Rigid Convolution Structures

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Summary

A monoidal category is called a convolution monoidal category if it arises from linearizing a 2-Segal space. The goal of this thesis is to study for which 2-Segal spaces the induced convolution monoidal category is a multi-fusion category.

With this aim, we show that multi-fusion categories admit an intrinsic description as rigid algebras in the symmetric monoidal 2-category of $\mathbb{C}$-linear additive categories. We use this observation to define, by analogy, a derived version of a multi-fusion category as a rigid algebra in the symmetric monoidal $(\infty, 2)$-category of stable ∞-categories. We show that examples of these arise as derived categories of multi-fusion categories and as categories of modules over smooth and proper $\mathbb{E}_2$-algebras.

Afterward, we show that rigid algebras in the $(\infty, 2)$-category of spans are precisely given by those 2-Segal objects that are Čech-nerves. Together with our previous result, we use this to provide an answer to our initial question. To prove this result, we provide a description of bimodules in the ∞-category of spans as birelative 2-Segal objects. Furthermore, we introduce a notion of morphism between birelative 2-Segal objects that extends this classification to an equivalence of ∞-categories.

We use this classification to construct examples of convolution monoidal structures that form derived multi-fusion categories and discuss some aspects of the associated fully extended TFTs. We finish by studying Grothendieck–Verdier-structures on convolution monoidal ∞-categories and by comparing them with rigid dualities.

Zusammenfassung

Wir nennen eine monoidale Kategorie ein Konvolutions monoidale Kategorie, wenn sie durch die Linearisierung eines 2-Segal Objekts entsteht. Das Ziel dieser Dissertation ist es zu verstehen für welche Klasse von 2-Segal Objekten die induzierte Konvolutions monoidale Kategorie eine Multi-Fusionskategorie ist.

Dafür zeigen wir, dass Multi-Fusionskategorie eine intrinsische Beschreibung als rigide Algebren in der symmetrisch monoidalen 2-Kategorie von $\mathbb{C}$-linearen additiven Kategorien besitzen. Wir verwenden diese Beobachtung, um analog eine derivierte Version einer Multi-Fusionskategorie als rigide Algebra in der symmetrischen monoidalen $(\infty, 2)$-Kategorie von stabilen ∞-Kategorien zu definieren. Wir zeigen, dass Beispiele solcher Kategorien sowohl als derivierte Kategorien von Multi-Fusionskategorien, und als Kategorien von Moduln über eigentlichen und glatten $\mathbb{E}_2$-algebren.

Danach zeigen wir, dass rigide Algebren in der $(\infty, 2)$-Kategorie von Korrespondenzen genau durch die 2-Segal Objekte gegeben sind die ein Čech-Nerve sind. Wir benutzen dieses Resultat, um unsere Ausgangsfrage zu beantworten. Um diese Klassifizierung zu beweisen, führen wir eine Beschreibung von Bimodulen in ∞-Kategorien von Korrespondezen mittels birelativen Segal Objekten ein. Außerdem führen wir einen Begriff von Morphismen zwischen birelativen 2-Segal Objekten ein, der die zuvor beschriebene Korrespondenz zu einer Äquivalenz von Kategorien erweitert.

Wir benutzen unsere vorherigen Resultate um Beispiele von Konvolutions monoidalen ∞-Kategorien zu definieren, die derivierten Multi-Fusionskategorien sind, und diskutieren deren Beziehung zu vollerweiterten TFTs. Zum Abschluss, betrachten wir Grothendieck–Verdier Strukturen auf konvolutions monoidalen Kategorien und vergleichen diese mit rigiden Dualitäten.

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Contents

1 Introduction

A fundamental conjecture in the area of Topological Field Theories (TFTs) is the Cobordism Hypothesis [BD95, Lur08]. This conjecture provides a classification of n-dimensional fully extended TFTs with target a symmetric monoidal (∞, n)-category $\mathcal{C}^{\otimes}$, in terms of objects in $\mathcal{C}$ satisfying certain finiteness conditions. The most basic example of this conjecture involves the classification of 1-dimensional TFTs with values in $\mathsf{Vect}_{\mathbb{C}}^{\otimes}$ (the symmetric monoidal category of $\mathbb{C}$-vector spaces) in terms of finite-dimensional vector spaces [Har12]. More interesting classes of examples arise in dimension 3 as so-called Turaev–Viro style TFTs [DSPS20, TV92]. These TFTs take values in a symmetric monoidal Morita 3-category of finite monoidal $\mathbb{C}$-linear abelian categories, and it has been shown in [DSPS20] that in this case sufficient finiteness conditions on such a monoidal category $\mathcal{A}^{\otimes}$ are given by *rigidity* and *semisimplicity*. Monoidal categories satisfying these conditions are known as *multi-fusion categories* [ENO05].
In recent years, there has been growing interest in constructing examples of (fully extended) $3d$-TFTs from non-semisimple monoidal categories. One approach is to construct these directly without using the cobordism hypothesis [CGPMV23]. This often leads to TFTs that are only partially defined, so-called *non-compact* TFTs [Lur08]. A different approach to this problem is through so-called *derived TFTs*. In this approach, one changes the target of the TFT and replaces its 1-categorical input by an ∞-category.
The latter approach has been successfully applied in dimension 2 for TFTs that take values in the 2-category of $\mathbb{C}$-linear abelian categories [Cos07]. In this case, the 1-categorical input is given by a $\mathbb{C}$-linear abelian category $\mathcal{A}$, which needs to be semisimple to induce a 2-dimensional fully extended TFT [Til98]. However, every abelian category $\mathcal{A}$ also has a natural ∞-category associated with it, called its *bounded derived ∞-category* $\mathcal{D}^{\flat}(\mathcal{A})$. This ∞-category naturally forms an object in the symmetric monoidal $(\infty, 2)$-category of $\mathbb{C}$-linear stable ∞-categories, and it has been shown in [Lur08, Cos07] that these only need to be *smooth and proper* to induce a fully extended TFT. These conditions are strictly weaker than being semisimple.
It therefore seems promising to apply this approach also in the case of Turaev–Viro style TFTs in dimension 3. However, this would require a derived version of a multi-fusion category as its input, a concept that has not been defined so far. To pursue the approach via derived TFTs, we first need to understand what a suitable definition of a derived multi-fusion category should be. For this, it is essential to systematically understand how examples of multi-fusion categories can be constructed. to be able to apply these techniques to the construction of derived multi-fusion categories. Let us hence look at some examples.

Multi-fusion categories from convolution

The simplest examples of multi-fusion categories arise from finite groups. For instance, for any finite group G the category Vect_G of *G-graded $\mathbb{C}$-vector spaces* and the category $\mathsf{Rep}_{\mathbb{C}}(G)$ of *$\mathbb{C}$-linear G-representations* are examples of multi-fusion categories [EGNO16]. A slight generalization of these examples is given by $\mathsf{Rep}_{\mathbb{C}}(\mathcal{G})$, the category of representations of a finite groupoid $\mathcal{G}$. These examples have in common that they all arise from the same underlying principle: all of them arise from the same linearization construction applied to a simplicial groupoid called the *Čech-nerve*. More precisely, the linearization construction is an assignment that associates to every finite groupoid $\mathcal{G}$ the category of functors $\mathsf{Fun}(\mathcal{G}, \mathsf{Vect}_{\mathbb{C}})$ and to every span of finite groupoids

$$\mathcal{G}_0 \longleftarrow \mathcal{G}_{01} \longrightarrow \mathcal{G}_1$$

a functor $\mathsf{Fun}(\mathcal{G}_0, \mathsf{Vect}_{\mathbb{C}}) \to \mathsf{Fun}(\mathcal{G}_1, \mathsf{Vect}_{\mathbb{C}})$, compatible with product and composition. In other words, it is a symmetric monoidal functor from the category of spans of finite groupoids to the category of $\mathbb{C}$-linear categories. The Čech-nerve then describes a specific algebra object in the category of spans, and the monoidal

structure arises as the image of this algebra structure under the symmetric monoidal functor.
Although the construction of these multi-fusion categories is of conceptional elegance, the associated TFTs are often of minor interest. The associated invariants can often be more easily calculated using a different TFT, known as Dijkgraaf–Witten theory [Pet06]. To describe more interesting invariants from knot theory, like the Jones polynomial of a knot [Wit89], we need to consider more elaborate examples like categories of representations of quantum groups [TV92, Wit89]. It would therefore be interesting to understand whether such multi-fusion categories could also arise from linearization constructions.
More generally, the above linearization construction associates a monoidal category to any simplicial groupoid $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Grpd}^f$ satisfying the so-called 2-*Segal conditions* [DK19]. These are 2-dimensional generalizations of the famous Segal conditions introduced in [Seg74]. These 2-Segal conditions capture the higher coherent associativity and unitality of the monoidal structure induced by linearization. More precisely, a 2-Segal groupoid can be viewed as an A_∞-algebra in the category of spans of groupoids [Ste21]. In the following, we will call the monoidal structures induced by a 2-Segal object *convolution monoidal structures*.
The class of 2-Segal groupoids contains the previously mentioned Čech-nerves, but also more general examples. The most interesting example is the so-called Waldhausen $S_\bullet$-construction of an abelian category [DK19]. The monoidal structures induced via linearization have been introduced by Walde [Wal16] under the name Hall monoidal structures [Wal16], since they categorify the algebra structures of Hall algebras [Sch06]. Moreover, these Hall algebras have a direct connection to the theory of quantum groups. Indeed, it is a famous result of Ringel [Rin90] that the Hall algebra of the $\mathsf{A_n}$ quiver is equivalent to the upper half of the quantum group of the corresponding Lie algebra $\mathfrak{sl}_n$. Therefore, it seems plausible that certain more interesting classes of multi-fusion categories might arise from linearizations of more general 2-Segal groupoids.
To investigate this, we restrict ourselves for simplicity to fusion categories and use their inherent combinatorial nature. Since these categories are semi-simple, each object is fully determined by its simple subobjects and their multiplicities. This data entirely characterizes the underlying abelian category in terms of combinatorial data. Moreover, the monoidal product can also be described in terms of the combinatorics of simple objects. Using these insights, we construct for every fusion category a candidate simplicial set. The 2-Segal conditions for this simplicial set then hold if a specific equation that we call the *parameterized set-theoretic pentagon equation* [KR01] is satisfied. By applying this approach, we can explicitly test for fusion categories with a small number of simple objects whether this equation admits a solution, and hence if our simplicial set is 2-Segal. A particularly simple and interesting example of a fusion category is the Ising category, as it describes the representation category of a quantum group [EGNO16]. We explicitly check the set-theoretic parameterized pentagon equation in this case and show that our construction fails. Surprisingly, it fails for every fusion category that is not equivalent to Vect_G:

Theorem 1 (2.12)**.** Let $(\mathcal{C}, \otimes, \alpha)$ be a fusion category. Then $\mathcal{C}$ arises as the linearization of a 2-Segal set $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Set}$ if and only if $\mathcal{C}$ is monoidally equivalent to Vect_G, i.e. it arises as the linearization of the nerve $\mathsf{N}(G)$ of G.

The proof of this theorem relies on an explicit combinatorial argument. The requirement that the monoidal structure induced by a 2-Segal set is rigid imposes further conditions on the components of the associator. We show that the only simplicial set that satisfies the 2-Segal conditions and these extra conditions is the nerve of a group. Therefore, rigidity turns out to be the main obstacle preventing convolution monoidal categories from being a fusion category.
This explicit argument is unfortunately limited to 2-Segal sets. Already for 2-Segal groupoids, the calculations become significantly more complex, requiring to understand the interplay of convolution with character theory. Consequently, a more conceptual explanation of Theorem 1 appears to be more appropriate. The main goal

of this thesis is to provide such a conceptual explanation using the theory of $(\infty, 2)$-categories.

The Main Strategy

Following the previous discussion, a question arises: are there 2-Segal objects, with values in other categories than sets, that can induce rigid convolution monoidal structures? However, proving that a specific monoidal structure is rigid can be quite challenging. Typically, no general principle for constructing the dual object exists, so in general one has to come up with an educated guess. Consequently, classifying all 2-Segal objects that induce rigid convolution monoidal structure appears to be a complicated task.
In his work [Gai15], Gaitsgory introduced a new perspective on rigidity while studying $\mathbb{C}$-linear rigid monoidal ∞-categories of sheaves in the context of derived algebraic geometry. His key insight can be summarized as follows:

Rigidity is a property of an algebra in a symmetric monoidal 2-category.

To unpack this, recall that every symmetric monoidal 2-category $(\mathbb{D}, \otimes)$ comes with a notion of algebra objects. For example, if $\mathbb{D}$ is the 2-category of categories, algebras are given by monoidal categories. Informally, the datum of such an algebra consists of an underlying object $A \in \mathbb{D}$, a unit morphism $\eta : \mathbb{1}_{\mathbb{D}} \to A$, a multiplication $\mu : A \otimes A \to A$, an associativity 2-morphism α, and higher coherences. Gaitsgory defined that such an algebra is *rigid* if A is dualizable, the unit admits a right adjoint, and the multiplication μ has a right adjoint $\mu^{\mathsf{R}} : A \to A \otimes A$ that satisfies the categorified Frobenius relations

$$(\mu \otimes id_A) \circ (id_A \otimes \mu^R) \simeq \mu^R \circ \mu \simeq (id_A \otimes \mu) \circ (\mu^R \otimes id_A),$$

where the isomorphisms are induced by the mates of α. Therefore, we can think of a rigid algebra as a categorified Frobenius algebra. A different way to interpret the last condition is that it describes the lowest instance of the structure of an A-A-bimodule morphism on μ^R.
The main advantages of this 2-categorical notion of rigid algebras for us are that they can be studied in any symmetric monoidal 2-category and are preserved by any symmetric monoidal 2-functor. An example of such a symmetric monoidal 2-functor has been constructed by Morton in [Mor11]. Morton proves that the linearization construction described in the previous section admits an extension to a symmetric monoidal 2-functor, with its source a 2-category of spans, and target a 2-category of $\mathbb{C}$-linear categories. When combined with Gaitsgory's observation, this implies that Mortons 2-functor constructs a convolution monoidal structure, satisfying Gaitsgory's rigidity, from every 2-Segal object that describes a rigid algebra in the 2-category spans. We call these 2-Segal objects *rigid*. Using this observation, we can formulate the leading question of this thesis:

Question. *Which rigid convolution monoidal structures arise from this construction?*

To answer this question, we will follow the following strategy:

1. Relate rigid algebras to the traditional notion of rigidity.
2. Classify rigid 2-Segal objects.
3. Construct rigid algebras through 2-categorical linearization constructions.

In the following sections, we will elaborate on these steps in more detail and present our main results.

Step 1: Rigid Categories

To ensure that Gaitsgory's 2-categorical notion of a rigid algebra is useful for the study of TFTs, it has to satisfy two main conditions:

1. It should recover the classical definition of rigidity.
2. It should be connected to fully extended TFTs.

For the symmetric monoidal $(\infty, 2)$-category $\mathbb{P}\mathsf{r}_{\mathbb{C}}^{\mathsf{L},\otimes}$ of $\mathbb{C}$-linear presentable ∞-categories, Gaitsgory has proven that a compactly generated monoidal $\mathbb{C}$-linear presentable ∞-category is a rigid algebra in $\mathbb{P}\mathsf{r}_{\mathbb{C}}^{\mathsf{L},\otimes}$ if the compact objects admit duals [Gai15]. In particular, its subcategory of compact objects is a rigid category in the original sense.
A similar result has been obtained for the symmetric monoidal 2-category $\mathbb{p}\mathsf{r}_{\mathbb{C}}^{\mathsf{L},\otimes}$ of $\mathbb{C}$-linear presentable 1-categories in [BZBJ18]. In this case, a monoidal $\mathbb{C}$-linear presentable 1-category is a rigid algebra if and only if every compact-projective object admits a dual. This definition of rigidity is more general than the one used in for Turaev–Viro style TFTs in [DSPS20] but is nevertheless a reasonable definition of rigidity. For instance, in the category rmod_R of modules over a commutative $\mathbb{C}$-algebra R, an R-module is dualizable if and only if it is finitely generated and projective, i.e. compact-projective. Therefore, rmod_R is a rigid algebra in the 2-categorical sense but not in the sense of [DSPS20].
In Section 4, we generalize the above approaches and study rigid algebras in the symmetric monoidal $(\infty, 2)$-category $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$, where $\mathcal{V}^{\otimes}$ is an arbitrary presentably symmetric monoidal ∞-category. The right generalizations of compact (resp. compact-projective) objects in this context are given by so-called $\mathcal{V}$-atomic objects introduced in [BMS24]. An object a in $\mathcal{A}$ is $\mathcal{V}$-atomic, if the internal Hom functor $\hom_{\mathcal{A}}(a, -) : \mathcal{A} \to \mathcal{V}$ for the $\mathcal{V}$-action on $\mathcal{A}$ commutes with colimits and is compatible with the $\mathcal{V}$-action. For $\mathcal{V}$-linear presentable ∞-categories, we then prove the following:

Theorem 2 (4.19)**.** Let $\mathcal{A}^{\otimes}$ be an atomically generated $\mathcal{V}$-linear presentably monoidal ∞-category. Then $\mathcal{A}^{\otimes}$ is rigid in $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ if and only if every $\mathcal{V}$-atomic object admits a dual.

When $\mathcal{V}^{\otimes}$ is the derived ∞-category of $\mathbb{C}$-vector space $\mathcal{D}(\mathbb{C})^{\otimes}$, the internal Hom recovers the mapping complex, and $\mathcal{V}$-atomic objects are precisely the compact objects. This recovers the result of [Gai15]. Similarly, by setting $\mathcal{V}^{\otimes}$ to be $\mathsf{Vect}_{\mathbb{C}}^{\otimes}$ (the category of $\mathbb{C}$-vector spaces), the classical case of [BZBJ18] can be recovered. A similar result has also been proven in [Ram24b].
The second requirement imposed on the 2-categorical notion of rigidity is that it has to be related to fully extended TFTs, in particular to those of Turaev–Viro style. For the symmetric monoidal $(\infty, 2)$-categories $\mathbb{P}\mathsf{r}_{\mathbb{C}}^{\mathsf{L},\otimes}$ and $\mathbb{p}\mathsf{r}_{\mathbb{C}}^{\mathsf{L},\otimes}$ such relations have been obtained in [BZGN19], and [BJS21] respectively. More precisely, the authors show that every rigid algebra defines a fully dualizable object in the respective Morita $(\infty, 2)$-category introduced in [Hau17]. We provide the following generalization:

Theorem 3 (4.26)**.** Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. Every rigid algebra in $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ is fully dualizable in the Morita $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}_{\mathcal{V}}^{\mathsf{L}})^{\otimes}$ of $\mathcal{V}$-linear presentable ∞-categories.

In this more general setting, the proof of this theorem becomes particularly simple. The definition of a rigid algebra naturally allows us to express the rigid category $\mathcal{A}^{\otimes}$ as a module over the so-called canonical algebra $\mathcal{F}_{\mathcal{A}}$ of $\mathcal{A}$. This algebra already appears prominently in various contexts in the TFT literature [SW21, KS22]. On the other hand, it is not true that any fully dualizable object in $\mathbb{M}\mathsf{or}(\mathsf{Pr}_{\mathcal{V}}^{\mathsf{L}})^{\otimes}$ is given by a rigid algebra. A counterexample has been constructed in [BZGN19], using the theory of D-modules. Therefore, the above result does not provide a classification of rigid algebras in terms of fully extended TFTs. However, we argue

that one can obtain such a classification in terms of *relative TFTs*.
Formally this means that we replace the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$ by its $(\infty, 2)$-category of oplax arrows $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})^{\to,\otimes}$ [JFS17]. The objects of the latter are given by 1-morphisms in $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$. In particular, for every algebra $\mathcal{A}^{\otimes} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$, the regular left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{A}$ yields an object of $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})^{\to}$. We claim that we can characterize rigid algebras in terms of this module:

Claim (4.30)**.** Let $\mathcal{A}^{\otimes}$ be a $\mathcal{V}$-linear presentably monoidal ∞-category. Then $\mathcal{A}^{\otimes}$ is rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$ if and only if the regular left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{A}$ is fully dualizable in the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})^{\to,\otimes}$.

We prove this claim using two unproven structural properties of the even higher Morita category [JFS17] that we explicitly state in Claim 4.1. We have no doubt that these categories have these properties. The proof would either require a careful ∞-categorical analysis of the construction of the even higher Morita categories in [JFS17], that is currently missing in the literature, or a different more intrinsic construction of these. We plan to follow the latter approach in future work. A similar result has been proven in the case of monoidal 1-categories in [Haï23].

Fusion Categories

Recall that our initial motivation was not just to construct rigid $\mathbb{C}$-linear categories, but specifically to construct $\mathbb{C}$-linear multi-fusion categories. Interestingly, when we apply the definition of a rigid algebra to the 2-category of small idempotent-complete $\mathbb{C}$-linear additive categories instead of large presentable categories, we arrive at the following extension of Gaitsgory's observation:

Theorem 4 (2.9)**.** A $\mathbb{C}$-linear idempotent-complete additive monoidal category is rigid in the symmetric monoidal 2-category of small idempotent-complete $\mathbb{C}$-linear additive categories $\mathbb{a}\mathsf{dd}^{\otimes}_{\mathbb{C}}$ if and only if it is a $\mathbb{C}$-linear multi-fusion category.

Note that, although it was not a requirement initially, it follows that a rigid algebra in $\mathbb{a}\mathsf{dd}^{\otimes}_{\mathbb{C}}$ has to be an abelian category. The key difference from the presentable case considered above is the following. Using the non-abelian derived category [Lur09a, Sect.5.5.8] We can associate to every additive category $\mathcal{A}$ a presentable category $\mathcal{P}^{\Sigma}_{1}(\mathcal{A})$, that contains $\mathcal{A}$ as its full subcategory of compact-projective objects. While this presentable category is always dualizable as a presentable category, the underlying small additive category $\mathcal{A}$ is only dualizable as a $\mathbb{C}$-linear additive category if it is semi-simple and has finite-dimensional Hom-spaces. Thus it has to be a multi-fusion category.
Since the above characterization of a multi-fusion is 2-categorical, we can study this condition within other 2-categories as well to obtain potential generalizations of multi-fusion categories. As mentioned earlier in this introduction, we are particularly interested in derived versions of multi-fusion categories. As these would arise for example as bounded derived categories, they naturally form algebras in the $(\infty, 2)$-category of $\mathbb{C}$-linear stable ∞-category $\mathbb{S}\mathsf{t}_{\mathbb{C}}$. Therefore, it is interesting to study these examples of rigid algebras, which we call by analogy with the additive case *derived multi-fusion categories*.
This terminology is purely motivated by analogy. In particular, we will *not* show that these satisfy any higher dualizability conditions. Nevertheless, we expect derived multi-fusion categories, in our sense, to induce non-compact fully extended 3-dimensional TFTs. This expectation arises from our analogy with multi-fusion categories, that in general, also only induce non-compact TFTs, as they satisfy all but one of the finiteness conditions required to define a fully dualizable object. Indeed over a general field, multi-fusion categories have to satisfy an extra condition called *separability* to be a fully dualizable object [DSPS20]. We anticipate that also in the derived setting a similar separability condition will be necessary to obtain a fully dualizable object in the Morita category of $\mathbb{C}$-linear stable ∞-categories generalizing Theorem 3 above.

To further motivate our definition of a derived multi-fusion category, we introduce two classes of examples of derived multi-fusion categories. As our first example, we show that this terminology is compatible with the 1-categorical case:

Proposition 1. Let $\mathcal{A}^{\otimes}$ be $\mathbb{C}$-linear multi-fusion category. Then its bounded derived ∞-category $\mathcal{D}^{\flat}(\mathcal{A})^{\otimes}$ is a $\mathbb{C}$-linear derived multi-fusion category.

As our second class of examples, we consider the ∞-category of modules over an $\mathbb{C}$-linear $\mathbb{E}_2$-algebra. This ∞-category naturally is an example of a $\mathbb{C}$-linear monoidal stable ∞-category. On the other hand, the $\mathbb{E}_2$-algebra itself can give rise to a TFT. Indeed, in [Lur08, Rem.4.1.27] Lurie sketches a criterion for such a $\mathbb{E}_2$-algebra to define a non-compact $3d$-TFT in terms of factorization homology. As a consistency check, we show that this criterion is compatible with our notion of derived multi-fusion category

Proposition 2. Let A be an $\mathbb{C}$-linear $\mathbb{E}_2$-algebra. Then A satisfies the assumptions of [Lur08, Rem.4.1.27] in the non-compact case if and only if its category of modules is a derived multi-fusion category.

In more elementary terms, the assumptions of [Lur08, Rem.4.1.2.17] demand the $\mathbb{E}_2$-algebra to be *smooth and proper*. Smooth and proper $\mathbb{E}_1$-algebras are ubiquitous in the study of homological mirror symmetry [HKK17]. Unfortunately, we are not aware of any example of a smooth and proper $\mathbb{E}_2$-algebra that does not satisfy the weaker assumption of being *separable*. Therefore, instead of giving explicit examples, we will discuss some possible construction methods.

Step 2: Rigid 2-Segal Spaces

For the second step, we investigate rigid algebras within symmetric monoidal $(\infty, 2)$-categories of spans of spaces. This requires first to understand algebras in these monoidal $(\infty, 2)$-categories that can be described using 2-Segal or decomposition spaces [DK19, GCKT18].

Higher Algebra in Spans

A 2-Segal space, also called a decomposition space [GCKT18], is a simplicial space $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ that satisfies a 2-dimensional analog of the famous Segal conditions introduced in [Seg68]. These concepts were initially introduced independently by Dyckerhoff and Kapranov [DK19] to describe the underlying structure of different Hall algebra constructions, and by Gálvez-Carrillo, Kock, and Tonks to describe various coalgebra structures appearing in combinatorics [GCKT18]. Since then, the subject has undergone substantial developments, including the introduction of n-Segal conditions for $n > 2$ [Pog17] and applications in different areas of mathematics like the theory of TFTs [DK18], Auslander–Reiten theory [DJW19], and Donaldson–Thomas theory [PS23].

The connection between 2-Segal spaces and Hall algebras arises from interpreting 2-Segal spaces as associative algebras in categories of spans. More precisely, for every $n \geq 0$, we think of the span

$$X_1 \times \cdots \times X_1 \xleftarrow{\{0,1\},\ldots,\{n-1,n\}} X_n \xrightarrow{\{0,n\}} X_1$$

as an n-ary multiplication and taking pullbacks as composition. The lowest dimensional 2-Segal conditions

$$\begin{array}{ccc} X_3 & \xrightarrow{\partial_3} & X_2 \\ \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} \\ X_2 & \xrightarrow{\partial_2} & X_1 \end{array} \qquad\qquad \begin{array}{ccc} X_3 & \xrightarrow{\partial_0} & X_2 \\ \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_1} \\ X_2 & \xrightarrow{\partial_0} & X_1 \end{array}$$

that demand the following squares to be pullback, then encode that, up to equivalence, different composites of the 2-ary multiplication coincide with the same 3-ary multiplication. In other words, this describes a coherent operadic version of associativity. This idea has been made precise by Stern [Ste21]. He has proven that the space of 2-Segal spaces is equivalent to the space of associative algebra objects in the ∞-category of spans of spaces.
But as the definition of a rigid algebra also uses more general algebraic structures, it does not suffice to only understand algebras. We also need to understand the Frobenius relations that require a good understanding of bimodules and bimodule maps.

Bimodules

Analogous to algebras, we introduce a similar description of bimodules using a 3-colored version (in the sense of operads) of the 2-Segal condition imposed on a birelative simplicial space $X_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{S}$. The objects corresponding to the 3-colors are given by $X_{\{0,0\}}$, $X_{\{0,1\}}$, and $X_{\{1,1\}}$, where the outer ones represent the underlying objects of the algebra, while the middle one corresponds to the underlying object of the bimodule. Similar to the case of 2-Segal objects, every object $f : [n] \to [1] \in \Delta_{/[1]}$ represents a span:

$$X_{f|_{0,1}} \times \cdots \times X_{f|_{n-1,n}} \longleftarrow X_f \longrightarrow X_{f|_{0,n}}$$

which we view as a multicolored n-ary operation, where two of them are again composed by taking pullbacks. Then the lowest birelative 2-Segal conditions demand the following square to be pullbacks

$$\begin{array}{ccc} X_f & \longrightarrow & X_{f|_{0,1,2}} \\ \downarrow & & \downarrow \\ X_{f|_{0,2,3}} & \longrightarrow & X_{f|_{0,2}} \end{array} \qquad \begin{array}{ccc} X_f & \longrightarrow & X_{f|_{1,2,3}} \\ \downarrow & & \downarrow \\ X_{f|_{0,1,3}} & \longrightarrow & X_{f|_{1,3}} \end{array}$$

These conditions simultaneously encode the associativity of the multiplications and module actions. Using this definition we can extend the result of [Ste21] to the case of bimodules:

Theorem 5 (8.2)**.** Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence of spaces

$$\mathrm{BiSeg}_\Delta(\mathcal{C})^{\simeq} \simeq \mathsf{BMod}(\mathsf{Span}(\mathcal{C}))^{\simeq}$$

between the subspace of $\mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})^{\simeq}$ generated by birelative 2-Segal objects and the space of bimodules in $\mathsf{Span}(\mathcal{C})^{\otimes}$ the ∞-category of spans in $\mathcal{C}$.

Similar conditions have been previously introduced for left modules in the context of 1-categories by Walde [Wal16] and Young [You18]. For bimodules the same conditions have been introduced independently by Carlier [Car20] in terms of augmented stable double Segal spaces. Carlier further shows that these conditions gives rise to non-coherent bicomodules. Our result generalizes these works into the context of ∞-categories.

Bimodule Maps

To use the formalism of 2-Segal spaces for the description of rigid algebras, we also need a description of bimodule maps in terms of morphisms between birelative Segal spaces. A natural first guess might be to define the corresponding notion of morphism between birelative 2-Segal spaces as a natural transformation between birelative simplicial spaces. Although this yields an interesting notion of morphism between birelative

2-Segal spaces, it does not correspond to that of a bimodule morphism. This fails for two reasons. On the one hand, a general morphism of birelative simplicial spaces only corresponds to a lax bimodule morphism [Wal16, Sect.4.2]. On the other hand, since the algebraic structure itself lives in an ∞-category of spans, a bimodule morphism should also be formalized in this context. These considerations lead to the following extension of Theorem 5:

Theorem 6 (9.1)**.** Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence of ∞-categories

$$\mathrm{BiSeg}_{\Delta}^{\leftrightarrow}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{BMod}(\mathsf{Span}(\mathcal{C}))$$

between the subcategory of $\mathsf{Span}(\mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}))$ with objects birelative 2-Segal objects and morphisms birelative 2-Segal spans on the left and the ∞-category of bimodule objects in the symmetric monoidal ∞-category $\mathsf{Span}(\mathcal{C})^{\otimes}$ on the right.

A span between birelative 2-Segal objects is called a *birelative* 2*-Segal span* if the legs of the span satisfy certain pullback conditions. Similar conditions have been previously introduced in the context of Segal spaces under the names CULF and IKEO in [GCKT18], where the authors show that such morphisms induce algebra morphisms. Restricted to the underlying simplicial objects these spans indeed recover the CULF and IKEO conditions and we can see the above theorem as a converse to their observation.

Rigidity for 2-Segal Spaces

The description of algebraic structures in ∞-categories of spans described in the last sections, allows us to finally systematically study rigid 2-Segal objects. For this end, we first need a suitable candidate for the $(\infty, 2)$-category of spans. The symmetric monoidal ∞-category $\mathsf{Span}(\mathcal{S})^{\otimes}$ admits two natural $(\infty, 2)$-categorical extensions. The first, $\mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes}$ has morphisms between spans as its 2-morphisms, whereas the second, $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$, has spans between spans as its 2-morphisms. For our purposes, the main difference between those choices lies in the existence of adjoints. In $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$, a morphism admits a right adjoint only if the left leg of the span is an equivalence. By contrast, in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ every morphism admits a left and right adjoint [Hau18].
Since for a 2-Segal object, the left leg of the multiplication span is only an equivalence if the simplicial object is already 1-Segal, we focus, to increase generality, on studying rigid 2-Segal objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$. In this context, the right adjoint of the multiplication span is given by the reversed span. Therefore, to understand rigid algebras it remains to understand the Frobenius relation. These force the diagrams

$$\begin{array}{ccc} X_3 & \xrightarrow{\partial_0} & X_2 \\ {\scriptstyle\partial_3}\downarrow & & \downarrow{\scriptstyle\partial_2} \\ X_2 & \xrightarrow{\partial_0} & X_1 \end{array} \qquad\qquad \begin{array}{ccc} X_3 & \xrightarrow{\partial_1} & X_2 \\ {\scriptstyle\partial_2}\downarrow & & \downarrow{\scriptstyle\partial_1} \\ X_2 & \xrightarrow{\partial_1} & X_1 \end{array}$$

to be pullbacks. A detailed analysis shows that the left-hand diagram can be interpreted as an extended 1-Segal condition, while the right-hand diagram corresponds to an invertibility condition. This observation leads to the main theorem of this thesis:

Theorem 7 (10.12)**.** Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ be a 2-Segal space. $X_\bullet$ is rigid in $2\mathbb{S}\mathsf{pan}(\mathcal{S})^{\otimes}$ if and only if it is equivalent to a Čech-nerve.

This Theorem provides an explanation for our earlier observation that all known examples of convolution monoidal structures that form multi-fusion categories are induced from Čech-nerves. Namely, these are the *only* examples of 2-Segal objects, that are rigid before linearization and therefore their linearizations are the generic examples of rigid convolution monoidal categories. We can summarize our discussion with the following slogan:

All rigid convolution monoidal structures arise from Čech-nerves.

However, it is important to recognize that this is only a slogan. Except for the case of 2-Segal sets, we have not proven that all rigid convolution structures arise from Čech-nerves. Indeed, the linearization functor does invert morphisms. So it is still possible, in principle, for rigid convolution monoidal structures to exist that are induced by 2-Segal spaces which are not Čech-nerves. This result should instead be interpreted as an indication that it is unlikely for a 2-Segal space to induce a rigid convolution monoidal structure unless it is a Čech-nerve.

This observation is further reinforced by examples. We are unaware of any example of a rigid convolution monoidal structure that does not originate from a Čech-nerve. Moreover, we can rule out certain classes of examples of convolution structures, such as those that originate from 1-Segal spaces that are not Čech-nerves, and those that originate from Waldhausen $S_\bullet$-construction of abelian categories. A more intriguing example is the Waldhausen $S_\bullet$-construction of a stable ∞-category, which appears to exhibit certain characteristics of rigid 2-Segal objects. These similarities arise from the interpretation of rigid algebras as Frobenius algebras. In Section 12 we show that the Waldhausen $S_\bullet$-construction admits a Frobenius algebra structure that differs from the rigid Frobenius algebra structure.

Step 3: Convolution Structures

In the final step, we connect our two examples of rigid algebras studied in the previous steps by linearization constructions. Although this approach does not lead to new examples of multi-fusion categories, it allows us to construct more general examples of rigid monoidal ∞-categories and derived multi-fusion categories, generalizing Vect_G and $\mathsf{Rep}_{\mathbb{C}}(G)$. Since the general theory of linearization functors with source $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ has not been developed so far, we restrict our discussion to linearizations of functors with source $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$.

A prototypical example of such a linearization construction has been constructed in [Mor11], where the author constructs a symmetric monoidal 2-functor from the 2-category $\mathbb{S}\mathsf{pan}_2(\mathsf{Grpd}^f)$ to the category of $\mathbb{pr}^{\mathsf{L}}_{\mathbb{C}}$. This functor assigns to any finite groupoid $\mathcal{G}$ the category $\mathsf{Fun}(\mathcal{G}, \mathsf{Vect}_{\mathbb{C}})$ and, to any morphism of finite groupoids, the functor of left Kan extension. It is a consequence of our characterization of rigid 2-Segal objects that rigid 2-Segal objects in $\mathbb{S}\mathsf{pan}_2(\mathsf{Grpd}^f)$ are precisely group objects. In particular, when applied to the nerve of a finite group G, we can use this functor to recover the convolution monoidal structures on Vect_G.

A similar construction can be carried out for every presentably symmetric monoidal ∞-category $\mathcal{V}$. For every such $\mathcal{V}$, the assignment that associates the $\mathcal{V}$-linear ∞-category $\mathsf{Fun}(X, \mathcal{V})$ to a space X and the functor of left Kan extension to a morphism $f : X \to Y$ of spaces admits an extension to a symmetric monoidal $(\infty, 2)$-functor from $\mathbb{S}\mathsf{pan}_2(\mathcal{S})$ to $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$. Applying this to a rigid algebra in $\mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes}$ that arises from an ∞-group, we obtain:

Theorem 8 (11.7)**.** Let $G_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ be an ∞-group. Then the categorified $\mathcal{V}$-linear group algebra $\mathcal{V}[G]$ is rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$.

Furthermore, using our 2-categorical characterization of multi-fusion categories, we can determine when the categorified $\mathcal{V}$-linear group algebra is a derived multi-fusion category. To do so, we need to understand when the underlying $\mathbb{C}$-linear stable ∞-categories are dualizable. The corresponding conditions are known under the name smooth and proper. For dg categories of quasi-coherent sheaves, these conditions have

already been studied by Orlov [Orl16]. Since the approach of Orlov [Orl16] only uses the formal properties of quasi-coherent sheaves, a similar criterion should apply for every linearization construction, and we provide an analogous criterion here.
According to this criterion, it remains to understand when certain right Kan extension functors preserve colimits. An interesting class of examples arises from the theory of ambidexterity, as developed in [HL13]. A space X is called π-finite if its homotopy groups are finite and $\pi_n(X,x)=0$ for n large enough. Under this condition, one can construct for maps between π-finite spaces a natural transformation between the functor of left and right Kan extension called the Norm map. This Norm map generalizes the Norm map between group homology and cohomology [Lur17]. Those ∞-categories $\mathcal{E}$ for which the norm map is an equivalence for every π-finite space are called ∞-semi-additive. One example of such an ∞-semi-additive ∞-category is $\mathcal{D}^\flat(\mathbb{C})$. This leads us to the following result generalizing our initial observation:

Theorem 9 (11.23)**.** Let G be a π-finite ∞-group. Then the categorified $\mathbb{C}$-linear group algebra $\mathcal{D}^\flat(\mathbb{C})[G]$ is a $\mathbb{C}$-linear derived multi-fusion category.

In case that G is a finite group, the categorified group algebra coincides with the $\mathcal{D}^\flat(\mathsf{Vect}_G)$ the bounded derived ∞-category of G-graded vector spaces. As this is the derived ∞-category of a multi-fusion category, this observation is consistent with Prop. 1. However, for a general π-finite ∞-group this result is more general. This theorem similarly applies to all other examples of ∞-semi-additive stable ∞-categories, leading to an even larger class of examples of derived multi-fusion categories. Additionally, we can further use our results for the construction of a derived variant of the multi-fusion category $\mathsf{Rep}_{\mathbb{C}}(G)^\otimes$.

Theorem 10 (11.21)**.** Let X be π-finite space. Then the $\mathbb{C}$-linear stable ∞-category $\mathsf{Fun}(X,\mathcal{D}^\flat(\mathbb{C}))^\otimes$ equipped with the pointwise monoidal structure is a derived multi-fusion category.

In conclusion, while the 2-categorical linearization construction may not directly produce quantum groups, it provides a useful framework for constructing examples of derived multi-fusion categories.

1.1 Outline

We conclude this introduction with a broad outline of this thesis. We describe in Section 2 how examples of fusion categories arise from Čech-nerves and describe the relation between fusion categories and 2-Segal sets. This discussion results in the proof of Theorem 1. In Section 3, we then introduce Gaitsgory's 2-categorical formulation of rigidity.

Sections 4 through 5 cover **Step 1**. We first study in Section 4 rigid algebras in presentable ∞-categories. In particular, we related these algebras to the classical notion of rigidity and to fully extended TFTs. In Section 5, we apply these results to the case of $\mathbb{C}$-linear categories and $\mathbb{C}$-linear ∞-categories, focusing on (derived) multi-fusion categories.

Sections 6 through 10 cover **Step 2**, where we classify rigid 2-Segal objects. These are the central sections of this thesis. To set the stage, we review various 2-Segal conditions in Section 6 that are relevant for describing homotopy coherent algebra in span categories. We then elaborate on these conditions by discussing different examples of 2-Segal objects in Section 7. In Sections 8 and Section 9, we relate these 2-Segal conditions to homotopy coherent algebra in the ∞-category of spans, showing an equivalence between the category of bimodules in spans and the category of birelative 2-Segal objects. In Section 10, we apply these results to classify rigid 2-Segal objects.

Finally, Sections 11 and 12 cover **Step 3**. In Section 11, we construct examples of linearization constructions and use them to construct rigid monoidal categories, leading to new examples of derived multi-fusion categories. We further compare the values of the corresponding (relative) TFTs on the circle for different examples of convolution monoidal categories. In Section 12, we discuss duality structures, that arise from Frobenius algebras, with a focus on the example of the Waldhausen construction.

The Appendices A to C cover various technical results. In Appendix A, we prove two technical lemmas, necessary for the proof in Section 8. Appendix 4 sketches the construction of the symmetric monoidal $(\infty, 2)$-category of $\mathcal{V}$-linear presentable ∞-categories, based on the approach of [GR19]. Additionally, in Appendix C, we review the construction of the Morita $(\infty, 2)$-category from [Hau23, Lur17] and classify the fully dualizable objects in this $(\infty, 2)$-category.

1.2 Notation

In this section, we collect some notation that will be used throughout this text. As usual, we pick a nested sequence $\mathbb{V}_0 \subset \mathbb{V}_1 \subset \mathbb{V}_2 \subset \mathbb{V}_3$ of Grothendieck universes and refer to sets in those as small, large, very large, and extremely large respectively.

- All statements about fully extended TFTs depend on the still conjectural Cobordism hypothesis [BD95, Lur08]. We consider all such statements under the additional assumption that the Cobordism hypothesis holds.
- We denote by Cat the large ∞-category of small ∞-categories, by CAT the very-large ∞-category of large ∞-categories, and by $\mathbf{CAT}_\infty$ the extremely large ∞-category of very large ∞-categories.
- We follow the convention to denote categories of ∞-categories by large letters and categories of 1-categories by small letters. For example, Cat denotes the ∞-category of small ∞-categories, and cat denotes the category of small 1-categories.
- Let $\mathcal{K}$ be a small collection of ∞-categories. We call an ∞-category $\mathcal{K}$-cocomplete if it admits $\mathcal{K}$-indexed colimits. Further, we call a functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{K}$-cocomplete ∞-categories $\mathcal{K}$-cocontinuous if it preserves $\mathcal{K}$-indexed colimits.
- We follow the convention to denote $(\infty, 2)$-categories by bold capital letters $\mathbb{C}, \mathbb{D}$, and 2-categories by bold small letters $\mathbbm{c}, \mathbbm{d}$.
- For the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$, we adopt the convention to denote a morphism f with source c_0 and target c_1 by $c_0 \leftarrow c_1 : f$.
- We often abuse notation and denote (symmetric) monoidal ∞-categories just by $\mathcal{C}^{\otimes}$ or $(\mathcal{C}, \otimes)$ instead of writing the full fibration as defined for example in [Lur17, Chapter 2.].

1.3 Previous Publications, Eigenanteilserklärung

I declare that the results presented in this dissertation are entirely my own research work unless stated otherwise. Critical ideas proposed by others will be marked or acknowledged. To the best of my knowledge, results attributed to others in the literature will be attributed to the primary sources or standard references. If the primary source is unknown to me, they will be attributed to "folklore".

- The text of Section 1 is entirely original. This Section is introductory and contains no new results.

- The text and results of Section 2 were produced entirely independently.
- The text and results of Section 3 were produced entirely independently. The main Definition 3.2 of this Section is due to [Gai15].
- The text and results of Section 4 and Section 5 were produced entirely independently.
- The text and results of Section 6, Section 7, Section 8, Section 9, and Appendix A.1 and A.2 were produced entirely independently by the author. The content of these Sections is available in preprint form at [Göd24].
- The text and results of Section 10, Section 11, and Section 12 were produced entirely independently by the author.
- The text and results of Appendix B and Appendix C were produced entirely independently by the author. The idea for the construction of Appendix B is due to [GR19], and the idea for the construction of Appendix C is due to [Lur17, Hau23].

2 Fusion Categories and 2-Segal objects

The main ingredient for 3-dimensional topological field theories of Turaev–Viro style are given by fusion categories [BW96, DSPS20]. Therefore and also for possible derived generalization of Turaev-Viro theories, it is important to understand how these can be constructed. As a start, let us recall the definition of a fusion category and look at some examples. Throughout this section, we assume that $\mathbb{K}$ is an algebraically closed field of possibly non-zero characteristic p.

Definition 2.1. A $\mathbb{K}$-linear monoidal abelian category $(\mathcal{C}, \otimes)$ is called a *tensor category* if:

(1) the monoidal structure is rigid, i.e. every object admits a left and a right dual

(2) the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is $\mathbb{K}$-linear and right exact in each variable

It is called a *finite tensor category* if the underlying abelian category is finite[1] and a *multi-fusion category* if the underlying abelian category is furthermore semisimple. A multi-fusion category $(\mathcal{C}, \otimes)$ is called a *fusion category* if the unit $\mathbb{1}_{\mathcal{C}}$ is simple.[2]

Example 2.1. Let G be a finite group. The category $\mathsf{Vect}_G^{\mathrm{fin}}$ of finite dimensional G-graded $\mathbb{K}$-vector spaces equipped with the monoidal product functor $*$ that maps two G-graded vector spaces $V_\bullet, W_\bullet \in \mathsf{Vect}_G^{\mathrm{fin}}$ to:

$$(V_\bullet * W_\bullet)_\bullet := \bigoplus_{h \in G} V_h \otimes W_{h^{-1}\bullet} \tag{1}$$

is a fusion category [EGNO16, Ex.4.1.2].

Example 2.2. Let G be a finite group. The category $\mathsf{Rep}(G)$ of finite dimensional G-representations can be equipped with a monoidal structure that maps G-representations V, W to the tensor product representation $V \otimes W$. This monoidal structure is rigid and the monoidal category $(\mathsf{Rep}(G), \otimes)$ is fusion if and only if $\mathrm{char}(\mathbb{K})$ does not divide the group order [EGNO16, Ex.4.1.2] .

[1] See [EGNO16, Def.1.8.6]

[2] We call an object $i \in \mathcal{C}$ simple if it admits no non-trivial subobjects. Since $\mathbb{K}$ is algebraically closed, all simple objects i are also absolutely simple, i.e. $\mathsf{End}(i) \simeq \mathbb{K}$.

Example 2.3. Let A be a semisimple finite dimensional $\mathbb{K}$-algebra. The category of finite dimensional A-bimodules $\mathsf{BMod}_A(\mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}})$ equipped with the relative tensor product is a multi-fusion category. Since $\mathbb{K}$ is algebraically closed, it follows that $A \simeq \oplus_{i=1}^{n}\mathbb{K}$ as a $\mathbb{K}$-algebra and we can identify the monoidal category $(\mathsf{BMod}_A(\mathsf{Vect}^{\mathrm{fin}}_{\mathbb{K}}), \otimes_A)$ with the monoidal category of matrices of finite dimensional vector spaces $\mathsf{Mat}(\mathsf{Vect}^{\mathrm{fin}}_{\mathbb{K}})$. This category has objects given by $n \times n$-matrices of finite dimensional $\mathbb{K}$-vector spaces and the monoidal structure is given by matrix multiplication. It follows, that this category is fusion if and only if the algebra A is simple. [EGNO16, Ex.4.1.3]

Remark 2.1. In the above examples of multi-fusion categories, we always chose the monoidal structure with the trivial associator, i.e. the one induced by the tensor product of vector spaces. For example the associators on the category $\mathsf{Vect}^{\mathrm{fd}}_G$ are parametrized by the group cohomology of $H^3(G, \mathbb{K}^*)$ of G. We expect that it is possible to modify the constructions given in this section to describe more interesting choices of associators using spans with local systems [Hau18]. Since this slight generalization is not relevant to the following discussion, we decided to keep the discussion simple and leave this generalization for future work.

All of the above examples of multi-fusion categories arise from the same so-called linearization construction, i.e. they arise from algebraic constructions in the symmetric monoidal 2-category of spans. The advantage of considering these examples from this perspective is that these are completely formal and therefore easily generalize to other contexts. The goal of this section is to study this cosntruction method in detail and to sort out, what kind of examples of fusion categories we can obtain.

2.1 Fusion Categories and Linearizations

To formally describe what we mean by a linearization construction, we first need to introduce some notation. We denote by $(\mathsf{Span}(\mathsf{Grpd}^f), \times)$ the symmetric monoidal $(2,1)$-category of *spans of finite groupoids* [Mor11] (see Definition 10.1) and by $(\mathrm{ab}^{\mathsf{rex}}_{\mathbb{K}}, \otimes)$ the symmetric monoidal $(2,1)$-category of *finite* $\mathbb{K}$*-linear abelian categories* and right exact functors [BDSPV15, Def.A.4] with monoidal structure given by the relative Deligne tensor product. This $(2,1)$-category is a full symmetric monoidal subcategory of the symmetric monoidal $(2,1)$-category $\mathsf{cat}^{\mathsf{rex}}_{\mathbb{K}}$ of finitely cocomplete $\mathbb{K}$-linear categories and $\mathbb{K}$-linear right exact functors equipped with the Kelly tensor product [Fra13] (see Definition 5.1). The following $(2,1)$-functor is what we will call in this section a *linearization construction*:

Proposition 2.1. *[Mor11] There exists a symmetric monoidal* $(2,1)$*-functor*

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(-) : \mathsf{Span}(\mathsf{Grpd}^f) \to \mathrm{ab}^{\mathsf{rex}}_{\mathbb{K}}$$

from the 2*-category of spans of finite*[3] *groupoids (see Definition 10.1) to the* 2*-category of finite* $\mathbb{K}$*-linear abelian categories. This functor maps a finite groupoid* $\mathcal{G}$ *to the category* $\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}) := \mathsf{Fun}(\mathcal{G}, \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}})$ *and a span of functors*

$$\begin{array}{ccccc} & & \mathcal{G}_1 & & \\ & \swarrow^{F_0} & & \searrow^{F_1} & \\ \mathcal{G}_0 & & & & \mathcal{G}_2 \end{array}$$

to the composite

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}_0) \xrightarrow{F_0^*} \mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}_1) \xrightarrow{F_{1,!}} \mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}_2)$$

where F_0^* *denotes the pullback functor along* F_0 *and* $F_{1,!}$ *the functor of left Kan extension along* F_1*.*

[3] A groupoid is called finite if it has finitely many isomorphism classes of objects and each object has finitely many automorphisms.

Remark 2.2. The $(2,1)$-category of groupoids is equivalent to the $(2,1)$-category of 1-truncated spaces $\tau_{\leq 1}\mathcal{S}$, i.e. those spaces that only admit non-trivial homotopy groups in degrees 0 and 1. For every groupoid $\mathcal{G}$ the category $\mathsf{Fun}(\mathcal{G}, \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}})$ identifies under this equivalence with the category of local systems of finite dimensional vector spaces on the space associated with $\mathcal{G}$. This justifies our notation (see Definition 11.2).

In particular, since the $(2,1)$-functor $\mathsf{Loc}_{\mathbb{K}}^{fd}(-)$ is symmetric monoidal, it preserves algebra objects:

Corollary 2.2. *Let (X_1, μ) be an algebra object in the symmetric monoidal $(2,1)$-category $\mathsf{Span}(\mathsf{Grpd}^f)$ with multiplication span:*

$$\begin{array}{ccc} & X_2 & \\ {}^{(\partial_2,\partial_0)}\swarrow & & \searrow^{\partial_1} \\ X_1 \times X_1 & & X_1 \end{array}$$

Then, the monoidal functor $\mathsf{Loc}_{\mathbb{K}}^{fd}(-)$ induces a monoidal structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(X_1)$, whose underlying monoidal product functor $*$ maps functors F, G to the functor $F * G$, obtained as the image of $F \otimes_{\mathbb{K}} G$ under the functor obtained by linearizing the span:

$$\begin{array}{ccc} & X_2 & \\ {}^{(\partial_2,\partial_0)}\swarrow & & \searrow^{\partial_1} \\ X_1 \times X_1 & & X_1 \end{array}$$

Here $F \otimes_{\mathbb{K}} G$ denotes the pointwise monoidal product of F and G, i.e. the composite

$$X_1 \times X_1 \xrightarrow{F \times G} \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}} \times \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}} \xrightarrow{\otimes} \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}}$$

We call the induced monoidal structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(X_1)$ a convolution monoidal structure.

More precisely, the monoidal product functor maps two functors $F, G \in \mathsf{Loc}_{\mathbb{K}}^{fd}(X_1)$ to the functor

$$\partial_{1,!}(\partial_2, \partial_0)^*(F \otimes_{\mathbb{K}} G) \in \mathsf{Loc}_{\mathbb{K}}^{fd}(X_1)$$

The datum of an associative algebra object in the monoidal category $\mathsf{Span}(\mathsf{Grpd}^f)$ admits a compact description in terms of certain simplicial groupoids $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Grpd}^f$ (see Definition 6.1). In this description, the groupoid X_1 describes the underlying object of the algebra and the groupoid X_2 together with its face maps describes the multiplication span as shown in Diagram (2.2). For such a simplicial groupoid to define an associative algebra it has to satisfy the so-called 2-Segal conditions (see Definition 6.1). We call a simplicial groupoid satisfying these conditions a 2-Segal groupoid.
So let us sort out, how we can describe the Examples 2.1-2.3 of multi-fusion categories as linearizations of 2-Segal groupoids. Recall, therefore, that the augmented simplex category Δ_+ is the category obtained from the simplex category Δ by freely adding an initial object denoted $[-1]$.

Construction 2.1. Let $F : \mathcal{G} \to \mathcal{H}$ be a functor between finite groupoids. We can describe F as a functor $\check{C}(F)_{\leq 0} : \Delta_{+,\leq 0}^{\mathrm{op}} \to \mathsf{Grpd}^f$ with source the full subcategory $\Delta_{+,\leq 0}$ of the augmented simplex category Δ_+ generated by the objects $[0]$ and $[-1]$. We denote by $\check{C}(F)_\bullet$ the simplicial groupoid obtained by restricting the right Kan extension of $\check{C}(F)_{\leq 0}$ along the inclusion $\Delta_{+,\leq 0} \subset \Delta_+$ to Δ. This simplicial groupoid is called the *Čech-nerve* of F. We can explicitly depict its lower dimensional simplices as:

$$\mathcal{G} \underset{\pi_2}{\overset{\pi_1}{\underset{\Delta}{\leftleftarrows\!\!\rightarrow}}} \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \quad \begin{matrix} \xleftarrow{\pi_{12}} \\ \xrightarrow{\Delta \times \mathrm{id}} \\ \xleftarrow{\pi_{13}} \\ \xrightarrow{\mathrm{id} \times \Delta} \\ \xleftarrow{\pi_{23}} \end{matrix} \quad \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \quad \begin{matrix} \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{matrix} \quad \cdots$$

where $\pi_{i,j}$ denotes the projection on the i'th and j'th factor. The Čech-nerve $\check{C}(F)_\bullet$ is an example of a groupoid object (see Definition 10.3) and so in particular 2-Segal.

Example 2.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between finite groupoids. We can apply Corollary 2.2 to the 2-Segal groupoid $\check{C}(F)_\bullet$. This induces a monoidal structure on the category $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})$. More explicitly, the monoidal product $H * G$ of two functors H and G is defined as the functor

$$\begin{array}{ccc} & \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \times_{\mathcal{H}} \mathcal{G} & \\ {\scriptstyle \pi_{1,2} \times \pi_{2,3}} \swarrow & & \searrow {\scriptstyle \pi_{1,3}} \\ (\mathcal{G} \times_{\mathcal{H}} \mathcal{G}) \times (\mathcal{G} \times_{\mathcal{H}} \mathcal{G}) & & \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \\ & & \\ & \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}} & \end{array} \tag{2}$$

obtained by first pulling back $H \otimes_{\mathbb{K}} G$ along $\pi_{1,2} \times \pi_{2,3}$ and then left Kan extending along the projection $\pi_{1,3}$. In formulas:

$$H * G := (\pi_{1,3})_!(\pi_{1,2} \times \pi_{2,3})^*(H \otimes_{\mathbb{K}} G).$$

We call the monoidal category $(\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G} \times_{\mathcal{H}} \mathcal{G}), *)$ the *categorified* $\mathbb{K}$*-linear Hecke algebra* of F and denote it $\mathcal{H}\mathsf{e}_{\mathbb{K}}(F)$.

Remark 2.3. The name categorified Hecke algebra arises from the following example. Let G be a finite group and H a not necessarily normal subgroup. Consider the functor between classifying groupoids $\mathsf{BH} \to \mathsf{BG}$ induced by the inclusion and denote by $\mathsf{fun}_{\mathbb{K}}(\mathsf{BH} \times_{\mathsf{BG}} \mathsf{BH})$ the $\mathbb{K}$-vector space of $\mathbb{K}$-valued functions on $\mathsf{BH} \times_{\mathsf{BG}} \mathsf{BH}$. The span in Diagram 2 induces an algebra structure on $\mathsf{fun}_{\mathbb{K}}(\mathsf{BH} \times_{\mathsf{BG}} \mathsf{BH})$ that coincides with the classical Hecke algebra of H and G [DK19, Sect.8.2].

In particular, we can recover our initial Examples 2.1-2.3 as linearizations of Čech-nerves:

Example 2.5. Let G be a finite group and consider the unique functor $F : * \to \mathsf{BG}$. The Čech-nerve $\check{C}(F)_\bullet$ of F is equivalent to the nerve $N(G)_\bullet$ of the group G. The associated monoidal structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(G)$ is monoidally equivalent to the one on $\mathsf{Vect}_G^{\mathrm{fd}}$ from Example 2.1.

Example 2.6. Let G be a finite group and consider the identity functor $\mathrm{id}_{\mathsf{BG}} : \mathsf{BG} \to \mathsf{BG}$. The Čech-nerve $\check{C}(\mathrm{id}_{\mathsf{BG}})_\bullet$ of $\mathrm{id}_{\mathsf{BG}}$ is equivalent to the constant simplicial groupoid $\mathsf{BG}_\bullet$ with value BG. The associated monoidal structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(BG)$ is monoidally equivalent to the one on $\mathsf{Rep}(G)$ from Example 2.2.

Example 2.7. Consider the functor $F : \{1, ..., n\} \to *$. The Čech-nerve $\check{C}(F)_\bullet$ of F has n-simplices given by the $(n + 1)$-fold Cartesian product of the set $\{1, ..., n\}$. The associated monoidal structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(\{1, ..., n\} \times \{1, ..., n\})$ is monoidally equivalent to the one on $\mathsf{Mat}(\mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}})$ from Example 2.3.

Note that under mild assumptions on the groupoids, all of the above examples are multi-fusion categories. This is true for all examples of categorified Hecke algebras. For a prime number p, we call a finite groupoid $\mathcal{G}$ *p-coprime*, if for all $g \in \mathcal{G}$ the cardinality of the finite group $\pi_1(g, \mathcal{G})$ is coprime to p. We denote by Grpd_p^f the full subcategory of Grpd^f spanned by the p-coprime groupoids.

Proposition 2.3. *Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Grpd}^f$ be a finite 2-Segal groupoid. Then the category $\mathsf{Loc}^{fd}_{\mathbb{K}}(X_1)$ is finite abelian semisimple if and only if X_1 is p-coprime.*

Proof. It follows from Proposition 2.1 that the category $\mathsf{Loc}^{fd}_{\mathbb{K}}(X_1)$ is finite abelian and that the monoidal product functor is $\mathbb{K}$-linear right exact. Hence, it remains to show that the category is semisimple. To do so, note that there exists an equivalence of groupoids:

$$X_1 \simeq \coprod_{x \in \pi_0(X_1)} \mathsf{B}\pi_1(X_1, x)$$

By functoriality, this induces an equivalence of categories:

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(X_1) \simeq \bigoplus_{x \in \pi_0(X_1)} \mathsf{Rep}_{\mathbb{K}}(\pi_1(X_1, x)).$$

Hence, it follows that the category is semisimple as it is a finite direct sum of semisimple categories if and only if X_1 is p-coprime. □

Proposition 2.4. *Let $F : \mathcal{G} \to \mathcal{H}$ be a functor between essentially finite groupoids. The categorified Hecke algebra $\mathcal{H}\mathsf{e}_{\mathbb{K}}(F)$ of F is a multi-fusion category if and only if $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is $\mathbb{K}$-coprime.*

Proof. It follows from the Proposition 2.3, that the category $\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})$ is finite abelian semisimple if and only if $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is p-coprime. Therefore, it remains to show that the monoidal structure on the categorified Hecke algebra is rigid. To do so, we construct an explicit dual. Denote by

$$\sigma : \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{G} \times_{\mathcal{H}} \mathcal{G}$$

the functor between groupoids that swaps the components of $\mathcal{G}$. Further, denote by $\mathbb{D}_{\mathbb{K}} : \mathsf{Vect}^{\mathrm{fd,op}}_{\mathbb{K}} \to \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$ the duality functor of the category of finite dimensional vector spaces. It is easy to check that the left and the right dual of a functor $F : \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \to \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$ are both given by the composite functor

$$\mathcal{G} \times_{\mathcal{H}} \mathcal{G} \xrightarrow{\sigma} \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \simeq (\mathcal{G} \times_{\mathcal{H}} \mathcal{G})^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathsf{Vect}^{\mathrm{fd,op}}_{\mathbb{K}} \xrightarrow{\mathbb{D}_{\mathbb{K}}} \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}} .$$

Hence, the categorified Hecke algebra is a multi-fusion category. □

Remark 2.4. Note that in case that the groupoid $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is not p-coprime, the categorified Hecke algebra is still rigid. Hence, it defines a finite tensor category.

Example 2.8. Let G be a finite group and denote by $F : * \to \mathsf{B}G$ the unique morphism. Unraveling the proof of Proposition 2.4 the dual $F^\vee$ of a functor $F : G \to \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$ takes the value $F^\vee(g) := \hom_{\mathbb{K}}(F(g^{-1}), \mathbb{K})$ at $g \in G$. Note that under the identification $\mathcal{H}\mathsf{e}_{\mathbb{K}}(F) \simeq \mathsf{Vect}^{\mathrm{fin}}_G$ the above formula for the dual coincides with the standard formula [EGNO16, Ex.2.10.14].

Summarizing the above discussion, categorified Hecke algebras provides us with a class of multi-fusion categories. Our next goal is to understand its size under an interesting equivalence relation. A necessary requirement on this equivalence relation is that equivalent multi-fusion categories should induce the same fully extended TFTs in the sense of [DSPS20]. The loosest equivalence relation that satisfies this assumption is called Morita equivalence, a notion that we now recall.
Therefore, denote by $\mathbb{M}\mathsf{or}(\mathsf{cat}^{\mathsf{rex}}_{\mathbb{K}})$ the Morita 2-category of $\mathbb{K}$-linear categories (see C.2). This 2-category has objects given by right exact $\mathbb{K}$-linear monoidal categories, 1-morphisms by right exact bimodule categories,

and 2-morphisms by right exact bimodule functors. We need the following subclass of 1-morphisms in $\mathbb{M}\mathsf{or}(\mathsf{cat}_{\mathbb{K}}^{\mathsf{rex}})$:

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories. A $\mathcal{C}-\mathcal{D}$-bimodule $\mathcal{M}$ in $\mathsf{cat}_{\mathbb{K}}^{\mathsf{rex}}$ is called *exact* if

(1) $\mathcal{M}$ is locally finite abelian,

(2) the action functors $\triangleright : \mathcal{C}\times\mathcal{M}\to\mathcal{M}$ and $\triangleleft : \mathcal{M}\times\mathcal{D}\to\mathcal{M}$ are exact in each variable separately,

(3) for every projective object $P_C\in\mathcal{C}$ (resp. $P_D\in\mathcal{D}$) and every object $M\in\mathcal{M}$ the object $P_C\triangleright M$ (resp. $M\triangleright P_D$) is projective in $\mathcal{M}$.

Definition 2.3. Let $(\mathcal{C},\otimes_{\mathcal{C}})$ and $(\mathcal{D},\otimes_{\mathcal{D}})$ be finite $\mathbb{K}$-linear tensor categories. Then $\mathcal{C}$ and $\mathcal{D}$ are called *categorical Morita equivalent* if there exist exact bimodule categories ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{C}}$ that exhibit $\mathcal{C}$ and $\mathcal{D}$ as isomorphic in $\mathbb{M}\mathsf{or}(\mathsf{cat}_{\mathbb{K}}^{\mathsf{rex}})$.

To determine the equivalence class of the categorified Hecke algebra under categorical Morita equivalence, we explicitly construct an exact bimodule category. Consider the category $(\mathsf{Grpd}^f)^{\mathrm{op}}$ equipped with its Cocartesian monoidal structure, i.e. the monoidal product is given by the coproduct [Lur17, Sect.2.4.3]. It follows from the universal property of this monoidal structure that every finite groupoid $\mathcal{G}$ has a unique structure of a commutative algebra with multiplication given by the diagonal functor $\mathcal{G}\times\mathcal{G}\leftarrow\mathcal{G}:\Delta$. Moreover, every functor $F:\mathcal{G}\to\mathcal{H}$ between finite groupoids uniquely extends to a morphism of commutative algebras in $(\mathsf{Grpd}^f)^{\mathrm{op}}$ [Lur17, Cor.2.4.3.9]. In particular, the functor F equips $\mathcal{G}$ with the structure of a left $\mathcal{H}$-module. Since the functor $\mathsf{Loc}_{\mathbb{K}}^{fd}(-)$ is symmetric monoidal, this module structure transports to an $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})$-module structure on $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G})$. The action of $H\in\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})$ maps an object $G\in\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G})$ to the functor

$$\mathcal{G}\xrightarrow{\Delta}\mathcal{G}\times\mathcal{G}\xrightarrow{F\times\mathrm{id}_{\mathcal{G}}}\mathcal{H}\times\mathcal{G}\xrightarrow{H\times G}\mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}}\times\mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}}\xrightarrow{\otimes}\mathsf{Vect}_{\mathbb{K}}^{\mathrm{fd}}$$

Note that this action functor is exact in each variable separately. If we further assume, that all groupoids are p-coprime, it follows that all categories involved are semisimple and $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G})$ is an exact left-module category over $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})$.[4] Further, it also admits an exact right $\mathcal{H}\mathsf{e}(F)$-module structure with action functor induced by linearizing the span

$$\mathcal{G}\times\mathcal{G}\times_{\mathcal{H}}\mathcal{G}\longleftarrow\mathcal{G}\times_{\mathcal{H}}\mathcal{G}\times_{\mathcal{H}}\mathcal{G}\longrightarrow\mathcal{G}$$

Proposition 2.5. *Let $F:\mathcal{G}\to\mathcal{H}$ be an essentially surjective functor between finite p-coprime groupoids. Then the exact $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})-\mathcal{H}\mathsf{e}_{\mathbb{K}}(F)$-module category $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G})$ induces a Morita equivalence between $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})$ and $\mathcal{H}\mathsf{e}(F)$.*

Proof. By [FGJS22, Prop.4.9] it suffices to show that there exists an equivalence of monoidal categories

$$\mathsf{Fun}_{\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})}(\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G}),\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{G}))\simeq\mathcal{H}\mathsf{e}(F),$$

where the left-hand side denotes the category of left $\mathsf{Loc}_{\mathbb{K}}^{fd}(\mathcal{H})$-module functors with monoidal structure given by composition. This is a consequence of Corollary 11.4. More precisely, the above equivalence associates to

[4] Condition (3) in Definition 2.2 is trivially satisfied since all objects are projective.

an object G in $\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}\times_{\mathcal{H}}\mathcal{G})$ its integral transform $\mathcal{I}_G$

$$\mathcal{I}_G:\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G})\xrightarrow{\pi_1^*}\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}\times_{\mathcal{H}}\mathcal{G})\xrightarrow{-\otimes G}\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}\times_{\mathcal{H}}\mathcal{G})\xrightarrow{\pi_{2,!}}\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G})\ .$$

It is easy to check that this defines a monoidal functor that it is an equivalence. □

Remark 2.5. Let G be a finite group. In the special case $*\to BG$, this result recovers the well-known fact that $\mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$ considered as a $\mathsf{Vect}^{\mathrm{fd}}_G$-$\mathsf{Rep}(G)$-bimodule category induces a Morita equivalence between $\mathsf{Rep}(G)$ and Vect_G [EGNO16, 7.12.19].

This class of fusion category deserves a special name:

Definition 2.4. [EGNO16, Def.9.7.1] A fusion category $(\mathcal{C},\otimes)$ is called

- *strongly pointed* if there exists a finite group G, such that $(\mathcal{C},\otimes)$ is monoidally equivalent to the fusion category $\mathsf{Vect}^{\mathrm{fd}}_G$
- *strongly group theoretical* if there exists a finite group G, such that $(\mathcal{C},\otimes)$ is Morita equivalent to the fusion category $\mathsf{Vect}^{\mathrm{fd}}_G$

Remark 2.6. This definition differs from the standard terminology for fusion categories. [EGNO16]. In general a fusion category is called pointed (resp. group theoretical) if it is monoidally (resp. Morita) equivalent to the monoidal category Vect^{ω}_G, where G is a finite group and $\omega\in H^3(G,\mathbb{K})$ denotes a non-trivial class in the group cohomology of G. Since as explained in Remark 2.1, we only consider categories of the form Vect_G with the trivial associator $\omega=1$, we adapted this terminology to our situation.

Corollary 2.6. *Let $F:\mathcal{G}\to\mathcal{H}$ a functor between p-coprime finite groupoids and assume that $\mathcal{H}$ is connected. Then $\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}\times_{\mathcal{H}}\mathcal{G})$ is group theoretical.*

Proof. It follows from Proposition 2.5 that there exists a Morita equivalence

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{G}\times_{\mathcal{H}}\mathcal{G})\simeq_{Mor}\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{H}).$$

Further, since by assumption $\mathcal{H}$ is connected, there exists a monoidal equivalence $\mathsf{Loc}^{fd}_{\mathbb{K}}(\mathcal{H})\simeq\mathsf{Rep}(\pi_1(\mathcal{H},h))$. The claim follows from Remark 2.5. □

We can therefore conclude that examples of fusion categories that arise as categorified Hecke algebras, and therefore as linearizations of Čech-nerves, are sums of strongly group theoretical fusion categories.

2.2 Fusion Categories from 2-Segal Sets

While still of interest in their own right, strongly group theoretical fusion categories form the most basic class of examples of multi-fusion categories. In particular, in the context of TFTs, one is usually more interested in representation categories of quantum groups that lead to more interesting invariants [TV92, Wit89]. As we see in the following example, these are usually not (strongly) group theoretical:

Example 2.9. The *Ising fusion category* $\mathscr{I}$ is an example of a Tambara-Yamagami $\mathbb{C}$-linear fusion category [TS98]. It has 3 isomorphism classes of simple objects $\{\mathbb{1},\epsilon,\sigma\}$ with fusion rules:

$\otimes$	$\mathbb{1}$	ϵ	σ
$\mathbb{1}$	$\mathbb{1}$	ϵ	σ
ϵ	ϵ	$\mathbb{1}$	σ
σ	σ	σ	$\mathbb{1}\oplus\epsilon$

The Ising category is known to be the representation category of a quantum group. More precisely, it can be obtained as the quotient of the category of tilting modules over Lusztig's quantum group $U_q^L(\mathfrak{sl}_2)$ at an eights root of unity by the so-called negligible modules [EGNO16].

Proposition 2.7. *The Ising fusion category $\mathscr{I}$ is not group theoretical. In particular, $\mathscr{I}$ is not the linearization of a Čech-nerve.*

Proof. Every group theoretical fusion category is integral [EGNO16, Rem.9.7.7], i.e. the Frobenius-Perron dimension of every simple object is an integer. The Frobenius-Perron dimension $\mathsf{FPdim}(\sigma)$ of the simple object σ is given by the largest eigenvalue of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

An easy calculation shows that the Eigenvalues are given by 0 and $\pm\sqrt{2}$. It follows that $\mathsf{FPdim}(\sigma) = \sqrt{2}$ and hence $\mathscr{I}$ is not integral. □

As a consequence of our discussion in the last section, the Ising category is not a categorified Hecke algebra. Thus it does not arise from the linearization of a Čech-nerve. The question remains, whether we can construct the Ising category by linearizing an algebra object in $\mathsf{Span}(\mathsf{Grpd}^f)$ that is not a Čech-nerve.
Evidence for a positive answer arises from the connection between 2-Segal groupoids and quantum groups. The most important example of a 2-Segal groupoid is the Waldhausen $S_\bullet$-construction (see Definition 7.1) of an exact category $\mathcal{C}$. These 2-Segal object describes an interesting class of algebras, known under the name Hall algebras [DK19, Sect.8]. Ringel [Rin90] has proven that there exists a connection between the theory of Hall algebras and quantum groups. For instance, he observed that one can recover the upper half of the quantum group $U_{\sqrt{q}}(\mathfrak{sl}_2)$ of the complex semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ at $\sqrt{q}$ as the Hall algebra of the category of vector spaces $\mathsf{Vect}_{\mathbb{F}_q}$ over the field with $q \in \mathbb{N}$ elements $\mathbb{F}_q$. Therefore, it seems promising that one can recover non-group theoretical fusion categories from linearization constructions as well. Further evidence is given by the following result:

Proposition 2.8. *Let $(\mathcal{C}, \otimes)$ be a fusion category and denote by I a finite set of isomorphism classes of simple objects of $\mathcal{C}$. Then there exists an equivalence of categories*

$$\phi : \mathcal{C} \to \mathsf{Loc}^{fd}_{\mathbb{K}}(I),$$

and a span

$$\begin{array}{ccc} & N & \\ {}^{(\partial_2,\partial_0)}\swarrow & & \searrow^{\partial_1} \\ I \times I & & I \end{array} \tag{3}$$

s.t. under the equivalence ϕ the functor

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(I) \times \mathsf{Loc}^{fd}_{\mathbb{K}}(I) \xrightarrow{\otimes_{\mathbb{K}}} \mathsf{Loc}^{fd}_{\mathbb{K}}(I \times I) \xrightarrow{(\partial_2,\partial_0)^*} \mathsf{Loc}^{fd}_{\mathbb{K}}(N) \xrightarrow{\partial_{1,!}} \mathsf{Loc}^{fd}_{\mathbb{K}}(I),$$

obtained by linearizing the Span (3) is naturally isomorphic to the monoidal product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

Proof. Consider the category $\mathsf{Loc}^{fd}_{\mathbb{K}}(I)$ and denote by δ_i the functor

$$\delta_i(j) = \begin{cases} 0 & i \neq j \\ \mathbb{K} & i = j \end{cases}.$$

Since $\mathcal{C}$ is semisimple, there exists a unique right-exact $\mathbb{K}$-linear functor

$$\mathsf{Loc}^{fd}_{\mathbb{K}}(I) \to \mathcal{C}$$

that maps the functor δ_i to the simple object $i \in I$. It is easy to see that this functor is an equivalence of categories. For the construction of the span let $i, j \in I$ be two simple objects. It follows from semisimplicity of $\mathcal{C}$ that the monoidal product

$$i \otimes j \simeq \bigoplus_{k \in I} N^k_{i,j} k$$

decomposes into a weighted sum of simples with finite-dimensional multiplicity vector spaces $N^k_{i,j} \in \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$. Given 3-simples $i, j, k \in I$, we define a set

$$\mathcal{N} := \{(n^k_{i,j})_l | i, j, k \in I, 1 \leq l \leq \mathsf{dim} N^k_{i,j}\}$$

together with morphisms $\partial_i : \mathcal{N} \to I$ that map

$$\partial_0((n^k_{i,j})_l) = j, \quad \partial_1((n^k_{i,j})_l) = k, \quad \partial_2((n^k_{i,j})_l) = i.$$

These assemble into a span

$$I \times I \xleftarrow{(\partial_2, \partial_0)} \mathcal{N} \xrightarrow{\partial_1} I$$

We denote the functor obtained by linearizing this span by $* : \mathsf{Loc}^{fd}_{\mathbb{K}}(I) \times \mathsf{Loc}^{fd}_{\mathbb{K}}(I) \to \mathsf{Loc}^{fd}_{\mathbb{K}}(I)$. Unraveling definitions, it is given by

$$\delta_i * \delta_j(k) = \mathbb{K}[\mathcal{N}^k_{i,j}]$$

where we denote by $\mathbb{K}[\mathcal{N}^k_{i,j}]$ the free $\mathbb{K}$-vector space generated by $\mathcal{N}^k_{i,j}$, the fiber of $\mathcal{N}$ over the simple objects $(i, j, k) \in I$.
It remains to show that this functor is naturally equivalent to the functor obtained from $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ under ϕ. It follows from [Mor11, Lem.3.2.2] that every right exact $\mathbb{C}$-linear functor

$$F : \mathsf{Loc}^{fd}_{\mathbb{K}}(I \times I) \to \mathsf{Loc}^{fd}_{\mathbb{K}}(I)$$

is determined up to natural isomorphism by the $(I \times I)$-I matrix of vector spaces with $((i, j), k)$-entry $F(\delta_{(i,j)})(k) \in \mathsf{Vect}^{\mathrm{fd}}_{\mathbb{K}}$. In particular, the functor

$$\tilde{\otimes} : \mathsf{Loc}^{fd}_{\mathbb{K}}(I \times I) \to \mathsf{Loc}^{fd}_{\mathbb{K}}(I)$$

induced by $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ under ϕ is uniquely determined by the $(I \times I) - I$-matrix with entries $N^k_{i,j}$. Moreover, it follows from [Mor11, Lem.3.2.3] that any $(I \times I) - I$-matrix of linear isomorphisms $\{\alpha^k_{i,j} : M^k_{i,j} \to L^k_{i,j}\}$ induces a natural isomorphism between the $\mathbb{C}$-linear functors represented by the matrices $\{M^k_{i,j}\}$ and $\{L^k_{i,j}\}$.

Therefore, choose for any triple of simples $i, j, k \in I$ a basis of the vector space $N_{i,j}^k$. Any such choice determines a matrix of isomorphisms $\{\alpha_{i,j}^k : \mathbb{K}[\mathcal{N}_{i,j}^k] \to N_{i,j}^k\}$ and therefore a natural isomorphism as required. □

The above proposition implies that certain parts of the data of a fusion category always arise from a linearization construction. Indeed, the sets I and $\mathcal{N}$ constructed in the proof describe the 1- and 2-simplices of a hypothetical 2-Segal set. To encode the remaining data of a 2-Segal set, we use an alternative description of those. We denote for every set X the category $\mathsf{Fun}(X, \mathsf{Set}^{\mathrm{fin}})$ of functors into the category of finite sets by $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X)$.

Definition 2.5. [DK19, Def.3.5.5] A monoidal category $(\mathcal{C}, \otimes)$ is called *finite* $\amalg$*-semisimple*, if:

(1) There exists a finite set X and an equivalence of categories $\mathcal{C} \simeq \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X)$.

(2) $\mathcal{C}$ admits finite coproducts and the monoidal product functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves finite coproducts separately in each argument.

A finite $\amalg$-semisimple monoidal category is a non-linear analog of a finite semisimple $\mathbb{K}$-linear monoidal category. Indeed, as observed in Proposition 2.8 the underlying category of every finite semisimple $\mathbb{K}$-linear monoidal category $\mathcal{C}$ is equivalent to the $\mathbb{K}$-linear category $\mathsf{Loc}_{\mathbb{K}}^{\mathrm{fd}}(I)$, where I denotes the set of isomorphism classes of simple objects and we think of finite coproducts as a non-linear analog of direct sums.
Their relation to 2-Segal sets is as follows. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{fin}}$ be a finite 2-Segal set with $X_0 \simeq *$. Analogously to Proposition 2.1, one can construct a symmetric monoidal $(2,1)$-functor with source the category $\mathsf{Span}(\mathsf{Set}^{\mathrm{fin}})$ that associates to any finite set I the category $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X)$. In particular, it associates with any finite 2-Segal set $X_\bullet$, a finite $\amalg$-semisimple monoidal category $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1)$. As described in Corollary 2.2 the underlying category is given by $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1)$ and the monoidal product is given the composite functor

$$\otimes : \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1) \times \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1) \to \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1 \times X_1) \xrightarrow{(\partial_0 \times \partial_2)^*} \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_2) \xrightarrow{\partial_{1,!}} \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(X_1).$$

The higher simplices of the 2-Segal set encode the higher associativity data of the monoidal category. Interestingly all examples of finite $\amalg$-semisimple monoidal categories arise from this construction:

Theorem 2.9. *[DK19, Thm. 3.5.8] The functor $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(-) : 2\mathrm{Seg}_\Delta(\mathsf{Set}^{fin}) \to \mathsf{cat}_{\amalg,\otimes}^{\mathrm{lax}}$ induces an equivalence between the category of $\amalg$-semisimple monoidal categories and coproduct preserving lax-monoidal functors and the category of finite 2-Segal sets with $X_0 \simeq *$ and simplicial maps.*

In the language of $\amalg$-semisimple monoidal categories, the construction of Proposition 2.8 associates to any fusion category $\mathcal{C}$ a unique candidate of a $\amalg$-semisimple category $\mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(I)$ and a $\amalg$-preserving functor

$$\otimes : \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(I) \times \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(I) \to \mathsf{Loc}_{\mathsf{Set}}^{\mathrm{fin}}(I).$$

The missing data is an associativity isomorphism that satisfies the pentagon condition. Therefore, it remains to understand how we can construct associators for $\amalg$-semisimple categories. Thinking about those as non-linear analogs of semisimple monoidal categories, we take our intuition from the well-known $\mathbb{K}$-linear case of fusion categories. For those, it is useful for computations to express everything in terms of simple objects. In this basis, the associator can be expressed by a family of matrices (called F-matrices or 6j-symbols) satisfying a system of algebraic equations called the Pentagon equation [FG23, Eq.A.11]. The problem of constructing an associator then reduces to the problem of constructing a solution of the Pentagon equation. We follow the analogous strategy in the non-linear case. Let X be a finite set and $(\mathsf{Loc}^{\mathrm{fin}}_{\mathsf{Set}}(X), \otimes)$ a $\amalg$-semisimple monoidal category. As in the proof of Proposition 2.8, we denote, for every $i \in X$, by δ_i the functor

$$\delta_i(j) = \begin{cases} 0 & i \neq j \\ * & i = j \end{cases}$$

These functors play the role of simple objects. In particular, we can express any functor $F : X \to \mathsf{Set}^{\mathrm{fin}}$ in terms of these δ_i as the functor $\coprod_{i \in X} F(i) \times \delta_i(-)$. This is in particular true for the monoidal product of two δ_i:

$$(\delta_i \otimes \delta_j)(-) \simeq \coprod_{k \in I} (\delta_i \otimes \delta_j)(k) \times \delta_k(-).$$

In analogy with fusion categories, we adopt the notation $\delta_i \otimes \delta_j(k) := \mathcal{N}^k_{i,j} \in \mathsf{Set}^{\mathrm{fin}}$. Finally, we can use this description to understand the associator. Unraveling the definitions, the datum of an associator can be expressed in terms of δ_i as a natural isomorphism:

$$\alpha : (\delta_i \otimes \delta_j) \otimes \delta_l \simeq \coprod_{n \in I} (\coprod_{k \in I} \mathcal{N}^k_{i,j} \times \mathcal{N}^n_{k,l}) \times \delta_n(-) \xrightarrow{\simeq} \coprod_{n \in I} (\coprod_{m \in I} \mathcal{N}^n_{i,m} \times \mathcal{N}^m_{j,l}) \times \delta_n(-) \simeq \delta_i \otimes (\delta_j \otimes \delta_l).$$

By definition of the δ_i, this morphism is uniquely determined by its components, i.e. the family of isomorphism of sets:

$$\alpha^n_{i,j,l} : \coprod_{k \in I} \mathcal{N}^k_{i,j} \times \mathcal{N}^n_{k,l} \to \coprod_{m \in I} \mathcal{N}^n_{i,m} \times \mathcal{N}^m_{j,l}.$$

As α is an associator of a monoidal category, it satisfies the pentagon condition [EGNO16, Def.2.1.1]. As in the case of fusion categories, this condition can be expressed in terms of the components $\alpha^n_{i,j,l}$. More precisely, the pentagon condition requires the following family of diagrams of sets to commute:

$$\begin{array}{ccccc}
 & & \coprod_{k,m \in I} \mathcal{N}^k_{i,j} \times \mathcal{N}^m_{k,l} \times \mathcal{N}^b_{m,a} & & \\
 & \swarrow^{\alpha^m_{i,j,l} \times id} & & \searrow^{id \times \alpha^b_{k,l,a}} & \\
\coprod_{k,m \in I} \mathcal{N}^m_{i,k} \times \mathcal{N}^k_{j,l} \times N^b_{m,a} & & & & \coprod_{k,m \in I} \mathcal{N}^k_{i,j} \times \mathcal{N}^b_{k,m} \times \mathcal{N}^m_{l,a} \\
\downarrow \simeq & & & & \downarrow \alpha^b_{i,j,m} \times id \\
\coprod_{m,k \in I} \mathcal{N}^m_{i,k} \times \mathcal{N}^b_{m,a} \times \mathcal{N}^k_{j,l} & & & & \coprod_{m,k \in I} \mathcal{N}^b_{i,m} \times \mathcal{N}^k_{j,m} \times \mathcal{N}^m_{l,a} \\
 & \searrow_{\alpha^b_{i,k,a} \times id} & & \nearrow_{id \times \alpha^m_{j,l,a}} & \\
 & & \coprod_{m,k \in I} \mathcal{N}^b_{i,m} \times \mathcal{N}^m_{k,a} \times \mathcal{N}^k_{j,l} & &
\end{array} \tag{4}$$

We call the family of diagrams in analogy with the case of fusion categories the *set-theoretic parametric pentagon equation.* This is a parameter dependent version of the set-theoretic pentagon from [KR01].

Remark 2.7. The difference between the pentagon equation for $\mathbb{K}$-linear and $\amalg$-semisimple monoidal categories is, that in the first case the morphisms $\alpha^n_{i,j,l}$ are isomorphisms of vector spaces, and in the latter isomorphisms of sets.

Consequently, the construction of an associator reduces to the construction of a solution of the *set-theoretic parametric pentagon equation* and our initial question reduces to the following combinatorial problem:

Question 2.10. *Does the set-theoretic parametric pentagon equation defined by a fusion category $\mathcal{C}$ via Proposition 2.8 admits a solution?*

We analyze this question explicitly for the example of the Ising category:

Theorem 2.11. *There does not exist a 2-Segal set $X_\bullet : \Delta^{\text{op}} \to \mathsf{Set}$, whose linearization is the Ising category $\mathscr{I}$.*

Proof. Let $I \simeq \{\mathbb{1}, \epsilon, \sigma\}$ be a set of isomorphism classes of simple objects of $\mathscr{I}$. By our above considerations, it suffices to show that there does not exist an associativity 2-isomorphism for the monoidal structure on $\mathsf{Loc}^{\text{fin}}_{\mathsf{Set}}(I)$ constructed in Proposition 2.8 that satisfies the set-theoretic parametric pentagon equation. To do so, we explicitly compute for any choice of associativity isomorphism the path of the element

$$(n^{\epsilon}_{\sigma,\sigma}, n^{1}_{\epsilon,\epsilon}, n^{\sigma}_{1,\sigma}) \in \mathcal{N}^{\epsilon}_{\sigma,\sigma} \times \mathcal{N}^{1}_{\epsilon,\epsilon} \times \mathcal{N}^{\sigma}_{1,\sigma}$$

in the set-theoretic parametric pentagon equation. For this particular component of the pentagon equation, all components of the associator are uniquely defined, except:

$$\alpha^{\sigma}_{\sigma,\sigma,\sigma} : \mathcal{N}^{1}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{1,\sigma} \coprod \mathcal{N}^{\epsilon}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{\epsilon,\sigma} \xrightarrow{\simeq} \mathcal{N}^{1}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{1,\sigma} \coprod \mathcal{N}^{\epsilon}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{\epsilon,\sigma}.$$

Since $\alpha^{\sigma}_{\sigma,\sigma,\sigma}$ has to be an isomorphism of sets, there exist two choices. The first one is uniquely determined by

$$\alpha^{\sigma}_{\sigma,\sigma,\sigma}((n^{1}_{\sigma,\sigma}, n^{\sigma}_{1,\sigma})) = (n^{\epsilon}_{\sigma,\sigma}, n^{\sigma}_{\epsilon,\sigma}),$$

and the second by

$$\beta^{\sigma}_{\sigma,\sigma,\sigma}((n^{1}_{\sigma,\sigma}, n^{\sigma}_{1,\sigma})) = (n^{1}_{\sigma,\sigma}, n^{\sigma}_{1,\sigma}).$$

We argue that the α-associator is not compatible with rigidity. Indeed, the zig-zag identity for σ implies that the composite

$$\sigma \simeq \sigma \otimes 1 \xrightarrow{\text{id}_\sigma \otimes \text{coev}_\sigma} \sigma \otimes (\sigma \otimes \sigma) \simeq (\sigma \otimes \sigma) \otimes \sigma \xrightarrow{\text{ev}_\sigma \otimes \text{id}_\sigma} 1 \otimes \sigma \simeq \sigma,$$

where the middle equivalence is the associator, is the identity functor. Unraveling this identity in terms of the multiplicity spaces for the choice of the α-associator, this composite reads as:

$$\mathbb{K} \xrightarrow{(1,0)} \mathbb{K}[\mathcal{N}^{1}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{1,\sigma}] \oplus \mathbb{K}[\mathcal{N}^{\epsilon}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{\epsilon,\sigma}] \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \mathbb{K}[\mathcal{N}^{1}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{1,\sigma}] \oplus \mathbb{K}[\mathcal{N}^{\epsilon}_{\sigma,\sigma} \times \mathcal{N}^{\sigma}_{\epsilon,\sigma}] \xrightarrow{(1,0)} \mathbb{K}$$

But this composite is the 0-map instead of the identity. Hence, it suffices to consider the set-theoretic parametric pentagon equation for the β-associator. This equation is represented in Diagram 1. Its value on $(n^{\epsilon}_{\sigma,\sigma}, n^{1}_{\epsilon,\epsilon}, n^{\sigma}_{1,\sigma})$ is given by:

$$(n^{\sigma}_{\sigma,1}, n^{1}_{\sigma,\sigma}, n^{\sigma}_{\epsilon,\sigma}) \neq (n^{\sigma}_{\sigma,\epsilon}, n^{\epsilon}_{\sigma,\sigma}, n^{\sigma}_{\epsilon,\sigma})$$

where the left hand side denotes the value of the counter clockwise direction and the right hand side of the clockwise. Consequently, all choices for the associator do not satisfy the set-theoretic parametric pentagon equation. Hence, there does not exist any associator on $\mathsf{Loc}^{\text{fin}}_{\mathsf{Set}}(I)$ for the Ising category. □

We invite the reader to do the same construction with their favorite example of a fusion category that is not

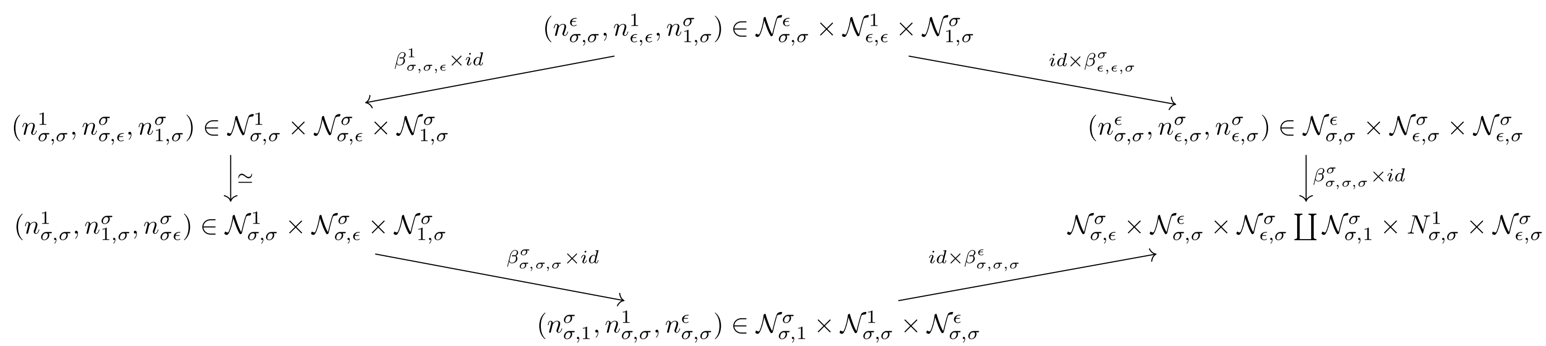


Figure 1: Pentagon equation for the β-associator

strongly pointed. Surprisingly, the Ising category is not an exception. It is a consequence of the following theorem that the construction presented in this section fails as soon as you start with a fusion category that is not pointed:

Theorem 2.12. *Let* $(\mathcal{C}, \otimes, \alpha)$ *be a fusion category. Then* $\mathcal{C}$ *is monoidally equivalent to the linearization of a* 2*-Segal set* $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Set}$ *if and only if* $\mathcal{C}$ *is strongly pointed.*

Proof. We have already seen in Example 2.5 that every strongly pointed fusion category arises as the linearization of a 2-Segal set. For the converse, assume that $\mathcal{C}$ is not strongly pointed. The proof we present here is a generalization of the above proof in Proposition 2.11 for the Ising category. Since $\mathcal{C}$ is not pointed there exists a non-invertible simple object $b \in \mathcal{C}$. Hence, there exists a simple object $1 \neq a \in \mathcal{C}$ s.t. the multiplicity set $\mathcal{N}^a_{b,{}^\vee b}$ is non-empty. Analogously to the case of the Ising category, we construct a component of the set-theoretic parametric Pentagon equation that does not admit a solution. Consider to this end the set

$$\mathcal{N}^a_{b,{}^\vee b} \times \mathcal{N}^1_{a,{}^\vee a} \times \mathcal{N}^b_{1,b} \subset \coprod_{i,j \in I} \mathcal{N}^i_{b,{}^\vee b} \times \mathcal{N}^j_{i,{}^\vee a} \times \mathcal{N}^b_{j,b}$$

By assumption this set is non-empty and we choose an element $(n^a_{b,{}^\vee b}, n^1_{a,{}^\vee a}, n^b_{1,b})$. Note that the second and third entry are unique. We claim that this element does not fulfill the set-theoretic parametric Pentagon equation. For this purpose, we compute the image of this element under the counter clockwise and clockwise path in diagram (4), starting with the counter clockwise one. The first map is given by

$$\mathcal{N}^a_{b,{}^\vee b} \times \mathcal{N}^1_{a,{}^\vee a} \times \mathcal{N}^b_{1,b} \xrightarrow{(\alpha^1_{b,{}^\vee b,{}^\vee a})_a \times \mathrm{id}} \textstyle\coprod_{j \in I} \mathcal{N}^1_{b,j} \times \mathcal{N}^j_{{}^\vee b,{}^\vee a} \times \mathcal{N}^b_{1,b}.$$

Note that for two simples $i, j \in I$ the set $\mathcal{N}^1_{i,j}$ has a unique element if $j \simeq {}^\vee i$ and is empty else. Hence, the above associator factors as:

$$\mathcal{N}^a_{b,{}^\vee b} \times \mathcal{N}^1_{a,{}^\vee a} \times \mathcal{N}^b_{1,b} \xrightarrow{(\alpha^1_{b,{}^\vee b,{}^\vee a})_a \times \mathrm{id}} \mathcal{N}^1_{b,{}^\vee b} \times \mathcal{N}^{{}^\vee b}_{{}^\vee b,{}^\vee a} \times \mathcal{N}^b_{1,b}.$$

We denote the image of $(n^a_{b,{}^\vee b}, n^1_{a,{}^\vee a}, n^b_{1,b})$ under this map by $(n^1_{b,{}^\vee b}, (n^{{}^\vee b}_{{}^\vee b,{}^\vee a})_i, n^b_{1,b})$. Since the set $\mathcal{N}^{{}^\vee b}_{{}^\vee b,{}^\vee a}$ may have more than one element, we have included a further index i in the notation. Note that the entries $n^1_{b,{}^\vee b}$ and $n^b_{1,b}$ are uniquely determined.
To analyze the second map, we need the following observation. Consider the zig-zag identity

$$b \simeq b \otimes \mathbb{1} \longrightarrow b \otimes ({}^\vee b \otimes b) \xrightarrow{\alpha_{b,{}^\vee b,b}} (b \otimes^\vee b) \otimes b \longrightarrow b$$

for the duality of the simple object b. Note, that for this composite to be the identity the component

$$\mathcal{N}^1_{b,{}^\vee b} \times \mathcal{N}^b_{1,b} \xrightarrow{(\alpha^b_{b,{}^\vee b,,b})^1_1} \mathcal{N}^b_{b,1} \times \mathcal{N}^1_{{}^\vee b,b}$$

of the associator has to be an isomorphism. This implies that the second map has to be given by the following morphism:

$$\mathcal{N}^1_{b,{}^\vee b} \times \mathcal{N}^{{}^\vee b}_{{}^\vee b,{}^\vee a} \times \mathcal{N}^b_{1,b} \simeq \mathcal{N}^{{}^\vee b}_{{}^\vee b,{}^\vee a} \times \mathcal{N}^1_{b,{}^\vee b} \times \mathcal{N}^b_{1,b} \xrightarrow{\mathrm{id} \times (\alpha^b_{b,{}^\vee b,,b})^1_1} \mathcal{N}^{{}^\vee b}_{{}^\vee b,{}^\vee a} \times \mathcal{N}^b_{b,1} \times \mathcal{N}^1_{{}^\vee b,b} \tag{5}$$

Since $(\alpha^b_{b,{}^\vee b,,b})^1_1$ is a morphism between sets with one element it has to be an isomorphism. Therefore the

image of $((n^{\vee b}_{\vee b,\vee a})_i, n^1_{b,\vee b}, n^b_{1,b})$ is uniquely given by $((n^{\vee b}_{\vee b,\vee a})_i, n^b_{b,1}, n^b_{\vee b,b})$.
By the same reasoning as in the first step, the component of the associator has to factor through the set

$$\mathcal{N}^{\vee b}_{\vee b,\vee a} \times \mathcal{N}^b_{b,1} \times \mathcal{N}^1_{\vee b,b} \simeq \mathcal{N}^b_{b,1} \times \mathcal{N}^{\vee b}_{\vee b,\vee a} \times \mathcal{N}^1_{\vee b,b} \xrightarrow{\mathrm{id} \times (\alpha^1_{\vee b,\vee a,b})^b_{\vee b}} \mathcal{N}^b_{b,1} \times \mathcal{N}^1_{\vee b,b} \times \mathcal{N}^b_{\vee a,b}.$$

We can therefore conclude that the image of $(n^a_{b,\vee b}, n^1_{a,\vee a}, n^b_{1,b})$ under the maps of the left hand side of the set-theoretic parametric pentagon equation has to be given by $(n^b_{b,1}, n^b_{\vee b,b}, (n^b_{\vee a,b})_l)$.
Next, we consider the clockwise direction. Note, that the last copy of $\mathcal{N}$ does not change under the second map. Hence, for the set theoretic parametric pentagon equation to be satisfied the first associator has to factor through

$$\mathcal{N}^a_{b,\vee b} \times \mathcal{N}^1_{a,\vee a} \times \mathcal{N}^b_{1,b} \xrightarrow{id \times (\alpha^b_{a,\vee a,b})^b_1} \mathcal{N}^a_{b,\vee b} \times \mathcal{N}^b_{a,b} \times \mathcal{N}^b_{\vee a,b},$$

and the next one has to factor as

$$\mathcal{N}^a_{b,\vee b} \times \mathcal{N}^b_{a,b} \times \mathcal{N}^b_{\vee a,b} \xrightarrow{(\alpha^b_{b,\vee b,b})^1_a \times \mathrm{id}} \mathcal{N}^1_{b,\vee b} \times \mathcal{N}^b_{1,b} \times \mathcal{N}^b_{\vee a,b}.$$

But recall that the components of the associator have to induce an isomorphism

$$\coprod_{i \in I} \mathcal{N}^i_{b,\vee} \times \mathcal{N}^b_{i,b} \xrightarrow{(\alpha^b_{b,\vee b,b})^j_i} \coprod_{j \in I} \mathcal{N}^b_{b,j} \times \mathcal{N}^j_{\vee b,b}.$$

Hence, since we already observed in the Diagram 5 that the component

$$(\alpha^b_{b,\vee b,,b})^1_1 : \mathcal{N}^1_{b,\vee b} \times \mathcal{N}^b_{1,b} \to \mathcal{N}^b_{b,1} \times \mathcal{N}^1_{\vee b,b}$$

maps isomorphically onto $\mathcal{N}^b_{b,1} \times \mathcal{N}^1_{\vee b,b}$ this is not true for the component $(\alpha^b_{b,\vee b,b})^1_a$. Therefore the object b has to be invertible and the fusion category is strongly pointed. □

Remark 2.8. The data constructed in Proposition 2.8 is enough to describe the fusion ring $\mathsf{K}_0[\mathcal{C}]$ [EGNO16, Sect.3], i.e. the Grothendieck ring, of the fusion category $\mathcal{C}$. In particular, fusion rings always arise from linearizations of spans. One of the major problems in the theory of fusion categories is to determine those fusion rings that can be extended to fusion categories, by constructing associators [LPR22]. The problem we have studied in this section is a non-linear avatar of this problem and our main theorem gives a complete answer for the case of 2-Segal sets.

