A homotopy-coherent calculus of lax matrices

Angus Hadrian Rush

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First Referee: Prof. Dr. Tobias Dyckerhoff
Second Referee: Prof. Dr. Rune Haugseng

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Fachbereich Mathematik

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Summary

This thesis consists of two chapters, together with two supporting appendices. The first chapter describes the construction of a homotopy-coherent calculus of lax matrices. The second chapter gives a construction of a lax monoidal structure on the functor implementing pull-push of local systems along spans of ∞-groupoids. The first appendix shows that the notion of internal left Kan extension can be captured by horn filling conditions, and the second proves a formula for reflective localization in terms of spans.

Zusammenfassung

Diese Doktorarbeit besteht aus zwei Kapiteln und zwei Anhängen. Das erste Kapitel beschreibt die Konstruktion eines homotopie-kohärenten Kalkül von laxen Matrizen. Das zweite Kapitel erläutert die Konstruktion einer lax monoidalen Struktur auf einem Pull-Push Funktor von lokalen Systemen entlang Korrespondenzen von ∞-Groupoiden. Im ersten Anhang wird gezeigt, dass interne Links-Kanerweiterungen durch Hornfüller beschrieben werden können, und im zweiten wird eine Formel für reflektive Lokalisierungen bezüglich Korrespondenzen hergeleitet.

Contents

Chapter 1

Introduction

1.1 Introduction

This thesis contains two very different answers to the same question: what higher-categorical structure generalizes the calculus of matrices? These two answers correspond to two different points of view as to the 'true nature' of a matrix:

(1) A matrix is a thing which encodes a linear map by breaking its source and target into constituent parts, and recording the action of the linear map on each part.

(2) A matrix is a rectangular array of numbers.

In Chapter 2, we will generalize (1) with respect to an appropriate higher-categorical analog of decomposition into constituent parts: lax (co)limits. We will refer to the objects encoding the morphisms between these decompositions as *lax matrices.* We will define a notion of lax matrix multiplication, yielding a homotopy-coherent calculus of lax matrices.

In Chapter 3, we will generalize (2) by replacing numbers by *spaces,* which we think of as encoding numbers via their homotopy type. We should then think of a matrix as a space parametrized by two other spaces, encoding the rows and columns. We can model this by a *span* of spaces. After recalling a calculus of matrices encoded via spans of spaces, we will show that it is in a certain sense *lax monoidal.*

1.1.1 Lax matrices

Matrix calculus

Consider the category $\mathrm{Vect}_{\mathbb{R}}$ of finite-dimensional real vector spaces, and let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a morphism there. Using that $\mathbb{R}^2$ is the cartesian product $\mathbb{R} \times \mathbb{R}$ in $\mathrm{Vect}_{\mathbb{R}}$, we can exchange our map into $\mathbb{R} \times \mathbb{R}$ for a cone over the product diagram

$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \qquad \rightsquigarrow \qquad \begin{tikzcd} & \mathbb{R} \\ \mathbb{R}^2 \arrow[ur, "A_0"] \arrow[dr, "A_1"'] & \\ & \mathbb{R} \end{tikzcd} .$$

Using that $\mathbb{R}^2$ is also the coproduct $\mathbb{R} \amalg \mathbb{R}$ in $\mathrm{Vect}_{\mathbb{R}}$, we can exchange each map out of $\mathbb{R}^2$ for a cone under the coproduct diagram.

$$\rightsquigarrow \qquad \begin{tikzcd} \mathbb{R} \arrow[r, "A_{00}"] \arrow[dr, "A_{10}" near start] & \mathbb{R} \\ \mathbb{R} \arrow[r, "A_{11}"'] \arrow[ur, "A_{01}" near start] & \mathbb{R} \end{tikzcd}$$

We recognize the resulting collection of maps (in this case, numbers) as the matrix representing A.

The essential ingredient of our decomposition was that since finite products and coproducts coincide in $\mathrm{Vect}_{\mathbb{R}}$, our linear map A could be interpreted as a map out of a coproduct and into a product. The rest followed trivially from the respective universal properties.

This argument works in great generality. Let $\mathbb{D}$ be an $(\infty, 2)$-category with sufficient lax limits and colimits. Let $F\colon \mathbb{A} \to \mathbb{D}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be lax functors. Let us denote the lax colimit of F by C_F, and the lax limit of G by L_G. A morphism $s\colon C_F \to L_G$ can be specified by any of the following equivalent data.

(1) A functor $\Delta^1 \to \mathbb{D}$ picking out the morphism s.

(2) A lax cone under F, i.e. a lax functor $\mathbb{A} \star \Delta^{\{1\}} \to \mathbb{D}$ whose restriction to $\mathbb{A}$ is given by F, and whose restriction to $\Delta^{\{1\}}$ is given by L_G; here, $- \star -$ denotes the *join* functor.

(3) A lax cone over G, i.e. a lax functor $\Delta^{\{0\}} \star \mathbb{B} \to \mathbb{D}$ whose restriction to $\Delta^{\{0\}}$ is given by C_F, and whose restriction to $\mathbb{B}$ is given by G.

(4) A lax functor $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ whose restriction to $\mathbb{A}$ is given by F, and whose restriction to $\mathbb{B}$ is given by G.

We refer to the data S in (4) as a *lax matrix* from F to G representing the morphism s.

Let us return briefly to our real vector spaces. Given a second map $B\colon \mathbb{R}^2 \to \mathbb{R}^2$, there is a formula for the matrix representing the composite $B \circ A$ in terms of the matrices representing A and B, which we draw here juxtaposed.

$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{B} \mathbb{R}^2 \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathbb{R} \arrow[r, "A_{00}"] \arrow[dr, "A_{10}" near start] & \mathbb{R} \arrow[r, "B_{00}"] \arrow[dr, "B_{10}" near start] & \mathbb{R} \\ \mathbb{R} \arrow[r, "A_{11}"'] \arrow[ur, "A_{01}" near start] & \mathbb{R} \arrow[r, "B_{11}"'] \arrow[ur, "B_{01}" near start] & \mathbb{R} \end{tikzcd}$$

To find the (i, j)th component of the matrix composition, we sum over all paths from the jth copy of $\mathbb{R}$ in the source to the ith copy of $\mathbb{R}$ in the target. Morally speaking, one can understand the origin of this sum as follows. Each matrix represents a map $\mathbb{R} \amalg \mathbb{R} \to \mathbb{R} \times \mathbb{R}$; a priori, these maps are not composable. To stitch together these maps, we need to supply a interpolating map $N^{-1}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \amalg \mathbb{R}$:

$$\mathbb{R} \amalg \mathbb{R} \xrightarrow{A} \mathbb{R} \times \mathbb{R} \xrightarrow{N^{-1}} \mathbb{R} \amalg \mathbb{R} \xrightarrow{B} \mathbb{R} \times \mathbb{R} .$$

We can think of the map N^{-1} as sending an object $(a, b) \in \mathbb{R} \times \mathbb{R}$ to the formal sum $a + b \in \mathbb{R} \amalg \mathbb{R}$. We note that this map is an isomorphism.

Again, we would like to find some analog of matrix multiplication for our lax matrices. Let $\mathbb{C}$ be another $(\infty, 2)$-category, $H\colon \mathbb{C} \to \mathbb{D}$ a lax functor with lax limit L_H, and $t\colon C_G \to L_H$ a morphism corresponding to a lax matrix $T\colon \mathbb{B} \star \mathbb{C} \to \mathbb{D}$ from G to H. Under certain conditions, there is a comparison map from the lax colimit of G to its lax limit. If this comparison map is an equivalence, then we can use its inverse to stitch together our morphisms s and t via the composition

$$L_F \xrightarrow{s} C_G \xleftarrow{\simeq} L_G \xrightarrow{t} C_H \ .$$

We can then translate this result to a matrix $(T \circ_G S)\colon \mathbb{A} \star \mathbb{C} \to \mathbb{D}$ from F to H. There is a formula for this matrix in terms of the matrices S and T, analogous to the case of ordinary matrix multiplication, in which the sum over paths is replaced by a *colimit.*

Our main goal in Chapter 2 is to formalize this notion of multiplication of lax matrices, and show that it is associative and unital up to coherent homotopy. We will this by constructing a $(\infty, 2)$-category $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ as the horizontal category underlying a double ∞-category whose objects are lax functors, whose horizontal morphisms are lax matrices, and whose horizontal composition is given by matrix multiplication.

We now describe what is to be gained from such a construction.

Semiadditivity

Let us again return to our world of linear maps. Our reasoning in constructing matrix multiplication was as follows.

(1) We can write a morphism out of a coproduct and into a product as a matrix by unraveling the universal properties.

(2) We know that finite products and coproducts in $\mathrm{Vect}_{\mathbb{R}}$ coincide. Therefore, we can chain maps from coproducts into products by composing with a comparison map, defining a notion of composition of matrices.

(3) We can work out an explicit description of the matrix representing this composition in terms of the original matrices by taking a sum in the hom-categories.

Suppose $\mathcal{D}$ is a 1-category in which finite products and coproducts coincide (in the sense that there is an invertible natural comparison map from the coproduct of a diagram to its product). Then precisely the reasoning above shows that we can construct a 1-category $\mathcal{M}\mathrm{at}(\mathcal{D})$, whose objects are n-tuples of objects of $\mathcal{D}$, whose morphisms are matrices of morphisms in $\mathcal{D}$ connecting each of the objects in the tuples, and whose composition is matrix composition, defined by composing with the comparison map. In fact, it turns out that this structure is sufficient to define a notion of addition on the hom-sets of $\mathcal{D}$, where the sum of f, $g\colon d \to d'$ is given by the matrix multiplication

$$\begin{tikzcd} & d' \arrow[dr, "\mathrm{id}"] & \\ d \arrow[ur, "f"] \arrow[dr, "g"'] & & d' \\ & d' \arrow[ur, "\mathrm{id}"'] & \end{tikzcd} \ .$$

This exhibits $\mathcal{D}$ as *enriched in commutative monoids.*

Note that (1) and (2) are not actually necessary to define a calculus of matrices: as long as

$\mathcal{D}$ is enriched in commutative monoids, we can construct a 1-category $\mathcal{M}\mathrm{at}(\mathcal{D})$ of matrices in $\mathcal{D}$ purely formally.

- The objects of $\mathcal{M}\mathrm{at}(\mathcal{D})$ are tuples of objects $(d_0, \ldots, d_m)$ of $\mathcal{D}$.
- A morphism from $(d_0, \ldots, d_m)$ to $(d'_0, \ldots, d'_n)$ consists of a collection of morphisms

  $$(f_{ij} \colon d_i \to d'_j)_{0 \le i \le m, 0 \le j \le n},$$

  which we will call a *matrix.*
- The composition of morphisms $(f_{ij} \colon d_i \to d'_j)$ and $(g_{jk} \colon d'_j \to d''_k)$ has components

  $$(g \circ f)_{ik} = \sum_{j=0}^{m} g_{jk} \circ f_{ij}.$$

A cone is a special case of a matrix, in the case that the source tuple is a singleton, and a cocone is a matrix in which the target tuple is a singleton. Playing around with the category of $\mathcal{M}\mathrm{at}(\mathcal{D})$, one notices something extraordinary: a cone is an isomorphism in $\mathcal{M}\mathrm{at}(\mathcal{D})$ if and only if it is a product cone, and a cocone is an isomorphism if and only if it is a coproduct cone. But since isomorphisms must have inverses (which are themselves isomorphisms), any coproduct cone admits an inverse product cone, and vice versa; thus products and coproduct coincide! Thus, in any 1-category which is enriched in commutative monoids, any products which exist are also coproducts, and vice versa.

In [CKW87], it is shown that an analogous result holds for lax matrices, at least in the truncated setting: if $\mathbb{D}$ is a 2-category which is enriched in categories with colimits, then any lax limits which happen to exist in $\mathbb{D}$ are also lax colimits, and vice versa. This result is shown by analogous reasoning to the above case: a lax (co)cone is a lax (co)limit cone if and only if it is an equivalence in the category of lax matrices; since equivalences have inverses, lax limits and colimits coincide. In [CDW24], a similar result is shown in the $(\infty, 2)$-categorical setting, for lax colimits indexed by strict functors out of $(\infty, 1)$-categories.

In Subsection 2.4.3, we detail the missing pieces necessary to show that this holds for arbitrary lax functors out of $(\infty, 2)$-categories.

Universality

A 1-category $\mathcal{D}$ which is enriched in commutative monoids, and possessing all finite products (which are therefore also coproducts) is known as *semiadditive.* A functor between semiadditive categories which preserves the enrichment and products is known as a *semiadditive functor.* One can consider a 2-category of semiadditive categories, semiadditive functors, and natural transformations.

There is a forgetful functor to the 2-category of categories enriched in commutative monoids, which forgets the existence of (co)products. It is natural to ask if this functor has a left adjoint. Put differently, given a category $\mathcal{D}$ enriched in commutative monoids, is there a free semiadditive category on $\mathcal{D}$? The answer is yes: the free semiadditive category on $\mathcal{D}$ is simply $\mathcal{M}\mathrm{at}(\mathcal{D})$.

In analogy, let us call an $(\infty, 2)$-category *lax semiadditive* if it is enriched in categories with colimits, and has all lax limits (which are by above lax colimits). In [GS16] it is shown that if $\mathbb{D}$ is a truncated 2-category enriched in categories with colimits, then $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ is the free lax semiadditive 2-category on $\mathbb{D}$ (this is not form in which this result is phrased there, but with sufficient work, their phrasing can be deformed to ours). In Subsubsection 2.4.3, we give a sketch of the proof there in our language, which we hope will generalize to the $(\infty, 2)$-categorical context.

Applications

The calculus of lax matrices has already been found applications as a powerful calculational tool, in the study of spherical functors in [DKS23] and [Dyc+21], and the *categorified homological algebra* program of [CDW23]. Until now, the definition of matrix multiplication was ad hoc, expressed in terms of a pullback along the fibrations encoding various twisted arrow categories. This definition did not admit a clear homotopy-coherent formulation, which meant that the lax matrix calculus could not be applied in certain situations, e.g. checking that complex diagrams are coherently commutative. Our definition of matrix multiplication is fully homotopy-coherent, and exists in more generality (for lax functors out of $(\infty, 2)$-categories). We also derive a simplified tool for calculating it in terms of *generalized coends,* which simplifies to an ordinary coend formula in the case of strict functors indexed by $(\infty, 1)$-categories.

1.1.2 Monoidal pull-push

Categorifying numbers by sets

At its most basic, a matrix is a rectangular array of numbers. One tried-and-true type of categorification replaces numbers by finite sets. We can then 'decategorify' by taking cardinality. Taking this approach to categorification, we can encode a matrix by a diagram

$$\begin{tikzcd} & \mathsf{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathsf{X} & & \mathsf{Z} \end{tikzcd}$$

of finite sets. Here, X is a set whose elements parametrize the indices of the input vector, and Z is a set whose elements parametrize the indices of the output vector. If x is an element of X and z is an element of Z, the (x, z)-entry of our matrix is encoded by the mutual fiber in Y over x and z.

We would like to view the above diagram of finite sets (a diagram of this shape is known as a *span*) as a linear map $k^{\mathsf{X}} \to k^{\mathsf{Z}}$ for some field k. One can recover the action of the matrix encoded in this way on a vector as follows. The vector space k^{X} is the free k-vector space on X, whose elements are functions $\phi\colon \mathsf{X} \to k$. The action of our matrix on the vector ϕ produces a new vector $g_! f^*\phi\colon \mathsf{Z} \to k$ via the following two-step process.

(1) We *pull* back ϕ along f, producing a function $f^*\phi\colon \mathsf{Y} \to k$.

(2) We *push* $f^*\phi$ forward along g, producing a function $g_! f^*\phi\colon \mathsf{Z} \to k$ defined by the formula

$$g_! f^*\phi(z) = \sum_{g(y)=z} f(\phi(y)).$$

Given matrices

$$\begin{tikzcd} & \mathsf{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathsf{X} & & \mathsf{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathsf{W} \arrow[dl, "h"'] \arrow[dr, "j"] & \\ \mathsf{Z} & & \mathsf{T} \end{tikzcd},$$

we define the *composition* to be the matrix given by taking the pullback over Z and then composing, yielding the diagram

$$\begin{tikzcd} & \mathsf{Y} \times_{\mathsf{Z}} \mathsf{W} \arrow[dl] \arrow[dr] & \\ \mathsf{X} & & \mathsf{T} \end{tikzcd}.$$

It is then easy to check that this process recreates the standard definition of matrix multiplication.

Categorifying numbers by spaces

This approach has the downside that it is not very general: we can only create matrices of natural numbers. As suggested by Baez in [BD01], we can do better by upgrading our sets to spaces; one can then decategorify by taking *homotopy cardinality* as introduced in [Kon88], defined for a space X by the formula

$$\chi(X) = \sum_{[x]\in\pi_0(X)} \prod_{n=1}^{\infty} |\pi_n(X,x)|^{(-1)^n}.$$

In this form, a categorified matrix can be specified by a diagram of spaces

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}.$$

The space X parametrizes the components of the input vector, and Z the output. Then the mutual fiber in Y over $x \in X$ and $z \in Z$ corresponds to the (x,z)-component of the matrix.

Again, we can define an action of the matrix above on a vector $\phi\colon X \to \mathbb{C}$, where ϕ is now required to be locally constant:

(1) We *pull* back ϕ along f, producing a function $f^*\phi\colon Y \to k$.

(2) We *push* $f^*\phi$ forward along g, producing a function $g_! f^*\phi\colon Z \to k$ defined by the formula
$$g_! f^*\phi(z) = \sum_{g([y])=[z]} f(\phi(y))\chi(Y_z).$$

Classical applications

Until now, we have not given any reason to think that this approach to categorifying matrices may actually be interesting. We now give two now classical examples as evidence that this is the case.

A very powerful technique in many areas of mathematics is to bestow an algebraic structure on objects which a priori do not have one. A common way to do this is by replacing a set by the free vector space on its elements, effectively considering not the elements of the set as fundamental, but rather their formal sums. There are more examples of this than one could possibly hope to write down; we content ourselves with the following two, which will serve as motivation.

(i) As first discovered by Steinitz in 1901 [Ste01], and independently by Hall in 1959 [Hal59], for any prime p one can form an associated algebra (now known as its *Hall algebra*) whose underlying vector space is the free vector space on the set of isomorphism classes of finite abelian p-groups, which we will denote Hall_p.

(ii) In [DW90], Dijkgraaf and Witten introduced, for any finite group G, a topological quantum field theory (now known as *Dijkgraaf-Witten theory*) given by a functor $\mathrm{Bord}_n \to \mathrm{Vect}_k$ which assigns to any $(n-1)$-dimensional surface Σ the free vector space on the set of equivalence classes of G-bundles on Σ. We will denote this vector space by $\mathrm{DW}_G(\Sigma)$.

Having succeeded in introducing vector spaces to the problem at hand, one should go of in search of structures which naturally induce linear maps between them. By the above discussion, in order to find a linear map between these structures, it suffices to find a span between the corresponding spaces.

(i) Let $\mathfrak{X}_1$ denote the groupoid of finite abelian p-groups, and let $\mathfrak{X}_2$ denote the groupoid of short exact sequences of finite abelian p-groups. Then Hall_p is the free vector space on $\pi_0(\mathfrak{X}_1)$. We can define a multiplication map $\mathrm{Hall}_p \otimes \mathrm{Hall}_p \to \mathrm{Hall}_p$ via pull-push along the span

$$\begin{tikzcd} & \mathfrak{X}_2 \arrow[dl] \arrow[dr] & \\ \mathfrak{X}_1 \times \mathfrak{X}_1 & & \mathfrak{X}_1 \end{tikzcd},$$

where the left-facing arrow sends a short exact sequence $0 \to A \to B \to C \to 0$ to (A, C), and the right-facing arrow sends it to B. This indeed defines an associative, unital algebra structure on Hall_p, reproducing the classical definition of a Hall algebra [Dyc+18].

(ii) Denote by BG the classifying space of G. Then $\mathrm{DW}_G(\Sigma)$ is the free vector space on the set $\pi_0(\mathrm{Fun}(\Sigma, BG))$. Any bordism M from Σ to Σ' gives a cospan of spaces

$$\begin{tikzcd} & M & \\ \Sigma \arrow[ur, hook] & & \Sigma' \arrow[ul, hook'] \end{tikzcd},$$

and in turn a span of groupoids

$$\begin{tikzcd} & \mathrm{Fun}(M, BG)^{\simeq} \arrow[dl] \arrow[dr] & \\ \mathrm{Fun}(\Sigma, BG)^{\simeq} & & \mathrm{Fun}(\Sigma', BG)^{\simeq} \end{tikzcd}.$$

We can thus define, for any bordism $M\colon \Sigma \to \Sigma'$, an associated linear map $\mathrm{DW}_G(\Sigma) \to \mathrm{DW}_G(\Sigma')$. This process is functorial, yielding a functor $\mathrm{DW}_G\colon \mathrm{Bord}_n \to \mathrm{Vect}_k$ [DW90] which reproduces (non-twisted) Dijkgraaf-Witten theory.

General coefficient systems

For any space X and any field k, the set of locally constant k-valued functions on X can be identified with the zeroth cohomology of X with coefficients in k. We might wish to replace k by some more general system of coefficients, perhaps varying along X; such a varying system of coefficients is called a *local system.* If $\mathcal{C}$ is an ∞-category, a *local system on X with values in* $\mathcal{C}$ is simply given by a functor from X to $\mathcal{C}$. Suppose $\mathcal{C}$ is cocomplete. Given our span of spaces

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd},$$

we can transport a $\mathcal{C}$-local system $\mathcal{F}\colon X \to \mathcal{C}$ to a $\mathcal{C}$-local system on Z in an analogous 2-step process:

(1) Pull back along f, producing the $\mathcal{C}$-local system $f^*\mathcal{F}\colon Y \to \mathcal{C}$.

(2) Push forward along g by taking a left Kan extension, producing a $\mathcal{C}$-local system on Z defined by the formula
$$g_! f^*\mathcal{F}(z) = \operatorname*{colim}_{y \in Y_z} \mathcal{F}(f(y)).$$
Note that we have simply replaced a weighted sum by a colimit.

In fact, denoting by $\mathrm{LS}(\mathcal{C})_X := \mathrm{Fun}(X, \mathcal{C})$ the ∞-category of $\mathcal{C}$-local systems, this construction produces a functor $\mathrm{LS}(\mathcal{C})_X \to \mathrm{LS}(\mathcal{C})_Z$.

We can say even more. Denote by $\mathrm{Span}(\mathcal{S})$ the ∞-category whose objects are spaces, whose morphisms are spans of spaces, and whose composition is given by taking pullbacks. Then this construction yields a functor $\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$ sending X to $\mathrm{LS}(\mathcal{C})_X$. This follows on abstract grounds from the universal property of Span as given in [Mac22], but in Section 3.3 we will give an explicit construction.

Let $\mathcal{F}\colon X \to \mathcal{C}$ and $\mathcal{G}\colon Y \to \mathcal{C}$ be $\mathcal{C}$-local systems. Suppose $\mathcal{C}$ is a monoidal ∞-category. Then given $\mathcal{C}$ local-systems on spaces X and Y, we can produce one on $X \times Y$ via the operation
$$(X \xrightarrow{\mathcal{F}} \mathcal{C}, Y \xrightarrow{\mathcal{G}} \mathcal{C}) \quad \mapsto \quad X \times Y \xrightarrow{\mathcal{F}\times\mathcal{G}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

If the tensor product on $\mathcal{C}$ preserves colimits in each slot, then this operation yields a lax monoidal structure on our functor $\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$. This is the main new feature of this treatment; we'll construct this explicitly in Section 3.4.

Motivation

Our interest in this construction stems from the case that $\mathcal{C} = \mathrm{Sp}$, the ∞-category of spectra, understood as a symmetric monoidal category via the smash product. A Sp-valued local system is nothing else but a parametrized spectrum; both contravariant and covariant functoriality along maps of spaces are well-studied in this context. Our construction packages all of this data, together with that of the smash product, in a fully homotopy-coherent way. We plan to use this as a stepping-stone result in order to better understand the Becker–Gottlieb transfer. There has been recent progress in this direction in [Car+22], where a potential obstruction for Becker–Gottlieb transfer is constructed. However, to the knowledge of the author, no counterexample to the functoriality at the level of the homotopy category is known.

Aside from possible applications to stable homotopy theory, we believe that a lax monoidal pull-push of local systems is of independent interest.

1.2 Structure of the thesis and main results

In this section, we give a very high-level overview of the arguments we give, and of the major results we intend to show along the way. We'll aim for terseness rather than completeness or precision. Each chapter begins with a more in-depth overview, and each section with a comprehensive one.

1.2.1 Lax matrices

In Chapter 2, we construct a homotopy-coherent calculus of lax matrices. Our first ingredient will be a model for $(\infty,2)$-categories as a special case of generalized ∞-operads which we'll call *Segal ∞-bicategories;* these will occupy our attention in Section 2.2. We'll note that the $(\infty,2)$-category $\mathbb{G}\mathrm{enOp}_\infty$ of generalized ∞-operads has a full subcategory $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ which models an $(\infty,2)$-category of $(\infty,2)$-categories (flagged by a set which we think of as specifying a set of objects, see Subsection 2.4.1), lax functors, and icons.

We then generalize the theory of operadic Kan extensions as laid out in [HA] and [GH15] to functors into Segal ∞-bicategories, giving an analogous formula in the framework of *generalized operadic colimits,* which we also define. Here, we have to be a bit careful: there are some results which we have not yet had time to check. However, we expect that the existing proofs in the non-generalized case should go through mutatis mutandis. Assuming this, we are able to show as a special case:

Theorem. Let $f\colon \mathbb{A} \to \mathbb{B}$ be a functor of $(\infty,2)$-categories which induces a bijection on objects, and let $\mathbb{D}$ be a locally cocomplete $(\infty,2)$-category. Then the notion of left Kan extension internal to $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ of functors into $\mathbb{D}$ along f is simply given by left Kan extension within hom-categories.

We then turn our attention in Section 2.3 to Segal ∞-bicategories which admit a map to the n-simplex. Given a Segal ∞-bicategory $\mathbb{A}$ together with a map $\mathbb{A} \to \Delta^n$, we define its *spine* to be the portion of $\mathbb{A}$ living over $\mathrm{Spine}(\Delta^n)$. Given a Segal ∞-bicategory $\mathbb{A}$ over the n-simplex, we study functors $\mathbb{A} \to \mathbb{D}$ which are *extended from their spine* in the sense of generalized operadic Kan extensions, and provide a simplified colimit formula in this case.

We then provide a universal characterization of the join of Segal ∞-bicategories.

Theorem. The functor $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{/\Delta^n} \to (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})^{n+1}$ which sends $\mathbb{A} \to \Delta^n$ to its fibers $(\mathbb{A}_0, \ldots, \mathbb{A}_n)$ over the vertices of Δ^n admits a right adjoint, sending $(\mathbb{A}_0, \ldots, \mathbb{A}_n) \mapsto \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \to \Delta^n$, which we interpret as defining a *join* operation on Segal ∞-bicategories.

A priori, one might only expect this adjunction to hold for strict functors. Our argument relies on being able to replace lax functors out of a Segal ∞-bicategory $\mathbb{A}$ by strict ones out of $\mathrm{Env}(\mathbb{A})$, the double-categorical envelope of $\mathbb{A}$.

Finally, we study functors $F\colon \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \to \mathbb{D}$ which are extended from their spine, and show that in this case, our colimit formula simplifies to a generalized coend formula:

Theorem. A functor

$$F\colon \mathbb{A} \star \mathbb{B} \star \mathbb{C} \to \mathbb{D}$$

is extended from its spine if and only if for each objects $a \in \mathbb{A}$ and $c \in \mathbb{C}$, we have

$$F_{a,c} = \int^{b:\mathbb{B}} F_{b,c} \circ F_{a,b};$$

here $\int^{b:\mathbb{B}}$ denotes the generalized coend of Definition 2.3.5.3, and $F_{a,c}$ the image of the *unique* morphism in $\mathbb{A} \star \mathbb{B} \star \mathbb{C}$ from a to c.

In Section 2.4, we finally get around to defining our double ∞-category $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ of lax matrices. We can give a conceptually clear definition by working internal to $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$:

Theorem. Let $\mathbb{D}$ be a locally cocomplete Segal ∞-bicategory. There is a double ∞-category $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ with the following description.

- An object of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ consists of a Segal ∞-bicategory $\mathbb{A}$ together with a lax functor $F\colon \mathbb{A} \to \mathbb{D}$.
- The vertical morphisms of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ are icons $\begin{tikzcd} \mathbb{A} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{D} \end{tikzcd}$.
- A horizontal morphism of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ from $F\colon \mathbb{A} \to \mathbb{D}$ to $G\colon \mathbb{B} \to \mathbb{D}$ is a lax functor $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ such that $S|\mathbb{A} = F$ and $S|\mathbb{B} = G$.
- A lax functor $U\colon \mathbb{A} \star \mathbb{B} \star \mathbb{C} \to \mathbb{D}$ exhibits $U|\mathbb{A} \star \mathbb{C}$ as the *composition* of $U|\mathbb{A} \star \mathbb{B}$ and $U|\mathbb{B} \star \mathbb{C}$ if and only if it is extended from its spine.

The underlying 'horizontal' $(\infty, 2)$-category of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$, denoted $\mathbb{L}\mathrm{axMat}(\mathbb{D})$, is a more general version of the $(\infty, 2)$-category of lax matrices hypothesized in [CDW24]. In particular, we provide a homotopy-coherent foundation for the results there, including a simplified formula for lax matrix composition in terms of *generalized coends,* and a conceptually transparent construction of the identity matrix. We note that there is some subtletly in this construction. The naive construction of the identity matrix fails, and to rectify this, we must apply a homotopy-coherent localization procedure; the technology to do this is built in Appendix A.2.

We also record a curiosity: $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ contains as a full subcategory the Morita category of monads in $\mathbb{D}$.

Finally, in Subsection 2.4.3, we sketch two applications of the tools we have built. Using the logic of Subsection 1.2.1, it should not be too difficult to show that in any locally cocomplete $(\infty, 2)$-category $\mathbb{D}$, lax limits and colimits coincide. The only tool that is missing is a tool for 'decomposing' a lax cone into its constituent simplices, which should be easy to construct. One immediate application is that given any monad in an locally cocomplete $(\infty, 2)$-category, its Kleisli object and Eilenberg-Moore object coincide if they exist.[1]

[1] In fact, one only needs that $\mathbb{D}$ is enriched in ∞-categories which admit geometric realizations, i.e. colimits over Δ^{op}.

Assuming this, we hope to be able to show, with a bit of effort, that $\mathbb{L}\mathrm{axMat}(\mathbb{D})$, the underlying 'horizontal' $(\infty, 2)$-category of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ is the free lax cocompletion of $\mathbb{D}$.

1.2.2 Monoidal pull-push

In Chapter 3, we construct a functorial pull-push of local systems, together with a lax monoidal structure. The first half of the chapter is mainly setting the stage, and recalling the necessary constructions.

In Section 3.2, we give an explicit combinatorial model for an ∞-category $\mathrm{LS}(\mathcal{C})$ of $\mathcal{C}$-local systems, where $\mathcal{C}$ is an ∞-category, in terms of the *enhanced twisted arrow category* of [GS20]. Much of the section consists of preparing the necessary combinatorial tools. Note that this is far from the most efficient way of encoding $\mathrm{LS}(\mathcal{C})$, but we will need the extraneous structure in our construction of the lax monoidal structure. Our main tool is the horn-filling property of left Kan extensions, described in Appendix A.1.

In Section 3.3, we use our explicit combinatorial model of $\mathrm{LS}(\mathcal{C})$ to construct a functor $\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$ sending a space X to the ∞-category $\mathrm{LS}(\mathcal{C})_X$ of $\mathcal{C}$-local systems on X, and a span $X \xleftarrow{f} Y \xrightarrow{g} Z$ to the pull-push functor $g_! f^*$. Again, this functor exists on theoretical grounds, but we will make use of our combinatorial model.

In Section 3.4, we prove the following result:

Theorem. Let $\mathcal{C}^\otimes$ and $\mathcal{D}^\boxtimes$ be symmetric monoidal ∞-categories (encoded as cartesian fibrations over $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$), and let $p^\otimes \colon \mathcal{C}^\otimes \to \mathcal{D}^\boxtimes$ be a monoidal functor. Denote the functor on the underlying ∞-categories by $p \colon \mathcal{C} \to \mathcal{D}$. Suppose:

- The functor p is a cartesian fibration.
- The tensor product $\otimes$ preserves p-cartesian morphisms in the sense that if f and g are p-cartesian morphisms in $\mathcal{C}$, then so is $f \otimes g$.

Then p classifies a lax monoidal functor $\mathcal{D}^\boxtimes \to \mathcal{C}\mathrm{at}_\infty^\times$, whose lax structure morphisms

$$\mathcal{C}_d \times \mathcal{C}_{d'} \to \mathcal{C}_{d \boxtimes d'}$$

are defined by restricting the tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to the fibers of p over d and d'.

We use this to give a more conceptual framing of the following result, which appears in [BGS19]:

Theorem. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category, and let $\mathcal{T}^\times$ be an ∞-category carrying the cartesian monoidal structure. Let $p^\otimes \colon \mathcal{C}^\otimes \to \mathcal{T}^\times$ be a monoidal functor, and let $p \colon \mathcal{C} \to \mathcal{T}$ be the underlying functor on ∞-categories. Suppose:

- The functor p is a Beck-Chevalley fibration.
- The tensor product $\otimes$ preserves both p-cartesian and p-cocartesian morphisms.

Then there is a lax monoidal functor

$$\hat{p} \colon \mathrm{Span}(\mathcal{T})^{\tilde{\times}} \to \mathcal{C}\mathrm{at}_\infty^\times$$

with the following description.

- On objects, the functor $\hat{p}$ sends $t \in \mathrm{Span}(\mathcal{T})$ to the fiber $\mathcal{C}_t \in \mathcal{C}\mathrm{at}_\infty$
- On morphisms, $\hat{p}$ sends a span $t \xleftarrow{g} s \xrightarrow{f} t'$ to the composition $f_! \circ g^* \colon \mathcal{C}_t \to \mathcal{C}_{t'}$.
- The structure morphisms
$$\mathcal{C}_t \times \mathcal{C}_{t'} \to \mathcal{C}_{t \times t'}$$
of the lax monoidal structure on $\hat{p}$ are given by the restriction of the tensor product $\otimes$ to the fibers over t and t'.

Applying this to our local systems construction, we find:

Theorem. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Further suppose that $\mathcal{C}$ is cocomplete, and that the tensor product $\otimes$ preserves colimits in each slot. Then there is a lax monoidal functor
$$\hat{r} \colon \mathrm{Span}(\mathcal{S})^{\tilde{\times}} \to \mathcal{C}\mathrm{at}_\infty^\times$$
with the following description.

- On objects, the functor $\hat{r}$ sends a space X to the ∞-category $\mathrm{LS}(\mathcal{C})_X$ of $\mathcal{C}$-local systems on X.
- On morphisms, the functor $\hat{r}$ sends a span of spaces $X \xleftarrow{g} Y \xrightarrow{f} X'$ to the pull-push
$$f_! \circ g^* \colon \mathrm{LS}(\mathcal{C})_X \to \mathrm{LS}(\mathcal{C})_{X'}.$$
- The structure morphisms of the lax monoidal structure are the maps
$$\mathrm{LS}(\mathcal{C})_X \times \mathrm{LS}(\mathcal{C})_{X'} \to \mathrm{LS}(\mathcal{C})_{X \times X'}$$
given by the composition
$$\mathrm{Fun}(X, \mathcal{C}) \times \mathrm{Fun}(Y, \mathcal{C}) \xrightarrow{\times} \mathrm{Fun}(X \times Y, \mathcal{C} \times \mathcal{C}) \xrightarrow{\otimes} \mathrm{Fun}(X \times Y, \mathcal{C}),$$
under the identification $\mathrm{LS}(\mathcal{C})_X \cong \mathrm{Fun}(X, \mathcal{C})$.

1.2.3 Appendices

We also include proofs of two results which are likely well known, but for which, to the knowledge of the author, no reference exists.

Horn filling via Kan extensions

In Appendix A.1, we show that the notion of Kan extension internal to an $(\infty, 2)$-category modelled as a coherent nerve can be expressed using horn filling conditions. The universal property of (global) Kan extension, phrased in terms of horn filling, is the following.

Definition. Let $\mathbb{A}$ be an $(\infty, 2)$-category in the sense of [Lur18, Tag 01W9] (i.e. a simplicial set with a collection of thin simplices with respect to which inner horn fillers exist). A not necessarily thin 2-simplex $\sigma \colon \Delta^2 \to \mathbb{A}$ is ***left Kan***, or simply ***Kan***, if for each $n \geq 3$, each solid diagram below admits a dashed filler.

$$\begin{tikzcd} \Delta^{\{0,1,n\}} \arrow[d, hook] \arrow[dr, "\sigma"] & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathbb{A} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

We then show the following.

Theorem. Let A be a quasicategory-enriched category, so that $\mathbb{A} = \mathrm{N}_{\mathrm{sc}}(\mathsf{A})$ is an $(\infty, 2)$-category. The following are equivalent.

(1) The pullback functor $f^*\colon \mathbb{A}(b, x) \to \mathbb{A}(a, x)$ admits a left adjoint at $F\colon a \to x$ in the sense of Definition A.1.2.10, given by a morphism $G\colon b \to x$, with local unit $\eta\colon G \Rightarrow F \circ f$.

(2) The left horn $\tau'\colon \Lambda^2_0 \to \mathbb{A}$ with that $\tau'|\{0,1\} = f$ and $\tau|\{0,2\} = F$ admits a Kan filler $\tau\colon \Delta^2 \to \mathbb{A}$ given by the 2-simplex

$$\tau = \begin{tikzcd} a \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & x \\ & b \arrow[ur, "G"'] & \end{tikzcd} .$$

This result forms the backbone of both of the main chapters.

Fiberwise localization

In Appendix A.2, we review basic definitions surrounding reflective localizations, and provide a formula for functorial reflective localization in terms of spans. This result is almost certainly well-known—many similar results appear in [HA]—but the author was not able to find a source for this form.

Theorem. Let $p\colon \mathcal{X} \to \mathcal{B}$ be a cartesian fibration, and suppose we are given, for each $b \in \mathcal{B}$, a reflective subcategory $\mathcal{Y}_b$ of the fiber $\mathcal{X}_b$ over b, witnessed by an inclusion $\rho_b\colon \mathcal{Y}_b \subseteq \mathcal{X}_b$. Further suppose that p has the following property:

($*$) Given a square of morphisms

$$\begin{tikzcd} x_{02} \arrow[r, "b"] \arrow[d] & x_{12} \arrow[d] \\ x_{01} \arrow[r, "a"] & y_{02} \end{tikzcd}$$

in $\mathcal{X}$ such that the downward-facing arrows are p-cartesian and a is a weak equivalence, then b is a weak equivalence.

Then there is a cocartesian fibration

$$\bar{p}\colon \mathrm{Span}^{\mathcal{Y}}(\mathcal{X}) \to \mathcal{B}^{\mathrm{op}},$$

where $\mathrm{Span}^{\mathcal{Y}}(\mathcal{X})$ is the category of spans of the form

$$y \xleftarrow{g} x \xrightarrow{f} y' ,$$

where g is p-cartesian and y and y' are in their respective reflective subcategories. If f is a weak equivalence (hence a reflector for x), then such a span is $\bar{p}$-cocartesian.

This is the linchpin of the construction of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$; the naive construction fails, roughly because the obvious definition of the identity matrix is not correct. However, we can exhibit the correct definition of the identity matrix as a localization of the obvious one, and the above theorem guarantees that this can be done in a homotopy-coherent way.

We have the immediate corollary, which is certainly well-known:

Corollary. Let $\mathcal{X} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a functor sending $b \mapsto \mathcal{X}_b$, and suppose for each object $b \in \mathcal{B}$ the ∞-category $\mathcal{X}_b$ has a reflective subcategory $\mathcal{Y}_b \subseteq \mathcal{X}_b$. Further suppose that for each $f \colon b \to b'$ in $\mathcal{B}$, the functor $\mathcal{X}_f \colon \mathcal{X}_{b'} \to \mathcal{X}_b$ preserves weak equivalences. Then there is an induced functor $\mathcal{Y} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending $b \mapsto \mathcal{Y}_b$.

1.2.4 Notation and conventions

This thesis consists of two essentially separate chapters. Both deal with very different structures, and thus with very different notational challenges. Due to this, the notation used in each chapter varies slightly. We will deal with this by including a notational primer at the beginning of each chapter.

We will not make any arguments which involve subtleties of issues of size. We thus avoid introducing nested Grothendieck universes, and keeping track explicitly of where we are operating. When potential issues appear, we will flag them; but the reader should assume that we would never attempt to take a (lax) (co)limit indexed by a large category.

1.3 Previous publications, Eigenständigkeitserklärung

The text of Chapter 1 is entirely original; this chapter is introductory and contains no new results.

The text and proven results of Chapter 2 were produced entirely independently.

The results of Chapter 3 were produced independently, with the exception of the proof of Lemma 3.2.3.6, which was joint work with Fernando Abellán García.

The material of Chapter 3 and Section A.1 are available in preprint form at [Rus22b].

1.4 Acknowledgements

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Chapter 2

Lax Matrices

2.1 Introduction

In this chapter, we construct, for any locally cocomplete ∞-category $\mathbb{D}$, a double ∞-category $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ of lax matrices in $\mathbb{D}$. A thorough description of the structure of $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ is given in Subsection 2.4.1.

2.1.1 Plan for the chapter

- In Section 2.2, we give an introduction to the model of $(\infty, 2)$-categories we will be using, *Segal ∞-bicategories,* as a special case of *generalized ∞-operads,* whose definition we also recall. We think of a Segal ∞-bicategory as having a set of objects; model independently, we should imagine an $(\infty, 2)$-category flagged by a set.

  Little of this material is original; much of it is either contained [Hau17], [GH15], [Hau16], or is a mild generalization of material found there. We define a notion of Kan extension between generalized ∞-operads as a straightforward generalization of the operadic Kan extension defined in [HA] and [Hau17], and as carefully explained in [Ara23]. Here we must assume that generalized operadic Kan extension has similar properties to ordinary operadic Kan extension; all of our further results must therefore be seen as contingent on these assumptions. For more information, see Subsection 2.1.3. With this caveat, our main new result, Proposition 2.2.4.4, is that generalized operadic Kan extension is a model for local Kan extension (i.e. Kan extension within hom categories) when restricted to functors between Segal ∞-bicategories. A model-independent phrasing of this result is that the notion of left Kan extension along extendable functors (in the sense of Definition 2.2.4.1) internal to the $(\infty, 2)$-category of $(\infty, 2)$-categories flagged by sets, lax functors, and icons is simply left Kan extension within hom-categories.

- In Section 2.3, we study $(\infty, 2)$-categories, modelled by Segal ∞-bicategories, admitting a map to the n-simplex. Although many results here are (to the knowledge of the author) technically new, we essentially follow [Hau16] and [GH15], generalizing the results there one-by-one. An exception is Subsection 2.3.4, where we define a model for the join of Segal ∞-bicategories generalizing the usual definition of the join of ∞-categories, and show that it satisfies a universal property.

- In Section 2.4, we use the material of Section 2.2 and Section 2.3 to define a double ∞-category $\mathfrak{LaxMat}(\mathbb{D})$ of lax matrices of diagrams in a locally cocomplete $(\infty,2)$-category $\mathbb{D}$. We note that the underlying 'horizontal' $(\infty,2)$-category $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ is a slightly more general form of the $(\infty,2)$-category of lax matrices hypothesized in [CDW24]. Our construction of $\mathfrak{LaxMat}(\mathbb{D})$ is modelled on the construction of the double ∞-category $\mathfrak{CAT}(\mathcal{V})$ of $\mathcal{V}$-enriched ∞-categories of [GH15], and the two are closely related: in the case that $\mathbb{D} = B\mathcal{V}^{\otimes}$ is the delooping of a symmetric monoidal ∞-category, the full subcategory of $\mathfrak{CAT}(\mathcal{B})$ on those enriched categories with an underlying *set* of objects is a full subcategory of $\mathfrak{LaxMat}(\mathbb{D})$. However, our construction requires an extra step, roughly because the naive construction of the identity matrix fails; we rectify this by exhibiting the proper identity matrix as a localization of the naive one.

  In Subsubsection 2.4.3, we sketch an argument that the results of [CDW24] regarding the equivalence of lax limits and colimits also hold in this generality.

  The original motivation for constructing $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ was to show that it satisfies a universal property: it is the free lax-cocomplete $(\infty,2)$-category on the cocomplete-enriched $(\infty,2)$-category $\mathbb{D}$. We sketch an argument that this might be the case, using essentially the arguments of [GS16].

2.1.2 General notation and conventions

We are dealing with two models for higher categories:

- An 'ambient' model, that of ∞-categories in the sense of [Lur18, Tag 003A] (often called *quasicategories* after Joyal), and $(\infty,2)$-categories in the sense of [Lur18, Tag 01W9] (which is really just a nicer phrasing of the definition of a fibrant scaled simplicial set).

- An 'internal' model, that of Segal ∞-categories and Segal ∞-bicategories (Definition 2.2.1.4), which are defined as cocartesian fibrations over Δ^{op}, the (opposite of the) simplex category, subject to certain conditions.

When working with objects living entirely within the ambient model, we do our level best to adhere to the following terminological and typographical practices regarding various types of categories.

- When we say '∞-groupoid' we will mean an $(\infty,0)$-category, here always modelled by a Kan complex. We will in general denote ∞-groupoids by capital roman letters coming from the end of the alphabet: X, Y, ...

- When we say '∞-category,' we will mean an $(\infty,1)$-category, modelled by a quasicategory. We will strive to denote ∞-categories by calligraphic letters typeset using the `eucal` package: $\mathcal{C}$, $\mathcal{D}$, ...

- When we say '$(\infty,2)$-category,' we mean it in the sense of [Lur18, Tag 01W9], modelled by a simplicial set satisfying certain lifting properties. We will denote $(\infty,2)$-categories by blackboard-bold letters typeset with the `mathbbol` package: $\mathbb{C}$, $\mathbb{D}$, ...

In particular, we follow Lurie in our definitions of categories of the following.

- The $(\infty, 1)$-category of spaces is $\mathcal{S}$. We model this as a quasicategory, constructed (as in [HTT]) as the homotopy-coherent nerve (as described in [HTT, Sec. 1.1.5], there called the simplicial nerve) of Kan, the Kan-enriched category of Kan complexes.
- The $(\infty, 1)$-category of $(\infty, 1)$-categories is $\mathcal{C}\mathrm{at}_\infty$. We model this as a quasicategory, defined to be the homotopy-coherent nerve of the Kan-complex-enriched category of quasicategories.
- The $(\infty, 2)$-category of $(\infty, 1)$-categories is $\mathbb{C}\mathrm{at}_\infty$. We model this as an $(\infty, 2)$-category, defined to be the coherent nerve of the quasicategory-enriched category of quasicategories.

In addition to the models described above, we will also need another model, that of *Segal ∞-bicategories* and their cousins (Definition 2.2.1.4), which is constructed internal to the above framework; roughly a Segal ∞-bicategory is cocartesian fibration $\mathbb{D} \to \Delta^{\mathrm{op}}$ classifying a functor $\Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ satisfying the Segal condition, such that $\mathbb{D}_{[0]}$ is a set. We should think of these as an $(\infty, 2)$-category with a *set* of objects; or more model independently, an $(\infty, 2)$-category flagged by a set of objects.

Here, we will use mathbbol-letters for the total space of our Segal ∞-bicategories to indicate that they are part of a model for flagged $(\infty, 2)$-categories, although they within the ambient space they are ∞-categories.

Working with one model internal to another introduces ambiguities, and potentially strenuous notation. To minimize this strain, we will adopt a model-free notation when we feel we can get away with it, in sections where we do not make explicit use of the specific properties of the Segal ∞-bicategory model. In the table on the next page, we give a brief dictionary to translate between the model-independent world and the world of Segal ∞-bicategories.

2.1.3 Loose ends

Due to time constraints, several results in this chapter are left unproven. In this section, we list them, and describe which later results depend on them.

(1) We define generalized operadic Kan extension (Definition 2.2.4.2) in analogy to operadic Kan extension [HA, Def. 3.1.2.2], but do not show that it is a model for a Kan extension, i.e. that it is left adjoint to pullback (Assumption 2.2.4.5). We strongly suspect that the proof in the non-generalized case will go through mutatis mutandis, but we have not yet checked that this is the case.

(2) We also assume that generalized operadic Kan extension is transitive (Assumption 2.2.4.6). In fact, we need only a weakened version of transitivity (the case that one of the functors is taking the generalized operadic Kan extension along a functor to a join, which amounts to taking the *colimit* in each hom-category), but assuming the full form allowed us to state several proofs much more cleanly.

(3) We assume that a simplified colimit formula for generalized operadic Kan extensions holds (Assumption 2.2.4.3), in analogy to [Ara23, Prop. 4.2]; again, we suspect that the proof there will generalized without any problems. If one does not wish to assume this, one can also check that the inclusions $\mathrm{Spine}(\mathbb{M}) \hookrightarrow \mathbb{M}$ (see Definition 2.3.2.1, in the case that $\mathcal{M} = \mathbb{M}$ is a Segal ∞-bicategory) are extendable in the sense of [Hau16, Def. 2.5].

Model independent	Segal ∞-bicategories	Reference
Set-flagged $(\infty, 2)$-category $\mathbb{A}$	Segal ∞-bicategory $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}$	Definition 2.2.1.4
Set of objects $\mathrm{ob}(\mathbb{A})$	Fiber $\mathbb{A}_{[0]}$	Definition 2.2.1.4
n-simplex Δ^n	$u_n\colon \Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$	Example 2.2.1.8
$\mathrm{Spine}(\Delta^n)$	$i_n\colon \Lambda^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$	Example 2.2.1.10
Lax functor $F\colon \mathbb{A} \to \mathbb{B}$	Morphism of generalized ∞-operads $\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "a"'] & & \mathbb{B} \arrow[dl, "b"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$	Definition 2.2.1.12
Strict functor $F\colon \mathbb{A} \to \mathbb{B}$	As above, F preserves cocartesian morphisms	Definition 2.2.1.12
Icon $\begin{tikzcd} \mathbb{A} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbb{B} \end{tikzcd}$	$\begin{tikzcd} \mathbb{A} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\alpha\Downarrow"] \arrow[dr] & & \mathbb{B} \arrow[dl] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$	Note 2.2.1.14
Hom-category $\mathbb{A}(a, b)$	$\mathbb{A}(a, b)$	Definition 2.2.1.15
Product $\mathbb{A} \times \mathbb{B}$	Fiber product $\mathbb{A} \times_{\Delta^{\mathrm{op}}} \mathbb{B} \to \Delta^{\mathrm{op}}$	Note 2.3.1.4
Local left Kan extension	Generalized operadic Kan extension	Proposition 2.2.4.4

2.2 Generalized operads and complete Segal categories

In this section, we recall the construction (given in this form in [Hau17], which is adapted from [HA]) of an $(\infty, 2)$-category $\mathbb{G}\mathrm{enOp}_\infty$ of generalized ∞-operads. Since the term 'generalized operad' is used in different ways in different areas of the literature (in particular, see Note 2.2.0.1), we should sketch what we mean by it. For us, a generalized ∞-operad $\mathcal{M}$ consists, roughly, of the following data:

- An ∞-category $\mathcal{M}_{[0]}$, whose objects we call the *objects* of $\mathcal{M}$. (We will mainly be interested in the case that $\mathcal{M}_{[0]}$ is a set, and for the remainder of this sketch we will assume this to be the case.)
- For each two objects m and m' of $\mathcal{M}$, an ∞-category $\mathcal{M}(m, m')$ of *morphisms* from m to m'.
- For each $n \geq 1$ and each n-tuple $(m_0, \dots, m_n)$ of objects of $\mathcal{M}$, a profunctor

$$S_{(m_i)_{0 \leq i \leq n}} \colon \mathcal{M}(m_{n-1}, m_n)^{\mathrm{op}} \times \cdots \times \mathcal{M}(m_0, m_1)^{\mathrm{op}} \times \mathcal{M}(m_0, m_n) \to \mathcal{S}.$$

  If $(f_n, \dots, f_1, f)$ is an object in the cartesian product above, we interpret the value

$S_{(m_i)_{0\leq i\leq n}}(f_n,\dots,f_1,f)$ as the space of n-ary operations from $(f_i)_{1\leq i\leq n}$ to f.

