads from Lemma 13.1.4. Moreover, the two maps $s_v, t_v \colon \mathsf{C}^{\downarrow} \to \mathsf{C}$ correspond to the maps

$$t_h, s_h \colon \left(\mathsf{C}^{\mathrm{odd\,op}\to}\right)^{\mathrm{odd\,op}} \to \mathsf{C}^{\mathrm{odd\,op}}.$$

Proof. We show that

$$\mathsf{C}^{\mathrm{odd\,op}\to} \simeq \left(\mathsf{C}^{\downarrow}\right)^{\mathrm{odd\,op}}.$$

Since $\Theta^{(1);\vec{k}}$ is glued from $\Theta^{(1);(j)}$ and similarly for the indices switched, we obtain isomorphisms

$$\left(\Theta^{(1);\vec{k}}\right)^{\mathrm{odd\,op}} \cong \Theta^{\vec{k};(1)}$$

from the isomorphism of computads from the following Lemma 13.1.4 and a tedious matching. These isomorphisms induce equivalences of spaces

$$\left(\mathsf{C}^{\mathrm{odd\,op}\to}\right)_{\vec{k}} \simeq \left(\mathsf{C}^{\downarrow}\right)_{\vec{k}}.$$

To see that these assemble to an equivalence of higher categories, observe that the face maps in $\mathsf{C}^{\to}$ and $\mathsf{C}^{\downarrow}$ are induced by the inclusions

$$s_v, t_v \colon \Theta^{(1);(j-1)} \hookrightarrow \Theta^{(1);(j)} \qquad \text{and} \qquad s_h, t_h \colon \Theta^{(j-1);(1)} \hookrightarrow \Theta^{(j);(1)},$$

respectively. In Lemma 13.1.4 we identified s_v, t_v on the left with s_h, t_h on the right, the ordering depending on the parity. □

Lemma 13.1.4. *There is an isomorphism of computads*

$$\left(\Theta^{(1);(j)}\right)^{\mathrm{odd\,op}} \cong \Theta^{(j);(1)},$$

given as follows. In both cases the top generators exchange, $\theta_{1;j} \leftrightarrow \theta_{j;1}$. If

j is odd

$$s_v\theta_{1;j-1} \leftrightarrow t_h\theta_{j-1;1} \qquad s_h\theta_{0;j} \leftrightarrow t_v\theta_{j;0}$$
$$t_v\theta_{1;j-1} \leftrightarrow s_h\theta_{j-1;1} \qquad t_h\theta_{0;j} \leftrightarrow s_v\theta_{j;0} \,.$$

j is even

$$s_v\theta_{1;j-1} \leftrightarrow s_h\theta_{j-1;1} \qquad s_h\theta_{0;j} \leftrightarrow t_v\theta_{j;0}$$
$$t_v\theta_{1;j-1} \leftrightarrow t_h\theta_{j-1;1} \qquad t_h\theta_{0;j} \leftrightarrow s_v\theta_{j;0} \,.$$

Here we use the same notation for the generators of $\Theta^{\mathrm{odd\,op}}_{(1);(j)}$ as for $\Theta_{(1);(j)}$.

Proof. The definition of $\theta_{1;j}$ in $\Theta^{(1);(j)}$ says, since 1 is odd,

$$t_v\theta_{1;j-1} \circ s_h\theta_{0;j} \xrightarrow{\theta_{1;j}} t_h\theta_{0;j} \circ s_v\theta_{1;j-1} \,. \tag{13.4}$$

If j is odd, since all morphisms appearing in the source and target are j-morphisms and hence are reversed in $\left(\Theta^{(1);(j)}\right)^{\mathrm{odd\,op}}$, we have in $\left(\Theta^{(1);(j)}\right)^{\mathrm{odd\,op}}$ that

$$s_h\theta_{0;j} \circ t_v\theta_{1;j-1} \xrightarrow{\theta_{1;j}} s_v\theta_{1;j-1} \circ t_h\theta_{0;j} \,.$$

On the other hand, if j is odd, in $\Theta^{(j);(1)}$ we have that

$$t_v\theta_{j;0} \circ s_h\theta_{j-1;1} \xrightarrow{\theta_{j;1}} t_h\theta_{j-1;1} \circ s_v\theta_{j;0} \,.$$

Comparing the last two expressions we find the exchange.

If j is even, in (13.4) the morphisms appearing in the source and target are j-morphisms and hence are not reversed in $\left(\Theta^{(1);(j)}\right)^{\mathrm{odd\,op}}$, but $\theta_{1;j}$ is a $(j+1)$-morphism and hence reversed, so the source and target switch:

$$t_h\theta_{0;j} \circ s_v\theta_{1;j-1} \xrightarrow{\theta_{1;j}} t_v\theta_{1;j-1} \circ s_h\theta_{0;j} \,.$$

On the other hand, in $\Theta^{(j);(1)}$ we have that

$$s_v\theta_{j;0} \circ s_h\theta_{j-1;1} \xrightarrow{\theta_{j;1}} t_h\theta_{j-1;1} \circ t_v\theta_{j;0} \,.$$

Now proceed by induction. □

Inspired by Proposition 13.1.3, we generalize the (op)lax arrow categories to general opposite functions. Recall from Proposition 11.3.3 the opposite function $\mathrm{op}^N_{a^n,b^n,k}$ associated to a pair of dexterity functions. In this section, we will always use $b^n = r^n$ and $k = 1$, hence we abbreviate notation to

$$\mathrm{op}_{a^n} \coloneqq \mathrm{op}_{a^n,r^n,1} \,.$$

Definition 13.1.5. Let C be a (∞, N)-category and a^n a dexterity function. Define the **a^n-lax arrow category** to be the $(\infty, N-1)$-category

$$\mathsf{C}^{a^n} := (\mathsf{C}^{\mathrm{op}_{a^n}\to})^{\mathrm{op}_{a^n}} \ .$$

We could also have used the other opposite function from Proposition 11.3.3 in the definition above and we would get an equivalent arrow category. To see this we use the following Proposition that we obtain similarly to Proposition 13.1.3.

Proposition 13.1.6. *Let $\mathcal{C}$ be an (∞, N)-category. Let* $\mathrm{oppp} \equiv \mathrm{op}$ *be the constant opposite function valued* op. *Then there is an equivalence of $(\infty, N-1)$-categories*

$$(\mathsf{C}^{\mathrm{oppp}\to})^{\mathrm{oppp}} \simeq \mathsf{C}^{\to}$$

induced on j-morphisms by isomorphisms of computads as indicated in (13.5).

Moreover, the two maps $s_h, t_h \colon \mathsf{C}^{\to} \to \mathsf{C}$ correspond to the maps

$$t_h, s_h \colon \ (\mathsf{C}^{\mathrm{oppp}\to})^{\mathrm{oppp}} \to \mathsf{C}^{\mathrm{oppp}} \, .$$

Proof. Recall that in $\Theta^{(1);(j)}$, we have that

$$t_v\theta_{1;j-1} \circ s_h\theta_{0;j} \xrightarrow{\theta_{1;j}} t_h\theta_{0;j} \circ s_v\theta_{1;j-1} \, .$$

Hence, in $(\Theta^{(1);(j)})^{\mathrm{oppp}}$ we have that

$$s_h\theta_{0;j} \circ t_v\theta_{1;j-1} \xleftarrow{\theta_{1;j}} s_v\theta_{1;j-1} \circ t_h\theta_{0;j} \, .$$

Switching s_v and t_v, and replacing s_h and t_h, we obtain an isomorphism of computads

$$(\Theta^{(1);(j)})^{\mathrm{oppp}} \cong (\Theta^{(1);(j)})^{\mathrm{oppp}} \, . \tag{13.5}$$

□

Since $\mathcal{D}^{-\mathrm{op}_{a^n,r^n,1}} = (\mathcal{D}^{\mathrm{op}_{a^n,r^n,1}})^{\mathrm{oppp}}$ holds for any (∞, M)-category D we immediately conclude the following.

Corollary 13.1.7. *Let C be an (∞, N)-category and a^n a dexterity function. Then there is an equivalence of $(\infty, N-1)$-categories*

$$\mathsf{C}^{a^n} \simeq \left(\mathsf{C}^{-\mathrm{op}_{a^n}\to}\right)^{-\mathrm{op}_{a^n}} .$$

Example 13.1.8. If $a^n = r^n$ we have from Example 11.3.4 that $\mathrm{op}_{a^n} \equiv \mathrm{id}$ is the constant opposite function valued id. In this case we have $\mathsf{C}^{a^n} = \mathsf{C}^{\to}$.

◇

Example 13.1.9. If $a^n = l^n$ and $n+1 = N$ we seen in Example 11.3.4 that $\mathrm{op}_{a^n} = \mathrm{even\,op}$. Hence $\mathsf{C}^{a^n} = (\mathsf{C}^{\mathrm{even\,op}\rightarrow})^{\mathrm{even\,op}}$ and by Corollary 13.1.7 and Proposition 13.1.3 we have

$$\mathsf{C}^{a^n} \simeq \left(\mathsf{C}^{\mathrm{odd\,op}\rightarrow}\right)^{\mathrm{odd\,op}} \simeq \mathsf{C}^{\downarrow}.$$

◇

Theorem 13.1.2 generalizes to these generalized arrow categories.

Corollary 13.1.10. *Let C be a symmetric monoidal (∞, N)-category, $f : X \to Y$ a 1-morphism in C, and a^n a dexterity function. The following are equivalent:*

1. *When viewed as an object in C^{a^n}, f is n-dualizable.*
2. *The objects X and Y are n-dualizable and f is n-times right adjunctible in $\mathsf{C}^{\mathrm{op}_{a^n}}$.*
3. *The objects X and Y are n-dualizable and f is a^n-adjunctible in C.*

Proof. Let $\mathrm{op}_{a^n} = \mathrm{op}_{a^n, r^n, 1}$ be the opposite function associated to a^n and r^n from Proposition 11.3.3. Since taking opposites does not change dualizability, the 1-morphism f is dualizable in C^{a^n} if and only if it is dualizable in $\mathsf{C}^{\mathrm{op}_{a^n}\rightarrow}$. This is the case if and only if X and Y are n-dualizable in $\mathsf{C}^{\mathrm{op}_{a^n}}$ and f is n-times right adjunctible in $\mathsf{C}^{\mathrm{op}_{a^n}}$, by applying Theorem 13.1.2 to $\mathsf{C}^{\mathrm{op}_{a^n}\rightarrow}$. By Proposition 11.3.3 the latter is equivalent to f being a^n-adjunctible in C. □

Our conclusion now is the following: we **could** define an a^n-lax twisted field theory to be a symmetric monoidal "a^n-lax natural transformation", i.e. a symmetric monoidal functor

$$\mathsf{Bord}_n \longrightarrow \mathsf{C}^{a^n},$$

and Corollary 13.1.10 characterizes the framed fully extended ones in terms of a^n-adjunctability. With this at hand, we have a (many!) notion(s) of a^n-lax twisted field theory which in terms of dualizability lies in the second equivalence of dexterity functions. However, this does not resolve the dichotomoy between (13.1) and (13.2):

If we want to satisfy (13.1), we must choose the **oplax** case, since the other choices involve taking opposites at certain levels. No other dexterity function can achieve (13.1).

As for the other desideratum (13.2), a variant of [JFS17, Theorem 7.4 & Remark 7.5] shows that since we are taking opposites at various levels, this is only satisfied for the **lax** case.