This is a surprising and subtle result. It is not true that there are no monoidal structures induced from 2-Segal sets. Indeed, there are plenty of 2-Segal sets, for example from rooted trees [BBD+25], that induce combinatorially tractable monoidal structures. The difference is that none of those will be rigid. As we have seen in the proof Theorem 2.12, the extra assumption of rigidity induces certain symmetries on the multiplicity sets of the 2-Segal set, that force it to be the nerve of a group. This behavior is subtle and only visible, due to the easy combinatorial structure of 2-Segal sets. Already in the case of 2-Segal groupoids the interplay between the groupoid and 2-Segal structure makes it hard to show similar results by hand. We will therefore spend the rest of this text to follow a different route and to provide a more abstract explanation of this result.

3 Rigidity

We have presented in the last section an explicit combinatorial proof of Theorem 2.12. Our main goal in this thesis is to provide a more abstract explanation of this result. Phrased more generally, we try to understand, why all examples of convolution monoidal structures that are multi-fusion categories that are known to us are induced by Čech-nerves. As we have observed in our proofs of Theorem 2.11 and 2.12, the main obstacle is, to determine whether the associated convolution monoidal category is rigid. For the case of the categorified $\mathbb{K}$-linear Hecke algebras, we were able to prove that they are rigid since we were able to construct the duals by hand. In general, this can be a challenging task, since one has to come up with an educated guess for the dual object. Especially, since we are only interested in the mere existence of duals and not their precise form, we would benefit from a more abstract characterization of rigidity.
Therefore, we introduce in this section an $(\infty, 2)$-categorical characterization of rigidity. This perspective on rigidity has been introduced by Gaitsgory [Gai15, D.1.1] in the context of dg-categories. He classifies rigid presentable dg-categories in terms of conditions on the adjoint of the monoidal product functor. The advantage of this perspective is twofold. On the one hand, it provides a criterion for the existence of duals without explicitly constructing them. On the other hand, this definition is extrinsic and can be formalized internally to any symmetric monoidal $(\infty, 2)$-category $\mathbb{C}$. As explained in the Introduction 1, we will in particular benefit from the second advantage and study rigid algebras in the $(\infty, 2)$-categories of spans in Section 10.3 that we call *rigid* 2-*Segal objects*. Let us now start to explain the definition:

Definition 3.1. [Lur17, Def.4.7.4.13] Let $\mathbb{D}$ be an $(\infty, 2)$-category. We call a diagram commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{G_1} & B \\ {\scriptstyle F_1}\downarrow & \simeq^{\alpha} \nearrow & \downarrow{\scriptstyle F_0} \\ C & \xrightarrow[G_0]{} & D \end{array}$$

vertically right adjointable, if the vertical functors F_0 and F_1 admit right adjoints in $\mathbb{D}$ (see Definition C.4) and the vertical right Beck–Chevalley transform of α

$$\mathsf{BC}_v^R[\alpha] : G_1 F_1^R \xRightarrow{\eta_{F_0}} F_0^R F_0 G_1 F_1^R \simeq^{\alpha} F_0^R G_1 F_1 F_1^R \xRightarrow{\epsilon_{F_1}} F_0^R G_1$$

is a 2-isomorphism. Here, we denote by η_{F_0} the counit and by ϵ_{F_1} the unit of the respective adjunction. Analogously, we define *horizontally right adjointable* as well as *horizontally and vertically left adjointable* diagrams.

Definition 3.2. Let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category. An algebra $(A, \mu, \alpha) \in \mathsf{Alg}(\mathbb{D})$ with multiplication 1-morphism μ and associativity 2-isomorphism α is called *locally rigid*, if it is dualizable, the multiplication $\mu : A \otimes A \to A$ admits a right adjoint and the associativity diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes id}\downarrow & \simeq^{\alpha} \nearrow & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{6}$$

is horizontally and vertically right adjointable. A is called *rigid* if further the unit morphism $\eta : \mathbb{1}_{\mathcal{C}} \to A$ admits a right adjoint.

In the following, we will often abuse notation and denote a rigid algebra (A, μ, α) only by its underlying object A. One can think of locally rigid algebras as a categorification of the concept of a Frobenius algebra. More precisely, we can interpret the right adjoint

$$\mu^R : A \to A \otimes A,$$

as the coproduct of a non-counital coalgebra structure on A. Under this interpretation, Diagram (6) is horizontally and vertically right adjointable if and only if the Frobenius relations

$$(\mu \otimes id) \circ (id \otimes \mu^R) \simeq \mu \circ \mu^R \simeq (id \otimes \mu) \circ (\mu^R \otimes id)$$

hold up to 2-isomorphism induced by the vertical and horizontal Beck-Chevalley transform of α [Koc04, Sect.2.2]. So an algebra is locally rigid if and only if the right adjoint of the multiplication canonically equips it with the structure of a non-counital Frobenius algebra.[5] We show next that in case A is rigid, the structure extends to a full Frobenius algebra structure on A. Therefore, we use the following definition of a Frobenius algebra:

Definition 3.3. [Lur17, Def.4.6.5.1] Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $A \in \mathsf{Alg}(\mathcal{C})$ an algebra object in $\mathcal{C}$ with multiplication $\mu : A \otimes A \to A$. We call a 1-morphism $\lambda : A \to \mathbb{1}_{\mathcal{C}}$ *non-degenerate* if the composite map

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\lambda} \mathbb{1}_{\mathcal{C}}$$

is the evaluation of a duality on A. We call the pair (A, λ) consisting of an algebra A and a non-degenerate morphism λ a *Frobenius algebra.*

Proposition 3.1. *Let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category and $A \in \mathsf{Alg}(\mathbb{D})$ a rigid algebra object in $\mathbb{D}$. The right adjoint η^R of the unit 1-morphism $\eta : \mathbb{1}_{\mathbb{D}} \to \mathcal{A}$ is non-degenerate and equips $\mathcal{A}$ with the structure of a Frobenius algebra.*

Proof. We can check this in the underlying homotopy 2-category $\mathsf{h}_2\mathbb{D}$ of $\mathbb{D}$. We claim that the coevaluation for the duality is given by the composite of right adjoints

$$\mathbb{1}_{\mathbb{D}} \xrightarrow{\eta} A \xrightarrow{\mu^R} A \otimes A$$

Indeed, it follows from the conditions imposed on a rigid algebra object that the diagram

$$\begin{array}{ccccccc}
A \simeq A \otimes \mathbb{1}_{\mathbb{D}} & \xrightarrow{\eta} & A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu^R} A \otimes (A \otimes A) \simeq (A \otimes A) \otimes A \xrightarrow{\mu \otimes \mathrm{id}_A} & A \otimes A & \xrightarrow{\eta^R \otimes \mathrm{id}_A} & A \\
& \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu^R} & \nearrow_{\mathrm{id}_A} & \\
& & A & \xrightarrow{\mathrm{id}_A} & A & &
\end{array}$$

commutes. But this is precisely one of the zig-zag identities of the duality. The other follows analogously. ☐

Remark 3.1. Note that in general there may exist different ways to extend an algebra to a Frobenius algebra. So a Frobenius algebra structure is extra data, whereas being (locally) rigid is a property. We return to this perspective in Section 12, when we discuss stable Grothendieck-Verdier categories.

Remark 3.2. Let us remark on how the above definition relates to the definition of rigid monoidal categories in terms of duals. Denote for a rigid algebra A in $\mathbb{D}^{\otimes}$ its dual in $\mathbb{D}$ by $A^{\vee}$. The above Frobenius algebra

[5] This relation is described in full detail in [KN24].

structure on A also exhibits A as a dual of itself. It follows that there exists a unique duality isomorphism $\mathbb{D}(-) : A^\vee \to A$ (see also Section 12). We will see in the next section that in the case of rigid monoidal categories this recovers the equivalence that associates to an object its dual.

Remark 3.3. The adjointability conditions in the definition of local rigidity also admit a third interpretation. Let $A \in \mathbb{D}^\otimes$ be an algebra object in $\mathbb{D}^\otimes$. Note that the objects $A \otimes A$ and A admit structures of $A-A$-bimodule objects in $\mathbb{D}^\otimes$. Let us analyze the left actions. These are given by

$$\triangleright_{A,A} : A \otimes (A \otimes A) \simeq (A \otimes A) \otimes A \xrightarrow{\mu \otimes id_A} A \otimes A, \tag{7}$$

and

$$\triangleright_A : A \otimes A \xrightarrow{\mu} A. \tag{8}$$

The algebra structure on A equips μ with the structure of a left module morphism with respect to these two actions. Note that by definition of the Beck–Chevalley transform Diagram (6) is vertically right adjointable if and only if the 2-morphism

$$(\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \mu^R) \Rightarrow \mu \circ \mu^R \tag{9}$$

is an equivalence. This data can be interpreted as the first instance of the structure of a left A-module morphism on μ^R with respect to the above left A-module structures. Similarly, the horizontal adjointability condition can μ^R be interpreted with the first instance of the structure of a right A-module morphism. For the study of locally rigid algebras in the symmetric monoidal $(\infty, 2)$-categories $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ and $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$ this perspective plays a central role. We make this more precise in Proposition 4.18 and Proposition 10.9.

We also record for latter use, how rigidity behaves under the inclusion of symmetric monoidal sub-$(\infty, 2)$-categories.

Definition 3.4. Let $F : \mathbb{C} \to \mathbb{D}$ be a functor of $(\infty, 2)$-categories. F is called a locally full inclusion, if

- the induced functor $\mathbb{C}(c, c') \to \mathbb{D}(F(c), F(c'))$ is fully faithful for all $c, c' \in \mathbb{C}$
- the induced functor $F^\simeq : \mathbb{C}^\simeq \to \mathbb{D}^\simeq$ on underlying spaces is a monomorphism

We then say that $\mathbb{C}$ is a locally full sub-$(\infty, 2)$-category of $\mathbb{D}$.

Proposition 3.2. *Let $\mathbb{C}^\otimes$ be a symmetric monoidal $(\infty, 2)$-category and $\mathbb{C}_0^\otimes$ locally full (see Section B). An algebra $A \in \mathbb{C}_0$ is locally rigid in $\mathbb{C}_0$ if and only if:*

(1) A is a rigid algebra in $\mathbb{C}^\otimes$,

(2) the evaluation and coevaluation of the duality on A lie in $\mathbb{C}_0$,

(3) the right adjoint of the multiplication $\mu^R : A \to A \otimes A$ lies in $\mathbb{C}_0$.

Proof. This follows from a simple unraveling of definitions. □

4 Presentable ∞-Categories

Let $\mathcal{V}^\otimes$ be a presentably symmetric monoidal ∞-category. In this section, we start to perform the first step of our strategy to understand rigid convolution structures. To this end, we relate (locally) rigid algebras in $(\infty, 2)$-categories of $\mathcal{V}$-linear presentable ∞-categories to the classical notion of rigidity, and study their relation to fully extended TFTs. We work in this generality since this context simultaneously describes 1-categorical rigidity as explained in Section 2 as well as rigidity for ∞-categories. In particular, we generalize results from the case of presentable dg-categories [Gai15, BZGN19] and $\mathbb{K}$-linear presentable 1-categories [BZBJ18, BJS21] to the case of $\mathcal{V}$-linear presentable ∞-category for an arbitrary presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$. Therefore, we first recall in Section 4.1 basic properties of ∞-categories with a specified set of colimits. In particular, we consider the ∞-category $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ of presentable ∞-categories linear over a presentably symmetric monoidal ∞-category $\mathcal{V}$. These ∞-categories also admit enhancements to $(\infty, 2)$-categories, that we explicitly construct in Appendix B following ideas from [GR19, Sect.I.1]. Here, we only recall the main results.

In Section 4.2, we recall the notion of atomic objects and atomically generated presentable $\mathcal{V}$-linear ∞-category [BM24, BMS24, RZ25] and study basic properties of these. Atomic objects form a generalization of the notion of a compact (resp. compact-projective) object from the theory of presentable dg-categories (resp. presentable linear 1-categories). In Section 4.3, we study (locally) rigid algebras in the symmetric monoidal $(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\otimes}_{\mathcal{V}}$. Our main goal is to relate for atomically generated $\mathcal{V}$-linear ∞-categories the abstract notion of local rigidity to the more familiar notion of rigidity in terms of duals for objects. In particular, we show that an atomically generated ∞-category is locally rigid if and only if all atomic objects are dualizable. This recovers the results of [Gai15] for compactly generated presentable dg-categories. Similar considerations and results have also been obtained in [Ram24b] with a view towards applications in algebraic K-theory.

In the final Section 4.4, we connect the definition of a locally rigid algebra to the theory of fully extended TFTs. More precisely, we show after recalling the main aspects of Morita $(\infty, 2)$-categories, that locally rigid presentable ∞-categories form fully dualizable objects in the symmetric monoidal Morita $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$ of $\mathcal{V}$-linear presentable ∞-categories. For completeness, we also recall in Appendix C the full construction of the Morita $(\infty, 2)$-category following [Hau23]. Finally, we provide in Section 4.5 a conjectural characterization of rigid algebras in terms of relative fully extended TFTs.

4.1 Basic Definition

Throughout this section, $\mathcal{K}$ denotes a set of small simplicial sets. The goal of this section is to recall the main aspects of the theory of ∞-categories that admit colimits indexed by the elements of $\mathcal{K}$. All constructions presented here, work completely analogously in the case of 1-categories. Our main source for the material presented here is [Lur17, Sect.4.8.1].

We call an ∞-category that admits $\mathcal{K}$-indexed colimits $\mathcal{K}$-*cocomplete* and a functor preserving $\mathcal{K}$-indexed colimits $\mathcal{K}$-*cocontinuous*. We denote by $\mathsf{Cat}^{\mathcal{K}}$ the corresponding ∞-category of small $\mathcal{K}$-cocomplete ∞-categories and $\mathcal{K}$-cocontinuous functors. For $\mathcal{K}$-cocomplete ∞-categories $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ we denote by $\mathsf{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ spanned by $\mathcal{K}$-cocontinuous functors, and by $\mathsf{Fun}^{\mathcal{K},\mathcal{K}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ the full subcategory of functors that are $\mathcal{K}$-cocontinuous in each argument separately.

The ∞-category $\mathsf{Cat}^{\mathcal{K}}$ can further be equipped with a symmetric monoidal structure called the Deligne-Lurie tensor product $\otimes$ [Lur17, Sect.4.8.1]. This comes equipped with a universal functor $\gamma : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ that is $\mathcal{K}$-cocontinuous in each argument separately, i.e. it satisfies the universal property that precomposition with γ induces an equivalence of ∞-categories

$$\mathsf{Fun}^{\mathcal{K},\mathcal{K}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}^{\mathcal{K}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}).$$

Analogously, one can construct a symmetric monoidal structure on the very-large ∞-category $\mathsf{CAT}^{\mathcal{K}}$ of large $\mathcal{K}$-cocomplete ∞-categories. A special example is given by the ∞-category $\mathsf{CAT}^{\mathsf{colim}}$ of large ∞-categories that admit all small colimits. We are especially interested in its full subcategory Pr^{L} generated by

presentable ∞-categories. This full subcategory inherits a symmetric monoidal structure from $\mathsf{CAT}^{\mathsf{colim},\otimes}$ [Lur17, Prop.4.8.1.15].
We can further relate the symmetric monoidal categories $\mathsf{Cat}^{\mathcal{K},\otimes}$ and $\mathsf{Cat}^{\mathcal{K}',\otimes}$ for nested $\mathcal{K} \subset \mathcal{K}'$. Indeed, for every collection of small ∞-categories $\mathcal{K} \subset \mathcal{K}'$ there exists a symmetric monoidal functor $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(-) : \mathsf{Cat}^{\mathcal{K},\otimes} \to \mathsf{Cat}^{\mathcal{K}',\otimes}$ that associates to a $\mathcal{K}$-cocomplete ∞-category its $\mathcal{K}'$-cocompletion relative to $\mathcal{K}$ [Lur09a, Sect.5.3.6]. For later use, we collect some examples

Example 4.1. Let ind be the collection of all small filtered ∞-categories [Lur09a, Def.5.3.1.7]. We denote the functor $\mathcal{P}^{\mathsf{ind}}(-)$ by $\mathrm{Ind}(-)$ and call it the *Ind-completion*.

Example 4.2. Let Σ be the set of all small sifted ∞-categories [Lur09a, Def.5.5.8.1]. We call the functor $\mathcal{P}^{\Sigma}(-)$ the *sifted completion*. Analogously, we denote by Σ_1 the set of all small sifted 1-categories and call the functor $\mathcal{P}^{\Sigma_1} := \mathcal{P}^{\Sigma}_1(-)$ the 1-categorical sifted completion.

Example 4.3. Let fin be the set of all small finite ∞-categories and the walking idempotent [Lur09a, Sect.4.4.5] and $\amalg$ consist of all discrete ∞-categories and the walking idempotent. We call the functor $\mathcal{P}^{\mathrm{fin}}_{\amalg}(-)$ the *finite cocompletion*. Analogously, we denote by $\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(-)$ the 1-categorical finite cocompletion.

We call a (commutative) algebra object in the symmetric monoidal ∞-category $\mathsf{Cat}^{\mathcal{K},\otimes}$ a $\mathcal{K}$-cocomplete (symmetric) monoidal ∞-category. Unraveling definitions, such an object consists of a (symmetric) monoidal ∞-category $\mathcal{C}^{\otimes}$, s.t. the underlying ∞-category $\mathcal{C}$ is $\mathcal{K}$-cocomplete and the monoidal product functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves $\mathcal{K}$-indexed colimits separately in each argument. In the following, we will mainly be interested in modules in $\mathsf{Cat}^{\mathcal{K},\otimes}$. We denote for every $\mathcal{K}$-cocomplete monoidal ∞-category $\mathcal{V}$ by $\mathsf{Cat}^{\mathcal{K}}_{\mathcal{V}}$ the ∞-category $\mathsf{RMod}_{\mathcal{V}}(\mathsf{Cat}^{\mathcal{K}})$ of right $\mathcal{V}$-modules. We call a right $\mathcal{V}$-modules $\mathcal{C}$, a $\mathcal{K}$-cocomplete $\mathcal{V}$-linear ∞-category. In case that $\mathcal{V}$ is symmetric monoidal, the ∞-category $\mathsf{Cat}^{\mathcal{K}}_{\mathcal{V}}$ admits a symmetric monoidal structure given by the relative tensor product $- \otimes_{\mathcal{V}} -$ [Lur17, Thm.4.5.2.1].
Interesting examples of ∞-categories of modules arise from the theory of idempotent algebras. A $\mathcal{K}$-cocomplete symmetric monoidal ∞-category $(\mathcal{V}, \mu) \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{K}})$ is called an *idempotent algebra* if the monoidal product functor

$$\mu : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

is an equivalence [Lur17, Def.4.8.2.8]. The main benefit of idempotent algebras is, that being a module over it is not a structure but a property. More formally the functor $L := - \otimes \mathcal{V} : \mathsf{Cat}^{\mathcal{K},\otimes} \to \mathsf{Cat}^{\mathcal{K},\otimes}$ is a symmetric monoidal localization functor [Lur17, Prop.4.8.2.10]. It follows that the symmetric monoidal forgetful functor $\mathsf{Cat}^{\mathcal{K},\otimes}_{\mathcal{V}} \to \mathsf{Cat}^{\mathcal{K},\otimes}$ identifies the symmetric monoidal ∞-category of $\mathcal{V}$-modules with the essential image of the localization functor L. Most known examples of idempotent algebras exist in Pr^{L} [CSY21, Sect.5]. Let us consider some examples that are relevant for the next sections:

Example 4.4. The ∞-category $\mathcal{S}$ of spaces is an idempotent algebra since $\mathcal{S}$ is the monoidal unit in $\mathsf{Pr}^{\mathsf{L},\otimes}$. It follows that there exists an equivalence $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{S}} \simeq \mathsf{Pr}^{\mathsf{L},\otimes}$. More generally the ∞-category $\mathcal{S}_{\leq n-1}$ of $(n-1)$-truncated spaces[6] forms an idempotent algebra in $\mathsf{Pr}^{\mathsf{L},\otimes}$. The ∞-category $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{S}_{\leq n-1}}$ is equivalent to the $(n+1,1)$-category $\mathsf{Pr}^{sL}_{(n,1)}$ of presentable $(n,1)$-categories [Lur17, Prop.4.8.2.15].

Example 4.5. The symmetric monoidal ∞-category of spectra $\mathcal{S}\mathrm{p}^{\otimes}$ forms an idempotent algebra in $\mathsf{Pr}^{\mathsf{L},\otimes}$ [Lur17, Prop.4.8.2.18]. The corresponding forgetful functor identifies the ∞-category $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{S}\mathrm{p}}$ with the full

[6] All homotopy groups above level $(n-1)$-vanish.

subcategory $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ of Pr^{L} of presentable stable ∞-categories.
Also the full symmetric monoidal subcategory $\mathcal{S}\mathrm{p}^{\otimes}_{\geq 0}$ of $\mathcal{S}\mathrm{p}$ generated by connective spectra forms an idempotent algebra [Lur18, Cor.C.4.1.4]. In this case, the forgetful functor identifies the ∞-category $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{S}\mathrm{p}_{\geq 0}}$ with the full subcategory $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{add}}$ of Pr^{L} generated by additive presentable ∞-categories [Lur18, Def.C.1.5.1].

Proposition 4.1. *[Lur17, Thm.7.1.3.1] Let $\mathcal{K}$ be a collection of small ∞-categories, $\mathcal{V}^{\otimes}$ a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category, and $\mathcal{W}^{\otimes}$ a $\mathcal{V}$-linear $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. There exists an equivalence of symmetric monoidal ∞-categories*

$$\mathsf{RMod}_{\mathcal{W}}(\mathsf{Cat}^{\mathcal{K}}_{\mathcal{V}})^{\otimes} \simeq \mathsf{Cat}^{\mathcal{K},\otimes}_{\mathcal{W}}.$$

Proof. This is a consequence of [Lur17, Thm.7.1.3.1] and [Lur17, Lem.4.8.4.2]. □

We record some useful consequences of the above proposition:

Example 4.6. It is easy to see that the tensor product of two idempotent algebras is again an idempotent algebra. We can use this to construct new examples of idempotent algebras. For example, the presentable ∞-category $\mathcal{S}\mathrm{p}_{\geq 0} \otimes \mathsf{Set}$ is equivalent to the category small abelian groups $\tau_{\leq 0}\,\mathcal{S}\mathrm{p}_{\geq 0} \simeq \mathsf{Ab}$ [Lur17, Ex.1.3.3.5]. Hence, the category of abelian groups is an idempotent algebra. It follows that we can identify the functor $\mathsf{Ab} \otimes -$ with the composite

$$\mathsf{Pr}^{\mathsf{L}} \xrightarrow{\mathcal{S}\mathrm{p}_{\leq 0}\otimes -} \mathsf{Pr}^{\mathsf{L}} \xrightarrow{\mathsf{Set}\otimes -} \mathsf{Pr}^{\mathsf{L}}.$$

Using Prop 4.1, we can identify its essential image with the $(2,1)$-category $\mathsf{pr}^{\mathsf{L}}_{\mathsf{add}}$ of additive presentable 1-category [BJS21, Def.2.7].

Example 4.7. Let $\mathcal{V}^{\otimes}$ be a symmetric monoidal presentable $(n,1)$-category. It follows from Example 4.4 that we can view $\mathcal{V}^{\otimes}$ as a commutative algebra in $\mathsf{Pr}^{\mathsf{L},\otimes}_{(n,1)} \simeq \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{S}_{\leq n-1}}$. In particular, we obtain an equivalence $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}} \simeq \mathsf{RMod}_{\mathcal{V}}(\mathsf{Pr}^{\mathsf{L}}_{(n,1)})^{\otimes}$, so that every $\mathcal{V}$-module is automatically a presentable $(n,1)$-category.

Example 4.8. For the study of finite tensor categories in Section 5.1, we are especially interested in the case of $\mathcal{V}^{\otimes} \simeq \mathsf{Ab}^{\otimes}$. It follows from Example 4.6 that $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Ab}}$ is equivalent to the $(2,1)$-category $\mathsf{pr}^{\mathsf{L}}_{\mathsf{add}}$ of presentable additive 1-categories. More generally, let R be a commutative ring and denote by $\mathsf{rmod}^{\otimes}_{R}$ the symmetric monoidal category of right R-modules. The ∞-category $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{rmod}_R}$ is equivalent to $\mathsf{RMod}_{\mathsf{rmod}_R}(\mathsf{Pr}^{\mathsf{L}}_{1}) \simeq \mathsf{pr}^{\mathsf{L}}_{R}$ the $(2,1)$-category of presentable R-linear 1-categories [BJS21, Def.2.7].

Example 4.9. For R an $\mathbb{E}_{\infty}$-algebra in $\mathcal{S}\mathrm{p}$, we denote by RMod_{R} the ∞-category of right R-module spectra and by $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st},R}$ the ∞-category $\mathsf{RMod}_{\mathsf{RMod}_R}(\mathsf{Pr}_{\mathrm{st}})$ of R-linear stable ∞-categories. These are of major importance for the study of derived multi-fusion categories in Subsection 5.3 .

The symmetric monoidal ∞-category $\mathsf{Cat}^{\mathcal{K},\otimes}_{\mathcal{V}}$ introduced above admits a natural enhancement to a symmetric monoidal $(\infty,2)$-category with $\mathcal{V}$-linear natural transformations as 2-morphisms. Further, all the results stated above admit a natural generalization to the level of symmetric monoidal $(\infty,2)$-categories. A construction of this is already given in [GR19, Sect.I.1.8.3] and [HSS17, Sect.4.4]. Due to their fundamental importance for this text, we have included a more detailed construction of these $(\infty,2)$-categories using complete double ∞-categories in Appendix B. Since these are technical and lengthy and we, at this point, only need their formal properties here, let us just summarize the main results:

Proposition 4.2 (Proposition B.16)**.** *Let $\mathcal{K}$ be a collection of small ∞-categories and $\mathcal{V}^{\otimes}$ a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. There exists a symmetric monoidal $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}_{\mathcal{V}}$ s.t.*

(1) its underlying symmetric monoidal ∞-category coincides with $\mathsf{Cat}_{\mathcal{V}}^{\mathcal{K},\otimes}$

(2) its 2-morphisms are given by $\mathcal{V}$-linear natural transformations.

We call $\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K},\otimes}$ the symmetric monoidal $(\infty,2)$-category of $\mathcal{V}$-linear $\mathcal{K}$-cocomplete ∞-categories.

Remark 4.1. Analogous results hold for large ∞-categories. In particular, we obtain for every presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ a symmetric monoidal $(\infty,2)$-category $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ of $\mathcal{V}$-linear presentable ∞-categories.

Proposition 4.3 (Proposition B.15)**.** *Let $\mathcal{K} \subset \mathcal{K}'$ be collections of small ∞-categories and $\mathcal{V}^{\otimes}$ a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. The $\mathcal{K}'$-cocompletion $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}$ extends to a symmetric monoidal $(\infty,2)$-functor $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}',\otimes} : \mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}\otimes} \to \mathbb{C}\mathsf{at}_{\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{V})}^{\mathcal{K}',\otimes}$*

Proposition 4.4 (Proposition B.16)**.** *Let $F : \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a $\mathcal{K}$-cocomplete symmetric monoidal functor between $\mathcal{K}$-cocomplete symmetric monoidal ∞-categories. The symmetric monoidal relative tensor product functor $- \otimes_{\mathcal{V}} \mathcal{W} : \mathsf{Cat}_{\mathcal{V}}^{\mathcal{K}} \to \mathsf{Cat}_{\mathcal{W}}^{\mathcal{K}}$ extends to a symmetric monoidal $(\infty,2)$-functor $- \otimes_{\mathcal{V}} \mathcal{W} : \mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K},\otimes} \to \mathbb{C}\mathsf{at}_{\mathcal{W}}^{\mathcal{K},\otimes}$*

Let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty,2)$-category with underlying symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$. We show in Appendix B.1 that for every symmetric monoidal subcategory $\mathcal{D}_0^{\otimes}$ of $\mathcal{D}^{\otimes}$, there exists a locally full symmetric monoidal sub-$(\infty,2)$-category $\mathbb{D}_0^{\otimes}$ of $\mathbb{D}^{\otimes}$ with underlying symmetric monoidal ∞-category $\mathcal{D}_0^{\otimes}$.

Example 4.10. $\mathsf{Pr}_{\mathrm{st}}^{\mathsf{L}\otimes}$ forms a symmetric monoidal subcategory of $\mathsf{Pr}^{\mathsf{L},\otimes}$ and we denote the associated locally full symmetric monoidal sub-$(\infty,2)$-category of $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}$ by $\mathbb{P}\mathsf{r}_{\mathrm{st}}^{\mathsf{L},\otimes}$.

4.2 Atomic Generation

Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. In this section, we introduce the basic notions for working with $\mathcal{V}$-linear ∞-categories. Many of the results presented in this section have already been proven by different authors [BMS24, BM24, RZ25] and we do not claim originality. The reason we are interested in $\mathcal{V}$-linear presentable ∞-categories is that we can think of these as being enriched over $\mathcal{V}$ without invoking the formalism of enriched ∞-category theory [GH15, RZ25]. Instead, we use the following construction of an internal Hom-functor:

Construction 4.1. Let $\mathcal{M} \in \mathsf{Pr}_{\mathcal{V}}^{\mathsf{L}}$ be a $\mathcal{V}$-linear presentable ∞-category. For every $m \in \mathcal{M}$ the colimit preserving functor

$$m \otimes - : \mathcal{V} \to \mathcal{M}$$

admits, by the adjoint functor theorem a right adjoint $\hom_{\mathcal{M}}^{\mathcal{V}}(m,-) : \mathcal{M} \to \mathcal{V}$ called the *internal Hom-functor*. As the right adjoint of a $\mathcal{V}$-linear functor, the internal Hom-functor is itself lax $\mathcal{V}$-linear [Lur17, Cor.7.3.2.7]. It follows from [Lur17, Rem.7.3.2.9] that this lax $\mathcal{V}$-linear structure is strict if and only if for every $v \in \mathcal{V}$ the morphism

$$\alpha_{m,v} : \hom_{\mathcal{M}}^{\mathcal{V}}(m,-) \otimes v \to \hom_{\mathcal{M}}^{\mathcal{V}}(m, - \otimes v)$$

adjoint to the counit

$$(m \otimes \hom_{\mathcal{M}}^{\mathcal{V}}(m,-)) \otimes v \to - \otimes v$$

is an equivalence. We say $m \in \mathcal{M}$ *preserves tensoring by objects* in $\mathcal{V}$ if for every $v \in \mathcal{V}$ the morphism $\alpha_{m,v}$ is an equivalence.

Remark 4.2. Let $\mathcal{M} \in \mathsf{Pr}_{\mathcal{V}}^{\mathsf{L}}$ be a $\mathcal{V}$-linear presentable ∞-category and $m \in \mathcal{M}$ an object of $\mathcal{M}$. In case that the presentably monoidal ∞-category $\mathcal{V}^{\otimes}$ is clear from the context, we will abuse notation and denote the internal Hom-functor for the $\mathcal{V}$-action by $\hom_{\mathcal{M}}(m,-)$.

Remark 4.3. The idea to use a homotopy coherent version of the internal Hom-functor to relate $\mathcal{V}$-linear presentable ∞-categories to $\mathcal{V}$-linear enriched ∞-categories has been made precise in [Hei23, MGS21, RZ25]. We will recall some aspects of this equivalence in Section 4.3.

Important concepts in the study of presentable stable ∞-categories are compact objects and compactly generated ∞-categories [Lur09a, Sect.5.5.7]. These notions allow one to reduce many statements about a large presentable ∞-category to its small subcategory of compact objects. An analogous role is played by compact-projective objects for additive presentable 1-categories [Lur18, Sect.C.1.5]. The correct $\mathcal{V}$-linear generalization is the notion of an atomic object:

Definition 4.1. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. An object $m \in \mathcal{M}$ is called $\mathcal{V}$-*atomic*, if the internal Hom-functor $\hom_{\mathcal{M}}(m, -)$ preserves colimits and tensoring by objects $v \in \mathcal{V}$. We denote the subcategory of $\mathcal{V}$-atomic objects by $\mathcal{M}^{\mathrm{atm}}$.

Definition 4.2. A full subcategory $\mathcal{M}_0 \subset \mathcal{M}$ is called $\mathcal{V}$-*generating* if the smallest full subcategory of $\mathcal{M}$ containing $\mathcal{M}_0$ that is closed under colimits and tensoring with $\mathcal{V}$ is $\mathcal{M}$. We call $\mathcal{M}$ $\mathcal{V}$-*atomically generated* if $\mathcal{M}^{\mathrm{atm}}$ is $\mathcal{V}$-generating. In particular, we call an object $\Omega \in \mathcal{M}$ a $\mathcal{V}$-*atomic generator*, if the full subcategory generated by Ω is $\mathcal{V}$-atomically generating.

Remark 4.4. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category and $\mathcal{M}$ a $\mathcal{V}$-linear ∞-category. When $\mathcal{V}^{\otimes}$ is clear from the context, we abuse notation and call a $\mathcal{V}$-atomic object of $\mathcal{M}$ atomic. Similarly, we call $\mathcal{M}$ atomically generated instead of $\mathcal{V}$-atomically generated.

Proposition 4.5. *Let $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ be a $\mathcal{V}$-linear presentable ∞-category. Then the full subcategory $\mathcal{M}^{\mathrm{atm}} \subset \mathcal{M}$ is small.*

Proof. We will show that there exists a regular cardinal κ, s.t. every atomic object is κ-compact. Therefore, note that for every atomic object $m \in \mathcal{M}^{\mathrm{atm}}$, we obtain an equivalence

$$\mathsf{Map}_{\mathcal{M}}(m, -) \simeq \mathsf{Map}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \hom_{\mathcal{M}}(m, -)).$$

Since m is atomic, it therefore suffices to show that $\mathbb{1}_{\mathcal{V}}$ is κ-compact for some regular cardinal κ. Since $\mathcal{V}$ is presentable, there exist regular cardinals λ, μ, a μ-small category I, and a λ-filtered diagram $F : I \to \mathcal{V}$ consisting of λ-compact objects, s.t. $\mathbb{1}_{\mathcal{V}} \simeq \mathsf{colim}_{i \in I} F(i)$. Setting $\kappa = \max\{\mu, \lambda\}$, it follows that $\mathbb{1}_{\mathcal{V}}$ is a κ-small colimit of κ-compact objects and hence itself κ-compact. It follows that $\mathcal{M}^{\mathrm{atm}} \subset \mathcal{M}^{\kappa}$, which is small since $\mathcal{M}$ is presentable. ☐

The classification of atomic objects is particularly simple in case that the presentably monoidal ∞-category $\mathcal{V}^{\otimes}$ is an idempotent algebra in Pr^{L}. As explained in the last section, in this case, the forgetful functor

$$\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}} \to \mathsf{Pr}^{\mathsf{L}}$$

is fully faithful. Hence, any functor that preserves colimits also preserves $\mathcal{V}$-tensorings. Consequently, an object $m \in \mathcal{M}$ is atomic if and only if the enriched Hom-functor $\hom_{\mathcal{M}}(m, -)$ preserves colimits.

Example 4.11. For $\mathcal{M} \in \mathsf{Pr}_{\mathcal{S}}$ the action of a space $X \in \mathcal{S}$ on an object $m \in \mathcal{M}$ is explicitly given by

$$m \otimes X \simeq \mathsf{colim}_X\, m$$

the colimit over the constant X-diagram with value m [Lur09a, Sect.4.4.4]. The internal Hom is given by the mapping space of the ∞-category $\mathcal{M}$ and an object $m \in \mathcal{M}$ is $\mathcal{S}$-atomic if and only if the functor

$\mathsf{Map}(m,-) : \mathcal{M} \to \mathcal{S}$ preserves all colimits.[7] In particular, $\mathcal{M}$ is $\mathcal{S}$-atomically generated if and only if there exists a small ∞-category $\mathcal{C}$ and an equivalence $\mathcal{P}(\mathcal{C}) \simeq \mathcal{M}$ [Lur09a, Cor.5.1.6.11]

In Section 5, we will discuss the atomic objects for the idempotent algebras $\mathcal{V}^{\otimes} \simeq \mathsf{Ab}^{\otimes}$ and $\mathcal{V}^{\otimes} \simeq \mathcal{S}\mathrm{p}^{\otimes}$. But in general the examples, we are interested in are not linear over an idempotent algebra. Indeed, for our applications to tensor categories, we are interested in examples of categories that are linear over a commutative ring R. To understand the atomic objects in these examples, we have to understand how the atomic objects behave under changing $\mathcal{V}^{\otimes}$. Therefore, recall that every monoidal cocontinous functor $F : \mathcal{W}^{\otimes} \to \mathcal{V}^{\otimes}$ in $\mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$ induces an adjunction on ∞-categories of modules:

$$F_! : \mathsf{Pr}^{\mathsf{L}}_{\mathcal{W}} \longleftrightarrow \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}} : F^*, \tag{10}$$

where $F_!(\mathcal{M}) \simeq \mathcal{M} \otimes_{\mathcal{W}} \mathcal{V}$ is given by extension of scalars and $F^*(\mathcal{N})$ by restriction of scalars along F. In particular, note that $F_!(\mathcal{W}) \simeq \mathcal{V}$ and that the unit of the adjunction induces a morphism $F_{\mathcal{W}} : \mathcal{W} \to F^*\mathcal{V}$ in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{W}}$.

Proposition 4.6. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and $A \in \mathsf{CAlg}(\mathcal{V})$ a commutative algebra in $\mathcal{V}$. We denote by*

$$F : \mathcal{V} \rightleftharpoons \mathsf{RMod}_A(\mathcal{V}) : U$$

the corresponding free forgetful adjunction [Lur17, Prop.4.2.4.8]. Let $\mathcal{M} \in \mathsf{Pr}_{\mathsf{RMod}_A(\mathcal{V})}$ be $\mathsf{RMod}_A(\mathcal{V})$-linear. Then an object $m \in \mathcal{M}$ is $\mathsf{RMod}_A(\mathcal{V})$-atomic if and only if it is $\mathcal{V}$-atomic in $F^*\mathcal{M}$. Further, $\mathcal{M}$ is $\mathsf{RMod}_A(\mathcal{V})$-atomically generated if and only if $F^*\mathcal{M}$ is $\mathcal{V}$-atomically generated.

Proof. Note that by definition the $\mathcal{V}$-action on an object $m \in F^*\mathcal{M}$ is given by the composite

$$\mathcal{V} \xrightarrow{F} \mathcal{W} \xrightarrow{m\otimes -} \mathcal{M}$$

Passing to right adjoints, we obtain an equivalence

$$\hom_{F^*\mathcal{M}}(m,-) \simeq U \circ \hom_{\mathcal{M}}(m,-). \tag{11}$$

Since $\mathcal{V}$ is presentably monoidal it follows from [Lur17, Cor.4.2.3.5] that the functor U preserves and creates colimits. Hence, we can conclude from (11) that for every $m \in \mathcal{M}$ the functor $\hom_{F^*\mathcal{M}}(m,-)$ preserves colimits if and only if the functor $\hom_{\mathcal{M}}(m,-)$ preserves colimits. Further note, that for every $m \in \mathcal{M}$ and $v \in \mathcal{V}$, we have a chain of equivalences:

$$\begin{aligned}\hom_{F^*\mathcal{M}}(m,-) \otimes v &\simeq U(\hom_{\mathcal{M}}(m,-)) \otimes v \\ &\simeq U(\hom_{\mathcal{M}}(m,-) \otimes_A F(v)).\end{aligned}$$

It follows from this chain that for every $m \in \mathcal{M}$ the functor $\hom_{F^*\mathcal{M}}(m,-)$ preserves $\mathcal{V}$-tensors, if and only if the functor $\hom_{\mathcal{M}}(m,-)$ preserves $\mathsf{RMod}_A(\mathcal{V})$-tensors. This implies the statement about atomic objects.
It remains to show the claim about atomic generation. By construction, it follows that when $F^*\mathcal{M}$ is $\mathcal{V}$-atomically generated, also $\mathcal{M}$ has to be $\mathsf{RMod}_A(\mathcal{V})$-atomically generated. On the other hand, the opposite direction follows from the observation that the ∞-category $\mathsf{RMod}_A(\mathcal{V})$ is generated under small colimits by the essential image of F, which follows analogously to [Lur17, Cor.7.1.4.14] □

[7] These are called completely compact in [Lur09a, Def.5.1.6.2]

Example 4.12. Every presentably monoidal ∞-category $\mathcal{V}$ is atomically generated over itself. Indeed, since we have for all $v, w \in \mathcal{V}$

$$\mathsf{Map}_{\mathcal{V}}(v, \hom_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, w)) \simeq \mathsf{Map}_{\mathcal{V}}(v \otimes \mathbb{1}_{\mathcal{V}}, w) \simeq \mathsf{Map}_{\mathcal{V}}(v, w), \tag{12}$$

it follows that $\hom_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -) \simeq \mathrm{id}_{\mathcal{V}}$. Consequently, $\mathbb{1}_{\mathcal{V}}$ is atomic and hence an atomic generator for $\mathcal{V}$.

For the study of locally rigid algebras in Section 4.3, we need to understand adjoints internal to the $(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$. An explicit description of these is given as follows:

Proposition 4.7. *[BM24, Sect.4.2] Let $\mathcal{V}$ be a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. A 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathbb{C}\mathsf{at}^{\mathcal{K}}_{\mathcal{V}}$ admits an internal right adjoint if and only if F admits a right adjoint F^R, s.t. F^R is $\mathcal{V}$-linear and $\mathcal{K}$-cocontinuous. We call F an internally left adjoint functor.*

Example 4.13. Let $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ be a $\mathcal{V}$-linear presentable ∞-category. An object $m \in \mathcal{M}$ is atomic, if and only if the unique cocontinuous $\mathcal{V}$-linear functor $\mathcal{V} \to \mathcal{M}$ that maps $\mathbb{1}_{\mathcal{V}}$ to m is an internally left adjoint functor in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$.

Thinking about $\mathcal{V}$-linear presentable categories as enriched ∞-categories via the internal Hom-functor, the following results show that we can think of internally left-adjoint functors in terms of enriched adjoints:

Lemma 4.8. *Let $\mathcal{N} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ be $\mathcal{V}$-linear. Then a morphism $f : n \to n'$ is an equivalence in $\mathcal{N}$ if and only if for every $n_0 \in \mathcal{N}$ the morphism*

$$\hom_{\mathcal{N}}(n_0, f) : \hom_{\mathcal{N}}(n_0, n) \to \hom_{\mathcal{N}}(n_0, n')$$

is an equivalence in $\mathcal{V}$.

Proof. The only if direction is obvious. For the converse direction, note we have an adjunction equivalence:

$$\mathsf{Map}_{\mathcal{N}}(-, -) \simeq \mathsf{Map}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \hom_{\mathcal{N}}(-, -)).$$

The claim follows from the Yoneda lemma. □

Proposition 4.9. *Let $F : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$. Then, there exists an equivalence*

$$\hom_{\mathcal{N}}(F(m), n) \simeq \hom_{\mathcal{M}}(m, F^R(n))$$

naturally in $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Further, after applying the functor $\mathsf{Map}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -)$ this equivalence recovers the underlying adjunction.

Proof. Let $v \in \mathcal{V}$ be an object. The first claim follows from the chain of adjunctions

$$\begin{aligned} \mathsf{Map}_{\mathcal{V}}(v, \hom_{\mathcal{N}}(F(m), n)) &\simeq \mathsf{Map}_{\mathcal{V}}(F(m) \otimes v, n) \\ &\simeq \mathsf{Map}_{\mathcal{V}}(F(m \otimes v), n) \\ &\simeq \mathsf{Map}_{\mathcal{V}}(m \otimes v, F^R(n)) \\ &\simeq \mathsf{Map}_{\mathcal{V}}(v, \hom_{\mathcal{M}}(m, F^R(n))) \end{aligned}$$

together with the Yoneda lemma. The second claim follows from unraveling the above equivalence for $v \simeq \mathbb{1}_{\mathcal{V}}$ □

It is well known, that a functor between compactly generated presentable stable ∞-categories preserves compact objects if and only if it admits a cocontinuous right adjoint [BZN09, Lem.3.5]. This gives a useful criterion to decide, whether a functor F in $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ admits an internal left adjoint. The following result can be seen as a generalization to arbitrary presentably monoidal $\mathcal{V}$:

Proposition 4.10. *Let $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ be atomically generated and $F : \mathcal{M} \to \mathcal{N}$ a cocontinuous $\mathcal{V}$-linear functor. Then F preserves atomic objects (i.e $F(\mathcal{M}^{\mathrm{atm}}) \subset \mathcal{N}^{\mathrm{atm}}$), if and only if it is internally left adjoint in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$.*

Proof. First, assume that F is internally left adjoint and let $m \in \mathcal{M}^{\mathrm{atm}}$ be atomic. It follows from Proposition 4.9 that the right adjoint of $F(m) \otimes -$ is given by $\hom_{\mathcal{M}}(m, F^R(-))$. Since F^R is cocontinuous and $\mathcal{V}$-linear and m is atomic, the functor $\hom_{\mathcal{M}}(m, F^R(-))$ is a composite of cocontinuous $\mathcal{V}$-linear functors. Hence $F^R(m)$ is atomic as well.
For the converse denote by $\mathcal{M}_0 \subset \mathcal{M}$ the subcategory generated by those objects $m \in \mathcal{M}$ s.t. the functor

$$\hom_{\mathcal{M}}(m, F^R(-)) : \mathcal{N} \to \mathcal{V}$$

is cocontinuous and $\mathcal{V}$-linear. Since F preserves atomic objects, it follows from Proposition 4.9 that $\mathcal{M}_0$ contains $\mathcal{M}^{\mathrm{atm}}$. But $\mathcal{M}_0$ is closed under colimits and $\mathcal{V}$-tensoring and hence $\mathcal{M}_0 \simeq \mathcal{M}$. Hence it follows from Lemma 4.8 that F is an internally left adjoint functor. □

Proposition 4.11. *Let $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ and $F^{\otimes} : \mathcal{W}^{\otimes} \to \mathcal{V}^{\otimes}$ a cocontinous monoidal functor, s.t. the associated functor $F_{\mathcal{W}} : \mathcal{W} \to F^*\mathcal{V}$ in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{W}}$ is an internal left adjoint. Then if m is $\mathcal{V}$-atomic in $\mathcal{M}$, m is also $\mathcal{W}$-atomic in $F^*\mathcal{M}$.*

Proof. Let m be an object of $\mathcal{M}$ and consider the chain of functors

$$\mathcal{W} \xrightarrow{F} \mathcal{V} \xrightarrow{m \otimes -} \mathcal{M}.$$

Passing to right adjoints it follows, that the internal Hom-functor for the induced action of $\mathcal{W}$ on $F^*\mathcal{M}$ is given by $F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, -))$. Now let $m \in \mathcal{M}$ be a $\mathcal{V}$-atomic object of $\mathcal{M}$. We need to show that $F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, -))$ is cocontinuous and commutes with the $\mathcal{V}$-action. Since F is an internal left adjoint and m is atomic, the composite functor $F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, -))$ preserves colimits. Further, compatibility with the action of $\mathcal{W}$ follows from the chain of equivalences

$$F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, - \otimes F(w))) \simeq F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, -) \otimes F(w)) \simeq F^R(\mathsf{hom}^{\mathcal{V}}_{\mathcal{M}}(m, -)) \otimes w$$

where we have used that m is atomic and $F_{\mathcal{W}}$ is an internal left adjoint. This finishes the proof. □

We finish this section with the following generalization of the famous Schwede-Shipley theorem [SS00]:

Proposition 4.12. *[RZ25, Cor.5.13, Generalized Schwede-Shipley] Let $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ and $\Omega \in \mathcal{M}$. Then Ω is an atomic generator of $\mathcal{M}$ if and only if there exists an algebra $A \in \mathsf{Alg}(\mathcal{V})$ as well as an equivalence $\mathsf{RMod}_A(\mathcal{V}) \to \mathcal{M}$ in $\mathsf{Pr}^{\mathsf{L}}_{V}$, carrying A to Ω.*

Proof. It is easy to see that for every algebra $A \in \mathcal{V}$ the category $\mathrm{LMod}_A(\mathcal{V})$ is atomically generated by A. To show the converse, we check the condition of [Lur17, Prop.4.5.8.5]. Conditions (1) to (3) are obviously satisfied. The functor $\hom_{\mathcal{M}}(\Omega, -)$ satisfies the conditions (4) and (6) since Ω is atomic. It remains to show that the functor $\hom_{\mathcal{M}}(\Omega, -)$ is conservative. By the Yoneda lemma, it suffices to show that the collection of functors $\{\mathsf{Map}(\Omega \otimes v, -)\}_{v \in \mathcal{V}}$ is jointly conservative. But this follows since Ω is a generator. □

4.3 Rigidity in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$

Throughout this section, $\mathcal{V}^{\otimes}$ denotes a presentably symmetric monoidal ∞-category. Since in this section our $\mathcal{V}$-linear ∞-categories additionally carry a monoidal structure, we adopt the notation and denote the monoidal product by $\otimes$ and the $\mathcal{V}$-action by $\triangleleft$. After we have introduced the necessary foundations to work with $\mathcal{V}$-linear presentable ∞-categories, we can now turn to study rigid algebra objects in the symmetric monoidal $(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$. By the universal property of the relative tensor product, there exists for all $\mathcal{V}$-linear ∞-categories $\mathcal{M}$ and $\mathcal{N}$ a chain of composable functors

$$\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}.$$

We denote objects in $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ that lie in the essential image of this functor by $m \otimes_{\mathcal{V}} n$.

Remark 4.5. Many of the results in this section have also been proven by Ramzi [Ram24b]. The main difference is that the text [Ram24b] only considers *commutative* locally rigid algebras and applies the results to the study of algebraic K-theory instead of fully extended TFTs.

Our first goal is to express the external definition of a locally rigid algebra object $\mathcal{M}^{\otimes} \in \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$ from Definition 3.2 in terms of the internal structure of the $\mathcal{V}$-linear presentably monoidal category $\mathcal{M}^{\otimes}$. In the case of a presentably monoidal dg-category $\mathcal{M}^{\otimes}$ this has been proven by Gaitsgory [Gai15, Appendix D]. More precisely he shows, that if $\mathcal{M}^{\otimes}$ is further compactly generated, it is locally rigid if and only if every compact object admits a dual. As we discussed in the last section, the right generalization of the notion of compact objects are atomic objects. Therefore, we first introduce the following definition:

Definition 4.3. Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be an atomically generated monoidal $\mathcal{V}$-linear ∞-category. Then $\mathcal{M}$ is called *locally atomically-rigid* if $\mathcal{M}$ is atomically generated and all atomic objects admit a left and a right dual (see Definition C.3). We call a locally atomically-rigid presentably monoidal ∞-category $\mathcal{M}^{\otimes}$ *atomically-rigid* if the unit object $\mathbb{1}_{\mathcal{M}}$ is atomic.

Remark 4.6. Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be a presentably monoidal $\mathcal{V}$-linear ∞-category and $m \in \mathcal{M}$ be right dualizable (resp. left dualizable) with right dual $m^{\vee}$ (resp. left dual ${}^{\vee}m$). Then the functor $m \otimes -$ (resp. $- \otimes m$) admits a $\mathcal{V}$-linear cocontinuous right adjoint given by $m^{\vee} \otimes -$ (resp. ${}^{\vee}m \otimes -$).

Remark 4.7. Let $\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be (locally) atomically-rigid. Denote by $\mathcal{M}^{\otimes\text{-}\mathrm{op}}$ the monoidal ∞-category $\mathcal{M}$ with the opposite monoidal structure. Then also the ∞-categories $\mathcal{M}^{\otimes\text{-}\mathrm{op}}$ and $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ are (locally) atomically-rigid.

We denote by $\mathcal{M}^{\mathrm{dbl}}$ the full subcategory of $\mathcal{M}$ generated by the dualizable objects. By definition, $\mathcal{M}$ is locally atomically-rigid if and only if $\mathcal{M}^{\mathrm{atm}} \subset \mathcal{M}^{\mathrm{dbl}}$. This inclusion is in fact an equivalence precisely if $\mathcal{M}$ is atomically-rigid:

Proposition 4.13. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be atomically-rigid and m an object of $\mathcal{M}^{\otimes}$. Then the following are equivalent:*

(1) m is atomic.

(2) m admits a left dual.

(3) m admits a right dual.

In particular, a locally atomically-rigid monoidal $\mathcal{V}$-linear ∞-category is atomically-rigid if and only if $\mathcal{M}^{\mathrm{atm}} \simeq \mathcal{M}^{\mathrm{dbl}}$.

Proof. It suffices to show that (2) implies (1). Therefore, let $m \in \mathcal{M}$ be left dualizable. We have to show that it is atomic. To do this, we show first that $\hom_{\mathcal{M}}(m, -)$ commutes with colimits. Consider a small ∞-category I and a diagram $F : I \to \mathcal{M}$. Applying Proposition 4.9 to the functor $m \otimes -$ we obtain a chain of equivalences

$$\begin{aligned}\hom_{\mathcal{M}}(m, \mathsf{colim}_{i\in I} F(i)) &\simeq \hom_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m^\vee \otimes \mathsf{colim}_{i\in I} F(i)) \\ &\simeq \mathsf{colim}_{i\in I} \hom_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m^\vee \otimes F(i)) \\ &\simeq \mathsf{colim}_{i\in I} \hom_{\mathcal{M}}(m, F(i)),\end{aligned}$$

where we have used that $\mathbb{1}_{\mathcal{M}}$ is atomic. Hence, $\hom_{\mathcal{M}}(m, -)$ commutes with colimits. Compatibility with $\mathcal{V}$-tensoring follows from the chain of equivalences

$$\begin{aligned}\hom_{\mathcal{M}}(m, - \triangleleft v) &\simeq \hom_{\mathcal{M}}(\mathbb{1}_{\mathcal{V}}, m^\vee \otimes (- \triangleleft v)) \\ &\simeq \hom_{\mathcal{M}}(\mathbb{1}_{\mathcal{V}}, (m^\vee \otimes -) \triangleleft v) \\ &\simeq \hom_{\mathcal{M}}(\mathbb{1}_{\mathcal{V}}, m^\vee \otimes -) \otimes v \\ &\simeq \hom_{\mathcal{M}}(m, -) \otimes v,\end{aligned}$$

where we have used again that $\mathbb{1}_{\mathcal{V}}$ is atomic. □

To show that atomic-rigidity is equivalent to rigidity, we first have to show that atomically generated ∞-categories are dualizable in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$. For the proof we need a bit of enriched ∞-category theory. Since we only need it at this point, we will not recall the theory of enriched ∞-categories here and instead only cite the results we need. For a complete introduction to enriched ∞-categories consider [GH15, Hei23, Hei24].
For a presentably symmetric monoidal ∞-category $\mathcal{V}$, we denote by $\mathsf{Enr}(\mathcal{V})^\otimes$ the symmetric monoidal ∞-category of small $\mathcal{V}$-enriched ∞-categories and $\mathcal{V}$-enriched functors [Hei24]. As shown in [Hei23, Thm.1.9] one can encode the datum of a $\mathcal{V}$-enriched ∞-category $\mathcal{M}$ in terms of a weak $\mathcal{V}$-tensoring on $\mathcal{M}$ [Hei23, Def.3.11]. In particular, for $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ the $\mathcal{V}$-tensoring on $\mathcal{M}$ that exists by definition, restricts to a weak $\mathcal{V}$-tensoring on the small ∞-category $\mathcal{M}^{\mathrm{atm}}$ and therefore exhibits it as $\mathcal{V}$-enriched. On the other hand, the enriched presheaf category constructions

$$\mathcal{P}_{\mathcal{V}}(-) : \mathsf{Enr}(\mathcal{V})^\otimes \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}} \tag{13}$$

gives a symmetric monoidal functor in the opposite direction [Hei24, Cor.5.5]. As the presheaf category does, the enriched presheaf category satisfies an enriched Yoneda lemma [Hin18, Prop.6.2.7]. It states that the $\mathcal{V}$-enriched functor $\mathcal{Y}^{\mathcal{M}}_{\mathcal{V}} : \mathcal{M} \to \mathcal{P}_{\mathcal{V}}(\mathcal{M})$ that associates to any $m \in \mathcal{M}$ the representable $\mathcal{V}$-enriched functor $\hom_{\mathcal{M}}(-, m) \in \mathcal{P}_{\mathcal{V}}(\mathcal{M})$ is fully faithful. This functor is called the $\mathcal{V}$-enriched Yoneda embedding. After this preliminary discussion, we can now proof the following:

Lemma 4.14. *[Ram24b, Obs.1.28] Let $\mathcal{M}$ be an atomically generated $\mathcal{V}$-linear ∞-category. There exists an equivalence of $\mathcal{V}$-linear ∞-categories $\mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}}) \simeq \mathcal{M}$.*

Proof. The $\mathcal{V}$ tensoring on $\mathcal{M}$ restricts to a weak $\mathcal{V}$ tensoring on the subcategory $\mathcal{M}^{\mathrm{atm}}$ of atomic objects. By [Hei23] the canonical functor $\mathcal{M}^{\mathrm{atm}} \to \mathcal{M}$ of weakly $\mathcal{V}$-tensored ∞-categories extends to a $\mathcal{V}$-linear cocontinuous functor $\iota : \mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}}) \to \mathcal{M}$. We claim that ι is an equivalence. It is essentially surjective, since by construction its essential image contains all atomic objects and is closed under colimits and $\mathcal{V}$-tensors.
It remains to show that the functor is fully faithful. Therefore, it suffices to show that for any $m_0, m_1 \in \mathcal{P}(\mathcal{M}^{\mathrm{atm}})$

the induced morphism in $\mathcal{V}$

$$\iota_{m_0,m_1} : \hom_{\mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}})}(m_0, m_1) \to \hom_{\mathcal{M}}(\iota(m_0), \iota(m_1))$$

is an equivalence. For any $m_0 \in \mathcal{P}(\mathcal{M}^{\mathrm{atm}})$, the collection of m_1, s.t. ι_{m_0,m_1} is an equivalence is closed under colimits and $\mathcal{V}$-tensors. Therefore, we can restrict to the case that $m_0 \in \mathcal{M}^{\mathrm{atm}}$. For fixed $m_0 \in \mathcal{M}^{\mathrm{atm}}$ the collection of m_1 for which ι_{m_0,m_1} is an equivalence is also closed under colimits and $\mathcal{V}$-tensors since m_0 is $\mathcal{V}$-atomic. Further, by the enriched Yoneda lemma, it also contains $\mathcal{M}^{\mathrm{atm}}$. Hence, since $\mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}})$ is $\mathcal{V}$-atomically generated by the essential image of the enriched Yoneda embedding, it follows that ι_{m_0,m_1} is an equivalence for all $m_0, m_1 \in \mathcal{M}$ and hence that ι is fully faithful. □

Proposition 4.15. *[Ram24a, Prop.1.40] Let $\mathcal{M}$ be an atomically generated $\mathcal{V}$-linear ∞-category. Then $\mathcal{M}$ is dualizable as an object in the symmetric monoidal ∞-category $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$*

Proof. It follows from Lemma 4.14 that $\mathcal{M}$ is equivalent to the enriched presheaf category $\mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}})$ on the small $\mathcal{V}$-enriched ∞-category of atomic objects. By [Ber20, Thm.1.7] enriched presheaf categories are dualizable □

Proposition 4.16. *Let $\mathcal{M}$ and $\mathcal{N}$ be atomically generated $\mathcal{V}$-linear ∞-categories. Then the relative tensor product $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ is atomically generated by the full subcategory generated by the objects $m \otimes_{\mathcal{V}} n$ with $m \in \mathcal{M}^{\mathrm{atm}}$ and $n \in \mathcal{N}^{\mathrm{atm}}$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be atomically generated $\mathcal{V}$-linear presentable ∞-categories. The $\mathcal{V}$-tensoring on $\mathcal{M}$ and $\mathcal{N}$ restricts to a $\mathcal{V}$-enrichment on the underlying categories of atomic objects $\mathcal{M}^{\mathrm{atm}}$ and $\mathcal{N}^{\mathrm{atm}}$. Since $\mathcal{M}$ and $\mathcal{N}$ are atomically generated, it follows from 4.14 that they are equivalent to the enriched presheaf categories $\mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}})$ and $\mathcal{P}_{\mathcal{V}}(\mathcal{N}^{\mathrm{atm}})$ respectively. Hence, the claim follows from the equivalence

$$\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N} \simeq \mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}}) \otimes_{\mathcal{V}} \mathcal{P}_{\mathcal{V}}(\mathcal{N}^{\mathrm{atm}}) \simeq \mathcal{P}_{\mathcal{V}}(\mathcal{M}^{\mathrm{atm}} \otimes_{\mathcal{V}} \mathcal{N}^{\mathrm{atm}})$$

induced by symmetric monoidality of $\mathcal{P}_{\mathcal{V}}(-)$. □

Proposition 4.17. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be a locally atomically-rigid $\mathcal{V}$-linear ∞-category and let $\mathcal{N}, \mathcal{L} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{M}}$ be $\mathcal{M}$-linear ∞-categories. Then every lax $\mathcal{M}$-linear functor $F : \mathcal{N} \to \mathcal{L}$ is $\mathcal{M}$-linear.*

Proof. We denote by $\mathcal{M}_0 \subset \mathcal{M}$ the full subcategory generated by those objects $m \in \mathcal{M}$, s.t. for every $n \in \mathcal{N}$ the morphism

$$F(n) \triangleleft m \to F(n \triangleleft m)$$

is an equivalence. Since F is $\mathcal{V}$-linear and commutes with colimits, it follows that the category $\mathcal{M}_0$ is closed under those. It hence suffices to show that every atomic object is in $\mathcal{M}_0$. But for $m \in \mathcal{M}^{\mathrm{atm}}$ an explicit inverse is given by the morphism

$$F(n \triangleleft m) \to F(n \triangleleft m) \triangleleft ({}^{\vee}m \otimes m) \to F(n \triangleleft (m \otimes {}^{\vee}m)) \triangleleft m \to F(n) \triangleleft m$$

Therefore F is $\mathcal{M}$-linear. □

Before we relate locally rigid algebras in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$ to locally-atomically rigid monoidal categories, we provide a more explicit description of local rigidity in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$. Therefore, recall from the discussion in Remark 3.3 that for every algebra object $(\mathcal{M}, \mu) \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ the monoidal product functor μ canonically is a $\mathcal{M}$-bimodule morphism. Hence, it follows from [Lur17, Ex.7.3.2.8] that its right adjoint μ^R carries a canonical structure of a lax $\mathcal{M}$-bimodule functor.

Proposition 4.18. *Let $(\mathcal{M}, \mu) \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be $\mathcal{V}$-linear presentably monoidal ∞-category. Then $\mathcal{M}$ is locally-rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$ if and only if*

(0) $\mathcal{M}$ is dualizable in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$

(1) the multiplication μ is an internal left adjoint in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$

(2) the structure of a lax $\mathcal{M}$-bimodule functor on μ^R from Remark 3.3 is strict

Further, it is rigid, if and only if the unit object $\mathbb{1}_{\mathcal{M}}$ is atomic.

Proof. It follows from [Lur17, Rem.7.3.2.9] that the $\mathcal{M}$-bimodule structure on μ^R from Remark 3.3 is strict if and only if the square in Diagram (6) is vertically and horizontally right adjointable. This shows that $\mathcal{M}$ is locally rigid if and only if it satisfies conditions $(0) - (2)$. For the second claim, note that it follows from Proposition 4.10 that the unit $\eta : \mathcal{V} \to \mathcal{M}$ is an internal left adjoint if and only if it preserves atomic objects. Since $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$ is a $\mathcal{V}$-atomic generator of $\mathcal{V}$ this is the case if and only if $\eta(\mathbb{1}_{\mathcal{V}}) \simeq \mathbb{1}_{\mathcal{M}}$ is atomic. □

After these preliminary considerations, we can now show that in case that $\mathcal{M}^{\otimes}$ is atomically generated the two notions of (local) rigidity coincide:

Proposition 4.19. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be an atomically generated $\mathcal{V}$-linear presentably monoidal ∞-category. Then $\mathcal{M}$ is locally atomically-rigid if and only it is locally rigid. Moreover, $\mathcal{M}$ is atomically-rigid, if and only if it is rigid.*

Proof. First, assume that $\mathcal{M}$ is locally rigid. We need to show that every object $m \in \mathcal{M}^{\mathrm{atm}}$ admits a left and right dual. For this, it suffices to show that for every atomic object $m \in \mathcal{M}^{\mathrm{atm}}$ the functor

$$\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{V}} \mathcal{V} \xrightarrow{\mathrm{id}_{\mathcal{M}} \otimes_{\mathcal{V}} m} \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \xrightarrow{\mu} \mathcal{M}$$

that maps $m' \in \mathcal{M}$ to $m' \otimes m$ admits a right adjoint functor that defines a morphism in $\mathrm{LMod}_{\mathcal{M}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$. Note, that since m is atomic the composite functor

$$\mathcal{M} \xrightarrow{\mu^R} \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \xrightarrow{\mathrm{id} \otimes \hom_{\mathcal{M}}(m,-)} \mathcal{M} \otimes_{\mathcal{V}} \mathcal{V} \simeq \mathcal{M}$$

provides a cocontinuous right adjoint of the above functor. Further, by Proposition 4.18 the right adjoint is a morphism in $\mathrm{LMod}_{\mathcal{M}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$. Hence, the right dual is given by $(\mathrm{id} \otimes_{\mathcal{V}} \hom_{\mathcal{M}}(m,-)) \circ \mu^R(\mathbb{1}_{\mathcal{M}})$. The left dual is constructed analogously.

Conversely assume $\mathcal{M}$ is locally atomically-rigid. We check the conditions of Proposition 4.18. By Proposition 4.15 $\mathcal{M}$ is dualizable. It remains to show that the monoidal product functor

$$\mu : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \to \mathcal{M}$$

admits an internal left adjoint in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$. To do so, we show that it preserves atomic objects. It follows from Proposition 4.16 that the category $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{M}$ is atomically generated by objects of the form $m \otimes_{\mathcal{V}} m'$ with $m, m' \in \mathcal{M}^{\mathrm{atm}}$. It therefore suffices to show that for every pair of atomic objects $m, m' \in \mathcal{M}^{\mathrm{atm}}$ the object $m \otimes m' \in \mathcal{M}$ is atomic. To show that, let I be a small category and $F : I \to \mathcal{M}$ be a diagram in $\mathcal{M}$. It follows from the chain of equivalences:

$$\begin{aligned}\hom_{\mathcal{M}}(m \otimes m', \mathsf{colim}_{i\in I} F(i)) &\simeq \hom_{\mathcal{M}}(m, m'^{\vee} \otimes \mathsf{colim}_{i\in I} F(i))\\ &\simeq \hom_{\mathcal{M}}(m, \mathsf{colim}_{i\in I}(m'^{\vee} \otimes F(i)))\\ &\simeq \mathsf{colim}_{i\in I} \hom_{\mathcal{M}}(m, (m'^{\vee} \otimes F(i)))\\ &\simeq \mathsf{colim}_{i\in I} \hom_{\mathcal{M}}(m \otimes m', F(i))\end{aligned}$$

that $\hom_{\mathcal{M}}(m \otimes m', -)$ commutes with small colimits. A similar calculation shows that $\hom_{\mathcal{M}}(m \otimes m', -)$ commutes with the $\mathcal{V}$-action. Hence, it follows from Proposition 4.11 that $\otimes$ is an internal left adjoint. The second claim about rigidity follows as in Proposition 4.18. $\square$

For our discussion in Section 5, we also record the following characterization of rigid $\mathcal{V}$-linear categories:

Proposition 4.20. *Let $\mathcal{E}^{\otimes}$ be an atomically generated $\mathcal{V}$-linear presentably symmetric monoidal ∞-category and consider the functor $F \coloneqq \mathbb{1}_{\mathcal{E}} \triangleleft - : \mathcal{V} \to \mathcal{E}$. The following are equivalent:*

(1) $\mathcal{E}^{\otimes}$ is $\mathcal{V}$-rigid

(2) for every $\mathcal{E}$-linear presentable ∞-category $\mathcal{A}$ an object $a \in \mathcal{A}$ is $\mathcal{E}$-atomic if and only if it is $\mathcal{V}$-atomic in $F^*\mathcal{A}$

(3) any object $e \in \mathcal{E}$ is $\mathcal{E}$-atomic if and only if it is $\mathcal{V}$-atomic in $F^*\mathcal{E}$

Proof. We first show (1) implies (2). Let $a \in \mathcal{A}$ be $\mathcal{E}$-atomic. Since $\mathbb{1}_{\mathcal{E}}$ is $\mathcal{V}$-atomic, it follows that the composite

$$\mathcal{V} \xrightarrow{\mathbb{1}_{\mathcal{E}}} \mathcal{E} \xrightarrow{a \triangleleft_{\mathcal{E}} -} \mathcal{A}$$

is an internal left adjoint in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ and hence a is $\mathcal{V}$-atomic. Assume now, that a is $\mathcal{V}$-atomic and denote by $\{e_i\}_{i\in I}$ a collection of $\mathcal{V}$-atomic generators of $\mathcal{E}$. We show that $\hom^{\mathcal{E}}_{\mathcal{A}}(a, -)$ commutes with small colimits. The claim for $\mathcal{E}$-tensoring is analogous. Let therefore J be a small ∞-category and $F : J \to \mathcal{A}$ a J-indexed diagram in $\mathcal{A}$. Since the e_i are generators, the functors $\{\hom_{\mathcal{E}}(e_i, -)\}_{i\in I}$ are jointly conservative. Hence, it suffices to show that for every $i \in I$, the canonical comparison map

$$\mathsf{colim}_{j\in J} \hom_{\mathcal{E}}(e_i, \hom_{\mathcal{A}}(a, F(j))) \to \hom_{\mathcal{E}}(e_i, \hom_{\mathcal{A}}(a, \mathsf{colim}_{j\in J} F(j)))$$

is an equivalence in $\mathcal{V}$. But this follows from the chain of equivalences

$$\begin{aligned}\mathsf{colim}_{j\in J} \hom_{\mathcal{E}}(e_i, \hom_{\mathcal{A}}(a, F(j))) &\simeq \mathsf{colim}_{j\in J} \hom_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, e_i^{\vee} \otimes \hom_{\mathcal{A}}(a, F(j)))\\ &\simeq \mathsf{colim}_{j\in J}\, e_i^{\vee} \triangleleft \hom_{\mathcal{A}}(a, F(j))\\ &\simeq e_i^{\vee} \triangleleft \hom_{\mathcal{A}}(a, \mathsf{colim}_{j\in J} F(j))\\ &\simeq \hom_{\mathcal{E}}(e_i, \hom_{\mathcal{A}}(a, \mathsf{colim}_{j\in J} F(j))),\end{aligned}$$

where we have used rigidity in the first step, the fact that $\mathbb{1}_{\mathcal{E}}$ is atomic in the second, and that a is $\mathcal{V}$-atomic in the last step. Afterward, we have applied the same reasoning in the opposite order.
It is clear that (2) implies (3). It remains to show that (3) implies that (1). This follows from the observation that an object of $\mathcal{E}$ is $\mathcal{E}$-atomic if and only if it is dualizable. $\square$

We finish this section with the following example of a rigid $\mathcal{V}$-linear presentably monoidal ∞-categories:

Proposition 4.21. *Let $V \in \mathsf{Alg}_{\mathbb{E}_2}(\mathcal{V})$ be an $\mathbb{E}_2$-algebra. Then the atomically generated $\mathcal{V}$-linear presentably monoidal ∞-category $\mathrm{LMod}_V(\mathcal{V})^{\otimes}$ equipped with the relative tensor product monoidal structure is rigid.*

Proof. It suffices to show that $\mathrm{LMod}_V(\mathcal{V})$ admits a collection of dualizable atomic generators and that the unit is atomic. By the generalized Schwede-Shipley theorem 4.12 $\mathrm{LMod}_V(\mathcal{V})$ is indeed atomically generated by V. Hence, it suffices to show that V is dualizable. But as V is the unit of the monoidal structure, this is clear. □

4.4 Implications for 2-dimensional TFTs

Let $\mathbb{K}$ be a field. The relevance of rigid $\mathbb{K}$-linear abelian categories for the study of fully extended TFTs comes from the result that these form 2-dualizable objects in the Morita 2-category of $\mathbb{K}$-linear presentable categories [BJS21]. This observation has been generalized in the ∞-categorical context of presentable dg-categories in [BZN09]. The goal of this section is to extend these results to the case of $\mathcal{V}$-linear presentable ∞-categories for an arbitrary presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$. This justifies our definition of locally rigid presentable ∞-categories from the perspective of fully extended TFTs.

To do so, we need an $(\infty, 2)$-categorical version of the Morita 2-category used in [BJS21] that serves as a target for the respective class of fully extended TFTs. Informally, for every sufficiently nice symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ the Morita $(\infty, 2)$-category has objects given by algebra objects, 1-morphisms by bimodule objects, and 2-morphisms by bimodule homomorphisms in $\mathcal{C}^{\otimes}$. We have already encountered a subcategory of a Morita category in Definition 2.3, when we discussed Morita equivalences for fusion categories.

An algebraic model of this symmetric monoidal $(\infty, 2)$-category has been constructed in [Hau17, Hau23] using non-symmetric ∞-operads. A major drawback of this model is that it is too rigid to describe the dualizability data of a fully dualizable object [Hau17, Conj.1.8]. A non-equivalent model for the Morita $(\infty, 2)$-category has been constructed in [GS18] using locally constant factorization algebras on stratified intervals. Although this model is flexible enough to classify the fully dualizable objects, it does not yield the expected result from [Lur08, Rem.4.1.27]. In fact, the only fully dualizable object is given by the unit [GS18, Thm.6.1]. The major problem is that a factorization algebra always comes with a canonical pointing. As a consequence, a locally-constant factorization algebra on a stratified interval encodes the datum of a pointed algebra (resp. bimodule) and keeping track of this additional pointing rules out any non-trivial examples of fully dualizable objects.

A way around this problem is to adapt the approach given in [Lur17, Sect.4.4] and to construct the Morita $(\infty, 2)$-category using symmetric ∞-operads. These extra symmetries provide a more flexible framework for the study of fully dualizable objects. To have a complete description, of this symmetric monoidal $(\infty, 2)$-category and its fully dualizable objects in this work, we have included a complete discussion in Appendix C. Since this is merely a reformulation of ideas from [Hau23] and [Lur17], let us here only collect the main results:

Proposition 4.22. *[GS18, Hau18, Lur17, Lur08] There exists a symmetric monoidal $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$ with*

- *(0) objects $\mathcal{V}$-linear presentably monoidal categories*
- *(1) 1-morphism $\mathcal{V}$-linear presentable bimodule categories*
- *(2) 2-morphism $\mathcal{V}$-linear cocontinuous bimodule functors*

and symmetric monoidal structure induced by the relative Deligne-Lurie tensor product $\otimes_{\mathcal{V}}$.

Before we can state the conditions for being fully dualizable, we need the following definition

Definition 4.4. [Lur17, Def.4.6.2.3] Let $\mathcal{A}^{\otimes}, \mathcal{B}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ and $\mathcal{M} \in \mathsf{BMod}_{A,B}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$. $\mathcal{M}$ is called *left dualizable* if there exists an object $\mathcal{N} \in \mathsf{BMod}_{\mathcal{B},\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ and morphisms

$$c : \mathcal{B} \to \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \text{ in } \mathsf{BMod}_{\mathcal{B},\mathcal{B}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}) \qquad e : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A} \text{ in } \mathsf{BMod}_{\mathcal{A},\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$$

such that the composites

$$\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{\mathrm{id} \otimes c} \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{e \otimes \mathrm{id}} \mathcal{M}$$

$$\mathcal{N} \simeq \mathcal{B} \otimes_{\mathcal{B}} \mathcal{N} \xrightarrow{c \otimes \mathrm{id}} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\mathrm{id} \otimes e} \mathcal{N}$$

are homotopic to the identity.

Definition 4.5. [Lur17, Def.4.6.4.2,4.6.4.13] A $\mathcal{V}$-linear presentably monoidal ∞-category $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ is called *proper*, if its underlying presentable ∞-category is dualizable as an object of $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$. It is called *smooth* if the evaluation bimodule $\mathcal{M} \in \mathrm{LMod}_{\mathcal{M}^e}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ is left-dualizable.

Proposition 4.23 (Thm. C.12)**.** *A $\mathcal{V}$-linear presentably monoidal ∞-category $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ is fully dualizable in $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$ if and only if it is smooth and proper.*

We show next that locally rigid algebras in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$ form fully dualizable in $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$. First observe, that by definition a locally rigid $\mathcal{V}$-linear ∞-category $\mathcal{M}^{\otimes}$ is dualizable in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$ and hence is proper. Consequently, we only need to show that $\mathcal{M}^{\otimes}$ is also smooth, i.e. dualizable as an $\mathcal{M}^e$-module. To do so, we exhibit $\mathcal{M}$ as an ∞-category of modules over an algebra in $\mathcal{M}^e$. As a first step, we show that these ∞-categories are in fact left-dualizable:

Proposition 4.24. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be a presentably monoidal $\mathcal{V}$-linear ∞-category and let $A \in \mathsf{Alg}(\mathcal{M})$ be an algebra object. Then $\mathsf{RMod}_A(\mathcal{M})$ is left dualizable with left dual $\mathrm{LMod}_A(\mathcal{M}) \in \mathsf{BMod}_{\mathcal{M},\mathcal{V}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$.*

Proof. We generalize the proof given in [Lur17, Rem.4.8.4.8]. To show that $\mathsf{RMod}_A(\mathcal{M})$ is left dualizable, we first need to define a unit morphism $c : \mathcal{V} \to \mathrm{LMod}_A(\mathcal{M}) \otimes_{\mathcal{M}} \mathsf{RMod}_A(\mathcal{M})$ in $\mathsf{BMod}_{\mathcal{V},\mathcal{V}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$. It follows from [Lur17, Thm.4.8.4.6] that there exists an equivalence of ∞-categories $\mathrm{LMod}_A(\mathcal{M}) \otimes_{\mathcal{M}} \mathsf{RMod}_A(\mathcal{M}) \simeq \mathsf{BMod}_{A,A}(\mathcal{M})$. By [Lur17, Thm.4.8.4.1] for every A-bimodule ${}_AM_A \in \mathsf{BMod}_{A,A}(\mathcal{M})$ there exists a unique $\mathcal{V}$-bimodule functor $F_M : \mathcal{V} \to \mathsf{BMod}_{A,A}(\mathcal{M})$ in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ that maps the unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$ to the bimodule M. We denote by $c := F_A$ the unique functor associated to the regular bimodule $A \in \mathsf{BMod}_{A,A}(\mathcal{M})$.
To show that this exhibits $\mathrm{LMod}_A(\mathcal{M})$ as the left dual of $\mathsf{RMod}_A(\mathcal{M})$, we have to show that for every $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ and $\mathcal{N} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{M}}$ the morphism c induces a homotopy equivalence

$$\mathsf{Map}_{\mathsf{Pr}^{\mathsf{L}}_{\mathcal{M}}}(\mathcal{C} \otimes \mathrm{LMod}_A(\mathcal{M}), \mathcal{N}) \to \mathsf{Map}_{\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}}(\mathcal{C}, \mathcal{N} \otimes_{\mathcal{M}} \mathsf{RMod}_A(\mathcal{M})).$$

It follows from [Lur17, Thm.4.8.4.6] that we can identify the right hand side with $\mathsf{Map}_{\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}}(\mathcal{C}, \mathsf{RMod}_A(\mathcal{N}))$. The claim follows from [Lur17, Thm.4.8.4.1] applied to the left-hand side. □

Construction 4.2. Let $(\mathcal{M}, \mu) \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be locally rigid. As discussed in Remark 3.3 $\mathcal{M}$ carries a natural bimodule structure over itself so that the monoidal product $\mu : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \to \mathcal{M}$ naturally extends to a $\mathcal{M}$-bimodule functor. Under the equivalence

$$\mathsf{BMod}_{\mathcal{M}}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}) \simeq \mathsf{RMod}_{\mathcal{M}^e}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$$

these induce the structure of a $\mathcal{M}^e$-module on $\mathcal{M}$ and of a $\mathcal{M}^e$-module morphism on μ. We denote by $\mathsf{hom}^{\mathcal{M}^e}_{\mathcal{M}}(-,-)$ the internal Hom-functor for this $\mathcal{M}^e$-action on $\mathcal{M}$. The object $\mathsf{hom}^{\mathcal{M}^e}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}})$ is an endo-

morphism object of $\mathbb{1}_{\mathcal{M}}$ for the $\mathcal{M}^e$-action [Lur17, Sect.4.7.1]. Hence, it admits a canonical lift to an algebra object in $\mathcal{M}^e$.
We compute its underlying object. Unraveling definitions, the functor $\mathsf{hom}_{\mathcal{M}}^{\mathcal{M}^e}(\mathbb{1}_{\mathcal{M}}, -)$ is given by the right adjoint of the monoidal product

$$\mu : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \to \mathcal{M}$$

Hence, the underlying object of the algebra $\mathsf{hom}_{\mathcal{M}}^{\mathcal{M}^e}(\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}})$ is given by $\mu^R(\mathbb{1}_{\mathcal{M}})$.

Definition 4.6. Let $(\mathcal{M}, \mu) \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be locally rigid. We will denote the algebra $\mu^R(\mathbb{1}_{\mathcal{M}}) \in \mathsf{Alg}(\mathcal{M}^e)$ by $\mathcal{F}_{\mathcal{M}}$ and call it the *canonical algebra* of $\mathcal{M}$.

Remark 4.8. The name canonical algebra is motivated by its relation to the canonical coend as defined in [SW21]. More precisely, this algebra can be equivalently described by the coend

$$\int^{m \in \mathcal{M}} m \otimes_{\mathcal{V}} m^{\vee} \in \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M}$$

of the functor $\otimes_{\mathcal{V}} \circ (\mathrm{id}_{\mathcal{M}} \times (-)^{\vee}) : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M}$. The canonical coend is then defined as the image of $\mathcal{F}_{\mathcal{M}}$ under the colimit preserving monoidal product functor $\mu : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \to \mathcal{M}$.

The relevance of this algebra object arises from the following observation:

Proposition 4.25. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be a $\mathcal{V}$-linear presentably monoidal ∞-category and consider it as an object of $\mathcal{M} \in \mathsf{Pr}^{\mathsf{L}}_{\mathcal{M}^{\mathsf{e}}}$. Then $\mathbb{1}_{\mathcal{M}}$ is a $\mathcal{M}^{\mathsf{e}}$-atomic generator of $\mathcal{M}$ if and only if $\mathcal{M}$ is locally rigid.*

Proof. Recall, that the multiplication functor $\mu : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{M} \to \mathcal{M}$ is $\mathcal{M}^{\mathsf{e}}$-linear for the regular $\mathcal{M}^{\mathsf{e}}$-bimodule structure on $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{M}$. Hence the functor μ is uniquely determined by its value on $\mathbb{1}_{\mathcal{M}^{\mathsf{e}}}$ [Lur17, Thm.4.8.4.1]. It follows from Example 4.13 that μ is an internal left adjoint in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{M}^{\mathsf{e}}}$ if and only if $\mathbb{1}_{\mathcal{M}}$ is $\mathcal{M}^{\mathsf{e}}$-atomic. This proves the claim. ∎

Theorem 4.26. *Let $\mathcal{M}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ be locally rigid. Then $\mathcal{M}^{\otimes}$ is smooth. In particular, $\mathcal{M}$ is fully dualizable in $\mathbb{M}\mathsf{or}(\mathsf{Pr}_{\mathcal{V}})^{\otimes}$.*

Proof. It follows from Proposition 4.25 that $\mathbb{1}_{\mathcal{M}}$ is a $\mathcal{M}^{\mathsf{e}}$-atomic generator of $\mathcal{M}$. Hence, by the generalized Schwede-Shipley theorem 4.12 $\mathcal{M}$ is equivalent to $\mathsf{RMod}_{\mathcal{F}_{\mathcal{M}}}(\mathcal{M}^e)$. The claim follows from Proposition 4.24. ∎

Remark 4.9. This generalizes results from [BJS21, DSPS20] and [BZN09] in multiple directions. On the one hand, it holds over every presentably symmetric monoidal ∞-category $\mathcal{V}$. On the other hand, it also holds for locally rigid presentable ∞-categories that are not necessarily atomically-generated.

We use the conjectural Cobordism Hypothesis [Lur08, Thm2.4.6] to interpret this result in the context of fully extended TFTs. Therefore we denote by $\mathbb{B}\mathsf{ord}_2^{\mathsf{fr},\otimes}$ the symmetric monoidal $(\infty, 2)$-category of framed cobordisms [Lur08, Sect.2.2][8]. Under the assumption that the Cobordism Hypothesis holds [Lur08, Thm2.4.6], we obtain as a Corollary of our above result:

Corollary 4.27. *Let $\mathcal{M} \in \mathsf{Alg}(\mathsf{Pr}_{\mathcal{V}})$ be locally rigid. Then $\mathcal{M}$ induces a fully extended framed TFT*

$$\mathsf{Z}_{\mathcal{M}} : \mathbb{B}\mathsf{ord}_2^{\mathsf{fr},\otimes} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$$

that maps the unique positively framed point + to $\mathcal{M}$.

[8]We will discuss this $(\infty, 2)$-category in more detail in Section 11.5

4.5 Rigidity as a relative Condition

We have shown in the last section that locally rigid $\mathcal{V}$-linear ∞-categories form examples of fully dualizable objects in $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$. However, it is not true that all fully dualizable objects are locally rigid $\mathcal{V}$-linear ∞-categories. An example of this phenomenon has been given in [BZGN19, Sect.1.2].

Example 4.14. Let $(G, m : G \times G \to G)$ be an affine group scheme of finite type and denote by $\mathsf{Dmod}(G) \coloneqq \mathrm{Ind}(G_{\mathrm{DR}})$ the presentable stable ∞-category of D-modules on G defined as the ∞-category of Ind-coherent sheaves on the de-Rham stack of G [BZN09, Sect. 4.1]. The group structures on G induces on $\mathsf{Dmod}(G)$ a monoidal structure via convolution, whose underlying monoidal product functor is given by $m_* : \mathsf{Dmod}(G) \otimes \mathsf{Dmod}(G) \to \mathsf{Dmod}(G)$.[9] The presentably monoidal stable ∞-category is not locally rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{st}}$ in general. Indeed, the monoidal product functor m_* only admits a right adjoint, given by $m^!$, if G is proper, which is only the case if G is finite.
Instead, $\mathsf{Dmod}(G)$ is equivalent in $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}})$ to the locally rigid stable ∞-category of Harish-Chandra bimodules [Ber17, Thm.2.3.12]. This is defined as the ∞-category $\mathsf{HC} \coloneqq \mathrm{Ind}(G\backslash G_{\mathrm{DR}}/G)$ of Ind-coherent sheaves on the double quotient stack $G\backslash G_{\mathrm{DR}}/G$. This ∞-category admits a monoidal structure via convolution. In fact, this monoidal structure is analogous to the discussion in Section 2 induced by the Čech-nerve of the morphism $BG \to BG_{\mathrm{DR}}$ of stacks.[10] This presentably monoidal stable ∞-category is locally rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{st}}$ [Ber17, Prop.2.3.11]. The reason is that although G is not proper itself it is proper relative to G_{DR}.

Remark 4.10. There does not exist an analogous example using local systems instead of D-modules as in Section 2. The main reason is in the case of local systems and a finite group $(G, m : G \times G \to G)$ the convolution monoidal product on $\mathsf{Fun}(G, \mathsf{Vect}_{\mathbb{C}})$ is given by left Kan-extension along m, which always admits a right adjoint.

In fact, it also follows from the above example that local-rigidity is not even a property that is invariant under Morita equivalence. The reason is that a locally rigid ∞-category $\mathcal{M}^{\otimes}$ admits a canonical pointing as a $\mathcal{M}^{\mathsf{e}}$-module, i.e a $\mathcal{M}^{\mathsf{e}}$-linear functor $\mathcal{M}^{\mathsf{e}} \to \mathcal{M}$ that maps the unit of $\mathcal{M}^{e}$ to the canonical algebra $\mathcal{F}_{\mathcal{M}}$. Therefore, to understand the precise relation between local rigidity and fully extended TFTs, one should also encode this morphism as datum in the target $(\infty, 2)$-category of the fully extended TFT. These classes of TFTs are known as relative TFTs [JFS17] and we will describe these TFTs in some more detail.
Relative TFTs describe so-called boundary conditions and defects of TFTs. Mathematically, we can think of these as morphisms between fully extended TFTs. The naive notion of a morphism between fully extended TFTs would be a symmetric monoidal natural transformation. Interestingly, this notion is too naive as the dualizability property of the cobordism category forces every such natural transformation to be invertible [Lur08, Rem.2.4.7]. A way to fix this is to consider lax or oplax natural transformations. This leads to two different versions of morphisms between fully extended TFTs and hence of relative TFTs. To classify relative TFTs one can again utilize the cobordism hypothesis by describing relative TFTs as fully dualizable objects in a symmetric monoidal $(\infty, 2)$-category of lax (resp. oplax transformations) [JFS17]. To use this approach for our study of locally rigid algebras, we first have to introduce the relevant symmetric monoidal $(\infty, 2)$-category of lax (resp. oplax) transformations.
For every pair of $(\infty, 2)$-categories $\mathbb{C}, \mathbb{D}$, we denote by $\mathbb{C} \otimes_{\mathrm{Gr}} \mathbb{D}$ their Gray tensor product as defined in [AGH24]. This construction assembles into a functor of ∞-categories $- \otimes_{\mathrm{Gr}} - : \mathsf{Cat}_2 \times \mathsf{Cat}_2 \to \mathsf{Cat}_2$ that preserves colimits in each argument separately. Hence, since Cat_2 is presentable, we obtain for every $(\infty, 2)$-category $\mathbb{C}$ adjunctions $- \otimes \mathbb{C} \dashv \mathsf{Fun}^{\mathrm{lax}}(\mathbb{C}, -)$ and $\mathbb{C} \otimes - \dashv \mathsf{Fun}^{\mathrm{oplax}}(\mathbb{C}, -)$.

[9] We describe the construction of this monoidal structure in more detail using local systems instead of D-modules in more detail in Section 11.2. The construction for D-modules is analogous.
[10] We will consider this construction in more detail in Section 11.3.

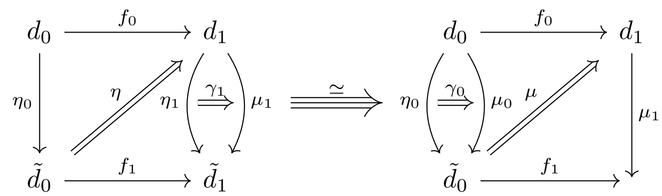
Definition 4.7. Let $\mathbb{D}$ be an $(\infty, 2)$-categories. We call the $(\infty, 2)$-category $\mathbb{D}^{\downarrow} := \mathsf{Fun}^{\mathrm{lax}}([1], \mathbb{D})$ (resp. $\mathbb{D}^{\rightarrow} := \mathsf{Fun}^{\mathrm{oplax}}([1], \mathbb{D})$) the $(\infty, 2)$-*category of lax arrows* (resp. *the* $(\infty, 2)$-*category of oplax arrows*)

Unraveling the definitions of the Gray tensor product, the $(\infty, 2)$-category $\mathbb{D}^{\rightarrow}$ has

- objects given by morphisms $f : d_0 \to d_1$ in $\mathbb{D}$.
- 1-morphisms given by oplax squares η :

$$\begin{array}{ccc} d_0 & \xrightarrow{f_0} & d_1 \\ {\scriptstyle \eta_0}\downarrow & \overset{\eta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle \eta_1} \\ \tilde{d}_0 & \xrightarrow[f_1]{} & \tilde{d}_1 \end{array}$$

- 2-morphisms given by oplax modifications $\gamma : \eta \to \mu$



Example 4.15. Let $\mathcal{D}$ be a $\otimes$-Gr-cocomplete symmetric monoidal ∞-category and consider the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathcal{D})^{\rightarrow}$. An object in this $(\infty, 2)$-category is represented by a bimodule $M \in \mathsf{BMod}_{A,B}(\mathcal{D})$ in $\mathcal{D}$. The datum of a morphism from a bimodule $M_0 \in \mathsf{BMod}_{A,B}(\mathcal{D})$ to a bimodule $M_1 \in \mathsf{BMod}_{A',B'}(\mathcal{D})$ consists of a pair of bimodules $N_0 \in \mathsf{BMod}_{A,A'}(\mathcal{D})$ and $N_1 \in \mathsf{BMod}_{B,B'}(\mathcal{D})$ together with a morphism

$$f_0 : N_0 \otimes_{A'} M_1 \to M_0 \otimes_B N_1$$

in $\mathsf{BMod}_{A,B'}(\mathcal{D})$. The datum of a 2-morphism from (N_0, N_1, f_0) to (K_0, K_1, f_1) consists of a pair, consisting of a morphism $g_0 : N_0 \to K_0$ in $\mathsf{BMod}_{A,A'}$ and a morphism $g_1 : N_0 \to N_1$ in $\mathsf{BMod}_{B,B'}$, such that the diagram

$$\begin{array}{ccc} N_0 \otimes_{A'} M_1 & \xrightarrow{f_0} & M_0 \otimes_B N_1 \\ {\scriptstyle g_0 \otimes_{A'} M_1}\downarrow & & \downarrow{\scriptstyle M_0 \otimes_B g_1} \\ K_0 \otimes_{A'} M_1 & \xrightarrow{f_1} & M_0 \otimes_B N_1 \end{array}$$

commutes.

As a right adjoint, the functor $(-)^{\rightarrow}$ preserves finite limits and hence $(-)^{\rightarrow}$ induces for every symmetric monoidal $(\infty, 2)$-category $\mathbb{D}^{\otimes}$ a symmetric monoidal structure on the $(\infty, 2)$-category $(\mathbb{D}^{\rightarrow})^{\otimes}$. In particular, we can study fully dualizable objects in $(\mathbb{D}^{\rightarrow})^{\otimes}$. To understand these objects, we need the following notation

Definition 4.8. Let $\mathcal{D}$ be an (∞, n)-category. A 1-morphism f in $\mathcal{C}$ is called 2-times right adjunctible, if $f : d_0 \to d_1$ admits a right adjoint f^{R} and the unit and counit transformations η and ϵ admit right adjoints as morphisms in the $(\infty, n-1)$-categories $\mathsf{Map}_{\mathcal{D}}(d_1, d_1)$ and $\mathsf{Map}\mathcal{D}(d_0, d_0)$ respectively.