- Further data encoding units, the composition of n-ary operations, etc.

Our interest in generalized ∞-operads stems from the special case that the profunctors $S_{(m_i)}$ are representable (as well as the profunctors encoding units), say by some functors

$$\circ_{i=1}^n\colon \mathcal{M}(m_{n-1},m_n)\times\cdots\times\mathcal{M}(m_0,m_1)\to\mathcal{M}(m_0,m_n);\qquad (f_n,\dots,f_1)\mapsto f_n\circ\cdots\circ f_1.$$

In this special case, $\mathcal{M}$ is a model for an $(\infty,2)$-category with set of objects $\mathcal{M}_{[0]}$ and hom-categories $\mathcal{M}(m,m')$. Relaxing the assumption that the above profunctors be representable allow generalized ∞-operads to model objects which behaves like $(\infty,2)$-categories, but where composition of morphisms is allowed to be undefined, or multiply-defined, or something more complicated.

Note 2.2.0.1. We are breaking with the naming conventions of [GH15] and [HA]:

- In [HA], Lurie uses the name *(generalized) ∞-operad* to refer to a construction in which the spaces of n-ary operations are acted on by the symmetric group S_n, permuting their inputs. In the literature, such beasts are often called *symmetric (∞-)operads.* This S_n action adds generality, but also complexity which we will not make use of. For this reason, we will consider a related construction without this action, as described by Haugseng [Hau17] and [Hau16], Gepner–Haugseng in [GH15], and elsewhere.

- In order to be consistent with Lurie, Haugseng and Gepner–Haugseng introduce in [Hau17], [GH15], and [Hau16] the terminology *(generalized) non-symmetric ∞-operad* for their version of the construction without the S_n action. Since we will not use make any use of the symmetric construction, we will consider the non-symmetric version to be the default in order to avoid having to write 'non-symmetric.'

We will see that if $\mathbb{A}$ and $\mathbb{B}$ are generalized ∞-operads which satisfy the above representability conditions and therefore model $(\infty,2)$-categories flagged by sets, a morphism of generalized operads from $\mathbb{A}$ to $\mathbb{B}$ models a lax functor, and a 2-morphism between such lax functors models an *icon.* Thus, $\mathbb{G}\mathrm{enOp}_\infty$ admits a full sub-$(\infty,2)$-category $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ whose objects are set-flagged $(\infty,2)$-categories, whose 1-morphisms are lax functors, and whose 2-morphisms are icons.

Our main new result is a partial description of left Kan extension internal to $\mathbb{G}\mathrm{enOp}_\infty$. Our main tool will be the following:

Assumption. Let $f\colon\mathcal{M}\to\mathcal{N}$ be a morphism of generalized ∞-operads which is extendable (Definition 2.2.4.1), and let $\mathbb{D}$ be an $(\infty,2)$-category which is locally cocomplete (Definition 2.2.3.7). Then the functor

$$f^*\colon \mathbb{G}\mathrm{enOp}_\infty(\mathcal{N},\mathbb{D})\to\mathbb{G}\mathrm{enOp}_\infty(\mathcal{M},\mathbb{D})$$

admits a left adjoint, given by *generalized operadic left Kan extension.*

The proof of this result rests on the assumption that Lurie's construction [HA, Thm. 3.1.2.3] of operadic Kan extensions continues to work in the generalized case. Unfortunately, we have not yet been able to verify this, so this result must be classified as a conjecture. In the special case that all the operads in question are actually Segal ∞-bicategories, we show the following simplification (Proposition 2.2.4.4).

Theorem. Let $\mathbb{D}$ be a locally cocomplete $(\infty,2)$-category and let $f\colon \mathbb{A}\to\mathbb{B}$ be a lax functor of set-flagged $(\infty,2)$-categories which is extendable as a functor of generalized ∞-operads. Let $G\colon \mathbb{B}\to\mathbb{D}$ be a lax functor, and $\alpha\colon F\Rightarrow G\circ f$ an icon.

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathbb{D} \\ & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd}$$

Then α exhibits G as a generalized operadic Kan extension of F along f if and only if for all objects a and a' of $\mathbb{A}$, the natural transformation $\alpha_{a,a'}$ exhibits $G_{fa,fa'}$ as the left Kan extension of $F_{a,a'}$ along $f_{a,a'}$.

$$\begin{tikzcd} \mathbb{A}(a,a') \arrow[rr, "F_{a,a'}"] \arrow[dr, "f_{a,a'}"'] & \arrow[d, phantom, "\Downarrow\alpha_{a,a'}"] & \mathbb{D}(Fa,Fa') \\ & \mathbb{B}(fa,fa') \arrow[ur, "G_{fa,fa'}=(f_{a,a'})_!F_{a,a'}"'] & \end{tikzcd}$$

That is, the notion of left Kan extension along extendable morphisms of operads internal to $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ is simply local left Kan extension.

Our plan for the section is as follows. Note that much of the material of this chapter is either already contained in the literature (especially [Hau17], [GH15], [Hau16], and [HA]), or a mild generalization of material found there.

- In Subsection 2.2.1, we define our basic objects of study — generalized ∞-operads — together with several special cases modelling $(\infty,2)$-categories and their cousins. Most notable is our preferred model for set-flagged $(\infty,2)$-categories, Segal ∞-bicategories. We also define $\mathbb{G}\mathrm{enOp}_\infty$, the $(\infty,2)$-category of generalized ∞-operads, and its full subcategory $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ of Segal ∞-bicategories, lax functors, and icons.

- In Section 2.2.2, we define the double-categorical envelope $\mathrm{Env}(\mathcal{M})$ of a generalized ∞-operad $\mathcal{M}$, and recall its universal property.

- In Subsection 2.2.3, we define generalized operadic colimits, which are a way to take colimits of diagrams spread out across the total space of a generalized ∞-operad, and show that in the case of generalized operadic colimits taken within the total space of a Segal ∞-bicategory, they can be reduced to ordinary colimits computed in a hom-category.

- In Subsection 2.2.4, we define generalized operadic Kan extensions via a colimit formula involving generalized operadic colimits. Here we must assume an analog of Lurie's proof of the universal property for operadic Kan extensions also holds in the generalized case, as well as a result of Arakawa which allows easy computation of generalized operadic Kan extensions via the colimit formula. We show that in the case that one takes generalized operadic Kan extensions of lax functors of ∞-bicategories, generalized operadic Kan extension models *local* left Kan extension, i.e. left Kan extensions taken in hom-categories.

2.2.1 Complete Segal categories, lax functors, and icons

In this section, we recall the definition of $\mathbb{G}\mathrm{enOp}_\infty$, the $(\infty, 2)$-category of generalized ∞-operads, together with some if its subcategories. None of this material is original, and the majority of it can be found in [GH15].

Definition 2.2.1.1. Call a morphism $\phi\colon [m] \to [n]$ in Δ^{op} ***inert*** if the corresponding morphism $\tilde{\phi}\colon [n] \to [m]$ in Δ has the property that $\tilde{\phi}(i+1) = \tilde{\phi}(i) + 1$ for all $0 \leq i < n$.

Definition 2.2.1.2 ([Hau17, Def. 3.1.3])**.** A ***generalized*** ∞***-operad*** is an inner fibration $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ such that

(1) Each inert map in Δ^{op} admit all q-cocartesian lifts.

(2) For each $n \geq 1$, the map

$$\mathcal{M}_{[n]} \to \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \cdots \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]}$$

induced by the inert maps $[n] \to [1]$ and $[n] \to [0]$ is an equivalence.

(3) Given $C \in \mathcal{M}_n$ and q-cocartesian lifts $\alpha_i\colon C \to C_i$ over the inert maps $[n] \to [1]$ and $[n] \to [0]$, the α_i exhibit C as the q-limit of the C_i.

We will call a morphism in $\mathcal{M}$ *inert* if it is a cocartesian lift of an inert morphism in Δ^{op}. A morphism from a generalized operad $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ to a generalized operad $q\colon \mathcal{N} \to \Delta^{\mathrm{op}}$ consists of a commuting triangle

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{N} \arrow[dl, "q"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

in $\mathcal{S}\mathrm{et}_\Delta$ such that f preserves inert morphisms.

Using Lurie's theory of categorical patterns ([HA, Appendix B]), Gepner and Haugseng display in [GH15, Eg. 3.2.4] a model structure describing generalized ∞-operads with the following properties.

Theorem 2.2.1.3. Consider Δ^{op} to be a marked simplicial set where all inert morphisms have been marked. There is a model structure on $(\mathcal{S}\mathrm{et}_\Delta^+)_{/\Delta^{\mathrm{op}}}$ with the following properties.

(1) An object $p\colon \overline{X} \to \Delta^{\mathrm{op}}$ is fibrant if and only if it is a generalized ∞-operad whose inert morphisms are marked.

(2) A morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ (over Δ^{op}) is a cofibration if and only if the underlying map of simplicial sets $f\colon X \to Y$ is a monomorphism.

(3) A morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ (over Δ^{op}) between fibrant objects is a fibration if and only if the underlying map of simplicial sets $f\colon X \to Y$ is a categorical fibration.

Proof. The first two are [GH15], the last follows from [HA, Prop. B.2.7]. □

Definition 2.2.1.4. We will make use of the following special cases of generalized ∞-operads.

- A ***double* ∞-*category*** is a generalized ∞-operad $\mathfrak{C} \to \Delta^{\mathrm{op}}$ which is also a cocartesian fibration. That is, a double ∞-category is the unstraightening of a simplicial object in $\mathcal{C}\mathrm{at}_\infty$ satisfying the Segal condition.

- A ***Segal* ∞-*bicategory*** is a double ∞-category $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ such that $\mathbb{D}_{[0]}$ is a set.

- A ***Segal* ∞-*category*** is a Segal ∞-bicategory $p\colon \mathcal{A} \to \Delta^{\mathrm{op}}$ such that q is a left fibration.

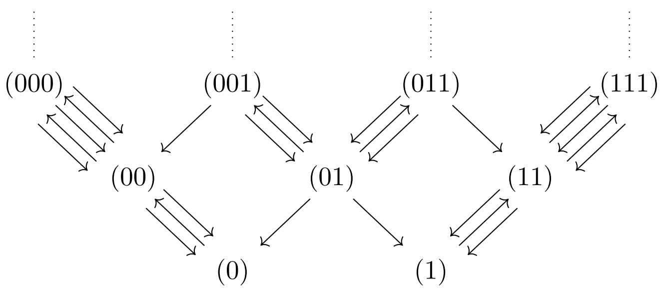
Note 2.2.1.5. Let $p\colon X \to \Delta^{\mathrm{op}}$ be any of the above. We will interpret the functor p as encoding some form of generalized operad, rather than simply a ∞-category living over Δ^{op}. Thus, when when we speak of an *object* of p, we mean an object of the generalized operad modelled by p, i.e. an object of $X_{[0]}$, the fiber of p over $[0] \in \Delta^{\mathrm{op}}$. When we say an object of X, we mean an object of the ∞-category X.

Note 2.2.1.6. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory. Then q models an $(\infty, 2)$-category with set of objects $\mathbb{D}_{[0]}$.

Notation 2.2.1.7. Following Haugseng, we will denote the category $(\Delta_{/[n]})^{\mathrm{op}}$ by $\Delta^{\mathrm{op}}_{/[n]}$.

Example 2.2.1.8. For each $n \geq 0$, the forgetful functor $u_n\colon \Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$ is a Segal ∞-category model of the n-simplex: the fiber of u_n over $[k]$, which one should interpret as the space of k-simplices, is the discrete space $\Delta^n([k]) = \Delta([k], [n])$.

Example 2.2.1.9. Denoting an object $\phi\colon [k] \to [1]$ of $\Delta^{\mathrm{op}}_{/[1]}$ by a string $(0 \ldots 01 \ldots 1)$ of $(k+1)$ 0s and 1s, we can sketch the category $\Delta^{\mathrm{op}}_{/[1]}$ as follows.



Here we have not drawn all morphisms between the pictured objects, only those which are sent to to face and degeneracy maps under the forgetful functor $u_1\colon \Delta^{\mathrm{op}}_{/[1]} \to \Delta^{\mathrm{op}}$. The ith row corresponds to the fiber of u_1 over $[i]$. In particular, we recognize two full subcategory inclusions $\Delta^{\mathrm{op}} \subseteq \Delta^{\mathrm{op}}_{/[1]}$ coming from to objects $\phi\colon [k] \to [1]$ which are constant on 0 and 1 respectively. In our model-independent language, these correspond to the inclusions $\Delta^{\{i\}} \hookrightarrow \Delta^1$, $i = 0, 1$.

Example 2.2.1.10. For $n \geq 2$, denote by $\Lambda^{\mathrm{op}}_{/[n]} \subseteq \Delta^{\mathrm{op}}_{/[n]}$ the full subcategory on those objects $\phi\colon [k] \to [n]$ whose image is a subinterval of $[n]$; or equivalently, such that $\phi(i+1) \leq \phi(i)+1$ for all $0 \leq i \leq n-1$. We should think of i_n as a generalized-operadic incarnation of the spine

$$\mathrm{Spine}(\Delta^n) := \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n.$$

The composition $i_n\colon \Lambda^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$ is *not* a Segal ∞-category, but it is a generalized ∞-operad; this reflects the fact that the spine of an n-simplex is not a category.

Definition 2.2.1.11. We define $\mathbb{G}\mathrm{enOp}_\infty \subset \mathbb{C}\mathrm{at}_{\infty/\Delta^{\mathrm{op}}}$ to be the sub-$(\infty,2)$-category whose objects are generalized ∞-operads $\mathcal{M} \to \Delta^{\mathrm{op}}$, and whose morphisms are morphisms of generalized operads.

The 2-simplices of $\mathbb{G}\mathrm{enOp}_\infty$ are then tetrahedra

$$\begin{tikzcd} & \mathcal{N} \arrow[dr, "G"] \arrow[d] & \\ \mathcal{M} \arrow[ur, "F"] \arrow[rr, "H"] \arrow[dr, "p"'] & \arrow[u, phantom, "\eta\Uparrow"] & \mathcal{P} \arrow[dl] \\ & \Delta^{\mathrm{op}} & \end{tikzcd},$$

in Cat_∞ whose 'side' faces are morphisms of generalized ∞-operads, and whose 'top' face is filled with a natural transformation $\eta\colon H \Rightarrow G \circ F$ which commutes with the projections to Δ^{op}.

Definition 2.2.1.12. Let $a\colon \mathfrak{A} \to \Delta^{\mathrm{op}}$ and $b\colon \mathfrak{B} \to \Delta^{\mathrm{op}}$ be double ∞-categories. A ***lax functor*** F from a to b is a morphism

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "F"] \arrow[dr, "a"'] & & \mathfrak{B} \arrow[dl, "b"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

of generalized ∞-operads. A lax functor F above is ***strict*** if it takes all a-cocartesian morphisms to b-cocartesian morphisms.

Definition 2.2.1.13. We define $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)} \subseteq \mathbb{G}\mathrm{enOp}_\infty$ to be the full $(\infty,2)$-category on Segal ∞-bicategories, $\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)} \subseteq \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ to be the underlying ∞-category, and $\mathcal{S}\mathrm{eg}_{(\infty,2)} \subseteq \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)}$ to be the subcategory on strict (non-lax) functors.

Note 2.2.1.14. A 2-simplex in $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ is then given by a 2-simplex

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "H"] \arrow[dr, "p"'] & \arrow[u, phantom, "\eta\Uparrow"] & \mathbb{C} \arrow[dl] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

in $\mathbb{G}\mathrm{enOp}_\infty$ whose vertices are Segal ∞-bicategories, and whose edges are lax functors. We interpret the 2-simplex above as an *icon* from H to $G \circ F$. This is ∞-bicategorical version of a classical 2-categorical concept explained, for example, in [JY21, Def. 4.6.2], as we now explain.

The condition that η be compatible with the projections to Δ^{op} implies that if D is an object in the fiber of p over $[n]$, then η_D is a morphism in the fiber of q over n. In low degrees:

(0) For each object D in the fiber of p over $[0]$ (i.e. each object of the Segal ∞-bicategory p), the natural transformation η has a component $\eta_D\colon H(D) \to (G \circ F)(D)$ which by the above must lie in the fiber of p over $[0]$. By our assumption that p is a Segal

∞-bicategory, the fiber of p over 0 is a set, so η_D identitfies $H(D)$ and $(G \circ F)(D)$ each D.

(1) Consider the restriction of the natural transformation η to $\mathbb{D}_{[1]}$.

$$\begin{tikzcd} \mathbb{D}_{[1]} \arrow[r, bend left, "H"] \arrow[r, bend right, "G\circ F"'] \arrow[r, phantom, "\Downarrow \eta|\mathbb{D}_{[1]}"] & \mathbb{E}_{[1]} \end{tikzcd}$$

By the decomposition given in Equation 2.2.1.1, this is the data of, for each d, $d' \in \mathbb{D}_{[0]}$, a natural transformation

$$\begin{tikzcd} & \mathbb{E}(Hd, Hd') \arrow[d, equal] \\ \mathbb{D}(d, d') \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow \eta_{dd'}"] & {} \\ & \mathbb{E}(GFd, GFd') \end{tikzcd} ;$$

here the right-hand morphism is an equality because H and $G \circ F$ agree on objects.

Definition 2.2.1.15. Let $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized operad such that $\mathcal{M}_{[0]}$ is a set, and let x and y be objects of p (i.e. objects in the fiber of p over $[0]$). We denote by $\mathcal{M}(x, y)$ the ***mapping category*** of p from x to y, defined as the full subcategory of $\mathcal{M}_{[1]}$ on those objects f admitting p-cocartesian lifts $f \to x$ and $f \to y$ of the 'source' and 'target' maps $[1] \to [0]$ in Δ^{op} respectively.

Note 2.2.1.16. We will refer to a generalized ∞-operad $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ such that $\mathcal{M}_{[0]}$ is a set a *generalized ∞-category with a set of objects.*

Note 2.2.1.17. Let $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad with a set of objects. Then the source and target maps $\mathcal{M}_{[1]} \to \mathcal{M}_{[0]}$ combine to a map $\mathcal{M}_{[1]} \to \mathcal{M}_{[0]} \times \mathcal{M}_{[0]}$. This gives us a decomposition

$$\mathbb{D}_{[1]} = \coprod_{x,y \in \mathbb{D}_{[0]}} \mathbb{D}(x, y). \tag{2.2.1.1}$$

2.2.2 Double-categorical envelopes

This section is a relatively mild generalization/rephrasing of [HA, Sec. 2.2.4]. Nothing proved here is original; the results here are also contained in [GH15].

Call a morphism $\phi\colon [m] \to [n]$ in Δ^{op} ***active*** if the corresponding morphism $\tilde{\phi}\colon [n] \to [m]$ in Δ has the property that $\tilde{\phi}(0) = 0$ and $\tilde{\phi}(n) = m$. Denote the subcategory of Δ^{op} on active morphisms by $\Delta^{\mathrm{op}}_{\mathrm{act}}$.

Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad. Denote by $\mathcal{M}_{\mathrm{act}} \to \Delta^{\mathrm{op}}_{\mathrm{act}}$ the pullback of q along the inclusion $\Delta^{\mathrm{op}}_{\mathrm{act}} \subset \Delta^{\mathrm{op}}$.

Note 2.2.2.1. Since we will frequently be dealing with objects encrusted with sub- and/or superscripts, it may sometimes be practical to write $\mathcal{M}^{\mathrm{act}}$ instead of $\mathcal{M}_{\mathrm{act}}$. We will use these two notations interchangeably.

Definition 2.2.2.2. Let $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad with a set of objects. For objects x and y of q, denote by $\mathcal{M}(x; y) \subseteq \mathcal{M}_{\mathrm{act}}$ the full subcategory on those objects

$M \in \mathcal{M}_{\mathrm{act}}$ which admit a cocartesian lift $s_!\colon M \to x$ of the 'source' map $s\colon q(M) \to [0]$, and a cocartesian lift $t_!\colon M \to y$ of the 'target' map $t\colon q(M) \to [0]$. We will refer to $\mathcal{M}(x;y)$ as the ***slack hom-category*** from x to y.

Note that

$$\mathcal{M}(x,y) = \mathcal{M}(x;y) \cap \mathcal{M}_{[1]},$$

and the inclusion $\mathcal{M}(x,y) \subseteq \mathcal{M}(x;y)$ is fully faithful.

Definition 2.2.2.3. Denote by $i\colon \mathrm{Act}(\Delta^{\mathrm{op}}) \subseteq \mathrm{Fun}(\Delta^1, \Delta^{\mathrm{op}})$ the full subcategory on active morphisms.

Definition 2.2.2.4. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad, and denote

$$\mathrm{Env}(\mathcal{M}) := \mathcal{M} \times_{\mathrm{Fun}(\{0\},\Delta^{\mathrm{op}})} \mathrm{Act}(\Delta^{\mathrm{op}}).$$

The ***double-categorical envelope*** of q is the functor $\mathrm{Env}(q)\colon \mathrm{Env}(\mathcal{M}) \overset{\mathrm{ev}_1}{\to} \Delta^{\mathrm{op}}$.

Note 2.2.2.5. By Note 2.3.1.4, the construction $\mathcal{M} \mapsto \mathrm{Env}(\mathcal{M})$ is given by a $\mathbb{G}\mathrm{enOp}_\infty$-pullback, and hence is functorial.

Note 2.2.2.6. An object of the total space $\mathrm{Env}(\mathcal{M})$ of $\mathrm{Env}(q)$ in the fiber over $[n]$ in Δ^{op} is a pair (M,ϕ), where M is an object in $\mathcal{M}$, and ϕ is an active morphism $q(M) \to [n]$ in Δ^{op}. A morphism from (M,ϕ) to (N,ψ) consists of a morphism $f\colon M \to N$ in $\mathcal{M}$, together with a commutative square

$$\begin{tikzcd} q(M) \arrow[r, "q(f)"] \arrow[d, "\phi"'] & q(N) \arrow[d, "\psi"] \\ {[k]} \arrow[r, "\alpha"] & {[l]} \end{tikzcd}$$

in Δ^{op} whose vertical morphisms are active. By [HA, Lemma 2.2.4.15], such a morphism is $\mathrm{Env}(q)$-cocartesian if and only if f is inert (i.e. if $q(f)$ is inert and f is q-cocartesian).

Proposition 2.2.2.7 ([GH15, Prop. A.1.2])**.** Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad. Then $\mathrm{Env}(q)$ is a double ∞-category with category of objects $\mathrm{Env}(\mathcal{M})_{[0]} = \mathcal{M}_{[0]}$.

Note 2.2.2.8. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad with a set of objects. Then we have

$$\mathcal{M}_{\mathrm{act}} = \coprod_{x,y \in \mathcal{M}_{[0]}} \mathcal{M}(x;y). \tag{2.2.2.1}$$

Thus, if K be a weakly connected simplicial set, then any morphism of simplicial sets $f\colon K \to \mathcal{M}_{\mathrm{act}}$ factors through $\mathcal{M}(x;y)$ for some objects x and y of q. In particular, for *any* simplicial set K, any morphism of simplicial sets $K^{\triangleright} \to \mathcal{M}_{\mathrm{act}}$ factors through $\mathcal{M}(x;y)$ for some objects x and y of q

Definition 2.2.2.9. The double category structure on $\mathrm{Env}(\mathcal{M})$ gives us a ***formal composition*** functor

$$- \odot -\colon \mathcal{M}_{\mathrm{act}} \times_{\mathcal{M}_{[0]}} \mathcal{M}_{\mathrm{act}} \to \mathcal{M}_{\mathrm{act}},$$

and in particular, for each morphisms $\alpha\colon y \to z$ and $\beta\colon w \to x$ in q, pre- and postcomposition functors

$$\mathcal{M}(x;y) \overset{-\odot\beta}{\to} \mathcal{M}(w;y)$$

and

$$\mathcal{M}(x;y) \overset{\alpha\odot-}{\to} \mathcal{M}(x;z).$$

Lemma 2.2.2.10. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad. Then the q-cocartesian morphisms in $\mathcal{M}_{\mathrm{act}}$ are preserved by the functor $- \odot -$. Concretely, let a, $b\colon \Delta^1 \to \mathcal{M}_{\mathrm{act}}$ be 1-cells whose source and targets are such that $b \odot a$ is well defined. If a and b are q-cocartesian when viewed as 1-cells in $\mathcal{M}$, then so is $b \odot a$.

Proof. As [HA, Rem. 2.2.4.8]. □

Proposition 2.2.2.11 ([GH15, Prop. A.1.3])**.** Let $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad, and let $q\colon \mathfrak{C} \to \Delta^{\mathrm{op}}$ be a double ∞-category. Then $\mathrm{Env}(p)\colon \mathrm{Env}(\mathcal{M}) \to \Delta^{\mathrm{op}}$ is a double ∞-category, and the inclusion $\mathcal{M} \hookrightarrow \mathrm{Env}(\mathcal{M})$ sending $M \mapsto (M, \mathrm{id}_{p(M)})$ induces an equivalence

$$\mathrm{Fun}(\mathrm{Env}(\mathcal{M}), \mathfrak{C}) \simeq \mathbb{G}\mathrm{enOp}_\infty(\mathcal{M}, \mathfrak{C}),$$

where $\mathrm{Fun}(\mathrm{Env}(\mathcal{M}), \mathfrak{C})$ denotes the full subcategory of $\mathbb{G}\mathrm{enOp}_\infty(\mathrm{Env}(\mathcal{M}), \mathfrak{C})$ on strict functors (Definition 2.2.1.12).

2.2.3 Generalized operadic colimits

Much of this section is a relatively mild generalization/rephrasing of [HA, Sec. 3.1.1].

Definition 2.2.3.1. Suppose $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ is a generalized ∞-operad with a set of objects, containing objects x and y, and let $\bar{p}\colon K^{\triangleright} \to \mathcal{M}(x; y)$ be a diagram. Write

$$\mathcal{M}(x, y)_{\bar{p}/} := \mathcal{M}(x; y)_{\bar{p}/} \times_{\mathcal{M}(x;y)} \mathcal{M}(x, y).$$

- The diagram $\bar{p}$ is a ***weak generalized operadic colimit diagram*** if the map $\mathcal{M}(x, y)_{\bar{p}/} \to \mathcal{M}(x, y)_{p/}$ is a categorical equivalence, where $p = \bar{p}|K$.
- The diagram $\bar{p}$ is a ***generalized operadic colimit diagram*** if for all morphisms $\alpha\colon y \to z$ and $\beta\colon w \to x$ in q, the composite diagrams

  $$K^{\triangleright} \xrightarrow{\bar{p}} \mathcal{M}(x; y) \xrightarrow{-\odot\beta} \mathcal{M}(w; y)$$

  and

  $$K^{\triangleright} \xrightarrow{\bar{p}} \mathcal{M}(x; y) \xrightarrow{\alpha\odot-} \mathcal{M}(x; z)$$

  are weak generalized operadic colimit diagrams.

Note 2.2.3.2. The map $\mathcal{M}(x, y)_{\bar{p}/} \to \mathcal{M}(x, y)_{p/}$ is always a left fibration, hence is a categorical equivalence if and only if it is a trivial fibration.

Example 2.2.3.3. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad with a set of objects, and let $\bar{p}\colon K^{\triangleright} \to \mathcal{M}(x; y)$ be a diagram. Suppose that the image of $\bar{p}$ is contained in the fiber of q over $[1]$, so that $\bar{p}$ factors through $\mathcal{M}(x, y) \subseteq \mathcal{M}(x; y)$. Then $\bar{p}$ is a weak generalized operadic colimit diagram if and only if it is an ordinary colimit in the ∞-category $\mathcal{M}(x, y)$.

Proposition 2.2.3.4. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized operad with a set of objects, containing objects x and y, and let $\bar{h}\colon K^{\triangleright} \times \Delta^1 \to \mathcal{M}(x; y)$ be a natural transformation from $\bar{h}_0 = \bar{h}|K^{\triangleright} \times \{0\}$ to $\bar{h}_1 = \bar{h}|K^{\triangleright} \times \{1\}$. Suppose that for each vertex $x \in K^{\triangleright}$, the edge $\bar{h}|\{x\} \times \Delta^1$ is q-cocartesian. Then:

(1) The diagram $\bar{h}_0$ is a weak generalized operadic colimit diagram if and only if the diagram $\bar{h}_1$ is a weak generalized operadic colimit diagram.

(2) Suppose q is a cocartesian fibration. Then the diagram $\bar{h}_0$ is a generalized operadic colimit diagram if and only if the diagram $\bar{h}_1$ is a generalized operadic colimit diagram.

Proof. This is based strongly on [HA, Prop. 3.1.1.15], but with enough differences that we give the full proof.

By Lemma 2.2.2.10, (1) implies (2), so we show (1). Consider the diagram

$$\begin{tikzcd} \mathcal{M}(x,y)_{\bar{h}_0/} \arrow[d, "v"'] & \mathcal{M}(x,y)_{\bar{h}/} \arrow[l, "(c)"'] \arrow[r, "(a)"] \arrow[d] & \mathcal{M}(x,y)_{\bar{h}_1/} \arrow[d, "u"] \\ \mathcal{M}(x,y)_{h_0/} & \mathcal{M}(x,y)_{h/} \arrow[l, "(d)"'] \arrow[r, "(b)"] & \mathcal{M}(x,y)_{h_1/} \end{tikzcd}.$$

We need to show that u is a categorical equivalence if and only if v is a categorical equivalence. Note:

(a) is a categorical equivalence because it is a pullback of the functor $\mathcal{M}(x;y)_{\bar{h}/} \to \mathcal{M}(x;y)_{h/}$, which is a trivial fibration by [HTT, Prop. 2.1.2.5] since the inclusion $K^{\triangleright} \times \{1\} \subseteq K^{\triangleright} \times \Delta^1$ is right anodyne, along the map $\mathcal{M}(x,y)_{/h} \to \mathcal{M}(x;y)_{/h}$.

(b) is a categorical equivalence by reasoning analogous to (a), because the inclusion $K \times \{1\} \subseteq K \times \Delta^1$ is right anodyne.

(c) is a categorical equivalence because it fits into the diagram

$$\begin{tikzcd} \mathcal{M}(x,y)_{\bar{h}/} \arrow[rr] \arrow[d, "(c)"'] & & \mathcal{M}(x;y)_{\bar{h}/} \arrow[d, "s"] \\ \mathcal{M}(x,y)_{\bar{h}_0/} \arrow[r] \arrow[d] & \mathcal{M}(x;y)_{\bar{h}_0/} \times_{(\Delta^{\mathrm{op}}_{\mathrm{act}})_{q\circ\bar{h}_0/}} (\Delta^{\mathrm{op}}_{\mathrm{act}})_{q\circ\bar{h}/} \arrow[r, phantom, ""] & {} \\ \{[1]\} \arrow[rr] & & \Delta^{\mathrm{op}}_{\mathrm{act}} \end{tikzcd},$$

where both squares are pullback, and s is a trivial fibration by [HTT, Prop. 3.1.1.12].

(d) is a categorical equivalence by reasoning analogous to (c).

Thus by 2/3, u is a categorical equivalence if and only if v is. □

Note 2.2.3.5. This is not direct translation of [HA, Prop. 3.1.1.15]; we demand only that the 1-cell $p|\{\infty\} \times \Delta^1$ is active and q-cocartesian, not that it is an equivalence. We can get away with this because:

- We are working relative to Δ^{op} rather than an arbitrary generalized operad, and
- The object $[1]$ is terminal in $\Delta^{\mathrm{op}}_{\mathrm{act}}$, so for any simplicial set K and any functor $p\colon K \to \Delta^{\mathrm{op}}_{\mathrm{act}}$, the fiber of the functor $(\Delta^{\mathrm{op}}_{\mathrm{act}})_{p/} \to \Delta^{\mathrm{op}}_{\mathrm{act}}$ over $[1]$ is contractible.

Definition 2.2.3.6. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ be a generalized ∞-operad with a set of objects. A diagram $p\colon K \to \mathcal{M}_{\mathrm{act}}$ is said to be ***admissible*** if K is small, and if p factors through some slack mapping category $\mathcal{M}(x;y)$ for some objects x and y of q.

Definition 2.2.3.7. A generalized ∞-operad $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ with a set of objects is ***locally cocomplete*** if it admits operadic colimits of all admissible diagrams.

The reason for this terminology is the following result, which we will prove at the end of this section.

Proposition 2.2.3.8. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory. The following are equivalent.

(1) The Segal ∞-bicategory q is locally cocomplete in the sense of Definition 2.2.3.7.

(2) Each hom-∞-category $\mathbb{D}(x, y)$ of q admits all small colimits, and the composition maps $- \circ -$ in q preserve small colimits in each slot.

Definition 2.2.3.9. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory. Covariant transport along q-cocartesian lifts of the active morphisms $[n] \to [1]$ yields a functor $T\colon \mathbb{D}_{\mathrm{act}} \to \mathbb{D}_{[1]}$, which we refer to as 'tightening'. This gives in particular for each pair of objects x, y in q a functor $T_{x,y}\colon \mathbb{D}(x; y) \to \mathbb{D}(x, y)$.

Tightening defines a monoidal functor $\mathrm{Env}(q) \to q$ for any Segal ∞-bicategory q. We will only need the following weaker version of this result.

Lemma 2.2.3.10. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory, and let x, y, and z be objects of q. Then the diagram

$$\begin{tikzcd}
\mathbb{D}_{\mathrm{act}} \times_{\mathbb{D}_{[0]}} \mathbb{D}_{\mathrm{act}} \arrow[r, "-\odot-"] \arrow[d, "T\times T"'] & \mathbb{D}_{\mathrm{act}} \arrow[d, "T"] \\
\mathbb{D}_{[1]} \times_{\mathbb{D}_{[0]}} \mathbb{D}_{[1]} \arrow[r, "-\circ-"'] & \mathbb{D}_{[1]}
\end{tikzcd}$$

commutes in $\mathrm{h}\mathcal{C}\mathrm{at}_\infty$.

Proof. Denote by $\gamma\colon [2] \to [1]$ the active map in Δ^{op}. We can identify the top horizontal morphism in the above square with the formal composition map $\mathrm{Env}(\mathbb{D})_{[2]} \xrightarrow{\gamma_!} \mathrm{Env}(\mathbb{D})_{[1]}$, and the bottom horizontal morphism with the actual composition map $\mathbb{D}_{[2]} \xrightarrow{\gamma_!} \mathbb{D}_{[1]}$. Using the definition of $\mathrm{Env}(\mathbb{D})$, the above square can then be identified with the square

$$\begin{tikzcd}
\mathbb{D} \times_{\mathrm{Fun}(\{0\},\mathbb{D})} \mathrm{Act}(\Delta^{\mathrm{op}})_{[2]} \arrow[r, "\gamma\circ-"] \arrow[d] & \mathbb{D} \times_{\mathrm{Fun}(\{0\},\mathbb{D})} \mathrm{Act}(\Delta^{\mathrm{op}})_{[1]} \arrow[d] \\
\mathbb{D}_{[2]} \arrow[r, "\gamma_!"] & \mathbb{D}_{[1]}
\end{tikzcd},$$

where the vertical maps send an object (D, ϕ) to $\phi_! D$ (and the values on higher-dimensional simplices are essentially uniquely determined by the universal property for q-cocartesian morphisms).

Composing the top horizontal map with the right-hand vertical map then gives the functor $(D, \phi) \mapsto (\gamma\circ\phi)_! D$, while composing the left-hand vertical map with the lower horizontal map gives $(D, \phi) \mapsto \gamma_!(\phi_! D)$. Both of these functors are therefore given by covariant transport along the map $\gamma \circ \phi$, and can be identified in $\mathrm{h}\mathcal{C}\mathrm{at}_\infty$. ☐

Proposition 2.2.3.11. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory containing objects x and y, and let $\bar{p}\colon K^{\triangleright} \to \mathbb{D}(x; y)$ be a diagram. The following are equivalent:

(1) The diagram $\bar{p}$ is a weak generalized operadic colimit diagram.

(2) The tightening of $\bar{p}$, i.e. the composition

$$K^{\triangleright} \to \mathbb{D}(x;y) \overset{T_{x,y}}{\to} \mathbb{D}(x,y),$$

is a colimit diagram.

Proof. Proposition 2.2.3.4 ensures that $\bar{p}$ is a weak generalized operadic colimit diagram if and only if its tightening is a weak generalized operadic colimit diagram. But since the tightening of $\bar{p}$ lives in the fiber of q over $[1]$, by Example 2.2.3.3 it is a weak generalized operadic colimit diagram if and only if it is an ordinary colimit diagram in $\mathbb{D}(x,y)$. □

Proposition 2.2.3.12. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory containing objects x and y, and let $\bar{p}\colon K^{\triangleright} \to \mathbb{D}(x;y)$ be a diagram. The following are equivalent:

(1) The diagram $\bar{p}$ is a generalized operadic colimit diagram.

(2) For each morphism $\alpha\colon y \to z$ and each morphism $\beta\colon w \to x$ of q, the compositions

$$K^{\triangleright} \to \mathbb{D}(x;y) \overset{T_{x,y}}{\to} \mathbb{D}(x,y) \overset{\alpha\circ-}{\to} \mathbb{D}(x,z)$$

and

$$K^{\triangleright} \to \mathbb{D}(x;y) \overset{T_{x,y}}{\to} \mathbb{D}(x,y) \overset{-\circ\beta}{\to} \mathbb{D}(w,y)$$

are colimit diagrams.

Proof. We need to show that $\bar{p}$ is a generalized operadic colimit diagram if and only if for any morphisms $\alpha\colon y \to z$ and $\beta\colon w \to x$, the 'upper' compositions in the diagrams

$$\begin{tikzcd}
 & & \mathbb{D}(x,y)\times\{\alpha\} \arrow[dr, "\alpha\circ-"] & \\
K^{\triangleright} \arrow[r, "\bar{p}"] & \mathbb{D}(x;y)\times\{\alpha\} \arrow[ur, "T_{x,y}"] \arrow[dr, "\alpha\odot-"'] & & \mathbb{D}(x,z) \\
 & & \mathbb{D}(x;z) \arrow[ur, "T_{x,z}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \{\beta\}\times\mathbb{D}(x,y) \arrow[dr, "-\circ\beta"] & \\
K^{\triangleright} \arrow[r, "\bar{p}"] & \{\beta\}\times\mathbb{D}(x;y) \arrow[ur, "T_{x,y}"] \arrow[dr, "-\odot\beta"'] & & \mathbb{D}(w,y) \\
 & & \mathbb{D}(w;y) \arrow[ur, "T_{w,y}"'] &
\end{tikzcd}$$

are colimit diagrams. The squares formed in each case are a restriction of the square of Lemma 2.2.3.10, and the 'upper' and 'lower' composition in each diagram are therefore equivalent as functors; thus, the 'upper' compositions are colimit diagrams if and only if the 'lower' compositions are colimit diagrams. But each 'lower' composition is a tightening of a cone in a slack hom-category, and by Proposition 2.2.3.11, a cone in a slack hom-category is weak generalized operadic colimit cones if and only if its tightening is a colimit cones in the corresponding ordinary hom-categories. □

Proof of Proposition 2.2.3.8. Suppose that (1) holds. Then each functor $p\colon K \to \mathbb{D}(x,y)$, where K is a small simplicial set and x and y are objects of q, is admissible, and therefore admits a generalized operadic colimit cone $\bar{p}\colon K^{\triangleright} \to \mathbb{D}(x;y)$ whose cone point we can assume also belongs to $\mathbb{D}(x,y)$, possibly after tightening. By Example 2.2.3.3, $\bar{p}$ is an ordinary colimit cone under p in $\mathbb{D}(x,y)$. Thus, each hom-category of q admits all colimits. By Proposition 2.2.3.12 generalized operadic colimit cones are preserved by pre- and postcomposition maps. Thus, (2) is satisfied.

Now suppose that (2) holds. Let $p\colon K \to \mathbb{D}_{\mathrm{act}}$ be an admissible diagram. Then p factors through some slack hom-category, say $p\colon K \to \mathbb{D}(x;y)$. Let $h\colon K \times \Delta^1 \to \mathbb{D}(x;y)$ be a natural transformation from $p = h|K \times \{0\}$ to a functor $h_1 := h|K \times \{1\}$, such that for each vertex k of K, $h|\{k\} \times \Delta^1$ is a q-cocartesian lift of the active morphism $[n] \to [1]$. Then h_1 is a tightening of p, and by assumption admits a colimit cone $\bar{h}_1\colon K^{\triangleright} \to \mathbb{D}(x,y)$.

Since the slack hom-category $\mathbb{D}(x;y)$ is an ∞-category, by [HTT, Prop. 2.1.2.3] the solid diagram

$$\begin{tikzcd} K \times \Delta^1 \coprod_{K\times\{1\}} K^{\triangleright} \arrow[r, "(h,\bar{h}_1)"] \arrow[d, hook] & \mathbb{D}(x;y) \\ (K\times\Delta^1)^{\triangleright} \arrow[ur, dashed, "\bar{h}'"'] & \end{tikzcd}$$

admits a dashed filler $\bar{h}'$. The composite

$$\bar{h}\colon K^{\triangleright} \times \Delta^1 \xrightarrow{s} (K \times \Delta^1)^{\triangleright} \xrightarrow{\bar{h}'} \mathbb{D}(x;y),$$

where s is the map fixing $K \times \Delta^1$ and collapsing $\{\infty\} \times \Delta^1$ to the cone point of $(K \times \Delta^1)^{\triangleright}$, satisfies the conditions of Proposition 2.2.3.4. By construction we have that $\bar{h}|K^{\triangleright} \times \{1\} = \bar{h}_1$, which is a generalized operadic colimit cone for p by assumption. Thus $\bar{h}_0 = \bar{h}|K^{\triangleright} \times \{0\}$ is a generalized operadic colimit cone for p, so (1) holds. □

2.2.4 Generalized operadic Kan extensions

In this section, we define generalized operadic Kan extension as a straightforward generalization of [HA, Def. 3.1.2.2]. Due to time constraints, we have not verified that some proofs given in the case of ordinary operadic Kan extension generalize; we therefore must include these as assumptions. The assumptions we need are Assumption 2.2.4.3, Assumption 2.2.4.5, and Assumption 2.2.4.6.

Definition 2.2.4.1. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a map of generalized ∞-operads (where the maps to Δ^{op} are suppressed). We say that σ' is ***extendable*** if

(1) Each of $\mathcal{M}$ and $\mathcal{N}$ has a set of objects, and the restriction $f|[0]\colon \mathcal{M}_{[0]} \to \mathcal{N}_{[0]}$ is a bijection.

(2) For each object $N \in \mathcal{N}$, the category $(\mathcal{M}_{\mathrm{act}})_{/N}$ is small.

Definition 2.2.4.2. Let $\sigma\colon \Delta^2 \to \mathbb{G}\mathrm{enOp}_\infty$ correspond to a diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\alpha"] & \mathbb{D} \\ & \mathcal{N} \arrow[ur, "G"'] & \end{tikzcd}.$$

such that f is extendable. We say that σ is a ***generalized operadic Kan extension*** if it has the following property.

(K) For each $n \geq 0$ and each object N of $\mathcal{N}_{[n]}$, the cone

$$((\mathcal{M}_{\mathrm{act}})_{/N})^{\triangleright} \to \mathbb{D}_{\mathrm{act}}$$

corresponding to the natural transformation coming from the pasting diagram

$$\begin{tikzcd}
(\mathcal{M}_{\mathrm{act}})_{/N} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{act}} \arrow[r, hook] \arrow[d] \arrow[dl, Rightarrow, shorten >=12pt, shorten <=12pt] & \mathcal{M} \arrow[r, "F"] \arrow[d, "f"'] & \mathbb{D} \arrow[dl, Rightarrow, shorten >=12pt, shorten <=12pt, "\alpha"'] \\
\{N\} \arrow[r] & \mathcal{N}_{\mathrm{act}} \arrow[r, hook] & \mathcal{N} \arrow[ur, "G"'] &
\end{tikzcd}$$

is a generalized operadic colimit diagram. Here, the left-hand square is partially-lax pullback.

In [Ara23, Prop. 4.2], it is shown that for ordinary operadic Kan extensions, it suffices to check that condition (K) is satisfied for $n = 1$. We must assume that this result is also applicable to the case of generalized operadic colimits.

Assumption 2.2.4.3 (cf. [Ara23, Prop. 4.2])**.** Let $\sigma\colon \Delta^2 \to \mathbb{G}\mathrm{enOp}_\infty$ be as above. Then property (K) is implied by the seemingly weaker condition

(K') For each object N of $\mathcal{N}_{[1]}$, the cone

$$((\mathcal{M}_{\mathrm{act}})_{/N})^{\triangleright} \to \mathbb{D}_{\mathrm{act}}$$

corresponding to the natural transformation coming from the pasting diagram

$$\begin{tikzcd}
(\mathcal{M}_{\mathrm{act}})_{/N} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{act}} \arrow[r, hook] \arrow[d] \arrow[dl, Rightarrow, shorten >=12pt, shorten <=12pt] & \mathcal{M} \arrow[r, "F"] \arrow[d, "f"'] & \mathbb{D} \arrow[dl, Rightarrow, shorten >=12pt, shorten <=12pt, "\alpha"'] \\
\{N\} \arrow[r] & \mathcal{N}_{\mathrm{act}} \arrow[r, hook] & \mathcal{N} \arrow[ur, "G"'] &
\end{tikzcd}$$

is a generalized operadic colimit diagram. Here, the left-hand square is partially-lax pullback.

Assuming the above for now, we can show that generalized operadic Kan extension of functors of Segal ∞-bicategories reduces to ordinary Kan extensions within the hom-categories.

Proposition 2.2.4.4. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}$ and $b\colon \mathbb{B} \to \Delta^{\mathrm{op}}$ be Segal ∞-bicategories, and let $f\colon \mathbb{A} \to \mathbb{B}$ be a lax functor of Segal ∞-bicategories which is also an extendable map of generalized ∞-operads. Let $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a locally cocomplete Segal ∞-bicategory, and let $F\colon \mathbb{A} \to \mathbb{D}$ be a functor. Then a triangle

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathbb{D} \\
& \mathbb{B} \arrow[ur, "G"'] &
\end{tikzcd}$$

in $\mathbb{G}\mathrm{enOp}_\infty$ is a generalized operadic Kan extension if and only if for all objects x and y of a, the natural transformation $\eta_{x,y}$ in the diagram

$$\begin{tikzcd} \mathbb{A}(x,y) \arrow[rr, "F_{x,y}"] \arrow[dr, "f_{x,y}"'] & {} \arrow[d, phantom, "\Downarrow \eta_{x,y}"] & \mathbb{D}(Fx,Fy) \\ & \mathbb{B}(fx,fy) \arrow[ur, "G_{Fx,Fy}"'] & \end{tikzcd}$$

exhibits $G_{Fx,Fy}$ as a left Kan extension of $F_{x,y}$ along $f_{x,y}$ in $\mathbb{C}\mathrm{at}_\infty$.

Proof. By assumption, the functor $b\colon \mathbb{B} \to \Delta^{\mathrm{op}}$ is a cocartesian fibration. This implies that $b^{\mathrm{act}}\colon \mathbb{B}^{\mathrm{act}} \to \Delta^{\mathrm{op}}_{\mathrm{act}}$ is also a cocartesian fibration since it is a pullback of b along $\Delta^{\mathrm{op}}_{\mathrm{act}} \hookrightarrow \Delta^{\mathrm{op}}$. By [HTT, Prop. 2.4.3.3], for any $\Xi \in \mathbb{B}_{[1]}$, the functor $b'\colon \mathbb{B}^{\mathrm{act}}_{/\Xi} \to (\Delta^{\mathrm{op}}_{\mathrm{act}})_{/[1]} \simeq \Delta^{\mathrm{op}}_{\mathrm{act}}$ is also a cocartesian fibration, and a morphism in $\mathbb{B}^{\mathrm{act}}_{/\Xi}$ is b'-cocartesian if and only if its image in $\mathbb{B}^{\mathrm{act}}$ is b^{act}-cocartesian. Thus, in the diagram

$$\begin{tikzcd} \mathbb{B}^{\mathrm{act}}_{/\Xi} \arrow[rr] \arrow[dr] & & \mathbb{B}^{\mathrm{act}} \arrow[dl] \\ & \Delta^{\mathrm{op}}_{\mathrm{act}} & \end{tikzcd}$$

both downward-facing maps are cocartesian fibrations and the horizontal map preserves cocartesian morphisms, hence is by the dual to [HTT, Prop. 2.4.1.3(3)] itself a cocartesian fibration. Thus the map $\mathbb{A}^{\mathrm{act}}_{/\Xi} \to \mathbb{A}^{\mathrm{act}}$ is a cocartesian fibration since it is by definition the pullback of b' along $\mathbb{A}^{\mathrm{act}} \to \mathbb{B}^{\mathrm{act}}$. Finally, the composition $\mathbb{A}^{\mathrm{act}}_{/\Xi} \to \mathbb{A}^{\mathrm{act}} \to \Delta^{\mathrm{op}}_{\mathrm{act}}$ is a cocartesian fibration since it is the composition of cocartesian fibrations.

Since $[1]$ is a terminal object of $\Delta^{\mathrm{op}}_{\mathrm{act}}$, the inclusion $\{[1]\} \hookrightarrow \Delta^{\mathrm{op}}_{\mathrm{act}}$ is cofinal. Thus, the pullback square

$$\begin{tikzcd} (\mathbb{A}_{[1]})_{/\Xi} \arrow[r, hook] \arrow[d] & \mathbb{A}^{\mathrm{act}}_{/\Xi} \arrow[d] \\ \{[1]\} \arrow[r, hook] & \Delta^{\mathrm{op}}_{\mathrm{act}} \end{tikzcd}$$

exhibits $(\mathbb{A}_{[1]})_{/\Xi} \hookrightarrow \mathbb{A}^{\mathrm{act}}_{/\Xi}$ as a pullback of a cofinal morphism along a cocartesian fibration, and therefore (combining [HTT, Prop. 4.1.2.15] and [HTT, Prop. 4.1.2.10]) as itself cofinal. Using the decomposition $\mathbb{A}_{[1]} = \coprod_{x',y' \in \mathbb{A}_{[0]}} \mathbb{A}(x',y')$, and the fact that f is extendable (and thus in particular a bijection on sets of objects) for any morphism $\Xi\colon fx \to fy$ in b, we have that $(\mathbb{A}_{[1]})_{/\Xi} \simeq \mathbb{A}(x,y)_{/\Xi}$.

