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Defects in non-semisimple 3d topological field theory and 2d logarithmic conformal field theory

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Summary

In this thesis, we study the connection between three-dimensional topological field theories (TFTs) and two-dimensional conformal field theories (CFTs). More specifically, we extend the "TFT construction of RCFT correlators" of Fuchs, Runkel, Schweigert and others from rational to so-called finite logarithmic CFTs. For these, the chiral data is encoded in a modular tensor category $\mathcal{C}$, which is not necessarily semisimple, but still finite.

Our first main result is the explicit construction of a 2-categorical version of Lyubashenko's modular functor in terms of the non-semisimple 3d TFT of De Renzi et al constructed from $\mathcal{C}$. We also extend this modular functor to a 2-category of "topological world sheets" in order to account for boundary conditions and topological defects. Based on this, our second main result consists of an explicit construction of a full CFT, in the form of a braided monoidal oplax natural transformation, using surface defects in the non-semisimple 3d TFT. As an example, we consider the case of the simplest surface defect, the transparent one, and show that our results match the expectations from the literature for the so-called diagonal or charge-conjugate CFT.

Zusammenfassung

In dieser Doktorarbeit wird der Zusammenhang zwischen drei dimensionalen topologischen Feldtheorien (TFTs) und zwei dimensionalen konformen Feldtheorien untersucht. Genauer gesagt erweitern wir die "TFT construction of RCFT correlators" von Fuchs, Runkel, Schweigert und weiteren Koautoren vom Fall rationaler zu so genannten endlichen logarithmischen konformen Feldtheorien. Für diese Theorien sind die chiralen Daten in einer modularen Tensorkategorie $\mathcal{C}$, welche endlich aber nicht unbedingt halbeinfach ist, verpackt.

Unser erstes Hauptresultat ist die explizite Konstruktion einer 2-kategorischen Version von Lyubashenko's modularen Funktor mittels der aus $\mathcal{C}$ konstruierten, nicht-halbeinfachen 3d TFT von De Renzi und Koautoren. Zudem erweitern wir diesen modularen Funktor auf eine 2-Kategorie von "topologischen Weltflächen" um sowohl Randbedingungen als auch topologische Defekte beschreiben zu können. Darauf aufbauend ist unser zweites Haupresultat die explizite Konstruktion einer vollen konformen Feldtheorie, axiomatisiert als eine oplax-natürliche Transformation, mittels Flächendefekten in der nicht halbeinfachen 3d TFT. Als Beispiel betrachten wir unsere Konstruktion im Fall für den einfachsten Flächendefekt, den transparenten, und zeigen, dass unsere Resultate die Erwartungen für die so genannte diagonale bzw. "charge-conjugate" Theorie reproduzieren.

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Relevant publications

This thesis is partially based on the following preprint:

- A. Hofer, I. Runkel. *Modular functors from non-semisimple* 3*d TFTs.* arXiv:2405.18038 [math.QA]

and the following as of yet unpublished work:

- A. Hofer, I. Runkel, *Non-semisimple CFT/TFT correspondence I: General setup*

These are listed in the bibliography at the end of this thesis as [HR1; HR2]. Both works are the result of a collaboration with my supervisor Ingo Runkel, whose contributions and ideas I fully acknowledge.

The paper [HR1] is the basis of Sections 3.1 and 4.1, as well as Chapter 5. The idea to use the non-semisimple TFT of [DGGPR1] to construct the modular functor of [Lyu2] was suggested by Ingo Runkel. My contributions consist of its 2-categorical formulation (resulting in Definition 4.1.1 below), its construction (given in Section 5.1), and proving that the construction satisfies the required axioms (resulting in Theorem 5.1.9). The idea to extend the modular functor, first to bordisms with involutions (see Section 4.1.2) and then to open-closed bordisms (see Section 4.1.3), was developed together during discussions and formalised by myself. In particular, the properties of the non-semisimple TFT (see Section 3.1.5) needed were proven by myself with the exception of Lemma 3.1.10, which is based on [TV2, Lem. 17.2], and was proven together. Using modular functors to give a topological proof of the adjunction between a modular tensor category and its Drinfeld centre (in the form of Proposition 5.2.3) was my idea.

The paper [HR2] is the basis of the rest of Chapter 4, as well as all of Chapter 6 and 7. The construction of a full 2d CFT in the spirit of [FFFS; FRSI; FjFRS] using the 3d TFT of [DGGPR1] was suggested by Ingo Runkel. Its 2-categorical formulation (in the form of Definition 4.3.1) was my idea and brought to its final form in discussions. The necessary modifications to the construction of [FFFS; FRSI; FjFRS] (resulting in Theorem 6.6.2) were obtained independently by myself and then refined in subsequent discussions. In particular, the idea to use connecting manifolds to also obtain the field content of the CFT (see Section 6.2) was mine. I also worked out the majority of the proof of Theorem 6.6.2 myself (forming the major part of Chapter 6). The idea to work locally and use the "wedge presentation" in Section 6.5 was suggested by Ingo Runkel and is inspired by [FjFRS, Sec. 5]. The results presented in Chapter 7 were obtained by myself.

Acknowledgements

First and foremost, I want to express my deepest gratitude to my supervisor, Ingo Runkel. Without his patient guidance and encouragement, I would not have been able to achieve the results presented in this thesis. I am especially grateful for the many discussions and for all his comments and remarks on earlier drafts. Similar thanks go to Nils Carqueville, who introduced me to this area of mathematics and continues to positively influence my academic journey. On this note, I would also like to thank Christoph Schweigert for his guidance over the last three years.

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Lastly, I am truly grateful for the unconditional support and love of my family. To my parents Peter and Veronika, and my sister Eva. Without their unwavering support, I would not have been able to focus on my studies of physics and mathematics. Finally, my deepest appreciation belongs to my wonderful partner Madeleine. Without her constant encouragement and support, I would not have been able to complete this thesis.

Contents

Chapter 1

Introduction

Two dimensional conformal field theories (CFTs) are one of the most well studied classes of quantum field theories, with applications ranging from string theory to critical systems in statistical physics. Their rich mathematical structure is illustrated by their close connection to several, distinct mathematical disciplines including complex analysis, probability theory, algebraic geometry, number theory, quantum topology, and representation theory.

Even though 2d CFTs have been studied by mathematicians for decades, a rigorous construction, in the form of a complete set of consistent correlation functions, still remains elusive for a large class of theories. Instead of tackling the problem of constructing a 2d CFT directly, the situation becomes more tractable by splitting the problem into a complex-analytic/algebro-geometric and a purely algebraic/topological part. For the class of so-called *rational* CFTs the work of [FFFS; FRSI; FjFRS] completely solved the second part, using three dimensional topological field theories (TFTs), over twenty years ago. Very recently, a great step towards combining the second with the first part was achieved in [DW].

In this thesis, we will present a rigorous solution of the second part for the class of so-called *finite logarithmic* CFTs, which includes all rational theories as well. More precisely, we will extend the TFT construction of rational CFT correlators of [FFFS; FRSI; FjFRS], to the setting where the algebraic input data is no longer semisimple but still finite. The next section is devoted to review the general ideas behind the TFT construction of CFT correlators and motivate the need to go beyond semisimplicity.

1.1 Motivation and background

1.1.1 Chiral CFT

Let us start with a brief overview of some of the tools used to study the complex-analytic/algebro-geometric part. In physical terms, this part corresponds to the study of *chiral* CFTs, which are defined on complex curves. There are several approaches to study chiral CFTs rigorously. For our purposes, the one based on vertex operator algebras (VOAs) will be the most suitable. More precisely, for us a chiral CFT will be encoded by a VOA $\mathcal{V}$, its representation category $\mathrm{Rep}(\mathcal{V})$, and a modular functor $\mathrm{Bl}_{\mathcal{V}}$. Let us briefly recall these notions.

Vertex operator algebras are a mathematical incarnation of the chiral symmetry algebras studied in physics. A VOA can roughly be thought of as something like a commutative algebra parametrised by the complex plane. In particular, as for algebras, there is also a notion of VOA-modules. The resulting representation categories are naturally equipped with extra structure, such as a non-trivial braiding and a ribbon twist. The study of VOAs and their modules is intimately related to algebraic geometry, number theory, and representation theory.

A modular functor is, colloquially speaking, a systematic assignment of mapping class group representations to surfaces which is compatible with gluing along boundaries. There are various versions of modular functors used in different contexts, with the main distinction between topological modular functors, complex-analytic, and algebro-geometric modular functors. Topological modular functors are closely related to 3-dimensional TFTs. This is because the TFT state spaces naturally carry mapping class group actions.

In the context of chiral CFTs/VOAs the modular functor is, depending on the specific formulation, complex-analytic or algebro-geometric in nature and encodes the monodromy and gluing behaviour of the so-called conformal block spaces. The conformal block spaces are solution spaces of certain differential equations the correlation functions need to obey [Seg; MS; TUY; FB; AU; DGT1; DGT2; GZ]. In this context, the surfaces are equipped with a complex structure and the conformal block spaces are expected to form a vector bundle with projectively flat connection over the moduli space of complex curves. The mapping class group action corresponds to the monodromy of this bundle by the Riemann-Hilbert correspondence. Finally, we want to mention that although the complex analytic and algebraic world are deeply intertwined, one still needs be a bit careful in the distinction between the corresponding modular functors, see e.g. [DW, Sec. 5.4] for an example of subtlety arising in this context. In this text we will not discuss any of these subtleties further and assume that the chiral side is under reasonable control. Instead, we will focus solely on the second part of the CFT construction from now on.

1.1.2 Full CFT

The algebraic/topological part is concerned with *full* CFTs, or, more precisely, with the construction of a full CFT starting with a chiral one. Full CFTs are defined on conformal manifolds, possibly with boundary. Moreover, one can further allow for stratifications of the manifold, i.e. embedded submanifolds of various codimensions. In this setting, 1-dimensional strata are used to describe (topological) line defects between, possibly different, full CFTs living on the 2-strata. We will call such a 2-manifold a *world sheet.* Topological defects generalise the notion of ordinary symmetries and can be used to understand further phenomena, such as dualities, on a conceptual level [FrFRS].

In a full CFT one deals with holomorphic and anti-holomorphic degrees of freedom. Accordingly, upon restriction to just the holomorphic sector, one should obtain a chiral CFT. On the other hand, one could imagine combining two chiral CFTs, in some way, to obtain a full CFT. This idea is often referred to as *holomorphic factorisation* or as *combing left-movers and right-movers.* In this thesis, we will address exactly this question, of combining two chiral CFTs to obtain a full CFT. More specifically, we will focus on the case where both the holomorphic and the anti-holomorphic sector are governed by the same chiral CFT.

Let us now get a bit more precise. A full CFT, with chiral sector given in the form of a VOA $\mathcal{V}$, its representation category $\mathrm{Rep}(\mathcal{V})$, and its modular functor $\mathrm{Bl}_{\mathcal{V}}$, consists of two extra pieces of data: the *field content*, and a *consistent system of correlators*. The field content describes the state space of the full CFT on the interval as well as on the circle. These spaces are equipped with an action of $\mathcal{V}$ for the interval or $\mathcal{V} \otimes_{\Bbbk} \overline{\mathcal{V}}$ for the circle, i.e. they are objects in $\mathrm{Rep}(\mathcal{V})$ or $\mathrm{Rep}(\mathcal{V} \otimes_{\Bbbk} \overline{\mathcal{V}})$. Here $\overline{\mathcal{V}}$ denotes the anti-holomorphic version of the VOA $\mathcal{V}$. A consistent system of correlators sends a world sheet $\mathfrak{S}$ to an element in the conformal block space $\mathrm{Bl}_{\mathcal{V}}(\widehat{\mathfrak{S}})$, where $\widehat{\mathfrak{S}}$ is the *Schottky* or *complex double* of $\mathfrak{S}$. Moreover, this assignment needs to satisfy mapping class group covariance and gluing properties. Note that the dependence on the conformal structure of the world sheet $\mathfrak{S}$ is already encoded in the modular functor $\mathrm{Bl}_{\mathcal{V}}$. Accordingly, only the topology of $\mathfrak{S}$ will be relevant which leads us to focus on the underlying *topological world sheet.*

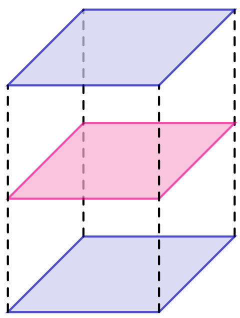
1.1.3 CFTs from TFTs

Typically, a chiral CFT does not fully determine a full CFT and one needs to specify an extra input datum. One way to obtain this extra input datum is via the connection to 3d TFTs. Since the 1980's, it has been known that there is a close relationship between chiral CFTs and 3d TFTs where the CFT is expected to be a boundary theory of the TFT [Wit]. It is further expected that the chiral conformal block spaces $\mathrm{Bl}_{\mathcal{V}}$ are isomorphic to the state spaces of the 3d TFT Z. Or more accurately, that the modular functor $\mathrm{Bl}_{\mathcal{V}}$ of the chiral CFT is isomorphic to the modular functor obtained from the TFT.

As mentioned above, this idea has been used to construct a full CFT from a given chiral one for the class of rational CFTs in the series of papers [FFFS; FRSI; FRSII; FRSIII; FRSIV; FjFRS; FjFSt]. Rational CFTs are characterised by the fact that the representation category $\mathcal{C} := \mathrm{Rep}(\mathcal{V})$ of the VOA $\mathcal{V}$ is a modular fusion category, i.e. a certain type of finitely *semisimple* ribbon tensor category, see Section 2.1 for the relevant definitions. This type of category is precisely the algebraic input datum used in the construction of 3d surgery TFTs in [RT; Tur]. Moreover, the mapping class group representations of the RT-TFT give rise to a topological modular functor. It has long been conjectured, and recently been shown [DW], that this topological modular functor reproduces the algebro-geomtric one constructed directly from the VOA $\mathcal{V}$ [FB; DGT2; DGT1]. From this we see that the correlators of the full CFT are automatically states of the TFT, which means we should be able to encode them completely topologically.

The main insight of Fuchs et al. was to construct a family of TFT states, which satisfies the consistency conditions for CFT correlators, by evaluating the 3d TFT on certain 3-manifolds, the so-called *connecting manifolds*. Moreover, they also identified the additional extra input datum with a symmetric special Frobenius algebra in $\mathcal{C}$ and obtained the field content via the representation theory of this algebra inside $\mathcal{C}$. Later it was pointed out in [KS], and further refined in [FSV], that this algebra corresponds to a surface defect in the TFT.

The connecting manifold of a world sheet $\mathfrak{S}$ with no boundaries is simply given by the cylinder $\mathfrak{S} \times [-1,1]$, with the world sheet as a surface defect embedded at $\mathfrak{S} \times \{0\}$:



The boundary of this cylinder is given by the so-called *orientation double* $\mathfrak{S} \sqcup -\mathfrak{S}$ of $\mathfrak{S}$, where $-\mathfrak{S}$ is the orientation reversal of $\mathfrak{S}$. The orientation double is the topological surface underlying the complex double $\widehat{\mathfrak{S}}$ from above. Thus by applying the TFT Z to $\mathfrak{S} \times [-1,1]$ we get an element in the state space $\mathrm{Z}(\widehat{\mathfrak{S}})$, which is, by the above discussion, isomorphic to the space of conformal blocks $\mathrm{Bl}_{\mathcal{V}}(\widehat{\mathfrak{S}})$.

For a general world sheet, one has to take a quotient in the construction of the orientation double and the connecting manifold. This is necessary to encode that the boundary should carry a conformal boundary condition, which identifies the chiral and anti-chiral sector.

The goal of this thesis is to extend this 'TFT construction of CFT correlators' to the class of *finite logarithmic* CFTs in the sense of [CG]. In this setting, the category of VOA-modules $\mathcal{C}$ is a modular tensor category, meaning that it is no longer semisimple, but still finite. These categories naturally generalise modular fusion categories and, in particular, are still rigid. The TFTs of [RT; Tur] have been generalised to these categories in [DGGPR1]. We will use an extension of these non-semisimple 3d TFTs, including surface defects, to construct the data of a full CFT. Along the way, we will propose a 2-categorical definition for a full CFT based on a modular functor, inspired by the ones used in [FFRS; KLR; FS2]. We want to mention here that in the non-semisimple setting the connection between topological and algebro-geometric modular functors is still unclear, see [DW, Sec. 5.6] for a discussion. See also [GZ], in particular Section 0.6, which makes a few comments on the connection between topological and complex-analytic modular functors in the non-semisimple setting.

From a mathematical perspective going beyond semisimplicity is motivated by the possibility of finding stronger topological invariants, see e.g. [DGGPR2; BGR; BCGP] as well as providing a topological interpretation of an algebraic structure. Moreover, once one leaves the semisimple realm it becomes interesting to apply methods from homological algebra [Shi4; LMSS; SW1]. From a physical perspective non-semisimple algebraic structures arise in variety of different contexts ranging from critical percolation (see e.g. [CR2, Sec. 2] and references therein) over lattice models [BGJSV; DHY; HK] to twists of supersymmetric field theories [CDGG; Gar; AGRS].

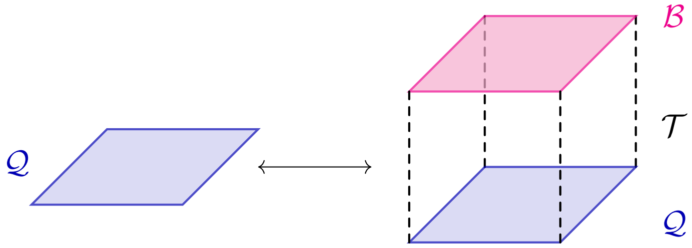
On the level of full CFTs, much less is known than in the rational setting. Most work has been focused on specific models, see e.g. [GR1; GR2]. Early work on the general theory based on ribbon Hopf algebras has been carried out in [FSSt1; FSSt2; FSSt3]. Some of the ideas developed there have been extended to general non-semisimple modular tensor categories to study full CFTs in a model-independent manner [FS2; FGSS; FS5], see also [FS3] for a concise review of some of these aspects. More recently operadic techniques have been used to study classification questions [BW; Woi] and to construct modular functors using string-net models [MSWY]. In the rational setting, a string-net construction of full CFTs has been achieved in [FSY].

We also want to mention, that a handful of full 2d CFTs have been constructed directly using probabilistic methods, see e.g. [GKR] and references therein for a review of recent work in this direction. These theories are not covered by our approach as the corresponding algebraic structures contain an infinite number of simple objects.

1.1.4 Aside: Relation to SymTFT

Before we explain our results, we want to motivate our considerations also from a slightly different perspective using the framework of *symmetry TFTs* which has recently gained popularity [PŠV; KLWZZ; GK; ABGHS; FMT; KOZ; BS], see also the lecture notes [SN; BBFGGPT] for further references, and in particular [CDR, Sec. 3.5] for the connection to the CFT/TFT correspondence reviewed below.

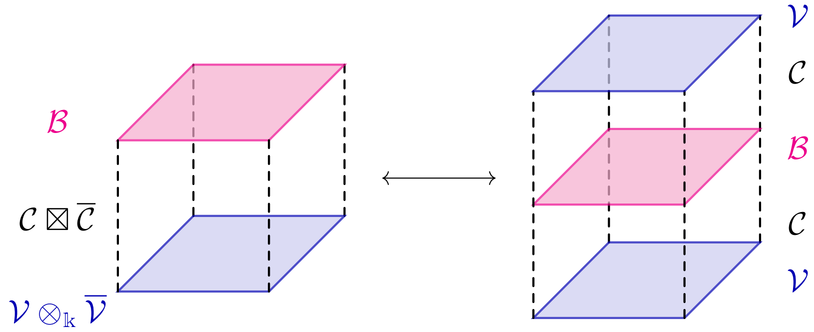
In the symmetry TFT framework the (generalised) symmetries of a d-dimensional quantum field theory $\mathcal{Q}$, in the form of topological defects of various codimensions, are governed by a $(d+1)$-dimensional TFT $\mathcal{T}$, together with a topological "Dirichlet" boundary condition $\mathcal{B}$, such that $\mathcal{Q}$ is boundary theory of $\mathcal{T}$. The idea is that $\mathcal{T}$ acts on $\mathcal{Q}$ via the topological defects of $\mathcal{Q}$ and that one can equivalently describe $\mathcal{Q}$ on a manifold M by instead considering $\mathcal{T}$ on the cylinder $M \times [0,1]$ with non-topological boundary condition $\mathcal{Q}$ at $M \times \{0\}$ and topological boundary condition $\mathcal{B}$ at $M \times \{1\}$:



Moreover, instead of considering the Dirichlet boundary condition $\mathcal{B}$ one can consider any topological boundary condition $\mathcal{B}'$, in this case one obtains a possible different QFT $\mathcal{Q}'$ by considering the cylinder above.

A mathematical analogy of the above ideas is to think of $\mathcal{T}$ as an algebra A, $\mathcal{Q}$ as a right A-module Q_A, and the Dirichlet boundary condition $\mathcal{B}$ as the left regular module ${}_AA$. In this analogy the cylinder idea is simply the natural isomorphism $Q \cong Q_A \otimes_A {}_AA$, where $\otimes_A$ denotes the tensor product over A. The same vector space may be equipped with actions from different algebras and the same is true for a given QFT, which can have many different symmetry TFTs acting.

Let us now discuss how the work of [FFRS; KLR; FS2] can be thought of as an early incarnation of this symmetry TFT framework. According to the above ideas, a full 2d CFT on a world sheet $\mathfrak{S}$ should be equivalently described by a 3d symmetry TFT on the cylinder $\mathfrak{S} \times [0,1]$, with a topological boundary condition on $\mathfrak{S} \times \{1\}$ and a conformal one at $\mathfrak{S} \times \{0\}$. For this we need to make a choice for the symmetry TFT. To this end, we assume that the symmetries of the full CFT contains two copies of a rational VOA $\mathcal{V}$ describing its holomorphic half. As before let us denote with $\mathcal{C} = \mathrm{Rep}(\mathcal{V})$ its modular fusion category of modules. In this case we can choose the symTFT to be the one constructed from $\mathrm{Rep}(\mathcal{V} \otimes_{\Bbbk} \overline{\mathcal{V}})$ by [RT; Tur]. Now we have $\mathrm{Rep}(\mathcal{V} \otimes_{\Bbbk} \overline{\mathcal{V}}) \simeq \mathrm{Rep}(\mathcal{V}) \boxtimes \mathrm{Rep}(\overline{\mathcal{V}}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$, where $\boxtimes$ is the Deligne tensor product and $\overline{\mathcal{C}}$ denotes the same underlying category as $\mathcal{C} = \mathrm{Rep}(\mathcal{V})$ but equipped with inverse braiding and twist. Moreover, the TFT for $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ has the special property that it factorises into the tensor product of the TFT for $\mathcal{C}$ and the one for $\overline{\mathcal{C}}$. Moreover, the TFT for $\overline{\mathcal{C}}$ is the orientation reversal of the TFT for $\mathcal{C}$. Using this we can "unfold the cylinder" and equivalently consider the TFT for $\mathcal{C}$ on $\mathfrak{S} \times [-1, 1]$ with the chiral CFT as a holomorphic boundary condition at $\mathfrak{S} \times \{1\}$, the anti-chiral CFT as an anti-holomorphic boundary condition at $\mathfrak{S} \times \{-1\}$, and a topological surface defect $\mathcal{B}$ at $\mathfrak{S} \times \{0\}$:



Now the right hand side is exactly the connecting manifold we already introduced above. From this point of view the TFT construction of CFT correlators of Fuchs et al. can be seen as an early incarnation of the symmetry TFT framework applied to study full 2d CFTs. Moreover, in this framework we can easily identify the extra input datum, needed to construct a full CFT from a given chiral one, with a topological surface defect for the 3d TFT. The Dirichlet, or regular, boundary condition from above is in this setting simply the transparent surface defect from $\mathcal{C}$ to $\mathcal{C}$. In particular, this choice is always possible, and the resulting full CFT is the one with the *charge-conjugate/diagonal* partition function, sometimes also known as the *Cardy case*.

As an aside, we want to mention that in this framework we could also describe full CFTs for which the chiral and anti-chiral part are not governed by the same VOA leading to so-called heterotic CFTs. In this case we have to consider a similar situation as the one sketched above with different 3d TFTs in in the regions $\mathfrak{S} \times [-1, 0]$ and $\mathfrak{S} \times [0, 1]$. However, one needs to be more careful here because the question of existence of surface defects poses a constraint on the TFTs, namely they have to be Witt-equivalent [DMNO; FSV], see also [MR] for implications on the VOA level. In the semisimple setting the relevant 3d defect TFT was constructed in [KMRS].

The second motivation of our work is to make the above ideas more precise for full 2d CFTs beyond the rational case. Below we will give an overview of our results along with some of their implications.

1.2 Modular functors from non-semisimple TFTs

As a first step we studied topological modular functors. A generalisation of the modular functor from the semisimple TFT of [RT; Tur] to not necessarily semisimple modular tensor categories was already achieved in [Lyu2] using an approach based on generators and relations. The semisimple TFT itself was generalised to the non-semisimple setting in [DGGPR1], building in particular on [CGP; BCGP; DGP]. These TFTs are non-compact in the sense that they are no longer defined on all bordisms [Lur; Haï]. However, they include all bordisms necessary to obtain a modular functor, and it was verified in [DGGPR2] that the resulting mapping class group actions agree with those given in [Lyu2].

The main goal of [HR1] was to explicitly construct the modular functor of [Lyu2] in terms of the non-semisimple 3d TFT of [DGGPR1], including in particular the gluing morphisms.

The variant of a modular functor that we will work with is defined to be a symmetric monoidal 2-functor. The source category is the bordism $(2,1)$-category $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ consisting of closed oriented 1-dimensional manifolds, 2-dimensional oriented bordisms, and isotopy classes of orientation preserving diffeomorphisms, together with extra decorations necessary to compensate for a gluing anomaly indicated by the superscript χ. The target 2-category is $\mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$ consisting of finite linear categories, left exact profunctors, and natural transformations, see Section 2.2.2 for details. This corresponds to the definition used in [FSY], related notions were studied in [Til] and [KL, Ch. 6].

Our first main result is

Theorem (Theorem 5.1.9)**.** For every modular tensor category $\mathcal{C}$, the 3d TFT

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk}$$

of [DGGPR1] induces a symmetric monoidal 2-functor

$$\mathrm{Bl}^{\chi}_{\mathcal{C}} \colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk},$$

where $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the 3-dimensional bordism category with $\mathcal{C}$ labelled ribbon graphs and extra decorations necessary to compensate a gluing anomaly and $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ is a subcategory of it which contains all objects but only bordism satisfying a certain admissibility condition, see Section 3.1.2.

We note that in our setting actually $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}} \simeq \mathcal{L}\mathrm{ex}^{\mathrm{op}}$ (see Section 2.2.2), so we could have equally well defined modular functors with $\mathcal{L}\mathrm{ex}$ as target. However, the present formulation is better adapted to the way the TFT is used to define $\mathrm{Bl}_{\mathcal{C}}^{\chi}$ on 1- and 2-morphisms and it is also closer to [Lyu2]. See also [GZ, Sec. 0.6] for a nice discussion on the relation to conformal blocks.