13.2. Dualizable relative Topological Field Theories

We now turn to comparing the adjunctibility conditions in the table in the beginning of this section. Leaving the first column aside, observe that by ambidexterity, i.e. Theorem 11.2.10 (a) the difference only lies in the adjunctibility of the appearing (co)unit n-morphisms. Hence, we only need to check that the appearing (co)unit n-morphisms also have the other adjoint. This can be guaranteed by asking for the twisted/relative field theory to be itself adjunctible, as was suggested to us by Constantin Teleman.

In the following propositions we assume that the Cobordism Hypothesis holds, that C is an (∞, N)-category and $n \geq 1$.

Proposition 13.2.1. *Let $\mathscr{Z}^{otw}$ be a framed fully extended n-dimensional oplax twisted field theory, i.e. a symmetric monoidal oplax natural transformation as above.*

If $\mathscr{Z}^{otw}$, when viewed as a 1-morphism in the (∞, n)-category $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$, is adjunctible, then the value of $\mathscr{Z}^{otw}$ at a point is a 1-morphism $f\colon X \to Y$ in $\mathcal{C}$ such that X and Y are n-dualizable and f is n-adjunctible.

The same statement is true for "oplax" replaced by "lax".

Proof. By the characterization [JFS17, Theorem 1.6] the value of $\mathscr{Z}^{otw}$ at a point is a 1-morphism $f\colon X \to Y$ in $\mathcal{C}$ such that X and Y are n-dualizable and f is n-times right adjunctible.

Evaluating the left adjoint $\mathscr{Z}^L$ of $\mathscr{Z}^{otw}$, we obtain a 1-morphism $g\colon Y \to X$ in $\mathcal{C}$, which again by [JFS17, Theorem 1.6] is n-times right adjunctible. Moreover, evaluating the adjunction data of the adjunction $\mathscr{Z}^L \dashv \mathscr{Z}^{otw}$ at a point, we see that g is a left adjoint of f. By Corollary 11.2.13 f is n-adjunctible. □

Proposition 13.2.2. *Let $j \geq 1$. Let $\mathscr{Z}^{otw}$ be a framed fully extended n-dimensional oplax twisted field theory and the 1-morphism $f\colon X \to Y$ in C its value at a point. Then, the following are equivalent:*

1. *In $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$ we have that the 1-morphism $\mathscr{Z}^{otw}$ is even^j-adjunctible.*

2. *In $\mathcal{C}$ we have that X and Y are n-dualizable and f is even^{n+j}-adjunctible.*

Let $\mathscr{Z}^{ltw}$ be a framed fully extended n-dimensional lax twisted field theory and the 1-morphism $g\colon X \to Y$ in C its value at a point. Then, the following are equivalent:

1. *In $\mathrm{Fun}^{lax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$ we have that the 1-morphism $\mathscr{Z}^{ltw}$ is even^j-adjunctible.*

2. *In $\mathcal{C}$ we have that X and Y are n-dualizable and*

$$g \text{ is } \begin{cases} \mathrm{even}^{n+j}\text{-adjunctible} & \text{if } n \text{ is even}, \\ \mathrm{odd}^{n+j}\text{-adjunctible} & \text{if } n \text{ is odd}. \end{cases}$$

The same is true with all appearances of "even" and "odd" reversed.

Proof. We consider the oplax case first. (1) ⇒ (2) If $\mathscr{Z}^{otw}$ is even^j-adjunctible, it has a right adjoint $\mathscr{Z}^R$. Evaluating $\mathscr{Z}^R$ at a point we obtain a 1-morphism f^R that is a right adjoint of f. The unit and counit of the adjunction $\mathscr{Z}^{otw} \dashv \mathscr{Z}^R$ are 2-morphisms $\mathscr{U}$ and $\mathscr{C}$ in $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$. By [JFS17, Corollary 7.7] evaluating $\mathscr{U}$ and $\mathscr{C}$ at a point we obtain 2-morphisms u and c in C which are n-times right adjunctible. Hence, f is $(n+1)$-times right adjunctible.

By induction, by iterating this argument for the adjunctibility data of the adjunction $\mathscr{Z}^{otw} \dashv \mathscr{Z}^R$, we have that this evaluates at a point to adjunctibility data exhibiting f as $(n+j)$-times right adjunctible, which is the same as even^{n+j}-adjunctible.

If $\mathscr{Z}^{otw}$ is odd^j-adjunctible we can use the same induction to get $(n+j-1)$-times right adjunctibility data for f. However, in the last step instead of right adjoints for the unit and counit j-morphisms of the data of $\mathscr{Z}^{otw}$, we have left adjoints. After evaluation at a point we obtain left adjoints of the unit and counit j-morphisms of the $(j-1)$-times right adjunctibility data for f that are themselves n-times right adjunctible by [JFS17, Corollary 7.7]. Combining these we get a^{n+j}-adjunctibility, with

$$a^{n+j} : \{1, \ldots, n+j\} \to \{L, R\}, \quad i \mapsto \begin{cases} L & i = j, \\ R & i \neq j \end{cases} \tag{13.6}$$

which by Theorem 11.2.6 is equivalent to odd^{n+j}-adjunctibility data for f.

(2) ⇒ (1) Let $f\colon X \to Y$ be an even^{n+j}-adjunctible. 1-morphism between n-dualizable objects. Then f is n-adjunctible by Theorem 11.2.10. Moreover, f has a right adjoint f^R which itself is n-adjunctible by Corollary 11.2.14, and in particular n-times right adjunctible. Hence, by [JFS17, Theorem 1.6] we obtain a framed fully extended n-dimensional oplax twisted field theory $\mathscr{Z}^R$. We claim that this is a right adjoint.

The unit u and counit c of $f \dashv f^R$ are n-times right adjunctible because f is $(n+1)$-times right adjunctible. So, invoking [JFS17, Corollary 7.7] there are 2-morphisms $\mathscr{U}$ and $\mathscr{C}$ in $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$ whose value at a point are u and c, respectively. We claim that they are the unit and counit of an adjunction. To see this, we check the compositions in the (zig) and (zag) at the point, where they are identities because u and c are the (co)units of an adjunction. Hence, by [JFS17, Corollary 7.7] they are identities in $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$.

We now repeat the argument inductively, replacing $f \dashv f^R$ by an adjunction appearing in the adjunctibility data of f to produce adjunctibility data of $\mathscr{Z}^{otw} \dashv \mathscr{Z}^R$ exhibiting j-times right adjunctibility.

If $f\colon X \to Y$ is odd^{n+j}-adjunctible we can construct $(j-1)$-times right adjunctibility data for $\mathscr{Z}^{otw}$ as above. For the last step we use that f by Theorem 11.2.6 is also a^{n+j}-adjunctible as defined in (13.6). From the a^{n+j}-adjunctibility data we get left adjoints of the unit and counit j-morphisms that are themselves n-times right adjunctible. By [JFS17, Corollary 7.7] and arguing as above we complete the $(j-1)$-times right adjunctibility data for $\mathscr{Z}^{otw}$ to odd^j-adjunctibility data.

The lax case works similar with appropriate dexterity functions given by Theorem 11.2.6. □

Combining the Proposition above with Theorem 11.2.10 we obtain the following.

Corollary 13.2.3. *Let $j \geq 1$. Let $\mathscr{Z}^{otw}$ be a framed fully extended n-dimensional oplax twisted field theory and the 1-morphism $f\colon X \to Y$ in C its value at a point. Then, the following are equivalent:*

1. *In $\mathrm{Fun}^{oplax}(\mathsf{Bord}_n^{fr}, \mathcal{C})$ we have that the 1-morphism $\mathscr{Z}^{otw}$ is j-adjunctible.*
2. *In $\mathcal{C}$ we have that X and Y are n-dualizable and f is $(n+j)$-adjunctible.*

The same statement is true for "oplax" replaced by "lax".

14. More General Notions of Higher Adjunctibility and Binary Trees

Contents

In this section we go one step further in generalizing the definition of mixed higher adjunctibility. The following example illustrates that in a higher Morita category we can have a kind of 2-adjunctibility that fits neither of the 4 possible dexterity functions 'LL', 'RR' 'RL' or 'LR'.

14.1. Dexterity Trees

Example 14.1.1. Let $\mathbb{K}$ be a field or a commutative ring. We have seen a description of the Morita 3-category Alg_2 in Chapter 12.

Let ${}_{\mathbb{K}}A_{\mathbb{K}}$ be a 1-morphism $\mathbb{K} \to \mathbb{K}$ in Alg_2. As discussed in Chapter 12 the right and left adjoints are given by the opposite algebra ${}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}}$. For the adjunction ${}_{\mathbb{K}}A_{\mathbb{K}} \dashv {}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}}$ we have

$$\text{unit} \qquad u : \mathrm{id}_{\mathbb{K}} \to {}_{\mathbb{K}}A_{\mathbb{K}} \otimes_{\mathbb{K}} {}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}} \;=\; {}_{\mathbb{K}}\mathbb{K}_{\mathbb{K}}A_{({}_{\mathbb{K}}A_{\mathbb{K}})\otimes_{\mathbb{K}}({}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}})} \qquad \text{and}$$

$$\text{counit} \qquad c : {}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}} \otimes_{\mathbb{K}} {}_{\mathbb{K}}A_{\mathbb{K}} \to \mathrm{id}_{\mathbb{K}} \;=\; {}_{({}_{\mathbb{K}}A^{\mathrm{op}}_{\mathbb{K}})\otimes_{\mathbb{K}}({}_{\mathbb{K}}A_{\mathbb{K}})}A_{\mathbb{K}}\mathbb{K}_{\mathbb{K}} \,.$$

By Proposition 12.1.3 the unit u is left adjunctible if and only if A is a finite dimensional $\mathbb{K}$-vector space and u is right adjunctible if and only if A is finitely presented projective as a $A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ module. Since a finitely presented module is projective if and only if it is flat, the latter condition is equivalent to A being a separable $\mathbb{K}$-algebra. Similarly, the counit c is right adjunctible if and only if A is finite dimensional and it is left adjunctible if and only if A is separable.

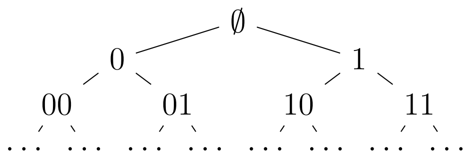
Choosing the dual numbers $A = \mathbb{K}[x]/(x^2)$ for A we have an example where A is finite dimensional over $\mathbb{K}$ but not separable. In this case the 1-morphism ${}_{\mathbb{K}}A_{\mathbb{K}}$ does not satisfy any a^2-adjunctibility since the unit is left but not right adjunctible, while the counit is right but not left adjunctible. ◇

To capture this phenomenon we introduce dexterity trees, that specify for each unit and counit individually whether it should have a left or right adjoint. The tree structure arises from the form of our data. For example, in 14.1.1 with $A = \mathbb{K}[x]/(x^2)$ we have adjunctions

$$\begin{array}{ccc} & (f^L \dashv f, u, c) & \\ & \swarrow \quad \searrow & \\ (u^L \dashv u, u_u, c_u) & & (c \dashv c^R, u_c, c_c) \end{array} \qquad \text{with dexterity tree} \qquad \begin{array}{ccc} & L & \\ / & & \backslash \\ L & & R \end{array}$$

Before we get to the precise Definition in 14.1.2 we recall some vocabulary regarding trees.