Again using that $\mathbb{A}_{[1]}$ is a disjoint union of hom-categories we can formulate condition (K') of Conjeture 2.2.4.3 as follows: for all objects x and y of a and all morphisms $\Xi\colon fx \to fy$ of b, the cone coming from the pasting diagram

$$\begin{tikzcd} \mathbb{A}^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathbb{A}^{\mathrm{act}} \arrow[r, hook] \arrow[d] \arrow[dl, phantom, "\Swarrow"] & \mathbb{A} \arrow[r, "F"] \arrow[d, "f"'] & \mathbb{D} \\ \{\Xi\} \arrow[r] & \mathbb{B}^{\mathrm{act}} \arrow[r, hook] & \mathbb{B} \arrow[ur, "G"'] \arrow[u, phantom, shift left=3, "\Swarrow \eta"] & \end{tikzcd}$$

is a generalized operadic colimit. Using that the inclusion $\mathbb{A}(x,y)_{/\Xi} \simeq (\mathbb{A}_{[1]})_{/\Xi} \hookrightarrow \mathbb{A}^{\mathrm{act}}_{/\Xi}$ is cofinal, we can equivalently demand that the cone coming from the pasting diagram

$$\begin{tikzcd}
\mathbb{A}(x,y)_{/\Xi} \arrow[r, hook] & \mathbb{A}^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathbb{A}^{\mathrm{act}} \arrow[r, hook] \arrow[d] \arrow[dl, Rightarrow] & \mathbb{A} \arrow[r, "F"] \arrow[d, "f"'] & \mathbb{D} \\
& \{\Xi\} \arrow[r] & \mathbb{B}^{\mathrm{act}} \arrow[r, hook] & \mathbb{B} \arrow[ur, "G"'] \arrow[u, phantom, shift left=3, "\Swarrow\eta"] &
\end{tikzcd}$$

be a generalized operadic colimit diagram. However, the image of this cone is completely contained in $\mathbb{D}(Fx,Fy)$: it is the cone given by the pasting diagram

$$\begin{tikzcd}
\mathbb{A}(x,y)_{/\Xi} \arrow[r] \arrow[d] & \mathbb{A}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, "f_{x,y}"'] \arrow[dl, Rightarrow] & \mathbb{D}(Fx,Fy) \\
\{\Xi\} \arrow[r] & \mathbb{B}(fx,fy) \arrow[ur, "G_{fx,fy}"'] \arrow[u, phantom, shift left=3, "\Swarrow\eta_{x,y}"] &
\end{tikzcd} .$$

Thus, by Example 2.2.3.3 it is equivalent to demand that for all objects x and y of a and all morphisms $\Xi\colon fx \to fy$ of b, this cone be an ordinary colimit in the ∞-category $\mathbb{D}(Fx,Fy)$, which is what we needed to show. □

We also must assume that the following result, a version of [HA, Prop. 3.1.2.3], holds in the setting of generalized ∞-operads.

Assumption 2.2.4.5 (cf. [HA, Prop. 3.1.2.3])**.** Let $f\colon \mathcal{M} \to \mathcal{N}$ be an extendable morphism of generalized ∞-operads, and let $\mathbb{D}$ be a locally cocomplete Segal ∞-bicategory. Then the pullback functor

$$f^*\colon \mathbb{G}\mathrm{enOp}_\infty(\mathcal{N},\mathbb{D}) \to \mathbb{G}\mathrm{enOp}_\infty(\mathcal{M},\mathbb{D})$$

admits a left adjoint, given by generalized operadic Kan extension.

At one point, we will assume that generalized operadic Kan extension is transitive.

Assumption 2.2.4.6 (cf. [HA, Prop. 3.1.4.1])**.** Let $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{P}$ be extendable morphisms of generalized ∞-operads, let F, G, H be morphisms of generalized operads, and let α and β be icons as in the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rrr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\alpha"] & & \mathbb{D} \\
& \mathcal{N} \arrow[urr, "G"] \arrow[dr, "g"'] & \arrow[d, phantom, "\Downarrow\beta"] & \\
& & \mathcal{P} \arrow[uur, "H"'] &
\end{tikzcd} .$$

Suppose α exhibits G as a generalized operadic Kan extension of G along f. Then β exhibits H as a generalized operadic Kan extension of H along g if and only if $\beta f \circ \alpha$ exhibits H as a generalized operadic Kan extension of F along $g \circ f$.

2.3 Generalized ∞-operads over the n-simplex

In this section, we consider generalized ∞-operads relative to an n-simplex: that is, morphisms of generalized ∞-operads

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "q"'] & & \Delta^{\mathrm{op}}_{/[n]} \arrow[dl, "u_n"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd},$$

where q is a generalized ∞-operad and u_n is our Segal ∞-category model of the n-simplex.

This generalized ∞-operadic picture is (at least to the author) less enlightening than a model-independent picture. Therefore, we give here a brief relatively model-independent tour of the results of this section, in the vernacular notation described in Subsection 2.1.2.

We mainly care about the case in which q is a Segal ∞-bicategory, so for the purposes of this explanation we will assume that our generalized ∞-operad above models a set-flagged $(\infty, 2)$-category $\mathbb{Q}$. Under this assumption, the model-independent picture of the above diagram is simply

$$\begin{tikzcd} \mathbb{Q} \arrow[d, "f"'] \\ \Delta^n \end{tikzcd},$$

where $\mathbb{Q}$ is a set-flagged $(\infty, 2)$-category, Δ^n is the n-simplex, and f is a functor.

Observation 2.3.0.1. The data of such a functor can be tabulated as follows:

(1) The fiber of f over the ith vertex of Δ^n is a set-flagged $(\infty, 2)$-category $\mathbb{D}_i$.

(2) The hom-functors of $\mathbb{Q}$ lying over each non-identity morphism $\alpha\colon i \to j$ of Δ^n form a profunctor
$$\mathbb{Q}_i^{\mathrm{op}} \times \mathbb{Q}_j \to \mathbb{C}\mathrm{at}_\infty; \qquad (x, y) \mapsto \mathbb{Q}(x, y).$$

(3) Further data interpolating between the compositions of these profunctors, which we will not make explicit use of.

Our plan for the section is as follows. Note that much of this chapter is a mild generalization of results found in [Hau16].

- In Subsection 2.3.1, we consider base change along morphisms $\phi\colon \Delta^m \to \Delta^n$ via pullback:
$$\begin{tikzcd} \phi^*(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \Delta^m \arrow[r, "\phi"] & \Delta^n \end{tikzcd}$$
The pasting lemma for pullbacks allows us to describe $\phi^*(\mathbb{Q})$ in terms of $\mathbb{Q}$.

- In Subsection 2.3.2, we discuss base change along spine inclusions $\mathrm{Spine}(\Delta^n) \hookrightarrow \Delta^n$:

$$\begin{tikzcd} \mathrm{Spine}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \mathrm{Spine}(\Delta^n) \arrow[r] & \Delta^n \end{tikzcd}.$$

  Note that $\mathrm{Spine}(\Delta^n)$ is not an $(\infty,1)$-category since it consists of a chain of *non-composable* morphisms, so $\mathrm{Spine}(\mathbb{Q})$ is not an $(\infty,2)$-category because it also contains such chains. This is the point at which we need the machinery of generalized ∞-operads.

  Denoting by $\mathbb{Q}_{i,j}$ the base change of f along the map $\Delta^{\{i,j\}} \hookrightarrow \Delta^n$, we show that

$$\mathrm{Spine}(\mathbb{Q}) \simeq \mathbb{Q}_{0,1} \amalg_{\mathbb{Q}_1} \cdots \amalg_{\mathbb{Q}_{n-1}} \mathbb{Q}_{n-1,n} \qquad \text{in } \mathbb{G}\mathrm{enOp}_\infty.$$

- In Subsection 2.3.3, we discuss generalized operadic Kan extension along maps of the form $\mathrm{Spine}(\mathbb{Q}) \hookrightarrow \mathbb{Q}$, and prove a colimit formula which we will frequent use of in later sections.

- Recall that $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ is our model for the $(\infty,2)$-category of set-flagged $(\infty,2)$-categories, lax functors, and icons. In Subsection 2.3.4, we show that the functor $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{/\Delta^n} \to (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})^{n+1}$ which remembers the fibers of f over the vertices of Δ^n, i.e. the functor

$$\begin{tikzcd} \mathbb{Q} \arrow[d, "f"'] \\ \Delta^n \end{tikzcd} \qquad \mapsto \qquad (\mathbb{Q}_0, \dots, \mathbb{Q}_n),$$

  admits a right adjoint

$$(\mathbb{Q}_0, \dots, \mathbb{Q}_n) \qquad \mapsto \qquad \begin{tikzcd} \mathbb{Q}_0 \star \cdots \star \mathbb{Q}_n \arrow[d] \\ \Delta^n \end{tikzcd},$$

  which we interpret as defining the $(n+1)$-fold join of the $\mathbb{Q}_i$. We show that the underlying adjunction on $(\infty,1)$-categories is a reflective localization.

- In Subsection 2.3.5, we investigate generalized operadic Kan extension along maps of the form

$$\mathrm{Spine}(\mathbb{Q}_0 \star \cdots \star \mathbb{Q}_n) \hookrightarrow \mathbb{Q}_0 \star \cdots \star \mathbb{Q}_n.$$

  Our main result is a formula for generalized operadic Kan extensions along such spine inclusions, given by a generalized version of the coend.

2.3.1 Generalized ∞-operads over the n-simplex

Definition 2.3.1.1. We will call a functor of generalized ∞-operads

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "q"'] & & \Delta^{\mathrm{op}}_{/[n]} \arrow[dl, "u_n"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

a ***generalized ∞-operad over the n-simplex***. We will suppress the maps to Δ^{op} if they are clear from context.

Lemma 2.3.1.2. let $f\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be generalized ∞-operad over the n-simplex as above. Then f is an inner fibration, and a morphism in $\mathcal{M}$ is f-cocartesian if and only if it is q-cocartesian.

Proof. That f is an inner fibration follows from the fact that u_n is a functor of ordinary 1-categories, and therefore admits *unique* lifts of inner horns. The rest follows from [HTT, Prop. 2.4.1.3(3)] and the fact that u_n is a left fibration. □

Construction 2.3.1.3. Denote by $\mathcal{C}\mathrm{at}_{/\Delta^{\mathrm{op}}}$ the 1-category of ordinary categories over Δ^{op}. There is a functor $\Delta \to \mathcal{C}\mathrm{at}_{/\Delta^{\mathrm{op}}}$ sending $[n]$ to the forgetful functor $u_n\colon \Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$, and a morphism $\phi\colon [m] \to [n]$ to the functor $\phi \circ -\colon \Delta^{\mathrm{op}}_{/[m]} \to \Delta^{\mathrm{op}}_{/[n]}$. Via the inclusion of ordinary categories into ∞-categories, this gives us a functor $\Delta \to \mathbb{G}\mathrm{enOp}_\infty$ sending $[n] \mapsto u_n$.

We can base change along morphisms $\phi\colon [k] \to [n]$ in Δ by forming $\mathbb{G}\mathrm{enOp}_\infty$-pullbacks

$$\begin{tikzcd} \phi^*(\mathcal{M}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \Delta^{\mathrm{op}}_{/[k]} \arrow[r, "\phi"] & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}, \tag{2.3.1.1}$$

giving functors

$$\phi^*\colon (\mathcal{G}\mathrm{enOp}_\infty)_{/\Delta^{\mathrm{op}}_{/[n]}} \to (\mathcal{G}\mathrm{enOp}_\infty)_{/\Delta^{\mathrm{op}}_{/[k]}}.$$

In fact, by [HTT, Lemma 6.1.1.1] this construction gives us a functor

$$\Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad [n] \mapsto (\mathcal{G}\mathrm{enOp}_\infty)_{/\Delta^{\mathrm{op}}_{/[n]}},$$

classified by the cartesian fibration

$$\mathrm{Fun}(\Delta^1, \mathcal{G}\mathrm{enOp}_\infty) \times_{\mathcal{G}\mathrm{enOp}_\infty} \Delta \to \Delta.$$

Note 2.3.1.4. By Lemma 2.3.1.2, any functor of generalized ∞-operads $f\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ is a categorical fibration, and thus by Theorem 2.2.1.3 a fibration with respect to the model structure describing generalized ∞-operads. Thus, taking the strict $\mathcal{S}\mathrm{et}_\Delta$-pullback in Equation 2.3.1.1 is a model for the homotopy pullback with respect to this model structure, hence also for the $\mathbb{G}\mathrm{enOp}_\infty$-pullback.

Note 2.3.1.5. It is not too difficult to check that base change preserves Segal ∞-bicategories, and thus restricts to a functor

$$\phi^*\colon (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \to (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[k]}}.$$

Notation 2.3.1.6. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be a generalized ∞-operad over the n-simplex.

- Denote by $i\colon [0] \to [n]$ the map $0 \mapsto i$. We denote the base change $i^*(\mathcal{M})$ by $q_i\colon \mathcal{M}_i \to \Delta^{\mathrm{op}}$, and refer to it as the ***fiber*** of q over i.
- Denote by $\psi_{ij}\colon [1] \to [n]$ the map sending $0 \mapsto i$ and $1 \mapsto j$. We denote the base change $\psi_{ij}^*(\mathcal{M})$ by $q_{ij}\colon \mathcal{M}_{ij} \to \Delta^{\mathrm{op}}_{/[1]}$.

If $p\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ is a generalized ∞-operad with set of objects, then we have a nice partial description for the generalized operad $\phi^*(\mathcal{M})$ which follows easily from the pasting lemma for pullbacks.

Lemma 2.3.1.7. Let $\mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be a generalized ∞-operad with a set of objects over the n-simplex, and let $i, j \in \{0, \dots, n\}$.

- The fiber of $\phi^*(\mathcal{M})$ over i agrees with the fiber of $\mathcal{M}$ over $\phi(i)$:
$$\phi^*(\mathcal{M})_i = \mathcal{M}_{\phi(i)}.$$
- The mapping categories of $\phi^*\mathcal{M}$ are given by the corresponding mapping categories of $\mathcal{M}$ in the following sense. Let $m \in \phi^*(\mathcal{M})_i$ and $m' \in \phi^*(\mathcal{M})_j$, and denote the corresponding object $m \in \mathcal{M}_{\phi(j)}$ by $\phi_*(m)$, and similarly $\phi_*(m')$. Then
$$\phi^*(\mathcal{M})(m, m') = \mathcal{M}(\phi_*(m), \phi_*(m')).$$

2.3.2 Spine inclusions

Recall Example 2.2.1.10: the forgetful functor $i_n\colon \Lambda^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$ is a generalized ∞-operad which we think of as modelling the spine of an n-simplex. The k-simplices of i_n are given by chains of k morphisms in the linearly ordered set $[n]$ which do not 'skip' any objects. In this section, we show that we can view a generalized ∞-operad $\mathcal{M}$ living over the spine of an n-simplex as an iterated pushout of the portions of $\mathcal{M}$ living over 1-simplices connecting adjacent objects of the linearly ordered set $[n]$.

Definition 2.3.2.1. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be a generalized ∞-operad over the n-simplex. Denote by $\mathrm{Spine}(q)\colon \mathrm{Spine}(\mathcal{M}) \to \Delta^{\mathrm{op}}$ the generalized ∞-operad given by composing the pullback of q along the inclusion $\Lambda^{\mathrm{op}}_{/[n]} \hookrightarrow \Delta^{\mathrm{op}}_{/[n]}$ with the forgetful functor to Δ^{op}.

$$\begin{tikzcd}
\mathrm{Spine}(\mathcal{M}) \arrow[r] \arrow[d] \arrow[dd, bend right, "\mathrm{Spine}(q)"'] & \mathcal{M} \arrow[d] \\
\Lambda^{\mathrm{op}}_{/[n]} \arrow[r, hook] \arrow[d] & \Delta^{\mathrm{op}}_{/[n]} \\
\Delta^{\mathrm{op}} &
\end{tikzcd}$$

Proposition 2.3.2.2. Let $q\colon \mathcal{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be a generalized ∞-operad over the n-simplex.

There is an equivalence of generalized ∞-operads

$$\begin{tikzcd} \mathcal{M}_{01} \amalg_{\mathcal{M}_1} \cdots \amalg_{\mathcal{M}_{n-1}} \mathcal{M}_{n-1,n} \arrow[rr, "\simeq"] \arrow[dr, "q_{01}\amalg_{q_1}\cdots\amalg_{q_{n-1}} q_{n-1,n}"'] & & \mathrm{Spine}(\mathcal{M}), \arrow[dl, "\mathrm{Spine}(q)"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

where the colimit of the left-hand object is taken in $\mathbb{G}\mathrm{enOp}_\infty$.

Proof. Mark the inert morphisms in each of the simplicial sets $\mathcal{M}_i$ and $\mathcal{M}_{ij}$, and denote by $\mathcal{M}^{\amalg}$ the $\mathcal{S}\mathrm{et}^+_\Delta$-colimit $\mathcal{M}_{01} \amalg_{\mathcal{M}_1} \cdots \amalg_{\mathcal{M}_{n-1}} \mathcal{M}_{n-1,n}$. By the logic of [Hau16, Lemma 6.18], $\mathcal{M}^{\amalg} \to \Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$ is a (non-fibrant) model for the $\mathbb{G}\mathrm{enOp}_\infty$-colimit we are interested in. By the logic of [Hau16, Cor. 6.19], the inclusion $\mathcal{M}^{\amalg} \subseteq \mathrm{Spine}(\mathcal{M})$ is a trivial cofibration in the model structure for generalized operads, exhibiting $\mathrm{Spine}(q)\colon \mathrm{Spine}(\mathcal{M}) \to \Delta^{\mathrm{op}}$ also as a model for the homotopy colimit with respect to the model structure for generalized operads, i.e. the $\mathbb{G}\mathrm{enOp}_\infty$-colimit. □

2.3.3 Extending from the spine

For this section, fix a locally cocomplete Segal ∞-bicategory $q\colon \mathbb{D} \to \Delta^{\mathrm{op}}$.

Definition 2.3.3.1. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex, and let $F\colon \mathbb{A} \to \mathbb{D}$ be a lax functor of Segal ∞-bicategories. We say that F is ***extended from its spine*** if it is a generalized operadic Kan extension of its restriction to $\mathrm{Spine}(\mathbb{A})$, i.e. if the (commuting) triangle

$$\begin{tikzcd} \mathrm{Spine}(\mathbb{A}) \arrow[rr, "F|\,\mathrm{Spine}(\mathbb{A})"] \arrow[dr, hook] & & \mathbb{D} \\ & \mathbb{A} \arrow[ur, "F"'] & \end{tikzcd}$$

is left Kan.

Our main result in this section, Theorem 2.3.3.9, is a colimit formula for spine extensions. In order to express this formula, we will need some notation.

Notation 2.3.3.2. Let $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex, and let $\psi \in \Delta^{\mathrm{op}}_{/[n]}$ be an object.

- Denote $\mathbb{A}_\psi := \mathbb{A} \times_{\Delta^{\mathrm{op}}_{/[n]}} \{\psi\}$.
- We will sometimes specify a morphism $\psi\colon [k] \to [n]$ of Δ by its values $(\psi(0), \dots, \psi(k))$. In this case we will write $\mathbb{A}_{(\psi(0),\dots,\psi(k))} := \mathbb{A}_\psi$.
- For any object $\psi\colon [k] \to [n]$ of $\Delta^{\mathrm{op}}_{/[n]}$, there is a unique active map from ψ to the object $(\psi(0), \psi(k))$ of $\Delta^{\mathrm{op}}_{/[n]}$ given by $(0,k)\colon [1] \to [k]$.

$$\begin{tikzcd} {[k]} \arrow[dr, "\psi"'] & & {[1]} \arrow[ll, "\exists!(0,k)"'] \arrow[dl, "(\phi(0),\phi(k))"] \\ & {[n]} & \end{tikzcd} .$$

Covariant transport along $(0,k)$ gives a functor $\mathbb{A}_\psi \to \mathbb{A}_{(\psi(0),\psi(k))}$. For any object $\Xi \in \mathbb{A}_{(\psi(0),\psi(k))}$, we will denote $(\mathbb{A}_\psi)_{/\Xi} := \mathbb{A}_\psi \times_{\mathbb{A}_{(\psi(0),\psi(k))}} (\mathbb{A}_{(\psi(0),\psi(k))})_{/\Xi}$.

Warning 2.3.3.3. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex. The symbols $\mathbb{A}_i$, $\mathbb{A}_{[i]}$ and $\mathbb{A}_{(i)}$ all have different meanings!

- The symbol $\mathbb{A}_i$ denotes the (total space of the) base change of a along the functor $\underline{i}\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}_{/[n]}$ sending $[k]$ to the constant morphism $[k] \to [n]$ with value i, as given by the pullback square
$$\begin{tikzcd} \mathbb{A}_i \arrow[r] \arrow[d, "a_i"'] & \mathbb{A} \arrow[d, "a"] \\ \Delta^{\mathrm{op}} \arrow[r, "\underline{i}"] & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd} .$$
- The symbol $\mathbb{A}_{[i]}$ denotes the category of i-simplices of the Segal ∞-bicategory $\mathbb{A}$, given by the diagram
$$\begin{tikzcd} \mathbb{A}_{[i]} \arrow[r] \arrow[d] & \mathbb{A} \arrow[d] \\ \Delta([i],[n]) \arrow[r] \arrow[d] & \Delta^{\mathrm{op}}_{/[n]} \arrow[d] \\ \{[i]\} \arrow[r] & \Delta^{\mathrm{op}} \end{tikzcd}$$
in which both squares are pullback.
- The symbol $\mathbb{A}_{(i)}$ denotes the portion of $\mathbb{A}$ lying over $(i) \in \Delta^{\mathrm{op}}_{/[n]}$.
$$\begin{tikzcd} \mathbb{A}_{[i]} \arrow[r] \arrow[d] & \mathbb{A} \arrow[d] \\ \{(i)\} \arrow[r] & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}$$

Although these symbols are not identical, they are related. For example, $\mathbb{A}_i$ is the total space of the cocartesian fibration classifying the diagram
$$\cdots \qquad \mathbb{A}_{(i,i,i)} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbb{A}_{(i,i)} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbb{A}_{(i)} \ ,$$

Note 2.3.3.4. Phrased in a model independent way:

- If $\psi = (i)$, i.e. if ψ is the map $\psi\colon [0] \to [n]$ sending $0 \mapsto i$, then $\mathbb{A}_\psi = \mathbb{A}_{(i)}$ should be thought of as the set of objects of a lying over the ith vertex of the n-simplex, i.e. $\mathbb{A}_{(i)} = (\mathbb{A}_i)_{[0]}$.
- Similarly, $\mathbb{A}_{(i,j)}$ is the category of morphisms $f\colon x_i \to x_j$ in a such that x_i is an object of $\mathbb{A}_i$ and x_j is an object of $\mathbb{A}_j$.
- The Segal condition implies that
$$\mathbb{A}_{(i,j,\cdots,k,\ell)} \simeq \mathbb{A}_{(i,j)} \times_{\mathbb{A}_{(j)}} \cdots \times_{\mathbb{A}_{(k)}} \mathbb{A}_{(k,\ell)}.$$
Therefore, $\mathbb{A}_{(i,j,\ldots,k,\ell)}$ is the ∞-category of chains of morphisms
$$x_i \xrightarrow{f_{ij}} x_j \longrightarrow \cdots \longrightarrow x_k \xrightarrow{f_{k\ell}} x_\ell$$

in a such that x_s is an object of a_s for all $s \in \{i, j, \dots, k, \ell\}$.

- For any object Ξ of $\mathbb{A}_{(i,\ell)}$, which we interpret as a morphism $\Xi\colon x_i \to x_\ell$ of a such that x_i is an object of a_i and x_ℓ is an object of a_ℓ, the category $(\mathbb{A}_{(i,j,\dots,k,\ell)})_{/\Xi}$ is the ∞-category of chains of morphisms as above together with a 2-morphism from their composite to Ξ:

$$\begin{tikzcd} x_i \arrow[r, "f_{ij}"] \arrow[rrrr, bend right, "\Xi"'] & x_j \arrow[r] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow"] & x_k \arrow[r, "f_{k\ell}"] & x_\ell \\ & & {} & & \end{tikzcd}$$

Lemma 2.3.3.5 (cf. [Hau16, Lemma 6.7(i)])**.** Let $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex. Let $\xi\colon [m] \to [n]$ be a morphism in Δ, and $\Xi \in \mathbb{A}$ an object lying over $\xi \in \Delta^{\mathrm{op}}_{/[n]}$. Then the projection

$$\mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \to (\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/\xi}$$

is a cocartesian fibration.

Proof. First consider the commutative diagram

$$\begin{tikzcd} \mathrm{Spine}(\mathbb{A})_{/\Xi} \arrow[r] \arrow[d] & \mathbb{A}_{/\Xi} \arrow[d] \\ \mathrm{Spine}(\mathbb{A}) \arrow[r] \arrow[d] & \mathbb{A} \arrow[d] \\ \Lambda^{\mathrm{op}}_{/[n]} \arrow[r] & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}.$$

Both squares are pullback by definition. Base change along the square

$$\begin{tikzcd} (\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}} \arrow[r] \arrow[d] & (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}} \arrow[d] \\ \Lambda^{\mathrm{op}}_{/[n]} \arrow[r] & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}$$

yields the diagram

$$\begin{tikzcd} \mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathbb{A}^{\mathrm{act}}_{/\Xi} \arrow[d] \\ \mathrm{Spine}(\mathbb{A})^{\mathrm{act}} \arrow[r] \arrow[d] & \mathbb{A}^{\mathrm{act}} \arrow[d] \\ (\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}} \arrow[r] & (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}} \end{tikzcd},$$

both of whose squares are therefore also pullback. Thus, in the diagram

$$\begin{tikzcd}
\mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d, "(2)"'] & \mathbb{A}^{\mathrm{act}}_{/\Xi} \arrow[d, "(1)"] \\
(\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/\xi} \arrow[r] \arrow[d] & (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/\xi} \arrow[d] \\
(\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}} \arrow[r] & (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}}
\end{tikzcd},$$

the outer square is pullback. The lower square is also pullback by definition, hence so is the upper square. The projection $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ is by Lemma 2.3.1.2 a cocartesian fibration, hence so is $\mathbb{A}^{\mathrm{act}} \to (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}}$. Thus, the map (1) is a cocartesian fibration by [HTT, Prop. 2.4.3.3(1)], so the map (2) is as well, which is what we needed to show. □

Definition 2.3.3.6. Let $\xi\colon [1] \to [n]$ be a morphism in Δ, and let $\xi(0) = i$ and $\xi(1) = i+k$. Suppose $k \geq 1$. Define a functor $\Phi\colon (\Delta^{\mathrm{op}})^{k-1} \to (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/\xi}$ by sending $([a_1], \dots, [a_{k-1}])$ to the object $[k + a_1 + \cdots + a_{k-1}] \to [n]$ with values $(i, i+1, \dots, i+1, \dots, i+(k-1), i+k)$, where there are $a_j + 1$ copies of $i+j$.

Lemma 2.3.3.7 (cf. [Hau16, Lemma 5.6])**.** Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex. Let $\xi\colon [1] \to [n]$ be a morphism in Δ sending $0 \mapsto i$ and $1 \mapsto i+k$ such that $k \geq 1$, and let Ξ be an object in $\mathbb{A}$ lying over $\xi \in \Delta^{\mathrm{op}}_{/[n]}$. Define an ∞-category $\mathcal{X}_\Xi$ via the pullback diagram

$$\begin{tikzcd}
\mathcal{X}_\Xi \arrow[r] \arrow[d] & \mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[d] \\
(\Delta^{\mathrm{op}})^{k-1} \arrow[r, "\Phi"] & (\Lambda^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/\xi}
\end{tikzcd}.$$

Then the top horizontal map $\mathcal{X}_\Xi \to \mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi}$ is cofinal.

Proof. The right-hand vertical map is a cocartesian fibration by Lemma 2.3.3.5. By [Hau17, Lemma 4.17], Φ is cofinal. Since the pullback of a cofinal map along a cocartesian fibration is cofinal, the result follows. □

Lemma 2.3.3.8 ([Hau16, Cor. 5.7])**.** Let $\mathcal{I}$ be an ∞-category and let $p\colon \mathcal{I} \to \mathcal{C}\mathrm{at}_\infty$ be a functor, classified by a cocartesian fibration $\mathcal{K} \to \mathcal{I}$, so that in particular $\mathcal{K}_\alpha \simeq p(\alpha)$. Suppose $\mathcal{D}$ is an ∞-category and $q\colon \mathcal{K} \to \mathcal{D}$ a functor such that:

(1) The functor q admits a colimit.

(2) For each $\alpha \in \mathcal{I}$, the restriction $q_\alpha = q|\mathcal{K}_\alpha$ admits a colimit.

Then we have an equivalence

$$\operatorname*{colim}_{\mathcal{K}} q \simeq \operatorname*{colim}_{\alpha \in \mathcal{I}} \operatorname*{colim}_{p(\alpha)} q_\alpha.$$

Theorem 2.3.3.9. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex, and let $F\colon \mathbb{A} \to \mathbb{D}$ be a lax functor of Segal ∞-bicategories. The functor F is extended from its spine if and only if for each $k \geq 2$, each $0 \leq i \leq n-k$, and each object Ξ of $\mathbb{A}_{(i,i+k)}$, the morphism

$$\operatorname*{colim}_{([a_0],\ldots,[a_{k-1}])\in(\Delta^{\mathrm{op}})^{k-1}} \operatorname*{colim}_{(\vec{f},\rho)\in(\mathbb{A}_{\Phi([a_0],\ldots,[a_{k-1}])})_{/\Xi}} F(f_{k+a_1+\ldots+a_{k-1}})\circ\cdots\circ F(f_0)\to F(\Xi)$$

induced by the structure morphisms encoding the laxness of F is an equivalence in $\mathbb{D}_{[1]}$.

Proof. Let $a\colon \mathbb{A}\to\Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex, and let $F\colon\mathbb{A}\to\mathbb{D}$ be a lax functor. According to the formula for generalized operadic Kan extensions, F is extended from its spine if and only if for each object Ξ of $\mathbb{A}_{[1]}$, the cone corresponding to the pasting diagram

$$\begin{tikzcd}
\mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathrm{Spine}(\mathbb{A})^{\mathrm{act}} \arrow[r] \arrow[d, hook] \arrow[dl, Rightarrow] & \mathbb{D}^{\mathrm{act}} \\
\{\Xi\} \arrow[r] & \mathbb{A}^{\mathrm{act}} \arrow[ur, "F|\mathbb{A}^{\mathrm{act}}"'] &
\end{tikzcd}$$

is a generalized operadic colimit. Since by assumption the Segal ∞-bicategory $q\colon\mathbb{D}\to\Delta^{\mathrm{op}}$ is locally cocomplete, any weak generalized operadic colimit taken in $\mathbb{D}$ is automatically a generalized operadic colimit, so by Corollary 2.2.3.11 the above cone is a weak generalized operadic colimit if and only if the cone corresponding to the pasting diagram

$$\begin{tikzcd}
\mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathrm{Spine}(\mathbb{A})^{\mathrm{act}} \arrow[r] \arrow[d, hook] \arrow[dl, Rightarrow] & \mathbb{D}^{\mathrm{act}} \arrow[r, "T"] & \mathbb{D}_{[1]} \\
\{\Xi\} \arrow[r] & \mathbb{A}^{\mathrm{act}} \arrow[ur, "F|\mathbb{A}^{\mathrm{act}}"'] & &
\end{tikzcd}$$

is an ordinary colimit cone. Note that if Ξ is contained in $\mathrm{Spine}(\mathbb{A})_{[1]}$, then the ∞-category parametrizing the base of this cone has a terminal object, namely id_Ξ, and the corresponding leg of the cone is sent to $F(\mathrm{id}_\Xi)$, which must be an equivalence, so this condition is vacuously satisfied. Since $\mathrm{Spine}(\mathbb{A})_{[1]}=\mathbb{A}_{(0,1)}\amalg\cdots\amalg\mathbb{A}_{(n-1,n)}$, if Ξ does not belong to $\mathrm{Spine}(\mathbb{A})_{[1]}$ then it must be an object of $\mathbb{A}_{(i,i+k)}$ for some $k\geq 2$ and $0\leq i\leq n-k$, i.e. a morphism of a whose source is some object x of a_i and whose target is some object z of a_{i+k}. Denote this morphism by $\Xi\colon x\to z$.

Since precomposition by a cofinal functor does not change the value of a colimit, Lemma 2.3.3.7 implies that the cone corresponding to the above diagram is a colimit cone if and only if the cone coming from the pasting diagram

$$\begin{tikzcd}
\mathcal{X}_\Xi \arrow[r] & \mathrm{Spine}(\mathbb{A})^{\mathrm{act}}_{/\Xi} \arrow[r] \arrow[d] & \mathrm{Spine}(\mathbb{A})^{\mathrm{act}} \arrow[r] \arrow[d, hook] \arrow[dl, Rightarrow] & \mathbb{D}^{\mathrm{act}} \arrow[r, "T"] & \mathbb{D}_{[1]} \\
& \{\Xi\} \arrow[r] & \mathbb{A}^{\mathrm{act}} \arrow[ur, "F|\mathbb{A}^{\mathrm{act}}"'] & &
\end{tikzcd} \tag{2.3.3.1}$$

is a colimit diagram. Unraveling the definitions, one sees that the map $\mathcal{X}_\Xi\to(\Delta^{\mathrm{op}})^{k-1}$ classifies the functor

$$(\Delta^{\mathrm{op}})^{k-1}\to\mathcal{C}\mathrm{at}_\infty;\qquad([a_1],\ldots,[a_{k-1}])\mapsto(\mathbb{A}_{\Phi([a_1],\ldots,[a_{k-1}])})_{/\Xi}.$$

We can describe $(\mathbb{A}_{\Phi([a_1],\ldots,[a_{k-1}])})_{/\Xi}$ as follows. Write

$$\Phi([a_1],\ldots,[a_{k-1}]) = (i, i+1, \ldots, i+1, \ldots, i+(k-1), i+k),$$

where there are $a_j + 1$ copies of $i+j$. We then have

$$\begin{aligned}(\mathbb{A}_{\Phi([a_1],\ldots,[a_{k-1}])})_{/\Xi} &\simeq (\mathbb{A}_{(i,i+1,\ldots,i+1,\ldots,i+(k-1),i+k)})_{/\Xi} \\ &\simeq (\mathbb{A}_{(i,i+1)} \times_{\mathbb{A}_{(i+1)}} \mathbb{A}_{(i+1,i+1)} \times_{\mathbb{A}_{(i+1)}} \cdots \times_{\mathbb{A}_{(i+(k-1))}} \mathbb{A}_{(i+(k-1),i+k)})_{/\Xi}.\end{aligned}$$

This is therefore the ∞-category of tuples $(\vec{f}, \rho)$, where $\vec{f} = (f_0, \ldots, f_{k+a_i+\cdots+a_{k-1}})$ is a tuple of composable morphisms in a starting at x and ending at z, and ρ is a morphism $f_{k+a_1+\cdots+a_{k-1}} \circ \cdots \circ f_0 \to \Xi$ in $\mathbb{A}_{(i,i+k)}$.

The top horizontal composition in the diagram of Equation 2.3.3.1 is the functor which sends the object $(\vec{f}, \rho)$ of $\mathcal{X}_\Xi$ to the object $F(f_{k+a_1+\cdots+a_{k-1}}) \circ \cdots \circ F(f_0)$. Using Lemma 2.3.3.8, we can rewrite the colimit of this functor as

$$\operatorname*{colim}_{([a_1],\ldots,[a_{k-1}])\in(\Delta^{\mathrm{op}})^{k-1}} \ \operatorname*{colim}_{(\vec{f},\rho)\in(\mathbb{A}_{\Phi([a_1],\ldots,[a_{k-1}])})_{/\Xi}} F(f_{k+a_1+\ldots+a_{k-1}}) \circ \cdots \circ F(f_0),$$

which is what we needed to show. □

Example 2.3.3.10. In the situation of Theorem 2.3.3.9, if $n = 0$ or $n = 1$, then every map $F\colon \mathbb{A} \to \mathbb{D}$ is extended from its spine, since the conditions above are vacuously satisfied.

Example 2.3.3.11. In the situation of Theorem 2.3.3.9, suppose $n = 2$. The functor F is extended from its spine if and only if for each object $\Xi \in \mathbb{A}_{(0,2)}$, the Δ^{op}-indexed diagram

$$\begin{tikzcd}
\vdots \\
\operatorname*{colim}_{\substack{(f,g,g',h)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,1,2)})_{/\Xi}}} F(h)\circ F(g)\circ F(g')\circ F(f) \arrow[d, shift left=4] \arrow[u, shift left=2] \arrow[d] \arrow[u, shift right=2] \arrow[d, shift right=4] \\
\operatorname*{colim}_{\substack{(f,g,h)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,2)})_{/\Xi}}} F(h)\circ F(g)\circ F(f) \arrow[d, shift left=2] \arrow[u] \arrow[d, shift right=2] \\
\operatorname*{colim}_{\substack{(f,h)\to\Xi \\ \in(\mathbb{A}_{(0,1,2)})_{/\Xi}}} F(h)\circ F(f) \arrow[d] \\
F(\Xi)
\end{tikzcd}$$

is a colimit diagram in $\mathbb{D}_{[1]}$.

Example 2.3.3.12. Suppose $n = 3$. The functor F is extended from its spine if and only if:

(1) For each object of $\mathbb{A}_{(0,2)}$ and $\mathbb{A}_{(1,3)}$, a formula analogous to Example 2.3.3.11 holds, and

(2) For each object $\Xi \in \mathbb{A}_{(0,3)}$, the morphisms coming from the lax structure on the functor F exhibit $F(\Xi)$ as the colimit of the $(\Delta^{\mathrm{op}})^2$-indexed diagram below.

$$
\begin{tikzcd}
\vdots & \vdots & \iddots \\
\underset{\substack{(f,g,g',h,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,1,2,3)})_{/\Xi}}}{\operatorname{colim}} F(j)\circ F(h)\circ F(g')\circ F(g)\circ F(f) \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[d, shift left=4] \arrow[d, shift left=2, leftarrow] \arrow[d] \arrow[d, shift right=2, leftarrow] \arrow[d, shift right=4] & \underset{\substack{(f,g,g',h,i,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,1,2,2,3)})_{/\Xi}}}{\operatorname{colim}} F(j)\circ F(i)\circ F(h)\circ F(g')\circ F(g)\circ F(f) \arrow[d, shift left=6] \arrow[d, shift left=4, leftarrow] \arrow[d, shift left=2] \arrow[d, leftarrow] \arrow[d, shift right=2] \arrow[d, shift right=4, leftarrow] \arrow[d, shift right=6] & \cdots \\
\underset{\substack{(f,g,h,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,2,3)})_{/\Xi}}}{\operatorname{colim}} F(h)\circ F(j)\circ F(g)\circ F(f) \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[d, shift left=2] \arrow[d, leftarrow] \arrow[d, shift right=2] & \underset{\substack{(f,g,h,i,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,1,2,2,3)})_{/\Xi}}}{\operatorname{colim}} F(j)\circ F(i)\circ F(h)\circ F(g)\circ F(f) \arrow[d, shift left=2] \arrow[d, leftarrow] \arrow[d, shift right=2] & \cdots \\
\underset{\substack{(f,h,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,2,3)})_{/\Xi}}}{\operatorname{colim}} F(j)\circ F(h)\circ F(f) \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \underset{\substack{(f,h,i,j)\to\Xi \\ \in(\mathbb{A}_{(0,1,2,2,3)})_{/\Xi}}}{\operatorname{colim}} F(j)\circ F(i)\circ F(h)\circ F(f) & \cdots
\end{tikzcd}
$$

The next proposition shows that spine extension behaves well with respect to base change.

Proposition 2.3.3.13 (cf. [Hau16, Cor. 6.15])**.** Let $\phi\colon [m] \to [n]$ be a morphism in Δ, let $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex, and let $F\colon \mathbb{A} \to \mathbb{D}$ be a lax functor of Segal ∞-bicategories which is extended from its spine. Then then upper composition in the diagram

$$
\begin{tikzcd}
\phi^*(\mathbb{A}) \arrow[r] \arrow[d] & \mathbb{A} \arrow[r, "F"] \arrow[d] & \mathbb{D} \\
\Delta^{\mathrm{op}}_{/[m]} \arrow[r, "\phi"] & \Delta^{\mathrm{op}}_{/[n]}
\end{tikzcd}
$$

is extended from its spine.

Proof. We define $\Lambda^{\mathrm{op}}_{/[n]}[\phi]$ as in [Hau17, Def. 4.36], and define $\mathbb{A}[\phi]$ by the pullback

$$
\begin{tikzcd}
\mathbb{A}[\phi] \arrow[r] \arrow[d] & \mathbb{A} \arrow[d] \\
\Lambda^{\mathrm{op}}_{/[n]}[\phi] \arrow[r] & \Delta^{\mathrm{op}}_{/[n]}
\end{tikzcd}
.
$$

We first follow the proof of [Hau16, Prop. 6.14]. Let Γ be an object of $\phi^*\mathbb{A}$ lying over $\gamma \in \Delta^{\mathrm{op}}_{/[m]}$ consider the diagram

$$
\begin{tikzcd}
\operatorname{Spine}(\phi^*\mathbb{A})^{\mathrm{act}}_{/\Gamma} \arrow[r] \arrow[d] & \mathbb{A}[\phi]^{\mathrm{act}}_{/\phi_*\Gamma} \arrow[d] \\
(\Lambda^{\mathrm{op}}_{/[m]})^{\mathrm{act}}_{/\gamma} \arrow[r] & (\Lambda^{\mathrm{op}}_{/[n]}[\phi])^{\mathrm{act}}_{/\phi\gamma}
\end{tikzcd}
.
$$

The method of proof of Lemma 2.3.3.5 shows that both vertical maps are cocartesian fibrations. The description of the cocartesian morphisms in [HTT, Prop. 2.4.3.3(2)] also makes it clear that the top horizontal map preserves cocartesian morphisms. Thus, so does the induced map from $\operatorname{Spine}(\phi^*\mathbb{A})^{\mathrm{act}}_{/\Gamma}$ into the pullback. It thus suffices to show that this map is an equivalence of cocartesian fibrations over $(\Lambda^{\mathrm{op}}_{/[m]})^{\mathrm{act}}_{/\gamma}$. We can show this by showing that the induced map on each fiber is an equivalence.

We first describe the fiber of the left-hand map over an object $\alpha\colon \psi \to \gamma$ in $(\Lambda^{\mathrm{op}}_{/[m]})^{\mathrm{act}}_{/\gamma}$, given by some diagram

$$\begin{tikzcd}
{[\ell]} \arrow[dr, "\psi"'] & & {[k]} \arrow[ll, "\alpha"'] \arrow[dl, "\gamma"] \\
& {[m]} &
\end{tikzcd}$$

in Δ. First we fix notation. Define a ∞-category $\mathcal{A}_\alpha$ together with a map $\mathcal{A}_\alpha \to \Delta^1$ by the diagram

$$\begin{tikzcd}
\mathcal{A}_\alpha \arrow[r] \arrow[d] & \phi^*\mathbb{A}^{\mathrm{act}} \arrow[r] \arrow[d] & \mathbb{A}^{\mathrm{act}} \arrow[d] \\
\Delta^1 \arrow[r, "\alpha"] & (\Delta^{\mathrm{op}}_{/[m]})^{\mathrm{act}} \arrow[r, "\phi"] & (\Delta^{\mathrm{op}}_{/[n]})^{\mathrm{act}}
\end{tikzcd},$$

where both squares are pullback. Then $\mathcal{A}_\alpha \to \Delta^1$ is a cocartesian fibration classifying the functor $\alpha_!\colon \mathbb{A}_{\phi\psi} \to \mathbb{A}_{\phi\gamma}$.

For any simplicial set K, a map from K in the fiber of the left-hand map over ψ is given by a commutative square

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathrm{Spine}(\phi^*\mathbb{A})^{\mathrm{act}}_{/\Gamma} \arrow[d] \\
\Delta^0 \arrow[r, "\alpha"] & (\Lambda^{\mathrm{op}}_{/[m]})^{\mathrm{act}}_{/\gamma}
\end{tikzcd},$$

which is equivalent to a commutative square

$$\begin{tikzcd}
K \star \Delta^0 \arrow[r] \arrow[d] & \phi^*\mathbb{A}^{\mathrm{act}} \arrow[d] \\
\Delta^0 \star \Delta^0 \arrow[r, "\alpha"] & (\Delta^{\mathrm{op}}_{/[m]})^{\mathrm{act}}
\end{tikzcd}.$$

Such a square is encoded by a map $K \to (\mathcal{A}_\alpha)_{/\Gamma} \times_{\mathcal{A}_\alpha} \mathbb{A}_{\phi\psi}$,

A similar analysis shows that the fiber of the fiber product over α is also given by $(\mathcal{A}_\alpha)_{/\Gamma} \times_{\mathcal{A}_\alpha} \mathbb{A}_{\phi\psi}$, and the map on fibers is simply the identity. Thus, the map $\mathrm{Spine}(\phi^*\mathbb{A})^{\mathrm{act}}_{/\Gamma} \to \mathbb{A}[\phi]^{\mathrm{act}}_{/\phi_*\Gamma}$ is cofinal. The proof then proceeds as in [Hau16, Cor. 6.15], i.e. by following the proof of [Hau17, Cor. 4.38], substituting $\mathrm{Spine}(\phi^*\mathbb{A})^{\mathrm{act}}_{/\Gamma} \to \mathbb{A}[\phi]^{\mathrm{act}}_{/\phi_*\Gamma}$ for $(\Lambda^{\mathrm{op}}_{/[m]})^{\mathrm{act}}_{/\gamma} \to (\Lambda^{\mathrm{op}}_{/[n]}[\phi])^{\mathrm{act}}_{/\phi\gamma}$. $\square$

Example 2.3.3.14. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}_{/[1]}$ be a Segal ∞-bicategory over the 1-simplex, and let $\phi\colon [2] \to [1]$ be given by $(0,0,1)$. By Proposition 2.3.3.13, $\phi^*\mathbb{A} \to \mathbb{A} \to \mathbb{D}$ is extended from its spine. Heuristically, we can also see this in a more direct way. First note that

$$(\phi^*\mathbb{A})_{(0,1,\ldots,1,2)} = \mathbb{A}_{(0,0,\ldots,0,1)}$$

where $(0,1,\ldots,1,2)$ contains n copies of 1, and $(0,0,\ldots,0,1)$ contains $n+1$ copies of 0. Plugging this into Example 2.3.3.11, we see that we need to show that for all $\Xi \in (\phi^*\mathbb{A})_{(0,2)} =$

$\mathbb{A}_{(0,1)}$ (representing a morphism $\Xi\colon x \to z$ in a), the portion of the diagram

$$\begin{tikzcd}
\vdots \\
\operatorname*{colim}\limits_{\substack{(f,f',g)\to\Xi \\ \in(\mathbb{A}_{(0,0,0,1)})_{/\Xi}}} F(g)\circ F(f')\circ F(f) \arrow[d, shift right=3, dotted] \arrow[d, shift left] \arrow[d, shift left=3] \\
\operatorname*{colim}\limits_{\substack{(f,g)\to\Xi \\ \in(\mathbb{A}_{(0,0,1)})_{/\Xi}}} F(g)\circ F(f) \arrow[u, shift right, dotted] \arrow[u, shift left=2] \arrow[d, shift right=2, dotted] \arrow[d, shift left=2] \\
\operatorname*{colim}\limits_{\substack{g\to\Xi \\ \in(\mathbb{A}_{(0,1)})_{/\Xi}}} F(g) \arrow[u, dotted] \arrow[r, dashed, "\simeq"] & F(\Xi)
\end{tikzcd}$$

consisting only of solid arrows is a colimit diagram in $\mathbb{D}(F(x), F(z))$; note that a priori the bottom most term is wrong, but because $\Xi \in \mathbb{A}_{(0,1)}$, the overcategory $(\mathbb{A}_{(0,1)})_{/\Xi}$ has terminal object id_Ξ, so we can replace the lower-most colimit by $F(\Xi)$.

We can augment the solid diagram to a simplicial object by adding the extra dotted face and degeneracy maps on the left side, which are induced by the maps

$$\mathbb{A}_{(0,0,\ldots,0,1)} \to \mathbb{A}_{(0,\ldots,0,1)}; \qquad (f_0, \cdots, f_n, g) \mapsto (f_1, \ldots, f_n, g)$$

and

$$\mathbb{A}_{(0,\ldots,0,1)} \to \mathbb{A}_{(0,0,\ldots,0,1)}; \qquad (f_1, \ldots, f_n, g) \mapsto (\mathrm{id}, f_1, \ldots, f_n).$$

respectively. These extra face and degeneracy maps exhibit the solid portion of the diagram as a décalage, and therefore a colimit diagram by [HTT, Lemma 6.1.3.17].

Lemma 2.3.3.15. Let $\mathbb{D}$ be a locally cocomplete Segal ∞-bicategory, and let $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ and $\mathbb{B} \to \Delta^{\mathrm{op}}_{/[n]}$ be Segal ∞-bicategories over the n-simplex. Let $f\colon \mathbb{A} \to \mathbb{B}$ be a map of generalized operads over the n-simplex, let $F\colon \mathbb{A} \to \mathbb{D}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be lax functors, and let $\alpha\colon F \Rightarrow G \circ f$ be an icon, as in the diagram

$$\begin{tikzcd}
\mathrm{Spine}(\mathbb{A}) \arrow[r, hook, "i"] \arrow[d, "\mathrm{Spine}(f)"'] & \mathbb{A} \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathbb{D} \\
\mathrm{Spine}(\mathbb{B}) \arrow[r, hook, "j"] & \mathbb{B} \arrow[ur, "G"'] &
\end{tikzcd}.$$

(1) Suppose α exhibits G as the generalized operadic Kan extension of F along f. Then the restriction of α to $\mathrm{Spine}(\mathbb{A})$ exhibits $G\circ j$ as the generalized operadic Kan extension of $F \circ i$ along $\mathrm{Spine}(f)$.

(2) Suppose the restriction of α to $\mathrm{Spine}(\mathbb{A})$ exhibits $G \circ j$ as the generalized operadic Kan extension of $F \circ i$ along $\mathrm{Spine}(f)$, and suppose F and G are extended from their spines. Then α exhibits G as the generalized operadic Kan extension of F along f.

Proof. To see that (1) holds, note that the diagrams which need to be colimit diagrams in order to show that α exhibits G as the generalized operadic Kan extension of F along f are a superset of those needed to show that the restriction of α to $\mathrm{Spine}(\mathbb{A})$ exhibits $G \circ j$ as the generalized operadic Kan extension of $F \circ i$ along $G \circ j$.

It follows from Assumption 2.2.4.6 that (2) holds. □

Lemma 2.3.3.16. Let $\mathbb{D}$ be a locally cocomplete Segal ∞-bicategory, and let $\mathbb{A} \to \Delta^{\mathrm{op}}_{/[n]}$ and $\mathbb{B} \to \Delta^{\mathrm{op}}_{/[n]}$ be Segal ∞-bicategories over the n-simplex. Let $f\colon \mathbb{A} \to \mathbb{B}$ be a map of generalized operads over the n-simplex, let $F\colon \mathbb{A} \to \mathbb{D}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be lax functors, and let $\alpha\colon F \Rightarrow G \circ f$ be an icon, as in the diagram

$$\begin{tikzcd} \mathrm{Spine}(\mathbb{A}) \arrow[r, hook, "i"] \arrow[d, "\mathrm{Spine}(f)"'] & \mathbb{A} \arrow[r, "F"] \arrow[d, "f"'] & \mathbb{D} \arrow[dl, phantom, "\Swarrow\alpha"] \\ \mathrm{Spine}(\mathbb{B}) \arrow[r, hook, "j"] & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd} .$$

Suppose α exhibits G as the generalized operadic Kan extension of F along f, and suppose F is extended from its spine. Then G is extended from its spine.

Proof. It follows from Lemma 2.3.3.15 that the restriction of α to $\mathrm{Spine}(\mathbb{A})$ exhibits $G \circ j$ as the generalized operadic Kan extension of $F \circ i$ along $\mathrm{Spine}(f)$, and the result follows by Assumption 2.2.4.6. □

2.3.4 Joins of Segal infinity-bicategories

In this subsection we define a generalization of the familiar join of simplicial sets to Segal ∞-bicategories. Our main result, Proposition 2.3.4.13, is that the expected universal property of of the join (Note 2.3.4.11), which a priori only holds for strict functors, also holds for lax functors. This follows from the observation that the ∞-category of lax functors of Segal ∞-bicategories from $\mathbb{A}$ to $\mathbb{D}$ is equivalent to the category of strict functors from $\mathrm{Env}(\mathbb{A})$ to $\mathbb{D}$, where $\mathrm{Env}(\mathbb{A})$ is the double categorical envelope of $\mathbb{A}$.