Proposition 4.28. *[JFS17, Thm.7.6] An object in* $(\mathbb{D}^{\rightarrow})^{\otimes}$ *represented by a morphism* $f : d_0 \to d_1$ *is*

- *dualizable, if and only if its source and target objects are dualizable in* $\mathbb{D}$ *and* f *is right adjunctible.*

- *fully dualizable if and only if its source and target morphisms are fully dualizable in $\mathbb{D}$ and f is 2-times right adjunctible.*

In particular, we can apply this proposition to the specific case, that $\mathbb{D}^{\otimes}$ is the Morita category of a $\otimes$-Gr-cocomplete symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$.

Proposition 4.29. *Let $\mathcal{D}^{\otimes}$ be an $\otimes$-Gr-cocomplete symmetric monoidal ∞-category. An object in $\mathbb{M}\mathsf{or}(\mathcal{D})^{\downarrow,\otimes}$ represented by a bimodule $M \in \mathsf{BMod}_{A,B}(\mathcal{D})$ is*

(1) dualizable, if and only if M is right dualizable (see Definition C.6)

(2) fully dualizable if and only if M defines an invertible morphism in $\mathbb{M}\mathsf{or}(\mathcal{D})$ between smooth and proper algebra objects

Proof. We apply Proposition 4.28 to the Morita category. It follows from Proposition C.10 that every object is dualizable in $\mathbb{M}\mathsf{or}(\mathcal{D})^{\otimes}$. It is a direct consequence of the definitions that M admits a right adjoint $N \in \mathsf{BMod}_{B,A}(\mathcal{D})$ in $\mathbb{M}\mathsf{or}(\mathcal{D})$ if and only if M is right dualizable. This proves (1). To prove (2), we denote the corresponding unit and counit morphisms by

$$c : N \otimes_A M \to B \quad \text{and} \quad u : A \to M \otimes_B N$$

Since $\mathbb{M}\mathsf{or}(\mathcal{D})$ only has invertible 3-morphisms, the above unit and counit morphisms admit right adjoints if and only if c and u are invertible and N defines an inverse of M in $\mathbb{M}\mathsf{or}(\mathcal{C})$. □

Example 4.16. Let $\mathcal{D}^{\otimes}$ be a $\otimes$-Gr-cocomplete symmetric monoidal ∞-category and $(A, \mu) \in \mathsf{Alg}(\mathcal{D})$ an algebra object. The regular right A-module A_A defines a 1-morphism $A_A : \mathbb{1}_{\mathcal{D}} \to A$ in $\mathbb{M}\mathsf{or}(\mathcal{D})$. We claim that A is right dualizable with dual $A \in \mathrm{LMod}_A(\mathcal{D})$. Indeed, the multiplication of A $\mu : A \otimes A \to A$ forms the counit and the unit of A forms $\eta : \mathbb{1}_{\mathcal{D}} \to A \simeq A \otimes_A A$ the unit of the corresponding adjunction. The triangle identities

$$A \simeq A \otimes \mathbb{1}_{\mathcal{D}} \xrightarrow{\mathrm{id}_A \otimes \eta} A \otimes A \xrightarrow{\ \mu\ } A$$

are then equivalent to the right and left unitality conditions.

Unfortunately, it follows from Prop 4.29 that there exist no interesting 2-dualizable objects in $\mathbb{M}\mathsf{or}(\mathcal{D})^{\rightarrow,\otimes}$. The reason for that is that the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathcal{D})$ only has invertible 3-morphisms. However, in case that the symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$ admits an extension to a symmetric monoidal $(\infty, 2)$-category $\mathbb{D}^{\otimes}$, we can define a non-trivial notion of 2-morphism between bimodules[11]. More precisely, if we describe the symmetric monoidal $(\infty, 2)$-category as a $(0, 1)$-fibration (see Definition B.10)

$$\mathbb{D}^{\otimes} \to \mathsf{Fin}_*$$

then a bimodule transformation can be described as a 2-functor

$$\begin{array}{ccc} \mathcal{BM}^{\otimes} \times [1]([1]) & \xrightarrow{\quad F \quad} & \mathbb{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathsf{Fin}_* & \end{array}$$

[11]For the case that $\mathbb{D}$ is an $(\infty, 2)$-category of ∞-categories consider Remark B.5

over Fin_* that preserves cocartesian lifts of inert morphisms. Here, $[1](1)$ denotes the $(\infty,2)$-category represented by the diagram

$$0 \underset{}{\overset{}{\Downarrow}} 1\ ,$$

and $\mathcal{BM}^{\otimes}$ denotes the symmetric ∞-operad parameterizing bimodule objects [Lur17, Sect.4.3.1]. More generally, these are the 2-morphisms of an $(\infty,2)$-category, denoted with underlying ∞-category $\mathsf{BMod}(\mathcal{D})$, that can be defined as the full sub 2-category of

$$\mathbb{B}\mathsf{mod}(\mathbb{D}) \subset \mathsf{Fun}_{\mathsf{Fin}_*}(\mathcal{BM}^{\otimes}, \mathbb{D}^{\otimes})$$

spanned by those $(\infty,2)$-functors that preserve cocartesian morphisms over inert maps in Fin_*.
Hence, in this case, the morphism ∞-categories of $\mathbb{M}\mathsf{or}(\mathcal{D})$ admit a natural 2-categorical structure. Hence, it should be possible to use this 2-categorical structure to extend the symmetric monoidal $(\infty,2)$-category $\mathbb{M}\mathsf{or}(\mathcal{D})^{\otimes}$ to an $(\infty,3)$-category denoted $\mathbb{M}\mathsf{or}(\mathbb{D})^{\otimes}$. This intuition has been made precise in [JFS17, Sect. 8] using iterated complete Segal spaces. The $(\infty,3)$-category obtained via this construction is called the *even higher Morita category.* In particular, for every presentably symmetric monoidal ∞-category $\mathcal{V}$, we can apply this construction to the symmetric monoidal $(\infty,2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$:

Claim 4.1. For every presentably symmetric monoidal ∞-category $\mathcal{V}$, the construction of [JFS17] yields a symmetric monoidal $(\infty,3)$-category $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$ with

- underlying symmetric monoidal $(\infty,2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}_{\mathcal{V}})^{\otimes}$.
- morphism $(\infty,2)$-category $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})(\mathcal{V},\mathcal{A}) \simeq \mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{A}}$ for every $\mathcal{A} \in \mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})$.

Remark 4.11. The construction of the even higher Morita category from [JFS17] is technically involved and the author was not able to come up with a proof of the above claim in their framework. Instead, we suggest to prove this claim using a construction of the even higher Morita category that is internally 2-categorical. We plan to develop this framework in future work.

In particular, it follows from Proposition 4.7 that for every presentably monoidal ∞-category $\mathcal{A}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})$ a 1-morphism $F \in \mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})(\mathcal{V},\mathcal{A}) \simeq \mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{A}}$ represented by a functor $F : \mathcal{M} \to \mathcal{N}$ of $\mathcal{A}$-modules admits a right adjoint if and only if the underlying functor admits a right adjoint F^{R} in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$ and the canonical $\mathcal{A}$-module structure on F^{R} is strong. We can use this to prove:

Proposition 4.30. *Let $\mathcal{A}^{\otimes}$ be a $\mathcal{V}$-linear presentably monoidal ∞-category. Under the hypothesis of Claim 4.1 the regular $\mathcal{A}$-module $\mathcal{A}$ represents a 2-dualizable object in $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}})^{\to,\otimes}$ if and only if $\mathcal{A}^{\otimes}$ is rigid.*

Proof. We check the conditions of Prop 4.28. It follows from Example 4.16 that the morphism represented by $\mathcal{A}$ is always 1-times right adjunctible with unit and counit given by

$$\mu :_{\mathcal{A}} \mathcal{A} \otimes \mathcal{A}_{\mathcal{A}} \to_{\mathcal{A}} \mathcal{A}_{\mathcal{A}} \quad \text{and} \quad \eta : \mathcal{V} \to \mathcal{A}.$$

In particular, it follows from Remark 3.3 that these admit right adjoints if and only if $\mathcal{A}$ is rigid. The claim then follows from Proposition 4.26. □

Remark 4.12. This Proposition provides a conjectural complete classification of rigid $\mathcal{V}$-linear ∞-categories in terms of 2-dimensional fully extended relative TFTs. A similar observation appears in the context of $\mathbb{C}$-linear braided monoidal 1-categories in [Haï23].

Remark 4.13. For every symmetric monoidal $(\infty, 2)$-category $\mathbb{D}^{\otimes}$ and every algebra object A in $\mathbb{D}$, it is possible to define an $(\infty, 2)$-category $\mathbb{D}_A$ with underlying ∞-category given by $\mathrm{LMod}_A(\mathbb{D})$. Using this construction, we expect Claim 4.1 to hold for every $\otimes$-Gr cocomplete symmetric monoidal $(\infty, 2)$-category. Since the proof of Proposition 4.30 is completely formal, it would provide a classification of general rigid algebras in the sense of Definition 3.2 in terms of TFTs.

5 Examples of Presentable ∞-Categories

Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. All ∞-categories that we consider in this section are idempotent-complete. After covering the general theory of locally rigid $\mathcal{V}$-linear presentable ∞-categories in the section 4, we focus in this section on explicit examples. In particular, we discuss how the notion of locally rigid algebras connects to more familiar notions from the theory of TFTs like tensor and multi-fusion categories introduced in Section 2.

As our first example, we study in Section 5.1 the case where $\mathcal{V}^{\otimes}$ is the category of modules over a commutative ring R, and in Section 5.3 the case where $\mathcal{V}^{\otimes}$ is the derived ∞-category of R-modules. In this context, we show that our notion of rigidity recovers the notion of cp-rigid monoidal categories from [BJS21, Def.1.3] and rigid R-linear dg-category from [Gai15]. The notion of cp-rigidity is more general than rigidity for tensor categories. Cp-rigid tensor categories include the class of tensor categories via the sifted completion construction (see Example 4.2), but also allow for more general examples. The main difference to the notion of rigidity for tensor categories is that cp-rigidity only requires the existence of duals for projective objects instead of duals for all objects.

Up to now, we have only discussed rigidity for large presentable categories, but our notion of rigidity is flexible enough to be applied in an arbitrary symmetric monoidal $(\infty, 2)$-category. Especially, since our objects of interest naturally form small instead of large categories, we will study in Section 5.2 rigidity in a 2-category of small R-linear categories. As these form a faithful sub-2-category of the 2-category of R-linear presentable categories, rigid algebras will satisfy additional finiteness conditions. The main difference to the presentable case is that, while a small additive category is always dualizable as a presentable category, it is only dualizable as an additive category if it is semisimple and the Hom-modules are finite and projective. Using this insight, we show that for an algebraically closed field $\mathbb{K}$, this notion of rigid algebra precisely recovers $\mathbb{K}$-linear multi-fusion categories.

Building on the above characterization of multi-fusion categories as rigid algebras, we study in Section 5.4 rigid algebras in small $\mathbb{K}$-linear stable ∞-categories. As in the 1-categorical case, these have to satisfy extra finiteness conditions. In particular, we see that in the derived context the semisimplicity condition gets replaced by a weaker condition called smoothness. By analogy, with the 1-categorical case of multi-fusion categories, we refer to these as derived multi-fusion categories. To justify this terminology, we first show that derived categories of fusion categories provide examples of these, and further relate our notion to a criterion for fully dualizable $\mathbb{E}_2$-algebras from [Lur08]. Unfortunately, we were not able to construct any interesting example of an $\mathbb{E}_2$-algebra that satisfies this criterion. To this end, we finish this section by discussing potential construction methods for these $\mathbb{E}_2$-algebras.

5.1 Additive Categories

Our goal in this section is to discuss the relation between locally rigid algebras and tensor categories. Therefore, we need to introduce different symmetric monoidal 2-categories:

- the symmetric monoidal 2-category $\mathsf{add}_{\mathrm{ic}}^{\amalg,\otimes}$ of idempotent-complete additive categories.

- the symmetric 2-category $\mathsf{add}^{\mathsf{rex},\otimes}$ of additive categories with finite colimits.
- the symmetric 2-category $\mathbb{pr}^{\mathsf{L},\otimes}_{\mathsf{add}}$ of presentable additive categories.

Although similar categories have partially been described, for example by Brochier–Jordan –Snyder [BJS21] using explicit constructions, we include a complete discussion here. To highlight the similarities with the derived approach presented in Section 5.3, we only use properties of idempotent-algebras for our construction. We denote by $\mathsf{cat}^{\mathsf{rex}}$ the $(2,1)$-category of finitely cocomplete small categories and right exact functors.[12] Using the machinery of Section 4.1, we can equip this category with a symmetric monoidal structure. The natural category for the study of abelian categories would be the full subcategory of $\mathsf{cat}^{\mathsf{rex}}$, whose objects are abelian categories. We denote this $(2,1)$-category by $\mathrm{ab}^{\mathsf{rex}}$. Unfortunately, this full subcategory is not symmetric monoidal, since the Deligne-Lurie tensor product of two abelian categories is not necessarily abelian [Fra13]. To circumvent this issue, we instead consider a slightly larger category.
Let $\mathsf{add}^{\mathsf{rex}}$ the full subcategory of $\mathsf{cat}^{\mathsf{rex}}$ generated by finitely cocomplete additive categories [Lur17, Def.1.1.2.1]. This $(2,1)$-category in particular includes the $(2,1)$-category $\mathrm{ab}^{\mathsf{rex}}$. Moreover, as we will see in Proposition 5.3, the symmetric monoidal structure on $\mathsf{cat}^{\mathsf{rex}}$ restricts to $\mathsf{add}^{\mathsf{rex}}$. To continue our discussion, we recall the definition of a projective object:

Definition 5.1. Let $\mathcal{C}$ be a small 1-category. An object $P \in \mathcal{C}$ is called *projective* if the functor

$$\mathsf{Map}(P,-) : \mathcal{C} \to \mathsf{Set}$$

preserves reflexive coequalizers (i.e sifted colimits). We say $\mathcal{C}$ has *enough projectives* if the smallest subcategory of $\mathcal{C}$ that contains the projective objects and is closed under finite colimits coincides with $\mathcal{C}$.

Remark 5.1. If $\mathcal{A}$ is abelian, this definition agrees with the classical definition of being projective. Indeed, for every object $P \in \mathcal{A}$ the functor $\mathsf{Map}(P,-)$ preserves epimorphisms if and only if it preserves reflective coequalizers.

For an additive category $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{p}}$ its full subcategory of projective objects. It follows from the definition that this category is itself idempotent-complete and closed under finite coproducts. Hence, it is additive. We denote by $\mathsf{cat}^{\amalg,\otimes}_{\mathrm{ic}}$ the symmetric monoidal $(2,1)$-category of idempotent-complete categories that admit finite coproducts and coproduct preserving functors, and by $\mathsf{add}^{\amalg}_{\mathrm{ic}}$ its full subcategory spanned by additive categories.
Note that an arbitrary right exact functor between right exact additive categories does not preserve projective objects. We denote by $\mathsf{add}^{\mathsf{rex}}_{\mathrm{p}}$ the (non-full) subcategory of $\mathsf{add}^{\mathsf{rex}}$ with objects right exact additive categories with enough projectives and morphisms right exact functors that preserve projective objects. Taking the full subcategory generated by projective objects then defines an equivalence

$$(-)^{\mathrm{p}} : \mathsf{add}^{\mathsf{rex}}_{\mathrm{p}} \to \mathsf{add}^{\amalg}_{\mathrm{ic}}$$

with inverse $\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(-) : \mathsf{add}_{\mathrm{ic}} \to \mathsf{add}^{\mathsf{rex}}_{\mathrm{p}}$ given by the 1-categorical finite colimit completion (see Example 4.3). For an idempotent-complete additive category, we can describe the category $\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(\mathcal{A})$ more explicitly. It is equivalent to the full subcategory of the category of additive functors $\mathsf{Fun}^{\mathsf{add}}(\mathcal{A},\mathsf{Ab})$ generated by functors that are given as coequalizers of representable functors. These are called finitely presented functors in [Kra21, Sect.2.1].
We further denote by $\mathsf{pr}^{\mathsf{L}}_{\mathsf{add}}$ the $(2,1)$-category of additive presentable 1-category. As explained in Example 4.8,

[12] Note that every category with finite colimits is idempotent-complete. This is different for ∞-categories.

this category is equivalent to the $(2,1)$-category $\mathsf{RMod}_{\mathsf{Ab}}(\mathsf{Pr}^{\mathsf{L}})$ of modules over the idempotent algebra Ab. In particular, it inherits a symmetric monoidal structure. To understand its relation to $\mathsf{add}^{\mathsf{rex}}$ and $\mathsf{add}^{\amalg}_{\mathrm{ic}}$, we need the following definitions:

Definition 5.2. Let $\mathcal{C}$ be a presentable 1-category. An object $C \in \mathcal{C}$ is called *compact* if the functor $\mathsf{Map}_{\mathcal{C}}(C, -) : \mathcal{C} \to \mathsf{Set}$ commutes with filtered colimits. An object $P \in \mathcal{C}$ is called is called *compact-projective* (cp) if it is compact and projective.
We say that $\mathcal{C}$ is *compactly generated* if the smallest subcategory of $\mathcal{C}$ that contains all compact objects and is closed under filtered colimits is $\mathcal{C}$ itself. $\mathcal{C}$ is called *cp-generated* if the smallest subcategory of $\mathcal{C}$ that contains the compact-projective objects and is closed under small colimits is $\mathcal{C}$ itself.

Note, that for an additive presentable category $\mathcal{A}$, its full subcategories of compact and compact-projective objects $\mathcal{A}^{\mathrm{c}}$ and $\mathcal{A}^{\mathrm{cp}}$ are themselves additive and idempotent complete. Furthermore, the category $\mathcal{A}^{\mathrm{c}}$ is closed under finite colimits. The sifted completion[13]

$$\mathcal{P}^{\Sigma}_{1}(-) : \mathsf{cat}^{\amalg}_{\mathrm{ic}} \to \mathsf{pr}^{\mathsf{L}}$$

identifies $\mathsf{cat}^{\amalg}_{\mathrm{ic}}$ with the subcategory $\mathsf{pr}^{\mathsf{L}}_{\mathrm{cp}}$ of pr^{L} generated by cp-generated presentable categories and compact-projective preserving cocontinuous functors. Similarly the inductive completion functor[14]

$$\mathrm{Ind}(-) : \mathsf{cat}^{\mathsf{rex}} \to \mathsf{pr}^{\mathsf{L}}$$

identifies $\mathsf{cat}^{\mathsf{rex}}$ with the category $\mathsf{pr}^{\mathsf{L}}_{\mathrm{c}}$ of compactly generated presentable categories and compact object preserving cocontinuous functors. The inverse is given by taking compact objects $(-)^{\mathrm{c}} : \mathsf{pr}^{\mathsf{L}}_{\mathrm{c}} \to \mathsf{cat}^{\mathsf{rex}}$.

Proposition 5.1. *The functors* $\mathcal{P}^{\Sigma}_{1}(-)$ *and* $\mathrm{Ind}(-)$ *restrict to equivalences* $\mathcal{P}^{\Sigma}_{1}(-) : \mathsf{add}^{\amalg}_{\mathrm{ic}} \to \mathsf{pr}^{\mathsf{L}}_{\mathrm{cp},\mathsf{add}}$ *and* $\mathrm{Ind}(-) : \mathsf{add}^{\mathsf{rex}} \to \mathsf{pr}^{\mathsf{L}}_{\mathrm{c},\mathsf{add}}$

Proof. We show the statement for the functor $\mathcal{P}^{\Sigma}_{1}(-)$. Given a cp-generated presentable additive category $\mathcal{A}$, the category $\mathcal{A}^{\mathrm{cp}}$ is closed under coproducts and idempotent-complete. Hence it is an object of $\mathsf{add}^{\amalg}_{\mathrm{ic}}$. On the other hand, let $\mathcal{B}$ be an idempotent complete additive category. It follows from the universal property of $\mathcal{P}^{\Sigma}_{1}(\mathcal{B})$ that there exists an equivalence $\mathcal{P}^{\Sigma}_{1}(\mathcal{B}) \simeq \mathsf{Fun}^{\amalg}(\mathcal{B}^{\mathrm{op}}, \mathsf{Set})$. It is easy to check that this category is additive. □

So far we have restricted ourselves to additive categories, instead of abelian categories, since the category of abelian categories, in general, does not admit symmetric monoidal structures. This is different for the category of presentable abelian categories:

Definition 5.3. A Grothendieck abelian category $\mathcal{A}$ is a presentable abelian category such that filtered colimits are exact in $\mathcal{A}$. We denote by Groth the full subcategory of $\mathsf{pr}^{\mathsf{L}}_{\mathsf{add}}$ consisting of Grothendieck abelian categories and colimit preserving functors.

Example 5.1. Let $\mathcal{A} \in \mathsf{add}^{\amalg}_{\mathrm{ic}}$ be a small additive category. Then $\mathcal{P}^{\Sigma}_{1}(\mathcal{A})$ is a Grothendieck abelian category. Indeed, since $\mathcal{P}^{\Sigma}_{1}(\mathcal{A})$ is presentable and additive, we can identify it with the category $\mathsf{Fun}^{\mathsf{add}}(\mathcal{A}^{\mathrm{op}}, \mathsf{Ab})$ of additive functors. This category is abelian and Grothendieck [Kra21, Sect.11.1, p.345].

We denote by $\mathsf{Groth}_{\mathrm{c}}$ and $\mathsf{Groth}_{\mathrm{cp}}$ the full subcategories of $\mathsf{pr}^{\mathsf{L}}_{\mathrm{c},\mathsf{add}}$ respectively $\mathsf{pr}^{\mathsf{L}}_{\mathrm{cp},\mathsf{add}}$ generated by Grothendieck abelian categories. The above result implies:

[13] See Example 4.2
[14] See Example 4.1

Proposition 5.2. *Every cp-generated presentable additive category is Grothendieck abelian. In particular, the inclusion restricts to an equivalence* $\mathsf{pr}^{\mathsf{L}}_{\mathrm{cp},\mathsf{add}} \simeq \mathsf{Groth}_{\mathrm{cp}}$.

Remark 5.2. While it is true that the sifted cocompletion $\mathcal{P}^{\Sigma}_{1}(\mathcal{A})$ of an idempotent-complete small additive category is abelian, the analogous statement does not hold for its finite cocompletion $\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(-)$ [Kra21, Lem.2.1.6]

Before we turn to the study of locally rigid algebras in these categories, let us show that the symmetric monoidal structures on $\mathsf{cat}^{\mathsf{rex},\otimes}$ and $\mathsf{cat}^{\amalg,\otimes}_{\mathrm{ic}}$ restrict to symmetric monoidal structures on $\mathsf{add}^{\mathsf{rex},\otimes}$ and $\mathsf{add}^{\otimes}_{\mathrm{ic}}$ respectively. To do so we exhibit them as categories of modules over idempotent algebras. We denote by $\mathsf{Ab}^{\mathrm{fg}}$ and $\mathsf{Ab}^{\mathrm{fg,p}}$ the symmetric monoidal categories of finitely generated (short f.g.) and f.g. projective abelian groups respectively. Note, that these arise as the respective categories of compact and compact projective objects in Ab. Since the tensor product of abelian groups preserves finitely generated (resp. finitely generated projective) abelian groups, it restricts to a symmetric monoidal structure on $\mathsf{Ab}^{\mathrm{fg}}$ and $\mathsf{Ab}^{\mathrm{fg,p}}$.

Proposition 5.3. *The following holds:*

(1) The symmetric monoidal category $\mathsf{Ab}^{\mathrm{fg},\otimes}$ *is an idempotent algebra in* $\mathsf{cat}^{\mathsf{rex},\otimes}$ *and its category of modules coincides with* $\mathsf{add}^{\mathsf{rex}}$.

(2) The symmetric monoidal category $\mathsf{Ab}^{\mathrm{fg,p},\otimes}$ *is an idempotent algebra in* $\mathsf{cat}^{\amalg,\otimes}_{\mathrm{ic}}$ *and its category of modules coincides with* $\mathsf{add}^{\amalg}_{\mathrm{ic}}$.

Proof. We prove (1). The proof of (2) is analogous. Note that the tensor product of abelian groups preserves finitely generated abelian groups. Hence, Ab also forms an idempotent algebra in the symmetric monoidal category $\mathsf{pr}^{\mathsf{L}}_{\mathrm{c}}$. In particular, since $(-)^{\mathrm{c}} : \mathsf{pr}^{\mathsf{L},\otimes}_{\mathrm{c}} \to \mathsf{cat}^{\mathsf{rex},\otimes}$ is a symmetric monoidal equivalence, $\mathsf{Ab}^{\mathrm{fg}}$ also forms an idempotent algebra in $\mathsf{cat}^{\mathsf{rex}}$. To determine the reflective subcategory that is classified by $\mathsf{Ab}^{\mathrm{fg}}$, we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathsf{Ab}}(\mathsf{pr}^{\mathsf{L}}_{\mathrm{cp}}) & \xrightarrow{\simeq} & \mathsf{pr}^{\mathsf{L}}_{\mathrm{c},\mathsf{add}} \\ \downarrow{\simeq} & & \downarrow{(-)^{\mathrm{c}}} \\ \mathrm{Mod}_{\mathsf{Ab}^{\mathrm{fg}}}(\mathsf{cat}^{\mathsf{rex}}) & \longrightarrow & \mathsf{cat}^{\mathsf{rex}} \end{array}$$

Since the diagram commutes, it follows that the subcategory classified by $\mathsf{Ab}^{\mathrm{fg}}$ coincides with the essential image of $\mathsf{pr}^{\mathsf{L}}_{\mathrm{c},\mathsf{add}}$ under $(-)^{\mathrm{c}}$. But this category is given by $\mathsf{add}^{\mathsf{rex}}$. Hence, the claim follows. ☐

We now turn to the study of locally rigid algebras in $\mathbb{pr}^{\mathsf{L},\otimes}_{\mathsf{add}}$. As a consequence of Proposition 4.19, it suffices to understand the Ab-atomic objects:

Proposition 5.4. *Let* $\mathcal{A}$ *be a presentable additive* 1*-category. An object* $A \in \mathcal{A}$ *is* Ab*-atomic if and only if it is compact-projective. Further,* $\mathcal{A}$ *is atomically-generated if and only if it is cp-generated.*

Proof. Let $\mathcal{A} \in \mathsf{pr}^{\mathsf{L}}_{\mathsf{add}}$ be a presentable additive 1-category. Since $\mathsf{Ab}^{\otimes}$ is an idempotent algebra, it suffices to understand for which $A \in \mathcal{A}$ the internal Hom-functor

$$\hom_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathsf{Ab}$$

preserves small colimits. Since finite products and coproducts coincide in Ab, it follows that for any $A \in \mathcal{A}$ the internal Hom-functor preserves finite coproducts. It suffices to show that it preserves filtered colimits and coequalizers if and only if A is compact-projective. Note, that the forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$ preserves filtered colimits and coequalizers. Consequently, A is atomic if and only if the composite

$$U \circ \hom_{\mathcal{A}}(A, -) \simeq \mathsf{Map}_{\mathcal{A}}(A, -)$$

preserves filtered colimits and coequalizers. But this is the definition of being compact-projective. □

We can finally conclude from the characterization of (locally) rigid algebras in Proposition 4.19:

Proposition 5.5. *A cp-generated presentably monoidal additive category $\mathcal{A}^{\otimes}$ is locally rigid in $\mathbb{pr}^{\mathsf{L},\otimes}_{\mathsf{add}}$ if and only if each compact-projective object admits a dual. It is rigid in $\mathbb{pr}^{\mathsf{L},\otimes}_{\mathsf{add}}$ if and only if it is locally rigid and the unit object is compact-projective.*

Definition 5.4. [BJS21, Def.1.3] A cp-generated presentably monoidal additive category $\mathcal{A}$ is called *locally cp-rigid* if every compact-projective object admits a dual. It is called *cp-rigid* if it is locally cp-rigid and the unit $\mathbb{1}_{\mathcal{A}}$ is compact-projective. We call a small monoidal additive category $\mathcal{A}$ *rigid*, if $\mathcal{P}^{\Sigma}_{1}(\mathcal{A})$ is cp-rigid. Equivalently it is rigid if every object of $\mathcal{A}$ admits a dual.

Remark 5.3. Our terminology is different from [BJS21]. Indeed, in their terminology, a locally cp-rigid presentable additive category would be called cp-rigid.

For a monoidal additive category $\mathcal{E}$, we abuse notation and denote $\mathbb{pr}^{\mathsf{L}}_{\mathcal{P}^{\Sigma}_{1}(\mathcal{E})}$ by $\mathbb{pr}^{\mathsf{L}}_{\mathcal{E}}$ and call a $\mathcal{P}^{\Sigma}_{1}(\mathcal{E})$-linear presentable additive category simply $\mathcal{E}$-linear. Similarly, we denote $\mathbb{add}^{\mathsf{rex}}_{\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(\mathcal{E})}$ by $\mathbb{add}^{\mathsf{rex}}_{\mathcal{E}}$ and call $\mathcal{P}^{\mathrm{fin}}_{1,\amalg}(\mathcal{E})$-linear finitely cocomplete additive categories $\mathcal{E}$-linear.

Example 5.2. Let R be a commutative ring. The symmetric monoidal category $\mathsf{rmod}^{\mathrm{fg,p},\otimes}_{R}$ of finitely generated projective R-modules equipped with the relative tensor product is a rigid additive category. We call $\mathsf{rmod}^{\mathrm{fg,p}}_{R}$-linear presentable (resp. finitely cocomplete) additive categories simply R-linear and denote $\mathbb{pr}^{\mathsf{L}}_{\mathsf{rmod}^{\mathrm{fg,p}}_{R}}$ (resp. $\mathbb{add}^{\mathsf{rex}}_{\mathsf{rmod}^{\mathrm{fg,p}}_{R}}$) by $\mathbb{pr}^{\mathsf{L}}_{R}$ (resp. $\mathbb{add}^{\mathsf{rex}}_{R}$).

The following is a consequence of Proposition 4.20:

Corollary 5.6. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category. Then a cp-generated $\mathcal{E}$-linear presentably monoidal additive category $\mathcal{A}^{\otimes}$ is (locally) rigid in $\mathbb{pr}^{\mathsf{L},\otimes}_{\mathcal{E}}$ if and only if $\mathcal{A}^{\otimes}$ is (locally) cp-rigid.*

To compare this new notion of (local) cp-rigidity to the familiar one for tensor categories introduced in Definition 2.1, we need the following large variant of a tensor category:

Definition 5.5. Let $\mathcal{A}^{\otimes}$ be a compactly generated presentably monoidal additive category. $\mathcal{A}^{\otimes}$ is called a *large tensor category* if every compact object admits a dual.[15]

Example 5.3. (1) Let $\mathbb{K}$ be a field and $\mathcal{A}^{\otimes}$ be a $\mathbb{K}$-linear finitely cocomplete monoidal abelian category. The presentably monoidal category $\mathrm{Ind}(\mathcal{A})^{\otimes}$ is a tensor category if and only if every object of $\mathcal{A}$ admits a left and right dual. This is the case if $\mathcal{A}^{\otimes}$ is a tensor category in the sense of Definition 2.1.

(2) Let $\mathcal{A}^{\otimes}$ be an $\mathbb{K}$-linear finite tensor category in the sense of Definition 2.1. Since $\mathcal{A}$ has enough projectives, it follows that $\mathrm{Ind}(\mathcal{A})^{\otimes} \simeq \mathcal{P}^{\Sigma}_{1}(\mathcal{A}^{\mathrm{p}})^{\otimes}$ is also locally cp-rigid. It is easy to see that it is cp-rigid if and only if $\mathcal{A}^{\otimes}$ is a multi-fusion category.

[15] This terminology agrees with our terminology from Section 2.

Example 5.4. It is shown in [BJS21] that local cp-rigidity is a strictly weaker condition than being a large tensor category. Indeed, let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. Then the category $\mathsf{BMod}_A(\mathsf{Vect_k})$ of A-bimodules is cp-generated by the bimodule $A \otimes A$, which is also dualizable since A is finite-dimensional. On the other hand, the trivial bimodule $\mathbb{C}$ is compact but not dualizable (since it is not projective). Hence, $\mathsf{BMod}_A(\mathsf{Vect_k})$ is an example of a category that is locally cp-rigid but not a large tensor category.

Remark 5.4. The main benefit of studying finite tensor categories over cp-rigid categories is that the stronger rigidity assumption forces the monoidal product functor of the finite tensor category to be exact. The above discussion shows that this is a rather unnatural, restrictive condition from the perspective of fully extended TFTs. The reason is that, as we discuss in Section 5.3, exact functors arise more naturally at the level of stable ∞-categories.

5.2 Fusion Categories

Throughout this section, $\mathbb{K}$ denotes an algebraically closed field. In the last section, we have classified locally rigid algebras in the symmetric monoidal 2-category of large $\mathbb{K}$-linear categories. As we are mainly interested in small $\mathbb{K}$-linear categories, our goal in this section is to provide a similar classification of rigid algebras in the symmetric monoidal 2-category of small $\mathbb{K}$-linear categories. Interestingly, these can be explicitly described in terms of multi-fusion categories:

Theorem 5.7. *Let $\mathbb{K}$ be an algebraically closed field. Then an additive idempotent complete $\mathbb{K}$-linear monoidal category $\mathcal{A}^{\otimes}$ is rigid in $\mathsf{add}_{\mathbb{K}}^{\amalg,\otimes}$ if and only if it is a $\mathbb{K}$-linear multi-fusion category (see Definition 2.1).*

To prove this Corollary, we first need to understand the dualizable objects in $\mathsf{add}_{\mathbb{K}}^{\amalg,\otimes}$. These have been classified for stable ∞-categories [HSS17, Sect.4]. The discussion for additive categories works completely analogous and we only recall the main steps.
Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category and $\mathcal{A}$ an $\mathcal{E}$-linear additive category. Consider the $\mathcal{E}$-linear functor

$$\mathcal{A} \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{add}}(\mathcal{E}, \mathcal{A}) \simeq \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L},\mathrm{cp}}(\mathcal{P}_1^{\Sigma}(\mathcal{E}), \mathcal{P}_1^{\Sigma}(\mathcal{A}))$$

that associates to every $a \in \mathcal{A}$ the $\mathcal{E}$-linear action functor $- \otimes a : \mathcal{E} \to \mathcal{A}$. After passing to the level of sifted completions, this functor admits a right adjoint $\hom_{\mathcal{A}}(a, -) : \mathcal{A} \to \mathcal{P}_1^{\Sigma}(\mathcal{E})$. Hence, we obtain a functor

$$\mathcal{A} \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L},\mathrm{cp}}(\mathcal{P}_1^{\Sigma}(\mathcal{E}), \mathcal{P}_1^{\Sigma}(\mathcal{A})) \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L}}(\mathcal{P}_1^{\Sigma}(\mathcal{A}), \mathcal{P}_1^{\Sigma}(\mathcal{E}))^{\mathrm{op}} \simeq \mathsf{Fun}_{\mathcal{E}}^{\mathsf{add}}(\mathcal{A}, \mathcal{P}_1^{\Sigma}(\mathcal{E}))^{\mathrm{op}},$$

where the second functor is obtained by passing to adjoints objectwise. Unraveling definitions, its opposite is the $\mathcal{E}$-linear functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{add}}(\mathcal{A}, \mathcal{P}_1^{\Sigma}(\mathcal{E}))$ that associates to every $a \in \mathcal{A}$ the $\mathcal{E}$-linear corepresentable functor $a' \mapsto \hom_{\mathcal{A}}(a, a') \in \mathcal{P}_1^{\Sigma}(\mathcal{E})$.

Definition 5.6. Let $\mathcal{E}$ be a symmetric monoidal rigid additive category and $\mathcal{A} \in \mathsf{add}_{\mathcal{E}}^{\amalg}$ an $\mathcal{E}$-linear additive category. The *$\mathcal{E}$-linear Yoneda embedding* is the $\mathcal{E}$-linear functor

$$\begin{aligned} \mathcal{Y}_{\mathcal{E}} : \mathcal{A} &\to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{add}}(\mathcal{A}^{\mathrm{op}}, \mathcal{P}_1^{\Sigma}(\mathcal{E})) \\ a &\mapsto \hom_{\mathcal{A}}(a, -), \end{aligned}$$

obtained from the above functor by passing through the tensor-hom-adjunction.

Unraveling definitions, the $\mathcal{E}$-linear Yoneda embedding maps $a \in \mathcal{A}$ to the $\mathcal{E}$-linear representable functor $\hom_{\mathcal{A}}(-, a) : \mathcal{A}^{\mathrm{op}} \to \mathcal{P}_1^{\Sigma}(\mathcal{E})$.

Definition 5.7. Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category and $\mathcal{A} \in \mathsf{add}_{\mathcal{E}}^{\amalg}$ an $\mathcal{E}$-linear additive category. Then $\mathcal{A}$ is called

(1) $\mathcal{E}$-*proper*, if the functor internal Hom functor $\hom_{\mathcal{A}}(-,-) : \mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{\mathrm{op}} \to \mathcal{P}_1^{\Sigma}(\mathcal{E})$ has its essential image contained in $\mathcal{E}$.

(2) $\mathcal{E}$-*smooth*, if the internal Hom functor $\hom_{\mathcal{A}}(-,-) \in \mathsf{Fun}_{\mathcal{E}}^{\mathsf{add}}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{op}, \mathcal{P}_1^{\Sigma}(\mathcal{E}))$ is compact-projective.

Example 5.5. Let $\mathcal{E}^{\otimes} \simeq \mathsf{Vect}_{\mathbb{K}}^{\mathrm{fin},\otimes}$. Then a $\mathbb{K}$-linear category $\mathcal{A}$ is $\mathsf{Vect}_{\mathbb{K}}^{\mathrm{fin}}$-proper if and only if for all objects a, b the Hom-space $\hom_{\mathcal{A}}(a,b)$ is finite-dimensional.

Using this definition, we can now classify dualizable $\mathcal{E}$-linear additive categories:

Proposition 5.8. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category and $\mathcal{A}$ an $\mathcal{E}$-linear additive category. Then $\mathcal{A}$ is dualizable in $\mathsf{add}_{\mathcal{E}}^{\amalg,\otimes}$, if and only if $\mathcal{A}$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper.*

Proof. The proof is completely analogous to the proof of Proposition 5.22. □

Therefore, to understand locally rigid algebras, it remains to understand adjoints in $\mathbb{add}_{\mathcal{E}}^{\amalg}$. It follows from Proposition 5.1 that the symmetric monoidal $(\infty, 2)$-functor

$$\mathcal{P}_1^{\Sigma}(-) : \mathbb{cat}_{\mathcal{E}}^{\amalg,\otimes} \to \mathbb{pr}_{\mathcal{E}}^{\mathsf{L},\otimes}$$

restricts to an equivalence of symmetric monoidal $(\infty,2)$-categories $\mathcal{P}_1^{\Sigma}(-) : \mathbb{add}_{\mathcal{E}}^{\amalg,\otimes} \to \mathbb{pr}_{\mathrm{cp},\mathcal{E}}^{\mathsf{L},\otimes}$. Therefore, instead of analyzing (locally) rigid algebras in $\mathbb{add}_{\mathcal{E}}^{\amalg,\otimes}$, we can analyze them in the symmetric monoidal $(\infty,2)$-category $\mathbb{pr}_{\mathrm{cp},\mathcal{E}}^{\mathsf{L},\otimes}$:

Theorem 5.9. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category. An $\mathcal{E}$-linear monoidal additive category $\mathcal{A}^{\otimes}$ is locally rigid in $\mathbb{add}_{\mathcal{E}}^{\amalg,\otimes}$ if and only if*

(1) $\mathcal{A}$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper.

(2) every compact-projective object admits a dual.

(3) $\mu^R(\mathbb{1}_{\mathcal{A}}) \in \mathcal{P}_1^{\Sigma}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A})$ is compact-projective.

Moreover, then $\mathcal{A}^{\otimes}$ is also a rigid algebra.

Proof. The first part follows directly from Proposition 3.2. It remains to prove the last statement. Therefore, it suffices to show that the right adjoint of the unit morphism

$$\eta : \mathcal{P}_1^{\Sigma}(\mathcal{E}) \to \mathcal{P}_1^{\Sigma}(\mathcal{A})$$

preserves compact-projective objects. But the right adjoint functor is given by the internal Hom-functor $\hom_{\mathcal{P}_1^{\Sigma}(\mathcal{A})}(\mathbb{1}_{\mathcal{A}}, -)$ that preserves compact-projective since $\mathcal{A}$ is $\mathcal{E}$-proper. □

Our next goal is to understand the relation between conditions (1) – (3) and multi-fusion categories more explicitly. The condition of $\mathcal{E}$-properness is easy to understand. Indeed, an $\mathcal{E}$-linear additive category $\mathcal{A}$ is $\mathcal{E}$-proper if and only if the $\mathcal{E}$-action on $\mathcal{A}$ is closed. The condition of $\mathcal{E}$-smoothness on the other hand is more subtle and related to separability and hence, semisimplicity.

Example 5.6. Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category and A be a smooth algebra[16] in $\mathcal{P}_1^{\Sigma}(\mathcal{E})$. It follows as in the proof of Proposition 5.29 that the category $\mathsf{RMod}_A(\mathcal{P}^{\Sigma}(\mathcal{E}))$ is $\mathcal{E}$-smooth if and only if A is a smooth algebra object in $\mathcal{P}^{\Sigma}(\mathcal{E})$. If $\mathcal{A}$ is smooth, then the regular bimodule is compact-projective and hence, the multiplication map

$$\mu : A \otimes_{\mathcal{E}} A \rightleftarrows A : \Delta$$

admits a section Δ as a morphism of A-bimodule. As a consequence, in the context of additive categories, smooth algebras are the same separable algebras. Analogously, it follows that the category is $\mathcal{E}$-proper if and only if the underlying object of the algebra A is compact-projective in $\mathcal{P}_1^{\Sigma}(\mathcal{E})$, i.e. an object of $\mathcal{E} \subset \mathcal{P}_1^{\Sigma}(\mathcal{E})$. In case, that $\mathcal{E}^{\otimes} \simeq \mathsf{rmod}_R^{\otimes}$ for a commutative ring R, the category rmod_A of modules over a separable R-algebra A is always semisimple and the algebra A is proper if and only if A is a finite projective R-module.

Proposition 5.10. *[Ste23, Lem.4.2.5] Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal additive category and let $\mathcal{A} \in \mathsf{pr}^{\mathsf{L}}_{\mathcal{E}}$ be an $\mathcal{E}$-smooth cp-generated presentable additive category. Then there exists a smooth algebra object $S \in \mathsf{Alg}(\mathcal{P}_1^{\Sigma}(\mathcal{E}))$ and an equivalence $\mathcal{A} \simeq \mathrm{LMod}_S(\mathcal{P}_1^{\Sigma}(\mathcal{E}))$.*

This proposition has already been proven in [Ste23, Lem.4.2.5]. For completeness, we provide a sketch of the proof here.

Proof Sketch. We will construct a compact-projective generator for $\mathcal{A}$. Since $\mathcal{A}$ is cp-generated, we can choose a collection of generators of compact projective objects $\{A_t\}_{t \in T}$. As T might be infinite, the sum $\bigoplus_{t \in T} A_t$ may not be projective. So we have to single out a finite collection of generators. Therefore, denote for any finite subset $S \subset T$ by $\mathcal{A}_S$ the smallest full subcategory containing $\{A_s\}_{s \in S}$ that is closed under colimits and the $\mathcal{E}$-action. Note that $\mathcal{A} \simeq \mathsf{colim}_{S \subset_{\mathsf{fin}} T} \mathcal{A}_S$ can be written as a filtered colimit of the $\mathcal{A}_S$. Since $\mathcal{A}$ is $\mathcal{E}$-smooth, it follows that the identity functor $\mathrm{id}_{\mathcal{A}} \in \mathsf{Fun}^{\mathsf{add}}_{\mathcal{E}}(\mathcal{A}, \mathcal{A})$ is compact. Hence, there exists some finite subset $S \subset T$, s.t. $\mathrm{id}_{\mathcal{E}}$ factors through $\mathcal{A}_S$. Hence, the inclusion of $\mathcal{A}_S$ admits a section and $\mathcal{A} \simeq \mathcal{A}_S$. Therefore, $\bigoplus_{s \in S} A_s$ is a compact-projective generator for $\mathcal{A}$. □

Recall that every cp-generated presentable additive category is in particular an abelian category (see Example 5.1). Since these are large abelian categories, we need a version of semisimplicity for large categories:

Definition 5.8. A cp-generated Grothendieck abelian category $\mathcal{A}$ is called *semisimple* if every short exact sequence in $\mathcal{A}$ splits.

Example 5.7. Let $\mathcal{A}$ be a semisimple small abelian category with enough projectives. Then the category $\mathrm{Ind}(\mathcal{A})$ is a semisimple Grothendieck abelian category. In particular, for every fusion category $\mathcal{C}$ the category $\mathrm{Ind}(\mathcal{C})$ is semisimple Grothendieck abelian.

Remark 5.5. Let $\mathcal{A}$ be a small semisimple abelian category. Then every object of $\mathcal{A}$ is projective and we have a chain of equivalences $\mathcal{A} \simeq \mathcal{A}^{\mathrm{p}} \simeq \mathcal{P}^{\mathrm{fin}}_{1,\amalg}(\mathcal{A})$. The latter makes sense, since every finite colimit in a semisimple abelian category $\mathcal{A}$ can be expressed as a retract of a coproduct. In particular, it follows that $\mathcal{P}_1^{\Sigma}(\mathcal{A}) \simeq \mathrm{Ind}(\mathcal{A})$.

Proposition 5.11. *Let $\mathcal{E}^{\otimes}$ be a semisimple rigid symmetric monoidal additive category and let $\mathcal{A}$ be a cp-generated $\mathcal{E}$-linear presentable additive category. If $\mathcal{A}$ is smooth, then it is semisimple.*

Proof. Since every short exact sequence between projective objects splits, it suffices to show that every object of $\mathcal{A}$ is projective. It follows from Proposition 5.10 that there exists a smooth algebra object $S \in \mathsf{Alg}(\mathcal{P}_1^{\Sigma}(\mathcal{E}))$,

[16]In the sense of Definition 4.5

s.t. $\mathcal{A} \simeq \mathrm{LMod}_S(\mathcal{P}^\Sigma_1(\mathcal{E}))$. Since S is smooth, the multiplication map $m : S \otimes_\mathcal{E} S \to S$ admits a section Δ as a S-bimodule map. It follows that for every S-module M the composite

$$M \simeq S \otimes_S M \to (S \otimes_\mathcal{E} S) \otimes_S M) \simeq S \otimes_\mathcal{E} M \to M \otimes_S S \simeq M$$

exhibits M as a retract of a free S-module. Since M is projective in the semisimple category $\mathcal{P}^\Sigma_1(\mathcal{E})$ also $M \otimes S$ is projective in $\mathsf{RMod}_S(\mathcal{P}^\Sigma_1(\mathcal{E}))$. Hence M is projective in $\mathsf{RMod}_S(\mathcal{P}^\Sigma_1(\mathcal{E}))$ as it is the retract of a free module. □

Corollary 5.12. *Let $\mathbb{K}$ be a perfect field, then a small $\mathbb{K}$-linear additive category $\mathcal{A}$ is dualizable in $\mathsf{add}^{\amalg,\otimes}_\mathbb{K}$ if and only if it is semisimple and all Hom-spaces are finite-dimensional.*

Proof. Since $\mathbb{K}$ is a perfect field, a $\mathbb{K}$-algebra is separable if and only if it is semisimple. Hence, it follows from the proof of Proposition 5.11 that every smooth and proper $\mathbb{K}$-linear additive category $\mathcal{A}$ is equivalent to the category of finite-dimensional modules $\mathsf{rmod}^{\mathrm{fin}}_A(\mathsf{Vect}_\mathbb{K})$ over a finite-dimensional, semisimple algebra A. On the other hand, it follows from Example 5.6 that for a $\mathbb{K}$-algebra A the category $\mathsf{rmod}^{\mathrm{fin}}_A(\mathsf{Vect}_\mathbb{K})$ is $\mathbb{K}$-smooth and $\mathbb{K}$-proper if and only if A is finite-dimensional and semisimple. □

Remark 5.6. A similar observation was made by Tillmann in the study of modular functors [Til98].

It remains to understand the underlying object of the canonical algebra $\mu^R(\mathbb{1}_\mathcal{A}) \in \mathcal{P}^\Sigma_1(\mathcal{A} \otimes_\mathcal{E} \mathcal{A})$. In case, that $\mathcal{P}^\Sigma_1(\mathcal{A})$ is semisimple, every object in $\mathcal{P}^\Sigma_1(\mathcal{A})$ can be written as a direct sum of simples indexed by a small set. In particular, these direct sums could be infinite. Those that arise as finite direct sums of simples admit the following interpretation:

Proposition 5.13. *[Ste23, Prop.2.3.2] Let $\mathcal{C}$ be a semisimple Grothendieck abelian category. Then an object in $\mathcal{C}$ is compact if and only if it is equivalent to a finite direct sum of simples.*

In case that $\mathcal{E}$ is the category of vector spaces over an algebraically closed field $\mathbb{K}$, we obtain the following explicit description:

Proposition 5.14. *Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}$ a cp-rigid semisimple $\mathbb{K}$-linear additive category. Then $\mu^R(\mathbb{1}_\mathcal{A})$ is compact-projective if and only if $\mathcal{A}$ has finitely many simples.*

Proof. Denote by I the small set of isomorphism classes of simple objects in $\mathcal{A}$. We claim that $\bigoplus_{i\in I} S_i \otimes S_i^*$ satisfies for all compact-projective objects $x, y \in \mathcal{A}$ the defining property of $\mu^R(\mathbb{1}_\mathcal{A})$. This follows from the chain of equivalences:

$$\begin{aligned}\hom_{\mathcal{A}\otimes_\mathbb{K}\mathcal{A}}(x \otimes_\mathbb{K} y, \bigoplus_{i\in I} S_i \otimes_\mathbb{K} S_i^*) &\simeq \bigoplus_{i\in I} \hom_\mathcal{A}(x, S_i) \otimes \hom_\mathcal{A}(y, S_i^*) \simeq \bigoplus_{i\in I} \hom_\mathcal{A}(x, S_i) \otimes \hom_\mathcal{A}(S_i, y^*) \\ &\simeq \hom_\mathcal{A}(x, y^*) \simeq \hom_\mathcal{A}(x \otimes y, \mathbb{1}_\mathcal{A}),\end{aligned}$$

where we have used the duality in the third step and semisimplicity in the fourth. The claim then follows from Proposition 5.13. □

Remark 5.7. We have used explicitly in the above calculation that we are working over an algebraically closed field. In general, the canonical algebra admits a description as a more complicated coend [SW21].

Proof of Theorem 5.7. We need to check that the conditions of Theorem 5.9. The claim then follows from combining Corollary 5.12 and Proposition 5.14. □

To finish our discussion of multi-fusion categories, we interpret our result in the context of fully extended TFTs. Let $\mathbb{K}$ be a field and $\mathcal{A}$ a $\mathbb{K}$-linear fusion category. If the field $\mathbb{K}$ is of characteristic zero, it has been shown in [DSPS20, Cor.3.4.8] that fusion categories form the fully dualizable objects in the symmetric monoidal 3-category $\mathcal{T}\mathsf{en}_{\mathbb{K}}^{\otimes}$ [DSPS20, Cor.2.6.8] of finite $\mathbb{K}$-linear tensor categories, a symmetric monoidal sub-3-category of the even higher Morita category $\mathbb{M}\mathsf{or}(\mathbb{pr}_{\mathbb{K}}^{\mathsf{L}})^{\otimes}$ [JFS17] of $\mathbb{K}$-linear presentable 1-categories.
However, this is only true for specific classes of fields. For a general perfect field $\mathbb{K}$ a (multi-)fusion category only defines a so-called *non-compact* 3-dimensional framed fully extended TFT [Lur08, Def.4.2.10]. Informally this means, that the corresponding fully extended 3-dimensional TFT is only defined on 3-manifolds that admit non-trivial boundary. The reason is that multi-fusion categories over a general field, don't satisfy the dualizability condition that is described in $(\infty, 3)$-category of framed 3-dimensional cobordisms $\mathsf{Bord}_3^{\mathrm{fr},\otimes}$ by the attachment of 3-handles to a 3-manifold.
To fix this, one needs an extra condition called *separability* to ensure that $\mathcal{A}$ is fully dualizable [DSPS20, Thm.3.4.10]. A multi-fusion category over a perfect field $\mathbb{K}$ is called *separable* if the canonical algebra $\mathcal{F}_{\mathcal{A}}$ is also smooth, i.e. dualizable in $\mathsf{BMod}_{\mathcal{F}_{\mathcal{A}}}(\mathcal{A}^{\mathsf{e}})$ [DSPS20, Def.2.5.8]. Equivalently $\mathcal{F}_A$ has to be a compact-projective object in $\mathsf{BMod}_{\mathcal{F}_{\mathcal{A}}}(\mathcal{A}^{\mathsf{e}})$. Over a field of characteristic 0, every multi-fusion category is separable. As the following example shows, this is not necessarily true in characteristic $p \neq 0$.

Example 5.8. Let $\mathbb{K} = \mathbb{F}_p$ be a finite field and $G = C_p$. The category $\mathsf{Vect}_{C_p}^{\mathrm{fin}}$ of C_p-graded vector spaces is a multi-fusion category, however it is not separable. For this, it suffices to show that $\mathsf{BMod}_{\mathcal{F}_{\mathcal{A}}}(\mathcal{A}^{\mathsf{e}})$ is not semisimple [DSPS20, Cor.2.5.10].
Therefore, note that $\mathsf{BMod}_{\mathcal{F}_{\mathcal{A}}}(\mathcal{A}^{\mathsf{e}})$ is equivalent to the Drinfeld center $\mathcal{Z}_{\mathbb{E}_1}(\mathsf{Vect}_{C_p}^{\mathrm{fin}})$. The Drinfeld center can be equivalently described as the category $\mathsf{Fun}(C_p/^{adj}C_p, \mathsf{Vect}_{\mathbb{F}_p}^{\mathrm{fin}})$, where $C_p/^{adj}C_p$ denotes the quotient groupoid of the adjoint action of C_p on itself. Pulling back along the projection $p : C_p/^{adj}C_p \to */C_p$ induces a fully faithful inclusion

$$p^* : \mathsf{Rep}_{\mathbb{F}_p}(C_p) \simeq \mathsf{Fun}(*/C_p. \mathsf{Vect}_{\mathbb{F}_p}^{\mathrm{fin}}) \hookrightarrow \mathsf{Fun}(C_p/^{adj}C_p, \mathsf{Vect}_{\mathbb{F}_p}).$$

But as the category $\mathsf{Rep}_{\mathbb{F}_p}(C_p)$ is not semisimple, the same is true for $\mathsf{BMod}_{\mathcal{F}_{\mathcal{A}}}(\mathcal{A}^{\mathsf{e}})$, and hence $\mathcal{F}_{\mathcal{A}}$ is not smooth.

5.3 Stable ∞-Categories

In the last section, we analyzed locally rigid additive 1-categories to understand rigid abelian categories and multi-fusion categories. A more refined invariant of an abelian category is its derived ∞-category [Lur17, Sect.1.3.2]. This ∞-category is most naturally described using the language of stable ∞-category [Lur17, Sect.1.1].
One big advantage of the theory of derived ∞-categories over the one of abelian categories is that all right/left exact functors between abelian categories induce exact functors between their derived ∞-categories. This is especially interesting for the study of 3-dimensional TFTs, where many restrictions on the input data are imposed, to ensure that all constructions are exact [EGNO16]. Therefore, these constructions could potentially be performed for a broader class of examples at the level of derived ∞-categories. Our goal in this section is to develop an analogue of the discussion presented in the last section for stable ∞-categories. Therefore, recall that an object c in a presentable ∞-category $\mathcal{C}$ is called *compact* if the representable functor

$$\mathsf{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathcal{S}$$

preserves filtered colimits. We denote by $\mathcal{C}^c$ its full subcategory of compact object, and call $\mathcal{C}$ compactly generated, if the smallest subcategory of $\mathcal{C}$ that contains $\mathcal{C}^c$ and is closed under colimits is $\mathcal{C}$ itself.

We denote by $\mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}$ the ∞-category of small, idempotent complete, finitely cocomplete ∞-categories and right exact functors and by $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{c}}$ the ∞-category of compactly generated presentable ∞-categories and compact object preserving cocontinuous functors. As explained in Section 4.1, these ∞-categories can be equipped with symmetric monoidal structures induced by the Deligne-Lurie tensor product. We denote by St the full subcategory of $\mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}$ generated by the idempotent complete, stable ∞-categories. Note that it follows from the definition of a stable ∞-category, that every right exact functor between stable ∞-categories is automatically exact [Lur17, Prop.1.4.1].

Our first goal in this section is to show that this subcategory is also closed under the symmetric monoidal structure by exhibiting it as a category of modules over an idempotent algebra.

Proposition 5.15. *The Ind-completion induces an equivalence* $\mathrm{Ind}(-) : \mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}} \to \mathsf{Pr}^{\mathsf{L}}_{\mathrm{c}}$ *with inverse given by mapping a compactly generated presentable ∞-category to its full subcategory of compact objects* $(-)^{\mathrm{c}} : \mathsf{Pr}^{\mathsf{L}}_{\mathrm{c}} \to \mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}$. *Moreover, these functors restrict to an equivalence* $\mathrm{Ind}(-) : \mathsf{St} \to \mathsf{Pr}^{\mathsf{L}}_{\mathrm{st,c}}$

Proof. The first part is [Lur17, Lem.5.3.2.9]. It follows from [Lur17, Prop.1.1.3.6] that for every stable ∞-category $\mathcal{C}$, the Ind-completion $\mathrm{Ind}(\mathcal{C})$ is also a stable ∞-category. The other direction follows from the observation that the subcategory of compact objects is closed under finite colimits and retracts. Hence, for every $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ the subcategory of compact objects $\mathcal{C}^c$ itself idempotent complete and stable. □

Proposition 5.16. *The ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$ of finite spectra [Lur17, Sect.1.4] is an idempotent algebra in* $\mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}$. *The forgetful functor* $\mathsf{RMod}_{\mathcal{S}\mathrm{p}^{\mathrm{fin}}}(\mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}) \to \mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}$ *is fully faithful with essential image* St.

Proof. Recall, that the ∞-category $\mathcal{S}\mathrm{p}$ is compactly generated with subcategory of compact objects given by the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$ and the smash-product monoidal structure on $\mathcal{S}\mathrm{p}^{\otimes}$ preserves compact objects. Hence, $\mathcal{S}\mathrm{p}^{\otimes}$ defines an idempotent algebra in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathrm{c}}$ and therefore $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$ is an idempotent algebra in $\mathsf{Cat}^{\mathsf{rex},\otimes}_{\mathrm{ic}}$. We, therefore, obtain a commutative square of symmetric monoidal functors

$$\begin{array}{ccc} \mathsf{RMod}_{\mathcal{S}\mathrm{p}}(\mathsf{Pr}^{\mathsf{L}}_{\mathrm{c}}) & \xrightarrow{\mathrm{fgt}} & \mathsf{Pr}^{\mathsf{L}}_{\mathrm{c,st}} \\ {\scriptstyle (-)^{\mathrm{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{c}}} \\ \mathsf{RMod}_{\mathcal{S}\mathrm{p}^{\mathrm{fin}}}(\mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}}) & \xrightarrow{\mathrm{fgt}} & \mathsf{Cat}^{\mathsf{rex}}_{\mathrm{ic}} \end{array}$$

All functors are fully faithful and the upper horizontal and left vertical functors are further essentially surjective. Hence, the functor fgt induces an equivalence on the essential image of the right vertical functor. But this is given by the ∞-category St. □

For later use, let us also record the $(\infty, 2)$-categorical variant of the above equivalence. Let $\mathcal{E}^{\otimes}$ be a symmetric monoidal stable ∞-category. As in the additive case, we adopt the convention and denote the symmetric monoidal $(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{Ind}(\mathcal{E})}$ by $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{E}}$ and call $\mathrm{Ind}(\mathcal{E})$-linear ∞-categories simply $\mathcal{E}$-linear.

We denote by $\mathbb{S}\mathsf{t}^{\otimes}_{\mathcal{E}}$ the locally full sub-$(\infty, 2)$-category of the symmetric monoidal $(\infty, 2)$-category $\mathbb{C}\mathsf{at}^{\otimes,\mathsf{rex}}_{\mathrm{ic},\mathcal{E}}$ generated by the $\mathcal{E}$-linear stable ∞-categories. Similarly, we denote by $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathrm{c},\mathcal{E}}$ the locally full sub-$(\infty, 2)$-category of $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{E}}$ generated by compactly generated $\mathcal{E}$-linear presentable ∞-categories and compact object preserving $\mathcal{E}$-linear functors. We can then proof the following extension of Proposition 5.15:

Proposition 5.17. *Let $\mathcal{E}^{\otimes}$ be a symmetric monoidal stable ∞-category. The symmetric monoidal $(\infty, 2)$-functor* $\mathrm{Ind}(-) : \mathbb{S}\mathsf{t}^{\otimes}_{\mathcal{E}} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{E}}$ *induces an equivalence of symmetric monoidal $(\infty, 2)$-categories* $\mathbb{S}\mathsf{t}^{\otimes}_{\mathcal{E}} \simeq \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{c},\mathcal{E}}$.

Proof. It suffices to show that it induces an equivalence on spaces of objects and an equivalence on ∞-categories of morphisms. It follows from Proposition 5.15 that the functor induces an equivalence on spaces of objects. For the equivalence on morphism ∞-categories note, that restriction functor

$$\mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A})) \to \mathsf{Fun}^{\mathrm{ex}}_{\mathcal{E}}(\mathcal{A}, \mathrm{Ind}(\mathcal{A}))$$

is an equivalence of ∞-categories. Since $\mathrm{Ind}(\mathcal{A})^{\mathrm{c}} \simeq \mathcal{A}$, this functor identifies the full subcategory of $\mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A}))$ generated by compact object preserving functors with those $\mathcal{E}$-linear exact functors $F : \mathcal{A} \to \mathrm{Ind}(\mathcal{A})$ whose essential image lies in $\mathcal{A}$. □

We now turn to the classification of locally rigid algebras in the $(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{st}}$. Therefore, we need to determine the atomic objects:

Proposition 5.18. *Let $\mathcal{C}$ be a presentable stable ∞-category. An object $C \in \mathcal{C}$ is $\mathcal{S}\mathrm{p}$-atomic if and only if it is compact. Further, $\mathcal{C}$ is atomically generated if and only if $\mathcal{C}$ is compactly generated.*

Proof. Since the ∞-category of spectra $\mathcal{S}\mathrm{p}^{\otimes}$ is an idempotent algebra, it follows that an object is atomic if and only if the internal Hom-functor $\hom_{\mathcal{C}}(c, -) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ preserves small colimits. Since $\mathcal{S}\mathrm{p}$ is a stable ∞-category, finite limits, and colimits coincide. Hence, for every $c \in \mathcal{C}$ the functor $\hom_{\mathcal{C}}(c, -) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ preserves finite colimits. Further, it follows from [BMS24, Prop. 2.8] $\hom_{\mathcal{C}}(c, -)$ preserves filtered colimits if and only if $\mathsf{Map}(c, -)$ preserves filtered colimits. In particular, a stable ∞-category is atomically-generated, if and only if it is compactly-generated [Lur09a, 5.5.7.1]. □

We can then conclude from Proposition 4.19:

Proposition 5.19. *Let $\mathcal{C}^{\otimes}$ be a compactly generated presentably monoidal stable ∞-category. Then, $\mathcal{C}^{\otimes}$ is locally rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{st}}$ if and only if every compact object is dualizable. It is rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathrm{st}}$ if and only if compact and dualizable objects coincide.*

Definition 5.9. Let $\mathcal{E}^{\otimes} \in \mathsf{CAlg}(\mathsf{St})$ be a symmetric monoidal stable ∞-category. We say $\mathcal{E}^{\otimes}$ is rigid if $\mathrm{Ind}(\mathcal{E})^{\otimes}$ is a rigid algebra in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathrm{st}}$. Equivalently, $\mathcal{E}^{\otimes}$ is rigid if any object of $\mathcal{E}$ admits a left and a right dual.

Proposition 5.20. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category. A compactly generated $\mathcal{E}$-linear presentable ∞-category $\mathcal{C}$ is locally rigid in $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{E}}$ if and only if every compact object admits a dual. Further, it is rigid if the dualizable and compact objects coincide.*

Proof. It follows from Proposition 4.20 that $\mathcal{C}$ is $\mathrm{Ind}(\mathcal{E})$-atomically generated and the $\mathcal{E}$-atomic objects coincide with the compact objects. Hence, it follows that $\mathcal{C}$ is locally rigid if every compact object admits a dual. The statement about rigidity follows analogously. □

Remark 5.8. For the rigid symmetric monoidal stable ∞-category $\mathcal{D}(\mathbb{C})^{\otimes}$ this recovers the notion of rigidity for presentable dg-categories from [Gai15].

5.4 Derived Multi-Fusion Categories

Our computations in Section 5.2 have shown that we can characterize multi-fusion categories over an algebraically closed field $\mathbb{K}$ as rigid algebras in the $(\infty, 2)$-category of $\mathbb{K}$-linear small additive categories. Therefore, it is an interesting question, what the analogue for stable ∞-categories would be. Following our intuition from the additive case, we adopt the following terminology:

Definition 5.10. Let $\mathcal{E}^\otimes$ be a rigid symmetric monoidal stable ∞-category. A rigid algebra in $\mathbb{S}\mathsf{t}_{\mathcal{E}}^{\otimes}$ is called an $\mathcal{E}$-linear *derived multi-fusion category.*

Notation 5.1. If $\mathcal{E}^\otimes$ is equivalent to the ∞-category of perfect complexes $\mathsf{Perf}(R)^\otimes$ over $R \in \mathsf{CAlg}(\mathcal{S}\mathrm{p})$, then we call a $\mathsf{Perf}(R)$-linear derived multi-fusion category simply R-linear.

Remark 5.9. For the sake of this text, the term derived multi-fusion category is completely motivated by our results in the additive case. In particular, we will not show that these objects satisfy any higher dualizability conditions as in the case of multi-fusion categories.

As classical fusion categories do, we expect that also derived multi-fusion categories satisfy dualizability in the symmetric monoidal $(\infty, 3)$-category $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}_{\mathcal{E}}^{\mathsf{L}})^\otimes$. This $(\infty, 3)$-category has objects presentably monoidal $\mathcal{E}$-linear ∞-categories, 1-morphisms $\mathcal{E}$-linear bimodule categories, 2-morphisms cocontinuous bimodule functors, and 3-morphisms bimodule transformations. We do not expect derived multi-fusion categories in the sense of Definition 5.10 to be fully dualizable in this symmetric monoidal $(\infty, 3)$-category. Instead, we expect them to define, as ordinary multi-fusion categories do, only non-compact $3d$-TFTs[17]. We expect that derived multi-fusion categories need to satisfy an extra smoothness condition to describe a fully dualizable object. More precisely, we expect that every derived multi-fusion category $\mathcal{A}^\otimes$, s.t. the canonical algebra $\mathcal{F}_{\mathcal{A}} \in \mathcal{A}^{\mathsf{e}}$ is a smooth algebra in $\mathcal{A}^{\mathsf{e}}$ defines a fully dualizable object in $\mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}_{\mathcal{E}}^{\mathsf{L}})^\otimes$.
After this motivational discussion, we now analyze the conditions imposed on a derived multi-fusion category in more detail. Using the equivalence $\mathbb{S}\mathsf{t}_{\mathcal{E}}^{\otimes} \simeq \mathbb{P}\mathsf{r}_{\mathrm{c},\mathcal{E}}$ and Proposition 3.2, we can rephrase Definition 5.10 as follows:

Proposition 5.21. *Let $\mathcal{E}^\otimes$ be a rigid symmetric monoidal stable ∞-category and $\mathcal{A}^\otimes$ an $\mathcal{E}$-linear monoidal stable ∞-category. Then $\mathcal{A}^\otimes$ is an $\mathcal{E}$-linear derived multi-fusion category if and only if*

(1) $\mathcal{A}$ is dualizable in $\mathsf{St}_{\mathcal{E}}^{\otimes}$.

(2) every object of $\mathcal{A}^\otimes$ has a left and right dual.

(3) the underlying object of the canonical algebra $\mu^R(\mathbb{1}_{\mathcal{A}}) \simeq \mathcal{F}_{\mathcal{A}} \in \mathrm{Ind}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A})$ is compact, i.e is an object of $\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}$.

The classification of dualizable objects in $\mathsf{St}_{\mathcal{E}}^{\otimes}$ proceeds analogously to that of dualizable objects in $\mathsf{add}_{\mathrm{ic}}^{\mathrm{II},\otimes}$, as discussed in Section 5.2, with the corresponding arguments carried out in detail in [HSS17, Sect.4]. For completeness, we sketch their arguments here.
For $\mathcal{A}$ an $\mathcal{E}$-linear stable ∞-category, we can consider the $\mathcal{E}$-linear functor

$$\mathcal{A} \to \mathsf{Fun}_{\mathcal{E}}^{\mathrm{ex}}(\mathcal{E}, \mathcal{A}) \simeq \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L},\mathrm{c}}(\mathrm{Ind}(\mathcal{E}), \mathrm{Ind}(\mathcal{A}))$$

that associates to every $a \in \mathcal{A}$ the $\mathcal{E}$-linear action functor $- \otimes a : \mathcal{E} \to \mathcal{A}$. Passing to right adjoints, we obtain an $\mathcal{E}$-linear functor

$$\Phi : \mathcal{A} \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L},\mathrm{c}}(\mathrm{Ind}(\mathcal{E}), \mathrm{Ind}(\mathcal{A})) \to \mathsf{Fun}_{\mathcal{E}}^{\mathsf{L}}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{E}))^{\mathrm{op}} \simeq \mathsf{Fun}_{\mathcal{E}}^{\mathrm{ex}}(\mathcal{A}, \mathrm{Ind}(\mathcal{E}))^{\mathrm{op}}$$

Note that for every stable ∞-category linear over a rigid symmetric monoidal stable ∞-category $\mathcal{E}$, also the opposite $\mathcal{A}^{\mathrm{op}}$ is $\mathcal{E}$-linear with $\mathcal{E}$-action given by the composite

$$\mathcal{E} \otimes \mathcal{A}^{\mathrm{op}} \xrightarrow{(-)^\vee \otimes \mathrm{id}_{\mathcal{A}^{\mathrm{op}}}} \mathcal{E}^{\mathrm{op}} \otimes \mathcal{A}^{\mathrm{op}} \xrightarrow{(-\otimes-)^{\mathrm{op}}} \mathcal{A}^{\mathrm{op}}$$

[17] Compare to the discussion at the end of Section 5.2

of first taking duals in $\mathcal{E}$ and then the opposite of the action functor. Hence, we may consider the opposite of the $\mathcal{E}$-linear functor Φ. The corresponding $\mathcal{E}$-linear functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Fun}^{\mathrm{ex}}_{\mathcal{E}}(\mathcal{A}, \mathrm{Ind}(\mathcal{E}))$ associates to every $a \in \mathcal{A}$ the functor $a' \mapsto \hom_{\mathcal{A}}(a, a') \in \mathrm{Ind}(\mathcal{E})$.

Definition 5.11. Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category and $\mathcal{A} \in \mathsf{St}_{\mathcal{E}}$ an $\mathcal{E}$-linear stable ∞-category. The *$\mathcal{E}$-linear Yoneda embedding* is the $\mathcal{E}$-linear functor

$$\begin{aligned} \mathcal{Y}_{\mathcal{E}} : \mathcal{A} &\to \mathsf{Fun}^{\mathrm{ex}}_{\mathcal{E}}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{E})) \\ a &\mapsto \hom_{\mathcal{A}}(a, -) \end{aligned}$$

adjoint to the functor constructed above.

Definition 5.12. Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category and $\mathcal{A} \in \mathsf{St}_{\mathcal{E}}$ an $\mathcal{E}$-linear stable ∞-category. The ∞-category $\mathcal{A}$ is called

(1) *$\mathcal{E}$-proper*, if the functor $\hom_{\mathcal{A}}(-,-) : \mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{E})$ has its essential image contained in $\mathcal{E}$.

(2) $\mathcal{E}$-smooth, if the functor $\hom_{\mathcal{A}}(-,-) \in \mathsf{Fun}^{\mathrm{ex}}_{\mathcal{E}}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{op}, \mathrm{Ind}(\mathcal{E}))$ is compact

We can then prove:

Proposition 5.22. *[HSS17, Prop.4.15] Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category, and let $\mathcal{A} \in \mathsf{St}_{\mathcal{E}}$ an $\mathcal{E}$-linear stable ∞-category. Then $\mathcal{A}$ is dualizable in $\mathsf{St}^{\otimes}_{\mathcal{E}}$ if and only if it is $\mathcal{E}$-smooth and $\mathcal{E}$-proper. In this case, the dual of $\mathcal{A}$ is given by $\mathcal{A}^{\mathrm{op}}$ equipped with the $\mathcal{E}$-action described above.*

Proof. It follows from [HSS17, Prop.4.10] that $\mathrm{Ind}(\mathcal{A})$ is dualizable in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{E}}$ with dual $\mathrm{Ind}(\mathcal{A}^{\mathrm{op}})$. Unraveling the construction, the evaluation of this duality

$$\mathrm{ev}_{\mathrm{Ind}(\mathcal{A})} : \mathrm{Ind}(\mathcal{A}) \otimes_{\mathrm{Ind}(\mathcal{E})} \mathrm{Ind}(\mathcal{A}^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{E})$$

is given by the Ind-extension of $\hom_{\mathcal{A}}(-,-) : \mathcal{A} \otimes_{\mathcal{E}} \mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ and the coevaluation

$$\mathrm{coev} : \mathrm{Ind}(\mathcal{E}) \to \mathrm{Ind}(\mathcal{A}) \otimes_{\mathrm{Ind}(\mathcal{E})} \mathrm{Ind}(\mathcal{A}^{\mathrm{op}}) \simeq \mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A}))$$

is given by the unique $\mathrm{Ind}(\mathcal{E})$-linear functor that maps $\mathbb{1}_{\mathcal{E}}$ to $\mathrm{id}_{\mathrm{Ind}(\mathcal{A})} \in \mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A}))$. Since the inductive completion $\mathrm{Ind}(-)$ induces an equivalence between the symmetric monoidal ∞-categories $\mathsf{St}^{\otimes}_{\mathcal{E}}$ $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathrm{c},\mathrm{Ind}(\mathcal{E})}$, it follows that $\mathcal{A}$ is dualizable in $\mathsf{St}^{\otimes}_{\mathcal{E}}$ if and only if the above evaluation and coevaluation functors preserve compact objects. However, the evaluation functor $\mathrm{ev}_{\mathrm{Ind}(\mathcal{A})}$ preserves compact objects precisely when $\mathcal{A}$ is $\mathcal{E}$-proper, while the coevaluation functor $\mathrm{coev}_{\mathrm{Ind}(\mathcal{A})}$ preserves compact objects if and only if $\mathcal{A}$ is $\mathcal{E}$-smooth. ☐

Remark 5.10. In case $\mathcal{E}^{\otimes} \simeq \mathcal{D}^{\flat}(\mathbb{C})^{\otimes}$, the notions of smooth and proper prominently appear in the study of dg-categories of quasi-coherent sheaves in non-commutative algebra geometry. The name originates from the fact that the category of perfect complexes on a separated scheme of finite type X is smooth and proper in the above sense if and only if the scheme is smooth and proper [Orl16].

We can then simplify our characterization of Proposition 5.21 to:

Proposition 5.23. *Let $\mathcal{E}$ be a rigid symmetric monoidal stable ∞-category and $\mathcal{A}$ an $\mathcal{E}$-linear monoidal stable ∞-category. Then $\mathcal{A}$ is an $\mathcal{E}$-linear derived multi-fusion category if and only if*

(1) $\mathcal{A}$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper.

(2) every object of $\mathcal{A}$ has a dual.

(3) the canonical algebra $\mu^R(\mathbb{1}_{\mathcal{A}}) \in \mathrm{Ind}(\mathcal{A} \otimes_{\mathcal{E}} \mathcal{A})$ is compact.

Fusion Categories

Having defined derived multi-fusion categories in the previous section, we now turn to the study of examples. One possible approach to construct monoidal stable ∞-categories is as bounded derived categories of monoidal abelian categories. In particular, we may consider the bounded derived category of tensor categories or more specifically, of a multi-fusion category. As a suitable derived generalization of a multi-fusion category should, in particular, describe bounded derived ∞-categories of multi-fusion categories, verifying the latter provides a natural first consistency test of our definition. Let us therefore determine, whether this holds for our definition of derived multi-fusion category.
We denote by $\mathsf{Add}^{\mathsf{rex},\otimes}$ the ∞-category of small, finitely cocomplete, idempotent complete, additive ∞-categories and right exact functors. As a start, we recall the construction of the bounded derived ∞-category:

Definition 5.13. Let $\mathcal{A}$ be an idempotent complete, additive category. We call the image $\mathsf{K}^\flat(\mathcal{A})$ of $\mathcal{A}$ under the composite functor

$$\mathsf{K}^\flat(-) : \mathsf{add}_{\mathrm{ic}}^{\mathrm{II}} \xrightarrow{\mathcal{P}^{\mathrm{fin}}} \mathsf{Add}^{\mathsf{rex}} \longleftrightarrow \mathsf{Cat}^{\mathsf{rex}} \xrightarrow{-\otimes \mathcal{S}\mathrm{p}^{\mathrm{fin}}} \mathsf{St}$$

the ∞-category of *complexes with bounded homology* in $\mathcal{A}$.

For an additive category $\mathcal{A}$, denote by $\mathrm{Ch}^\flat(\mathcal{A})$ the dg-category of bounded complexes in $\mathcal{A}$. Its dg-nerve $\mathsf{N}_{\mathrm{dg}}(\mathrm{Ch}^\flat(\mathcal{A}))$ [Lur17, Constr. 1.3.1.6] is a quasi-category, whose homotopy category $\mathsf{hN}_{\mathrm{dg}}(\mathrm{Ch}^\flat(\mathcal{A}))$ coincides with the 1-category of complexes with bounded homology in $\mathcal{A}$. To justify our above notation, let us show that these two ∞-categories are equivalent:

Proposition 5.24. *Let $\mathcal{A}$ be an additive category. The stable ∞-category $\mathsf{K}^\flat(\mathcal{A})$ is equivalent to $\mathsf{N}_{\mathrm{dg}}(\mathrm{Ch}^\flat(\mathcal{A}))$. In particular, its homotopy category $\mathsf{hK}^\flat(\mathcal{A})$ coincides with the 1-category of complexes with bounded homology in $\mathcal{A}$.*

Proof. Denote by Add the category of idempotent complete, additive ∞-categories, and additive functors. It follows from [LMGR+24, Cor.3.4.10] that the ∞-category $\mathsf{N}_{\mathrm{dg}}(\mathrm{Ch}^\flat(\mathcal{A}))$ arises as the image of $\mathcal{A}$ under the restriction of the left adjoint of the inclusion $\mathsf{St} \hookrightarrow \mathsf{Add}$ to the subcategory $\mathsf{add}_{\mathrm{ic}}^{\mathrm{II}}$ of idempotent complete additive 1-categories. Therefore, it suffices to show that $\mathsf{K}^\flat(-)$ also provides such a left adjoint. Let $\mathcal{B}$ be an idempotent complete, stable ∞-category. The claim follows from the chain of equivalences

$$\mathsf{Fun}^{\mathrm{ex}}(\mathsf{K}^\flat(\mathcal{A}), \mathcal{B}) \simeq \mathsf{Fun}^{\mathsf{rex}}(\mathcal{P}_{\mathrm{II}}^{\mathrm{fin}}(\mathcal{A}) \otimes \mathcal{S}\mathrm{p}^{\mathrm{fin}}, \mathcal{B}) \simeq \mathsf{Fun}^{\mathsf{rex}}(\mathcal{P}_{\mathrm{II}}^{\mathrm{fin}}(\mathcal{A}), \mathcal{B}) \simeq \mathsf{Fun}^{\mathrm{II}}(\mathcal{A}, \mathcal{B}),$$

where we have used the universal property of $\mathcal{P}_{\mathrm{II}}^{\mathrm{fin}}(-)$ and $- \otimes \mathcal{S}\mathrm{p}^{\mathrm{fin}}$. □

For any abelian category $\mathcal{A}$, we can apply the functor $\mathsf{K}^\flat(-)$ to the full subcategory $\mathcal{A}^{\mathrm{p}}$ of projective objects. This ∞-category provides a model for the bounded derived ∞-category of $\mathcal{A}$ if $\mathcal{A}$ satisfies the following finiteness condition:

Definition 5.14. Let $\mathcal{A}$ be an abelian category. The *homological dimension* of an object $A \in \mathcal{A}$ is the least number $n \in \mathbb{N} \cup \{\infty\}$ s.t. there exists a projective resolution

$$P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A$$

with $n+1$-terms. The *homological dimension* of $\mathcal{A}$ is the supremum of the homological dimensions of all of its objects.

In case that $\mathcal{A}$ is the category of modules over an algebra A, we call the homological dimension of rmod_A the homological dimension of the algebra A.

Example 5.9. Let $\mathcal{A}$ be a semisimple, abelian category. Then every object of $\mathcal{A}$ is projective and hence, $\mathcal{A}$ has homological dimension 0.

Definition 5.15. Let $\mathcal{A}$ be an abelian category with enough projectives and of finite homological dimension. We call the ∞-category $\mathsf{K}^\flat(\mathcal{A}^{\mathrm{p}})$ the *bounded derived ∞-category* $\mathcal{D}^\flat(\mathcal{A})$ of $\mathcal{A}$.

For later use, we also record the Ind-completion of the functor $\mathsf{K}^\flat(-)$. Therefore, we consider the composite of symmetric monoidal functors

$$\mathsf{add}_{\mathrm{ic}}^{\amalg,\otimes} \xrightarrow{\mathcal{P}^{\Sigma}(-)} \mathsf{Pr}^{\mathsf{L},\otimes} \xrightarrow{-\otimes \mathcal{S}\mathrm{p}} \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathrm{st}}.$$

It follows that for any additive category $\mathcal{A}$ there exists a chain of equivalences

$$\mathrm{Ind}(\mathsf{K}^\flat(\mathcal{A})) \simeq \mathrm{Ind}(\mathcal{S}\mathrm{p}^{\mathrm{fin}} \otimes \mathcal{P}^{\mathrm{fin}}_{\amalg}(\mathcal{A})) \simeq \mathcal{S}\mathrm{p} \otimes \mathrm{Ind}(\mathcal{P}^{\mathrm{fin}}(\mathcal{A})) \simeq \mathcal{S}\mathrm{p} \otimes \mathcal{P}^{\Sigma}(\mathcal{A}).$$

Hence, the above functor associates to $\mathcal{A}$ the Ind-completion of $\mathsf{K}^\flat(\mathcal{A})$. In case, that $\mathcal{A}^{\mathrm{p}}$ arises as the subcategory of projective objects in an abelian category $\mathcal{A}$ with enough projectives, we can describe the ∞-categories $\mathcal{P}^{\Sigma}(\mathcal{A}^{\mathrm{p}})$ and $\mathcal{S}\mathrm{p} \otimes \mathcal{P}^{\Sigma}(\mathcal{A}^{\mathrm{p}})$ more explicitly.
For an abelian category $\mathcal{A}$ denote by $\mathrm{Ch}(\mathcal{A})$ the dg-category of unbounded complexes in $\mathcal{A}$. Its unbounded derived ∞-category $\mathcal{D}(\mathcal{A})$ is defined as the localization of the ∞-category of unbounded complexes $\mathsf{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ at the collection of quasi-isomorphisms. In particular, it follows from the formal properties of ∞-categorical localizations, that its homotopy 1-category agrees with the ordinary unbounded derived category $\mathsf{h}_1\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$. Further, denote by $\mathcal{D}_{\geq 0}(\mathcal{A})$ the full subcategory of $\mathcal{D}(\mathcal{A})$ spanned by complexes with homology concentrated in positive degrees. If $\mathcal{A}$ is equivalent to the category of modules over a ring R, we abuse notation and denote the corresponding derived category by $\mathcal{D}(R)$.

Proposition 5.25. *[LMGR+24, Prop.3.6.6] Let $\mathcal{A}$ be a cp-generated Grothendieck abelian category. Then*

(1) the presentable additive ∞-category $\mathcal{D}_{\geq 0}(\mathcal{A})$ is equivalent to $\mathcal{P}^{\Sigma}(\mathcal{A}^{\mathrm{cp}})$.

(2) the presentable stable ∞-category $\mathcal{D}(\mathcal{A})$ is equivalent to $\mathrm{Ind}(\mathsf{K}^\flat(\mathcal{A}^{\mathrm{cp}}))$.

After these preliminary observations, we can now prove that derived categories of multi-fusion categories indeed form derived multi-fusion categories.

Proposition 5.26. *Let $\mathcal{E}^{\otimes}$ be a rigid, semisimple, symmetric monoidal, additive category, and $\mathcal{A}^{\otimes}$ be a rigid algebra in $\mathsf{add}_{\mathcal{E}}^{\amalg,\otimes}$. Then its bounded derived category $\mathcal{D}^\flat(\mathcal{A})$ is a $\mathcal{D}^\flat(\mathcal{E})$-linear derived multi-fusion category.*

Proof. As a consequence of Proposition B.15 and Proposition B.16, all functors in Definition 5.13 extend to symmetric monoidal $(\infty,2)$-functors. Hence, the $(\infty,2)$-functor $\mathsf{K}^\flat(-)$ preserves rigid algebras. Since $\mathcal{E}$ is assumed to be semisimple, it follows from Proposition 5.11 that $\mathcal{A}$ is semisimple, and hence $\mathsf{K}^\flat(\mathcal{A}) \simeq \mathcal{D}^\flat(\mathcal{A})$. □

Corollary 5.27. *Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}$ a $\mathbb{K}$-linear multi-fusion category. Then $\mathcal{D}^\flat(\mathcal{A})$ is a $\mathbb{K}$-linear derived multi-fusion category.*

This serves as a first consistency check for our definition of derived multi-fusion category. In general, we expect the existence of examples of *non-semisimple* (finite) tensor categories whose bounded derived ∞-category forms a derived multi-fusion category. However, since these typically have infinite homological dimension, constructing their bounded derived ∞-category would require different techniques.

Moreover, it is a subtle question to determine the right notion of a derived category of a finite $\mathbb{K}$-linear tensor category $\mathcal{A}^{\otimes}$. The most straightforward approach is to consider the unbounded derived ∞-category $\mathcal{D}(\mathcal{A})^{\otimes}$. However, the underlying category of every finite tensor category is equivalent to the category of finite-dimensional modules over a finite-dimensional $\mathbb{K}$-algebra A [EGNO16, Def.1.8]. In particular, its unbounded derived ∞-category $\mathcal{D}(\mathcal{A})$ is equivalent to the ∞-category of modules $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(A) \simeq \mathrm{LMod}_A(\mathcal{D}(\mathbb{K}))$. However, the latter is only $\mathbb{K}$-smooth if the algebra A, and hence the category $\mathcal{A}$, has finite homological dimension [Orl16]. But a finite tensor category with finite homological dimension, necessarily has homological dimension 0 [EGNO16, Rem.6.1.4], and hence is a fusion category.

This shows that, to construct derived multi-fusion categories from finite tensor categories, that are *not* fusion categories, one has to consider a more refined version of the derived ∞-category construction. We intend to investigate this question systematically in a future work.

We conclude by highlighting an application of derived categories of multi-fusion categories. Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}^{\otimes}$ a $\mathbb{K}$-linear multi-fusion category. As $\mathcal{A}^{\otimes}$ is already fully dualizable in the symmetric monoidal 3-category $\mathcal{T}\mathsf{en}_{\mathbb{K}}^{\otimes}$, we expect that the fully extended TFT associated to $\mathcal{D}^{\flat}(\mathcal{A})^{\otimes}$ can be computed directly from the one associated to $\mathcal{A}^{\otimes}$. From the perspective of fully extended TFTs, the category $\mathcal{D}^{\flat}(\mathcal{A})^{\otimes}$ is therefore of limited interest.

More intriguing, however, are the TFTs that are relative to $\mathcal{D}^{\flat}(\mathcal{A})^{\otimes}$ (compare Section 4.5). A TFT relative to $\mathcal{D}^{\flat}(\mathcal{A})^{\otimes}$ is determined by an $\mathrm{Ind}(\mathcal{D}^{\flat}(\mathcal{A}))$-linear presentable stable ∞-category $\mathcal{C}$ satisfying the conditions of Proposition 4.28. Even in the special case where $\mathcal{A}^{\otimes} \simeq \mathsf{Vect}_{\mathbb{C}}^{\mathrm{fin},\otimes}$, one finds many interesting TFTs that are relative to $\mathcal{D}(\mathbb{C})^{\otimes}$ but not relative to $\mathsf{Vect}_{\mathbb{C}}^{\otimes}$.

Unraveling the conditions of Proposition 4.28 in these contexts, we see that a TFT relative to $\mathsf{Vect}_{\mathbb{C}}$ is induced by a finite, semisimple, $\mathbb{C}$-linear category, while a TFT relative to $\mathcal{D}(\mathbb{C})$ corresponds to a smooth, and proper dg-category. Such dg-categories play a central role in homological mirror symmetry [HKK17]. We expect that TFTs relative to $\mathrm{Ind}(\mathcal{D}^{\flat}(\mathcal{A}))$ provide the appropriate framework for describing fusion categorical symmetries in the context of homological mirror symmetry [DHL23].

$\mathbb{E}_2$-algebras

As a second consistency check, we compare our definition of a derived multi-fusion category with the definition of a fully dualizable $\mathbb{E}_2$-algebra from [Lur08].

Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. In [Lur08, Rem.4.1.27], the author classifies fully dualizable $\mathbb{E}_n$-algebras in the Morita $(\infty, n+1)$-category $\mathbb{M}\mathsf{or}_{\mathbb{E}_n}(\mathcal{V})^{\otimes}$ of $\mathbb{E}_n$-algebras [Hau17] using the theory of factorization homology.

To state this classification criterion, we recall the main properties of factorization homology relevant to our setting and refer the reader to [AF20] for a general introduction.

We denote by $\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}}$ the ∞-category whose objects are finitary smooth framed n-manifolds and whose morphisms are smooth framed embeddings [AF20, Def.2.5]. This ∞-category admits a symmetric monoidal structure given by the disjoint union of smooth manifolds. The monoidal unit is given by the empty manifold $\emptyset$, regarded as a framed n-manifold. We denote by D^n the n-dimensional open disk equipped with its standard framing. We write $\mathcal{D}\mathrm{isk}_n^{\mathrm{fr},\otimes} \subset \mathcal{M}\mathrm{fld}_n^{\mathrm{fr},\otimes}$ for the full symmetric monoidal subcategory generated by D^n.

Given any presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, a symmetric monoidal functor

$$\mathfrak{B} : \mathcal{D}\mathrm{isk}_n^{\mathrm{fr},\otimes} \to \mathcal{V}^{\otimes}$$

is called a $\mathcal{D}\mathrm{isk}_n^{\mathrm{fr}}$-*algebra*. It follows from [AF20] that there exists an equivalence of ∞-categories

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V}) \simeq \mathsf{Fun}^{\otimes}(\mathcal{D}\mathrm{isk}_n^{\mathrm{fr},\otimes}, \mathcal{V}^{\otimes}).$$

Hence, any $\mathbb{E}_n$-algebra $B \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V})$ determines an essentially unique $\mathcal{D}\mathrm{isk}_n^{\mathrm{fr}}$-algebra $\mathfrak{B}$. Factorization homology with values in B is defined as the symmetric monoidal functor

$$\begin{array}{ccc} \mathcal{D}\mathrm{isk}_n^{\mathrm{fr},\otimes} & \xrightarrow{\mathfrak{B}} & \mathcal{V}^{\otimes} \\ \big\downarrow & \nearrow_{\int_{-} B} & \\ \mathcal{M}\mathrm{fld}_n^{\mathrm{fr},\otimes} & & \end{array}$$

obtained via operadic left Kan extension. For any framed n-manifold M, the object $\int_M B$ is called the *factorization homology of M with values in B.*
We now draw some consequences of this definition. It follows from the definition that

$$\int_{\emptyset} B \simeq \mathbb{1}_{\mathcal{V}} \quad \text{and} \quad \int_{D^n} B \simeq B.$$

Further, for any $0 \leq k \leq n$ and any framed k-manifold N^k, the manifold $N^k \times D^{n-k}$ equipped with the product framing defines a framed n-manifold. Further, this construction can be applied to every $\mathcal{D}\mathrm{isk}_{n-k}$-algebra

$$\mathfrak{N}^k : \mathcal{D}\mathrm{isk}_{n-k}^{\mathrm{fr},\otimes} \xrightarrow{N\times -} \mathcal{M}\mathrm{fld}_n^{\mathrm{fr},\otimes}$$

and hence induces the structure of a $\mathbb{E}_{n-k}$-algebra on

$$\int_{N^k \times D^{n-k}} B \in \mathsf{Alg}_{\mathbb{E}_{n-k}}(\mathcal{V}).$$

One can analogously construct actions of these algebras by applying factorization on manifolds with boundary. To this end, we define analogously the symmetric monoidal ∞-category $\mathcal{M}\mathrm{fld}_1^{\partial,\mathrm{fr},\otimes}$ of finitary framed 1-manifolds with boundary. We further denote by $\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{fr},\otimes}$ the full symmetric monoidal subcategory generated by $\mathbb{R}^1$ and $\mathbb{R}_{\leq 0}$. Consider the fiber product

$$(\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{fr}})^{\otimes}_{/(-1,1]} := \mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{fr},\otimes} \times_{\mathcal{M}\mathrm{fld}_1^{\partial,\mathrm{fr},\otimes}} (\mathcal{M}\mathrm{fld}_1^{\partial,\mathrm{fr}})^{\otimes}_{/(-1,1]}$$

of ∞-operads. According to [AF20, Lem.3.21] the data of an algebra $\mathfrak{F}$ over the ∞-operad $(\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{fr}})^{\otimes}_{/(-1,1]}$ corresponds to the data of

- an algebra given by $\mathfrak{F}(\mathbb{R}^1)$
- and a right $\mathfrak{F}(\mathbb{R})$-module given by $\mathfrak{F}(\mathbb{R}_{\leq 0})$.

Let M be a framed smooth n-manifold and $f : M \to (-1,1]$ a continuous map, s.t. the restriction $f| : M|_{(-1,1)} \to (-1,1)$ is a smooth fiber bundle. We call such data a smooth manifold with *collar boundary.*

This structure induces a decomposition

$$M \simeq (\mathbb{R} \times M_{01}) \coprod M_1$$

with $\mathbb{R} \times M_{01} = f^{-1}((-1,1))$ and $M_1 = f^{-1}((-0.5,1]$. In particular, we obtain for any such $f : M \to (-1,1]$ a morphism of ∞-operads [AF20, Const.2.40]

$$(\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{fr}})^{\otimes}_{/(-1,1]} \xrightarrow{f^{-1}} (\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}})^{\otimes}_{/M} \longrightarrow \mathcal{M}\mathrm{fld}_n^{\mathrm{fr},\otimes}$$

that exhibits M_1 as a right module over $M_{01} \times \mathbb{R}$ in $\mathcal{M}\mathrm{fld}_n^{\mathrm{fr},\otimes}$. Pictorially, the module action is realized by embedding the cylinder $M_{01} \times \mathbb{R}$ into the open boundary of M_1. Finally, by the functoriality of factorization homology, this construction shows that for every $\mathbb{E}_n$-algebra B the factorization homology $\int_{M_1} B$ inherits the structure of a module over the factorization homology of $\int_{M_{01}} B$.
In particular, we can interpret for every $0 \leq k \leq n$ the n-disk D^n as the manifold $D^k \times D^{n-k}$ with collar boundary given by

$$S^{k-1} \times D^1 \times D^{n-k},$$

and n-framing given by the product of the bounding framing on $S^{k-1} \times D^1$ and the trivial framing on D^{n-k}. This exhibits for every $0 \leq k \leq n$ and every $\mathbb{E}_n$-algebra B the factorization homology $\int_{D^n} B \simeq B$ as a module over $\int_{S^{k-1} \times D^{n-k+1}} B$. We are now in the position to state the criterion of Lurie:

Claim 5.1. [Lur08, Rem.4.1.27] Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category, and $B \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V})$ an $\mathbb{E}_n$-algebra in $\mathcal{V}$. Then B is fully dualizable in the Morita category of $\mathbb{E}_n$-algebras $\mathsf{Mor}_{\mathbb{E}_n}(\mathcal{V})^{\otimes}$ if and only if, for every $0 \leq k \leq n$, B is dualizable as an $\int_{S^{k-1} \times D^{n-k+1}} B$-module.

Remark 5.11. Moreover, B induces a non-compact fully extended $n+1$-dimensional TFT if and only if it satisfies the above dualizability criterion for all $k \neq n$.

Since our main focus lies on $\mathbb{E}_2$-algebras, let us now discuss this case in detail:

- For $k = 0$ the factorization homology is given by

$$\int_{S^{-1}} B \simeq \int_{\emptyset} B \simeq \mathbb{1}_{\mathcal{E}}$$

  and B is dualizable as a $\mathbb{1}_{\mathcal{E}}$-module if and only if it is dualizable in $\mathcal{V}^{\otimes}$.