The complete rooted binary tree $\mathsf{T}_{\mathsf{n+1}}$ of height $n+1$ is given by words on the alphabet $\{0,1\}$ of length $\leq n$. The **root** is $\emptyset$ and the **parent** of a word w of length l is the length $l-1$ prefix of w. We refer to a direct successor of a node as **child** and to two children with the same parent as **siblings**.



The length $|w|$ of a word w is the **depth** of the corresponding node in the tree. Note that we use the convention that the height of a rooted tree is the number of nodes in the longest downward path starting at the root. Thus T_n has n layers and $2^n - 1$ nodes. This makes the number of entries n of a dexterity function a^n align with the height of the corresponding dexterity tree from Definition 14.1.2. In this section, by tree we always mean a complete rooted binary tree of finite height.

Definition 14.1.2. Let C be an (∞, N)-category. Let

$$t^n : \mathsf{T}_\mathsf{n} \to \{L, R\}.$$

be a complete rooted binary tree labelled by $\{L, R\}$ of height n. Let f be a k-morphism in C. A set of t^n**-adjunctibility data** for f with **dexterity tree** t^n is defined inductively as

- step 1: the data of a $\begin{cases} \text{left adjoint,} & \text{if} \quad t^n(\emptyset) = L, \\ \text{right adjoint,} & \text{if} \quad t^n(\emptyset) = R, \end{cases}$
  together with unit and counit witnessing the adjunction.

- step $j = 2, \dots, n$: for all units from step $j-1$, witnessing the adjunction corresponding to the length $j-1$ word w,
$$\text{the data of a } \begin{cases} \text{left adjoint,} & \text{if} \quad t^n(w\,0) = L, \\ \text{right adjoint,} & \text{if} \quad t^n(w\,0) = R, \end{cases}$$
together with units and counits witnessing the adjunctions. Similarly, data for all counits depending on the value of $t^n(w\,1)$.

A k-morphism f is t^n**-adjunctible** if there exists a set of t^n-adjunctibility data for f.

14.2. Equivalent Tree-Adjunctibilities

Corresponding to Lemma 11.2.16, that allowed us to exchange 'LL' $\sim$ 'RR' and 'LR' $\sim$ 'RL' in the entries of dexterity functions to get equivalent notions of a^n-adjunctibility, we now can exchange any subtrees of the form

$$\begin{array}{c} | \\ L \\ \diagup \quad \diagdown \\ L \qquad L \\ t_1\ t_2 \quad t_3\ t_4 \end{array} \sim \begin{array}{c} | \\ R \\ \diagup \quad \diagdown \\ R \qquad R \\ t_3\ t_4 \quad t_1\ t_2 \end{array} \quad \text{and} \quad \begin{array}{c} | \\ L \\ \diagup \quad \diagdown \\ R \qquad R \\ t_1\ t_2 \quad t_3\ t_4 \end{array} \sim \begin{array}{c} | \\ R \\ \diagup \quad \diagdown \\ L \qquad L \\ t_3\ t_4 \quad t_1\ t_2 \end{array}. \tag{14.1}$$

to get equivalent notions of t^n-adjunctibility. Note that we swapped the subtrees of the two children since Lemma 11.1.3 exchanges the role of the unit and counit adjunctions in the data. The relations (14.1) generate an equivalence relation $\sim$ on the set of dexterity trees.

Theorem 14.2.1. *Let C be an (∞, N)-category and $t^n, s^n : \mathsf{T_n} \rightrightarrows \{L, R\}$ be two dexterity trees that are equivalent with respect to $\sim$. Then a k-morphism f is t^n-adjunctible if and only if it is s^n-adjunctible.*

Proof. If $t^n \sim s^n$ are equivalent, there is a series of exchanges of the form (14.1) to transform t^n into s^n. Given a k-morphism f, all of the intermediate dexterity trees as well as t^n and s^n require equivalent adjunctibility data for f by Lemma 11.1.3. □

Example 14.2.2. For 3-times left adjunctibility given by the constant 'L' dexterity function l^3 we have the corresponding constant 'L' dexterity tree of height 3. Some equivalent trees are

$$\begin{array}{c} L \\ \diagup \quad \diagdown \\ L \qquad R \\ L\ L \quad R\ R \end{array}, \quad \begin{array}{c} L \\ \diagup \quad \diagdown \\ R \qquad R \\ R\ R \quad R\ R \end{array}, \quad \begin{array}{c} R \\ \diagup \quad \diagdown \\ R \qquad R \\ L\ L \quad L\ L \end{array} \quad \text{and} \quad \begin{array}{c} R \\ \diagup \quad \diagdown \\ L \qquad R \\ R\ R \quad L\ L \end{array}.$$

The second and third tree are induced by the dexterity functions 'LRR' and 'RRL', while the first and fourth tree are not induced by any dexterity function.

◇

We saw that for dexterity functions and a^n-adjunctibility the a priori 2^n different definitions reduced to two equivalence classes represented by even^n and odd^n defined in (11.10) and (11.11). For dexterity trees and t^n-adjunctibility we start with 2^{2^n-1} a priori different notions. Let $\mathcal{T}_n$ be the set of dexterity trees and $\mathscr{T}_n := \mathcal{T}_n/\sim$ be the set of equivalence classes generated by the relations (14.1). The question of determining an upper bound for non-equivalent definitions of dexterity trees and t^n-adjunctibility amounts to a calculation of $|\mathscr{T}_n|$, which is one of the goals for the remainder of this section. The second one is to find representatives that replace even^n and odd^n in this more general setup. We start with the latter, and we will achieve the former goal by counting the representatives.

First we give a name to those t^n, that have no other equivalent dexterity trees.

Definition 14.2.3. Let t^n be a dexterity tree. We say t^n is a **fixed tree** if every pair of siblings in t^n has distinct labels.

In a fixed tree we cannot apply any of the relations (14.1) to any subtree.

Definition 14.2.4. Let t^n be a dexterity tree. We say t^n is in **normal form** if every subtree of t^n is either a fixed tree or has a root r with label 'R'.

Any tree in normal form is a constant 'R' tree with some subtrees replaced by fixed trees.

Theorem 14.2.5. *Every dexterity tree t^n is equivalent to exactly one tree in normal form τ^n. In particular, we have*

$$|\mathscr{T}_1| = 2 \quad \textit{and} \quad |\mathscr{T}_n| = |\mathscr{T}_{n-1}|^2 + 2^{2^{n-1}-1} \quad \textit{for} \quad n \geq 2.$$

We will prove this theorem in a series of Lemmas. But before that let us take a look at the example of trees of height 2.

Example 14.2.6. For height $n = 2$ we have 8 different dexterity trees that contain two equivalent pairs.

t^2	L / \\ L L ∼ R / \\ R R	L / \\ R R ∼ R / \\ L L	L / \\ L R	L / \\ R L	R / \\ L R	R / \\ R L

Only the first and third dexterity tree are not in normal form, but they are equivalent to one in normal form.

◇

Lemma 14.2.7. *Let t^n be a dexterity tree. Then there is a dexterity tree τ^n in normal form with $t^n \sim \tau^n$.*

Proof. We can transform a dexterity tree t^n into its equivalent normal form by calling the function TransferIntoNormalform(t^n) defined in algorithm 2. □

Algorithm 1 Change the label of the root in a tree t if t is not a fixed tree

```
function ChangeRoot(tree t)
    if t is a fixed tree then
        return t
    else
        Find a node N of minimal depth, which has a sibling with the same label
        while N is not the root of t do
            Apply suitable (14.1) to the parent of N and its children in t
            N ← parent of N
        end while
        return t
    end if
end function
```

Algorithm 2 Change a tree t into normal form

```
function TransformIntoNormalform(tree t)
    if t is a fixed tree then
        return t
    else
        if the root of t has label 'L' then
            t ← ChangeRoot(t)                              ▷ (see algorithm 1)
        end if
        tree t_l := left subtree of the root of t
        tree t_r := right subtree of the root of t
        t_l ← TransferIntoNormalform(t_l)
        t_r ← TransferIntoNormalform(t_r)
        return t
    end if
end function
```

Lemma 14.2.8. *Let $s_1^n, s_2^n, s_3^n, s_4^n$ be dexterity trees and let*

$$t_1^{n+1} = \begin{array}{c} r \\ / \quad \backslash \\ s_1^n \quad s_2^n \end{array}, \quad t_2^{n+1} = \begin{array}{c} r \\ / \quad \backslash \\ s_3^n \quad s_4^n \end{array}$$

be two dexterity trees with roots with the same label $r \in \{L, R\}$. Then $t_1^{n+1} \sim t_2^{n+1}$ if and only if $s_1^n \sim s_3^n$ and $s_2^n \sim s_4^n$.

Proof. If $s_1^n \sim s_3^n$ and $s_2^n \sim s_4^n$ then we can transform the subtrees of t_1^{n+1} separately into the corresponding subtrees of t_2^{n+1} and we get $t_1^{n+1} \sim t_2^{n+1}$.

For the converse direction note that we have the following commutativity for dexterity trees. Applying a rule from (14.1) to the root and then changes to the left and right subtrees is the same as first applying the subtree changes to the respective other tree and then a suitable rule from (14.1) to the root. If $t_1^{n+1} \sim t_2^{n+1}$ there is a sequence of equivalent trees given by applications of (14.1). Since t_1^{n+1} and t_2^{n+1} have a root labelled the same way and by the above argument, we can write this sequence of equivalent trees as a sequence of changes to the left and right subtree followed by an even number of applications of (14.1) to the root. The latter cancel out and we have a sequence of equivalent subtrees that shows $s_1^n \sim s_3^n$ and $s_2^n \sim s_4^n$. □

Lemma 14.2.9. *Let τ_1^n, τ_2^n be two dexterity trees in normal form with $\tau_1^n \neq \tau_2^n$. Then $\tau_1^n \nsim \tau_2^n$.*

Proof. If one of τ_1^n or τ_2^n is a fixed tree then no other dexterity tree is equivalent to that one and we are done. If both τ_1^n and τ_2^n are not fixed trees both of their roots are labelled by R. Therefore either their left or right subtrees (or both) differ from each other. Pass to such a pair of subtrees and repeat this process. Since the trees have finite height this terminates and we find two non equivalent subtrees. By Lemma 14.2.8 all of the previous pairs of trees and in particular τ_1^n and τ_2^n are non equivalent. □

Proof of Theorem 14.2.5. By Lemma 14.2.7 and Lemma 14.2.9 we see that there is exactly one tree in normal form in each equivalence class in $\mathscr{T}_n$.

Next, we determine $|\mathscr{T}_n|$. For $n = 1$ the two equivalence classes are 'L' and 'R' and we have $|\mathscr{T}_1| = 2$.