We first recall a few basic results from [Abe23], specialized to the situation we are interested in (functors indexed by a fully-marked $(\infty, 1)$-category). We will not operate in full rigor, referring the reader to the paper for more details.

Definition 2.3.4.1 ([Abe23, Def. 4.8]**.** Let $\mathcal{A}$ be a quasicategory. We define $\mathbb{F}\mathrm{un}(\mathcal{A}, \mathbb{C}\mathrm{at}_\infty)$ to be the $(\infty, 2)$-category whose objects are functors $F\colon \mathcal{A} \to \mathbb{C}\mathrm{at}_\infty$, whose morphisms are natural transformations, and whose 2-morphisms are modifications.

Definition 2.3.4.2 ([Abe23, Def. 4.7])**.** Let $\mathcal{A}$ be a quasicategory. We define $\mathbb{F}\mathrm{ib}_0(\mathcal{A})$ to be the $(\infty, 2)$-category whose objects are cocartesian fibrations over $\mathcal{A}$, whose 1-morphisms are commutative triangles

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathcal{Y} \arrow[dl, "p"] \\ & \mathcal{A} & \end{tikzcd}$$

such that f sends q-cocartesian morphisms in $\mathcal{X}$ to p-cocartesian morphisms in $\mathcal{Y}$, and whose 2-morphisms are natural transformations which commute with the projections down to $\mathcal{A}$.

Example 2.3.4.3. There is a full subcategory inclusion $\mathbb{S}\mathrm{eg}^{\mathrm{ic}}_{(\infty,2)} \subseteq \mathbb{F}\mathrm{ib}_0(\Delta^{\mathrm{op}})$ on Segal ∞-bicategories; here $\mathbb{S}\mathrm{eg}^{\mathrm{ic}}_{(\infty,2)}$ denotes the sub-$(\infty, 2)$-category of $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ on strict functors.

Theorem 2.3.4.4 ([Abe23, Thm. 4.11])**.** Let $\mathcal{A}$ be a quasicategory. Then there exists a natural equivalence of $(\infty, 2)$-categories

$$\mathrm{St}_{\mathcal{A}} : \mathbb{F}\mathrm{ib}_0(\mathcal{A}) \xleftrightarrow{\simeq} \mathbb{F}\mathrm{un}(\mathcal{A}, \mathbb{C}\mathrm{at}_\infty) : \mathrm{Un}_{\mathcal{A}}.$$

Theorem 2.3.4.5 ([Abe23, Rem. 4.31])**.** Let $f\colon \mathcal{A} \to \mathcal{B}$ be a functor of quasicategories. Then there is an adjunction of $(\infty, 2)$-categories

$$f^* : \mathbb{F}\mathrm{un}(\mathcal{B}, \mathbb{C}\mathrm{at}_\infty) \longleftrightarrow \mathbb{F}\mathrm{un}(\mathcal{A}, \mathbb{C}\mathrm{at}_\infty) : f_*,$$

where f_* is is given by right Kan extension along f, computed pointwise via the usual limit formula for Kan extensions.

Definition 2.3.4.6. Let $n \geq 0$, and let $p_i\colon \mathbb{C}_i \to \Delta^{\mathrm{op}}$ be a Segal ∞-bicategory for each $0 \leq i \leq n$. Denote the straightening of p_i by $P_i\colon \Delta^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_\infty$. A ***join*** of the Segal ∞-bicategories p_i is a cocartesian fibration $p_0 \star \cdots \star p_n\colon \mathbb{C}_0 \star \cdots \star \mathbb{C}_n \to \Delta^{\mathrm{op}}_{/[n]}$ classifying the right Kan extension

$$\begin{tikzcd} \Delta^{\mathrm{op}} \amalg \cdots \amalg \Delta^{\mathrm{op}} \arrow[r, "{(P_0,\dots P_n)}"] \arrow[d, hook, "i"'] & \mathbb{C}\mathrm{at}_\infty \\ \Delta^{\mathrm{op}}_{/[n]} \arrow[ur, "{i_*(P_0,\dots P_n)}"'] & \end{tikzcd}.$$

Here i is the full subcategory inclusion sending $[k]$ in the jth copy of Δ^{op} to the object $\underline{j}\colon [k] \to [n]$ which is constant on j.

Observation 2.3.4.7. It follows easily from the limit formula for right Kan extensions that the functor $i_*(P_0, \dots P_n)$ sends an object $\psi\colon [k] \to [n]$ to the product $\prod_{i=0}^n (\mathbb{C}_i)_{[\psi_i]}$, where $[\psi_i] = \psi^{-}\{i\}$.

Composing a join $\mathbb{C}_0 \star \cdots \star \mathbb{C}_n \to \Delta^{\mathrm{op}}_{/[n]}$ with the forgetful functor $\Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$, one finds a triangle

$$\begin{tikzcd} \mathbb{C}_0 \star \cdots \star \mathbb{C}_n \arrow[rr, "p_0 \star \cdots \star p_n"] \arrow[dr, "p"'] & & \Delta^{\mathrm{op}}_{/[n]} \arrow[dl, "q"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}.$$

Since q and $p_0 \star \cdots \star p_n$ are cocartesian fibrations, p is as well. The fiber of p over $[k] \in \Delta^{\mathrm{op}}$ is the ∞-category

$$\coprod_{[s_0] \oplus \cdots \oplus [s_n] = [k]} (\mathbb{C}_0)_{s_0} \times \cdots \times (\mathbb{C}_n)_{s_n}, \tag{2.3.4.1}$$

where each $s_i \in \Delta^{\mathrm{op}}_+$, the (opposite of the) augmented simplex category. The functoriality in Δ^{op} is inherited from the functoriality of each $\mathbb{C}_i$ over s_i, and that p satisfies the Segal condition easily follows, as does completeness. Thus, the join $p_0 \star \cdots \star p_n$ is a Segal ∞-bicategory over the n-simplex.

Example 2.3.4.8. Let $a\colon \mathbb{A} \to \Delta^{\mathrm{op}}$ and $b\colon \mathbb{B} \to \Delta^{\mathrm{op}}$ be Segal ∞-bicategories. Then the join of a and b is a Segal ∞-bicategory $a \star b\colon \mathbb{A} \star \mathbb{B} \to \Delta^{\mathrm{op}}_{/[1]}$ over the 1-simplex. Straightening, we can view the join $\mathbb{A} \star \mathbb{B} \to \Delta^{\mathrm{op}}_{/[1]}$ as a diagram $\Delta^{\mathrm{op}}_{/[1]} \to \mathbb{C}\mathrm{at}_\infty$, which we sketch below. (For a recollection of the category $\Delta^{\mathrm{op}}_{/[1]}$, recall Example 2.2.1.9).

$$\begin{tikzcd}[column sep=tiny]
\vdots && \vdots && \vdots && \vdots \\
\mathbb{A}_{[2]} \arrow[dr, shift left=3] \arrow[dr, shift left=1.5] \arrow[dr] \arrow[dr, shift right=1.5, leftarrow] \arrow[dr, shift right=3, leftarrow] && \mathbb{A}_{[1]} \times \mathbb{B}_{[0]} \arrow[dl] \arrow[dr, shift left=1.5] \arrow[dr, leftarrow] \arrow[dr, shift right=1.5] && \mathbb{A}_{[0]} \times \mathbb{B}_{[1]} \arrow[dl, shift left=1.5] \arrow[dl, leftarrow] \arrow[dl, shift right=1.5] \arrow[dr] && \mathbb{B}_{[2]} \arrow[dl, shift left=3] \arrow[dl, shift left=1.5] \arrow[dl] \arrow[dl, shift right=1.5, leftarrow] \arrow[dl, shift right=3, leftarrow] \\
& \mathbb{A}_{[1]} \arrow[dr, shift left=1.5] \arrow[dr, leftarrow] \arrow[dr, shift right=1.5] && \mathbb{A}_{[0]} \times \mathbb{B}_{[0]} \arrow[dl] \arrow[dr] && \mathbb{B}_{[1]} \arrow[dl, shift left=1.5] \arrow[dl, leftarrow] \arrow[dl, shift right=1.5] \\
&& \mathbb{A}_{[0]} && \mathbb{B}_{[0]}
\end{tikzcd}$$

For x and y objects of $a \star b$, it follows easily from the formula in Equation 2.3.4.1 that

$$(\mathbb{A} \star \mathbb{B})(x, y) = \begin{cases} \mathbb{A}(x, y), & x, y \in \mathrm{ob}(a) \\ *, & x \in \mathrm{ob}(a) \text{ and } y \in \mathrm{ob}(b) \\ \emptyset, & x \in \mathrm{ob}(b) \text{ and } y \in \mathrm{ob}(a) \\ \mathbb{B}(x, y), & x, y \in \mathrm{ob}(b) \end{cases}.$$

More generally, if $a_i \colon \mathbb{A}_i \to \Delta^{\mathrm{op}}$ are Segal ∞-bicategories for $0 \leq i \leq n$ and x is an object of $\mathbb{A}_i$ and y is an object of $\mathbb{A}_j$, then

$$(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n)(x, y) = \begin{cases} *, & i < j \\ \mathbb{A}_i(x, y), & i = j \\ \emptyset, & i > j \end{cases}.$$

Observation 2.3.4.9. Let $a_i \colon \mathbb{A}_i \to \Delta^{\mathrm{op}}$ are Segal ∞-bicategories for $0 \leq i \leq n$, and let $a_0 \star \cdots \star a_n \colon \mathbb{A}_0 \star \cdots \mathbb{A}_n \to \Delta^{\mathrm{op}}_{/[n]}$ be their join. Let $\phi \colon [k] \to [n]$ be an injective morphism of Δ. Then the base change of $a_0 \star \cdots \star a_n$ along ϕ yields a join:

$$\begin{tikzcd}
\mathbb{A}_{\phi(0)} \star \cdots \star \mathbb{A}_{\phi(k)} \arrow[r, hook] \arrow[d, "a_{\phi(0)} \star \cdots \star a_{\phi(k)}"'] & \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \arrow[d, "a_0 \star \cdots \star a_n"] \\
\Delta^{\mathrm{op}}_{/[k]} \arrow[r, "\phi"] & \Delta^{\mathrm{op}}_{/[n]}
\end{tikzcd}.$$

Note that this is in general *not* the case if ϕ is not injective! It follows from Observation 2.3.4.7 and the pasting law for pullbacks that in the general case, the base change is given by

$$\begin{tikzcd}
(\mathbb{A}_{\phi(0)} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{/[\phi_0]}) \star \cdots \star (\mathbb{A}_{\phi(k)} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{/[\phi_k]}) \arrow[r] \arrow[d] & \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \arrow[d, "a_0 \star \cdots \star a_n"] \\
\Delta^{\mathrm{op}}_{/[k]} \arrow[r, "\phi"] & \Delta^{\mathrm{op}}_{/[n]}
\end{tikzcd},$$

where $\phi_i = |\phi^{-1}\{i\}| - 1$.

The universal property for right Kan extension of Theorem 2.3.4.5, together with the above observations, immediately imply the following.

Proposition 2.3.4.10. For each $n \geq 0$, there is an adjunction

$$\mathbb{L}_n : (\mathbb{S}\mathrm{eg}^{\mathrm{ic}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \longleftrightarrow (\mathbb{S}\mathrm{eg}^{\mathrm{ic}}_{(\infty,2)})^{n+1} : \mathbb{R}_n,$$

where $\mathbb{L}_n$ is given by base change along the inclusion $\Delta^{\mathrm{op}} \amalg \cdots \amalg \Delta^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}_{/[n]}$, sending a Segal ∞-bicategory $\mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ to its fibers $(\mathbb{M}_0, \ldots, \mathbb{M}_n)$, and $\mathbb{R}_n$ sends a tuple of Segal ∞-bicategories $(\mathbb{C}_0, \ldots, \mathbb{C}_n)$ to the join $\mathbb{C}_0 \star \cdots \star \mathbb{C}_n \to \Delta^{\mathrm{op}}_{/[n]}$.

Note 2.3.4.11. Let $\mathbb{M}$ and $\mathbb{C}_i$ be as above. The above proposition shows that the data of a *strict* (=non-lax) functor

$$\begin{tikzcd} \mathbb{M} \arrow[rr, "F"] \arrow[dr] & & \mathbb{C}_0 \star \cdots \star \mathbb{C}_n \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd}$$

over the n-simplex is completely determined by its restrictions $(F_i \colon \mathbb{M}_i \to \mathbb{C}_i)_{i=0}^n$ to the fibers. Proposition 2.3.4.13 shows that this result also holds for lax functors.

Lemma 2.3.4.12. Let $p \colon \mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ be a Segal ∞-bicategory over the n-simplex. Then there exists a functor $\mathrm{Env}(\mathbb{M}) \to \Delta^{\mathrm{op}}_{/[n]}$ exhibiting $\mathrm{Env}(p)$ as a Segal ∞-bicategory over the n-simplex, and the fiber of $\mathrm{Env}(\mathbb{M})$ over i is $\mathrm{Env}(\mathbb{M}_i)$.

Proof. We first show that $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ is a Segal ∞-bicategory over the n-simplex with fiber over i given by $\mathrm{Act}(\Delta^{\mathrm{op}})$. By Proposition 2.2.2.11 and Proposition 2.2.2.7, $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \to \Delta^{\mathrm{op}}$ is a double ∞-category such that $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})_{[0]}$ is a set, and therefore a Segal ∞-bicategory. We thus need to find a map of ∞-operads $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \to \Delta^{\mathrm{op}}_{/[n]}$.

Unraveling the definitions, $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ is the nerve of a 1-category whose objects are composable pairs $[k] \xrightarrow{\phi} [\ell] \xrightarrow{\psi} [n]$ of morphisms in Δ such that ϕ is active, and whose morphisms are commuting diagrams

$$\begin{tikzcd} {[k]} & & {[k']} \arrow[ll] \arrow[d, "\phi'"] \\ {[\ell]} \arrow[from=u, "\phi"'] \arrow[dr, "\psi"'] & & {[\ell']} \arrow[ll] \arrow[dl, "\psi'"] \\ & {[n]} & \end{tikzcd}$$

in Δ. Composing vertically gives a functor $s \colon \mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \to \Delta^{\mathrm{op}}_{/[n]}$ such that the diagram

$$\begin{tikzcd} \mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \arrow[rr, "s"] \arrow[dr, "\mathrm{Env}(u_n)"'] & & \Delta^{\mathrm{op}}_{/[n]} \arrow[dl, "u_n"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

commutes. Since u_n is a left fibration, s preserves cocartesian morphisms, and hence exhibits $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ as a pre-Segal ∞-bicategory over the n-simplex. The fiber of $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ over i is the full subcategory of $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ on those objects $[k] \xrightarrow{\phi} [\ell] \xrightarrow{\psi} [n]$ such that ψ is constant with value i. Since there is always a unique such ψ, this fiber is equivalent (in fact, isomorphic) to $\mathrm{Act}(\Delta^{\mathrm{op}})$.

Now consider the diagram

$$\begin{tikzcd}
\mathrm{Env}(\mathbb{M}_i) \arrow[r] \arrow[d] & \mathrm{Env}(\mathbb{M}) \arrow[d] \\
\mathrm{Act}(\Delta^{\mathrm{op}}) \arrow[r] \arrow[d] & \mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \arrow[d] \\
\Delta^{\mathrm{op}} \arrow[r, "i"] & \Delta^{\mathrm{op}}_{/[n]}
\end{tikzcd} .$$

We have just shown that the lower square is pullback, so in order to show that the outer square is pullback it suffices to show that the upper square is pullback. To do this, it suffices to show that the induced map from $\mathrm{Env}(\mathbb{M}_i)$ into the pullback is an equivalence. Since $\mathrm{Env}(\mathbb{M}) \to \mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]})$ is the pullback of $\mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ along $\mathrm{Env}(\Delta^{\mathrm{op}}_{/[n]}) \to \Delta^{\mathrm{op}}_{/[n]}$, it is a cocartesian fibration; the same logic shows that $\mathrm{Env}(\mathbb{M}_i) \to \mathrm{Env}(\Delta^{\mathrm{op}})$ is a cocartesian fibration, and it follows from Note 2.2.2.6 that the induced map from $\mathrm{Env}(\mathbb{M}_i)$ to the pullback is a morphism of cocartesian fibrations. Thus, it suffices to show that it is fiberwise an equivalence. Over the object $\phi\colon [k] \to [\ell]$ in $\mathrm{Act}(\Delta^{\mathrm{op}})$, it is the map

$$(\mathbb{M}_i)_{[k]} \to \mathbb{M}_{(i,\ldots,i)}; \qquad M \mapsto M$$

where $(i, \ldots, i)$ contains $k+1$ copies of i; this is even an isomorphism of simplicial sets. □

Proposition 2.3.4.13. For each $n \geq 0$, there is an adjunction of $(\infty, 2)$-categories

$$\mathbb{L}_n : (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \longleftrightarrow (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})^{n+1} : \mathbb{R}_n,$$

where $\mathbb{L}_n$ is given by base change along the inclusion $\Delta^{\mathrm{op}} \amalg \cdots \amalg \Delta^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}_{/[n]}$, sending a Segal ∞-bicategory $\mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ to its fibers $(\mathbb{M}_0, \ldots, \mathbb{M}_n)$, and $\mathbb{R}_n$ sends a tuple of Segal ∞-bicategories $(\mathbb{C}_0, \ldots, \mathbb{C}_n)$ to the join $\mathbb{C}_0 \star \cdots \star \mathbb{C}_n \to \Delta^{\mathrm{op}}_{/[n]}$.

Proof. By [Abe23, Cor. 5.12], it suffices to show that for each $\mathbb{C}_0, \ldots, \mathbb{C}_n$ as above, the top horizontal map in the diagram

$$\begin{tikzcd}
(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}}(\mathbb{M}, \mathbb{C}_0 \star \cdots \star \mathbb{C}_n) \arrow[r] & \prod_{i=0}^n \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}(\mathbb{M}_i, \mathbb{C}_i) \\
\mathrm{Fun}_{/\Delta^{\mathrm{op}}_{/[n]}}(\mathrm{Env}(\mathbb{M}), \mathbb{C}_0 \star \cdots \star \mathbb{C}_n) \arrow[r] \arrow[u, "\simeq"] & \prod_{i=0}^n \mathrm{Fun}(\mathrm{Env}(\mathbb{M}_i), \mathbb{C}_i) \arrow[u, "\simeq"']
\end{tikzcd}$$

is an equivalence, natural in $\mathbb{M}$. Pulling back along the inclusions $\mathbb{M} \hookrightarrow \mathrm{Env}(\mathbb{M})$ and $\mathbb{M}_i \hookrightarrow \mathrm{Env}(\mathbb{M}_i)$ (which is natural because by Note 2.2.2.5, $\mathrm{Env}(-)$ is functorial) and applying Lemma 2.3.4.12, it suffices to show that the bottom arrow is an equivalence, which follows from Proposition 2.3.4.10. □

For the majority of this chapter, we will only need the underlying $(\infty, 1)$-adjunction:

Proposition 2.3.4.14. For each $n \geq 0$, there is an adjunction of ∞-categories

$$\lambda_n : (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \longleftrightarrow (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1} : \rho_n,$$

where λ_n is given by base change along the inclusion $\Delta^{\mathrm{op}} \amalg \cdots \amalg \Delta^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}_{/[n]}$, sending a Segal ∞-bicategory $\mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ to its fibers $(\mathbb{M}_0, \ldots, \mathbb{M}_n)$, and ρ_n sends a tuple of Segal ∞-bicategories $(\mathbb{C}_0, \ldots, \mathbb{C}_n)$ to the join $\mathbb{C}_0 \star \cdots \star \mathbb{C}_n \to \Delta^{\mathrm{op}}_{/[n]}$.

Proposition 2.3.4.15. The map ρ_n is fully faithful.

Proof. It suffices to show that each component of the counit is an equivalence, which follows from the observation that

$$(\mathbb{C}_0 \star \cdots \star \mathbb{C}_n)_i \cong \mathbb{C}_i, \qquad 0 \leq i \leq n.$$

□

Note 2.3.4.16. A functor which admits a fully faithful right adjoint is called a reflective localization; we have just shown that λ_n is a reflective localization for all n.

Definition 2.3.4.17. We say that a Segal ∞-bicategory over the n-simplex is ***of join type*** if it is in the essential image of the functor ρ_n.

Example 2.3.4.18. By Proposition 2.3.4.13 and Proposition 2.3.4.15, the full subcategory of $(\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}}$ on those objects of join type is a reflective subcategory. A reflector for an object $\mathbb{M} \to \Delta^{\mathrm{op}}_{/[n]}$ is given by the unit

$$\begin{tikzcd} \mathbb{M} \arrow[rr, "r_{\mathbb{M}}"] \arrow[dr] & & \mathbb{M}_0 \star \cdots \star \mathbb{M}_n \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd},$$

which collapses the mapping categories not contained within a fiber. A morphism

$$\begin{tikzcd} \mathbb{M} \arrow[rr, "f"] \arrow[dr] & & \mathbb{N} \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd}$$

in $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}}$ is a weak equivalence if and only if f is a fiberwise equivalence, i.e. if $f_i \colon \mathbb{M}_i \to \mathbb{N}_i$ is an equivalence of Segal ∞-bicategories for each $0 \leq i \leq n$.

2.3.5 Coend calculus

In this section, we will denote by $q \colon \mathbb{D} \to \Delta^{\mathrm{op}}$ a locally cocomplete Segal ∞-bicategory.

Notation 2.3.5.1. Let $a \colon \mathbb{A} \to \Delta^{\mathrm{op}}$ and $b \colon \mathbb{B} \to \Delta^{\mathrm{op}}$ be Segal ∞-bicategories, and let $a \star b \colon \mathbb{A} \star \mathbb{B} \to \Delta^{\mathrm{op}}_{/[1]}$ be their join. Let $F \colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ be a lax functor. Then for each object x of a and each object y of b, there is a unique morphism in $a \star b$ from x to y. Denote the image of this morphism in d by $F_{x,y} \colon F(x) \to F(y)$.

Proposition 2.3.5.2. Let $a \colon \mathbb{A} \to \Delta^{\mathrm{op}}$, $b \colon \mathbb{B} \to \Delta^{\mathrm{op}}$, and $c \colon \mathbb{C} \to \Delta^{\mathrm{op}}$ be Segal ∞-bicategories, let $a \star b \star c \colon \mathbb{A} \star \mathbb{B} \star \mathbb{C} \to \Delta^{\mathrm{op}}_{/[2]}$ be their join, and let $F \colon \mathbb{A} \star \mathbb{B} \star \mathbb{C} \to \mathbb{D}$ be a lax functor of ∞-bicategories. Then F is extended from its spine if and only if for each object x of a and z of c, the diagram

$$
\begin{array}{c}
\vdots \\[2ex]
\operatorname*{colim}_{\substack{(f\colon y\to y', g\colon y'\to y'')\\ \in \mathbb{B}_2}} F_{y'',z}\circ F(g)\circ F(f)\circ F_{x,y} \\
\downarrow\uparrow\downarrow\uparrow\downarrow \\
\operatorname*{colim}_{f\colon y\to y'\in\mathbb{B}_1} F_{y',z}\circ F(f)\circ F_{x,y} \\
\downarrow\uparrow\downarrow \\
\operatorname*{colim}_{y\in\mathbb{B}_0} F_{y,z}\circ F_{x,y} \\
\downarrow \\
F_{x,z}
\end{array}
$$

is a colimit cone for $F_{x,z}$.

Proof. It follows from Observation 2.3.4.7 that

$$(\mathbb{A}\star\mathbb{B}\star\mathbb{C})_{(0,1,\ldots,1,2)} = \mathbb{A}_{[0]}\times\mathbb{B}_{[n]}\times\mathbb{C}_{[0]},$$

where $(0,1,\ldots,1,2)$ has $n+1$ copies of 1. Thus, an object $\Xi\in(\mathbb{A}\star\mathbb{B}\star\mathbb{C})_{(0,2)}$ is given by a pair $(x,z)\in\mathbb{A}_{[0]}\times\mathbb{C}_{[0]}$, and we have

$$
\begin{aligned}
((\mathbb{A}\star\mathbb{B}\star\mathbb{C})_{(0,1,\ldots,1,2)})_{/\Xi} &\simeq (\mathbb{A}_{[0]})_{/x}\times\mathbb{B}_{[n]}\times(\mathbb{C}_{[0]})_{/z}\\
&\simeq \mathbb{B}_{[n]},
\end{aligned}
$$

since $\mathbb{A}_{[0]}$ and $\mathbb{C}_{[0]}$ are by assumption ∞-groupoids. The result follows from Example 2.3.3.11. $\square$

Definition 2.3.5.3. We will abbreviate the colimit condition above by

$$F_{x,z} = \int^{y\colon\mathbb{B}} F_{y,z}\circ F_{x,y},$$

and call the symbol $\int^{y\colon\mathbb{B}}$ the ***generalized coend*** over $\mathbb{B}$.

Generalized coends commute with generalized coends:

Proposition 2.3.5.4. Let $a_i\colon\mathbb{A}_i\to\Delta^{\mathrm{op}}$ be a Segal ∞-bicategory for all $0\le i\le 3$ and let $F\colon\mathbb{A}_0\star\mathbb{A}_1\star\mathbb{A}_2\star\mathbb{A}_3\to\mathbb{D}$ be a lax functor of Segal ∞-bicategories. We will denote a generic object of a_i by x_i for each $i\in\{0,\ldots,3\}$. Then F is extended from its spine if and only if

$$F_{x_0,x_2} = \int^{x_1\colon\mathbb{A}_1} F_{x_1,x_2}\circ F_{x_0,x_1}, \qquad F_{x_1,x_3} = \int^{x_2\colon\mathbb{A}_2} F_{x_2,x_3}\circ F_{x_1,x_2},$$

and

$$F_{x_0,x_3} = \int^{x_1:\mathbb{A}_1} F_{x_1,x_3}\circ F_{x_0,x_1} = \int^{x_2:\mathbb{A}_2} F_{x_2,x_3}\circ F_{x_0,x_2}.$$

In particular, both of the latter coends agree (in the sense that they satisfy the same universal property), and we will denote them by

$$\int^{x_1\colon \mathbb{A}_1}\int^{x_2\colon \mathbb{A}_2} F_{x_2,x_3}\circ F_{x_1,x_2}\circ F_{x_0,x_1} = \int^{x_2\colon \mathbb{A}_2}\int^{x_1\colon \mathbb{A}_1} F_{x_2,x_3}\circ F_{x_1,x_2}\circ F_{x_0,x_1}.$$

Proof. We check the conditions of Example 2.3.3.12. The first two integrals correspond to condition (1).

We now show that the latter integrals correspond to condition (2). It follows from Observation 2.3.4.7 that

$$(\mathbb{A}_0\star\mathbb{A}_1\star\mathbb{A}_2\star\mathbb{A}_3)_{(0,1,\dots,1,2,\dots,2,3)} = (\mathbb{A}_0)_{[0]}\times(\mathbb{A}_1)_{[m]}\times(\mathbb{A}_2)_{[n]}\times(\mathbb{A}_4)_{[0]},$$

where there are $m+1$ copies of 1 and $n+1$ copies of 2, so

$$\big((\mathbb{A}_0\star\mathbb{A}_1\star\mathbb{A}_2\star\mathbb{A}_3)_{(0,1,\dots,1,2,\dots,2,3)}\big)_{/(x_0,x_3)} \simeq (\mathbb{A}_1)_{[m]}\times(\mathbb{A}_2)_{[n]}.$$

Condition (2) is thus that each F_{x_0,x_3} is the colimit of the diagram

$$\begin{tikzcd}
\vdots & \vdots & \iddots \\
\operatorname*{colim}\limits_{\substack{f\colon x_1\to x_1'\in(\mathbb{A}_1)_{[1]}\\ x_2\in(\mathbb{A}_2)_{[0]}}} F_{x_2,x_3}\circ F_{x_1',x_2}\circ F(f)\circ F_{x_0,x_1} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & \operatorname*{colim}\limits_{\substack{f\colon x_1\to x_1'\in(\mathbb{A}_1)_{[1]}\\ g\colon x_2\to x_2'\in(\mathbb{A}_2)_{[1]}}} F_{x_2',x_3}\circ F(g)\circ F_{x_1',x_2}\circ F(f)\circ F_{x_0,x_1} \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & \cdots \\
\operatorname*{colim}\limits_{\substack{x_1\in(\mathbb{A}_1)_{[0]}\\ x_2\in(\mathbb{A}_2)_{[0]}}} F_{x_2,x_3}\circ F_{x_1,x_2}\circ F_{x_0,x_1} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \operatorname*{colim}\limits_{\substack{x_1\in(\mathbb{A}_1)_{[0]}\\ g\colon x_2\to x_2'\in(\mathbb{A}_2)_{[1]}}} F_{x_2',x_3}\circ F(g)\circ F_{x_1,x_2}\circ F_{x_0,x_1} & \cdots
\end{tikzcd}.$$

Since $q\colon \mathbb{D}\to\Delta^{\mathrm{op}}$ is by assumption locally cocomplete, it follows from Proposition 2.2.3.12 that the composition in $\mathbb{D}$ preserves colimits, so we can pull each 'component' colimit inside the composition. Since colimits commute with colimits, we can compute the colimit of the entire diagram by first taking the colimits of the columns and then taking the colimit of the resulting diagram, or first taking the colimits of the rows. The former process yields

$$\int^{x_1:\mathbb{A}_1} F_{x_1,x_3}\circ F_{x_0,x_1},$$

and the latter yields

$$\int^{x_2:\mathbb{A}_2} F_{x_2,x_3}\circ F_{x_0,x_2}.$$

□

2.4 Lax matrices

In this section, we finally use the tools developed in the previous chapters (and the appendices) to construct a double ∞-category $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ of lax matrices in a cocomplete $(\infty,2)$-category $\mathbb{D}$, whose underlying 'horizontal' $(\infty,2)$-category $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ has

- Objects given by lax functors into $\mathbb{D}$,
- Morphisms given by lax matrices, and

- Composition given by lax matrix composition.

We also discuss some of the proven and expected properties of $\mathbb{L}\mathrm{axMat}(\mathbb{D})$. More precisely:

- In Subsection 2.4.1, we provide an elementary introduction to the basic structure of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ at the level of the homotopy double-category, and the underlying horizontal 2-category $\mathbb{L}\mathrm{axMat}(\mathbb{D})$.
- In Subsection 2.4.2, we finally construct $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ and $\mathbb{L}\mathrm{axMat}(\mathbb{D})$.
- In Subsection 2.4.3, we provide a brief overview of possible paths forward, including building bridges to previous work.

Note 2.4.0.1. Except in Subsection 2.4.2, where the actual construction of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$ takes place, we will use the model-independent vernacular described in Subsection 2.1.2.

2.4.1 Lax matrix calculus

In this subsection, we describe the rough structure of the double ∞-category $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$.

Lax functors

Recall that a Segal ∞-bicategory models an $(\infty, 2)$-category $\mathbb{A}$ with set of objects $\mathrm{ob}(A)$. Model-independently we should view this as an $(\infty, 2)$-category $\mathbb{A}$ together with an essentially surjective map $\mathrm{ob}(A) \to \mathbb{A}$. We describe below our reasons for considering flagged $(\infty, 2)$-categories rather than non-flagged ones.

Let $\mathbb{A}$ and $\mathbb{B}$ be $(\infty, 2)$-categories flagged by sets. A lax functor $F\colon \mathbb{A} \to \mathbb{B}$ consists, roughly, of the following data.

- A map $\mathrm{ob}(\mathbb{A}) \to \mathrm{ob}(\mathbb{B})$ sending $a \mapsto Fa$.
- For each objects a and a' of $\mathbb{A}$, a functor $F_{a,a'}\colon \mathbb{A}(a, a') \to \mathbb{B}(Fa, Fa')$.
- For each commuting triangle

$$\begin{tikzcd} & a' \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & a'' \end{tikzcd}$$

  in $\mathbb{A}$, a structure 2-morphism

$$\begin{tikzcd} & F(a') \arrow[dr, "F(g)"] \arrow[d, phantom, "\Downarrow"] & \\ F(a) \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"'] & {} & F(a'') \end{tikzcd}$$

  in $\mathbb{B}$.

- For each object a of $\mathbb{A}$, a structure 2-morphism $\mathrm{id}_{F(a)} \Rightarrow F\mathrm{id}_a$ in $\mathbb{B}(Fa, Fa)$.

- Higher data witnessing that the structure morphisms are associative and unital, together with higher coherences relating these, etc.

Several special cases of lax functors are of independent interest.

Example 2.4.1.1. Ordinary (non-lax) functors are a special case of lax functors in which the structure 2-morphisms are invertible.

Example 2.4.1.2. Unraveling the definition, a lax functor $\Delta^0 \to \mathbb{D}$ consists of the following data.

- An object $d \in \mathbb{D}$.
- An object $T \in \mathbb{D}(d, d)$, i.e. a morphism $T\colon d \to d$.
- A structure 2-morphism $\mu\colon T \circ T \Rightarrow T$.
- A structure 2-morphism $\epsilon\colon \mathrm{id}_d \Rightarrow T$
- Invertible 2-morphisms witnessing that μ is associative and that ϵ acts as a unit, together with higher coherences.

Thus, a lax functor $\Delta^0 \to \mathbb{D}$ is the same as a monad in $\mathbb{D}$.

Example 2.4.1.3. Let X be a set, and let ∇_X be the contractible groupoid with set of objects X. Let $(\mathcal{V}, \otimes)$ be a monoidal category, and let $B\mathcal{V}^{\otimes}$ be the corresponding $(\infty, 2)$-category with a single object. Then a lax functor $\nabla_X \to B\mathcal{V}^{\otimes}$ consists of:

- For each elements x, x' of X, an object $V(x, x')$ of $\mathcal{V}$.
- For each elements x, x' and x'' of X, a structure 2-morphism $V(x', x'') \otimes V(x, x') \to V(x, x'')$.
- For each element x of X, a structure 2-morphism $I \to V(x, x)$, where I is the unit object of $\mathcal{V}^{\otimes}$.
- Equivalences in $\mathcal{V}$ enforcing associativity, unitality, etc.

Thus, a lax functor $F\colon \nabla_X \to B\mathcal{V}^{\otimes}$ is the same thing as a $\mathcal{V}$-enriched category with set of objects X. This is the situation considered in [Hau16]; the generalized ∞-operad Δ_X^{op} is a Segal ∞-category modelling ∇_X in the case that X is a set, and otherwise models something like a contractible groupoid with space of objects X.

Note 2.4.1.4. Working with flagged $(\infty, 2)$-categories rather than 'bare' $(\infty, 2)$-categories allows us to talk about 'the contractible groupoid with set of objects A,' which model independently does not make any sense.

Let $F\colon \mathbb{A} \to \mathbb{D}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be lax functors. An *icon* from F to G consists, roughly, of the following data.

- The condition that F and G agree on objects (for a more precise phrasing, see Note 2.2.1.14).

- For each objects a and a' of $\mathbb{A}$, a natural transformation

$$\begin{tikzcd} \mathbb{A}(a,a') \arrow[r, bend left, "F_{a,a'}"] \arrow[r, bend right, "G_{a,a'}"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{B}(Fa,Fa') \end{tikzcd} .$$

- Higher data exhibiting compatibility of the icons with the structure morphisms of F and G encoding unitality and composition.

Lax matrices

Let $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{D}$ be set-flagged $(\infty,2)$-categories such that $\mathbb{D}$ is locally cocomplete, and let $F\colon \mathbb{A} \to \mathbb{D}$ and $G\colon \mathbb{B} \to \mathbb{D}$ be lax functors.

Definition 2.4.1.5. A ***lax matrix*** from F to G is a lax functor

$$\mathbb{A} \star \mathbb{B} \xrightarrow{S} \mathbb{D}$$

such that $S|\mathbb{A} = F$ and $S|\mathbb{B} = G$. If S is a lax matrix from F to G, we will write $S\colon F \rightsquigarrow G$.

Note 2.4.1.6. It is evocative to imagine $\mathbb{A}\star\mathbb{B}$ as living over a copy of Δ^1, such that the fiber over $\{0\}$ is $\mathbb{A}$ and the fiber over $\{1\}$ is $\mathbb{B}$. One then recovers F and G by base changing along the inclusion $\Delta^{\{i\}} \hookrightarrow \Delta^1$.

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[r, hook] \arrow[d] & \mathbb{A}\star\mathbb{B} \arrow[u, "S"] \arrow[d] & \mathbb{B} \arrow[ul, "G"'] \arrow[l, hook'] \arrow[d] \\ \Delta^{\{0\}} \arrow[r, hook] & \Delta^1 & \Delta^{\{1\}} \arrow[l, hook'] \end{tikzcd}$$

A *lax matrix* S from F to G then consists of the following data.

- For each object a of $\mathbb{A}$ and each object b of $\mathbb{B}$, a morphism $S_{a,b}\colon F(a) \to G(b)$.
- For each morphism $f\colon a' \to a$ in $\mathbb{A}$, a 2-morphism $S_{a',b} \Rightarrow S_{a,b} \circ F(f)$.
- For each morphism $g\colon b \to b'$ in $\mathbb{B}$, a 2-morphism $S_{a,b'} \Rightarrow F(g) \circ S_{a,b}$.
- Higher coherences...

Example 2.4.1.7. Let $X = \{0,1\}$ be a set with two elements, and let F, $G\colon X \to \mathbb{D}$ be strict functors. Then a lax matrix S from F to G consists of four morphisms

$$\begin{tikzcd} F(0) \arrow[r, "a"] \arrow[dr, "c"] & G(0) \\ F(1) \arrow[r, "d"'] \arrow[ur, "b"] & G(1) \end{tikzcd} ,$$

which we can write in the more familiar form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Example 2.4.1.8. Let F, $F'\colon \Delta^0 \to \mathbb{D}$ be lax functors modelling monads (d,T) and (d',T'). Since $\Delta^0 \star \Delta^0 = \Delta^1$, a lax matrix $S\colon F \rightsquigarrow F'$ is given by a lax functor $S\colon \Delta^1 \to \mathbb{D}$ such that $S|\{0\} = F$ and $S|\{1\} = F'$. This consists of the data of the monads (d,T) and (d',T'), together with a morphism $M\colon d \to d'$ in $\mathbb{D}$, 2-morphisms $M \circ T \Rightarrow M$ and $T' \circ M \Rightarrow M$, coherences $(M \circ T) \circ T \simeq M \circ (T \circ T)$, etc. That is, a lax matrix from F to F' is the same thing as a (T,T')-bimodule.

Construction 2.4.1.9. Given a functor $F\colon \mathbb{A} \to \mathbb{D}$, we construct the ***identity matrix*** I_F on F as follows.

- We view $\mathbb{A}$ as living over a copy of Δ^0, and pull it back to Δ^1.[1]

$$\begin{tikzcd} \mathbb{A} \times \Delta^1 \arrow[r] \arrow[d] & \mathbb{A} \arrow[r] \arrow[d] & \mathbb{D} \\ \Delta^1 \arrow[r] & \Delta^0 & \end{tikzcd}$$

- We take a local left Kan extension along the map $\mathbb{A} \times \Delta^1 \to \mathbb{A} \star \mathbb{A}$:

$$\begin{tikzcd} \mathbb{A} \times \Delta^1 \arrow[r, "F \circ \mathrm{pr}_{\mathbb{A}}"] \arrow[d, "r"'] & \mathbb{D} \\ \mathbb{A} \star \mathbb{A} \arrow[ur, "I_F"'] \arrow[u, phantom, shift left=4, "\overset{\eta}{\Leftarrow}"] & \end{tikzcd} .$$

Definition 2.4.1.10. Let $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ be a lax matrix from $F = S|\{0\}$ to $G = S|\{1\}$. For any object a of $\mathbb{A}$, and any object b of $\mathbb{B}$, we denote the image of the unique morphism $a \to b$ in $\mathbb{A} \star \mathbb{B}$ under F by $S_{a,b}\colon F(a) \to G(b)$, and call it the ***component*** of S from a to b.

Example 2.4.1.11. Using Proposition 2.2.4.4 we can compute that the lax matrix I_F has components

$$(I_F)_{a,a'} = \operatorname*{colim}_{\substack{f\colon a \to a' \\ \in \mathbb{A}(a,a')}} F(f).$$

Definition 2.4.1.12. Let $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ and $T\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ be lax matrices from F to G. A ***morphism of lax matrices*** is given by an icon $S \Rightarrow T$. Unravelling the definition, this is the data of:

- For each $(a,b) \in \mathrm{ob}(\mathbb{A}) \times \mathrm{ob}(\mathbb{B})$, a 2-morphism $S_{a,b} \Rightarrow T_{a,b}$ in $\mathbb{B}(F(a), F(b))$.
- Higher data exhibiting compatibility between the 2-morphisms of the icon and the pre- and postcomposition within $\mathbb{A}$ and $\mathbb{B}$.

Heuristically, a morphism of lax matrices is simply a component-wise morphism.

[1] In Segal ∞-bicategories, the 1-simplex is modelled by $u_1\colon \Delta^{\mathrm{op}}_{/[1]} \to \Delta^{\mathrm{op}}$, and the product $\mathbb{A} \times \Delta^1$ by the fiber product $\mathbb{A} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{/[1]}$, where $\mathbb{A} \to \Delta^{\mathrm{op}}$ is a Segal ∞-bicategorical model for $\mathbb{A}$. For more information, see the dictionary in Subsection 2.1.2.

Matrix multiplication

In this section, let $\mathbb{D}$ denote a locally cocomplete set-flagged $(\infty, 2)$-category.

Construction 2.4.1.13. Let $F\colon \mathbb{A} \to \mathbb{D}$, $G\colon \mathbb{B} \to \mathbb{D}$, and $H\colon \mathbb{C} \to \mathbb{D}$ be lax functors, and let $S\colon F \rightsquigarrow G$ and $T\colon G \rightsquigarrow H$ be lax matrices. We define the ***composition*** of S and T, denoted $T \circ_G S\colon F \rightsquigarrow H$ by the functor $\mathbb{A} \star \mathbb{C} \to \mathbb{D}$ constructed in the following three-step process.

(1) We glue the diagrams defining the lax matrices S and T together. Note that since each join admits a natural map to Δ^1, the corresponding pushout naturally lives over $\mathrm{Spine}(\Delta^2) = \Lambda^2_1$.

$$\begin{tikzcd} \mathbb{A} \star \mathbb{B} \coprod_{\mathbb{B}} \mathbb{B} \star \mathbb{C} \arrow[r, "(S,T)"] \arrow[d] & \mathbb{D} \\ \Lambda^2_1 & \end{tikzcd} .$$

(2) Using local left Kan extension, we extend the corresponding pushout to a full 3-fold join $\mathbb{A} \star \mathbb{B} \star \mathbb{C}$, which we should imagine as living over Δ^2.

$$\begin{tikzcd} \mathbb{A} \star \mathbb{B} \coprod_{\mathbb{B}} \mathbb{B} \star \mathbb{C} \arrow[rr, "S \amalg_G T"] \arrow[d] \arrow[dr, hook, "i"'] & & \mathbb{D} \\ \Lambda^2_1 \arrow[dr, hook] & \mathbb{A} \star \mathbb{B} \star \mathbb{C} \arrow[ur, "i_!(S \amalg_G T)"'] \arrow[d] & \\ & \Delta^2 & \end{tikzcd}$$

(3) We restrict the resulting functor to $\mathbb{A} \star \mathbb{C} \subseteq \mathbb{A} \star \mathbb{B} \star \mathbb{C}$, which we imagine as restricting to $\Delta^{\{0,2\}} \subseteq \Delta^2$.

$$\begin{tikzcd} & \mathbb{A} \star \mathbb{B} \coprod_{\mathbb{B}} \mathbb{B} \star \mathbb{C} \arrow[rr, hook, "(S,T)"] \arrow[dd] \arrow[dr, hook, "i"'] & & \mathbb{D} \\ \mathbb{A} \star \mathbb{C} \arrow[rr] \arrow[dd] & & \mathbb{A} \star \mathbb{B} \star \mathbb{C} \arrow[ur, "i_!(S,T)"'] \arrow[dd] & \\ & \Lambda^2_1 \arrow[dr, hook] & & \\ \Delta^{\{0,2\}} \arrow[rr, hook] & & \Delta^2 & \end{tikzcd}$$

Denoting the components of the lax matrices S and T by $S_{a,b}$ and $T_{b,c}$ respectively, It follows from Proposition 2.3.5.2 that the lax matrix $T \circ_G S$ has components

$$(T \circ_G S)_{a,c} = \int^{b\colon \mathbb{B}} T_{b,c} \circ S_{a,b}. \tag{2.4.1.1}$$

Here the symbol $\int^{b\colon \mathbb{B}}$ denotes the *generalized coend* of Definition 2.3.5.3.

Example 2.4.1.14. Let $F_i\colon \Delta^{\{i\}} \to \mathbb{D}$ be lax functors for $0 \leq i \leq 2$, modelling monads (d_i, T_i).

Let $M_{01}\colon \Delta^{\{0,1\}} \to \mathbb{D}$ be a lax matrix modelling a (T_0, T_1)-bimodule, and let $M_{12}\colon \Delta^{\{1,2\}} \to \mathbb{D}$ be a lax matrix modelling a (T_1, T_2)-bimodule. The matrix multiplication $M_{12} \circ_{T_1} M_{01}$ is then a (T_0, T_2)-bimodule with component $M_{02} = \int^{1:\Delta^{\{1\}}} M_{12} \circ M_{01}$. According to Proposition 2.3.5.2, this coend can be computed as the colimit of the solid simplicial diagram

$$\begin{tikzcd}
\vdots \\
M_{12} \circ T_1 \circ T_1 \circ M_{01} \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] \\
M_{12} \circ T_1 \circ M_{01} \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[u, shift left=2] \arrow[u, shift right=2] \\
M_{12} \circ M_{01} \arrow[u] \arrow[d, dashed] \\
M_{02}
\end{tikzcd}.$$

This is the standard definition of the tensor product $M_{01} \otimes_T M_{12}$ of the bimodules M_{01} and M_{12}.

Proposition 2.4.1.15. Matrix multiplication is unital: for any lax matrix $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$ between $F = S|\{0\}$ and $G = S|\{1\}$, we have that

$$S \circ_F I_F \simeq S \qquad \text{and} \qquad I_G \circ_G S \simeq S$$

Proof. We prove the first; the second is similar. Consider the diagram

$$\begin{tikzcd}
(\mathbb{A} \times \Delta^1) \amalg_{\mathbb{A}} (\mathbb{A} \star \mathbb{B}) \arrow[r, hook, "i"] \arrow[d, "s"'] & (\mathbb{A} \times \Delta^1) \star \mathbb{B} \arrow[r, "\pi"] \arrow[d, "t"] & \mathbb{A} \star \mathbb{B} \arrow[r, "F"] \arrow[dl, phantom, "\Leftarrow\alpha"] & \mathbb{D} \\
(\mathbb{A} \star \mathbb{A}) \amalg_{\mathbb{A}} (\mathbb{A} \star \mathbb{B}) \arrow[r, hook, "j"] & \mathbb{A} \star \mathbb{A} \star \mathbb{B} \arrow[urr, "G"'] & &
\end{tikzcd},$$

where the maps i and j are spine inclusions (so $s = \mathrm{Spine}(t)$), π is induced by the canonical projection, and α exhibits G as the generalized operadic Kan extension of $F \circ \pi$ along t.

By Lemma 2.3.3.15, the restriction of α to $(\mathbb{A} \times \Delta^1) \amalg_{\mathbb{A}} (\mathbb{A} \star \mathbb{B})$ exhibits $G \circ j$ as a generalized operadic Kan extension of $F \circ \pi \circ i$ along s. Thus, by definition of the identity matrix, $G \circ j = I_F \amalg_F F$.

By Lemma 2.3.3.16, G is extended from its spine, so by the definition of matrix composition, $S \circ_F I_F$ is given by the restriction of G along the inclusion $k\colon \mathbb{A} \star \mathbb{B} \hookrightarrow \mathbb{A} \star \mathbb{A} \star \mathbb{B}$ lying over $\Delta^{\{0,2\}} \hookrightarrow \Delta^2$. But using Proposition 2.2.4.4, we can also compute G directly via local left Kan extension; the matrix component of $S \circ_F I_F$ at (a, b) is given by left Kan extension of the map $\Delta^0 \to \mathbb{D}(F(a), G(b))$ picking out $S_{a,b}$ along the map id_{Δ^0}, and is therefore also given by $S_{a,b}$. ☐

Proposition 2.4.1.16. Lax matrix multiplication is associative: let $S\colon \mathbb{A} \star \mathbb{B} \to \mathbb{D}$, $T\colon \mathbb{B} \star \mathbb{C} \to \mathbb{D}$, and $U\colon \mathbb{C} \star \mathbb{M} \to \mathbb{D}$ be composable lax matrices $F \rightsquigarrow G \rightsquigarrow H \rightsquigarrow J$. Then

$$(U \circ_H T) \circ_G S \simeq U \circ_H (T \circ_G S).$$

Proof. We have

$$\begin{aligned}((U \circ_H T) \circ_G S)_{a,m} &\simeq \int^{b\colon \mathbb{B}} \left(\int^{c\colon \mathbb{C}} U_{c,m} \circ T_{b,c} \right) \circ S_{a,b} \\ &\simeq \int^{c\colon \mathbb{C}} U_{c,m} \circ \left(\int^{b\colon \mathbb{B}} T_{b,c} \circ S_{a,b} \right) \\ &\simeq (U \circ_H (T \circ_G S))_{a,m},\end{aligned}$$

where the second equivalence follows from Proposition 2.3.5.4. □

2.4.2 The double ∞-category of lax matrices

In this section, we show the following.

Theorem 2.4.2.1. Let $p\colon \mathbb{D} \to \Delta^{\mathrm{op}}$ be a locally cocomplete Segal ∞-bicategory. Then there exists a double ∞-category $\mathfrak{LaxMat}(\mathbb{D})$ of lax matrices in $\mathbb{D}$, which agrees with the truncated description given in Subsection 2.4.1.

Construction 2.4.2.2. Let $\mathbb{D} \to \Delta^{\mathrm{op}}$ be a locally cocomplete Segal ∞-bicategory. Denote by $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}}$ the *oplax overcategory,*[2] i.e. the $(\infty, 2)$-category whose objects are morphisms $\mathbb{A} \to \mathbb{D}$ in $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$, and whose morphisms are diagrams

$$\begin{tikzcd} \mathbb{A} \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathbb{B} \arrow[dl] \\ & \mathbb{D} & \end{tikzcd} .$$

Then the forgetful functor $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}} \to \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ is a $(1,0)$-*fibration* in the sense of [AM24, Def. 2.2.5], and it follows easily from the discussion there that the projection $\pi\colon (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)} \to \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)}$ is a cartesian fibration, where an edge as pictured above is π-cartesian if and only if the 2-morphism α is an equivalence.

Define an ∞-category $\mathcal{X}'$ together with a functor $p\colon \mathcal{X}' \to \Delta$ as a limit of the solid diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[rrr, dashed] \arrow[dd, dashed, "p'"'] & & & (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}} \arrow[d] \\ & \mathrm{Fun}(\Delta^1, \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)}) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)} \arrow[r, hook] & \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)} \\ \Delta \arrow[r, "{[n] \mapsto \Delta^{\mathrm{op}}_{/[n]}}"] & \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)} & & \end{tikzcd} .$$

Then p' is a cartesian fibration classifying a functor

$$\Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad [n] \mapsto \mathcal{X}'_n = \left(\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)}\right)_{/\Delta^{\mathrm{op}}_{/[n]}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} \left(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}\right)_{\nearrow \mathbb{D}} .$$

[2] To be completely rigorous, we should allow that the hom-∞-categories of $\mathbb{D}$ are large, and demand that the $(\infty, 2)$-categories $\mathbb{A}$, $\mathbb{B}$,... be locally small, so that all of the necessary colimits exist. To avoid introducing extraneous notation encoding this information, only never to make explicit use of it, we state it only here, for once and for all.