- For $k = 1$, the factorization homology is

$$\int_{S^0 \times D^1 \times D^1} B \simeq B^{\mathsf{e}}$$

  equivalent to the enveloping algebra of B. As described in [Lur08, Rem.4.1.27], the action on

$$\int_{D^1 \times D^1} B \simeq B$$

  corresponds, under this identification, to the regular B^{e}-action on B. In particular, B is dualizable as an $\int_{S^0 \times D^2} B$-module if and only if B is *smooth*. Pictorially, this action corresponds to gluing rectangles, as shown in Figure 2.

- For $k = 2$, the factorization homology

$$\int_{S^1 \times D^1} B$$

can be interpreted as a twisted version of Hochschild homology [Lur17, Thm.5.5.3.11]. Pictorially, the action on B can be visualized by gluing a blackboard framed cylinder to the boundary of D^2, as depicted in Figure 3.

In case, that $\mathcal{V}^{\otimes} \simeq \mathrm{Ind}(\mathcal{E})^{\otimes}$ for some rigid symmetric monoidal stable ∞-category $\mathcal{E}^{\otimes}$, we can compare Lurie's criterion 5.1 to our definition of a derived multi-fusion category. We can associate to any $\mathbb{E}_2$-algebra E in $\mathrm{Ind}(\mathcal{E})^{\otimes}$ its category of modules $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$. This forms a compactly generated stable ∞-category. As we have observed in Proposition 4.21, the $\mathbb{E}_2$-algebra structure on E naturally equips $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ with a rigid monoidal structure. In particular, its subcategory of compact objects $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))^{\mathrm{c},\otimes}$ forms a rigid stable ∞-category. To compare our Definition with the criterion of Lurie, we need the following alternative characterization of the ∞-category $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))^{\mathrm{c},\otimes}$:

Proposition 5.28. *Denote by $\mathsf{Perf}_{\mathcal{E}}(E)$ the smallest stable subcategory of $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ that contains E, is idempotent complete, and is closed under the $\mathcal{E}$-action. Then the inclusion $\mathsf{Perf}_{\mathcal{E}}(E) \to \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ induces an $\mathcal{E}$-linear equivalence $\mathsf{Perf}_{\mathcal{E}}(E) \simeq \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))^{\mathrm{c}}$*

Proof. The category $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))^{\mathrm{c}}$ is by construction stable, idempotent complete, and closed under the $\mathcal{E}$-action. Hence, it contains $\mathsf{Perf}_{\mathcal{E}}(E)$, since E is compact in $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$. It follows from [Lur09a, Prop.5.3.5.11] that the inclusion $\mathsf{Perf}_{\mathcal{E}}(E) \to \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ extends to a fully faithful $\mathcal{E}$-linear functor

$$F : \mathrm{Ind}(\mathsf{Perf}_{\mathcal{E}}(E)) \to \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$$

It suffices to show that F is essentially surjective. If F is not essentially surjective, then there exists a non-zero E-module M, s.t. for all $N \in \mathsf{Perf}_{\mathcal{E}}(E)$ the internal Hom $\hom_E(N, M) \simeq 0 \in \mathrm{Ind}(\mathcal{E})$. But for $N \simeq E$,

$$\hom_E(E, M) \simeq M \simeq 0$$

which contradicts the assumption that M is not 0. ☐

Proposition 5.29. *Let $\mathcal{E}$ be a rigid stable symmetric monoidal ∞-category and $E \in \mathsf{Alg}_{\mathbb{E}_2}(\mathrm{Ind}(\mathcal{E}))$ an $\mathbb{E}_2$-algebra. Then the ∞-category $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))^{\otimes}$ is $\mathcal{E}$-proper (resp. $\mathcal{E}$-smooth) if and only E is proper (resp. smooth)*

Proof. We first show the statement about properness. To show that $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ is $\mathcal{E}$-proper, it suffices to show that for all compact object $A, B \in \mathsf{Perf}_{\mathcal{E}}(E)$ the internal Hom $\hom_E(A, B) \in \mathrm{Ind}(\mathcal{E})$ is compact. Since A is compact and hence $\mathrm{Ind}(\mathcal{E})$-atomic, the functor $\hom_E(A, -)$ commutes with small colimits and the action of $\mathcal{E}$. It, therefore, suffices by Proposition 5.28, to show the case that $B \simeq E$. Doing an analogous argument for B instead of A, we can also reduce to the case that $A \simeq E$. It follows that $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ is $\mathcal{E}$-proper if and only if

$$\hom_E(E, E) \simeq E \in \mathrm{Ind}(\mathcal{E})$$

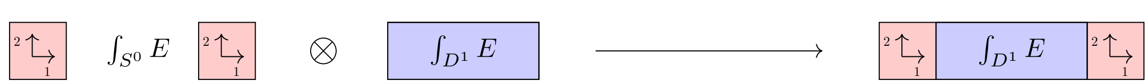


Figure 2: A pictorial description of the $\int_{S^0} E$-module structure on $\int_{D^1} E$. Here, the arrows represent the 2-framing.

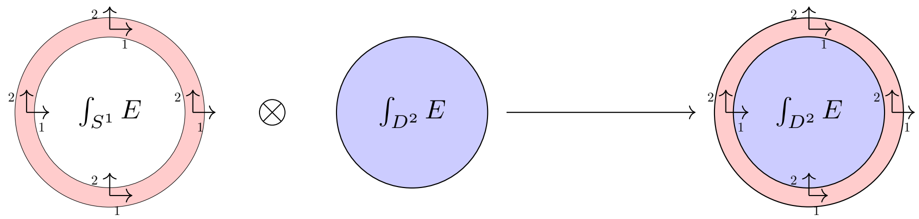


Figure 3: A pictorial description of the $\int_{S^1} E$-module structure on $\int_{D^2} E$. Here, the arrows represent the framing vectors.

is compact.

For the claim about smoothness, observe that under the equivalence

$$\mathsf{Fun}^{\mathsf{L}}_{\mathcal{E}}(\mathrm{LMod}_E(\mathcal{E}), \mathrm{LMod}_E(\mathcal{E})) \simeq \mathsf{BMod}_E(\mathcal{E})$$

the identity functor gets identified with the regular bimodule $E \in \mathsf{BMod}_E(\mathrm{Ind}(\mathcal{E}))$. Hence, $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ is $\mathcal{E}$-smooth if and only if the regular bimodule E is compact and consequently E is smooth. □

Proposition 5.30. *Let $E \in \mathsf{Alg}_{\mathbb{E}_2}(\mathrm{Ind}(\mathcal{E}))$ be an $\mathbb{E}_2$-algebra. Then $\mathsf{Perf}_{\mathcal{E}}(E)^{\otimes}$ is an $\mathcal{E}$-linear derived multi-fusion category if and only if E is smooth and proper.*

Proof. It follows from Proposition 4.21 that every object of $\mathsf{Perf}_{\mathcal{E}}(E)$ is dualizable and from Proposition 5.29 that $\mathsf{Perf}_{\mathcal{E}}(E)$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper if and only if E is so. Hence, it suffices to show that the canonical algebra is compact. Note that the monoidal product of $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ identifies under the equivalence

$$\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E})) \otimes \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E})) \simeq \mathrm{LMod}_{E\otimes E}(\mathrm{Ind}(\mathcal{E}))$$

with the functor of extension of scalars

$$\mu_! : \mathrm{LMod}_{E\otimes E}(\mathrm{Ind}(\mathcal{E})) \to \mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$$

along the multiplication $\mu : E \otimes E \to E$. In particular, the right adjoint is given by the restriction of scalars μ^*, and the underlying object of the canonical algebra is given by $\mu^*(E)$. Observe that under the equivalence

$$\mathrm{LMod}_{E\otimes E}(\mathrm{Ind}(\mathcal{E})) \simeq \mathsf{BMod}_E(\mathrm{Ind}(\mathcal{E}))$$

the $E \otimes E$-module $\mu^* E$ gets identified with the regular bimodule ${}_E E_{\sigma^* E}$, with right action twisted by the braiding automorphism

$$\sigma : E^{\mathrm{op}} \to E$$

Consequently, it follows from smoothness of E and [Lur17, Rem.4.6.2.12] that $\mu^*(E)$ is compact in $\mathsf{BMod}_E(\mathrm{Ind}(\mathcal{E}))$. □

Corollary 5.31. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category and $E \in \mathsf{Alg}_{\mathbb{E}_2}(\mathrm{Ind}(\mathcal{E}))$ an $\mathbb{E}_2$-algebra in $\mathrm{Ind}(\mathcal{E})$. Then is $\mathsf{Perf}_{\mathcal{E}}(E)^{\otimes}$ a derived multi-fusion category if and only if E satisfies the criterion 5.1 for $k = 0, 1$.*

Proof. Since $\mathcal{E}$ is rigid, it follows that E is dualizable in $\mathrm{Ind}(\mathcal{E})$ if and only if it is compact. The claim then follows from the above discussion. □

Remark 5.12. A more abstract perspective on this result is the following. We expect that the assignment that assigns to an $\mathbb{E}_2$-algebra $E \in \mathsf{Alg}(\mathrm{Ind}(\mathcal{E}))$ the monoidal ∞-category $\mathrm{LMod}_E(\mathrm{Ind}(\mathcal{E}))$ extends to a symmetric monoidal $(\infty, 3)$-functor

$$\mathrm{LMod}_{-}(\mathrm{Ind}(\mathcal{E})) : \mathbb{M}\mathsf{or}_{\mathbb{E}_2}(\mathrm{Ind}(\mathcal{E}))^{\otimes} \to \mathbb{M}\mathsf{or}(\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{E}})^{\otimes}$$

from the Morita category of $\mathbb{E}_2$-algebras to the even Higher Morita category of presentable $\mathcal{E}$-linear ∞-categories[18]. As it is symmetric monoidal, this functor would preserve fully dualizable objects.

After this general discussion, let us consider some examples of smooth and proper $\mathbb{E}_2$-algebras in rigid stable ∞-categories:

Example 5.10. Separable algebras form a well-behaved subclass of smooth algebras. As in the 1-categorical setting, an algebra (A, μ) in a monoidal stable ∞-category $\mathcal{C}^{\otimes}$ is called *separable* if the multiplication

$$\mu : A \otimes A \to A : \Delta$$

admits a section Δ as a map of A-bimodules. These algebras have been studied extensively in [Ram23].
Let R be a commutative ring. The most basic example of a separable algebra admitting the structure of an $\mathbb{E}_2$-algebra is a separable commutative R-algebra A, which defines a separable algebra in $\mathcal{D}(R)^{\otimes}$. If A is moreover perfect as an R-module, then it defines a smooth and proper $\mathbb{E}_2$-algebras in $\mathcal{D}(R)^{\otimes}$. Interestingly, every separable $\mathbb{E}_2$-algebra in any symmetric monoidal stable ∞-category is already $\mathbb{E}_\infty$ [Ram23, Thm. 3.25]. Other higher categorical examples of proper and separable $\mathbb{E}_2$-algebras arise from the theory of higher group algebras, as discussed in [Ram23, Thm.5.16]. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal stable ∞-category. For any ∞-category with finite limits, we denote by $\mathsf{Span}(\mathcal{C})^{\otimes}$ the symmetric monoidal ∞-category of spans in $\mathcal{C}$ (see Definition 10.1). Every symmetric monoidal functor

$$F : \mathsf{Span}(\mathcal{C})^{\otimes} \to \mathcal{V}^{\otimes}$$

associates to any $\mathbb{E}_2$-group object G in $\mathcal{C}$ an $\mathbb{E}_2$-algebra $F(A)$ in $\mathcal{V}^{\otimes}$. It follows from [Ram23, Thm.5.16] that the image $F(A)$ is a proper and separable $\mathbb{E}_\infty$-algebra in $\mathcal{V}$ if and only if F inverts the span

$$* \leftarrow G \rightarrow *$$

Examples of this phenomenon come from the theory of higher semi-additivity that we discuss in Section 11.3.

However, as described in Section 5.2, separability is a natural finiteness condition for additive categories. It would therefore be more interesting to single out examples of smooth and proper $\mathbb{E}_2$-algebras that are not separable.
Smooth and proper $\mathbb{E}_1$-algebras are well-studied and ubiquitous in mathematics as they play a fundamental role in the field of homological mirror symmetry. They appear in areas ranging from geometry to representation theory to topology (see [BD21, Sect.5] for some examples). However, it turns out to be a subtle question to identify a class of examples of smooth and proper $\mathbb{E}_2$-algebras. The author is unaware of a *single example* of such a smooth and proper $\mathbb{E}_2$-algebra that is not separable. Therefore, instead of describing an example, we will describe some possible approaches for the construction of $\mathbb{E}_2$-algebras.

[18]Compare Section 4.5

Example 5.11. Let $\mathbb{K}$ be a field. The historically first examples of $\mathbb{E}_2$-algebra arise in topology, specifically in the theory of loop spaces. Given a pointed topological space $X \in \mathcal{S}_*$, one can form its based loop space, defined as the pullback

$$\Omega_* X \simeq * \times_X *.$$

It is well known that the *double loop space* $\Omega^2_* X$ canonically carries the structure of an $\mathbb{E}_2$-algebra, with multiplication given by concatenation of loops [Lur17, Thm.5.2.6.15]. In particular, its algebra of singular chains

$$C_*(\Omega^2_* X, \mathbb{K}) \in \mathcal{D}(\mathbb{K})$$

naturally carries the structure of an $\mathbb{E}_2$-algebra in $\mathcal{D}(\mathbb{K})^{\otimes}$. From the definition, it follows that if

$$\dim(H_*(\Omega^2_* X, \mathbb{K})) < \infty,$$

then $C_*(\Omega^2 X, \mathbb{K})$ is proper. Furthermore, it has been shown in [BD19, Prop.5.1] that if $\Omega_* X$ is weakly homotopy equivalent to a finite CW-complex, then $C_*(\Omega^2_* X)$ is *smooth.* Thus, requiring smoothness and properness induces strong finiteness conditions on X and its loop spaces simultaneously. Already in elementary examples, it becomes apparent that these conditions are rarely compatible. For example, consider the case

$$X \simeq \mathsf{B}^2\mathsf{A}, \quad \Omega_* X \simeq \mathsf{BA}, \quad \Omega^2_* X \simeq A$$

for an abelian group A. Then $C_*(A, \mathbb{K})$ is proper if and only if A is finite, but in this case, BA is never homotopy equivalent to a finite CW-complex, and so $C_*(\Omega^2_* X, \mathbb{K})$ can not be smooth.
Conversely, if A is infinite, BA may be equivalent to a finite CW-complex. For instance, if $A \simeq \mathbb{Z}$, then $\mathsf{B}\mathbb{Z} \simeq S^1$ is a finite CW complex. Hence, in this case, $C_*(A, \mathbb{K})$ is smooth, but not proper.
In general, the homology of X and its loop space are strongly related by the *Serre spectral sequence*, and we expect that stronger structural results may be derived from it.

Example 5.12. Let $\mathcal{A}$ be a $\mathbb{K}$-linear monoidal abelian category. Then it's derived ∞-category

$$\mathcal{D}(\mathcal{A}) \in \mathsf{Pr}^{\mathsf{L}}_{\mathbb{K}}$$

is a presentably monoidal stable ∞-category. It is a consequence of the Eckmann-Hilton argument that the *algebra of derived endomorphisms*

$$\mathbb{R}\mathsf{End}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})$$

carries the structure of an $\mathbb{E}_2$-algebra [Lur18, Sect.D.1.3]. Furthermore, if $\mathcal{D}(\mathcal{A})$ is generated by the monoidal unit, we obtain a monoidal equivalence

$$\mathcal{D}(\mathcal{A})^{\otimes} \simeq \mathrm{LMod}^{\otimes}_{\mathbb{R}\mathsf{End}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})}.$$

In particular, if $\mathcal{D}(\mathcal{A})$ is smooth and proper, the same would follow for the $\mathbb{E}_2$-algebra $\mathbb{R}\mathsf{End}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})$. However, we are not aware of any interesting example where the monoidal unit generates the full derived ∞-category. One way to circumvent this issue is to consider the monoidal $\mathbb{K}$-linear stable subcategory

$$\mathcal{D}(\mathbb{1}_{\mathcal{A}})^{\otimes} \subset \mathcal{D}(\mathcal{A})^{\otimes}$$

generated by the monoidal unit $\mathbb{1}_{\mathcal{A}}$. In this case, we always have an equivalence

$$\mathcal{D}(\mathbb{1}_{\mathcal{A}})^{\otimes} \simeq \mathrm{LMod}^{\otimes}_{\mathbb{R}\mathsf{End}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})}.$$

In particular, if $\mathcal{D}(\mathcal{A})$ is proper, the same is true for the full subcategory $\mathcal{D}(\mathbb{1}_{\mathcal{A}})$. Whether smoothness descends to subcategories is a subtler issue. A sufficient condition is that $\mathcal{D}(\mathbb{1}_{\mathcal{A}})$ is an admissible subcategory of $\mathcal{D}(\mathcal{A})$, i.e. the inclusion

$$i : \mathcal{D}(\mathbb{1}_{\mathcal{A}}) \hookrightarrow \mathcal{D}(\mathcal{A})$$

admits a left and a right adjoint [Orl16]. It is conceivable that this approach yields examples of smooth and proper $\mathbb{E}_2$-algebras, though we are currently not aware of any such example.

Example 5.13. Given smooth and proper $\mathbb{E}_1$-algebras $A, B \in \mathsf{Alg}(\mathcal{D}(\mathbb{K}))$ and a perfect $A-B$-bimodule M, we can construct new examples of smooth and proper $\mathbb{E}_1$-algebra via *gluing*. More precisely, if we represent A and B by dg-algebras and M by a dg-bimodule [Lur17, Prop.7.1.4.6], we can form the dg-algebra of *upper triangular matrices*

$$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

This dg-algebra has elements given by formal upper triangular matrices

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \quad \text{with } a \in A, m \in M, \text{ and } b \in B,$$

and multiplication defined by matrix multiplication

$$\begin{pmatrix} a_0 & m_0 \\ 0 & b_0 \end{pmatrix}\begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix} \coloneqq \begin{pmatrix} a_0a_1 & a_0m_1 + m_0b_1 \\ 0 & b_0b_1 \end{pmatrix}$$

It has been shown by Lunts–Schnürer [LS14, Thm.3.24] that under the above conditions, the algebra of upper triangular matrices is itself smooth and proper. In particular, if A and B are classical separable $\mathbb{K}$-algebras and M is a perfect $A-B$-bimodule, this construction yields non-trivial smooth and proper dg-algebras. Unfortunately, even if we assume A and B to be commutative, the inherent asymmetry in the formula for matrix multiplication prevents the associated upper triangular matrix algebra from being commutative or $\mathbb{E}_2$. A more abstract perspective on upper triangular matrix algebras is the following. The data of dg-algebras A and B, together with a dg-bimodule M determines a diagram

$$F : [1] \to \mathbb{M}\mathsf{or}(\mathcal{D}(\mathbb{C}))$$

in the Morita $(\infty, 2)$-category. The upper triangular matrix algebra can then be interpreted as the lax limit of this functor. Therefore, it might be fruitful in our context to compute similar lax-limits in the $(\infty, 3)$-category $\mathbb{M}\mathsf{or}_{\mathbb{E}_2}(\mathcal{D}(\mathbb{C}))$.

Example 5.14. Let R be a commutative ring spectrum, and let $A \in \mathsf{Alg}(\mathsf{RMod}_R)$ be an R-algebra. Then the *Hochschild Cohomology*

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A) \in \mathsf{RMod}_R$$

of A admits the structure of an $\mathbb{E}_2$-algebra [Lur17, Rem.5.1.13], a result commonly referred to as the *Deligne-Conjecture*. It is therefore natural to ask under which conditions on A its Hochschild Cohomology is smooth and proper.

Properness is easy. If the algebra A is smooth and proper, then its Hochschild Cohomology is also *proper*. Thus, it remains to understand when the Hochschild cohomology is smooth. Since the functoriality of Hochschild cohomology is hard to control, this turns out to be a subtle problem. We therefore focused on computing explicit examples.
Computable examples arise from the theory of *quadratic monomial algebras*. Given a quiver

$$Q = (Q_0, Q_1)$$

with quadratic monomial relations

$$R \subset Q_1 \times_{Q_0} Q_1,$$

the associated quadratic monomial algebra is given by $\mathbb{K}[Q]/\langle R\rangle$, where $\mathbb{K}[Q]$ denotes the path algebra of the quiver. If the quiver Q contains, no cyclic paths, then $\mathbb{K}[Q]/R$ is smooth and proper [HKK17, Prop.3.4]. Moreover, one can explicitly resolve the diagonal bimodule and thus obtain an explicit complex that computes Hochschild Cohomology [HKK17, Prop.3.4]. We computed some simple examples but were not able to find one where the Hochschild cohomology is smooth and proper.
The hope that this approach might lead to examples arises from the case of separable algebras. In this setting, it has been shown by Ramzi [Ram23, Thm 6.45] that the Hochschild Cohomology of a large class of separable algebras is itself separable. We hope that this approach might give rise to interesting new examples.

In conclusion, this discussion suggests that with the current techniques, it is difficult to find examples of smooth and proper $\mathbb{E}_2$-algebras that are not separable. This does not rule out their existence, but constructing such an example would likely require an educated guess. As the research on derived phenomena in TFTs in dimensions higher than two is still in its early stages, we expect such examples to emerge in the future.

6 2-Segal Conditions

Throughout this section, $\mathcal{C}$ denotes an ∞-category with finite limits. After studying rigid presentable ∞-categories in the previous sections, we now continue with the second step of our strategy. to this end, we shift our focus to the study of locally rigid algebras in symmetric monoidal $(\infty, 2)$-categories of spans. Before we can turn to the discussion of locally rigid algebras, it is necessary to understand how to encode algebraic structures within this ∞-category. For algebra objects, this has been accomplished in [DK19, Ste21] in terms of so-called 2-*Segal objects*. As we will see in this section, more general algebraic structures in this ∞-category can be described in terms of conditions similar to the 2-Segal. Therefore, we introduce in this section a variety of 2-*Segal type* conditions generalizing the original 2-Segal conditions. In Section 8, we relate these conditions to homotopy coherent algebra in span categories. Before stating the formal definitions, let us motivate them for the example of the 2-Segal condition of [DK19].
There exists an ∞-category of spans $\mathsf{Span}(\mathcal{C})$, whose objects are the objects of $\mathcal{C}$ and whose 1-morphisms are spans

$$X \longleftarrow W \longrightarrow Y$$

of 1-morphisms in $\mathcal{C}$. Since it is an ∞-category, it also has invertible n-morphisms for all $n \geq 2$. For example, an invertible 2-morphism is given by a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & \uparrow \simeq & \searrow & \\ X & & Z & & Y \\ & \nwarrow & \downarrow \simeq & \nearrow & \\ & & V & & \end{array}$$

where the middle arrows are equivalences. We read this diagram as a morphism from the upper to the lower span. The ∞-category $\mathsf{Span}(\mathcal{C})$ further inherits a monoidal structure from the Cartesian product $\times$ on $\mathcal{C}$. We can, therefore, study homotopy coherent associative algebras in $\mathsf{Span}(\mathcal{C})^{\otimes}$. Informally, the datum of such an algebra consists of

- an underlying object $X_1 \in \mathcal{C}$,
- a multiplication span
$$\begin{array}{ccccc} & & X_2 & & \\ & \swarrow^{(\partial_0,\partial_2)} & & \searrow^{\partial_1} & \\ X_1 \times X_1 & & & & X_1 \end{array}$$
- a unit span
$$\begin{array}{ccccc} & & X_0 & & \\ & \swarrow^{p_{X_0}} & & \searrow^{s_0} & \\ * & & & & X_1 \end{array}$$
- and higher morphisms that describe homotopy coherent associativity and unitality.

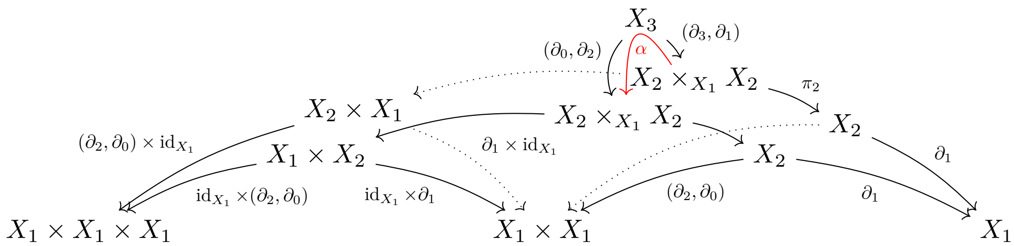
As the notations suggest, this data organizes into a simplicial object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$, whose n-simplices, for $n > 2$ are encoded in the associativity data. For example, the lowest instance of higher associativity is the existence of an associativity 2-isomorphism

$$\alpha : X_2 \times_{X_1} X_2 \overset{\simeq}{\Longrightarrow} X_2 \times_{X_1} X_2$$

in $\mathsf{Span}(\mathcal{C})$. This data is given by an object $X_3 \in \mathcal{C}$, together with 2 invertible morphisms

$$\begin{array}{ccccc} & & X_3 & & \\ & \swarrow^{(\partial_0,\partial_2)} & & \searrow^{(\partial_3,\partial_1)} & \\ X_2 \times_{X_1} X_2 & & & & X_2 \times_{X_1} X_2 \end{array} \tag{14}$$

that fit into a commutative diagram:



Since the diagram commutes, the components ∂_i satisfy the simplicial identities

$$\partial_i \circ \partial_j = \partial_{j-1} \partial_i \text{ for } i < j.$$

The lowest dimensional 2-Segal condition is the requirement that the morphisms in Equation (14) are invertible (see Definition 6.1) and therefore encodes the lowest dimensional instance of associativity. Analogously, the higher-dimensional 2-Segal conditions encode the higher associativity of the algebra.

We now summarize the content of this section. In Section 6.1, we introduce the definitions of 2-Segal objects and 2-Segal spans. They assemble into an ∞-category $2\mathrm{Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$ that we will show in Corollary 9.3 to be equivalent to the ∞-category $\mathsf{Alg}(\mathsf{Span}(\mathcal{C}))$ of algebra objects and algebra morphisms in the ∞-category $\mathsf{Span}(\mathcal{C})^{\otimes}$. Afterward, we extend the discussion in 6.2 and introduce 2-Segal and Segal span conditions for the indexing categories $\Delta^{\mathrm{op}}_{/[1]}$, $\Delta^{\mathrm{op}}_{\geq}$ and $\Delta^{\mathrm{op}}_{\leq}$ generalizing the ones on Δ. We show in Section 8 that these admit a similar interpretation in terms of bi-, left, and right module objects in the symmetric monoidal ∞-category $\mathsf{Span}(\mathcal{C})^{\otimes}$ respectively. Finally, we provide in Subsection 6.3 a different characterization of these new 2-Segal conditions in terms of active-inert pullbacks. This generalizes the equivalence between 2-Segal and decomposition spaces from [GCKT18] to this more general class of indexing categories.

6.1 2-Segal Objects

To simplify the exposition, we frequently abuse notation and identify a simplicial object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ with its extension to $\mathsf{Fin}^{\mathrm{op}}_{\geq}$ the category of all finite non-empty linearly ordered sets. Let us recall, as a start, the definition of a 2-Segal object.

Definition 6.1. [DK19, Def.2.3.1] Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. $X_\bullet$ is called 2-*Segal* if for every $n \geq 3$ and $0 \leq i < j \leq n$ the map

$$X_n \to X_{\{0,1,\cdots,i,j,j+1,\cdots,n\}} \times_{X_{\{i,j\}}} X_{\{i,i+1,\ldots,j\}}$$

is an equivalence. We denote by $2\text{-}\mathsf{Seg}_{\Delta}(\mathcal{C})$ the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ generated by 2-Segal objects.

The morphisms in the ∞-category $2\text{-}\mathsf{Seg}_{\Delta}(\mathcal{C})$ are morphisms of simplicial objects. Under the equivalence between 2-Segal objects and algebra objects in Spans [Ste21], a general simplicial map between 2-Segal objects induces a lax-algebra morphism between the corresponding algebra objects [Wal16, Sect.4.2]. Let us sort out what a strong algebra morphism looks like in terms of 2-Segal objects:

Definition 6.2. Let $\sigma : \Delta^1 \to \mathsf{Span}(2\text{-}\mathsf{Seg}_{\Delta}(\mathcal{C}))$ be a 1-morphism in the ∞-category of spans of 2-Segal objects.[19] The morphism σ can be represented by a span

$$\begin{array}{ccc} & M_\bullet & \\ {}^{s}\swarrow & & \searrow^{t} \\ X_\bullet & & Y_\bullet \end{array}$$

with $X_\bullet, Y_\bullet$ and $M_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ 2-Segal. The morphism σ is called a 2-*Segal span* if for every $n \geq 0$ the induced diagrams

$$\begin{array}{ccc} X_n & \longleftarrow & M_n \\ \downarrow & & \downarrow \\ X_{\{0,n\}} & \longleftarrow & M_{\{0,n\}} \end{array}$$

and

$$\begin{array}{ccc} Y_n & \longleftarrow & M_n \\ \downarrow & & \downarrow \\ \prod_{i=0}^{n-1} Y_{\{i,i+1\}} & \longleftarrow & \prod_{i=0}^{n-1} M_{\{i,i+1\}} \end{array}$$

are pullback diagrams. We call s an *active equifibered* Δ^{op}*-morphism* and t a *relative Segal* Δ^{op}*-morphism*.

Notation 6.1. We introduce in Subsection 6.2 similar notions for other indexing categories than Δ. If the indexing category is clear from the context, we frequently abuse notation and drop the indexing category in the notation. E.g, we will call an active equifibered Δ^{op}-morphism simply an active equifibered morphism when the indexing category is clear from the context.

Remark 6.1. The conditions imposed on the individual legs have previously been studied under the name CULF [GCKT18, Sect.4] and IKEO in [GCKT18, Sect.8.5]. The authors further demonstrate that these types of morphisms induce algebra morphisms in the ∞-category of spans. Our terminology is motivated by the theory of algebraic patterns [BHS22], where similar notions appear.

It is worth noting that in the definition of a 2-Segal span, we explicitly require the tip of the span to be 2-Segal as well. This is not an additional requirement. Indeed, as demonstrated in [GCKT18, Lem.4.6], the source of an active equifibered morphism $f : X_\bullet \to Y_\bullet$ with target 2-Segal is itself 2-Segal.
In the following, we interpret 2-Segal spans as the morphisms of a category and therefore have to study their behavior under composition of spans:

Lemma 6.1. *Let* $\sigma : \Lambda^2_1 \to \mathsf{Span}(2\text{-}\mathsf{Seg}_\Delta(\mathcal{C}))$ *be given by a composable pair of spans*

$$\begin{array}{ccccc} & M_\bullet & & N_\bullet & \\ \swarrow & & \searrow\ \swarrow & & \searrow \\ X_\bullet & & Y_\bullet & & Z_\bullet \end{array}$$

[19] See Construction 8.3

s.t. each individual span is a 2-Segal span. Then also the composite span

$$\begin{array}{ccc} & M_\bullet \times_{Y_\bullet} N_\bullet & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ X_\bullet & & Z_\bullet \end{array}$$

is a 2-Segal span.

Proof. First, we show that s is active equifibered. For $n \neq 0$, consider the diagram:

$$\begin{array}{ccc} N_{\{0,n\}} \times_{Y_{\{0,n\}}} M_{\{0,n\}} & \longleftarrow & N_n \times_{Y_n} M_n \\ \downarrow & & \downarrow \\ M_{\{0,n\}} & \longleftarrow & M_n \\ \downarrow & & \downarrow \\ X_{\{0,n\}} & \longleftarrow & X_n \end{array}$$

We need to prove that the exterior diagram is a pullback diagram. By assumption, the lower square is a pullback diagram. Additionally, the upper square is a pullback square if and only if the outer rectangle in the diagram

$$\begin{array}{ccccc} N_{\{0,n\}} & \longleftarrow & N_n & \longleftarrow & N_n \times_{Y_n} M_n \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\{0,n\}} & \longleftarrow & Y_n & \longleftarrow & M_n \end{array}$$

is a pullback square. But this follows from the pasting law.
On the other hand, we show that t is relative Segal. We need to show that for every $n \geq 0$, the exterior rectangle in the diagram:

$$\begin{array}{ccccc} Z_n & \longleftarrow & N_n & \longleftarrow & M_n \times_{Y_n} N_n \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{i=0}^{n-1} Z_{\{i,i+1\}} & \longleftarrow & \prod_{i=0}^{n-1} N_{\{i,i+1\}} & \longleftarrow & \prod_{i=0}^{n-1} M_{\{i,i+1\}} \times_{Y_{\{i,i+1\}}} N_{\{i,i+1\}} \end{array}$$

is a pullback. By assumption, the left square is a pullback square. Further, the right square is a pullback square if and only if the outer rectangle in the diagram:

$$\begin{array}{ccccc} \prod_{i=0}^{n-1} M_{\{i,i+1\}} & \longleftarrow & M_n & \longleftarrow & N_n \times_{Y_n} M_n \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{i=0}^{n-1} Y_{\{i,i+1\}} & \longleftarrow & Y_n & \longleftarrow & N_n \end{array}$$

is a pullback square. But this is again a consequence of the pasting law. □

Definition 6.3. We define the ∞-*category of* 2-*Segal objects* $2\mathrm{Seg}_\Delta^{\leftrightarrow}(\mathcal{C})$ as the wide subcategory of $\mathsf{Span}(\text{2-}\mathsf{Seg}_\Delta(\mathcal{C}))$ with morphisms given by 2-Segal spans.

6.2 Birelative 2-Segal Objects

To describe more general algebraic structures in categories of spans, we need to consider more general indexing categories than Δ. In this subsection, we extend the framework of the previous sections to encompass categories of functors with source $\Delta_{/[1]}$. We call such a functor $M_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ a *birelative simplicial object*. The corresponding birelative 2-Segal condition is a multi-coloured version of the 2-Segal condition from Definition 6.1:

Definition 6.4. Let $M_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ be a birelative simplicial object. $M_\bullet$ is called *birelative* 2*-Segal*, if for every $n \geq 3$, $f : [n] \to [1]$ and $0 \leq i < j \leq n$ the diagram

$$\begin{array}{ccc} M_f & \longrightarrow & M_{f|_{i,\dots,j}} \\ \downarrow & & \downarrow \\ M_{f|_{0,\dots,i,j,\dots,n}} & \longrightarrow & M_{f|_{i,j}} \end{array}$$

is Cartesian. We denote the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})$ generated by the birelative 2-Segal objects by $\mathrm{Bi2Seg}_\Delta(\mathcal{C})$.

As 2-Segal objects encode algebra objects, we will see in Theorem 9.1 that birelative 2-Segal objects encode bimodule objects. For completeness, we also introduce a 2-Segal type condition describing left and right modules. These have already been studied in the case of 1-categories in [Wal16] and [You18]. We denote by $\Delta_\leq$ (resp. $\Delta_\geq$) the full subcategory of $\Delta_{/[1]}$ generated by the objects $f : [n] \to [1]$ that take the value 0 (resp. 1) at least and the value 1 (resp. 0) at most ones. We call a functor with source $\Delta_\leq$ (resp. $\Delta_\geq$) a *left (resp. right) relative simplicial object*. The corresponding 2-Segal condition reads as:

Definition 6.5. Let $M_\bullet : \Delta^{\mathrm{op}}_\leq \to \mathcal{C}$ be a left relative simplicial object. $M_\bullet$ is called *left relative* 2*-Segal*, if for every $n \geq 3$, $f : [n] \to [1] \in \Delta_\leq$ and $0 \leq i < j \leq n$ the diagram

$$\begin{array}{ccc} M_f & \longrightarrow & M_{f|_{i,\dots,j}} \\ \downarrow & & \downarrow \\ M_{f|_{0,\dots,i,j,\dots,n}} & \longrightarrow & M_{f|_{i,j}} \end{array}$$

is Cartesian. We denote the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}_\leq, \mathcal{C})$ generated by the left relative 2-Segal objects by $\mathrm{L2Seg}_\Delta(\mathcal{C})$. Similarly, we define *right relative* 2*-Segal objects* and the ∞-category of right relative 2-Segal objects $\mathrm{R2Seg}_\Delta(\mathcal{C})$.

Remark 6.2. The categories $\Delta_\leq$ and $\Delta_\geq$ defined above are equivalent to the categories, denoted with the same symbols, defined in [Wal16, Def.3.2.5]. Further, our notions of left and right relative 2-Segal objects coincide with the notion of a relative 2-Segal object in [Wal21, Def.3.5.1.].

To handle the cases of bi-, left, and right relative 2-Segal conditions at once, we introduce the following notation:

Notation 6.2. For $\# \in \{/[1], \leq, \geq\}$ we call a functor $M_\bullet : \Delta^{\mathrm{op}}_\# \to \mathcal{C}$ a #-relative simplicial object. Further, we call a #-relative simplicial object a #-relative 2-Segal object if it satisfies the corresponding 2-Segal conditions. We denote the full subcategory generated by #-relative 2-Segal objects by $\#\text{-}2\mathrm{Seg}_\Delta(\mathcal{C})$.

Example 6.1. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be 2-Segal. We can associate to $X_\bullet$ a birelative, left and right relative 2-Segal object

(1) $X^b_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \Delta^{\mathrm{op}} \to \mathcal{C}$

(2) $X^l_\bullet : \Delta^{\mathrm{op}}_{\leq} \to \Delta^{\mathrm{op}} \to \mathcal{C}$

(3) $X^r_\bullet : \Delta^{\mathrm{op}}_{\geq} \to \Delta^{\mathrm{op}} \to \mathcal{C}$

via precomposition with the corresponding forgetful functor. It follows from the 2-Segal conditions for $X_\bullet$ that these objects fulfill the respective (bi)relative Segal conditions. Under the equivalence of Corollary 9.2 and Theorem 9.1, these correspond to the regular bimodule (resp. left, right module) associated to the algebra object corresponding to $X_\bullet$.

We also introduce, in analogy with Definition 6.2, a notion of morphism between #-relative 2-Segal objects:

Definition 6.6. Let $A_\bullet, B_\bullet$ and $M_\bullet$ be #-relative 2-Segal. A *#-relative 2-Segal span* from $A_\bullet$ to $B_\bullet$ is given by a span

$$\begin{array}{ccccc} & & M_\bullet & & \\ & \swarrow^{s} & & \searrow^{t} & \\ A_\bullet & & & & B_\bullet \end{array}$$

s.t. for every $n \geq 0$ and $f : [n] \to [1] \in \Delta_\#$ the diagrams

$$\begin{array}{ccc} A_f & \longleftarrow & M_f \\ \downarrow & & \downarrow \\ A_{f|_{0,n}} & \longleftarrow & M_{f|_{0,n}} \end{array}$$

and

$$\begin{array}{ccc} B_f & \longleftarrow & M_f \\ \downarrow & & \downarrow \\ \prod_{i=0}^{n-1} B_{f|_{i,i+1}} & \longleftarrow & \prod_{i=0}^{n-1} M_{f|_{i,i+1}} \end{array}$$

are pullback diagrams. We call s an *active equifibered $\Delta^{\mathrm{op}}_\#$-morphism* and t a *relative Segal $\Delta^{\mathrm{op}}_\#$-morphism.*

Analogously to Lemma 6.1, one can prove the following:

Lemma 6.2. *Let $\sigma : \Lambda^2_1 \to \mathsf{Span}(\#\text{-}2\mathrm{Seg}_\Delta(\mathcal{C}))$ be given by a composable pair of spans*

$$\begin{array}{ccccccc} & M_\bullet & & & & N_\bullet & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X_\bullet & & & Y_\bullet & & & Z_\bullet \end{array}$$

s.t. the individual spans are #-relative 2-Segal spans. Then also the composite span

$$\begin{array}{ccccc} & & M_\bullet \times_{Y_\bullet} N_\bullet & & \\ & \swarrow & & \searrow & \\ X_\bullet & & & & Z_\bullet \end{array}$$

is a #-relative 2-Segal span.

Proof. Similar to Lemma 6.1. □

Definition 6.7. We define the ∞-*category of birelative* 2-*Segal objects* $\mathrm{Bi2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$ to be the wide subcategory of $\mathsf{Span}(\mathrm{Bi2Seg}_{\Delta}(\mathcal{C}))$ with morphisms birelative 2-Segal spans.
Similarly, we define the ∞-*category of left (resp. right) relative* 2-*Segal objects* $\mathrm{L2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$ (resp. $\mathrm{R2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$)) as the wide subcategory of $\mathsf{Span}(\mathrm{L2Seg}_{\Delta}(\mathcal{C}))$ (resp. $\mathsf{Span}(\mathrm{R2Seg}_{\Delta}(\mathcal{C}))$) with morphisms given by left (resp. right) relative 2-Segal spans.

It has been shown in [Wal16, Prop.3.5.10] for 1-categories that left and right relative 2-Segal objects admit a different description in terms of morphisms of simplicial objects, called relative 2-Segal objects. This description is useful in the study of examples. Let us, therefore, lift this description into the realm of ∞-categories. We do this for left relative 2-Segal objects. The case of right relative 2-Segal objects is analogous.

Definition 6.8. [You18, Wal16, Def.2.2,Prop.3.5.10] A morphism $\pi : X_\bullet \to Y_\bullet$ of simplicial objects is called *relative* 2-*Segal*, if

1) the source object $X_\bullet$ is 1-Segal,

2) the target object $Y_\bullet$ is 2-Segal,

3) and for every $0 \le i < j \le n$ the following square is Cartesian

$$\begin{array}{ccc} X_n & \longrightarrow & Y_{\{i,..j\}} \\ \downarrow & & \downarrow \\ X_{\{0,\dots,i,j,\dots,n\}} & \longrightarrow & Y_{\{i,j\}} \end{array} \tag{15}$$

We denote the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}))$ generated by relative 2-Segal morphisms by $\mathrm{Rel2Seg}_{\Delta}(\mathcal{C})$.

Remark 6.3. We can extend the square in Equation (15) to the following diagram

$$\begin{array}{ccccc} X_n & \longrightarrow & Y_n & \longrightarrow & Y_{\{i,\dots,j\}} \\ \downarrow & & \downarrow & & \downarrow \\ X_{\{0,\dots i,j,\dots,n\}} & \longrightarrow & Y_{\{0,\dots,i,j,\dots,n\}} & \longrightarrow & Y_{\{i,j\}} \end{array}$$

Since $Y_\bullet$ is 2-Segal, the right square is a pullback square. By the pasting law, it follows that the outer square is a pullback square if and only if the right square is a pullback square. Unraveling the definitions, this means that a morphism $X_\bullet \to Y_\bullet$ is relative 2-Segal if and only if it is an active equifibered Δ^{op}-morphism from a 1-Segal to a 2-Segal object.

Remark 6.4. Let us unpack the definition of a relative 2-Segal object $\pi : X_\bullet \to Y_\bullet$. The 2-Segal object $X_\bullet$ encodes an algebra object, and the object Y_0 encodes the underlying object of a left module. The remaining data encodes the module-action. For example, the module action of X_1 onto Y_0 is given by the span

$$Y_1 \times X_0 \xleftarrow{(\pi_1,\partial_1)} X_1 \xrightarrow{\partial_0} X_0$$

The relative 2-Segal conditions again encode the higher coherences of the left module action.

We also introduce a notion of morphism between relative 2-Segal objects:

Definition 6.9. Let $\pi^i_\bullet : X^i_\bullet \to Y^i_\bullet$ for $0 \leq i \leq 2$ be relative 2-Segal objects. A *relative* 2-*Segal span* from $\pi^0_\bullet$ to $\pi^2_\bullet$ is given by a span

$$\begin{array}{ccccc} X^0_\bullet & \xleftarrow{s^X_\bullet} & X^1_\bullet & \xrightarrow{t^X_\bullet} & X^2_\bullet \\ \downarrow{\scriptstyle \pi^0_\bullet} & & \downarrow{\scriptstyle \pi^1_\bullet} & & \downarrow{\scriptstyle \pi^2_\bullet} \\ Y^0_\bullet & \xleftarrow[s^Y_\bullet]{} & Y^1_\bullet & \xrightarrow[t^Y_\bullet]{} & Y^2_\bullet \end{array}$$

of relative 2-Segal objects s.t.

(1) the span $Y^0_\bullet \xleftarrow{s^Y_\bullet} Y^1_\bullet \xrightarrow{t^Y_\bullet} Y^2_\bullet$ is 2-Segal,

(2) for every $n \geq 1$ the square

$$\begin{array}{ccc} X^1_n & \xrightarrow{s^X_n} & X^0_n \\ \downarrow & & \downarrow \\ X^1_{\{0\}} & \xrightarrow{s^X_1} & X^0_{\{0\}} \end{array}$$

is Cartesian,

(3) and for every $n \geq 1$ the diagram

$$\begin{array}{ccc} X^1_n & \xrightarrow{\qquad\qquad t^X_n \qquad\qquad} & X^2_n \\ \downarrow & & \downarrow \\ Y^1_{\{0,1\}} \times \cdots \times Y^1_{\{n-2,n-1\}} \times X^1_{\{n\}} & \longrightarrow & Y^2_{\{0,1\}} \times \cdots \times Y^2_{\{n-2,n-1\}} \times X^2_{\{n\}} \end{array}$$

is Cartesian.

We call the pair $(s^X_\bullet, s^Y_\bullet)$ *active equifibered* and the pair $(t^X_\bullet, t^Y_\bullet)$ *relative Segal*.

Definition 6.10. We define the ∞-category of relative 2-Segal objects as the wide subcategory $\mathrm{Rel2Seg}^{\leftrightarrow}_\Delta(\mathcal{C})$ of the ∞-category $\mathsf{Span}(\mathrm{Rel2Seg}_\Delta(\mathcal{C}))$ with morphisms given by relative 2-Segal spans.

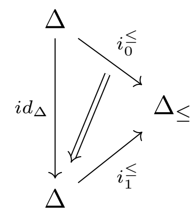
We now turn to the construction of an equivalence $\Theta^{\leftrightarrow}_L : \mathrm{L2Seg}^{\leftrightarrow}_\Delta(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Rel2Seg}^{\leftrightarrow}_\Delta$. To do so, we construct an equivalence

$$\Theta_L : \mathrm{L2Seg}_\Delta(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Rel2Seg}_\Delta(\mathcal{C})$$

that induces $\Theta^{\leftrightarrow}_L$ on ∞-categories of spans.
For the construction of Θ_L, we use observation [Wal16, Rem.3.2.8]. First, note that the category $\Delta_\leq$ contains two copies of the simplex category as full subcategories. We denote the corresponding fully faithful inclusion by $i^\leq_0 : \Delta \to \Delta_\leq$ and $i^\leq_1 : \Delta \to \Delta_\leq$. These are given on objects by

$$i^\leq_0([n])(k) = 0 \quad \forall k \in [n],$$
$$i^\leq_1([n])(k) = \begin{cases} 0 & \forall k \neq n+1 \\ 1 & k = n+1 \end{cases} .$$

Further, we observe that the morphisms $d_{n+1} : i_0^{\leq}([n]) \to i_1^{\leq}([n])$ assemble into a natural transformation $d_{\bullet+1} : i_0^{\leq}(-) \Rightarrow i_1^{\leq}(-)$. This datum is equivalent to the datum of a lax cocone



(16)

and therefore induces a unique map from the lax colimit. It follows from [GHN15, Thm.7.4] that the lax colimit is given by the total space of the cocartesian Grothendieck construction $\int_{\Delta^1} id_\Delta \to \Delta$, where we identify the functor $id_\Delta : \Delta \to \Delta$ with a morphism in Cat. The morphism

$$\theta_L : \int_{\Delta^1} id_\Delta \to \Delta_{\leq}$$

induced by Diagram 16 is given on the fiber over 0 by $i_0^{\leq} : \Delta \to \Delta_{\leq}$, on the fiber over 1 by $i_1^{\leq} : \Delta \to \Delta_{\leq}$ and on morphisms $f : ([n], 0) \to ([m], 1)$ by

$$i_0^{\leq}([n]) \xrightarrow{i_0^{\leq}(f)} i_0^{\leq}(m) \xrightarrow{d_{m+1}} i_1^{\leq}(m).$$

It follows from the explicit description that this functor is an equivalence.

Proposition 6.3. *The functor* $\Theta_L := \theta_L^* : \mathsf{Fun}(\Delta_{\leq}^{\mathrm{op}}, \mathcal{C}) \to \mathsf{Fun}(\int_{\Delta^1} id_\Delta^{\mathrm{op}}, \mathcal{C})$ *restricts to an equivalence of* ∞*-categories*

$$\Theta_L : \mathrm{L2Seg}_\Delta(\mathcal{C}) \to \mathrm{Rel2Seg}_\Delta(\mathcal{C}).$$

Proof. It follows from the universal property of the lax colimit that we have an equivalence of ∞-categories

$$\mathsf{Fun}(\int_{\Delta^1} id_\Delta^{\mathrm{op}}, \mathcal{C}) \simeq \mathsf{Fun}(\Delta^1 \times \Delta^{\mathrm{op}}, \mathcal{C}) \simeq \mathsf{Fun}(\Delta^1, \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})).$$

Under this equivalence, the functor Θ_L maps a left relative simplicial object $X_\bullet : \Delta_{\leq}^{\mathrm{op}} \to \mathcal{C}$ to the natural transformation

$$(i_1^{\leq})^* X_\bullet \xrightarrow{\partial_{n+1}} (i_0^{\leq})^* X_\bullet$$

between simplicial objects. The claim follows from the observation that the functor Θ_L maps left relative 2-Segal objects to relative 2-Segal objects. □

It follows from the discussion after Construction 8.3 that the construction of the ∞-category of spans induces a functor:

$$\mathsf{Span}(-) : \mathsf{Cat}_\infty^{\mathsf{lex}} \to \mathsf{Cat}_\infty$$

with source the ∞-category of small ∞-categories with finite limits and finite limit preserving functors. We use this functor to conclude our comparison:

Corollary 6.4. *The functor* $\Theta_L^{\leftrightarrow} := \mathsf{Span}(\Theta_L) : \mathsf{Span}(\mathrm{L2Seg}_\Delta(\mathcal{C})) \to \mathsf{Span}(\mathrm{Rel2Seg}_\Delta(\mathcal{C}))$ *restricts to an equivalence of* ∞*-categories*

$$\Theta_L^{\leftrightarrow} : \mathrm{L2Seg}_\Delta^{\leftrightarrow}(\mathcal{C}) \to \mathrm{Rel2Seg}_\Delta^{\leftrightarrow}(\mathcal{C})$$

Proof. It suffices to show that $\Theta_L^{\leftrightarrow}$ maps left relative 2-Segal spans to relative 2-Segal spans. But this follows from unraveling the definitions. ◻

Remark 6.5. Analogously to the proof of Corollary 6.4 one can construct an equivalence

$$\Theta_R^{\leftrightarrow} : \mathrm{R2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Rel2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$$

between the ∞-category of right relative 2-Segal objects and relative 2-Segal objects.

We use Corollary 6.4 to construct examples of left relative 2-Segal objects and spans. This is particularly fruitful for the example of the hermitian Waldhausen construction that we discuss in Section 7.2.

Example 6.2. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object. We denote by

$$- * [0] : \Delta \to \Delta$$

the endofunctor of Δ that adds a maximal element. The simplical object $P^{\triangleright}(X)_\bullet := (- * [0])^*(X_\bullet)$ is called the final path space [DK19, Section 6.2]. It comes equipped with a morphism of simplicial sets

$$p_{\triangleright} : P^{\triangleright}(X)_\bullet \to X_\bullet$$

This morphism is a relative 2-Segal object. Indeed, it follows from [DK19, Thm.6.3.2] that $P^{\triangleright}(X)_\bullet$ is 1-Segal and it is easy to see that the relative 2-Segal conditions for $p_{\triangleright}$ reduce to the 2-Segal conditions on $X_\bullet$. This relative 2-Segal object describes the regular left action of the 2-Segal object $X_\bullet$ on itself. Similarly, one defines the initial path space $P^{\triangleleft}(X)_\bullet$ that describes the regular right action of $X_\bullet$ on itself.

Example 6.3. Let $X_\bullet$ be a 2-Segal object in $\mathcal{C}$. Recall, that the edgewise subdivision functor $e : \Delta \to \Delta$ is defined on objects as $[n] \mapsto [n] * [n]^{\mathrm{op}}$. We denote the functor given by precomposition with e by

$$\mathsf{Tw}(-) := e^* : \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

and call it the twisted arrow construction. Further, we call for every simplicial object $X_\bullet$ the associated simplicial object $\mathsf{Tw}(X)_\bullet$ the twisted arrow simplicial object. It has been shown in [BOO+20, Thm.2.9], that for every 2-Segal object $X_\bullet$, the simplicial object $\mathsf{Tw}(X)_\bullet$ is 1-Segal. Moreover, it is easy to check that the morphism

$$\mathsf{Tw}(X)_\bullet \to X_\bullet \times X_\bullet^{\mathrm{op}}$$

defines a relative 2-Segal object. This encodes the regular bimodule action of $X_\bullet$ on itself.

6.3 Birelative Decomposition Spaces

An alternative, but equivalent, way to formalize the 2-Segal conditions are the *decomposition space conditions* of [GCKT18]. For the definition of these recall that the simplex category Δ admits a factorization system $(\Delta^{\mathrm{act}}, \Delta^{\mathrm{int}})$ generated by the active (depicted $\twoheadrightarrow$) and inert morphisms (depicted $\rightarrowtail$) [Lur17]. This factorization system has the special property that Δ admits active-inert-pushouts, i.e. every cospan

$$\begin{array}{ccc} n & \overset{f}{\rightarrowtail} & m \\ {\scriptstyle g}\big\downarrow\!\!\!\downarrow & & \\ l & & \end{array}$$

with f inert and g active admits an extension to a pushout diagram:

$$\begin{array}{ccc} n & \rightarrowtail & m \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ l & \rightarrowtail & k \end{array}$$

in Δ. A simplicial space $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ is then called a decomposition space if it maps active-inert pullbacks in Δ^{op} to pullback diagrams of spaces. The goal of this section is to extend the definition of a decomposition space to the category $\Delta_{/[1]}$. Therefore we first recall the definition of the active-inert factorization system on $\Delta_{/[1]}$

Notation 6.3. Given an object $f : [n] \to [1] \in \Delta_{/[1]}$, we will frequently denote it by f_n to indicate its dependence on the source. Given $i < j$ in $[n]$ we denote by $[i, j] := \{i, ..., j\} \subset [n]$ the subinterval from i to j and by $f_{[i,j]}$ the restriction of f_n to this subinterval.

Definition 6.11. A morphism $f : g^0_n \to g^1_m$ in $\Delta_{/[1]}$ is called

(1) *active* if the underlying morphism of linearly ordered sets $f : [n] \to [m]$ is endpoint preserving. We depict active morphisms by $\twoheadrightarrow$.

(2) *inert* if the underlying morphism of linearly ordered sets $f : [n] \to [m]$ is a subinterval inclusion. We depict inert morphisms by $\rightarrowtail$

We denote the wide subcategory of $\Delta_{/[1]}$ spanned by the active (resp. inert) morphisms by $\Delta^{\mathrm{act}}_{/[1]}$ (resp. $\Delta^{\mathrm{int}}_{/[1]}$). Similarly, we call a morphism f in $\Delta^{\mathrm{op}}_{/[1]}$ active (resp. inert) if it is active (resp. inert) as a morphism in $\Delta_{/[1]}$.

Proposition 6.5. *Every morphism $f : g^0_n \to g^1_m$ in $\Delta_{/[1]}$ admits an unique factorization $f = f'' \circ f'$ with f'' inert and f' active. In particular the subcategories $(\Delta^{\mathrm{act}}_{/[1]}, \Delta^{\mathrm{int}}_{/[1]})$ form a factorization system on $\Delta_{/[1]}$ [Lur09a, Def.5.2.8.8].*

Proof. This is a consequence of the dual of [Lur17, Prop.2.1.2.5] applied to the Cartesian fibration $\Delta_{/[1]} \to \Delta$. Alternatively, one can see this more directly as follows. The underlying morphism of linearly ordered sets $f : [n] \to [m]$ admits an active-inert factorization given by the active map $f' : [n] \twoheadrightarrow [f(0), f(n)]$ and the inert map $f'' : [f(0), f(n)] \hookrightarrow [m]$. These morphisms admit unique extensions to morphisms in $\Delta_{/[1]}$. $\square$

To characterize the active-inert pullback diagrams in $\Delta^{\mathrm{op}}_{/[1]}$, we need the following definition:

Definition 6.12. Let $g : [n] \to [1]$ and $f : [m] \to [1]$ be two objects in $\Delta_{/[1]}$. We will call g and f composable, if $g(n) = f(0)$. The concatenation of g with f denoted $g * f : [n + m] \to [1]$ is defined as the map:

$$(g * f)(k) = \begin{cases} g(k) & k \leq n \\ f(k - n) & k \geq n \end{cases}$$

Similarly, we denote for two linearly ordered sets S, T by $S * T$ the partially ordered set $S \coprod T/(s_{max} \sim t_{min})$, where s_{max} denotes the maximal element of S and t_{min} denotes the minimal element of T.

Given an active morphism $f^2 : h^0_k \twoheadrightarrow h^1_l$ and an inert morphism $e^1 : g^0_n \rightarrowtail h^0_k$ in $\Delta_{/[1]}$ we can consider an active-inert factorization of its composite $f^2 \circ e^1$. This factorization can be organized into a commutative square

$$\begin{array}{ccc} g^0_n & \xrightarrow{f^1} & g^1_m \\ {\scriptstyle e^1}\downarrow & & \downarrow{\scriptstyle e^2} \\ h^0_k & \xrightarrow{f^2} & h^1_l \end{array}$$

It follows from the definition of active and inert morphisms that this square is equivalent to the square

$$\begin{array}{ccc} g_n^0 & \overset{f^1}{\twoheadrightarrow} & g_m^1 \\ \downarrow & & \downarrow \\ h_a^0 * g_n^0 * h_b^0 & \underset{f_a^2 * f^1 * f_b^2}{\twoheadrightarrow} & h_a^1 * g_m^1 * h_b^1 \end{array}$$

In analogy with [GCKT18, Sect.2.6], we call the squares of the form:

$$\begin{array}{ccc} g_n^0 & \overset{f^1}{\twoheadrightarrow} & g_m^1 \\ \downarrow & & \downarrow \\ h_a^0 * g_n^0 * h_b^0 & \underset{\mathrm{id} * f^1 * \mathrm{id}}{\twoheadrightarrow} & h_a^0 * g_m^1 * h_b^0 \end{array}$$

identity extension squares. These are precisely the active-inert pullbacks in $\Delta^{\mathrm{op}}_{/[1]}$.

Proposition 6.6. *Let $\sigma : \Delta^1 \times \Delta^1 \to \Delta_{/[1]}$ be an identity extension square:*

$$\begin{array}{ccc} g_n^0 & \overset{f^1}{\twoheadrightarrow} & g_m^1 \\ \downarrow & & \downarrow \\ h_a^0 * g_n^0 * h_b^0 & \underset{\mathrm{id} * f^1 * \mathrm{id}}{\twoheadrightarrow} & h_a^0 * g_m^1 * h_b^0 \end{array}$$

Then σ is a pushout square in $\Delta_{/[1]}$.

In analogy with [GCKT18], we introduce the following definition:

Definition 6.13. Let $X_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ be a birelative simplicial object. $X_\bullet$ is called a *birelative decomposition space* if it sends every active-inert pullback in $\Delta^{\mathrm{op}}_{/[1]}$ to a pullback square in $\mathcal{C}$, i.e

$$\begin{array}{ccc} g_n^0 & \overset{f^1}{\twoheadrightarrow} & g_m^1 \\ {\scriptstyle e^1}\downarrow & & \downarrow{\scriptstyle e^2} \\ h_k^0 & \underset{f^2}{\twoheadrightarrow} & h_l^1 \end{array} \quad \overset{X}{\longmapsto} \quad \begin{array}{ccc} X_{g_n^0} & \overset{X_{f^1}}{\longleftarrow} & X_{g_m^1} \\ {\scriptstyle X_{e^1}}\uparrow & & \uparrow{\scriptstyle X_{e^2}} \\ X_{h_k^0} & \underset{X_{f^2}}{\longleftarrow} & X_{h_l^1} \end{array}$$

As in the case of 2-Segal spaces [GCKT18], we show that the birelative decomposition space condition is equivalent to the birelative 2-Segal condition:

Proposition 6.7. *Let $X_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ be a birelative simplicial object. The following are equivalent:*

(1) *$X_\bullet$ is birelative 2-Segal.*

(2) *$X_\bullet$ is a birelative decomposition space.*

Proof. Assume first that $X_\bullet$ is a birelative decomposition space. Let $f : [n] \to [1]$ with $n \geq 3$ be an object of $\Delta_{/[1]}$ and $0 \leq i < j \leq n$. Consider the square

$$\begin{array}{ccc} f|_{i,j} & \twoheadrightarrow & f_{i,\dots,j} \\ \downarrow & & \downarrow \\ f_{0,\dots,i,j,\dots,n} & \twoheadrightarrow & f \end{array}$$

in $\Delta_{/[1]}$. Note that $X_\bullet$ satisfies the birelative 2-Segal conditions if and only if it maps each such square to a pullback square. But this square is an identity extension square in $\Delta_{/[1]}$ and hence an active-inert pullback.
Conversely assume that $X_\bullet$ is birelative 2-Segal. We need to show that the image under $X_\bullet$ of every identity extension square

$$\begin{array}{ccc} g^0_n & \xrightarrow{f^1} & g^1_m \\ \downarrow & & \downarrow \\ h^0_a * g^0_n * h^0_b & \xrightarrow[\mathrm{id} * f^1 * \mathrm{id}]{} & h^0_a * g^1_m * h^0_b \end{array}$$

is a pullback diagram. We can extend every such to a rectangle:

$$\begin{array}{ccccc} g^0_{\{0,n\}} = g^1_{\{0,m\}} & \twoheadrightarrow & g^0_n & \xrightarrow{f^1} & g^1_m \\ \downarrow & & \downarrow & & \downarrow \\ h^0_a * g^1_{\{0,m\}} * h^0_b & \twoheadrightarrow & h^0_a * g^0_n * h^0_b & \xrightarrow[\mathrm{id} * f^1 * \mathrm{id}]{} & h^0_a * g^1_m * h^0_b \end{array}$$

Since $X_\bullet$ is birelative 2-Segal it maps the left square and the outer rectangle to pullback diagrams. The claim follows from the pasting law for pullbacks. □

Remark 6.6. The active-inert factorization system on $\Delta_{/[1]}$ restricts to an active inert factorization system on $\Delta_{\leq}$ and $\Delta_{\geq}$. One can analogously to Definition 6.13 define a notion of left and right relative decomposition space and show that it is equivalent to the left and right relative 2-Segal conditions.

7 Examples of higher Segal Objects

Our main goal in this section is to provide examples of the various types of 2-Segal conditions introduced in the last section. The fundamental example of a 2-Segal space is the Waldhausen $S_\bullet$-construction[20] of an exact ∞-category, whose space of n-simplices $S_n(\mathcal{C})$ can be described as the space of length n-flags in $\mathcal{C}$. To show that this simplicial space satisfies the 2-Segal conditions one has to compare different pasting of flags. The reason that these are equivalent is a consequence of the third isomorphism theorem that holds in any exact ∞-category $\mathcal{C}$. It is therefore natural to expect examples of other types of 2-Segal objects to arise from similar constructions with exact ∞-categories.
Indeed, as our first example, we consider exact functors $F : \mathcal{C} \to \mathcal{D}$ between exact ∞-categories. Every such

[20] See Definition 7.1

induces a morphism of simplicial spaces $S_\bullet(F) : S_\bullet(\mathcal{C}) \to S_\bullet(\mathcal{D})$. We show that under the assumptions of Proposition 7.2, this morphism gives rise to examples of active equifibered and relative Segal morphisms.
Moreover, we also present a different construction of an active equifibered and relative Segal morphism from an exact functor F. Instead of a morphism between 2-Segal spaces, we construct from $F : \mathcal{C} \to \mathcal{D}$ a 2-Segal space $S_\bullet^{\mathsf{rel}}(F)$ itself, called the relative $S_\bullet$-construction [Wal06]. This 2-Segal space fits into a sequence of simplicial spaces

$$\mathcal{D}_\bullet \xrightarrow{\iota_\bullet} S_\bullet^{\mathsf{rel}}(F) \xrightarrow{\pi_\bullet} S_\bullet(\mathcal{C}) \tag{17}$$

that is important in algebraic K-theory since it induces the long exact sequence of relative K-theory [Wal06]. After recalling the definition of the Waldhausen and relative Waldhausen construction in the setting of exact ∞-categories [Bar15], we show that the morphisms $\iota_\bullet$ and $\pi_\bullet$ are examples of a relative Segal and an active equifibered morphism respectively.
These constructions admit applications to the theory of Hall algebras. Indeed, every such morphism induces a (co)algebra morphism between the corresponding Hall algebras. In particular, the morphism induced by the relative $S_\bullet$-construction has been applied for the construction of derived Hecke actions [KV22, Sect.5]. This is especially advantageous in the realm of ∞-categories, where constructing homotopy coherent algebra morphisms can be challenging.
This discussion admits an analog for birelative simplicial spaces in the setting of hermitian K-theory. For exact 1-categories with exact duality $(\mathcal{C}, \mathsf{D}_\mathcal{C})$ it has been proven by Young [You18, Thm.3.6] that the hermitian $\mathcal{R}_\bullet$-construction of $(\mathcal{C}, \mathsf{D}_\mathcal{C})$ describes a relative 2-Segal space over $S_\bullet(\mathcal{C})$. Young uses this observation to construct representations of Hall algebras and to apply it in orientifold Donaldson–Thomas-theory [You20]. In Section 7.2, we extend Young's result to the setting of exact ∞-categories with duality. Our proof is inspired by the construction of the real Waldhausen $S_\bullet$-construction as presented in [HSV19]. For an exact ∞-category $\mathcal{C}$ with duality functor $\mathsf{D}_\mathcal{C}$ the relative 2-Segal space from Example 6.3

$$\mathsf{Tw}(S_\bullet(\mathcal{C})) \to S_\bullet(\mathcal{C}) \times S_\bullet(\mathcal{C})^{\mathrm{rev}} \tag{18}$$

admits an extension to a morphism of simplicial spaces with C_2-action. For $n = 1$, we can explicitly describe the C_2-action by the commutative diagram:

$$\begin{array}{ccc}
\begin{array}{ccc} C_{0,1} & \longrightarrow & C_{0,2} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{1,2} \end{array} & \longmapsto & \begin{array}{ccc} \mathsf{D}_\mathcal{C}(C_{1,2}) & \longrightarrow & \mathsf{D}_\mathcal{C}(C_{0,2}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{D}_\mathcal{C}(C_{0,1}) \end{array} \\
\downarrow & & \downarrow \\
(C_{0,1}, C_{1,2}) & \longmapsto & (\mathsf{D}_\mathcal{C}(C_{1,2}), \mathsf{D}_\mathcal{C}(C_{0,1}))
\end{array}$$

The hermitian $\mathcal{R}_\bullet$-construction then arises as the simplicial space of homotopy fixed points. Furthermore, the induced morphism $\mathcal{R}_\bullet(\mathcal{C}) \to S_\bullet(\mathcal{C})$ is a relative 2-Segal space. Using this method we finally construct examples of active equifibered and relative Segal maps between relative 2-Segal objects from exact duality preserving functors $F : \mathcal{C} \to \mathcal{D}$.
These results can be applied to the construction of ∞-categorical Hall algebra representations and their morphisms. In particular, these constructions are essential for the construction of representations of categorified Hall algebras [PS23, DPS22].

7.1 2-Segal Spans from Exact ∞-Functors

In this subsection, we construct examples of active equifibered and relative Segal morphisms between Waldhausen $S_\bullet$-construction. For the reader's convenience, we first recall the construction of the Waldhausen $S_\bullet$-construction of an exact ∞-categories.

An *exact ∞-category* consists of a triple $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ of an additive ∞-category $\mathcal{C}$ and two wide Waldhausen subcategories $\mathcal{C}^{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{eg}}$ satisfying a list of compatibility conditions [Bar15, Def.3.1]. We call the morphism in $\mathcal{C}^{\mathrm{in}}$ *ingressive* denoted $\rightarrowtail$ and the morphism in $\mathcal{C}^{\mathrm{eg}}$ *egressive* denoted $\twoheadrightarrow$. For an exact ∞-category $\mathcal{C}$, a bicartesian square

$$\begin{array}{ccc} C_1 & \rightarrowtail & C_2 \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & C_3 \end{array}$$

in $\mathcal{C}$ is called an *extension square*. A functor

$$F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$$

between exact ∞-categories is called an *exact ∞-functor* [Bar15, Def.4.1] if it preserves ingressive and egressive morphisms, zero objects, and extension squares. We denote by Exact_∞ the subcategory of $\mathsf{Fun}(\Lambda^2_2, \mathsf{Cat})$ spanned by exact ∞-categories and exact functors.

Consider for every $n \geq 0$ the poset $[n]$ as an ∞-category and let $\mathsf{Ar}([n]) := \mathsf{Fun}([1], [n])$ be the associated ∞-category of arrows. These assemble into a cosimplicial ∞-category $\mathsf{Ar}([-]) : \Delta \to \mathsf{Cat}$.

Definition 7.1. Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ be an exact ∞-category. We denote by $S_n(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \subset \mathsf{Map}(\mathsf{Ar}([n]), \mathcal{C})$ the full subspace spanned by those functors $F : \mathsf{Ar}([n]) \to \mathcal{C}$ that satisfy the following conditions:

(1) for every $0 \leq i \leq n$ we have $F(i, i) = 0$.

(2) for every $0 \leq i \leq k \leq j \leq n$ the morphism $F(i, j) \rightarrowtail F(k, j)$ is ingressive.

(3) for every $0 \leq i \leq j \leq l \leq n$ the morphism $F(i, j) \twoheadrightarrow F(i, l)$ is egressive

(4) for every $0 \leq i \leq k \leq j \leq l \leq n$ the square

$$\begin{array}{ccc} F(i,j) & \rightarrowtail & F(k,j) \\ \downarrow & & \downarrow \\ F(i,l) & \rightarrowtail & F(k,l) \end{array}$$

is bicartesian.

When n varies, the spaces $S_n(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ assemble into a simplicial space called the *Waldhausen $S_\bullet$-construction*. When the classes of ingressive and egressive morphisms are clear from the context, we abuse notation and denote the Waldhausen $S_\bullet$-construction by $S_\bullet(\mathcal{C})$. For completeness, we include a proof of the following statement. The original proof in the more general context of proto-exact ∞-categories is given in [DK19, Thm.7.3.3].

Proposition 7.1. *[DK19, Thm.7.3.3] Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ be an exact ∞-category. The simplicial space $S_\bullet(\mathcal{C})$ is 2-Segal.*

Proof. By the path space criterion [DK19, Thm.6.3.2] it suffices to show that the two path spaces $(P^{\triangleright}S)_{\bullet}$ and $(P^{\triangleleft}S)_{\bullet}$ are 1-Segal. It follows from [Lur17, Lem.1.2.2.4.] that for every $n \geq 0$ the projection maps

$$\mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{C}^{\mathrm{eg}}) \xleftarrow{p^v} S_n(\mathcal{C}) \xrightarrow{p_h} \mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{C}^{\mathrm{in}})$$

that are induced by restricting to the subposet

$$\{(0,n) < (1,n) < ... < (n-1,n)\} \hookrightarrow \mathsf{Ar}([n]) \hookleftarrow \{(0,1) < (0,2) < ... < (0,n)\}$$

are equivalences. Through these equivalences, the 1-Segal conditions for the two path spaces translate into the conditions that for every $n \geq 2$, the morphisms

$$\mathsf{Map}_{\mathsf{Cat}}([n], \mathcal{C}^{\mathrm{eg}}) \to \mathsf{Map}_{\mathsf{Cat}}([1] \amalg_{[0]} \cdots \amalg_{[0]} [1], \mathcal{C}^{\mathrm{eg}})$$

and

$$\mathsf{Map}_{\mathsf{Cat}}([n], \mathcal{C}^{\mathrm{in}}) \to \mathsf{Map}_{\mathsf{Cat}}([1] \amalg_{[0]} \cdots \amalg_{[0]} [1], \mathcal{C}^{\mathrm{in}})$$

are equivalences. But this condition is satisfied, since the functor $[1] \amalg_{[0]} \cdots \amalg_{[0]} [1] \to [n]$ is an equivalence of ∞- categories. □

Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ and $(\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be exact ∞-categories and $F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be an exact ∞-functor. It follows from the construction of the Waldhausen construction that F induces a functor $S_{\bullet}(F) : S_{\bullet}(\mathcal{C}) \to S_{\bullet}(\mathcal{D})$ between the corresponding Waldhausen constructions. We analyze the conditions under which this morphism is active equifibered (resp. relative Segal). To do so, we need to introduce some definitions:

Definition 7.2. Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ be an exact ∞-category and let $\mathcal{C}_0 \subset \mathcal{C}$ be a full exact[21] subcategory. We call $\mathcal{C}_0$

(1) *extension closed* if every exact-sequence $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$

$$\begin{array}{ccc} C_1 & \rightarrowtail & C_2 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 0 & \rightarrowtail & C_3 \end{array}$$

in $\mathcal{C}$ with $C_1, C_3 \in \mathcal{C}_0$, factors through $\mathcal{C}_0$.

(2) *closed under quotients (resp. subobjects)* if for every egressive morphism $C_2 \twoheadrightarrow C_3$ (resp. ingressive morphism $C_1 \rightarrowtail C_2$) in $\mathcal{C}$ with $C_2 \in \mathcal{C}_0$ also $C_3 \in \mathcal{C}_0$ (resp. $C_1 \in \mathcal{C}_0$)

We can then prove the following:

Proposition 7.2. *Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ and $(\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be exact ∞-categories and $F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be a fully faithful exact ∞-functor. The induced morphism $S_{\bullet}(F)$ of simplicial spaces is*

(1) active equifibered if and only if the essential image of F is closed under quotients and subobjects.

(2) relative Segal if and only if the essential image of F is extension closed.

[21]A subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is called exact, if the triple $(\mathcal{C}_0, \mathcal{C}^{\mathrm{in}} \cap \mathcal{C}_0, \mathcal{C}^{\mathrm{eg}} \cap \mathcal{C}_0)$ is an exact ∞-category.

Proof. To show (1), we need to show that for every $n \geq 1$ the morphism

$$\eta_1 : \mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{C}^{\mathrm{in}}) \to \mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{D}^{\mathrm{in}}) \times_{\mathcal{D}^{\simeq}} \mathcal{C}^{\simeq}$$

is an equivalence of spaces. It follows from the 2-out-of-3 property for fully faithful functors that η_1 is fully faithful. To prove the claim, it suffices to show that η_1 is essentially surjective. But this follows since $\mathcal{C} \subset \mathcal{D}$ is closed under subobjects.

The proof of (2) is similar. We need to show that for every $n \geq 1$ the functor

$$\eta_2 : \mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{C}^{\mathrm{in}}) \to \mathsf{Map}_{\mathrm{in}}([n-1], \mathcal{D}^{\mathrm{in}}) \times_{\mathcal{D}^{\simeq} \times \cdots \times \mathcal{D}^{\simeq}} (\mathcal{C}^{\simeq} \times \cdots \times \mathcal{C}^{\simeq})$$

is an equivalence. It follows again from the 2-out-of-3 property for fully faithful functors that this functor is fully faithful. Moreover, since the essential image is closed under extensions, it is easy to see that the functor is also essentially surjective. □

Remark 7.1. Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}})$ be an exact ∞-category, and $\mathcal{C}_0 \subset \mathcal{C}$ be full exact subcategories. The inclusion functor

$$i : \mathcal{C}_0 \to \mathcal{C}$$

induces a $\mathbb{K}$-linear map between the corresponding vector spaces of groupoid functions

$$i_* : \mathrm{Hom}_{\mathsf{Set}}(\pi_0(\mathcal{C}_0^{\simeq}), \mathbb{K}) \to \mathrm{Hom}_{\mathsf{Set}}(\pi_0(\mathcal{C}^{\simeq}), \mathbb{K})$$

that maps the constant function $[C]_{\mathcal{C}_0}$ for $C \in \mathcal{C}_0$ to the corresponding constant function $[C]_{\mathcal{C}}$ in $\mathcal{C}$. If $\mathcal{C}_0$ is closed under extensions (resp. under quotients and subobjects), then, according to the definition, this map naturally extends to a morphism between the corresponding Hall algebras [Rin90] (resp. Hall coalgebras). These classes of morphisms are precisely the ones described by the above class of active equifibered and relative Segal morphisms.

As explained in the introduction of this section, we can also construct different examples of active equifibered and relative Segal maps, using the Sequence (17). We therefore first recall all the necessary ingredients for this construction. Let $F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be an exact functor between exact ∞-categories. Following [DKSS24], we define the *relative $S_\bullet$-construction* of F as the simplicial space denoted $S_\bullet^{\mathsf{rel}}(F)$ whose ∞-category of n-simplices is defined as the pullback

$$\begin{array}{ccc} S_n^{\mathsf{rel}}(F) & \longrightarrow & S_{n+1}(\mathcal{D}) \\ \downarrow & & \downarrow \partial_{n+1} \\ S_n(\mathcal{C}) & \xrightarrow{F} & S_n(\mathcal{D}) \end{array}$$

Alternatively, we can also describe $S_n^{\mathsf{rel}}(F)$ as the full subspace of the space of sections of the Grothendieck construction of the functor

$$\mathsf{Fun}([n-1], \mathcal{C}^{\mathrm{in}}) \xrightarrow{\mathrm{ev}_{n-1}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

generated by the ingressive sections. Since the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ generated by 2-Segal objects is closed under limits, it follows that $S_\bullet^{\mathsf{rel}}(F)$ is itself 2-Segal. The simplicial space $S_\bullet^{\mathsf{rel}}(F)$ sits in a sequence of simplicial spaces

$$\mathcal{D}_\bullet \xrightarrow{\iota_\bullet} S_\bullet^{\mathsf{rel}}(F) \xrightarrow{\pi_\bullet} S_\bullet(\mathcal{C}),$$

where the maps $\iota_n : \mathcal{D} \to S_n(F)$ and $\pi_n : S_n(F) \to S_n(\mathcal{C})$ are given by

$$d \mapsto (0 \rightarrowtail \dots \rightarrowtail 0 \rightarrowtail b)$$

and

$$(a_0 \rightarrowtail \dots \rightarrowtail \dots \rightarrowtail a_{n-1} \rightarrowtail b) \mapsto (a_0 \rightarrowtail \dots \rightarrowtail a_{n-1})$$

respectively. These maps have the following properties:

Proposition 7.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between exact ∞-categories. Then:*

(1) the morphism $\pi_\bullet$ is active equifibered.

(2) the morphism $\iota_\bullet$ is relative Segal.

Proof. Note that $S_n^{\mathsf{rel}}(F)$ can be written as the pullback

$$\begin{array}{ccc} S_n^{\mathsf{rel}}(F) & \longrightarrow & \mathsf{Map}_{\mathsf{Cat}}([n-1], \mathcal{C}^{\mathrm{in}}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} \\ S_1^{rel}(F) & \longrightarrow & \mathcal{C}^{\simeq} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Map}_{\mathsf{Cat}}([1], \mathcal{D}^{\mathrm{in}}) & \longrightarrow & \mathcal{D}^{\simeq} \end{array}$$

of the outer rectangle. To show the claim for $\pi_\bullet$, we need to show that the upper diagram is a pullback as well. But this follows from the pasting law for pullbacks. For (2) note that the pullback of spaces

$$\mathcal{D}^{\simeq} \times_{S_1^{\mathsf{rel}}(F) \times_{S_0^{\mathsf{rel}}(F)} \cdots \times_{S_0^{\mathsf{rel}}(F)} S_1^{\mathsf{rel}}(F)} S_n^{\mathsf{rel}}(F)$$

can be identified with the full subspace of $S_n^{\mathsf{rel}}(F) \simeq S_n(\mathcal{C}) \times_{S_n(\mathcal{D})} S_{n+1}(\mathcal{D})$ generated by those objects $(F_{\mathcal{C}}.G_{\mathcal{D}}, \alpha)$, s.t.

(1) the restriction of $G_{\mathcal{D}}$ to the subposet $\{(0,n) < \dots < (n-1,n)\}$ is constant

(2) the restriction of $F_{\mathcal{C}}$ to the subposet $\{(0,1) < (1,2) < \dots < (n-1,n)\}$ is constant at 0

It then follows from an iterated Kan extension argument similar to [Lur17, Lem.1.2.2.4] that this subspace is equivalent to $\mathcal{D}^{\simeq}$. □

As an application, we use the above results for the construction of the so-called derived Hecke actions from [KV22].

Proposition 7.4. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $p : Y_\bullet \to X_\bullet$ be a relative 2-Segal object. Let further, $f : X_\bullet \to Z_\bullet$ be an active equifibered, and $g : W_\bullet \to X_\bullet$ be a relative Segal morphism. Then*

(1) the composite $f \circ p : Y_\bullet \to Z_\bullet$ is a relative 2-Segal object.

(2) the pullback $\pi : Y_\bullet \times_{X_\bullet} W_\bullet \to W_\bullet$ is a relative 2-Segal object.

Proof. Since $Y_\bullet$ is 1-Segal by definition, for the first case we only need to show that the composite $f \circ p$ is active equifibered. But this follows since active equifibered morphisms are closed under composition.
For the second case, note that it follows from the proof of Lemma 6.1 that the pullback of an active equifibered morphism along a relative Segal morphism is active equifibered. We therefore only need to show that $Y_\bullet \times_{X_\bullet} W_\bullet$ is 1-Segal. Consider the morphisms:

$$Y_\bullet \times_{X_\bullet} W_\bullet \to Y_\bullet \to *_\bullet.$$

Since both of them are relative Segal, their composite is so. In particular, it follows that $Y_\bullet \times_{X_\bullet} W_\bullet$ is 1-Segal. □

As a consequence, we obtain:

Corollary 7.5. *Let $F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be an exact fully faithful functor between exact ∞-categories whose essential image is extension closed. Then the simplicial space $S_\bullet(F)$ is 1-Segal.*

Proof. This follows from an application of Proposition 7.4 to the pullback diagram that defines $S_\bullet(F)$. □

Example 7.1. Let $F : (\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{eg}}) \to (\mathcal{D}, \mathcal{D}^{\mathrm{in}}, \mathcal{D}^{\mathrm{eg}})$ be a fully faithful exact functor between exact ∞-categories whose essential image is closed under extensions. Then the morphism

$$S_\bullet(F) \to S_\bullet(\mathcal{C})$$

defines a relative 2-Segal object by Corollary 7.5. Unraveling the definitions, this equips $\mathcal{D}$ with the structure of an $S_\bullet(\mathcal{C})$-module with action given by

$$\mathcal{C} \times \mathcal{D} \leftarrow S_1(F) \rightarrow \mathcal{D}$$

This module structure is responsible for the so-called *derived Heck actions* as defined in [KV22, Sect.5.2]. Indeed, the analysis of [DK18, Sect.8] for the theory with transfer given by Borel–Moore homology applied to the above module structure recovers their derived Hecke actions. Furthermore, it can be used to extend their derived Hecke action to the categorified Hall algebra of [PS23, DPS22].

7.2 Relative 2-Segal Spans from Duality preserving ∞-Functors

Our goal in this section is to construct examples of relative 2-Segal spaces and relative 2-Segal spans. to this end, we present a general construction of relative 2-Segal objects from 2-Segal spaces with duality. For this construction, we first have to recall some facts about ∞-categories with duality. Since we only work with simplicial objects in $\mathcal{S}$, we drop it from the notation.

Definition 7.3. Let $C \in \mathcal{C}$ be an object in an ∞-category. A C_2-*action* on C is a functor $\mathsf{BC}_2 \to \mathcal{C}$ that maps the unique point to C.

Recall that the category Δ admits a canonical C_2-action. This action maps an object $[n]$ to $[n]$ and a morphism $f : [n] \to [m]$ to the composite morphism

$$[n] \simeq [n]^{op} \xrightarrow{f^{op}} [m]^{op} \simeq [m]$$

where the equivalence $[n] \simeq [n]^{op}$ is given by $i \mapsto n - i$. By functoriality of taking presheaves, the C_2-action on Δ induces one on $\mathcal{P}(\Delta)$ that maps a simplicial space $X_\bullet$ to $X_\bullet^{\mathrm{rev}}$. The restriction of this action to the full subcategory $\mathsf{Cat} \subset \mathcal{P}(\Delta)$ maps an ∞-category to its opposite.

Definition 7.4. A *simplicial space with duality* is a homotopy fixed point with respect to the above C_2-action on $\mathcal{P}(\Delta)$. Similarly, an ∞-*category with duality* is a homotopy fixed point with respect to the C_2-action on Cat given by taking opposites.

More precisely an ∞-category with duality is a section of the cocartesian fibration $\widetilde{\mathsf{Cat}_\infty} \to \mathsf{BC}_2$ encoding the C_2-action on Cat. We define the ∞-category $\mathsf{Cat}_\infty^{\mathsf{C}_2}$ of ∞-categories with duality and the ∞-category of simplicial spaces with duality $\mathcal{P}(\Delta)^{\mathsf{C}_2}$ as the respective ∞-categories of homotopy fixed points.

Remark 7.2. The ∞-categories Δ, $\mathcal{P}(\Delta)$ and Cat also admit a trivial C_2-action. We denote the respective category of homotopy fixed points with respect to this trivial action by $\Delta[\mathsf{C}_2]$, $\mathcal{P}(\Delta)[\mathsf{C}_2]$ and $\mathsf{Cat}[\mathsf{C}_2]$. Unraveling definitions, these are given by the ∞-categories of functors $\mathsf{Cat}[\mathsf{C}_2] \simeq \mathsf{Fun}(\mathsf{BC}_2, \mathsf{Cat})$. Hence, a homotopy fixed point with respect to this trivial action describes an object with C_2-action.

Note that the non-trivial C_2-action on $\mathcal{P}(\Delta)$ restricts to a C_2-action on $2\text{-}\mathsf{Seg}_\Delta$.

Definition 7.5. A 2-*Segal space with duality* is a homotopy fixed point with respect to the non-trivial C_2-action on $2\text{-}\mathsf{Seg}_\Delta$.

We denote by $2\text{-}\mathsf{Seg}_\Delta^{\mathsf{C}_2}$ the ∞-category of homotopy fixed points. By construction, this ∞-category comes equipped with a forgetful functor to $2\text{-}\mathsf{Seg}_\Delta$. We call a morphism $f : X_\bullet \to Y_\bullet$ in $2\text{-}\mathsf{Seg}_\Delta^{\mathsf{C}_2}$ active equifibered (resp. relative Segal) if its image in $2\text{-}\mathsf{Seg}_\Delta$ is active equifibered (resp. relative Segal). Similarly, we call a span

$$\begin{array}{ccc} & Y_\bullet & \\ {\scriptstyle s_\bullet}\swarrow & & \searrow{\scriptstyle t_\bullet} \\ X_\bullet & & Z_\bullet \end{array}$$

in $2\text{-Seg}_\Delta^{\mathsf{C}_2}$ a 2-*Segal span with duality* if $s_\bullet$ is active equifibered and $t_\bullet$ is relative Segal. Since the ∞-category $2\text{-}\mathsf{Seg}_\Delta^{\mathsf{C}_2}$ admits small limits, it makes sense to define:

Definition 7.6. We define the ∞-category $2\text{-Seg}_\Delta^{\leftrightarrow,\mathsf{C}_2}$ of 2-*Segal spaces with duality* as the subcategory of $\mathsf{Span}(2\text{-Seg}_\Delta^{\mathsf{C}_2})$ with morphisms 2-Segal spans with duality.

Similarly, note that the trivial C_2-action on $\mathcal{P}(\Delta)$ induces the trivial C_2-action on $\mathrm{Rel2Seg}_\Delta$.

Definition 7.7. A *relative* 2-*Segal space with* C_2-*action* is a homotopy fixed point with respect to the trivial C_2-action on Rel2Seg. We denote by $\mathrm{Rel2Seg}_\Delta[\mathsf{C}_2]$ the ∞-category of homotopy fixed points.

As above the ∞-category $\mathrm{Rel2Seg}[\mathsf{C}_2]$ comes equipped with a forgetful functor to the ∞-category $\mathrm{Rel2Seg}_\Delta$. We define active equifibered and relative Segal morphisms as for $2\text{-}\mathsf{Seg}_\Delta^{\mathsf{C}_2}$. The associated version of 2-Segal span is called a *relative* 2-*Segal span with* C_2-*action*.

Definition 7.8. We define the ∞-*category of relative* 2-*Segal spaces with* C_2-*action* $\mathrm{Rel2Seg}_\Delta^{\leftrightarrow}[\mathsf{C}_2]$ as the subcategory of $\mathsf{Span}(\mathrm{Rel2Seg}_\Delta[\mathsf{C}_2])$ with morphisms relative 2-Segal spans with C_2-action.

Our first goal in this section is the construction of a functor

$$\mathsf{Tw}^{\leftrightarrow}(\text{-})^{\mathsf{C}_2} : 2\text{-Seg}_\Delta^{\leftrightarrow,\mathsf{C}_2} \to \mathrm{Rel2Seg}_\Delta^{\leftrightarrow},$$

extending the functor $\mathsf{Tw}(-)$, that associates to every 2-Segal space with duality a relative 2-Segal space. Recall the definition of the edgewise subdivision functor $e : \Delta \to \Delta$ from Example 6.3. We call a morphism in $\mathsf{Cat}[\mathsf{C}_2]$ a C_2-equivariant functor. This functor becomes C_2-equivariant if we equip the source with the trivial and the target with the non-trivial C_2-action. By functoriality, this lifts to a C_2-equivariant functor

$$\mathsf{Tw} := e^* : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$$

that induces a functor $\mathsf{Tw} : \mathcal{P}(\Delta)^{\mathsf{C}_2} \to \mathcal{P}(\Delta)[\mathsf{C}_2]$ on homotopy fixed points. The canonical inclusions $[n] \hookrightarrow [n] * [n]^{\mathrm{op}}$ and $[n]^{\mathrm{op}} \hookrightarrow [n] * [n]^{\mathrm{op}}$ induce a natural transformation

$$\mathsf{Tw} \Rightarrow id \times (-)^{op} : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$$

It follows from [HSV19, Lem.2.23] that the natural transformation is C_2-equivariant and hence induces a natural transformation

$$\mathsf{Tw} \Rightarrow id \times (-)^{\mathrm{op}} : \mathcal{P}(\Delta)^{\mathsf{C}_2} \to \mathcal{P}(\Delta)[\mathsf{C}_2]$$

on homotopy fixed points. In total, we can interpret this construction as a functor

$$\mathcal{P}(\Delta)^{\mathsf{C}_2} \xrightarrow{\mathsf{Tw}^{\to}(\text{-})} \mathsf{Fun}(\Delta^1, \mathcal{P}(\Delta)[\mathsf{C}_2]) \xrightarrow{(-)^{\mathsf{C}_2}} \mathsf{Fun}(\Delta^1, \mathcal{P}(\Delta))$$

that associates to a simplicial space with duality $X_\bullet$ the morphism

$$\mathsf{Tw}(X)_\bullet^{\mathsf{C}_2} \to (X_\bullet \times X_\bullet^{\mathrm{rev}})^{\mathsf{C}_2}$$

Proposition 7.6. *Let $X_\bullet \in \mathcal{P}(\Delta)^{\mathsf{C}_2}$ be a simplicial space with duality and let $X_\bullet \times X_\bullet^{\mathrm{rev}}$ be the induced simplicial space with C_2-action. There exists an equivalence of simplicial spaces*

$$(X_\bullet \times X_\bullet^{\mathrm{rev}})^{\mathsf{C}_2} \simeq X_\bullet$$

Proof. For any simplicial space $X_\bullet$, we can construct a C_2-action on $X_\bullet \times X_\bullet$ via right Kan extension

$$\begin{array}{ccc} * & \xrightarrow{X_\bullet} & \mathcal{P}(\Delta) \\ \downarrow & \nearrow_{X_\bullet \times X_\bullet} & \\ \mathsf{BC}_2 & & \end{array}$$

By adjunction, the projection onto the first factor $X_\bullet \times X_\bullet^{\mathrm{rev}} \to X_\bullet$ induces a morphism of simplicial spaces with C_2-action

$$\kappa : X_\bullet \times X_\bullet^{\mathrm{rev}} \to X_\bullet \times X_\bullet$$

It follows from the construction that the functor underlying κ is given by

$$\mathrm{id}_{X_\bullet} \times \mathsf{D} : X_\bullet \times X_\bullet^{\mathrm{rev}} \to X_\bullet \times X_\bullet,$$

where D denotes the duality on $X_\bullet$. Hence, it is an equivalence. We therefore obtain an equivalence on homotopy fixed points. The claim follows from the transitivity of right Kan extensions $\square$

Proposition 7.7. *The functor* $\mathsf{Tw}^{\rightarrow}(\text{-})$ *constructed above restricts to a functor*

$$\mathsf{Tw}^{\rightarrow}(\text{-}) : 2\text{-Seg}_{\Delta}^{\mathsf{C}_2} \to \text{Rel2Seg}_{\Delta}[\mathsf{C}_2]$$

Furthermore, it preserves active equifibered and relative Segal morphisms.

Proof. It follows from Example 6.3 that the map $\mathsf{Tw}(X)_\bullet \to X_\bullet \times X_\bullet^{\text{rev}}$ is a relative 2-Segal object with C_2-action. The second claim can be checked directly by looking at the corresponding pullback diagrams. ☐

Proposition 7.8. *The functor* $(-)^{\mathsf{C}_2} : \mathsf{Fun}(\Delta^1, \mathcal{P}(\Delta)[\mathsf{C}_2]) \to \mathsf{Fun}(\Delta^1, \mathcal{P}(\Delta))$ *restricts to a functor*

$$(-)^{\mathsf{C}_2} : \text{Rel2Seg}_{\Delta}[\mathsf{C}_2] \to \text{Rel2Seg}_{\Delta}$$

Further, it preserves active equifibered and relative Segal morphisms.

Proof. The claim follows from the fact that the functor $\mathsf{lim}_{\mathsf{BC}_2}(-)$ preserves small limits. ☐

Combining Proposition 7.8 and 7.7 we can obtain our first goal of this section

Proposition 7.9. *The composite* ∞*-functor*

$$2\text{-Seg}_{\Delta}^{\mathsf{C}_2} \xrightarrow{\mathsf{Tw}^{\rightarrow}(\text{-})} \text{Rel2Seg}_{\Delta}[\mathsf{C}_2] \xrightarrow{(-)^{\mathsf{C}_2}} \text{Rel2Seg}_{\Delta}$$

induces an ∞*-functor on the level of spans*

$$\mathsf{Tw}^{\leftrightarrow}(\text{-})^{\mathsf{C}_2} : 2\text{-Seg}_{\Delta}^{\leftrightarrow,\mathsf{C}_2} \to \text{Rel2Seg}_{\Delta}^{\leftrightarrow}[\mathsf{C}_2] \to \text{Rel2Seg}_{\Delta}^{\leftrightarrow}$$

We use this Proposition for the construction of examples of relative 2-Segal spaces and spans. In the previous section, we have shown that algebraic K-theory is a rich source of examples of active equifibered and relative Segal Δ^{op}-morphisms. The analogue for relative 2-Segal spans is hermitian K-theory. The ideas behind hermitian K-theory originate from the fundamental work of Hesselholt and Madsen [HM15] on real algebraic K-theory. An ∞-categorical formulation of these ideas is given in [HSV19]. Analogous to algebraic K-theory, hermitian K-theory is described by a hermitian analogue of the Waldhausen construction. The hermitian Waldhausen construction associates to every exact ∞-category with duality a simplicial space with duality. For the construction of the hermitian Waldhausen construction, we follow the presentation from [HSV19, Sect.8.2]. To do so, we first recall some facts about exact ∞-categories with duality and $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-categories.
Recall from Subsection 7.1 that the ∞-category Exact_∞ of exact ∞-categories is defined as the subcategory of $\mathsf{Exact}_\infty \subset \mathsf{Fun}(\Lambda_2^2, \mathsf{Cat})$ with objects exact ∞-categories and morphisms exact functors. The ∞-category Λ_2^2 carries a natural C_2-action. This action combines with the C_2-action on Cat to an action on $\mathsf{Fun}(\Lambda_2^2, \mathsf{Cat})$ that is defined on objects as

$$(\mathcal{C}_0 \to \mathcal{C}_2 \leftarrow \mathcal{C}_1) \mapsto (\mathcal{C}_1^{op} \to \mathcal{C}_2^{op} \leftarrow \mathcal{C}_0^{op}).$$

It is easy to see that this action restricts to a C_2-action on the subcategory Exact_∞.

Definition 7.9. An *exact* ∞*-category with duality* is a homotopy fixed point with respect to the above C_2-action on Exact_∞. We call the ∞-category of homotopy fixed points $\mathsf{Exact}_\infty^{\mathsf{C}_2}$ the ∞-category of exact ∞-categories with duality. A morphism in this ∞-category is called an exact duality preserving functor.

Informally, an exact ∞-category with duality $\mathcal{C}$ is a Waldhausen ∞-category such that the underlying ∞-category admits the structure of an ∞-category with duality and the class of ingressive morphisms $\mathcal{C}_{in}$

together with the opposites of the ingressive morphisms form the structure of an exact ∞-category on $\mathcal{C}$. These serve as an input for the hermitian Waldhausen construction.
In [HSV19], the authors work with real exact ∞-categories and equip the real Waldhausen $S_\bullet$-construction with the structures of a real simplicial space. Since the ∞-category of exact ∞-categories with duality forms a full subcategory of the ∞-category of real exact ∞-categories [HSV19, Rem.2.33], we can apply the construction of [HSV19, Sect 8.2] to our situation.
The key ingredient in the construction [HSV19, Sect.8.2] is enriched ∞-category theory. In our situation, we are interested in ∞-categories enriched over the ∞-category $\mathcal{S}[\mathsf{C}_2]$ of spaces with C_2-action. The amount of enriched ∞-category theory necessary for this construction is described in [HSV19, App. A].[22]

Notation 7.1. In the following, we adopt the convention to denote $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-categories by $\underline{\mathcal{C}}$ to distinguish them from their underlying ∞-category $\mathcal{C}$.