For $n+1$ with $n \geq 1$ we have seen above that $|\mathscr{T}_{n+1}|$ is the number of dexterity trees of height $n+1$ that are in normal form. The normal form trees of height $n+1$ with a root labelled 'L' are the fixed trees where each pair of siblings is labelled either ('L', 'R') or ('R', 'L'). The number of such trees is $2^{2^{n-1}-1}$. The normal form trees of height $n+1$ with a root labelled 'R' are of the form where both the left and right subtree of the root is a normal form tree of height n. By recursion the number of each of these trees is $|\mathscr{T}_n|$. Overall we get the desired recursion formula

$$|\mathscr{T}_{n+1}| = |\mathscr{T}_n|^2 + 2^{2^{n-1}-1}.$$

□

Remark 14.2.10. The sequence in Theorem 14.2.5 can be found in the OEIS as A332757 [SI23]. It also describes the number of involutions in the n-fold iterated wreath product of $\mathbb{Z}/2\mathbb{Z}$ or the number of involutory automorphisms of $\mathsf{T}_{\mathsf{n+1}}$. The first five terms are $2, 6, 44, 2064$ and 4292864. ◇

Remark 14.2.11. It is possible to make choices other than the normal forms for the representatives of $\mathscr{T}_n$. Recall that a normal form tree consists of the constant 'R' tree with some subtrees swapped with fixed subtrees. But one can also choose a different dexterity tree t^n and set the representatives to be of the form t^n with some subtrees swapped with fixed subtrees. Algorithm 2 only needs a small modification which tracks t^n and when it needs to change the label of the current root, to deliver the new representatives. ◇

Appendix

Appendix - Contents

A. Cardinals independent of ZFC

A.1. Cardinals and Ordinals

We work with **Z**ermelo-**F**raenkel set theory and the axiom of **c**hoice (ZFC). In this framework we have the classical von Neumann definition of ordinal and cardinal numbers. In particular, there is a transfinite sequence of cardinal numbers

$$0, 1, 2, 3, \ldots; \aleph_0, \aleph_1, \ldots; \aleph_\omega, \ldots$$

indexed by ordinals.

An ordinal can either be zero, a successor ordinal or a limit ordinal. A cardinal that is neither zero nor a successor cardinal is a **weak limit cardinal**.

To define strong limit cardinals we need cardinal exponentiation.

A.2. Cardinal Arithmetic

Consider two cardinalities $\kappa = \mathrm{card}(S), \lambda = \mathrm{card}(T)$ of sets S and T. The sum is given by disjoint union

$$\kappa + \lambda = \mathrm{card}(S \amalg T).$$

For multiplication we take the cartesian product

$$\kappa \cdot \lambda = \mathrm{card}(S \times T).$$

Exponentiation is done by looking at the function set

$$\lambda^\kappa = \mathrm{card}\left(\mathrm{Map}(S, T)\right).$$

Definition A.2.1. A cardinal number κ is a **strong limit cardinal** if for all cardinals $\lambda < \kappa$ we have $2^\lambda < \kappa$.

With this definition $\aleph_0$ is a strong limit cardinal, and we refer to all other ones as uncountable strong limit cardinals.

Using the description of cardinalities of sets its easy to see, that for any two ordinals α, β we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha,\beta)}.$$

We can use this to show that $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, since

$$2^{\aleph_0} \le \aleph_0^{\aleph_0} \le \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}. \tag{A.1}$$

For exponentiation of larger aleph-numbers we have the following formula proven by Hausdorff in 1904 [Hau04].

Theorem A.2.2 (Hausdorff's Formula, see also 9.3.11 in [HJ17])**.** *For all ordinals α, β we have*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}\,.$$

Corollary A.2.3. *For all $d \in \mathbb{N}$ we have*

$$\aleph_d^{\aleph_0} = 2^{\aleph_0} \cdot \aleph_d\,.$$

Proof. Apply Theorem A.2.2 d-times and use $\aleph_0 \cdot \aleph_1 \cdot \ldots \cdot \aleph_d = \aleph_d$ and (A.1). □

A.3. The Continuum Hypothesis

For any cardinal number κ we have $2^\kappa > \kappa$. In particular, $2^{\aleph_0}$ is uncountable, meaning it is at least $\aleph_1$. The question of whether $2^{\aleph_0}$ is exactly $\aleph_1$ is known as the Continuum Hypothesis.

Continuum Hypothesis A.3.1. $2^{\aleph_0} = \aleph_1$.

In 1963 Paul Joseph Cohen developed a new technique, called forcing, to show that the Continuum hypothesis cannot be answered in ZFC [Coh63].

Theorem A.3.2 ([Coh63])**.** *Let α be an ordinal. It is consistent with ZFC that $2^{\aleph_0} > \aleph_\alpha$.*

Iterating the exponential of $\aleph_0$ we get another transfinite sequence of cardinal numbers: The beth numbers, defined via transfinite recursion

$$\beth_0 = \aleph_0, \qquad \beth_{\alpha+1} = 2^{\beth_\alpha}, \qquad \beth_\lambda = \sup\{\,\beth_\alpha \mid \alpha \le \lambda\,\}\,.$$

Strengthening the Continuum Hypothesis we get the following statement.

Generalized CH A.3.3. *For all ordinals α one has $\beth_\alpha = \aleph_\alpha$.*

It is consistent with ZFC that the $\beth$-sequence is exactly the $\aleph$-sequence. But its is also consistent with ZFC that it is a very sparse subsequence.

A.4. Cofinality and the Dominating Number

A concept that we will use frequently is the cofinality of a partially ordered set. For a poset $(P, \leq)$ we say that $C \subset P$ is **cofinal** if for all $p \in P$ there is a $c \in C$ with $p \leq c$. The **cofinality** of P is

$$\mathrm{cf}(P) := \min_{C \subset A \text{ cofinal}} \mathrm{card}(C).$$

Using the axiom of choice we have the following well-known result.

Proposition A.4.1. *Let $(P, \leq)$ be a a partially ordered set and $Q \subset P$ a cofinal subset. Then $\mathrm{cf}(Q) = \mathrm{cf}(P)$.*

Proof. Since being cofinal is transitive we have $\mathrm{cf}(Q) \geq \mathrm{cf}(P)$. Let $C \subset P$ be a cofinal subset. Since Q is cofinal in P we can choose a $q_c \in Q$ for every $c \in C$, such that $c \leq q_c$. The set $\{ q_c \mid c \in C \}$ has the same cardinality as C and is a subset of Q. Since C is cofinal in P there is a $c_q \in C$ for all $q \in Q$, such that $q \leq c_q$. By construction $q \leq q_{c_q}$ for all $q \in Q$, which shows that $\{ q_c \mid c \in C \}$ is cofinal in Q. Thus, $\mathrm{cf}(Q) \leq \mathrm{cf}(P)$ which completes the proof. □

An infinite cardinal number λ is **regular** if $\mathrm{cf}\,\lambda = \lambda$. We do not consider finite cardinals to be regular. The cardinal $\aleph_0$ is regular and using the axiom of choice one can prove the following.

Theorem A.4.2 (Theorem 9.2.4 in [HJ17])**.** *For every infinite cardinal, its successor κ^+ is regular.*

Consider the set $\prod_{\mathbb{N}} \mathbb{N}$ of countable sequences of natural numbers. Given $x = (x_i)_{\mathbb{N}}, y = (y_i)_{\mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{N}$ we define $x \leq y$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$. We get a partially ordered set $(\prod_{\mathbb{N}} \mathbb{N}, \leq)$. The cofinality of $(\prod_{\mathbb{N}} \mathbb{N}, \leq)$ is called **dominating number** and is denoted by $\mathfrak{d} := \mathrm{cf}\,(\prod_{\mathbb{N}} \mathbb{N})$.

The following are the only provable restrictions on $\mathfrak{d}$ in ZFC.

Theorem A.4.3 (Theorem 2.4 in [Bla09])**.** *For the dominating number we have*

$$\aleph_1 \;\leq\; \mathfrak{d} \;\leq\; \beth_1 = 2^{\aleph_0}.$$

The optimality is a consequnce of a result by Hechler [Hec74]. See also Theorem 2.5 in [Bla09].

B. Ind-objects and Categories

B.1. Diagrams and Formal Colimits

Definition B.1.1. A category C is **small** if the collection of objects of C forms a set. A category is **essentially small** if it is equivalent to a small category.

Definition B.1.2. Let κ be an infinite regular cardinal. A **κ-filtered category** is a category D in which every diagram with $< \kappa$ arrows has a cocone. A **filtered category** is a category D, that is $\aleph_0$-filtered, meaning it is a category in which every finite diagram has a cocone.

We mostly work with diagrams given by posets. From the definition we get the easy criterion that a poset $(P, \leq)$ is filtered if and only if for all $a, b \in P$ there is a $c \in P$ with $a \leq c$ and $b \leq c$.

Now, let us first state the main definition and then discuss it in Remark B.1.5.

Definition B.1.3. Let C be a category.

- Objects: An object F in $\mathsf{Ind}(\mathsf{C})$ is a **formal colimit** given by a diagram $F : \mathsf{D} \to \mathsf{C}$, where D is a small filtered category. We use the notation
$$F \;=\; \text{“}\operatorname*{colim}_{d \in D}\text{”} F(d).$$
- Morphisms: The set of morphism in $\mathsf{Ind}(\mathsf{C})$ between objects $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{E} \to \mathsf{C}$ is defined as
$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G) \;:=\; \lim_{d \in \mathsf{D}} \operatorname*{colim}_{e \in \mathsf{E}} \mathrm{Hom}_{\mathsf{C}}(F(d), G(e)) \tag{B.1}$$

Example B.1.4. Let FinSet be the category of finite sets. Then $\mathsf{Ind}(\mathsf{FinSet}) \cong \mathsf{Set}$ is the category of all sets. Let $\mathsf{FinVect}$ be the category of finite dimensional $\mathbb{K}$-vector spaces. Then $\mathsf{Ind}(\mathsf{FinVect}) \cong \mathsf{Vect}$ is the category of all $\mathbb{K}$-vector spaces. ◇

Remark B.1.5. The idea behind Ind-categories is to freely add filtered colimits to a category by considering all inductive systems of objects and morphisms from the original category. This is done to either pass to a category with better categorical properties or to study existing categories via their “small” objects as in Example B.1.4.

To be precise given C, there are three desiderata for $\mathsf{Ind}(\mathsf{C})$ that lead to the defintion:

(i) The original category C embeds fully faithful in $\mathsf{Ind}(\mathsf{C})$. That is, the inclusion

$$\mathsf{C} \to \mathsf{Ind}(\mathsf{C}), \qquad c \mapsto (* \mapsto c)\,,$$

that maps an object c to the corresponding cocone over the one-object diagram, is full and faithful.

(ii) The formal colimits $F : \mathsf{D} \to \mathsf{C}$ are the actual colimits of themselves as diagrams in $\mathsf{Ind}(\mathsf{C})$ via the embedding in (i). That is,

$$\text{“}\operatorname*{colim}_{d\in D}\text{”} F(d) \;\cong\; \operatorname*{colim}_{d\in D} F(d).$$

(iii) The objects of C are compact in $\mathsf{Ind}(\mathsf{C})$. That is, for any $c \in \mathsf{C}$ regarded as an object of $\mathsf{Ind}(\mathsf{C})$ via (i), the corepresentable functor

$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(c, \cdot) : \; \mathsf{Ind}(\mathsf{C}) \to \mathsf{Set}$$

preserves filtered colimits.