An object of $\mathcal{X}'$ lying over $[n] \in \Delta$ is a diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "a"'] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}$$

in $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$ such that a is a Segal ∞-bicategory over the n-simplex and F is a lax functor of Segal ∞-bicategories, and a morphism in $\mathcal{X}'$ lying over $\phi\colon [m] \to [n]$ is a diagram

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "H"'] \arrow[d, "a"'] & \arrow[phantom, u, "\overset{\alpha}{\Longrightarrow}"] & \mathbb{B} \arrow[ul, "G"'] \arrow[d, "b"] \\ \Delta^{\mathrm{op}}_{/[m]} \arrow[rr, "\phi_*"] & & \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}$$

such that H is a lax functor of Segal ∞-bicategories and $\alpha\colon F \Rightarrow G \circ H$ is an icon. Such a morphism is p'-cartesian if and only if the lower square is pullback and α is an equivalence (i.e. an icon each of whose components is a natural equivalence).

Definition 2.4.2.3. For each $n \geq 0$, denote by $\mathcal{X}_n \subseteq \mathcal{X}'_n$ the smallest subcategory containing:

(1) Those objects

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "a"'] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} \end{tikzcd}$$

such that F is extended from its spine.

(2) Those morphisms Ξ given by a diagram

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "H"'] \arrow[dr, "a"'] & \arrow[phantom, u, "\overset{\alpha}{\Longrightarrow}"] & \mathbb{B} \arrow[ul, "G"'] \arrow[dl, "b"] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd}$$

such that H is a fiberwise equivalence. That is, such that for each $0 \leq i \leq n$ and each map $\phi_i\colon [0] \xrightarrow{i} [n]$ in Δ, the morphism $\phi^*(\Xi)$ in $\mathcal{X}_0$ is given by a diagram

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A}_i \arrow[ur, "F_i"] \arrow[rr, "H_i"'] \arrow[dr, "a_i"'] & \arrow[phantom, u, "\overset{\alpha_i}{\Longrightarrow}"] & \mathbb{B}_i \arrow[ul, "G_i"'] \arrow[dl, "b"] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

such that H_i is an equivalence of Segal ∞-bicategories.

Proposition 2.4.2.4. For each morphism $\phi\colon [m] \to [n]$ in Δ, the map $\phi^*\colon \mathcal{X}'_n \to \mathcal{X}'_m$ has the property that $\phi^*(\mathcal{X}_n) \subseteq \mathcal{X}_m$, so the cartesian fibration $p'\colon \mathcal{X}' \to \Delta$ restricts to a cartesian fibration $p\colon \mathcal{X} \to \Delta$.

Proof. By Proposition 2.3.3.13, base change preserves property (1). That base change preserves property (2) follows easily from Lemma 2.3.1.7. □

Definition 2.4.2.5. For each $n \geq 0$, denote by $\mathcal{Y}_n \subseteq \mathcal{X}_n$ the full subcategory on those objects

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "a"'] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd}$$

such that $\mathbb{A}$ is of join type.

Proposition 2.4.2.6. For each $n \geq 0$, the inclusion $\mathcal{Y}_n \subseteq \mathcal{X}_n$ exhibits $\mathcal{Y}_n$ as a reflective subcategory of $\mathcal{X}_n$.

Proof. By item (2) of Note A.2.1.3, we need to show that each object in $\mathcal{X}_n$ admits a reflector, i.e. that for each object $x \in \mathcal{X}_n$, the category $(\mathcal{Y}_n)_{x/}$ has an initial object. Suppose x is given by the object

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "a"'] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} .$$

We claim that there is an initial object $\rho_x\colon x \to y_x$ in $(\mathcal{Y}_n)_{x/}$, given by a morphism

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "r_{\mathbb{A}}"'] \arrow[dr] & \arrow[phantom, u, "\overset{\eta}{\Longrightarrow}"] & \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \arrow[ul, "\bar{F}"'] \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} ,$$

in $\mathcal{X}_n$ where the lower triangle is a reflector in the sense of Example 2.3.4.18, and the upper triangle is a generalized operadic Kan extension (this is guaranteed to exist by Conjecture 2.2.4.5, together with the assumption that $\mathbb{D}$ is locally cocomplete). The functor $r_{\mathbb{A}}$ is clearly a fiberwise equivalence, and that $\bar{F}$ is extended from its spine is a consequence of Lemma 2.3.3.16, so ρ_x really is an object of $(\mathcal{Y}_n)_{x/}$.

Unraveling the definitions, it suffices to show that for each $m \geq 2$ and each solid diagram

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "\rho_x"] & \\
\Lambda^m_0 \arrow[r, "u"] \arrow[d, hook] & (\mathbb{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}} \\
\Delta^m \arrow[ur, dashed, "\bar{u}"'] &
\end{tikzcd},$$

a dashed filler $\bar{u}$ exists, and that any such dashed filler necessarily specifies an $(m-2)$-simplex in $(\mathcal{Y}_n)_{x/}$. Let us first concentrate on finding the fillers. To specify $\bar{u}$, we should find fillers into each factor of the fiber product which agree when mapped down to $\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}$. We can find a filler in $(\mathbb{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}}$ under our assumption that the lower triangle of ρ_x is a reflector. We thus need to solve the lifting problem

$$\begin{tikzcd}
\Lambda^m_0 \arrow[r] \arrow[d, hook] & (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow \mathbb{D}} \arrow[d] \\
\Delta^m \arrow[r, "u'"] \arrow[ur, dashed] & \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}
\end{tikzcd}$$

where u' comes from our filler on the first factor. Equivalently, we should solve the filling problem

$$\begin{tikzcd}
\Delta^{\{0,1,m+1\}} \arrow[d, hook] \arrow[dr, "w"] & \\
\Lambda^{m+1}_0 \arrow[r] \arrow[d, hook] & \mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)} \\
\Delta^{m+1} \arrow[ur, dashed] &
\end{tikzcd},$$

where w is the upper triangle of r_x. Such a filler exists by our assumption that w is an operadic left Kan extension together with Proposition A.1.0.3.

We now need to check that our filler specifies an $(m-2)$-simplex in $(\mathcal{Y}_n)_{x/}$, by checking that $\bar{u}$ satisfies the conditions of Definition 2.4.2.3, and that $\bar{u}|\Delta^{\{1,\ldots,m\}}$ is contained within $\mathcal{Y}_n$.

- Condition (1) of Definition 2.4.2.3 is a condition on the 0-simplices of $\bar{u}$, which are already contained in Λ^m_0 for all $m \geq 2$.
- Condition (2) of Definition 2.4.2.3 is satisfied because equivalences satisfy 2/3.
- The condition that $\bar{u}|\Delta^{\{1,\ldots,m\}}$ belong to $\mathcal{Y}_n$ is also a condition on the 0-simplices of $\bar{u}$, which are already contained in Λ^m_0 for all $m \geq 2$.

□

We thus have the existence of enough reflectors. By the material of Appendix A.2.1, the inclusion $\mathcal{Y}_n \subseteq \mathcal{X}_n$ has a left adjoint $\lambda_n \colon \mathcal{X}_n \to \mathcal{Y}_n$ with the following description.

- The functor λ_n is defined on objects by sending an object $x \in \mathcal{X}_n$ to the target y_x of a reflector $r_x \colon x \to y_x$, i.e.

$$\lambda_n \colon \left(\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \arrow[r, "\lambda_n F"] \arrow[d] & \mathbb{D} \\ \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} \right)$$

The functor $\lambda_n F$ has the following description. By Proposition 2.2.4.4, $\lambda_n F$ agrees with F on the objects of $\mathbb{A}$, and for each $m \in \mathbb{A}_i$ and $m' \in \mathbb{A}_j$ has the following description on mapping categories

- For $i < j$, the mapping category $(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n)(m, m')$ can be identified with the contractible space $*$, and $(\lambda_n F)_{m,m'} \colon * \to \mathbb{D}(Fm, Fm')$ picks out the object $\operatorname{colim}_{a \in \mathbb{A}(m,m')} Fa$ since it is given by the left Kan extension of $F_{m,m'}$ along the map $\mathbb{A}(m, m') \to *$.
- For $i = j$, the objects m and m' live in the same fiber, so the map $\mathbb{A}(m, m') \to (\mathbb{A}_0 \star \cdots \star \mathbb{A}_n)(m, m')$ is an equivalence of ∞-categories, and $(\lambda_n F)_{m,m'}$ agrees with $F_{m,m'}$.
- For $i > j$, the mapping category $(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n)(m, m')$ is empty.

• The functor λ_n is defined on morphisms by

$$\lambda_n \colon \left(\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "f"'] \arrow[dr] & \arrow[d, phantom, "\Rightarrow \alpha"] & \mathcal{N} \arrow[ul, "G"'] \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A}_0 \star \cdots \star \mathbb{A}_n \arrow[ur, "\lambda_n F"] \arrow[rr, "f_0 \star \cdots \star f_n"'] \arrow[dr] & \arrow[d, phantom, "\Rightarrow \lambda_n \alpha"] & \mathcal{N}_0 \star \cdots \star \mathcal{N}_n \arrow[ul, "\lambda_n G"'] \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} \right). \tag{2.4.2.1}$$

Note that since f is by assumption a fiberwise equivalence, the map $f_0 \star \cdots \star f_n$ is an equivalence. The only new information here is the icon $\lambda_n \alpha$, whose component natural transformations we now describe. For each $m \in \mathbb{A}_i$, and $m' \in \mathbb{A}_j$,

- If $i < j$, the component $(\lambda_n \alpha)_{m,m'}$ is a natural transformation between functors whose domain is a contractible space, determining an essentially unique (and slightly abusively named) morphism

$$(\lambda_n \alpha)_{m,m'} \colon \operatorname*{colim}_{a \in \mathbb{A}(m,m')} F(a) \to \operatorname*{colim}_{b \in \mathcal{N}(fm, fm')} G(b)$$

  in $\mathbb{D}(Fm, Fm')$.
- If $i = j$, them m and m' belong to the same fiber, so the component $(\lambda_n \alpha)_{m,m'}$ agrees with $\alpha_{m,m'}$, and is therefore a natural equivalence by assumption.
- For $i > j$, $(\lambda_n \alpha)_{m,m'}$ is a natural transformation between functors out of the mapping space $(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n)(m, m')$, which is empty, and hence carries no information.

A morphism in $\mathcal{X}_n$ is a weak equivalence if its image under λ_n is an equivalence. The morphism on the right-hand side of Equation 2.4.2.1 is an equivalence if and only if each component of the icon $\lambda_n\alpha$ is a natural equivalence, i.e. if each morphism $(\lambda_n\alpha)_{m,m'}$ is an equivalence. Thus:

Proposition 2.4.2.7. A morphism in $\mathcal{X}_n$ given by a diagram

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "f"'] \arrow[dr] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathbb{B} \arrow[ul, "G"'] \arrow[dl] \\ & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd} \tag{2.4.2.2}$$

is a weak equivalence if and only if for all objects m and m' of $\mathbb{A}$, the component $\alpha_{m,m'}$ of the icon α induces an equivalence from the colimit of the source functor to the colimit of the target functor.

$$\begin{tikzcd} \mathbb{A}(m,m') \arrow[r, bend left, "F_{m,m'}"{name=U}] \arrow[r, bend right, "(G\circ f)_{m,m'}"'{name=D}] & \mathbb{D}(Fm,Fm') \arrow[from=U, to=D, phantom, "\Downarrow \alpha_{m,m'}"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \operatorname{colim} F_{m,m'} \arrow[d, dashed, "\simeq"] \\ \operatorname{colim}\,(G\circ f)_{m,m'} \end{tikzcd}$$

Corollary 2.4.2.8. The weak equivalences in each $\mathcal{X}_n$ are preserved by base change.

Proof. Consider a morphism in $\mathcal{X}_n$ as in Equation 2.4.2.2 (the back right face of the below diagram), and its base change under $\phi\colon [k] \to [n]$ (the front left-top face). Suppose that the original morphism is a weak equivalence, i.e. that each component of the icon α induces an equivalence from the colimit of its source to the colimit of its target.

$$\begin{tikzcd} & & & & \mathbb{D} & \\ \phi^*\mathbb{A} \arrow[urrrr, bend left, "F\circ a"] \arrow[rrr, "a"] \arrow[drr, "\phi^* f"'] \arrow[ddr] & & & \mathbb{A} \arrow[ur, "F"] \arrow[drr, "f"] \arrow[ddr] & & \\ & & \phi^*\mathbb{B} \arrow[uurr, bend left, "G\circ b"] \arrow[rrr, "b"] \arrow[dl] \arrow[u, phantom, "\Downarrow \phi^*\alpha"] & & & \mathbb{B} \arrow[uul, "G"'] \arrow[dl] \arrow[ul, phantom, "\overset{\alpha}{\Rightarrow}"] \\ & \Delta^{\mathrm{op}}_{/[k]} \arrow[rrr] & & & \Delta^{\mathrm{op}}_{/[n]} & \end{tikzcd}$$

We would like to show that each component of the icon $\phi^*\alpha$ yields an equivalence upon taking colimits. But each component of the icon $\phi^*\alpha$, say between objects m and m' in $\phi^*\mathbb{A}$, can be identified with the component of the icon α between $a(m)$ and $a(m')$, and the components of α yield equivalences upon taking colimits by assumption. $\square$

Thus, we are in a position to read off the following corollary of Proposition A.2.2.2.

Corollary 2.4.2.9. There is a functor

$$\mathcal{Y}\colon \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad [n] \mapsto \mathcal{Y}_n,$$

classified by a cocartesian fibration $\mathfrak{L}\mathrm{axMat}(\mathbb{D}) \to \Delta^{\mathrm{op}}$.

Proposition 2.4.2.10. The cocartesian fibration $\mathfrak{L}\mathrm{axMat}(\mathbb{D}) \to \Delta^{\mathrm{op}}$ is a double ∞-category.

Proof. We will show that the Segal map

$$\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[n]} \to \mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[1]} \times_{\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[0]}} \cdots \times_{\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[0]}} \mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[1]}$$

is an equivalence by showing that it is an equivalence of cartesian fibrations over $\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[0]}^{n+1}$, so we first show that both of the necessary maps really are cartesian fibrations.

We first note that for each $n \geq 0$ the composite map

$$\mathcal{X}'_n = (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow\mathbb{D}} \to (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \xrightarrow{\lambda_n} (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1}$$

is a cartesian fibration. That the first map is a cartesian fibration follows from the fact that $(\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow\mathbb{D}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)} \to \mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)}$ is a cartesian fibration, and that λ_n is a cartesian fibration follows from the fact that λ_n admits a fully faithful right adjoint and hence (identifying $(\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1}$ with its image under ρ_n) a section.

We need to show that for each $m \geq 1$ we can always produce dashed lifts as in the diagram

$$\begin{tikzcd}
\Lambda^m_m \arrow[r] \arrow[d, hook] & \mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[n]} \arrow[r, hook, "i"] \arrow[d, "r'_n"] & (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})_{/\Delta^{\mathrm{op}}_{/[n]}} \times_{\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)}} (\mathbb{S}\mathrm{eg}^{\mathrm{lax,ic}}_{(\infty,2)})_{\nearrow\mathbb{D}} \arrow[d, "r_n"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & ((\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{\simeq})^{n+1} \arrow[r, hook] & (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1}
\end{tikzcd}$$

such that the image of $\Delta^{\{m-1,m\}}$ is cartesian. We can do this by producing lifts in the outer diagram, and then showing that they factor through the inclusion i, which in all cases follows easily by checking the produced lifts always satisfy the conditions of Definition 2.4.2.3 and Definition 2.4.2.5.

This shows that $\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[n]} \to (\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1}$ is a cartesian fibration for all $n \geq 0$. Thus, the Segal map

$$\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[n]} \to \mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[1]} \times_{\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[0]}} \cdots \times_{\mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[0]}} \mathfrak{L}\mathrm{axMat}(\mathbb{D})_{[1]}$$

is a map of cartesian fibrations over $(\mathcal{S}\mathrm{eg}^{\mathrm{lax}}_{(\infty,2)})^{n+1}$ (which manifestly preserves cartesian edges), so in order to show it is an equivalence, it suffices to show that the induced map on fibers is an equivalence. The fiber of the first map over $(\mathbb{A}_0, \ldots, \mathbb{A}_n)$ is given by the full subcategory of $\mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n, \mathbb{D})$ on those lax functors which are extended from their spine; we will denote this category by $\mathbb{G}\mathrm{enOp}^{\mathrm{ext}}_\infty(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n, \mathbb{D})$.

The map on the fiber over $(\mathbb{A}_0, \ldots, \mathbb{A}_n)$ is then given the composition

$$\begin{aligned}
\mathbb{G}\mathrm{enOp}^{\mathrm{ext}}_\infty(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n, \mathbb{D}) &\xrightarrow{\simeq} \mathbb{G}\mathrm{enOp}_\infty(\mathrm{Spine}(\mathbb{A}_0 \star \cdots \star \mathbb{A}_n), \mathbb{D}) \\
&\xrightarrow{\simeq} \mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_0 \star \mathbb{A}_1 \amalg_{\mathbb{A}_1} \cdots \amalg_{\mathbb{A}_{n-1}} \mathbb{A}_{n-1} \star \mathbb{A}_n, \mathbb{D}) \\
&\xrightarrow{\simeq} \mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_0 \star \mathbb{A}_1, \mathbb{D}) \times_{\mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_1, \mathbb{D})} \cdots \times_{\mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_{n-1}, \mathbb{D})} \mathbb{G}\mathrm{enOp}_\infty(\mathbb{A}_{n-1} \star \mathbb{A}_n, \mathbb{D}),
\end{aligned}$$

where the first map is an equivalence by definition of spine extension, and the second and third by Proposition 2.3.2.2. $\square$

One recovers the $(\infty,2)$ category $\mathbb{L}\mathrm{axMat}(\mathbb{D}) \to \Delta^{\mathrm{op}}$ by taking the underlying 'horizontal' $(\infty,2)$-category of $\mathfrak{L}\mathrm{axMat}(\mathbb{D})$.

2.4.3 Outlook

The homotopy category of lax matrices; lax limits and lax colimits

In [CDW24], a calculus of lax matrices between diagrams in a locally cocomplete $(\infty,2)$-category $\mathbb{D}$ parametrized by strict functors of $(\infty,1)$-categories is considered. The definition of a lax matrix given there is a priori somewhat different from ours: given strict functors $F\colon \mathcal{A} \to \mathbb{D}$ and $G\colon \mathcal{B} \to \mathbb{D}$, the ∞-category of lax matrices from F to G is

$$\mathrm{LaxMat}^{\mathrm{CDW}}(\mathbb{D})(F,G) = \operatorname*{laxlim}_{(a,b)\colon \mathcal{A}^{\mathrm{op}}\times\mathcal{B}} \mathbb{D}(F(a),G(b)).$$

Here, the lax limit is taken in $\mathbb{C}\mathrm{at}_\infty$. There is an explicit formula for computing lax limits in $\mathbb{C}\mathrm{at}_\infty$, given by taking sections of the covariant Grothendieck construction; unraveling the definitions, one sees that this definition of a lax matrix encodes the same information as Definition 2.4.1.5.

Also in [CDW24], a version of matrix multiplication is constructed by manipulating the Grothendieck constructions encoding the lax matrices. The end result amounts to taking a hom-wise coend, and reproduces the formula of Equation 2.4.1.1. This matrix multiplication allows the definition of a 1-category whose objects are functors $F\colon \mathcal{A} \to \mathbb{D}$, and whose morphisms are equivalence classes of lax matrices in $\mathrm{LaxMat}^{\mathrm{CDW}}(\mathbb{D})(F,G)$. It is mentioned there that this 1-category is a shadow of an $(\infty,2)$-category of lax matrices. The purpose of this project was to construct this $(\infty,2)$-category.

The use of $\mathrm{hoLaxMat}^{\mathrm{CDW}}(\mathbb{D})$ in [CDW24] is to prove, in analogy to the case of semiadditive categories, that a calculus of lax matrices encodes lax limits and colimits: that a lax cone is a lax (co)limit cone if and only if it is an isomorphism in $\mathrm{hoLaxMat}^{\mathrm{CDW}}(\mathbb{D})$, and that since isomorphisms by definition admit inverses, that lax limits and colimits of strict functors indexed by $(\infty,1)$-categories coincide.

We hope to be able to use our definition of $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ to show that this result generalizes to lax functors indexed by $(\infty,2)$-categories, but we defer this to another work. The difficulty lies in the identification

$$\mathbb{L}\mathrm{axMat}(F,G) \simeq \operatorname*{laxlim}_{(a,b)\colon \mathbb{A}^{\mathrm{op}}\times\mathbb{B}} \mathbb{D}(F(a),G(b)). \tag{2.4.3.1}$$

Already in the case of strict functors out of $(\infty,1)$-categories, the ad hoc argument above is difficult to make rigorous; the connection between the lax limit taken above and the category of lax matrices defined in terms of the join is mediated by the explicit description of lax limits in $\mathbb{C}\mathrm{at}_\infty$ given by taking sections of the Grothendieck construction. To the knowledge of the author, no analogous formula for the lax limit of a lax functor indexed by $(\infty,2)$-category exists. However, one possible way of formalizing this identification is that on the left-hand side, a lax matrix is seen as a sort of (F,G)-bimodule, and on the right-hand side an $(F^{\mathrm{op}}\times G)$-module; tools developed by Hinich in [Hin20] could be used to bridge the gap.

However, assuming the identification of Equation 2.4.3.1 can be made, the logic of [CDW24] generalizes essentially verbatim to imply:

(∗) For any locally cocomplete $(\infty, 2)$-category $\mathbb{D}$ and any lax functor of $(\infty, 2)$-categories $F\colon \mathbb{A} \to \mathbb{D}$, the lax limit and the lax colimit coincide if they exist.

We note that this result is known in the truncated case, see [CKW87].

Following [CDW24], we call a locally cocomplete $(\infty, 2)$-category with lax limits (which are therefore also lax colimits) *lax semiadditive,* and a functor which preserves these structures by the same name.

The universal property of $\mathbb{L}\mathrm{axMat}(\mathbb{D})$

Assuming that (∗) holds, the logic of [GS16] should, with some effort, generalize to allow us to show that if $\mathbb{D}$ is a locally cocomplete category, then $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ is the free lax semiadditive category on $\mathbb{D}$, in the sense that for any other lax semiadditive category $\mathbb{E}$ and any locally cocontinuous functor $F\colon \mathbb{D} \to \mathbb{E}$, there is an essentially unique lax semiadditive functor $\bar{F}\colon \mathbb{L}\mathrm{axMat}(\mathbb{D}) \to \mathbb{E}$ extending F.

$$\begin{tikzcd} \mathbb{D} \arrow[r, "F"] \arrow[d, hook] & \mathbb{E} \\ \mathbb{L}\mathrm{axMat}(\mathbb{D}) \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$

Roughly, the argument given in [GS16, Thm. 15.18] is as follows. Denote by $\mathbb{P}\mathrm{sh}^{\mathrm{colim}}(\mathbb{D}) \subseteq \mathbb{F}\mathrm{un}(\mathbb{D}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}})$ the full subcategory on locally cocontinuous functors. By general abstract nonsense, the full subcategory $\Phi(\mathbb{D}) \subseteq \mathbb{P}\mathrm{sh}^{\mathrm{colim}}(\mathbb{D})$ generated by lax colimits of representables should satisfy the above universal property; it therefore suffices to exhibit an equivalence $\mathbb{L}\mathrm{axMat}(\mathbb{D}) \simeq \Phi(\mathbb{D})$.

To this end, we attempt to define a functor

$$\mathbb{L}\mathrm{axMat}(\mathbb{D}) \to \Phi(\mathbb{D}); \qquad F \mapsto \operatorname*{laxcolim}_{a:\mathbb{A}} \mathbb{D}(-, F(a)); \tag{2.4.3.2}$$

here, the lax colimit is being taken in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}}$. This clearly defines some sort of bijective correspondence between equivalence classes of objects of $\mathbb{L}\mathrm{axMat}(\mathbb{D})$ and $\Phi(\mathbb{D})$. We now argue that this assignment on objects yields a similar correspondence on morphisms. It will be easiest to show this correspondence in the other direction, i.e. starting on the right-hand side of Equation 2.4.3.2.

The data of a locally cocontinuous functor

$$\operatorname*{laxcolim}_{a\colon \mathbb{A}} \mathbb{D}(-, F(a)) \to \operatorname*{laxcolim}_{b\colon \mathbb{B}} \mathbb{D}(-, G(b))$$

on the right-hand side is the same (expanding the lax colimit on the left-hand side) as a coherent system of locally cocontinuous functors

$$\left\{ \mathbb{D}(-, F(a)) \to \operatorname*{laxcolim}_{b\colon \mathbb{B}} \mathbb{D}(-, G(b)) \right\}_{a\colon \mathbb{A}}.$$

By the $\mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}}$-enriched Yoneda lemma, we can replace a morphism out of a representable presheaf $\mathbb{D}(-,d)$ into some presheaf X by an element $X(d)$, giving the equivalent data

$$\left\{ S_a \in \operatorname*{laxcolim}_{b\colon \mathbb{B}} \mathbb{D}(F(a), G(b)) \right\}_{a\colon \mathbb{A}^{\mathrm{op}}}.$$

Using that $\mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}}$ is $\mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}}$-enriched, and therefore by $(*)$ that the lax colimit and lax limit coincide, we can equivalently write

$$\left\{ S_a \in \operatorname*{laxlim}_{b\colon \mathbb{B}} \mathbb{D}(F(a), G(b)) \right\}_{a\colon \mathbb{A}^{\mathrm{op}}}.$$

Using the explicit description of lax limits in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{colim}}$, such a coherent system of objects in $\mathrm{laxlim}_{b\colon \mathbb{B}} \mathbb{D}(F(a), G(b))$ is the same as an element

$$S \in \operatorname*{laxlim}_{(a,b)\colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B}} \mathbb{D}(F(a), G(b).$$

Therefore, under the assumption that the identification given in Equation 2.4.3.1 holds, the data of a morphism on the right-hand side is the same as the data of a morphism on the left-hand side.

Chapter 3

Monoidal pull-push

3.1 Introduction

This chapter represents a natural continuation of the author's Master's thesis, available in revised form at [Rus22a]. There, we provided a corrected statement and a simplified proof of a theorem of Barwick [Bar17, Thm. 12.2], providing several sufficient conditions for a functor of quasicategories $p\colon \mathcal{C} \to \mathcal{D}$ to yield a cocartesian fibration between ∞-categories of spans $\mathrm{Span}(p)\colon \mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$, and hence a functor $\hat{r}\colon \mathrm{Span}(\mathcal{D}) \to \mathcal{C}\mathrm{at}_\infty$ via the Grothendieck construction; we note that the same proof was arrived at independently in [Hau+23, Thm. 3.1]. One such sufficient condition is that p be a so-called *Beck–Chevalley fibration* (Definition 3.3.0.1). Roughly speaking, a Beck–Chevalley fibration is a bicartesian fibration satisfying a straightened version of the Beck–Chevalley condition. More information about Beck–Chevalley fibrations is given in the introduction to Section 3.3 of this work.

3.1.1 Plan for the chapter

In Section 3.2, with an eye to the future construction of a lax monoidal structure, we produce a somewhat complicated model for the ∞-category of $\mathcal{C}$-local systems in some cocomplete category $\mathcal{C}$. The majority of the section consists of setting up the necessary scaffolding in order to show explicitly that this model does what it says on the tin.

It is an easy consequence of the theory of Beck–Chevalley fibrations that there exists a functor $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$, which sends a space X to the category $\mathrm{LS}(\mathcal{C})_X$ of $\mathcal{C}$-local systems on X, and a morphism in $\mathrm{Span}(\mathcal{S})$ represented by a span

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd}$$

to the functor $f_! \circ g^*\colon \mathrm{LS}(\mathcal{C})_X \to \mathrm{LS}(\mathcal{C})_{X'}$. This construction is not original; we recall it in Section 3.3.

In Section 3.4, we introduce a generalization of Beck–Chevalley fibrations, which we call *monoidal Beck–Chevalley fibrations.* Roughly speaking, a monoidal Beck–Chevalley fibration $\tilde{p}\colon \mathcal{C}_{\otimes} \to \mathcal{D}_{\boxtimes}$ is an enhancement of an ordinary Beck–Chevalley fibration $p\colon \mathcal{C} \to \mathcal{D}$, which takes into account monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, and provides a lax monoidal structure structure on the functor $\mathrm{Span}(\mathcal{D}) \to \mathcal{C}\mathrm{at}_{\infty}$.

To this end, we first give in Subsection 3.4.1 a streamlined list of conditions to check that a monoidal functor which is also a cocartesian fibration straightens to a lax monoidal functor into $\mathcal{C}\mathrm{at}_{\infty}$. We then note in Subsection 3.4.2 that these conditions easily generalize to the case that the functor is Beck-Chevalley. In Subsection 3.4.3, we construct a monoidal version of the enhanced twisted arrow category of Abellán–Stern, and in Subsection 3.4.4, we use this to construct a monoidal structure on the ∞-category $\mathrm{LS}(\mathcal{C})$ of $\mathcal{C}$-local systems.

Given $\mathcal{C}$-local systems $\mathcal{F}\colon X \to \mathcal{C}$ and $\mathcal{G}\colon Y \to \mathcal{C}$, we can produce a $\mathcal{C}$-local system on $X \times Y$ as the composition

$$X \times Y \xrightarrow{\mathcal{F}\times\mathcal{G}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}\ .$$

This construction extends to a functor $\mathrm{LS}(\mathcal{C})_X \times \mathrm{LS}(\mathcal{C})_Y \to \mathrm{LS}(\mathcal{C})_{X\times Y}$. Finally, in Subsection 3.4.4, we use the theory of monoidal Beck–Chevalley fibrations to show that the above functors endow our pull-push functor $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_{\infty}$ constructed in Section 3.3 with a lax monoidal structure.

3.1.2 Relation to previous work

That pull-push of local systems can be written as a symmetric monoidal functor out of a category of spans is far from a new idea. Our approach differs from previous ones, however, in that our aim is to provide an explicit, elementary construction of monoidal pull-push of local systems in terms of basic building blocks. We do not aim to work in complete generality when it does not, in our view, lead to greater conceptual clarity.

There have been (to the knowledge of the author) two main works with results similar to those in this chapter.

- In [BGS19], similar results to those in Subsection 3.4.2, about monoidal Beck-Chevalley fibrations, are stated in a rather different situation. The application to local systems is not considered. We have treated these results in less generality.
- In [GR19], results about extending a functor out of a category of spans are proved, but more generally, more abstractly, and for the most part model-independently; there, spans come in $(\infty, 2)$-categories, modelled when necessary by complete 2-fold Segal spaces. Our approach is more explicit, based on hands-on combinatorial computations done in a quasicategorical twisted arrow category.

3.1.3 General conventions and set-theoretic size issues

We do our best to adhere to the following terminological and typographical practices regarding various types of categories.

- When we say '∞-groupoid,' we will mean an $(\infty, 0)$-category, here always modelled by a Kan complex. We will in general denote ∞-groupoids by capital roman letters coming from the end of the alphabet: X, Y, ...
- When we say '∞-category,' we will mean an $(\infty, 1)$-category, usually modelled by a

quasicategory. We will strive to denote ∞-categories by calligraphic letters typeset using the eucal package: $\mathcal{C}$, $\mathcal{D}$, ...

- When we say '∞-bicategory,' we will mean an $(\infty, 2)$-category in the sense of [Lur09], modelled by a fibrant scaled or marked-scaled[1] simplicial set; which one we mean should be clear from context. We will denote $(\infty, 2)$-categories by blackboard-bold letters typeset with the mathbbol package: $\mathbb{C}$, $\mathbb{D}$, ...

We follow Lurie in our definitions of categories of ∞-categories (except for a few size-related transgressions explained below). We also adhere to the above typographical conventions in doing so. Thus:

- The $(\infty, 1)$-category of spaces is $\mathcal{S}$. We model this as a quasicategory, constructed (as in [HTT]) as the homotopy-coherent nerve (as described in [HTT, Sec. 1.1.5], there called the simplicial nerve) of Kan, the Kan-enriched category of Kan complexes.
- The $(\infty, 1)$-category of $(\infty, 1)$-categories is $\mathcal{C}\mathrm{at}_\infty$. We model this as a quasicategory, defined to be the homotopy-coherent nerve of QCat, the Kan-enriched category of quasicategories.
- The $(\infty, 2)$-category of $(\infty, 1)$-categories is $\mathbb{C}\mathrm{at}_\infty$. We model this as an ∞-bicategory, defined to be the scaled nerve (as described in [HTT, Sec. 3.1]) of the $\mathcal{S}\mathrm{et}_\Delta^+$-enriched category of quasicategories.
- The $(\infty, 1)$-category of $(\infty, 2)$-categories is $\mathcal{C}\mathrm{at}_{(\infty,2)}$. This is used only briefly, and will be modelled as the simplicial localization of $(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}})^f$, the full subcategory of marked-scaled simplicial sets on fibrant objects, at the class of bicategorical equivalences.

For the most part, our conventions regarding set-theoretic size issues are standard. However, we will at several points point need to consider an ∞-category whose objects are large ∞-categories. The rigorous solution would be to introduce a series of nested Grothendieck universe, keep track of which one we are currently in, and carry this around as extra notation. However, there are no arguments in this paper which hinge on any set-theoretic size-issues, and the author feels that clarity is lost, rather than gained, by introducing this extraneous notation. Therefore, we will use the same notation for the large ∞-category of small ∞-categories and the 'huge' ∞-category of large ∞-categories.

Suppose K is a simplicial set and $\mathcal{C}$ is an ∞-category. By $\mathrm{Fun}(K, \mathcal{C})$, we mean the ∞-category of maps $K \to \mathcal{C}$. By $\mathrm{Map}(K, \mathcal{C})$, we mean the ∞-groupoid of such maps; that is,

$$\mathrm{Map}(K, \mathcal{C}) = \mathrm{Fun}(K, \mathcal{C})^{\simeq},$$

where $(-)^{\simeq} \colon \mathcal{S}\mathrm{et}_\Delta \to \mathrm{Kan}$ denotes the *core* functor, i.e. the functor associating to a simplicial set K the largest Kan complex contained in it.

3.1.4 The marked-scaled model structure

We will need two different models for $(\infty, 2)$-categories: Lurie's theory of scaled simplicial sets, as laid out in [Lur09]; and Abellán–Stern's theory of marked-scaled simplicial sets, as

[1]The theory of marked-scaled simplicial sets is given in [AS22], and summarized in Subsection 3.1.4.

explained in [AS22]. We will assume a knowledge of scaled simplicial sets. We give a basic outline of the portions of the theory of marked-scaled simplicial sets which we will need.

A marked-scaled simplicial set is a triple (X, E_X, T_X), where $E_X \subseteq X_1$ is a collection of edges of X containing all degenerate edges, and $T_X \subseteq X_2$ is a collection of triangles of X containing all degenerate triangles. We will use the following notation.

- We will denote the category of marked-scaled simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}$.
- To save on notation, we will sometimes denote the marked-scaled simplicial set (X, E_X, T_X) by $X_{T_X}^{E_X}$, particularly in the case that E_X or T_X are $\sharp$ or $\flat$, the maximum (resp. minimum) markings and scalings. For example, the bimarked simplicial set $X_\flat^\sharp$ has all 1-simplices marked and only degenerate 2-simplices scaled.
- Let (Δ^n, E, T) be a marked-scaled n-simplex. For any simplicial subset $S \subseteq \Delta^n$, we will denote by (S, E, T) the simplicial subset S together with the inherited marking and scaling.

There is a set of marked-scaled anodyne morphisms, which have the left lifting property with respect to marked-scaled fibrations. We will not need to use the full power of the marked-scaled anodyne morphisms, so we content ourselves with an incomplete description.

Definition 3.1.4.1. The set of ms-anodyne morphisms is a saturated set of morphisms between marked-scaled simplicial sets containing the following classes of morphisms:

(A1) Inner horn inclusions

$$(\Lambda_i^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}),$$

for $n \geq 2$ and $0 < i < n$.

(A2) Outer horn inclusions

$$(\Lambda_n^n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}),$$

for $n \geq 1$.

Example 3.1.4.2. The marked-scaled anodyne morphisms encapsulate both left- and marked-anodyne morphisms.

- For any right anodyne morphism $A \hookrightarrow B$, the morphism $A_\sharp^\sharp \hookrightarrow B_\sharp^\sharp$ is marked-scaled anodyne.
- For any marked anodyne morphism $(A, \mathcal{E}) \to (B, \mathcal{F})$, the morphism $A_\sharp^{\mathcal{E}} \to B_\sharp^{\mathcal{F}}$ is marked-scaled anodyne.

Theorem 3.1.4.3. There is a model structure on the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}$, whose trivial cofibrations are contain the set of marked-scaled anodyne maps, and whose fibrant objects are ∞-bicategories with the equivalences marked and thin simplices scaled.

Theorem 3.1.4.4. There is a Quillen equivalence

$$(-)_\flat : \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}} : G,$$

where $(-)^\flat$ endows any scaled simplicial set with the flat marking, and G forgets the marking.

3.1.5 Selected results from Monoidal Pull-Push I

We will at several points make use of the results of the authors Master's thesis, available in revised form at [Rus22a]. We reproduce here our main results here, which we will need in the later chapters. These results, originally due to Barwick, give sufficient conditions on a functor $p\colon \mathcal{C} \to \mathcal{D}$ so that the induced functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$ is a cocartesian fibration. There, one of the conditions was incorrectly listed; we give here a corrected version, as found independently by [Hau+23].

Here, the subcategories $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$ pick out distinguished classes of morphisms to which the legs of the spans of $\mathrm{Span}(\mathcal{C})$ are allowed to belong; the 'forwards-facing' legs of $\mathrm{Span}(\mathcal{C})$ are restricted to come from the subcategory $\mathcal{C}_\dagger$, and the 'backwards-facing' legs from $\mathcal{C}^\dagger$; and similarly for $\mathcal{D}_\dagger$ and $\mathcal{D}^\dagger$.

Theorem 3.1.5.1 ([Bar17, Thm.12.2])**.** Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

(1) Each morphism $g \in \mathcal{D}_\dagger$ admits a lift to a morphism in $\mathcal{C}_\dagger$ (given a lift of the source) which is both p-cocartesian and $p_\dagger$-cocartesian.

(2) Consider a commutative square

$$\sigma = \begin{tikzcd} y' \arrow[r, tail, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ where g' belongs to $\mathcal{C}^\dagger$, and f and f' belong to $\mathcal{C}_\dagger$. Suppose that f is p-cocartesian. Then f' is p'-cocartesian if and only if σ is an ambigressive pullback square (and in particular $g \in \mathcal{C}^\dagger$).

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Theorem 3.1.5.2. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

(1) The subcategory $\mathcal{C}^\dagger \subseteq \mathcal{C}$ consists of all p-cartesian morphisms in $\mathcal{C}$; that is, an n-simplex in $\mathcal{C}$ belongs to $\mathcal{C}^\dagger$ if and only if each 1-simplex it contains is p-cartesian.

(2) The map $p^\dagger\colon \mathcal{C}^\dagger \to \mathcal{D}^\dagger$ is a cartesian fibration.

(3) Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ where g and g' belong to $\mathcal{C}^\dagger$, and f belongs to $\mathcal{C}_\dagger$. Further suppose that f is p-cocartesian. Then f' belongs to $\mathcal{C}_\dagger$, and is both p-cocartesian and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

3.1.6 Basic facts about Kan extensions

In this section, we recall a few basic facts about Kan extensions. These are mostly results found in [Lur18] which we will need. Since Kan extensions are defined by the colimit formula, we will start by defining colimits. Clasically, colimits in ∞-categories are defined using colimit cones.

Definition 3.1.6.1. Let $F\colon K \to \mathcal{C}$ be a diagram, where $\mathcal{C}$ is an ∞-category. A cocone $\tilde{F}\colon K^{\triangleright} \to \mathcal{C}$ is a ***colimit cone*** if it is an initial object in the ∞-category $\mathcal{C}_{F/}$.

It is sometimes more convenient to define colimits via natural transformations to a constant functor.

Definition 3.1.6.2. Let $f\colon K \to \mathcal{C}$ be a map between simplicial sets, where $\mathcal{C}$ is an ∞-category. Let $c \in \mathcal{C}$ be an object, and denote by $\underline{c}$ the constant functor $K \to \mathcal{C}$ with value c. A natural transformation $f \Rightarrow \underline{c}$ ***exhibits*** c ***as the colimit of*** f if it is an initial object in the ∞-category $\mathcal{C}^{F/}$.

Fortunately, these definitions are compatible: an object $K^{\triangleright} \to \mathcal{C}$ in $\mathcal{C}_{/F}$ is a colimit cone with cone tip c if and only if the composite map

$$K \times \Delta^1 \to K \times \Delta^1 \amalg_{K\times\{1\}} \Delta^0 \to K \star \Delta^0 \to \mathcal{C} \tag{3.1.6.1}$$

exhibits c as a colimit of F. This follows from the fact that for any ∞-category $\mathcal{C}$ and any functor $F\colon K \to \mathcal{C}$, the comparison map $\mathcal{C}_{/F} \to \mathcal{C}^{/F}$ of [HTT, Prop. 4.2.1.5] is a categorical equivalence.

Note 3.1.6.3. It follows immediately from this description that colimit cones are homotopically invariant: if a natural transformation η exhibits some object as a colimit of some diagram, than any natural transformation η' which is homotopic to η will do just as well.

Notation 3.1.6.4. Let $f\colon X \to Y$ be a map of simplicial sets, and $y \in Y$ an object. We will use the following notation.

- Denote by $X_{/y}$ the fiber product $X \times_Y Y_{/y}$.
- Denote by $\pi\colon X_{/y} \to X$ the projection map.
- Denote by $\alpha\colon f \circ \pi \Rightarrow \underline{y}$ the natural transformation $X_{/y} \times \Delta^1 \to X$ coming from the comparison map $X_{/y} \to X^{/y}$.

Definition 3.1.6.5 ([Lur18, Tag 02YC])**.** Let X, Y, and $\mathcal{C}$ be ∞-categories, $f\colon X \to Y$, $F\colon X \to \mathcal{C}$ and $G\colon Y \to \mathcal{C}$ functors, and $\eta\colon F \Rightarrow G \circ f$ a natural transformation.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "G"'] & \end{tikzcd}$$

We say that η ***exhibits*** G ***as the left Kan extension of*** F ***along*** f if for each object $y \in Y$, the natural transformation $F \circ \pi \Rightarrow \underline{G(y)}$ furnished by the pasting diagram

$$
\begin{tikzcd}
X_{/y} \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\eta}{\Swarrow}"] & \mathcal{C} \\
\{y\} \arrow[r, hook] & Y \arrow[ur, "G"'] &
\end{tikzcd}
$$

exhibits $G(y)$ as the colimit of the functor $F \circ \pi$. Here, $\underline{G(y)}$ is the 'counterclockwise' path from $X_{/y}$ to $\mathcal{C}$.

The following guarantees the existence of local left Kan extensions.

Theorem 3.1.6.6 ([Lur18, Tag 0300])**.** Suppose that $f \colon X \to Y$ is a map of simplicial sets, and suppose $F \colon X \to \mathcal{C}$ is a map of simplicial sets such that $\mathcal{C}$ is a quasicategory. The functor F admits a left Kan extension along f if and only if for all objects $x \in X$, the colimit of the functor

$$X_{/y} \xrightarrow{\pi} X \xrightarrow{F} \mathcal{C}$$

exists in $\mathcal{C}$.

The following is a combination of special cases of [Lur18, Tag 02YL] and [Lur18, Tag 02YM]

Example 3.1.6.7. For any homotopy-commuting diagram of simplicial sets

$$
\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta \Downarrow \simeq"] & \mathcal{C} \\
& Y \arrow[ur, "G"'] &
\end{tikzcd}
$$

where X, Y, and $\mathcal{C}$ are ∞-categories and $f \colon X \to Y$ is a categorical equivalence, η exhibits G as a left Kan extension of F along f.

The following guarantees existence of global left Kan extensions.

Theorem 3.1.6.8 ([Lur18, Tag 030C])**.** Let $f \colon X \to Y$ be a map of simplicial sets, and suppose that $\mathcal{C}$ is a quasicategory which admits $X_{/y}$-shaped colimits for all $y \in Y$. Then the restriction functor $f^* \colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ admits a left adjoint $f_!$, sending a functor $F \colon X \to \mathcal{C}$ to $f_! F \colon Y \to \mathcal{C}$, its left Kan extension along f.

Proposition 3.1.6.9 ([Lur18, Tag 031N])**.** Consider ∞-categories and functors

$$
\begin{tikzcd}
X \arrow[rrr, "F"] \arrow[dr, "f"'] & & & \mathcal{C} \\
& Y \arrow[urr, "G"] \arrow[dr, "g"'] & & \\
& & Z \arrow[uur, "H"'] &
\end{tikzcd},
$$

and natural transformations $\alpha \colon F \Rightarrow G \circ f$ and $\beta \colon H \circ g \Rightarrow G$, and suppose that α exhibits G as the left Kan extension of F along f. Then β exhibits H as the left Kan extension of G along g if and only if $\beta f \circ \alpha$ exhibits H as the left Kan extension of F along $g \circ f$.

Proposition 3.1.6.10 ([Lur18, Tag 030C])**.** Suppose $H\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, $f\colon X \to Y$ is a map of simplicial sets, $G\colon Y \to \mathcal{C}$ is a functor, and $\eta\colon F \Rightarrow G \circ f$ is a natural transformation. Then η exhibits G as a left Kan extension of F along f if and only if $H\eta\colon H \circ F \Rightarrow H \circ G \circ f$ exhibits $H \circ G$ as a left Kan extension of $H \circ F$ along f.

Proposition 3.1.6.11 ([Lur18, Tag 02YG])**.** The property that η exhibits G as a left Kan extension of F along f is homotopy-invariant; any homotopic η will do just as well.

The following is an easy consequence of [Lur18, Tag 030D], to be explained later.

Proposition 3.1.6.12. Let $f\colon X \to Y$ be a map of simplicial sets, and let $\mathcal{C}$ be a category such that all functors $X \to \mathcal{C}$ admit left Kan extensions along f, so that we have an adjunction $f_! \dashv f^*$. Let $G\colon Y \to \mathcal{C}$ and $\eta\colon F \Rightarrow G \circ f$. The natural transformation η exhibits G as a left Kan extension of F along f if and only if it is adjunct (in the sense of Definition A.1.2.3) to an equivalence in $\mathrm{Fun}(Y, \mathcal{C})$ relative to $s = \mathrm{id}_{\Delta^1}$.

Corollary 3.1.6.13. For any 2-simplex $\sigma\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ given by a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "G"'] & \end{tikzcd}$$

in $\mathbb{C}\mathrm{at}_\infty$ such that $\mathcal{C}$ is cocomplete and $\mathcal{A}$ and $\mathcal{B}$ are small, the natural transformation η exhibits G as a left Kan extension of F along f if and only if σ is left Kan in the sense of Definition A.1.0.1.

We end this section with a few miscellaneous tricks which we can play with maps $\Delta^n \to \mathbb{C}\mathrm{at}_\infty$ involving left Kan simplices.

Lemma 3.1.6.14. Denote by $\mathcal{E} \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices together with the simplices $\Delta^{\{0,2,3\}}$ and $\Delta^{\{1,2,3\}}$. Let $\sigma\colon \Delta^3_{\mathcal{E}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{2,3\}}$ is mapped to an equivalence. Then $\sigma|\Delta^{\{0,1,2\}}$ is left Kan if and only if $\sigma|\Delta^{\{0,1,3\}}$ is left Kan.

Proof. Replacing thin simplices by strict compositions using Proposition 3.1.6.11, this statement reduces to that of Proposition 3.1.6.10. □

The next lemma is similar but easier.

Lemma 3.1.6.15. Denote by $\mathcal{E} \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices together with the simplices $\Delta^{\{0,2,3\}}$ and $\Delta^{\{1,2,3\}}$. Let $\sigma\colon \Delta^3_{\mathcal{E}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{2,3\}}$ is mapped to an equivalence. Then $\sigma|\Delta^{\{0,1,2\}}$ is thin if and only if $\sigma|\Delta^{\{0,1,3\}}$ is thin.

Lemma 3.1.6.16. Denote by $\mathcal{E}' \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices, together with the simplex $\Delta^{\{0,1,2\}}$. Let $\sigma\colon \Delta^3_{\mathcal{F}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\sigma|\Delta^{\{0,1,3\}}$ is left Kan. Then $\sigma|\Delta^{\{0,2,3\}}$ is left Kan if and only if $\sigma|\Delta^{\{1,2,3\}}$ is left Kan.

Proof. Replacing thin simplices by strict compositions using Proposition 3.1.6.11, this reduces to Propsition 3.1.6.9. □

3.2 Local systems

For any ∞-category $\mathcal{C}$, a $\mathcal{C}$-local system on some space X is simply a functor $X \to \mathcal{C}$. This leads us to define the ∞-category of $\mathcal{C}$-local systems on X simply by $\mathrm{LS}(\mathcal{C})_X := \mathrm{Fun}(X, \mathcal{C})$. The ∞-category $\mathrm{LS}(\mathcal{C})$ of $\mathcal{C}$-local systems is then the total space of a cartesian fibration classifying the functor

$$X \mapsto \mathrm{LS}(C)_X.$$

It will be helpful to have an explicit description of $\mathrm{LS}(\mathcal{C})$. Given a cartesian fibration

$$p' \colon \textstyle\int \mathrm{Fun}(-,-) \to \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$$

which classifies the functor

$$\mathrm{Fun}(-,-) \colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty; \qquad (\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The strict pullback p of p' in the diagram

$$\begin{tikzcd}
\mathrm{LS}(\mathcal{C}) \arrow[r] \arrow[d, "p"'] & \int \mathrm{Fun}(-,-) \arrow[d, "p'"] \\
\mathcal{S} \times \{\mathcal{C}\} \arrow[r] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}
\end{tikzcd}$$

will then classify the functor

$$\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{LS}(\mathcal{C})_X.$$

The total space $\mathrm{LS}(\mathcal{C})$ of p will thus be our candidate for our ∞-category of local systems. The fibration p remembers the source of the local system.

Our goal in this section is to give a comprehensive, explicit account of the construction sketched above. In [GS20], a suitable model for $\int \mathrm{Fun}(-,-)$ is given: the so-called *enhanced twisted arrow category* $\mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)$. In Subsection 3.2.1, we reproduce the pertinent points of [GS20], defining the enhanced twisted arrow category. In Subsection 3.2.3, we use the enhanced twisted arrow category to define the category $\mathrm{LS}(\mathcal{C})$ of local systems, together with a cartesian fibration $p \colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ classifying the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \to \mathrm{Fun}(X, \mathcal{C}).$$

We note that this functor sends a map of spaces $f \colon X \to Y$ to the pullback functor

$$f^* \colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

If $\mathcal{C}$ is cocomplete, each functor f^* has a left adjoint $f_!$ given by left Kan extension. By abstract nonsense, the cartesian fibration p is also a cocartesian fibration, whose cocartesian edges correspond to left Kan extension. In Subsection 3.2.4 we will show this explicitly, using results built up in Subsection 3.2.2 and Subsection 3.2.3; if the reader is willing to take this on faith, these sections can be safely skipped.

Note 3.2.0.1. The reader may notice that we are being inefficient here. If we were only interested in the construction outlined above, then using the twisted arrow category would be much more combinatorially strenuous than necessary; the objects of the twisted arrow category $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)$ are functors of ∞-categories $\mathcal{D} \to \mathcal{C}$, with $\mathcal{C}$ and $\mathcal{D}$ both allowed to vary. If, in the very next breath, we fix the target $\mathcal{C}$, then it makes more sense to use some sort of lax overcategory $(\mathbb{C}\mathrm{at}_\infty)_{\nearrow\mathcal{C}}$ from the beginning. There is a method to our madness; in Section 3.4, we will introduce a monoidal version of this construction, and here we will need to allow the targets of our functors to vary.