For an $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-category $\underline{\mathcal{C}}$, the underlying ∞-category is obtained by taking homotopy fixed points on Hom-spaces [HSV19, Sect.3.1]. It follows from [HSV19, Cor.2.8] that the ∞-categories $\mathsf{Cat}_\infty^{\mathsf{C}_2}$ and $\mathcal{P}(\Delta)^{\mathsf{C}_2}$ are Cartesian closed and are therefore enriched over $\mathcal{S}[\mathsf{C}_2]$. Informally, the C_2-action maps a duality preserving functor

$$F : (\mathcal{C}, \mathsf{D}_\mathcal{C}) \to (\mathcal{D}, \mathsf{D}_\mathcal{D})$$

to the duality preserving functor

$$\mathcal{C} \xrightarrow{\mathsf{D}_\mathcal{C}} \mathcal{C}^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}} \xrightarrow{\mathsf{D}_\mathcal{D}^{\mathrm{op}}} \mathcal{D} \ .$$

This enrichment restricts along the inclusion $\Delta^{\mathsf{C}_2} \subset \mathsf{Cat}_\infty^{\mathsf{C}_2}$ to an enrichment of Δ. We denote the $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-category Δ by $\underline{\Delta}$. Further, the ∞-category $\mathcal{S}[\mathsf{C}_2]$ is naturally enriched over itself.
It follows from [HSV19, Prop.2.12] that there exists an equivalence $\underline{\mathcal{P}}(\Delta)^{\mathsf{C}_2} \simeq \underline{\mathsf{Fun}}_{\mathcal{S}[\mathsf{C}_2]}(\underline{\Delta}^{op}, \mathcal{S}[\mathsf{C}_2])$ of $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-categories between the ∞-category of simplicial spaces with duality and the ∞-category of $\mathcal{S}[\mathsf{C}_2]$-enriched functors. We will use this equivalence for the construction of the duality structures on the Waldhausen $S_\bullet$-construction.
The ∞-category $\mathsf{Exact}_\infty^{\mathsf{C}_2}$ naturally admits the structure of a $\mathcal{S}[\mathsf{C}_2]$-enriched ∞-category $\underline{\mathsf{Exact}}_\infty^{\mathsf{C}_2}$ [HSV19, Sect.7.1] that is cotensored over $\underline{\mathsf{Cat}}_\infty^{\mathsf{C}_2}$ [HSV19, Sect.8.2]. This cotensoring induces an $\mathcal{S}[\mathsf{C}_2]$-enriched functor:

$$(-)^- : \underline{\mathsf{Exact}}_\infty^{\mathsf{C}_2} \times \underline{\mathsf{Cat}}_\infty^{\mathsf{C}_2, \mathrm{op}} \to \underline{\mathsf{Exact}}_\infty^{\mathsf{C}_2}.$$

For an ∞-category with duality $\mathcal{I}$ and an exact ∞-category with duality $\mathcal{C}$ the underlying exact ∞-category of the cotensor $\mathcal{C}^I$ is given by the functor category $\mathsf{Fun}(I, \mathcal{C})$. The exact structure is defined objectwise, and the induced duality structure maps a functor $F : I \to \mathcal{C}$ to the composite functor

$$I \xrightarrow{\mathsf{D}_I^{\mathrm{op}}} I^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathsf{D}_\mathcal{C}} \mathcal{C}.$$

As the first step of our construction of the $S_\bullet$-construction, we define an $\mathcal{S}[\mathsf{C}_2]$-enriched version of the ∞-category of arrows. Analogously to the non-enriched case, we define the $\mathcal{S}[\mathsf{C}_2]$-enriched functor

$$\underline{\mathsf{Ar}}(-) : \underline{\Delta} \subset \underline{\mathsf{Cat}}_\infty^{\mathsf{C}_2} \xrightarrow{\underline{\mathrm{hom}}([1],-)} \underline{\mathsf{Cat}}_\infty^{\mathsf{C}_2}$$

where $\underline{\mathrm{hom}}(-,-)$ denotes the $\mathcal{S}[\mathsf{C}_2]$-enriched internal Hom-functor of $\mathsf{Cat}_\infty^{\mathsf{C}_2}$. For every $[n]$ this induces a

[22] For a more general discussion of enriched ∞-category theory see [GH15]

duality structure on the ∞-category of $\mathsf{Ar}([n])$. Note that for every $[n] \in \underline{\Delta}$ the induced duality structure on $\underline{\mathcal{C}^{\mathsf{Ar}([n])}}$ restricts to a duality structure on $S_n(\mathcal{C})$. It follows that the $\mathcal{S}[\mathsf{C}_2]$-enriched functor

$$(-)^{-} : \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2} \times \underline{\Delta}^{op} \to \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2}$$

restricts to an $\mathcal{S}[\mathsf{C}_2]$-enriched functor

$$\underline{S}_{\bullet}(-) : \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2} \times \underline{\Delta}^{op} \to \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2}.$$

Such a functor is transpose to a $\mathcal{S}[\mathsf{C}_2]$-enriched functor

$$\underline{S}_{\bullet}^{\simeq}(-) : \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2} \to \mathsf{Fun}_{\mathcal{S}[\mathsf{C}_2]}(\underline{\Delta}^{\mathrm{op}}, \underline{\mathsf{Exact}}_{\infty}^{\mathsf{C}_2}) \xrightarrow{(-)^{\simeq}} \mathsf{Fun}_{\mathcal{S}[\mathsf{C}_2]}(\underline{\Delta}^{\mathrm{op}}, \underline{\mathcal{S}}[\mathsf{C}_2]) \simeq \underline{\mathcal{P}}(\Delta)^{\mathsf{C}_2}.$$

We can finally define:

Definition 7.10. Let $\mathcal{C}$ be an exact ∞-category with duality. The 2-Segal space with duality $\underline{S}_{\bullet}^{\simeq}(\mathcal{C})$ is called the *Waldhausen $S_{\bullet}$-construction with duality.*

The underlying simplicial space of $\underline{S}_{\bullet}$ coincides with the Waldhausen $S_{\bullet}$-construction of an exact ∞-category as introduced in Definition 7.1. In particular, the Waldhausen $S_{\bullet}$-construction with duality is an example of a 2-Segal space with duality.
We can now apply Proposition 7.9 to the Waldhausen $S_{\bullet}$-construction with duality.

Definition 7.11. [HS04, Sect.1.8] Let $\mathcal{C}$ be an exact ∞-category with duality. We call the simplicial object $\mathsf{Tw}(\underline{S}_{\bullet}^{\simeq}(\mathcal{C}))^{\mathsf{C}_2}$ the *hermitian $S_{\bullet}$-construction* and denote it by $\mathcal{R}_{\bullet}(\mathcal{C})$.

Remark 7.3. It is known by the fundamental work of Waldhausen [Wal06] that for every exact ∞-category $\mathsf{Tw}(S_{\bullet}(\mathcal{C}))$ is equivalent to Quillens Q-construction $Q(\mathcal{C})$. The authors constructed a hermitian version of the Q-construction [CDH$^+$20]. We expect that for every exact ∞-category with duality $\mathcal{C}$, the equivalence between the $\mathsf{Tw}(S_{\bullet}(\mathcal{C}))$ and the Q-construction $Q(\mathcal{C})$ extends to an equivalence of simplicial spaces with C_2-action.

The following result extends [You18, Thm.3.6] into the realm of ∞-categories:

Corollary 7.10. *Let $\mathcal{C}$ be an exact ∞-category with duality. The morphism*

$$\mathcal{R}_{\bullet}(\mathcal{C}) \to S_{\bullet}(\mathcal{C})$$

is a relative 2-Segal object.

Proof. Apply Proposition 7.9 to $\underline{S}_{\bullet}^{\simeq}(\mathcal{C})$. □

We can further use Proposition 7.9 for the construction of relative 2-Segal spans. Let $F : \mathcal{C} \to \mathcal{D} \in \mathsf{Exact}_{\infty}^{\mathsf{C}_2}$ be a duality preserving exact functor. It follows from the construction of $\underline{S}_{\bullet}(-)$ that it induces a morphism

$$\underline{S}_{\bullet}(F) : \underline{S}_{\bullet}(\mathcal{C}) \to \underline{S}_{\bullet}(\mathcal{D})$$

of simplicial spaces with duality.

Corollary 7.11. *Let $F : \mathcal{C} \to \mathcal{D} \in \mathsf{Exact}_\infty^{\mathsf{C}_2}$ be a fully faithful duality preserving exact functor and consider the induced diagram*

$$\begin{array}{ccc} \mathcal{R}_\bullet(\mathcal{C}) & \xrightarrow{\mathcal{R}_\bullet(F)} & \mathcal{R}_\bullet(\mathcal{D}) \\ \downarrow & & \downarrow \\ S_\bullet(\mathcal{C}) & \xrightarrow{S_\bullet(F)} & S_\bullet(\mathcal{D}) \end{array}$$

Then the morphism $(\mathcal{R}_\bullet(F), S_\bullet(F))$ is

(1) active equifibered if F is closed under extensions and subobjects.

(2) relative Segal if the essential image of F is closed under extensions.

Remark 7.4. The author is not aware of a hermitian analog of the relative $S_\bullet$-construction constructed in Subsection 7.1. An analog of the construction given in Subsection 7.1 for exact ∞-categories does not exist for exact ∞-categories with duality. The problem is that the defining pullback diagram does not lift to a diagram of simplicial spaces with duality.

Remark 7.5. The relative 2-Segal objects described in this section describe representations of Hall algebras, which have previously studied in the 1-categorical context in [You18, You20]. As in the case of Remark 7.1, the active equifibered and relative Segal morphisms constructed above, correspond on the level of representations of Hall algebras, to the inclusion of subrepresentations.

8 Modules in Span Categories

Let $\mathcal{C}$ be an ∞-category with finite limits. The Cartesian product on $\mathcal{C}$ induces a symmetric monoidal structure on the ∞-category $\mathsf{Span}(\mathcal{C})$. The goal of this section is to provide a characterization of bimodule objects in the monoidal ∞-category $\mathsf{Span}(\mathcal{C})^\otimes$ in terms of birelative 2-Segal objects in $\mathcal{C}$ (see Definition 6.4). The proof we present here is a multicolored version of the proof provided in [Ste21, Sect.2], with many ideas drawn from there. For this, we use an explicit combinatorial model for the monoidal ∞-category $\mathsf{Span}(\mathcal{C})^\otimes$, denoted $\mathsf{Span}_\Delta(\mathcal{C}^\times)$, constructed in [DK19] using quasi-categories. We recall this construction in Section 8.1. After this preliminary Section, we prove the claimed equivalence in Section 8.2.
Before we start, we sketch the strategy of the proof here. Since the ∞-category of spans is equivalent to its opposite, studying bicomodules instead of bimodules suffices. Let the functor

$$F : \Delta_{/[1]} \to \mathsf{Span}_\Delta(\mathcal{C}^\times)$$

over Δ represent a bicomodule object. Unraveling the definitions, we identify F in Corollary 8.3 with a functor

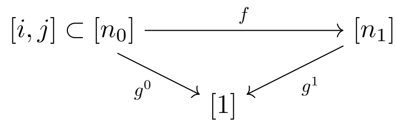
$$F : \mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg} \to \mathcal{C}$$

satisfying a list of conditions. In particular, F inverts a class of morphisms that we will denote by E. To deal with these conditions, we construct a localization functor

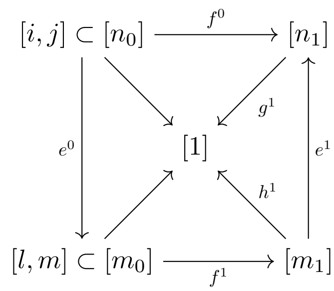
$$\mathcal{L} : \mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg} \to \Delta_1^*$$

along the set of morphisms E. The category Δ_1^* contains $\Delta_{/[1]}^{\mathrm{op}}$ as a full subcategory and the remaining conditions translate under the restriction $\mathsf{Fun}(\Delta_1^*, \mathcal{C}) \to \mathsf{Fun}(\Delta_{/[1]}^{\mathrm{op}}, \mathcal{C})$ to the birelative 2-Segal conditions (see Definition 6.4).

More explicitly we can describe this identification as follows. The category $\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg}$ has objects $(f, \{i, j\})$ represented by diagrams of the form:



in Δ. Similarly a 1-morphism $(e^0, e^1) : (f_0, [i, j]) \to (f_1, [l, m])$ in $\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg}$ can be represented by a commutative diagram in Δ of the form:



s.t. $e_0(i) \leq l \leq m \leq e_1(j)$. We denote by $X_\bullet : \Delta^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ the birelative simplicial object associated to F under the equivalence of Theorem 8.2. In terms of $X_\bullet$ the value of F on $(f, [i, j])$ admits an interpretation as:

$$X_{g^1_{[(f(i),f(i+1)]}} \times \cdots \times X_{g^1_{[(f(j-1),f(j)]}}.$$

Similarly, in terms of $X_\bullet$ the value of F on the morphism (e^0, e^1) translates to the morphism:

$$X_{g^1_{[(f^0(i),f^0(i+1)]}} \times \cdots \times X_{g^1_{[(f^0(j-1),f^0(j)]}} \to X_{h^1_{[(f^1(l),f^1(l+1)]}} \times \cdots \times X_{h^1_{[(f^1(m-1),f^1(m)]}},$$

whose composition with the projection onto $X_{h^1_{[(f^1(l),f^1(l+1)]}}$ is given by:

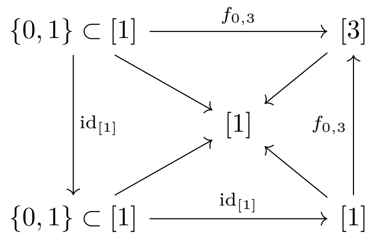
$$X_{g^1_{[(f^0(i),f^0(i+1)]}} \times \cdots \times X_{g^1_{[(f^0(j-1),f^0(j)]}} \to X_{g^1_{[(f^0(i),f^0(i+1)]}} \xrightarrow{X_{e^1_{[f^1(l),f^1(l+1)]}}} X_{h^1_{[(f^1(l),f^1(l+1)]}}.$$

We unravel this in a specific example.

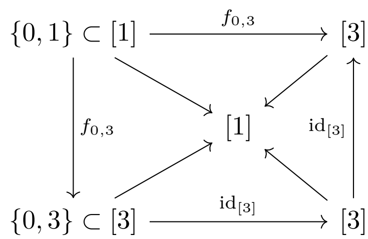
Example 8.1. Consider the objects $g : [3] \xrightarrow{0011} [1]$ and $\mathrm{id}_{[1]} : [1] \to [1]$ in $\Delta_{/[1]}$ and denote by $X_\bullet$ the birelative simplicial object associated to the functor F. We denote by $f_{i,j} : [1] \to [3]$ the unique morphism with image $\{i, j\}$ for $i \leq j$. The morphism $f_{0,3}$ extends to an object $(f_{0,3} : \mathrm{id}_{[1]} \to g, \{0, 1\})$ in $\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg}$. Under the identification of the theorem the value of F on $(f_{0,3}, \{0, 1\})$ identifies with

$$X_{g_{[f(0),f(1)]}} = X_g =: X_{(0,0,1,1)}.$$

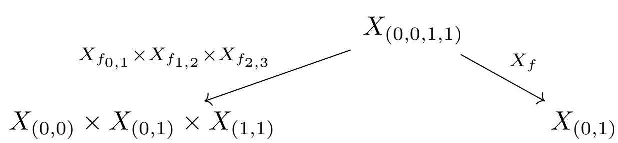
Next, consider the morphisms



and



in $\mathsf{Tw}(\Delta_{/[1]}) \times_{\Delta} \Delta^{\amalg}$. Under the identification with $X_\bullet$, we can interpret the two morphisms $F(f_{0,3}, \{0,1\}) \to F(\mathrm{id}_{[3]}, \{0,3\})$ and $F(f_{0,3}, \{0,1\}) \to F(\mathrm{id}_{[1]}, \{0,1\})$ in $\mathcal{C}$ as the span



This span describes the simultaneous action of the algebra objects $X_{(0,0)}$ and $X_{(1,1)}$ on the bimodule $X_{(0,1)}$.

8.1 Preliminaries

In this section, we introduce some notation and definitions essential for the combinatorics needed in the next section. Particularly, we recall an explicit construction of a monoidal structure on the ∞-category of spans in the model of quasi-categories. We will use this model in the following sections. First, we introduce some notation:

Notation 8.1. Let $f : [n] \to [1]$ be an object in $\Delta_{/[1]}$. We will say that f is *supported at* $S \subset \{0, 1\}$, if $\mathsf{Im}(f) = S$. We can uniquely represent a morphism f by a sequence $(0, 0, ..., 0, 1, ..., 1)$ with $n + 1$-entries.

Definition 8.1. The *interval category* ∇ is the subcategory of Δ with objects given by those $[n]$ with $n \geq 1$, and morphisms are maps that preserve maximal and minimal elements, also called active maps. The *augmented interval category* ∇^+ is the wide subcategory of Δ with morphisms being the active maps.

Definition 8.2. Let S be a linearly ordered set. An *inner interstice* of S is a pair $(n, n+1) \in S \times S$, where $n + 1$ denotes the successor of n in S. We denote the set of inner interstices of S by $\mathbb{I}(S)$. This set comes equipped with a canonical linear order induced from S. This construction can be enhanced to a functor

$$\mathbb{I} : \nabla_+^{\mathrm{op}} \to \Delta_+$$

that associates to a morphism $f : S \to T \in \nabla_+$ the morphism $\mathbb{I}(f) : \mathbb{I}(T) \to \mathbb{I}(S)$

$$\mathbb{I}(f)(j, j+1) = (k, k+1) \quad ; f(k) \leq j < j+1 \leq f(k+1).$$

Definition 8.3. We define $\Delta^{\rm II}$ to be the category with objects $([n], i \leq j)$ consisting of a linearly ordered set $[n]$ and a pair of objects $i \leq j \in [n]$. A morphism $f : ([n], i \leq j) \to ([m], k \leq l)$ is given by a morphism f in Δ, s.t. $f(i) \leq k \leq l \leq f(j)$. We think of an object $([n], i \leq j)$ as a subinterval $\{i, ..., j\} \subset [n]$.
We will frequently denote an object $([n], i \leq j)$ in $\Delta^{\rm II}$ by $([n], [i, j])$ It is easy to check that the forgetful functor

$$\pi_{\rm II} : \Delta^{\rm II} \to \Delta$$

is a Cocoartesian fibration. For every morphism $f : [n] \to [m]$ a $\pi_{\rm II}$-cocartesian lift is given by the morphism $f : ([n], i \leq j) \to ([m], f(i) \leq f(j))$.

Construction 8.1. Given a morphism $f : [m] \to [n]$ over $[1]$ we can uniquely decompose it as follows. We can decompose the source into a composite of $[m] := \{0, 1\} * \{1, 2\} * ... * \{m-1, m\}$. Restricting f to each individual interval yields a morphism

$$f_i := f|_{\{i-1,i\}} : \{i-1, i\} := [1_i] \to \{f(i-1) \leq f(i)\} := [n_i]$$

in $\Delta_{/[1]}$ that preserves endpoints. We further denote $\{0 \leq f(0)\} := [n_l]$ and $\{f(m) \leq n\} := [n_r]$. Using this decomposition, we can uniquely reconstruct f as a morphism in $\Delta_{/[1]}$:

$$f = f_1 * ... * f_m : [1_1] * ... * [1_m] \to [n_1] * ... * [n_m] \hookrightarrow [n_l] * [n_1] * ... * [n_m] * [n_r].$$

We call this process *decomposition.*

Next, we recall the construction of the Cartesian monoidal structure and the ∞-category of spans in the quasi-categorical model as presented in [DK19, Chapt.10]. In the following, we implicitly identify every 1-category with its nerve.

Construction 8.2. [Ste21, Constr.1.29] Let $\mathcal{C}$ be an ∞-category with finite products. We define a simplicial set over Δ via the adjunction formula

$$\mathsf{Map}_\Delta(K, \bar{\mathcal{C}}^\times) \simeq \mathsf{Map}_{\mathsf{Set}_\Delta}(K \times_\Delta \Delta^{\rm II}, \mathcal{C}).$$

This defines a Cartesian fibration $p : \bar{\mathcal{C}}^\times \to \Delta$. We define $\mathcal{C}^\times \subset \bar{\mathcal{C}}^\times$ to be the full subcategory on those objects $F : \Delta^0 \times_\Delta \Delta^{\rm II} \to \mathcal{C}$, that display $F(\{i \leq j\})$ as a product of $F(\{k \leq k+1\})$ for $i \leq k < j$. The restricted functor $p : \mathcal{C}^\times \to \Delta$ is also a Cartesian fibration and exhibits the Cartesian monoidal structure on $\mathcal{C}$.
A morphism $\tilde{\Phi} : \Delta^1 \to \mathcal{C}^\times$ represented by a map $\Phi : \{[n] \to [m]\} \times_\Delta \Delta^{\rm II} \to \mathcal{C}$ is Cartesian if and only if Φ carries all $\pi_{\rm II}$-cocartesian edges of $\{[n] \xrightarrow{f} [m]\} \times_\Delta \Delta^{\rm II} \to \Delta$ to equivalences [Lur09a, Cor.3.2.2.12]. This happens if and only if Φ maps all maps of the form $f : ([i, j] \subset [n]) \to ([f(i), f(j)] \subset [m])$ induced by a weakly monotone map $f : [n] \to [m]$ to equivalences.

Construction 8.3. [Ste21, Constr.1.33] Given $X_\bullet \in \mathsf{Set}_\Delta$. There exists an adjunction:

$$\mathsf{Tw}_X : (\mathsf{Set}_\Delta)_{/X} \longleftrightarrow (\mathsf{Set}_\Delta)_{/X} : \overline{\mathsf{Span}}_X$$

by setting $\mathsf{Tw}_X(f : S \to X) = \mathsf{Tw}(S) \to \mathsf{Tw}(X) \to X$ and $\overline{\mathsf{Span}}_X(S \to X)$ to be the left vertical arrow of the pullback:

$$\begin{array}{ccc} \overline{\mathsf{Span}}_X(S) & \longrightarrow & \overline{\mathsf{Span}}(S) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \overline{\mathsf{Span}}(X) \end{array}$$

Let $p : S \to X$ be a map of simplicial sets. An n-simplex in $\overline{\mathsf{Span}}_X(S)$ represented by a map $\phi : \mathsf{Tw}(\Delta^n) \to S$ is called a *Segal simplex* if, for every $\Delta^k \subset \Delta^n$ the composite diagram:

$$\{0, k\} * \mathsf{Tw}(\mathsf{Sp}(\Delta^k)) \subset \mathsf{Tw}(\Delta^k) \subset \mathsf{Tw}(\Delta^n) \xrightarrow{\phi} S$$

is a p-limit diagram. Here, $\mathsf{Sp}(\Delta^k)$ denotes the spine of Δ^k and we call the join $\{0, k\} * \mathsf{Tw}(\mathsf{Sp}(\Delta^k))$ the *Segal cone*. We denote by $\mathsf{Span}_X(S) \subset \overline{\mathsf{Span}}_X(S)$ the simplicial subset consisting of Segal simplices. In case $X \simeq *$ we adopt the notation $\mathsf{Span}(\mathcal{C}) \coloneqq \mathsf{Span}_*(\mathcal{C})$.

It follows from [DK19, Thm.10.2.6] that for every ∞-category $\mathcal{C}$ the simplicial set $\mathsf{Span}(\mathcal{C})$ is itself an ∞-category.

Proposition 8.1. *[DK19, Prop.10.2.31] Let $\mathcal{C}$ be an ∞-category with finite limits. Then the functor:*

$$\mathsf{Span}_\Delta(\mathcal{C}^\times) \to \Delta$$

is a monoidal ∞-category.

Note that the construction of the ∞-category $\mathsf{Span}(\mathcal{C})$ is functorial. Indeed, every finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ induces a functor $\mathsf{Span}(F) : \mathsf{Span}(\mathcal{C}) \to \mathsf{Span}(\mathcal{D})$ between ∞-categories of spans. In particular, after passing to homotopy coherent nerves the construction $\mathsf{Span}(-)$ induces an ∞-functor

$$\mathsf{Span}(-) : \mathsf{Cat}_\infty^{\mathsf{lex}} \to \mathsf{Cat}$$

where $\mathsf{Cat}_\infty^{\mathsf{lex}}$ denotes the ∞-category of ∞-categories with finite limits and finite limit preserving functors.

8.2 Characterization of Bimodules

Let $\mathcal{C}$ be an ∞-category with finite limits and denote by $\mathcal{C}^\times$ the associated Cartesian monoidal structure as constructed in Construction 8.2. Our main goal in this section is to prove the following:

Theorem 8.2. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence of spaces:*

$$\mathsf{BMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))^\simeq \simeq \mathsf{BiSeg}_\Delta(\mathcal{C})^\simeq.$$

For better readability of the proof, we have included some technical lemmas in the Appendix A. As a start, let us unravel the datum of a cobimodule in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$. Such an object is given by a commutative triangle:

$$\begin{array}{ccc} \Delta_{/[1]} & \xrightarrow{F'} & \mathsf{Span}_\Delta(\mathcal{C}^\times) \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

s.t F preserves inert morphisms and the adjoint morphism $\tilde{F} : \mathsf{Tw}(\Delta_{/[1]}) \to \mathcal{C}^\times$ maps every n-simplex $\Delta^n \to \Delta_{/[1]}$ to a Segal simplex. Unraveling the definitions, F' corresponds by adjunction to a morphism

$$F : \Theta_1 \coloneqq \mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}} \to \mathcal{C}.$$

We have included a precise discussion of these conditions in the Appendix A.1 and state here only the main result.

Proposition 8.3. *A functor $F : \Theta_1 \to \mathcal{C}$ defines a bicomodule object, if and only if*

(1) F sends degenerate[23] intervals to terminal objects.

(2) F sends every object $(\phi : g^0_{[n_0]} \to g^1_{[n_1]}, [i,j])$ together with its projection to subintervals to a product diagram[24].

(3) F sends every morphism

$$\sigma \simeq \left\{ \begin{array}{ccc} [i,j] \subset g^0_{n_0} & \xrightarrow{g} & g^1_{n_1} \\ \big\downarrow f & & \big\uparrow \tilde{f} \\ [\tilde{i},\tilde{j}] \subset g^0_{m_0} & \xrightarrow{\tilde{g}} & g^1_{m_1} \end{array} \right\}$$

s.t f restricts to an isomorphism $\{i, ..., j\} \to \{\tilde{i}, ..., \tilde{j}\}$ and $\tilde{f}$ to an isomorphism $\{g(i), ..., g(j)\} \to \{\tilde{g}(\tilde{i}), ..., \tilde{g}(\tilde{j})\}$ to an equivalence. We denote by E the set of morphisms of the above form.

(4) F maps all Segal cone diagrams from Definition A.1 to limit diagrams.

Definition 8.4. We define several full subcategories of $\mathsf{Fun}(\Theta_1, \mathcal{C})$. We denote by

- $\mathsf{BMod}_{Sp}(\mathcal{C})$ the full subcategory of $\mathsf{Fun}(\Theta_1, \mathcal{C})$ generated by those functors that satisfy the conditions of Proposition 8.3.
- $\mathsf{Fun}^*(\Theta_1, \mathcal{C})$ the full subcategory generated by those functors that map degenerate intervals to terminal objects.
- $\mathsf{BMod}_{\pi}(\mathcal{C})$ the full subcategory of $\mathsf{Fun}(\Theta_1, \mathcal{C})$ generated functors that satisfy conditions (1) and (2) of Proposition 8.3.

Furthermore, we denote by Ω_1 the full subcategory of Θ_1 with objects those morphisms $([i,j] \subset g^0_{n_0} \to g^1_{n_1})$, s.t. the interval $[i,j]$ is non-degenerate.

We first analyse condition (1):

Proposition 8.4. *The restriction functor induces an equivalence of ∞-categories*

$$\mathsf{Fun}^*(\Theta_1, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Fun}(\Omega_1, \mathcal{C})$$

Proof. The proof is the same as [Ste21, Cor.2.9]. □

To handle condition (3) of Proposition 8.3, we explicitly construct a localization of Ω_1 along the class of morphisms E.

Definition 8.5. The category $\Delta^{\star}_1$ has objects consisting of pairs of a finite ordered set $[k]$ and a $[k]$-indexed sequence of composable[25] objects $(f_{n_0}, ..., f_{n_k})$ in $\Delta_{/[1]}$. By definition, these define a morphism

$$f := f_{n_0} * ... * f_{n_k} : [n_0] * [n_1] * ... * [n_k] \to [1].$$

A morphism $(g, \theta) : ((f^1_{n_0}, ...f^1_{n_k}), [k]) \to ((f^2_{m_0}, ..., f^2_{m_l}), [l])$ consists of a

[23] An interval $[i,j] \subset [n]$ is called degenerate if $i = j$.
[24] See Construction 8.2
[25] See Definition 8.1

(1) a morphism $\theta : [l] \to [k]$.

(2) a commutative diagram

$$\begin{array}{ccc} [m_0] * ... * [m_l] & \xrightarrow{g} & [n_0] * ... * [n_k] \\ & {\scriptstyle f^2} \searrow \quad \swarrow {\scriptstyle f^1} & \\ & [1] & \end{array}$$

s.t. for any $i \in [k]$ with $\theta^{-1}(i) = \{j_1, ..., j_p\}$, the restriction

$$g_i : [m_{j_1}] * ... * [m_{j_p}] \to [n_0] * ... * [n_k]$$

has image contained in $[n_i]$.

Construction 8.4. We define a functor $\mathcal{L} : \Omega_1 \to \Delta_1^*$ as follows:

- It maps an object $[i, j] \subset g^0_{n_0} \xrightarrow{f} g^1_{n_1}$ to the object $(g^1_{[f(i),f(i+1)]}, ..., g^1_{[f(j-1),f(j)]})$. The image is indexed by the inner interstices[26] $\mathbb{I}([i, j])$ of the linearly ordered set $\{i, i + 1, ..., j\}$.
- A morphism in Ω_1 is given by

$$\sigma \simeq \left\{ \begin{array}{ccc} [i,j] \subset g^0_{n_0} & \xrightarrow{g} & g^1_{n_1} \\ \Big\downarrow f & & \Big\uparrow \tilde{f} \\ [\tilde{i}, \tilde{j}] \subset g^0_{m_0} & \xrightarrow{\tilde{g}} & g^1_{m_1} \end{array} \right\}.$$

The functor $\mathcal{L}$ maps σ to the map $\mathcal{L}(\sigma) = (h_\sigma, \mathbb{I}(f))$, whose second component is given by

$$\mathbb{I}(f) \; : \; \mathbb{I}([\tilde{i}, \tilde{j}]) \to \mathbb{I}([i, j]).$$

We define h_σ componentwise. Consider an inner interstice $\{p, p + 1\} \in \mathbb{I}([i, j])$. The restriction of h_σ to the component indexed by $\mathbb{I}(f)^{-1}(\{p, p + 1\})$ is given by:

$$\begin{array}{ccc} [\tilde{g}(f(p)), \tilde{g}(f(p) + 1)] * ... * [\tilde{g}(f(p + 1) - 1), \tilde{g}(f(p + 1))] & \xrightarrow{\tilde{f}} & [g(p), g(p + 1)] \\ & {\scriptstyle g^1_{[\tilde{g}(f(p)),\tilde{g}(f(p+1))]}} \searrow \quad \swarrow {\scriptstyle g^1_{[g(p),g(p+1)]}} & \\ & [1] & \end{array}$$

It follows from the commutativity of σ that this yields a morphism in Δ_1^*.

The technical proof of the following proposition is included in Appendix A.2:

Proposition 8.5. *The functor $\mathcal{L} : \Omega_1 \to \Delta_1^\star$ is an ∞-categorical localization at the morphisms E defined in Proposition 8.3 (3).*

Definition 8.6. Let $\mathcal{C}$ be an ∞-category with finite limits. We define the category $\mathsf{Fun}^{\mathrm{bim}}(\Delta_1^\star, \mathcal{C}) \subset \mathsf{Fun}(\Delta_1^\star, \mathcal{C})$ as the full subcategory with objects those functors, that

[26] See Definition 8.2

(I) send diagrams of the form

$$\begin{array}{ccccc} & & (g_0, ..., g_k) & & \\ \swarrow & \swarrow & & \searrow & \searrow \\ g_0 & g_1 & \dots & g_{k-1} & g_k \end{array}$$

to equivalences.

(II) send diagrams of the form

$$\begin{array}{ccc} & (*_{j \in [n_1]} g_j^2, ..., *_{j \in [n_l]} g_j^2) & \\ \swarrow & & \searrow \\ (g^1_{n_1}, ..., g^1_{n_l}) & & (g_1^2, ..., g_k^2) \\ \searrow & & \swarrow \\ & (g_1^1|_{\{0,1\}}, ..., g_1^l|_{\{n_l-1, n_l\}}) & \end{array}$$

to equivalences.

Proposition 8.6. *Restriction along the functor $\mathcal{L} : \Omega_1 \to \Delta_1^*$ induces an equivalence of ∞-categories*

$$\mathsf{BMod}_{Sp}(\mathcal{C}) \simeq \mathsf{Fun}^{\mathrm{bim}}(\Delta_1^\star, \mathcal{C}).$$

Proof. It follows from Proposition 8.5 that restriction along $\mathcal{L}$ induces an equivalence

$$\mathsf{Fun}_{E^{-1}}(\Omega_1, \mathcal{C}) \simeq \mathsf{Fun}(\Delta_1^\star, \mathcal{C}),$$

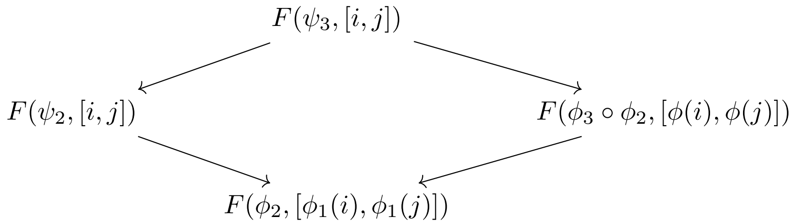
where we denote by $\mathsf{Fun}_{E^{-1}}(\Omega_1, \mathcal{C})$ the full subcategory of $\mathsf{Fun}(\Omega_1, \mathcal{C})$ on those functors, that map all morphisms in E to equivalences. To conclude, we only have to match the remaining conditions of Proposition 8.3. By construction, condition (2) on the left matches with condition (I) on the right. For condition (4) let σ be a 3-simplex

$$g^0_{n_0} \xrightarrow{\phi_1} g^1_{n_1} \xrightarrow{\phi_2} g^2_{n_2} \xrightarrow{\phi_3} g^3_{n_3}$$

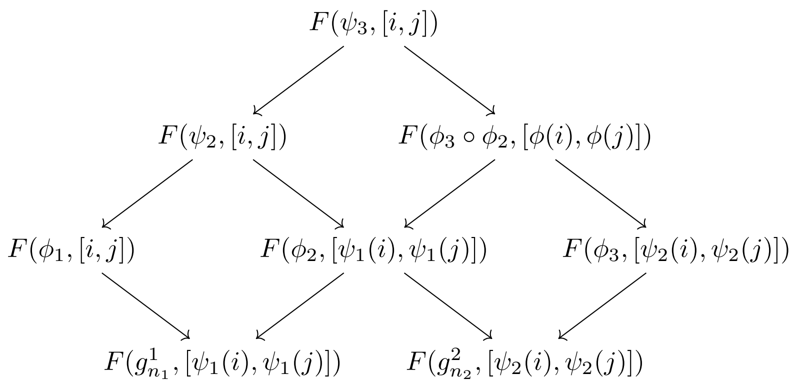
in Ω_1 and F an object of $\mathsf{BMod}_{Sp}(\mathcal{C})$. Further let $[i, j] \subset [n_0]$ be a subinterval. The corresponding limit diagram for condition (4) reads as

$$\begin{array}{ccccc} & & F(\psi_3, [i,j]) & & \\ & \swarrow & \downarrow & \searrow & \\ F(\phi_1, [i,j]) & & F(\phi_2, [\psi_1(i), \psi_1(j)]) & & F(\phi_3, [\psi_2(i), \psi_2(j)]) \\ & \searrow \ \swarrow & & \searrow \ \swarrow & \\ & F(g^1_{n_1}, [\psi_1(i), \psi_1(j)]) & & F(g^2_{n_2}, [\psi_2(i), \psi_2(j)]) & \end{array}$$

It follows from the dual of [Lur09a, Prop.4.4.2.2] that this diagram is a limit diagram, if and only if the diagram



is a pullback diagram. Combining this with the previous diagram, we obtain the following diagram:



It follows from the pasting property for pullbacks, that the diagram corresponding to any sub 2-simplex is a limit diagram. Iterating this argument, we see that condition (4) is satisfied if and only if it is satisfied on 2-simplices. The claim follows from the observation that condition (II) is precisely the image of condition (4) on 2-simplices under $\mathcal{L}$. $\square$

Using the above proposition, the problem has shifted to analyzing the conditions of Definition 8.6. To this end, consider the full subcategory $i : \Delta^{\mathrm{op}}_{/[1]} \to \Delta^\star_1$ generated by objects $f : [n] \to [1] \in \Delta_{/[1]}$ with $n \geq 0$. Restriction and right Kan extension induce an adjunction

$$i^* : \mathsf{Fun}(\Delta^\star_1, \mathcal{C}) \rightleftarrows \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}) : i_*.$$

We denote by $\mathsf{Fun}^\times(\Delta^\star_1, \mathcal{C}) \subset \mathsf{Fun}(\Delta^\star_1, \mathcal{C})$ the full subcategory on those functors that satisfy condition (I) of Definition 8.6.

Proposition 8.7. *The functor* $i_* : \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}) \to \mathsf{Fun}(\Delta^\star_1, \mathcal{C})$ *factors through the full subcategory* $\mathsf{Fun}^\times(\Delta^\star_1, \mathcal{C})$. *Moreover, the induced functor*

$$i_* : \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}) \to \mathsf{Fun}(\Delta^\star_1, \mathcal{C})$$

is an equivalence of ∞*-categories.*

Proof. Let $F \in \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})$ be a functor and let $(f_{m_1}, f_{m_2}, ..., f_{m_k})$ be an object of $\Delta^\star_1$. The value of i_*F on $(f_{m_1}, f_{m_2}, ..., f_{m_k})$ can be computed as the limit of F over the diagram indexed by the category

$$\mathcal{D} := (\Delta^{\mathrm{op}}_{/[1]})_{/(f_{m_1}, f_{m_2}, ..., f_{m_k})}.$$

An object of the category $\mathcal{D}$ consists of an element $i \in \{1, ..., k\}$ together with a morphism $h_l \to f_{m_i}$ in $\Delta_{/[1]}$. Note that a morphism from $(h_l \to f_{m_i})$ to $(e_p \to f_{m_j})$ in $\mathcal{D}$ only exists if $i = j$. In this case, it is given by a commutative diagram

$$\begin{array}{ccc} e_p & \longrightarrow & h_l \\ & \searrow \quad \swarrow & \\ & f_{m_i} & \end{array}$$

in $\Delta_{/[1]}$. It follows, that the value of $i_*F(f_{m_1}, ..., f_{m_k})$ is given by the limit

$$\begin{array}{ccccc} & & i_*F(f_{m_1}, ..., f_{m_k}) & & \\ \swarrow & \swarrow & & \searrow & \searrow \\ F(f_{m_1}) & F(f_{m_2}) & ... & F(f_{m_{k-1}}) & F(f_{m_k}) \end{array}$$

This proves the first claim. The second claim follows from [Lur09a, Prop.4.3.2.15]. $\square$

We need the following auxiliary lemma:

Lemma 8.8. *Let $F \in \mathsf{Fun}(\Delta_1^\star, \mathcal{C})$ be a functor. Then F satisfies condition (II), if and only if it satisfies condition (II) where all but one of the g_2^i have source equal to* $[1]$*. We will call this condition (II').*

Proof. It follows from the assumptions that (II) implies (II'). We assume that F satisfy condition (II'). We consider the diagram in $\Delta_1^\star$ displayed in Figure 4. Since F satisfies (II'), it follows that F maps the bottom

$$\begin{array}{ccccc} (g_1^1, ..., g_l^1) & \longrightarrow & (g_1^1|_{\{0,1\}}, ..., g_l^1|_{\{n_l-1,n_l\}}) & & \\ \uparrow & & \uparrow & & \\ (g_1^2 \star g_1^1|_{\{1,2\}} \star ...g_1^1|_{\{n_1-1,n_1\}}, g_2^1|_{\{0,1\}}, ..., g_l^1|_{\{n_l-1,n_l\}} & \longrightarrow & (g_1^2, g_1^1|_{\{1,2\}}, ..., g_l^1|_{\{n_l-1,n_l\}}) & \longrightarrow & (g_1^2|_{\{0,1\}}, ..., g_1^2|_{\{m_1-1,m_1\}}, g_1^1|_{\{1,2\}}, ..., g_l^1|_{\{n_l-1,n_l\}}) \\ \uparrow & & \uparrow & & \uparrow \\ (g_1^2 \star g_2^2 \star g_1^1|_{\{2,3\}} \star ...g_1^1|_{\{n_1-1,n_1\}}, g_2^1|_{\{0,1\}}, ..., g_1^l|_{\{n_l-1,n_l\}}) & \longrightarrow & (g_1^2, g_2^2, g_1^1|_{\{2,3\}}, ..., g_l^1|_{\{n_l-1,n_l\}}) & \longrightarrow & (g_1^2|_{\{0,1\}}, ..., g_1^2|_{\{m_1-1,m_1\}}, g_2^2, g_1^1|_{\{2,3\}}, ..., g_l^1|_{\{n_l-1,n_l\}}) \\ \uparrow & & \uparrow & & \uparrow \\ ... & & ... & & ... \end{array}$$

Figure 4: Proof of Lemma 8.8

right square and the exterior right rectangle of the diagram to pullback diagrams. By the pasting law, it follows that the functor F also sends the bottom left square to a pullback square. Hence, it follows from (II') that the upper left square is also a pullback square. Another application of the pasting law implies that the vertical left square is a pullback square. Iterating this argument yields the claim. $\square$

After all these intermediate steps, we can finally prove:

Proposition 8.9. *The restriction functor $i^* : \mathsf{Fun}(\Delta_1^*, \mathcal{C}) \to \mathsf{Fun}(\Delta_{/[1]}^{\mathrm{op}}, \mathcal{C})$ descends to an equivalence*

$$\mathsf{Fun}^{\mathrm{bim}}(\Delta_1^\star, \mathcal{C}) \simeq \mathsf{BiSeg}_\Delta(\mathcal{C})$$

where $\mathsf{BiSeg}_\Delta(\mathcal{C}) \subset \mathsf{Fun}(\Delta_{/[1]}^{\mathrm{op}}, \mathcal{C})$ denotes the full subcategory generated by birelative 2-Segal objects.

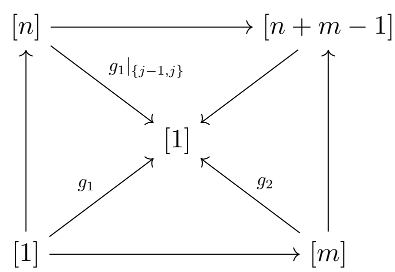
Proof. Let $F \in \mathsf{Fun}^\times(\Delta_1^\star, \mathcal{C})$ be a functor and let

$$\begin{array}{ccc} g_1 & \longleftarrow & *_{0 \le k < j-1} g_1|_{\{k,k+1\}} * g_2 * (*_{j \le k < n} g_1|_{k,k+1}) \\ \downarrow & & \downarrow \\ (g_1|_{\{0,1\}}, ..., g_1|_{\{n-1,n\}}) & \longleftarrow & (g_1|_{\{0,1\}}, ..., g_1|_{\{j-2,j-1\}}, g_2, g_1|_{\{j,j+1\}}, ..., g_1|_{\{n-1,n\}}) \end{array}$$

be a diagram of type (II'). We can expand this diagram

$$\begin{array}{ccc}
g_1 & \longleftarrow & *_{0\leq k<j-1} g_1|_{\{k,k+1\}} * g_2 * (*_{j\leq k<n} g_1|_{k,k+1}) \\
\downarrow & & \downarrow \\
(g_1|_{\{0,1\}}, \ldots, g_1|_{\{n-1,n\}}) & \longleftarrow & (g_1|_{\{0,1\}}, \ldots, g_1|_{\{j-2,j-1\}}, g_2, g_1|_{\{j,j+1\}}, \ldots, g_1|_{\{n-1,n\}}) \\
\downarrow & & \downarrow \\
g_1|_{\{j-1,j\}} & \longleftarrow & g_2
\end{array}$$

Since F satisfies condition (I) of Definition 8.6, it follows that F maps the lower square to a pullback diagram. By the pasting lemma, the upper square is a pullback if and only if the outer rectangle is a pullback. But the latter can be identified with the opposite of the diagram



which precisely recovers the birelative Segal conditions. ☐

As a corollary, we obtain analogous results for left and right module objects in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$:

Corollary 8.10. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exist equivalences of spaces*

$$\mathrm{LMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))^\simeq \simeq \mathsf{LSeg}_\Delta(\mathcal{C})^\simeq,$$

and

$$\mathsf{RMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))^\simeq \simeq \mathsf{RSeg}_\Delta(\mathcal{C})^\simeq.$$

9 Module Morphisms in Span Categories

In the last section, we have constructed for every ∞-category with finite limits $\mathcal{C}$ an equivalence between the space of bimodule objects in the monoidal ∞-category $\mathsf{Span}_\Delta(\mathcal{C}^\times)$ and the space of birelative 2-Segal objects. The goal of this section is to extend this to an equivalence of ∞-categories:

Theorem 9.1. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence of ∞-categories*

$$\mathsf{BiMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times)) \simeq \mathrm{Bi2Seg}_\Delta^{\leftrightarrow}(\mathcal{C})$$

between the ∞-category of bimodule objects in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$ and the ∞-category of birelative 2-Segal objects and birelative 2-Segal spans.

This characterization will play a major role in our discussion of locally rigid algebras in ∞-categories of spans in the following section. As a corollary of the above characterization, we obtain a similar characterization for the ∞-category of left (resp. right) module objects and algebra objects in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$.

Corollary 9.2. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence*

$$\mathrm{LMod}(\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})) \simeq \mathrm{L2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$$

between the ∞-category of left modules in $\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})$ and the ∞-category of left relative 2-Segal objects and left relative 2-Segal spans.

Corollary 9.3. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence*

$$\mathsf{Alg}(\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})) \simeq \mathrm{2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$$

between the ∞-category of algebras in $\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})$ and the ∞-category of 2-Segal objects and 2-Segal spans.

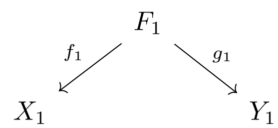
Combining Corollary 9.2 with Corollary 6.4, we obtain

Corollary 9.4. *Let $\mathcal{C}$ be an ∞-category with finite limits. There exists an equivalence*

$$\mathrm{LMod}(\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})) \simeq \mathrm{Rel2Seg}_{\Delta}^{\leftrightarrow}(\mathcal{C})$$

between the ∞-category of left modules in $\mathsf{Span}_{\Delta}(\mathcal{C}^{\times})$ and the ∞-category of relative 2-Segal objects.

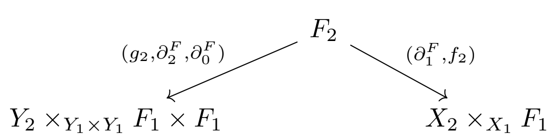
Before we prove these statements, let us explicitly describe the relation between algebra morphisms and Segal spans at the level of lowest dimensional coherence. To this end, consider two algebra objects (X_1, μ^X) and (Y_1, μ^Y) in $\mathsf{Span}(\mathcal{C}^{\times})$ with associated 2-Segal objects $X_\bullet$ (resp. $Y_\bullet$). To construct the data of an algebra morphism $F : (X_1, \mu^X) \to (Y_1, \mu^Y)$ between those, we first have to provide a morphism on underlying objects. This is given by a span:



with tip denoted F_1. For this span to be part of an algebra morphism, we have to provide higher coherence data. At the lowest level this is given by an invertible 2-morphism

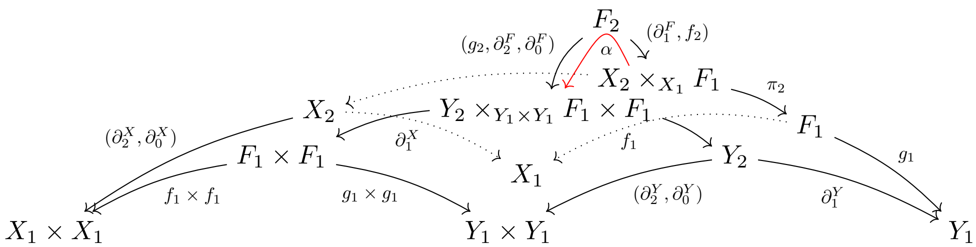
$$\alpha : F_1 \circ \mu^X \simeq \mu^Y \circ (F_1 \times F_1)$$

in the ∞-category $\mathsf{Span}(\mathcal{C})$, i.e there has to exist an object $F_2 \in \mathcal{C}$ together with two invertible 1-morphisms:



(19)

fitting into a commutative diagram



As the notation suggests, these data define the 2-simplices, and the 2-dimensional face maps of the simplicial object $F_\bullet$, and the 2-dimensional part of the maps $f_\bullet : F_\bullet \to X_\bullet$, and $g_\bullet : F_\bullet \to Y_\bullet$. The equivalences in (19) form the lowest dimensional instances of the active equifibered and relative Segal conditions, respectively (see Definition 6.2). Similarly, the higher simplices of the simplicial object $F_\bullet$, as well as the higher active equifibered and relative Segal conditions, imposed on morphisms $f_\bullet$ and $g_\bullet$ are encoded in the higher coherence data.

After these initial considerations, we now turn to the proof of Theorem 9.1. For this, we will need the following Lemma:

Lemma 9.5. *Let $\mathcal{C}$ be an ∞-category with finite limits and consider a morphism:*

$$F : \mathsf{Tw}(\Delta^n) \times (\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}) \to \mathcal{C}$$

such that the associated morphism $\tilde{F} : \mathsf{Tw}(\Delta^n) \to \mathsf{Fun}((\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}), \mathcal{C})$ factors through $\mathsf{BMod}_{Sp}(\mathcal{C})$. Then F defines a morphism:

$$F : \Delta_{/[1]} \times \Delta^n \to \mathsf{Span}_\Delta(\mathcal{C})$$

if and only if for every 2-simplex $\Delta^2 \to \Delta^n \times \Delta_{/[1]}$ of the form

(1) $(f_0 : [n_0] \to [1], i) \xrightarrow{\phi_1} (f_1 : [n_1] \to [1], i) \xrightarrow{\phi_2} (f_2 : [n_2] \to [1], i)$

(2) $(f_0 : [n_0] \to [1], i) \xrightarrow{i<j} (f_0 : [n_0] \to [1], j) \xrightarrow{j<k} (f_0 : [n_0] \to [1], k)$

(3) $(f_0 : [n_0] \to [1], i) \xrightarrow{\phi_1} (f_1 : [n_1] \to [1], i) \xrightarrow{i<j} (f_2 : [n_2] \to [1], j)$

(4) $(f_0 : [n_0] \to [1], i) \xrightarrow{i<j} (f_0 : [n_0] \to [1], j) \xrightarrow{\phi_1} (f_1 : [n_1] \to [1], j)$

the restriction $F|_{\Delta^2}$ is a Segal simplex.

Proof. The only if condition follows from the assumption. For the other direction, we first observe that for every morphism $\phi : f_{n_0} \to f_{n_1}$ in $\Delta_{/[1]}$, every $i < j < k$, and every fixed subinterval $[l, m] \subset [n_0]$ the diagram

$$\begin{array}{ccccc} & & F(\phi, i<k, [l,m]) & & \\ & \swarrow & & \searrow & \\ F(\phi, i<j, [l,m]) & & & & F(\phi, j<k, [l,m]) \\ & \searrow & & \swarrow & \\ & & F(\phi, j, [l,m]) & & \end{array}$$

is a limit diagram. Indeed, since $\tilde{F}$ factors through $\mathsf{Bim}_{Sp}(\mathcal{C})$, the diagram is equivalent to a product of diagrams of the form

$$\begin{array}{ccccc} & & F(f_{n_1}|_{\phi(p),\phi(p+1)}, i<k, [\phi(p), \phi(p+1)]) & & \\ & \swarrow & & \searrow & \\ F(f_{n_1}|_{\phi(p),\phi(p+1)}, i<j, [\phi(p), \phi(p+1)]) & & & & F(f_{n_1}|_{\phi(p),\phi(p+1)}, j<k, [\phi(p), \phi(p+1)]) \\ & \searrow & & \swarrow & \\ & & F(f_{n_1}|_{\phi(p),\phi(p+1)}, j, [\phi(p), \phi(p+1)]) & & \end{array}$$

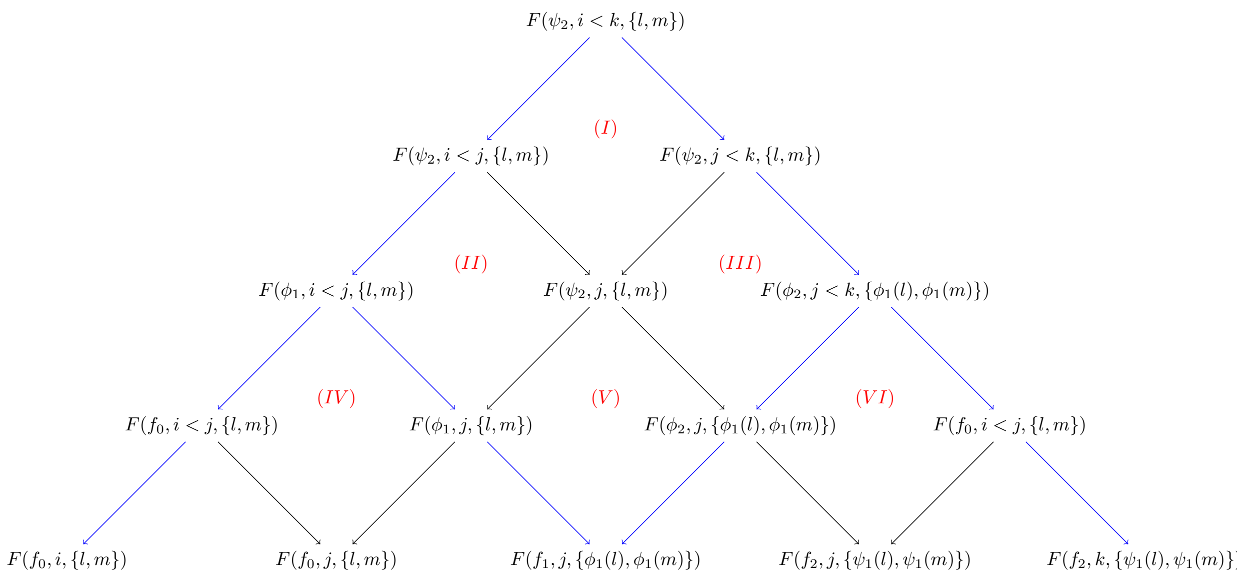
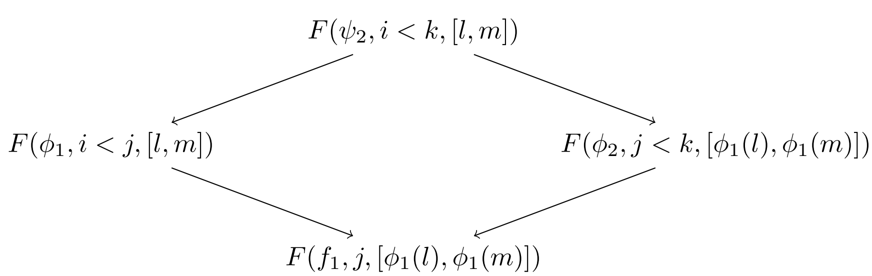


Figure 5: Image under F of the decomposition of the 2-simplex σ into a 4-simplex. The original 2-simplex is colored blue.

These are pullback diagrams due to condition (II). Let $\sigma : \Delta^2 \to \Delta_{/[1]} \times \Delta^n$ be a 2-simplex represented by a composable pair of morphisms:

$$(f_0 : [n_0] \to [1], i) \xrightarrow{\phi_1} (f_1 : [n_1] \to [1], j) \xrightarrow{\phi_2} (f_2 : [n_2] \to [1], k).$$

By Proposition A.1 we need to check that for every interval $[l, m] \subset [n_0]$ the following square is a pullback diagram:



To check this, we include this diagram in a fourfold composite of spans shown in Figure 5. The square is depicted by the blue part in the diagram. We denote the composite rectangles, obtained by pasting two squares, by the sum of their labels. It follows from (4), that the rectangles (IV), $(IV) + (II)$, from (3) that the rectangles $(III) + (VI)$, (VI), and from (1) that the square (V) are all pullback squares. Moreover, as shown above, the square (I) is a pullback diagram. It follows from an iterated application of the pasting lemma that the blue square is a pullback square. □

Proof of Theorem 9.1. We prove the equivalence $\mathsf{BicoMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times)) \simeq \mathrm{Bi2Seg}_\Delta^\leftrightarrow(\mathcal{C})^{\mathrm{op}}$. The claimed result then follows from the equivalence $\mathsf{BicoMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times)) \simeq \mathsf{BiMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))^{\mathrm{op}}$.
Unraveling definitions and using Proposition A.1, an n-simplex $\eta \in \mathsf{BicoMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))_n$ is represented by

a morphism

$$\eta : \mathsf{Tw}(\Delta^n) \times (\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}) \to \mathcal{C}$$

such that

(1) for every $0 \leq k \leq n$ the restriction $\eta_{\{k\}} : \{\mathrm{id}_k\} \times (\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}) \to \mathcal{C}$ preserves inert morphisms, i.e. defines a bicomodule in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$.

(2) for every 2-simplex $\Delta^2 \to \Delta_{/[1]} \times \Delta^n$ of the form of Lemma A.2 depicted

$$(f_0 : [n_0] \to [1], i) \xrightarrow{\phi_1} (f_1 : [n_1] \to [1], j) \xrightarrow{\phi_2} (f_2 : [n_2] \to [1], k)$$

with $0 \leq i \leq j \leq k \leq n$ and every subinterval $[p, q] \subset [n_0]$ the associated Segal cone diagram[27] is a limit diagram:

$$\begin{array}{ccc} & \eta(\psi_2, [p,q], ik) & \\ \swarrow & & \searrow \\ \eta(\phi_2, [\phi_1(p), \phi_1(q)], jk) & & \eta(\phi_1, [p,q], ij) \\ \searrow & & \swarrow \\ & \eta(f_1, [\phi_1(p), \phi_1(q)], j) & \end{array} \tag{20}$$

The datum of an n-simplex in $\mathsf{BicoMod}(\mathsf{Span}_\Delta(\mathcal{C}^\times))$ corresponds via adjunction to a functor

$$\eta \ : \ \mathsf{Tw}(\Delta^n) \ \to \ \mathsf{Fun}(\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}, \mathcal{C}).$$

We claim that η factors through the full subcategory

$$\mathsf{BMod}_{Sp}(\mathcal{C}) \subset \mathsf{Fun}(\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}, \mathcal{C})),$$

as defined in Definition 8.4. For the evaluation of η on objects of the form id_k, this follows from condition (1) and (2). Hence, let $i < j$ be an object of $\mathsf{Tw}(\Delta^n)$. We need to check the conditions of Definition 8.4. All conditions except condition (3) of Definition 8.4 are satisfied by assumption. For this condition to hold it suffices to check that for every inert morphism $\phi : f^0_{n_0} \to f^1_{n_1} \in \Delta_{/[1]}$ and every subinterval $[k, l] \subset [n_0]$ the morphisms

(i) $\eta(\phi, [k, l], i < j) \to \eta(f^1_{n_1}, [\phi(k), \phi(l)], i < j)$

(ii) $\eta(\phi, [k, l], i < j) \to \eta(f^0_{n_0}, [k, l], i < j)$

are equivalences. To this end, consider the following 2-simplices:

(i) $(f^0_{n_0}, i) \xrightarrow{\phi} (f^1_{n_1}, i) \xrightarrow{i<j} (f^1_{n_1}, j)$

(ii) $(f^0_{n_0}, i) \xrightarrow{i<j} (f^0_{n_0}, j) \xrightarrow{\phi} (f^1_{n_1}, j)$

in $\Delta_{/[1]} \times_\Delta \Delta^n$. Since $\eta(-, -, i) \in \mathsf{Bim}_{Sp}(\mathcal{C})$ for all $i \in [n]$, the pullback Diagrams 20 associated by η to these 2-simplices exhibit the morphisms in (i) and (ii) as pullbacks of equivalences. Hence, they are themselves equivalences.

[27] See Definition A.1

Consequently, we can apply Lemma 9.5. It therefore, suffices to analyze condition $(I)-(IV)$ of Lemma 9.5. Condition (I) and (II) imply that η descends to a morphism

$$\eta : \Delta^n \to \mathsf{Span}(\mathsf{BMod}_{Sp}(\mathcal{C})).$$

Under the equivalence $\mathsf{BMod}_{Sp}(\mathcal{C}) \simeq \mathsf{Fun}^{\mathrm{bim}}(\Delta_1^{\star}, \mathcal{C})$ of Proposition 8.6 condition (III) reads as

$$\begin{array}{ccccc} & & \eta_{\phi_1(p)\leq\phi_1(p+1)}(ij)\times \dots \times \eta_{\phi_1(q-1)\leq\phi_1(q)}(ij) & & \\ & \swarrow & & \searrow & \\ \eta_{p,p+1}(ij)\times\dots\times\eta_{q-1,q}(ij) & & & & \eta_{\phi_1(p)\leq\phi_1(p+1)}(ij)\times\dots\times\eta_{\phi_1(q-1)\leq\phi_1(q)}(j) \\ & \searrow & & \swarrow & \\ & & \eta_{p,p+1}(j)\times\dots\times\eta_{q-1,q}(j) & & \end{array}$$

where the maps are the image under η of the diagram

$$\begin{array}{ccccc} & & f^1_{\{\phi_1(p)\leq\phi_1(p+1)\}},\dots,f^1_{\{\phi_1(q-1)\leq\phi_1(q)\}},ij) & & \\ & \swarrow & & \searrow & \\ (f^0_{\{p,p+1\}},\dots,f^0_{\{q-1,q\}},ij) & & & & (f^1_{\{\phi_1(p)\leq\phi_1(p+1)\}},\dots,f^1_{\{\phi_1(q-1)\leq\phi_1(q)\}},j) \\ & \searrow & & \swarrow & \\ & & (f^0_{\{p,p+1\}},\dots,f^0_{\{q-1,q\}},j) & & \end{array}$$

in the category $\Delta_1^{*}\times\mathsf{Tw}(\Delta)$. Similarly, condition (IV) is given by

$$\begin{array}{ccccc} & & \eta_{\phi_1(p)\leq\phi_1(p+1)}(ij)\times \dots \times \eta_{\phi_1(q-1)\leq\phi_1(q)}(ij) & & \\ & \swarrow & & \searrow & \\ \eta_{\phi_1(p)\leq\phi_1(p+1)}(i)\times\dots\times\eta_{\phi_1(q-1)\leq\phi_1(q)}(i) & & & & \eta_{\phi_1(p),\phi_1(p)+1}(ij)\times\dots\times\eta_{\phi_1(q)-1,\dots,\phi_1(q)}(ij) \\ & \searrow & & \swarrow & \\ & & \eta_{\phi_1(p),\phi_1(p)+1}(i)\times\dots\times\eta_{\phi_1(q)-1,\phi_1(q)}(i) & & \end{array}$$

Observe that the above diagrams are products of individual square-shaped diagrams. It, therefore, suffices to show that each such square is a pullback diagram. For condition (III), we therefore need to check that for every $p \leq r \leq q-1$ the diagram

$$\begin{array}{ccc} & \eta_{\phi_1(r)\leq\phi_1(r+1)}(ij) & \\ \swarrow & & \searrow \\ \eta_{\{r,r+1\}}(ij) & & \eta_{\phi_1(r)\leq\phi_1(r+1)}(j) \\ \searrow & & \swarrow \\ & \eta_{\{r,r+1\}}(j) & \end{array}$$

is a pullback square. Similarly, for condition (IV), we need to check that the diagram:

$$\begin{array}{ccc} & \eta_{\phi_1(r)\leq\phi_1(r+1)}(ij) & \\ \swarrow & & \searrow \\ \eta_{\phi_1(r)\leq\phi_1(r+1)}(i) & & \eta_{\{\phi_1(r),\phi_1(r)+1\}}(ij)\times...\eta_{\{\phi_1(r+1)-1,\phi_1(r+1)\}}(ij) \\ \searrow & & \swarrow \\ & \eta_{\{\phi_1(r),\phi_1(r)+1\}}(i)\times...\eta_{\{\phi_1(r+1)-1,\phi_1(r+1)\}}(i) & \end{array}$$

is a pullback diagram. Under the equivalence of Proposition 8.9, the map η identifies with a functor $\eta : \mathsf{Tw}(\Delta^n) \to \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})$. We denote the value of η on an object $i \leq j \in \mathsf{Tw}(\Delta^n)$ by $M^{i,j}_{\bullet}$, if $i < j$ and by $X^i_{\bullet}$, if $i = j$.

Under this equivalence condition (III) translates into the condition that for every $f_l \in \Delta^{\mathrm{op}}_{/[1]}$ and $0 \leq i < j \leq n$ the diagram

$$\begin{array}{ccc} & M^{i,j}_{f_l} & \\ \swarrow & & \searrow \\ M^{i,j}_{f_{\{0,l\}}} & & X^j_{f_l} \\ \searrow & & \swarrow \\ & X^j_{f_{\{0,l\}}} & \end{array}$$

is a pullback diagram. Similarly, condition (IV) translates into the condition that for every $f_l \in \Delta^{\mathrm{op}}_{/[1]}$ and $0 \leq i < j \leq n$ the diagram:

$$\begin{array}{ccc} & M^{i,j}_{f_l} & \\ \swarrow & & \searrow \\ M^{i,j}_{f_{\{0,1\}}} \times ... \times M^{i,j}_{f_{\{l-1,l\}}} & & X^i_{f_l} \\ \searrow & & \swarrow \\ & X^i_{f_{\{0,1\}}} \times ... \times X^i_{f_{\{l-1,l\}}} & \end{array}$$

is pullback diagrams. By Lemma 9.6 and Lemma 9.7, the conditions suffice to hold for $j = i + 1$. But these are precisely the active equifibered and relative Segal conditions. This finishes the proof. $\square$

Lemma 9.6. *Let $\tilde{\eta} : \Delta^n \to \mathsf{Span}(\mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}))$ be a functor with adjoint $\eta : \mathsf{Tw}(\Delta^n) \to \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})$ and denote by $M^{i,j}_{\bullet}$ the value of η on the object $i \to j \in \mathsf{Tw}(\Delta^n)$ if $i < j$ and by $X^i_{\bullet}$, if $i = j$. Further, let $f_l : [l] \to [1]$ be an object of $\Delta_{/[1]}$. Then the following are equivalent:*

(1) the diagram

$$\begin{array}{ccccc} & & M^{i,j}_{f_l} & & \\ & \swarrow & & \searrow & \\ M^{i,j}_{f_{\{0,l\}}} & & & & X^j_{f_l} \\ & \searrow & & \swarrow & \\ & & X^j_{f_{\{0,l\}}} & & \end{array}$$

is a pullback diagram for all $0 \leq i < j \leq n$.

(2) the diagram

$$\begin{array}{ccccc} & & M^{i,i+1}_{f_l} & & \\ & \swarrow & & \searrow & \\ M^{i,i+1}_{f_{\{0,l\}}} & & & & X^{i+1}_{f_l} \\ & \searrow & & \swarrow & \\ & & X^{i+1}_{f_{\{0,l\}}} & & \end{array}$$

is a pullback diagram for all $0 \leq i < n$.

Proof. (1) implies (2) by assumption. To prove the converse, we do induction on the difference $p = j - i$. The claim is true for $p = 1$ by assumption. Assume that the result holds for $p - 1$. Let $0 \leq i < j \leq n$ be objects of Δ^n with $j - i = p$. We denote $j - 1$ by k. Consider the diagram:

$$\begin{array}{ccccc} X^j_f & \longleftarrow & M^{k,j}_f & \longleftarrow & M^{i,j}_f \\ \downarrow & & \downarrow & & \downarrow \\ X^j_{f_{\{0,l\}}} & \longleftarrow & M^{k,j}_{f_{\{0,l\}}} & \longleftarrow & M^{i,j}_{f_{\{0,l\}}} \\ & & \downarrow & & \downarrow \\ & & X^k_{f_{\{0,l\}}} & \longleftarrow & M^{i,k}_{f_{\{0,l\}}} \end{array}$$

The upper left square is a pullback diagram by the induction hypothesis, and the lower square is a pullback diagram by the definition of η. To show that the big horizontal rectangle is a pullback diagram, it, therefore suffices to show that the big vertical rectangle is a pullback diagram. To show this, we factor the vertical rectangle as

$$
\begin{array}{ccc}
M_f^{k,j} & \longleftarrow & M_f^{i,j} \\
\downarrow & & \downarrow \\
X_f^{k} & \longleftarrow & M_f^{i,k} \\
\downarrow & & \downarrow \\
X_{f_{\{0,l\}}}^{k} & \longleftarrow & M_{f_{\{0,l\}}}^{i,k}
\end{array}
$$

The lower square is a pullback square by the induction hypothesis, and the upper square is a pullback square by the definition of η. It follows that the outer square is a pullback square. This finishes the proof. ☐

Lemma 9.7. *Let* $\tilde{\eta} : \Delta^n \to \mathsf{Span}(\mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C}))$ *be a functor with adjoint* $\eta : \mathsf{Tw}(\Delta^n) \to \mathsf{Fun}(\Delta^{\mathrm{op}}_{/[1]}, \mathcal{C})$ *and denote the value of* η *on the object* $i \to j \in \mathsf{Tw}(\Delta^n)$ *by* $M_{\bullet}^{i,j}$ *if* $i < j$ *and by* $X_{\bullet}^{i}$ *if* $i = j$. *Further, let* $f_l : [l] \to [1]$ *be an object of* $\Delta_{/[1]}$. *Then the following are equivalent:*

(1) the diagram

$$
\begin{array}{ccccc}
 & & M_{f_l}^{i,j} & & \\
 & \swarrow & & \searrow & \\
M_{f_{\{0,1\}}}^{i,j} \times \dots \times M_{f_{\{l-1,l\}}}^{i,j} & & & & X_{f_l}^{i} \\
 & \searrow & & \swarrow & \\
 & & X_{f_{\{0,1\}}}^{i} \times \dots \times X_{f_{\{l-1,l\}}}^{i} & &
\end{array}
$$

is a pullback diagram for all $0 \leq i < j \leq n$.

(2) the diagram

$$
\begin{array}{ccccc}
 & & M_{f_l}^{i,i+1} & & \\
 & \swarrow & & \searrow & \\
M_{f_{\{0,1\}}}^{i,i+1} \times \dots \times M_{f_{\{l-1,l\}}}^{i,i+1} & & & & X_{f_l}^{i} \\
 & \searrow & & \swarrow & \\
 & & X_{f_{\{0,1\}}}^{i} \times \dots \times X_{f_{\{l-1,l\}}}^{i} & &
\end{array}
$$

is a pullback diagram for all $0 \leq i < n$.

Proof. The proof is analogous to the one of Lemma 9.6. By assumption, (1) implies (2). For the converse, we do induction on the difference $p = j - i$. The case $p = 1$ follows by assumption. Assume therefore the result holds for $p-1$. Let $0 \leq i < j \leq n$ be objects of Δ^n and denote $j-1$ by k. Consider the diagram:

$$\begin{array}{ccccc}
\prod_{m=1}^{l} M^{k,j}_{f_{\{m-1,m\}}} & \longleftarrow & \prod_{m=1}^{l} M^{i,j}_{f_{\{m-1,m\}}} & \longleftarrow & M^{i,j}_{f} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{m=1}^{l} X^{k}_{f_{\{m-1,m\}}} & \longleftarrow & \prod_{m=1}^{l} M^{i,k}_{f_{\{m-1,m\}}} & \longleftarrow & M^{i,k}_{f} \\
 & & \downarrow & & \downarrow \\
 & & \prod_{m=1}^{l} X^{i}_{f_{\{m-1,m\}}} & \longleftarrow & X^{i}_{f}
\end{array}$$

It follows from the inductive hypothesis that the bottom square is a pullback diagram. Further, the upper left square is a pullback by definition of η. Therefore, to show that the big vertical rectangle is a pullback diagram, it suffices to show that the big horizontal rectangle is a pullback. The horizontal rectangle factors as

$$\begin{array}{ccccc}
\prod_{m=1}^{l} M^{k,j}_{f_{\{m-1,m\}}} & \longleftarrow & M^{k,j}_{f} & \longleftarrow & M^{i,j}_{f} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{m=1}^{l} X^{k}_{f_{\{m-1,m\}}} & \longleftarrow & X^{k}_{f} & \longleftarrow & M^{i,k}_{f}
\end{array}$$

The left square is a pullback square by the induction hypothesis and the right square is a pullback square by definition of η. Consequently, also the big horizontal square is a pullback diagram. □

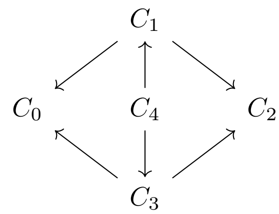
10 Rigid 2-Segal Objects

Throughout this section, let $\mathcal{C}$ be an ∞-category with finite limits. Building on our understanding of homotopy-coherent algebra in ∞-categories of spans from the previous sections, we can now finally perform the second step of our strategy, and classify locally rigid algebras in symmetric monoidal $(\infty, 2)$-categories of spans.

Therefore, we need a suitable symmetric monoidal $(\infty, 2)$-category of spans. Interestingly, there are two canonical choices, the $(\infty, 2)$-category of spans $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$, and the $(\infty, 2)$-category of 2-spans $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$. The key difference between these $(\infty, 2)$-categories is that $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ contains more 2-morphisms and therefore potentially allows for a larger class of locally rigid algebras. Indeed, whereas a 2-morphism in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$ is given by a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccc}
 & & C_1 & & \\
 & \swarrow & \downarrow & \searrow & \\
C_0 & & \downarrow & & C_2 \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & C_3 & &
\end{array}$$

a 2-morphism in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ is given by a diagram of the form

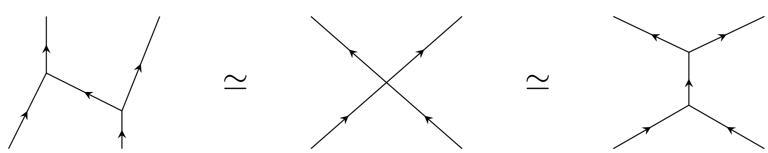


Nevertheless, it is important to consider both $(\infty, 2)$-categories, as we will see in Section 11 that more linearization functors are defined on $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ than on $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$. Therefore, we begin by analyzing locally rigid algebra objects in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ in Subsection 10.1. Afterwards, we turn to the study of locally rigid algebra objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$, where we explicitly calculate mates of 2-morphisms in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$. These calculations yield explicit additional limit conditions that a 2-Segal object has to fulfill to define a locally rigid algebra object.

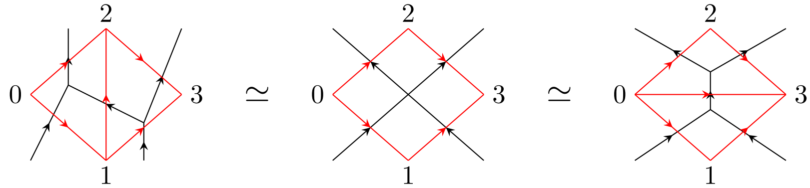
In Section 10.3, we finally characterize locally rigid 2-Segal objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$. To motivate the conditions we obtain, recall that locally rigid algebras form special examples of Frobenius algebras. These admit a graphical calculus, whose basics we recall now [Koc04].

For a Frobenius algebra, we visualize its multiplication and resp. comultiplication by the following graphs:

that we read from bottom to top. Further, composed operations are visualized by vertically stacking these basic graphs. In this graphical notation, the homotopy coherent Frobenius relation can be represented by the following pictures:



Here, the middle diagram represents a 4-ary operation with two inputs and two outputs, mediating between the two outer composites. This describes the homotopy coherence in the Frobenius relation. The relation to simplicial objects becomes more apparent in the Poincaré dual picture:



The direction on the red edges is determined by the condition that the red and black arrows, read in this order, form a right-handed coordinate system. The labels are chosen so that the edges point from the smaller to the larger values. When evaluating a 2-Segal object $X_\bullet$ on this diagram, we obtain the equation:

$$X_{\{0,1,2\}} \times_{X_{\{1,2\}}} X_{\{1,2,3\}} \simeq X_{\{0,1,2,3\}} \simeq X_{\{0,2,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}}$$

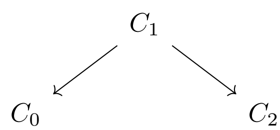
We will reencounter this equation as the condition that classifies locally rigid 2-Segal spaces. Let us now start with the explicit study of locally rigid 2-Segal objects.

10.1 Rigidity in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$

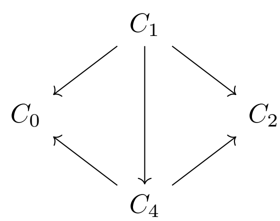
Let $\mathcal{C}$ be an ∞-category with finite limits. Our first goal is to understand locally rigid algebras in the $(\infty,2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$. To begin, we first recall its definition:

Proposition 10.1. *[Hau18] Let $\mathcal{C}$ be an ∞-category with finite limits. There exists a symmetric monoidal $(\infty,2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ called the symmetric monoidal $(\infty,2)$-category of spans, with*

- *objects given by objects in $\mathcal{C}$,*
- *1-morphisms given by spans in $\mathcal{C}$*



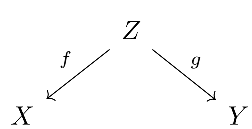
- *2-morphisms given by morphism between spans in $\mathcal{C}$*



and symmetric monoidal product functor $\otimes$ that maps two objects $C_0, C_1 \in \mathcal{C}$ to the Cartesian product in $\mathcal{C}$.

Remark 10.1. This is an $(\infty,2)$-categorical version of the 2-category from Theorem 2.1. Inserting in the above definition the $(2,1)$-category Grpd^f of finite groupoids, one can recover the symmetric monoidal 2-category from Section 2 as the homotopy 2-category of the above symmetric monoidal $(\infty,2)$-category.

Notation 10.1. Let $\mathcal{C}$ be an ∞-category with finite limits, and let



be a span from X to Y. We often abuse notation and denote a span by a map $(f,g) : Z \to X \times Y$. In this notation, we implicitly assume that the first map is the backward pointing and the second map is the forward pointing leg of the span.

As a first step to study locally rigid algebras in the above symmetric monoidal $(\infty,2)$-category, we need to determine the dualizable objects:

Proposition 10.2. *[Hau18, Cor.12.5] Let $\mathcal{C}$ be an ∞-category with finite limits, and let C be an object in the symmetric monoidal $(\infty, 2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$. We denote by $\Delta : C \to C \times C$ the diagonal map. The spans*

$$C \times C \xleftarrow{\Delta} C \xrightarrow{p_C} * \quad \text{and} \quad * \xleftarrow{p_C} C \xrightarrow{\Delta} C \times C$$

exhibit C as its own left and right dual.

Proof. A straightforward calculation confirms that these indeed form the evaluation and coevaluation of a duality on C. □

Consequently, the condition of being dualizable does not impose any restriction on our class of locally rigid algebras. Next, we need to understand adjoints for 1-morphisms.

Proposition 10.3. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let*

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

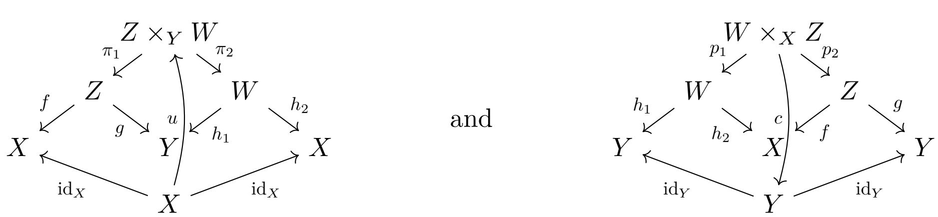
be a span from X to Y. Then (f, g) admits a right adjoint in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$ if and only f is an equivalence. In this case, the right adjoint of (f, g) is given by the reversed span

$$Y \xleftarrow{g} Z \xrightarrow{f} X$$

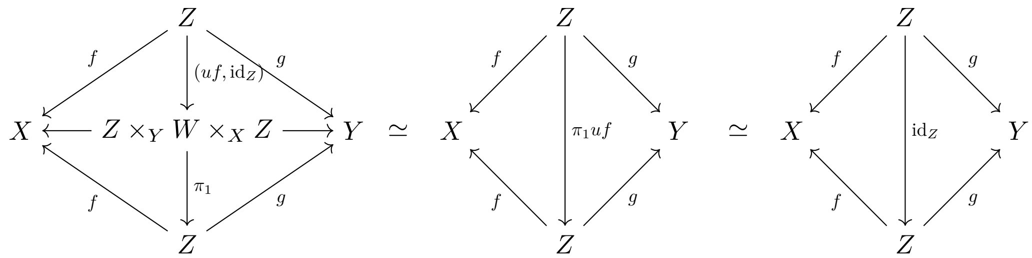
Proof. We will prove this claim by an explicit calculation in the homotopy 2-category $\mathsf{h}_2\mathbb{S}\mathsf{pan}_2(\mathcal{C})$. It is easy to see that if f is an equivalence, there exists an adjunction

$$\left(X \xleftarrow{f} Z \xrightarrow{g} Y\right) \dashv \left(Y \xleftarrow{g} Z \xrightarrow{f} X\right).$$

We assume, therefore, that the span (f, g) has an adjoint $(h_1, h_2) : W \to Y \times X$ together with 2-morphisms



It follows from the first diagram that f has a left inverse given by $\pi_1 \circ u$. We need to show that f also has a right inverse. The first of the zig-zag identities of the adjunction is given by the equivalence



This implies that $\pi_1 u$ is also a right inverse of f and hence that f is an equivalence. ☐

Consequently, a 2-Segal object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ can only define a locally rigid algebra in $\mathbb{S}\mathsf{pan}_2(\mathcal{S})^\otimes$ if the morphism

$$X_2 \xrightarrow{(\partial_2, \partial_0)} X_1 \times X_1$$

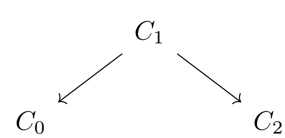
is an equivalence of spaces. In particular, if $X_\bullet$ is further assumed to be rigid, X_0 must be the final object, and consequently, the 2-Segal object $X_\bullet$ has to be *Segal*. Unfortunately, this excludes many interesting examples of 2-Segal spaces like the Waldhausen construction from Example 7.1. Therefore, we will explore in the next section whether allowing for more general 2-morphisms can yield more interesting examples of rigid 2-Segal spaces.

10.2 Rigidity in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$

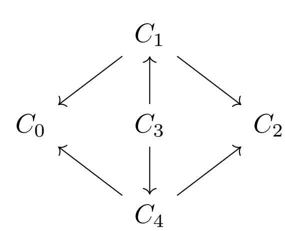
As we have discussed in the previous section, the main issue with the $(\infty, 2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^\otimes$ for the study of locally rigid 2-Segal objects is that this $(\infty, 2)$-category lacks sufficient adjoints. Interestingly, there exists a second variant of a symmetric monoidal $(\infty, 2)$-category of spans in which every 1-morphism has both adjoints. This $(\infty, 2)$-category is called the $(\infty, 2)$-category of 2-spans:

Proposition 10.4. *[Hau18] Let $\mathcal{C}$ be an ∞-category with finite limits. There exists a symmetric monoidal $(\infty, 2)$-category $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$ called the symmetric monoidal $(\infty, 2)$-category of 2-spans with*

- *objects given by objects in $\mathcal{C}$,*
- *1-morphisms given by spans in $\mathcal{C}$*



- *2-morphisms given by 2-spans in $\mathcal{C}$*



Further, the underlying monoidal product functor of the symmetric monoidal structure maps two objects C_0, C_1 to their Cartesian product $C_0 \times C_1$ in $\mathcal{C}$.

For the study of locally rigid algebra objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$, we again need to determine the dualizable objects and the 1-morphisms with adjoints. Note that the symmetric monoidal $(\infty, 2)$-categories $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$ and $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ have the same underlying symmetric monoidal ∞-category. Hence, it follows from Proposition 10.2 that every object in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ is self-dual. Moreover, we can explicitly determine adjoints of 1-morphisms:

Proposition 10.5. *[Hau18, Lem.12.3] Let $\mathcal{C}$ be an ∞-category with finite limits, and let σ be a 1-morphism in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ given by the span*

$$\begin{tikzcd} & C_1 \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ C_0 & & C_2 \end{tikzcd}$$

Then σ has a right (resp. left) adjoint given by the opposite span

$$\begin{tikzcd} & C_1 \arrow[dl, "f_1"'] \arrow[dr, "f_0"] & \\ C_2 & & C_0 \end{tikzcd}$$

Proof. The unit and counit of the adjunction are given by the 2-spans

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl] \arrow[dr] & \\ C_2 & C_1 \arrow[u, "\Delta_{C_0}"] \arrow[d] & C_2 \\ & C_2 \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_0 \arrow[dl] \arrow[dr] & \\ C_0 & C_1 \arrow[u] \arrow[d, "\Delta_{C_2}"] & C_0 \\ & C_1 \times_{C_2} C_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

respectively, where we denote by Δ_C the respective relative diagonal morphisms. It is straightforward to verify that these indeed satisfy the triangle identities of an adjunction. The fact that it is also a left adjoint follows from reading all diagrams upside down. □

Note that as a consequence, every locally rigid 2-Segal object is automatically rigid. To check that a 2-Segal object in $\mathcal{C}$ is rigid in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$, it remains to understand the adjointability of Diagram 6. More generally, we follow the spirit of Remark 3.3 and study these conditions for arbitrary module morphisms. To do so, we need the following variant of Definition 6.9:

Definition 10.1. Let $\pi^0_\bullet : X^0_\bullet \to Y_\bullet$ and $\pi^2_\bullet : X^2_\bullet \to Y_\bullet$ be $Y_\bullet$-relative 2-Segal objects. A *$Y_\bullet$-relative 2-Segal span* from $\pi^0_\bullet$ to $\pi^2_\bullet$ is given by a span

$$\begin{tikzcd} X^0_\bullet \arrow[d, "\pi^0_\bullet"'] & X^1_\bullet \arrow[l, "s^X_\bullet"'] \arrow[r, "t^X_\bullet"] \arrow[d, "\pi^1_\bullet"] & X^2_\bullet \arrow[d, "\pi^2_\bullet"] \\ Y_\bullet & Y_\bullet \arrow[l, "s^Y_\bullet"] \arrow[r, "t^Y_\bullet"'] & Y_\bullet \end{tikzcd}$$

of $Y_\bullet$-relative 2-Segal objects s.t.:

(1) The morphisms $s^Y_\bullet$ and $t^Y_\bullet$ are equivalences.

(2) For every $n \geq 1$ the square

$$\begin{array}{ccc} X^1_n & \xrightarrow{s^X_n} & X^0_n \\ \downarrow & & \downarrow \\ X^1_{\{0\}} & \xrightarrow{s^X_1} & X^0_{\{0\}} \end{array}$$

is Cartesian.

(3) For every $n \geq 1$ the diagram

$$\begin{array}{ccc} X^1_n & \xrightarrow{t^X_n} & X^2_n \\ \downarrow & & \downarrow \\ X^1_{\{n\}} & \longrightarrow & X^2_{\{n\}} \end{array}$$

is Cartesian.

We call the pair $(s^X_\bullet, s^Y_\bullet)$ $Y_\bullet$*-active equifibered* and the pair $(t^X_\bullet, t^Y_\bullet)$ $Y_\bullet$*-relative Segal.* Further, we call the ∞-category $\mathrm{Rel2Seg}^{\leftrightarrow}_{\Delta,\mathrm{Y}_\bullet}(\mathcal{C})$ whose objects are $Y_\bullet$-relative 2-Segal objects and whose morphisms are $Y_\bullet$-relative 2-Segal spans the *∞-category of $Y_\bullet$-relative 2-Segal objects.*

This definition can be reformulated using more familiar concepts from the theory of simplicial spaces:

Definition 10.2. A map of simplicial objects $\pi_\bullet : X_\bullet \to Y_\bullet$ is called a *right (resp. left) fibration*, if for all $n \geq 1$ the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\partial_0} & X_{n-1} \\ {\scriptstyle\pi_n}\downarrow & & \downarrow{\scriptstyle\pi_{n-1}} \\ Y_n & \xrightarrow[\partial_0]{} & Y_{n-1} \end{array} \quad \text{resp.} \quad \begin{array}{ccc} X_n & \xrightarrow{\partial_n} & X_{n-1} \\ {\scriptstyle\pi_n}\downarrow & & \downarrow{\scriptstyle\pi_{n-1}} \\ Y_n & \xrightarrow[\partial_n]{} & Y_{n-1} \end{array}$$

is Cartesian.

Proposition 10.6. *Let $\pi^i_\bullet : X^i_\bullet \to Y_\bullet$ for $0 \leq i \leq 2$ be $Y_\bullet$-relative 2-Segal objects. A span*

$$\begin{array}{ccccc} X^0_\bullet & \xleftarrow{s^X_\bullet} & X^1_\bullet & \xrightarrow{t^X_\bullet} & X^2_\bullet \\ {\scriptstyle\pi^0_\bullet}\downarrow & & \downarrow{\scriptstyle\pi^1_\bullet} & & \downarrow{\scriptstyle\pi^2_\bullet} \\ Y_\bullet & \xleftarrow[\simeq]{} & Y_\bullet & \xrightarrow{\simeq} & Y_\bullet \end{array}$$

is a $Y_\bullet$-relative 2-Segal span if and only if the map $s^X_\bullet$ is a left fibration and the map $t^X_\bullet$ is a right fibration.

Proof. We need to show that for every $n \geq 1$ all diagrams of the form

$$\begin{array}{ccc} X^1_n & \longrightarrow & X^0_n \\ \downarrow & & \downarrow \\ X^1_{\{0\}} & \longrightarrow & X^0_{\{0\}} \end{array} \qquad \begin{array}{ccc} X^1_n & \longrightarrow & X^2_n \\ \downarrow & & \downarrow \\ X^1_{\{n\}} & \longrightarrow & X^2_{\{n\}} \end{array}$$

are Cartesian if and only if all diagrams of the form

$$\begin{array}{ccc} X^1_n & \xrightarrow{s^X_n} & X^0_n \\ {\scriptstyle d^1_n}\downarrow & & \downarrow{\scriptstyle d^0_n} \\ X^1_{n-1} & \xrightarrow{s^X_{n-1}} & X^0_{n-1} \end{array} \qquad \begin{array}{ccc} X^1_n & \xrightarrow{t^X_n} & X^2_n \\ {\scriptstyle d^1_0}\downarrow & & \downarrow{\scriptstyle d^2_0} \\ X^1_{n-1} & \xrightarrow{t^X_{n-1}} & X^2_{n-1} \end{array}$$

are. But this follows by induction from the pasting lemma applied to the pasted diagrams

$$\begin{array}{ccc} X_n^1 & \xrightarrow{s_n^X} & X_n^0 \\ {\scriptstyle d_n^1}\downarrow & & \downarrow{\scriptstyle d_n^0} \\ X_{n-1}^1 & \xrightarrow{s_{n-1}^X} & X_{n-1}^0 \\ \downarrow & & \downarrow \\ X_{\{0\}}^1 & \xrightarrow{s_0^X} & X_{\{0\}}^0 \end{array} \qquad \begin{array}{ccc} X_n^1 & \xrightarrow{t_n^X} & X_n^2 \\ {\scriptstyle d_0^1}\downarrow & & \downarrow{\scriptstyle d_0^2} \\ X_{n-1}^1 & \xrightarrow{t_{n-1}^X} & X_{n-1}^2 \\ \downarrow & & \downarrow \\ X_{\{n\}}^1 & \xrightarrow{t_0^X} & X_{\{n\}}^2 \end{array}$$

□

Whereas Definition 6.9 describes the datum module morphisms between modules over different algebras, Definition 10.1 describes module morphisms between modules over the same algebra. In particular, we can derive the following result as a consequence of Corollary 9.4:

Theorem 10.7. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $Y_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object. There exists an equivalence*

$$\mathrm{LMod}_{Y_1}(\mathsf{Span}_\Delta(\mathcal{C}^\times)) \simeq \mathrm{Rel2Seg}^{\leftrightarrow}_{\Delta,\mathrm{Y}_\bullet}(\mathcal{C}) \tag{21}$$

between the ∞-category of left Y_1-modules in the ∞-category of spans $\mathsf{Span}_\Delta(\mathcal{C}^\times)$ and the ∞-category of $Y_\bullet$-relative 2-Segal objects and $Y_\bullet$-relative 2-Segal spans.

Using the above theorem, we can now efficiently describe module morphisms in categories of spans.

Notation 10.2. Let $\mathbb{D}$ be a symmetric monoidal $(\infty, 2)$-category and let $A \in \mathsf{Alg}(\mathcal{D})$ be an algebra object in $\mathbb{D}^\otimes$. Furthermore, let $F : M \to N$ in $\mathrm{LMod}_A(\mathbb{D})$ be a morphism of left A-modules in $\mathbb{D}$. We call the commutative square

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\rhd_M} & M \\ {\scriptstyle id_A \otimes F}\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle F} \\ A \otimes N & \xrightarrow{\rhd_N} & N \end{array} \tag{22}$$

the *A-equivariance square associated to F* and the morphism α the *A-equivariance 2-isomorphism.*

Example 10.1. Let $\mathcal{C}$ be an ∞-category with finite limits and let $\phi : M_\bullet \to X_\bullet$ and $\psi : N_\bullet \to X_\bullet$ be $X_\bullet$-relative 2-Segal objects representing X_1-modules in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$. Furthermore, let

$$\begin{array}{ccccc} M_\bullet & \xleftarrow{f_\bullet} & F_\bullet & \xrightarrow{g_\bullet} & N_\bullet \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \psi} \\ X_\bullet & \xleftarrow[id]{} & X_\bullet & \xrightarrow[id]{} & X_\bullet \end{array}$$

be a $X_\bullet$-relative 2-Segal span representing a X_1-module homomorphism in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$. The X_1-equivariance square associated to the above relative 2-Segal span is given by the diagram

$$\begin{array}{ccccc} X_1 \times M_0 & \xleftarrow{(\phi_1, \partial_1^M)} & M_1 & \xrightarrow{\partial_0^M} & M_0 \\ {\scriptstyle id \times f_0}\uparrow & & {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle f_0} \\ X_1 \times F_0 & \xleftarrow{(\pi, \partial_1^F)} & F_1 & \xrightarrow{\partial_0^F} & F_0 \\ {\scriptstyle id \times g_0}\downarrow & & {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_0} \\ X_1 \times N_0 & \xleftarrow[(\psi_1, \partial_1^N)]{} & N_1 & \xrightarrow[\partial_0^N]{} & N_0 \end{array} \tag{23}$$

and the X_1-equivariance 2-isomorphism α by the span:

$$F_0 \times_{N_0} N_1 \xleftarrow{(\partial_1^F, g_1)} F_1 \xrightarrow{(f_1, \partial_0^F)} M_1 \times_{M_0} F_0 \tag{24}$$

Note, that it follows from the conditions imposed on a $X_\bullet$-relative 2-Segal span from Definition 10.1 that the legs of the span α are indeed equivalences.