If we add the standard fact, that as in every category,

(iv) for all $G \in \mathsf{Ind}(\mathsf{C})$ the Hom-functor

$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(\cdot, G) : \; \mathsf{Ind}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Set}$$

sends colimits to limits,

we can recover the definition of morphisms in $\mathsf{Ind}(\mathsf{C})$ given by (B.1) as follows.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G) &\overset{(ii)}{=} \mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(\operatorname*{colim}_{d\in\mathsf{D}} F(d), \operatorname*{colim}_{e\in\mathsf{E}} G(e)) \\
&\overset{(iv)}{\cong} \lim_{d\in\mathsf{D}} \mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F(d), \operatorname*{colim}_{e\in\mathsf{E}} G(e)) \\
&\overset{(iii)}{\cong} \lim_{d\in\mathsf{D}} \operatorname*{colim}_{e\in\mathsf{E}} \mathrm{Hom}_{\mathsf{C}}(F(d), G(e)) \\
&\overset{(i)}{\cong} \lim_{d\in\mathsf{D}} \operatorname*{colim}_{e\in\mathsf{E}} \mathrm{Hom}_{\mathsf{C}}(F(d), G(e)).
\end{aligned}$$

The last line translates to the following description of a morphism $f : F \to G$ in $\mathsf{Ind}(\mathsf{C})$. It is a collection of maps $\{f(d) : F(d) \to G(e_d)\}_{d\in\mathsf{D}}$, where each $e_d \in E$ is large enough. ◇

Remark B.1.6. The dual notion is that of Pro-categories, where the objects are formal cofiltered limits. One can either define them directly or via the duality

$$\mathsf{Pro}(\mathsf{C}) \;:=\; \mathsf{Ind}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}.$$

As an example we saw that $\mathsf{Ind}(\mathsf{FinSet}) \cong \mathsf{Set}$, but $\mathsf{Pro}(\mathsf{FinSet})$ are the profinite sets, which are equivalent to compact totally disconnected Hausdorff spaces by Proposition 6.1.2. ◇

Remark B.1.7. Note, that we did define $\mathsf{Ind}(\mathsf{C})$ also for non-small categories C. The diagrams however are still small. This is different from considering two Grothendieck universes $\mathbb{U} \in \mathbb{V}$, such that C is not $\mathbb{U}$- but $\mathbb{V}$-small and taking the Ind-completion in $\mathbb{V}$. One example, where this difference manifests, is that $\mathsf{Ind}(\mathsf{C})$ for small abelian C does satisfy (AB5*) and has a generator. One can apply Grothendiecks Theorem 1.10.1 from [Tohoku] and show that the abelian category $\mathsf{Ind}(\mathsf{C})$ has enough injectives. If C is large, $\mathsf{Ind}(\mathsf{C})$ no longer necessarily has a generator and $\mathsf{Ind}(\mathsf{C})$ might not have enough injectives. ◇

B.2. Essentially monomorphic and epimorphic objects

Definition B.2.1. Let C be a category. A formal filtered colimit $F : \mathsf{D} \to \mathsf{C}$ in $\mathsf{Ind}(\mathsf{C})$ is **monomorphic** if for all morphisms d in D the image $F(d)$ is a monomorphism in C. A formal filtered colimit is **essentially monomorphic** if it is isomorphic to a monomorphic objects in $\mathsf{Ind}(\mathsf{C})$. Dually, we define **(essentially) epimorphic** objects that require epimorphisms instead of monomorphisms. We denote the full subcategories of $\mathsf{Ind}(\mathsf{C})$ of essentially monomorphic objects with $\mathsf{Ind}_{\rightarrowtail}(\mathsf{C})$, and of essentially epimorphic objects with $\mathsf{Ind}_{\twoheadrightarrow}(\mathsf{C})$.

Remark B.2.2. Similarly, one can define essentially monomorphic and epimorphic objects in Pro-categories. Under the duality from Remark B.1.6 we have

$$\mathsf{Ind}_{\rightarrowtail}(\mathsf{C}) = \mathsf{Pro}_{\twoheadrightarrow}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} \qquad \text{and} \qquad \mathsf{Ind}_{\twoheadrightarrow}(\mathsf{C}) = \mathsf{Pro}_{\rightarrowtail}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}.$$

◇

B.3. Small Ind-Categories

Let κ be an infinite regular cardinal in this section. In Chapter 4 and Chapter 5 it will be convenient to not consider κ-small but rather $\leq\kappa$-small categories.

Definition B.3.1. Let D be a filtered category. We say that D is $\leq\kappa$-small if D contains $\leq \kappa$ morphisms.

Proposition B.3.2. *A poset $(P, \leq)$ as a filtered category is $\leq\kappa$-small if and only if* $\mathrm{card}(P) \leq \kappa$.

Proof. Since P has at least as many morphisms as objects, P being $\leq \kappa$-small implies that $\mathrm{card}(P) \leq \kappa$. Conversely, for any two objects $a, b \in P$ there is at most one morphism $a \to b$ or $b \to a$. Thus, the number of morphisms in P is bounded by

$$\mathrm{card}(P) \cdot \mathrm{card}(P) \leq \kappa \cdot \kappa = \kappa,$$

where the last equality follows from κ being infinite. □

Definition B.3.3. Let C be a category. Define the **$\leq \kappa$-small Ind-category** of C, to be the full subcategory of $\mathsf{Ind}(\mathsf{C})$, consisting of formal filtered colimits $F : \mathsf{D} \to \mathsf{C}$, where D is κ-small. We denote it by $\mathsf{Ind}^{\kappa}(\mathsf{C})$ and also use $\mathsf{Ind}^{\kappa}_{\rightarrowtail}(\mathsf{C})$ for the full subcategory of essentially monomorphic κ-small objects of $\mathsf{Ind}(\mathsf{C})$.

Remark B.3.4. There are several different ways that cardinal numbers can appear in the definition of an Ind-category. In Part I, mostly Section 4.4, we will use small Ind-Categories from Definition B.3.3. In Part II we consider Ind-categories of κ-bounded spaces, such as Ind- κ-small compact Hausdorff spaces $\mathsf{Ind}(\mathsf{cHaus}_{\kappa})$. In this case the restrictions apply to the category of which we consider formal filtered colimits, but the Ind-category is given by the standard Definition B.1.3. Note that, $\mathsf{Ind}^{\kappa}(\mathsf{cHaus})$ and $\mathsf{Ind}(\mathsf{cHaus}_{\kappa})$ are very different. For example, the latter is cocomplete, while the former only has κ-small colimits. On the other Hand a compact Hausdorff space of cardinality $> \kappa$ has a representation as a one-object diagram in $\mathsf{Ind}^{\kappa}(\mathsf{cHaus})$ but not in $\mathsf{Ind}(\mathsf{cHaus}_{\kappa})$.

One can also define Ind-categories consisting of formal κ-filtered objects, but they only appear in the proof of Proposition 4.2.49 in this thesis. ◇

C. Some General Topology

In this appendix, we compile a collection of definitions, results and examples from classical point-set topology that appear throughout this thesis. The primary focus is on compact generation, as it plays a significant role in the comparison between condensed or compactological sets and classical topological spaces. Additionally, we aim to clarify the differences between competing definitions found in the literature.

The category cgWHaus of compactly generated weak Hausdorff spaces is the "convenient category of topological spaces" that is often used by homotopy theorists. The idea that the category cgHaus of compactly generated Hausdorff spaces is better behaved than Top already appears in Ronald Brown's 1961 PhD thesis [Bro61]. His 1963 paper [Bro63] follows up on this idea and shows that cgHaus is cartesian closed. In 1967 Norman Steenrod wrote the well-known paper "A convenient category of topological spaces" [Ste67], where he worked with compactly generated Hausdorff spaces. In 1969 Michael Campbell McCord in [McC69] slightly enlarged this category by weakening the Hausdorff assumption to the appropriately named weak Hausdorff spaces. He attributes the idea to John Coleman Moore. In 2019 Dustin Clausen and Peter Scholze introduced condensed sets in [Condensed] and showed that cgWHaus are the topological spaces that embed fully faithfully in condensed sets.

We use the term **compact** to mean covering compact and do not assume compact spaces to be Hausdorff.

C.1. Locally compact spaces

Before we turn to compactly generated and weak Hausdorff spaces, we recall locally compact spaces and some properties

Definition C.1.1. A topological space X is **locally compact** if every point $x \in X$ has a neighborhood basis $\mathcal{B}$, such that all elements of $\mathcal{B}$ are compact.

In terms of separation we have the following result.

Theorem C.1.2. *Every Hausdorff locally compact space is $T_{3\frac{1}{2}}$ (Tychonoff).*

Theorem 3.3.1 in [Eng77]. □

Theorem C.1.3 (Theorem 3.3.12 and 3.3.13 in [Eng77])**.** *Let $(X_i)_{i\in I}$ be a family of locally compact spaces.*

1. *The disjoint union $\coprod_{i\in I} X_i$ is locally compact if and only if all spaces X_i are locally compact.*
2. *The product $\prod_{i\in I} X_i$ is locally compact if and only if all spaces X_i are locally compact and all but a finite number are compact.*

C.2. Weak Hausdorff spaces

Definition C.2.1 (McCord, Moore)**.** A topological space X is **weak Hausdorff** or **weak** T_2 if for any continuous map $f : S \to X$, where S is compact Hausdorff, the image $f(S)$ is closed.

Proposition C.2.2 (Lemma 2.1 in [McC69])**.** *Let X be a topological space X that is weak Hausdorff. Then for any continuous map $f : S \to X$, where S is compact Hausdorff, the image $f(S)$ is compact Hausdorff.*

Proposition C.2.3. *Let $\{X_i\}_{i\in I}$ be a collection of weak Hausdorff spaces. Then their product $X = \prod_{i\in I} X_i$ with the product topology is also weak Hausdorff.*

Proof. Suppose S is a compact Hausdorff space and $f : S \to X$ a continuous map. The projections $\pi_j : \prod_{i\in I} X_i \to X_j$ are continuous and it follows from Proposition C.2.2, that $\pi_j \circ f(S)$ are closed and Hausdorff for all $j \in I$. Thus the set $P := \prod_{i\in I} \pi_i \circ f(S)$ is a closed Hausdorff subset of X. Since $f(S)$ is a compact subset of P, we have that $f(S)$ is closed in P and therefore also in X. □

To find examples that are used in the proof of Theorem C.2.6 the notion of one-point compactification of a space X, the **Alexandroff extension** X^*, will be very useful. The open subsets of $X^* = X \cup \{\infty\}$ are the open subsets of X and sets U, such that $X^* \setminus U$ is a closed compact subset of X.

Theorem C.2.4 (Alexandroff, see Chapter 5 Theorem 21 in [Kel75])**.** *The Alexandroff extension X^* of a topological space X is compact and has X as a subspace. The space X^* is Hausdorff if and only if X is Hausdorff and locally compact.*

Lemma C.2.5. *Let X be a Hausdorff space. Then all compact subsets of its Alexandroff extension X^* are closed.*

Proof. Let X be a Hausdorff space, $X^* = X \cup \{\infty\}$ its Alexandroff extension and $K \subset X^*$ a compact subset. If $\infty \notin K$ then $K \subset X$, is a compact subset of a Hausdorff space X and therefore closed in X. Hence, $X^* \setminus K$ is an open and K a closed subset of X^*.