Our main technical contribution is the realisation of the morphisms needed for the gluing of surfaces by evaluating $\widehat{\mathrm{V}}_{\mathcal{C}}$ on certain 3-dimensional bordisms:

Proposition (Proposition 5.1.7)**.** Let Σ be a surface with at least one incoming and one outgoing boundary component, and let Σ_{gl} be the surface obtained from gluing these boundaries. Then there is a natural isomorphism

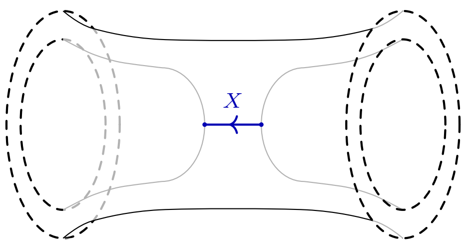
$$\mathrm{Bl}_{\mathcal{C}}^{\chi}(\Sigma_{\mathrm{gl}}) \cong \oint^{X\in\mathcal{C}} \mathrm{Bl}_{\mathcal{C}}^{\chi}(\Sigma)(X,X)$$

induced by a 3-dimensional bordism. The symbol $\oint^{X\in\mathcal{C}}$ denotes a type of coend over $\mathcal{C}$ which is well behaved for left exact functors.

To be more precise, the isomorphism in the proposition is induced by the dinatural family

$$\mathrm{Bl}_{\mathcal{C}}^{\chi}(\Sigma)(X,X) \to \mathrm{Bl}_{\mathcal{C}}^{\chi}(\Sigma_{\mathrm{gl}})$$

which is obtained by evaluating the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ on a family of bordisms M_X from $\underline{\Sigma}$ to $\underline{\Sigma_{\mathrm{gl}}}$ in $\widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C})$, where $\underline{\Sigma}$ and $\underline{\Sigma_{\mathrm{gl}}}$ are obtained from Σ and Σ_{gl} by replacing boundary components with marked discs, respectively, see Section 5.1.3 for details. Locally M_X can be visualised as:



where the inner boundary corresponds to a part of $\underline{\Sigma}$ and the outer boundary to a part of $\underline{\Sigma_{\mathrm{gl}}}$.

We refer to the modular functor defined on the 2-category $\mathrm{Bord}_{2+\varepsilon,2,1}^{\chi}$, which includes the anomaly compensating decorations, as *chiral*. Another interesting source 2-category is $\mathrm{Bord}_{2+\varepsilon,2,1}^{\circlearrowleft}$, which consists of oriented bordisms equipped with an orientation reversing involution, but which does not carry the extra decorations for anomaly cancellation. It contains open-closed bordisms as a full subcategory, see Section 4.1.2.

There is a symmetric monoidal 2-functor $U\colon \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ which assigns decorations coming from the involution in such a way that pullback along U produces an "anomaly-free" modular functor, see Section 4.1.2 for details. In the converse direction, there is a symmetric monoidal 2-functor $\widehat{(-)}\colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ obtained by sending any oriented manifold M to its orientation double $\widehat{M} := M \sqcup -M$ with the natural orientation reversing involution. These two 2-functors are neither inverses nor adjoints to each other, however combining them to a 2-endofunctor of $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ is still quite interesting. For instance, pulling back the chiral modular functor $\mathrm{Bl}^{\chi}_{\mathcal{C}}$, obtained in the above theorem, along this 2-endofunctor gives a topological relation between $\mathcal{C}$ and its Drinfeld centre $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$:

Proposition (Proposition 5.2.1)**.** Let $\mathcal{C}$ be a modular tensor category. There exists a braided monoidal 2-natural isomorphism filling the following diagram of symmetric monoidal 2-functors

$$\begin{tikzcd}
\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \arrow[r, "\widehat{(-)}"] \arrow[drr, "\mathrm{Bl}^{\chi}_{\mathcal{Z}(\mathcal{C})}"'] & \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \arrow[r, "U"] & \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \arrow[d, "\mathrm{Bl}^{\chi}_{\mathcal{C}}"] \arrow[dl, phantom, "\overset{\cong}{\Leftarrow}"] \\
 & & \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}
\end{tikzcd}.$$

To prove this proposition we study the behaviour of the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ under orientation reversal as well as the Deligne tensor product, see Section 3.1.5 for detailed statements. Finally, by factoring the orientation double 2-functor $\widehat{(-)}$ over the 2-category of open-closed oriented bordisms $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ we realise $\mathrm{Bl}^{\chi}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}$ as the closed sector of a full modular functor.

1.3 TFT construction of full 2d CFT

The second main result of this thesis is the extension of the TFT construction of CFT correlators to the non-semisimple setting. Since surface defects play a key role in semisimple setting we will need to extend the 3d TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ to include surface defects. However, we will not work with an explicit construction of such a defect TFT. Instead, we will consider any defect TFT

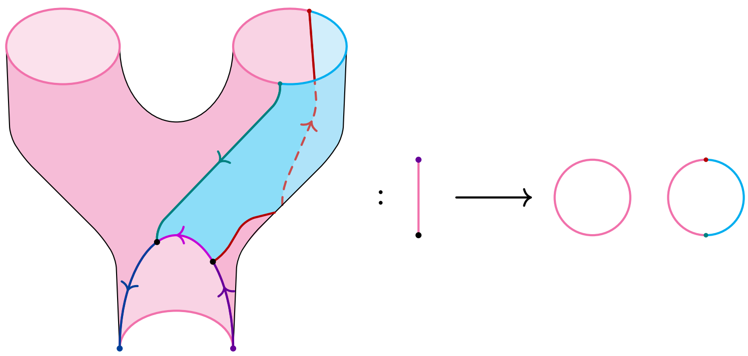
$$\mathrm{Z}_{\mathcal{C}}\colon \mathrm{Bord}^{\chi,\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}}) \to \mathrm{vect}_{\Bbbk},$$

as in [DKR1; CMS; CRS2], extending the 3d TFT with embedded ribbon graphs

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk}$$

of [DGGPR1] in the sense of Definition 6.1.1. This roughly means that we want to view $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ as a subcategory of $\mathrm{Bord}^{\chi,\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}})$ and the restriction of $\mathrm{Z}_{\mathcal{C}}$ to this subcategory should be naturally isomorphic to $\widehat{\mathrm{V}}_{\mathcal{C}}$. The 3-dimensional defect bordism category $\mathrm{Bord}^{\chi,\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}})$ consists of stratified and labelled bordisms and depends on the so-called *defect data* $\mathbb{D}_{\mathcal{C}}$, which encodes the combinatorial data needed to label the various strata, see Section 3.2 for details. Generically, the defect TFT $\mathrm{Z}_{\mathcal{C}}$ might also only be defined on a subcategory of "admissible bordisms", this will however not play a role for our considerations see Remark 6.1.2. As a product of this general setup we will see that the construction of CFT correlators is achieved independently of the specifics of the defect TFT $\mathrm{Z}_{\mathcal{C}}$ as long as certain algebraic conditions are satisfied, see Chapter 6 for precise statements.

Using the defect data $\mathbb{D}_{\mathcal{C}}$ of $\mathrm{Z}_{\mathcal{C}}$ we define the $(2,1)$-*category* $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ *of topological world sheets* consisting of compact, stratified and labelled 1-manifolds, stratified and labelled bordisms with corners between them, and isotopy classes of diffeomorphisms. This is done by introducing stratifications to the open-closed bordism $(2,1)$-category $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ and comes with an obvious forgetful 2-functor $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \to \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$, see Section 4.2 for details. An example of a 1-morphism from an interval to the disjoint union of two defect circles in $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ is given by:



Here the different colours represent different labels for the strata. By pulling back the full modular functor constructed from $\widehat{\mathrm{V}}_{\mathcal{C}}$ we obtain a symmetric monoidal 2-functor

$$\mathrm{Bl}_{\mathcal{C}} \colon \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}.$$

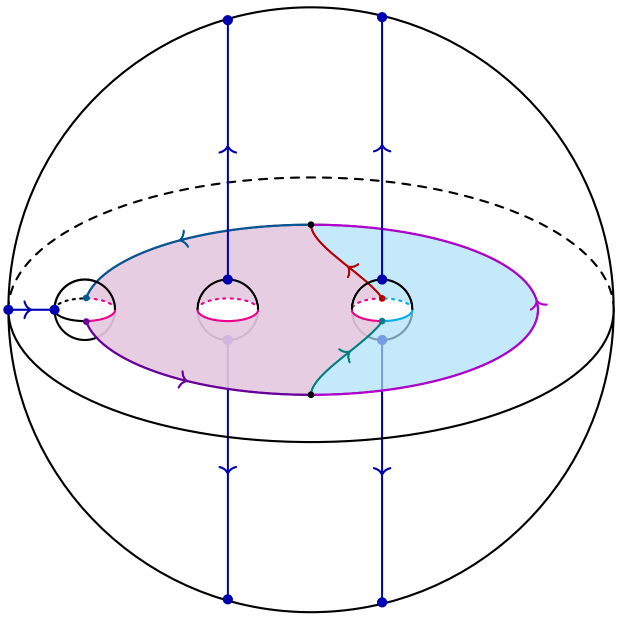
A *full conformal field theory based on* $\mathrm{Bl}_{\mathcal{C}}$ is now defined as a braided monoidal oplax natural transformation

$$\begin{tikzcd}
\mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \arrow[rr, bend left, "\Delta_{\Bbbk}"] \arrow[rr, bend right, "\mathrm{Bl}_{\mathcal{C}}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Cor}"] & & \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}},
\end{tikzcd}$$

where $\Delta_{\Bbbk} \colon \mathfrak{WS} \to \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ is the constant symmetric monoidal 2-functor sending every object to $\mathrm{vect}_{\Bbbk}$, see Definition 4.3.1. This definition captures the algebraic structure of a full CFT with the 1-morphism components of Cor determining the field content and the 2-morphism components being the actual correlators as elements in vector spaces of conformal blocks. Moreover, the axioms of an oplax natural transformation correspond precisely to the diffeomorphism covariance and the compatibility of correlators under gluing of surfaces Section 4.3. This formulation is also closely related to the notions of twisted or relative field theories considered in [ST; FT; JFS].

As in the original construction of [FFFS; FRSI; FjFRS], the idea is to obtain the correlator of a world sheet $\mathfrak{S}$ by evaluating the defect TFT $\mathrm{Z}_{\mathcal{C}}$ on its connecting manifold. It turns out that in the non-semisimple setting we need to slightly modify the connecting manifold, leading to more boundary components than just the orientation double $\widehat{\mathfrak{S}}$. These extra boundary components are themselves connecting manifolds in one dimension lower and will encode the field content of the full CFT.

For the example of the pair-of-pants 1-morphism from above, now drawn as a disc with two discs and a half disc removed, the connecting manifold is given by



In this example the double of $\mathfrak{S}$ corresponds to the big exterior sphere, while the extra boundary components are the three stratified 2-spheres in the interior. As mentioned above, one should think of the extra boundary components as representing the field content of the full CFT. Moreover, the blue 1-strata connecting them to the double of $\mathfrak{S}$, play the role of their "chiral and anti-chiral labels" and are, accordingly, labelled with objects in $\mathcal{C}$ and endowed with a canonical choice of framing. The orientation of the blue 1-strata arises from the source and target of $\mathfrak{S}$ as a 1-morphism.

Under some technical assumptions on the defect TFT $\mathrm{Z}_{\mathcal{C}}$ we prove our main result:

Theorem (Theorem 6.6.2)**.** Evaluating the 3d defect TFT $\mathrm{Z}_{\mathcal{C}}$ on the connecting manifolds induces a braided monoidal oplax natural transformation

$$\begin{tikzcd} \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \arrow[rr, bend left, "\Delta_{\Bbbk}"] \arrow[rr, bend right, "\mathrm{Bl}_{\mathcal{C}}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Cor}"] & & \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}. \end{tikzcd}$$

To illustrate that this abstract result can be used to compute with we will test our construction in the simplest possible setting for $\mathrm{Z}_{\mathcal{C}}$ the TFT

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \mathrm{Bord}_{3,2}^{\chi}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk}$$

of [DGGPR1]. This corresponds to the above mentioned diagonal CFT and we will show that our construction reproduces expectations from the literature for various quantities of physical interest including boundary states and annulus partition functions, see Chapter 7 for details.

1.4 Structure of the thesis

Excluding the introduction this thesis consists of seven chapters. The second and third chapter consist mainly of prerequisites. The fourth chapter introduces and motivates the 2-categorical structures studied in the later parts. Chapters 5 and 6 constitute the main part of this thesis and are largely devoted to proving our two main theorems. Finally, the last two chapters are dedicated to more explicit computations in examples.

A more detail overview is as follows:

- Chapter 2 introduces and summarizes most of the algebraic concepts and structures used throughout the thesis and is roughly divided into two parts.

In the first part we review notions from tensor category theory with a particular focus on modular tensor categories and fix our notation and conventions, mostly following [BK; EGNO; TV2]. The second part is devoted to introducing the 2-category $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ of left exact profunctors and is mostly based on [Lor; FS1; FSY].

- Chapter 3 serves as an introduction to the relevant TFTs. First, we mainly review the construction of the non-semisimple 3d TFT of [DGGPR1]. There we also include a discussion of its behaviour under orientation reversal and the Deligne tensor product. Afterwards, we discuss 3d defect TFTs in the sense of [DKR1; CMS; CRS2] and how the non-semisimple TFT fits into this framework.

- Chapter 4 is dedicated to the 2-categorical structures underlying full CFTs. First we introduce different notions of modular functors as symmetric monoidal 2-functors with source various two-dimensional bordism categories and study how they are related under some topological operations. Afterwards, we define the 2-category of topological world sheets. Finally, we propose the definition of a full CFTs as an oplax natural transformation and discuss in detail how this 2-categorical notion captures the relevant physical features.

- Chapter 5 contains the first main result of this thesis: the explicit construction of a chiral modular functor from the non-semisimple 3d TFT of [DGGPR1]. It further consists of a discussion of how this chiral modular functor behaves under the topological operations studied in Chapter 4 and is largely based on [HR1].

- Chapter 6 contains our second main result: the construction of a full CFT for the full modular functor obtained from the non-semisimple 3d TFT of [DGGPR1]. All of this chapter is devoted to proving Theorem 6.6.2 and can be considered the heart of this thesis. This and the following chapter are largely based on [HR2].

- Chapter 7 is devoted studying the full CFT obtained from our construction applied to the non-semisimple 3d TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ itself. This corresponds to the setting with only transparent surface defects and thus the diagonal CFT. We first check that $\widehat{\mathrm{V}}_{\mathcal{C}}$ satisfies the technical assumptions mentioned above. Afterwards we compute various quantities of physical interest, including two point functions as well as boundary states and compare our results to the ones the proposed in [FGSS]. We also study the action of line defects on the bulk fields and show that the algebra generated by defect operators is isomorphic to the linearised Grothendieck ring of $\mathcal{C}$ in the form of Proposition 7.3.2.

- Chapter 8 serves as an outlook. In the first part we illustrate some of the algebraic structures our construction produces in the presence of non-trivial surface defects. For this we apply the CFT construction from Chapter 6 to the defect TFT obtained by performing the generalised orbifold construction of [CRS1] for the non-semisimple TFT of [DGGPR1]. Afterwards, we will comment on various further directions and open questions.

Chapter 2

Algebraic preliminaries

In this chapter we summarize some algebraic concepts and structures used throughout the thesis and fix our notation and conventions. We will always work over an algebraically closed field $\Bbbk$ of characteristic zero. Unless otherwise noted, functors between abelian (and linear) categories will always be assumed to be additive (and linear). Throughout this thesis $\mathrm{vect}_\Bbbk$ will denote the category of finite dimensional $\Bbbk$-vector spaces.

In this thesis a 2-category will always mean a weak 2-category otherwise known as a bicategory. Analogously a 2-functor will always mean a weak 2-functor with coherence isomorphisms otherwise known as a pseudofunctor.

The rest of this chapter is structured as follows: Section 2.1 is devoted to modular tensor categories which are the linear categorical basis for most of the constructions in this thesis. In Section 2.2 we will first discuss some general results for coends in functor categories and afterwards introduce the 2-category of left exact profunctors which will later become the target 2-category for our constructions.

2.1 Modular tensor categories

In this section, we collect definitions and results related to ribbon categories with a particular emphasis on modular tensor categories $\mathcal{C}$. Moreover, we collect a number of results on relations between $\mathcal{C}$, its Drinfeld centre $\mathcal{Z}(\mathcal{C})$, and the Deligne product $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ in the modular setting.

2.1.1 Finite ribbon categories

A linear category is called *finite* if it is equivalent, as a linear category, to the category A-mod of finite dimensional modules of some finite dimensional algebra

A. In particular, a finite linear category is abelian, every object has a projective cover, and its Hom sets are finite dimensional vector spaces. For the complete intrinsic definition of finite linear categories see [EGNO, Sec. 1.8]. By a *finite tensor category* we mean a finite linear category which is in addition a rigid monoidal category such that the monoidal product $\otimes$ is bilinear and the monoidal unit $\mathbb{1}$ is simple.

A *finite ribbon category* $\mathcal{C}$ is a finite tensor category which is also ribbon. We will employ the following conventions for structure morphisms in $\mathcal{C}$. Every object X in $\mathcal{C}$ has a two-sided dual X^*, with duality morphisms denoted by:

$$\begin{aligned} \mathrm{ev}_X &: X^* \otimes X \to \mathbb{1}, & \mathrm{coev}_X &: \mathbb{1} \to X \otimes X^*, \\ \widetilde{\mathrm{ev}}_X &: X \otimes X^* \to \mathbb{1}, & \widetilde{\mathrm{coev}}_X &: \mathbb{1} \to X^* \otimes X. \end{aligned} \tag{2.1.1}$$

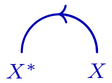
The components of the braiding and twist isomorphisms will be denoted by

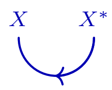
$$\beta_{X,Y} \colon X \otimes Y \to Y \otimes X, \qquad \vartheta_X \colon X \to X. \tag{2.1.2}$$

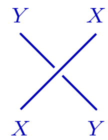
The twist ϑ satisfies

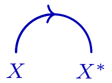
$$\vartheta_{X\otimes Y} = \beta_{Y,X} \circ \beta_{X,Y} \circ (\vartheta_X \otimes \vartheta_Y), \quad \text{and} \quad (\vartheta_X)^* = \vartheta_{X^*} \tag{2.1.3}$$

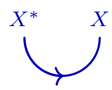
for all $X, Y \in \mathcal{C}$. A direct computation shows that the twist ϑ endows $\mathcal{C}$ with a pivotal structure, which is even spherical [EGNO, Sec. 8.10]. Moreover, we will appeal to the standard coherence results and assume $\mathcal{C}$ to be strictly monoidal and strictly pivotal. In diagrammatic notation the structural morphisms of $\mathcal{C}$ will be represented as

$\mathrm{ev}_X =$ 

$\mathrm{coev}_X =$ 

$\beta_{X,Y} =$ 

$\widetilde{\mathrm{ev}}_X =$ 

$\widetilde{\mathrm{coev}}_X =$ 

$\vartheta_X =$ 

(2.1.4)

Note that we read such diagrams from the bottom to the top. We will use the same conventions as [TV2] and denote with $\overline{\mathcal{C}}$ the same underlying pivotal tensor category, but equipped with the inverse braiding and twist, i.e.

$$\overline{\beta}_{X,Y} := \beta^{-1}_{Y,X} \colon X \otimes Y \to Y \otimes X \qquad \overline{\vartheta}_X := \vartheta^{-1}_X \colon X \to X \tag{2.1.5}$$

see [TV2, Sec. 1.2.2 & Ex. 3.1.7]. We will call $\overline{\mathcal{C}}$ the *mirrored* category of $\mathcal{C}$.

Next recall that $\mathcal{C}$ admits an inner Hom functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ which sends $(X, Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $X^* \otimes Y$. Due to our finiteness assumptions the coend of this functor exists and can be explicitly described as a cokernel, see [KL, Ch. 5] for details. We will denote this coend as

$$\mathrm{L} := \int^{X \in \mathcal{C}} X^* \otimes X \tag{2.1.6}$$

with universal dinatural transformation

$$\iota_X : X^* \otimes X \to \mathrm{L}. \tag{2.1.7}$$

The object L is referred to as the *canonical coend* and naturally carries the structure of a Hopf algebra with Hopf pairing in the braided monoidal category $\mathcal{C}$. For Hopf algebras in braided monoidal categories we will follow the convention of [TV2, Ch. 6]. The structural morphisms of L are induced from the universal property of the coend as shown in Figure 2.1. If we do not assume $\mathcal{C}$ to be strictly pivotal the canonical isomorphisms $(X \otimes Y)^* \cong Y^* \otimes X^*$ for the product, $\mathbb{1}^* \otimes \mathbb{1} \cong \mathbb{1}$ for the unit, and $X \cong X^{**}$ for the antipode are needed, see [TV2, Sec. 6.4 & 6.5] for details. Exchanging the over and under braidings in the definition of ω gives a pairing $\overline{\omega} : \mathrm{L} \otimes \mathrm{L} \to \mathbb{1}$ which satisfies

$$\omega \circ (S \otimes \mathrm{id}_{\mathrm{L}}) = \overline{\omega} = \omega \circ (\mathrm{id}_{\mathrm{L}} \otimes S) \tag{2.1.8}$$

see [KL, Eq. 5.2.8].

Definition 2.1.1. A finite ribbon category $\mathcal{C}$ is called *modular* if the canonical Hopf pairing ω of the coend L is non-degenerate.

There are other equivalent definitions of modularity, one of them will be discussed below, see [Shi3, Thm. 1.1] for the full list. From now on we will always assume $\mathcal{C}$ to be modular. It can be shown that $\mathcal{C}$ is unimodular and that the Hopf algebra L admits a unique-up-to-scalar two-sided integral $\Lambda : \mathbb{1} \to \mathrm{L}$ and cointegral $\lambda : \mathrm{L} \to \mathbb{1}$, see [DGGPR1, Sec. 2] and references therein for more details.

We will normalise the integral and cointegral in terms of the *modularity parameter* $\zeta \in \Bbbk^\times$ as

$$\lambda \circ \Lambda = \mathrm{id}_{\mathbb{1}} \quad , \quad \omega \circ (\mathrm{id}_{\mathrm{L}} \otimes \Lambda) = \zeta \lambda \ . \tag{2.1.9}$$

That the second condition is possible is in fact equivalent to non-degeneracy of ω, see [Ker, Thm. 5] or [TV2, Lem. 6.2]. The following statement is analogous to the case of classical Hopf algebras [KL, Cor. 4.2.13].

Proposition 2.1.2. The cointegral λ induces a non-degenerate pairing $\kappa := \lambda \circ \mu : \mathrm{L} \otimes \mathrm{L} \to \mathbb{1}$ which equips L with the structure of a Frobenius algebra in $\mathcal{C}$. The copairing is given by $(S \otimes \mathrm{id}_{\mathrm{L}}) \circ \Delta \circ \Lambda : \mathbb{1} \to \mathrm{L} \otimes \mathrm{L}$.

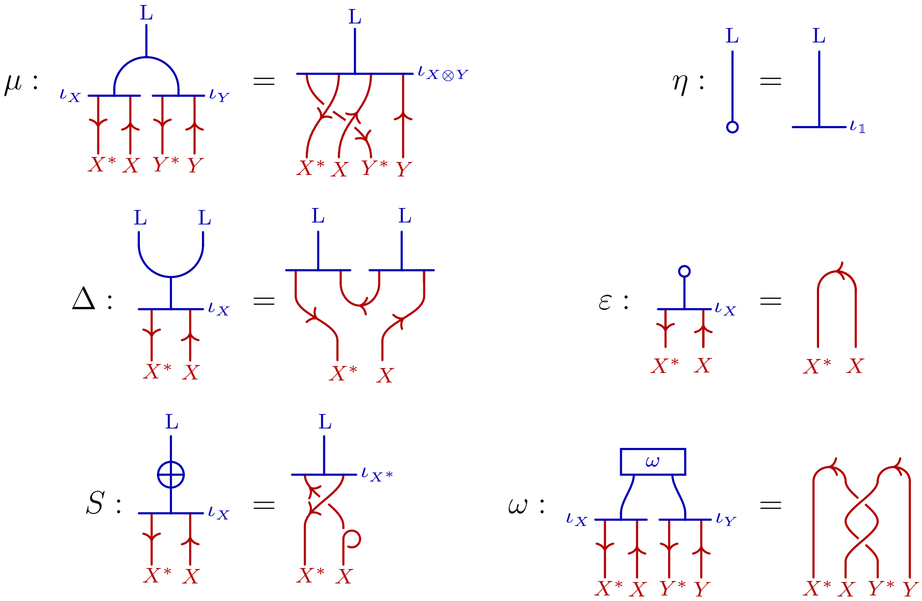


Figure 2.1: The dinatural transformations defining the Hopf algebra structure morphisms and the Hopf pairing on L. We use red for the ribbons labelled with $X; Y \in \mathcal{C}$ here to highlight that the structural morphism of L do not depend on the specific labels.

The pairing κ is called the *Radford pairing*. Note that this convention for κ keeps the algebra structure of L fixed while the one we used in [HR1] keeps the coalgebra structure. This difference in convention makes some of the computations in chapter 7 a bit more straightforward. Since the Hopf pairing is also non-degenerate, the composition

$$\mathcal{S} := (\omega \otimes \mathrm{id}_{\mathrm{L}}) \circ (\mathrm{id}_{\mathrm{L}} \otimes ((S \otimes \mathrm{id}_{\mathrm{L}}) \circ \Delta \circ \Lambda)) \colon \mathrm{L} \to \mathrm{L} \tag{2.1.10}$$

is invertible and satisfies

$$\kappa \circ (\mathcal{S} \otimes \mathrm{id}_{\mathrm{L}}) = \omega = \kappa \circ (\mathrm{id}_{\mathrm{L}} \otimes \mathcal{S}) \tag{2.1.11}$$

by definition. We can define another endomorphism of L by the universal property via $\mathcal{T} \circ \iota_X = \iota_X \circ (\mathrm{id}_{X^*} \otimes \vartheta_X)$. This morphism is invertible with inverse $\mathcal{T}^{-1} \circ \iota_X = \iota_X \circ (\mathrm{id}_{X^*} \otimes \vartheta_X^{-1})$. Moreover the constants $\Delta^{\pm}$ defined via

$$\varepsilon \circ \mathcal{T}^{\pm 1} \circ \Lambda = \Delta^{\pm}\, \mathrm{id}_{\mathbb{1}} \tag{2.1.12}$$

are non-zero and satisfy $\zeta = \Delta^+\Delta^-$, see [DGGPR1, Prop. 2.6 & Cor. 4.6]. Using this and the normalisation of Λ and λ a direct computation shows that

$$\mathcal{S}^2 = \zeta S^{-1} \tag{2.1.13}$$

where S^{-1} is the inverse of the antipode of L. The name *modular* tensor category is justified by the following lemma:

Lemma 2.1.3 ([Lyu1, Thm. 2.1.9])**.** The morphisms $\mathcal{T}$ and $\mathcal{S}$ induce a projective $\mathrm{SL}(2,\mathbb{Z})$-action on the morphism spaces $\mathrm{Hom}_{\mathcal{C}}(\mathrm{L},\mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathrm{L})$ via pre- and postcompostion, respectively.

Accordingly, we will call $\mathcal{S}\colon \mathrm{L} \to \mathrm{L}$ the modular S-transformation of $\mathcal{C}$. The modular group action on $\mathrm{Hom}_{\mathcal{C}}(\mathrm{L},\mathbb{1})$ is actually a part of the chiral modular functor discussed in Section 5.1.2.

2.1.2 (Co)modules of the canonical Hopf algebra

Let us now discuss how the categories of L-modules and comodules are related to more well-known categories.

To this end first recall the *Drinfeld centre* of a monoidal category $\mathcal{A}$ as the braided monoidal category $\mathcal{Z}(\mathcal{A})$ with objects pairs (X,γ), where $X \in \mathcal{A}$, and $\gamma\colon X \otimes - \Rightarrow - \otimes X$ is a natural isomorphism, called a *half braiding*, satisfying a hexagon type axiom [EGNO, Def. 7.13.1]. A key property of the Drinfeld centre is that for suitable $\mathcal{A}$ it is modular:

Proposition 2.1.4 ([Shi7, Thm. 5.11])**.** Let $\mathcal{A}$ be finite tensor category satisfying the sphericality condition of [DSS2, Def. 3.5.2]. Then $\mathcal{Z}(\mathcal{A})$ is a modular tensor category.