3.2.1 The enhanced twisted arrow category

For a 2-category C, the twisted arrow 1-category $\mathrm{Tw}(\mathsf{C})$ has the following description.

- The objects of $\mathrm{Tw}(\mathsf{C})$ are the morphisms of C:
$$f\colon c \to c'.$$
- For two objects $(f\colon c \to c')$ and $(g\colon d \to d')$, the morphisms $f \to g$ are given by diagrams
$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] & d \arrow[d, "g"] \arrow[l, phantom, "\xRightarrow{\alpha}" description, shift right=6] \\ c' & d' \arrow[l, "a'"] \end{tikzcd},$$
where α is a 2-morphism $f \Rightarrow a' \circ g \circ a$.
- The composition of morphisms is given by concatenating the corresponding diagrams.
$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & d \arrow[r, "b"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\xRightarrow{\beta}"] & e \arrow[d, "h"] \\ c' & d' \arrow[l, "a'"] & e' \arrow[l, "b'"] \end{tikzcd}$$

In [GS20], a homotopy-coherent version of the twisted arrow category of an ∞-bicategory is defined. For any ∞-bicategory $\mathbb{C}$, the twisted arrow ∞-category of $\mathbb{C}$, denoted $\mathrm{Tw}(\mathbb{C})$, is a quasicategory whose n-simplices are diagrams $\Delta^n \star (\Delta^n)^{\mathrm{op}} \to \mathbb{C}$, together with a scaling to ensure that the information encoded in an n-simplex is determined, up to contractible choice, by data

n squares
$$\begin{tikzcd} c_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\xRightarrow{\alpha_0}"] & c_1 \arrow[r, dotted] \arrow[d, dashed, "f_1"] & \cdots \arrow[r, dotted] & c_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, dashed, "f_{n-1}"'] \arrow[dr, phantom, "\xRightarrow{\alpha_{n-1}}"] & c_n \arrow[d, "f_n"] \\ c_0' & c_1' \arrow[l, "a_0'"] & \cdots \arrow[l, dotted] & c_{n-1}' \arrow[l, dotted] & c_n' \arrow[l, "a_{n-1}'"] \end{tikzcd}$$

in $\mathbb{C}$. Here, the top row of morphisms corresponds to the spine of the n-simplex $\Delta^n \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$, and the bottom row of morphisms to the spine of $(\Delta^n)^{\mathrm{op}} \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$. To make this correspondence more clear, we will introduce the following notation.

Notation 3.2.1.1. For $i \in [n] \subset [2n+1]$, we will write $\bar{i} := 2n+1-i$. Thus, $\bar{0} = 2n+1$, $\bar{1} = 2n$, etc.

Thus, the ith column in the above diagram corresponds to the image of the morphism $i \to \bar{i}$ in $\Delta^n \star (\Delta^n)^{\mathrm{op}}$.

The scaling mentioned above is defined in the following way.

Definition 3.2.1.2. We define a cosimplicial object $\tilde{Q}\colon \Delta \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}$ by sending

$$\tilde{Q}([n]) = (\Delta^n \star (\Delta^n)^{\mathrm{op}}, \dagger),$$

where $\dagger$ is the scaling consisting of all degenerate 2-simplices, together with all 2-simplices of the following kinds:

(1) All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through Δ^n.

(2) All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through $(\Delta^n)^{\mathrm{op}}$.

(3) All simplices $\Delta^{\{i,j,\bar{k}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.

(4) All simplices $\Delta^{\{k,\bar{j},\bar{i}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.

Note that $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$. We will use this identification freely.

This extends to a nerve-realization adjunction

$$Q : \mathcal{S}\mathrm{et}_{\Delta} \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}} : \mathrm{Tw},$$

where the functor Q is the extension by colimits of the functor $\tilde{Q}$, and for any scaled simplicial set X, the simplicial set $\mathrm{Tw}(X)$ has n-simplices

$$\mathrm{Tw}(\mathbb{C})_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}}(Q(\Delta^n), X).$$

Notation 3.2.1.3. For any simplicial set X, $Q(X)$ carries the scaling given simplex-wise by that described above. We will denote this also by $\dagger$. Using the identification $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$, we can write $Q(\Delta^n)$ explicitly and compactly as $\Delta^{2n+1}_{\dagger}$, which we will often do.

The inclusion $\Delta^n \amalg (\Delta^n)^{\mathrm{op}} \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ provides, for any ∞-bicategory $\mathbb{C}$ with underlying ∞-category $\mathcal{C}$, a morphism of simplicial sets

$$\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

In [GS20], the following is shown.

Theorem 3.2.1.4. Let $\mathbb{C}$ be an ∞-bicategory (presented as a fibrant scaled simplicial set), and let $\mathcal{C}$ be the underlying ∞-category (presented as a quasicategory). Then the map

$$p_{\mathbb{C}}\colon \mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

is a cartesian fibration between quasicategories, and a morphism $f\colon \Delta^1 \to \mathrm{Tw}(\mathbb{C})$ is $p_{\mathbb{C}}$-cartesian if and only if the morphism $\sigma\colon \Delta^3_{\dagger} \to \mathbb{C}$ to which it is adjunct is fully scaled, i.e. factors through the map $\Delta^3_{\dagger} \to \Delta^3_{\sharp}$. Furthermore, the cartesian fibration $p_{\mathbb{C}}$ classifies the functor

$$\mathrm{Map}_{\mathbb{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}\mathrm{at}_{\infty}.$$

3.2.2 Two lemmas about marked-scaled anodyne morphisms

In the section following this one, we provide a fairly explicit construction of a family of marked-scaled anodyne inclusions. Constructing these inclusions directly from the classes of generating marked-scaled anodyne inclusions given in Definition 3.1.4.1 would be possible but tedious. To lighten this load somewhat, we reproduce in this section two lemmas from [AS22]. The majority of this section is a retelling of [AS22, Sec. 2.3 and 2.4]. Note however that our needs will be rather different than those of [AS22], and this is reflected in some minor differences of notation.

We first need a compact notation for specifying simplicial subsets of Δ^n. Denote by $P\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ the power set functor.

Definition 3.2.2.1. Let T be a finite linearly ordered set, so $T \cong [n]$ for some $n \in \mathrm{N}$. For any subset $\mathcal{A} \subseteq P(T)$, define a simplicial subset

$$\mathcal{S}_T^{\mathcal{A}} := \bigcup_{S \in \mathcal{A}} \Delta^{T \smallsetminus S} \subseteq \Delta^T.$$

Notation 3.2.2.2. If the linearly ordered set T is clear from context, we will drop it, writing $\mathcal{S}^{\mathcal{A}}$.

Note 3.2.2.3. The assignment $\mathcal{A} \mapsto \mathcal{S}^{\mathcal{A}}$ does not induce is not a one-to-one correspondence between subsets $\mathcal{A} \subseteq P([n])$ and simplicial subsets $S^{\mathcal{A}} \subseteq \Delta^n$, since for any two subsets $S \subsetneq S' \subset [n]$, we have that $\Delta^{[n]\smallsetminus S'} \subsetneq \Delta^{[n]\smallsetminus S}$.

Example 3.2.2.4. Take $T = [2]$. Taking $\mathcal{A} = \{\{2\}, \{1, 2\}\}$ gives the simplicial subset $\mathcal{S}_T^{\mathcal{A}} = \Delta^{\{0,1\}} \cup \Delta^{\{0\}} = \Delta^{\{0,1\}} \subset \Delta^2$, as does taking $\mathcal{A} = \{\{2\}\}$.

It will be useful to consider two subsets $\mathcal{A}$ and $\mathcal{A}' \subseteq P(T)$ to be equivalent if they produce the same simplicial subset of Δ^T.

Definition 3.2.2.5. We will write $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{S}^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}'}$.

The relation $\sim$ is an equivalence relation, but we will not use this.

The set $P([n])$ forms a poset, ordered by inclusion, and any $\mathcal{A} \subseteq P([n])$ a subposet. An element q of a poset Q is said to be *minimal* if there are no elements of Q which are strictly less than q. By Note 3.2.2.3 only the minimal elements of $\mathcal{A}$ contribute to the union defining $\mathcal{S}^{\mathcal{A}}$. We have just shown the following.

Lemma 3.2.2.6. For any $\mathcal{A} \subseteq P([n])$, we have $\mathcal{A} \sim \min(\mathcal{A})$, where $\min(\mathcal{A}) \subseteq \mathcal{A}$ is the set of minimal elements of $\mathcal{A}$.

Simplicial subsets of the form $\mathcal{S}_T^{\mathcal{A}}$ enjoy the following easily-proved calculation rules.

Lemma 3.2.2.7. Suppose $\mathcal{A} \subseteq P([n])$ is a subset such that for all S, $T \in \mathcal{A}$, it holds that $S \cap T = \emptyset$. Then $\mathcal{S}^{\mathcal{A}}$ contains each k-simplex of Δ^n for all $k < |\mathcal{A}| - 1$.

Proof. Let $X \subseteq [n]$. The simplex $\Delta^X \to \Delta^n$ factors through $\mathcal{S}^{\mathcal{A}}$ if and only if it factors through $\Delta^{[n]\smallsetminus T}$ for some (possibly not unique) $T \in \mathcal{A}$. This in turn is true if and only if X and T do not have any elements in common. For $|X| < |\mathcal{A}|$, there is always a set $T \in \mathcal{A}$ which does not have any elements in common with X. □

Lemma 3.2.2.8. Let $\mathcal{A} \subseteq P([n])$ and $T \subseteq [n]$. Then

$$\mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T = \mathcal{S}_{[n]}^{\mathcal{A} \cup \{[n]\smallsetminus T\}}.$$

The next lemma will be particularly useful in building inclusions $\mathcal{S}_{[n]}^{\mathcal{A}} \hookrightarrow \Delta^n$ simplex-by-simplex.

Lemma 3.2.2.9. Let $\mathcal{A} \subset P([n])$, and let $T \subset [n]$. Then the square

$$\begin{tikzcd} \mathcal{S}_T^{\mathcal{A}|T} \arrow[r, hook] \arrow[d, hook] & \Delta^T \arrow[d, hook] \\ \mathcal{S}_{[n]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T \end{tikzcd}$$

is bicartesian, where $\mathcal{A}|T$ is the subset of $P(T)$ given by

$$\mathcal{A}|T = \{S \cap T \mid S \in \mathcal{A}\}.$$

Proof. We have an equality

$$\mathcal{S}_T^{\mathcal{A}|T} = \mathcal{S}_{[n]}^{\mathcal{A}} \cap \Delta^T.$$

as subsets of Δ^n. □

Notation 3.2.2.10. For any marked-scaled n-simplex (Δ^n, E, T) we will by minor abuse of notation reuse the same letters E and T to denote the restriction of the markings and scalings to any simplicial subset $S \subseteq \Delta^n$.

In the remainder of this section we reproduce two lemmas from [AS22] which provide criteria for the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \subseteq (\Delta^n, E, T)$ to be marked-scaled anodyne.

Definition 3.2.2.11. Let $\mathcal{A} \subseteq P([n])$. We call $X \in P([n])$ an ***$\mathcal{A}$-basal set*** if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of all $\mathcal{A}$-basal sets by $\mathrm{Bas}(\mathcal{A})$.

Definition 3.2.2.12 ([GS20], Definition 1.3)**.** We will call a subset $\mathcal{A} \subseteq P([n])$ ***inner dull*** if it satisfies the following conditions.

- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- There exists $0 < i < n$ such that for all $S \in \mathcal{A}$, we have $i \notin S$.
- For every S, $T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.
- For each $\mathcal{A}$-basal set $X \in P([n])$, there exist u, $v \in X$ such that $u < i < v$.

The element $i \in [n]$ is known as the *pivot point.*

Note 3.2.2.13. The last condition of Definition 3.2.2.12 is always satisfied if $\mathcal{A}$ contains two singletons $\{u\}$ and $\{v\}$ such that $u < i < v$.

Definition 3.2.2.14. Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i, and let $X \in \mathcal{A}$. We will denote the adjacent elements of X surrounding i by $\ell^X < i < u^X$.

Lemma 3.2.2.15 (The Pivot Trick: [AS22], Lemma 2.3.5)**.** Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i, and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that the following conditions hold:

(1) Every marked edge $e \in E$ which does not contain i factors through $\mathcal{S}^{\mathcal{A}}$.

(2) For all scaled simplices $\sigma = \{a < b < c\}$ of Δ^n which do not factor through $\mathcal{S}^{\mathcal{A}}$, and which do not contain the pivot point i, we have $a < i < c$, and $\sigma \cup \{i\}$ is fully scaled.

(3) For all $X \in \mathrm{Bas}(\mathcal{A})$ and all $r,\, s \in [n]$ such that $\ell^X \leq r < i < s \leq u^X$, the triangle $\{r, i, s\}$ is scaled

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

Definition 3.2.2.16. We will call a subset $\mathcal{A} \subseteq P([n])$ ***right dull*** if it satisfies the following conditions.

- It is nonempty; $\mathcal{A} \neq \emptyset$.
- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- For all $S \in \mathcal{A}$, we have $n \notin S$.
- For every $S,\, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.

In this case, we will refer to n as the pivot point.

Lemma 3.2.2.17 (Right-anodyne pivot trick)**.** Let $\mathcal{A} \subset P([n])$ be a right dull subset (whose pivot point is by definition n), and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that the following conditions hold:

- Every marked edge $e \in E$ which does not contain n factors through $\mathcal{S}^{\mathcal{A}}$.
- Let $\sigma = \{a < b < c\}$ be a scaled simplex not containing n. Then either σ factors through $\mathcal{S}^{\mathcal{A}}$, or $\sigma \cup \{n\}$ is fully scaled, and $c \to n$ is marked.
- For all Z in $\mathrm{Bas}(\mathcal{A})$, For all $r,\, s \in [n]$ with $r \leq \min(Z) \leq \max(Z) \leq s < n$, the triangle $\Delta^{\{r,s,n\}}$ is scaled, and the simplex $\Delta^{\{s,n\}}$ is marked.

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

3.2.3 The infinity-category of local systems

In Subsection 3.2.1, we defined the twisted arrow category $\mathrm{Tw}(\mathbb{C})$ of an ∞-bicategory $\mathbb{C}$, and noted that the natural cartesian fibration $\mathrm{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ classifies the mapping functor $\mathrm{Map}_{\mathbb{C}}(-,-)$. Thus, the objects $\mathrm{Tw}(\mathbb{C})$ are simply the morphisms in $\mathbb{C}$.

We can use this to define, for any cocomplete ∞-category $\mathcal{C}$, the ∞-category of $\mathcal{C}$-local systems; we simply consider $\mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)$, whose objects are all functors between ∞-categories, and restrict to those functors whose domain is an ∞-groupoid, and whose codomain is $\mathcal{C}$.

Definition 3.2.3.1. For any cocomplete ∞-category $\mathcal{C}$, we define the ∞-category of $\mathcal{C}$-local systems $\mathrm{LS}(\mathcal{C})$ together with a map of simplicial sets $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ by the following pullback diagram.

$$\begin{tikzcd} \mathrm{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} \tag{3.2.3.1}$$

Note 3.2.3.2. It is natural to wonder if we are making life unnecessarily difficult by not simply considering the category $\mathcal{S} \times_{\mathcal{C}\mathrm{at}_\infty} (\mathbb{C}\mathrm{at}_\infty)_{/\mathcal{C}}$, which is after all also an ∞-category of functors from spaces into $\mathcal{C}$. The reason for the more complicated construction given here will become apparent in Subsection 3.4.3, when we define the monoidal structure on $\mathrm{LS}(\mathcal{C})$.

It is shown in [GS20] that p', hence also p, is a cartesian fibration. The cartesian fibration p classifies the functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending

$$f\colon X \to Y \qquad \longmapsto \qquad f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ is cocomplete, each pullback map f^* has a left adjoint $f_!$, given by left Kan extension. It follows on abstract grounds that p is also a cocartesian fibration, whose cocartesian edges represent left Kan extension. In Subsection 3.2.4, we prove a more precise version of this result which we will need later. Our goal in this section is to do some legwork to facilitate the proof of this result.

In investigating the map $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$, it will be useful to factor the pullback square in Equation 3.2.3.1 into the two pullback squares

$$\begin{tikzcd}
\mathrm{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hook] \arrow[d, "p''"] & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\
\mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}] \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty{}^{\mathrm{op}}
\end{tikzcd},$$

where $[\mathcal{C}]$ denotes the path component of $\mathcal{C}$ in $(\mathcal{C}\mathrm{at}_\infty{}^{\mathrm{op}})^{\simeq}$. In order to show that p is a cocartesian fibration, it will help us to understand the map p''. The n-simplices of $\mathcal{R}$ are given by maps

$$Q(\Delta^n) = (\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger \to \mathcal{C}\mathrm{at}_\infty$$

such that

- each object in $(\Delta^n)^{\mathrm{op}} \subseteq Q(\Delta^n)$ is sent to a quasicategory which is equivalent to $\mathcal{C}$ (and in particular cocomplete), and
- each morphism in $(\Delta^n)^{\mathrm{op}} \subseteq Q(\Delta^n)$ is mapped to an equivalence in $\mathcal{C}\mathrm{at}_\infty$.

We can more usefully encode the second condition by endowing $\mathcal{C}\mathrm{at}_\infty$ and Q with a marking.

Definition 3.2.3.3. We define the following markings.

- We denote by $\mathcal{C}\mathrm{at}_\infty^\natural$ the marked-scaled simplicial set whose underlying scaled simplicial set is $\mathcal{C}\mathrm{at}_\infty$, and where all equivalences have been marked.
- We denote by $\heartsuit$ the marking on $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ consisting of all morphisms belonging to $(\Delta^n)^{\mathrm{op}}$, and by $(\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger^\heartsuit$ the corresponding marked-scaled simplicial set.

Definition 3.2.3.4. We define a cosimplicial object

$$\tilde{R}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}; \qquad [n] \mapsto (\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger^\heartsuit.$$

We denote the extension of $\tilde{R}$ by colimits by

$$R\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} R(n).$$

The marked-scaled simplicial sets $R(\Delta^n)$ are rather complicated, and showing that p'' is a cartesian fibration by solving the necessary lifting problems explicitly would be impractical. We will instead replace $R(\Delta^n)$ by something simpler, considering the simplicial subsets coming from the inclusions

$$\Delta^n \star \Delta^{\{\overline{0}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}, \tag{3.2.3.2}$$

which we understand to inherit the marking and scaling.

Definition 3.2.3.5. We will denote by $\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}$ the cosimplicial object

$$\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}; \qquad [n] \mapsto (\Delta^n \star \Delta^{\{\overline{0}\}})_\dagger^\heartsuit,$$

and by J the extension by colimits

$$J\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} \tilde{J}(n).$$

The inclusions of Equation 3.2.3.2 induce for each $n \geq 0$ an inclusion of marked-scaled simplicial sets

$$v_n\colon \tilde{J}(n) \to \tilde{R}(n).$$

Lemma 3.2.3.6. For all $n \geq 0$, the map v_n is marked-scaled anodyne, hence a weak equivalence in the model structure on marked-scaled simplicial sets.

We postpone the proof of Lemma 3.2.3.6 until the end of this section.

Note 3.2.3.7. Some care is warranted: the v_n are not the components of a natural transformation $\tilde{J} \Rightarrow \tilde{R}$! The necessary squares simply do not commute for morphisms $\phi\colon [m] \to [n]$ in Δ such that $\phi(0) \neq 0$. This means that we do not, for a general simplicial set X, get a canonical weak equivalence of marked-scaled simplicial sets $J(X) \to R(X)$.

Despite the warning given in Note 3.2.3.7, it is still possible to get maps $J(X) \to R(X)$ in some cases. In the remainder of this section, we check that we can produce a marked-scaled weak equivalence $J(\Lambda_0^n) \to R(\Lambda_0^n)$.

Notation 3.2.3.8. Denote by $\mathring{\Delta}$ the subcategory of Δ on morphisms $[m] \to [n]$ which send $0 \mapsto 0$. Denote by I the inclusion $\mathring{\Delta} \hookrightarrow \Delta$.

It is easy to check the following.

Lemma 3.2.3.9. The morphisms v_n form a natural transformation $v\colon \tilde{J} \circ I \Rightarrow \tilde{R} \circ I$.

Lemma 3.2.3.10. For all $n \geq 1$, there is a marked-scaled equivalence $J(\Lambda_0^n) \to R(\Lambda_0^n)$.

Proof. We can write $J(\Lambda_0^n)$ as a colimit of the composition $\tilde{J} \circ b$ coming from the bottom of the diagram

$$\begin{tikzcd} P_n \arrow[r, "a"] \arrow[d, hook] & \mathring{\Delta} \arrow[d, "I"'] \arrow[dr, "\tilde{J}\circ I"] & \\ (\Delta \downarrow \Lambda_0^n)^{\mathrm{nd}} \arrow[r, "b"] & \Delta \arrow[r, "\tilde{J}"] & \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}} \end{tikzcd},$$

where $(\Delta \downarrow \Lambda_0^n)^{\mathrm{nd}}$ is the category of nondegenerate simplices of Λ_0^n. Denote by P_n the poset of proper subsets of $[n]$ such that $0 \in S$. There is an obvious inclusion $P_n \hookrightarrow (\Delta \downarrow \Lambda_0^n)^{\mathrm{nd}}$, and one readily checks using Quillen's Theorem A that this inclusion is cofinal. Further note that the functor $P_n \to (\Delta \downarrow \Lambda_0^n) \to \Delta$ factors through $\mathring{\Delta}$ via a map $a\colon P_n \to \mathring{\Delta}$. Thus, we can equally express $J(\Lambda_0^n)$ as the colimit of the functor $\tilde{J} \circ I \circ a$. Precisely the same reasoning tells us that we can express $R(\Lambda_0^n)$ as the colimit of $\tilde{R} \circ I \circ a$. The result now follows from Lemma 3.2.3.9, Lemma 3.2.3.6, and the fact that each strict colimit is the model for the homotopy colimit. ∎

The author is indebted to Fernando Abellán García for his help with the following proof.

Proof of Lemma 3.2.3.6. In this proof, all simplicial subsets of Δ^{2n+1} will be assumed to carry the marking $\heartsuit$ and the scaling $\dagger$.

We can write each v_n as a composition

$$\Delta^{\{0,\dots,n,\overline{0}\}} \overset{v''_n}{\hookrightarrow} \Delta^{\{0,\dots,n,\overline{0}\}} \cup \Delta^{\{\overline{n},\dots,\overline{0}\}} \overset{v'_n}{\hookrightarrow} \Delta^{2n+1}.$$

Here, the map v''_n is a pushout along the inclusion $(\Delta^{\{n\}})^\sharp_\sharp \hookrightarrow (\Delta^n)^\sharp_\sharp$, which is marked-scaled anodyne by Proposition 3.1.4.2. It remains to show that each of the maps v'_n is marked-scaled anodyne.

To this end, we introduce some notation. Let $M^n_0 = \Delta^{\{0,\dots,n,\overline{0}\}} \cup \Delta^{\{\overline{n},\dots\overline{0}\}} \subset \Delta^{2n+1}$, and for $1 \leq k \leq n$, define

$$M^n_k := M^n_0 \cup \left(\bigcup_{\ell=1}^{k} \Delta^{[2n+1]\smallsetminus\{\ell,\overline{\ell}\}} \right).$$

There is an obvious filtration

$$M^n_0 \overset{i^n_0}{\hookrightarrow} M^n_1 \overset{i^n_1}{\hookrightarrow} \cdots \overset{i^n_{n-1}}{\hookrightarrow} M^n_n \overset{i^n_n}{\hookrightarrow} \Delta^{2n+1}. \tag{3.2.3.3}$$

Define

$$j^n_k := i^n_n \circ \cdots \circ i^n_k \colon M^n_k \hookrightarrow \Delta^{2n+1}, \qquad 0 \leq k \leq n.$$

In particular, note that $j^n_0 = v'_n$.

This allows us to replace our goal by something superficially more difficult: we would like to show that, for each $n \geq 0$ and each $0 \leq k \leq n$, the map i^n_k is marked-scaled anodyne. We proceed by induction. We take as our base case $n = 0$, where we have the trivial filtration

$$M^0_0 \overset{i^0_0}{=} \Delta^1.$$

This is an equality of subsets of Δ^1, hence certainly an equivalence.

We now suppose that the result holds true for $n-1$; that is, that the maps i^{n-1}_k are marked-scaled anodyne for all $0 \leq k \leq n-1$. We aim to show that each i^n_k is marked-scaled anodyne for each $0 \leq k \leq n$.

For $0 \leq k < n$, we can write i^n_k as the inclusion

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \hookrightarrow \mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{[2n+1]\smallsetminus\{k,\overline{k}\}}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,\dots n\} \\ \{1,\overline{1}\} \\ \vdots \\ \{k-1,\overline{k-1}\} \end{matrix} \right\}.$$

By Lemma 3.2.2.9, we have a pushout square

$$\begin{tikzcd}
\mathcal{S}^{\mathcal{A}|([2n+1]\smallsetminus\{k,\overline{k}\})}_{[2n+1]\smallsetminus\{k,\overline{k}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{[2n+1]\smallsetminus\{k,\overline{k}\}} \arrow[d, hook] \\
\mathcal{S}^{\mathcal{A}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{[2n+1]\smallsetminus\{k,\overline{k}\}}
\end{tikzcd}.$$

Therefore, it suffices to show that the top morphism is marked-scaled anodyne. One checks that this map is of the form j_k^{n-1}, and is thus marked-scaled anodyne by the inductive hypothesis. Therefore, it remains only to show that i_n^n is marked-scaled anodyne.

The case $n = 0$ is an isomorphism, so there is nothing to show. We treat the case $n = 1$ separately. In this case, i_1^1 takes the form

$$\Delta^{\{0,1,\overline{0}\}} \cup \Delta^{\{\overline{1},\overline{0}\}} \overset{v_1'}{\hookrightarrow} \Delta^3,$$

which we construct in the following way.

(1) First we fill the simplex $\Delta^{\{1,\overline{1},\overline{0}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A2).

(2) Then we fill the simplex $\Delta^{\{0,1,\overline{1}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A1).

(3) Finally, we fill the full simplex Δ^3 (together with its marking and scaling) as a pushout along a morphism of type (A1).

Now we may assume that $n \geq 2$. In this case, we can write $i_n^n \colon M_n^n \hookrightarrow \Delta^{2n+1}$ as an inclusion

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \subseteq \Delta^{2n+1}, \qquad \mathcal{A} = \left\{ \begin{smallmatrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,\ldots n\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{smallmatrix} \right\}.$$

We will express this inclusion as the following composition of fillings:

(1) We first add the simplices $\Delta^{\{n,\overline{n},\ldots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\ldots,\overline{0}\}}$, ..., $\Delta^{\{2,\ldots,n,\overline{n},\ldots,\overline{0}\}}$.

(2) We next add $\Delta^{\{0,\ldots,n,\overline{n},\overline{0}\}}$, $\Delta^{\{0,\ldots,n,\overline{n},\overline{n-1},\overline{0}\}}$, ..., $\Delta^{\{0,\ldots,n,\overline{n},\ldots,\overline{2},\overline{0}\}}$

(3) We next add $\Delta^{\{1,\ldots n,\overline{n},\ldots,\overline{0}\}}$.

(4) We finally add $\Delta^{\{0,\ldots n,\overline{n},\ldots,\overline{0}\}}$.

We proceed.

(1) • Using Lemma 3.2.2.9, we see that the square

$$\begin{tikzcd}
\mathcal{S}^{\mathcal{A}|\{n,\overline{n},\ldots,\overline{0}\}}_{\{n,\overline{n},\ldots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} \arrow[d, hook] \\
\mathcal{S}^{\mathcal{A}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\ldots,\overline{0}\}}
\end{tikzcd}$$

is pushout. In order to show that the bottom inclusion is marked-scaled anodyne, it thus suffices to show that the top inclusion is marked-scaled anodyne. We have

$$\mathcal{A}|\{n,\overline{n},\ldots,\overline{0}\} = \left\{ \begin{smallmatrix} \{\overline{n},\ldots,\overline{1}\} \\ \{n\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{n-1}\} \\ \{n,\overline{n}\} \end{smallmatrix} \right\} \sim \left\{ \begin{smallmatrix} \{n\} \\ \{\overline{n-1}\} \\ \vdots \\ \{\overline{1}\} \end{smallmatrix} \right\} =: \mathcal{A}' \ .$$

This is a dull subset of $\{n, \overline{n}, \dots, \overline{0}\}$ with pivot $\overline{n}$. The only $\mathcal{A}'$-basal set is $\{n, \overline{n-1}, \dots, \overline{1}\}$. One checks that $\mathcal{A}'$, together with the marking $\heartsuit$ and scaling $\dagger$, satisfies the conditions of the Pivot Trick:

- For $n = 2$, the only the marked edge not containing the pivot $\overline{2}$ is $\overline{1} \to \overline{0}$, which belongs to $\mathcal{S}^{\mathcal{A}'}_{\{n,\overline{n},\dots,\overline{0}\}}$. The only scaled simplex which does not contain $\overline{2}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{2 < \overline{1} < \overline{0}\}$. The simplex $\sigma \cup \{\overline{2}\}$ is fully scaled.
- For $n = 3$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'}$. The only scaled triangle which does not contain the pivot $\overline{3}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{3 < \overline{2} < \overline{1}\}$. The simplex $\sigma \cup \{\overline{3}\}$ is fully scaled.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 3.2.2.7.

In each case, the simplex $\{n, \overline{n}, \overline{n-1}\}$ is scaled, so the top inclusion is marked-scaled anodyne by the Pivot Trick. We can write

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\dots,\overline{0}\}} = \mathcal{S}^{\mathcal{A}\cup([2n+1]\smallsetminus\{n,\overline{n},\dots,\overline{0}\})}_{[2n+1]},$$

We see that

$$\mathcal{A} \cup ([2n+1] \smallsetminus \{n, \overline{n}, \dots, \overline{0}\}) \sim \left\{ \begin{smallmatrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,\dots n-1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{smallmatrix} \right\},$$

which we denote by $\mathcal{A}_{n-1}$.

- We proceed inductively. Suppose we have added the simplices $\Delta^{\{n,\overline{n},\dots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\dots,\overline{0}\}}$, $\dots$, $\Delta^{\{k+1,\dots,n,\overline{n},\dots,\overline{0}\}}$, for $2 \leq k \leq n-1$. Using Lemma 3.2.2.8 and Lemma 3.2.2.6 can write the result of these additions as

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \left(\bigcup_{i=k+1}^{n} \Delta^{\{i,\dots,n,\overline{n},\dots,\overline{0}\}} \right) = \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]},$$

where

$$\mathcal{A}_{k+1} = \left\{ \begin{smallmatrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,\dots,k\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{smallmatrix} \right\}.$$

Using Lemma 3.2.2.9, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{A}_{k+1}|\{k,\dots,n,\overline{n},\dots,\overline{0}\}}_{\{k,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{k,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \cup \Delta^{\{k,\dots,n,\overline{n},\dots,\overline{0}\}} \end{tikzcd}$$

is pushout, so in order to show that the bottom morphism is marked-scaled anodyne, it suffices to show that the top morphism is. We see that

$$\mathcal{A}_{k+1}|\{k,\dots,n,\overline{n},\dots,\overline{0}\} = \left\{\begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k,\overline{k}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\} \sim \left\{\begin{matrix} \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k+1,\overline{k+1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\} =: \mathcal{A}'_{k+1}.$$

This is a dull subset of $P(\{k,\dots,n,\overline{n},\dots,\overline{0}\})$ with pivot $\overline{k}$, and one checks that the conditions of the Pivot Trick are satisfied:

- For $n = 3$, where the only value of k is $k = 2$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'_{k+1}}$ by Lemma 3.2.2.7, and one can check that each scaled triangle factors through $\mathcal{S}^{\mathcal{A}'_{k+1}}$.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 3.2.2.7.

The basal sets are of the form

$$\{k, a_1, \dots, a_{n-k}, \overline{k-1}, \dots, \overline{1}\},$$

where each a_1, ..., a_{n-k} is of the form ℓ or $\overline{\ell}$ for $k+1 \leq \ell \leq n$. In each case, the simplex $\{a_{n-k}, \overline{k}, \overline{k-1}\}$ is scaled. Thus, the conditions of the Pivot Trick, the top morphism is marked-scaled anodyne.

We have now added the simplices promised in part 1., and are left with the simplicial subset $\mathcal{S}^{\mathcal{A}_2}_{[2n+1]}$, where

$$\mathcal{A}_2 = \left\{\begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\}.$$

(2) Each step in this sequence is solved exactly like those above. The calculations are omitted. The end result is the simplicial subset

$$\mathcal{B}_2 = \left\{\begin{matrix} \{\overline{1}\} \\ \{0,1\} \\ \{2,\overline{2}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\}.$$

(3) Using Lemma 3.2.2.9, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{B}_2|\{1,\dots,n,\overline{n},\dots,\overline{0}\}}_{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{B}_2}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{B}_2}_{[2n+1]} \cup \Delta^{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \end{tikzcd}$$

is pushout. We thus have to show that the top morphism is marked-scaled anodyne. We have

$$\mathcal{B}_2|\{1,\dots,n,\overline{n},\dots,\overline{0}\} \sim \left\{\begin{matrix} \{\overline{1}\} \\ \{2,\overline{2}\} \\ \vdots \\ \{n,\overline{n}\} \\ \{\overline{1}\} \end{matrix}\right\}.$$

One readily sees that this is a right-dull subset, and that the conditions of the Right-Anodyne Pivot Trick are satisfied.

(4) One solves this as before, checking that the conditions of the Pivot Trick are satisfied, with pivot point 2.

□

3.2.4 The map governing local systems is a cocartesian fibration

Our aim is to show that the map of quasicategories $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ defined by the diagram

$$\begin{tikzcd} \mathrm{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hookleftarrow] \arrow[d, "p''"] & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}] \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd},$$

in which both squares are pullback, is a cocartesian fibration. We now define the class of morphisms in $\mathrm{LS}(\mathcal{C})$ which we claim are p-cocartesian.

Definition 3.2.4.1. A morphism $\tilde{\sigma}\colon \Delta^1 \to \mathrm{LS}(\mathcal{C})$ is said to be ***left Kan*** if the simplex $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ to which it is adjoint has the property that the restriction $\sigma|\Delta^{\{0,1,\overline{0}\}}$ is left Kan in the sense of Corollary 3.1.6.13.

We draw the 'front' and 'back' of a general 3-simplex $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ corresponding to some morphism $\Delta^1 \to \mathrm{LS}(\mathcal{C})$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, "\mathcal{H}"] & Y \arrow[d, "\mathcal{G}"] \\ \mathcal{C} \arrow[ur, phantom, "\Rightarrow\zeta"] & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\Rightarrow\eta"] & Y \arrow[d, "\mathcal{G}"] \arrow[dl, "\mathcal{G}'"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd}$$

Here, the 2-simplices which are notated without natural transformations are constrained by the scaling $\dagger$ to be thin. The morphism $\tilde{\sigma}$ is left Kan if and only if $\mathcal{G}'$ is a left Kan extension of $\mathcal{F}$ along f, and η is a unit map.

Notation 3.2.4.2. We endow the quasicategory $\mathrm{LS}(\mathcal{C})$ with the marking $\clubsuit \subseteq \mathrm{LS}(\mathcal{C})_1$ consisting of all left Kan edges.

Theorem 3.2.4.3. The map $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ is a cocartesian fibration, and an edge $\Delta^1 \to \mathrm{LS}(\mathcal{C})$ is p-cocartesian if and only if it is left Kan.

Proof. By [HTT, Prop. 3.1.1.6], it suffices to show that the map $\mathrm{LS}(\mathcal{C})^\clubsuit \to \mathcal{S}^\sharp$ has the right lifting property with respect to each of the classes of generating marked anodyne morphisms. We check these one-by-one.

(1) We need to check that all lifting problems

$$\begin{tikzcd} (\Lambda^n_i)^\flat \arrow[r] \arrow[d] & \mathrm{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ (\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp \end{tikzcd}, \qquad n \geq 2, \quad 0 < i < n$$

admit solutions. This follows from Theorem 3.2.1.4.

(2) We need to show that each of the lifting problems

$$\begin{tikzcd}
(\Lambda^n_0)^{\mathcal{L}} \arrow[r] \arrow[d] & \mathrm{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\
(\Delta^n)^{\mathcal{L}} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp}
\end{tikzcd}, \qquad n \geq 1.$$

has a solution. Given any such lifting problem, it suffices to find a solution to the outer lifting problem

$$\begin{tikzcd}
(\Lambda^n_0)^{\mathcal{L}} \arrow[r] \arrow[d] & \mathrm{LS}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{R} \arrow[d] \\
(\Delta^n)^{\mathcal{L}} \arrow[r] \arrow[urr, dashed] & \mathcal{S} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}]
\end{tikzcd}.$$

Passing to the adjoint lifting problem, we need to find a solution to the lifting problem

$$\begin{tikzcd}
R(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp \arrow[r, "\sigma"] \arrow[d, hook] & \mathcal{C}\mathrm{at}_\infty \\
R(\Delta^n) \arrow[ur, dashed] &
\end{tikzcd} \tag{3.2.4.1}$$

such that $\sigma|\{0, 1, \overline{0}\}$ is left Kan. Here R is the functor of Definition 3.2.3.4.

We note the existence of a commutative square

$$\begin{tikzcd}
J(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp \arrow[r, hook, "b"] \arrow[d, hook] & R(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp \arrow[d, hook] \\
J(\Delta^n) \arrow[r, hook, "v_n"] & R(\Delta^n)
\end{tikzcd},$$

where J is the functor defined in Definition 3.2.3.5. Here, v_n is the morphism of Lemma 3.2.3.6. The morphism b is defined component-wise via the following morphisms.

- The morphism $J(\Lambda^n_0) \to R(\Lambda^n_0)$ comes from Lemma 3.2.3.10, where it is proven to be a weak equivalence in the marked-scaled model structure.
- The map $(\Delta^{\{\overline{0}\}})^\sharp_\sharp \to ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp$ is marked-scaled anodyne by Proposition 3.1.4.2.
- The maps $(\Delta^{\{\overline{0}\}})^\sharp_\sharp \to ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp$ is marked-scaled anodyne by Proposition 3.1.4.2.
- The rest of the morphisms connecting the components are isomorphisms.

We showed in Lemma 3.2.3.6 that the lower morphism in this square is a marked-scaled equivalence. We would now like to show that the upper morphism is a marked-scaled equivalence. Recall that, since the cofibrations in the model structure on marked-scaled simplicial sets are simply those morphisms whose underlying morphism of simplicial sets is a monomorphism, the colimits defining b are models for the homotopy colimits, so the result follows from the observation that each component is a weak equivalence in the marked-scaled model structure.

Using [HTT, Prop. A.2.3.1], we see that in order to solve the lifting problem of Equation 3.2.4.1, it suffices to show that the lifting problem

$$\begin{tikzcd} J(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp \arrow[r, "b\circ\sigma"] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ J(\Delta^n) \arrow[ur, dashed, "\ell"'] & \end{tikzcd}$$

has a solution whose restriction $\ell|\{0,1,\overline{0}\}$ is left Kan. However, we note that there exists a pushout square

$$\begin{tikzcd} (\Lambda_0^{\{0,\dots,n,\overline{0}\}})^\flat_\flat \arrow[r] \arrow[d] & J(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp \arrow[d] \\ (\Delta^{\{0,\dots,n,\overline{0}\}})^\flat_\flat \arrow[r] & J(\Delta^n) \end{tikzcd}$$

in which the rightward-facing morphisms are isomorphisms on underlying simplicial sets, and where the only new decorations being added are $(\Delta^{\{0,\dots,n\}})^\flat_\flat \hookrightarrow (\Delta^{\{0,\dots,n\}})^\flat_\sharp$. Therefore, it suffices to solve the lifting problems

$$\begin{tikzcd} (\Lambda_0^{\{0,\dots,n,\overline{0}\}})^\flat_\flat \arrow[r] \arrow[d] & \mathbb{C}\mathrm{at}_\infty \\ (\Delta^{\{0,\dots,n,\overline{0}\}})^\flat_\flat \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \quad , \qquad n \geq 1$$

such that $\ell'|\{0,1,\overline{0}\}$ is left Kan. That these lifting problems admit solutions for $n \geq 2$ follows. The case $n = 1$ is the statement that left Kan extensions of functors into cocomplete categories exist along functors between small categories.

(3) We need to show that every lifting problem of the form

$$\begin{tikzcd} (\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\sharp \arrow[r] \arrow[d] & \mathrm{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ (\Delta^2)^\sharp \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp \end{tikzcd}$$

has a solution. Considering the adjoint lifting problem, we find that it suffices to show that for any $\sigma\colon \Delta^5_\dagger \to \mathbb{C}\mathrm{at}_\infty$ such that the restrictions $\sigma|\{0,1,\overline{0}\}$ and $\sigma|\{1,2,\overline{1}\}$ are left Kan and the morphisms belonging to $\sigma|\{\overline{2},\overline{1},\overline{0}\}$ are equivalences, the restriction $\sigma|\{0,2,\overline{0}\}$ is left Kan. Applying Lemma 3.1.6.14 to $\sigma|\{1,2,\overline{1},\overline{0}\}$, we see that $\sigma|\{1,2,\overline{1}\}$ is left Kan if and only if $\sigma|\{1,2,\overline{0}\}$ is left Kan. Applying Lemma 3.1.6.16 to $\sigma|\{0,1,2,\overline{0}\}$ guarantees that $\sigma|\{0,2,\overline{0}\}$ is left Kan as required.

(4) We need to show that for all Kan complexes K, the lifting problem

$$\begin{tikzcd} K^\flat \arrow[r] \arrow[d] & \mathrm{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ K^\sharp \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp \end{tikzcd}$$

has a solution. To see this, note that each morphism in K must be mapped to an equivalence in $\mathrm{LS}(\mathcal{C})$, and a morphism $a\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{LS}(\mathcal{C})$ is an equivalence and only if it is p-cartesian, and lies over an equivalence in $\mathcal{S}$.

- By Theorem 3.2.1.4, the morphism a is p-cartesian if and only if the map $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ to which it is adjoint factors through the map $\Delta^3_\dagger \to \Delta^3_\sharp$. In particular, the restriction $\sigma|\Delta^{\{0,1,\overline{0}\}}$ is thin.
- The image of a in $\mathcal{S}$ is the restriction $\sigma|\Delta^{\{0,1\}}$, which is therefore an equivalence.

It follows from Example 3.1.6.7 that the morphism a is automatically left Kan. Thus, each morphism in K is mapped to a left Kan morphism in $\mathrm{LS}(\mathcal{C})$, and we are justified in marking them; our lifting problems admit solutions.

□

3.3 Pull-push of local systems

For any cocomplete ∞-category $\mathcal{C}$, we have constructed an ∞-category $\mathrm{LS}(\mathcal{C})$ of local systems on $\mathcal{C}$, together with a map $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$. We have further shown that this map is a bicartesian fibration.

- As a cartesian fibration, it classifies the pullback functor $(-)^*\colon \mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$; for any morphism $X \to Y$ in $\mathcal{S}$, this functoriality gives us a map
$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$
- As a cocartesian fibration, it classifies the left Kan extension functor $(-)_!\colon \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$; for any morphism $X \to Y$, in $\mathcal{S}$, this functoriality gives us a map
$$f_!\colon \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(Y, \mathcal{C}).$$

We can combine these functorialities. Given a *span* of morphisms in $\mathcal{S}$, i.e. a diagram in $\mathcal{S}$ of the form

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd}, \tag{3.3.0.1}$$

we can pull back along g and push forward along f, giving a map $f_! \circ g^*\colon \mathrm{Fun}(X, \mathcal{C}) \to$

$\mathrm{Fun}(X', \mathcal{C})$ via the composition

$$\begin{tikzcd} & \mathrm{Fun}(Y, \mathcal{C}) \arrow[dr, "f_!"] & \\ \mathrm{Fun}(X, \mathcal{C}) \arrow[ur, "g^*"] \arrow[rr, dashed, "f_! \circ g^*"'] & & \mathrm{Fun}(X', \mathcal{C}) \end{tikzcd} .$$

In this section we recall a construction which upgrades this construction to a functor $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$, where $\mathrm{Span}(\mathcal{S})$ is an ∞-category with the following rough description.

- The objects of $\mathrm{Span}(\mathcal{S})$ are the same as the objects of $\mathcal{S}$, i.e. spaces X, Y, etc.
- The morphisms from X to X' are given by spans in $\mathcal{S}$, i.e. diagrams of the form given in Equation 3.3.0.1.
- The 2-simplices witnessing the composition of two morphisms

  $$X \xleftarrow{h} Y \xrightarrow{f} X' \quad \text{and} \quad X' \xleftarrow{g} Y \xrightarrow{j} X''$$

  are diagrams of the form

  $$\begin{tikzcd} & & Z \arrow[dl, "g'"'] \arrow[dr, "f'"] & & \\ & Y \arrow[dl, "h"'] \arrow[dr, "f"] & & Y' \arrow[dl, "g"'] \arrow[dr, "j"] & \\ X & & X' & & X'' \end{tikzcd} , \tag{3.3.0.2}$$

  where the square formed is pullback. The corresponding composition is then given by the span

  $$\begin{tikzcd} & Z \arrow[dl, "h \circ g'"'] \arrow[dr, "j \circ f'"] & \\ X & & X'' \end{tikzcd} .$$

The non-trivial part of constructing such a functor $\hat{r}$ will be showing that it respects composition in a homotopy-coherent way; we need that for any diagram of the form given in Equation 3.3.0.2, both ways of composing morphisms from left to right in the diagram

$$\begin{tikzcd} & & \mathrm{Fun}(Z, \mathcal{C}) \arrow[dr, "(f')_!"] & & \\ & \mathrm{Fun}(Y, \mathcal{C}) \arrow[ur, "(g')^*"] \arrow[dr, "f_!"] & & \mathrm{Fun}(Y', \mathcal{C}) \arrow[dr, "j_!"] & \\ \mathrm{Fun}(X, \mathcal{C}) \arrow[ur, "h^*"] & & \mathrm{Fun}(X', \mathcal{C}) \arrow[ur, "g^*"] & & \mathrm{Fun}(X'', \mathcal{C}) \end{tikzcd} ,$$

agree up to a specified natural equivalence. The arrows h^* and $j_!$ are the same in both cases, and can be ignored. This condition can thus be distilled down to the so-called *Beck-Chevalley condition:*

- For any pullback square in $\mathcal{S}$

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd},$$

  the comparison map

$$f_! \circ g^* \overset{\eta}{\Rightarrow} f_! \circ g^* \circ (f')^* \circ (f')_! \overset{\simeq}{\Rightarrow} f_! \circ f^* \circ (g')^* \circ (f')_! \overset{\epsilon}{\Rightarrow} (g')^* \circ (f')_!$$

  is an equivalence.

One can phrase the Beck-Chevalley condition at the level of fibrations rather than functors into $\mathcal{C}\mathrm{at}_\infty$. This is a standard definition, here more or less lifted from [HL13].

Definition 3.3.0.1. A bicartesian fibration of quasicategories $p\colon \mathcal{X} \to \mathcal{T}$ such that $\mathcal{T}$ admits pullbacks is called a ***Beck-Chevalley fibration*** if it has the following property.

(BC) For any commuting square in $\mathcal{X}$

$$\begin{tikzcd} z \arrow[r, "f'"] \arrow[d, "g'"'] & y \arrow[d, "g"] \\ y' \arrow[r, "f"] & x \end{tikzcd}$$

lying over a pullback square in $\mathcal{T}$, if the morphism f is p-cocartesian and the morphisms g and g' are p-cartesian, then the morphism f' is p-cocartesian.

In fact, it turns out that this condition is sufficient to guarantee that the the pull-push procedure is functorial.

Proposition 3.3.0.2. Let $p\colon \mathcal{X} \to \mathcal{T}$ be a Beck-Chevalley fibration. Then there is a functor $\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_\infty$ sending an object $t \in \mathrm{Span}(\mathcal{T})$ to the fiber $\mathcal{X}_t$, and a span $t \overset{b}{\leftarrow} s \overset{a}{\rightarrow} t'$ to the composition $a_! \circ b^*\colon \mathcal{X}_t \to \mathcal{X}_{t'}$.

Proof. It follows immediately from Theorem 3.1.5.2 that there is a cocartesian fibration

$$\tilde{p}\colon \mathrm{Span}^{\mathrm{cart}}(\mathcal{X}) \to \mathrm{Span}(\mathcal{T}),$$

where $\mathrm{Span}^{\mathrm{cart}}$ denotes the category of spans whose backwards-facing legs are constrained to be p-cartesian; that is, $\mathcal{X}^\dagger = \mathcal{X}^{\mathrm{cart}}$. Straightening gives the result that we want. □

We would like to show that pull-push of local systems is functorial. According to Proposition 3.3.0.2, it suffices to show the following, the proof of which will come at the end of this section.

Proposition 3.3.0.3. The functor $p\colon \mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ is a Beck-Chevalley fibration.

We first prove a helpful lemma. This is simply a reformulation of the definition of a left Kan extension in the special case that we consider left Kan extensions along maps of Kan complexes.

Lemma 3.3.0.4. Let $f\colon X \to Y$ be a map between Kan complexes, let $F\colon X \to \mathcal{C}$ and $G\colon Y \to \mathcal{C}$ be functors, and let $\eta\colon F \to G \circ y$ be a natural transformation. Then η exhibits G as a left Kan extension of F along f if and only if, for all $y \in Y$, the natural transformatin $F \circ \pi \Rightarrow \underline{G(y)}$ defined by the pasting diagram

$$\begin{tikzcd} X_{/y} \arrow[r, "\pi"] \arrow[d] & X \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \eta"] & \mathcal{C} \\ \{y\} \arrow[r, hook] & Y \arrow[ur, "G"'] & \end{tikzcd},$$

exhibits $G(y)$ as the colimit of $F \circ \pi$, where the left-hand square is homotopy pullback.

Proof. We note that we can factor the above square into three squares

$$\begin{tikzcd} X_{/y} \arrow[r, "\simeq"] \arrow[d] & X^{/y} \arrow[r] \arrow[d] & Y^{\Delta^1} \times_Y X \arrow[r, "\simeq"] \arrow[d] & X \arrow[d, "f"] \\ \{y\} \arrow[r, equal] & \{y\} \arrow[r] & Y \arrow[r, equal] & Y \end{tikzcd},$$

where the middle square is a strict pullback. Since the map $Y^{\Delta^1} \times_Y X \to Y$ is a Kan fibration, the middle square is also a homotopy pullback. We note that the left and right squares are also homotopy pullbacks, since the horizontal morphisms are weak equivalences. Thus, the outer square is a homotopy pullback. $\square$

Proof of Theorem 3.3.0.3. We have already shown that p is a bicartesian fibration, and we know that $\mathcal{S}$ admits pullbacks. Therefore, it suffices to show that p has Property (BC).

We consider a square $\sigma\colon \Delta^1 \times \Delta^1 \cong \Delta^{\{0,1,2\}} \amalg_{\Delta^{\{0,2\}}} \Delta^{\{0,1',2\}} \to \mathrm{LS}(\mathcal{C})$ with the following properties.

(1) The restriction $\sigma|\Delta^{\{0,1\}}$ is p-cartesian.

(2) The restriction $\sigma|\Delta^{\{1',2\}}$ is p-cartesian.

(3) The restriction $\sigma|\Delta^{\{1,2\}}$ is p-cocartesian.

(4) The square σ lies over a pullback square

$$p(\sigma) = \begin{tikzcd} X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[d, "g"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

in $\mathcal{S}$.

We need to show that $\sigma|\Delta^{\{0,1'\}}$ is p-cocartesian.