Let $K \subset X^*$ that is compact and such that $\infty \in K$. Assume that $K \setminus \{\infty\}$ is not closed in X. Then there is a point $x \in X \setminus K$, and a sequence $(x_n)_{n\in\mathbb{N}}$ in $K \setminus \{\infty\}$ that converges to x. The set $L := \{x\} \cup \{\, x_n \mid n \in \mathbb{N}\,\}$ is a compact subset of X. Since X is Hausdorff, L is closed in X. So $X^* \setminus L$ is open in X^*. Furthermore, for all $n \in \mathbb{N}$ the set $\{x_n\} \cup (X \setminus L)$ is open in X and X^*. Thus,

$$\{X^* \setminus L\} \;\cup\; \{\, \{x_n\} \cup (X \setminus L) \mid n \in \mathbb{N}\,\}$$

is an open cover of K with no finite subcover, which is a contradiction. Thus, $X \setminus K$ is open in X and X^* showing that $K \subset X^*$ is closed. □

Theorem C.2.6. *We have the following strict inclusions of classes of topological spaces*

$$\left\{\begin{matrix} T_2 \\ \textit{spaces} \end{matrix}\right\} \subsetneq \left\{\begin{matrix} \textit{Spaces, where} \\ \textit{every compact} \\ \textit{subset is closed} \end{matrix}\right\} \subsetneq \left\{\begin{matrix} \textit{weak } T_2 \\ \textit{spaces} \end{matrix}\right\} \subsetneq \left\{\begin{matrix} \textit{Spaces, where} \\ \textit{every compact } T_2 \\ \textit{subset is closed} \end{matrix}\right\} \subsetneq \left\{\begin{matrix} T_1 \\ \textit{spaces} \end{matrix}\right\}$$

Proof. We start by proving or referencing proofs for all the inclusions.

- It is well-known that compact subsets of Hausdorff spaces are closed. For example, refer to Theorem 3.1.8 in [Eng77].
- Let X be a space, where all compact subsets are closed. Let $f : S \to X$ a continuous map and S be compact and Hausdorff. Then $f(S)$ as an image of a compact set under a continuous map is compact and therefore closed.
- Let X be a weak Hausdorff space and $K \subset X$ be compact and Hausdorff. Then the embedding $K \hookrightarrow X$ and X being weak Hausdorff shows that K is closed.
- Let X be a space, where all compact Hausdorff subsets are closed. Then singletons subsets are closed, which is equivalent to T_1. See e.g. [Eng77] section 1.5 for the latter.

Now we turn to finding examples that prove that all of the inclusions in the statement are strict.

- Let X be an uncountable set, equipped with the cocountable topology. That is $\emptyset \neq U \subset X$ is open if and only if $X \setminus U$ is countable. No two non-empty open sets in X are disjoint. Thus, X is not Hausdorff.

Let $K \subset X$ be a compact subset. Then K is finite. Otherwise, there is a countable infinite, subset $N \subset K$ and the open covering $\{\{n\} \cup X \setminus \{N\}\}_{n \in N}$ has no finite subcovering. Hence, K is closed.

- Let Y be a Hausdorff space that is not locally compact, such as $\prod_{n=1}^{\infty} \mathbb{R}$. Here $\mathbb{R}$ has the Euclidean topology and Y the product topology. It is not locally compact by Theorem 3.1.31.

  Its Alexandroff extension Y^* is weak Hausdorff by Lemma C.2.5 and one of the implications we have shown. By Proposition C.2.3 the space $X := Y^* \times Y^*$ is also weak Hausdorff. However, the diagonal Δ_{Y^*} is compact as a product of compact space but not closed, since Y^* is not Hausdorff by Theorem C.2.4.

- We consider $X = \mathbb{N}$, equipped with the cofinite topology. That is $\emptyset \neq U \subset \mathbb{N}$ is open if and only if $\mathbb{N} \setminus U$ is finite.

  Let $H \subset X$ be a Hausdorff subset. Then H has to be finite, since the subspace topology is also the cofinite one, which is non-Hausdorff on infinite sets. Therefore, H is closed.

  Let $S = \mathbb{N} \cup \{\infty\}$ be the Alexandroff extension of $\mathbb{N}$ with the discrete topology. That is, $U \subset S$ is open if and only if $\infty \notin U$ or $S \setminus U$ is finite. This space is also called countable Fort space. The discrete topology on $\mathbb{N}$ is Hausdorff and locally compact. By Theorem C.2.4 it follows that S is compact Hausdorff. Define

$$f : S \to X, \quad \mathbb{N} \ni n \mapsto 2n, \quad \infty \mapsto 2.$$

  For $\emptyset \neq U \subset X$ open, $X \setminus U$ in X and therefore $S \setminus f^{-1}(U)$ are finite. Hence, f is continuous. The image $f(S) = 2\mathbb{N}$ is not closed in X, which shows that X is not weak Hausdorff

- Let X be the real line with two origins. That is, we consider $\mathbb{R}$ with its Euclidean topology and let X be the quotient space of $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$, where we identify all (x, a) and (x, b), whenever $x \neq 0$. We denote the two origins with 0_a and 0_b and identify all the non-zero elements with their corresponding value $\mathbb{R} \setminus \{0\}$. The singletons $\{0_a\}$ and $\{0_b\}$ are closed as complements of the open sets $\{0_b\} \cup (\mathbb{R} \setminus \{0\})$ and $\{0_a\} \cup (\mathbb{R} \setminus \{0\})$. Since every other singleton is also closed, we have that X is T_1.

  The subset $\{0_a\} \cup ([-1, 1] \setminus \{0\})$ has its usual Euclidean topology and is therefore compact and Hausdorff. But its closure in X also contains $\{0_b\}$.

□

Remark C.2.7. Topological spaces, where all compact subsets are closed, are often called **KC-spaces**. ◇

Remark C.2.8. We have seen in Proposition 3.1.30 that every topological group that is T_0 is already $T_{3\frac{1}{2}}$. So for topological vector spaces the classes in Theorem C.2.6 coincide. ◇

C.3. Compactly generated spaces

As mentioned in the introduction the categorical properties of cgHaus were studied in the sixties by Ronald Brown [Bro61, Bro63], Norman Steenrod [Ste67] and others. However, the notion of compactly generated or k-space seems to be older. In the 1950 paper "Compact sets of functions and function rings" [Gal50] David Gale defines k-spaces and attributes the definition to Witold Hurewicz.

Different Definitions and their hierarchy Unfortunately, the notion of compactly generated space has multiple distinct definitions in the literature. Most of them are equivalent on Hausdorff spaces, but not in general.

To discuss the differences between 3 possible definitions we use the naming convention from [CD24] - CG 1, CG 2 and CG 3. Here CG 2 is what we will refer to as 'compactly generated'. In the literature one also find the name k-space for CG 1 or CG 2, but we will avoid this terminology. Here is an overview:

CG 1	Generated by compact subspaces
CG 2	Generated by maps from compact Hausdorff spaces
CG 3	Generated by compact Hausdorff subspaces

Remark C.3.1. All of this property will be defined using the final topology. Note, that in ZFC we can take the final topology not only with respect to a set but also with respect to a proper class of maps. ◇

In [Ste67] Steenrod only considers Hausdorff spaces. But the definition that he uses is CG 1.

Definition C.3.2 (Steenrod)**.** A topological space X is CG 1 if its topology coincides with the final topology with respect to the family of all inclusions from compact subspaces. That is, a subset $U \subset X$ is open if and only if $U \cap K$ is open in K for all compact subspaces $K \subset X$.

Example C.3.3. Consider the product topology on $X = \prod_I \mathbb{R}$ for an uncountable set I. Then X is not CG1. See chapter 7 exercise J (b) in [Kel75]. ◇

The next definition is the one that we refer to as 'compactly generated'. Mainly because it is the definition used in the condensed mathematics literature.

Definition C.3.4. A topological space X is CG 2 or **compactly generated** if its topology coincides with the final topology with respect to the family of all continuous maps from arbitrary compact Hausdorff spaces. That is, a subset $U \subset X$ is open if and only if $f^{-1}(U)$ is open in S for all compact Hausdorff spaces S and all continuous maps $f : S \to X$.

Lastly, similar to Theorem C.2.6 we have a variant that replaces 'compact' with 'compact Hausdorff'.

Definition C.3.5. A topological space X is CG 3 if its topology coincides with the final topology with respect to the family of all inclusions from compact Hausdorff subspaces. That is, a subset $U \subset X$ is open if and only if $U \cap K$ is open in K for all compact Hausdorff subspaces $K \subset X$.

Lemma C.3.6. *Let X be a topological space and* CG 2*. Every open subset $A \subset X$ is also* CG 2*, when equipped with its subspace topology.*

To see that CG 3 is strictly stronger than CG 2 the following Lemma will be useful in Theorem C.3.8.

Lemma C.3.7. *Let X be a topological space that is* CG 3*. Then X is T_1.*

Proof. Let $\{x\} \subset X$ be a singleton in X and $S \subset X$ a compact Hausdorff subspace. Then $\{x\} \cap S$ is closed in S. Thus, $\{x\}$ is closed in X and X is T_1. □

Without any assumptions on separability we have the following.

Theorem C.3.8. *We have the following strict inclusions of classes of topological spaces*

$$\{\text{CG 3 } \textit{spaces}\} \subsetneq \{\text{CG 2 } \textit{spaces}\} \subsetneq \{\text{CG 1 } \textit{spaces}\}$$

Proof. That CG 3 implies CG 2 follows from considering inclusions of its compact Hausdorff subspaces. That CG 2 implies CG 1 is a consequence of continuous functions mapping compact sets to compact sets.

Now, we provide examples to show that the inclusions in the statement are strict.

- The space $X = \prod_I \mathbb{R}$ from Example C.3.3 is not CG 1 and therefore also not CG 2. Its Alexandroff extension X^* is compact and therefore CG 1. By Theorem C.2.4 X is an open subspace of X^*. It follows from Lemma C.3.6 that X^* is not CG 2.

- Consider the Sierpinski space $X = \{0, 1\}$ with topology $\{\emptyset, \{1\}, X\}$. It is not T_1 and therefore by Lemma C.3.7 not CG 3. But X is the continuous image under the identity of the compact Hausdorff space $\{0, 1\}$ with discrete topology. Also both singletons are continuous images of compact Hausdorff space and therefore the Sierpinski space is CG 2.

□

However, most of the time one considers CG 1, CG 2 or CG 3 spaces, they also will be (weak) Hausdorff.

Theorem C.3.9. *Let X be a topological space.*

(i) If X is CG 2 *and weak Hausdorff, then X is* CG 3*.*

(ii) If X is CG 1 *and Hausdorff, then X is* CG 2*.*

Proof. (i) Let X be a CG 2 and weak Hausdorff. For all compact Hausdorff spaces S and continuous maps $f : S \to X$ the image $f(S)$ is compact Hausdorff by Proposition C.2.2. Thus, X being CG 3 follows directly from X being CG 2.

(ii) Let X be a CG 1 and Hausdorff. Let K be a compact subset of X. Then K is also Hausdorff and the image of the inclusion $K \hookrightarrow X$. Thus, X being CG 2 follows directly from X being CG 1.

□

K-ification There is a way to turn a topological space into a compactly generated space. Since the latter are sometimes referred to as k-spaces this process is known as k-ification.

Definition C.3.10. Let $X \in \mathsf{Top}$ be a topological space. A subset A of X is **k-closed** if for all continuous maps $f : S \to X$, where S is compact Hausdorff, the set $f^{-1}(A)$ is closed in S. Its **k-ification** kX is the underlying set of X equipped with the topology with closed set given by all k-closed subsets of X.

In other words, the k-ification is assigning to a space a new topology generated by maps from compact Hausdorff spaces.

Proposition C.3.11. *Let X be a topological space. A subset $A \subset X$ with its subspace topology is compact if and only if A with the subspace topology of kX is compact.*

Proposition C.3.12. *Let $X \in \mathsf{Top}$ be a topological space. If X is compactly generated, we have $X = kX$. In particular, we always have $k^2X = kX$. If X is not compactly generated, the topology of kX is strictly finer than the topology of X.*

Lemma C.3.13 (Corollary 1.10 in [Str09])**.** *Let X be a compactly generated space and $Y \in \mathsf{Top}$. Then a map $f : X \to Y$ is continuous if and only if $f : X \to kY$ is continuous if and only if its restriction to all compact Hausdorff subspaces is continuous.*

Given a morphism $f : X \to Y$ in Top It follows from Proposition C.3.12 and Lemma C.3.13 that $kf : kX \to kY$ is continuous. So $k : \mathsf{Top} \to \mathsf{cTop}$ defines a functor from topological spaces to compactly generated spaces.