In particular a modular tensor category $\mathcal{C}$ satisfies this sphericality condition which implies that $\mathcal{Z}(\mathcal{C})$ is a modular tensor category as well. To see this, combine [SS, Thm. 1.3] and [GR4, Cor. 4.7]. A generalisation of this result to pivotal tensor categories can be found in [MW, Cor. 2.13].

The forgetful functor $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ which sends a pair (X,γ) to X has a left adjoint $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. In the modular setting F is actually also a right adjoint as we will see below. The corresponding monad $U \circ F$ is naturally isomorphic to the central monad Z on $\mathcal{C}$

$$(U \circ F)(-) \cong Z(-) := \int^{X\in\mathcal{C}} X^* \otimes - \otimes X \tag{2.1.14}$$

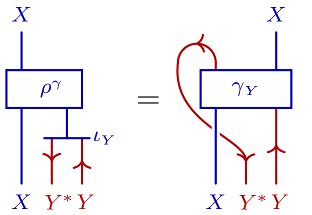
and the corresponding Eilenberg-Moore category is precisely $\mathcal{Z}(\mathcal{C})$ [BV2, Sec. 5.6] (see also [TV2, Ch. 9] for a textbook account). We will discuss a topological proof of this statement in Proposition 5.2.3 using modular functors.

From the braiding in $\mathcal{C}$ we get an isomorphism of monads $Z(-) \cong - \otimes \mathrm{L}$, see e.g. [Shi3, Lem. 3.7], which gives rise to the following:

Lemma 2.1.5. The functor

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &\to \mathcal{C}_{\mathrm{L}} \\ (X, \gamma) &\mapsto (X, \rho^{\gamma}) \end{aligned} \tag{2.1.15}$$

where the right L-action ρ^{γ} is obtained from the half-braiding γ via



(2.1.16)

is an equivalence categories.

This lemma is a special case of [Maj, Thm. 3.2].

Under the equivalence $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}_{\mathrm{L}}$ the forgetful functor $U \colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ simply corresponds to the functor forgetting the L-module structure. In particular, its left adjoint is given by the free module functor $- \otimes \mathrm{L} \colon \mathcal{C} \to \mathcal{C}_{\mathrm{L}}$. Moreover, since L is a Frobenius algebra this is automatically also a right adjoint for U, as claimed above.

There is also a different functor $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ given by sending an object $X \in \mathcal{C}$ to the object $(X, \beta_{X,-}) \in \mathcal{Z}(\mathcal{C})$ with half braiding given by the braiding $\beta_{X,-}$. In particular, we can equip every object in $\mathcal{C}$ with a L-action by combining this functor with the above equivalence. We will call the resulting action the *canonical* L-*action on* X and denote it with $\rho_X \colon X \otimes \mathrm{L} \to X$. By taking the (left) partial trace $\mathrm{tr}_{\mathrm{L}}(\rho_X) \colon \mathrm{L} \to \mathbb{1}$ we get an element in $\chi_X \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}, \mathbb{1})$ which is called the *internal character of* X [Shi2, Sec. 3.5].[1]

Next, recall the Deligne tensor product of two finite linear categories $\mathcal{A}$ and $\mathcal{B}$ is the finite linear category $\mathcal{A} \boxtimes \mathcal{B}$ together with a functor $\boxtimes \colon \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$ which is left exact in each argument and satisfies the following universal property: Let $\mathcal{D}$ be another finite linear category and let $F \colon \mathcal{A} \times \mathcal{B} \to \mathcal{D}$ be a functor left exact in each argument, then there is a unique (up to unique natural isomorphism) left exact functor $\widehat{F} \colon \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{D}$ together with an equivalence $F \cong \widehat{F} \circ \boxtimes$. See [EGNO, Sec. 1.11] for the existence of the Deligne product as well as more properties.[2] For $\mathcal{C}$ and $\mathcal{D}$ modular tensor categories, also $\mathcal{C} \boxtimes \mathcal{D}$ is a modular tensor category. This

[1]Shimizu defines internal characters using the canonical end $\mathcal{E} = \int_{X \in \mathcal{C}} X \otimes X^*$ which can be identified with the dual of the coend L via the Hopf pairing ω and under this identification the two versions of internal characters coincide.

[2]Usually one defines the Deligne tensor product for right exact functors, however by the equivalence of left exact and right exact functors for finite linear categories [FSS, Thm. 3.2] this does not make a difference, see also the discussion in [BW, Sec. 2.4].

can for example be seen by noting that the isomorphism $\mathrm{L}_{\mathcal{C}\boxtimes\mathcal{D}} \cong \mathrm{L}_{\mathcal{C}} \boxtimes \mathrm{L}_{\mathcal{D}}$ [FSS, Cor. 3.12] where $\mathrm{L}_{\mathcal{C}}$ and $\mathrm{L}_{\mathcal{D}}$ are the canonical Hopf algebra of $\mathcal{C}$ and $\mathcal{D}$, respectively, is an isomorphism of Hopf algebras.

Let us now turn to the question of comodules of L. First note that we can equip any object $X \in \mathcal{C}$ with the structure of a right L-comodule with coaction δ_X given by

$$\delta_X := (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \colon X \to X \otimes \mathrm{L}. \tag{2.1.17}$$

We will call this the *canonical* L-*coaction on* X and the (left) partial trace $\check{\chi}_X := \mathrm{tr}_{\mathrm{L}}(\delta_X) = \iota_X \circ \widetilde{\mathrm{coev}}_X \colon \mathbb{1} \to \mathrm{L}$ the *internal cocharacter of* X.

Lemma 2.1.6. The functor

$$\begin{aligned} \mathcal{C} \boxtimes \overline{\mathcal{C}} &\to \mathcal{C}^{\mathrm{L}} \\ X \boxtimes Y &\mapsto (X \otimes Y, \mathrm{id}_X \otimes \delta_Y) \end{aligned} \tag{2.1.18}$$

is an equivalence of categories.

This is a special case of [Lyu3, Cor. 2.7.2], see also [EGNO, Prop. 7.18.4] or [Shi5, Rem. 3.6] for a proof based on the theory of exact module categories.

Using the Hopf pairing $\omega \colon \mathrm{L} \otimes \mathrm{L} \to \mathbb{1}$ we can turn comodules into modules yielding a functor $\mathcal{C}^{\mathrm{L}} \to \mathcal{C}_{\mathrm{L}}$. Under the equivalences of Lemma 2.1.5 and 2.1.6 this is precisely the ribbon functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$ sending $X \boxtimes Y \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ to $(X \otimes Y, ((\beta_{X,-} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \overline{\beta}_{Y,-}))$. Moreover, since $\mathcal{C}$ is assumed to be modular, i.e. ω is non-degenerate this functor is an equivalence. Shimizu proved that the converse also holds:

Proposition 2.1.7 ([Shi3, Thm. 1.1])**.** Let $\mathcal{B}$ be a finite ribbon tensor category. Then $\mathcal{B}$ is modular if and only if the canonical ribbon functor

$$\mathcal{B} \boxtimes \overline{\mathcal{B}} \to \mathcal{Z}(\mathcal{B}) \tag{2.1.19}$$

is an equivalence.

Note that the functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{C}$ induced by the monoidal product $\otimes$ on $\mathcal{C}$ corresponds to the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ and thus also has a two-sided adjoint $\mathcal{C} \to \mathcal{C}\boxtimes\overline{\mathcal{C}}$ which in this form is given by $F(X) = \int^{Y\in\mathcal{C}} (Y^* \otimes X) \boxtimes Y \in \mathcal{C}\boxtimes\overline{\mathcal{C}}$.

For an object $X \in \mathcal{C}$ the canonical action ρ_X and coaction δ_X satisfy $\rho_X = (\mathrm{id}_X \otimes \omega) \circ (\delta_X \otimes \mathrm{id}_{\mathrm{L}})$ which immediately gives $\chi_X = \omega \circ (\check{\chi}_X \otimes \mathrm{id}_{\mathrm{L}})$.

Finally, using yet another characterisation of modularity[3] one can show that the endomorphisms obtained by combining the canonical coaction δ with the cointegral λ has the following cutting property:

[3]Namely the triviality of the Müger centre [Shi3, Sec. 4].

Lemma 2.1.8 ([GR4, Lem. 6.3])**.** For each $X \in \mathcal{C}$ there exist $m \geqslant 0$, $a\colon X \to \mathbb{1}^{\oplus m}$, $b\colon \mathbb{1}^{\oplus m} \to X$ such that

$$\begin{aligned}(\mathrm{id}_X \otimes \lambda) \circ \delta_X &= b \circ a \\ &= \sum_{\alpha=1}^{m} b_\alpha \circ a_\alpha\end{aligned} \tag{2.1.20}$$

where $a_\alpha\colon X \to \mathbb{1}$, $b_\alpha\colon \mathbb{1} \to X$ are the components of a, b.

In particular, for non-semisimple $\mathcal{C}$, the case $m = 0$ appears for example for $X = \mathbb{1}$ since $(\mathrm{id}_{\mathbb{1}} \otimes \lambda) \circ \delta_{\mathbb{1}} = \lambda \circ \eta = 0$ by a monadic extension of Maschke's theorem [BV1, Thm. 6.5]. Using (2.1.9) we get a similar statement for the integral Λ as $\rho_X \circ (\mathrm{id}_X \otimes \Lambda) = \zeta(\mathrm{id}_X \otimes \lambda) \circ \delta_X$.

2.1.3 Tensor ideals and modified traces

In order to construct a TFT from a modular tensor category we will need to introduce one more notion which is hidden in the semisimple setting. Let us denote with $\mathrm{Proj}(\mathcal{C})$ the full subcategory of projective objects of $\mathcal{C}$. This forms a *tensor ideal* in $\mathcal{C}$, i.e. it is closed under retracts, and absorbent with respect to monoidal products with arbitrary objects of $\mathcal{C}$ [EGNO, Prop. 4.2.12]. In fact, $\mathrm{Proj}(\mathcal{C})$ is the smallest non-zero tensor ideal of $\mathcal{C}$ [GKP1, Lem. 4.4.1]. In particular, if $\mathcal{C}$ is semisimple then $\mathcal{C} = \mathrm{Proj}(\mathcal{C})$ and it has no non-trivial proper ideals. This simple observation has drastic consequences if one wants to define invariants of closed ribbon graphs using $\mathcal{C}$:

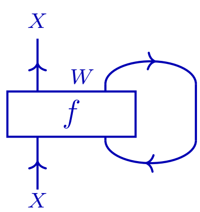
Lemma 2.1.9 ([BGR, Lem. 2.6])**.** Let $\mathcal{I}$ be a non-trivial proper ideal of $\mathcal{C}$. Then the categorical trace of $\mathcal{C}$ vanishes identically on $\mathcal{I}$, i.e. for any $X \in \mathcal{I} \subset \mathcal{C}$ and any $f \in \mathrm{End}_{\mathcal{C}}(X)$ we have $\mathrm{tr}(f) = 0$.

To circumvent this problem of vanishing traces, and thus also quantum dimensions, in the non-semisimple setting the notion of a *modified trace* was introduced and studied in [GPT; GKP1; GKP2; GPV].

For this recall that the (right) partial trace of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(X \otimes Y)$ is defined as the endomorphism

$$\mathrm{tr}_{\mathrm{R}}(f) = (\mathrm{id}_X \otimes \widetilde{\mathrm{ev}}_Y) \circ (f \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Y) \in \mathrm{End}_{\mathcal{C}}(X), \tag{2.1.21}$$

or graphically

$\mathrm{tr}_{\mathrm{R}}(f) =$  (2.1.22)

A *modified trace* t on $\mathrm{Proj}(\mathcal{C})$ is a family of linear maps

$$\{\mathrm{t}_P : \mathrm{End}_{\mathcal{C}}(P) \to \Bbbk\}_{P \in \mathrm{Proj}(\mathcal{C})} \tag{2.1.23}$$

satisfying the following conditions:

1) *Cyclicity*: For all $P, Q \in \mathrm{Proj}(\mathcal{C})$ and $f : P \to Q$, $g : Q \to P$ we have

$$\mathrm{t}_Q(f \circ g) = \mathrm{t}_P(g \circ f). \tag{2.1.24}$$

2) *Right partial trace*: For all $P \in \mathrm{Proj}(\mathcal{C})$, $X \in \mathcal{C}$ and $h \in \mathrm{End}_{\mathcal{C}}(P \otimes X)$,

$$\mathrm{t}_{P \otimes X}(h) = \mathrm{t}_P(\mathrm{tr}_{\mathrm{R}}(h)); \tag{2.1.25}$$

In general, there is also a notion of left partial trace tr_{L} and a modified trace would need to satisfy a condition analogous to 2) above for the left partial trace. However, since $\mathcal{C}$ is ribbon in our setting, it suffices to only consider one of them [GKP1, Thm 3.3.1].

Proposition 2.1.10 ([GKP3, Thm 5.5 & Cor. 5.6])**.** There exists a unique-up-to-scalar non-zero modified trace t on $\mathrm{Proj}(\mathcal{C})$. Moreover, for any $P \in \mathrm{Proj}(\mathcal{C})$ and $X \in \mathcal{C}$ the pairing

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(P, X) \times (X, P) &\to \Bbbk \\ (f, g) &\mapsto \mathrm{t}_P(g \circ f) \end{aligned} \tag{2.1.26}$$

is non-degenerate.

In the semisimple setting this unique-up-to-scalar non-zero modified trace is simply the categorical trace tr of $\mathcal{C}$.

2.2 Left exact profunctors

Let us now come to the second main algebraic ingredient, the symmetric monoidal 2-category $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ of finite linear categories, left exact profunctors, and natural transformations. This 2-category will play a pivotal role throughout this thesis as the target of our modular functors. After reviewing some general facts about coends in functor categories, we will recall the notion of *left exact coends* introduced in [Lyu2]. We will then consider left exact profunctors and introduce the 2-category $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$. Finally, we will comment on how it is related to the more familiar 2-categories $\mathcal{L}\mathrm{ex}_{\Bbbk}$ and $\mathcal{R}\mathrm{ex}_{\Bbbk}$ of finite linear categories, left/right exact functors, and natural transformations and discuss how this is related to adjunctions of the 1-morphisms of $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$.

2.2.1 Coends in functor categories

We start by recalling some general results on coends, in particular in functor categories. See [Lor; FS1] for a more detailed exposition and proofs. In this section $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ will denote finite ribbon categories, albeit these conditions can be drastically relaxed in general. Moreover, all functors we will consider will be assumed to be linear unless stated otherwise.

First we recall the "delta distribution" property of the Hom functor, which is a reformulation of the Yoneda lemma.

Lemma 2.2.1 ([FS1, Prop. 4])**.** Let $F\colon \mathcal{A} \to \mathrm{vect}_{\Bbbk}$ be a linear functor. For any object $Y \in \mathcal{A}$ the coend of the functor

$$\mathrm{Hom}_{\mathcal{A}}(-,Y) \otimes_{\Bbbk} F(-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{vect}_{\Bbbk} \tag{2.2.1}$$

exists and can be realised as the vector space $F(Y)$ with the family of linear maps

$$\begin{aligned} i_X\colon \mathrm{Hom}_{\mathcal{A}}(X,Y) \otimes_{\Bbbk} F(X) &\to F(Y) \\ (f,x) &\mapsto F(f)(x) \end{aligned} \tag{2.2.2}$$

for $X \in \mathcal{A}$. Moreover, the isomorphism $F(Y) \cong \int^{X\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(X,Y) \otimes_{\Bbbk} F(X)$ is natural in Y.

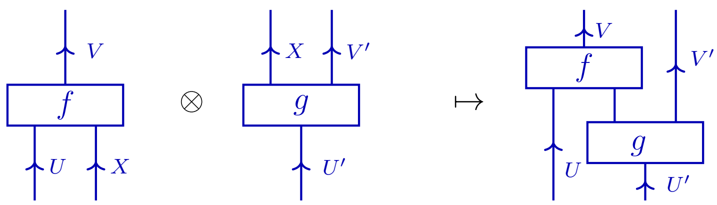
In practice this allows one to explicitly compute many coends as can be seen from the following useful corollary.

Corollary 2.2.2. For any $U, V, U', V' \in \mathcal{A}$ the coend of the functor

$$\mathrm{Hom}_{\mathcal{A}}(U \otimes (-), V) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(U', (-) \otimes V')\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{vect}_{\Bbbk} \tag{2.2.3}$$

exists and can be realised by $(\mathrm{Hom}_{\mathcal{A}}(U \otimes U', V \otimes V'), i)$ with the dinatural transformation

$$i_X\colon \mathrm{Hom}_{\mathcal{A}}(U \otimes X, V) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(U', X \otimes V') \to \mathrm{Hom}_{\mathcal{A}}(U \otimes U', V \otimes V')$$



(2.2.4)

Moreover, the isomorphism between $\mathrm{Hom}_{\mathcal{A}}(U \otimes U', V \otimes V')$ and the coend over X of $\mathrm{Hom}_{\mathcal{A}}(U \otimes X, V) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(U', X \otimes V')$ is natural in U, V, U', and V'.

More generally let $G\colon \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D}$ be a functor. We can fix an object $Y \in \mathcal{A}$, often called a *parameter* in this context, to obtain a functor

$$G_Y := G(Y, -, -)\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D}. \tag{2.2.5}$$

Let us assume the coend of G_Y exists and denote it with

$$e_Y := \int^{X \in \mathcal{B}} G_Y(X, X) \in \mathcal{D}. \tag{2.2.6}$$

The fact that G is functorial in Y as well implies that the assignment

$$Y \longmapsto e_Y \tag{2.2.7}$$

defines a functor from $\mathcal{A}$ to $\mathcal{D}$, which we will denote by $\int^{X \in \mathcal{B}} G(-, X, X)$. On the other hand we can also view G as a functor

$$\widetilde{G}\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Fun}(\mathcal{A}, \mathcal{D}). \tag{2.2.8}$$

Let us further assume that the coend of $\tilde{G}$ exists as an object in the functor category $\mathrm{Fun}(\mathcal{A}, \mathcal{D})$ and denote it by

$$\Big(\int^{X \in \mathcal{C}} \widetilde{G}(X, X) \Big)(-)\colon \mathcal{A} \to \mathcal{D}. \tag{2.2.9}$$

The parameter theorem for coends explains how these two constructions are related:

Theorem 2.2.3 ([Mac, Sec. IX.7.])**.** The functor

$$\int^{X \in \mathcal{B}} G(-, X, X)\colon \mathcal{A} \to \mathcal{D} \tag{2.2.10}$$

has a natural structure of a coend for the functor

$$\tilde{G}\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Fun}(\mathcal{A}, \mathcal{D}), \tag{2.2.11}$$

provided that all coends $\int^{X \in \mathcal{B}} G(Y, X, X)$ exist, i.e.

$$\int^{X \in \mathcal{B}} G(-, X, X) \cong \Big(\int^{X \in \mathcal{B}} \widetilde{G}(X, X) \Big)(-) \tag{2.2.12}$$

as objects in $\mathrm{Fun}(\mathcal{A}, \mathcal{D})$.

We now turn to results that are specific to the setting of finite linear categories, in particular instead of general functors we will from now on only consider left exact functors. We denote the category of left exact functors from $\mathcal{A}$ to $\mathcal{B}$ by $\mathcal{L}\mathrm{ex}(\mathcal{A},\mathcal{B})$, and analogously with $\mathcal{R}\mathrm{ex}(\mathcal{A},\mathcal{B})$ the category of right exact functors. The following idea is due to [Lyu2].

Definition 2.2.4. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{D}$ be finite linear categories and let

$$G\colon \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D} \tag{2.2.13}$$

be a functor left exact in each argument. The *left exact coend* $\oint^{X\in\mathcal{B}} G(-,X,X)$ is the coend of G over $\mathcal{B}$ which is universal with respect to functors in the category $\mathrm{Lex}(\mathcal{A},\mathcal{D})$.

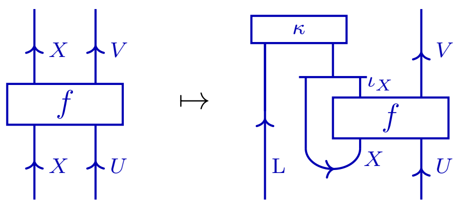
The notation $\oint$ indicates that this is in general not the same thing as the standard coend in the full functor category $\mathrm{Fun}(\mathcal{A},\mathcal{D})$. For our purposes left exact coends will be instrumental because they work well with the Hom functor of a modular tensor category $\mathcal{C}$ in the following sense.

Proposition 2.2.5 ([FS1, Prop. 9])**.** The coend (over the first and third argument) of the functor

$$\mathrm{Hom}_{\mathcal{C}}\big((-)\otimes(-),(-)\otimes(-)\big): \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C} \to \mathrm{vect}_{\Bbbk} \tag{2.2.14}$$

exists in the functor category $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{vect}_{\Bbbk})$ and is given by $\big(\mathrm{Hom}_{\mathcal{C}}(\mathrm{L}\otimes(-),(-)),i\big)$ with the family, natural in U and V, of dinatural transformations

$$i_X^{U,V}: \mathrm{Hom}_{\mathcal{C}}(X\otimes U, X\otimes V) \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}\otimes U, V)$$



where κ is the Radford pairing on the coend L from above.

Using the rigidity of $\mathcal{C}$ we can also express this as

$$\oint^{X\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X^*\otimes X\otimes -,-) \cong \mathrm{Hom}_{\mathcal{C}}\left(\left(\int^{X\in\mathcal{C}} X^*\otimes X\right)\otimes -,-\right). \tag{2.2.15}$$

Or in other words: the left exact coend "commutes" with the Hom functor. It should be noted here that from a formal point of view, there is no reason to consider left exact instead of right exact functors. Indeed, for finite linear categories the resulting functor categories are actually equivalent [FSS, Thm. 3.2] and there is a similar structure known as right exact ends. However, for our purposes it turns out that left exact functors are more natural (cf. Section 5.1). One reason for this is that left exact functors are representable:

Lemma 2.2.6 ([DSS1, Cor. 1.10])**.** Let $\mathcal{A}$ be a finite linear category. A functor $F\colon \mathcal{A} \to \mathrm{vect}_{\Bbbk}$ is representable, i.e. there exists an object $A_F \in \mathcal{A}$ such that $F(-) \cong \mathrm{Hom}_{\mathcal{A}}(A_F, -)$, if and only if F is left exact.

There is an analogous statement for contravariant functors $G\colon \mathcal{A} \to \mathrm{vect}_{\Bbbk}$. Thus, taking the left exact coend instead of the regular coend ensures that representability is preserved.

We can also understand the appearance of left exact coends better by recalling the following result from homological algebra (which however will not be used in the following).

Lemma 2.2.7. Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathcal{A}'$ be an abelian category. Let us denote with $\mathrm{Inj}(\mathcal{A})$ the full subcategory of injective objects in $\mathcal{A}$. The restriction functor $\mathcal{L}\mathrm{ex}(\mathcal{A}, \mathcal{A}') \to \mathrm{Fun}(\mathrm{Inj}(\mathcal{A}), \mathcal{A}')$ is an equivalence.

Proof. We claim that $F \mapsto R_0F$ is an inverse to the restriction functor with $R_0F\colon \mathcal{A} \to \mathcal{A}'$ the zeroth right derived functor of $F\colon \mathrm{Inj}(\mathcal{A}) \to \mathcal{A}'$. Recall that this is defined on objects $X \in \mathcal{A}$ as $R_0F(X) = \ker\big(F(I_0) \to F(I_1)\big)$ with $0 \to X \to I_0 \to I_1 \to \cdots$ an injective resolution of X. Note that R_0F is well defined up to isomorphism since every choice of injective resolution leads to isomorphisms in homology. Using standard techniques in homological algebra it is straightforward to verify that R_0F is indeed left exact and that for a left exact functor G we have $R_0G \cong G$. □

Analogously it can be shown that $\mathcal{R}\mathrm{ex}(\mathcal{A}, \mathcal{A}') \simeq \mathrm{Fun}(\mathrm{Proj}(\mathcal{A}), \mathcal{A}')$. Combining the equivalences $\mathrm{Lex}(\mathcal{A}, \mathcal{A}') \simeq \mathrm{Fun}(\mathrm{Inj}(\mathcal{A}), \mathcal{A}')$ and $\mathcal{R}\mathrm{ex}(\mathcal{A}, \mathcal{A}') \simeq \mathrm{Fun}(\mathrm{Proj}(\mathcal{A}), \mathcal{A}')$ with the fact that in a finite tensor category, every projective object is also injective and vice versa [EGNO, Prop. 6.1.3] we immediately get an equivalence $\mathcal{R}\mathrm{ex}(\mathcal{C}, \mathcal{D}) \simeq \mathcal{L}\mathrm{ex}(\mathcal{C}, \mathcal{D})$. Moreover, we also have $\mathcal{L}\mathrm{ex}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Proj}(\mathcal{C}), \mathcal{D})$. With this we can reinterpret the definition of the left exact coend as the requirement of having a projective object present.

Remark 2.2.8. The equivalence $\mathcal{R}\mathrm{ex}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{L}\mathrm{ex}(\mathcal{A}, \mathcal{B})$ constructed above is different from the one given in [FSS, Thm. 3.2] mentioned before. This can be seen by considering where for example the identity functor is sent under both equivalences. In the above equivalence the identity gets sent to itself while in the one by Fuchs et. al. it gets sent to the Nakayama functor [FSS, Sec. 3.5].

2.2.2 The 2-category of left exact profunctors

In this section we will define the symmetric monoidal 2-category of left exact profunctors and discuss some of its properties. For the relevant 2-categorical notions we refer to [JY] and to [SP, Ch. 2] for details on symmetric monoidal 2-categories. For an introduction to profunctors in general we refer to [Lor, Ch. 5]. Let $\mathcal{A}$ and $\mathcal{B}$ be finite linear categories. A *left exact profunctor* P from $\mathcal{A}$ to $\mathcal{B}$, denoted as $P\colon \mathcal{A} \nrightarrow \mathcal{B}$ is a left exact functor $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B} \to \mathrm{vect}_{\Bbbk}$.[4] With this we can define the symmetric monoidal 2-category $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ of *left exact profunctors*:

- objects: finite linear categories;
- 1-morphisms: for objects $\mathcal{A}$ and $\mathcal{B}$ a 1-morphism from $\mathcal{A}$ to $\mathcal{B}$ is a left exact linear profunctor from $\mathcal{A} \nrightarrow \mathcal{B}$;
- 2-morphisms: natural transformations of the underlying functors;
- horizontal composition: for 1-morphisms $F\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $G\colon \mathcal{B} \nrightarrow \mathcal{C}$ the horizontal composition is the left exact coend

$$G \diamond F(-,\sim) := \oint^{B\in\mathcal{B}} G(B,\sim) \otimes_{\Bbbk} F(-,B)\colon \mathcal{A} \nrightarrow \mathcal{C} \tag{2.2.16}$$

  where $-$ stands for an argument from the source category (in this case $\mathcal{A}$) and $\sim$ for an argument from the target category (in this case $\mathcal{C}$);
- identity 1-morphism: for an object $\mathcal{A}$ the identity 1-morphism $\mathcal{A} \nrightarrow \mathcal{A}$ is given by the unique functor induced from the universal property of the Deligne product applied to the Hom functor $\mathrm{Hom}_{\mathcal{A}}(-,\sim)$, in order to avoid confusion with the identity functor $\mathrm{id}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ we will denote the identity profunctor by $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A} \nrightarrow \mathcal{A}$;
- vertical composition: standard vertical composition of natural transformations;
- identity 2-morphism: identity natural transformations;
- Deligne's tensor product of finite linear categories as the symmetric monoidal structure;

In the following we will often use $-$ for both the source and target argument as it will be clear from the context which one is which, e.g. we will write $\mathrm{Hom}_{\mathcal{A}}(-,-)$ instead of $\mathrm{Hom}_{\mathcal{A}}(-,\sim)$.