The map σ adjunct to a map

$$\tau\colon \Delta_\dagger^{\{0,1,2,\overline{2},\overline{1},\overline{0}\}} \amalg_{\Delta_\dagger^{\{0,2,\overline{2},\overline{0}\}}} \Delta_\dagger^{\{0,1',2,\overline{2},\overline{1}',\overline{0}\}} \to \mathbb{C}\mathrm{at}_\infty$$

such that $\tau|\Delta^{\{\bar{2},\bar{1},\bar{0}\}} \amalg_{\Delta^{\{\bar{2},\bar{0}\}}} \Delta^{\{\bar{2},\bar{1}',\bar{0}\}}$ is the constant functor with value $\mathcal{C}$. We can read off the following further properties of τ, corresponding to the properties of σ above.

(1) The simplices $\tau|\Delta^{\{0,1,\bar{0}\}}$ and $\tau|\Delta^{\{0,\bar{1},\bar{0}\}}$ are thin.

(2) The simplices $\tau|\Delta^{\{1',2,\bar{1}'\}}$ and $\tau|\Delta^{\{1',\bar{2},\bar{1}'\}}$ are thin.

(3) The simplex $\tau|\Delta^{\{1,2,\bar{1}\}}$ is left Kan.

(4) The restriction $\tau|\Delta^{\{0,1,2\}} \amalg_{\Delta^{\{0,2\}}} \Delta^{\{0,1',2\}}$ is equal to $p(\sigma)$.

The condition that $\sigma|\Delta^{\{0,1'\}}$ is p-cocartesian corresponds to the condition that the simplex $\tau|\Delta^{\{0,1,\bar{0}\}}$ is left Kan.

The diagram τ contains a lot of redundant data, which we would now like to consolidate. We first shuffle some data around our diagram. Applying Lemma 3.1.6.14 to the simplex $\tau|\Delta^{\{1,2,\bar{1},\bar{0}\}}$, we see that $\tau|\Delta^{\{1,2,\bar{0}\}}$ is left Kan. Applying Lemma 3.1.6.15 to the simplex $\tau|\Delta^{\{1',2,\bar{1}',\bar{0}\}}$, we see that $\tau|\Delta^{\{1',2,\bar{0}\}}$ is thin.

We can now study a subdiagram which contains all the information we need. We consider the restriction

$$\tau|\Delta^{\{0,1,2,\bar{0}\}} \amalg_{\Delta\{0,2,\bar{0}\}} \Delta^{\{0,1',2,\overline{0'}\}}.$$

This is determined, up to specific choices of compositions, by the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[r, "F"] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow\eta"] & \mathcal{C} \\
X_1 \arrow[r, "f"] & X_2 \arrow[ur, "G"'] &
\end{tikzcd} .$$

Here, the square pictured is σ, and the triangle is $\tau|\Delta^{\{1',2,\bar{0}\}}$.

Since the simplices $\tau|\Delta^{\{0,1,\bar{0}\}}$ and $\tau|\Delta^{\{1,2,\bar{0}\}}$ are thin, the pasting of the square and the triangle above is homotopic to the restriction $\tau|\Delta^{\{0,1,\bar{0}\}}$. This is the triangle which we wish to show is left Kan. According to Lemma 3.3.0.4 it suffices to show that for all $x \in X_1$, the pasting diagram

$$\begin{tikzcd}
(X_0)_{/x} \arrow[r, "\pi"] \arrow[d] & X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[r, "F"] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow\eta"] & \mathcal{C} \\
\{x\} \arrow[r, hook] & X_1 \arrow[r, "f"] & X_2 \arrow[ur, "G"'] &
\end{tikzcd}$$

exhibits $G(f(x))$ as the colimit of $F \circ f' \circ \pi$. But applying the pasting law for homotopy pullbacks, this follows directly from the assumption that η exhibits G as a left Kan extension of F along f. $\square$

3.4 Monoidal pull-push of local systems

The classical Grothendieck construction provides an equivalence between pseudofunctors $\hat{p}\colon \mathcal{D} \to \mathcal{C}at$ and cartesian fibrations $p\colon \int \hat{p} \to \mathcal{D}^{\mathrm{op}}$. There are many conditions one can impose upon the functor $\hat{p}$, and many structures one can endow it with; it is natural to wonder whether these properties and structures can be captured in the fibration p.

An answer is known in the case that $\hat{p}$ is lax monoidal (the author learned about this fact from [MV20]). Suppose $p\colon \mathcal{C} \to \mathcal{D}$ is a cartesian fibration of 1-categories which classifies a pseudofunctor $\hat{p}\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$. Further suppose the following:

- The category $\mathcal{C}$ carries a symmetric monoidal structure $(\otimes, I_{\mathcal{C}}, \ldots)$.
- The category $\mathcal{D}$ carries a symmetric monoidal structure $(\boxtimes, I_{\mathcal{D}}, \ldots)$.
- The functor p is strong monoidal: for objects x, $y \in \mathcal{C}$, we have that $p(x \otimes y) = p(x) \boxtimes p(y)$, and $p(I_{\mathcal{C}}) = I_{\mathcal{D}}$.[2]
- The monoidal product $\otimes$ preserves p-cartesian morphisms in the sense that if f and g are p-cartesian morphisms, then $f \otimes g$ is also p-cartesian.

Under these conditions, the pseudofunctor $\hat{p}$ carries a lax monoidal structure $(\mathcal{D}^{\mathrm{op}}, \boxtimes) \to (\mathcal{C}\mathrm{at}, \times)$, described as follows.

- Restricting the tensor product
$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
to the fibers $\mathcal{C}_d$ and $\mathcal{C}_{d'}$ of p over d and d' yields the structure maps
$$\hat{p}(d) \times \hat{p}(d') \to \hat{p}(d \boxtimes d').$$
That these structure maps form a pseudonatural transformation, i.e. that the necessary squares commute up to specified homotopy, follows from the assumption that the monoidal product preserves p-cartesian morphisms in each slot.
- The requirement that $p(I_{\mathcal{C}}) = I_{\mathcal{D}}$ can be rephrased to say that $I_{\mathcal{D}} \in \hat{p}(I_{\mathcal{C}})$. This gives us the structure map $* \to \hat{p}(I_{\mathcal{D}})$, where $*$ is the unit object in $\mathcal{C}\mathrm{at}$.

If we want to study covariant rather than contravariant pseudofunctors, we should replace cartesian fibrations by cocartesian fibrations; of course, now we must assume that $\otimes$ preserves cocartesian edges rather than cartesian. The rest of the theory remains unchanged. These results also remain true in the ∞-categorical case, as we will show in Subsection 3.4.1.

As we saw in the last section, if we have a bicartesian fibration which satisfies the Beck-Chevalley condition (i.e. a Beck-Chevalley fibration), we can combine both cartesian and cocartesian functorialities into pull-push functoriality. Our aim in Subsection 3.4.2 will be to show that the theory of Beck-Chevalley fibrations admits a monoidal generalization. More specifically, we will show the following. Consider a Beck-Chevalley fibration $r\colon \mathcal{X} \to \mathcal{T}$ such that $\mathcal{X}$ carries a symmetric monoidal structure $\otimes$, and $\mathcal{T}$ carries a symmetric monoidal structure $\boxtimes$. Under the assumption that r is strong monoidal, and that $\otimes$ preserves both cartesian and cocartesian edges, the induced functor
$$\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_\infty$$
constructed in the previous section carries a lax monoidal structure. Note that this statement is not entirely original. A similar result in a somewhat different context is proved in [BGS19].

[2]The reason for the strict equality in this condition is that we have chosen specific monoidal functors $\otimes$ and $\boxtimes$, and specific unit objects $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$. Later, we will replace these strict choices by fibrations which give the same data up to coherent homotopy.

We will then apply our results to local systems. We will show that if $\mathcal{C}$ is a symmetric monoidal ∞-category, then the ∞-category of $\mathcal{C}$-local systems carries a symmetric monoidal structure, defined on objects by

$$(\mathcal{F}\colon X \to \mathcal{C}) \otimes (\mathcal{G}\colon Y \to \mathcal{C}) \quad = \quad X \times Y \xrightarrow{\mathcal{F}\times\mathcal{G}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

and that the functor $\mathrm{LS}(\mathcal{C}) \to \mathcal{S}$ is can be given the structure of a monoidal Beck-Chevalley fibration, thus classifying a lax monoidal functor

$$(\mathrm{Span}(\mathcal{S}), \tilde{\times}) \to (\mathcal{C}\mathrm{at}_\infty, \times).$$

3.4.1 The lax monoidal Grothendieck construction

Our aim in this section is to show an ∞-categorical version of the statement that a monoidal cartesian fibration $p\colon (\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$ such that $\otimes$ preserves p-cartesian morphisms straightens to a lax monoidal pseudofunctor $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ (as explained more concretely in the introduction to Section 3.4). We will assume familiarity with the theory of symmetric monoidal ∞-categories as laid out in [HA, Chap. 2]; roughly, a symmetric monoidal ∞-category is defined to be a cocartesian fibration classifying a commutative monoid in $\mathcal{C}\mathrm{at}_\infty$. Our first goal will be to rephrase some of the results there in terms of cartesian fibrations rather than cocartesian fibrations.

Notation 3.4.1.1. We will frequently refer to the following maps in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$.

- Denote by $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ the map in $\mathcal{F}\mathrm{in}_*$ sending $i \mapsto 1$ and everything else to $*$. Denote the same map in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ by ρ_i.
- Denote by $\mu\colon \langle 2\rangle \to \langle 1\rangle$ the active map in $\mathcal{F}\mathrm{in}_*$. We will denote the same map in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ also by μ.

Definition 3.4.1.2. A ***CSMC*** (contravariantly-presented symmetric monoidal ∞-category) is a cartesian fibration $p\colon \mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ such that the contravariant transport maps $\rho_i^*\colon (\mathcal{C}_\otimes)_{\langle n\rangle} \to (\mathcal{C}_\otimes)_{\langle 1\rangle}$ are the canonical projections exhibiting $(\mathcal{C}_\otimes)_{\langle n\rangle}$ as an n-fold homotopy product.

Notation 3.4.1.3. For any CSMC $\mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, we will denote the fiber $(\mathcal{C}_\otimes)_{\langle 1\rangle}$ simply by $\mathcal{C}$.

The equivalence $(\mathcal{C}_\otimes)_{\langle n\rangle} \simeq \mathcal{C}^n$ allow us to trade maps into $(\mathcal{C}_\otimes)_{\langle n\rangle}$ for n maps into $\mathcal{C}$, well-defined up to equivalence. Given a diagram $a\colon K \to (\mathcal{C}_\otimes)_{\langle n\rangle}$, we will often abuse this terminology by calling any such corresponding diagrams $a_i\colon K \to \mathcal{C}$ 'the' components of a, as long as we are making reference only to properties of these components which are preserved under equivalence.

Definition 3.4.1.4. A ***map*** between CSMCs $q\colon \mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ and $p\colon \mathcal{D}_\boxtimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is a functor r making the diagram

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

commute. Given a map of CSMCs as above, we will further make use of the following terminology.

- The map r is a ***monoidal functor*** if it sends q-cartesian morphisms to p-cartesian morphisms.

  This implies, for example, that $r(x \otimes y) \simeq r(x) \boxtimes r(y)$, and that $r(I_{\mathcal{C}}) \simeq I_{\mathcal{D}}$. It also automatically implies that the diagrams encoding associativity, etc., commute up to coherent homotopy.

- An edge f in $(\mathcal{C}_{\otimes})_{\langle n\rangle}$ is ***componentwise cartesian (resp. componentwise cocartesian)*** if for each $1 \leq i \leq n$, the transport $\rho_i^*(f)$ is $r|\langle 1\rangle$-cartesian (resp. cocartesian).

  We can think of a morphism f in $(\mathcal{C}_{\otimes})_{\langle n\rangle}$ as an n-tuple of morphisms f_i in $\mathcal{C}$. We say that f is componentwise (co)cartesian if each component f_i is (co)cartesian as an edge of the underlying fibration $\mathcal{C} \to \mathcal{D}$.

- The tensor product $\otimes$ ***preserves cartesian (resp. cocartesian) edges*** if for all $\phi\colon \langle n\rangle \leftarrow \langle m\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the associated functor $\phi^*\colon (\mathcal{C}_{\otimes})_{\langle m\rangle} \to (\mathcal{C}_{\otimes})_{\langle n\rangle}$ sends componentwise cartesian (resp. componentwise cocartesian) morphisms contained in the fiber $(\mathcal{C})_{\langle m\rangle}$ to componentwise cartesian (resp. componentwise cocartesian) morphisms contained in the fiber $(\mathcal{C}_{\otimes})_{\langle n\rangle}$.

  We will show in Lemma 3.4.1.5 that this is equivalent to demanding that if morphisms f and g in $\mathcal{C}$ are $r|\langle 1\rangle$-(co)cartesian, then $f \otimes g$ is as well.

Lemma 3.4.1.5. Let

$$\begin{tikzcd} \mathcal{C}_{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_{\boxtimes} \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

be a map between CSMCs. The tensor product $\otimes$ preserves cartesian (resp. cocartesian) morphisms if and only if for all $r|\langle 1\rangle$-cartesian (resp. $r|\langle 1\rangle$-cocartesian) morphisms f and g in $\mathcal{C}$, the morphism $f \otimes g$ is also $r|\langle 1\rangle$-cartesian (resp. $r|\langle 1\rangle$-cocartesian).

Proof. We consider the cartesian case. The cocartesian case is identical.

Suppose that $\otimes$ preserves r-cartesian morphisms, and let f and g be $r|\langle 1\rangle$-cartesian morphisms in $\mathcal{C}$. Then there is a morphism $[f, g]$ in $(\mathcal{C}_{\otimes})_{\langle 2\rangle}$ with components f and g. Taking ϕ to be the active map $\langle 2\rangle \to \langle 1\rangle$ shows that $f \otimes g$ is $r|\langle 1\rangle$-cartesian.

Conversely, suppose that for any $r|\langle 1\rangle$-cartesian morphisms f and g, the morphism $f \otimes g = \phi^*([f, g])$ in $\mathcal{C}$ is $r|\langle 1\rangle$-cartesian. By associativity of the tensor product and induction, we have that the n-ary product of $r|\langle 1\rangle$-cartesian morphisms is again $r|\langle 1\rangle$-cartesian for $n \geq 2$. The image of the map α^*, for $\alpha\colon \langle 0\rangle \to \langle 1\rangle$, i.e. the 0-ary tensor product, is equivalent to id_I, where I is the unit object of $\mathcal{C}$, and is thus an equivalence, hence also $r|\langle 1\rangle$-cartesian.

Now let $\psi\colon \langle m\rangle \to \langle n\rangle$ be a general map in $\mathcal{F}\mathrm{in}_*$, and let $f = [f_1, \ldots, f_m]$ be a componentwise cartesian map. The ith component of $\psi^*(f)$ is given by $\bigotimes_{\psi(j)=i} f_j$, which is $r|\langle 1\rangle$-cartesian no matter the cardinality of $\psi^{-1}(i)$. Thus, $\psi^*(f)$ is r-cartesian. $\square$

Lemma 3.4.1.6. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, and let $f\colon \Delta^1 \to \mathcal{C}$ pick out an equivalence in $\mathcal{C}$.

(1) Let $n \geq 1$. Any solid diagram

$$\begin{tikzcd}
\{0\} \times \Delta^1 \arrow[d, hook] \arrow[drr, "f"] & & \\
(\partial\Delta^n \times \Delta^1) \coprod_{(\partial\Delta^n \times \{0\})} (\Delta^n \times \{0\}) \arrow[rr] \arrow[d, hook] & & \mathcal{C} \arrow[d, "p"] \\
\Delta^n \times \Delta^0 \arrow[rr] \arrow[urr, dashed] & & \mathcal{D}
\end{tikzcd}$$

admits a dashed filler.

(2) Let $n \geq 2$. Any solid diagram

$$\begin{tikzcd}
\{0\} \times \Delta^1 \arrow[d, hook] \arrow[drr, "f"] & & \\
(\Lambda^n_n \times \Delta^1) \coprod_{(\Lambda^n_n \times \{0\})} (\Delta^n \times \{0\}) \arrow[rr] \arrow[d, hook] & & \mathcal{C} \arrow[d, "p"] \\
\Delta^n \times \Delta^0 \arrow[rr] \arrow[urr, dashed] & & \mathcal{D}
\end{tikzcd}$$

admits a dashed filler.

Proof. Case (1) follows immediately from [HTT, Prop. 2.4.1.8]. To solve (2), notice that in order to produce the dashed lift we can first fill $\Delta^{\{0,\dots n-1\}} \times \Delta^1$ and then the whole filler, both of which are of shape (1). $\square$

Lemma 3.4.1.7. Let $q\colon \mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}^{\mathrm{op}}$ and $p\colon \mathcal{D}_\otimes \to \mathcal{F}\mathrm{in}^{\mathrm{op}}_*$ be CSMCs, and let r be an inner fibration $\mathcal{C}_\otimes \to \mathcal{D}_\boxtimes$ such that the diagram

$$\begin{tikzcd}
\mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}^{\mathrm{op}}_* &
\end{tikzcd}$$

commutes. Suppose r is a monoidal functor, and that the tensor product $\otimes$ preserves cartesian morphisms. Let $f\colon c_0 \to c_1$ be a morphism in $(\mathcal{C}_\otimes)_{\langle t\rangle}$. Then f is r-cartesian if and only if it is componentwise-cartesian.

Proof. First, suppose that f is r-cartesian. Let $\phi\colon \langle t\rangle \to \langle s\rangle$ be any morphism in $\mathcal{F}\mathrm{in}_*$. For a square in $\mathcal{C}_\otimes$

$$\begin{tikzcd}
c_0' \arrow[r, "\phi^*(f)"] \arrow[d, "u"'] & c_1' \arrow[d, "v"] \\
c_0 \arrow[r, "f"] & c_1
\end{tikzcd}$$

in which u and v are q-cartesian (and hence r-cartesian) and f is r-cartesian, $\phi^*(f)$ is also r-cartesian. Applying this result to the inert maps $\rho^i\colon \langle t\rangle \to \langle 1\rangle$ implies that if a morphism f in $(\mathcal{C}_\otimes)_{\langle t\rangle}$ is r-cartesian, then the components f_i are r-cartesian, hence also $r|\langle 1\rangle$-cartesian.

Conversely, suppose that f is a componentwise-cartesian morphism in $(\mathcal{C}_\otimes)_{\langle t\rangle}$ (so that f is $r|\langle t\rangle$-cartesian). We should show that f is r-cartesian, i.e. that for any $n \geq 2$ and any solid diagram

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[d, hook] \arrow[dr, "f"] & \\ \Lambda^n_n \arrow[r, "\tau"] \arrow[d, hook] & \mathcal{C}_\otimes \arrow[d] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dashed, "\sigma'"] & \mathcal{D}_\boxtimes \end{tikzcd},$$

we can always find a dashed filler. Let $\langle m\rangle = p(\sigma(0))$, and let $h\colon \Delta^n \times \Delta^1 \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ be a natural transformation from the constant map $\Delta^n \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with value $\langle m\rangle$ to $p\circ\sigma$, so that in particular $h|\Delta^1 \times \{0\} = \mathrm{id}_{\langle m\rangle}$. Let $h' = h|\Lambda^n_n \times \Delta^1$. Then we can find a lift

$$\begin{tikzcd} \Lambda^n_n \times \{1\} \arrow[r, "\tau"] \arrow[d, hook] & \mathcal{C}_\otimes \arrow[d, "q"] \\ \Lambda^n_n \times \Delta^1 \arrow[r, "h'"] \arrow[ur, dashed, "\tilde{h}"] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \end{tikzcd}$$

such that for each $k \in \Lambda^n_n$, the morphism $h_k := \tilde{h}|\{k\} \times \Delta^1$ is q-cartesian. Note that by construction:

(i) The restriction of $\tilde{h}$ to $\Lambda^n_n \times \{0\}$ lies entirely within the fiber $(\mathcal{C}_\otimes)_{\langle m\rangle}$.

(ii) The restriction of $\tilde{h}$ to $\Delta^{\{n-1,n\}} \times \{0\}$ is a cartesian transport of f, and therefore componentwise-cartesian by our assumption that $\otimes$ preserves cartesian morphisms, and in particular $r|\langle m\rangle$-cartesian.

(iii) Since for each $0 \le k \le n$ the morphism h_k is q-cartesian and r preserves cocartesian morphisms, the morphism $r(h_k)$ is p-cartesian.

(iv) By (iii) and [HTT, Prop. 2.4.1.3(3)], each h_k is also r-cartesian.

(v) In particular, the morphism h_0 is an equivalence.

Using (iii), we can produce a dashed lift

$$\begin{tikzcd} \Lambda^n_n \times \Delta^1 \coprod_{\Lambda^n_n\times\{1\}} \Delta^n \times \{1\} \arrow[r, "(r\circ\tilde{h},\sigma)"] \arrow[d, hook] & \mathcal{D}_\boxtimes \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[ur, dashed, "\bar{h}"] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \end{tikzcd}.$$

All in all, we have assembled the data

$$\begin{tikzcd} \Lambda^n_n \times \Delta^1 \arrow[r, "\tilde{h}"] \arrow[d, hook] & \mathcal{C}_\otimes \arrow[d, "r"'] \\ \Delta^n \times \Delta^1 \arrow[r, "\bar{h}"] \arrow[ur, dashed, "\bar{\sigma}"] & \mathcal{D}_\boxtimes \end{tikzcd}.$$

We construct a dashed lift $\bar{\sigma}$ in 2 steps:

(1) By (i) and (ii) above, we can fill $\Delta^n \times \{0\}$ in $\mathcal{C}_\otimes$.

(2) The remaining portion th be filled is of the form Lemma 3.4.1.6(2), which we can fill by (v).

The restriction of $\bar{\sigma}$ to $\Delta^n \times \{1\}$ is the filling σ' we needed. $\square$

Lemma 3.4.1.8. Let $q\colon \mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}^{\mathrm{op}}$ and $p\colon \mathcal{D}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ be CSMCs, and let r be an inner fibration $\mathcal{C}_\otimes \to \mathcal{D}_\boxtimes$ such that the diagram

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

commutes. Then the following are equivalent.

(1) The map r is a cartesian fibration.

(2) The map r has the following properties.

  (a) The restriction $r|\langle 1\rangle$ is a cartesian fibration.

  (b) The map r is a monoidal functor.

  (c) The tensor product $\otimes$ preserves cartesian edges.

Proof. Suppose that 1. holds, i.e. that r is a cartesian fibration. Then a) holds: $r|\langle 1\rangle$ is a cartesian fibration because the pullback of a cartesian fibration is again a cartesian fibration.

We now show that b) holds, i.e. that the map r sends q-cartesian morphisms to p-cartesian morphisms. To this end, let $f\colon c \to c' \in \mathcal{C}_\otimes$ be a q-cartesian morphism, and consider the image $p(f)\colon r(c) \to r(c')$ in $\mathcal{D}_\otimes$. Let $g\colon \tilde{d} \to r(c')$ be a p-cartesian lift of $q(f) \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, and $\hat{g}\colon \hat{d} \to c'$ a r-cartesian lift of g. By [HTT, Prop. 2.4.1.3], $\hat{g}$ is a q-cartesian lift of $q(f)$, so f and $\hat{g}$ are equivalent as morphisms in $\mathcal{C}_\otimes$. Thus $r(f)$ and g are equivalent as morphisms in $\mathcal{D}_\otimes$, so $r(f)$ is p-cartesian since g is. This proves b).

We have now shown that under the assumption that (1) holds, q-cartesian transport preserves r-cartesian morphisms. This, together with Lemma 3.4.1.7, proves c).

Now, suppose that (2) holds. We immediately note that (a) implies that for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the restriction $r|\langle n\rangle$ is a cartesian fibration, and that by Lemma 3.4.1.7 an edge is $r|\langle n\rangle$-cartesian if and only if it is r-cartesian.

We now show that r admits cartesian lifts. Let $f\colon d \to d'$ be an edge in $\mathcal{D}_\boxtimes$ lying over an edge $\phi\colon \langle n\rangle \leftarrow \langle m\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, and let c' be a lift of d' to $\mathcal{C}_\otimes$. We can take a q-cartesian lift $g\colon c'' \to c$ of f, whose image in $\mathcal{D}_\boxtimes$ is by b) a p-cartesian map $h\colon d'' \to d'$. This gives us the solid data

$$\begin{tikzcd} & c'' \arrow[dr, "g"] & \\ c \arrow[ur, dashed, "k"] \arrow[rr, dashed, "\ell"'] & & c' \end{tikzcd} \qquad \text{in } \mathcal{C}_\otimes,$$

$$\begin{tikzcd} & d'' \arrow[dr, "h"] & \\ d \arrow[ur, dashed, "j"] \arrow[rr, "f"'] & & d' \end{tikzcd} \qquad \text{in } \mathcal{D}_\boxtimes$$

$$\begin{tikzcd} & \langle n\rangle \arrow[dl, equal] & \\ \langle n\rangle & & \langle m\rangle \arrow[ul, "\phi"'] \arrow[ll, "\phi"] \end{tikzcd} . \qquad \text{in } \mathcal{F}\mathrm{in}_*^{\mathrm{op}}.$$

Using the fact that h is p-cartesian, we can fill the 2-simplex in $\mathcal{D}_\boxtimes$, giving us in particular a map $j\colon d \to d''$. We can lift j to an $r|\langle n\rangle$-cartesian morphism $k\colon c \to c''$, which is therefore also r-cartesian. Using that r is an inner fibration, we can compose g and k relative to the simplex in $\mathcal{D}_\boxtimes$, giving us a lift ℓ of f. But ℓ is the composition of two r-cartesian morphisms, and hence itself r-cartesian. □

Note 3.4.1.9. Lemma 3.4.1.8 gives us an ∞-categorical version of the monoidal Grothendieck construction of [MV20] discussed at the begininning of this section. There, a monoidal structure is modelled as tuple $(\mathcal{C}, \otimes, \dots)$. Because of the specific choice of a bifunctor $\otimes$ and the surrounding corresponding coherence data, strong conditions must be placed on the functor r: it must be monoidal 'on the nose.' Here, we model our monoidal structure as a cartesian fibration, leaving these choices unmade. The strong notion of monoidality is thus replaced by the usual one.

The connection between the above result and the monoidal Grothendieck construction, somewhat explicitly, is as follows. Lemma 3.4.1.8 tells us that the data of a functor $(\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$ satisfying conditions analogous to those given in [MV20] is equivalent to a cartesian fibration between the corresponding CSMCs. A cartesian fibration r between CSMCs $p\colon \mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ and $q\colon \mathcal{D}_\boxtimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ as above straightens to a $\mathcal{D}_\boxtimes^{\mathrm{op}}$-monoid in $\mathcal{C}\mathrm{at}_\infty$. But a $\mathcal{D}_\boxtimes^{\mathrm{op}}$-monoid in $\mathcal{C}\mathrm{at}_\infty$ can be essentially uniquely extended to an $\mathcal{D}_\boxtimes^{\mathrm{op}}$-algebra object in $\mathcal{C}\mathrm{at}_{\infty,\times}$ [HA, Prop 2.4.2.4–2.4.2.6], which is to say, a lax monoidal functor $(\mathcal{D}^{\mathrm{op}}, \boxtimes) \to (\mathcal{C}\mathrm{at}_\infty, \times)$.

In the remainder of this section, we study a special type of CSMC, those whose tensor product is given by the cartesian product.

Definition 3.4.1.10. A ***CSMC*** $p\colon \mathcal{D}_\boxtimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is ***cartesian*** if the unit object $*$ is final, and if for all objects X and Y in $\mathcal{D}$, the canonical maps $X \times * \leftarrow X \boxtimes Y \to * \times Y$ exhibit $X \boxtimes Y$ as the product of X and Y.

Example 3.4.1.11. Let $\mathcal{T}$ be a category with finite products. Then $\mathcal{T}^{\mathrm{op}}$ admits finite coproducts, and we can consider the cocartesian monoidal structure

$$(\mathcal{T}^{\mathrm{op}})^{\amalg} \to \mathcal{F}\mathrm{in}_*$$

of [HA, Construction 2.4.3.1]. Taking the opposite of this functor gives us a cartesian fibration

$$\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}.$$

That this is a cartesian CSMC follows immediately from the fact that $(\mathcal{T}^{\mathrm{op}})^{\amalg} \to \mathcal{F}\mathrm{in}_*$ is a cocartesian symmetric monoidal category.

In any category $\mathcal{T}$ with pullbacks, one can form a category $\mathrm{Span}(\mathcal{T})$ of spans in $\mathcal{T}$. If $\mathcal{T}$ also has a terminal object, hence all finite limits (including, of course, finite products), then the category of spans inherits a monoidal structure via

$$\left(\begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \right) \otimes \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl] \arrow[dr] & \\ X' & & Y' \end{tikzcd} \right) = \begin{tikzcd}[column sep=small] & Z \times Z' \arrow[dl] \arrow[dr] & \\ X \times X' & & Y \times Y' \end{tikzcd}$$

We now construct the monoidal structure on this category of spans explicitly, starting from any cartesian CSMC $\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$.

Proposition 3.4.1.12. Suppose $\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is a cartesian CSMC whose underlying category $\mathcal{T}$ admits pullbacks. Then there exists a (cocartesian-presented) symmetric monoidal category $\mathrm{Span}(\mathcal{T})^\times \to \mathcal{F}\mathrm{in}_*$.

Proof. We upgrade this functor to a functor of triples. We will consider the following triples.

- We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where
  - $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
  - $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
  - $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

  This is obviously adequate.

- We define a triple $(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$ as follows.
  - $\mathcal{T} = \mathcal{T}_\times$
  - $\mathcal{T}_\dagger = \mathcal{T}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
  - $\mathcal{T}^\dagger = \mathcal{T}_\times$

To see that this is adequate, we first note that p admits relative pullbacks. To see this, note that each fiber admits pullbacks by virtue our assumption that $\mathcal{T}$ admits pullbacks, and the identifications $(\mathcal{T}_\times)_{\langle n\rangle} \simeq \mathcal{T}^n$. The functoriality coming from p implements products, and therefore commute with pullbacks. Thus, $\mathcal{T}_\times$ admits pullbacks, and p preserves pullbacks. This immediately implies that $(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$ is adequate:

(1) Pullbacks of this form are simply squares with horizontal morphisms given by equivalences.

(2) Any pullback square lies over a pullback square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, so this reduces to the lemma about cartesian morphisms.

We thus consider the map of triples

$$\pi\colon (\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger) \to (\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger).$$

One readily checks that π satisfies the conditions of Theorem 3.1.5.1:

- The first condition holds because cocartesian morphisms lying over equivalences are themselves equivalences, hence also cartesian, so we can solve the relevant lifting problems using cartesian lifts.
- The second condition holds because a square whose bottom-horizontal morphism is an equivalence is pullback if and only if its top-horizontal morphism is an equivalence.

This gives us a functor $\mathrm{Span}(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger) \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$. Pulling back along the equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ gives us the cocartesian fibration functor $\mathrm{Span}(\mathcal{T})^\times \to \mathcal{F}\mathrm{in}_*$. It remains only to check that the maps $\rho^i_*\colon \mathrm{Span}(\mathcal{T})^\times_{\langle n\rangle} \to \mathrm{Span}(\mathcal{T})^\times_{\langle 1\rangle}$ are the canonical projections exhibiting

$$\mathrm{Span}(\mathcal{T})^\times_{\langle n\rangle} \simeq \left(\mathrm{Span}(\mathcal{T})^\times_{\langle 1\rangle}\right)^n.$$

In order to show this, we should show that for each simplicial set K, the map a in the diagram

$$\begin{tikzcd}
\mathrm{Map}(K, \mathrm{Span}(\mathcal{T})^{\times}_{\langle n\rangle}) \arrow[r, "a"] \arrow[d, "\simeq"'] & \prod_{i=1}^{n} \mathrm{Map}(K, \mathrm{Span}(\mathcal{T})^{\times}_{\langle 1\rangle}) \arrow[d, "\simeq"] \\
\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(K), (\mathcal{T}_{\times})_{\langle n\rangle}) \arrow[r, "b"] \arrow[d, hook, "i"'] & \prod_{i=1}^{n} \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(K), (\mathcal{T}_{\times})_{\langle 1\rangle}) \arrow[d, hook, "j"] \\
\mathrm{Map}(\mathrm{sd}(K), (\mathcal{T}_{\times})_{\langle n\rangle}) \arrow[r, "c", "\simeq"'] & \prod_{i=1}^{n} \mathrm{Map}(\mathrm{sd}(K), (\mathcal{T}_{\times})_{\langle 1\rangle})
\end{tikzcd}$$

is an isomorphism in the category hKan. In order to show this, it suffices to show that the map b is an isomorphism in hKan. We note that the morphisms i and j are inclusions of connected components, and the map c is an isomorphism because $\mathcal{T}_{\times} \to \mathcal{F}\mathrm{in}_{*}^{\mathrm{op}}$ is a CSMC, so in order to show that b is an isomorphism in hKan, it suffices to show that it is essentially surjective. This follows from the fact that a square in $(\mathcal{T}_{\times})_{\langle n\rangle} \simeq ((\mathcal{T}_{\times})_{\langle 1\rangle})^{n}$ is pullback if and only if each component is pullback. □

3.4.2 Monoidal Beck-Chevalley fibrations

In this section, we combine several results from earlier sections.

- In Section 3.3, we showed that a bicartesian fibration $p\colon \mathcal{X} \to \mathcal{T}$ satisfying the Beck-Chevalley condition allows us to combine the functoriality $\mathcal{T} \to \mathcal{C}\mathrm{at}_{\infty}$ and $\mathcal{T}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{\infty}$ into push-pull functoriality $\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_{\infty}$. We called such bicartesian fibrations *Beck-Chevalley fibrations.*
- In Subsection 3.4.1 we showed that a cartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$ between monoidal categories whose tensor products were subject to certain compatibility conditions classifies a lax monoidal functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{\infty}$; and dually, that a cocartesian fibration p satisfying dual compatibility conditions classifies a lax monoidal functor $\mathcal{D} \to \mathcal{C}\mathrm{at}_{\infty}$.

We will now show that a Beck-Chevalley fibration $p\colon \mathcal{X} \to \mathcal{T}$ between monoidal categories whose tensor products satisfy appropriate compatibility conditions classifies a lax monoidal functor $\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_{\infty}$, whose lax structure morphisms are given by

$$\mathcal{X}_t \times \mathcal{X}_{t'} \to \mathcal{X}_{t\times t'}.$$

We will call such fibrations *monoidal Beck-Chevalley fibrations.*

Definition 3.4.2.1. A ***monoidal Beck-Chevalley fibration*** is a functor r of CSMCs

$$\begin{tikzcd}
\mathcal{X}_{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{T}_{\times} \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_{*}^{\mathrm{op}} &
\end{tikzcd},$$

where $\mathcal{T}_{\times}$ is a cartesian CSMC, with the following characteristics.

(M1) The map $r|\langle 1\rangle$ is a Beck-Chevalley fibration.

(M2) The map r is monoidal.

(M3) The tensor product $\otimes$ preserves r-cartesian morphisms.

(M4) The tensor product $\otimes$ preserves r-cocartesian morphisms.

Lemma 3.4.2.2. Let p and q be CSMCs as below. Then for any monoidal functor r, a morphism f in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is r-cocartesian if and only if it is r-cocartesian.

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

Proof. Consider the following diagram of pullback squares.

$$\begin{tikzcd} (\mathcal{C}_\otimes)_{\langle n\rangle} \arrow[r] \arrow[d, "r|\langle n\rangle"'] & \mathcal{C}_\otimes \arrow[d, "r"] \\ (\mathcal{D}_\otimes)_{\langle n\rangle} \arrow[r] \arrow[d, "p|\langle n\rangle"'] & \mathcal{D}_\otimes \arrow[d, "p"] \\ \{\langle n\rangle\} \arrow[r] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \end{tikzcd}$$

It follows from [HTT, Cor. 4.3.1.15][3] that a morphism in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is $r|\langle n\rangle$-cocartesian if and only if its image in $\mathcal{C}_\otimes$ is r-cocartesian. But a morphism is $r|\langle n\rangle$-cocartesian if and only if each component is $r|\langle 1\rangle$-cocartesian. $\square$

The conditions in Definition 3.4.2.1 are to do only with the properties of the functor $r|\langle 1\rangle$, together with properties of the tensor product $\otimes$. We can also express these properties in terms directly in terms of the fibrations p, q, and r.

Lemma 3.4.2.3. Consider a functor r of CSMCs, where $\mathcal{T}_\times$ is a cartesian CSMC.

$$\begin{tikzcd} \mathcal{X}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{T}_\times \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

The following are equivalent.

(1) The map r is a monoidal Beck-Chevalley fibration.

(2) The map r has the following properties.

  (a) The category $\mathcal{T}$ underlying $\mathcal{T}_\times$ admits pullbacks.

  (b) The map r is a cartesian fibration.

  (c) The functor $r|\langle 1\rangle$ is a cocartesian fibration.

[3]Note an unfortunate notational clash: our maps p, q, and r do not agree with Lurie's.

(d) The functor r obeys the following interchange law: for any diagram

$$\begin{tikzcd}
\vec{z} \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{y} \arrow[d, "g"] \\
\vec{y}' \arrow[r, "f"] & \vec{x}
\end{tikzcd} \tag{3.4.2.1}$$

in $\mathcal{X}_\otimes$ whose image in $\mathcal{T}_\boxtimes$ is pullback, and which lies over a square

$$\begin{tikzcd}
\langle n \rangle & \langle n \rangle \arrow[l, "\alpha", "\simeq"'] \\
\langle m \rangle \arrow[u, "\phi"] & \langle m \rangle \arrow[l, "\beta", "\simeq"'] \arrow[u, "\psi"']
\end{tikzcd} , \tag{3.4.2.2}$$

in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ such that α and β are equivalences, if g and g' are r-cartesian and f is r-cocartesian, then f' is r-cocartesian.

Proof. That 1. implies a) and c) is clear, and b) follows from Lemma 3.4.1.8.

We now show 1. implies d). Consider the diagram

$$\begin{tikzcd}
\langle n \rangle \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, equal] & & \langle n \rangle \arrow[dd, equal] & \\
& \langle n \rangle & & \langle n \rangle \arrow[ll, "\alpha"] \arrow[ul, "\alpha"'] \\
\langle n \rangle \arrow[rr, equal] & & \langle n \rangle & \\
& \langle m \rangle \arrow[uu, "\phi"] \arrow[ul, "\phi"] & & \langle m \rangle \arrow[ll, "\beta"] \arrow[uu, "\psi"'] \arrow[ul, "\phi\circ\beta"']
\end{tikzcd} \qquad \text{in } \mathcal{F}\mathrm{in}_*^{\mathrm{op}},$$

which we should think of as a natural transformation from Diagram 3.4.2.2 to a constant diagram.

Beginning with the solid diagram below coming from Diagram 3.4.2.1, we can find q-cartesian lifts of the diagonal arrows. Filling we find a dashed cube

$$\begin{tikzcd}
\vec{u} \arrow[rr, dashed, "h'"] \arrow[dd, dashed] \arrow[dr, dashed, "\simeq"] & & \vec{v} \arrow[dd, dashed] \arrow[dr, dashed, "\simeq"] & \\
& \vec{x} \arrow[rr, "f'"] \arrow[dd] & & \vec{y} \arrow[dd] \\
\vec{v}' \arrow[rr, dashed, "h"] \arrow[dr, dashed] & & \vec{w} \arrow[dr, dashed] & \\
& \vec{y}' \arrow[rr, "f"] & & \vec{z}
\end{tikzcd} \qquad \text{in } \mathcal{X}_\otimes$$

lying over the cube in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, whose diagonal arrows are q-cartesian, and with equivalences as marked. Mapping this cube down to $\mathcal{T}_\times$, we find a cube

$$\begin{tikzcd}
r(\vec{u}) \arrow[rr, dashed] \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & & r(\vec{v}) \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & \\
& r(\vec{x}) \arrow[rr] \arrow[dd] & & r(\vec{y}) \arrow[dd] \\
r(\vec{v}') \arrow[rr, dashed] \arrow[dr, dashed] & & r(\vec{w}) \arrow[dr, dashed] & \\
& r(\vec{y}') \arrow[rr] & & r(\vec{z})
\end{tikzcd} \qquad \text{in } \mathcal{T}_\times$$

lying over the cube in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, whose diagonal morphisms are p-cartesian, and whose front face is pullback by assumption. The bottom face is pullback since it lies over a pullback square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ and the diagonal morphsims are p-cartesian, and the top face is pullback because the diagonal morphisms are equivalences. The cube lemma thus implies that the back square is pullback. Note that the back square is entirely contained in the fiber over $\langle n\rangle$, and thus can be thought of as consisting of n component squares in $\mathcal{T}$; by the equivalence $(\mathcal{T}_\otimes)_{\langle n\rangle} \simeq (\mathcal{T})^n$, these component squares are themselves pullback.

We now return our attention to the diagram in $\mathcal{X}_\otimes$. The morphism f is r-cocartesian by assumption. That the tensor product $\otimes$ preserves r-cocartesian morphisms implies that h is r-cocartesian. Applying the Beck-Chevalley condition componentwise to the back face yields that h' is componentwise cocartesian, hence r-cocartesian. Since f' is equivalent to h', f' is also r-cocartesian. Thus, d) holds.

We now show that 2. implies 1. Restricting each of the conditions of 2. to the fibers over $\langle 1\rangle$ immediately implies that $r|\langle 1\rangle$ is a Beck-Chevalley fibration, and b) implies that r is monoidal. It remains to show that the tensor product $\otimes$ preserves r-cartesian and r-cocartesian morphisms. To this end, consider a square

$$\sigma = \begin{tikzcd}
\vec{x} \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{y} \arrow[d, "g"] \\
\vec{y}' \arrow[r, "f"] & \vec{z}
\end{tikzcd} \qquad \text{in } \mathcal{X}_\otimes$$

such that g and g' are q-cartesian, lying over a square

$$\begin{tikzcd}
\langle n\rangle \arrow[r, equal] & \langle n\rangle \\
\langle m\rangle \arrow[u, "\phi"] \arrow[r, equal] & \langle m\rangle \arrow[u, "\phi"']
\end{tikzcd} \qquad \text{in } \mathcal{F}\mathrm{in}_*^{\mathrm{op}}. \tag{3.4.2.3}$$

Note that since r is monoidal, the image of σ in $\mathcal{T}_\times$ is automatically pullback. We now note that if f is r-cartesian, then f' is r-cartesian by [HTT, Prop. 2.4.1.7], and if f is r-cocartesian, then f' is r-cocartesian (hence $r|\langle 1\rangle$-cocartesian) by the interchange law. $\square$

Proposition 3.4.2.4. For any monoidal Beck-Chevalley fibration, there is a diagram

$$\begin{tikzcd} \mathrm{Span}'(\mathcal{X})^{\otimes} \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & \mathrm{Span}(\mathcal{T})^{\otimes} \arrow[dl, "\pi"] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd}$$

where π and ϖ are symmetric monoidal categories and ρ exhibits $\mathrm{Span}'(\mathcal{X})^{\otimes}$ as a $\mathrm{Span}(\mathcal{T})^{\otimes}$-monoidal category. Straightening, one finds a lax monoidal functor

$$\hat{r}\colon (\mathrm{Span}(\mathcal{T}), \widetilde{\times}) \to (\mathcal{C}\mathrm{at}_\infty, \times)$$

with the following description up to equivalence.

- On objects, the functor $\hat{r}$ sends $t \in \mathrm{Span}(\mathcal{T})$ to the fiber $\mathcal{X}_t \in \mathcal{C}\mathrm{at}_\infty$
- On morphisms, $\hat{r}$ sends a span $t \xleftarrow{g} s \xrightarrow{f} t'$ to the composition $f_! \circ g^* : \mathcal{X}_t \to \mathcal{X}_{t'}$.
- The structure morphisms
$$\mathcal{X}_t \times \mathcal{X}_{t'} \to \mathcal{X}_{t \times t'}$$
of the lax monoidal structure on $\hat{r}$ are given by the restriction of the tensor product $\otimes$ to the fibers over t and t'.

Proof. Recall the triple structure $(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$ from the proof of Proposition 3.4.1.12. We further define a triple structure $(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger)$ as follows.

- $\mathcal{X} = \mathcal{X}_\otimes$.
- $\mathcal{X}_\dagger = (\mathcal{X}_\otimes)_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$.
- $\mathcal{X}^\dagger$ consists only of r-cartesian morphisms.

One sees that this is adequate, since pullbacks of the necessary form exist by the usual procedure:

- Map the diagram down to $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, take the pullback there.
- Take a relative pullback in $\mathcal{T}_\times$.
- Take an r-cartesian lift to produce an r-relative pullback in $\mathcal{X}_\times$. This lies over a pullback in $\mathcal{T}_\times$, hence is a pullback.

Thus, we have a map of adequate triples $(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \to (\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$. The fact that this map satisfies the conditions of Theorem 3.1.5.2 follows immediately from Lemma 3.4.2.3. This gives us a cocartesian fibration

$$\rho\colon \mathrm{Span}(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \to \mathrm{Span}(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger).$$

Combining this with the map π of Proposition 3.4.1.12 and pulling back along the map

$\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ gives us the triangle

$$\begin{tikzcd}
\mathrm{Span}'(\mathcal{X})^{\otimes} \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & \mathrm{Span}(\mathcal{T})^{\otimes} \arrow[dl, "\pi"] \\
& \mathcal{F}\mathrm{in}_* &
\end{tikzcd},$$

where π is the map shown in Proposition 3.4.1.12 to be a cocartesian fibration. It remains only to show that $\varpi = \pi \circ \rho$ is a monoidal category. The same argument as in Proposition 3.4.1.12 shows that it suffices to show that square in $(\mathcal{X}_\otimes)_{\langle n \rangle}$ is ambigressive pullback if and only if each component is ambigressive pullback, which bolds because of the equivalence $(\mathcal{X}_\otimes)_{\langle n \rangle} \simeq \mathcal{X}^n$. □

3.4.3 The monoidal twisted arrow category

We have seen that for any ∞-bicategory $\mathbb{C}$ (presented as a fibrant scaled simplicial set), there is an ∞-category $\mathrm{Tw}(\mathbb{C})$, the *twisted arrow category* of $\mathbb{C}$. If $\mathbb{C}$ carries a monoidal structure, this structure is inherited by the twisted arrow ∞-category $\mathrm{Tw}(\mathbb{C})$ by defining

$$\left(\begin{tikzcd} c_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "=\eta\Rightarrow"] & c_2 \arrow[d] \\ d_1 & d_2 \arrow[l] \end{tikzcd}\right) \otimes \left(\begin{tikzcd} c_1' \arrow[r] \arrow[d] \arrow[dr, phantom, "=\eta'\Rightarrow"] & c_2' \arrow[d] \\ d_1' & d_2' \arrow[l] \end{tikzcd}\right) = \begin{tikzcd} c_1 \otimes c_1' \arrow[r] \arrow[d] \arrow[dr, phantom, "=\eta\otimes\eta'\Rightarrow"] & c_2 \otimes c_2' \arrow[d] \\ d_1 \otimes d_1' & d_2 \otimes d_2' \arrow[l] \end{tikzcd}$$

In this section, we construct this monoidal structure explicitly.

For our construction, will need to upgrade the twisted arrow category construction to a functor

$$\mathrm{Tw}\colon \mathcal{C}\mathrm{at}_{(\infty,2)} \to \mathcal{C}\mathrm{at}_{\infty},$$

where $\mathcal{C}\mathrm{at}_{(\infty,2)}$ is the ∞-category of ∞-bicategories. To this end, we note that we can certainly express the twisted arrow category as an ordinary functor

$$\mathrm{Tw}'\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathcal{S}\mathrm{et}_\Delta^{+}.$$

Here we take $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ and $\mathcal{S}\mathrm{et}_\Delta^{+}$ to carry their standard model structures.

Lemma 3.4.3.1. The functor Tw' preserves weak equivalences between fibrant objects.

Proof. Let $f\colon \mathbb{C} \to \mathbb{D}$ be a weak equivalence between fibrant objects in the scaled model structure. Thus, f is a bicategorical equivalence between ∞-bicategories. Denote the map on the underlying quasicategories by $\mathring{f}\colon \mathcal{C} \to \mathcal{D}$. We note that $\mathring{f}$ is an equivalence of quasicategories.

We consider the diagram

$$\begin{tikzcd}
\mathrm{Tw}'(\mathbb{C}) \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\mathrm{Tw}'(f)"] \arrow[ddr, bend right, "q"'] & & \\
& \mathcal{P} \arrow[r, "g"] \arrow[d, "r"] & \mathrm{Tw}'(\mathbb{D}) \arrow[d, "p"] \\
& \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \arrow[r, "\mathring{f} \times \mathring{f}^{\mathrm{op}}"] & \mathcal{D} \times \mathcal{D}^{\mathrm{op}}
\end{tikzcd}$$

where the square formed is pullback, and h is the map guaranteed us by the universal property of the pullback. We note that since p is a cartesian fibration, g is a weak equivalence since $\mathring{f} \times \mathring{f}^{\mathrm{op}}$ is. Thus, in order to show that $\mathrm{Tw}'(f)$ is a weak equivalence, it suffices to show that h is.

Since h is a morphism $q \to r$ of cartesian fibrations, in order to show that it is an equivalence it suffices to check that it is a fiberwise equivalence. Fix some object $C = (c, c') \in \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. The restriction of h to the fibers of $\mathrm{Tw}'(\mathbb{C})$ and $\mathcal{P}$ over C is the map

$$\mathrm{Map}_{\mathbb{C}}(c, c') \to \mathrm{Map}_{\mathbb{D}}(f(c), f(c')).$$

This is a weak equivalence for all C because f is an equivalence of ∞-bicategories. $\square$

By Theorem 3.1.4.4, there is a right Quillen equivalence $G\colon \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ms}} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}$. Composing this with the functor Tw' above gives us a functor

$$\mathrm{Tw}' \circ G\colon \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ms}} \to \mathcal{S}\mathrm{et}_{\Delta}^{+}.$$

Since G is a right Quillen functor, it preserves weak equivalences between fibrant objects. Thus, the composite $\mathrm{Tw}' \circ G$ does as well. Restricting to fibrant objects and taking simplicial localizations yields a functor

$$\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ms}}[W^{-1}] \to \mathcal{S}\mathrm{et}_{\Delta}^{+}[W^{-1}].$$

By [HA, Example 1.3.4.8], we have equivalences of ∞-categories $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ms}}[W^{-1}] \simeq \mathcal{C}\mathrm{at}_{(\infty,2)}$ and $\mathcal{S}\mathrm{et}_{\Delta}^{+}[W^{-1}] \simeq \mathcal{C}\mathrm{at}_{\infty}$, giving us our functor

$$\mathrm{Tw}\colon \mathcal{C}\mathrm{at}_{(\infty,2)} \to \mathcal{C}\mathrm{at}_{\infty}.$$

We are now ready to construct a model for the monoidal twisted arrow category.

Construction 3.4.3.2. The category $\mathcal{C}\mathrm{at}_{\infty}$ admits a Cartesian monoidal structure, which can be expressed as a commutative monoid $\mathcal{C}\mathrm{at}_{\infty}^{\times}\colon \mathcal{F}\mathrm{in}_{*} \to \mathcal{C}\mathrm{at}_{(\infty,2)}$. The starting point of our construction is the composition

$$\mathcal{F}\mathrm{in}_{*} \xrightarrow{\mathcal{C}\mathrm{at}_{\infty}{}^{\times}} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathrm{Tw}} \mathcal{C}\mathrm{at}_{\infty} \ .$$

Note that this composition yields a commutative monoid in $\mathcal{C}\mathrm{at}_{\infty}$ since the functor Tw preserves products. The relative nerve of this composition is a CSMC $\mathrm{Tw}(\mathcal{C}\mathrm{at}_{\infty})_{\otimes} \to \mathcal{F}\mathrm{in}_{*}^{\mathrm{op}}$

with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^I \arrow[r] & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)^i \end{tikzcd}$$

commutes.