Proposition C.3.14. *There is an adjunction*

$$\mathsf{cTop} \underset{k}{\overset{\iota}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Top}$$

between the inclusion functor ι and k-ification.

The inclusion as a left adjoint commutes with colimits. It does not commute with limits. In fact, the product of compactly generated spaces is not necessary compactly generated. See [Eng77] example 3.3.29. The categorical product in cTop is given by the k-ification of the usual product topology. See [Str09] for the details.

Relation to other topological notions We collect some properties that guarantee compact generation.

Theorem C.3.15 (Proposition 1.6 and 1.7 in [Str09])**.** *The following statements describe some subclasses of compactly generated spaces.*

(i) Every locally compact Hausdorff space is compactly generated,

(ii) Every sequential space is compactly generated,

(iii) Every first countable space is compactly generated,

Remark C.3.16. The Hausdorff assumption is vital in Item (i). An example for a locally compact but not compactly generated space is constructed by Isbell in [Isb87]. The construction relies on the axiom of choice. ◇

Corollary C.3.17. *Metric spaces are compactly generated.*

Proof. Every metric space is Hausdorff and first countable. By Theorem C.3.15 Item (iii) this implies compact generation. □

Assuming compact generation also gives us a characterization of weak Hausdorff spaces. It is analogous to the well-known statement that a space X is Hausdorff if and only if its diagonal δ_X is closed in $X \times X$.

Proposition C.3.18 (Proposition 2.14 in [Str09])**.** *Let X be compactly generated. Then X is weak Hausdorff if and only if its diagonal $\delta_X = \{ (x, x) \mid x \in X \}$ is closed in $X \times X$.*

Stability We already noted, that compact generation is not stable under products. But what about some of the other standard constructions in topology?

Theorem C.3.19 (Theorem 3.3.25 - 3.3.27 in [Eng77]). *We have the following.*

(i) Closed and open subsets with subspace topology of compactly generated spaces are compactly generated

(ii) The disjoint union $\bigsqcup_{i \in I} X_i$ is compactly generated if and only if all spaces X_i are compactly generated.

(iii) The direct product $X \times Y$ of a locally compact space X and a compactly generated space Y is compactly generated.

Proposition C.3.20 (Proposition 2.1 in [Str09]). *Let X be a compactly generated space and E an equivalence relation on X Then X/E is compactly generated.*

Small compact generation In view of the set theoretic details of condensed sets we will also use the following variant of compact generation. Let κ be a cardinal number for the rest of this section.

Definition C.3.21. A topological space X is **κ-compactly generated** if its topology coincides with the final topology with respect to the family of all continuous maps from κ-small compact Hausdorff spaces. That is, a subset $U \subset X$ is open if and only if $f^{-1}(U)$ is open in S for all κ-small compact Hausdorff spaces S and all continuous maps $f : S \to X$.

Proposition C.3.22 (Remark 4.1.2 in [Ás21]). *A κ-small topological space X is compactly generated if and only if it is κ-compactly generated.*

C.4. The category cgWHaus

Proposition C.4.1 (Theorem 3.1 in [Str09]). *We have the following*

(i) A morphism in cgWHaus *is a monomorphism if and only if it is injective,*

(ii) A morphism in cgWHaus *is an epimorphism if and only if has dense image.*

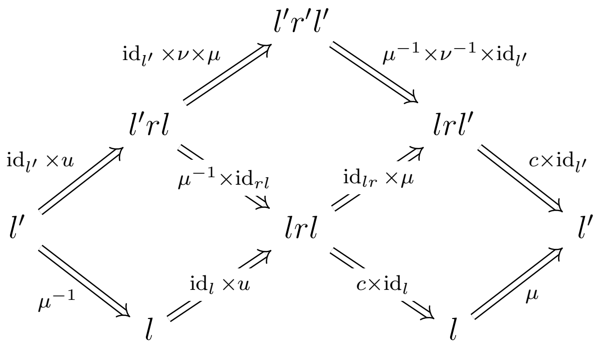
Proposition C.4.2 (Corollary 2.21 in [Str09]). *Let X be compactly generated and E an equivalence relation on X. Then $X/E \in$* cgWHaus *if and only if $E \subset k(X \times X)$ is closed.*

D. More Adjunctions and Dualizability

D.1. Some basic Lemmas about Adjunctions

In this appendix we collect some proofs of statements in Chapter 10 about well-known properties of adjoints for the reader's convenience.

Proof of Lemma 10.1.9. The first zig-zag identity (zig) is given by

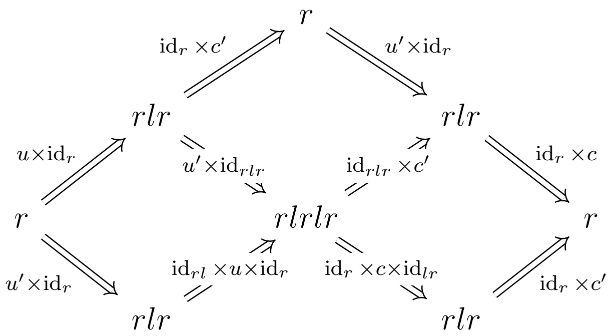


where all three squares commute by the interchange law in a bicategory. The bottom is equal to $\mathrm{id}_{l'}$ by (zig) of $(l \dashv r, u, c)$. Similarly, (zag) follows from the interchange law and (zag) of $(l \dashv r, u, c)$. □

Proof of Lemma 10.1.10. We define the two 2-morphisms

$$\varphi : \left(\; r \xRightarrow{u \times \mathrm{id}_r} r \circ l \circ r \xRightarrow{\mathrm{id}_r \times c'} r \; \right),$$
$$\psi : \left(\; r \xRightarrow{u' \times \mathrm{id}_r} r \circ l \circ r \xRightarrow{\mathrm{id}_r \times c} r \; \right).$$

To see that they are inverse to each other we consider the diagram

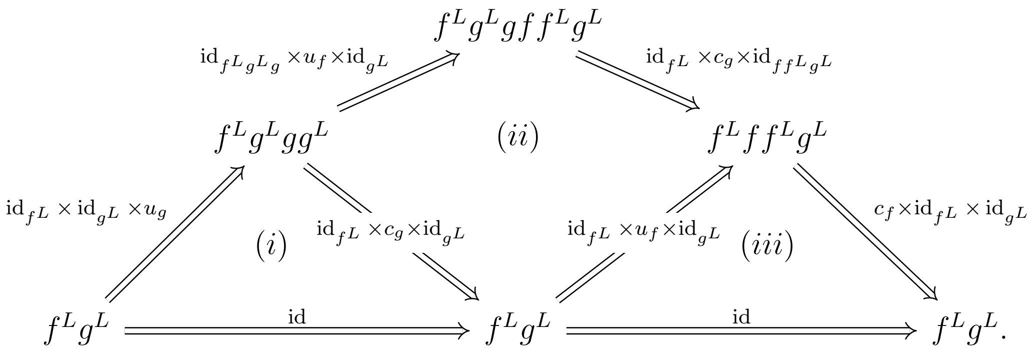


where all three squares commute by the interchange law in a bicategory. The bottom path equals id_r by (zig) of the two adjunctions. Similarly, we see that $\varphi \circ \psi = \mathrm{id}_r$, where we need (zag) of the two adjunctions. Using the interchange law as above it also follows that

$$(\psi \times \mathrm{id}_l) \circ u = u' \qquad \text{and} \qquad (\mathrm{id}_l \times c) \circ c = c'.$$

□

Proof of Lemma 10.1.11. The first zig-zag identity (zig) is given by the diagram



Here (i) commutes as (zig) of $g^L \dashv g$ composed with $(f^L \circ -)$. The triangle (iii) is (zig) of $f^L \dashv f$ composed with $(- \circ g^L)$ and (ii) commutes by the interchange law in a bicategory. The second zig-zag identity (zag) is similar. □

D.2. Dualizability

In this Appendix we review some of the definitions of higher dualizability that can be found in the literature and see that they agree with n-dualizability as defined in Definition 10.3.2.

Let C be an (∞, N)-category. A k-morphism f **has all adjoints** in C if there exists a **tower of adjunctions**

$$\ldots \dashv f^{LLL} \dashv f^{LL} \dashv f^L \dashv f \dashv f^R \dashv f^{RR} \dashv f^{RRR} \dashv \ldots$$

in C. This is the notion of adjunctibility that one gets from Lurie's definition of fully dualizable objects in [Lur09]. The corresponding data in C can be defined as follows.

Definition D.2.1. Let C be a symmetric monoidal (∞, N)-category and X an object in C. A **set of full 1-dualizability data** for X is a dual object X^* together with unit u and counit c witnessing the duality. For $n \geq 3$ a **set of full n-dualizability data** is a set of full $(n-1)$-dualizability data together with towers of adjunctions for all the unit and counit n-morphisms in the set of full $(n-1)$-dualizability data. An object X is **fully n-dualizable** if there exists a set of full n-dualizability data for X.

Unwrapping this definition for $n \geq 3$ we have the following data.

- A dual object X^* of X together with unit u and counit c,
- Towers of adjunctions for u and c

$$\ldots \dashv u^{LL} \dashv u^{L} \dashv u \dashv u^{R} \dashv u^{RR} \dashv \ldots\ , \tag{D.1}$$

$$\ldots \dashv c^{LL} \dashv c^{L} \dashv c \dashv c^{R} \dashv c^{RR} \dashv \ldots\ , \tag{D.2}$$

- Towers of adjunctions for all the units and counits in the adjunctions in (D.1) and (D.2) as well as the corresponding units and counits.
- ...
- Towers of adjunctions for all the unit and counit n-morphisms.

Remark D.2.2. The above definition is the data in C that one gets from Lurie's definition of **fully dualizable objects**. To see this, first discard the non-invertible k-morphisms for $k > n$. Therein take the **fully dualizable** part C^{fd}, which can be obtained by discarding k-morphisms that are not adjunctible and objects that are not dualizable. An object is fully dualizable in Lurie's sense if it lies in the essential image of the natural symmetric monoidal functor $\mathsf{C}^{\mathrm{fd}} \to \mathsf{C}$. Furthermore, the (∞, n)-category C^{fd} is said to **have duals**, meaning that every object is fully n-dualizable and every morphism has a left and right adjoint. See section 2.3 of [Lur09]. ◇

If C is merely a monoidal (∞, N)-category we need to differentiate between left and right duals for objects. We adapt the definition of a set of full n-dualizability data of an object X to start with a tower of left and right duals for X of infinite length

$$\ldots \dashv X^{LL} \dashv X^{L} \dashv X \dashv X^{R} \dashv X^{RR} \dashv \ldots\ .$$

In Definition D.2.1 this tower is hidden behind the braiding.

Lemma D.2.3. *Let C be a symmetric monoidal category. Let X be a dualizable object in C with dual X^*, unit u and counit c. Then we have the adjunctions*

$$(X^* \dashv X, u, c) \qquad \text{and} \qquad (X \dashv X^*, u \circ B_{X,X^*}, c \circ B_{X,X^*}),$$

in the delooping bicategory $\mathbf{B}\mathsf{C}$, where B is the braiding of C.