[4] Alternatively we could also consider functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{vect}_{\Bbbk}$ left exact in each argument using the universal property of the Deligne product.

Next, recall the symmetric monoidal 2-category $\mathcal{L}\mathrm{ex}_{\Bbbk}$ of finite linear categories, left exact functors, and natural transformations again with the Deligne tensor product as symmetric monoidal structure. There are two identity-on-objects 2-functors $h_{-}, h^{-}: \mathcal{L}\mathrm{ex}_{\Bbbk} \to \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$: The first one sends a functor $F: \mathcal{A} \to \mathcal{B}$ to the *representable* profunctor $h_F: \mathcal{B} \nrightarrow \mathcal{A}$ induced by $\mathrm{Hom}_{\mathcal{B}}(-, F(-)): \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{vect}_{\Bbbk}$ on the Deligne product $\mathcal{B}^{\mathrm{op}} \boxtimes \mathcal{A}$. The second one sends a functor $F: \mathcal{A} \to \mathcal{B}$ to the *corepresentable* profunctor $h^F: \mathcal{A} \nrightarrow \mathcal{B}$ induced by $\mathrm{Hom}_{\mathcal{B}}(F(-), -): \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{vect}_{\Bbbk}$ on the Deligne product $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B}$. To be more precise h_{-} is a 2-functor contravariant on 1-morphisms while h^{-} is contravariant on 2-morphisms. Compatibility of h_{-} and h^{-} with horizontal composition, i.e. functoriality, can be proved by directly using the results on coends from the previous section, see e.g. [Lor, Sec. 5.2].

For a fixed left exact functor $F: \mathcal{A} \to \mathcal{B}$ the two profunctors h_F and h^F are closely related:

Lemma 2.2.9. Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact functor. The profunctor h_F is left adjoint to the profunctor h^F as 1-morphisms in the 2-category $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$.

Proof. This is a variation of the proof given in [Lor, Rem. 5.2.1]. For later use we will repeat some of the argument and indicate where one needs to be more careful.

To obtain the unit $\eta: \mathrm{Hom}_{\mathcal{A}} \Rightarrow h^F \diamond h_F$ first note that $h^F \diamond h_F$ is induced from $\oint^{B \in \mathcal{B}} \mathrm{Hom}_{\mathcal{B}}(B, F(-)) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(F(-), B) \cong \mathrm{Hom}_{\mathcal{B}}(F(-), F(-))$ where we used the Yoneda lemma. We define η to be induced from the natural transformation $\mathrm{Hom}_{\mathcal{A}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{B}}(F(-), F(-))$ given by the action of F on morphisms.

The counit $\varepsilon: h_F \diamond h^F \to \mathrm{Hom}_{\mathcal{B}}$ is obtained as follows: First note that the natural transformation $\mathrm{Hom}_{\mathcal{B}}(F(A), -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(-, F(A)) \Rightarrow \mathrm{Hom}_{\mathcal{B}}(-,-)$ coming from the composition map is dinatural in A and thus factorises over the coend $\oint^{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(F(A), -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(-, F(A))$. We can now define ε to be the natural transformation $h_F \diamond h^F \to \mathrm{Hom}_{\mathcal{B}}$ induced from this natural transformation.

To prove the triangle identities it suffices to prove them for the natural transformations from which η and ε are induced instead. This can now be done completely analogously as in [Lor, Rem. 5.2.1] by working component wise. $\square$

Finally, we want to mention that under our finiteness assumptions the 2-functors h_{-} and h^{-} are actually 2-equivalences by the Eilenberg-Watts theorem, see [Shi1, Lem. 3.2]. The reason we use $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ as a target rather than $\mathcal{L}\mathrm{ex}_{\Bbbk}$ is that the former appears naturally when we define the modular functor on 1- and 2-morphisms. Moreover, the coends appearing in the definition of horizontal composition can physically be interpreted as a way of *summing over intermediate states*, see [FS1, Sec. 1.2] for more details on this interpretation.

Chapter 3

Non-semisimple 3d defect TFTs

The goal of this thesis is to use non-semisimple 3d defect TFTs to construct full 2d CFTs. In this chapter we will introduce the former. To this end will first review the construction of the non-semisimple TFTs of [DGGPR1]. Afterwards we will review the notion of 3d defect TFT and explain how the non-semisimple TFTs of [DGGPR1] fits into this framework.

Let us first fix our conventions regarding manifolds. With manifold we will always mean a compact smooth manifold, however since we are exclusively working in dimensions less then four we will sometimes work in the topological category instead and tacitly assume that everything has been smoothed already. Every manifold we will consider will be oriented, and every diffeomorphism will be orientation preserving unless explicitly stated otherwise. For any manifold M we will denote the manifold with reversed orientation by $-M$. The interval $[0, 1]$ will be denoted by I and the unit circle by S^1. Finally, by a closed manifold we mean a compact manifold without boundary.

3.1 Non-semisimple surgery TFT

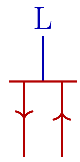
In this section we are going to review the non-semisimple TFTs of [DGGPR1]. To this end, we begin with recalling bichrome graphs as well as the corresponding 3-manifold invariants. Then we will define a 3-dimensional bordism category and introduce two subcategories of so-called admissible bordisms and study their relation. We then apply the universal TFT construction to obtain two partially defined TFTs[1] and discuss an algebraic model for the state spaces of these TFTs, as well as their behaviour under orientation reversal and Deligne products. Our exposition mostly follows [DGGPR2, Sec. 2], except that we include the aforementioned

[1]In the literature partially defined TFTS are sometimes referred to as non-compact TFTs, see e.g. [Haï].

results on orientation reversal and Deligne products.

3.1.1 3-manifold invariants

Bichrome graphs are a generalisation of the ribbon graphs of [Tur, Sec. I.2] where two kinds of edges are present, *red* edges without labels, and *blue* edges labelled by objects of $\mathcal{C}$. Coupons can be, according to the edges intersecting them, either *bichrome* and unlabelled, or *blue* and labelled as usual by morphisms of $\mathcal{C}$. Furthermore, there are only two possible configurations of bichrome coupons allowed, from which we will only need the following one (which we just draw as a horizontal line):

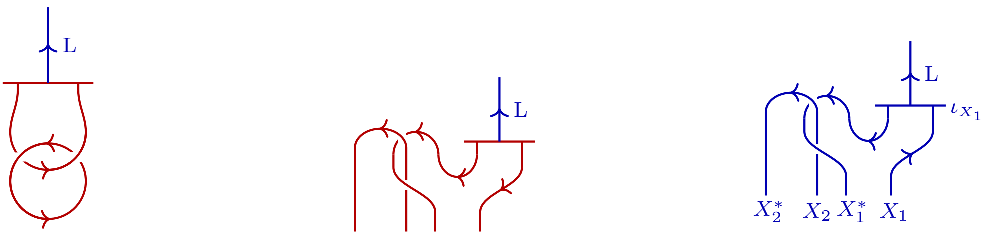


See [DGGPR1, Sec. 3.2] for the other one, which involves an end instead of the coend, and [DGGPR1, Rem. 3.6] on why it suffices to only consider one. Red coupons are generally forbidden.[2]

There is a category $\mathcal{B}_\mathcal{C}$ of $\mathcal{C}$-coloured bichrome graphs. Objects $(\underline{V}, \underline{\varepsilon})$ are finite sequences $((V_1, \varepsilon_1), \ldots, (V_n, \varepsilon_n))$ where every V_k is an object of $\mathcal{C}$ and $\varepsilon_k \in \{+, -\}$. From every object $(\underline{V}, \underline{\varepsilon})$ we get a set of $\mathcal{C}$-labelled framed, oriented blue points located at the fixed points $0, 1, 2, 3, \ldots$ on the real axis in $\mathbb{R}^2$. A morphism $T\colon (\underline{V}, \underline{\varepsilon}) \to (\underline{W}, \underline{\nu})$ is an isotopy class of bichrome graphs in $\mathbb{R}^2 \times I$ between the corresponding standard sets of framed, oriented blue points such that the framings, orientations, and labels match. The subcategory $\mathcal{R}_\mathcal{C}$ consisting of all objects but only blue graphs is the familiar category of $\mathcal{C}$-coloured ribbon graphs of [Tur]. In [DGGPR1, Sec. 3.1] it is shown how to extend the Reshetikhin-Turaev functor $F_\mathcal{C}\colon \mathcal{R}_\mathcal{C} \to \mathcal{C}$ to all bichrome graphs. For this construction a non-zero integral Λ of L needs to be chosen, and for this reason the corresponding functor will be denoted by $F_\Lambda\colon \mathcal{B}_\mathcal{C} \to \mathcal{C}$. We will not recall the construction in full detail, instead we will illustrate how to evaluate the functor $F_\Lambda : \mathcal{B}_\mathcal{C} \to \mathcal{C}$ in an example in Figure 3.1.

Starting from the bichrome graph in Figure 3.1(a), we form the *cut presentation* in Figure 3.1(b) where each red loop is cut once (bichrome coupons are considered part of the red graph). Now we choose objects $X_1, \ldots, X_n$ and label the red edges correspondingly by the X_k and X_k^*, e.g. in Figure 3.1(b) we have $n = 2$. The bichrome coupons meeting these edges will be labelled by the universal dinatural

[2] In principle one could use natural transformations to introduce red coupons as well without interfering with the evaluation procedure of bichrome graphs described below. We will however not pursue this idea further here as it will not be needed.



(a) Morphism in $\mathcal{B}_\mathcal{C}$ from $\varnothing$ to $(\mathrm{L}, +)$

(b) Cut presentation of the morphism

(c) $\mathcal{C}$-coloured ribbon graph for the morphism

Figure 3.1: Schematic algorithm for the evaluation of the functor $F_\Lambda : \mathcal{B}_\mathcal{C} \to \mathcal{C}$ on a morphism.

morphism ι_{X_k}. At this point, all edges and coupons are labelled and thus blue, see Figure 3.1(c). We have now obtained a $\mathcal{C}$-coloured ribbon graph, i.e. a morphism in $\mathcal{R}_\mathcal{C}$. To this we can apply the Reshetikhin-Turaev functor $F_\mathcal{C}$ to obtain a morphism in $\mathcal{C}$. By construction, the resulting family of morphisms in $\mathcal{C}$ is dinatural in the labels $X_1, \dots, X_n$. Thus, by the universal property of L we get a morphism out of $\mathrm{L}^{\otimes n}$, possibly tensored with the objects coming from blue boundary vertices. As the final step we precompose our morphism with the n-fold tensor power of the integral Λ, again tensored with the identity on the objects coming from blue boundary vertices. In our example we obtain

$$F_\Lambda(T) = (\overline{\omega} \otimes \mathrm{id}_\mathrm{L}) \circ (\Lambda \otimes (\Delta \circ \Lambda)) = \mathcal{S} \circ S \circ \Lambda = \mathcal{S} \circ \Lambda \tag{3.1.1}$$

where $\overline{\omega} = \omega \circ (S \otimes \mathrm{id}_\mathrm{L})$ from (2.1.8) and $\mathcal{S}$ is the modular S-morphism from (2.1.10).

It can be shown that this construction gives a functor F_Λ called the *Lyubashenko-Reshetikhin-Turaev functor*, we refer to [DGGPR1, Sec. 3.1] for more details on the definition and well-definedness of F_Λ. Moreover, by construction we have a commuting diagram of functors

$$\begin{tikzcd} \mathcal{R}_\mathcal{C} \arrow[r, "F_\mathcal{C}"] \arrow[d] & \mathcal{C} \\ \mathcal{B}_\mathcal{C} \arrow[ur, "F_\Lambda"'] & \end{tikzcd} \tag{3.1.2}$$

where the functor $\mathcal{R}_\mathcal{C} \to \mathcal{B}_\mathcal{C}$ acts as the identity on objects and as the inclusion of purely blue ribbon graphs into bichrome graphs at the level of morphisms.

We can now use this functor to define invariants of closed bichrome graphs, i.e. endomorphisms of $\varnothing$ in $\mathcal{B}_\mathcal{C}$, and of 3-manifolds. For this we need so-called *admissible bichrome graphs*, which are bichrome graphs with at least one blue edge labelled by a projective object of $\mathcal{C}$. Otherwise we could still define invariants as usual but due to Lemma 2.1.9 many of them would automatically be zero.

For an admissible closed bichrome graph T we define its *cutting presentation* as a bichrome graph T_P featuring a single incoming boundary vertex and a single outgoing one, both positive and labelled by $P \in \mathrm{Proj}(\mathcal{C})$, and whose trace closure is T. For any T a closed admissible bichrome graph and T_P a cutting presentation of T, the scalar

$$F'_\Lambda(T) := \mathrm{t}_P(F_\Lambda(T_P)) \tag{3.1.3}$$

is a topological invariant of T, so in particular it is independent of the choice of cutting presentation T_P, see [DGGPR1, Thm. 3.3].

We can now use F'_Λ to define invariants of *decorated* 3*-manifolds*, i.e. pairs (M, T), where M is a connected closed 3-manifold, and where $T \subset M$ is a closed bichrome graph. We call a pair (M, T) *admissible* if T is.

Let now (M, T) be an admissible decorated 3-manifold, and let L be a surgery presentation of M given by a red framed oriented link in S^3 with ℓ components and signature σ. Assume further that the bichrome graph T is contained in the exterior of the surgery link L, so that we can think of them as simultaneously embedded in S^3. Moreover let $\mathcal{D}$ be a choice of square root of the modularity parameter ζ, and

$$\delta = \frac{\mathcal{D}}{\Delta_+} = \frac{\Delta_-}{\mathcal{D}}. \tag{3.1.4}$$

Modular tensor categories with $\delta = 1$ are sometimes called *anomaly free.* The scalar

$$\mathrm{L}'_\mathcal{C}(M, T) := \mathcal{D}^{-1-\ell}\delta^{-\sigma}F'_\Lambda(L \cup T) \tag{3.1.5}$$

is a topological invariant of the pair (M, T), see [DGGPR1, Thm. 3.8], and called the *renormalised Lyubashenko invariant* of the admissible decorated 3-manifold (M, T).

Remark 3.1.1. The construction of [DGGPR1] also allows for other tensor ideals of $\mathcal{C}$ as long as they permit a modified trace, see [BGR] for examples of this. In particular by choosing $\mathcal{C}$ as tensor ideal with the categorical trace one recovers the original Lyubashenko invariants and thus also the Reshetikhin-Turaev invariants for $\mathcal{C}$ semisimple.[3] This can also be seen by noting that in the semisimple case the unit $\mathbb{1}$ is projective and thus every graph is automatically admissible by adding a closed $\mathbb{1}$-labelled subgraph. The construction of renormalised 3-manifold invariants via modified traces was introduced in [CGP].

Let us now discuss how $\mathrm{L}'_\mathcal{C}$ behaves under orientation reversal. To do so, we need to be more specific on the orientation data contained in bichrome graph. Namely, a bichrome graph T (in a 3-manifold M or in $\mathbb{R}^2 \times I$) is a ribbon graph with

[3] One needs to be a bit careful with the normalisation of the integral Λ in this case in order to reproduce the standard Kirby colour, see [DGGPR1, Sec. 2.9] for more details on this point.

blue and red components, and each of these consists of ribbons and coupons. A ribbon is an embedded annulus or rectangle with a one-dimensional core which carries a 1-orientation and gives the direction of the ribbon, as well as a 2-orientation on the ribbon surface. In pictures like the one in Figure 3.1(a), we draw the core and its orientation, while the ribbon surface is not shown and is assumed to be parallel to the paper-plane $\mathbb{R} \times \{0\} \times I$ and carries its 2-orientation. A coupon is an embedded rectangle with a 2-orientation and a choice of bottom edge and top edge to which ribbons attach, and which determine the source and target object of the labelling morphism (for a blue coupon).

For a bichrome graph T in $\mathbb{R}^2 \times I$ we define the *mirrored* bichrome graph $\overline{T}$ as the image of T under reflection along the plane $\mathbb{R} \times \{0\} \times I$ (the paper-plane in our drawings). In particular, if a ribbon lies in this plane (or is parallel to the plane), the reflection does not affect its 1- or 2-orientation. Note that we can obtain a diagram of $\overline{T}$, i.e. a projection of the embedding $\overline{T} \subset \mathbb{R}^2 \times I$ to the plane $\mathbb{R} \times \{0\} \times I$, from a diagram of T by exchanging all overcrossings with undercrossings and vice-versa.

For a decorated 3-manifold (M, T) we denote by $(-M, T)$ the decorated 3-manifold obtained by reversing the orientation of M and keeping the 1- and 2-orientations of T as they are. For example, if M is the 3-sphere written as $\mathbb{R}^3 \cup \{\infty\}$ and T a closed bichrome graph in $\mathbb{R}^3$, then $(-M, T)$ is orientation-preserving diffeomorphic to $(M, \overline{T})$ via the above reflection.

Lemma 3.1.2. For an admissible decorated 3-manifold (M, T) we have

$$\mathrm{L}'_{\mathcal{C}}(-M, T) = \mathrm{L}'_{\overline{\mathcal{C}}}(M, T), \tag{3.1.6}$$

with $\overline{\mathcal{C}}$ the mirrored category of $\mathcal{C}$.

Proof. First recall that if L is a surgery presentation of M, then $\overline{L}$ is a surgery presentation of $-M$ [Sav, Ch. 3.4]. Moreover, for an admissible pair $(-M, T)$ a surgery representation is given by $(S^3, \overline{L} \cup \overline{T})$. In particular the signature of $\overline{L}$ is minus the signature of L. Let us denote with $F'_{\overline{\Lambda}}$ the Lyubashenko-Reshetikhin-Turaev functor for $\overline{\mathcal{C}}$. On bichrome graphs T with no red components we immediately get

$$F'_{\overline{\Lambda}}(\overline{T}) = F'_{\Lambda}(T) \tag{3.1.7}$$

by the same argument as in the semisimple setting [Tur, Cor. II.2.8.4].

Next note that since $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equal as pivotal categories, the coend L is given by the same object and dinatural transformation in either case. This remains true for the coalgebra structure and the unit morphism as their definition in Figure 2.1 does not involve the ribbon structure. The product $\overline{\mu}$ and antipode $\overline{S}$ are in general different when computed in $\overline{\mathcal{C}}$, and we denote by $\overline{\mathrm{L}}$ the corresponding Hopf algebra in $\overline{\mathcal{C}}$. We stress that $\overline{\mathrm{L}}$ is in general not a Hopf algebra in $\mathcal{C}$ as the braiding enters the Hopf algebra axioms. It can however be directly checked from the definition that $\overline{\mu} = \mu \circ \beta^{-1}_{\mathrm{L},\mathrm{L}} : \mathrm{L} \otimes \mathrm{L} \to \mathrm{L}$ as morphisms in $\mathcal{C}$. Using this it follows that Λ is also a two-sided integral for $\overline{\mathrm{L}}$, and we set $\overline{\Lambda} = \Lambda$. Then in particular $\Delta^{\overline{\mathcal{C}}}_{\pm} = \Delta^{\mathcal{C}}_{\mp}$. Using this the claim follows. $\square$

Next we turn to 3-manifold invariants for Deligne products of modular tensor categories. Recall that for $\mathcal{C}$ and $\mathcal{D}$ modular tensor categories, also $\mathcal{C} \boxtimes \mathcal{D}$ is a modular tensor category.

Let M be a 3-manifold and let $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$ be $\mathcal{C}$ and $\mathcal{D}$ labelled, admissible bichrome graphs in M, respectively, with the same underlying unlabelled graph T. We define $T_{\mathcal{C}\boxtimes\mathcal{D}}$ as the bichrome graph obtained by labelling every ribbon and every coupon of T with the Deligne product of the labels used for $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$.

Lemma 3.1.3. Let $(M, T_{\mathcal{C}})$, $(M, T_{\mathcal{D}})$, and $(M, T_{\mathcal{C}\boxtimes\mathcal{D}})$ be as above. Then

$$\mathrm{L}'_{\mathcal{C}\boxtimes\mathcal{D}}(M, T_{\mathcal{C}\boxtimes\mathcal{D}}) = \mathrm{L}'_{\mathcal{C}}(M, T_{\mathcal{C}}) \cdot \mathrm{L}'_{\mathcal{D}}(M, T_{\mathcal{D}}). \tag{3.1.8}$$

Proof. First recall the isomorphism $\mathrm{L}_{\mathcal{C}\boxtimes\mathcal{D}} \cong \mathrm{L}_{\mathcal{C}} \boxtimes \mathrm{L}_{\mathcal{D}}$ from [FSS, Cor. 3.12]. This is actually an isomorphism of Hopf algebras because the Hopf algebra structures come from the universal property of the relevant coends, thus we can choose $\Lambda_{\mathcal{C}\boxtimes\mathcal{D}} = \Lambda_{\mathcal{C}} \boxtimes \Lambda_{\mathcal{D}}$. Therefore the invariants of purely red graphs are the same. For blue graphs note that the unique-up-to-scalar modified trace on $\mathcal{C} \boxtimes \mathcal{D}$ is canonically determined by the modified traces on $\mathcal{C}$ and $\mathcal{D}$. To see this combine the relation between trivialisations of the Nakayama functor and modified traces on a finite tensor category [SW2, Thm. 3.6] with the behaviour of the Nakayama functor under the Deligne tensor product [FSS, Prop. 3.20]. $\square$

3.1.2 Admissible bordism categories

Before coming to the definition of the relevant 3-dimensional bordism categories, we need to adapt a few definitions from above to a more general setting. A *blue set P inside a closed surface Σ* is a finite set of blue points of Σ, each endowed with a 0-orientation $\pm$, a non-zero tangent vector, and a label given by an object of $\mathcal{C}$. A *bichrome graph T inside a 3-dimensional bordism $M \colon \Sigma \to \Sigma'$* is a bichrome graph embedded inside M such that its boundary vertices are given by blue sets inside the boundary ∂M. Moreover the boundary identification $\partial M \cong -\Sigma \sqcup \Sigma'$ needs to be compatible with the blue sets. With this terminology in place, we can define the symmetric monoidal category $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ of 3-dimensional bordisms with $\mathcal{C}$-coloured bichrome graphs:

- An *object $\underline{\Sigma}$ of* $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is a triple (Σ, P, λ) where:

1. Σ is a closed surface;
2. $P \subset \Sigma$ is a blue set;
3. $\lambda \subset H_1(\Sigma; \mathbb{R})$ is a Lagrangian subspace with respect to the intersection pairing.[4]

- A *morphism* $\underline{M} \colon \underline{\Sigma} \to \underline{\Sigma}'$ is an equivalence class of triples (M, T, n) where:

  1. M is a 3-dimensional bordism from Σ to Σ';
  2. $T \subset M$ is a bichrome graph from P to P';
  3. $n \in \mathbb{Z}$ is an integer called the *signature defect* of M.

  Two triples (M, T, n) and (M', T', n') are equivalent if $n = n'$ and if there exists an isomorphism of bordisms $f : M \to M'$ satisfying $f(T) = T'$.

- The *identity morphism* $\mathrm{id}_{\underline{\Sigma}} : \underline{\Sigma} \to \underline{\Sigma}$ *associated with an object* $\underline{\Sigma} = (\Sigma, P, \lambda)$ *of* $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the equivalence class of the triple

$$(\Sigma \times I, P \times I, 0). \tag{3.1.9}$$

- The *composition* $\underline{M}_2 \circ \underline{M}_1 : \underline{\Sigma}_1 \to \underline{\Sigma}_2$ *of morphisms* $\underline{M}_1 \colon \underline{\Sigma}_1 \to \underline{\Sigma}$ *and* $\underline{M}_2 \colon \underline{\Sigma} \to \underline{\Sigma}_2$ in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the equivalence class of the triple

$$\big(M_2 \cup_{\Sigma} M_1 \, , \, T_1 \cup_P T \, , \, n_1 + n_2 - \mu((M_1)_*(\lambda_1), \lambda, (M_2)^*(\lambda_2)) \big), \tag{3.1.10}$$

  where $(M_1)_*(\lambda_1)$ and $(M_2)^*(\lambda_2)$ are certain Lagrangian subspaces of $H_1(\Sigma; \mathbb{R})$ and μ denotes the Maslov index, see [Tur, Sec. IV. 3-4] for details.

- The *monoidal product* $\underline{\Sigma} \sqcup \underline{\Sigma}'$ *of objects* $\underline{\Sigma}$, $\underline{\Sigma}'$ in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the triple

$$(\Sigma \sqcup \Sigma', P \sqcup P', \lambda \oplus \lambda'). \tag{3.1.11}$$

  The *unit of* $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the object whose surface is the empty set, and it will be denoted $\varnothing$. The *monoidal product* $\underline{M} \sqcup \underline{M}' : \underline{\Sigma} \sqcup \underline{\Sigma}' \to \underline{\Sigma}'' \sqcup \underline{\Sigma}'''$ *of morphisms* $\underline{M} : \underline{\Sigma} \to \underline{\Sigma}''$, $\underline{M}' : \underline{\Sigma}' \to \underline{\Sigma}'''$ in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is the equivalence class of the triple

$$(M \sqcup M', T \sqcup T', n + n'). \tag{3.1.12}$$

It is straightforward to see that $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ can be equipped with the standard symmetric braiding, and moreover with a pivotal structure coming from orientation reversal. To be more precise, the dual object of $\underline{\Sigma}$ in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ is given by

[4] This is needed to precisely formulate the gluing anomaly, see [Tur, Ch. IV]. See also [Haï, Sec. 4] for an interpretation of the origin of λ in the context of relative TFTs.

$\underline{\Sigma}^* = (-\Sigma, -P, \lambda)$ where $-P$ is the same set of blue points with 0-orientation and tangent vector reversed. The 3-manifold underlying the dual of a morphism $[(M, T, n)]$ is again M and not $-M$, but with in-going and out-going boundary component exchanged, see [TV2, Sec. 10.1.4] for more details on the pivotal structure. As usual we will say a morphism $[(M, T, n)]$ has a certain topological property if the 3-manifold underlying M has this property, e.g. we say $[(M, T, n)]$ is connected if and only if M is connected. The superscript χ indicates that we are working with an extension of the standard bordism category $\mathrm{Bord}_{3,2}(\mathcal{C})$ (which does not include Lagrangian subspaces and signature defects).

Since the invariants discussed in the previous section are only defined for admissible pairs (M, T) the corresponding TFTs will also only be defined on a subcategory of the bordism category which is generically no longer rigid. There are two choices for this subcategory:

Definition 3.1.4. The *admissible bordism (sub)categories* $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ and $\widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ are the symmetric monoidal subcategories of $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ with the same objects but featuring only morphisms $[(M, T, n)]$ which satisfy one of the following *admissibility conditions*:

1. $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$: Every connected component of M disjoint from the *outgoing* boundary contains an admissible bichrome subgraph of T, i.e. at least one edge of T in that component is labelled with a projective object.

2. $\widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$: Every connected component of M disjoint from the *incoming* boundary contains an admissible bichrome subgraph of T.

Note that there are morphisms in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ which are neither in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ nor $\widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$, e.g. $(M, \emptyset, 0)$ for any closed 3-manifold M. As mentioned above, neither $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ nor $\widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ are rigid in general, in fact the dualisable objects are exactly the surfaces with at least one projectively labelled marked point. The following lemma is clear:

Lemma 3.1.5. The duality functor $(-)^* \colon \mathrm{Bord}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})^{\mathrm{op}}$ induces an equivalence of symmetric monoidal categories $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \simeq \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})^{\mathrm{op}}$.