Let $(A, \mu, \alpha) \in \mathsf{Alg}(\mathbb{D})$ be an algebra object. If we equip the multiplication 1-morphism

$$\mu : A \otimes A \to A$$

with the structure of a left A-module morphism, as explained in Remark 3.3, then the Diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes id} \downarrow & \overset{\simeq^{\alpha}}{\nearrow\!\!\!\nearrow} & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

featuring in the definition of a locally rigid algebra is just the A-equivariance square of the left module structure on μ. To classify locally rigid algebra objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$, we therefore need to compute the Beck–Chevalley transforms of the respective equivariance 2-isomorphism as in Diagram 24:

Proposition 10.8. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $\phi : M_\bullet \to X_\bullet$ and $\psi : N_\bullet \to X_\bullet$ be $X_\bullet$-relative 2-Segal objects representing X_1-modules in $\mathsf{Span}(\mathcal{C})^{\otimes}$. Furthermore, let*

$$\begin{array}{ccccc} M_\bullet & \xleftarrow{f_\bullet} & F_\bullet & \xrightarrow{g_\bullet} & N_\bullet \\ \downarrow {\scriptstyle \phi} & & \downarrow & & \downarrow {\scriptstyle \psi} \\ X_\bullet & \xleftarrow[id]{} & X_\bullet & \xrightarrow[id]{} & X_\bullet \end{array}$$

be a $X_\bullet$-relative 2-Segal span representing a X_1-module homomorphism in $\mathsf{Span}(\mathcal{C})^{\otimes}$. The vertical and horizontal Beck–Chevalley transformation of the X_1-equivariance 2-isomorphism α are both given by the span

$$F_0 \times_{M_0} M_1 \xleftarrow{(\partial_0, f_1)} F_1 \xrightarrow{(g_1, \partial_1)} N_1 \times_{N_0} F_0$$

Proof. We explicitly compute the vertical Beck-Chevalley transform. The computation of the horizontal Beck-Chevalley transform is analogous. Recall that by Definition 3.1 the vertical Beck-Chevalley transform of the equivariance square (22) is given by the 2-morphism:

$$\triangleright_M \circ (id_{X_1} \times F^R) \Rightarrow F^R \circ F \circ \triangleright_M \circ (id_{X_1} \times F^R) \simeq F^R \triangleright_N \circ (id_{X_1} \times F) \circ (id_{X_1} \circ F^R) \Rightarrow F^R \circ \triangleright_N . \tag{25}$$

We translate the components into our particular situation. Therefore, we first compute its source and target. Note, that in the case at hand the source of the Beck-Chevalley transform $\triangleright_M \circ (id_{X_1} \times F^R)$ is given by the span

$$
\begin{array}{ccccccc}
 & & & X_1 \times F_0 \times_{X_1 \times M_0} M_1 & & & \\
 & & \swarrow & & \searrow & & \\
 & M_1 & & & & X_1 \times F_0 & \\
 {\scriptstyle (\pi_1, g_0\pi_2)} \swarrow & & \searrow {\scriptstyle (\pi_1, f_0\pi_2)} & & {\scriptstyle (\phi_1, \partial_0)} \swarrow & & \searrow \\
X_1 \times N_0 & & & X_1 \times M_0 & & & M_0
\end{array}
$$

and the target $F^R \circ \triangleright_N$ is given by the span

$$
\begin{array}{ccccccc}
 & & & N_1 \times_{N_0} F_0 & & & \\
 & & \swarrow & & \searrow & & \\
 & N_1 & & & & F_0 & \\
 {\scriptstyle (\psi_1, \partial_0^N)} \swarrow & & \searrow {\scriptstyle \partial_1^N} & & {\scriptstyle g_0} \swarrow & & \searrow \\
X_1 \times N_0 & & & N_0 & & & M_0
\end{array}
$$

Observe that the source 1-morphism of the Beck-Chevalley transform is equivalent to

$$
\begin{array}{ccc}
 & F_0 \times_{M_0} M_1 & \\
 {\scriptstyle (\phi_1 p_2^1, g_0 p_1^1)} \swarrow & & \searrow {\scriptstyle \partial_1^M p_2^1} \\
X_1 \times N_0 & & M_0
\end{array}
$$

Let us now collect the individual components of the Beck-Chevalley transform. It follows from Proposition 10.5 that the unit of the adjunction $u : \mathrm{id}_{M_0} \to F^R \circ F$ is given by the span over $M_0 \times M_0$

$$
\begin{array}{ccc}
 & F_0 & \\
 {\scriptstyle f_0} \swarrow & & \searrow {\scriptstyle \Delta_{N_0}} \\
M_0 & & F_0 \times_{N_0} F_0
\end{array}
$$

and the counit for the adjunction $(\mathrm{id}_{X_1} \times F) \circ (\mathrm{id}_{X_1} \times F^R) \Rightarrow \mathrm{id}_{X_1 \times N_0}$ is given by the span over $X_1 \times N_0$

$$
\begin{array}{ccc}
 & X_1 \times F_0 & \\
 {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{M_0}} \swarrow & & \searrow {\scriptstyle \mathrm{id}_{X_1} \times g_0} \\
X_1 \times (F_0 \times_{M_0} F_0) & & X_1 \times N_0
\end{array}
\tag{26}
$$

Hence, the first 2-morphism in the BC-transform is given by the following span in $\mathcal{C}_{/(X_1 \times N_0) \times M_0}$

$$\begin{array}{ccc} & (F_0 \times_{M_0} M_1) \times_{M_0} F_0 & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle (\pi_1, \Delta_{N_0}\pi_2)} \\ F_0 \times_{M_0} M_1 & & (F_0 \times_{M_0} M_1) \times_{M_0} (F_0 \times_{N_0} F_0) \end{array}$$

Next, we need to identify the equivalence induced by the 2-morphism in the X_1-equivariance square of $F_\bullet$. It follows from Equation (24) that it is given by the span:

$$\begin{array}{ccc} & F_0 \times_{M_0} F_1 \times_{N_0} F_0 & \\ {\scriptstyle id_{F_0} \times (f_1, \partial_1) \times id_{F_0}} \swarrow & & \searrow {\scriptstyle id_{M_0} \times (\partial_0, g_1) \times id_{F_0}} \\ F_0 \times_{M_0} (M_1 \times_{M_0} F_0) \times_{N_0} F_0 & & F_0 \times_{M_0} (F_0 \times_{N_0} N_1) \times_{N_0} F_0 \end{array}$$

in $\mathcal{C}_{/(X_1 \times N_0) \times M_0}$. Finally, the last 2-morphism in the BC-transform is induced by the counit from Equation 26. Note, that it is given by the span

$$\begin{array}{ccc} & F_0 \times_{N_0} N_1 \times_{N_0} F_0 & \\ {\scriptstyle \Delta_{M_0} \times id_{N_1} \times id_{F_0}} \swarrow & & \searrow {\scriptstyle \pi_{N_1 \times_{N_0} F_0}} \\ (F_0 \times_{M_0} F_0) \times_{N_0} N_1 \times_{N_0} F_0 & & N_1 \times_{N_0} F_0 \end{array}$$

in $\mathcal{C}_{/(X_1 \times N_0) \times M_0}$. To obtain the BC-transform, we have to compose all these 2-morphisms. We can compute the composition of the first two 2-morphisms to be given by

$$\begin{array}{ccc} & (F_0 \times_{M_0} M_1) \times F_0 \times_{(F_0 \times_{M_0} M_1) \times_{M_0} (F_0 \times_{N_0} F_0} (F_0 \times_{M_0} F_1 \times_{N_0} F_0) & \\ {\scriptstyle h_1} \swarrow & & \searrow {\scriptstyle h_2} \\ F_0 \times_{M_0} (F_0 \times_{N_0} N_1) \times_{N_0} F_0 & & F_0 \times_{M_0} M_1 \end{array}$$

Next, we simplify this span. To this end, note that the tip of the above span is the limit of the following diagram:

$$\begin{array}{ccccccc} F_0 & & & F_0 \times_{M_0} F_1 \times_{N_0} F_0 & & & F_0 \\ & \searrow {\scriptstyle \mathrm{id}_{F_0 \times_{M_0} M_1}} & {\scriptstyle (\partial_0 \pi_2, \pi_3)} \swarrow & & \searrow {\scriptstyle (\pi_1, f_1 \pi_2)} & {\scriptstyle \Delta_{N_0}} \swarrow & \\ & & F_0 \times_{M_0} M_1 & & F_0 \times_{N_0} F_0 & & \end{array}$$

It follows from the dual of [Lur09a, Prop.4.4.2.2] that this limit is also given by $F_0 \times_{M_0} F_1$, so that the above span is equivalent to

$$
\begin{array}{ccc}
 & F_0 \times_{M_0} F_1 & \\
{\scriptstyle (\pi_1, f_1\pi_2)} \swarrow & & \searrow {\scriptstyle \mathrm{id}_{F_0} \times (\partial_0, g_1, \partial_1)} \\
F_0 \times_{M_0} \times M_1 & & F_0 \times_{M_0} F_0 \times_{N_0} N_1 \times_{N_0} F_0
\end{array}
$$

To finish the proof, we have to compute the composite of the above span with the span in Diagram (10.2). Direct computation yields:

$$
\begin{array}{ccc}
 & (F_0 \times_{M_0} F_1) \times_{(F_0 \times_{M_0} F_0 \times_{N_0} N_1 \times_{N_0} F_0)} (F_0 \times_{N_0} N_1 \times_{N_0} F_0) & \\
\swarrow & & \searrow \\
F_0 \times_{M_0} M_1 & & N_1 \times_{N_0} F_0
\end{array}
$$

To simplify pullback, note that the tip of this span is equivalently given by the limit of the diagram:

$$
\begin{array}{ccccccc}
N_1 \times_{N_0} F_0 & & & F_0 \times_{M_0} F_1 & & & F_0 \\
 & \searrow {\scriptstyle \mathrm{id}_{N_1 \times_{N_0} F_0}} & {\scriptstyle (g_1, \partial_1)\pi_2} \swarrow & & \searrow {\scriptstyle (\pi_1, \partial_0\pi_2)} & {\scriptstyle (g_1, \partial_1)\pi_2} \swarrow & \\
 & & N_1 \times_{N_0} F_0 & & F_0 \times_{M_0} F_0 & &
\end{array}
.
$$

But the limit of this diagram is also equivalent to F_1. Hence, we can conclude that the BC-transform is given by the span in $\mathcal{C}_{/(X_1 \times N_0) \times M_0}$

$$
\begin{array}{ccc}
 & F_1 & \\
{\scriptstyle (\partial_0, f_1)} \swarrow & & \searrow {\scriptstyle (g_1, \partial_1)} \\
F_0 \times_{M_0} M_1 & & N_1 \times_{N_0} F_1
\end{array}
$$

□

Note that we can also read every $X_\bullet$-relative 2-Segal span, as described above, in the opposite direction:

$$
\begin{array}{ccccc}
N_\bullet & \xleftarrow{g_\bullet} & F_\bullet & \xrightarrow{f_\bullet} & M_\bullet \\
\downarrow {\scriptstyle \psi} & & \downarrow & & \downarrow {\scriptstyle \phi} \\
X_\bullet & \xleftarrow[id]{} & X_\bullet & \xrightarrow[id]{} & X_\bullet
\end{array}
$$

This yields a span of $X_\bullet$-relative 2-Segal objects with source and target given by

$$
N_\bullet \to X_\bullet, \quad \text{and} \quad M_\bullet \to X_\bullet
$$

respectively. However, the conditions imposed on a relative 2-Segal span are not symmetric. As a result, this reversed diagram does not, in general, represent a relative 2-Segal span itself and hence does not induce a module morphism. On the other hand, note that the reversed span satisfies the lowest instance of the relative 2-Segal span conditions, if the Beck-Chevalley transform computed above is an equivalence. This observation admits the following generalization:

Proposition 10.9. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $\phi : M_\bullet \to X_\bullet$ and $\psi : N_\bullet \to X_\bullet$ be relative 2-Segal objects representing X_1-modules in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$. Furthermore, let:*

$$\begin{array}{ccccc} M_\bullet & \xleftarrow{f_\bullet} & F_\bullet & \xrightarrow{g_\bullet} & N_\bullet \\ {\scriptstyle\phi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\psi} \\ X_\bullet & \xleftarrow[id]{} & X_\bullet & \xrightarrow[id]{} & X_\bullet \end{array}$$

be a relative 2-Segal span representing a X_1-module homomorphism in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$. The following are equivalent:

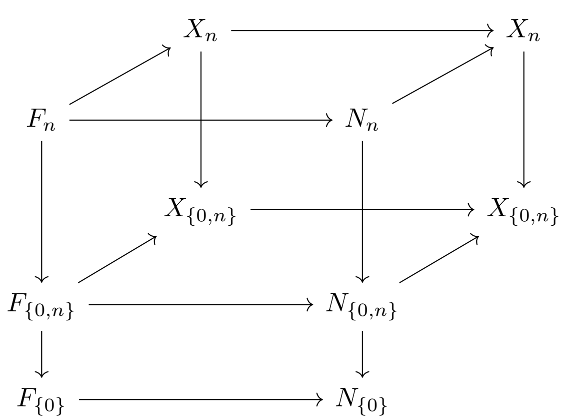
(1) The vertical (resp. horizontal) BC-transform of the X_1-equivariance square associated to $F_\bullet$ is an equivalence.

(2) The reversed diagram:

$$\begin{array}{ccccc} N_\bullet & \xleftarrow{g_\bullet} & F_\bullet & \xrightarrow{f_\bullet} & M_\bullet \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\phi} \\ X_\bullet & \xleftarrow[id]{} & X_\bullet & \xrightarrow[id]{} & X_\bullet \end{array} \tag{27}$$

is a relative 2-Segal span.

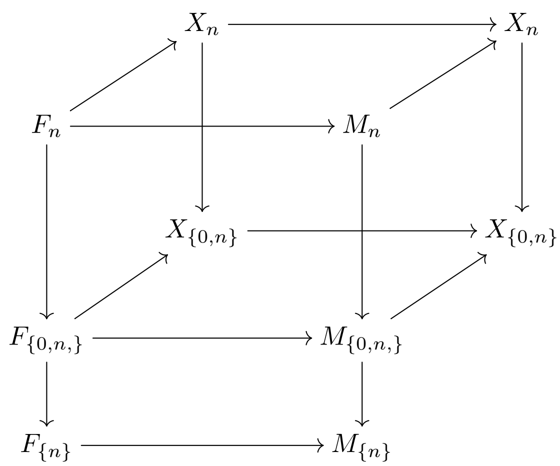
Proof. By assumption (2) $\Rightarrow$ (1). To show the converse, we have to show that the morphisms in Diagram (27) satisfy the conditions of Definition 10.1. Let therefore $n \geq 2$ and consider the following diagram



(28)

We need to show that the large front rectangle is a pullback diagram. It follows from the assumption that the bottom square is a pullback diagram. It therefore suffices to show that the top square is a pullback diagram. Observe, that since $\pi_\bullet : F_\bullet \to X_\bullet$ and $\psi_\bullet : N_\bullet \to X_\bullet$ are relative 2-Segal objects, the side faces of the cube are pullback squares. Further, the back face of the cube is a pullback square. Consequently, another application of the pasting lemma implies that the front square is a pullback square as desired.

The analogous conditions for $f_\bullet : F_\bullet \to M_\bullet$ follow from a similar argument applied to the diagram



(29)

□

Remark 10.2. Let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category. As sketched in Section 4.5, one can define for every algebra object $A \in \mathsf{Alg}(\mathbb{D})$ an $(\infty, 2)$-category

$$\mathbb{L}\mathsf{Mod}_A(\mathbb{D}),$$

whose objects are left A-modules, 1-morphisms are left A-module morphisms and 2-morphisms are A-module 2-morphisms. In particular, for any ∞-category $\mathcal{C}$ with finite limits, and any algebra object $A \in \mathsf{Alg}(2\mathbb{S}\mathsf{pan}(\mathcal{C}))$, one can define an $(\infty, 2)$-category

$$\mathbb{L}\mathsf{Mod}_A(2\mathbb{S}\mathsf{pan}(\mathcal{C})).$$

We expect that there exists a notion of 2-morphism between relative 2-Segal spans, that extends the equivalence of Theorem 10.7 to an equivalence of $(\infty, 2)$-categories. We expect that under this equivalence a left A-module morphism

$$F : M \to N$$

admits an adjoint internal to $\mathbb{L}\mathsf{Mod}_A(2\mathbb{S}\mathsf{pan}(\mathcal{C}))$ if and only if the associated relative 2-Segal span satisfies the hypothesis of 10.9 and that in this case the adjoint is given by the reversed relative 2-Segal span. We plan to address this question in future work.

For every 2-Segal object $X_\bullet$, we can apply the above results to the X_1-equivariance squares that appear in the definition of rigidity 3.2.

Corollary 10.10. *Let $\mathcal{C}$ be an ∞-category with finite limits and $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ a 2-Segal object in $\mathcal{C}$. Then the following are equivalent:*

(1) $X_\bullet$ defines a rigid algebra object in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$

(2) the span

$$X_{\{0,1,2\}} \times_{X_{\{1,2\}}} X_{\{1,2,3\}} \longleftarrow X_3 \longrightarrow X_{\{0,1,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}}$$

is an equivalence in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$

(3) the diagram

$$\begin{array}{ccccc} (P^{\triangleright}X)_\bullet \simeq X_{\bullet+1_2} & \longleftarrow & P^{\triangleright}(P^{\triangleright}(X_\bullet)) \simeq X_{\bullet+1_1+1_2} & \longrightarrow & P^{\triangleright}(X_\bullet) \times X_{1_1+1_2} \simeq X_{\bullet+1_1} \times X_{1_1+1_2} \\ \downarrow & & \downarrow & & \downarrow \\ X_\bullet & \longleftarrow & X_\bullet & \longrightarrow & X_\bullet \end{array}$$

is a relative 2*-Segal span, where we denote by* 1_1 *(resp.* 1_2*) the first (resp. second) shift by* 1*.*

(4) the diagram

$$\begin{array}{ccccc} (P^{\triangleleft}X)_\bullet \simeq X_{1_2+\bullet} & \longleftarrow & P^{\triangleleft}(P^{\triangleleft}(X_\bullet)) \simeq X_{1_2+1_1+\bullet} & \longrightarrow & P^{\triangleleft}(X_\bullet) \times X_{1_2+1_1} \simeq X_{1_1+\bullet} \times X_{1_2+1_1} \\ \downarrow & & \downarrow & & \downarrow \mathrm{id}_{X_\bullet} \\ X_\bullet & \longleftarrow & X_\bullet & \longrightarrow & X_\bullet \end{array}$$

is a relative 2*-Segal span, where we denote by* 1_1 *(resp.* 1_2*) the first (resp. second) shift by* 1*.*

Proof. Recall from Example 6.2 that the regular X_1 left (resp. right) module structure on X_1 is described by the relative 2-Segal object $P^{\triangleright}X_\bullet \to X_\bullet$ (resp. $P^{\triangleleft}X_\bullet \to X_\bullet$). An easy calculation shows that the relative 2-Segal span:

$$\begin{array}{ccccc} P^{\triangleright}(X_\bullet) \times X_{1_1+1_2} \simeq X_{\bullet+1_1} \times X_{1_1+1_2} & \longleftarrow & P^{\triangleright}(P^{\triangleright}(X_\bullet)) \simeq X_{\bullet+1_1+1_2} & \longrightarrow & (P^{\triangleright}X)_\bullet \simeq X_{\bullet+1_2} \\ \downarrow & & \downarrow & & \downarrow \\ X_\bullet & \longleftarrow & X_\bullet & \longrightarrow & X_\bullet \end{array}$$

encodes the left-module structure on the multiplication $\mu : X_1 \times X_1 \to X_1$ as described in Remark 3.3. The equivalence of (1) – (3) then follows from Proposition 10.9 and Proposition 10.8. The equivalence with (4) follows analogously, when one considers the right module structure on the multiplication (see Remark 3.3). □

10.3 Rigid 2-Segal Objects

Corollary 10.10 equips us with a precise criterion for identifying rigid 2-Segal objects. Our main goal in this section is to explore some of its implications.

Recall from Example 2.4 that for every morphism between finite groupoids $F : \mathcal{G} \to \mathcal{H}$ the convolution monoidal structure induced by the Čech-nerve $\check{C}(F)$ on $\mathsf{Fun}(\mathcal{G} \times_{\mathcal{H}} \mathcal{G}, \mathsf{Vect}_{\mathbb{K}})$ from Section 2 is rigid. Thus, it is reasonable to expect that Čech-nerves form examples of rigid 2-Segal objects. In fact, we show that this holds for a slightly more general class of simplicial objects:

Definition 10.3. [Lur09a, Def 6.1.2.7] Let $\mathcal{C}$ be an ∞-category with finite products and let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. $X_\bullet$ is called a *groupoid object*, if for every $[n] \geq 0$ and every partition $[n] = S_1 \cup S_2$ into sets S_1, S_2, s.t. the intersection $S_1 \cap S_2$ consists of one element s, the diagram

$$\begin{array}{ccc} X_n & \longrightarrow & X_{S_1} \\ \downarrow & & \downarrow \\ X_{S_2} & \longrightarrow & X_s \end{array} \tag{30}$$

is a pullback diagram.

Example 10.2. Let $\mathcal{C}$ be an ∞-category with finite limits, and let $\check{C}(F)_\bullet$ be the Čech-nerve of a morphism $F : C \to D$ a morphism in $\mathcal{C}$. Then, for every $[n] \geq 0$ and every partition $[n] = S_1 \cup S_2$, as above, the commutative diagram

$$\begin{array}{ccc} C^{\times_D n} & \longrightarrow & C^{\times_D S_1} \\ \downarrow & & \downarrow \\ C^{\times_D S_2} & \longrightarrow & D \end{array}$$

is a pullback. Hence, $\check{C}(F)_\bullet$ is a groupoid object.
Conversely, if $\mathcal{C}$ is an ∞-topos, e.g. the ∞-category $\mathcal{S}$, then every groupoid object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ is equivalent to the Čech-nerve of

$$X_0 \to \mathsf{colim}_{[n]\in\Delta^{\mathrm{op}}} X_n$$

the induced morphism into the colimit of $X_\bullet$.

Proposition 10.11. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a groupoid object in $\mathcal{C}$. Then $X_\bullet$ is locally rigid $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$.*

Proof. Using Corollary 10.10, it suffices to show that both maps in the span

$$X_{\{0,1,2\}} \times_{X_{\{1,2\}}} X_{\{1,2,3\}} \longleftarrow X_3 \longrightarrow X_{\{0,1,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}}$$

are equivalences. For the left morphism, this follows from the pasting lemma applied to the diagram:

$$\begin{array}{ccccc} X_{\{0,1\}} & \longleftarrow & X_{\{0,1,2\}} & \longleftarrow & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ X_{\{1\}} & \longleftarrow & X_{\{1,2\}} & \longleftarrow & X_{\{1,2,3\}} \end{array}$$

The statement for the opposite morphism follows analogously. □

Surprisingly, this class turns out to be the only class of rigid 2-Segal objects:

Theorem 10.12. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object in $\mathcal{C}$. Then $X_\bullet$ is locally rigid in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$ if and only if $X_\bullet$ is a groupoid object.*

Before we present a proof of this theorem, we first recall some general results about the relation between groupoids and 2-Segal objects. One of the main ingredients of the proof is the following lemma:

Lemma 10.13. *[Lur09b, Prop 1.1.8] Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ be a Segal space. The following are equivalent:*

(1) *$X_\bullet$ is a groupoid object*

(2) *the homotopy category $\mathsf{hX}_\bullet$ is a groupoid*

Proof. It follows from the definition of a groupoid object that (1) implies (2). We now show the converse direction. Since $X_\bullet$ is a Segal space, there exists a pullback diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{p} & X_{\{0,1\}} \times_{X_{\{0\}}} X_{\{0,2\}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} & \longrightarrow & X_{\{0,1\}} \times X_{\{2\}} \end{array}$$

Since $X_\bullet$ is a Segal space by assumption, the map q is an equivalence, and $X_\bullet$ is a groupoid object if and only if the morphism p is an equivalence. It therefore suffices to show that the composite map $p \circ q^{-1}$ is an equivalence in $\mathcal{S}_{/X_{\{0,1\}} \times X_{\{2\}}}$. This can be checked fiberwise over $X_{\{0,1\}} \times X_{\{2\}}$.
Therefore, it suffices to show that for every pair $(f, x) \in X_{\{0,1\}} \times X_{\{2\}}$ the induced map on fibers

$$(X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}}) \times_{X_{\{0,1\}} \times X_{\{2\}}} \{(f, x)\} \longrightarrow (X_{\{0,1\}} \times_{X_{\{0\}}} X_{\{0,2\}}) \times_{X_{\{0,1\}} \times X_{\{2\}}} \{(f, x)\}$$

is an equivalence of spaces. Denote by $y, z \in X_0$ the images of f under the face maps of $X_\bullet$ Unwinding the definitions, the above map identifies with the morphism

$$\mathsf{Map}_{\mathsf{hX}_\bullet}(z, x) \longrightarrow \mathsf{Map}_{\mathsf{hX}_\bullet}(y, x)$$

given by precomposition with the class of $[f] \in \mathsf{hX}_\bullet$. But this map is an equivalence if and only if $[f]$ is invertible in $\mathsf{hX}_\bullet$. But this is the case, since $\mathsf{hX}_\bullet$ is a groupoid. □

Lemma 10.14. *Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ be a Segal space. The following are equivalent:*

(1) *the homotopy category $\mathsf{hX}_\bullet$ is a groupoid*

(2) *the diagram*

$$\begin{array}{ccc} X_3 & \xrightarrow{\partial_2} & X_{\{0,1,3\}} \\ {\scriptstyle\partial_1}\downarrow & & \downarrow{\scriptstyle\partial_1} \\ X_{\{0,2,3\}} & \xrightarrow[\partial_1]{} & X_{\{0,3\}} \end{array}$$

is Cartesian

Proof. Using Lemma 10.13 it is easy to see that (1) ⇒ (2). Therefore, it remains to prove that (2) ⇒ (1). Let $f : x \to y \in X_1$ represent a morphism in $\mathsf{hX}_\bullet$. The image of f under the map

$$X_1 \xrightarrow{(s_1, s_0)} X_0 \times_{X_{\{1,3\}}} X_{\{0,1,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}} \times_{X_{\{0,2\}}} X_0 \simeq X_0 \times_{X_{\{1,3\}}} X_3 \times_{X_{\{0,2\}}} X_0 \qquad (31)$$

yields a 3-simplex $\sigma_f \in X_3$ with the property that

$$\partial_0 \sigma_f = \begin{array}{ccccc} & & x & & \\ & {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} & \\ y & & \xrightarrow{\mathrm{id}_y} & & y \end{array} \qquad (32)$$

and

$$\partial_3 \sigma_f = \begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ x & & \xrightarrow{\mathrm{id}_x} & & x \end{array} \qquad (33)$$

Hence g is an inverse of f in $\mathsf{hX}_\bullet$. This implies that the homotopy category $\mathsf{hX}_\bullet$ is a groupoid. □

Lemma 10.15. *Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. The following are equivalent:*

(1) *$X_\bullet$ is Segal*

(2) *$X_\bullet$ is 2-Segal and the map $X_2 \xrightarrow{(\partial_2, \partial_0)} X_1 \times_{X_0} X_1$ is an equivalence.*

Proof. It has been shown in [DK19, Prop.2.3.3] that (1) implies (2) . For the other direction, we show by induction on $n \geq 2$ that the Segal maps

$$X_n \to X_{\{0,1\}} \times_{X_{\{1\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

are equivalences. The base case is fulfilled by assumption. The inductive step is a consequence of the following chain of equivalences

$$\begin{aligned} X_n &\simeq X_{\{0,\dots,n-2,n\}} \times_{X_{\{n-2,n\}}} X_{\{n-2,n-1,n\}} \\ &\simeq X_{\{0,1\}} \times_{X_{\{1\}}} \dots . \times_{X_{\{n-2\}}} X_{\{n-2,n\}} \times_{X_{\{n-2,n\}}} X_{\{n-2,n-1\}} \times_{X_{\{n-1\}}} X_{\{n-1,n\}} \\ &\simeq X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \dots \times_{X_{\{n-1\}}} X_{\{n-1,n\}} \end{aligned}$$

where we used the 2-Segal condition in the first step and the inductive hypothesis in the second. □

After this preparation, we can now prove our main result:

Proof of Theorem 10.12. We have already seen in Proposition 10.11 that every groupoid object defines a locally rigid algebra in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$.
For the converse, we use that by Corollary 10.10 the legs of the span

$$X_{\{0,1,2\}} \times_{X_{\{1,2\}}} X_{\{1,2,3\}} \longleftarrow X_3 \longrightarrow X_{\{0,1,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}} \tag{34}$$

are equivalences. We first show that $X_\bullet$ satisfies the Segal conditions. For this purpose, we consider the retract diagram:

$$\begin{array}{ccccc} X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} & \xrightarrow{(s_1,s_0)} & X_{\{0,1,2\}} \times_{X_{1,2}} X_{\{1,2,3\}} & \xrightarrow{(\partial_1,\partial_0)} & X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \\ \uparrow{\scriptstyle (\partial_2,\partial_0)} & & \uparrow{\scriptstyle (\partial_3,\partial_0)} & & \uparrow{\scriptstyle (\partial_2,\partial_0)} \\ X_2 & \xrightarrow{s_1} & X_3 & \xrightarrow{\partial_1} & X_2 \end{array} \tag{35}$$

Since the middle arrow is an equivalence, as it is the left leg of the span in Equation (34), also the outside arrows are equivalences. It follows from Lemma 10.15 that $X_\bullet$ is Segal.
To show that $X_\bullet$ is a groupoid object, we use Lemma 10.13 together with Lemma 10.14. To use these results, we first reduce to the case that $X_\bullet$ is a Segal space. Note that since Yoneda reflects and preserves all limits, it suffices to show that the composite with the Yoneda embedding $よ \circ X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a groupoid object. We can therefore reduce to the case that $X_\bullet$ is a Segal object in a presheaf category $\mathcal{C} \simeq \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$. Further, note that the conditions imposed on a groupoid object can be checked pointwise. Hence, this reduces to the case $\mathcal{C} = \mathcal{S}$ and the result follows from Lemma 10.13 together with Lemma 10.14 □

Remark 10.3. In terms of the interpretation of rigid algebras as Frobenius algebras, the above theorem says that a 2-Segal object $X_\bullet$ with multiplication span

$$X_1 \times X_1 \xleftarrow{(\partial_2,\partial_0)} X_2 \xrightarrow{\partial_1} X_1$$

and comultiplication span given by the reversed span

$$X_1 \xleftarrow{\partial_1} X_2 \xrightarrow{(\partial_2,\partial_0)} X_1 \times X_1$$

defines a Frobenius algebra, if and only if $X_\bullet$ is a groupoid object. This interpretation was mentioned to the author by Joachim Kock in private conversation.

For an ∞-category $\mathcal{C}$ in which all groupoid objects are effective [Lur09a, 6.1.2.14], for instance, if $\mathcal{C}$ is an ∞-topos, we can even strengthen this result:

Corollary 10.16. *Let $\mathcal{C}$ be an ∞-category with finite limits in which all groupoid objects are effective and let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object. Then $X_\bullet$ is locally rigid if and only if it is equivalent to a Čech nerve.*

Proof. This follows from Theorem 10.12 together with the definition of an effective groupoid object in [Lur09a, Def. 6.1.2.14] □

Combining this result with our discussion in Section 10.1 we can further classify rigid algebras in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$.

Definition 10.4. Let $\mathcal{C}$ be an ∞-category with finite limits. A groupoid object $G_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$ is called an ∞-*group object* if $G_0 \simeq *$. If $\mathcal{C}$ is the ∞-category $\mathcal{S}$, we call an ∞-group object in $\mathcal{S}$ an ∞-group.

Remark 10.4. We will often abuse notation and denote an ∞-group object $G_\bullet$ in $\mathcal{C}$ by its 1-simplices $G := G_1$.

Corollary 10.17. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object s.t. $X_0 \simeq *$. Then the following are equivalent:*

(1) $X_\bullet$ is rigid in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^\otimes$

(2) $X_\bullet$ is rigid in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$

(3) $X_\bullet$ is a ∞-group object in $\mathcal{C}$

Remark 10.5. Let $\mathcal{C}$ be an ∞-category with finite limits. The ∞-category $\mathsf{Span}(\mathcal{C})^\otimes$ does not admit geometric realizations of simplicial objects. We can, therefore, not use the construction of Appendix C and define a Morita category of $\mathsf{Span}(\mathcal{C})^\otimes$.
A way to solve this issue for $\mathcal{C} \simeq \mathcal{S}$ is the ∞-category of spaces is the following. It is a consequence of Theorem 11.1 that the presheaf category construction extends to a symmetric monoidal $(\infty, 2)$-functor

$$\mathcal{P}(-) : \mathbb{S}\mathsf{pan}_2(\mathcal{S})^\otimes \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}$$

In particular, it follows from the universal property of the presheaf category

$$\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(X), \mathcal{P}(Y)) \simeq \mathsf{Fun}(X, \mathcal{P}(Y)) \simeq \mathsf{Fun}(X \times Y, \mathcal{S}) \simeq \mathcal{S}_{/X\times Y}$$

that the $(\infty, 2)$-functor $\mathcal{P}(-)$ is full faithful. The symmetric monoidal $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}})^\otimes$, therefore, forms a natural candidate for the Morita category of $\mathsf{Span}(\mathcal{C})^\otimes$. As we discuss in Section 11.3, we do not expect that the $(\infty, 2)$-functor $\mathcal{P}(-)$ can be extended to an $(\infty, 2)$-functor with source $2\mathbb{S}\mathsf{pan}(\mathcal{C})^\otimes$.

The results above explain our results from Section 2 and clarify why, to the author's knowledge, all known examples of rigid convolution monoidal structures arise from Čech-nerves. However, it is important to note that Theorem 10.12 does *not* imply that all rigid convolution monoidal structures have to originate from Čech-nerves. Indeed, we have only proven this in the case of 2-Segal sets.
In fact, since the linearization functor from Proposition 2.1 inverts certain 2-morphisms, it remains possible that rigid convolution monoidal structures could be induced by 2-Segal spaces that are not groupoid objects. Nevertheless, we can characterize precisely when this occurs:

Corollary 10.18. *Let $\mathcal{C}$ be an ∞-category with finite limits, and $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object in $\mathcal{C}$. For every symmetric monoidal $(\infty, 2)$-category $\mathbb{D}^{\otimes}$, and every symmetric monoidal $(\infty, 2)$-functor*

$$\mathsf{D} : 2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes} \to \mathbb{D}^{\otimes}$$

the algebra $\mathcal{D}(X_1) \in \mathsf{Alg}(\mathbb{D})$ is rigid if and only if $\mathcal{D}$ maps the 2-morphism

$$\begin{array}{ccccc} & \swarrow & X_{\{0,1,3\}} \times_{X_{\{0,3\}}} X_{\{0,2,3\}} & \searrow & \\ & & \uparrow & & \\ X_{\{0,2\}} \times X_{\{2,3\}} & & X_3 & & X_{\{0,1\}} \times X_{\{1,3\}} \\ & & \downarrow & & \\ & \nwarrow & X_{\{0,1,2\}} \times_{X_{\{1,2\}}} X_{\{1,2,3\}} & \nearrow & \end{array}$$

to an equivalence.

11 Convolution structures and TFTs

The goal of this thesis is to understand more systematically how examples of fusion categories arise from linearization functors. As discussed in Section 2, typical examples of fusion categories that arise this way are of the form Vect_G and $\mathsf{Rep}(G)$ for a finite group G. The goal of this section is to extend these examples to the realm of ∞-categories and describe the associated derived multi-fusion categories. This chapter forms a natural ∞-categorical enhancement of our discussion in Section 2.1, generalizing many of the results presented there.
In particular, we will perform the third and final step of our strategy by connecting our results from Section 5 and Section 10 with each other by using linearization $(\infty, 2)$-functors. To do so, we first note that our reformulation of rigidity in terms of locally rigid algebra objects in an $(\infty, 2)$-category implies that such objects are preserved by any symmetric monoidal $(\infty, 2)$-functors. Therefore, to construct rigid monoidal structures (and more generally fusion categories) from 2-Segal objects, we first need to understand symmetric monoidal $(\infty, 2)$-functors of the form

$$\mathsf{D}(-) : \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes} \to \mathbb{P}\mathsf{r}^{\otimes}_{\mathcal{V}}.$$

Variants of such $(\infty, 2)$-functors are ubiquitous in mathematics. They appear in the theory of quasi-coherent (and ind-coherent) sheaves in algebraic geometry [GR19, BZN09], the theory of mixed sheaves in geometric representation theory [HL23], as well as in the context of sheaf theory in topology [Vol21]. The reason for this is that they encode the 6-functor formalism of a sheaf theory. Consequently, every sheaf theory should induce a variant of such a functor (see [GR19, Man22]).

Let us elaborate on this relation. Such an $(\infty, 2)$-functor D maps every span of the form

$$\begin{array}{ccc} & C_0 & \\ \swarrow & & \searrow^{f} \\ C_0 & & C_1 \end{array} \qquad \longrightarrow \qquad f_! : \mathsf{D}(C_0) \to \mathsf{D}(C_1)$$

to a functor that describes the proper pushforward, and every span

$$\begin{array}{ccc} & C_0 & \\ {}^{f}\swarrow & & \searrow \\ C_1 & & C_0 \end{array} \qquad \longrightarrow \qquad f^* : \mathsf{D}(C_1) \to \mathsf{D}(C_0)$$

to a functor that describes the pullback. As D takes values in Pr^{L}, these functors admit right adjoints

$$f_! \dashv f^! \quad \text{and} \quad f^* \dashv f_*$$

that describe the exceptional pullback and ordinary pushforward. Since D is monoidal and every object in $\mathcal{C}$ admits the unique structure of a commutative algebra object in $(\mathcal{C}^{\mathrm{op}})^{\times} \subset \mathsf{Span}(\mathcal{C})^{\otimes}$, the functor D equips $\mathsf{D}(C)$ with a monoidal product

$$\otimes : \mathsf{D}(C) \otimes_{\mathcal{V}} \mathsf{D}(C) \simeq \mathsf{D}(C \times C) \xrightarrow{\Delta^*} \mathsf{D}(C)$$

that describes the pointwise product of sheaves. Again, as D takes values in Pr^{L}, we obtain an adjunction

$$\otimes \dashv \mathrm{hom}_{\mathcal{C}}(-,-)$$

and hence, all six functors. Further, many compatibilities between these six functors, such as the base-change identity, are encoded in the symmetric monoidal $(\infty, 2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$.
The main example of a linearization construction we are interested in is a generalized version of the linearization construction from Proposition 2.1 and describes the higher representation theory of groups. This construction has already been considered in different places in the theory of stable homotopy theory [CSY21, CCRY22]. Indeed, for every presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, we construct following [CCRY22] a symmetric monoidal $(\infty, 2)$-functor

$$\mathsf{Loc}_{\mathcal{V}}(-) : \mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes} \to \mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$$

from the $(\infty, 2)$-category of spans of spaces that associates to every space X the ∞-category $\mathsf{Fun}(X, \mathcal{V})$. The corresponding ∞-category describes $\mathcal{V}$-valued representations of the ∞-groupoid X. In particular, for a connected space $X \simeq \mathsf{B}G$ this ∞-category describes $\mathcal{V}$-valued representations of a topological group G. For the special case $\mathcal{V} \simeq \mathsf{Vect}_{\mathbb{C}}$, this recovers the category of groupoid representations and the linearization construction is the one from Proposition 2.1. Conversely, by using more general stable ∞-categories as an input, we obtain interesting derived examples by the same method.
The main examples of rigid monoidal structures that we describe are linearizations of ∞-groups in Section 11.2 and Čech-nerves in Section 11.3. These serve as generalizations of the category of G-graded vector spaces and categorified Hecke algebras from Examples 2.1 and 2.4. Afterwards, in Section 11.4, we analyse the conditions under which these categories form derived multi-fusion categories and relate them to the classical

examples of fusion categories studied in Section 2. Finally in Section 11.5, we compare the (relative) fully extended framed TFTs induced by these rigid convolution structures by computing the value on S^1.

11.1 Linearization Functors

In this Section, we generalize our notion of a *linearization construction* from Proposition 2.1. These concepts are known under various names in the literature, like a theory with transfer [DK19], or 6-functor formalism [GR19]. As in Section 2, we use linearization constructions for the construction of rigid convolution monoidal structures in the next Section. Let us start by defining what we mean by a linearization construction:

Definition 11.1. Let $\mathcal{C}$ be an ∞-category with finite limits and let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. A *linearization construction* on $\mathcal{C}$ is a symmetric monoidal $(\infty, 2)$-functor

$$\mathsf{D} : \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}. \tag{36}$$

Remark 11.1. It is possible to modify the definition of a linearization construction, by restricting the classes of morphisms that appear as the backward (resp. forward) pointing leg of a span. Using this more general notion, one could recover the examples from [GR19, BZN09, HL23, Vol21]. Since this modification is not necessary for the example of local systems, we stick to this less technical notion.

We denote by $\iota_{\mathsf{b}} : (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ and $\iota_{\mathsf{f}} : \mathcal{C}^{\otimes} \to \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ the symmetric monoidal inclusions that associate to a morphism $f : C_0 \to C_1$ the span where the forward (resp. backward) pointing leg of the span is the identity. In particular, D restricts to symmetric monoidal functors

$$\mathsf{D}_{\mathsf{b}} : (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}} \tag{37}$$

and

$$\mathsf{D}_{\mathsf{f}} : \mathcal{C}^{\otimes} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}} \tag{38}$$

For f a morphism in $\mathcal{C}$, we denote the functor $\mathsf{D}_{\mathsf{b}}(f)$ by f^* and the functor $\mathsf{D}_{\mathsf{f}}(f)$ by $f_!$. It follows from Proposition 10.3 that f^* forms a right adjoint of $f_!$. Further, since D_{f} takes values in $\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}}$ also f^* itself admits a right adjoint that we denote by $f_* := f^{*,\mathsf{R}}$.

Remark 11.2. For a general sheaf theory, one expects an adjunction $f_! \dashv f^*$ only for a certain class of morphisms f called *smooth morphisms*. One can embed this condition in our framework by considering linearization functors that are defined on a sub-$(\infty, 2)$-category of $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$ with fewer 2-morphisms.

Notation 11.1. Let D be a linearization construction with target $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$ and denote by $\mathbb{1}_{\mathcal{V}}$ the unit of $\mathcal{V}^{\otimes}$. For an object $C \in \mathcal{C}$ denote the unique morphism to the terminal object by $p_C : C \to *$. We denote the object $p_C^*(\mathbb{1}_{\mathcal{V}})$ by $\mathbb{1}_C$ and denote for every $\mathcal{F} \in \mathsf{D}(X)$ the object $p_!(\mathcal{F})$ by $C_*(X, \mathcal{F})$.

Next, we turn to the construction of our main example of a linearization functor. To do so, we use the universal property of the symmetric monoidal $(\infty, 2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$.

Definition 11.2. [Mac22, Def.4.4.3] Let $\mathcal{C}$ be an ∞-category with finite limits and let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category. We equip $\mathcal{C}^{\mathrm{op}}$ with the cocartesian monoidal structure $(\mathcal{C}^{\mathrm{op}})^{\amalg}$ [Lur17, Def.2.4.3.7]. We call a symmetric monoidal functor $F^{\otimes} : (\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathbb{D}^{\otimes}$ a *symmetric monoidal bivariant functor*, if

(1) for every morphism $f : c \to d$ the morphism $F(f)$ admits a left adjoint in $\mathbb{D}$.

(2) for every pullback square

$$\begin{array}{ccc} c_0 \times_{c_1} c_2 & \longrightarrow & c_0 \\ \downarrow & & \downarrow{\scriptstyle g} \\ c_2 & \longrightarrow & c_1 \end{array}$$

in $\mathcal{C}$, its image under F

$$\begin{array}{ccc} F(c_1) & \longrightarrow & F(c_0) \\ \downarrow & & \downarrow \\ F(c_2) & \longrightarrow & F(c_0 \times_{c_1} c_2) \end{array}$$

is vertically left adjointable in the sense of Definition 3.1.

Example 11.1. Let $\mathcal{C}$ be an ∞-category with finite limits. The symmetric monoidal functor $\iota_{\mathsf{b}} : (\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$, that associates to a morphism $f : C_0 \to C_1$ in $\mathcal{C}$ the span

$$C_0 \xleftarrow{f} C_1 \xrightarrow{\mathrm{id}_{C_1}} C_1$$

is an example of a symmetric monoidal bivariant functor.

In fact, the above example is the universal example:

Theorem 11.1. *[Mac22, Thm.4.4.6] Let $\mathcal{C}$ be an ∞-category with finite limits and let $\mathbb{D}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category. The symmetric monoidal functor $\iota : (\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathsf{Biv}^{\otimes}((\mathcal{C}^{\mathrm{op}})^{\amalg}, \mathbb{D}^{\otimes}) \simeq \mathsf{Fun}_2^{\otimes}(\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}, \mathbb{D}^{\otimes})$$

between the ∞-category of symmetric monoidal bivariant functors on the left and the ∞-category of symmetric monoidal $(\infty, 2)$-functors on the right.

We can now use this theorem to construct our main example of a linearization functor. The following construction follows [CCRY22, Sect.4.1]:

Example 11.2. We first construct the universal local systems functor. To this end, denote by

$$\mathsf{Loc}_{\mathcal{S}}(-)^{\otimes} : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathsf{Pr}^{\mathsf{L},\otimes}$$

the symmetric monoidal functor of *space valued local systems*. This functor associates to a space X the ∞-category $\mathsf{Fun}(X, \mathcal{S})$ of $\mathcal{S}$-valued local systems and to a morphism of spaces $f : X \to Y$ the pullback functor

$$f^* : \mathsf{Fun}(Y, \mathcal{S}) \to \mathsf{Fun}(X, \mathcal{S}).$$

Note that since $\mathcal{S}$ is cocomplete, the pullback functor f^* admits for any morphism of spaces $f : X \to Y$ a left-adjoint given by left Kan extension. Hence, it follows from [HL13, Thm.4.3.3] that this functor is bivariant. Since $\mathcal{S}$ is presentable, for any space X also the ∞-category $\mathsf{Fun}(X, \mathcal{S})$ is presentable. Consequently, it follows from Proposition 11.1 that the functor $\mathsf{Loc}_{\mathcal{S}}(-)^{\otimes}$ extends to a symmetric monoidal $(\infty, 2)$-functor

$$\mathsf{Loc}_{\mathcal{S}}(-)^{\otimes} : \mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes} \to \mathbb{P}\mathsf{r}^{L,\otimes}.$$

The above linearization functor is the universal local systems functor in the sense that every other local system functor factors through it:

Example 11.3. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and consider the symmetric monoidal $(\infty, 2)$-functor

$$\mathcal{V} \otimes_{\mathcal{S}} - : \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$$

from Proposition B.16. Since post composition with a symmetric monoidal $(\infty, 2)$-functor preserves symmetric monoidal bivariant theories, it follows that the composite functor:

$$\mathcal{V} \otimes_{\mathcal{S}} \mathsf{Loc}_{\mathcal{S}}(-) : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$$

is a symmetric monoidal bivariant functor. Hence, by Theorem 11.1 it extends to a linearization functor

$$\mathsf{Loc}_{\mathcal{S}}(-)^{\otimes} : \mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$$

We call the value of this functor on a space X the ∞-category of $\mathcal{V}$-valued local systems on X.

Example 11.4. Let $\mathcal{V}$ be a presentably symmetric monoidal $(\infty, 2)$-category. There exists a symmetric monoidal functor

$$\mathsf{Fun}(-, \mathcal{V}) : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathsf{Pr}^{\mathsf{L}\otimes}_{\mathcal{V}}$$

that associates to a space X the ∞-category $\mathsf{Fun}(X, \mathcal{V})$ and to a morphism of spaces $f : X \to Y$ the pullback functor f^*. It follows from [CCRY22, Prop.4.11] that

$$\mathcal{V} \otimes_{\mathcal{S}} \mathsf{Loc}_{\mathcal{S}}(-) \simeq \mathsf{Fun}(-, \mathcal{V}).$$

Hence, the two linearization constructions coincide.

Example 11.5. Let $\mathcal{C} \simeq \mathsf{Vect}_{\mathbb{C}}$ be the presentable 1-category of complex vector spaces. Denote by $\mathcal{S}^{\leq 1}$ the ∞-category of 1-truncated spaces [Lur09a, Def.5.5.6.1]. This category is equivalent to the $(2, 1)$-category of groupoids Grpd. Abusing notation, we denote by

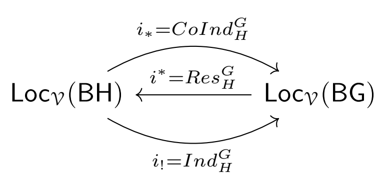
$$\mathsf{Loc}_{\mathbb{C}}(-) : \mathbb{S}\mathsf{pan}_2(\mathsf{Grpd})^{\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathsf{Vect}_{\mathbb{C}}}$$

the restriction of $\mathsf{Loc}_{\mathbb{C}}(-)^{\otimes}$ to the full subcategory of spans of groupoids. On the level of homotopy bicategories, this symmetric monoidal $(\infty, 2)$-functor induces the 2-functor from Proposition 2.1.

Example 11.6. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. We give an algebraic description of the data encoded in the functor $\mathsf{Loc}_{\mathcal{V}}(-)$. Consider, therefore, a pair consisting of a group and a subgroup $H \subset G$. The ∞-category

$$\mathsf{Loc}_{\mathcal{V}}(\mathsf{BG})$$

can be interpreted as the ∞-category of representations of G in $\mathcal{V}$. Moreover, the embedding $i : \mathsf{BH} \to \mathsf{BG}$ induces functors



which may be interpreted as functors of inducing, restricting, and coinducing representations along i. In particular, for the map $p : \mathsf{BG} \to *$, the corresponding functors

$$i_* \coloneqq C^*_{\mathcal{V}}(BG,-) : \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG}) \longrightarrow \mathcal{V} \quad \text{and} \quad i_! \coloneqq C^{\mathcal{V}}_*(\mathsf{BG},-) : \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG}) \longrightarrow \mathcal{V}$$

may be interpreted as $\mathcal{V}$-valued versions of the functor of group homology and cohomology of G-representations. In this example, the monoidal product

$$\otimes : \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG}) \otimes_{\mathcal{V}} \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG}) \to \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG})$$

describes the product of representations, and the associated internal $\hom_{\mathcal{V}}(-,-)$ describes morphisms of G-representations.

We also need a relative version of this construction. By the universal property of the Cocartesian monoidal structure every space X admits an essentially unique structure of a commutative algebra object in $(\mathcal{S}^{\mathrm{op}})^{\amalg}$ with multiplication given by

$$X \times X \leftarrow X : \Delta$$

Moreover, every morphism of spaces $f : X \to Y$ uniquely induces a morphism of commutative algebra objects and hence equips X with the structure of a Y-module. Hence, the symmetric monoidal functor

$$\mathsf{Loc}_{\mathcal{V}}(-) : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$$

induces for every space X a $\mathcal{V}$-linear presentably symmetric monoidal structure on the ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$. Unraveling definitions the monoidal product is given by the composite

$$\otimes : \mathsf{Loc}_{\mathcal{V}}(X) \otimes_{\mathcal{V}} \mathsf{Loc}_{\mathcal{V}}(X) \simeq \mathsf{Loc}_{\mathcal{V}}(X \times X) \xrightarrow{\Delta^*} \mathsf{Loc}_{\mathcal{V}}(X)$$

that maps a pair of functors $F, G : X \to \mathcal{V}$ to the composite

$$F \otimes G : X \xrightarrow{\ \Delta\ } X \times X \xrightarrow{F \times G} \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{V}$$

We call this monoidal structure the *pointwise monoidal structure* on $\mathsf{Loc}_{\mathcal{V}}(X)$. For every morphism of spaces $f : X \to Y$ it equips the ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ with the structure of a $\mathsf{Loc}_{\mathcal{V}}(Y)$-module with action functor given by

$$\mathsf{Loc}_{\mathcal{V}}(Y) \otimes_{\mathcal{V}} \mathsf{Loc}_{\mathcal{V}}(X) \simeq \mathsf{Loc}_{\mathcal{V}}(Y \times X) \xrightarrow{(f \times \mathrm{id}_X)^*} \mathsf{Loc}_{\mathcal{V}}(X \times X) \xrightarrow{\Delta^*} \mathsf{Loc}_{\mathcal{V}}(X).$$

Hence, it makes sense to consider the following construction:

Proposition 11.2. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and let Y be a space. The symmetric monoidal bivariant functor*

$$\mathsf{Loc}_{\mathcal{V}}(-)^{\otimes} : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathbb{P}\mathsf{r}^{\otimes,\mathsf{L}}_{\mathcal{V}}$$

induces a symmetric monoidal bivariant functor

$$\mathsf{Loc}_{\mathcal{V}}(-)^{\otimes}_{Y} : (\mathcal{S}^{\mathrm{op}}_{Y/})^{\amalg} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathsf{Loc}_{\mathcal{V}}(Y)}$$

that associates to any space X over Y the ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ with the above $\mathsf{Loc}_{\mathcal{V}}(Y)$-module structure.

Proof. It follows from the functoriality of the relative tensor product symmetric monoidal structure that the colimit preserving symmetric monoidal functor

$$\mathsf{Loc}_{\mathcal{V}}(-)^{\otimes} : (\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}}$$

induces a symmetric monoidal functor

$$\mathrm{Mod}(\mathsf{Loc}_{\mathcal{V}}(-))^{\otimes} : \mathrm{Mod}_Y(\mathcal{S}^{\mathrm{op}})^{\otimes} \to \mathrm{Mod}_{\mathsf{Loc}_{\mathcal{V}}(Y)}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$$

Since the symmetric monoidal structure on $\mathcal{S}^{\mathrm{op}}$ is cocartesian, it follows from [Lur17, Prop.2.4.3.9] that there exists an equivalence of ∞-categories

$$\mathrm{Mod}_Y(\mathcal{S}^{\mathrm{op}}) \simeq (\mathcal{S}_{/Y})^{\mathrm{op}}.$$

We equip $(\mathcal{S}_{/Y})^{\mathrm{op}}$ with the Cocartesian monoidal structure. To show that this equivalence extends to a symmetric monoidal equivalence, it suffices to show that the monoidal structure on $\mathrm{Mod}_Y(\mathcal{S}^{\mathrm{op}})$ is Cocartesian [Lur17, Def.2.4.0.1]. The unit of $\mathrm{Mod}_Y(\mathcal{S}^{\mathrm{op}})$ is given by Y as a module over itself. Under the above equivalence, this module gets mapped to the identity morphism $\mathrm{id}_Y : Y \to Y$, which is the initial object in $(\mathcal{S}_{/Y})^{\mathrm{op}}$.
It therefore suffices to show that for all objects $X, Z \in \mathrm{Mod}_Y(\mathcal{S}^{op})$ the relative tensor product $X \otimes_Y Z$ is equivalent to the pullback $X \times_Y Z$. Unraveling definitions, we obtain that the relative tensor product can be computed as the limit of the diagram

$$X \otimes_Y Z \longrightarrow X \times Z \rightrightarrows X \times Y \times Z \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots$$

It follows from the construction of the Bar-resolution that this limit is obtained as the right Kan extension of the truncated cosimplicial object

$$X \times Z \underset{(\mathrm{id}_X, f_X) \times \mathrm{id}_Z}{\overset{\mathrm{id}_X \times (f_Z, \mathrm{id}_Z)}{\rightrightarrows}} X \times Y \times Z$$

where (f_Z, id_Z) (resp. (id_X, f_Z)) describes the action functor of the left (resp. right) action of Y on Z (resp. X). By transitivity of right Kan extensions it suffices to calculate the limit of the truncated cosimplicial object, which is given by $X \times_Y Z$.
The symmetric monoidal functor obtained under this equivalence is denoted $\mathsf{Loc}_{\mathcal{V}}(-)^{\otimes}_Y$. It remains to show that this functor is a symmetric monoidal bivariant functor. We first establish condition (1). For a morphism

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Z \\ & {\scriptstyle g_X} \searrow \quad \swarrow {\scriptstyle g_Z} & \\ & Y & \end{array}$$

over Y, its image under $\mathsf{Loc}_{\mathcal{V}}(-)_Y$ is given by the $\mathsf{Loc}_{\mathcal{V}}(Y)$-module functor $f^* : \mathsf{Loc}_{\mathcal{V}}(X) \to \mathsf{Loc}_{\mathcal{V}}(Z)$. The underlying functor has a left adjoint $f_!$. It therefore suffices to show that the canonical lax $\mathsf{Loc}_{\mathcal{V}}(Y)$-linear structure on $f_!$ is strong [Lur17, Rem.7.3.2.9]. For $\mathcal{F} \in \mathsf{Loc}_{\mathcal{V}}(Y)$ and $\mathcal{G} \in \mathsf{Loc}_{\mathcal{V}}(X)$ this follows from the base change equivalence

$$f_!(g_X^*\mathcal{F} \otimes \mathcal{G}) \simeq f_!(f^* g_Z^* \mathcal{F} \otimes \mathcal{G}) \simeq g_Z^* \mathcal{F} \otimes f_! \mathcal{G}$$

and [Lur17, Rem.7.3.2.9]. We now show that this functor satisfies condition (2) of Definition 11.2. But this follows from the corresponding property for the functor $\mathsf{Loc}_{\mathcal{V}}(-)$ and the fact that pullbacks in slice categories are computed as pullbacks in the underlying category. □

Corollary 11.3. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and let Y be a space. There exists a symmetric monoidal $(\infty, 2)$-functor*

$$\mathsf{Loc}_{\mathcal{V}}(-)^{\otimes} : \mathbb{S}\mathsf{pan}_2(\mathcal{S}_{/Y})^{\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathsf{Loc}_{\mathcal{V}}(Y)}.$$

Construction 11.1. Let $g_X : X \to Y$ and $g_Z : Z \to Y$ be spaces over Y and let $\mathcal{F} \in \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z)$ be a local system on $X \times_Y Z$. The *integral transform functor* associated to $\mathcal{F}$ is the $\mathsf{Loc}_{\mathcal{V}}(Y)$-linear cocontinuous functor defined as the composite

$$\mathcal{I}_{\mathcal{F}} : \mathsf{Loc}_{\mathcal{V}}(X) \xrightarrow{\pi_X^*} \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z) \xrightarrow{\mathcal{F}\otimes -} \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z) \xrightarrow{\pi_{Z,!}} \mathsf{Loc}_{\mathcal{V}}(Z)$$

This functor maps to $\mathcal{G} \in \mathsf{Loc}_{\mathcal{V}}(X)$ the local system $\pi_{Z,!}(\mathcal{F} \otimes \pi_X^* \mathcal{G}) \in \mathsf{Loc}_{\mathcal{V}}(Z)$

Corollary 11.4. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and let X, Z be spaces over Y. There exists an equivalence of ∞-categories*

$$\mathsf{Loc}_{\mathcal{V}}(X \times_Y Z) \simeq \mathsf{Fun}_{\mathsf{Loc}_{\mathcal{V}}(Y)}(\mathsf{Loc}_{\mathcal{V}}(X), \mathsf{Loc}_{\mathcal{V}}(Z))$$

that associates to every local system $\mathcal{F} \in \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z)$ it's Integral transform $\mathcal{I}_{\mathcal{F}}$.

Proof. It follows from Proposition 11.2 that the ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ is self dual in $\mathsf{Pr}^{\mathsf{L},\otimes}_{\mathsf{Loc}_{\mathcal{V}}(Y)}$. In particular, we obtain an equivalence

$$\mathsf{Loc}_{\mathcal{V}}(X \times_Y Z) \simeq \mathsf{Loc}_{\mathcal{V}}(X) \otimes_{\mathsf{Loc}_{\mathcal{V}}(Y)} \mathsf{Loc}_{\mathcal{V}}(Z) \simeq \mathsf{Fun}_{\mathsf{Loc}_{\mathcal{V}}(Y)}(\mathsf{Loc}_{\mathcal{V}}(X), \mathsf{Loc}_{\mathcal{V}}(Z))$$

Unraveling this equivalence it maps a local system $\mathcal{F} \in \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z)$ to it's Integral transform $\mathcal{I}_{\mathcal{F}}$. □

Remark 11.3. This equivalence is a special feature of the linearization functor $\mathsf{Loc}_{\mathcal{V}}(-)$. For a general linearization functor, there may exist non equivalent integration kernels that induce the same linearization functor.

The above corollary will be of major importance in Section 11.4 when we try to understand the existence of adjoints for functors between categories of local systems. Before we move on, let us understand some examples of ∞-categories of local systems:

Example 11.7. Let G be a group and R a commutative ring. The ∞-category $\mathsf{Loc}_{\mathsf{rmod}_R}(BG)$ is equivalent to the 1-category $\mathsf{Fun}(BG, \mathsf{rmod}_R)$ of R-linear representations of the group G as studied in Section 2.

Example 11.8. For a commutative ring R the ∞-category $\mathsf{Loc}_{\mathcal{D}(R)}(\mathsf{BG})$ is given by the ∞-category $\mathsf{Fun}(\mathsf{BG}, \mathcal{D}(R))$. We call the ∞-category $\mathsf{Loc}_{\mathcal{D}(R)}(G)$ the ∞-category of *derived representations* of the group G.

A nice property of ∞-categories of local systems is that they are atomically generated:

Notation 11.2. For every space X, the category $\mathsf{Loc}_{\mathcal{V}}(X)$ is naturally $\mathcal{V}$-linear. We will, in general, abuse notation and denote the internal Hom for this $\mathsf{Loc}_{\mathcal{V}}(X)$-action by $\hom_X(-,-)$ instead of $\hom_{\mathsf{Loc}_{\mathcal{V}}(X)}(-,-)$.

Proposition 11.5. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. For every space X the $\mathcal{V}$-linear presentable ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ is atomically generated.*

Proof. Denote by $\pi_0(X)$ the set of connected components of X and denote by $\{x_i\} \subset \pi_0(X)$ a set of representatives. We denote by $j_i : * \to X$ the inclusion of the point x_i. Note, that for every $x_i \in \pi_0(X)$ the functor $j_{i,!}$ is an internal left adjoint in $\mathbb{P}\mathsf{r}^{\mathsf{L}}_{\mathcal{V}}$. Indeed, the right adjoint j_i^* preserves colimits and is $\mathcal{V}$-linear by the projection formula. Hence, it follows from Proposition 4.10 that for all $x_i \in \pi_0(X)$ the local system $j_{i,!}(\mathbb{1}_{\mathcal{V}})$ is atomic. We claim that they also generate.
For this, it suffices to show that a morphism $\alpha : \mathcal{L}_0 \to \mathcal{L}_1$ of local systems on X is an equivalence if and only if for every $x_i \in \pi_0(X)$ the morphism

$$\hom_X(j_{i,!}(\mathbb{1}_{\mathcal{V}}), \mathcal{L}_0) \to \hom_X(j_{i,!}(\mathbb{1}_{\mathcal{V}}), \mathcal{L}_1)$$

in $\mathcal{V}$ induced by α is an equivalence. By adjunction, this is the case if and only if for every i the morphism

$$\alpha_{x_i} : \mathcal{L}_0(x_i) \simeq \hom_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, j_i^*(\mathcal{L}_0)) \to \hom_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, j_i^*(\mathcal{L}_1)) \simeq \mathcal{L}_1(x_i)$$

is an equivalence. But this is the definition of an equivalence of local systems. □

In particular, if X is a connected space, the ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ admits an atomic generator. Hence, it follows from the generalized Schwede–Shipley Theorem 4.12:

Corollary 11.6. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category and X a connected space. There exists an algebra $A \in \mathsf{Alg}(\mathcal{V})$, whose underlying object are the chains on the based loop space of X with values in the trivial local system,*

$$C_*^{\mathcal{V}}(\Omega_*(X), \mathbb{1}_{\mathcal{V}}),$$

such that there exists an equivalence of ∞-categories

$$\mathsf{Loc}_{\mathcal{V}}(X) \simeq \mathsf{RMod}_A(\mathcal{V}).$$

Proof. It follows from the generalized Schwede–Shipley Theorem 4.12 that the $\mathcal{V}$-linear ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)$ is equivalent to the ∞-category of right modules in $\mathcal{V}$ over $\hom_{\mathsf{Loc}_{\mathcal{V}}(X)}(j_!(\mathbb{1}_{\mathcal{V}}), j_!(\mathbb{1}_{\mathcal{V}}))$. To finish the proof, it therefore suffices to identify the underlying object of the algebra $\hom_{\mathsf{Loc}_{\mathcal{V}}(X)}(j_!(\mathbb{1}_{\mathcal{V}}), j_!(\mathbb{1}_{\mathcal{V}}))$. Note that by adjunction there exists an equivalence

$$\hom_X(j_!(\mathbb{1}_{\mathcal{V}}), j_!(\mathbb{1}_{\mathcal{V}})) \simeq j^* j_! \, \mathbb{1}_{\mathcal{V}} \, .$$

Since the functor $\mathsf{Loc}_{\mathcal{V}}(-)$ is bivariant, it satisfies base-change. Applied to the pullback diagram

$$\begin{array}{ccc} \Omega(X) & \xrightarrow{p_{\Omega(X)}} & * \\ {\scriptstyle p_{\Omega(X)}}\downarrow & & \downarrow{\scriptstyle j} \\ * & \xrightarrow[j]{} & X \end{array}$$

this yields a chain of equivalences

$$\hom_X(j_!(\mathbb{1}_{\mathcal{V}}), j_!(\mathbb{1}_{\mathcal{V}})) \simeq j^* j_!(\mathbb{1}_{\mathcal{V}}) \simeq p_{\Omega(X),!} p^*_{\Omega(X)}(\mathbb{1}_{\mathcal{V}}) \simeq C_*(\Omega(X), \mathbb{1}_{\mathcal{V}}).$$

identifying the internal Hom with the object of chains on the based loop space of X with values in the trivial local system. □

Remark 11.4. Using a more elaborate argument, one can further show that the algebra structure on $\hom_X(j_!(\mathbb{1}_\mathcal{V}), j_!(\mathbb{1}_\mathcal{V}))$ coincides with the algebra structure on $C_*^\mathcal{V}(\Omega(X))$ induced by the concatenation of loops.

Example 11.9. Let G be a group and BG it's classifying space. The based loop space of BG is given by G and the equivalence of Corollary 11.6 induces the familiar equivalence

$$\mathsf{Rep}_R(G) \simeq \mathsf{Fun}(\mathsf{BG}, \mathsf{rmod}_R) \simeq \mathsf{rmod}_{R[G]}$$

between G-representations and modules over the group algebra $R[G]$.

Let us also consider the derived analog of the above example:

Example 11.10. Let G be a group and BG its associated one object groupoid. There exists an equivalence of R-linear presentable ∞-categories

$$\mathsf{Fun}(BG, \mathsf{D}(R)) \simeq \mathsf{RMod}_{R[G]}(\mathsf{D}(R)) \simeq \mathsf{D}(R[G])$$

where $\mathsf{D}(R[G])$ denotes the derived ∞-category of modules over the group algebra $R[G]$.

11.2 Categorified Group Algebras

Throughout this section, $\mathcal{C}$ denotes an ∞-category with finite limits. After we have discussed basic properties of linearization functors in the last section, we can now use them as in Section 2 for the construction of convolution monoidal structures:

Definition 11.3. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object and D a linearization functor on $\mathcal{C}$. We call the induced monoidal structure $*$ on the ∞-category $\mathsf{D}(X_1)$ obtained by linearizing the 2-Segal object $X_\bullet$ along D the *convolution monoidal structure* associated to $X_\bullet$ and the monoidal category $(\mathsf{D}(X_1), *) \in \mathsf{Alg}(\mathsf{Cat})$ a *convolution monoidal category.*

The convolution monoidal structure on $\mathsf{D}(X_1)$ arises via transporting the algebra structure on X_1 along the monoidal functor D. For example, the underlying tensor product functor of the convolution monoidal structure on $\mathsf{D}(X_1)$ is given by the composite:

$$* : \mathsf{D}(X_1) \otimes \mathsf{D}(X_1) \simeq \mathsf{D}(X_1 \times X_1) \xrightarrow{(\partial_2, \partial_0)^*} \mathsf{D}(X_2) \xrightarrow{\partial_{1,!}} \mathsf{D}(X_1)$$

Note that a linearization functor is a symmetric monoidal $(\infty, 2)$-functor. Therefore, it does not only preserve algebra objects, but also the property of being locally rigid. Hence, as a consequence of Corollary 10.17 we obtain the following:

Corollary 11.7. *Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a group object in $\mathcal{C}$ and*

$$\mathsf{D} : \mathbb{S}\mathsf{pan}_2(\mathcal{C})^\otimes \to \mathbb{P}\mathsf{r}_\mathcal{V}^{\mathsf{L},\otimes}$$

be a linearization functor on $\mathcal{C}$. Then the associated convolution monoidal ∞-category $(\mathsf{D}(X_1), *)$ is rigid in $\mathbb{P}\mathsf{r}_\mathcal{V}^{\mathsf{L},\otimes}$. In particular, $(\mathsf{D}(X_1), *)$ induces a fully extended 2-dimensional framed TFT with values in $\mathbb{M}\mathsf{or}(\mathsf{Pr}_\mathcal{V}^{\mathsf{L}})^\otimes$.

Proof. This is a consequence of Corollary 11.7 and Corollary 4.27. ☐

Remark 11.5. This result explains why categories of sheaves on group objects that appear at different places in mathematics are generally rigid monoidal [HL23][BZN09].

In the rest of this section, we analyse this rigid monoidal structure in more detail for the examples of the linearization functor $\mathsf{Loc}_{\mathcal{V}}(-)$ introduced in the last section. These ∞-categories form generalizations of the category of G-graded vector spaces as introduced in Example 2.5 in the context of higher algebra.

Definition 11.4. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ be an ∞-group object in spaces with multiplication $\mu : G \times G \to G$. We call the associated $\mathcal{V}$-linear monoidal ∞-category $(\mathsf{Loc}_{\mathcal{V}}(X_1), *)$ the *categorified $\mathcal{V}$-linear group algebra* and denote it by $\mathcal{V}[G]$. Its monoidal product functor is given by

$$* : \mathsf{Loc}_{\mathcal{V}}(G) \otimes_{\mathcal{V}} \mathsf{Loc}_{\mathcal{V}}(G) \simeq \mathsf{Loc}_{\mathcal{V}}(G \times G) \xrightarrow{\mu_!} \mathsf{Loc}_{\mathcal{V}}(G)$$

the functor of left Kan extension along the multiplication functor.

Example 11.11. Let G be a finite group. The nerve $\mathsf{N}(\mathsf{B}G)_\bullet$ defines a group object in spaces. For a field $\mathbb{K}$, the categorified $\mathsf{Vect}_{\mathbb{K}}$-linear group algebra $\mathsf{Vect}_{\mathbb{K}}[G]$ is the category of G-graded vector spaces from Example 2.5

Example 11.12. Let G be a finite group and $\mathbb{K}$ a field. Since the derived ∞-category construction commutes with finite products, we obtain an equivalence

$$\mathcal{D}(\mathbb{K})[G] \simeq \prod_{g \in G} \mathcal{D}(\mathbb{K}) \simeq \mathcal{D}(\mathsf{Vect}_G)$$

between $\mathcal{D}(\mathbb{K})[G]$ and the derived ∞-category of the abelian category Vect_G.

Example 11.13. More generally, we obtain a rigid convolution monoidal structure from every group object in spaces. This class, for example, includes the class of connective spectra [Seg74].

Let $F : \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a cocontinuous symmetric monoidal functor between presentably symmetric monoidal ∞-categories $\mathcal{V}^{\otimes}$ and $\mathcal{W}^{\otimes}$. Since the functor

$$- \otimes_{\mathcal{V}} \mathcal{W} : \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{V}} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathcal{W}}$$

is symmetric monoidal, it maps the categorified $\mathcal{V}$-linear group algebra to the $\mathcal{W}$-linear group algebra

$$\mathcal{V}[G] \otimes_{\mathcal{V}} \mathcal{W} \simeq (\mathcal{S}[G] \otimes_{\mathcal{S}} \mathcal{V}) \otimes_{\mathcal{V}} \otimes \mathcal{W} \to \mathcal{S}[G] \otimes_{\mathcal{S}} \mathcal{W} \simeq \mathcal{W}[G]$$

In particular, since $\mathcal{S}$ is the monoidal unit in $\mathsf{Pr}^{\mathsf{L},\otimes}$, there exists for every presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ a unique cocontinuous symmetric monoidal functor:

$$U : \mathcal{S}^{\otimes} \to \mathcal{V}^{\otimes}$$

which is determined by its value on a contractible space

$$U(*) \simeq \mathbb{1}_{\mathcal{V}} .$$

It follows that for every ∞-group G and every presentably symmetric monoidal ∞-category $\mathcal{V}$, we obtain a monoidal equivalence

$$\mathcal{V}[G] \simeq \mathcal{S}[G] \otimes_{\mathcal{S}} \mathcal{V}.$$

In particular, this has the following consequence on the associated fully extended TFT:

Theorem 11.8. *Let G be an ∞-group and $\mathcal{V}^{\otimes}$ a presentably symmetric monoidal ∞-category. Denote by*

$$\mathsf{Z}_{\mathcal{S}[G]} : \mathsf{Bord}_2^{\mathrm{fr},\otimes} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{S}})^{\otimes}$$

the fully extended 2d-TFT associated to the rigid presentably monoidal ∞-category $\mathcal{S}[G]$. Then the fully extended 2d-TFT associated to the $\mathcal{V}$-linear categorified group algebra $\mathcal{V}[G]$ factors as:

$$\mathsf{Z}_{\mathcal{V}[G]}(-) : \mathsf{Bord}_2^{\mathrm{fr},\otimes} \xrightarrow{\mathsf{Z}_{\mathcal{S}[G]}} \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{S}})^{\otimes} \xrightarrow{\mathbb{M}\mathsf{or}(\mathcal{V}\otimes_{\mathcal{S}}-)} \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}.$$

Proof. By the cobordism hypothesis, it suffices to compare the values of the fully extended TFT's on the positive framed point $+$. By our above analysis, there exists an equivalence

$$\mathsf{Z}_{\mathcal{V}[G]}(+) \simeq \mathcal{V}[G] \simeq \mathcal{S}[G] \otimes_{\mathcal{S}} \mathcal{V} \simeq (\mathbb{M}\mathsf{or}(- \otimes_{\mathcal{S}} \mathcal{V}) \circ \mathsf{Z}_{\mathcal{S}[G]})(+)$$

of $\mathcal{V}$-linear monoidal ∞-categories. This implies the claim. ☐

Consequently, to understand the fully extended $2d$-TFT associated to the $\mathcal{V}$-linear categorified group algebra $\mathcal{V}[G]$ for general $\mathcal{V}$, it suffices to consider the case $\mathcal{V} \simeq \mathcal{S}$. In particular, the TFT is independent of $\mathcal{V}$. As we will see in the next sections, this no longer holds if we consider categorified Hecke algebras or extensions to dimension 3. In Section 11.5, we will compute the value of this TFT on S^1.

11.3 Categorified Hecke Algebras

As observed in Section 10.3, the class of locally rigid 2-Segal objects in $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$ is slightly broader than in $\mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes}$. These additional objects are precisely the Čech-nerves.
In this section, we study the associated convolution monoidal structures which provide an ∞-categorical generalization of the categorified Hecke algebras studied in Section 2:

Example 11.14. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category, and $F : X \to Y$ a morphism of spaces. We denote by $\check{C}(F)_\bullet$ its Čech-nerve.
We call the associated $\mathcal{V}$-linear presentably monoidal ∞-category $(\mathsf{Loc}_{\mathcal{V}}(\check{C}(F)_1), *)$ the *categorified $\mathcal{V}$-linear Hecke algebra* and denote it by $\mathcal{H}\mathsf{e}(F)$. Its monoidal product functor is given by

$$\mathcal{H}\mathsf{e}(F) \otimes_{\mathcal{V}} \mathcal{H}\mathsf{e}(F) \xrightarrow{(\pi_{1,2}\times\pi_{2,3})^*} \mathcal{H}\mathsf{e}(X \times_Y X \times_Y X) \xrightarrow{\pi_{1,3,!}} \mathcal{H}\mathsf{e}(X \times_Y X)$$

first pulling back and then left Kan extension along the appropriate projections.

Since in our notation, linearization functors, like $\mathsf{Loc}_{\mathcal{V}}(-)$, are, in general, only defined on the sub-$(\infty, 2)$-category $\mathbb{S}\mathsf{pan}_2(\mathcal{C})$ of $2\mathbb{S}\mathsf{pan}(\mathcal{C})$, it does not follow as for categorified group algebras that the convolution monoidal structure on the categorified Hecke algebra is rigid. Instead, we need an extension of the notion of a linearization functor that is defined on the symmetric monoidal $(\infty, 2)$-category $2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes}$:

Definition 11.5. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal $(\infty, 2)$-category. An *extended linearization functor* is a symmetric monoidal $(\infty, 2)$-functor

$$\mathsf{D}_{\mathrm{ex}} : 2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes} \to \mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}_{\mathcal{V}}$$

from the $(\infty, 2)$-category of 2-spans in $\mathcal{C}$. Further, we call D_{ex} an *extension* of the linearization functor D defined as the composite

$$\mathsf{D} : \mathbb{S}\mathsf{pan}_2(\mathcal{C})^{\otimes} \longhookrightarrow 2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes} \xrightarrow{\mathsf{D}_{\mathrm{ex}}} \mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$$

Corollary 11.9. *Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category, and $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ a groupoid object in $\mathcal{C}$. For every extended linearization functor*

$$\mathsf{D}_{\mathrm{ex}} : 2\mathbb{S}\mathsf{pan}(\mathcal{C})^{\otimes} \to \mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$$

is the associated convolution monoidal structure $(\mathsf{D}(X_1), *) \in \mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ rigid.

Example 11.15. Morton [Mor11] explicitly constructs a symmetric monoidal 2-functor

$$\mathsf{Loc}_{\mathsf{Vect}_{\mathbb{C}}}(-) : 2\mathbb{S}\mathsf{pan}(\mathsf{Grpd}^{f})^{\otimes} \to \mathbb{pr}_{\mathsf{Vect}_{\mathbb{C}}}^{\mathsf{L},\otimes}$$

to the 2-category of $\mathbb{C}$-linear presentable 1-categories that extends the symmetric monoidal 2-functor from Example 11.5. It follows that for every functor between finite groupoids $F : \mathcal{G} \to \mathcal{H}$ the associated categorified Hecke algebra $\mathcal{H}\mathsf{e}_{\mathbb{C}}(F)$ from Example 2.4 is rigid in $\mathbb{pr}_{\mathbb{C}}^{\mathsf{L}}$. In this case, we have also shown this by different methods in Proposition 2.4.

To the author's knowledge, the construction in [Mor11] is the only example of an extended linearization functor that is documented in the literature and we will not construct a new example in this text. However, let us try to understand, at least in principle, how such functors can arise. The main ideas for this discussion are taken from [HL13].

Recall that for every span

$$\begin{array}{ccccc} & & C_0 & & \\ & \swarrow & & \searrow^{f} & \\ C_0 & & & & C_1 \end{array}$$

viewed as a morphism in the $(\infty, 2)$-category $2\mathbb{S}\mathsf{pan}(\mathcal{C})$, the reversed span

$$\begin{array}{ccccc} & & C_0 & & \\ & {}^{f}\swarrow & & \searrow & \\ C_1 & & & & C_0 \end{array}$$

is an ambidextrous adjoint[28]. Therefore, an extended linearization functor D_{ex} has to associate to any morphism $f : C_0 \to C_1$ in $\mathcal{C}$ a natural equivalence

$$\mathrm{Nm}_f : f_! \xrightarrow{\simeq} f_*.$$

Equivalently, it has to associate to f a natural transformation

$$\mu_f : \mathrm{id}_{\mathsf{D}(C_1)} \Rightarrow f_! f^*$$

that is the unit of an adjunction between $f_!$ and f^*. The datum of the natural isomorphism Nm_f is called a

[28]This means it is simultaneously a left and a right adjoint.

Norm map. Conversely, for every non-extended linearization functor D, we can construct from a 2-morphism

$$\begin{array}{ccccc} & & B_0 & & \\ & {}^{g_0}\swarrow & \uparrow{\scriptstyle u} & \searrow^{f_0} & \\ A & & B_2 & & C \\ & {}_{g_1}\nwarrow & \downarrow{\scriptstyle v} & \nearrow_{g_1} & \\ & & B_1 & & \end{array}$$

in $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ and a natural transformation

$$\mu_u : \mathrm{id}_{B_0} \Rightarrow u_! u^*,$$

a natural transformation:

$$f_{0,!}g_0^* \xRightarrow{\mu_u} f_{0,!}(u_! u^*)g_0^* \simeq f_{1,!}(v_! v^*)g_1^* \Longrightarrow f_{1,!}g_1^*$$

from the image of the upper span under D to the image of the lower span under D. The idea is to view this natural transformation as the value of an extension of D to $2\mathbb{S}\mathsf{pan}(\mathcal{C})$ constructed from the original 2-functor D and the natural transformation μ_u.
To understand extended linearization functors, it is therefore essential to understand Norm maps. A canonical collection of Norm maps for the linearization functor $\mathsf{Loc}_{\mathcal{V}}(-)$ has been constructed [HL13, Har20]. Since we need it in the following, we recall the basics of the construction here:

Definition 11.6. [HL13, Def.4.4.1] Let X be a space. We say that X is a *finite n-type* if the following conditions are satisfied:

(1) The space X is n-truncated.

(2) For every point $x \in X$ and every integer m the set $\pi_m(X, x)$ is finite and $\pi_0(X)$ is a finite set.

A space X is called *π-finite*, if it is a finite n-type for some n. The *∞-category of n-finite types (resp. π-finite spaces)* $\mathcal{S}^n$ (resp. $\mathcal{S}^\pi$) is the full subcategory of $\mathcal{S}$ generated by finite n-types (resp. π-finite spaces). More generally a morphism of spaces $f : X \to Y$ is called *n-truncated* if all its fibers are n-truncated.

For every presentable ∞-category $\mathcal{V}$, the authors of [HL13, Constr.4.1.8] construct by induction on n a subset of the set of n-truncated morphisms $f : X \to Y$, called $\mathcal{V}$-ambidextrous morphisms. For these morphisms, they construct a collection of Norm isomorphisms $\mathrm{Nm}_f : f_! \to f_*$. In particular, we call a finite n-type X $\mathcal{V}$-ambidextrous if the projection map $p_X : X \to *$ is $\mathcal{V}$-ambidextrous. Conversely, the following holds:

Proposition 11.10. *[HL13, Prop.4.3.5] Let $\mathcal{V}$ be a presentable ∞-category, and let $f : X \to Y$ be a morphism of spaces. Then f is $\mathcal{V}$-ambidextrous if and only if f is n-truncated and each fiber of f is $\mathcal{V}$-ambidextrous.*

As we see in the following examples, the notion of $\mathcal{V}$-ambidexterity highly depends on the presentable ∞-category $\mathcal{V}$. To quantify this behavior, we use the following definition:

Definition 11.7. [HL13, Def.4.4.2] Let $\mathcal{V}$ be a presentable ∞-category, and let $n \geq -2$ be an integer. We say that $\mathcal{V}$ is *n-semiadditive* if every finite n-type is $\mathcal{V}$-ambidextrous. It is called ∞-semiadditive if it is n-semiadditive for all $n \in \mathbb{N}$.

Let us study this notion in some examples:

Example 11.16. By definition a space is (-2)-truncated if it is contractible. Consequently every presentable ∞-category $\mathcal{V}$ is (-2)-*semiadditive*.

Example 11.17. Let $\mathcal{V}$ be (-2)-semiadditive. A space X is a *finite* (-1)-*type* if it is either empty or contractible. For the $\emptyset$ the norm map is given by the unique map

$$p_{\emptyset,!}(*) \simeq \emptyset \to * \simeq p_{\emptyset,*}(*)$$

from the initial object to the final object of $\mathcal{V}$. It follows that $\mathcal{V}$ is (-1)-*semiadditive* if and only if it is pointed. For example, the ∞-category $\mathcal{S}$ is not pointed, but the ∞-category $\mathcal{S}_*$ of pointed spaces is.

Example 11.18. Let $\mathcal{V}$ be (-1)-semiadditive. Note that, since $\mathcal{V}$ is pointed, there exists for every pair of objects $C, D \in \mathcal{V}$ a zero map between these defined as the composite

$$C \to 0 \to D.$$

A space X is a *finite* 0-*type* if it is a finite set. A $\mathcal{V}$-valued local system $F : X \to \mathcal{V}$ on X is determined by an X-indexed collection of objects $\{V_x\}_{x\in X}$. The Norm map is defined as the map

$$\mathrm{Nm}_X : \coprod_{x\in X} V_x \to \prod_{x\in X} V_x$$

whose component $(\mathrm{Nm}_X)_{x,y} : V_x \to V_y$ are given by id_{V_x} if $x = y$ and the zero map else. An ∞-category satisfying this condition is called *semiadditive*. Examples include all presentable additive categories and stable ∞-categories, but not the ∞-category $\mathcal{S}_*$.

Example 11.19. Let $\mathcal{V}$ be 0-semiadditve. A *finite* 1-*type* is equivalent to a finite groupoid. In particular, if it is connected, it is equivalent to BG for a finite group G. The norm map then defines for every G-representation $V : \mathsf{BG} \to \mathcal{V}$ in $\mathcal{V}$ a map

$$\mathrm{Nm}_{\mathsf{BG}} : \mathsf{colim}_{\mathsf{BG}} V := V^{\mathsf{BG}} \to V_{\mathsf{BG}} =: \mathsf{lim}_{\mathsf{BG}} V$$

from the homotopy coinvariants V^{BG} to the homotopy invariants V_{BG}.

Example 11.20. Let R be a ring and $\mathcal{V} \simeq \mathsf{rmod}_R$ be the category of R-modules. In this case, the homotopy (co)invariants of a G-representation V coincide with the classical (co)invariants and the norm map is defined for every finite group G as

$$\mathrm{nm}_{\mathsf{BG}} : V^{\mathsf{BG}} \to V_{\mathsf{BG}}, \quad [v] \mapsto \sum_{g\in G} g.v.$$

This map admits an inverse if and only if multiplication by $|G|$ is invertible. In this case, the inverse is given by multiplication by $|G|^{-1}$. It follows that the category rmod_R is 1-semiadditive if and only if for every finite group $|G|$ multiplication by $|G|$ is invertible in R.

Example 11.21. Let R be a ring and let $\mathcal{D}(R)$ be its derived ∞-category. In this case, the homotopy coinvariants are given by group homology, and the homotopy invariants are given by group cohomology. For an ordinary G-equivariant R-module V, the norm map is given by the composite

$$\mathrm{Nm}_{\mathsf{BG}} : C_*(V, G) \to V^{\mathsf{BG}} \xrightarrow{\mathrm{nm}_{\mathsf{BG}}} V_{\mathsf{BG}} \to C^*(V, G)$$

where $\mathrm{nm}_{\mathsf{BG}}$ denotes the norm map from Example 11.20 [Lur17, Rem.6.1.6.23]. The ∞-category $\mathcal{D}(R)$ is 1-semiadditive if for every finite group G multiplication by $|G|$ is invertible in R.

More generally the following holds for stable ∞-categories:

Proposition 11.11. *Let R be an $\mathbb{E}_1$-ring spectrum, s.t. $\pi_0(R)$ is a $\mathbb{Q}$-vector space and let $\mathcal{V}$ be an R-linear presentable ∞-category. Then $\mathcal{V}$ is ∞-semiadditive.*

Proof. Since $\mathcal{V}$ is R-linear, it follows for all objects $C \in \mathcal{V}$ the mapping spectrum $\hom_{\mathcal{V}}(C, C)$ is an R-algebra. In particular $\pi_0 \hom_{\mathcal{V}}(C, C) := \mathsf{Ext}_{\mathcal{V}}(C, C)$ is an $\pi_0(R)$-algebra. The claim then follows from [HL13, Cor.4.4.2.1]. ☐

Example 11.22. It follows that for every $\mathbb{C}$-linear abelian category $\mathcal{A}$, its derived ∞-category $\mathcal{D}(\mathcal{A})$ is ∞-semiadditive.

As a consequence of the above discussion, we note that while (-1)-semiadditivity and 0-semiadditivity force us to work in an algebraic category, 1- and higher semiadditivity impose strong restrictions on the characteristic of the ring we are considering. Indeed, whereas every $\mathbb{Q}$-linear presentable ∞-category is ∞-semiadditive, this is never the case for $\mathbb{F}_q$-linear ∞-categories. The defects are precisely those finite groups $|G|$, whose characteristic divides the group order.

Remark 11.6. The theory of higher semi-additivity has been developed in [HL13] to describe the additivity phenomena in chromatic homotopy theory. The main example considered in [HL13] is the presentable stable ∞-category $\mathcal{S}\mathrm{p}_{K(n)}$ of $K(n)$-local spectra [HL13, Def.2.1.13], where $K(n)$ denotes the n-th Morava K-theory spectrum.
While this ∞-category admits a symmetric monoidal structure induced from the smash product of spectra, it is not rigid and thus it is not relevant to our discussion of derived multi-fusion categories in the next Section.

We summarize our discussion in the following conjecture, generalizing our results from Proposition 2.4:

Conjecture 11.12. *[HL13, Rem.4.2.5] Let $\mathcal{V}^{\otimes}$ be an n-semiadditive presentably symmetric monoidal ∞-category. Then the linearization functor $\mathsf{Loc}_{\mathcal{V}}(-)$ extends to a symmetric monoidal $(\infty, 2)$-functor*

$$\mathsf{Loc}_{\mathcal{V}}(-) : 2\mathbb{S}\mathsf{pan}(\mathcal{S}^n)^{\otimes} \to \mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$$

In particular for every morphism of n-finite spaces $F : X \to Y$ the categorified $\mathcal{V}$-linear Hecke algebra $\mathcal{H}\mathsf{e}_{\mathcal{V}}(F)$ is rigid.

Remark 11.7. In [BZN09] and [HL23], the authors construct Hecke-type convolution monoidal structures using linearization functors arising in algebraic and mixed geometry, respectively. Furthermore, they show that these monoidal structures are rigid by explicitly constructing duals, as we did in Proposition 2.4. Since their argument only uses the properties of the non-extended linearization combined with the ambidexterity of certain adjunctions, we expect that these examples are also induced from an extended linearization functor.

11.4 Derived Fusion Categories from Convolution

In this section, we analyze under which conditions the rigid monoidal structures on categories of local systems discussed in the last section form examples of derived multi-fusion categories. This heavily depends on the notion of semiadditivity discussed in the last Section. As in Section 5.4, we work linearly over an idempotent complete rigid symmetric monoidal small stable ∞-category $\mathcal{E}^{\otimes}$.

Proposition 11.13. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditive rigid symmetric monoidal stable ∞-category. For every m-finite space X the pointwise monoidal structure on $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)^{\otimes}$ is rigid in $\mathbb{P}\mathsf{r}_{\mathcal{E}}^{\mathsf{L},\otimes}$.*

Proof. It follows from Proposition 11.5 that the ∞-category $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ is dualizable. Under the equivalence $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X) \otimes_{\mathcal{E}} \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X) \simeq \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X \times X)$ the monoidal product functor $\otimes$ identifies with

$$\Delta^* : \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X \times X) \to \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$$

Its right adjoint is given by right Kan extension Δ_*. Since X is $\mathcal{E}$-ambidextrous by assumption, it follows that Δ_* preserves colimits. It remains to show that it is a $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$-bimodule functor. But this follows from the projection formula for right Kan extension [CSY21, Lem.3.1.2]

$$\Delta_*(\mathcal{F} \otimes \mathcal{G}) \simeq \Delta_*(\Delta^* \pi_1^* \mathcal{F} \otimes \mathcal{G}) \simeq \pi_1^* \mathcal{F} \otimes \Delta_* \mathcal{G}$$

where $\mathcal{F}, \mathcal{G} \in \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$. The proof for the right action is analogous. By a similar argument, it follows that the right adjoint p_* of the unit $p^* : \mathrm{Ind}(\mathcal{E}) \to \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ is cocontinuous and $\mathrm{Ind}(\mathcal{E})$-linear, where p denotes the unique map $p : X \to *$. □

Remark 11.8. Since the pointwise monoidal structure on $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ is closed with internal Hom denoted $\mathcal{H}\mathsf{om}_X(-,-)$, it follows that for a compact object $\mathcal{F} \in \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ the dual is given by

$$\mathcal{H}\mathsf{om}_X(\mathcal{F}, p_X^*(\mathbb{1}_{\mathcal{E}})).$$

For $X \simeq \mathsf{BG}$, the local system $p_{\mathsf{BG}}^*(\mathbb{1}_{\mathcal{E}})$ corresponds to the trivial G-representation and we recover the formula for the dual from Example 2.2.

Lemma 11.14. *Let $\mathcal{E}^{\otimes}$ be a rigid symmetric monoidal stable ∞-category and X a space. Let $i_x : * \to X$ be a point in X and denote by $p_X : X \to *$ the unique morphism. For all $\mathcal{F}, \mathcal{G} \in \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ there exists equivalences*

$$\hom_X(\mathcal{F}, \mathcal{G}) \simeq p_{X,*}\mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G}) \quad \textit{and} \quad \hom_{\mathcal{E}}(\mathcal{F}(x), \mathcal{G}(x)) \simeq i_x^* \mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})$$

in $\mathcal{V}$.

Proof. The first claim follows from the chain of equivalences

$$\begin{aligned} \hom_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, p_{X,*}\mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})) &\simeq \hom_X(p_X^*(\mathbb{1}_{\mathcal{E}}), \mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})) \\ &\simeq \hom_X(p_X^*(\mathbb{1}_{\mathcal{E}}) \otimes \mathcal{F}, \mathcal{G}) \\ &\simeq \hom_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, \hom_X(\mathcal{F}, \mathcal{G})), \end{aligned}$$

where we have used that all functors are $\mathrm{Ind}(\mathcal{E})$-linear and the universal property of the internal Hom for the $\mathrm{Ind}(\mathcal{E})$-action on $\mathsf{Loc}_{\mathcal{E}}(X)$. The second claim follows similarly from the chain of equivalences:

$$\begin{aligned} \hom_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, i_x^*\mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})) &\simeq \hom_X(i_{x,!}(\mathbb{1}_{\mathcal{E}}), \mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})) \\ &\simeq \hom_X(i_{x,!}(\mathbb{1}_{\mathcal{E}}) \otimes \mathcal{F}, \mathcal{G}) \\ &\simeq \hom_X(i_{x,!}(\mathbb{1}_{\mathcal{E}} \otimes \mathcal{F}(x)), \mathcal{G}) \\ &\simeq \hom_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, \hom_{\mathcal{E}}(\mathcal{F}(x), \mathcal{G}(x))). \end{aligned}$$

□

Corollary 11.15. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditve, rigid symmetric monoidal stable ∞-category. For every m-finite space X, there exists an equivalence of ∞-categories*

$$\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)^{\mathrm{c}} \simeq \mathsf{Fun}(X, \mathcal{E}) \coloneqq \mathsf{Loc}_{\mathcal{E}}(X)$$

between the full subcategory of compact objects in $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)^{\mathrm{c}}$ *and the ∞-categories of functors* $\mathsf{Fun}(X, \mathcal{E})$.

Proof. Since $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ is atomically rigid, a local system is compact if and only if it is dualizable. Since for every point $i_x : * \to X$ the functor

$$i_x^* : \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X) \to \mathrm{Ind}(\mathcal{E})$$

is symmetric monoidal, it follows that every dualizable local system is pointwise dualizable. In particular, since $\mathrm{Ind}(\mathcal{E})^{\otimes}$ is atomically rigid, every compact local system lies in $\mathsf{Fun}(X, \mathcal{E})$. This shows that $\mathsf{Loc}_{\mathcal{E}}(X)^{\mathrm{c}} \subset \mathsf{Fun}(X, \mathcal{E})$.
It remains to show that every object in $\mathsf{Fun}(X, \mathcal{E})$ is dualizable. For this, it suffices to show that for every $\mathcal{F} \in \mathsf{Fun}(X, \mathcal{E})$ the canonical map

$$\eta : \mathcal{F} \otimes \mathcal{H}\mathsf{om}_X(\mathcal{F}, p_X^*(\mathbb{1}_{\mathcal{E}})) \to \mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{F})$$

is an equivalence. This can be checked pointwise. But for every point $i_x : * \to X$ in X, the map $i_x^*\eta$ identifies with the canonical evaluation map

$$\eta_x : \mathcal{F}(x) \otimes \hom_{\mathcal{E}}(\mathcal{F}(x), \mathbb{1}_{\mathcal{E}}) \to \hom_{\mathcal{E}}(\mathcal{F}(x), \mathcal{F}(x))$$

in $\mathcal{E}$. This map is an equivalence since $\mathcal{F}$ is pointwise dualizable. □

As a consequence, under the conditions of the Proposition 11.13, the pointwise monoidal structure on $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)^{\otimes}$ preserves the full subcategory $\mathsf{Loc}_{\mathcal{E}}(X)$. Hence, it restricts to a symmetric monoidal structure on $\mathsf{Loc}_{\mathcal{E}}(X)^{\otimes}$. It is therefore natural to ask under which conditions the monoidal stable ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)^{\otimes}$ forms a derived multi-fusion category.
For this, we need to determine under which conditions on the space X the $\mathcal{E}$-linear ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)$ of $\mathcal{E}$-valued local systems is further $\mathcal{E}$-smooth and $\mathcal{E}$-proper. A criterion for smoothness and properness for dg-categories of quasi-coherent sheaves has been presented by Orlov [Orl16]. As his criterion only depends on the geometry of the underlying scheme, we adopt his approach to our setup:

Proposition 11.16. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditive rigid symmetric monoidal stable ∞-category. For every m-finite space X the $\mathcal{E}$-linear stable ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)$ is $\mathcal{E}$-proper.*

Proof. To show that $\mathsf{Loc}_{\mathcal{E}}(X)$ is $\mathcal{E}$-proper, we need to show that for all compact objects $\mathcal{F}, \mathcal{G} \in \mathsf{Loc}_{\mathcal{E}}(X)$ the internal Hom object $\hom_{\mathcal{E}}(\mathcal{F}, \mathcal{G})$ is compact in $\mathrm{Ind}(\mathcal{E})$. Since $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$ is rigid by Proposition 11.13, it follows that for every pair of compact objects $\mathcal{F}, \mathcal{G}$ the object $\mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G})$ is compact. The claim follows from the equivalence

$$\hom_{\mathcal{E}}(\mathcal{F}, \mathcal{G}) \simeq p_{X,*}\mathcal{H}\mathsf{om}_X(\mathcal{F}, \mathcal{G}) \tag{39}$$

and the fact that since X is $\mathcal{E}$-ambidextrous the functor $p_{X,*} \simeq p_{X,!}$ preserves compact objects as its right adjoint p_X^* is cocontinuous. □

Proposition 11.17. *Let $\mathcal{E}$ be a rigid symmetric monoidal stable ∞-category and X be a space. Then $\mathsf{Loc}_{\mathcal{E}}(X)$ is smooth if and only if the local system*

$$\Delta_! p_X^*(\mathbb{1}_{\mathcal{E}}) \in \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$$

is compact.