Example 3.4.3.3. An object of $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$.

Example 3.4.3.4. A morphism in $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[4] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.
- A morphism $F \to G_1 \times G_2$ in $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)$ corresponding to a diagram $\Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ with front and back

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \arrow[dl] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd} .$$

A morphism of the above form is cartesian if and only if the corresponding simplex $\Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ is thin.

Lemma 3.4.3.5. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let F, $G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

(1) For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.

(2) For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

[4]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

(1) The maps Φ, Γ, and ρ are all cartesian fibrations.

(2) The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.

Proof. We first prove 2. We already know [HTT, Lemma 3.2.5.11] that ρ is an inner fibration, so it remains only to show that we can solve lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^{\mathrm{op}} \end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0)) \end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition. $\square$

Since the category $\mathcal{C}\mathrm{at}_\infty$ admits products, it admits a cartesian monoidal structure, which we can write as a commutative monoid $G_1\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$. We can equally view the cartesian monoidal structure as a cocartesian monoidal structure on ${\mathcal{C}\mathrm{at}_\infty}^{\mathrm{op}}$, giving us a commutative monoid $G_0\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$. Taking these together gives a commutative monoid

$$G = G_0 \times G_1 \colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto ({\mathcal{C}\mathrm{at}_\infty}^{\mathrm{op}})^n \times {\mathcal{C}\mathrm{at}_\infty}^n.$$

For each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\alpha_n\colon \mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \to ({\mathcal{C}\mathrm{at}_\infty}^n)^{\mathrm{op}} \times {\mathcal{C}\mathrm{at}_\infty}^n$, which form the components of a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto \mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor G.

We now apply the relative nerve to the data $\alpha\colon F \Rightarrow G$. Because the pointwise product of any number of thin 1-simplices in $\mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)$, the conditions of Lemma 3.4.3.5 are satisfied. Unrolling, we find a commuting triangle of cartesian fibrations

$$\begin{tikzcd} \mathrm{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes \arrow[rr, "r'"] \arrow[dr, "q'"'] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}, \tag{3.4.3.1}$$

where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is the relative nerve (as a cartesian fibration) of the functor F_0
- $(\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times) \to \mathcal{F}\mathrm{in}_*$ is the relative nerve (as a *cocartesian* fibration) of the same.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a fiberwise equivalence. Composing a commutative monoid $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this equivalence allows us to express symmetric monoidal ∞-categories as commutative monoids in $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

3.4.4 The monoidal category of local systems

In this subsection, we show that the monoidal structure on the twisted arrow category, defined in Subsection 3.4.3, can be used to construct a monoidal structure on the category of local systems.

Construction 3.4.4.1. We will denote the full subcategory of $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ on those objects $[\mathcal{D}_1, \ldots, \mathcal{D}_n]$ such that $\mathcal{D}_i$ is an ∞-groupoid for all $1 \le i \le n$ by $\widetilde{\mathcal{S}}_\times$. Note that $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*$ can be understood as the relative nerve (as a cartesian fibration) of a commutative monoid $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{S}$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the tensor product $\otimes\colon \mathcal{C} \to \mathcal{C}$ preserves colimits in each slot. We express this monoidal ∞-category as a commutative monoid $\mathcal{C}^\otimes\colon \mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$. Using this, we define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is the inclusion $\widetilde{\mathcal{S}}_\times \hookrightarrow \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ on the first component of the product, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{(\mathcal{C}^\otimes)^{\mathrm{op}}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

on the second. Forming the pullback square

$$\begin{tikzcd} \mathrm{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\ \widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}^{\times}_{\infty})^{\mathrm{op}} \end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd} \mathrm{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} .$$

We claim that $\mathrm{LS}(\mathcal{C})_\otimes$ is a CSMC, $\widetilde{\mathcal{S}}_\times$ is a cartesian CSMC. Since the (cartesian) relative nerve construction produces a cartesian fibration by definition, the map p is a cartesian fibration; by its definition, it is even a CSMC. Furthermore, because the map r is a pullback of the horizontal map in Diagram 3.4.3.1, it is a cartesian fibration. Hence q is also a cartesian fibration. It remains only to show that q is a CSMC. It suffices to show that r is a cartesian fibration of ∞-operads. Note that because both p' and q' are CSMCs, r' is a cartesian fibration of ∞-operads. The claim follows because r is a pullback of r'.

Example 3.4.4.2. An object of $\mathrm{LS}(\mathcal{C})_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $X \to \mathcal{C}$, $i \in \langle n\rangle^\circ$.

Example 3.4.4.3. A morphism in $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists of

- A 'source' object $F\colon X \to \mathcal{C}$
- A pair of 'target' objects $G_i\colon Y_i \to \mathcal{C}$, $i = 1, 2$.
- A morphism $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)$ corresponding to a diagram $\Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ with front and back

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr] & Y_1 \times Y_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C} & \mathcal{C}\times\mathcal{C} \arrow[l, "\otimes"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "F"'] & Y_1 \times Y_2 \arrow[d, "G_1\times G_2"] \arrow[dl] \\ \mathcal{C} & \mathcal{C}\times\mathcal{C} \arrow[l, "\otimes"] \end{tikzcd} .$$

A morphism of the above form is r-cartesian if and only if the corresponding simplex $\Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ is thin, and q-cartesian if and only if the corresponding simplex is thin, and α is an equivalence.

Example 3.4.4.4. Examining the q-cartesian case more closely, one sees that the tensor product of two local systems $F\colon X \to \mathcal{C}$ and $G\colon Y \to \mathcal{C}$ is the local system given the composition

$$X \times Y \xrightarrow{F\times G} \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C} .$$

Lemma 3.4.4.5. The tensor product of local systems preserves cocartesian edges (in the sense of Definition 3.4.1.4).

Proof. In order to show that the tensor product of local systems preserves cocartesian edges, it suffices by Lemma 3.4.1.5 to check that the tensor product of two $r|\langle 1\rangle$-cocartesian edges is again $r|\langle 1\rangle$-cocartesian. Let $e\colon F \to G$ and $e'\colon F' \to G'$ be $r|\langle 1\rangle$-cocartesian edges of $\mathrm{LS}(\mathcal{C})_{\otimes}$.

$$e = \begin{tikzcd} X \arrow[r] \arrow[d, "F"'] & Y \arrow[d, "G"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}"] \end{tikzcd}, \qquad e' = \begin{tikzcd} X' \arrow[r] \arrow[d, "F'"'] & Y' \arrow[d, "G'"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}"] \end{tikzcd}.$$

We wish to show that $e \otimes e'$ is $r|\langle 1\rangle$-cocartesian.

The tensor product $e \otimes e'$ is given, up to homotopy, by the pasting diagram

$$\begin{tikzcd} X \times X' \arrow[r] \arrow[d] & Y \times Y' \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"'] & \mathcal{C} \times \mathcal{C} \arrow[l] \arrow[d, "\otimes"] \\ \mathcal{C} & \mathcal{C} \arrow[l] \end{tikzcd}$$

In order to show that this map is r-cocartesian, we need to show that the outer triangle

$$\begin{tikzcd} X \times X' \arrow[rr, "F\times F'"] \arrow[dr, "f\times f'"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] & \mathcal{C} \\ & Y \times Y' \arrow[ur, "G\times G'"'] \arrow[urr, bend right, "\otimes\circ(G\times G')"'] & & \end{tikzcd}$$

exhibits $\otimes \circ (G \times G')$ as the left Kan extension of $\otimes \circ (F \times F')$ along $f \times f'$. But this follows from the fact that $\otimes$ preserves colimits in both slots. □

Proposition 3.4.4.6. The map r above is a monoidal Beck-Chevalley fibration.

Proof. We need to check the conditions (M1)–(M4) of Definition 3.4.2.1. The condition (M1) is the content of Proposition 3.3.0.3. Conditions (M2) and (M3) follow immediately from Lemma 3.4.1.8, using the fact that r is a cartesian fibration. That (M4) holds is the content of Lemma 3.4.4.5. □

Corollary 3.4.4.7. There is a lax monoidal functor

$$\hat{r}\colon (\mathrm{Span}(\mathcal{S}), \widetilde{\times}) \to (\mathcal{C}\mathrm{at}_\infty, \times),$$

with the following description up to equivalence.

- On objects, the functor $\hat{r}$ sends a space X to the ∞-category $\mathrm{LS}(\mathcal{C})_X$ of $\mathcal{C}$-local systems on X.

- On morphisms, the functor $\hat{r}$ sends a span of spaces $X \xleftarrow{g} Y \xrightarrow{f} X'$ to the pull-push

$$f_! \circ g^*\colon \mathrm{LS}(\mathcal{C})_X \to \mathrm{LS}(\mathcal{C})_{X'}.$$

- The structure morphisms of the lax monoidal structure are the maps

  $$\mathrm{LS}(\mathcal{C})_X \times \mathrm{LS}(\mathcal{C})_{X'} \to \mathrm{LS}(\mathcal{C})_{X \times X'}$$

  given by the composition

  $$\mathrm{Fun}(X, \mathcal{C}) \times \mathrm{Fun}(Y, \mathcal{C}) \xrightarrow{\times} \mathrm{Fun}(X \times Y, \mathcal{C} \times \mathcal{C}) \xrightarrow{\otimes} \mathrm{Fun}(X \times Y, \mathcal{C}),$$

  under the identification $\mathrm{LS}(\mathcal{C})_X \cong \mathrm{Fun}(X, \mathcal{C})$.

Appendix A

Appendices

A.1 Horn filling via internal Kan extensions

In this appendix, we will show that the notion of a Kan extension internal to any $(\infty, 2)$-category can be captured by a horn-filling property. This material appeared in [Rus22a], in the special case that $\mathbb{A} = \mathbb{C}\mathrm{at}_\infty$.

Definition A.1.0.1. Let $\mathbb{A}$ be an $(\infty, 2)$-category in the sense of [Lur18, Tag 01W9] (i.e. a simplicial set with a collection of thin simplices with respect to which inner horn fillers exist). A not necessarily thin 2-simplex $\sigma\colon \Delta^2 \to \mathbb{A}$ is ***left Kan***, or simply ***Kan***, if for each $n \geq 3$, each solid diagram below admits a dashed filler.

$$\begin{tikzcd} \Delta^{\{0,1,n\}} \arrow[d, hook] \arrow[dr, "\sigma"] & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathbb{A} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{A.1.0.1}$$

Note A.1.0.2. The reason to prefer the name 'Kan' rather than 'left Kan' is that the above notion is actually the correct one in either case. When one uses the convention that a 2-simplex in $\mathbb{A}$ corresponds to a diagram

$$\begin{tikzcd} a \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & x \\ & b \arrow[ur, "G"'] & \end{tikzcd}$$

with the 2-morphism pointing *down,* then the above notion models left Kan extension; when one uses the convention that the 2-morphism points *up,* then it models right Kan extension. However, we will always use the former convention.

The goal of this section is to prove the following. For simplicity, we assume that $\mathbb{A}$ is the coherent nerve of a quasicategory-enriched category.

Theorem A.1.0.3. Let A be a quasicategory-enriched category, so that $\mathbb{A} = \mathrm{N}_{\mathrm{sc}}(\mathsf{A})$ is an $(\infty, 2)$-category [Lur18, Tag 01YL]. The following are equivalent.

(1) The pullback functor $f^*\colon \mathbb{A}(b,x) \to \mathbb{A}(a,x)$ admits a left adjoint at $F\colon a \to x$ in the sense of Definition A.1.2.10, given by a morphism $G\colon b \to x$, with local unit $\eta\colon G \Rightarrow F \circ f$.

(2) The left horn $\tau'\colon \Lambda^2_0 \to \mathbb{A}$ with that $\tau'|\{0,1\} = f$ and $\tau|\{0,2\} = F$ admits a Kan filler $\tau\colon \Delta^2 \to \mathbb{A}$ given by the 2-simplex

$$\tau = \begin{tikzcd} a \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & x \\ & b \arrow[ur, "G"'] & \end{tikzcd}.$$

The "(1) $\Rightarrow$ (2)" direction of Proposition A.1.0.3 says that a 2-simplex τ as above exhibiting G as a left Kan extension of F along f is Kan, i.e. allows the solution of the lifting problems of Equation A.1.0.1. Let us consider the lifting problem corresponding to $n = 3$ to see why this might be the case.

Suppose we are given a map $\sigma\colon \Lambda^3_0 \to \mathbb{A}$ such that $\sigma|\Delta^{\{0,1,3\}}$ is given by τ. This is the data of objects and morphisms

$$\begin{tikzcd} & b \arrow[dd, "g"] \arrow[drr, "G"] & & \\ a \arrow[ur, "f"] \arrow[dr, "h"'] \arrow[rrr, "F"] & & & x \\ & c \arrow[urr, "H"'] & & \end{tikzcd}$$

together with 2-morphisms $\eta\colon F \Rightarrow G \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow H \circ h$, making up the sides of Λ^3_0. In order to fill this to a full 3-simplex in $\mathbb{A}$, we need to produce a 2-morphism $\alpha\colon G \Rightarrow H \circ g$ and a filling of the full 3-simplex, which is the data of a homotopy-commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & G \circ f \arrow[d, "\alpha f"] \\ H \circ h \arrow[r, "H\beta"] & H \circ g \circ f \end{tikzcd}$$

in $\mathbb{A}(a,x)$.[1]

We can rephrase this as follows. The data contained in $\sigma\colon \Lambda^3_0 \to \mathbb{A}$ can be drawn

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & G \circ f \\ H \circ h \arrow[r, "H\beta"] & H \circ g \circ f \end{tikzcd}}^{\text{in } \mathbb{A}(a,x)} \qquad \overbrace{\begin{tikzcd} G \\ H \circ g \end{tikzcd}}^{\text{in } \mathbb{A}(b,x)}$$

where the left-hand diagram is a map $LC^2 \to \mathbb{A}(a,x)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ excluding the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathbb{A}(b,x)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

(1) We first can fill the lower-left half of the diagram on the left simply by taking the

[1] In particular, if we take $h = f$, $g = \mathrm{id}_b$, and β to be the identity $f \Rightarrow f$, we recover the classical universal property satisfied by left Kan extension.

composition $H\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & G \circ f \\ & H \circ g \circ f \end{tikzcd}}^{\text{in } \mathbb{A}(a,x)} \qquad \overbrace{\begin{tikzcd} G \\ H \circ g \end{tikzcd}}^{\text{in } \mathbb{A}(b,x)} .$$

(2) Using the adjunction $f_! \dashv f^*$, we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left (in the sense to be described in Subsection A.1.2):

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & G \circ f \\ & H \circ g \circ f \end{tikzcd}}^{\text{in } \mathbb{A}(a,x)} \qquad \overbrace{\begin{tikzcd} f_! F \arrow[r, "\xi"] \arrow[dr] & G \\ & H \circ g \end{tikzcd}}^{\text{in } \mathbb{A}(b,x)}$$

(3) Since η is by assumption a local unit, ξ is an equivalence, so the diagram on the right has a filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & G \circ f \arrow[d, "\alpha f"] \\ & H \circ g \circ f \end{tikzcd}}^{\text{in } \mathbb{A}(a,x)} \qquad \overbrace{\begin{tikzcd} G \arrow[r, "\xi"] \arrow[dr] & G \arrow[d, "\alpha"] \\ & H \circ g \end{tikzcd}}^{\text{in } \mathbb{A}(b,x)}$$

The higher lifting problems which we have to solve in Equation A.1.0.1 for $n > 3$ amount to replacing $\Delta^1 \times \Delta^1$ by $(\Delta^1)^{n-1}$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved.

- In Subsection A.1.1 we explain the combinatorics of filling cubes relative to their boundaries.
- In Subsection A.1.2, we give a formalization the concept of adjunct data, and provide a means of for filling partial adjunct data to total adjunct data, analogous to solving a lifting problem by passing to the adjunct lifting problem.
- In Subsection A.1.3, we show that we can always solve the lifting problems of Equation A.1.0.1.

A.1.1 Filling cubes relative to their boundaries

Our main goal in Appendix A.1 is to prove Proposition A.1.0.3. To do this, we must understand what data we need to specify in order to solve a lifting problem of the form

$$\begin{tikzcd} (\Lambda^n_0) \arrow[r] \arrow[d, hook] & \mathbb{A} \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} .$$

Under our assumption that $\mathbb{A}$ is given by the homotopy coherent nerve of a quasicategory-enriched category $\mathbb{A}$, solving the above lifting problem is equivalent to solving the adjunct lifting problem

$$\begin{tikzcd} \mathfrak{C}^{\mathrm{sc}}[\Lambda^n_0] \arrow[r] \arrow[d, hook] & \mathsf{A} \\ \mathfrak{C}^{\mathrm{sc}}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

where the diagram in question is now a diagram of quasicategory-enriched categories. The inclusion $\mathfrak{C}^{\mathrm{sc}}[\Lambda^n_0] \hookrightarrow \mathfrak{C}^{\mathrm{sc}}[\Delta^n]$ is a bijection on objects, so in order to solve such lifting problems, we need to fill the missing data in each mapping space. As we will see in Subsection A.1.3, these fillings take the form of filling a cube $(\Delta^1)^{n-1}$ relative to its boundary missing one face.

In this subsection, we give the combinatorics of filling the n-cube. Our main result is Proposition A.1.1.21, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map.

Definition A.1.1.1. The ***n-cube*** is the simplicial set $C^n := (\Delta^1)^n$.

Note A.1.1.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Our first step is to understand the nondegenerate simplices of the n-cube.

Definition A.1.1.3. Denote by $a^i \colon \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\ldots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\ldots,n\}}$ to $\{1\}$.

Note A.1.1.4. The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0,\ldots,0)$ and ending at $(1,\ldots,1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1,\ldots,n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition A.1.1.5. For any permutation $\tau \colon \{1,\ldots,n\} \to \{1,\ldots,n\}$, we define an n-simplex

$$\phi(\tau) \colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)},\ldots,a^{\tau(n)}).$$

Note A.1.1.6. It is easy to check that the above definition really results in a bijection between S_n and the nondegenerate simplices of C^n.

Notation A.1.1.7. We will denote any permutatation $\tau \colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ by the corresponding n-tuple $(\tau(1),\ldots,\tau(n))$. Thus, the identity permutation is $(1,\ldots,n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2,1,3,\ldots,n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)},\ldots,a^{\tau(n)}) \colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i \colon \Delta^n \to \Delta^1$ and any face map $\partial_j \colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\dots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)}, \dots, a^{\tau(n)}) := (\partial_i^* a^{\tau(1)}, \dots, \partial_i^* a^{\tau(n)}).$$

Example A.1.1.8. Consider the 3-simplex (a^1, a^2, a^3) in C^3. This has spine

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation A.1.1, we then calculate that

$$d_2(a^1, a^2, a^3) = (a^1, a^2, a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0,0,0) \to (1,0,0) \to (1,1,1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition A.1.1.9. The ***boundary of the n-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{A.1.1.1}$$

The following is easy to see.

Proposition A.1.1.10. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \dots, \gamma_n) \colon \Delta^{n-1} \to C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition A.1.1.11. The ***left box of the n-cube***, denoted LC^n, is the simplicial subset of C^n given by the union of all of the faces in Equation A.1.1.1 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example A.1.1.12. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma A.1.1.13. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

(1) The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.

(2) For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.

(3) The nth face $d_n\phi$ always belongs to LC^n.

Proof. (1) We have

$$d_0\phi = (a^{\tau(0)-1}, \ldots a^{\tau(n)-1})$$

since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition A.1.1.10 guarantees that $d_0\phi$ is contained in the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

This face belongs to LC^n except when $j = n$.

(2) In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.

(3) In this case,

$$d_n(a^{\tau(1)}, \ldots, a^{\tau(n)}) = (a^{\tau(1)}, \ldots, a^{\tau(n)})$$

since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation A.1.1.14. For any subset $T \subseteq [n]$, write

$$\Lambda^n_T := \bigcup_{t \in T} d_t \Delta^n \subset \Delta^n.$$

Lemma A.1.1.15. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n - 1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \ldots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma A.1.1.15 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition A.1.1.16. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) < \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example A.1.1.17. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation A.1.1.18. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$\begin{aligned} LC^n &\hookrightarrow LC^n \cup \phi(1,\dots,n) \\ &\hookrightarrow \cdots \\ &\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\dots,n,1)} \phi(\tau) \end{aligned} \tag{A.1.1.2}$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2,\dots,n,1)$, the intersection

$$B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{A.1.1.3}$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition A.1.1.19. Each inclusion in Equation A.1.1.2 is inner anodyne.

Proof. The first inclusion

$$LC^n \hookrightarrow LC^n \cup \phi(1,\dots,n)$$

is inner anodyne because by Lemma A.1.1.13 the intersection $LC^n \cap \phi(1,\dots,n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma A.1.1.13), so by Lemma A.1.1.15, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (1,\dots,n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),$$

which implies that $\tau = (1,\dots,n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. □

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$. Thus, each of their spines begins

$$(0,\dots,0,0) \to (0,\dots,0,1) \to \cdots,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that

$(0, \dots, 0, 0) \to (0, \dots, 0, 1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition A.1.1.19. One continues to fill simplices in the order prescribed in Definition A.1.1.16, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma A.1.1.15.

Lemma A.1.1.20. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition A.1.1.21. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0, \dots, 0, 0) \to (0, \dots, 0, 1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \xhookrightarrow{i} J^n \xhookrightarrow{j} C^n \ ,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\ J^n \arrow[r, hook] & C^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} ,$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0, \dots, 0, 0) \to (0, \dots, 0, 1)$.

A.1.2 Adjunct data

When confronted with a lifting problem, one frequently finds a lift by passing to an adjoint lifting problem which has a solution, and transporting the solution back along the adjunction. This relies on the fact that providing data on one side of an adjunction is, in a certain sense, equivalent to providing adjunct data on the other side. In this section, we give a formalization of the the notion of adjunct data in the ∞-categorical context. Our main result is Proposition A.1.2.8, which shows that under certain conditions, we can use data on one side of an adjunction of an ∞-categories to fill in data on the other side. First, we recall somewhat explicitly the definition of an adjunction given in [HTT, Def. 5.2.2.1].

Definition A.1.2.1. An ***adjunction*** is a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$. We say that p is ***associated*** to functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ if there exist equivalences $h_0\colon \mathcal{C} \to \mathcal{M}_0$

and $h_1 \colon \mathcal{D} \to \mathcal{M}_1$, and a commutative diagram

$$\begin{tikzcd} \mathcal{C} \times \Delta^1 \arrow[r, "u"] \arrow[dr, "\mathrm{pr}"'] & \mathcal{M} \arrow[d, "p"] & \mathcal{D} \times \Delta^1 \arrow[l, "v"'] \arrow[dl, "\mathrm{pr}"] \\ & \Delta^1 & \end{tikzcd}$$

such that the following conditions are satisfied.

- *The map u is associated to the functor f:* the restriction $u|\mathcal{C} \times \{0\} = h_0$, the restriction $u|\mathcal{C} \times \{1\} = f \circ h_1$, and for each $c \in \mathcal{C}$, the edge $u|\{c\} \times \Delta^1$ is p-cocartesian.

- *The map v is associated to the functor g:* the restriction $v|\mathcal{D} \times \{0\} = g \circ h_0$, the restriction $u|\mathcal{D} \times \{1\} = h_1$, and for each $d \in \mathcal{D}$, the edge $u|\{d\} \times \Delta^1$ is p-cartesian.

If f and g are functors to which an adjunction is associated as above, then we say that f is left adjoint to g, and equivalently that g is right adjoint to f.

Note A.1.2.2. It is shown in [HTT, Prop. 5.2.1.3] that if f and g are functors such that f is left adjoint to g, then we can choose an adjunction $p \colon \mathcal{M} \to \Delta^1$ and data (h_0, h_1, u, v) such that h_0 and h_1 are isomorphisms. We can use this to identify $\mathcal{M}_0$ with $\mathcal{C}$, and $\mathcal{M}_1$ with $\mathcal{D}$. In what follows we will always assume that we have chosen such data, and will thus leave the isomorphisms h_0 and h_1 implicit.

Definition A.1.2.3. Let $s \colon K \to \Delta^1$ be a map of simplicial sets whose fibers we denote by K_0 and K_1, and let $p \colon \mathcal{M} \to \Delta^1$ be a bicartesian fibration associated to adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via data $(u \colon \mathcal{C} \times \Delta^1 \to \mathcal{M}, v \colon \mathcal{D} \times \Delta^1 \to \mathcal{M})$. We say that a map $\alpha \colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha} \colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A \colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

(1) The restriction $A|_{K \times \{0\}} = \alpha$.

(2) The restriction $A|_{K \times \{1\}} = \tilde{\alpha}$.

(3) The restriction $A|K_0 \times \Delta^1$ is equal to the composition

$$K_0 \times \Delta^1 \hookrightarrow K \times \Delta^1 \stackrel{\alpha \circ \times \mathrm{id}}{\to} \mathcal{C} \times \Delta^1 \stackrel{u}{\to} \mathcal{M}.$$

(4) The restriction $A|K_1 \times \Delta^1$ is equal to the composition

$$K_1 \times \Delta^1 \hookrightarrow K \times \Delta^1 \stackrel{\tilde{\alpha} \times \mathrm{id}}{\to} \mathcal{D} \times \Delta^1 \stackrel{v}{\to} \mathcal{M}.$$

In the next examples, fix adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$ via data (u, v).

Example A.1.2.4. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$ of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is $u|\{c\}\times\Delta^1$ and b is $v|\{d\}\times\Delta^1$. This corresponds to the map $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

Example A.1.2.5. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id}\colon fC \to fC$ in $\mathcal{D}$. This morphism is adjunct to the component of the unit map $\eta_C\colon C \to gfC$ relative to $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

Example A.1.2.6. For any simplicial sets K and K', any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g. This will follow from Proposition A.1.2.8.

Lemma A.1.2.7. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\partial\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^1, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{A.1.2.1}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition A.1.2.8. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, u and v be as in Definition A.1.2.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s\circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a monomorphim such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{A.1.2.2}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s. Furthermore, any two such lifts are equivalent as functors $K \to \mathcal{D}$.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. From the map A and the map $\alpha\colon K \to \mathcal{C} \cong \mathcal{M}_0$, we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd} \left(K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1\right)^{\heartsuit} \arrow[r] \arrow[d, "w"'] & \mathcal{M}^{\natural} \arrow[d] \\ (K \times \Delta^1)^{\heartsuit} \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^{\sharp} \end{tikzcd},$$

where $(K \times \Delta^1)^{\heartsuit}$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\} \times \Delta^1$ for $k \in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. We can write the morphism of simplicial sets underlying w as a smash-product $(K' \hookrightarrow K) \wedge (\Delta^{\{0\}} \hookrightarrow \Delta^1)$. Building $K' \hookrightarrow K$ simplex by simplex, in increasing order of dimension, we can write w as a transfinite composition of pushouts along inclusions of the form given in Equation A.1.2.1 (by assumption each simplex σ we adjoin to K is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). We can thus also build our lift ℓ by lifting against each of these in turn, and we can always choose each lift so that it is compatible with v. We can then take $\tilde{\alpha} = \ell|K \times \{1\}$.

It is manifest from this technique that the space of lifts is contractible; in particular, any two such lifts are equivalent in $\mathrm{Map}(K \times \Delta^1, \mathcal{M})$, so their restrictions to $K \times \{1\}$ are equivalent as functors $K \to \mathcal{D}$.

□

Example A.1.2.9. Let $f : \mathcal{C} \leftrightarrow \mathcal{D} : g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K' = \Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \arrow[r, hook, "i"] \arrow[dr] & \Delta^3 = K \arrow[d] \\ & \Delta^1 \end{tikzcd},$$

where both downwards-facing arrows take $0,\ 1 \mapsto 0$ and $2,\ 3 \mapsto 1$, and fix a map $\tilde{\alpha}\colon K \to N(\mathcal{C})$ and a map $\alpha'\colon K' \to N(\mathcal{D})$ such that $\tilde{\alpha} \circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition A.1.2.8 implies that the solution to the lifting problem on the left can be transported along the adjunction $f \dashv g$, yielding a solution to the lifting problem on the right. Similarly, its dual implies the converse. Thus, Proposition A.1.2.8 implies in particular that such lifting problems are equivalent.

We note that in the proof of Proposition A.1.2.8, we did not need to use the full power of the statement that f was left adjoint to g (i.e. that $p\colon \mathcal{M} \to \Delta^1$ was a bicartesian fibration), only that f was *locally* left adjoint to g, (i.e. that p admitted cocartesian lifts at certain objects). This allows us to generalize Proposition A.1.2.8 somewhat.

Definition A.1.2.10. Let $g\colon \mathcal{D} \to \mathcal{C}$ be a functor between quasicategories, and pick some cartesian fibration $p\colon \mathcal{M} \to \Delta^1$ which classifies it. We assume without loss of generality that $h_0\colon \mathcal{M}_0 \cong \mathcal{C}$ and $h_1\colon \mathcal{M}_1 \cong \mathcal{D}$ are isomorphisms, and we will notationally suppress them.

We say that g ***admits a left adjoint at*** $c \in \mathcal{C}$ if p admits a cocartesian lift of the morphism id_{Δ^1} with source $c \in \mathcal{C}$.

Note A.1.2.11. Clearly, g admits a left adjoint if and only admits a left adjoint at all objects $c \in \mathcal{C}$.

Definition A.1.2.12. Let $s\colon K \to \Delta^1$ be a map of simplicial sets whose fibers we denote by K_0 and K_1, and let $p\colon \mathcal{M} \to \Delta^1$ be a cartesian fibration associated to $f\colon \mathcal{D} \to \mathcal{C}$ via a map $v\colon \mathcal{D} \times \Delta^1 \to \mathcal{M}$. We say that a map $\alpha\colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

(1) The restriction $A|_{K \times \{0\}} = \alpha$.

(2) The restriction $A|_{K \times \{1\}} = \tilde{\alpha}$.

(3) For each vertex $k \in K_0$, the image of $\{k\} \times \Delta^1$ under A in $\mathcal{M}$ is p-cartesian.

(4) The restriction $A|K_1 \times \Delta^1$ is equal to the composition

$$K_1 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\tilde{\alpha} \times \mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{v} \mathcal{M}.$$

The proof of Proposition A.1.2.8 can be used as is to show the following statement.

Proposition A.1.2.13. Let $p\colon \mathcal{M} \to \Delta^1$ classify $g\colon \mathcal{D} \to \mathcal{C}$ via $v\colon \mathcal{D} \times \Delta^1 \to \mathcal{M}$, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s \circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a monomorphim such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Further suppose that g admits a left adjoint at all vertices belonging to the image $\alpha(K_0) \subseteq \mathcal{C}$.

Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{A.1.2.3}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s. Furthermore, any two such lifts are equivalent as functors $K \to \mathcal{D}$.

Definition A.1.2.14. Let $g\colon \mathcal{D} \to \mathcal{C}$ be a functor, and suppose that g admits a left adjoint $f(c) \in \mathcal{D}$ at some object $c \in \mathcal{C}$. A morphism $c \to g(d)$ in $\mathcal{C}$ is a ***local unit*** if it is adjunct to an equivalence $f(c) \xrightarrow{\simeq} d$ in $\mathcal{D}$.

Note that local units in this sense always exist, given by any adjunct to the map $\mathrm{id}_{f(c)}$.

A.1.3 Kan implies locally left adjoint

In this subsection, we will prove Theorem A.1.0.3. In order to do that, the discussion in Subsection A.1.1 shows that we will have to solve lifting problems of the form

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_0] \arrow[r] \arrow[d, hook] & \mathsf{A} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}, \tag{A.1.3.1}$$

where A is a quasicategory-enriched enriched category with $\mathrm{N}_{\mathrm{sc}}(\mathsf{A}) = \mathbb{A}$. We expect that the reader is familiar with the basics of rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category $\mathfrak{C}[\Delta^n]$ are given by the set $\{0, \ldots, n\}$, and the mapping spaces are defined by

$$\mathfrak{C}[\Delta^n](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases},$$

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \ldots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda^n_0]$ has the same objects as $\mathfrak{C}[\Delta^n]$, and each morphism space $\mathfrak{C}[\Lambda^n_0](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^n](i,j)$, as we shall soon describe.

The map $\mathfrak{C}[\Lambda^n_0](i,j) \hookrightarrow \mathfrak{C}[\Delta^n](i,j)$ is an isomorphism except for $(i,j) = (1,n)$ and $(i,j) = (0,n)$; in these cases, it is an inclusion. The missing data corresponds in the case of $(1,n)$ to the missing face $d_0\Delta^n$ of Λ^n_0, and in the case of $(0,n)$ to the missing interior. Thus, to find a filling as in Equation A.1.3.1, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_0](1,n) \arrow[r] \arrow[d, hook] & \mathsf{A}(\mathcal{F}(1), \mathcal{F}(n)) \\ \mathfrak{C}[\Delta^n](1,n) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{A.1.3.2}$$

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_0](0,n) \arrow[r] \arrow[d, hook] & \mathsf{A}(\mathcal{F}(0),\mathcal{F}(n)) \\ \mathfrak{C}[\Delta^n](0,n) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{A.1.3.3}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^n](1,n) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \mathsf{A}(\mathcal{F}(1),\mathcal{F}(n)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^n](0,n) \arrow[r, "\ell"] & \mathsf{A}(\mathcal{F}(0),\mathcal{F}(n)) \end{tikzcd} \tag{A.1.3.4}$$

commute.

Notation A.1.3.1. Recall our desired filling $\mathcal{F}\colon \mathfrak{C}[\Delta^n] \to \mathsf{A}$ of Equation A.1.3.1. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \ldots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \mathsf{A}(X_{i_1}, X_{i_k})$ by $f_{i_k \cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

We now turn our attention to the proof of Theorem A.1.0.3. The author would like to offer the friendly recommendation that in reading the proof below, it is helpful to follow along with the explanation given at the beginning of Section A.1.

Proof of Theorem A.1.0.3. Suppose that 1. holds. We need to check that the given 2-simplex τ is Kan. To do this, we need to solve the lifting problems of Equation A.1.3.2 and Equation A.1.3.3, and check that our solutions satisfy the condition of Equation A.1.3.4. In the discussuion surrounding these equations, we ignored that the simplex $\tau\colon \Delta^{\{0,1,n\}} \to \mathbb{A}$ was Kan. We now reintroduce this information. The information that the simplex $\tau\colon \Delta^{\{0,1,n\}} \to \mathbb{A}$ is Kan tells us the following.

- Concretely, it tells us that the map $f_{n,1}$ is the Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha^{\{0,1,n\}}_{\{0,n\}}\colon f_{n,0} \to f_{n,1} \circ f_{1,0}$ is the unit map.
- In particular, it tells us that the map $f^*_{1,0}$ admits a left adjoint at $f_{n,0}$.

The lifting problems as described above are somewhat unwieldy, given in terms of mapping spaces of simplicial categories $\mathfrak{C}[\Delta^n]$ and $\mathfrak{C}[\Lambda^n_0]$. It will be useful to give these mapping spaces more down-to-earth descriptions, in terms of simplicial sets with which we are familiar. We have the following succinct descriptions of the inclusions of Equation A.1.3.2 and Equation A.1.3.3 along which we have to extend.

- There is an isomorphism of simplicial sets

  $$\mathfrak{C}[\Delta^n](0,n) \xrightarrow{\cong} C^{n-1}$$

  specified completely by sending a subset $S \subseteq [n]$ to the point

  $$(z_1, \ldots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

The inclusion $\mathfrak{C}[\Lambda^n_0](0,n) \hookrightarrow \mathfrak{C}[\Delta^n](0,n)$ corresponds to the simplicial subset $LC^{n-1} \hookrightarrow C^{n-1}$.

- There is a similar isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^n](1,n) \xrightarrow{\cong} C^{n-2}$$

specified completely by sending $S \subseteq \{1,\dots,n\}$ to the point

$$(z_1,\dots,z_{n-2}) \in C^{n-2}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$

The inclusion $\mathfrak{C}[\Lambda^n_0](1,n) \hookrightarrow \mathfrak{C}[\Delta^n](1,n)$ corresponds to the inclusion $\partial C^{n-2} \hookrightarrow C^{n-2}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^{n-1} \arrow[r] \arrow[d, hook] & \mathsf{A}(X_0,X_n) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-2} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathsf{A}(X_1,X_n) \\ C^{n-2} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$(\star) = \begin{tikzcd} C^{n-2} \arrow[r] \arrow[d, hook] & \mathsf{A}(X_1,X_n) \arrow[d, "f^*_{1,0}"] \\ C^{n-1} \arrow[r] & \mathsf{A}(X_0,X_n) \end{tikzcd}$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition A.1.1.21, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$(*') = \begin{tikzcd} \Delta^0 * \partial C^{n-2} \arrow[r, "\alpha"] \arrow[d, hook] & \mathsf{A}(X_0,X_n) \\ \Delta^0 * C^{n-2} \arrow[ur, dashed] & \end{tikzcd}.$$

The image of the 1-simplex $(0,\dots,0) \to (0,\dots,0,1)$ of $\Delta^0 * \partial C^{n-2} \subseteq C^{n-2}$ under α is the unit map $\alpha^{\{0,1,n\}}_{\{0,n\}} \colon f_{n,0} \to f_{n,1} \circ f_{1,0}$. This is not in general an equivalence, so we cannot use Proposition A.1.1.21 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to an equivalence in $\mathsf{A}(X_1,X_n)$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-2} is in the image of $f^*_{1,0}$, so by Proposition A.1.2.13 and our assumption that f^*_{10} admits a left adjoint at $f_{n,0}$, we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-2} \to \mathsf{A}(X_1,X_{n-2})$ which is adjunct to α relative to the map

$$s \colon \Delta^0 * \partial C^{n-2} \to \Delta^1$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-2} to $\Delta^{\{1\}}$; in particular, the image of the morphism $(0,\dots,0) \to (0,\dots,0,1)$ under $\tilde{\alpha}'$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$(**') = \begin{tikzcd} \Delta^0 * \partial C^{n-2} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \mathsf{A}(X_1, X_{n+1}) \\ \Delta^0 * C^{n-2} \arrow[ur, dashed, "\tilde{\beta}"'] & \end{tikzcd}$$

However, since image of $(0,\dots 0) \to (0,\dots,1)$ is an equivalence, Proposition A.1.1.21 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition A.1.2.13, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-2}}$ be the image of $\tilde{\beta}|_{C^{n-2}}$ under $f^*_{1,0}$, which is true by construction.

Now, suppose that 2. holds, i.e. that τ is Kan. The map r in the pullback diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d, "r"'] & \mathbb{A}_{\nearrow c} \arrow[d] \\ \Delta^1 \arrow[r, "f"] & \mathbb{A} \end{tikzcd},$$

is a cartesian fibration classifying the functor $f^* \colon \mathbb{A}(b,c) \to \mathbb{A}(a,c)$; here $\mathbb{A}_{\nearrow c}$ is the *oplax overcategory* of [AM24, Def. 2.2.5]. Denote by $u \colon \Delta^1 \to \mathcal{M}$ the morphism whose image in $\mathbb{A}_{\nearrow c}$ corresponds to the 2-simplex τ in $\mathbb{A}$. We would like to show that u is r-cocartesian, i.e. that any solid lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "u"] & & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{M} \arrow[r] \arrow[d, "r"] & \mathbb{A}_{\nearrow c} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^1 \arrow[r, "f"] & \mathbb{A} \end{tikzcd}$$

admits a dashed solution. Using the fact that the right-hand square is pullback, we can instead show that the 'outer' lifting problem admits a solution. Using the universal property of the overcategory, this is equivalent to showing that the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ \Lambda^{n+1}_0 \arrow[r] \arrow[d, hook] & \mathbb{A} \\ \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

has a solution, which follows because τ is Kan. $\square$

A.2 Fiberwise localization

A.2.1 Reflective subcategories

In this section, we review basic definitions of reflective subcategories, mainly to fix notation. None of this is new, and most of it can be found in [HTT, Sec. 5.2.7].

Definition A.2.1.1. Let $\mathcal{X}$ be a ∞-category. A subcategory $\rho\colon \mathcal{Y} \subseteq \mathcal{X}$ is ***reflective*** if the inclusion functor ρ is (1) full and faithful and (2) admits a left adjoint $\lambda\colon \mathcal{X} \to \mathcal{Y}$.

Definition A.2.1.2. Let $\mathcal{X}$ be any ∞-category, and $\rho\colon \mathcal{Y} \subseteq \mathcal{X}$ be any inclusion of a full subcategory, and $x \in \mathcal{X}$. A ***reflector*** for x is a morphism $r_x\colon x \to y_x$, where $y_x \in \mathcal{Y}$, such that r_x is initial in $\mathcal{Y}_{x/}$.

Note A.2.1.3. Here are a few immediate consequences of the definitions.

(1) If a reflector for an object $x \in \mathcal{X}$ exists, then it is unique up to contractible choice.

(2) A full subcategory inclusion $\rho\colon \mathcal{Y} \subseteq \mathcal{X}$ is reflective if and only if each $x \in \mathcal{X}$ admits a reflector $r\colon x \to y_x$. In this case, the left adjoint $\lambda\colon \mathcal{X} \to \mathcal{Y}$ sends $x \mapsto \lambda(x) := y_x$.

(3) We can rephrase the definition of a reflector as follows: $r_x\colon x \to y_x$ is a reflector for x if composition with r_x induces a homotopy equivalence of spaces $\mathcal{Y}(y_x, y) \to \mathcal{X}(x, y)$ for all $y \in \mathcal{Y}$.

For all that follows, let $\rho\colon \mathcal{Y} \to \mathcal{X}$ be a reflective subcategory with left adjoint $\lambda\colon \mathcal{X} \to \mathcal{Y}$.

Definition A.2.1.4. Call a morphism $a\colon x \to x'$ in $\mathcal{X}$ a ***weak equivalence*** if $\lambda(a)\colon \lambda(x) \to \lambda(x')$ is an equivalence.

Note that each equivalence is a weak equivalence, and that weak equivalences satisfy 2/3.

Definition A.2.1.5. Call an object $\bar{x} \in \mathcal{X}$ ***local*** if it has the property that for all weak equivalences $s\colon x \to x'$, the map $s^*\colon \mathcal{X}(x', \bar{x}) \to \mathcal{X}(x, \bar{x})$ given by pulling back along s is a homotopy equivalence of spaces.

Proposition A.2.1.6. (1) An object $\bar{x} \in \mathcal{X}$ is local if and only if it is in the essential image of the inclusion $\rho\colon \mathcal{Y} \subseteq \mathcal{X}$.

(2) A morphism $f\colon x \to y$, where $y \in \mathcal{Y}$, is a reflector if and only if it is a weak equivalence.

Proof. (1) [HTT, Prop. 5.5.4.2(1)].

(2) We can compute $\lambda(f)$ by picking reflectors for x and y and pushing f forward along them; since $y \in \mathcal{Y}$, the identity map id_y is a reflector for y.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "r_x"'] & y \arrow[d, "\mathrm{id}_y"] \\ y_x \arrow[r, "\lambda(f)"] & y \end{tikzcd}$$

Thus the morphism $\lambda(f)$ is an equivalence if and only if r_x and f are equivalent as objects in $\mathcal{Y}_{x/}$, which in turn is true if and only if f is a reflector.

∎

A.2.2 Fiberwise localization

Let $\mathcal{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a functor sending $b \mapsto \mathcal{X}_b$, and suppose we are given the data of, for each $b \in \mathcal{B}$, a reflective subcategory $\rho_b\colon \mathcal{Y}_b \subseteq \mathcal{X}_b$ with localization functor $\lambda_b\colon \mathcal{X}_b \to \mathcal{Y}_b$. We might hope to construct from this data a functor $\mathcal{Y}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ by 'restricting' the functor $\mathcal{X}$ to the subcategories $\mathcal{Y}_b$, but this is doomed to fail since $\mathcal{X}$ does not necessarily fix these subcategories. However, we can define, for any morphism $f\colon b \to c$ in $\mathcal{B}$, a functor $\mathcal{Y}_f\colon \mathcal{Y}_c \to \mathcal{Y}_b$ via

$$\mathcal{Y}_f\colon \mathcal{Y}_c \overset{\rho_c}{\subseteq} \mathcal{X}_c \overset{\mathcal{X}_f}{\to} \mathcal{X}_b \overset{\lambda_b}{\to} \mathcal{Y}_b. \tag{A.2.2.1}$$

Proposition A.2.2.1. Let $\mathcal{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ and $\mathcal{Y}_b \subseteq \mathcal{X}_b$ be as above. Suppose for each morphism $f\colon b \to c$ in $\mathcal{B}$, the functor $\mathcal{X}_f\colon \mathcal{X}_c \to \mathcal{X}_b$ preserves weak equivalences. Then the procedure of Equation A.2.2.1 extends to a functor $\mathcal{Y}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending $b \mapsto \mathcal{Y}_b$ and $f \mapsto \mathcal{Y}_f$.

We will postpone the proof of this proposition to the end of the section. The idea is the following. Consider morphisms $f\colon b \to c$ and $g\colon c \to d$ in $\mathcal{B}$. Let $y \in \mathcal{Y}_d$ be an object. According to Equation A.2.2.1 we define $\mathcal{Y}_g(y)$ to be the target of a reflector $r\colon \mathcal{X}_g(y) \to \mathcal{Y}_g(y)$. We then define $\mathcal{Y}_f(\mathcal{Y}_g(y))$ to be the target of a reflector $r'\colon \mathcal{X}_f(\mathcal{Y}_g(y)) \to \mathcal{Y}_f(\mathcal{Y}_g)(y)$. Consider the morphisms

$$\mathcal{X}_g(\mathcal{X}_f(y)) \xrightarrow{\mathcal{X}_g(r)} \mathcal{X}_g(\mathcal{Y}_f(y)) \xrightarrow{r'} \mathcal{Y}_g(\mathcal{Y}_f(y)) \ .$$

Since r is a reflector, it is in particular a weak equivalence, so $\mathcal{X}_g(r)$ is a weak equivalence since $\mathcal{X}_g$ preserves weak equivalences by assumption. Since r' is a reflector, we know that it is an equivalence, and that its target is local. Thus, the composition $r' \circ \mathcal{X}_g(r)$ is a weak equivalence whose target is local, hence a reflector exhibiting $\mathcal{Y}_g(\mathcal{Y}_f(y)) \simeq \mathcal{Y}_{g\circ f}(y)$.

However, the above procedure does not admit a simple coherent formulation. To remedy this, we will prove an unstraightened version of this result. Our proof will be a modification of proofs given independently in [Hau+23] and in [Rus22a].

Proposition A.2.2.2. Let $p\colon \mathcal{X} \to \mathcal{B}$ be a cartesian fibration, and suppose we are given, for each $b \in \mathcal{B}$, a reflective subcategory $\rho_b\colon \mathcal{Y}_b \subseteq \mathcal{X}_b$. Further suppose that p has the following property:

($*$) Given a square of morphisms

$$\begin{tikzcd} x_{02} \arrow[r, "b"] \arrow[d] & x_{12} \arrow[d] \\ x_{01} \arrow[r, "a"] & y_{02} \end{tikzcd}$$

in $\mathcal{X}$ such that the downward-facing arrows are p-cartesian and a is a weak equivalence, then b is a weak equivalence.

Then there is a cocartesian fibration

$$\bar{p}\colon \mathrm{Span}^{\mathcal{Y}}(\mathcal{X}) \to \mathcal{B}^{\mathrm{op}},$$

where $\mathrm{Span}^{\mathcal{Y}}(\mathcal{X})$ is the category of spans of the form

$$y \xleftarrow{g} x \xrightarrow{f} y' \ ,$$

where g is p-cartesian and y and y' are in their respective reflective subcategories. If f is a weak equivalence (hence a reflector for x), then such a span is $\bar{p}$-cocartesian.

Proof. Recall that a morphism in a fiber $\mathcal{X}_b$ of p is said to be a weak equivalence if it is sent to an equivalence by the localization functor $\lambda_b \colon \mathcal{X}_b \to \mathcal{Y}_b$. We will say that a morphism in $\mathcal{X}$ not necessarily contained in a fiber of p is a weak equivalence if and only if it is equvialent (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{X})$ of morphisms in $\mathcal{X}$) to a weak equivalence in some fiber $\mathcal{X}_b$, and that an object in $\mathcal{X}$ is p-local if it is a local object in some fiber. By [Bar17], we are assured the existence of:

- A complete Segal space $\mathrm{Span}'(\mathcal{X})$ of spans in $\mathcal{X}$ whose backwards-facing leg is p-cartesian, and whose forwards-facing leg is unconstrained.
- A complete Segal space $\mathrm{Span}^{\simeq}(\mathcal{B})$ of spans in $\mathcal{B}$ whose forwards-facing leg is an equivalence, and whose backwards-facing leg is unconstrained, together with an equivalence $\mathcal{B}^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Span}^{\simeq}(\mathcal{B})$.

By [Rus22a, Prop. 3.5.2], there is a Reedy fibration $\bar{p}' \colon \mathrm{Span}'(\mathcal{X}) \to \mathcal{B}^{\mathrm{op}}$. Denote by $\mathrm{Span}^{\mathcal{Y}}(\mathcal{X}) \subseteq \mathrm{Span}'(\mathcal{X})$ the full subcategory on local objects. Then the map $\mathrm{Span}^{\mathcal{Y}}(\mathcal{X}) \to \mathcal{B}^{\mathrm{op}}$ is also a Reedy fibration. We claim that any morphism of the form

$$x \xleftarrow{f} y \xrightarrow{g} x' \ ,$$

where f is p-cartesian and g is a weak equivalence, is p-cocartesian in the sense of [Rus22a, Def. 2.4.1]. To see this, we need to show that given a morphism of that form we can produce an essentially unique dashed filling of the solid diagram

$$\begin{tikzcd}
 & & \mathbf{y}' & & \\
 & x \arrow[ur, "g", "\circ" description] \arrow[dl, "f"', "\bullet" description] & & x' \arrow[ul, dashed, "(2)", "\bullet" description] \arrow[dr, dashed, "(4)"] & \\
\mathbf{y} & & z \arrow[ll, "\bullet" description] \arrow[ul, dashed, "(1)", "\bullet" description] \arrow[ur, dashed, "(3)", "\circ" description] \arrow[rr] & & \mathbf{y}''
\end{tikzcd} ,$$

lying over a filled diagram in the base. Here:

- The morphisms marked $\xrightarrow{\bullet}$ are p-cartesian.
- The morphisms marked $\xrightarrow{\circ}$ are weak equivalences.
- The bold-faced objects $\mathbf{y}$ are local.

We can fill (1) because f is p-cartesian, and the result will be p-cartesian by [HTT, Prop. 2.4.1.7]. Taking (2) to be a p-cartesian lift, we can complete (3), which will be a weak equivalence by property $(*)$. Note that by our assumption that the forwards-facing legs of the spans in $\mathcal{B}$ are equivalences, the bottom-right triangle which remains to be filled lies entirely in a single homotopy fiber. Since (3) is a weak equivalence and $\mathbf{y}''$ is a local object, by Definition A.2.1.5 we can fill (4). Each of these fillings was essentially unique, and the square formed is a pullback square because it is a p-pullback square lying over a pullback square.

Thus we have shown that $\bar{p}$ is a Reedy fibration of complete Segal spaces, and that the spans of the promised form are $\bar{p}$-cocartesian. Since we can always form such a $\bar{p}$-cocartesian span by first taking a p-cartesian lift and then a reflector of the source, we have a sufficient supply of $\bar{p}$-cocartesian morphisms, so $\bar{p}$ is a cocartesian fibration of complete Segal spaces. Taking the 'first row' of the above functor of complete Segal spaces yields a cocartesian fibration of quasicategories as promised. $\square$

Proof of Proposition A.2.2.1. Unstraightening our functor $\mathcal{X}$ to a cartesian fibration, we apply Proposition A.2.2.2, and then straighten the resulting cocartesian fibration again. $\square$

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