3.1.3 Universal TFT construction

We extend the renormalised Lyubashenko invariant to closed morphisms $\underline{M} = [(M, T, n)]$, i.e. endomorphisms of the monoidal unit, of $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ by setting

$$\mathrm{L}'_{\mathcal{C}}(\underline{M}) := \delta^n \mathrm{L}'_{\mathcal{C}}(M, T) \tag{3.1.13}$$

for M connected and by setting

$$\mathrm{L}'_{\mathcal{C}}(\underline{M}_1 \sqcup \ldots \sqcup \underline{M}_k) := \prod_{i=1}^{k} \mathrm{L}'_{\mathcal{C}}(\underline{M}_i) \tag{3.1.14}$$

for $\underline{M}$ a finite disjoint union of closed connected morphisms $\underline{M}_1, \ldots, \underline{M}_k$.

The universal TFT construction of [BHMV] allows us to extend $\mathrm{L}'_{\mathcal{C}}$ to a functor from bordisms to vector spaces as follows. For $\underline{\Sigma} \in \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ define $\widehat{\mathcal{V}}(\underline{\Sigma})$ to be the free vector space generated by the set of morphisms $\underline{M_{\Sigma}} : \varnothing \to \underline{\Sigma}$ of $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$, and $\widehat{\mathcal{V}}'(\underline{\Sigma})$ the free vector space generated by the set of morphisms $\underline{M'_{\Sigma}} : \underline{\Sigma} \to \varnothing$ of $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$. Next, consider the bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\underline{\Sigma}} \colon \widehat{\mathcal{V}}'(\underline{\Sigma}) \times \widehat{\mathcal{V}}(\underline{\Sigma}) &\to \Bbbk \\ (\underline{M'_{\Sigma}}, \underline{M_{\Sigma}}) &\mapsto \mathrm{L}'_{\mathcal{C}}(\underline{M'_{\Sigma}} \circ \underline{M_{\Sigma}}). \end{aligned} \tag{3.1.15}$$

Let $\widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma})$ be the quotient vector space of $\widehat{\mathcal{V}}(\underline{\Sigma})$ with respect to the right radical of the bilinear form $\langle \cdot, \cdot \rangle_{\underline{\Sigma}}$, and similarly let $\widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma})$ be the quotient vector space of the $\widehat{\mathcal{V}}'(\underline{\Sigma})$ with respect to the left radical of the bilinear form $\langle \cdot, \cdot \rangle_{\underline{\Sigma}}$. We will abuse notation by denoting both projections as $[\,\cdot\,]$, i.e. $[\,\cdot\,] : \mathcal{V}(\underline{\Sigma}) \to \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma})$ and $[\,\cdot\,] : \mathcal{V}'(\underline{\Sigma}) \to \widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma})$, and by also denoting the pairing induced from (3.1.15) by

$$\langle \cdot, \cdot \rangle_{\underline{\Sigma}} \colon \widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma}) \otimes \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma}) \to \Bbbk. \tag{3.1.16}$$

This pairing is non-degenerate by construction.

For $\underline{M} \colon \underline{\Sigma} \to \underline{\Sigma}'$ a morphism in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ let $\widehat{\mathrm{V}}_{\mathcal{C}}(\underline{M})$ be the linear map defined by

$$\begin{aligned} \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{M}) \colon \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma}) &\to \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma}') \\ [\underline{M_{\Sigma}}] &\mapsto [\underline{M} \circ \underline{M_{\Sigma}}], \end{aligned} \tag{3.1.17}$$

and similarly let $\widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{M})$ be the linear map defined by

$$\begin{aligned} \widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{M}) \colon \widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma}') &\to \widehat{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma}) \\ [\underline{M'_{\Sigma'}}] &\mapsto [\underline{M'_{\Sigma'}} \circ \underline{M}]. \end{aligned} \tag{3.1.18}$$

The construction clearly defines functors

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk}, \qquad \widehat{\mathrm{V}}'_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Vect}_{\Bbbk}, \tag{3.1.19}$$

moreover from the equivalence $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})^{\mathrm{op}} \simeq \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ we further get

$$\check{\mathrm{V}}_{\mathcal{C}} \colon \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk}, \qquad \check{\mathrm{V}}'_{\mathcal{C}} \colon \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Vect}_{\Bbbk}, \tag{3.1.20}$$

where $\check{\mathrm{V}}_{\mathcal{C}} = \widehat{\mathrm{V}}'_{\mathcal{C}} \circ (-)^*$ and $\check{\mathrm{V}}'_{\mathcal{C}} = \widehat{\mathrm{V}}_{\mathcal{C}} \circ (-)^*$ and $(-)^*$ is the duality functor. Note that we can restrict our attention to the covariant functors $\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk}$ and $\check{\mathrm{V}}_{\mathcal{C}} \colon \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk}$ because the contravariant ones can be recovered from them by precomposition with $(-)^*$.

In fact, both functors define TFTs, with the non-trivial step being the proof of monoidality:

Theorem 3.1.6 ([DGGPR1, Thm. 4.12])**.** Let $\mathcal{C}$ be a modular tensor category. The functors

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk} \quad \text{and} \quad \check{\mathrm{V}}_{\mathcal{C}} \colon \widetilde{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk} \tag{3.1.21}$$

are symmetric monoidal.

Remark 3.1.7. (1) By the universal construction for any $\underline{\Sigma} \in \mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ the state space $\widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma})$ and its linear dual $\widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma})^*$ are spanned by the vectors obtained from evaluating $\widehat{\mathrm{V}}_{\mathcal{C}}$ on all bordisms $\varnothing \to \underline{\Sigma}$ and $\underline{\Sigma} \to \varnothing$, respectively.

(2) For any $\underline{\Sigma} \in \mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ we have

$$\check{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma})^* \cong \check{\mathrm{V}}'_{\mathcal{C}}(\underline{\Sigma}) = \widehat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma}^*) \tag{3.1.22}$$

where $\underline{\Sigma}^*$ is the dual object of $\underline{\Sigma}$ in $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$, and where the first isomorphism is induced by the non-degenerate pairing (3.1.16) (but for $\check{\mathrm{V}}_{\mathcal{C}}$). In this sense we get a pair of TFTs dual to each other.

3.1.4 Algebraic state spaces

We will now recall an algebraic model for the state spaces of $\widehat{\mathrm{V}}_{\mathcal{C}}$ associated with connected objects of $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$.

For this let $g, q, p \in \mathbb{Z}_{\geqslant 0}$, we consider a standard closed connected surface Σ_g of genus g, a handlebody H_g with $\partial H_g = \Sigma_g$, and the Lagrangian subspace $\lambda_g \subset H_1(\Sigma_g)$ given by the kernel of the inclusion of Σ_g into H_g. For a $(p+q)$-tuple of objects $(\underline{X};\underline{Y}) \equiv (X_1, \dots, X_p, Y_1, \dots, Y_q) \in \mathcal{C}^{\times(p+q)}$, we denote with

$$\underline{\Sigma}_g^{(\underline{X};\underline{Y})} = (\Sigma_g, P_{(\underline{X};\underline{Y})}, \lambda_g) \tag{3.1.23}$$

the object of $\mathrm{Bord}^{\chi}_{3,2}(\mathcal{C})$ with p negatively oriented marked points decorated using the X_i and q positively oriented marked points decorated using the Y_j. For $f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes X_1 \otimes \dots \otimes X_p, Y_1 \otimes \dots \otimes Y_q)$ we consider the admissible bordism $[(H_g, T_f, 0)] : \varnothing \to \underline{\Sigma}_g^{(\underline{X};\underline{Y})}$ shown in Figure 3.2.

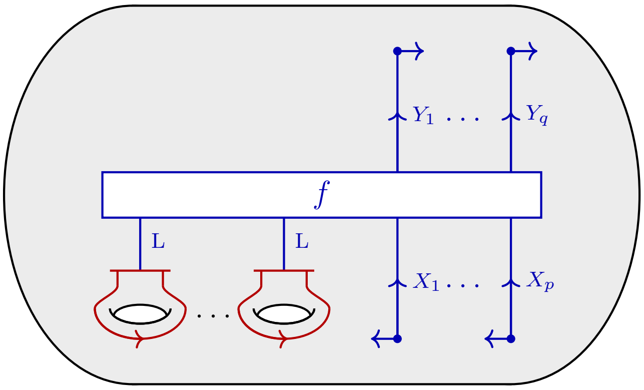


Figure 3.2: The bordism used to identify Hom-spaces in $\mathcal{C}$ and state spaces of $\widehat{\mathrm{V}}_{\mathcal{C}}$.

Theorem 3.1.8 ([DGGPR1, Prop. 4.17])**.** The map

$$\Phi_{\Sigma}^{(\underline{X};\underline{Y})} \colon \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes X_1 \otimes \ldots \otimes X_p, Y_1 \otimes \ldots \otimes Y_q) \to \widehat{\mathrm{V}}_{\mathcal{C}}\left(\underline{\Sigma}_g^{(\underline{X};\underline{Y})}\right)$$
$$f \mapsto \widehat{\mathrm{V}}_{\mathcal{C}}\left([(H_g, T_f, 0)]\right)(1_{\Bbbk}) \tag{3.1.24}$$

is a linear isomorphism.

Remark 3.1.9. (1) In [DGGPR1, Sec. 4] only positively oriented marked points are considered. Our setting is obtained from theirs by first declaring negatively oriented points to be labelled with the dual object and then using the isomorphism (3.1.22) together with the natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes X_1 \otimes \cdots \otimes X_p, Y_1 \otimes \cdots \otimes Y_q) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes X_1 \otimes \cdots \otimes X_p \otimes Y_q^* \otimes \cdots \otimes Y_1^*, \mathbb{1})$$

induced by the rigidity and braiding of $\mathcal{C}$.

(2) A different, but related model of the state spaces can be given in terms skein modules [DGGPR2, Sec. 2.3.3].

3.1.5 Orientation reversal and Deligne products

Finally, let us study how the behaviour of the 3-manifold invariants under orientation reversal and the Deligne product – discussed at the end of Section 3.1.1 – gives corresponding statements about the TFTs. For this we will need the following technical lemma, based on [TV2, Lem. 17.2].[5]

[5]That lemma cannot be used in our setting directly because the admissible bordism categories are not rigid.

Lemma 3.1.10. Let $\mathcal{A}$ be a category with a fixed object $\mathbb{1}$. Let $F, G \colon \mathcal{A} \to \mathrm{vect}_{\Bbbk}$ be functors such that:

(1) for $H = F, G$ one has

  (a) there exists a linear isomorphism $H^0 \colon \Bbbk \to H(\mathbb{1})$;

  (b) for all $A \in \mathcal{A}$ one has

$$H(A) = \mathrm{span}_{\Bbbk}\big\{ H(f)(H^0(1_{\Bbbk})) \,\big|\, f \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)\big\}; \tag{3.1.25}$$

  (c) for all $A \in \mathcal{A}$ one has

$$H(A)^* = \mathrm{span}_{\Bbbk}\big\{ (H^0)^{-1} \circ H(g) \,\big|\, g \in \mathrm{Hom}_{\mathcal{A}}(A, \mathbb{1})\big\}; \tag{3.1.26}$$

(2) for all $u \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \mathbb{1})$ the following diagram commutes,

$$\begin{tikzcd} & F(\mathbb{1}) \arrow[r, "F(u)"] & F(\mathbb{1}) \arrow[dr, "(F^0)^{-1}"] & \\ \Bbbk \arrow[ur, "F^0"] \arrow[dr, "G^0"'] & & & \Bbbk \\ & G(\mathbb{1}) \arrow[r, "G(u)"'] & G(\mathbb{1}) \arrow[ur, "(G^0)^{-1}"'] & \end{tikzcd} . \tag{3.1.27}$$

Then for all $A \in \mathcal{A}$ there exists a unique linear map $\varphi_A \colon F(A) \to G(A)$ such that for all $f \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)$ the following diagram commutes,

$$\begin{tikzcd} & F(\mathbb{1}) \arrow[r, "F(f)"] & F(A) \arrow[dd, dashed, "\varphi_A"] \\ \Bbbk \arrow[ur, "F^0"] \arrow[dr, "G^0"'] & & \\ & G(\mathbb{1}) \arrow[r, "G(f)"'] & G(A) \end{tikzcd} . \tag{3.1.28}$$

Furthermore, the collection $(\varphi_A)_{A \in \mathcal{A}}$ is a natural isomorphism $F \Rightarrow G$.

Proof. If φ_A exists, then it is an isomorphism by (1b). The proof of existence of φ_A will be very similar to the one in [TV2, Lem. 17.2]. Namely, let $\Bbbk\langle \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)\rangle$ be the vector space freely generated by $\mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)$ and let

$$\pi_A^H \colon \Bbbk\langle \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)\rangle \to H(A) \tag{3.1.29}$$

be the linear extension of the map

$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A) \to H(A), \quad f \mapsto H(f)(H^0(1_{\Bbbk})). \tag{3.1.30}$$

The maps π_A^F and π_A^G are surjective by (1b). We want to define φ_A such that $\varphi_A \circ \pi_A^F = \pi_A^G$, which is well defined if $\pi_A^G|_{\ker(\pi_A^F)} = \{0\}$. To check this, let $v = \sum_{i=1}^n v_i f_i \in \ker(\pi_A^F)$ with $v_i \in \Bbbk$ and $f_i \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, A)$. By (1c) it is enough to check that for all $g \in \mathrm{Hom}_{\mathcal{A}}(A, \mathbb{1})$ we have that

$$\sum_{i=1}^n v_i \left((G^0)^{-1} \circ G(g) \circ G(f_i)\right)(G^0(1_{\Bbbk})) = 0. \tag{3.1.31}$$

By (2) and functoriality of F and G the left hand side is equal to

$$\sum_{i=1}^n v_i \left((F^0)^{-1} \circ F(g) \circ F(f_i)\right)(F^0(1_{\Bbbk})), \tag{3.1.32}$$

but this is zero, as already $\sum_{i=1}^n v_i\, F(f_i)(F^0(1_{\Bbbk})) = \pi_A^F(v) = 0$. Hence φ_A is well defined and satisfies (3.1.28). The proof of naturality is now the same as in [TV2, Lem. 17.2]. □

For the actual application to TFT functors we will use the following corollary, which includes a slightly stronger uniqueness statement as commutativity of (3.1.28) will be implied by monoidality.

Corollary 3.1.11. If $\mathcal{A}$ is monoidal, $\mathbb{1} \in \mathcal{A}$ is the monoidal unit, and F, G are monoidal functors satisfying the conditions in Lemma 3.1.10, then there exists a unique monoidal natural isomorphism $F \Rightarrow G$.

Proof. The natural isomorphism is the one constructed in Lemma 3.1.10 with F^0 and G^0 coming from the monoidal structure of F and G, respectively. The proof of monoidality is the same as in [TV2, Lem. 17.2] as that part of the proof only uses (3.1.28). Uniqueness follows as in the proof of [CMRSS, Lem. 4.2]. □

Let us now consider the effect of orientation reversal. First note that $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\overline{\mathcal{C}}) = \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ as $\mathcal{C}$ and $\overline{\mathcal{C}}$ have the same underlying pivotal monoidal category. Consider the following symmetric monoidal functor

$$\begin{aligned} \overline{(-)} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) &\to \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \\ (\Sigma, P, \lambda) &\mapsto (-\Sigma, P, \lambda) \\ [(M, T, n)] &\mapsto [(-M, T, n)] \end{aligned} \quad . \tag{3.1.33}$$

Note that this functor is not the same as the duality functor $(-)^*$, in particular it does not change the direction of composition. By combining Remark 3.1.7 with Lemma 3.1.2, we can use Corollary 3.1.11 to get:

Corollary 3.1.12. There is a unique natural monoidal isomorphism[6]

$$\widehat{\mathrm{V}}_{\overline{\mathcal{C}}} \cong \widehat{\mathrm{V}}_{\mathcal{C}} \circ \overline{(-)}. \tag{3.1.34}$$

Next we turn to the effect of taking Deligne products. Let $\mathcal{C}$ and $\mathcal{D}$ be modular tensor categories. We define $\widehat{\mathrm{Bord}}_{3,2}^{\chi,\mathrm{fact}}(\mathcal{C}\boxtimes\mathcal{D})$ as the following symmetric monoidal subcategory of $\widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C}\boxtimes\mathcal{D})$:

- objects: (Σ, P, λ) such that the labels of P are of the form $X \boxtimes Y$ for some $X \in \mathcal{C}$ and $Y \in \mathcal{D}$;
- morphisms: $[(M, T, n)]$ such that all coupons of T are pure tensors under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(- \boxtimes -, - \boxtimes -) \cong \mathrm{Hom}_{\mathcal{C}}(-,-) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(-,-); \tag{3.1.35}$$

By forgetting either the label in $\mathcal{C}$ or in $\mathcal{D}$, one obtains symmetric monoidal functors $\widehat{\mathrm{Bord}}_{3,2}^{\chi,\mathrm{fact}}(\mathcal{C}\boxtimes\mathcal{D}) \to \widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C})$ and $\widehat{\mathrm{Bord}}_{3,2}^{\chi,\mathrm{fact}}(\mathcal{C}\boxtimes\mathcal{D}) \to \widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{D})$. Pulling back along these functors, it is straightforward to define

$$\widehat{\mathrm{V}}_{\mathcal{C}} \otimes_{\Bbbk} \widehat{\mathrm{V}}_{\mathcal{D}} \colon \widehat{\mathrm{Bord}}_{3,2}^{\chi,\mathrm{fact}}(\mathcal{C}\boxtimes\mathcal{D}) \to \mathrm{vect}_{\Bbbk}. \tag{3.1.36}$$

Combining Corollary 3.1.11 with Lemma 3.1.3 we conclude:

Corollary 3.1.13. Let $\widehat{\mathrm{V}}^{\mathrm{fact}}_{\mathcal{C}\boxtimes\mathcal{D}}$ be the restriction of $\widehat{\mathrm{V}}_{\mathcal{C}\boxtimes\mathcal{D}}$ to $\widehat{\mathrm{Bord}}_{3,2}^{\chi,\mathrm{fact}}(\mathcal{C}\boxtimes\mathcal{D})$. Then there is a unique monoidal natural isomorphism

$$\widehat{\mathrm{V}}^{\mathrm{fact}}_{\mathcal{C}\boxtimes\mathcal{D}} \cong \widehat{\mathrm{V}}_{\mathcal{C}} \otimes_{\Bbbk} \widehat{\mathrm{V}}_{\mathcal{D}}. \tag{3.1.37}$$

3.2 Defect TFTs

In this section we will review the notion of *defect TFTs* of [CMS; CRS1]. A 3-*dimensional defect TFT* is a symmetric monoidal functor

$$\mathrm{Z} \colon \mathrm{Bord}_{3,2}^{\mathrm{def}}(\mathbb{D}) \to \mathrm{vect}_{\Bbbk}, \tag{3.2.1}$$

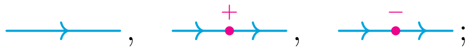
where the source category consists of stratified and labelled bordisms.

To make this precise, first recall that a n-dimensional *stratified* manifold M consists of an n-dimensional manifold together with a filtration $M = M_n \supset M_{n-1} \supset \cdots \supset M_0 \supset M_{-1} = \varnothing$ satisfying a number of technical conditions including that

[6] Recall that there is no extra condition for a monoidal natural transformation to preserve the braiding as this is automatic.

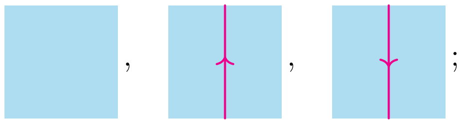
M_i/M_{i-1} is an i-dimensional submanifold of M for $0 \leqslant i \leqslant n$. The connected components of M_i/M_{i-1} will be called the *i-strata* of M. We will not make the other conditions explicit here as we will focus on a specific class of stratified manifolds and instead refer to [CMS, Sec. 2.1] for the full definition. The class of stratified manifolds we will work with is the one discussed in [CRS2, Sec. 2.2.1], called defect manifolds there, for which local neighbourhoods are defined inductively. Instead of repeating the full definition here we will instead roughly go through it up to dimension $n \leqslant 3$ as this will cover all cases of interest for us:

For $n = 1$ the induction starts and we have three possible local neighbourhoods: The open oriented interval $(-1, 1)$, as well as the interval $(-1, 1)$ with a single 0-stratum at 0 which is either positively or negatively oriented:

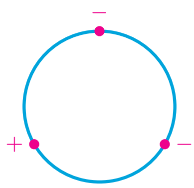


(3.2.2)

For $n = 2$ we have the first induction step leading us to consider two types of local neighbourhoods: The first type is given by cylinders over the one dimensional local neighbourhoods with the induced orientation:

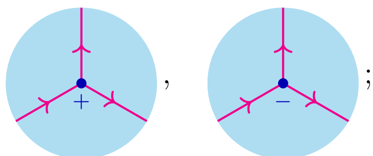

(3.2.3)

Note that the last two are equivalent since there is an orientation preserving diffeomorphism between them. The second type is given by cones over 1-dimensional stratified circles such as



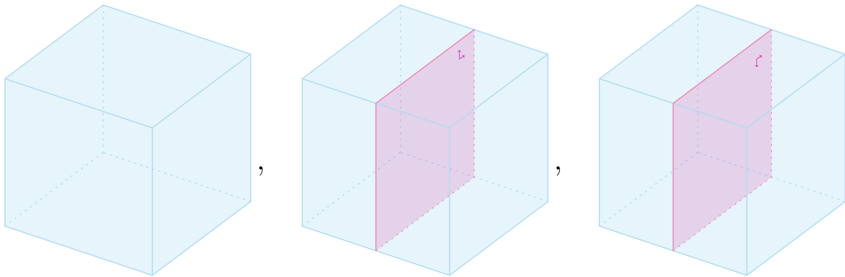
(3.2.4)

(with an arbitrary number of 0-strata) with induced orientation on the 1- and 2-strata and two possible orientations on the 0-stratum given by the cone point:



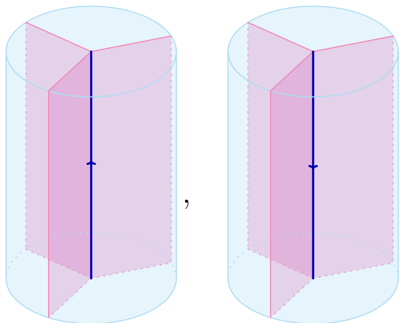
(3.2.5)

Finally, for $n = 3$ we repeat the previous step and again consider cylinders over the two dimensional local neighbourhoods as well as cones over stratified 2-spheres, with the induced orientation. The cylinders can further be distinguished as cylinders over cylinders over the one dimensional local neighbourhoods:



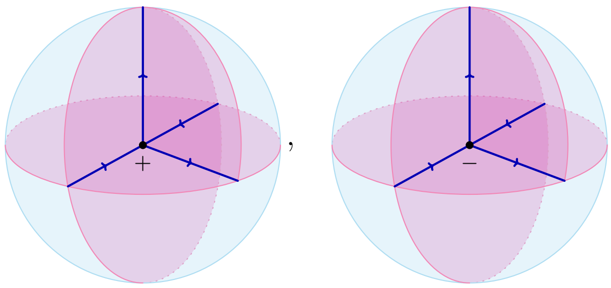
$$, \quad , \quad , \tag{3.2.6}$$

where the little coordinate systems in the second and third example illustrate the differing orientation on the 2-stratum.[7] As well as cylinders over the two dimensional cones:



$$, \quad ; \tag{3.2.7}$$

Finally, the last type of local neighbourhoods are cones over stratified 2-spheres with the induced orientation on the 1-, 2-, and 3-strata and two possible orientations on the 0-stratum given by the cone point:



$$, \quad ; \tag{3.2.8}$$

With the underlying topology out of the way let us now come to the labelling data: The labels for the strata are encoded in the so-called *defect data*

$$\mathbb{D} = (D_3, D_2, D_1; s, t, j). \tag{3.2.9}$$

Here, the D_i, for $i \in \{1, 2, 3\}$, are sets whose elements will label the i-dimensional strata of the 3-dimensional stratified manifolds while the source, target, and junction map $s, t\colon D_2 \times \{\pm\} \to D_3$ and $j\colon D_1 \times \{\pm\} \to$ (ordered lists of elements of D_2)

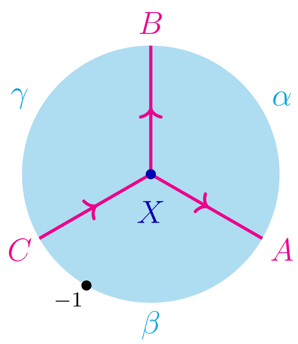
[7] To be a bit more precise we consider the orientations on the 2-strata such that the vertically draw arrows correspond to the second tangent direction.

encode the adjacency conditions for the labels including possible orientations, see [CMS; CRS2] for the detailed definition. Note that in contrast to [CMS; CRS2] we do not use a *cyclic* order for the junction map j but a linear order instead. This extra choice of order will be reflected in the labelling below.

To account for 0-strata we would need a fourth set D_0 together with an 'adjacency map' out of it to continue, see [CRS2, Def. 2.4]. However, it turns out that that for a given defect TFT, there is a canonical way to add this extra data by considering certain invariant vectors in the state spaces of defect spheres [CRS2, Sect. 2.4]. This process is referred to as "D_0-completion" and from now on we will always assume that our defect data $\mathbb{D}$ has been D_0-completed. Due to this we will not discuss the combinatorial description of labelling 0-strata in a 3-manifold.

Since we want to construct a bordism category we need to label stratified surfaces as well as stratified 3-manifolds with boundary. In particular, as explained above, the only local neighbourhoods we have to label are cylinders over 2-dimensional local neighbourhoods which is the same as labelling the 2-dimensional local neighbourhoods up to an index shift.

For a stratified surface we label every i-stratum for $i \in \{0, 1, 2\}$ with an element in the set D_{i+1}, such that the labels of the strata in its neighbourhood are compatible with the source, target, and junction map. Let us explain how this works in the example of the first 2-dimensional cone above:



(3.2.10)

The 2-strata inherit the orientation of the whole underlying manifold, which in this case we take to be the standard orientation of the unit disc in $\mathbb{R}^2$. The 0-stratum is positively oriented and we label it with $X \in D_1$, then the labels $A, B, C \in D_2$ of the three 1-strata need to satisfy $j(X, +) = ((C, -), (B, +), (A, +))$, where the signs are chosen in a way to indicate if the corresponding 1-stratum is oriented towards or away from the 0-stratum. The linear order of the list $((C, -), (B, +), (A, +))$ is obtained by going in clockwise direction along the stratified circle used to obtain the cone starting from the image of the south pole $-1 \in S^1$, illustrated as a black dot above. The labels need also be compatible with 2-strata in the form of $s(A, +) = t(C, -) = \beta$, $t(A, +) = s(B, +) = \alpha$, and $t(B, +) = s(C, -) = \gamma$. We want to note here that which side of the 1-stratum is the source and which one is the target is a convention, see [CMS, Sect. 2.3] for details.

To label bordisms between stratified and labelled surfaces we require the labels of the i-strata in the interior to match the $(i-1)$-strata on the boundary. The symmetric monoidal category $\mathrm{Bord}_{3,2}^{\mathrm{def}}(\mathbb{D})$ of 3-*dimensional defect bordisms* with defect data $\mathbb{D}$ now consists of stratified and labelled surfaces and stratified and equivalence classes of labelled bordisms between them, see [CRS2, Def. 2.4] for details. As in Section 3.1.2 it will be necessary to extend this bordism category to account for a possible gluing anomaly by adding Lagrangian subspaces and signature defects. We will denote the so obtained category by $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D})$.