Proof. By Definition 4.5 the $\mathcal{E}$-linear ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)$ is $\mathcal{E}$-smooth, if and only if the identity functor is a compact object in

$$\mathrm{id} \in \mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X), \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)).$$

By Corollary 11.4 the Integral transform construction induces an equivalence

$$\mathsf{Fun}^{\mathsf{L}}_{\mathrm{Ind}(\mathcal{E})}(\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X), \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)) \simeq \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X \times X)$$

between the ∞-category of functors and the ∞-category of integration kernels. Therefore, it suffices to show that the Integral transform of $\Delta_!(\mathbb{1}_X)$ is naturally equivalent to the identity. But it follows from the projection formula that we have an equivalence

$$\pi_{2,!}(\Delta_! \mathbb{1}_X \otimes \pi_1^* -) \simeq \pi_{2,!}\Delta_!(p_X^* \mathbb{1}_{\mathcal{E}} \otimes \Delta^* \pi_1^* -) \simeq \mathbb{1}_X \otimes - \simeq \mathrm{id}_{\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)}.$$

Hence, id is compact. □

Corollary 11.18. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditive rigid symmetric monoidal stable ∞-category. For every m-finite space the $\mathcal{E}$-linear stable ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper.*

Proof. Proposition 11.16 already shows that the ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)$ is $\mathcal{E}$-compact. It remains to establish $\mathcal{E}$-smooth. Since X is m-finite and $\mathcal{E}$ is rigid, it follows that

$$\mathbb{1}_X = p_X^*(\mathbb{1}_{\mathcal{E}})$$

is compact in $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$. The claim then follows from the fact that the functor $\Delta_!$ preserves compact objects as it admits a cocontinuous right adjoint. □

A similar analysis for additive instead of stable ∞-categories yields:

Corollary 11.19. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditive rigid symmetric monoidal additive 1-category. For every m-finite space X the $\mathcal{E}$-linear additive category*

$$\mathsf{Loc}_{\mathcal{E}}(X) \simeq \mathsf{Loc}_{\mathcal{E}}(\tau_{\leq 1} X) \in \mathsf{add}^{\amalg}_{\mathcal{E}}$$

is $\mathcal{E}$-smooth and $\mathcal{E}$-proper.

It remains to understand the canonical algebra $\mathcal{F}_X$ for the pointwise monoidal structure on $\mathsf{Loc}_{\mathcal{E}}(X)$.

Proposition 11.20. *Let $\mathcal{E}$ be an m-semiadditive rigid symmetric monoidal stable ∞-category and let X be an m-finite space. Then, the canonical algebra $\mathcal{F}_X \in \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X \times X)$ is compact.*

Proof. Recall that the canonical algebra is defined as $\mu^R(\mathbb{1}_X)$, where we denote by μ the monoidal product of $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(X)$. It follows from unraveling the defining equivalence

$$\mathsf{Map}_{X \times X}(\mathcal{F} \otimes \mathcal{G}, \mathbb{1}_X) \simeq \mathsf{Map}_{X \times X}(\Delta^*(\mathcal{F} \otimes_{\mathcal{E}} \mathcal{G}), \mathbb{1}_X) \simeq \mathsf{Map}_X(\mathcal{F} \otimes_{\mathcal{E}} \mathcal{G}, \mathcal{F}_X)$$

that the underlying object of the canonical algebra $\mathcal{F}_X$ is equivalent to $\Delta_*(\mathbb{1}_X)$. Since X is m-finite, the functor Δ_* preserves the compact object $\mathbb{1}_X$, and hence $\mathcal{F}_X$ is compact. □

Corollary 11.21. *Let $\mathcal{E}^{\otimes}$ be a m-semiadditive rigid symmetric monoidal stable ∞-category and X a π-finite space. Then the $\mathcal{E}$-linear monoidal stable ∞-category $\mathsf{Loc}_{\mathcal{E}}(X)^{\otimes}$ is an $\mathcal{E}$-linear derived multi-fusion category.*

Example 11.23. Let $\mathcal{A}^{\otimes}$ be a $\mathbb{C}$-linear symmetric monoidal multi-fusion category and $\mathcal{D}^{\flat}(\mathcal{A})$ its bounded derived ∞-category. This ∞-category is ∞-semiadditive and hence, for any π-finite space X, the monoidal ∞-category $\mathsf{Loc}_{\mathcal{D}^{\flat}(\mathcal{A})}(X)^{\otimes}$ is a $\mathcal{D}^{\flat}(\mathcal{A})$-linear derived multi-fusion category.

In particular, we can recover by an analogous reasoning in the additive case our result from Section 2:

Corollary 11.22. *Let $\mathbb{K}$ an algebraically closed field of characteristic 0 and $\mathcal{G}$ a finite groupoid. Then the category $\mathsf{Loc}_{\mathbb{K}}(\mathcal{G})$ is a $\mathbb{K}$-linear multi-fusion category.*

As our second example, we consider categorified group algebras. Recall that for every ∞-group G the associated $\mathrm{Ind}(\mathcal{E})$-linear categorified group algebra $\mathrm{Ind}(\mathcal{E})[G]^{\otimes}$ is rigid. If we assume that

$$G_{\bullet} : \Delta^{\mathrm{op}} \to \mathcal{S}^m$$

takes values in m-finite spaces, it follows from the above discussion that the monoidal structure on $\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(G)$, equips $\mathsf{Loc}_{\mathcal{E}}(G)$ with an $\mathcal{E}$-linear monoidal structure, that is further rigid. We denote the underlying monoidal category by $\mathcal{E}[G]^{\otimes}$ and call it the $\mathcal{E}$-linear group algebra.
We now study under which conditions on the m-finite ∞-group G does $\mathcal{E}[G]$ form a derived multi-fusion category. It follows from Corollary 11.18 that this ∞-category is always $\mathcal{E}$-smooth and $\mathcal{E}$-proper.
Therefore, it remains to determine the canonical algebra $\mathcal{F}_{\mathcal{E}[G]}$. Recall that the multiplication on the categorified group algebra is given by the functor:

$$\mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(G \times G) \xrightarrow{\mu_!} \mathsf{Loc}_{\mathrm{Ind}(\mathcal{E})}(G).$$

In particular, this functor admits a right adjoint. It follows that the underlying object of the canonical algebra is given by

$$\mathcal{F}_{X_{\bullet}} \simeq \mu^* e_!(\mathbb{1}_{\mathcal{V}}),$$

where $e : * \to G$ is the inclusion of the unit object of G.

Example 11.24. Let $\mathbb{K}$ be an algebraically closed field and G be a finite group with categorified group algebra $\mathsf{Loc}_{\mathsf{Vect}_{\mathbb{K}}}(G) \simeq \mathsf{Vect}_G$. Note that there exists a pullback diagram

$$\begin{array}{ccccc} G & \xrightarrow{\mathrm{id} \times \tau} & G \times G & \xrightarrow{\mathrm{id}_{G \times G}} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \\ * & \xrightarrow{e} & G & & \end{array}$$

where τ denotes the map that associates to a group element its inverse. Under the associated base change equivalence, the canonical algebra $\mathcal{F}_G \simeq \mu^* e_! \mathbb{K}$ identifies with the functor

$$F : G \times G \to \mathsf{Vect}_{\mathbb{K}}, \quad F(g,h) = \begin{cases} \mathbb{K} & \text{if } h = g^{-1} \\ 0 & \text{else} \end{cases} .$$

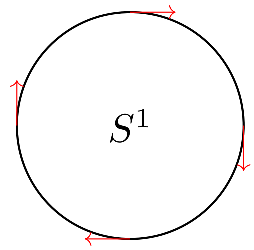


Figure 6: 1-framed circle. The blue arrow denotes the framing vector

Under the equivalence

$$\mathsf{Fun}(G \times G, \mathsf{Vect}_{\mathbb{K}}) \simeq \mathsf{Vect}_{G\times G}$$

this object gets mapped to

$$\bigoplus_{g\in G} \mathbb{K}_{(g,g^{-1})} \in \mathsf{Vect}_{G\times G}$$

which is compact-projective if and only if G is finite or equivalently if Vect_G has finitely many simples. This coincides with our observation from Proposition 5.14.

The correct ∞-categorical generalization of the Example 11.24 is the following:

Proposition 11.23. *Let $\mathcal{E}^{\otimes}$ be an m-semiadditive rigid symmetric monoidal stable ∞-category and $G_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}^m$ an m-finite ∞-group. Then the categorified $\mathcal{E}$-linear group algebra $\mathcal{E}[G]$ is an $\mathcal{E}$-linear derived multi-fusion category.*

Proof. Since G is m-finite by assumption, it follows from Corollary 11.18 that the category $\mathsf{Loc}_{\mathcal{E}}(G)$ is $\mathcal{E}$-smooth and $\mathcal{E}$-proper. Hence, the claim follows sine the multiplication μ of G is $\mathcal{E}$-ambidextrous if G is m-finite. ☐

Example 11.25. Let G be a finite group and $\mathcal{A}^{\otimes}$ a symmetric monoidal multi-fusion category. It follows from the above discussion that the ∞-category $\mathcal{D}^\flat(\mathcal{A})[G]$ is an example of a derived $\mathcal{D}^\flat(\mathcal{A})$-linear multi-fusion category. This provides a natural ∞-categorical generalization of the fusion category Vect_G of G-graded vector spaces.

11.5 TFTs from Convolution Structures

We conclude our discussion of rigid convolution monoidal structures by exploring the associated fully extended framed TFTs. In the case of convolution structures, some interesting structural properties already become apparent at the level of 1-manifolds. Therefore, we focus in this section on computing the value of these TFTs on the framed circle S^1 (Figure 6). Similar calculations for other sheaves theories have been described in [BZFN10, BZGN19].

Let $\mathcal{D}^{\otimes}$ be a symmetric monoidal ∞-category and $D \in \mathcal{D}^{\otimes}$ a fully dualizable object. Hence, by the cobordism hypothesis there exists a fully extended framed 1-dimensional TFT

$$\mathsf{Z}_D : \mathsf{Bord}_1^{\mathrm{fr},\otimes} \to \mathcal{D}^{\otimes}, \quad + \mapsto D.$$

The framed circle S^1 defines a 1-morphism

$$S^1 : \emptyset \to \emptyset$$

in $\mathsf{Bord}_1^{\mathrm{fr}}$ and it is well known that the value $\mathsf{Z}_D(S^1)$ can be computed using the duality data on D as:

$$\mathsf{Z}(S^1_0) : \mathbb{1}_{\mathbb{D}} \xrightarrow{\mathrm{coev}_D} D \otimes D^\vee \simeq^\tau D^\vee \otimes D \xrightarrow{\mathrm{ev}_D} \mathbb{1}_{\mathbb{D}} \,.$$

Here, we denoted by

$$\mathrm{ev}_D : D^\vee \otimes D \to \mathbb{1}_{\mathbb{D}} \quad \text{and} \quad \mathrm{coev}_D : \mathbb{1}_{\mathbb{D}} \to D \otimes D^\vee$$

the evaluation and coevaluation of the dualizable object D. This formula arises from the decomposition of S^1 as a composite of 1-morphisms in $\mathsf{Bord}_1^{\mathrm{fr}}$ as represented in Figure 7.

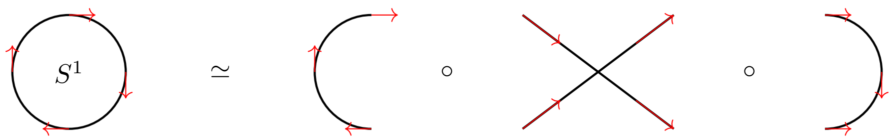


Figure 7: Decomposition of the framed circle as a composite of basic 1-morphisms in $\mathsf{Bord}_1^{\mathrm{fr}}$ consisting of coevaluation, switch, and evaluation (from left to right). The red arrows denote the framing.

Let us evaluate this in the case of Corollary 4.27. For this, let $\mathcal{V}^\otimes$ be a presentably symmetric monoidal ∞-category and $\mathcal{A}^\otimes$ a rigid algebra in $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$. We denote by

$$\mathsf{Z}_{\mathcal{A}} : \mathsf{Bord}_1^{\mathrm{fr},\otimes} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}, \quad + \mapsto \mathcal{A}^\otimes$$

the framed fully extended 1-dimensional TFT induced by $\mathcal{A}$. It follows from Proposition C.10 that the evaluation and coevaluation of $\mathcal{A}$ are given by

$$\mathcal{A} : \mathcal{A}^{\otimes\text{-}\mathrm{op}} \otimes_{\mathcal{V}} \mathcal{A} \to \mathcal{V} \quad \text{and} \quad \mathcal{A} : \mathcal{V} \to \mathcal{A} \otimes \mathcal{A}^{\otimes\text{-}\mathrm{op}}$$

the algebra $\mathcal{A}$ viewed as an $\mathcal{A} \otimes \mathcal{A}^{\otimes\text{-}\mathrm{op}}$ left (resp. right module). In particular, we can compute

$$\mathsf{HH}(\mathcal{A}) \coloneqq \mathsf{Z}_{\mathcal{A}}(S^1) \simeq \mathcal{A} \otimes_{\mathcal{A} \otimes_{\mathcal{V}} \mathcal{A}^{\otimes\text{-}\mathrm{op}}} \mathcal{A},$$

where $\mathsf{HH}(\mathcal{A})$ denotes the *trace* of $\mathcal{A}^\otimes$. This comes together with a *universal trace morphism*

$$\mathsf{tr} : \mathcal{A} \simeq \mathcal{A} \otimes_{\mathcal{V}} \mathcal{V} \to \mathcal{A} \otimes \mathcal{A} \to \mathsf{HH}(\mathcal{A})$$

that is given on objects by

$$\mathsf{tr}(A) = A \otimes_{\mathcal{A}^e} \mathbb{1}_{\mathcal{A}} \,.$$

Let us evaluate this TFT for the convolution structures constructed in the previous sections:

Proposition 11.24. *Let $\mathcal{V}^\otimes$ be a presentably symmetric monoidal ∞-category, and let $X \in \mathcal{S}$ be a space. Then there exists an equivalence of $\mathcal{V}$-linear ∞-categories*

$$\mathsf{HH}(\mathsf{Loc}_{\mathcal{V}}(X)) \simeq \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, \mathcal{V})).$$

Moreover, the universal trace map identifies with the functor

$$p^* : \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, X)) \to \mathsf{Loc}_{\mathcal{V}}(X)$$

obtained by pulling along $p : \mathsf{Map}(S^1, X) \simeq X \times_{X \times X} X \to X$.

Proof. It follows from the proof of Proposition 11.2 that for every space $Y \in \mathcal{S}$, there exists an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Mod}_Y(\mathcal{S}^{\mathrm{op}})^{\otimes} \simeq (\mathcal{S}_{/Y})^{\mathrm{op},\otimes},$$

where the left-hand side is equipped with the monoidal structure given by the relative tensor product and the right-hand side with the cocartesian monoidal structure. In particular, it follows from the monoidality of the functor

$$\mathsf{Loc}_{\mathcal{V}}(-) : (\mathcal{S}_{/Y})^{\mathrm{op},\otimes} \to \mathsf{Pr}^{\mathsf{L},\otimes}_{\mathsf{Loc}_{\mathcal{V}}(Y)}$$

that we obtain for all $X, Z \to Y$ an equivalence

$$\mathsf{Loc}_{\mathcal{V}}(X) \otimes_{\mathsf{Loc}_{\mathcal{V}}(Y)} \mathsf{Loc}_{\mathcal{V}}(Z) \simeq \mathsf{Loc}_{\mathcal{V}}(X \times_Y Z).$$

Hence, if we apply this to the diagonal map $\Delta : X \to X \times X$, we obtain an equivalence

$$\mathsf{HH}(\mathsf{Loc}_{\mathcal{V}}(X)) \simeq \mathsf{Loc}_{\mathcal{V}}(X) \otimes_{\mathsf{Loc}_{\mathcal{V}}(X \times X)} \mathsf{Loc}_{\mathcal{V}}(X) \simeq \mathsf{Loc}_{\mathcal{V}}(X \times_{X \times X} X) \simeq \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, X))$$

as desired. Moreover, in the symmetric monoidal ∞-category $(\mathcal{S}^{\mathrm{op}})^{\otimes}$ the universal trace map is given by the diagram

$$p : X \times_{X \times X} X \leftarrow X \times X \leftarrow X \times * \simeq X.$$

Hence, we obtain under the above equivalence the claimed form of the trace map:

$$p^* : \mathsf{Loc}_{\mathcal{V}}(X) \to \mathsf{Loc}_{\mathcal{V}}(X \times_{X \times X} X) \simeq \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, X))$$

□

The trace of the categorified group and Hecke algebra have already been calculated in [BZFN10, Thm.5.3] in the case of quasi-coherent sheaves and in [HL23, Thm.2.7.2] in the case of constructible sheaves. We claim that an analogous result holds in our case

Claim 11.1. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category, and $F : X \to Y$ a morphism of spaces. Then there exists an equivalence of ∞-categories

$$\mathsf{HH}(\mathcal{H}\mathsf{e}(F)) \simeq \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, Y)).$$

Moreover, the universal trace map identifies with the functor $\pi_! q^*$ obtained by linearizing the span

$$\begin{array}{ccc} & X \times_{X \times Y} X \simeq \mathsf{Map}(S^1, Y) \times_Y X & \\ {}^{q}\swarrow & & \searrow^{\pi} \\ X \times_Y X & & \mathsf{Map}(S^1, Y) \end{array}$$

where π is the projection map.

In particular, in case $F \simeq \mathrm{id}_X$, the convolution monoidal category $\mathcal{H}\mathsf{e}(\mathrm{id}_X)$ is equivalent to the pointwise monoidal structure and the above claim recovers our result from Proposition 11.24. As a consequence, we see that for every morphism of spaces $F : X \to Y$ we have an equivalence

$$\mathsf{Z}_{\mathcal{H}\mathsf{e}(F)}(S^1) \simeq \mathsf{Z}_{\mathsf{Loc}_{\mathcal{V}}(Y)}(S^1).$$

We expect that this is not a coincidence. Indeed, we have observed in the 1-categorical case in Proposition 2.5 that under certain conditions the monoidal categories are Morita equivalent, and hence induce equivalent fully extended TFTs. We expect a similar result to hold in this greater generality. A sketch of a proof in the context of quasi-coherent sheaves has already been given in [BZFN12], and we expect that this proof can be extended to our set-up.

Remark 11.9. Let X be a π-finite space and $\mathcal{V}$ be ∞-semiadditive. We expect that there exists a different way to compute the fully extended framed $2d$-TFT associated to the rigid $\mathcal{V}$-linear monoidal ∞-category $\mathsf{Loc}_{\mathcal{V}}(X)^{\otimes}$ as a *finite gauge or Dijkgraaf-Witten theory* as discussed in [FHLT09].
We have observed in Section 10.2 that every object in the symmetric monoidal $(\infty,2)$-category $2\mathbb{S}\mathsf{pan}(\mathcal{S})^{\otimes}$ admits a dual and every morphism admits both adjoints. In particular, every space X is fully-dualizable in the symmetric monoidal $(\infty,2)$-category $2\mathbb{S}\mathsf{pan}(\mathcal{S})^{\otimes}$ (see Definition C.5) and therefore, by the Cobordism Hypothesis induces a fully extended TFT

$$\mathsf{Z}_X : \mathsf{Bord}_2^{\mathrm{fr},\otimes} \to 2\mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes}$$

A computation shows that this $(\infty,2)$-functor associates to S^1 with the product framing the span

$$\ast \simeq \mathsf{Map}(\emptyset, X) \leftarrow \mathsf{Map}(S^1, X) \rightarrow \mathsf{Map}(\emptyset, X) \simeq \ast$$

More generally, we can view any framed 1-manifold M with boundary decomposed $\partial M = \partial M_{in} \coprod \partial M_{out}$ as a cospan

$$\partial M_{in} \hookrightarrow M \hookleftarrow \partial M_{out}$$

and the value of the TFT is given by applying $\mathsf{Map}(-, X)$ to the diagram:

$$\mathsf{Map}(M_{in}, X) \leftarrow \mathsf{Map}(M, X) \rightarrow \mathsf{Map}(M_{out}, X)$$

This procedure extends similarly to framed 2-manifolds with corners. In this context, we expect that the functor $\mathsf{Loc}_{\mathcal{V}}(-)$ can be used to construct a symmetric monoidal $(\infty,2)$-functor

$$\mathsf{Loc}_{\mathcal{V}} : 2\mathbb{S}\mathsf{pan}(\mathcal{S})^{\otimes} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$$

so that we obtain a fully extended framed TFT

$$\mathsf{Bord}_2^{\mathrm{fr},\otimes} \xrightarrow{\mathsf{Z}_X(-)} 2\mathbb{S}\mathsf{pan}_2(\mathcal{S})^{\otimes} \xrightarrow{\mathsf{Loc}_{\mathcal{V}}(-)} \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}$$

that maps $+$ to $\mathsf{Loc}_{\mathcal{V}}(X)$, and hence is equivalent to the fully extended TFT $\mathsf{Z}_{\mathsf{Loc}_{\mathcal{V}}(X)}$ constructed in Corollary 4.27. The benefit of this construction is that it simplifies the computation of the fully extended framed TFT.

It follows from the above discussion, that for a morphism of spaces $F : X \to Y$, to compute the fully extended framed TFT $\mathsf{Z}_{\mathcal{H}\mathsf{e}(F)}(-)$, one can consider the simpler pointwise monoidal category on $\mathsf{Loc}_{\mathcal{V}}(Y)^{\otimes}$. But as we have seen in Section 4.5 these $\mathcal{V}$-linear monoidal ∞-categories also define 1-dimensional relative fully extended framed TFTs

$$\mathsf{Z}^{\to}_{\mathcal{H}\mathsf{e}(F)} : \mathsf{Bord}_1^{\mathrm{fr}} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}, \quad + \mapsto (\mathcal{H}\mathsf{e}(F) : \mathcal{H}\mathsf{e}(F) \to \mathcal{V})$$

and

$$\mathsf{Z}^{\to}_{\mathsf{Loc}_{\mathcal{V}}(Y)} : \mathsf{Bord}_1^{\mathrm{fr}} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}, \quad + \mapsto (\mathsf{Loc}_{\mathcal{V}}(Y) : \mathsf{Loc}_{\mathcal{V}}(Y) \to \mathcal{V})$$

So let us also compute the value of these TFTs on S^1.

Example 11.26. Let $\mathcal{D}$ be an $\otimes$-Gr-cocomplete symmetric monoidal ∞-category and $(\mathcal{A}, \mu, \eta) \in \mathbb{M}\mathsf{or}(\mathcal{D})$ an algebra object in $\mathcal{D}^{\otimes}$. It follows from Example 4.16 that $\mathcal{A}$ viewed as a right module over itself induces a 1-dimensional fully extended TFT with

$$\mathsf{Z}^{\to}_{\mathcal{D}} : \mathsf{Bord}_1^{\mathrm{fr},\otimes} \to \mathbb{M}\mathsf{or}(\mathcal{D})^{\to,\otimes}.$$

To compute the value on S^1, we have to determine the evaluation and coevaluation of the duality of $\mathcal{A} : \mathbb{1}_{\mathcal{D}} \to \mathcal{A}$. It follows from the discussion in [JFS17, Sect.7] and Example 4.16 that the evaluation is given by the commutative diagram

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \xrightarrow{\mathcal{A}\otimes\mathcal{A}} & \mathcal{A}^{\mathsf{e}} \\ \downarrow & \overset{\mathrm{ev}_{\mathcal{A}}}{\Nearrow} & \downarrow{\scriptstyle \mathcal{A}} \\ \mathbb{1}_{\mathcal{D}} & \longrightarrow & \mathbb{1}_{\mathcal{D}} \end{array}$$

where the morphism $\mathrm{ev}_{\mathcal{A}} : \mathbb{1}_{\mathcal{D}} \to (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}^{\mathsf{e}}} \mathcal{A} \simeq \mathcal{A}$ is given by the unit of the algebra $\mathcal{A}$ and the coevaluation is given by the commutative diagram

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \longrightarrow & \mathbb{1}_{\mathcal{D}} \\ \downarrow & \overset{\mathrm{coev}_{\mathcal{A}}}{\Nearrow} & \downarrow{\scriptstyle \mathcal{A}} \\ \mathbb{1}_{\mathcal{D}} & \xrightarrow{\mathcal{A}\otimes\mathcal{A}} & \mathcal{A}^{\mathsf{e}} \end{array}$$

where the morphism $\mathrm{coev}_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is given by the multiplication of $\mathcal{A}$. In particular, we can compute the value on the circle as the commutative diagram

$$\mathsf{Z}^{\to}_{\mathcal{D}}(S^1) \simeq \begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \longrightarrow & \mathbb{1}_{\mathcal{D}} \\ \downarrow & \overset{\mathrm{coev}_{\mathcal{A}}}{\Nearrow} & \downarrow{\scriptstyle \mathcal{A}} \\ \mathbb{1}_{\mathcal{D}} & \xrightarrow{\mathcal{A}\otimes\mathcal{A}} & \mathcal{A}^{\mathsf{e}} \\ \downarrow & \overset{\mathrm{ev}_{\mathcal{A}}}{\Nearrow} & \downarrow{\scriptstyle \mathcal{A}} \\ \mathbb{1}_{\mathcal{D}} & \longrightarrow & \mathbb{1}_{\mathcal{D}} \end{array}$$

where the composite 2-morphism is given by

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\eta} \mathcal{A} \xrightarrow{\mathsf{tr}_{\mathcal{A}}} \mathcal{A} \otimes_{\mathcal{A}^{\mathsf{e}}} \mathcal{A} \simeq \mathsf{HH}(\mathcal{A}).$$

Example 11.27. We compute the relative TFT corresponding to our examples of convolution monoidal structures. Let $F: X \to Y$ be a morphism of spaces and consider the associated Hecke category $\mathcal{H}\mathsf{e}(F)$ with associated TFT

$$\mathsf{Z}^{\rightarrow}_{\mathcal{H}\mathsf{e}(F)} : \mathsf{Bord}_1^{\mathrm{fr},\otimes} \to \mathbb{M}\mathsf{or}(\mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}})^{\otimes}.$$

It follows from Claim 11.1 that the value of the TFT on S^1 is given by the composite

$$\mathcal{V} \xrightarrow{p_X^* \Delta_{X,!}} \mathcal{H}\mathsf{e}(F) \xrightarrow{\pi_! q^*} \mathsf{Loc}_{\mathcal{V}}(\mathsf{Map}(S^1, Y)).$$

We analyse this more carefully in the case, where $F: \mathsf{BH} \to \mathsf{BG}$ is induced by the inclusion of groups $H \subset G$. In this case, the universal trace map is induced by linearizing the span

$$H\backslash G/H \longleftarrow G/^{adj}H \longrightarrow G/^{adj}G$$

where we denote by $G/^{adj}H$ the groupoid quotient by the adjoint action. This span is usually called the *Horrocycle correspondence* [BZGN19, Sect.6.2]. Further, the unit morphism is induced by the span

$$* \longleftarrow \mathsf{BH} \longrightarrow H\backslash G/H$$

Hence, we obtain by linearizing the composite span

$$\begin{array}{ccccc} & & H/^{adj}H & & \\ & \swarrow & & \searrow & \\ & \mathsf{BH} & & G/^{adj}H & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ * & & H\backslash G/H & & G/^{adj}G \end{array}$$

$$p_{H/H}: H/^{adj}H \to *, \qquad i_{G/H}: H/^{adj}H \to G/^{adj}G$$

that the value of the fully extended relative TFT on the circle is given by the $\mathcal{V}$-linear functor

$$\mathsf{Z}^{\rightarrow}_{H\backslash G/H}(S^1) : \mathcal{V} \xrightarrow{p^*_{H/H}} \mathsf{Loc}_{\mathcal{V}}(H/^{adj}H) \xrightarrow{i_{G/H,!}} \mathsf{Loc}_{\mathcal{V}}(G/^{adj}G).$$

In particular, this $\mathcal{V}$-linear cocontinuous functor is fully determined by the value of $\mathbb{1}_{\mathcal{V}}$. Since the ∞-category $\mathsf{Loc}_{\mathcal{V}}(G/^{adj}G)$ categorifies the algebra of class functions on G, we think of objects in this ∞-category as categorified characters of G. In particular, we call

$$\mathfrak{Ch}(G, H) \coloneqq \mathsf{Z}_{H\backslash G/H}(S^1)(\mathbb{1}_{\mathcal{V}}) \simeq i_{G/H,!} p^*_{H/H}(\mathbb{1}_{\mathcal{V}})$$

the *character of* G *and* H.

Remark 11.10. Given a group G and a subgroup H, one obtains a $\mathbb{C}$-linear G-representation on $\mathbb{C}[G/H]$ the space of functions on G/H. In particular, this representation has a character $\xi_{G,H}$, One should think of $\mathfrak{Ch}(G, H)$ as a categorified version of this character.

Let us compare the values for different choices of H.

Example 11.28. We consider the extreme cases. In case $H \simeq *$ is the trivial group, the categorified Hecke algebra is given by $\mathcal{H}\mathsf{e}(F) \simeq \mathcal{V}[G]$, and the value of the TFT on the circle is induced by the span

$$* \longleftarrow * \xrightarrow{\; i_G \;} G/^{adj}G$$

The character of G and $*$ is then given by

$$\mathfrak{Ch}(G,*) \simeq i_{G,!}(\mathbb{1}_{\mathcal{V}})$$

the skyscraper local system supported at the neutral element. On the other hand, if $H = G$ is the whole group, then the categorified Hecke algebra is given by $\mathcal{H}\mathsf{e}(F) \simeq \mathsf{Loc}_{\mathcal{V}}(\mathsf{BG})$ with the pointwise monoidal structures, and the value of the TFT on the circle is induced by the span

$$* \xleftarrow{\; p_{G/G} \;} G/^{adj}G \longrightarrow G/^{adj}G$$

The character of G and G is then given by

$$\mathfrak{Ch}(G,G) \simeq p^*_{G/G}(\mathbb{1}_{\mathcal{V}})$$

the constant local system on $G/^{adj}G$. For non-trivial G these two local systems have different support and in particular

$$\mathfrak{Ch}(G,G) \not\simeq \mathfrak{Ch}(G,*).$$

Therefore, we can conclude that

$$\mathsf{Z}^{\rightarrow}_{\mathcal{V}[G]} \not\simeq \mathsf{Z}^{\rightarrow}_{\mathsf{Loc}_{\mathcal{V}}(\mathsf{BG})}.$$

12 Frobenius Algebras in Spans

In the main part of this text, we have classified rigid 2-Segal spaces and have shown that these induce rigid convolution monoidal structures. Nevertheless, our result does not imply the converse. One example of a 2-Segal space, whose convolution monoidal structure behaves similarly to that of a rigid category, is the Waldhausen $S_\bullet$-constructions of a stable ∞-category $\mathcal{C}$. The reason is that the Waldhausen construction of a stable ∞-category naturally carries the structure of a Frobenius algebra in the ∞-category $\mathsf{Span}(\mathcal{S})^{\otimes}$ that is different from that of a rigid algebra. These Frobenius algebra structures induce duality structures that are more general than rigid dualities. These are known as Grothendieck–Verdier (short GV) dualities [BD13]. Symmetric Frobenius algebras, or more precisely, Calabi–Yau-algebras (CY-algebras) in the monoidal ∞-categories $\mathsf{Span}(\mathcal{S})^{\otimes}$ have already been studied in [Ste21]. As algebra objects in $\mathsf{Span}(\mathcal{S})^{\otimes}$ can be described by 2-Segal simplicial spaces

$$X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S},$$

interestingly, also CY-algebras admit a similar description as 2-Segal cyclic spaces. A cyclic space is a functor

$$Y_\bullet : \Lambda^{\mathrm{op}} \to \mathcal{S}$$

with source given by Connes cyclic category Λ [Con83], an extension of the simplex category. Informally, the cyclic category is obtained from the simplex category by adding for every object $[n]$ an extra $(n+1)$-periodic automorphism. These extra symmetries on the source category encode the dualities of the CY-algebra.
The main difference between the datum of an ordinary and a symmetric Frobenius algebra is that the datum of a symmetric Frobenius algebra also encodes a homotopy coherent trivialization of the square of the duality automorphism. On the level of indexing categories, this is reflected by replacing the cyclic Λ category by the paracyclic category Λ_∞ (see Definition 12.3). The main difference between these is that, while the extra automorphisms in the cyclic category admit a certain periodicity, the duality functor of the paracyclic category may not be periodic. This reflects the missing coherent trivialization of the duality automorphism. In the 1-categorical context, the relation between Frobenius algebras and paracyclic 2-Segal objects has been described [CMS25].
It has been described by Lurie [Lur15] that we can extend the Waldhausen construction of a stable ∞-category $\mathcal{C}$ to a paracyclic object. In this case, the corresponding duality automorphism is induced by the shift functor of the stable ∞-category $\mathcal{C}$. Our main goal in this section is to analyze the duality structures that arise from linearizing Frobenius algebras in $\mathsf{Span}(\mathcal{S})^{\otimes}$ and to apply this analysis to the example of the Waldhausen $S_\bullet$-construction. As a start, let us therefore first generalize the notion of a GV-duality from [BD13] to the context of stable ∞-categories and introduce some terminology.
Recall, that the spectral Co-Yoneda embedding $\mathcal{Y}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{S}\mathrm{p})$ is the functor that maps $C \in \mathcal{C}$ to the mapping spectrum $\hom_{\mathcal{C}}(C, -)$ (See Definition 5.11). We say that a functor $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ is stably corepresentable if it lies in the essential image of $\mathcal{Y}_{\mathcal{C}}$. After this preliminary discussion, we can now define:

Definition 12.1. Let $\mathcal{C}$ be a monoidal stable ∞-category. An object $K \in \mathcal{C}$ is called a *weak dualizing object* if for every $Y \in \mathcal{C}$ the mapping spectrum $\hom_{\mathcal{C}}(K, - \otimes Y) : \mathcal{C} \to \mathcal{S}\mathrm{p}$ is stably corepresentable.

For every weak dualizing object K it follows that the exact functor

$$\hom_{\mathcal{C}}(K, - \otimes -) : \mathcal{C} \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{S}\mathrm{p})$$

factors through the essential image of the spectral Co-Yoneda embedding. Its composite with the essential inverse induces an exact duality functor

$$\mathbb{D} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$$

that satisfies the universal property $\hom_{\mathcal{C}}(K, X \otimes Y) \simeq \hom_{\mathcal{C}}(\mathbb{D}(X), Y)$. We call K a *dualizing object* if the duality functor $\mathbb{D}$ is an equivalence of ∞-categories.

Example 12.1. Let $\mathcal{C}$ be a rigid monoidal stable ∞-category. Then the unit object $\mathbb{1}_{\mathcal{C}}$ is a dualizing object for $\mathcal{C}$ and the associated duality functor maps $X \in \mathcal{C}$ to its dual X^*.

Definition 12.2. A *stable Grothendieck–Verdier category* is a pair consisting of monoidal stable ∞-category $\mathcal{C}$ and a dualizing object $K \in \mathcal{C}$.

We often abuse notation and drop the dualizing object from the notation of a stable GV-category $(\mathcal{C}, K)$. In the following, we call a GV duality rigid if the dualizing object is the monoidal unit and the GV structure coincides with the rigid duality. Our next goal is to provide an alternative characterization of GV-structures in terms of Frobenius algebras (Definition 3.3) in the ∞-category of presentable stable ∞-categories. The key for this is to understand the relation between different duality structures on a presentable stable ∞-category.

Lemma 12.1. *Let* $\mathcal{C}^{\otimes}$ *be a monoidal* ∞*-category and let* X *be a dualizable object with evaluation* $\mathrm{ev}_X : X \otimes {}^{\vee}X \to \mathbb{1}$ *and coevaluation* $\mathrm{coev}_X : \mathbb{1} \to {}^{\vee}X \otimes X$*. Further, let* $\alpha : X' \to {}^{\vee}X$ *be an isomorphism. Then the twisted evaluation* $\mathrm{ev}_X \circ \mathrm{id}_X \otimes \alpha$ *and twisted coevaluation* $\alpha^{-1} \otimes \mathrm{id}_X \circ \mathrm{coev}_X$ *exhibit* X' *as a dual of* X*.*

Proof. The proof is a simple diagram chase. □

Lemma 12.2. *Let* $\mathcal{C}^{\otimes}$ *be a monoidal* ∞*-category and let* X *be an object. Further, let* $(\mathrm{ev}_0, \mathrm{coev}_0, X_0)$ *and* $(\mathrm{ev}_1, \mathrm{coev}_1, X_1)$ *be dualizability data that exhibit* X_0 *and* X_1 *as right duals of* X*. Then there exists a unique isomorphism* $\mathbb{D} : X_0 \to X_1$*, such that* $\mathrm{ev}_0 \simeq \mathrm{ev}_1 \circ (\mathrm{id}_X \otimes \mathbb{D})$ *and* $\mathrm{coev}_0 \simeq (\mathbb{D}^{-1} \otimes \mathrm{id}_X) \circ \mathrm{coev}_1$

Proof. One can easily check that the composite map

$$\mathbb{D} : X_0 \xrightarrow{\mathrm{coev}_1 \otimes \mathrm{id}_{X_0}} X_1 \otimes X \otimes X_0 \xrightarrow{\mathrm{id}_{X_1} \otimes \mathrm{ev}_0} X_1$$

is an isomorphism and satisfies the required hypothesis. □

As explained in Section 4.1 there exists for every stable ∞-category $\mathcal{C}$ an exact internal Hom functor $\mathrm{hom}_{\mathcal{C}}(-,-) : \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ which is usually called the mapping spectrum. We denote by

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathcal{S}\mathrm{p}$$

its unique cocontinuous extension. We call this functor the *Hom-bimodule* of $\mathcal{C}$. For a compact object $C \in \mathrm{Ind}(\mathcal{C})$ the functor $\mathrm{Hom}_{\mathcal{C}}(C,-)$ coincides with the mapping spectrum functor of the stable ∞-category $\mathrm{Ind}(\mathcal{C})$. To increase readability, we will in the following denote the presentable stable ∞-category $\mathrm{Ind}(\mathcal{C}^{c,\mathrm{op}})$ by $\mathcal{C}^{\diamond}$. For every functor $F : \mathcal{C} \to \mathcal{D}$ we denote by $F^{\diamond}$ the functor induced functor $\mathrm{Ind}(F^{\mathrm{op}}) : \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})$.

Proposition 12.3. *Let* $(\mathcal{C}, K)$ *be a stable GV-category. Then the cocontinuous functor* $\mathrm{Hom}_{\mathcal{C}}(K,-) : \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{p}$ *equips* $\mathrm{Ind}(\mathcal{C})$ *with the structure of a Frobenius algebra in* $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$*.*

Proof. We need to show that the composite functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(K, - \otimes -) : \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{p}$$

defines the evaluation of a duality on $\mathcal{C}$. By the universal property of the duality functor $\mathbb{D}$, it follows that the cocontinuous functor $\mathrm{Hom}_{\mathcal{C}}(K, - \otimes -)$ is equivalent to the composite $\mathrm{Hom}_{\mathcal{C}}(-,-) \circ (\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \otimes \mathrm{Ind}(\mathbb{D}))$. Hence, it follows from Lemma 12.1 that $\mathrm{Hom}_{\mathcal{C}}(K,-)$ is non-degenerate. □

Proposition 12.4. *Let* $\mathcal{C}^{\otimes}$ *be a monoidal stable* ∞*-category and let* $\lambda : \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{p}$ *be a non-degenerate* 1*-morphism. If* λ *admits a left adjoint then* $\lambda^{\mathsf{L}}(\mathbb{1}_{\mathcal{S}\mathrm{p}}) \in \mathcal{C}$ *is a dualizing object.*

Proof. By assumption $\lambda(- \otimes -) : \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{p}$ induces an evaluation of a duality on $\mathrm{Ind}(\mathcal{C})$. It follows from Proposition 12.2 that there exists a duality equivalence $\mathbb{D}(-) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ such that for all objects $C, D \in \mathrm{Ind}(\mathcal{C})$

$$\lambda(C \otimes D) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{D}(C), D).$$

The claim then follows from the chain of equivalences

$$\mathrm{hom}_{\mathcal{C}}(\mathbb{D}(C), D) \simeq \lambda(C \otimes D) \simeq \mathrm{hom}_{\mathcal{S}\mathrm{p}}(\mathbb{1}_{\mathcal{S}\mathrm{p}}, \lambda(C \otimes D)) \simeq \mathrm{hom}_{\mathcal{C}}(\lambda^{\mathsf{L}}(\mathbb{1}_{\mathcal{S}\mathrm{p}}), C \otimes D).$$

□

Remark 12.1. The hypothesis of the above Proposition is always satisfied if $\mathcal{C}$ is a smooth stable ∞-category. Indeed, since $\mathcal{S}\mathrm{p}^{\otimes}$ is rigid this follows from [BD21, Prop.2.4.(3)].

The above analysis shows that under certain assumptions on a monoidal stable ∞-category $\mathcal{C}^{\otimes}$, the datum of a dualizing object is equivalent to a Frobenius algebra structure on $\mathrm{Ind}(\mathcal{C})$. To obtain stable GV-duality structures on convolution monoidal categories, we therefore need to understand how to extend algebra objects in $\mathsf{Span}(\mathcal{S})^{\otimes}$ to Frobenius algebras. Similar to algebra objects, which can be encoded by 2-Segal spaces, Frobenius algebra objects can be encoded using so called 2-*Segal paracyclic spaces*:

Definition 12.3. The *paracyclic category* Λ_∞ is the category with objects the standard linearly ordered sets $[n]$ for $n \geq 0$ and morphisms from $[m]$ to $[n]$ given by weakly monotone maps $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy the periodicity condition $f(i+m+1) = f(i) + n + 1$ for all $i \in \mathbb{Z}$.

Note that any morphism $f \in \mathsf{Map}_{\Lambda_\infty}([m],[n])$ is uniquely determined by its restriction to $\{0,...,m\}$ and conversely every morphism $f : \{0,...,m\} \to \mathbb{Z}$ that satisfies $f(m) \leq f(0) + n + 1$ uniquely extends to a morphism in Λ_∞. It follows that the simplex category Δ embeds faithfully in Λ_∞ by identifying it with the wide subcategory on those morphisms $f \in \mathsf{Map}_{\Lambda_\infty}([m],[n])$ that map $\{0,...,m\}$ into $\{0,...,n\}$. Additionally, the category Λ_∞ admits for each $[n]$ an automorphism

$$t_n : [n] \to [n], \quad i \mapsto i+1.$$

This morphism is called the *paracyclic shift*. For later use, we record its relations with the face and degeneracy maps:

$$t_n d_i^n = \begin{cases} d_{i+1}^n t_{n-1}, & 0 \leq i < n; \\ d_0^n, & i = n \end{cases} \tag{40}$$

$$t_n \sigma_i^n = \begin{cases} \sigma_{i+1}^n t_{n+1}, & 0 \leq i < n \\ \sigma_0^n (t_{n+1})^2, & i = n \end{cases} \tag{41}$$

Definition 12.4. We call a presheaf $X_\bullet : \Lambda^{\mathrm{op}} \to \mathcal{S}$ a *paracyclic space*. $X_\bullet$ is called a *paracyclic* 2-*Segal space*, if the restriction $X_\bullet|_{\Delta^{\mathrm{op}}}$ is a 2-Segal space

For a paracyclic space $X_\bullet$, we denote the image of the paracyclic shift by $X(t_n) := \tau_n$. This functor will play the role of a duality equivalence for the corresponding Frobenius algebra in $\mathsf{Span}(\mathcal{S})^{\otimes}$. Let us consider some examples of paracyclic 2-Segal spacea:

Example 12.2. [Ste19, Sect.3.3.2.3] Let H, G be groups and $F : \mathsf{BH} \to \mathsf{BG}$ be a functor. For every element z in G, we can extend the Čech-nerve to a paracyclic groupoid $\check{C}(F)^z_\bullet : \Lambda_\infty^{\mathrm{op}} \to \mathsf{Grpd}^f$. In this case, the paracyclic shift is given by the functor

$$\mathsf{BH} \times_{\mathsf{BG}} \cdots \times_{\mathsf{BG}} \mathsf{BH} \xrightarrow{\tau_n^z} \mathsf{BH} \times_{\mathsf{BG}} \cdots \times_{\mathsf{BG}} \mathsf{BH}$$

that maps an object $(g_1,...,g_n) \in \mathsf{BH} \times_{\mathsf{BG}} \cdots \times_{\mathsf{BG}} \mathsf{BH}$ to the object $(z(\prod_{i=1}^n g_i)^{-1}, g_1, \ldots, g_{n-1})$. In particular, if $\mathsf{BH} \simeq *$ this equips $\mathsf{N}(G)^z_\bullet$ with the structure of a paracyclic 2-Segal set.

Example 12.3. Let $\mathcal{C}$ be a stable ∞-category and denote by $S_\bullet : \Delta^{\mathrm{op}} \to \mathcal{S}$ its Waldhausen $S_\bullet$-construction. The simplicial space $S_\bullet(\mathcal{C})$ admits an extension to a paracyclic object [Lur15, Def.4.3.4]. We describe the paracyclic shift in low dimensions. In degree 1, the paracyclic shift is given by the shift functor $[1] : \mathcal{C}^{\simeq} \to \mathcal{C}^{\simeq}$ of the stable ∞-category $\mathcal{C}$. In degree n an object of $S_n(\mathcal{C})$ is given by a flag:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n.$$

The paracyclic shift maps such a flag to the rotated flag

$$X_2/X_1 \longrightarrow X_3/X_1 \longrightarrow \dots \longrightarrow X_1[1].$$

To relate paracyclic 2-Segal spaces to Frobenius algebras, recall that every space X is dualizable in $\mathsf{Span}(\mathcal{S})^{\otimes}$ and the evaluation of the duality is given by the span

$$X \times X \xleftarrow{\ \Delta\ } X \xrightarrow{\ p_X\ } * \tag{42}$$

We can then show:

Proposition 12.5. *Let $X_\bullet : \Lambda_\infty^{\mathrm{op}} \to \mathcal{C}$ be a paracyclic 2-Segal object in $\mathcal{C}$. Then, the span*

$$X_1 \xleftarrow{\ \tau_1 s_0\ } X_0 \xrightarrow{\ p_{X_0}\ } *$$

equips the associative algebra associated to the 2-Segal object $X_\bullet$ with the structure of a Frobenius algebra. Further, the paracyclic shift $\tau : X_1 \to X_1$ induces the duality equivalence between the evaluation of the Frobenius algebra and the one from Diagram 42.

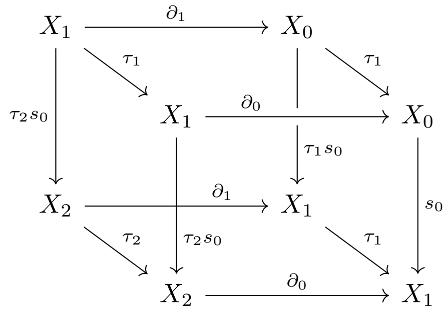
Proof. We need to show that the composite span

$$\begin{array}{ccccccc} & & & X_2 \times_{X_1} X_0 & & & \\ & & \swarrow & & \searrow & & \\ & X_2 & & & & X_0 & \\ \swarrow & & \searrow^{\partial_1} & & \swarrow^{\tau_1 s_0} & & \searrow \\ X_1 \times X_1 & & & X_1 & & & * \end{array}$$

is the evaluation of a duality on X_1. Therefore, we first show that the diagram

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow^{\tau_2 s_0} & & \searrow^{\partial_1} & \\ X_2 & & & & X_0 \\ & \searrow_{\partial_1} & & \swarrow_{\tau_1 s_0} & \\ & & X_1 & & \end{array}$$

is a pullback diagram. It follows from the identities of a paracyclic space that the diagram



commutes. Note that by the 2-Segal conditions, the front face is a pullback diagram. Since τ is an equivalence, we can read the cube as an equivalence between the front and the back face. Hence, the back face is also a pullback diagram. It follows that our initial span is equivalent to the span

$$X_1 \times X_1 \xleftarrow{(\mathrm{id}\,,\tau_1)} X_1 \xrightarrow{p_{X_1}} *$$

But this is the canonical evaluation span from Diagram 42 twisted by the automorphism τ. The claim follows from Lemma 12.1. ☐

We can now apply this result to the construction of Frobenius algebra structures on convolution monoidal categories.

Proposition 12.6. *Let R be a commutative ring spectrum and let $\mathrm{ev} : \mathcal{C} \otimes_R \mathcal{D} \to \mathsf{RMod}_R$ be a duality datum in $\mathsf{Pr}_R^{\mathsf{L},\otimes}$, then also the composite*

$$\mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{C} \otimes_R \mathcal{D} \xrightarrow{\mathrm{ev}} \mathsf{RMod}_R \longrightarrow \mathcal{S}\mathrm{p}$$

induces a duality datum in $\mathsf{Pr}_{\mathrm{st}}^{\mathsf{L},\otimes}$.

Proof. The symmetric monoidal ∞-category $\mathsf{RMod}_R^{\otimes}$ defines rigid algebra in $\mathbb{P}\mathsf{r}_{\mathrm{st}}^{\mathsf{L},\otimes}$ and hence a Frobenius algebra. The claim follows from [Lur17, Cor.4.6.5.14]. ☐

Corollary 12.7. *Let $X_\bullet : \Lambda_\infty^{\mathrm{op}} \to \mathcal{S}$ be a paracyclic 2-Segal space and R a commutative ring spectrum. Then, the paracyclic structure induces on the convolution monoidal category $(\mathsf{Loc}_R(X_1), *)$ the structure of a Frobenius algebra in $\mathbb{P}\mathsf{r}_{\mathrm{st}}^{\mathsf{L},\otimes}$. Moreover, if*

$$(\partial_2, \partial_0)^* : \mathsf{Loc}_R(X_1 \times X_1) \to \mathsf{Loc}_R(X_2)$$

preserves compact objects and X_0 is R-ambidextrous then

$$\tau_{1,!} s_{0,!} p_{X_0}^*(R) \in \mathsf{Loc}_R(X_1)^{\mathrm{c}}$$

is a dualizing object.

Proof. It follows from Proposition 12.5 and Proposition 12.6 that the cocontinuous functor

$$\mathsf{Loc}_R(X_1) \xrightarrow{s_0^*\tau_1^*} \mathsf{Loc}_R(X_0) \xrightarrow{p_{X_0,!}} \mathsf{RMod}_R \xrightarrow{\mathsf{Fgt}} \mathcal{S}\mathrm{p}$$

is a non-degenerate 1-morphism in $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$. This implies the first claim. For the second claim, note that the convolution monoidal structure on $\mathsf{Loc}_R(X_1)$ restricts to a monoidal structure on $\mathsf{Loc}_R(X_1)^{\mathrm{c}}$ if and only if $(\partial_2, \partial_0)^*$ preserves compact objects. The claim then follows from Proposition 12.4 and the observation that if X_0 is R-ambidextrous then $p_{X_0,!}$ admits a left adjoint. $\square$

Our next goal is to understand the duality equivalence associated to the above Frobenius structure. To this end, we need the following construction from [GR19, Sect.II.4]:

Construction 12.1. Let R be a commutative ring spectrum. It follows from Proposition 11.5 that for every space X the R-linear ∞-category $\mathsf{Loc}_R(X)$ is compactly generated. Hence, the Hom-bimodule

$$\mathrm{Hom}_X(-,-) : \mathsf{Loc}_R(X) \otimes_R \mathsf{Loc}_R(X)^{\diamond} \to \mathsf{RMod}_R$$

is an evaluation of a duality on $\mathsf{Loc}_R(X)$ (Proposition 5.22). On the other hand, the image of the canonical evaluation on X from Diagram (42) in $\mathsf{Span}(\mathcal{S})^{\otimes}$ under the symmetric monoidal functor $\mathsf{Loc}_R(-)$ induces a different evaluation

$$\mathsf{Loc}_R(X) \otimes_R \mathsf{Loc}_R(X) \simeq \mathsf{Loc}_R(X \times X) \xrightarrow{\Delta^*} \mathsf{Loc}_R(X) \xrightarrow{p_!} \mathsf{RMod}_R$$

on $\mathsf{Loc}_R(X)$. It follows from Lemma 12.1 that there exists a unique R-linear equivalence

$$\mathbb{D}_X(-) : \mathsf{Loc}_R(X)^{\diamond} \to \mathsf{Loc}_R(X)$$

that satisfies the universal property $\mathrm{Hom}_{\mathcal{C}}(\mathbb{D}_X(\mathcal{F}), \mathcal{G}) \simeq p_!\Delta^*(\mathcal{F} \otimes_R \mathcal{G}) \simeq p_!(\mathcal{F} \otimes \mathcal{G})$. In particular, if $\mathcal{F}$ is compact it induces an equivalence

$$p_!(\mathcal{F} \otimes \mathcal{G}) \simeq \mathrm{hom}_X(\mathbb{D}_X(\mathcal{F}), \mathcal{G})$$

in RMod_R.

Before we apply this to our study of Frobenius algebras, let us record some formal properties of the functor $\mathbb{D}_X(-)$:

Proposition 12.8. *Let R be a commutative ring spectrum and let $f : X \to Y$ be a morphism of spaces, then*

(1) there exists a natural equivalence of functors $\mathbb{D}_Y f_!^{\diamond} \simeq f_!\mathbb{D}_X : \mathsf{Loc}_R(X)^{\diamond} \to \mathsf{Loc}_R(Y)$.

(2) if f is R-ambidextrous, there is a natural equivalence $\mathbb{D}_X f^{*,\diamond} \simeq f^*\mathbb{D}_Y$.

Proof. Since all functors are cocontinuous and preserve compact objects, it suffices to show this equivalence on compact objects. Consider the chain of equivalences of spectra

$$\begin{aligned}
\mathrm{hom}_Y(\mathbb{D}_Y(f_!\mathcal{F}), \mathcal{G})) &\simeq \mathrm{hom}_R(R, p_!^Y(f_!\mathcal{F} \otimes \mathcal{G})) \\
&\simeq \mathrm{hom}_R(R, f_!p_!^Y(\mathcal{F} \otimes f^*\mathcal{G})) \\
&\simeq \mathrm{hom}_R(R, p_!^X(\mathcal{F} \otimes f^*\mathcal{G})) \\
&\simeq \mathrm{hom}_Y(f_!\mathbb{D}_X(\mathcal{F}), \mathcal{G})
\end{aligned}$$

where we have used the defining property of the duality and the projection formula. This implies statement (1) by Lemma 4.8. Statement (2) follows analogously for the chain of equivalences:

$$\begin{aligned}\hom_X(\mathbb{D}_X(f^*\mathcal{G}),\mathcal{F}) &\simeq \hom_R(R, p^X_!(f^*\mathcal{G}\otimes\mathcal{F}))\\ &\simeq \hom_R(R, f_! p^Y_!(f^*\mathcal{G}\otimes\mathcal{F}))\\ &\simeq \hom_R(R, p^Y_!(\mathcal{G}\otimes f_!(cF))\\ &\simeq \hom_Y(\mathbb{D}(\mathcal{G}), f_!\mathcal{F}) \simeq \hom_X(f^*\mathbb{D}(\mathcal{G}),\mathcal{F}),\end{aligned}$$

where we have used in the last step that f is ambidextrous. □

Example 12.4. Let R be a commutative ring spectrum and $V \in \mathsf{Perf}(R)$ be a perfect R-module. It follows from the defining property of $\mathbb{D}(-)$, that

$$\hom_R(R, V\otimes W) \simeq \hom_R(\mathbb{D}(V), W).$$

Since V is dualizable with dual $\hom_R(V,R)$, it follows that $\mathbb{D}(V)$ is equivalent to $\hom_R(V,R)$. In particular, it follows that $\mathbb{D}_*(-)$ is equivalent to $\mathrm{Ind}(\hom_R(-,R))$. More generally if RMod_R is m-semiadditive and X is m-finite, it follows from Remark 11.8 that for every compact object $\mathcal{F} \in \mathsf{Loc}_R(X)^{\mathrm{c}}$ there exists an equivalence $\mathbb{D}_X(\mathcal{F}) \simeq \mathcal{H}\mathsf{om}_X(\mathcal{F}, p_X^* R)$ and $\mathbb{D}_X(-)$ is naturally equivalent to $\mathrm{Ind}(\mathcal{H}\mathsf{om}_X(-, p_X^* R))$

After these preparations, we can now compute the duality functor of the GV-duality:

Theorem 12.9. *Let $X_\bullet : \Lambda_\infty^{\mathrm{op}} \to \mathcal{S}$ be a paracyclic 2-Segal object, such that $(\partial_2,\partial_0)^*$ preserves compact objects and X_0 is R-ambidextrous. Then the composite duality*

$$\mathbb{D}\circ\tau_1^* : \mathsf{Loc}_R(X_1)^{\mathrm{c,op}} \to \mathsf{Loc}_R(X_1)^{\mathrm{c}}$$

is the duality equivalence for the stable GV-category $(\mathsf{Loc}_R(X_1)^{\mathrm{c}}, \tau_{1,!}s_{0,!}p^*_{X_0}(R))$.

Proof. Let $\mathcal{F},\mathcal{G}$ be compact objects. Since by assumption p_{X_0} and s_1 are R-ambidextrous, we obtain an equivalence

$$\hom_R(\tau_{1,!}s_{0,!}p^*_{X_0}(R), \mathcal{F}*\mathcal{G}) \simeq \hom_R(R, p_{X_0,!}s_0^*\tau_1^*(\mathcal{F}*\mathcal{G})).$$

It follows from our considerations in Proposition 12.5 that

$$p_{X_0,!}s_0^*\tau_1^*(-*-) \simeq p_{X_1!}(\tau^*(-)\otimes -).$$

The claim follows from the chain of equivalences

$$\hom_R(R, p_{X_0,!}s_0^*\tau_1^*(\mathcal{F}*\mathcal{G})) \simeq \hom_R(R, p_{X_1,!}(\tau^*(\mathcal{F})\otimes\mathcal{G})) \simeq \hom_R(\mathbb{D}_X(\tau^*(\mathcal{F})),\mathcal{G}),$$

where we have used the universal property of $\mathbb{D}_X(-)$ in the last step. □

Example 12.5. Let $F : \mathsf{BH} \to \mathsf{BG}$ be a functor between finite groupoids and let $z \in \mathcal{Z}(G)$ be a central element. We consider the paracyclic 2-Segal object $\check{C}(F)^z_\bullet$. For an ∞-semiadditive commutative ring spectrum R, the R-linear categorified Hecke algebra fulfills the assumptions of Theorem 12.9.
It follows that the stable ∞-category $\mathcal{H}\mathsf{e}_R(\mathsf{BH}\times_{\mathsf{BG}}\mathsf{BG})^{\mathrm{c}}$ admits a stable GV-duality with duality equivalence

$$\mathcal{H}\mathsf{om}_{\mathsf{BH}\times_{\mathsf{BG}}\mathsf{BH}}(\tau_z^*-, R_{\mathsf{BH}\times_{\mathsf{BG}}\mathsf{BH}}) : \mathcal{H}\mathsf{e}_R(\mathsf{BH}\times_{\mathsf{BG}}\mathsf{BG})^{\mathrm{c}} \to \mathcal{H}\mathsf{e}_R(\mathsf{BH}\times_{\mathsf{BG}}\mathsf{BG})^{\mathrm{c,op}}.$$

Note that in case $z = 1$ the paracyclic shift

$$\tau_1 : \mathsf{BH} \times_{\mathsf{BG}} \mathsf{BH} \to \mathsf{BH} \times_{\mathsf{BG}} \mathsf{BH}$$

swaps the components of the fiber product, and the stable GV-duality recovers the rigid duality from Proposition 2.4.

Example 12.6. Let G be a finite group with multiplication μ and $z \in G$ an element. We consider the paracyclic 2-Segal object $\mathsf{N}(\mathsf{BG})^z_\bullet$. For an ∞-semiadditive commutative ring spectrum R, the R-linear categorified group algebra $\mathsf{Perf}(R)[G]$ fulfills the assumption of Theorem 12.9. We describe the duality in more detail.
Consider the morphisms $\mu_z : G \to G$ and $\iota : G \to G$ given by multiplication by z and inversion respectively. By construction the paracyclic shift is given by

$$\tau = \mu_z \circ \iota : G \to G.$$

Consequently, the dualizing object is given by $(z^{-1})_! R \in \mathsf{Perf}(R)[G]$, where $z : * \to G$ denotes the unique morphism with image z. Moreover, the duality equivalence is given by

$$\mathbb{D}_G \simeq \mathcal{H}\mathsf{om}(\iota^* \mu_z^*(-), R_G).$$

For $z = 1$ this recovers the rigid duality from Example 2.5.

For a general stable ∞-category $\mathcal{C}$, its Waldhausen $S_\bullet$-construction $S_\bullet(\mathcal{C})$ generally does not satisfy the assumption of Theorem 12.9. The reason for that is that the homotopy groups of the fiber of the multiplication map $\partial_1 : S_2(\mathcal{C}) \to S_1(\mathcal{C})$ are given by the Ext-groups of the stable ∞-category $\mathcal{C}$ which are for a general stable ∞-category neither finite groups nor bounded. A similar problem appears in the study of derived Hall algebras [Toë06]. Töen solved this problem by considering stable ∞-categories that are linear over the finite field with q-elements $\mathbb{F}_q$ and $\mathbb{F}_q$-proper:

Proposition 12.10. *Let $\mathcal{C}$ be a $\mathbb{F}_q$-linear proper stable ∞-category and $S_\bullet(\mathcal{C})$ its Waldhausen $S_\bullet$-construction. Then for every ∞-semiadditive commutative ring spectrum R the convolution product*

$$* : \mathsf{Loc}_R(S_1(\mathcal{C})) \otimes_R \mathsf{Loc}_R(S_1(\mathcal{C})) \to \mathsf{Loc}_R(S_1(\mathcal{C}))$$

preserves compact objects.

Proof. It suffices to show that the functor preserves compact generators. Let therefore be $C, D : * \to \mathcal{C}^\simeq$ objects and $C_! R, D_! R$ the corresponding compact generators of $\mathsf{Loc}_R(S_1(\mathcal{C}))$. Denote by $\mathsf{Ext}(C, D)$ the mapping space $\mathsf{Map}(D, C[1])$ in $\mathcal{C}$. It follows from base change along the pullback diagram

$$\begin{array}{ccccc} \mathsf{Ext}(C,D) & \xrightarrow{i} & S_2(\mathcal{C}) & \xrightarrow{\partial_1} & S_1(\mathcal{C}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (\partial_2, \partial_0)} & & \\ \{C, D\} & \xrightarrow[(C,D)]{} & \mathcal{C}^\simeq \times \mathcal{C}^\simeq & & \end{array}$$

that we have an equivalence

$$C_! R * D_! R \simeq (\partial_1 \circ i)_! p^*(R).$$

Hence, it suffices to show that $p^*(R)$ is compact in $\mathsf{Loc}_R(\mathsf{Ext}(C, D))$. We do so by showing that $\mathsf{Ext}(C, D)$ is

R-ambidextrous. Note that the i-th homotopy group of the space $\mathsf{Ext}(C, D)$ is given by the abelian group underlying the finite-dimensional $\mathbb{F}_q$-vector space $\mathsf{Ext}^{2-i}_{\mathcal{C}}(C, D)$, i.e the Ext-group of the stable ∞-category $\mathcal{C}$. Since $\mathcal{C}$ is $\mathbb{F}_q$-proper, there exists an $i \in \mathbb{N}$ such that

$$\mathsf{Ext}^{2-j}(C, D) \simeq 0 \quad \text{for all } j > i.$$

Further, every non-zero homotopy group is the abelian group underlying a finite-dimensional $\mathbb{F}_q$-vector space and is therefore finite. Since R is ∞-semiadditive, this implies the claim. □

Remark 12.2. Let $\mathsf{Perf}_{\mathbb{F}_q}$ be the ∞-category of perfect complexes of $\mathbb{F}_q$-vector spaces. Haiden has established a geometric interpretation of a monoidal sub-∞-category of $\mathsf{Loc}_R(S_1(\mathsf{Perf}_{\mathbb{F}_q}))$ using Legendrian tangles in $\mathbb{R}^3$ [Hai21]. In this setup, the GV-duality admits a geometric interpretation in terms of bending Legendrian tangles.

Example 12.7. Let $\mathcal{C}$ be a $\mathbb{F}_q$-linear proper stable ∞-category and denote by $S_\bullet(\mathcal{C})$ its associated Waldhausen $S_\bullet$-construction. It follows from Proposition 12.10 that the paracyclic object $S_\bullet(\mathcal{C})$ satisfies the assumptions of Theorem 12.9. We can conclude that $0_!(S)$ is a dualizing object for $\mathsf{Loc}_R(S_1(\mathcal{C}))^{\mathrm{c}}$ and that the dualizing equivalence is given by $[1]^*\mathbb{D}_{\mathcal{C}^\simeq}(-)$. In particular, we obtain for every pair of objects $C, D \in \mathcal{C}^\simeq$,

$$\hom_{\mathcal{C}^\simeq}(0_!R, C_!R * D_!R) \simeq \hom_{\mathcal{C}^\simeq}(C[1]_!R, D_!R).$$

Note that although the dualizing object is the monoidal unit, the monoidal category is not rigid. Indeed, a rigid monoidal structure would induce for every object $B \in \mathcal{C}^\simeq$ an equivalence

$$\hom_{\mathcal{C}^\simeq}(B_!R, C_!R * D_!R) \simeq \hom_{\mathcal{C}^\simeq}(C[1]_!R * B_!R, D_!R).$$

A short calculation shows that this would induce an equivalence

$$C_*(\mathsf{Ext}(C, D)_B, R) \simeq \hom_R(C_*(\mathsf{Ext}(C[1], B)_D, R), R)$$

where we denote by $\mathsf{Ext}(C, D)_B$ the fiber over $B \in \mathcal{C}^\simeq$ along the map ∂_1. These are the spaces classifying extensions of the form

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & D \end{array} \quad \text{and} \quad \begin{array}{ccc} C[1] & \longrightarrow & D \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array},$$

respectively, which are in general only equivalent if $B \simeq 0$. The failure of this more general equivalence is precisely the difference between a GV and a rigid duality.

A Auxiliary Statements

The goal of this section is to present the technical proofs of Proposition 8.3 and 8.5. We have extracted these technical proofs from Section 8 to maintain the flow of the argument presented there.

A.1 Proof of Proposition 8.3

The goal of this subsection is to prove Proposition 8.3. As a start, we unravel the datum of a bicomodule in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$. Such an object is given by a commutative triangle:

$$\begin{array}{ccc} \Delta_{/[1]} & \xrightarrow{\quad F \quad} & \mathsf{Span}_\Delta(\mathcal{C}^\times) \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

such that F preserves inert morphisms and the adjoint morphism $\tilde{F} : \mathsf{Tw}(\Delta_{/[1]}) \to \mathcal{C}^\times$ maps every n-simplex $\Delta^n \to \Delta_{/[1]}$ to a Segal simplex.[29]

We analyze the second condition first. By the universal property of the Cartesian monoidal structure, $\tilde{F}$ is equivalent to a functor

$$\bar{F} : \mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg} \to \mathcal{C}.$$

Similarly, an n-morphism $\eta : \Delta^n \to \mathsf{Fun}_\Delta(\Delta_{/[1]}, \mathsf{Span}_\Delta(\mathcal{C}^\times))$ is equivalent to a functor

$$\bar{\eta} : \mathsf{Tw}(\Delta_{/[1]} \times \Delta^n) \times_\Delta \Delta^{\amalg} \simeq \mathsf{Tw}(\Delta^n) \times (\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\amalg}) \to \mathcal{C}.$$

We put $\Theta_{n+1} := \mathsf{Tw}(\Delta_{/[1]} \times \Delta^n) \times_\Delta \Delta^{\amalg} \simeq \mathsf{Tw}(\Delta^n) \times \Theta_1$. The objects of this category are given by

$$[i,j] \subset (g_p : [p] \to [1], k) \xrightarrow{f} (g_q : [q] \to [1], l),$$

where $0 \leq k \leq l \leq n$, $f : g_p \to g_q$ is a morphism in $\Delta_{/[n]}$ and $[i,j]$ is a subinterval of $[p]$.

Definition A.1. Let $\bar{\eta} : \mathsf{Tw}(\Delta^n \times \Delta_{/[1]}) \times_\Delta \Delta^{\amalg} \to \mathcal{C}$ be a morphism of simplicial sets, $\sigma : \Delta^k \to \Delta^n \times \Delta_{/[1]}$ a k-simplex

$$(g_{n_0}, i_0) \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_k} (g_{n_k}, i_k) \ ,$$

and $[i,j] \subset [n_0]$ a subinterval. The *Segal cone* of $\bar{\eta}$ associated to σ and $[i,j]$ is the diagram

$$\begin{array}{ccccccc} & & & \bar{\eta}(\psi_n, [i,j], i_0 < i_k) & & & \\ & & \swarrow & & \searrow & & \\ & \bar{\eta}(\phi_1, [i,j], i_0 < i_1) & & \cdots & & \bar{\eta}(\phi_k, [\psi_{k-1}(i), \psi_{k-1}(j)], i_{k-1} < i_k) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \bar{\eta}(g_{n_0}, [i,j], i_0) & & \bar{\eta}(g_{n_1}, [\psi(i), \psi(j)], i_1) & & \bar{\eta}(g_{n_{k-1}}, [\psi_{k-1}(i), \psi_{k-1}(j)], i_{k-1}) & & \bar{\eta}(g_{n_k}, [\psi_k(i), \psi_k(j)], i_k) \end{array}$$

where we put $\psi_i := \phi_i \circ \phi_{i-1} \circ \ldots \circ \phi_1$.

Proposition A.1. *Let $\bar{\eta} : \mathsf{Tw}(\Delta^n \times \Delta_{/[1]}) \times_\Delta \Delta^{\amalg} \to \mathcal{C}$ be a morphism of simplicial sets. Its adjoint η defines an n-morphism in $\mathsf{Fun}_\Delta(\Delta_{/[1]}, \mathsf{Span}_\Delta(\mathcal{C}^\times))$ if and only if for every k-simplex*

$$\begin{array}{ccc} ([n_0], i_0) \xrightarrow{\phi_1} & \ldots & \xrightarrow{\phi_k} ([n_k], i_k) \\ \searrow & & \swarrow \\ & [1] & \end{array}$$

in $\Delta_{/[1]} \times \Delta^n$ and every interval $[i,j] \subset [n_0]$ the associated Segal cone of $\bar{\eta}$ is a limit diagram in $\mathcal{C}$.

[29]See Construction 8.3

Proof. By definition of $\mathsf{Span}_\Delta(\mathcal{C}^\times)$, we need to show that for every k-simplex $\Delta^k \to \Delta^n \times \Delta_{/[1]}$ given by

$$(g_{n_0}, i_0) \to (g_{n_1}, i_1) \to \dots \to (g_{n_k}, i_k), y$$

the restriction of $\bar{\eta}$ to $\mathsf{Tw}(\Delta^k) \subset \mathsf{Tw}(\Delta_{/[1]} \times \Delta^n)$ is a Segal simplex in $\mathcal{C}^\times$. According to [DK19, Lem.10.2.13] there exists a functor

$$\tilde{H}_t : (\Delta^1 \times \mathsf{Tw}(\Delta^k)) \times_\Delta \Delta^{\mathrm{II}} \to \mathcal{C}$$

adjoint to a homotopy

$$H_t : (\Delta^1 \times \mathsf{Tw}(\Delta^k)) \to \mathcal{C}^\times$$

such that $H_1 = \bar{\eta}$, the components of this homotopy are Cartesian morphisms and H_0 has image contained in $\mathcal{C}^\times_{g_0}$. Consequently, the restriction of $\bar{\eta}$ to the Segal cone[30] is a relative limit in $\mathcal{C}^\times$ if and only if the restriction of H_0 to the Segal cone is a limit in $\mathcal{C}^\times_{g_0}$ [Lur09a, Prop. 4.3.1.9].
We can check the latter for each interval $[i, j] \subset [n_0]$ individually. The restriction of $\tilde{H}_0$ to the Segal cone is represented in $\mathcal{C}$ by the diagram

$$\begin{array}{ccccccc}
 & & & \tilde{H}_0(\psi_n, [i,j], i_0 < i_n) & & & \\
 & \swarrow & & & & \searrow & \\
 & \tilde{H}_0(\phi_1, [i,j], i_0 < i_1) & & \cdots & & \tilde{H}_0(\phi_k, [i,j], i_{k-1} < i_k) & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\tilde{H}_0(g_{n_0}, [i,j], i_0) & & \tilde{H}_0(g_{n_1}, [\psi_1(i), \psi_1(j)], i_1) & & \tilde{H}_0(g_{n_{k-1}}, [\psi_{k-1}(i), \psi_{k-1}(j)], i_{k-1}) & & \tilde{H}_0(g_{n_k}, [\psi_k(i), \psi_k(j)], i_k)
\end{array}$$

Since the components of the homotopy are Cartesian, it restricts to an equivalence between this diagram and the Segal cone diagram of $\bar{\eta}$ associated to σ and $[i, j]$. Therefore the simplex is a Segal simplex, if for all simplices σ and all subintervals $[i, j]$, the Segal cone diagram of $\bar{\eta}$ associated to σ and $[i, j]$ are limit diagrams. □

In fact, it turns out to be sufficient to check this condition for 2-simplices:

Lemma A.2. *Let $\mathcal{C}$ be an ∞-category with finite limits. For an n-simplex*

$$F : \Delta^n \to \mathsf{Fun}_\Delta(\Delta_{/[1]}, \overline{\mathsf{Span}_\Delta}(\mathcal{C}^\times))$$

the following are equivalent:

(1) F restricts to a functor $F : \Delta^n \to \mathsf{Fun}_\Delta(\Delta_{/[1]}, \mathsf{Span}_\Delta(\mathcal{C}^\times))$.

(2) for every non-degenerate 2-simplex $\Delta^2 \to \Delta_{/[1]} \times \Delta^n$ the restriction $F|_{\Delta^2}$ is a Segal simplex.

Proof. This follows from the iterated application of [Lur09a, Prop.4.2.3.8]. □

To study the first condition, recall that a morphism $\phi : \Delta^1 \to \Delta_{/[1]}$ is called inert if its image in Δ is inert. If we depict a morphism in $\Delta_{/[1]}$ by a commutative diagram

$$\begin{array}{ccc}
[n_0] & \xrightarrow{f} & [n_1] \\
 & \searrow \quad \swarrow & \\
 & [1] &
\end{array}$$

[30] See Construction 8.3

then the morphism is inert when f is the inclusion of a subinterval. Let $\phi : \Delta^1 \to \Delta_{/[1]}$ be an inert morphism and F a bicomodule object in $\mathsf{Span}_\Delta(\mathcal{C}^\times)$. By definition $F : \Delta_{/[1]} \to \mathsf{Span}_\Delta(\mathcal{C}^\times)$ has to map inert morphisms to Cartesian morphisms. Unraveling the definition of the Cartesian fibration $\mathsf{Span}_\Delta(\mathcal{C}^\times)$, it follows that the image of ϕ is Cartesian, if and only if the adjoint map $\widetilde{F \circ \phi} : \mathsf{Tw}(\Delta^1) \to \mathcal{C}^\times$ maps all morphisms in $\mathsf{Tw}(\Delta^1)$ to Cartesian morphisms. In other words for any morphism $\phi : g^0_{n_0} \to g^1_{n_1}$ and any interval $[i,j] \subset [n_0]$

(1) the morphism $F(\phi, \{i,j\}) \to F(g^0_{n_0}, \{i,j\})$ induced by the source map $\phi \to g^0_{n_0}$ in $\mathsf{Tw}(\Delta_{/[1]})$ is an equivalence.

(2) the morphism $F(\phi, \{i,j\}) \to F(g^1_{n_1}, \{\phi(i), \phi(j)\})$ induced by the target map $\phi \to g^1_{n_1}$ in $\mathsf{Tw}(\Delta_{/[1]})$ is an equivalence.

We derive some consequences of this:

Lemma A.3. *Suppose $F : \Delta_{/[1]} \to \mathsf{Span}_\Delta(\mathcal{C}^\times)$ represents a cobimodule. Let $f : g^0_{n_0} \to g^1_{n_1}$ be a morphism in $\Delta_{/[1]}$ viewed as an object in $\mathsf{Tw}(\Delta_{/[1]})$*

(1) *Denote for $[i,j] \subset [n_0]$ by $f|_{[i,j]}$ the restriction of f to $g^0_{[i,j]} \subset g^0_{n_0}$. Then the induced morphism*

$$F(f|_{[i,j]}, [i,j]) \to F(f, [i,j])$$

is an equivalence.

(2) *Let $\tilde{f} : g^0_{n_0} \to g^1_{[l,k]}$ be a morphism with $[l, ..., k] \subset [n_1]$, such that the composite with the inert morphism $\phi : g^1_{[l,k]} \hookrightarrow g^1_{n_1}$ is f. Then the induced morphism*

$$F(f, [l,k]) \to F(\tilde{f}, [l,k])$$

is an equivalence.

Proof. The proof is analogous to [Ste21, Prop.2.2] □

Lemma A.4. *Suppose $F : \Delta_{/[1]} \to \mathsf{Span}_\Delta(\mathcal{C}^\times)$ represents a cobimodule. Let σ be a morphism in the ∞-category $\mathsf{Tw}(\Delta_{/[1]}) \times_\Delta \Delta^{\mathrm{II}}$ of the form:*

$$\sigma = \left\{ \begin{array}{ccc} [i,j] \subset g^0_{n_0} & \xrightarrow{g} & g^1_{n_1} \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \tilde{f}} \\ [\tilde{i},\tilde{j}] \subset g^0_{m_0} & \xrightarrow[\tilde{g}]{} & g^1_{m_1} \end{array} \right\}$$

such that f restricts to an isomorphism $[i, ..., j] \to [\tilde{i}, ..., \tilde{j}]$ and $\tilde{f}$ restricts to an isomorphism

$$[g(i), ..., g(j)] \to [\tilde{g}(\tilde{i}), ..., \tilde{g}(\tilde{j})].$$

Then F sends σ to an equivalence.

Proof. We decompose σ into the diagram

$$\begin{array}{ccc} g^0_{[i,j]} & \xrightarrow{g_{[i,j]}} & g^1_{n_1} \\ {\scriptstyle\phi}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ g^0_{n_0} & \xrightarrow{g} & g^1_{m_1} \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \bar{f}} \\ g^0_{m_0} & \xrightarrow[\bar{g}]{} & g^1_{m_1} \end{array}$$

Note that the upper square gets mapped by F to an equivalence and that the composite of the left vertical maps is inert. So by 2-out-of-3 we can restrict to those σ, such that the morphism f is inert. Such a diagram σ can be further decomposed as:

$$\begin{array}{ccc} g^0_{[i,j]} & \xrightarrow{g_{[i,j]}} & g^1_{n_1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \bar{f}} \\ g^0_{[i,j]} & \xrightarrow{\bar{g}\circ f} & g^1_{m_1} \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ g^0_{m_0} & \xrightarrow[\bar{g}]{} & g^1_{m_1} \end{array}$$

Using the same argument as above applied to the lower square, we can reduce to the case that $f = \mathrm{id}$. So let σ be a diagram of the form

$$\begin{array}{ccc} g^0_{n_0} & \xrightarrow{g_{[i,j]}} & g^1_{n_1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \bar{f}} \\ g^0_{n_0} & \xrightarrow[\bar{g}]{} & g^1_{m_1} \end{array}$$

such that $\bar{f}$ induces an isomorphism $[\bar{g}(0), \bar{g}(n_0)] \to [g(0), g(n_0)]$. In particular, there exists a decomposition

$$\begin{array}{ccc} g^0_{n_0} & \xrightarrow{g} & g^1_{n_1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \bar{f}} \\ g^0_{n_0} & \xrightarrow{\bar{g}} & g^1_{m_1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \psi} \\ g^0_{n_0} & \xrightarrow[\bar{g}]{} & g^1_{[\bar{g}(0),\bar{g}(n_0)]} \end{array}$$

with ψ inert. It follows by assumption that $f \circ \psi$ is itself inert. Hence the claim follows from 2-out-of-3 again. $\square$

Putting these results together, we have proven Proposition 8.3:

Proposition A.5. *A functor $F : \Theta_1 \to \mathcal{C}$ defines a bicomodule object if and only if*

(1) F sends degenerate intervals to terminal objects.

(2) F sends every object $(\phi : g^0_{[n_0]} \to g^1_{[n_1]}, [i,j])$ together with its projection to subintervals to a product diagram.[31]

(3) F sends morphisms of the form

$$\sigma \simeq \left\{ \begin{array}{ccc} [i,j] \subset g^0_{n_0} & \xrightarrow{g} & g^1_{n_1} \\ \downarrow f & & \uparrow \tilde{f} \\ [\tilde{i},\tilde{j}] \subset g^0_{m_0} & \xrightarrow{\tilde{g}} & g^1_{m_1} \end{array} \right\}$$

s.t. the morphism f restricts to an isomorphism $\{i,...,j\} \to \{\tilde{i},...,\tilde{j}\}$ and the morphism $\tilde{f}$ restricts to an isomorphism $\{g(i),...,g(j)\} \to \{\tilde{g}(\tilde{i}),...,\tilde{g}(\tilde{j})\}$ to equivalences. We denote by E the wide subcategory of Θ_1 with morphisms of the above form.

(4) F maps all Segal cone diagrams from Definition A.1 to limit diagrams.

A.2 Proof of Proposition 8.5

The goal of this subsection is to prove the following:

Proposition A.6. *The functor $\mathcal{L} : \Omega_1 \to \Delta_1^\star$ from Construction 8.4 is an ∞-categorical localization at the morphisms E, defined in Corollary 8.3 (3).*

To prove this, we show that this functor satisfies the assumptions of [Wal21, Lem.3.1.1]. To this end, we need to consider for any object $N = (e^0_{m_0}, ..., e^{k-1}_{m_{k-1}}) \in \Delta_1^\star$ the strict fiber of the functor $\mathcal{L}$ at N. We denote the strict fiber by $\Omega_{1,N}$ and we denote by $\Omega^E_{1,N}$ the subcategory of $\Omega_{1,N}$ with morphisms in E. To apply [Wal21, Lem.3.1.1], we first construct an explicit initial object in $\Omega^E_{1,N}$:

Construction A.1. For any $N = (e^0_{m_0}, ..., e^{k-1}_{m_{k-1}}) \in \Delta_1^\star$ as above we define the object $I_N \in \Omega_{1,N}$ as

$$\begin{array}{ccc} [0,k] \subset [k] & \xrightarrow{f_N} & [m_0] * ... * [m_{k-1}] \\ & {\scriptstyle e_k} \searrow \quad [1] \quad \swarrow {\scriptstyle e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}} & \end{array}$$

where the morphism f_N is defined by

$$f_N(i) = \begin{cases} 0 \in [m_i] & i \leq k \\ m_{k-1} \in [m_{k-1}] & i = k \end{cases},$$

and e_k is the unique morphism that makes the triangle commute. If $e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}$ is supported at $\{0\}$, we denote by $I_N * \mathrm{id}_{[1]}$ the object

$$\begin{array}{ccc} [0,k] \subset [k] & \xrightarrow{f_N} & [m_0] * ... * [m_{k-1}] * [1] \\ & \searrow \quad [1] \quad \swarrow {\scriptstyle e^0_{m_0} * ... * e^{k-1}_{m_{k-1}} * \mathrm{id}_{[1]}} & \end{array}$$

Similarly, we denote by $\mathrm{id}_{[1]} * I_N$ the analogously defined object if $e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}$ is supported at $\{1\}$.

[31] See Construction 8.2

Lemma A.7. *For every N the category $\Omega^E_{1,N}$ has an initial object given by*

(1) $I_N * \mathrm{id}_{[1]}$ if $e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}$ is supported at 0.

(2) I_N if $e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}$ is surjective.

(3) $\mathrm{id}_{[1]} * I_N$ if $e^0_{m_0} * ... * e^{k-1}_{m_{k-1}}$ is supported at 1.

Proof. We will prove that the respective object is initial. Let

$$[i,j] \subset [n] \xrightarrow{\alpha} [l] \xrightarrow{h} [1]$$

be an object in $\Omega^E_{1,N}$. We will construct a unique morphism Φ in $\Omega^E_{1,N}$ from the claimed initial object. In any of the above cases, we define the morphism $\phi : [k] \to [n]$ as the inert map that includes $[k]$ as the interval $\{i, ..., j\}$. This is uniquely defined since every morphism in E has to induce an isomorphism between the chosen subintervals. We can decompose[32] the object $[l] \xrightarrow{h} [1]$ into

$$[l_{left}] * \{\alpha(i), ..., \alpha(j)\} * [l_{right}] \xrightarrow{h_{left} * h * h_{right}} [1].$$

Since any morphism in E has to map $\{\alpha(i), ..., \alpha(j)\}$ isomorphically onto $[f_N(0), f_N(k)] = [n]$, the restriction of Φ to $\{\alpha(i), ..., \alpha(j)\}$ is uniquely determined. We show that our assumptions allow us to extend this arrow uniquely to the outer part. In case (2), we can uniquely extend. We only discuss case (1), since (3) is analogous. In case (1), we can uniquely extend Φ to $[l_{left}]$ by setting it constantly 0. An extension to $[l_{right}]$ is equivalent to a morphism to $\mathrm{id}_{[1]}$ in $\Delta_{/[1]}$. This morphism exists uniquely since $\mathrm{id}_{[1]}$ is a final object of $\Delta_{/[1]}$. □

We furthermore need to show that the inclusion $\Omega^E_{1,N} \subset \Omega^1_{/N}$ of the strict into the lax fiber is cofinal. To show this, we make the following preliminary considerations. By Quillen's theorem A [Lur09a, Thm.4.1.3.1] this amounts to showing, that for every object

$$\{i,j\} \subset [n] \xrightarrow{\alpha} [l] \xrightarrow{h} [1],$$

whose image under $\mathcal{L}$ is given by $(h_{l_{i+1}}, ..., h_{l_j})$, and every morphism

$$g : (h_{l_{i+1}}, ..., h_{l_j}) \to (e_{m_0}, ..., e_{m_{k-1}})$$

in $\Delta^\star_1$, the category $\Omega^E_{N,1} \times_{(\Omega_1)_{/N}} ((\Omega_1)_{/N})_{g/}$ is contractible. We do so by constructing an explicit initial object. By definition, the morphism g amounts to a pair consisting of a morphism $\gamma : [k-1] \to \{i+1, ..., j\}$ in Δ and a morphism

$$\bar{g} : [m_0] * ... * [m_{k-1}] \to [l_{i+1}] * ... * [l_j] \text{ in } \Delta_{/[1]}.$$

We denote by $[n_c] := \{p, ..., q\} \subset \{i,j\} \subset [n]$ the unique linearly ordered set such that γ factors as

$$\gamma : [k-1] \to [n_c] \hookrightarrow [n_l] * [n_c] * [n_r].$$

By construction $\bar{g}$ has image contained in $[l_c] := [l_{p+1}] * ... * [l_q]$. We introduce the following decompositions:

$$[l] = [l_l] * [l_c] * [l_r],$$
$$[m] = [m_l] * [m_c] * [m_r].$$

[32] See Construction 8.1

Lemma A.8. *There exists a morphism in* Ω_1

$$\begin{array}{ccc} [p,q] \subset d_{[p,\dots,q]} & \longrightarrow & h_{l_c} \\ \downarrow & & \uparrow \\ [0,k] \subset h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} & \longrightarrow & h_{l^1} * e_m * h_{l^2} \end{array}$$

that extends to a morphism

$$\mu_{Z,N} := \left\{ \begin{array}{ccc} Z = [i,j] \subset d_{n_l} * d_{n_c} * d_{n_r} & \longrightarrow & h_{l_l} * h_{l_c} * h_{l_r} \\ \downarrow & & \uparrow \\ Z_N = [0,k] \subset d_{n_l} * h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} * d_{n_r} & \longrightarrow & h_{l_l} * h_{l^1} * e_m * h_{l^2} * h_{l_r} \end{array} \right\}$$

covering g*, such that* $\mu_{Z,N}$ *is an initial object in* $\Omega^E_{N,1} \times_{(\Omega_1)_{/N}} ((\Omega_1)_{/N})_{g/}$.

Proof. We define a morphism $\nu : \{p+1,...,q-1\} \to [k]$ as $\mathbb{I}(\gamma)$ and extend it to a morphism

$$\nu : \{p,...,q\} \to [1] * [k] * [1]$$

by mapping endpoints to endpoints. We decompose $[l_c] = [l^1] * [l^m_c] * [l^2]$, where we denote by $[l^m_c]$ the subinterval of $[l]$, that contains the image of $\bar{g}$. By construction, the map $\bar{g} : e_m \to h_{[l^m_c]}$ hits both endpoints. Therefore, we can extend it to a morphism

$$\bar{g}' := \mathrm{id}_{h_{l^1}} * g * id_{h_{l^2}} : h_{l^1} * e_m * h_{l^2} \to h_{l^c}.$$

By construction $\bar{g}'$ also hits both endpoints and extends uniquely to a morphism over [1]. Similarly, we define

$$\bar{f}_N : h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} \to h_{l^1} * e_m * h_{l^2}$$

to be f_N on e_k and to send endpoints to endpoints. This morphism also extends uniquely to a morphism over [1]. By decomposing the morphisms ν, h and $\bar{f}_N \circ \nu$, we obtain a not necessarily commuting diagram

$$\begin{array}{ccc} [p,q] \subset d_{[1_{p+1}]} * ... * d_{[1_p]} & \xrightarrow{\alpha} & h_{l_{p+1}} * ... * h_{l_q} \\ \downarrow \nu & & \uparrow \bar{g}' \\ [0,k] \subset h_{\{0,l^1\}} * e_{k_{p+1}} * ... * e_{k_q} * h_{\{0,l^2\}} & \xrightarrow[\bar{f}_N]{} & h_{l^1} * e_{m_{p+1}} * ... * e_{m_q} * h_{l^2} \end{array}$$

This diagram commutes in $\Delta_{/[1]}$ if and only if it commutes restricted to each individual $[1_r]$ with $r \in \{p{+}1 \leq q\}$. Since all maps preserve endpoints, this is clear for $r = p+1$ and $r = q$. In the other cases, it suffices to show that $\bar{g}'$ sends the endpoints of $[n_r]$ to the endpoints of $[m_r]$. This follows since $\bar{g}'$ is induced by a morphism g in $\Delta^\star_1$.

Since all morphisms preserve endpoints, we can extend the diagram by taking star products with the morphisms $\mathrm{id}_{d_{n_l}}$, $\mathrm{id}_{d_{n_r}}$, $\mathrm{id}_{h_{l_l}}$, $\mathrm{id}_{h_{l_r}}$, $\alpha|_{d_{n_l}} : d_{n_l} \to d_{l_l}$ and $\alpha|_{d_{n_r}} : d_{n_r} \to d_{l_r}$

$$\begin{tikzcd} Z = \{i,j\} \subset d_{n_l} * d_{n_c} * d_{n_r} \arrow[r] \arrow[d] & h_{l_l} * h_{l_c} * h_{l_r} \\ Z_N = \{0,k\} \subset d_{n_l} * h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} * d_{n_r} \arrow[r] & h_{l_l} * h_{l^1} * e_m * h_{l^2} * h_{l_r} \arrow[u] \end{tikzcd}$$

By construction, this diagram defines a morphism in Ω_1 covering g. We call this morphism $\mu_{Z,N}$. Let us show that $\mu_{Z,N}$ is initial. Suppose, we are given another morphism in Ω_1

$$\begin{tikzcd} Z = [i,j] \subset d_n \arrow[r, "\alpha"] \arrow[d, "\rho"] & h_l \\ X = [0,k] \subset x_a \arrow[r, "\beta"'] & y_b \arrow[u, "w"'] \end{tikzcd}$$

covering g. We can decompose it as follows

$$\begin{tikzcd} Z = [i,j] \subset d_{n_l} * d_{n_c} * d_{n_r} \arrow[r, "\alpha"] \arrow[d, "\rho"] & h_{l_l} * h_{l_c} * h_{m_r} \\ X = [0,k] \subset x_{a_l} * x_{a_c} * x_{a_r} \arrow[r, "\beta"'] & y_{b_l} * y_{b_c} * y_{b_r} \arrow[u, "w"'] \end{tikzcd} \tag{43}$$

It follows from [Ste21, Lem. A.3] that ρ is uniquely determined except at the endpoints by γ. Therefore, the restriction of diagram Diagram (43) to d_{n_c} looks as follows:

$$\begin{tikzcd} d_{n_c} \arrow[r, "h"] \arrow[d, "\rho"] & h_{l_c} \\ x_{a_c^1} * e_k * x_{a_c^2} \arrow[r, "\beta_c^1 * f_N * \beta_c^2"] & y_{b_c^1} * e_n * y_{b_c^2} \arrow[u, "w"'] \end{tikzcd}$$

Every morphism $Z_N \to X$ in E commuting with the above morphism $Z \to X$ and $\mu_{Z,M}$, must in particular restrict to a commutative diagram

$$\begin{tikzcd} d_{n_c} \arrow[rr] \arrow[d] \arrow[drrr] & & h_{l_c} & \\ h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} \arrow[rr, dashed] \arrow[drr] & & h_{l^1} * e_m * h_{l^2} \arrow[u] & \\ & & x_{a_c^1} * e_k * x_{a_2^c} \arrow[r] & y_{b_c^1} * e_m * y_{b_c^2} \arrow[ul] \arrow[uul] \end{tikzcd} \tag{44}$$

Moreover, since the bottom square is in E, it must restrict to a commutative diagram

$$\begin{tikzcd} e_k \arrow[r] \arrow[d] & e_m \\ e_k \arrow[r] & e_m \arrow[u] \end{tikzcd}$$

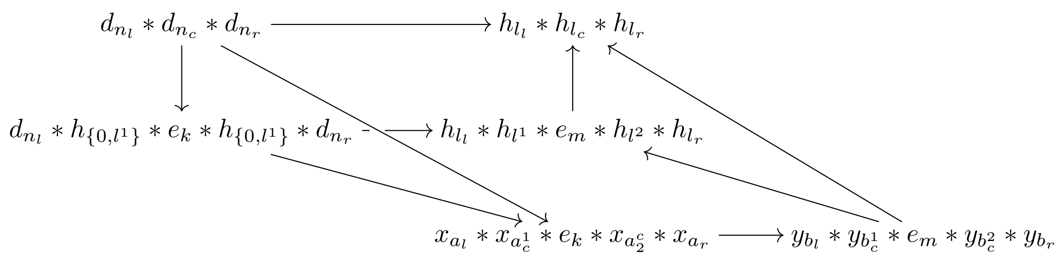
It follows that the component $h_{\{0,l^1\}} * e_k * h_{\{0,l^2\}} \to x_{a_c^1} * e_k * x_{a_c^2}$ is uniquely determined by the left hand triangle in Diagram 44. We can further decompose w, since it must restrict to $\bar{g}$ on e_m, into

$$w = w^1 * g * w^2 : y_{b_c^1} * e_m * y_{b_c^2} \to h_{l^1} * h_{l_c^m} * h_{l^2}$$

Hence, the component of the bottom morphism in the right hand triangle is uniquely determined and must be given by

$$w^1 * \mathrm{id}_{[m]} * w^2 : y_{b_c^1} * e_m * y_{b_c^2} \to h_{l^1} * e_m * h_{l^2}.$$

We can now extend back to the full diagram



Note that the vertical arrows in the back square are given by identities when restricted to h_{l_l}, h_{l_r}, e_{m_l}, and e_{m_r}. Hence, the bottom square is uniquely determined by the morphisms $h_{l_l} \to x_{a_l}$, $h_{l_r} \to x_{a_r}$, $y_{b_l} \to e_{m_l}$, and $y_{b_r} \to e_{m_r}$. So there is a unique morphism $Z_N \to X$ with the desired properties.

□

Proof of Proposition 8.5. The result follows from combining Lemma A.8 and Lemma A.7 with [Wal21, Lem.3.1.1.]. □

B Presentable ∞-Categories

In Section 4.1, we have recalled basic constructions with ∞-categories that are linear over a fixed symmetric monoidal ∞-category $\mathcal{V}$. The main goal of this section is to extend these constructions to the context of $(\infty, 2)$-categories. The constructions that we present here have already been sketched in [GR19, Sect.I.1], and we provide a more detailed discussion. Therefore, we first construct in Appendix B.1 for every collection of small ∞-categories $\mathcal{K}$ a large $(\infty, 2)$-category of small $\mathcal{K}$-cocomplete ∞-categories. Afterward, we equip in Appendix B.2 these with symmetric monoidal structures by extending the construction of [Lur17, Sect.4.8.1] to the context of $(\infty, 2)$-categories. Finally, we use these preliminary constructions to construct symmetric monoidal $(\infty, 2)$-categories of $\mathcal{V}$-linear ∞-categories. A different construction of this $(\infty, 2)$-category has been presented using enriched ∞-categories in [RZ25, Sect.2].

B.1 $(\infty, 2)$-Categories of ∞-Categories

The goal of this section is to construct symmetric monoidal $(\infty, 2)$-categories of small and large ∞-categories using complete Segal objects in Cat. We follow the convention to denote large ∞-categories of small ∞-categories by small and very large ∞-categories of large ∞-categories by capital letters. For example, the large ∞-category of small ∞-categories is denoted by Cat, whereas the very large ∞-category of large ∞-categories is denoted by CAT. We start by recalling the definition of complete Segal ∞-categories:

Definition B.1. Let $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Cat}$ be a simplicial ∞-category. $X_\bullet$ is called

(1) a *double ∞-category*, if for every $n \geq 2$ and every $0 < i \leq n$ the inclusions $\rho_i : [1] \simeq \{i-1, i\} \hookrightarrow [n]$ induce an equivalence of ∞-categories.

$$X_n \longrightarrow X_{\{0,1\}} \times_{X_1} \cdots \times_{X_{n-1}} X_{\{n-1,n\}}.$$

(2) *essentially constant* if the ∞-category X_0 is an ∞-groupoid.

(3) *complete* if the map $s_0 : X_0 \to X_1$ induces an equivalence onto the full subcategory X_1^{inv} of invertible morphisms.