In [Ara17] Araújo showed that for full n-dualizablity one does not need to construct a set of full n-dualizability data, but a partial set of data suffices. For bicategories, this statement was already proven in [Lur09] and as Theorem 3.9 in [Pst14]. With our definitions we can formulate the slight generalization to not necessarily braided monoidal categories as follows.

Definition D.2.4. Let C be a monoidal (∞, N)-category and X an object. A **set of partial 1-dualizability data** for X is a left or right dual object X^* together with unit u and counit c witnessing the duality. For $n \geq 2$ a **set of partial n-dualizability data** is a set of partial 1-dualizability data together with sets of a^n-adjunctibility data for u and c for a dexterity map $a^n : \{1, 2, \ldots, n\} \to \{L, R\}$.

Proposition D.2.5 (Araújo, Theorem 4.1.19 in [Ara17])**.** *Let C be a symmetric monoidal (∞, N)-category and $n \geq 1$. An object X in C is fully n-dualizable (Definition D.2.1) if and only if it is n-dualizable (Definition 10.3.2) if and only if it has a set of partial n-dualizability data.*

Proof. We construct full n-dualizability data from a set of partial n-dualizability data.

Consider the delooping $\mathbf{B}\mathsf{C}$ which is an $(\infty, N+1)$-category with a single object and 1-morphisms given by objects in C with composition given by the symmetric monoidal product. The partial n-dualizability data of X in C gives us a^{n+1}-adjunctibility data for X in $\mathbf{B}\mathsf{C}$ with a dexterity function $a^{n+1} : \{1, 2, \ldots, n+1\} \to \{L, R\}$. By Lemma D.2.3 we also have b^{n+1}-adjunctibility data for X, where b^{n+1} agrees with a^{n+1} except for $b^{n+1}(1) = -a^{n+1}(1)$ with the opposite value. By Theorem 11.2.10 we see that X in $\mathbf{B}\mathsf{C}$ is ambidextrous n-adjunctible and we can extend the ambidextrous n-adjunctibility data to $(n+1)$-adjunctibility data.

By ambidexterity we have towers of adjunctions up to the level of n-morphisms. Consider the unit u and counit c of an adjunction of n-morphisms $f \dashv g$. From the n-adjunctibility data we have two-step towers

$$u^L \dashv u \dashv u^R \qquad \text{and} \qquad c^L \dashv c \dashv c^R.$$

Applying the Interchange Lemma 11.1.3 to $f \dashv g$ gives us an adjunction $(g \dashv f, c^L, u^L)$. But we also have an adjunction $g \dashv f$ from ambidexterity which has a unit and counit that are part of the $n+1$-adjunctibility data and therefore have left and right adjoints. It follows now from Proposition 10.2.4 that u^L and c^L have left adjoints u^{LL} and c^{LL}. Continuing with $(g \dashv f, c^L, u^L)$ and two-step towers $c^{LL} \dashv c^L \dashv c$ and $u^{LL} \dashv u^L \dashv u$ we can inductively craft an infinite tower of left adjoints for u and c. Analogously, we build towers of right adjoints for u and c. Altogether we get full n-dualizability data for X in C. □

D.3. Proofs of Dualizability in the Morita Bicategory

Proof of Proposition 12.1.2. Let A be a $\mathbb{K}$-algebra. The identity morphism of A in Alg_1 is given by ${}_AA_A$. We need to show that the zig-zag identities hold up to equivalence. For the first one

$$\left(A \xrightarrow{{}_{\mathbb{K}}A_{A\otimes A^{\mathrm{op}}} \otimes_{\mathbb{K}} {}_AA_A} A \otimes_{\mathbb{K}} A^{\mathrm{op}} \otimes_{\mathbb{K}} A \xrightarrow{{}_AA_A \otimes_{\mathbb{K}} {}_{A^{\mathrm{op}}\otimes A}A_{\mathbb{K}}} A \right) \cong {}_AA_A,$$

we need a bimodule isomorphism

$$({}_{\mathbb{K}}A_{A\otimes_{\mathbb{K}}A^{\mathrm{op}}} \otimes_{\mathbb{K}} {}_AA_A) \bigotimes_{A\otimes_{\mathbb{K}}A^{\mathrm{op}}\otimes_{\mathbb{K}}A} ({}_AA_A \otimes_{\mathbb{K}} {}_{A^{\mathrm{op}}\otimes_{\mathbb{K}}A}A_{\mathbb{K}}) \cong {}_AA_A\,. \tag{D.3}$$

This isomorphism is given by mapping an element $(a\otimes b)\otimes(c\otimes d)$ of the left hand side in (D.3) to the product $bdac$. Similarly, we get an isomorphism for the other zig-zag identity

$$\left(A^{\mathrm{op}} \xrightarrow{{}_{A^{\mathrm{op}}}A^{\mathrm{op}}_{A^{\mathrm{op}}} \otimes_{\mathbb{K}} {}_{\mathbb{K}}A_{A\otimes A^{\mathrm{op}}}} A^{\mathrm{op}} \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A^{\mathrm{op}} \xrightarrow{{}_{A^{\mathrm{op}}\otimes A}A_{\mathbb{K}} \otimes_{\mathbb{K}} {}_{A^{\mathrm{op}}}A^{\mathrm{op}}_{A^{\mathrm{op}}}} A^{\mathrm{op}} \right) \cong {}_{A^{\mathrm{op}}}A^{\mathrm{op}}_{A^{\mathrm{op}}}.$$

□

Before we turn to the general question, which bimodules ${}_AM_B$ have adjoints, consider the special case that $A = B = \mathbb{K}$ is a field. Then ${}_AM_B$ is just a vector space and it has a left/right adjoint precisely when it has a left/right dual as an object in the category Vect of vector spaces and linear maps. To see this, we use a finite basis to construct the unit morphism. We will see that we need a finiteness condition for the general case of bimodules to replace the finite basis. Similar to the vector space case the left and right adjoint of ${}_AM_B$ will then be given by $\mathrm{Hom}_A(M, A)$ and $\mathrm{Hom}_B(M, B)$, respectively.

Let ${}_AL_B$ be a bimodule. Assume we have a right adjoint ${}_BR_A$ in Alg_1. Then there is a unit 2-morphism

$$u : {}_AA_A \to {}_AL_B \otimes_B {}_BR_A$$

of (A, A)-bimodules and a counit 2-morphism

$$c : {}_BR_A \otimes_A {}_AL_B \to {}_BB_B$$

of (B,B)-bimodules satisfying the zig-zag identities. The unit u is completely determined by

$$u(1) = \sum_{i,j} l_i \otimes_B r_j \qquad \text{for some} \quad l_i \in L,\ r_j \in R, \tag{D.4}$$

and c is given by a bilinear map

$$R \times L \to B, \qquad (r,l) \mapsto c(r,l). \tag{D.5}$$

The first zig-zag identity is

$$l \overset{u\otimes \mathrm{id}}{\longmapsto} \sum_{i,j} l_i \otimes r_j \otimes l \overset{\mathrm{id}\otimes c}{\longmapsto} \sum_{i,j} l_i \otimes c(r_j,l) \overset{\sim}{\mapsto} \sum_{i,j} l_i c(r_j,l) \;=\; l \tag{D.6}$$

for all $l \in L$ and the second zig-zag identity is

$$r \overset{\mathrm{id}\otimes u}{\longmapsto} \sum_{i,j} r \otimes l_i \otimes r_j \overset{c\otimes \mathrm{id}}{\longmapsto} \sum_{i,j} c(r,l_i) \otimes r_j \overset{\sim}{\mapsto} \sum_{i,j} c(r,l_i) r_j \;=\; r \tag{D.7}$$

for all $r \in R$. Recall that for finite dimensional vector spaces one uses the fact that there is a finite dual basis to determine the l_i and r_j. For modules we want finitely presented projectiveness to replace the finite dimensionality. We will use the fact that a right A-module P is finitely presented projective if and only if there is a retract of a free A-module A^n of finite rank n. The following Lemma and Proposition and their proofs are mainly taken from [Yua].

Lemma D.3.1 (Dual basis Lemma). *A left A-module P is finitely presented projective if and only if there exist $e_1, \ldots, e_n \in P$ and elements $e_1^*, \ldots, e_n^*$ in the right A-module $P^* := \mathrm{Hom}_A(P,A)$ such that*

$$p = \sum_{i=1}^{n} e_i(e_i^*(p)) \qquad \textit{for all} \quad p \in P. \tag{D.8}$$

Furthermore, in this case we have

$$p^* = \sum_{i=1}^{n} p^*(e_i)e_i^* \qquad \textit{for all} \quad p^* \in P^*. \tag{D.9}$$

Proof. The elements $e_1, \ldots, e_n \in P$ define a morphism $r : A^n \to P$ and the $e_1^*, \ldots, e_n^* : P \to A$ define a morphism $\iota : P \to A^n$. Equation (D.9) implies that r is a retract $r \circ \iota = \mathrm{id}_P$. Conversely, for a retract $r : A^n \to P$ and map $\iota : P \to A^n$ such that $r \circ \iota = \mathrm{id}_P$ we can define $e_1, \ldots, e_n \in P$ as images of the unit vectors in A^n and $e_1^*, \ldots, e_n^*$ as components of ι.

Equation (D.9) follows from applying $\mathrm{Hom}_A(\_, P)$ to get dual morphisms $r^* : P^* \to A^n$ and $\iota^* : A^n \to P^*$ with $\iota^* \circ r^* = \mathrm{id}_{P^*}$. So P^* is finitely presented projective and the e_i^* together with the images of e_i under the canonical map $(P^*)^* \to P$ form a dual basis. □

Proof of Proposition 12.1.3. Let ${}_AR_B$ be a bimodule that is finitely presented projective as a left A-module. Define $L = \mathrm{Hom}_A(R, A)$. Then L is (B, A)-bimodule. By Lemma D.3.1 there are $e_1, \ldots, e_n \in R$ and $e_1^*, \ldots, e_n^* \in L$ such that

$$r = \sum_{i=1}^{n} e_i(e_i^*(r)) \qquad \text{for all} \quad r \in R. \tag{D.10}$$

and

$$l = \sum_{i=1}^{n} l(e_i)e_i^* \qquad \text{for all} \quad l \in L. \tag{D.11}$$

We define a unit as in (D.4) with $r_i = e_i$ and $l_i = e_i^*$ and a counit as in (D.5) with $c(r, l) = l(r)$ for $r \in R$ and $l \in L$. Both zig-zag identities (D.6) and (D.7) are satisfied by (D.10) and (D.11).

Conversely, let ${}_AR_B$ be a bimodule with left adjoint ${}_BL_A$. Let u be the unit and c be the counit given by (D.4) and (D.5). The first zig-zag identity is equation (D.6) and gives us $l = \sum_{i,j} l_i c(r_j, l)$ for all $l \in L$. So $l_1, \ldots, l_n$ together with $r_1, \ldots, r_n$ are a dual basis as in Lemma D.3.1. Therefore ${}_AR_B$ is finitely presented projective as a left A-module. Furthermore $L \cong \mathrm{Hom}_A(R, A)$ by the uniqueness of adjoints up to isomorphism from Lemma 10.1.7.

Dually, the statement for right adjoints can be proven analogously or by using the opposite category. □

List of Figures

List of Symbols

Symbols with description and first appearance outside of the introduction.

List of Symbols

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