Definition 3.2.1. A 3-dimensional defect TFT with defect data $\mathbb{D}$ is a symmetric monoidal functor

$$\mathrm{Z}\colon \mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}) \to \mathrm{vect}_{\Bbbk}. \tag{3.2.11}$$

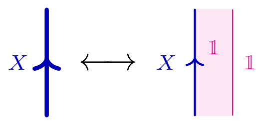
For a 3d defect TFT Z we will sometimes refer to the labelled i-dimensional strata for $i \in \{0,1,2\}$ as the *point, line*, and *surface defects* of Z and the labelled 3-strata as the bulk phases of Z.

Remark 3.2.2. To any 3d defect TFT Z with defect data $\mathbb{D}$ one can associate a certain type of 3-category $\mathcal{T}_{\mathrm{Z}}$. The objects of $\mathcal{T}_{\mathrm{Z}}$ are given by the elements of D_3, the 1-morphisms by (lists of) elements of D_2, the 2-morphisms by (lists of) elements of D_1, and the 3-morphisms from state spaces of defect spheres as in the D_0-completion mentioned above. The rest of the data is obtained from the source, target, and junction maps. For more details see [CMS, Sect. 3]. We will refer to $\mathcal{T}_{\mathrm{Z}}$ as the *defect* 3-category of Z

Let us now briefly describe how the 3d TFT with embedded ribbon graphs

$$\widehat{\mathrm{V}}_{\mathcal{C}}\colon \widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{3.2.12}$$

discussed in the previous section can be understood as a defect TFT in analogy to [CRS1, Rem. 5.9 (ii)] or [CMS, Sec. 4.2]. The defect data is given by $D_3 = \{*\}$, $D_2 = \{\mathbb{1}\}$, $D_1 = \mathrm{ob}(\mathcal{C})$, and $D_0 = \mathrm{Mor}(\mathcal{C})$. The source target and junction map are trivial in this case. However, the map one obtains from D_0-completion is encoded via the structural morphisms of $\mathcal{C}$. A bordism with embedded ribbon graph is turned into a bordism with defects by viewing an $X \in \mathcal{C}$ labelled ribbon as a $\mathbb{1}$-labelled surface defect bounded by an X-labelled line defect on one side and by a $\mathbb{1}$-labelled line defect on the other side:



(3.2.13)

The coupons can be included via point defects in a similar way. From now on we will often view $\widehat{\mathrm{V}}_{\mathcal{C}}$ as a defect TFT in this way without further comment. The corresponding 3-category $\mathcal{T}_{\widehat{\mathrm{V}}_{\mathcal{C}}}$ is $B^2\mathcal{C}$, i.e. the 3-category with one object, only the identity 1-morphism, and $\mathcal{C}$ as 2-endomorphism category [CMS, Prop. 4.4].

Chapter 4

Categorical structures underlying a full 2d CFT

In this chapter we are going to introduce the categorical structures underlying a full two-dimensional CFT. First, we will review the 2-categorical definitions of various versions of modular functors, corresponding to different source bordism 2-categories. Afterwards, we introduce the symmetric monoidal 2-category of *topological world sheets* by adding topological defects as well as boundary conditions to one of the bordism 2-categories. Finally, we are going to propose a 2-categorical definition of a full CFT and discuss why this definition encodes the data expected of a full CFT. For standard 2-categorical notions such as 2-functors we refer to [JY]. For details on symmetric monoidal 2-categories we refer to [SP, Ch. 2], see also [De1, App. D] for a concise review.

4.1 Modular functors

A modular functor is, colloquially speaking, a systematic assignment of mapping class group representations to surfaces which is compatible with gluing along boundaries. As mentioned in the introduction modular functors were introduced to axiomatise aspects of two-dimensional CFTs [Seg]. Consequently there are two main variants of modular functors: *chiral* modular functors and *full* modular functors corresponding to chiral and full CFTs, respectively. Both of these are defined on different oriented bordism 2-categories. Moreover, we will also briefly discuss modular functors on surfaces with involution which can be used to include unoriented bordisms. For our conventions on manifolds see the last paragraph in the introduction of chapter 3.

Before coming to the definitions modular functors we want to note that there are several distinct axiomatisations of modular functors used in the literature. For our purposes the 2-categorical version due to [FSY] will be the most natural.

4.1.1 Chiral modular functors

Let us start with chiral modular functors, which will be one of the main objects of interest in this thesis and are the modular functors most directly related to the 3d TFTs of the previous chapter. The symmetric monoidal 2-category $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ of two dimensional *chiral bordisms* consists of:

- objects: closed one-dimensional manifolds, i.e. finite disjoint unions of the unit circle S^1;
- 1-morphisms: for objects Γ and Γ', a 1-morphism from Γ to Γ', denoted as $\Gamma \to \Gamma'$, is a tuple (Σ, λ) where Σ is two-dimensional bordism[1] $\Sigma\colon \Gamma \to \Gamma'$ and $\lambda \subset H_1(\Sigma; \mathbb{R})$ is a Lagrangian subspace with respect to the intersection pairing;[2]
- 2-morphisms: for 1-morphisms $(\Sigma, \lambda)\colon \Gamma \to \Gamma'$ and $(\Sigma', \lambda')\colon \Gamma \to \Gamma'$, a 2-morphism $(\Sigma, \lambda) \Rightarrow (\Sigma', \lambda')$ is a tuple $([f], n)$ where $n \in \mathbb{Z}$, and $[f]$ is the isotopy class of a diffeomorphism $f\colon \Sigma \to \Sigma'$ which is compatible with the boundary parametrisations;
- horizontal composition: for 1-morphisms $(\Sigma, \lambda)\colon \Gamma \to \Gamma'$ and $(\Sigma', \lambda')\colon \Gamma' \to \Gamma''$ the horizontal composition $(\Sigma', \lambda') \diamond (\Sigma, \lambda)\colon \Gamma \to \Gamma''$ is given by
$$(\Sigma' \sqcup_{\Gamma'} \Sigma, \lambda' + \lambda), \tag{4.1.1}$$
where we view λ and λ' as subspaces of $H_1(\Sigma' \sqcup_{\Gamma'} \Sigma; \mathbb{R})$ via the inclusions $H_1(\Sigma; \mathbb{R}) \hookrightarrow H_1(\Sigma' \sqcup_{\Gamma'} \Sigma; \mathbb{R})$ and $H_1(\Sigma'; \mathbb{R}) \hookrightarrow H_1(\Sigma' \sqcup_{\Gamma'} \Sigma; \mathbb{R})$, respectively;
- identity 1-morphism: for an object Γ the identity 1-morphism $\mathrm{id}_\Gamma\colon \Gamma \to \Gamma$ is given by
$$(\Gamma \times I, \{0\}); \tag{4.1.2}$$
- vertical composition: for 2-morphisms $([f], n)\colon (\Sigma, \lambda) \Rightarrow (\Sigma', \lambda')$ and $([g], m)\colon (\Sigma', \lambda') \Rightarrow (\Sigma'', \lambda'')$, the vertical composition is given by the 2-morphism
$$([g \circ f], n + m - \mu((C_f)_*\lambda, \lambda', (C_g)^*\lambda''));\,[3] \tag{4.1.3}$$

[1]Note that we have an explicit bordism here, not its diffeomorphism class.
[2]The intersection pairing is only non-degenerate for $\Gamma = \Gamma' = \varnothing$.
[3]Here (C_f) and (C_g) denote the diffeomorphism class of the three dimensional mapping cylinder bordisms coming from $[f]$ and $[g]$, and the new signature defect is computed as in (3.1.10).

- identity 2-morphism: for a 1-morphism $(\Sigma, \lambda)\colon \Gamma \to \Gamma'$ the identity 2-morphism $\mathrm{id}_{(\Sigma,\lambda)}\colon (\Sigma, \lambda) \Rightarrow (\Sigma, \lambda)$ is given by

$$([\mathrm{id}], 0); \tag{4.1.4}$$

- disjoint union as symmetric monoidal structure;

There is more coherence data which needs to be specified such as the associativity 2-morphisms for horizontal composition, however we will skip these details, see for example [De1, App. B & C] as well as [FSY, Rem. 2.2] for more details. Alternatively, we could also define $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ as the subcategory of the bordism category defined in [De1, Sec. 2.2] with trivial colourings and only mapping cylinders as 2-morphisms.

The endomorphisms of a surface Σ form a central extension of the pure mapping class group of Σ by $\mathbb{Z}$, where the extension corresponds to the signature defects, see also [DGGPR2, Sec. 3.1]. Note here that we could define the 2-morphisms as diffeomorphism classes of mapping cylinders for diffeomorphisms instead. This is because elements in the isotopy class of f are in one to one correspondence to elements in the diffeomorphism class of the mapping cylinder C_f of f.

In the following we will often suppress the Lagrangian subspaces and signature defects when they are not directly relevant.

Definition 4.1.1. A *chiral modular functor* is a symmetric monoidal 2-functor

$$\mathrm{Bl}^{\chi}\colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk} \tag{4.1.5}$$

from the symmetric monoidal 2-category of two-dimensional chiral bordisms to the symmetric monoidal 2-category of left exact profunctors.

Let us briefly discuss which aspects of a chiral CFT are captured in this definition by explaining how one should be able to get a chiral modular functor $\mathrm{Bl}^{\chi}_{\mathcal{V}}$ from a suitable VOA $\mathcal{V}$.

Since every object in $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ is a finite disjoint union of circles and Bl^{χ} is symmetric monoidal it suffices to specify $\mathrm{Bl}^{\chi}_{\mathcal{V}}(S^1) := \mathrm{Rep}(\mathcal{V})$. This models the idea that fields of the chiral CFT should transform under the action of the VOA. Now for a bordism $\Sigma\colon \Gamma \to \Gamma'$ we want to assign a (left exact) functor $\mathrm{Bl}^{\chi}_{\mathcal{V}}(\Sigma)\colon \mathrm{Bl}^{\chi}_{\mathcal{V}}(\Gamma)^{\mathrm{op}} \boxtimes \mathrm{Bl}^{\chi}_{\mathcal{V}}(\Gamma') \to \mathrm{vect}_{\Bbbk}$. The idea is to obtain this functor by sending the product $\mathcal{V}$-modules to the vector space of conformal blocks $\mathrm{Bl}_{\mathcal{V}}(C)$ on a complex curve C with underlying surface Σ and with field insertions coming from the $\mathcal{V}$-modules. The 2-morphism level of $\mathrm{Bl}^{\chi}_{\mathcal{V}}$ arises to keep track of the complex structure via some form of the Riemann-Hilbert correspondence. Finally, the compatibility of $\mathrm{Bl}^{\chi}_{\mathcal{V}}$ with gluing of surfaces in the form of functoriality corresponds to the idea that the coend performs a "sum over intermediate states". For a more detailed exposition to these ideas we refer to [FS1; FSWY] and the introduction of [GZ] for more on the relation between gluing and coends. We will sometimes call the 1-morphism components of a chiral modular functor the *conformal block functors*.

4.1.2 Modular functors on surfaces with involution

Next we will extend the chiral modular functors from the previous section to a larger 2-category of bordisms with orientation-reversing involutions. To this end, we will discuss two functors, one in each direction, between the chiral bordism 2-category and the one with orientation reversing involutions.

The symmetric monoidal 2-category $\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ of two-dimensional *bordisms with involution* consists of:

- objects: are tuples (Γ,τ), where Γ is a closed one-dimensional manifold, and τ is an orientation reversing involution of Γ;
- 1-morphisms: for objects (Γ,τ) and (Γ',τ'), a 1-morphism from (Γ,τ) to (Γ',τ'), denoted as $(\Gamma,\tau)\to(\Gamma',\tau')$, is a tuple (Σ,σ), where Σ is a two-dimensional bordism $\Sigma\colon\Gamma\to\Gamma'$, and σ is an orientation reversing involution of Σ which restricts to τ (τ') on Γ (resp. Γ');
- 2-morphisms: for 1-morphisms $(\Sigma,\sigma)\colon(\Gamma,\tau)\to(\Gamma',\tau')$ and $(\Sigma,\sigma)\colon(\Gamma,\tau)\to(\Gamma',\tau')$ a 2-morphism $[f]\colon(\Sigma,\lambda)\Rightarrow(\Sigma',\lambda')$ is the isotopy class $[f]$ of a diffeomorphism $f\colon\Sigma\to\Sigma'$ which is compatible with the boundary parametrisations and commutes with the involutions;
- horizontal composition: for 1-morphisms $(\Sigma,\sigma)\colon(\Gamma,\tau)\to(\Gamma',\tau')$ and $(\Sigma',\sigma')\colon(\Gamma',\tau')\to(\Gamma'',\tau'')$, the horizontal composition $(\Sigma',\sigma')\diamond(\Sigma,\sigma)\colon(\Gamma,\tau)\to(\Gamma'',\tau'')$ is given by $(\Sigma'\sqcup_{\Gamma'}\Sigma,\sigma'\sqcup_{\Gamma'}\sigma)$;
- identity 1-morphism: for an object (Γ,τ) the identity 1-morphism $\mathrm{id}_{(\Gamma,\tau)}\colon(\Gamma,\tau)\to(\Gamma,\tau)$ is $(\Gamma\times I,\tau\times\mathrm{id}_I)$;

The rest of the structure is defined analogous to $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$, see Section 4.1.1 for details.

Definition 4.1.2. An *anomaly free modular functor* is a symmetric monoidal 2-functor

$$\mathrm{Bl}^{\circlearrowleft}\colon\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}\to\mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}\tag{4.1.6}$$

from the symmetric monoidal 2-category of two-dimensional bordisms with orientation reversing involution to the symmetric monoidal 2-category of left exact profunctors.

The name *anomaly free* can be justified as follows: There is a functor from $\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ to $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ which forgets most of the data contained in the involutions. To construct this functor let (Σ,σ) be a 1-morphism in $\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ and denote with λ_σ the eigenspace of the induced map $\sigma_*\colon H_1(\Sigma,\mathbb{R})\to H_1(\Sigma,\mathbb{R})$ for the eigenvalue -1. By [FFFS, Lem. 3.5] λ_σ is a Lagrangian subspace of $H_1(\Sigma,\mathbb{R})$.

Lemma 4.1.3. The assignment

$$\begin{aligned}\mathcal{U}\colon \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} &\to \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}\\ (\Gamma,\tau)&\mapsto \Gamma\\ (\Sigma,\sigma)&\mapsto(\Sigma,\lambda_\sigma)\\ [f]&\mapsto([f],0)\end{aligned}\tag{4.1.7}$$

defines a symmetric monoidal functor.

Proof. Functoriality for horizontal compositions as well as symmetric monoidality are clear by definition. With this particular choice of Lagrangian subspaces the choice of 0 as signature defect is preserved under composition of 2-morphisms by [FFFS, Prop. 3.6], which in turn guarantees the functoriality with respect to vertical composition. □

There is also a symmetric monoidal 2-functor in the converse direction called the *orientation double* functor, or just *double* functor constructed as follows:

- For an object $\Gamma\in\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ we define
$$\widehat{\Gamma}=\Gamma\sqcup-\Gamma.\tag{4.1.8}$$
This is a closed oriented manifold which naturally carries an orientation reversing involution $\tau_\Gamma\colon\widehat{\Gamma}\to\widehat{\Gamma}$.
- For a 1-morphism $(\Sigma,\lambda)\colon\Gamma\to\Gamma'$ we use the surface Σ to define
$$\widehat{\Sigma}=\Sigma\sqcup-\Sigma.\tag{4.1.9}$$
This is a two dimensional manifold with boundary $\partial\widehat{\Sigma}=\partial(\Sigma)\sqcup\partial(-\Sigma)$. Thus we can use the boundary parametrisation of $\Sigma\colon\Gamma\to\Gamma'$ to equip $\widehat{\Sigma}$ with the structure of a bordism from $\widehat{\Gamma}$ from $\widehat{\Gamma}'$. Moreover $\widehat{\Sigma}$ again comes with an orientation reversing involution $\sigma_\Sigma\colon\widehat{\Sigma}\to\widehat{\Sigma}$. It is easy to check that σ_Σ is compatible with the boundary parametrisations.
- For a 2-morphism $F=([f],n)\colon(\Sigma,\lambda)\to(\Sigma',\lambda')$ we set $\widehat{F}=[f\sqcup f]$.

The following lemma is clear:

Lemma 4.1.4. The assignment

$$\begin{aligned} \widehat{(-)} \colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} &\to \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \\ \Gamma &\mapsto (\widehat{\Gamma}, \tau_\Gamma) \\ (\Sigma, \lambda) &\mapsto (\widehat{\Sigma}, \sigma_\Sigma) \\ ([f], n) &\mapsto \widehat{F} \end{aligned} \tag{4.1.10}$$

defines a symmetric monoidal 2-functor

4.1.3 Full modular functors

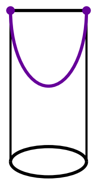
The last notion of modular functors we will discuss are so-called *full modular functors* of [FSY, Sec. 2.1]. The source category is the symmetric monoidal 2-category $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ of two dimensional oriented *open-closed bordisms*. This 2-category is defined as a kind of categorification of the category of open-closed bordisms defined in [LP, Sec. 3], see [BCR] for a detailed account. The basic idea is to also allow compact one dimensional manifolds with boundaries as objects, consequentially bordisms between them need to have an underlying manifold with corners, more precisely a $\langle 2\rangle$-manifold. Recall here that a two dimensional $\langle 2\rangle$-manifold Σ is a 2-dimensional compact manifold with corners together with a decomposition $\partial\Sigma = \partial^{\mathrm{g}}\Sigma \cup \partial^{\mathrm{f}}\Sigma$ of its boundary into a *gluing* boundary $\partial^{\mathrm{g}}\Sigma$ and a *free* boundary $\partial^{\mathrm{f}}\Sigma$ such that $\partial^{\mathrm{g}}\Sigma \cap \partial^{\mathrm{f}}\Sigma$ consists of the corner points of Σ.[4] We define $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ as follows:

- objects: compact one-dimensional manifolds, i.e. finite disjoint unions of the standard interval $I = [0, 1]$ and the unit circle S^1;
- 1-morphisms: for objects Γ and Γ', a 1-morphism from Γ to Γ', denoted as $\Gamma \to \Gamma'$, is a two-dimensional open-closed bordism $\Sigma \colon \Gamma \to \Gamma'$, i.e. a $\langle 2\rangle$-manifold together with a parametrisation of its gluing boundary $\partial^{\mathrm{gl}}\Sigma \cong -\Gamma \sqcup \Gamma'$;
- 2-morphisms: for 1-morphisms $\Sigma \colon \Gamma \to \Gamma'$ and $\Sigma' \colon \Gamma \to \Gamma'$ a 2-morphism $[f] \colon \Sigma \Rightarrow \Sigma'$ is the isotopy class of a diffeomorphism $f \colon \Sigma \to \Sigma'$ of $\langle 2\rangle$-manifolds which is compatible with the boundary parametrisations.

The rest of the structure is defined again analogously to $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$, see Section 4.1.1 for details.

[4] The gluing and free boundaries are called black and coloured boundaries, respectively, in [LP, Sec. 3].

An example of a 1-morphism W from S^1 to I is given by the so-called whistle bordism



where the purple line indicates the free boundary, while the thick black ones correspond to the gluing boundary. Later on the free boundaries will carry labels corresponding to different boundary conditions for the CFT.

Definition 4.1.5. A *full modular functor* is a symmetric monoidal 2-functor

$$\mathrm{Bl}^{\mathrm{oc}} \colon \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk} \tag{4.1.11}$$

from the symmetric monoidal 2-category of two-dimensional open-closed bordisms to the symmetric monoidal 2-category of left exact profunctors.

Finally, let us discuss the connection between $\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ and $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$. For this we first extend the orientation double functor to $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ as follows:

- For an object $\Gamma \in \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ we define

  $$\widehat{\Gamma} = \Gamma \sqcup -\Gamma/\sim \quad \text{with} \quad (p,+) \sim (p,-) \quad \text{for} \quad p \in \partial\Gamma. \tag{4.1.12}$$

  This is a closed oriented manifold which naturally carries an orientation reversing involution $\tau_\Gamma \colon \widehat{\Gamma} \to \widehat{\Gamma}$ coming from the orientation reversing involution on $\Gamma \sqcup -\Gamma$.

- For a 1-morphism $\Sigma \colon \Gamma \to \Gamma'$ we define

  $$\widehat{\Sigma} = \Sigma \sqcup -\Sigma/\sim \quad \text{with} \quad (p,+) \sim (p,-) \quad \text{for} \quad p \in \partial^f\Sigma. \tag{4.1.13}$$

  This is a two dimensional manifold with boundary

  $$\begin{aligned} \partial\widehat{\Sigma} &= \partial^{\mathrm{g}}(\Sigma) \sqcup \partial^{\mathrm{g}}(-\Sigma)/\sim \\ &\cong (-\Gamma \sqcup \Gamma') \sqcup -(-\Gamma \sqcup \Gamma')/\sim \\ &\cong -\widehat{\Gamma} \sqcup \widehat{\Gamma}'. \end{aligned} \tag{4.1.14}$$

  Thus we can use the boundary parametrisation of $\Sigma \colon \Gamma \to \Gamma'$ to equip $\widehat{\Sigma}$ with the structure of a bordism from $\widehat{\Gamma}$ to $\widehat{\Gamma}'$. Moreover $\widehat{\Sigma}$ again comes with an orientation reversing involution $\sigma_\Sigma \colon \widehat{\Sigma} \to \widehat{\Sigma}$. It is easy to check that σ_Σ is compatible with the boundary parametrisations.

- For a 2-morphism $[f]\colon \Sigma \to \Sigma'$ we can extend the underlying diffeomorphism f to a homeomorphism $\widehat{f}\colon \widehat{\Sigma} \to \widehat{\Sigma'}$ by the universal property of the quotient. Moreover $\widehat{f}$ commutes with the induced involutions by definition. Since the smooth and the topological mapping class group are isomorphic [FM, Sec. 2.1] it doesn't matter which smooth extension we choose and we can set $\widehat{[f]} := [\widehat{f}]$.

Lemma 4.1.6. The assignment

$$\begin{aligned} \widehat{(-)}\colon \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} &\to \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \\ \Gamma &\mapsto (\widehat{\Gamma}, \tau_\Gamma) \\ \Sigma &\mapsto (\widehat{\Sigma}, \sigma_\Sigma) \\ [f] &\mapsto [\widehat{f}\,] \end{aligned} \tag{4.1.15}$$

defines a symmetric monoidal 2-functor.

We employ the same notation for the orientation double functor here as in the previous section since there is a commutative diagram

$$\begin{tikzcd} \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \arrow[rr, "\widehat{(-)}"] \arrow[dr] & & \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \\ & \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \arrow[ur, "\widehat{(-)}"'] & \end{tikzcd} \tag{4.1.16}$$

where $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ forgets the Lagrangian subspace and signature defect.

Remark 4.1.7. Note that $\widehat{(-)}\colon \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ is not locally essentially surjective as it only gives surfaces whose quotient by the involution is again orientable, e.g. S^2 with antipodal involution is not in its essential image as the quotient space is the crosscap. This also explains in which sense $\mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ contains unorientable bordisms.

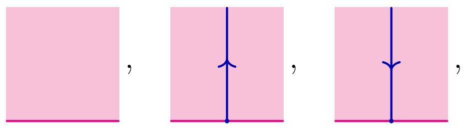
4.2 Topological world sheets

In principle we could now define the notion of a full CFT based on a modular functor. However, in order to include boundary conditions as well as topological defects directly, we will first define *topological world sheets* and their corresponding symmetric monoidal 2-category. The basic idea is to enhance the 2-category of open-closed bordisms from the previous section to a 2-category of open-closed *defect* bordisms as we did in the Section 3.2 with the ordinary bordism category.

Let us start with a description of the relevant topology. As in Section 3.2 we will fix a class of stratified, open-closed surfaces by describing the allowed local neighbourhoods. We will call these *unlabelled topological world sheets*, or often also just *topological world sheets*.

In the interior, of a topological world sheet, we allow for the 2-dimensional local neighbourhoods (3.2.3) and (3.2.5) discussed in Section 3.2. Near the boundary components, we will need different local models, depending on the type of boundary.

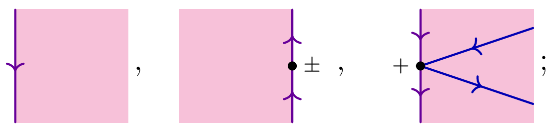
Firstly, near the gluing boundary we allow for:



(4.2.1)

where the boundary 1-strata, indicated with a darker colouring, inherit the orientation of the neighbouring 2-strata in the interior. The 0-stratum also inherits the orientation of the interior 1-stratum ending on it. Moreover, a 0-stratum, on a gluing boundary, is only allowed as the endpoint of exactly one 1-stratum in the interior.

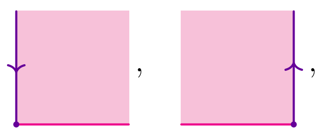
For free boundaries we remove the second restriction and allow for an arbitrary (finite) number of 1-strata ending on a 0-stratum:



(4.2.2)

It is useful to think of these neighbourhoods as a special case of the allowed 2-dimensional neighbourhoods in the interior, where one 2-stratum is "trivial" and the adjacent 1-strata are part of the free boundary.[5]

Finally, near the corner point we allow for the following two possibilities:



(4.2.3)

where the 2-stratum has the standard orientation of $\mathbb{R}^2$.

A *morphism between (unlabelled) topological world sheets* is a continuous map between open-closed surfaces which sends j-strata to j-strata in a smooth and orientation preserving manner.

[5] In principle we could also allow for an independent orientation on the free boundary 1-strata, making this analogy even more apparent. However, in this case the the labelling, discussed below, becomes a bit more involved.

Remark 4.2.1. This definition of unlabelled world sheet is closely related, but differs slightly from the one given in [FSY, Def. 2.4]. They only consider stratifications with at least one 0-stratum for every boundary component. This is extra 0-stratum seems to be necessary to make the ambient world sheet from [FSY, Sec. 2.4] well-defined.

Next, let us discuss the labelling data. As in Section 3.2, this will be encoded in what we will call 2-*dimensional defect data*

$$\mathbb{D}^{2\mathrm{d}} = (D_2^{2\mathrm{d}}, D_1^{2\mathrm{d}}, D_0^{2\mathrm{d}}; s^{2\mathrm{d}}, t^{2\mathrm{d}}, j^{2\mathrm{d}}). \tag{4.2.4}$$

As before, the elements of the sets $D_i^{2\mathrm{d}}$, for $i \in \{0, 1, 2\}$, will be used to label the i-strata while the source, target, and junction map $s^{2\mathrm{d}}, t^{2\mathrm{d}} \colon D_1^{2\mathrm{d}} \times \{\pm\} \to D_2^{2\mathrm{d}}$ and $j^{2\mathrm{d}} \colon D_0 \times \{\pm\} \to$ (ordered lists of elements of $D_1^{2\mathrm{d}}$) encode the adjacency conditions, see [DKR1, Sec. 2.3] or [CRS2, Ex. 2.5]. In order to include the labels for the free boundaries, i.e. the boundary conditions, we assume that the set D_2 contains a special element T, which we think of as the "trivial phase", and label free boundaries with the elements in $D_1^{2\mathrm{d}}$ which have T as their source. We will call T the *transparent* element in $D_2^{2\mathrm{d}}$.

Remarks 4.2.2.

1. Alternatively, we also could have introduced an extra set of boundary conditions $D_{1,\partial}^{2\mathrm{d}}$ and boundary point defects $D_{0,\partial}^{2\mathrm{d}}$ together with a target $t_\partial^{2\mathrm{d}}$ and junction map $j_\partial^{2\mathrm{d}}$. To see how this extra data is already encoded in the above, note that we can simply set $D_{1,\partial}^{2\mathrm{d}} := s^{-1}((T, +)) \subset D_1^{2\mathrm{d}}$. Moreover, this process is reversible by taking the union of the corresponding sets for 1- and 0-dimensional objects and adding an extra element to $D_2^{2\mathrm{d}}$.