We denote by $\mathsf{Seg}(\mathsf{Cat})$, $\mathsf{Seg}_{\mathsf{ec}}(\mathsf{Cat})$, and $\mathsf{CSS}(\mathsf{Cat})$ the full sub ∞-categories of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ spanned by double ∞-categories, essentially constant double ∞-categories, and essentially constant, complete double ∞-categories respectively.

The ∞-category $\mathsf{CSS}(\mathsf{Cat})$ is equivalent to the large ∞-category of small $(\infty, 2)$-categories Cat_2 and we frequently identify these two. In particular, all $(\infty, 2)$-categories that we construct in this text are described by essentially constant complete double ∞-categories.
To every double ∞-category $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Cat}$, one can associate a Segal space by composing with the functor $(-)^{\simeq} : \mathsf{Cat} \to \mathcal{S}$ that associates to an ∞-category its underlying ∞-groupoid. In particular, if $X_\bullet$ is an $(\infty, 2)$-category, this functor associates to $X_\bullet$ a complete Segal space and hence an ∞-category. This ∞-category is called the *underlying ∞-category of $X_\bullet$* and is denoted by $\iota_1 X$.
As shown in [Lur09b], there exists functors $I : \mathsf{Seg}(\mathsf{Cat}) \to \mathsf{Seg}_{\mathsf{ec}}(\mathsf{Cat})$ and $L : \mathsf{Seg}_{\mathsf{ec}}(\mathsf{Cat}) \to \mathsf{CSS}(\mathsf{Cat})$ that are right (resp. left adjoint) to the corresponding inclusion. For every double ∞-category $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Cat}$. we denote the $(\infty, 2)$-category represented by $L(I(X_\bullet))$ by $\mathbb{X}$ and call it the *underlying $(\infty, 2)$-category of $X_\bullet$*. The functor I explicitly sends a double ∞-category to the pullback

$$\begin{array}{ccc} I(\mathcal{C})_\bullet & \longrightarrow & \mathcal{C}_\bullet \\ \downarrow & & \downarrow \\ (\mathcal{C}_0^{\simeq})^{\times \bullet + 1} & \longrightarrow & \mathcal{C}_0^{\times \bullet + 1} \end{array}$$

in $\mathsf{Seg}(\mathsf{Cat})$, where we denote by $\mathcal{C}_0^{\times \bullet + 1}$ the simplicial ∞-category whose n-simplices are given by $\mathcal{C}_0^{\times n+1}$.
For every pair of object $x, y \in X_0$ in a double ∞-category $X_\bullet \in \mathsf{Seg}(\mathsf{Cat})$, one can extract an *$\infty$-category of horizontal morphisms* as the pullback of ∞-categories

$$\begin{array}{ccc} X(x,y) & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ * & \xrightarrow{(x,y)} & X_0 \times X_0 \end{array}$$

Analogously, we define the ∞-category of morphisms of an $(\infty, 2)$-category. It follows from the construction of the underlying $(\infty, 2)$-category $\mathbb{X}$ of the double ∞-category $X_\bullet$, that the morphism ∞-categories are equivalent. Note that since the functor $(-)^{\simeq}$ preserves limits the morphism space of the underlying ∞-category $\iota_1 \mathbb{X}$ of an $(\infty, 2)$-category $\mathbb{X}$ is the underlying space of the morphism ∞-category of $\mathbb{X}$.
We also need a notion of full sub-$(\infty, 2)$-category. Let $\mathbb{X}$ be an $(\infty, 2)$-category represented as an essentially constant complete double ∞-category, and let $Y \subset \mathbb{X}_0$ be a subspace of the space of objects $\mathbb{X}_0$. The *full*

sub-$(\infty,2)$*-category* $\mathbb{Y}$ with space of object Y_0 is defined as the pullback of double ∞-categories:

$$\begin{array}{ccc} \mathbb{Y}_\bullet & \longrightarrow & \mathbb{X}_\bullet \\ \downarrow & & \downarrow \\ Y_0^{\times \bullet+1} & \longrightarrow & X_0^{\times \bullet+1} \end{array}$$

Further, we need the following local version of a full subcategory. We call an $(\infty,2)$-functor $F:\mathbb{X}\to\mathbb{Y}$ a *locally full inclusion* if it induces a monomorphism $\mathbb{X}_0\to\mathbb{Y}_0$ on spaces of objects and for every pair of objects X,X' it induces a fully faithful functor $\mathbb{X}(X,X')\to\mathbb{Y}(F(X),F(X'))$ on mapping categories. We then call $\mathbb{X}$ a locally full sub-$(\infty,2)$-category. In particular, for any $(\infty,2)$-category $\mathbb{X}$, we can associate to any pair consisting of a subspace $\mathbb{X}_0'\subset\mathbb{X}_0$ and a full subcategory $\mathbb{X}_1'\subset\mathbb{X}_1$, such that the restriction of the source and target map to $\mathbb{X}_1'$ have image in $\mathbb{X}_0'$ and $\mathbb{X}_1'$ is closed under composition a locally full $(\infty,2)$-category $\mathbb{X}'\subset\mathbb{X}$ [GR19, 10.2.3.6].
After these preliminary discussions, we now turn to the main construction of this section. Let $\mathcal{K}$ be a set of small ∞-categories. Our plan is to construct the $(\infty,2)$-category of $\mathcal{K}$-cocomplete ∞-categories as a locally full sub-$(\infty,2)$-category of the $(\infty,2)$-category of ∞-categories. For this, we first need the following definition:

Definition B.2. Let $\mathcal{K}$ be a collection of ∞-categories and $p:\mathcal{C}\to\mathcal{D}$ a cocartesian fibration of ∞-categories. We call the fibration *fiberwise* $\mathcal{K}$*-cocomplete* if

(1) for every $d\in\mathcal{D}$ the fiber $\mathcal{C}_d$ admits $\mathcal{K}$-indexed colimits.

(2) for every morphism $f:d\to d'$ the cocartesian transport functor $f_!:\mathcal{C}_d\to\mathcal{C}_{d'}$ preserves $\mathcal{K}$-indexed colimits.

For fiberwise $\mathcal{K}$-cocomplete cocartesian fibrations $\mathcal{C}^0\to\mathcal{D}$ and $\mathcal{C}^1\to\mathcal{D}$, we denote by $\mathsf{Fun}^{\mathcal{K}}_{\mathcal{D}}(\mathcal{C}^0,\mathcal{C}^1)$ the full subcategory of $\mathsf{Fun}_{\mathcal{D}}(\mathcal{C}^0,\mathcal{C}^1)$ spanned by those functors $F:\mathcal{C}^0\to\mathcal{C}^1$, such that for every $d\in\mathcal{D}$ the restriction $F_d:\mathcal{C}^0_d\to\mathcal{C}^1_d$ preserves $\mathcal{K}$-indexed colimits. We call such a functor F *fiberwise* $\mathcal{K}$*-cocontinuous*.

Let $\mathcal{K}$ be a collection of small ∞-categories. For any small ∞-category $\mathcal{C}$, we denote by $\mathsf{Cocart}^{\mathcal{K}}_{\mathcal{C}}$ the subcategory of the large ∞-category $\mathsf{Cat}^{\mathcal{K}}_{/\mathcal{C}}$ with objects fiberwise $\mathcal{K}$-cocomplete cocartesian fibrations and morphisms fiberwise $\mathcal{K}$-cocontinuous functors. Further, we denote by $\mathsf{Cocart}^{\mathcal{K},\simeq}_{\mathcal{C}}$ the wide subcategory of $\mathsf{Cocart}^{\mathcal{K}}_{\mathcal{C}}$ with morphisms those fiberwise $\mathcal{K}$-cocontinuous functors $F:\mathcal{D}\to\mathcal{E}$ over $\mathcal{C}$, such that for any $c\in\mathcal{C}$ the restriction of F to the fiber over c is an equivalence of ∞-categories.
These constructions assemble into ∞-functors

$$\mathsf{Cocart}^{\mathcal{K}}_{-},\mathsf{Cocart}^{\mathcal{K},\simeq}_{-}:\mathsf{Cat}^{\mathrm{op}}\to\mathsf{CAT}$$

with target the ∞-category of large ∞-categories. On morphisms this ∞-functor maps a functor $F:\mathcal{C}\to\mathcal{D}$ to the functor $F^*:\mathsf{Cocart}_{\mathcal{D}}\to\mathsf{Cocart}_{\mathcal{C}}$ that pulls back cocartesian fibrations along F.
In particular, we can restrict this functor under the inclusion of the simplex category $\Delta^{\mathrm{op}}\hookrightarrow\mathsf{Cat}$. This yields a large simplicial ∞-category $\mathsf{Cocart}^{\mathcal{K}}_{\bullet}$ with ∞-category of n-simplices given by $\mathsf{Cocart}^{\mathcal{K}}_{[n]}$. It is easy to check that this simplicial ∞-category defines a large double ∞-category.

Remark B.1. All of the above analogously works for Cartesian fibrations instead of cocartesian fibrations. As in the case of cocartesian fibrations we define, for every set of small simplicial sets $\mathcal{K}$ functors

$$\mathsf{Cart}^{\mathcal{K}}_{-},\mathsf{Cart}^{\mathcal{K},\simeq}_{-}:\mathsf{Cat}^{op}\to\mathsf{CAT}$$

that associate to a small ∞-category $\mathcal{C}$ the ∞-category $\mathsf{Cart}_{\mathcal{C}^{\mathrm{op}}}$ of Cartesian fibrations over $\mathcal{C}^{\mathrm{op}}$.

We will need the following simple swapping lemma:

Lemma B.1. *Let $\mathcal{C}, \mathcal{D}$ be small ∞-categories. The Straightening-Unstraightening equivalence induces an equivalence of spaces*

$$\mathsf{Map}_{\mathsf{CAT}}(\mathcal{C}, \mathsf{Cocart}_{\mathcal{D}}) \simeq \mathsf{Map}_{\mathsf{CAT}}(\mathcal{D}, \mathsf{Cart}_{\mathcal{C}^{\mathrm{op}}}).$$

Proposition B.2. *The large double ∞-category $\mathsf{Cocart}_{\bullet}^{\mathcal{K},\simeq}$ of is essentially constant and complete. Hence, it defines a large $(\infty, 2)$-category.*

Proof. Note that the large ∞-category of 0-simplices $\mathsf{Cocart}_{0}^{\mathcal{K},\simeq}$ is equivalent to the large space $\mathsf{Cat}^{\mathcal{K},\simeq}$ of small $\mathcal{K}$-cocomplete ∞-categories. Hence, it is essentially constant. To show that it is complete, it suffices to show that its underlying Segal space is complete. Under the Straightening-Unstraightening equivalence, this reduces to the statement that the Segal space

$$\iota_1 \mathsf{Cocart}_{\bullet}^{\mathcal{K},\simeq} \simeq \mathsf{MAP}([-], \mathsf{Cat}^{\mathcal{K}})$$

is complete. But this is the Rezk-nerve of the ∞-category CAT. Hence it is complete. ☐

Definition B.3. We call the $(\infty, 2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K}}$ corresponding to the essentially constant, complete double category $\mathsf{Cocart}_{\bullet}^{\mathcal{K},\simeq}$ the $(\infty, 2)$-*category of $\mathcal{K}$-cocomplete ∞-categories.*

Remark B.2. Note that under the Straightening-Unstraightening equivalence, the objects identify with $\mathcal{K}$-cocomplete ∞-categories $\mathcal{C}$ and the morphisms with functors $\mathcal{K}$-cocontinuous functors $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories. The 2-morphisms are given by commutative triangles

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & \Delta^1 & \end{array}$$

such that the restrictions $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ are equivalences. Unraveling definitions, this datum encodes a natural transformation between the $\mathcal{K}$-cocontinuous functors associated to the cocartesian fibrations $\mathcal{C}$ and $\mathcal{D}$.

More concretely, this can be seen as follows:

Proposition B.3. *Let $\mathcal{C}, \mathcal{D} \in \mathbb{C}\mathsf{at}^{\mathcal{K}}$ be $\mathcal{K}$-cocomplete ∞-categories. The mapping ∞-category $\mathbb{C}\mathsf{at}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})$ is equivalent to the ∞-category of $\mathcal{K}$-cocontinuous functors $\mathsf{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})$.*

Proof. We need to compute the pullback of large ∞-categories

$$\begin{array}{ccc} \mathbb{C}\mathsf{at}^{\mathcal{K}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathsf{Cocart}_{1}^{\mathcal{K},\simeq} \\ \downarrow & & \downarrow \\ * & \xrightarrow{(\mathcal{C}, \mathcal{D})} & \mathsf{Cocart}_{0}^{\mathcal{K},\simeq} \times \mathsf{Cocart}_{0}^{\mathcal{K},\simeq} \end{array}$$

By the Yoneda lemma it suffices to show that for every ∞-category $\mathcal{E}$, there exists a natural equivalence of ∞-categories

$$\mathsf{Map}(\mathcal{E}, \mathbb{C}\mathsf{at}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})) \simeq \mathsf{Map}(\mathcal{E}, \mathsf{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})).$$

In fact, since Δ is dense in CAT [Lur09b, Ex.4.4.9], it suffices to show this for all $[n] \in \mathsf{CAT}$. We obtain a sequence of natural equivalences:

$$\begin{aligned}\mathsf{Map}([n], \mathsf{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})) &\simeq \mathsf{Map}_{\mathsf{Cart}^{\mathcal{K}}_{[n]^{\mathrm{op}}}}([n]^{\mathrm{op}} \times \mathcal{C}, [n]^{\mathrm{op}} \times \mathcal{D})) \\ &\simeq \mathsf{Map}([1], \mathsf{Cart}^{\mathcal{K}}_{[n]^{op}}) \times_{\iota_0 \mathsf{Cart}^{\mathcal{K}}_{[n]^{\mathrm{op}}} \times \iota_0 \mathsf{Cart}^{\mathcal{K}}_{[n]^{\mathrm{op}}}} \{[n]^{\mathrm{op}} \times \mathcal{C}, [n]^{\mathrm{op}} \times \mathcal{D}\} \\ &\simeq \mathsf{Map}([n], \mathsf{Cocart}^{\simeq, \mathcal{K}}_{[1]}) \times_{\mathsf{Map}([n], \iota_0 \mathsf{CAT}) \times \mathsf{Map}([n], \iota_0 \mathsf{CAT})} \{\mathcal{C}, \mathcal{D}\} \\ &\simeq \mathsf{Map}([n], \mathbb{C}\mathsf{at}(\mathcal{C}, \mathcal{D}))\end{aligned}$$

as desired. □

Remark B.3. Everything we have done so far works analogously for large ∞-categories. For every collection of small ∞-categories $\mathcal{K}$ and any small ∞-category $\mathcal{C}$, we denote by $\mathsf{COCART}^{\mathcal{K}}_{\mathcal{C}}$ the ∞-category of large $\mathcal{K}$-cocomplete cocartesian fibrations over $\mathcal{C}$. Further, we denote by $\mathbb{C}\mathsf{AT}^{\mathcal{K}}_{\infty}$ the $(\infty, 2)$-category of large $\mathcal{K}$-cocomplete ∞-categories.

Let $\mathcal{K}$ be the large set of all small ∞-categories. We denote by $\mathbb{C}\mathsf{AT}^{\mathsf{colim}}_{\infty}$ the ∞-category of *large ∞-categories with small colimits*. We are particularly interested in the full sub $(\infty, 2)$-category spanned by presentable ∞-categories.

Definition B.4. We call the full sub-$(\infty, 2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L}}$ of the $(\infty, 2)$-category of cocomplete large ∞-categories $\mathbb{C}\mathsf{AT}^{\mathsf{colim}}_{\infty}$ spanned by presentable ∞-categories the $(\infty, 2)$-category of presentable ∞-categories.

B.2 Symmetric Monoidal Structures on $\mathbb{C}\mathsf{at}^{\mathcal{K}}_{\infty}$

In the last subsection, we have constructed for every collection of small ∞-categories $\mathcal{K}$ an $(\infty, 2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K}}$ of $\mathcal{K}$-cocomplete ∞-categories and $\mathcal{K}$-cocontinuous functors. Our goal in this section is to equip these $(\infty, 2)$-categories with symmetric monoidal structures. To do so, we follow a similar route as in the $(\infty, 1)$-categorical case [Lur17, Sect.4.8.1]. First, let us recall the definition of a symmetric monoidal $(\infty, 2)$-category:

Definition B.5. A *symmetric monoidal double ∞-category* is a Segal object $X^{\otimes}_{\bullet} : \Delta^{\mathrm{op}} \to \mathsf{Cat}^{\otimes}$ in the ∞-category of symmetric monoidal ∞-categories. The *underlying double ∞-category* is the composite

$$X_{\bullet} : \Delta^{\mathrm{op}} \xrightarrow{X^{\otimes}_{\bullet}} \mathsf{Cat}^{\otimes} \xrightarrow{\mathrm{ev}_{\langle 1 \rangle}} \mathsf{Cat}$$

with the functor $\mathrm{ev}_{\langle 1 \rangle}$ that associates to a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathsf{Fin}$ the fiber over $\langle 1 \rangle$.

Definition B.6. A *symmetric monoidal $(\infty, 2)$-category* $X^{\otimes}_{\bullet}$ is a commutative monoid object

$$X^{\otimes}_{\bullet} : \mathsf{Fin}_* \to \mathsf{CSS}(\mathsf{Cat})$$

in the ∞-category of essential constant, complete double ∞-categories. Equivalently, a symmetric monoidal $(\infty, 2)$-category is a symmetric monoidal double $(\infty, 2)$-category $X_{\bullet} : \Delta^{\mathrm{op}} \to \mathsf{Cat}^{\otimes}$, whose underlying double ∞-category is essentially constant and complete.

We also need to be able to describe symmetric monoidal structures on sub-$(\infty, 2)$-categories:

Definition B.7. Let $\mathbb{X}_\bullet^\otimes$ be a symmetric monoidal $(\infty,2)$-category and let $Y_0^\otimes \subset \mathbb{X}_0^\otimes$ be a symmetric monoidal subspace of the symmetric monoidal space of objects of $\mathbb{X}_\bullet$. The full symmetric sub-$(\infty,2)$-category $\mathbb{Y}_\bullet^\otimes$ with space of objects Y_0 is defined as the pullback of symmetric monoidal $(\infty,2)$-categories

$$\begin{array}{ccc} \mathbb{Y}_\bullet & \longrightarrow & \mathbb{X}_\bullet \\ \downarrow & & \downarrow \\ (\iota_* Y_0^\otimes)_\bullet & \longrightarrow & (\iota_* X_0)_\bullet^\otimes \end{array}$$

where i_* denotes the functor of right Kan extension along the inclusion $i : \{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$.

We now turn to the construction of the symmetric monoidal structure on $\mathbb{C}\mathsf{at}^{\mathcal{K}}$. Therefore, we first extend the Cartesian symmetric monoidal structure on Cat to the $(\infty,2)$-category $\mathbb{C}\mathsf{at}$. Note that for every small ∞-category $\mathcal{C}$, the ∞-category $\mathsf{Cat}_{/\mathcal{C}}$ admits finite limits, and that for every functor $F : \mathcal{C} \to \mathcal{D}$ the pullback functor $F^* : \mathsf{Cat}_{\mathcal{D}} \to \mathsf{Cat}_{\mathcal{C}}$ preserves finite limits. Further, the full subcategory $\mathsf{Cocart}_{\mathcal{C}} \subset \mathsf{Cat}_{\mathcal{C}}$ is closed under finite limits. Consequently, the functors $\mathsf{Cocart}_{-}, \mathsf{Cocart}_{-}^{\simeq}$ factor:

$$\mathsf{Cocart}_{-}, \mathsf{Cocart}_{-}^{\simeq} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}^{\mathsf{lex}}$$

through the ∞-category $\mathsf{CAT}^{\mathsf{lex}}$ of large ∞-categories with finite limits and finite limit preserving functors. We use this factorization for the construction of a symmetric monoidal structure on the $(\infty,2)$-category $\mathbb{C}\mathsf{at}$. Indeed, it follows from [Lur17, Cor.2.4.1.9] that there exists a limit preserving functor

$$\Theta : \mathsf{CAT}^{\mathsf{lex}} \to \mathsf{CAT}^{\otimes}$$

that associates to a large ∞-category with finite limits $\widehat{\mathcal{C}}$ the Cartesian monoidal structure $\widehat{\mathcal{C}}^{\times} \to \mathsf{Fin}_*$ on $\widehat{\mathcal{C}}$. Hence, the composite

$$\mathsf{Cocart}_\bullet^{\simeq,\times} : \Delta^{\mathrm{op}} \xrightarrow{\mathsf{Cocart}_{-}^{\simeq}} \mathsf{Cat}^{\mathsf{lex}} \xrightarrow{\Theta} \mathsf{CAT}^{\otimes}$$

defines a simplicial symmetric monoidal ∞-category.
We can describe these symmetric monoidal ∞-categories more precisely as follows. An object of $\mathsf{Cocart}_{[n]}^{\times}$ over $\langle m \rangle \in \mathsf{Fin}$ consists of tuples $(\mathcal{C}_1, ..., \mathcal{C}_m)$ of cocartesian fibrations over $[n]$. A morphism of such pairs over a morphism $\alpha : \langle m \rangle \to \langle k \rangle$ in Fin_* consists for every $i \in \langle k \rangle$ of a morphism

$$\prod_{j \in \alpha^{-1}(i)} \mathcal{C}_j \to \mathcal{D}_i$$

in the ∞-category $\mathsf{Cocart}_{[n]}$.
It follows from the construction of Θ that the composite $\mathrm{ev}_{\langle 1 \rangle} \circ \Theta$ is homotopic to the identity functor. Hence, it follows from Proposition B.2 that the functor $\mathsf{Cocart}_\bullet^{\simeq,\times} : \Delta^{op} \to \mathsf{Cat}^{\otimes}$ defines a symmetric monoidal $(\infty,2)$-category:

Definition B.8. We call the large symmetric monoidal $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\otimes}$ associated to $\mathsf{Cocart}_\bullet^{\simeq,\times}$ *the symmetric monoidal $(\infty,2)$-category of small ∞-categories.*

Lurie uses in [Lur17, Sect.4.8.1] the theory of cocompletions developed [Lur09a, Sect.5.3.6] to construct symmetric monoidal structures on the ∞-categories $\mathsf{Cat}^{\mathcal{K}}$. We now use the same strategy to extend this structure to the level of $(\infty,2)$-categories. For this purpose, we need a description of symmetric monoidal $(\infty,2)$-categories using fibrations.

So far, we have described symmetric monoidal $(\infty,2)$-categories as commutative monoids in the ∞-category Cat_2. As we can describe symmetric monoidal ∞-categories as cocartesian fibrations

$$\mathcal{C}^{\otimes} \to \mathsf{Fin}_*,$$

we can equivalently describe symmetric monoidal $(\infty,2)$-categories as $(0,1)$-fibrations [AGH24, Sect.3]

$$\mathbb{D}^{\otimes} \to \mathsf{Fin}_* .$$

Let us recall the Definition of a $(0,1)$-fibration over an ∞-category $\mathcal{C}$:

Definition B.9. Let $p : \mathbb{D} \to \mathbb{C}$ be an $(\infty,2)$-functor, A morphism $f : d \to d'$ in $\mathbb{D}^{\otimes}$ lying over $p(f) : p(d) \to p(d')$ is called *p-cocartesian*, if for every $e \in \mathbb{D}$ and $c \in \mathbb{C}$ the commutative diagram

$$\begin{array}{ccc} \mathsf{Map}_{\mathbb{D}}(d', e) & \xrightarrow{f^*} & \mathsf{Map}_{\mathbb{D}}(d, e) \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathbb{C}}(p(d'), c) & \xrightarrow{p(f)^*} & \mathsf{Map}_{\mathbb{C}}(p(d), c) \end{array}$$

on morphism ∞-categories is a pullback in Cat.

Since Fin_* is an ∞-category, the following simplified definition of a $(0,1)$-fibration suffices:

Definition B.10. Let $\mathcal{C}$ be an ∞-category. An $(\infty,2)$-functor $p : \mathbb{D} \to \mathcal{C}$ is called a $(0,1)$-*fibration*, if for every object $d \in \mathbb{D}$ and every morphism $g : p(d) \to c'$ in $\mathcal{C}$, there exists a p-cocartesian morphism $f : d \to d'$ lying over g.

For our discussion, we also need the following analog of a local fibration:

Definition B.11. Let $\mathcal{C}$ be an ∞-category. An $(\infty,2)$-functor $p : \mathbb{D} \to \mathcal{C}$ is called a local $(0,1)$-fibration, if for every morphism $f : c \to c'$ in $\mathcal{C}$, the induced functor

$$\mathbb{D} \times_{\mathcal{C}} [1] \to [1]$$

is a $(0,1)$-fibration.

Let $p : \mathbb{D} \to \mathcal{C}$ be a $(0,1)$-fibration. As in the case of cocartesian fibrations of ∞-categories, every $(0,1)$-fibration can be unstraightened [AGH24, Sect.3] to an $(\infty,2)$-functor

$$\mathsf{Un}(p) : \mathcal{C} \to \mathsf{Cat}_2$$

with target the ∞-category of $(\infty,2)$-categories. In particular, we obtain a description of symmetric monoidal $(\infty,2)$-categories in terms of $(0,1)$-fibrations.

Definition B.12. Let $p : \mathbb{D}^{\otimes} \to \mathsf{Fin}_*$ be a $(0,1)$-fibration. Then p is called a *symmetric monoidal* $(\infty,2)$-*category*, if the unstraightening of p

$$\mathsf{Un}(p) : \mathsf{Fin}_* \to \mathsf{Cat}_2$$

is a commutative monoid object.

Let us now use this approach to define symmetric monoidal structures on the $(\infty,2)$-categories $\mathbb{C}\mathsf{at}^{\mathcal{K}}$. Following [Lur17], we consider the power set $\mathsf{P}(\mathsf{Set}_{\Delta})$ of the large set of small simplicial sets. Equipped with the partial order given by inclusion, this defines an ∞-category denoted by P. We consider the $(\infty, 2)$-functor

$$\mathbb{C}\mathsf{at}^{\otimes} \times \mathsf{P} \to \mathsf{Fin}_* \times \mathsf{P},$$

and denote by $\mathcal{M}$ the locally full sub-$(\infty, 2)$-category of $\mathbb{C}\mathsf{at}^{\otimes} \times \mathsf{P}$ such that

(1) an object $(\mathcal{C}_1, \dots \mathcal{C}_m, \mathcal{K})$ belongs to $\mathcal{M}$ if each of the $\mathcal{C}_i$ admits $\mathcal{K}$-indexed colimits

(2) Let $f : (\mathcal{C}_1, \dots \mathcal{C}_m, \mathcal{K}) \to (\mathcal{D}_1, \dots \mathcal{D}_l, \mathcal{K}')$ be a morphism in $\mathbb{C}\mathsf{at}^{\times} \times \mathsf{P}$ covering $\alpha : \langle m \rangle \to \langle l \rangle$ in Fin_* and $\mathcal{K} \subset \mathcal{K}'$. Then f belongs to $\mathcal{M}$ if for every $j \in \langle l \rangle$ the induced functor

$$\prod_{i \in \alpha^{-1}(j)} \mathcal{C}_i \to \mathcal{D}_j$$

preserves $\mathcal{K}$-indexed colimits separately in each variable.

For every $\mathcal{K} \in \mathsf{P}$ we denote by $\mathbb{C}\mathsf{at}^{\mathcal{K}, \otimes}$ the fiber of $\mathcal{M}$ over $\mathcal{K}$. We can use the results from [Lur09a, Sect.5.3.6] to prove:

Proposition B.4. *The $(\infty, 2)$-functor $q : \mathcal{M} \to \mathsf{Fin}_* \times \mathsf{P}$ is a $(0, 1)$-fibration. In particular, for every collection of simplicial sets $\mathcal{K}$, the $(\infty, 2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K}, \otimes}$ is symmetric monoidal.*

Proof. We follow the strategy of [Lur17, Prop.4.8.1.3]. We show first that q is a local $(0, 1)$-fibration. Therefore, suppose given an object $(\mathcal{C}_1, \dots, \mathcal{C}_n, \mathcal{K})$ in $\mathcal{M}$ and a morphism $\alpha : (\langle n \rangle, \mathcal{K}) \to (\langle m \rangle, \mathcal{K}')$ in $\mathsf{Fin}_* \times \mathsf{P}$. Supplying a locally cocartesian lift of α requires to choose a collection of functor

$$f_j : \prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j$$

for $1 \leq j \leq m$, such that $\mathcal{D}_j$ admits $\mathcal{K}'$ indexed colimits, and the f_j preserve $\mathcal{K}$-indexed colimits separately in each variable. For $1 \leq i \leq n$, we denote by $\mathcal{R}_i$ the collection of all colimit diagrams in $\mathcal{C}_i$ indexed by simplicial sets in $\mathcal{K}$. We now set

$$\mathcal{D}_j \coloneqq \mathcal{P}_{\mathcal{R}}^{\mathcal{K}'}(\prod_{\alpha(i)=j} \mathcal{C}_i),$$

where $\mathcal{R}$ is the product of all $\mathcal{R}_i$ with $i \in \alpha^{-1}(\{j\})$, as defined in [Lur09a, Sect.5.3.6]. It follows from the universal property of the $\mathcal{K}'$-cocompletion [Lur09a, Prop.5.3.6.2] that the respective inclusion functors define a locally cocartesian morphism.

It remains to show that q is also a $(0, 1)$-fibration. Therefore, it suffices to show by [GR19, Lem.11.3.1.5] that the induced functor on underlying ∞-categories is a cocartesian fibration of ∞-categories. But this is precisely the content of [Lur17, Prop.4.8.1.3]. □

We can summarize this discussion with the following:

Definition B.13. For every collection of small simplicial sets $\mathcal{K}$, the symmetric monoidal $(\infty, 2)$-category

$$\mathbb{C}\mathsf{at}^{\mathcal{K}, \otimes} \coloneqq \mathcal{M}_{\mathcal{K}}$$

obtained as the fiber of q at $\mathcal{K}$, is caled the *symmetric monoidal $(\infty, 2)$-category of $\mathcal{K}$-cocomplete ∞-categories.*

Corollary B.5. *Let $\mathcal{K} \subset \mathcal{K}'$ be collections of small simplicial sets. The symmetric monoidal functor $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(-) : \mathsf{Cat}^{\mathcal{K},\otimes} \to \mathsf{Cat}^{\mathcal{K}',\otimes}$ extends to a symmetric $(\infty,2)$-functor*

$$\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(-) : \mathbb{C}\mathsf{at}^{\mathcal{K},\otimes} \to \mathbb{C}\mathsf{at}^{\mathcal{K}',\otimes}.$$

Let $\mathcal{K}$ be a collection of small ∞-categories. The construction of the symmetric monoidal structure on the $(\infty,2)$-category of small $\mathcal{K}$-cocomplete ∞-categories $\mathbb{C}\mathsf{at}^{\mathcal{K}}$ can be analogously performed on the $(\infty,2)$-category of large $\mathcal{K}$-cocomplete ∞-categories $\mathbb{C}\mathsf{AT}_{\infty}^{\mathcal{K}}$. In particular, we obtain a symmetric monoidal structure on the $(\infty,2)$-category $\mathbb{C}\mathsf{AT}_{\infty}^{\mathsf{colim}}$. It follows from [Lur17, Prop.4.8.1.15] that the subspace $\mathsf{Pr}^{\mathsf{L},\otimes,\simeq} \subset \mathsf{CAT}^{\mathsf{colim},\otimes,\simeq}$ of presentable ∞-categories is closed under the symmetric monoidal structure on $\mathsf{CAT}^{\mathsf{colim}}$. Hence, we can define:

Definition B.14. We call the full symmetric monoidal sub $(\infty,2)$-category $\mathbb{P}\mathsf{r}^{\mathsf{L},\otimes}$ of the symmetric monoidal ∞-category $\mathbb{C}\mathsf{AT}_{\infty}^{\mathsf{colim},\otimes}$ the *symmetric monoidal $(\infty,2)$-category of presentable ∞-categories.*

For the discussion in the following section, we also need a description of these symmetric monoidal $(\infty,2)$-category in the language of symmetric monoidal double ∞-categories. We denote for every $[n] \in \Delta$ by

$$h_n : \mathsf{Cocart}_{[n]}^{\mathcal{K},\otimes} \subset \mathsf{Cocart}_{[n]}^{\sim,\times} \to \mathsf{Fin}_*$$

the subcategory such that

- an objects $(p_1, \dots p_m)$ belongs to $\mathsf{Cocart}_{[n]}^{\mathcal{K},\otimes}$ if the fibrations are fiberwise $\mathcal{K}$-cocomplete.
- A morphism $f : (p_1, \dots, p_m) \to (q_1, \dots, q_l)$ in $\mathsf{Cocart}_{[n]}^{\sim,\times}$ covering $\alpha : \langle m \rangle \to \langle l \rangle$ belongs to $\mathsf{Cocart}_{[n]}^{\mathcal{K},\otimes}$, if for every $j \in \langle l \rangle$ the induced functor

$$\begin{array}{ccc} \prod_{\alpha(i)=j} \mathcal{C}_i & \xrightarrow{f_j} & \mathcal{D}_j \\ & {\scriptstyle \prod p_i} \searrow \quad \swarrow {\scriptstyle q_j} & \\ & [n] & \end{array}$$

  preserves fiberwise $\mathcal{K}$-indexed colimits separately in each variable.

These ∞-categories assemble into a simplicial object

$$\mathsf{Cocart}_{\bullet}^{\mathcal{K},\otimes} : \Delta^{\mathrm{op}} \to \mathsf{Cat}_{/\,\mathsf{Fin}_*}.$$

Unwinding definitions, we see that this simplicial object identifies with the symmetric monoidal double category associated to the $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}$. In particular, for every $[n]$ the functor

$$h_n : \mathsf{Cocart}_{n}^{\mathcal{K},\otimes} \to \mathsf{Fin}_*$$

is a cocartesian fibration, and the functor

$$\mathsf{Cocart}_{\bullet}^{\mathcal{K},\otimes} : \Delta^{\mathrm{op}} \to \mathsf{Cat}^{\otimes} \subset \mathsf{Cat}_{/\,\mathsf{Fin}_*}$$

factors through the ∞-category $\mathsf{Cat}^{\otimes}$. Hence, the simplicial object defines a symmetric monoidal double category. We will make use of this description in the following sections.

Remark B.4. The symmetric monoidal double category underlying the symmetric monoidal $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}$ can be analogously constructed using the double ∞-category $\mathsf{Cart}^{\mathcal{K}}_{\bullet}$.

B.3 $(\infty,2)$-Categories of Linear ∞-Categories

The goal of this section is to construct for every set of small simplicial sets $\mathcal{K}$ and every $\mathcal{K}$-cocomplete symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ a symmetric monoidal $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}_{\mathcal{V}}$, whose underlying symmetric monoidal ∞-category is given by $\mathsf{Cat}^{\mathcal{K},\otimes}_{\mathcal{V}}$ with symmetric monoidal structure induced from the relative Deligne-Lurie product. The discussion here is an expansion of the argument sketched in [GR19, Sect.1.1.6]. As a start, we recall the construction of the symmetric monoidal structures on categories of modules from [Lur17].

Let $\mathcal{C}$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. We denote by $\mathsf{RMod}(\mathcal{C})$ the ∞-category of right module objects in $\mathcal{C}$ [Lur17, Sect.4.5]. This ∞-category admits forgetful functor $\mathsf{RMod}(\mathcal{C}) \to \mathsf{Alg}(\mathcal{C})$ to the ∞-category of algebra objects in $\mathcal{C}$. This ∞-functor associates to a pair (A, C_A) consisting of an algebra object $A \in \mathsf{Alg}(\mathcal{C})$ and a right A-module $C_A \in \mathsf{RMod}_A(\mathcal{C})$ the algebra object A. It follows from [Lur17, Prop.3.2.4.3] that the symmetric monoidal structure on $\mathcal{C}$ induces the pointwise symmetric monoidal structure on $\mathsf{RMod}(\mathcal{C})$ and $\mathsf{Alg}(\mathcal{C})$. More precisely, the monoidal product of (A, C_A) and (B, D_B) is given by $C_A \otimes D_B$ considered as a right $A \otimes B$-module. Further, the functor forgetful functor p extends to a symmetric monoidal functor $p : \mathsf{RMod}(\mathcal{C})^{\otimes} \to \mathsf{Alg}(\mathcal{C})^{\otimes}$.

Since the symmetric monoidal ∞-category $\mathcal{C}$ is $\otimes$-GR-cocomplete, the symmetric monoidal functor p is also a cocoartesian fibration [Lur17, Thm.4.5.2.1]. The cocoartesian transport along an algebra morphism $f : A \to B$ associates to (A, C_A) the right B-module $(B, C \otimes_A B_B)$. We use this to construct the monoidal structure given by the relative tensor product. Let $A \in \mathsf{CAlg}(\mathcal{C})$ be a commutative algebra object. Then A defines a commutative algebra $A^{\otimes}$ object in $\mathsf{Alg}(\mathcal{C})^{\otimes}$. We consider the pullback diagram

$$\begin{array}{ccc} \mathsf{RMod}_A(\mathcal{C})^{\otimes} & \longrightarrow & \mathsf{RMod}(\mathcal{C})^{\otimes} \\ \big\downarrow & & \big\downarrow p \\ \mathsf{Fin}_* & \xrightarrow{A} & \mathsf{CAlg}(\mathcal{C})^{\otimes} \end{array}$$

in the ∞-category of symmetric ∞-operads $\mathsf{Op}^{\mathsf{Fin}}_{\infty}$. Since p is a cocoartesian fibration, it follows that the left vertical map is a cocartesian fibration as well and hence that $\mathsf{RMod}_A(\mathcal{C})^{\otimes}$ is a symmetric monoidal ∞-category. Unraveling definitions, the monoidal product maps two right A-modules C_A, D_A to the relative tensor product

$$(C_A \otimes D_A) \otimes_{A \otimes A} A \simeq C \otimes_A D.$$

This construction recovers the symmetric monoidal structure from [Lur17, Thm.4.5.2.1]. Let us now apply this to construct the symmetric monoidal $(\infty,2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}_{\mathcal{V}}$. To do so, we extend the defining pullback diagram to a pullback diagram of symmetric monoidal double $(\infty,2)$-categories. For this, we first need to construct for every symmetric ∞-operad $\mathcal{O}^{\otimes}$ a symmetric monoidal double ∞-category of $\mathcal{K}$-cocomplete $\mathcal{O}^{\otimes}$-monoidal ∞-categories. Note that the functor $\mathsf{Alg}_{\mathcal{O}^{\otimes}}(-) : \mathsf{Cat}^{\otimes} \to \mathsf{Cat}^{\otimes}$ preserves finite limits. Hence, the composite

$$\mathsf{Alg}^{\mathrm{oplax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} : \Delta^{\mathrm{op}} \xrightarrow{\mathsf{Cocart}^{\mathcal{K},\otimes}_{\bullet}} \mathsf{Cat}^{\otimes} \xrightarrow{\mathsf{Alg}_{\mathcal{O}^{\otimes}}(-)} \mathsf{Cat}^{\otimes}$$

defines a symmetric monoidal double ∞-category. Similarly, the composite

$$\mathsf{Alg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} : \Delta^{\mathrm{op}} \xrightarrow{\mathsf{Cart}^{\mathcal{K},\otimes}_{\bullet}} \mathsf{Cat}^{\otimes} \xrightarrow{\mathsf{Alg}_{\mathcal{O}^{\otimes}}(-)} \mathsf{Cat}^{\otimes}$$

defines a symmetric monoidal double ∞-category.

Proposition B.6. *Let $\mathcal{K}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ a symmetric monoidal ∞-operad. The essentially constant symmetric monoidal double ∞-categories $\mathsf{Alg}^{\mathrm{oplax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$ and $\mathsf{Alg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$ are complete.*

Proof. We prove the result for $\mathsf{Alg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$. The other one is analogous. It suffices to show that the underlying Segal space $\mathsf{Alg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\simeq}_{\bullet}$ is complete. It follows from the universal property of the Cartesian monoidal structure that there exists an equivalence of simplicial spaces

$$\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\simeq}_{\bullet} \simeq \mathsf{Mon}^{\simeq}(\mathcal{O}^{\otimes}, \mathsf{Cart}^{\mathcal{K}}_{[-]}),$$

where the right hand side denotes the simplicial space of $\mathcal{O}^{\otimes}$-monoid objects in $\mathsf{Cart}^{\mathcal{K}}_{[-]}$. An application of the swapping lemma yields an equivalence

$$\mathsf{Mon}^{\simeq}(\mathcal{O}^{\otimes}, \mathsf{Cart}^{\mathcal{K}}_{[-]^{\mathrm{op}}}) \simeq \mathsf{Map}([-], \mathsf{Alg}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})).$$

But the last simplicial space is the Rezk-nerve of the ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})$ and hence is complete. □

Definition B.15. Let $\mathcal{K}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ a symmetric ∞-operad. We define the *symmetric monoidal* $(\infty, 2)$*-category of* $\mathcal{K}$*-cocomplete* $\mathcal{O}^{\otimes}$*-monoidal* ∞*-categories an oplax* $\mathcal{O}^{\otimes}$*-monoidal functors* $\mathbb{A}\mathsf{lg}^{\mathrm{oplax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}$ as the symmetric monoidal $(\infty, 2)$-category underlying the symmetric monoidal double ∞-category $\mathsf{Alg}^{\mathrm{oplax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$.
Similarly, we define the *symmetric monoidal* $(\infty, 2)$*-category of* $\mathcal{K}$*-cocomplete* $\mathcal{O}^{\otimes}$*-monoidal* ∞*-categories a lax* $\mathcal{O}^{\otimes}$*-monoidal functors* $\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}(\mathcal{K}))^{\otimes}$ as the symmetric monoidal $(\infty, 2)$-category underlying the symmetric monoidal double ∞-category $\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$

Remark B.5. To justify this definition, let us unravel it for the associative operad $\mathsf{Ass}^{\otimes}$. An object of the $(\infty, 2)$-category $\mathbb{A}\mathsf{lg}^{\mathrm{oplax}}(\mathsf{Cat}^{\mathcal{K}})$ consists of a $\mathcal{K}$-cocomplete monoidal ∞-category $\mathcal{C}^{\otimes}$. A morphism is given by an associative algebra object $(\mathcal{C}^{\otimes}, \mu)$ in $\mathsf{Cocart}^{\mathcal{K},\otimes}_{1}$. The underlying object consists of a cocartesian fibration $\mathcal{C} \to [1]$ encoding a $\mathcal{K}$-cocontinuous functor $F : \mathcal{C}_0 \to \mathcal{C}_1$. The multiplication functor

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\ \mu\ } & \mathcal{C} \\ & \searrow \quad \swarrow & \\ & [1] & \end{array}$$

encodes a monoidal product on the fibers $\mu_i : \mathcal{C}_i \otimes \mathcal{C}_i \to \mathcal{C}_i$. Moreover, we obtain for every $(c_0, c_1) \in \mathcal{C}_0$ a commutative diagram

$$\begin{array}{ccccc} (m_0, m_1) & \xrightarrow{\ \mu_0\ } & m_0 \otimes m_1 & & \\ \downarrow{\scriptstyle F\times F} & & \downarrow & \searrow^{\circ} & \\ (F(m_0), F(m_1)) & \xrightarrow{\ \mu_1\ } & F(m_0) \otimes F(m_1) & \xleftarrow[\eta_{c_0,c_1}]{} & F(m_0 \otimes m_1) \end{array}$$

The morphism η_{c_0,c_1} is part of a natural transformation $\eta : F(- \otimes -) \to F(-) \otimes F(-)$ that encodes part of the coherence data of an oplax monoidal structure on F. This justifies the name oplax. If we instead consider a morphism in the $(\infty, 2)$-category $\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})$ with underlying object a Cartesian fibration $\mathcal{D} \to [1]^{op}$. The analogous diagram is given by

$$\begin{array}{ccc} (F(m_0), F(m_1)) & & F(m_0)\otimes F(m_1) \xrightarrow{\phi_{d_0,d_1}} F(m_0\otimes m_1) \\ \Big\downarrow{\scriptstyle F\times F} & \mapsto & \Big\downarrow \qquad \swarrow_{\circ} \\ (m_0, m_1) & & m_0\otimes m_1 \end{array}$$

The morphism ϕ_{d_0,d_1} is part of a natural transformation $\phi : F(-)\otimes F(-) \to F(-\otimes -)$ that encodes part of the coherence data of a lax monoidal structure on F. This justifies the name lax.
A 2-morphism consists of a map between algebra objects in $\mathsf{Cocart}_{[1]}$. This datum contains a map $\psi : \mathcal{C} \to \mathcal{D}$ of cocartesian fibrations over $[1]$, whose restrictions to the fibers ψ_i are equivalences. The first coherence datum is given by the commutative diagram

$$\begin{array}{ccc} \mathcal{C}\otimes\mathcal{C} & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}\otimes\mathcal{D} & \longrightarrow & \mathcal{D} \end{array}$$

For every $(c_0, c_1) \in \mathcal{C}_0$ this encodes the datum of a commutative diagram

$$\begin{array}{ccc} F(c_0\otimes c_1) & \xrightarrow{\eta_{c_0,c_1}} & F(c_0)\otimes F(c_1) \\ \big\downarrow{\scriptstyle \psi_{c_0\otimes c_1}} & & \big\downarrow{\scriptstyle \psi_{c_0}\otimes\psi_{c_1}} \\ G(c_0\otimes c_1) & \xrightarrow{\phi_{c_0,c_1}} & G(c_0)\otimes G(c_1) \end{array}$$

This is the lowest instance of the structure of a monoidal natural transformation between lax-monoidal functors.

Ultimately, we are interested in a version of the above construction, where the 1-morphisms are given by strongly monoidal functors. For this, note that the discussion above suggests that the lax-monoidal functor induced by an algebra object in $\mathsf{Cocart}_{[1]}^{\mathcal{K},\otimes}$ is strong monoidal if the algebra action preserves cocartesian morphisms. Following this intuition, we denote by $\mathsf{Alg}_{\mathcal{O}^\otimes}^{\mathrm{strg}}(\mathsf{Cocart}_n^{\mathcal{K}})$ the full subcategory of $\mathsf{Alg}_{\mathcal{O}^\otimes}^{\mathrm{lax}}(\mathsf{Cocart}_n^{\mathcal{K}})$ spanned by those algebra objects $\mathcal{C}^\otimes$, s.t. the induced $\mathcal{O}^\otimes$-monoid object $\mathcal{C}^\otimes : \mathcal{O}^\otimes \to \mathsf{Cocart}_n^{\mathcal{K}}$ factors through $\mathsf{Cocart}_n^{\mathrm{strg},\mathcal{K}}$. These assemble into a symmetric monoidal double ∞-category

$$\mathsf{Alg}_{\mathcal{O}^\otimes}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})_\bullet^\otimes : \Delta^{\mathrm{op}} \to \mathsf{Cat}^\otimes.$$

To obtain the underlying $(\infty,2)$-category, we need the following:

Proposition B.7. *The essentially constant, symmetric monoidal double ∞-category $\mathbb{A}\mathsf{lg}_{\mathcal{O}^\otimes}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})_\bullet^\otimes$ is complete*

Proof. It suffices to show that the underlying Segal space $\mathsf{Alg}_{\mathcal{O}^\otimes}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})_\bullet^\simeq$ is complete. It follows from the universal property of the Cartesian monoidal structure that there exists an equivalence of simplicial spaces

$$\mathbb{A}\mathsf{lg}_{\mathcal{O}^\otimes}^{\mathrm{strg}}(\mathsf{Cat}(\mathcal{K}))_\bullet^\simeq \simeq \mathsf{Mon}^\simeq(\mathcal{O}^\otimes, \mathsf{Cart}_{[-]^{\mathrm{op}}}^{\mathrm{strg},\mathcal{K}})$$

with the simplicial space of $\mathcal{O}^\otimes$-monoid objects in $\mathsf{Cart}_{[-]^{\mathrm{op}}}^{\mathcal{K}}$. An application of the swapping lemma yields an equivalence

$$\mathsf{Mon}^{\simeq}(\mathcal{O}^{\otimes}, \mathsf{Cart}^{\mathrm{strg},\mathcal{K}}_{[-]^{\mathrm{op}}}) \simeq \mathsf{Map}([-], \mathsf{Mon}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})).$$

But the last simplicial space is the Rezk-nerve of the ∞-category $\mathsf{Mon}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})$ of the ∞-category of $\mathcal{K}$-cocomplete $\mathcal{O}^{\otimes}$ monoidal ∞-categories and $\mathcal{O}^{\otimes}$ monoidal functors. Hence, it is complete. $\square$

We have already computed the underlying symmetric monoidal ∞-categories of the above symmetric monoidal $(\infty, 2)$-categories. For completeness, we also compute their morphism ∞-categories.

Proposition B.8. *Let $\mathcal{K}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ a symmetric monoidal ∞-operad. For all $\mathcal{O}^{\otimes}$-monoidal ∞-categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$, there exists an equivalence of ∞-categories:*

$$\mathsf{Map}_{\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})}(\mathcal{C}, \mathcal{D}) \simeq \mathsf{Fun}^{\mathrm{lax},\mathcal{K}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}).$$

Proof. It suffices to construct for every $[n] \in \Delta$ a natural equivalence

$$\mathsf{Map}([n], \mathsf{Map}_{\mathbb{A}\mathsf{lg}^{\mathrm{lax}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})}(\mathcal{C}, \mathcal{D})) \simeq \mathsf{Map}([n], \mathsf{Fun}^{\mathrm{lax},\mathcal{K}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})).$$

To do so, we first note that by definition there exists an equivalence

$$\mathsf{Map}([n], \mathsf{Fun}^{\mathrm{lax},\mathcal{K}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})) \simeq \mathsf{Map}_{\mathsf{Cocart}^{\mathcal{K}}_{\mathcal{O}^{\otimes}\times[n]}}(\mathcal{C}^{\otimes} \times [n], \mathcal{D}^{\otimes} \times [n]).$$

Using the definition of the mapping space in $\mathsf{Cocart}^{\mathcal{K}}_{\mathcal{O}^{\otimes}\times[n]}$, we obtain an equivalence

$$\mathsf{Map}_{\mathsf{Cocart}^{\mathcal{K}}_{\mathcal{O}^{\otimes}\times[n]}}(\mathcal{C}^{\otimes} \times [n], \mathcal{D}^{\otimes} \times [n]) \simeq \mathsf{Map}([1], \mathsf{Cocart}^{\mathcal{K}}_{\mathcal{O}^{\otimes}\times[n]}) \times_{(\mathsf{Cocart}^{\mathcal{K},\simeq}_{\mathcal{O}^{\otimes}\times[n]})^{\times 2}} \{(\mathcal{C}^{\otimes} \times [n], \mathcal{D}^{\otimes} \times [n])\}.$$

Using the swapping lemma, we can identify the above space with

$$\mathsf{Map}([n], \mathsf{Seg}_{\mathcal{O}^{\otimes}}(\mathsf{Cart}^{\mathcal{K}}_{[1]})) \times_{\mathsf{Seg}(\mathsf{Cat}^{\mathcal{K}})^{\simeq,\times 2}} \{\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\}.$$

The claim follows from [Lur17, Thm.2.4.2.5] $\square$

Corollary B.9. *Let $\mathcal{K}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ a symmetric monoidal ∞-operad. For all $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \in \mathsf{Alg}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}(\mathcal{K}))$ there exists an equivalence of ∞-categories:*

$$\mathsf{Map}_{\mathbb{A}\mathsf{lg}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}}}(\mathcal{C}, \mathcal{D}) \simeq \mathsf{Fun}^{\mathcal{K}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}).$$

We can now use these symmetric monoidal $(\infty, 2)$-categories to finish our construction of the symmetric monoidal $(\infty, 2)$-category $\mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}_{\mathcal{V}}$. Therefore, recall that we can encode a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ as a morphism of symmetric ∞-operads

$$\begin{array}{ccc} \mathsf{Fin}_* & \xrightarrow{\mathcal{V}^{\otimes}} & \mathsf{Alg}(\mathsf{Cat}^{\mathcal{K}})^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathsf{Fin}_* & \end{array}$$

The symmetric monoidal ∞-category $\mathsf{Alg}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}$ is the symmetric monoidal category of 0-simplices of the symmetric monoidal double ∞-category $\mathsf{Alg}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} \in \mathsf{Cat}^{\otimes}_{\Delta}$. We denote by $\mathsf{Fin}_{*,\bullet}$ the constant symmetric monoidal double ∞-category on Fin_*. By the universal property of the constant simplicial object we can uniquely extend the morphism $\mathcal{V}^{\otimes} : \mathsf{Fin}_* \to \mathsf{Alg}(\mathsf{Cat}^{\mathcal{K}})^{\otimes} \in \mathsf{Cat}^{\otimes}$ to a morphism of symmetric monoidal double ∞-categories

$$\mathcal{V}^{\otimes} \times [-] : \mathsf{Fin}_{*,\bullet} \to \mathsf{Alg}^{\mathrm{strg}}(\mathsf{Cocart}^{\mathcal{K}})^{\otimes}_{\bullet}$$

By construction for every $n \geq 0$ the underlying object of the algebra $\mathcal{V}^{\otimes} \times [n] \in \mathsf{Alg}(\mathsf{Cocart}^{\mathcal{K}}_n)$ is given by the cocartesian fibration $\mathcal{V} \times [n] \xrightarrow{\mathsf{pr}_2} [n]$ and the action is given by

$$\mathcal{V} \times [n] \times_{[n]} \mathcal{V} \times [n] \simeq \mathcal{V} \times \mathcal{V} \times [n] \xrightarrow{\mu \times \mathrm{id}_{[n]}} \mathcal{V} \times [n]$$

Remark B.6. In terms of monoidal functors, $\mathcal{V} \times [n]$ encodes the functor $[n] \to \mathsf{Cat}$ that is constant on $\mathcal{V}$ and the action is pointwise.

We further need the following property of constant simplicial objects. Recall that for every ∞-category and every object $C \in \mathcal{C}$ there exists an equivalence $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}_{/C}) \simeq \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})_{/C_\bullet}$, where $C_\bullet$ denotes the constant simplicial object on C. Hence, under this equivalence, we can identify every simplicial object over a constant simplicial object with a simplicial object in the slice.

Proposition B.10. *Let $\mathcal{V}^{\otimes}$ be a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. We define the simplicial symmetric ∞-operad $\mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cocart}(\mathcal{K})^{\otimes}_{\bullet})$ as the pullback*

$$\begin{array}{ccc}
\mathsf{RMod}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} & \longrightarrow & \mathsf{RMod}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} \\
\downarrow{\scriptstyle \pi_\bullet} & & \downarrow \\
\mathsf{Fin}_\bullet & \longrightarrow & \mathsf{Alg}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}
\end{array}$$

of simplicial symmetric ∞-operads. The simplicial symmetric ∞-operad

$$\pi : \mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} : \Delta^{\mathrm{op}} \to \mathsf{Op}^{\mathsf{Fin}_*}_{\infty}$$

associated to the left vertical morphism $\pi_\bullet$ is a symmetric monoidal double ∞-category.

Proof. It suffices to show that for every $n \geq 0$ the ∞-operad $\mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{[n]}$ is a symmetric monoidal ∞-category and the simplicial maps preserve cocartesian morphism. The first claim follows from the observation that for every n the functor $\pi_n : \mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{[n]} \to \mathsf{Fin}$ is the pullback of a cocartesian fibration and hence itself cocartesian. The second claim follows from the observation that every morphism

$$\mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{[n]} \to \mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{[m]}$$

arises as the pullback of a functor that preserves cocartesian morphism. Hence, it preserves cocartesian morphisms itself. □

Proposition B.11. *The double ∞-category $\mathsf{RMod}^{\mathrm{strg}}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})_\bullet$ is complete.*

Proof. It suffices to show that the underlying Segal space is complete. Since the functor $(-)^{\simeq}$ is a right adjoint, this space can be computed as the pullback

$$\begin{array}{ccc}
\mathsf{RMod}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\simeq}_{\bullet} & \longrightarrow & \mathsf{RMod}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})^{\simeq}_{\bullet} \simeq \mathsf{Map}([-], \mathsf{RMod}(\mathsf{Cat}^{\mathcal{K}})) \\
\downarrow{\scriptstyle \pi_\bullet} & & \downarrow \\
* & \xrightarrow{\mathcal{V}} & \mathsf{Alg}^{\mathrm{strg}}(\mathsf{Cat}^{\mathcal{K}})^{\simeq}_{\bullet} \simeq \mathsf{Map}([-], \mathsf{Alg}(\mathsf{Cat}^{\mathcal{K}}))
\end{array}$$

Hence, it is given by $\mathsf{Map}([-], \mathsf{RMod}_{\mathcal{V}}(\mathsf{Cat}^{\mathcal{K}}))$ and is therefore complete. $\square$

Definition B.16. Let $\mathcal{K}$ be a collection of ∞-categories and $\mathcal{V}^{\otimes}$ a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. We define the *symmetric monoidal* $(\infty, 2)$*-category of* $\mathcal{K}$*-cocomplete* $\mathcal{V}$*-linear* ∞*-categories* $\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K},\otimes}$ as the symmetric monoidal $(\infty, 2)$-category underlying $\mathsf{RMod}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$.

Proposition B.12. *Let* $\mathcal{V}^{\otimes}$ *be a* $\mathcal{K}$*-cocomplete monoidal* ∞*-category and* $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_{\mathcal{V}}^{\mathcal{K}}$ *be* $\mathcal{K}$*-cocomplete* $\mathcal{V}$*-linear* ∞*-categories. Then there exists an equivalence of* ∞*-categories* $\mathsf{Map}_{\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}}}(\mathcal{C}, \mathcal{D}) \simeq \mathsf{Fun}_{\mathcal{V}}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})$ *between the* ∞*-category of morphisms in* $\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}}$ *and the* ∞*-category of* $\mathcal{K}$*-cocontinuous* $\mathcal{V}$*-linear functors.*

Proof. Unraveling the definition of morphism ∞-categories, it follows that the morphism ∞-category of $\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}}$ can be computed as the pullback

$$\begin{array}{ccc}
\mathsf{Map}_{\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathsf{Map}_{\mathbb{R}\mathsf{Mod}(\mathsf{Cat}_{\mathcal{V}}^{\mathcal{K}})}((\mathcal{C}, \mathcal{V}), (\mathcal{D}, \mathcal{V})) \\
\downarrow & & \downarrow \\
* & \xrightarrow{\mathrm{id}_{\mathcal{V}}} & \mathsf{Map}_{\mathbb{A}\mathsf{lg}(\mathsf{Cat}^{\mathcal{K}})}(\mathcal{V}, \mathcal{V})
\end{array}$$

But this diagram can be identified with the diagram

$$\begin{array}{ccc}
\mathsf{Map}_{\mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K}}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathsf{Fun}^{\mathcal{K}}_{\mathsf{RMod}}((\mathcal{C}, \mathcal{V}), (\mathcal{D}, \mathcal{V})) \\
\downarrow & & \downarrow \\
* & \xrightarrow{\mathrm{id}_{\mathcal{V}}} & \mathsf{Fun}^{\mathcal{K}}_{\mathsf{Ass}}(\mathcal{V}, \mathcal{V})
\end{array}$$

whose pullback is given by $\mathsf{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$. $\square$

Remark B.7. We can describe the homotopy category of the $(\infty, 2)$-category $\mathbb{A}\mathsf{lg}^{\mathrm{strg}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})$ as follows. This 2-category has

- objects: $\mathcal{K}$-cocomplete $\mathcal{O}^{\otimes}$-monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$.
- 1-morphisms: functors over $\mathcal{O}^{\otimes}$

  $$\begin{array}{ccc}
  \mathcal{C}^{\otimes} & \xrightarrow{F} & \mathcal{D}^{\otimes} \\
  & \searrow \quad \swarrow & \\
  & \mathcal{O}^{\otimes} &
  \end{array}$$

  that preserves cocartesian morphisms and are fiberwise $\mathcal{K}$-cocontinuous.
- 2-morphisms: natural transformations over $\mathcal{O}^{\otimes}$ between $\mathcal{K}$-cocontinuous functors, i.e. commutative diagrams

  $$\begin{array}{ccc}
  \mathcal{C}^{\otimes} \times [1] & \xrightarrow{F} & \mathcal{D}^{\otimes} \\
  & \searrow \quad \swarrow & \\
  & \mathcal{O}^{\otimes} &
  \end{array}$$

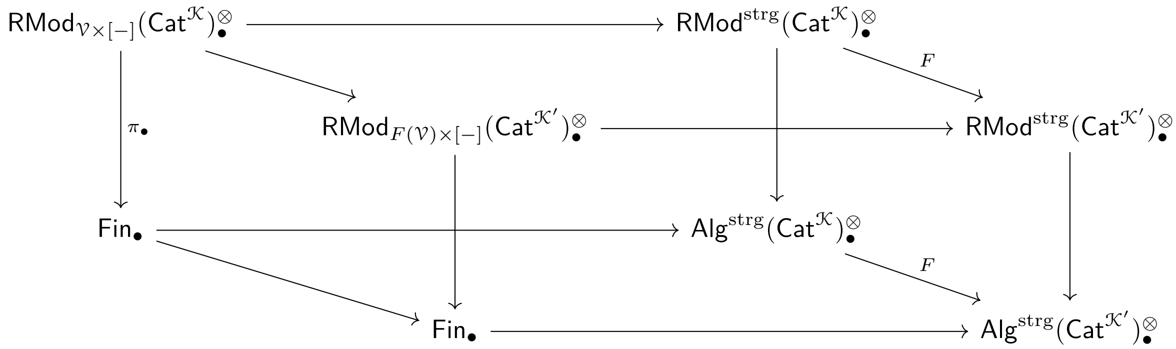
  such that $F(-, 0)$ and $F(-, 1)$ are $\mathcal{K}$-cocontinuous $\mathcal{O}^{\otimes}$-monoidal functors.

Analogously to [BM24, Prop.4.7] and [BM24, Prop.4.7] one proves:

Proposition B.13. *Let* $\mathcal{K}$ *be a collection of small* ∞*-categories and* $\mathcal{O}^{\otimes}$ *a symmetric* ∞*-operad. A 1-morphism* $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *in* $\mathbb{A}\mathsf{lg}^{\mathrm{strg}}_{\mathcal{O}^{\otimes}}(\mathsf{Cat}^{\mathcal{K}})$ *is a left adjoint if and only if it admits a relative right adjoint that is* $\mathcal{K}$*-cocontinuous and* $\mathcal{O}^{\otimes}$*-monoidal.*

Corollary B.14. *Let $\mathcal{V}$ be a $\mathcal{K}$-cocomplete monoidal ∞-category. A 1-morphism in $\mathbb{C}\mathsf{at}^{\mathcal{K}}_{\mathcal{V}}$ represented by a $\mathcal{K}$-cocontinuous $\mathcal{V}$-linear functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ admits a right adjoint if and only if the underlying functor $F : \mathcal{C} \to \mathcal{D}$ admits a $\mathcal{K}$-cocontinuous right adjoint and the associated $\mathcal{V}$-linear structure is strong.*

Let $F : \mathsf{Cocart}^{\mathcal{K},\otimes}_{\bullet} \to \mathsf{Cocart}^{\mathcal{K}',\otimes}_{\bullet}$ be a symmetric monoidal functor between symmetric monoidal double ∞-categories that preserves geometric realizations. It follows, that F induces a commutative cube



where the right face of the cube consists of symmetric monoidal functors between symmetric monoidal double ∞-categories. Hence, also the induced arrow

$$\mathsf{RMod}_{\mathcal{V}^{\otimes}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} \to \mathsf{RMod}_{F(\mathcal{V})^{\otimes}\times[-]}(\mathsf{Cat}^{\mathcal{K}'})^{\otimes}_{\bullet}$$

is a symmetric monoidal functor between symmetric monoidal double ∞-categories. In particular, we can use this for the following:

Proposition B.15. *Let $\mathcal{K} \subset \mathcal{K}'$ be collections of small ∞-categories and let $\mathcal{V}^{\otimes} \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{K}})$ be a $\mathcal{K}$-cocomplete symmetric monoidal ∞-category. The symmetric monoidal ∞-functor $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \mathsf{Cat}^{\mathcal{K},\otimes} \to \mathsf{Cat}^{\mathcal{K}',\otimes}$ extends to a symmetric monoidal $(\infty, 2)$-functor*

$$\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \mathbb{C}\mathsf{at}^{\mathcal{K},\otimes}_{\mathcal{V}} \to \mathbb{C}\mathsf{at}^{\mathcal{K}',\otimes}_{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(V)}.$$

Let us finally construct a base change functor. Let $F : \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a morphism of $\mathcal{K}$-cocomplete symmetric monoidal ∞-categories. By the universal property of the cocartesian symmetric monoidal structure on $\mathsf{CAlg}(\mathsf{Cat}^{\mathcal{K}})$, it extends to a symmetric monoidal functor $F : \mathsf{Fin}_* \times \Delta^1 \to \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}$. By a similar argument as above, this functor uniquely extends to a morphism

$$F \times [-] : \mathsf{Fin}_{*,\bullet} \times \Delta^1_{\bullet} \to \mathsf{CAlg}^{\mathrm{strg}}(\mathsf{Cocart}(\mathcal{K}))^{\otimes}_{\bullet}$$

of simplicial objects in $\mathsf{Cat}_{/\,\mathsf{Fin}_*}$, such that each $F \times [n]$ is a morphism of ∞-operads. It follows that the pullback

$$\mathsf{RMod}_{\mathcal{F}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} : \mathsf{RMod}_{\mathcal{V}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet} \to \mathsf{RMod}_{\mathcal{W}\times[-]}(\mathsf{Cat}^{\mathcal{K}})^{\otimes}_{\bullet}$$

induces a morphism of symmetric monoidal double ∞-categories. Passing to underlying $(\infty, 2)$-categories we obtain:

Proposition B.16. *Let $F : \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes} \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{K}})$ be a colimit preserving functor between $\mathcal{K}$-cocomplete symmetric monoidal ∞-categories. The symmetric monoidal ∞-functor $- \otimes_{\mathcal{V}} \mathcal{W} : \mathsf{Cat}^{\mathcal{K}}_{\mathcal{V}} \to \mathsf{Cat}^{\mathcal{K}}_{\mathcal{W}}$ extends to a* *symmetric monoidal* $(\infty, 2)$*-functor*

$$- \otimes_{\mathcal{V}} \mathcal{W} : \mathbb{C}\mathsf{at}_{\mathcal{V}}^{\mathcal{K},\otimes} \to \mathbb{C}\mathsf{at}_{\mathcal{W}}^{\mathcal{K},\otimes}.$$

C Morita Categories

In Section 4.3, we have shown that for every presentably symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, rigid algebras in $\mathbb{P}\mathsf{r}_{\mathcal{V}}^{\mathsf{L},\otimes}$ form fully dualizable objects in the corresponding Morita $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathsf{Pr}_{\mathcal{V}}^{\mathsf{L}})^{\otimes}$. Since a complete description of this $(\infty, 2)$-category, together with a classification of its fully dualizable objects is missing in the literature, we have included a discussion in this appendix. To this end, let us recall first the construction of the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(-)$ from [Lur17, Sect.4.4.]. We adopt the model-independent presentation given in [Hau23, Sect.5] to the setting of symmetric ∞-operads.
We denote by $\mathcal{G}$ a set of small ∞-categories that contains Δ^{op}. It follows from [Lur17, Cor.4.8.1.4] that the inclusion $\mathsf{Cat}^{\mathcal{G}} \hookrightarrow \mathsf{Cat}$ lifts to a lax symmetric monoidal functor. Following [JFS17, Def.8.1], we call an object $\mathcal{C}^{\otimes} \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{G}})$ a $\otimes$*-GR-cocomplete symmetric monoidal* ∞*-category.* These are symmetric monoidal ∞-categories that admit geometric realizations and whose tensor product functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations in each argument separately. These will serve as an input of the construction of the Morita category.
The main technical ingredient is the generalized ∞-operad $\mathsf{Tens}^{\otimes}$ constructed in [Lur17, 4.4.1.1]. It comes equipped with a functor

$$\mathsf{Tens}^{\otimes} \xrightarrow{\mathrm{ev}_{\Delta^{\mathrm{op}}} \times \mathrm{ev}_{\mathrm{fin}}} \Delta^{op} \times \mathsf{Fin}_*$$

that equips $\mathsf{Tens}^{\otimes}$ with the structure of a Δ^{op}-family of ∞-operads. In particular, for every $[n] \in \Delta^{\mathrm{op}}$ its fiber $\mathsf{Tens}^{\otimes}_{[n]}$ is an ∞-operad. The colors of this ∞-operad are denoted by a_i for $i \in [n]$ and $m_{i,i+1}$ for $0 \leq i < n$. Unraveling definitions, the datum of an algebra over the ∞-operad $\mathsf{Tens}^{\otimes}_{[n]}$ is given by the datum of

(1) an associative algebra A_i for every $i \in [n]$,

(2) an $A_i - A_{i+1}$-bimodule $M_{i,i+1}$ for every $0 \leq i < n$.

Let α be a morphism in Δ^{op}, we denote the fiber of $\mathsf{Tens}^{\otimes}$ over α by $\mathsf{Tens}^{\otimes}_{\alpha}$. Of particular importance are the inner face maps $d_i : [m-1] \to [m]$ for $0 < i < m$ that leave out the i-th element. The prototypical example of such an inner face map is the morphism $d_1 : [1] \to [2]$. Because of its importance for the forthcoming discussion, we denote the generalized ∞-operad $\mathsf{Tens}^{\otimes}_{d_1}$ by $\mathsf{Tens}^{\otimes}_{\succ}$.
Let us now start with the construction of the Morita $(\infty, 2)$-category following [Hau23] and [Lur17]. We therefore need the following property of the generalized ∞-operad $\mathsf{Tens}^{\otimes}$:

Proposition C.1. *[Lur17, Thm. 4.4.3.1] The forgetful functor* $\mathsf{Tens}^{\otimes} \to \Delta^{\mathrm{op}}$ *is an exponential fibration.*

Corollary C.2. *For any* ∞*-category* $\mathcal{C}$ *over* Δ^{op} *there exists an* ∞*-category*[33]

$$\overline{\mathsf{Mor}}(\mathcal{C}) := \mathrm{ev}_{\Delta^{\mathrm{op}},*}\, \mathrm{ev}^*_{\mathsf{Fin}_*}\, \mathcal{C}$$

over Δ^{op}*, such that*

$$\mathsf{Fun}_{/\Delta^{\mathrm{op}}}(K, \overline{\mathsf{Mor}}(\mathcal{C})) \simeq \mathsf{Fun}_{/\,\mathsf{Fin}_*}(K \times_{\Delta^{\mathrm{op}}} \mathsf{Tens}^{\otimes}, \mathcal{C}).$$

[33] $\mathrm{ev}_{\Delta^{\mathrm{op}},*}$ denotes the right adjoint of the functor $\mathrm{ev}^*_{\Delta^{\mathrm{op}}}$ given by pulling back along the functor $\mathrm{ev}_{\Delta^{\mathrm{op}}} : \mathsf{Tens}^{\otimes} \to \Delta^{\mathrm{op}}$.

Proof. Let $\mathcal{D}$ be an ∞-category. Observe that there exists a chain of equivalences

$$\begin{aligned}\mathsf{Map}(\mathcal{D}, \mathsf{Fun}_{/\Delta^{\mathrm{op}}}(K, \overline{\mathbb{M}\mathsf{or}}(\mathcal{C}))) &\simeq \mathsf{Map}_{/\Delta^{\mathrm{op}}}(\mathcal{D} \times K, \overline{\mathbb{M}\mathsf{or}}(\mathcal{C})) \\ &\simeq \mathsf{Map}_{/\,\mathsf{Fin}_*}((\mathcal{D} \times K) \times_{\Delta^{\mathrm{op}}} \mathsf{Tens}^{\otimes}, \mathcal{C}) \\ &\simeq \mathsf{Map}_{/\,\mathsf{Fin}_*}(\mathcal{D} \times (K \times_{\Delta^{\mathrm{op}}} \mathsf{Tens}^{\otimes}), \mathcal{C})) \\ &\simeq \mathsf{Map}(\mathcal{D}, \mathsf{Fun}_{/\,\mathsf{Fin}_*}(K \times_{\Delta^{\mathrm{op}}} \mathsf{Tens}^{\otimes}, \mathcal{C})).\end{aligned}$$

The claim then follows from the Yoneda lemma. □

Note by construction an object in the fiber of $\overline{\mathbb{M}\mathsf{or}}(\mathcal{C})$ over $[k] \in \Delta^{\mathrm{op}}$ is given by an arbitrary functor

$$\begin{array}{ccc}\mathsf{Tens}^{\otimes}_{[k]} & \longrightarrow & \mathcal{C} \\ & {\scriptstyle \mathrm{ev}_{\mathsf{Fin}_*}} \searrow \quad \swarrow & \\ & \mathsf{Fin}_* & \end{array}$$

over Fin_*. It therefore makes sense to define:

Definition C.1. Let $\mathcal{C}^{\otimes}$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. We define $\mathcal{M}\mathrm{or}(\mathcal{C})$ to be the full subcategory of $\overline{\mathbb{M}\mathsf{or}}(\mathcal{C})$ spanned by the $\mathsf{Tens}^{\otimes}_{[k]}$-algebras.

This ∞-category satisfies the following useful universal property:

Lemma C.3. *For any functor $K \to \Delta^{\mathrm{op}}$ there exists an equivalence*

$$\mathsf{Fun}_{/\Delta^{\mathrm{op}}}(K, \mathcal{M}\mathrm{or}(\mathcal{C})) \simeq \mathsf{Alg}_{K \times_{\Delta^{\mathrm{op}}} \mathsf{Tens}^{\otimes, \mathrm{op}}}(\mathcal{C}).$$

Proof. The proof is analogous to [Hau23, Prop.4.7] □

It follows from the construction of $\mathsf{Tens}^{\otimes}$ that every morphism $\alpha : [n] \to [m]$ in Δ induces morphisms of generalized ∞-operads

$$\mathsf{Tens}^{\otimes}_{[m]} \to \mathsf{Tens}^{\otimes}_{\alpha} \leftarrow \mathsf{Tens}^{\otimes}_{[n]}.$$

We denote in the following the colors of $\mathsf{Tens}^{\otimes}_{\alpha}$ over $[m]$ (resp. over $[n]$) by a_i and $m_{i,i+1}$ (resp. b_j and $n_{j,j+1}$). Our next goal is to show that the functor $\mathcal{M}\mathrm{or}(\mathcal{C}) \to \Delta^{\mathrm{op}}$ defines a double ∞-category. A morphism $\beta : [n] \to [m]$ in Δ is called convex, if the image of β is convex. We note that every convex map induces a morphism of ∞-operads

$$\mathsf{Tens}^{\otimes}_{[n]} \to \mathsf{Tens}^{\otimes}_{[m]},$$

and hence for every symmetric ∞-operad $\mathcal{C}^{\otimes}$ a restriction functor

$$v_{\beta} : \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[m]}}(\mathcal{C}) \to \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[n]}}(\mathcal{C}).$$

These restriction functors are fundamental for the proof of the following:

Theorem C.4. *[Lur17, Sect.4.4] Let $\mathcal{C}^{\otimes}$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. Then the functor*

$$p : \mathcal{M}\mathrm{or}(\mathcal{C}) \to \Delta^{\mathrm{op}}$$

is a double ∞-category. If $\bar{\alpha}$ is an edge of $\mathcal{M}\mathrm{or}(\mathcal{C})$ over $[n] \leftarrow [m] : \alpha$ in Δ^{op}, then $\bar{\alpha}$ is p-cocartesian if and only if the corresponding functor

$$F : \mathsf{Tens}^{\otimes}_{\alpha} \to \mathcal{C}$$

is an operadic left Kan extension of $F|_{\mathsf{Tens}^{\otimes}_{[n]}}$.

Proof. The fact that p is a cocartesian fibration and the description of cocartesian morphisms follow from [Lur17, Cor.4.4.3.2]. It remains to show that $\mathbb{M}\mathsf{or}(\mathcal{C})$ satisfies the Segal condition, i.e the functor

$$\mathcal{M}\mathrm{or}(\mathcal{C})_{[n]} \xrightarrow{\simeq} \mathcal{M}\mathrm{or}(\mathcal{C})_{[1]} \times_{\mathcal{M}\mathrm{or}(\mathcal{C})_{[0]}} \cdots \times_{\mathcal{M}\mathrm{or}(\mathcal{C})_{[0]}} \mathcal{M}\mathrm{or}(\mathcal{C})_{[1]}$$

induced by cocartesian transport along the maps $\rho_i : [1] \simeq \{i-1, i\} \hookrightarrow [n]$ for $1 \leq i \leq n$ is an equivalence. Note that all the ρ_i are convex. It follows from [Lur17, Prop.4.4.3.5] that for a convex morphism : $[n] \leftarrow [m] : \beta$ in Δ^{op} the cocartesian transport functor $\beta_!$ is given by the corresponding restriction functor

$$\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[n]}}(\mathcal{C}) \xrightarrow{v_\beta} \mathsf{Alg}_{\mathsf{Tens}^{\otimes}[m]}(\mathcal{C}).$$

Therefore the Segal condition reduces to the claim that the product of the corresponding restriction functors induces an equivalence

$$\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[n]}}(\mathcal{C}) \to \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}) \times_{\mathsf{Alg}_{\mathsf{Tens}^{\otimes}[0]}(\mathcal{C})} \cdots \times_{\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})} \mathsf{Alg}_{\mathsf{Tens}^{\otimes}[1]}(\mathcal{C}).$$

But this is a consequence of [Lur17, Prop.4.4.1.11]. ☐

For our purposes, we also need to understand the functorial properties of this construction. For this, note that for any inner face map $\partial_i : [m-1] \to [m]$ in Δ there exists a commutative diagram

$$\begin{array}{ccc} ([1]) \simeq \{i-1, i+1\} & \longrightarrow & ([m-1]) \\ \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_i} \\ ([2]) \simeq \{i-1, i, i+1\} & \longrightarrow & ([m]) \end{array}$$

The functoriality of $\mathsf{Tens}^{\otimes}$ induces for each such diagram a map of generalized ∞-operads

$$\xi : \mathsf{Tens}^{\otimes}_{\succ} \to \mathsf{Tens}^{\otimes}_{\partial_i}.$$

We can then prove:

Proposition C.5. *The construction of $\mathcal{M}\mathrm{or}(-)$ extends to define an ∞-functor*

$$\mathcal{M}\mathrm{or}(-) : \mathsf{Cat}^{\mathcal{G}} \to \mathsf{Seg}(\mathsf{Cat})$$

into the ∞-category of double ∞-categories.

Proof. It remains to show that for every functor $G : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ in $\mathsf{Cat}^{\mathcal{G}}$ the induced morphism of cocartesian fibrations

$$\begin{array}{ccc} \mathcal{M}\mathrm{or}(\mathcal{C}) & \xrightarrow{\mathcal{M}\mathrm{or}(G)} & \mathcal{M}\mathrm{or}(\mathcal{D}) \\ & \searrow \quad \swarrow & \\ & \Delta^{\mathrm{op}} & \end{array}$$

preserves all cocartesian edges. Since every morphism in Δ^{op} decomposes into a convex morphism and a composite of inner face maps, it suffices to consider those cases separately. The convex case follows from the explicit description of the cocartesian transport functor and the assumption that G preserves inert morphisms. Therefore let $\partial_i : [m-1] \to [m]$ in Δ be an inner face map. It follows from Theorem C.4 that a functor

$$F : \mathsf{Tens}^{\otimes}_{\partial_i} \to \mathcal{C}$$

over Fin_* representing an arrow in $\mathcal{M}\mathrm{or}(\mathcal{C})$ is cocartesian, if and only F is an operadic left Kan extension of its restriction $F|_{\mathsf{Tens}^{\otimes}_{[m]}} := F_0$. We depict this in the following diagram:

$$\begin{array}{ccc} \mathsf{Ten}^{\otimes}_{[m]} & \xrightarrow{F_0} & \mathcal{C}^{\otimes} \\ \downarrow & \overset{F}{\nearrow} & \downarrow \\ \mathsf{Ten}^{\otimes}_{\partial_i} & \longrightarrow & \mathrm{fin} \end{array}$$

Unraveling [Lur17, Lem.4.4.3.8] F is an operadic left Kan extension if and only if

(1) for $j \in [m-1]$ the morphism $F(a_{\alpha(j)}) \to F(b_j)$ is cocartesian,

(2) for $0 < j < i$ the morphism $F(m_{j-1,j}) \to F(n_{j-1,j})$ is cocartesian,

(3) for $i < j < m$ the morphism $F(m_{j,j+1}) \to F(n_{j-1,j})$ is cocartesian,

(4) the functor F induces an equivalence $\mathsf{colim}_{\Delta^{op}} \mathsf{Bar}_{a_i}(m_{i-1,i}, m_{i,i+1}) \simeq F(n_{i-1,i})$, where $\mathsf{Bar}_{-}(-,-)$ denotes the 2-sided Bar construction [Lur17, 4.4.2.7].

Since G by assumption preserves Δ^{op}-indexed colimits and cocartesian morphisms, it follows that $G \circ F$ is an operadic left Kan extension of $G \circ F_0$. Hence $\mathcal{M}\mathrm{or}(G)$ preserves cocartesian morphisms. ☐

In the following, we will identify $\mathcal{M}\mathrm{or}(-)$ with its composite with the Straithening-Unstraightening equivalence. More precisely, we consider

$$\mathcal{M}\mathrm{or}(-) : \mathsf{Cat}^{\mathcal{G}} \to \mathsf{Seg}(\mathsf{Cat})$$

as a functor to the ∞-category of Segal objects in Cat.

Proposition C.6. *The functor* $\mathcal{M}\mathrm{or}(-)$ *preserves products. In particular, it induces a functor*

$$\mathcal{M}\mathrm{or}(-)^{\otimes} : \mathsf{Cat}^{\mathcal{G},\otimes} \to \mathsf{CAlg}(\mathsf{Seg}(\mathsf{Cat})) \simeq \mathsf{Seg}(\mathsf{Cat}^{\otimes}_{\infty})$$

to the ∞*-category of category objects in* $\mathsf{Cat}^{\otimes}_{\infty}$

Proof. To see that $\mathcal{M}\mathrm{or}(-)$ preserves products, it suffices by Lemma C.3 to note that for any ∞-operad $\mathcal{O}^{\otimes}$ the functor $\mathsf{Alg}_{\mathcal{O}^{\otimes}}(-)$ preserves products and that products are computed pointwise in $\mathsf{Seg}(\mathsf{Cat})$. ☐

In particular, we can consider the composition

$$\mathcal{M}\mathrm{or}(-) : \mathsf{Cat}^{\mathcal{G}} \xrightarrow{\mathcal{M}\mathrm{or}(-)} \mathsf{Seg}(\mathsf{Cat}) \xrightarrow{I} \mathsf{Seg}_{\mathrm{ec}}(\mathsf{Cat}) \xrightarrow{L} \mathsf{Cat}_{(\infty,2)}$$

with the product preserving functors $I : \mathsf{Seg}(\mathsf{Cat}) \to \mathsf{Seg}_{\mathrm{ec}}(\mathsf{Cat})$ and $L : \mathsf{Seg}_{\mathrm{ec}}(\mathsf{Cat}) \to \mathsf{Cat}_{(\infty,2)}$ that associates to any Segal object in Cat an essentially constant Segal object and to any essentially constant Segal object its completion.

Definition C.2. Let $\mathcal{C}^{\otimes} \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{G}})$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. We call the symmetric monoidal $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathcal{C})^{\otimes}$ obtained by applying the symmetric monoidal functor $\mathbb{M}\mathsf{or}(-)$ to $\mathcal{C}^{\otimes}$ the *Morita* $(\infty, 2)$*-category of* $\mathcal{C}$.

Moreover, the next result follows from the functoriality of the construction of the Morita category:

Corollary C.7. *Let* $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *be a morphism in* $\mathsf{CAlg}(\mathsf{Cat}^{\mathcal{G}})$. *Then* F *induces a symmetric monoidal* $(\infty, 2)$*-functor* $\mathbb{M}\mathsf{or}(F) : \mathbb{M}\mathsf{or}(\mathcal{C})^{\otimes} \to \mathbb{M}\mathsf{or}(\mathcal{D})^{\otimes}$.

To get a better understanding of this $(\infty, 2)$-category, let us analyze the morphism ∞-categories.

Proposition C.8. *Let* $\mathcal{C}^{\otimes}$ *be a* $\otimes$*-GR-cocomplete symmetric monoidal* ∞*-category. There exists an equivalence of* ∞*-categories*

$$\mathbb{M}\mathsf{or}(\mathcal{C})(A, B) \simeq \mathsf{BMod}_{A,B}(\mathcal{C}).$$

Under this equivalence, the composition gets identified with

$$\otimes_B : \mathsf{BMod}_{A,B}(\mathcal{C}) \times \mathsf{BMod}_{B,C}(\mathcal{C}) \to \mathsf{BMod}_{A,C}(\mathcal{C})$$

the relative tensor product functor [Lur17, Sect.4.4.2].

Proof. It follows from [Lur09b, Thm.1.2.13] that for every essentially constant Segal object $\mathcal{D}_\bullet : \Delta^{op} \to \mathbb{D}$ the unit morphism $\mathcal{D}_\bullet \to L(\mathcal{D})_\bullet$ of the adjunction

$$L : \mathsf{Seg}_{\mathrm{ec}}(\mathsf{Cat}) \rightleftarrows \mathsf{Cat}_{(\infty,2)} : \iota$$

is a fully faithful and essentially surjective morphism of double categories [Lur09b, 1.2.12]. Hence, it suffices to calculate the mapping ∞-categories and composition functors for $I(\mathcal{M}\mathrm{or}(\mathcal{C}))_\bullet$. The mapping ∞-categories are defined as the pullback of ∞-categories

$$\begin{array}{ccc} I(\mathcal{M}\mathrm{or}(\mathcal{C}))(A,B) & \longrightarrow & I(\mathcal{M}\mathrm{or}(\mathcal{C}))_{[1]} \\ \downarrow & & \downarrow \\ * & \xrightarrow{(A;B)} & \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})^{\simeq} \times \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})^{\simeq} \end{array}$$

By definition of the functor I this diagram sits in a larger rectangle

$$\begin{array}{ccccc} I(\mathcal{M}\mathrm{or}(\mathcal{C}))(A,B) & \longrightarrow & I(\mathcal{M}\mathrm{or}(\mathcal{C}))_{[1]} & \longrightarrow & \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{(A,B)} & \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})^{\simeq} \times \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})^{\simeq} & \longrightarrow & \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C}) \times \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C}) \end{array}$$

where all sub-squares are pullback diagrams. It follows that $I(\mathcal{M}\mathrm{or}(\mathcal{C})(A, B)) \simeq \mathcal{M}\mathrm{or}(\mathcal{C})(A, B)$ for all $A, B \in \mathcal{C}_0$. Using the equivalences $\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}) \simeq \mathsf{BMod}(\mathcal{C})$ and $\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C}) \simeq \mathsf{Alg}(\mathcal{C})$ we can obtain an equivalence of the pullback with $\mathbb{M}\mathsf{or}(\mathcal{C})(A, B) \simeq \mathsf{BMod}_{A,B}(\mathcal{C})$.
Similarly, the composition functor for the double ∞-category $\mathcal{M}\mathrm{or}(\mathcal{C})$ is given by the functor

$$\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}) \times_{\mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[0]}}(\mathcal{C})} \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}) \simeq \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[2]}}(\mathcal{C}) \xrightarrow{\alpha_{1,!}} \mathsf{Alg}_{\mathsf{Tens}^{\otimes}_{[1]}}(\mathcal{C}).$$

After passing to fibers over A, B, C, this functor becomes identified under the equivalence of mapping ∞-categories $I(\mathcal{M}\mathrm{or}(\mathcal{C})(A,B)) \simeq \mathbb{M}\mathsf{or}(\mathcal{C})(A,B)$ with the relative tensor produce functor [Lur17, Sect. 4.4.2]

$$- \otimes_B - : \mathsf{BMod}_{A,B}(\mathcal{C}) \times_{\mathsf{Alg}(\mathcal{C})} \mathsf{BMod}_{B,C}(\mathcal{C}) \to \mathsf{BMod}_{A,C}(\mathcal{C}).$$

This proves the claimed description of the composition functor. □

After we have now finished the construction of the Morita $(\infty,2)$-category, let us now explicitly determine the fully dualizable objects. To formulate the definition of a fully dualizable object, we first need to recall the definitions of dual objects and adjoint morphisms.

Definition C.3. [Lur17, 4.6.1.1] Let $\mathcal{C}^\otimes \in \mathsf{Alg}(\mathsf{Cat})$ be a monoidal ∞-category. An object $C \in \mathcal{C}$ is called *right dualizable* if there exists an object $C^\vee \in \mathcal{C}$ and a pair of morphisms

$$c : \mathbb{1}_\mathcal{C} \to C \otimes C^\vee \qquad\qquad e : C^\vee \otimes C \to \mathbb{1}_\mathcal{C}$$

such that the compositions

$$C \xrightarrow{c\otimes \mathrm{id}} C \otimes C^\vee \otimes C \xrightarrow{\mathrm{id}\otimes e} C$$
$$C^\vee \xrightarrow{\mathrm{id}\otimes c} C^\vee \otimes C \otimes C^\vee \xrightarrow{e\otimes \mathrm{id}} C$$

are homotopic to the identity. We then call $C^\vee$ the *right dual* of C, and C the *left dual* of $C^\vee$. The object $C^\vee$ is called *left dualizable*.

Remark C.1. Note that in case that $\mathcal{C}^\otimes$ is symmetric monoidal, an object $C \in \mathcal{C}$ has a right dual $C^\vee$ if and only if it has a left dual ${}^\vee C$ and that there exists a canonical equivalence between the left and the right dual. Therefore, we will in the following abuse notation and call in case that $\mathcal{C}^\otimes$ is symmetric monoidal the right (respectively left) dual of an object simply the dual.

Definition C.4. Let $\mathbb{D}$ be an $(\infty,2)$-category. A 1-morphism $f : C \to D$ in $\mathbb{D}$ admits a *left adjoint* if there exists a 1-morphism $f^L : D \to C$, and a 2-morphism $u : \mathrm{id}_D \to f \circ f^L$ that induces for every object $E \in \mathbb{C}$ and every pair of 1-morphisms $g : E \to D$ and $h : E \to C$ an equivalence

$$\mathsf{Map}_{\mathbb{D}(E,C)}(f^L \circ g, h) \to \mathsf{Map}_{\mathbb{D}(E,D)}(f \circ f^l \circ g, f \circ h) \to \mathsf{Map}_{\mathbb{D}(E,D)}(g, f \circ h)$$

of mapping ∞-categories.

We can now state the definition of fully dualizability in an $(\infty,2)$-category:

Definition C.5. Let $\mathbb{D}^\otimes$ be a symmetric monoidal $(\infty,2)$-category. An object $C \in \mathbb{D}$ is called *fully dualizable*

(1) if C admits a dual $C^\vee$,

(2) the evaluation morphism $ev_C : C^\vee \otimes C \to \mathbb{1}_\mathbb{D}$ admits both adjoints.

Remark C.2. It follows from [Lur08, Prop.4.2.3] that this definition coincides with the usual definition of a fully dualizable object as given in [Lur08, Def.2.3.21].

Let us now unpack this definition for the example of the symmetric monoidal $(\infty,2)$-category $\mathbb{M}\mathsf{or}(\mathcal{C})^\otimes$. To classify the dualizable objects, we use the following alternative characterization:

Proposition C.9. *[Lur17, Lem.4.6.1.6] Let $\mathcal{C}^\otimes \in \mathsf{Alg}(\mathsf{Cat})$ be a monoidal ∞-category. The following are equivalent:*

(1) *$A \in \mathcal{C}$ admits a right dual $A^\vee$.*

(2) *There exists a map $e : A^\vee \otimes A \to \mathbb{1}_\mathcal{C}$, s.t. for every objects $B, C \in \mathcal{C}$ the map e induces a homotopy equivalence*
$$\mathsf{Map}_\mathcal{C}(B, C \otimes A^\vee) \to \mathsf{Map}_\mathcal{C}(D, C \otimes A^\vee \otimes A) \to \mathsf{Map}_\mathcal{C}(D \otimes A, C). \tag{45}$$

Notation C.1. Let $\mathcal{D}^\otimes \in \mathsf{CAlg}(\mathsf{Cat})$ be a symmetric monoidal ∞-category and $A \in \mathsf{Alg}(\mathcal{D})$ be an algebra object in $\mathcal{D}$. The ∞-category $\mathsf{Alg}(\mathcal{D})$ itself inherits a monoidal structure from $\mathcal{D}$. We denote the monoidal product of the algebras A and $A^{\otimes\text{-}\mathrm{op}}$ by A^e and call it the enveloping algebra of A.

Proposition C.10. *Let $\mathcal{C}^\otimes \in \mathsf{CAlg}(\mathsf{Cat}^\mathcal{G})$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. The evaluation bimodule $A \in \mathsf{BMod}_{A\otimes A^{\otimes\text{-}\mathrm{op}}, \mathbb{1}_\mathcal{C}}(\mathcal{C})$ considered as a 1-morphism*

$$A \otimes A^{\otimes\text{-}\mathrm{op}} \xrightarrow{A} \mathbb{1}_\mathcal{C} \ \ \textit{in } \mathbb{M}\mathsf{or}(\mathcal{C})$$

exhibits $A^{\otimes\text{-}\mathrm{op}}$ as a dual of A in $\mathbb{M}\mathsf{or}(\mathcal{C})^\otimes$.

Proof. It follows from [Lur17, Prop.4.6.3.11] that the 1-morphism $A \otimes A^{\otimes\text{-}\mathrm{op}} \xrightarrow{A^e} \mathbb{1}_\mathcal{C}$ induces an equivalence of ∞-categories
$$\mathsf{BMod}_{B, A^{\otimes\text{-}\mathrm{op}}\otimes C}(\mathcal{C}) \to \mathsf{BMod}_{B\otimes A, A\otimes A^{\otimes\text{-}\mathrm{op}}\otimes C}(\mathcal{C}) \to \mathsf{BMod}_{B\otimes A, C}(\mathcal{C})$$
that is given on objects as
$$N \to (A \otimes N) \otimes_{A\otimes A^{\otimes\text{-}\mathrm{op}}\otimes C} (A^e \otimes C).$$
The induced equivalence on underlying spaces is the map from Equation (45) and hence the claim follows from Proposition C.9. ☐

Next, let us study adjoints in the $(\infty, 2)$-category $\mathbb{M}\mathsf{or}(\mathcal{C})$:

Definition C.6. [Lur17, Def.4.6.2.3] Let $\mathcal{C}^\otimes$ be a GR-$\otimes$-cocomplete symmetric monoidal ∞-category, and let $A, B \in \mathsf{Alg}(\mathcal{C})$ be algebra objects in $\mathcal{C}$. A bimodule $M \in \mathsf{BMod}_{A,B}(\mathcal{C})$ is called *left dualizable* if there exists an object $N \in \mathsf{BMod}_{B,A}(\mathcal{C})$ and morphisms

$$c : B \to N \otimes_A M \text{ in } \mathsf{BMod}_{B,B}(\mathcal{C}) \qquad e : M \otimes_B N \to A \text{ in } \mathsf{BMod}_{A,A}(\mathcal{C})$$

such that the composites
$$M \simeq M \otimes_B B \xrightarrow{\mathrm{id}\otimes c} M \otimes_B N \otimes_A M \xrightarrow{e\otimes \mathrm{id}} M$$
$$N \simeq B \otimes_B N \xrightarrow{c\otimes \mathrm{id}} N \otimes_A M \otimes_B N \xrightarrow{\mathrm{id}\otimes e} N$$

are homotopic to the identity. In that case, we call N the *left dual* of M.

Proposition C.11. *Let $\mathcal{C} \in \mathsf{CAlg}(\mathsf{Cat}^G)$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. Let $A, B \in \mathsf{Alg}(\mathcal{C})$ be algebra objects of $\mathcal{C}$. Further, let $X \in \mathsf{BMod}_{A,B}(\mathcal{C})$, $Y \in \mathsf{BMod}_{B,A}(\mathcal{C})$ be bimodule objects and let $u : X \otimes_B Y \to A$ be a morphism in $\mathsf{BMod}_{A,A}(\mathcal{C})$. Then the following are equivalent:*

(1) u exhibits Y as a left dual of X in the sense of Definition C.6.

(2) u exhibits Y as a left adjoint of X internal to $\mathbb{M}\mathsf{or}(\mathcal{C})$.

Proof. This is a reformulation of [Lur17, Prop.4.6.2.1]. ☐

Because of their importance for the study of fully dualizable objects in $\mathbb{M}\mathsf{or}(\mathcal{C})^{\otimes}$ the adjoints of the evaluation bimodules A^e deserve a special name.

Definition C.7. [Lur17, Def.4.6.4.2] An algebra object $A \in \mathsf{Alg}(\mathcal{C})$ is called *proper* if it is dualizable as an object of $\mathcal{C}$.

Definition C.8. An algebra $A \in \mathsf{Alg}(\mathcal{C})$ is called *smooth* if the evaluation bimodule $A \in \mathsf{BMod}_{A^e,\mathbb{1}_{\mathcal{C}}}(\mathcal{C})$ is left-dualizable.

We can now use all these results to classify the fully dualizable objects as follows:

Theorem C.12 ([Lur08])**.** *Let $\mathcal{C}^{\otimes} \in \mathsf{CAlg}(\mathsf{Cat}^{\mathcal{G}})$ be a $\otimes$-GR-cocomplete symmetric monoidal ∞-category. Then an algebra object $A \in \mathsf{Alg}(\mathcal{C})$ is fully dualizable in $\mathbb{M}\mathsf{or}(\mathcal{C})^{\otimes}$ if and only if it is smooth and proper.*

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