2. Analogously to Remark 3.2.2, one can construct a 2-category from the defect data $\mathbb{D}^{2\mathrm{d}}$ with objects the elements in $D_2^{2\mathrm{d}}$, the 1-morphisms (lists) of elements in $D_1^{2\mathrm{d}}$, and 2-morphisms (lists) of elements in $D_0^{2\mathrm{d}}$, see [DKR1] for details. The transparent element T plays the role of a distinguished object in this 2-category similarly to a monoidal unit.

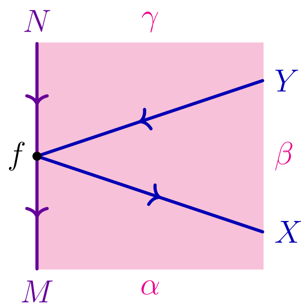
Definition 4.2.3. Let $\mathbb{D}^{2\mathrm{d}}$ be a set of 2-dimensional defect data. A $\mathbb{D}^{2\mathrm{d}}$-labelled world sheet $\mathfrak{S}$ consists of an unlabelled world sheet $\check{\mathfrak{S}}$ together with the following assignments of labels to the strata of $\check{\mathfrak{S}}$:

- to any 2-stratum an element in $D_2^{2\mathrm{d}}$ which we will call its *phase*;
- to any 1-stratum in the interior of $\check{\mathfrak{S}}$ an element in $D_1^{2\mathrm{d}}$ compatible with its source and target map;
- to any gluing boundary 1-stratum the element in $D_2^{2\mathrm{d}}$ labelling the neighbouring 2-stratum in the interior;

- to any gluing 0-stratum the element in D_1^{2d} which labels the interior 1-stratum ending on it;
- to any free boundary 1-stratum an element in D_1^{2d} such that its source is the transparent element $T \in D_2^{\mathrm{2d}}$ and its target matches the neighbouring 2-stratum in the interior;
- to any free boundary 0-stratum an element in D_0^{2d} compatible with the labels of the surrounding strata as in Section 3.2;
- to any corner point the same element in D_1^{2d} as the unique free boundary 1-stratum in its neighbourhood;

A morphism of $\mathbb{D}^{\mathrm{2d}}$-labelled world sheets $\mathfrak{S}$ and $\mathfrak{S}'$ is a morphism of unlabelled world sheets $\check{\mathfrak{S}}$ and $\check{\mathfrak{S}}'$ compatible with the labelling maps.

An example for an allowed labelling for the third local neighbourhood in (4.2.2) is given by:



(4.2.5)

Here $\alpha, \beta, \gamma \in D_2^{\mathrm{2d}}$, $M, N, X, Y \in D_1^{\mathrm{2d}}$, and $f \in D_0^{\mathrm{2d}}$ are such that $j^{\mathrm{2d}}(f,+) = ((M,+),(X,+),(Y,-),(N,-))$ and $t(M,+) = s(X,+) = \alpha$, $t(X,+) = s(Y,-) = \beta$, $t(Y,-) = s(N,-) = \gamma$, and $t(N,-) = s(M,+) = T$. The linear order of the list $j^{\mathrm{2d}}(f,+) = ((N,-),(Y,-),(X,+),(M,+))$ is chosen as always starting from the free boundary 1-stratum which goes out of the 0-stratum, labelled by M in this case, and then going counter-clockwise. This choice is always possible because every free boundary 0-stratum will have exactly one incoming and one outgoing free boundary 1-stratum.

By an analogous index shift as in the definition of $\mathrm{Bord}_{3,2}^{\mathrm{def}}(\mathbb{D})$ we can also decorate compact stratified 1-manifolds using the 2d defect data $\mathbb{D}^{\mathrm{2d}}$. Putting this together leads to the following bordism 2-category:

Definition 4.2.4. Let $\mathbb{D}^{\mathrm{2d}}$ be a set of 2-dimensional defect data as above. The 2-category of $\mathbb{D}^{\mathrm{2d}}$-labelled *topological world sheets* $\mathfrak{WS}(\mathbb{D}^{\mathrm{2d}})$ is the symmetric monoidal 2-category consisting of:

- objects: $\mathbb{D}^{\mathrm{2d}}$-decorated compact stratified 1-manifolds;

- 1-morphisms: for objects $\mathfrak{C}$ and $\mathfrak{C}'$, a 1-morphism from $\mathfrak{C}$ to $\mathfrak{C}'$, denoted as $\mathfrak{C} \to \mathfrak{C}'$, is a topological world sheet $\mathfrak{S}$, together with a parametrisation of its gluing boundary $\partial^{\mathrm{gl}}\mathfrak{S} \cong -\mathfrak{C} \sqcup \mathfrak{C}'$ as defect 1-manifolds with boundary;

- 2-morphisms: for 1-morphisms $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}'$ and $\mathfrak{S}'\colon \mathfrak{C} \to \mathfrak{C}'$ a 2-morphism $[f]\colon \mathfrak{S} \Rightarrow \mathfrak{S}'$ is the isotopy class of an isomorphism $f\colon \mathfrak{S} \to \mathfrak{S}'$ of topological world sheets which is compatible with the boundary parametrisations;

- horizontal composition: for 1-morphisms $\mathfrak{S}_1\colon \mathfrak{C}_1 \to \mathfrak{C}$ and $\mathfrak{S}_2\colon \mathfrak{C} \to \mathfrak{C}_2$ the horizontal composition $\mathfrak{S}_2 \diamond \mathfrak{S}_1\colon \mathfrak{C}_1 \to \mathfrak{C}_2$ is given by the topological world sheet
$$\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1 \tag{4.2.6}$$
obtained by gluing along $\mathfrak{C}$ using the parametrisation of the gluing boundaries of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively.

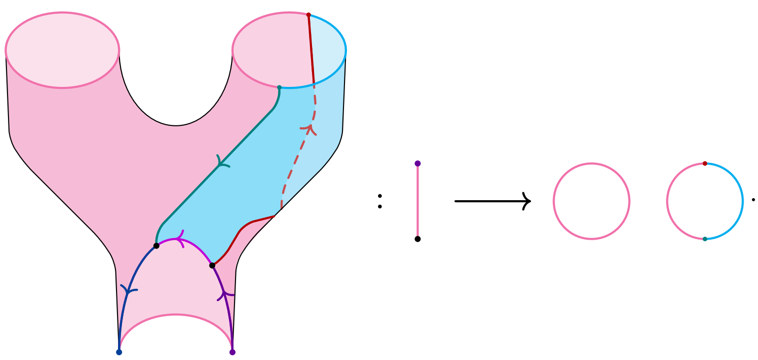
- identity 1-morphism: for an object $\mathfrak{C}$ the identity 1-morphism $\mathrm{id}_{\mathfrak{C}}\colon \mathfrak{C} \to \mathfrak{C}$ is
$$\mathfrak{C} \times I \tag{4.2.7}$$
with stratification and decoration induced from $\mathfrak{C}$;

- vertical composition: for 2-morphisms $[f]\colon \mathfrak{S} \Rightarrow \mathfrak{S}'$ and $[g]\colon \mathfrak{S}' \Rightarrow \mathfrak{S}''$, the vertical composition is given by the 2-morphism
$$[g \circ f]; \tag{4.2.8}$$

- identity 2-morphism: for a 1-morphism $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}'$ the identity 2-morphism $\mathrm{id}_{\mathfrak{S}}\colon \mathfrak{S} \Rightarrow \mathfrak{S}$ is
$$[\mathrm{id}]; \tag{4.2.9}$$

- disjoint union as symmetric monoidal structure;

An example of a 1-morphism from an interval to the disjoint union of two defect circles is given by



(4.2.10)

where we used different colours as an indication of different labels.

Forgetting the labels and stratification gives a symmetric monoidal 2-functor

$$U\colon \mathfrak{WS}(\mathbb{D}^{2\mathrm{d}}) \to \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \tag{4.2.11}$$

which sends a defect manifold to its underlying manifold. In the following we will denote the image under this functor simply by using the corresponding capital Greek letter, i.e. $U(\mathfrak{C}) \equiv \mathit{\Gamma}$ and $U(\mathfrak{S}) \equiv \mathit{\Sigma}$. Note that for a world sheet $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}'$ its automorphism group $\mathrm{Aut}(\mathfrak{S}) = \mathrm{End}_{\mathfrak{WS}(\mathfrak{C},\mathfrak{C}')}(\mathfrak{S})$ is by definition the subgroup of the mapping class group of $\mathit{\Sigma}$ which fixes the defects. This coincides with the notion of the mapping class group of a world sheet from [FSY, Def. 2.17] as a morphism between stratified manifolds sending strata to strata.

4.3 Correlators and field content

With this we can finally define the second main object of interest of this thesis: a full conformal field theory.

Definition 4.3.1. Let $\mathrm{Bl}^{\chi}\colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$ be a chiral modular functor and let $\mathbb{D}^{2\mathrm{d}}$ be a set of 2-dimensional defect data. Denote with $\mathrm{Bl}\colon \mathfrak{WS}(\mathbb{D}^{2\mathrm{d}}) \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$ the composition

$$\mathfrak{WS}(\mathbb{D}^{2\mathrm{d}}) \xrightarrow{U} \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \xrightarrow{\widehat{(-)}} \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \xrightarrow{\mathrm{Bl}^{\chi}} \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}. \tag{4.3.1}$$

A *full conformal field theory*, with chiral data governed by Bl^{χ} and boundary conditions and topological defects encoded in $\mathbb{D}^{2\mathrm{d}}$, is a braided monoidal oplax natural transformation

$$\begin{tikzcd} \mathfrak{WS}(\mathbb{D}^{2\mathrm{d}}) \arrow[r, bend left, "\Delta_{\Bbbk}"{name=U}] \arrow[r, bend right, "\mathrm{Bl}"'{name=D}] & \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk} \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{Cor}"] \end{tikzcd}$$

where $\Delta_{\Bbbk}\colon \mathfrak{WS}(\mathbb{D}^{2\mathrm{d}}) \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$ is the constant symmetric monoidal 2-functor sending every object to $\mathrm{vect}_{\Bbbk}$.

The rest of this section will be devoted to unwrapping this definition. An oplax natural transformation Cor as above consists of the following data [JY, Def. 4.3.1]:

- 1-morphism components: A left exact profunctor $\mathrm{Cor}_{\mathfrak{C}}\colon \Delta_{\Bbbk}(\mathfrak{C}) \nrightarrow \mathrm{Bl}(\mathfrak{C})$;

- 2-morphism components: A natural transformation

$$\begin{tikzcd}
\Delta_{\Bbbk}(\mathfrak{C}) \arrow[r, "\Delta_{\Bbbk}(\mathfrak{S})"] \arrow[d, "\mathrm{Cor}_{\mathfrak{C}}"'] & \Delta_{\Bbbk}(\mathfrak{C}') \arrow[d, "\mathrm{Cor}_{\mathfrak{C}'}"] \arrow[dl, Rightarrow, "\mathrm{Cor}_{\mathfrak{S}}"] \\
\mathrm{Bl}(\mathfrak{C}) \arrow[r, "\mathrm{Bl}(\mathfrak{S})"'] & \mathrm{Bl}(\mathfrak{C}')
\end{tikzcd}$$

for every world sheet $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}'$ in $\mathfrak{WS}(\mathbb{D}^{2\mathrm{d}})$;

A braided monoidal oplax natural transformation further includes [SP, Def. 2.7]:

- A modification Π with components for any $\mathfrak{C}, \mathfrak{C}' \in \mathfrak{WS}(\mathbb{D}^{2\mathrm{d}})$ given by natural isomorphisms

$$\begin{tikzcd}
 & \mathrm{Bl}(\mathfrak{C}) \boxtimes \Delta_{\Bbbk}(\mathfrak{C}') \arrow[dr, "\mathrm{id}_{\mathrm{Bl}(\mathfrak{C})} \boxtimes \mathrm{Cor}_{\mathfrak{C}'}"] & \\
\Delta_{\Bbbk}(\mathfrak{C}) \boxtimes \Delta_{\Bbbk}(\mathfrak{C}') \arrow[ur, "\mathrm{Cor}_{\mathfrak{C}} \boxtimes \mathrm{id}_{\Delta_{\Bbbk}(\mathfrak{C}')}"] \arrow[d, "\cong"'] & & \mathrm{Bl}(\mathfrak{C}) \boxtimes \mathrm{Bl}(\mathfrak{C}') \arrow[d, "\cong"] \arrow[dll, Rightarrow, "\Pi_{\mathfrak{C},\mathfrak{C}'}"] \\
\Delta_{\Bbbk}(\mathfrak{C} \sqcup \mathfrak{C}') \arrow[rr, "\mathrm{Cor}_{\mathfrak{C} \sqcup \mathfrak{C}'}"'] & & \mathrm{Bl}(\mathfrak{C} \sqcup \mathfrak{C}')
\end{tikzcd}$$

where the unlabelled isomorphism are part of the symmetric monoidal structure of $\Delta_{\Bbbk}$ and Bl;

- A natural isomorphism

$$\begin{tikzcd}
\mathrm{vect}_{\Bbbk} \arrow[r, "\cong"] \arrow[d, equal] & \Delta_{\Bbbk}(\varnothing) \arrow[d, "\mathrm{Cor}_{\varnothing}"] \arrow[dl, Rightarrow, "\Upsilon"] \\
\mathrm{vect}_{\Bbbk} \arrow[r, "\cong"'] & \mathrm{Bl}(\varnothing)
\end{tikzcd}$$

where the unlabelled isomorphism are part of the symmetric monoidal structure of $\Delta_{\Bbbk}$ and Bl;

This data needs to satisfy the following axioms:

(i) Naturality of 2-morphism components: For any 2-morphism $[f]\colon \mathfrak{S} \Rightarrow \mathfrak{S}'$ in $\mathfrak{W}\mathfrak{S}(\mathbb{D}^{2\mathrm{d}})$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cor}_{\mathfrak{C}'} \diamond \Delta_{\Bbbk}(\mathfrak{S}) \arrow[r, "\mathrm{Cor}_{\mathfrak{S}}"] \arrow[d, "\mathrm{Cor}_{\mathfrak{C}'} \diamond \Delta_{\Bbbk}([f])"'] & \mathrm{Bl}(\mathfrak{S}) \diamond \mathrm{Cor}_{\mathfrak{C}} \arrow[d, "\mathrm{Bl}([f]) \diamond \mathrm{Cor}_{\mathfrak{C}}"] \\
\mathrm{Cor}_{\mathfrak{C}'} \diamond \Delta_{\Bbbk}(\mathfrak{S}) \arrow[r, "\mathrm{Cor}_{\mathfrak{S}}"'] & \mathrm{Bl}(\mathfrak{S}) \diamond \mathrm{Cor}_{\mathfrak{C}}
\end{tikzcd}$$

(ii) Oplax naturality: For any pair of composable 1-morphisms $\mathfrak{S}_1\colon \mathfrak{C}_1 \to \mathfrak{C}$ and $\mathfrak{S}_2\colon \mathfrak{C} \to \mathfrak{C}_2$ in $\mathfrak{W}\mathfrak{S}(\mathbb{D}^{2\mathrm{d}})$ the following diagram commutes:

$$\begin{tikzcd}[column sep=tiny]
& (\mathrm{Bl}(\mathfrak{S}_2) \diamond \mathrm{Cor}_{\mathfrak{C}}) \diamond \Delta_{\Bbbk}(\mathfrak{S}_1) \arrow[dr, "\cong"] & \\
(\mathrm{Cor}_{\mathfrak{C}_2} \diamond \Delta_{\Bbbk}(\mathfrak{S}_2)) \diamond \Delta_{\Bbbk}(\mathfrak{S}_1) \arrow[ur, "\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{id}"] \arrow[d, "\cong"'] & & \mathrm{Bl}(\mathfrak{S}_2) \diamond (\mathrm{Cor}_{\mathfrak{C}} \diamond \Delta_{\Bbbk}(\mathfrak{S}_1)) \arrow[d, "\mathrm{id} \diamond \mathrm{Cor}_{\mathfrak{S}_1}"] \\
\mathrm{Cor}_{\mathfrak{C}_2} \diamond (\Delta_{\Bbbk}(\mathfrak{S}_2) \diamond \Delta_{\Bbbk}(\mathfrak{S}_1)) \arrow[d, "\cong"'] & & \mathrm{Bl}(\mathfrak{S}_2) \diamond (\mathrm{Bl}(\mathfrak{S}_1) \diamond \mathrm{Cor}_{\mathfrak{C}_1}) \arrow[d, "\cong"] \\
\mathrm{Cor}_{\mathfrak{C}_2} \diamond (\Delta_{\Bbbk}(\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1)) \arrow[dr, "\mathrm{Cor}_{\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1}"'] & & (\mathrm{Bl}(\mathfrak{S}_2) \diamond \mathrm{Bl}(\mathfrak{S}_1)) \diamond \mathrm{Cor}_{\mathfrak{C}_1} \arrow[dl, "\cong"] \\
& (\mathrm{Bl}(\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1)) \diamond \mathrm{Cor}_{\mathfrak{C}_1} &
\end{tikzcd}$$

where the isomorphisms correspond to either the associators of horizontal composition in $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ or the 2-functor data of $\Delta_{\Bbbk}$ and Bl, respectively.

(iii) Oplax unitality:

$$\begin{tikzcd}
\mathrm{Cor}_{\mathfrak{C}} \diamond \mathrm{id}_{\Delta_{\Bbbk}(\mathfrak{C})} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Cor}_{\mathfrak{C}} \arrow[r, "\cong"] & \mathrm{id}_{\mathrm{Bl}(\mathfrak{C})} \diamond \mathrm{Cor}_{\mathfrak{C}} \arrow[d, "\cong"] \\
\mathrm{Cor}_{\mathfrak{C}} \diamond \Delta_{\Bbbk}(\mathrm{id}_{\mathfrak{C}}) \arrow[rr, "\mathrm{Cor}_{\mathrm{id}_{\mathfrak{C}}}"'] & & \mathrm{Bl}(\mathrm{id}_{\mathfrak{C}}) \diamond \mathrm{Cor}_{\mathfrak{C}}
\end{tikzcd}$$

where the isomorphisms correspond to either the unitors of horizontal composition in $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ or the 2-functor data of $\Delta_{\Bbbk}$ and Bl, respectively.

As well as four other axioms involving Π and Υ which we will not discuss further, see [SP, Def. 2.7].

Let us take a moment to see that this data actually corresponds to something which deserves to be called a full conformal field theory. As mentioned above we will denote the manifolds underlying a defect 1-manifold and a topological world sheet with the corresponding capital Greek letter. First, note that $\mathrm{Cor}_{\mathfrak{C}}$ is given by a (left exact) functor $\mathrm{Bl}(\mathfrak{C}) \to \mathrm{vect}_{\Bbbk}$ because $\Delta_{\Bbbk}(\mathfrak{C}) = \mathrm{vect}_{\Bbbk}$ for any $\mathfrak{C} \in \mathfrak{W}\mathfrak{S}(\mathbb{D}^{2\mathrm{d}})$ and $\mathrm{vect}_{\Bbbk}^{\mathrm{op}} \simeq \mathrm{vect}_{\Bbbk}$. Thus we get an essentially unique object $\mathbb{F}_{\mathfrak{C}} \in \mathrm{Bl}(\mathfrak{C}) = \mathrm{Bl}(\mathit{\Gamma})$ such that $\mathrm{Cor}_{\mathfrak{C}}(-) \cong \mathrm{Hom}_{\mathrm{Bl}(\mathit{\Gamma})}(\mathbb{F}_{\mathfrak{C}}, -)$ by left exactness of $\mathrm{Cor}_{\mathfrak{C}}$. Moreover, these objects factorise as $\mathbb{F}_{\mathfrak{C}\sqcup\mathfrak{C}'} \cong \mathbb{F}_{\mathfrak{C}} \boxtimes \mathbb{F}_{\mathfrak{C}'}$ using the monoidality provided by $\Pi_{\mathfrak{C},\mathfrak{C}'}$. Thus we can restrict our attention to $\mathfrak{C}$ being a defect interval or circle. As explained in Section 4.1.1 the category $\mathrm{Bl}(I) = \mathrm{Bl}^{\chi}(S^1)$ should correspond to the representation category of the chiral symmetry algebra while $\mathrm{Bl}(S^1) \simeq \mathrm{Bl}^{\chi}(S^1) \boxtimes \overline{\mathrm{Bl}^{\chi}(S^1)}$ corresponds to its double. This means that the objects $\mathbb{F}_{\mathfrak{C}}$ are automatically equipped with an action of the correct symmetry algebra. This observation suggests to interpret the 1-morphism component of Cor as the *field content* of the full CFT.

Now by combining the isomorphism $\mathrm{Cor}_{\mathfrak{C}}(-) \cong \mathrm{Hom}_{\mathrm{Bl}(\mathit{\Gamma})}(\mathbb{F}_{\mathfrak{C}}, -)$ with the Yoneda lemma twice we get the following isomorphism of vector spaces

$$\mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}'} \diamond \Delta_{\Bbbk}(\mathfrak{S}), \mathrm{Bl}(\mathfrak{S}) \diamond \mathrm{Cor}_{\mathfrak{C}}) \cong \mathrm{Bl}(\mathfrak{S})(\mathbb{F}_{\mathfrak{C}}; \mathbb{F}_{\mathfrak{C}'}) \tag{4.3.2}$$

for every world sheet $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}'$. Thus the 2-morphism component $\mathrm{Cor}_{\mathfrak{S}}$ is automatically a vector in the space of conformal blocks of $\mathfrak{S}$ with field insertions given by the field content. This suggests to think of the 2-morphism components of Cor as the actual correlators of the theory.

Before we explain how the axioms Cor needs to satisfy should be interpreted, it will be beneficial to consider the following chain of abstract isomorphisms

$$\begin{aligned}\mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}'} \diamond \Delta_{\Bbbk}(\mathfrak{S}), \mathrm{Bl}(\mathfrak{S}) \diamond \mathrm{Cor}_{\mathfrak{C}}) &\cong \mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}'}, \mathrm{Bl}(\mathfrak{S}) \diamond \mathrm{Cor}_{\mathfrak{C}}) \\ &\cong \mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}'} \diamond \mathrm{Cor}_{\mathfrak{C}}^{\dagger}, \mathrm{Bl}(\mathfrak{S})) \\ &\cong \mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}'} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger}, \mathrm{Bl}(\mathfrak{S}))\end{aligned} \tag{4.3.3}$$

where $\mathrm{Cor}_{\mathfrak{C}}^{\dagger}(-) \cong \mathrm{Hom}_{\mathrm{Bl}(\mathit{\Gamma})}(-, \mathbb{F}_{\mathfrak{C}})$ is the (right) adjoint of $\mathrm{Cor}_{\mathfrak{C}}$ in $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$ from Lemma 2.2.9. In the first step we used that $\Delta_{\Bbbk}$ acts trivially, in the second one the adjunction, and in the final one that horizontal composition over $\mathrm{vect}_{\Bbbk}$ is just the tensor product. From now on will suppress the isomorphisms (4.3.2) and (4.3.3) from our notation and will always denote $\mathrm{Cor}_{\mathfrak{S}}$ for the corresponding element in any of the three vector spaces as it will be clear from the context.

Under the isomorphism (4.3.3) the axioms Cor needs to satisfy become:

(i) Naturality of 2-morphism components reduces to commutativity of

$$\begin{tikzcd}
 & \mathrm{Bl}(\mathfrak{S}) \arrow[dd, "\mathrm{Bl}([f])"] \\
\mathrm{Cor}_{\mathfrak{C}'} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \arrow[ur, "\mathrm{Cor}_{\mathfrak{S}}"] \arrow[dr, "\mathrm{Cor}_{\mathfrak{S}'}"'] & \\
 & \mathrm{Bl}(\mathfrak{S}')
\end{tikzcd} \cdot \tag{4.3.4}$$

This is the *covariance of correlators* and can be seen as a more general version of mapping class group action invariance.

(ii) For simplicity we will only consider the oplax naturality axiom in a strictified version of $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}$, i.e. with trivial associators and unitors. In this setting we oplax naturality becomes commutativity of

$$\begin{tikzcd}
\mathrm{Cor}_{\mathfrak{C}_2} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \diamond \mathrm{Cor}_{\mathfrak{C}_1} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \arrow[r, "\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}"] & \mathrm{Bl}(\mathfrak{S}_2) \diamond \mathrm{Bl}(\mathfrak{S}_1) \arrow[d, "\cong"] \\
\mathrm{Cor}_{\mathfrak{C}_2} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}_1}^{\dagger} \arrow[u, "\mathrm{id} \otimes_{\Bbbk} \eta_{\mathrm{Cor}_{\mathfrak{C}}} \otimes_{\Bbbk} \mathrm{id}"] \arrow[r, "\mathrm{Cor}_{\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1}"'] & \mathrm{Bl}(\mathfrak{S}_2 \sqcup_{\mathfrak{C}} \mathfrak{S}_1)
\end{tikzcd} \tag{4.3.5}$$

where $\eta_{\mathrm{Cor}_{\mathfrak{C}}} \colon \mathrm{id}_{\mathrm{vect}_{\Bbbk}} \Rightarrow \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \diamond \mathrm{Cor}_{\mathfrak{C}}$ denotes the unit of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}_{\mathfrak{C}}^{\dagger}$. This encodes the behaviour of the 2-morphism components under gluing of world sheets and should be interpreted as the *factorisation of correlators.*

(iii) Under the same strictification assumption as above, the oplax unitality axioms corresponds to

$$\mathrm{Cor}_{\mathrm{id}_{\mathfrak{C}}} = \varepsilon_{\mathrm{Cor}_{\mathfrak{C}}} \tag{4.3.6}$$

with $\varepsilon_{\mathrm{Cor}_{\mathfrak{C}}} \colon \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \Rightarrow \mathrm{Bl}(\mathrm{id}_{\mathfrak{C}})$ the counit of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}_{\mathfrak{C}}^{\dagger}$. This can be interpreted as a non-degeneracy axiom for 2-point correlators.

The axioms involving Π essentially correspond to the statement that the correlator of the disjoint union of two world sheets should be the tensor product of the individual correlators. Finally, the axiom for Υ guarantees that the correlator for $\varnothing$, viewed as a world sheet $\varnothing \colon \varnothing \to \varnothing$ is trivial. These are precisely the consistency conditions one would expect from a consistent system of correlators.

In summary we see that the 1-morphism component of Cor corresponds to the field content while the 2-morphism component are the actual correlators of the full CFT.

Our definition of full CFT is closely related to the notions of twisted or relative field theories considered in [ST; FT; JFS]. More precisely a full CFT, as defined above, can be understood as an open-closed, defect variant of a not fully extended relative field theory.

Chapter 5

TFT construction of modular functors

In this chapter we will discuss how the non-semisimple 3d TFT of [DGGPR1] reviewed in Section 3.1 can be used to obtain the various versions of modular functors discussed in Section 4.1. In particular we will discuss our first main result: The TFT-construction of Lyubashenko's modular functor [Lyu2] as a chiral modular functor.

Throughout the rest of this thesis let $\mathcal{C}$ be a fixed modular tensor category and denote with

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{5.0.1}$$

the version of the TFT constructed from $\mathcal{C}$ with admissibility condition on bordism components disjoint from the outgoing boundary as in Section 3.1.

5.1 Chiral modular functors

We will now discuss the first main result of this thesis, Theorem 5.1.9: The construction of a chiral modular functor using the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$. We will call this the chiral modular functor of $\mathcal{C}$ and denote it by

$$\mathrm{Bl}^{\chi}_{\mathcal{C}} \colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}. \tag{5.1.1}$$

To this end we will first define the action of $\mathrm{Bl}^{\chi}_{\mathcal{C}}$ on objects, 1-morphisms, and 2-morphisms, in Section 5.1.1 and Section 5.1.2. We will then show that these assignments assemble into a symmetric monoidal 2-functor by studying the gluing of surfaces in Section 5.1.3.

We want to emphasise here that the following considerations are 2-categorical versions of the ones discussed in [BK, Ch. 5] if we restrict to the semisimple case.

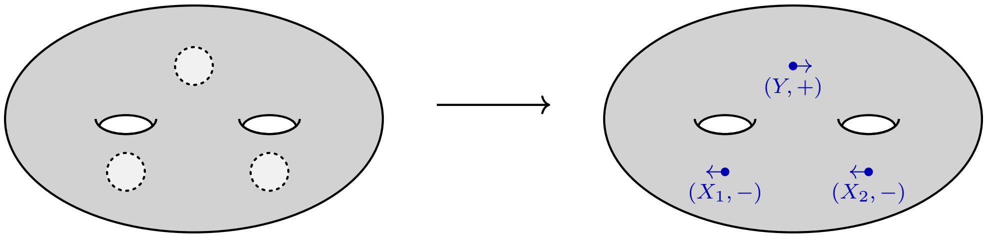


Figure 5.1: Turning a genus two surface with two incoming and one outgoing boundary circles $\Sigma_2^{(2,1)}$ into the object $\underline{\Sigma}_2^{(X_1,X_2;Y)}$.

On a more technical level the main difference to [BK, Ch. 5] lies in the treatment of gluing which is more subtle here due to the appearance of coends.

5.1.1 Block functors

Definition 5.1.1 (Chiral block functor on objects)**.** For an object $\Gamma \in \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ we define $\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Gamma)$ to be the Deligne product over its connected components

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Gamma) := \mathcal{C}^{\boxtimes \pi_0(\Gamma)}, \tag{5.1.2}$$

where we index the factors directly by the connected components $\pi_0(\Gamma)$ of Γ.

In particular $\mathrm{Bl}^{\chi}_{\mathcal{C}}(S^1) = \mathcal{C}$, and by convention the empty product is $\mathrm{vect}_{\Bbbk}$, so that $\mathrm{Bl}^{\chi}_{\mathcal{C}}(\varnothing) := \mathrm{vect}_{\Bbbk}$.

Next we construct the profunctors describing the value on 1-morphisms, the conformal block functors. Let $(\Sigma, \lambda)\colon \Gamma \to \Gamma'$ be a 1-morphism in $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$. We want to construct a left exact functor

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}\big((\Sigma,\lambda)\big)\colon (\mathcal{C}^{\mathrm{op}})^{\boxtimes \pi_0(\Gamma)} \boxtimes \mathcal{C}^{\boxtimes \pi_0(\Gamma')} \to \mathrm{vect}_{\Bbbk}. \tag{5.1.3}$$

First let us fix lists of objects $\underline{X} \in \mathcal{C}^{\times \pi_0(\Gamma)}$ and $\underline{Y} \in \mathcal{C}^{\times \pi_0(\Gamma')}$, where we again index the factors directly by the connected components. We want to turn (Σ, λ) into an object of $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ by gluing $|\pi_0(\Gamma)| + |\pi_0(\Gamma')|$ disks with specific marked points into the boundary of Σ. To this end, it suffices to consider $\Gamma = (S^1)^{\sqcup |\pi_0(\Gamma)|}$ and $\Gamma' = (S^1)^{\sqcup |\pi_0(\Gamma')|}$. This restriction amounts to working with a skeleton of $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$, the general setting can then be obtained from [SP, Lem. 2.22].

Let us now consider standard unit disks $D^2 \subset \mathbb{C}$ with the following marked points: For any incoming boundary component $\gamma \in \pi_0(\Gamma)$, we mark the origin of the disk with a *negatively oriented* point labelled with the corresponding element X_γ in $\underline{X}$. For any outgoing boundary component $\gamma' \in \pi_0(\Gamma')$, we mark the origin of the disk with a *positively oriented* point labelled with $Y_{\gamma'}$ in $\underline{Y}$. As tangent vectors at these points we use the unit vector along the positive real axis for

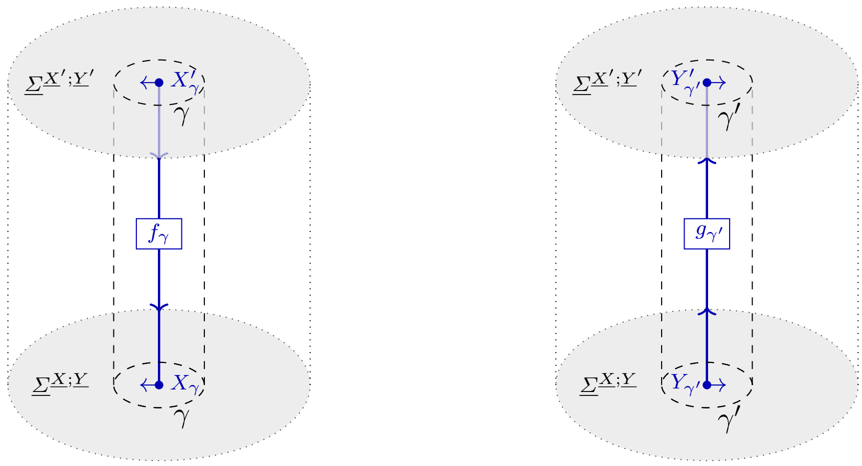


Figure 5.2: Local ribbon graphs for (a) incoming boundary γ and (b) outgoing boundary components γ'.

positively oriented points, and the one along the negative real axis for negatively oriented points. For the Lagrangian subspace $\lambda \in H_1(\Sigma;\mathbb{R})$ note that $\iota_*(\lambda)$, where $\iota_*\colon H_1(\Sigma;\mathbb{R}) \to H_1(\overline{\Sigma};\mathbb{R})$ is the map obtained from the inclusion $\iota\colon \Sigma \to \overline{\Sigma}$, is a Lagrangian subspace of $H_1(\overline{\Sigma};\mathbb{R})$ by [De1, Prop. C.2.]. Altogether this procedure produces an object

$$\underline{\Sigma}^{(\underline{X};\underline{Y})} := (\overline{\Sigma}, P_{(\underline{X},\underline{Y})}, \iota_*(\lambda)) \in \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \ , \tag{5.1.4}$$

see Figure 5.1 for an illustration.

Now for lists of objects $\underline{X}' \in \mathcal{C}^{\times\pi_0(\Gamma)}$ and $\underline{Y}' \in \mathcal{C}^{\times\pi_0(\Gamma')}$ and morphisms $f_\gamma\colon X'_\gamma \to X_\gamma$ and $g_{\gamma'}\colon Y_{\gamma'} \to Y'_{\gamma'}$ with $\gamma \in \pi_0(\Gamma)$ and $\gamma' \in \pi_0(\Gamma')$ we define a bordism $\underline{C}^{(\underline{f};\underline{g})}$ in $\widehat{\mathrm{Bord}}_{3,2}(\mathcal{C})$ as the class of $(\overline{\Sigma} \times [0,1], T_{(\underline{f};\underline{g})}, 0)$ with embedded ribbon graph $T_{(\underline{f};\underline{g})}$ given by locally embedding f_γ (respectively $g_{\gamma'}$) in a cylinder $D_2 \times I$ over the boundary component γ (respectively γ'), see Figure 5.2 for a local illustration. Note that $\underline{C}^{(\underline{f};\underline{g})}$ is indeed admissible.

Lemma 5.1.2. The assignment

$$\begin{aligned}\mathrm{bl}^{\chi}_{\mathcal{C}}\big((\Sigma,\lambda)\big)\colon (\mathcal{C}^{\mathrm{op}})^{\times\pi_0(\Gamma)} \times \mathcal{C}^{\times\pi_0(\Gamma')} &\to \mathrm{vect}_{\Bbbk} \\ (\underline{X};\underline{Y}) &\mapsto \hat{\mathrm{V}}_{\mathcal{C}}(\underline{\Sigma}^{(\underline{X};\underline{Y})}), \\ (\underline{f};\underline{g}) &\mapsto \hat{\mathrm{V}}_{\mathcal{C}}(\underline{C}^{(\underline{f};\underline{g})}).\end{aligned} \tag{5.1.5}$$

defines a functor left exact in each argument.

Proof. Functoriality follows from the functoriality of $\widehat{\mathrm{V}}_{\mathcal{C}}$, note that contravariance in the incoming components comes from the orientation of the ribbons.

For left exactness let us first consider the case where $\varSigma$ is connected. Let $p = |\pi_0(\varGamma)|$ and $q = |\pi_0(\varGamma')|$ and let us fix an explicit choice of ordering of boundary components. Next recall the family of isomorphisms

$$\Phi_{\varSigma}^{(\underline{X};\underline{Y})} \colon \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes X_1 \otimes \cdots \otimes X_p \, , Y_1 \otimes \cdots \otimes Y_q) \to \widehat{\mathrm{V}}_{\mathcal{C}} \left(\underline{\varSigma}_g^{(\underline{X};\underline{Y})} \right), \tag{5.1.6}$$

from Section 3.1.4. A straightforward computation shows that this family defines a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{L}^{\otimes g} \otimes - \otimes \cdots \otimes -, - \otimes \cdots \otimes -) \Longrightarrow \mathrm{bl}_{\mathcal{C}}^{\chi}\big((\varSigma,\lambda)\big)(-,\ldots,-;-,\ldots,-). \tag{5.1.7}$$

Thus $\mathrm{bl}_{\mathcal{C}}^{\chi}\big((\varSigma,\lambda)\big)$ is representable and therefore in particular left exact in each argument. For the general case it suffices to consider two connected components, i.e. $\varSigma = \varSigma_1 \sqcup \varSigma_2$ with $\varSigma_1$ and $\varSigma_2$ connected. From the monoidality of $\widehat{\mathrm{V}}_{\mathcal{C}}$ we get

$$\mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma) = \mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma_1 \sqcup \varSigma_2) \cong \mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma_1) \otimes_{\Bbbk} \mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma_2). \tag{5.1.8}$$

As $\mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma_1)$ and $\mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma_2)$ are left exact by the above argument so is $\mathrm{bl}_{\mathcal{C}}^{\chi}(\varSigma)$. □

Definition 5.1.3 (Chiral block functor on 1-morphisms)**.** For $(\varSigma,\lambda)\colon \varGamma \to \varGamma'$ we define

$$\mathrm{Bl}_{\mathcal{C}}^{\chi}\big((\varSigma,\lambda)\big) \colon (\mathcal{C}^{\mathrm{op}})^{\boxtimes \pi_0(\varGamma)} \boxtimes \mathcal{C}^{\boxtimes \pi_0(\varGamma')} \to \mathrm{vect}_{\Bbbk} \tag{5.1.9}$$

to be the (left exact) functor induced by $\mathrm{bl}_{\mathcal{C}}^{\chi}\big((\varSigma,\lambda)\big)$ in (5.1.5) on the Deligne product.

Corollary 5.1.4. For any pair of 1-morphisms $\varSigma$ and $\varSigma'$ the chiral block functor satisfies

$$\mathrm{Bl}_{\mathcal{C}}^{\chi}(\varSigma \sqcup \varSigma') \cong \mathrm{Bl}_{\mathcal{C}}^{\chi}(\varSigma) \otimes_{\Bbbk} \mathrm{Bl}_{\mathcal{C}}^{\chi}(\varSigma'). \tag{5.1.10}$$

5.1.2 Mapping class group actions

Let us now turn to the 2-morphism level. Let $(\varSigma,\lambda)$ and $(\varSigma',\lambda')$ be 1-morphisms from $\varGamma$ to $\varGamma'$ and let $([f],n)$ be a 2-morphism from $(\varSigma,\lambda)$ to $(\varSigma',\lambda')$. We construct decorated surfaces $\underline{\varSigma}^{(\underline{X};\underline{Y})}$ and $\underline{\varSigma'}^{(\underline{X};\underline{Y})}$ as above for $\underline{X} \in \mathcal{C}^{\times \pi_0(\varGamma)}$ and $\underline{Y} \in \mathcal{C}^{\times \pi_0(\varGamma')}$. The diffeomorphism f underlying $([f],n)$ gives rise to a 3-manifold with corners, its mapping cylinder C_f. This only depends on the isotopy class of $[f]$ if we are considering the diffeomorphism class of the 3-manifold C_f. Analogously to the construction of $\mathrm{Bl}_{\mathcal{C}}^{\chi}$ on surfaces we can glue in solid cylinders with embedded ribbon graph labelled by the $\underline{X} \in \mathcal{C}^{\times \pi_0(\varGamma)}$ and $\underline{Y} \in \mathcal{C}^{\times \pi_0(\varGamma')}$ to obtain a family of morphisms $(\underline{C_f^{(X;Y)}}, n)\colon \underline{\Sigma^{(X;Y)}} \to \underline{\Sigma'^{(X;Y)}}$ in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$. Applying the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ gives us a family of linear maps

$$\widehat{\mathrm{V}}_{\mathcal{C}}((\underline{C_f^{(X;Y)}}), n)\colon \mathrm{bl}^{\chi}_{\mathcal{C}}(\Sigma)(\underline{X};\underline{Y}) \to \mathrm{bl}^{\chi}_{\mathcal{C}}(\Sigma')(\underline{X};\underline{Y}), \tag{5.1.11}$$

which is natural in the labels by construction. We now set

$$\mathrm{bl}^{\chi}_{\mathcal{C}}\big(([f],n)\big) := \Big(\widehat{\mathrm{V}}_{\mathcal{C}}((\underline{C_f^{(X;Y)}}), n)\Big)_{\underline{X}\in\mathcal{C}^{\times\pi_0(\Gamma)},\underline{Y}\in\mathcal{C}^{\times\pi_0(\Gamma')}}. \tag{5.1.12}$$

Definition 5.1.5 (Chiral block functor on 2-morphisms)**.** For a 2-morphism $([f],n)\colon (\Sigma,\lambda) \Rightarrow (\Sigma',\lambda')$,

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}\big(([f],n)\big)\colon \mathrm{Bl}^{\chi}_{\mathcal{C}}\big((\Sigma,\lambda)\big) \Rightarrow \mathrm{Bl}^{\chi}_{\mathcal{C}}\big((\Sigma',\lambda')\big) \tag{5.1.13}$$

is defined as the natural isomorphism induced by $\mathrm{bl}^{\chi}_{\mathcal{C}}\big(([f],n)\big)$ on the Deligne product.

Remark 5.1.6. (1) Let us briefly discuss how this is related to [DGGPR2]. First note that any 2-endomorphism $[f]$ of Σ gives an element in (a central extension of) the mapping class group of the decorated surface $\underline{\Sigma^{(X;Y)}}$ discussed in [DGGPR2, Sec. 3.1] because the underlying diffeomorphism f needs to be compatible with the boundary parametrisation, see also [BK, Prop. 5.1.8]. Thus we can associate it to a mapping cylinder endomorphism of $\underline{\Sigma^{(X;Y)}}$ defined in [DGGPR2, Sec. 3.2], however this needs to be in the same diffeomorphism class as $(\underline{C_f^{(X;Y)}}, n)$ by [BK, Prop. 5.1.8]. Thus our construction and the one of [DGGPR2] are compatible with each other.

(2) Recall from Section 3.1.3 that the signature defect n enters in the construction of $\widehat{\mathrm{V}}_{\mathcal{C}}$ via the coefficient δ, defined in equation 3.1.4. If $\delta = 1$, i.e. if $\mathcal{C}$ is anomaly free, then $\widehat{\mathrm{V}}_{\mathcal{C}}$, and thus $\mathrm{Bl}^{\chi}_{\mathcal{C}}$, will give genuine linear representations of the unextended mapping class group, see also [DGGPR2, Sec. 3.2].

5.1.3 Gluing of surfaces

We now turn to the compatibility of $\mathrm{Bl}^{\chi}_{\mathcal{C}}$ with horizontal composition, which means for $\Sigma_1\colon \Gamma_1 \to \Gamma$ and $\Sigma_2\colon \Gamma \to \Gamma_2$ composable 1-morphisms we want to show that the profunctor $\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_2 \sqcup_\Gamma \Sigma_1)$ is naturally isomorphic to the composition $\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_2) \diamond \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_1)$. Note that to include situations where we do not want to glue along the whole outgoing boundary of Σ_1 or the whole incoming boundary of Σ_2 we can always modify the Σ_i by taking the disjoint union with sufficiently many copies of the cylinder $S^1 \times I$.

Let $\underline{W} \in \mathcal{C}^{\times\pi_0(\Gamma_1)}$, $\underline{X} \in \mathcal{C}^{\times\pi_0(\Gamma)}$, and $\underline{Y} \in \mathcal{C}^{\times\pi_0(\Gamma_2)}$. We will employ the shorthand notation $\underline{\Sigma_1} \equiv \underline{\Sigma_1}^{(W;X)}$, $\underline{\Sigma_2} \equiv \underline{\Sigma_2}^{(X;Y)}$, and $\underline{\Sigma_2 \sqcup_\Gamma \Sigma_1} \equiv \underline{\Sigma_2 \sqcup_\Gamma \Sigma_1}^{(W;Y)}$.

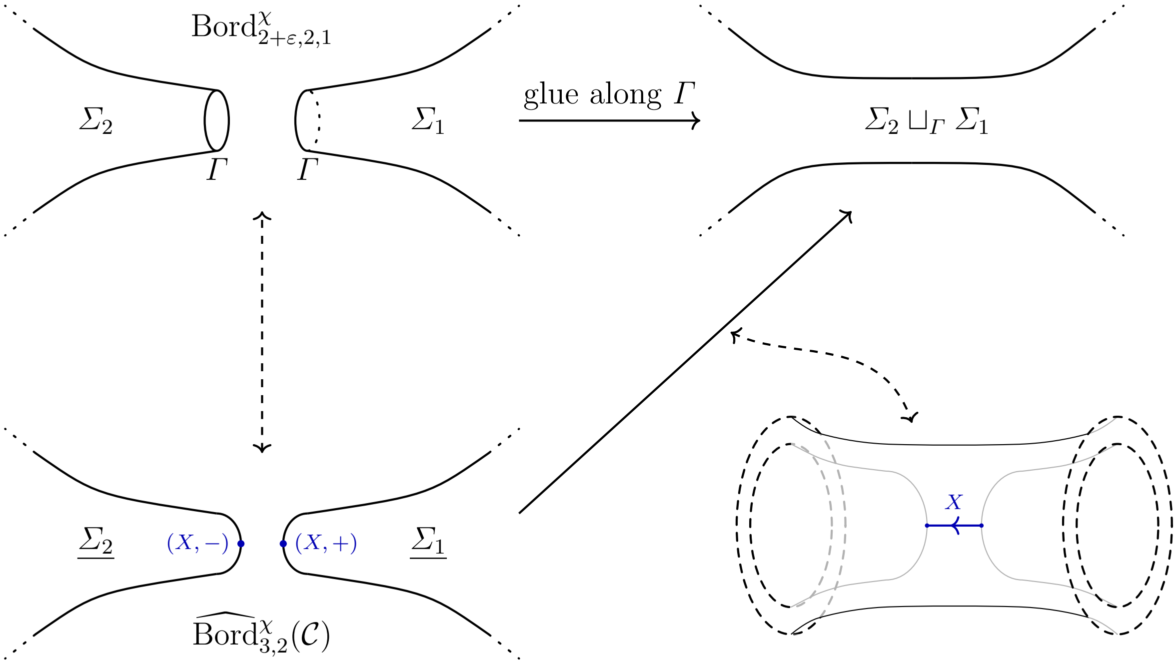


Figure 5.3: Schematic visualisation of the gluing procedure to obtain a family of morphisms in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$. The apparent mismatch between the orientations of the marked points and the blue ribbon is explained by the fact that we take incoming boundaries of bordisms with negative orientation.

We define a family of bordisms

$$G_{\underline{X}} \colon \underline{\Sigma_2} \sqcup \underline{\Sigma_1} \to \underline{\Sigma_2 \sqcup_\Gamma \Sigma_1} \tag{5.1.14}$$

in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$ with underlying 3-manifold

$$G_{\underline{X}} = ((\underline{\Sigma_2} \sqcup \underline{\Sigma_1}) \times I)/\sim \tag{5.1.15}$$

where $\sim$ identifies on $\underline{\Sigma_2} \sqcup \underline{\Sigma_1} \times \{1\}$ the discs glued into the components of Γ on Σ_1 and Σ_2.

On handlebodies the action of the bordism $G_{\underline{X}}$ is the same as attaching a handle with an embedded X_γ-labelled ribbon to the neighbourhood of the marked points labelled with X_γ for any $\gamma \in \pi_0(\Gamma)$. See Figure 5.3 for a local visualisation.

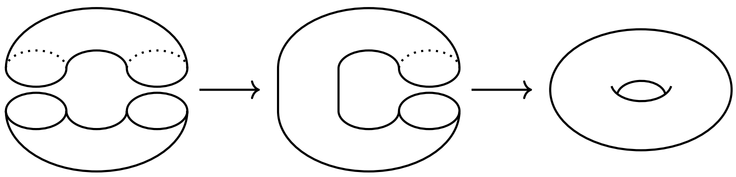
This family of bordisms is by definition dinatural in the labels $\underline{X}$ as well as natural in the $\underline{W}$ and $\underline{Y}$. In the rest of this section we will study the image of the so obtained family under the TFT $\widehat{V}_{\mathcal{C}}$.

Since gluing is a local procedure we can obtain $\Sigma_2 \sqcup_\Gamma \Sigma_1$ by consecutively gluing along each connected component of Γ one at a time. Moreover the order in which these gluings are performed is irrelevant. Formally this amounts to the functoriality and associativity of horizontal composition in $\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$. We can thus restrict our attention to one gluing. Let us fix an order of gluings and let Σ denote the surface obtained after gluing Σ_1 and Σ_2 along $(|\pi_0(\Gamma)|-1)$ components of Γ. We now need to distinguish the following two scenarios:

1. We glue boundary components on two different components of Σ.

2. We glue boundary components on a connected component of Σ.

Both of these scenarios are different from a global perspective, as can be seen by the example of gluing two disjoint cylinders to a torus:



Gluing along the left pair of incoming and outgoing boundary circles results in a cylinder so that the other two boundary components now lie on the same connected component. The second gluing then identifies the two ends of the cylinder to arrive at the torus.

We can now formulate the main result of this section.

Proposition 5.1.7. Let Σ be a surface with at least one incoming and one outgoing boundary, and let Σ_{gl} be the surface obtained from gluing these boundaries. Then the dinatural family

$$\eta_X \colon \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma)(X,X) \to \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_{\mathrm{gl}}) \tag{5.1.16}$$

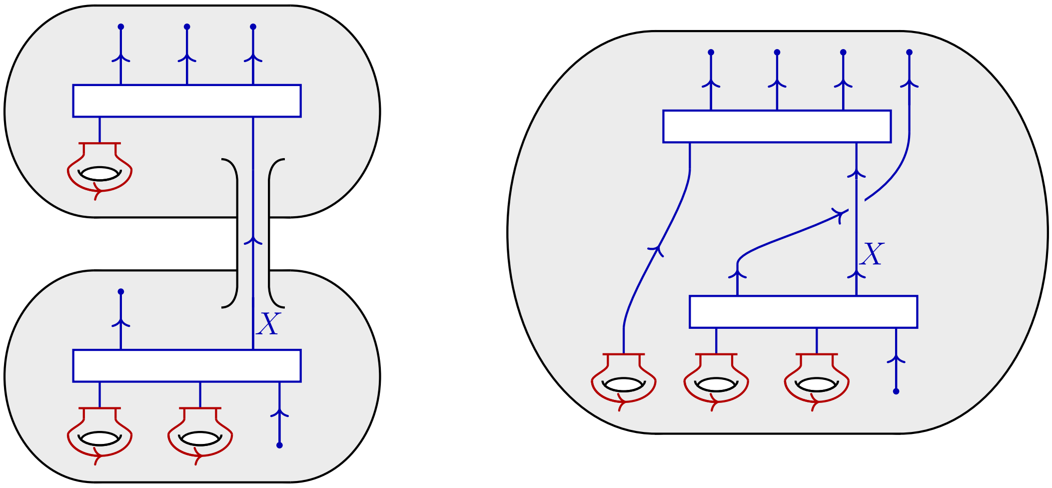
obtained from the gluing bordisms G_X is universal in the category of left exact functors, in other words

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_{\mathrm{gl}}) \cong \oint^{X\in\mathcal{C}} \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma)(X,X). \tag{5.1.17}$$

Corollary 5.1.8. The chiral block functors are compatible with horizontal composition

$$\mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_2 \sqcup_{\Gamma} \Sigma_1) \cong \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_2) \diamond \mathrm{Bl}^{\chi}_{\mathcal{C}}(\Sigma_1). \tag{5.1.18}$$

To prove Proposition 5.1.7 we will now study the two scenarios discussed above separately.



(a) Action of M_X on handlebodies bounding $\Sigma_2^{(1,2)}$ and $\Sigma_1^{(1,3)}$, respectively.

(b) Isotoped handlebody with embedded ribbon graph as in Corollary 2.2.2.

Figure 5.4: Illustration of the handlebody obtained after applying M_X for gluing $\Sigma_2^{(1,2)}$ and $\Sigma_1^{(1,3)}$ to $\Sigma_3^{(1,4)}$.

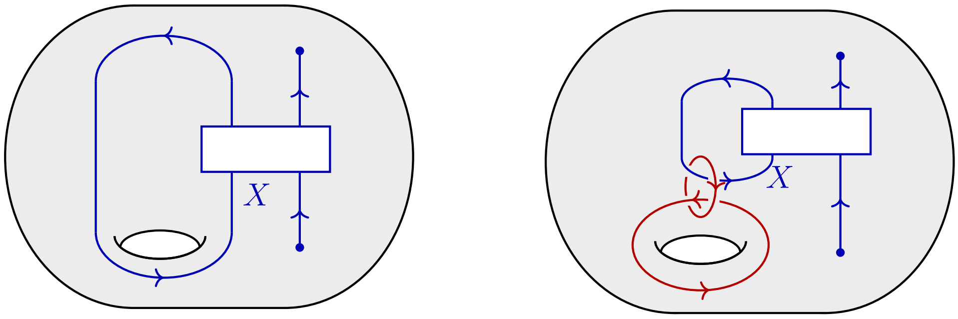
Proof of Proposition 5.1.7: disconnected case

For the first case we can assume without loss of generality that Σ has only two connected components, i.e. $\Sigma \cong \Sigma_{g_1}^{(p_1,q_1)} \sqcup \Sigma_{g_2}^{(p_2,q_2)}$ where $\Sigma_g^{(p,q)}$ denotes a connected surfaces of genus g with p incoming and q outgoing boundary components. After gluing along one boundary component we obtain a connected surface of the form $\Sigma_{\mathrm{gl}} \cong \Sigma_{g_1+g_2}^{(p_1+p_2-1,q_1+q_2-1)}$. Using the identification of the state space with the morphism space in $\mathcal{C}$ from Section 3.1.4 it can easily be verified that the composition of G_X with the standard handlebodies is given by a handlebody with embedded ribbon graph induced from composition and the braiding. This is illustrated in Figure 5.4 for the case of gluing $\Sigma_2^{(1,2)}$ and $\Sigma_1^{(1,3)}$ to $\Sigma_3^{(1,4)}$.

Using the isomorphisms identifying state spaces of the TFT with Hom spaces of $\mathcal{C}$ it immediately follows that the dinatural family η_X corresponds to the universal family from Corollary 2.2.2.

Proof of Proposition 5.1.7: connected case

For the second case let us assume for simplicity that Σ is a genus zero surface with two incoming and two outgoing boundary components, i.e. $\Sigma \cong \Sigma_0^{(2,2)}$, the general case works analogously. After gluing two of the boundary components we obtain a surface of the form $\Sigma_{\mathrm{gl}} \cong \Sigma_1^{(1,1)}$. Following the same line of arguments as above



(a) Action of G_X on handlebody bounding $\Sigma_0^{(2,2)}$.

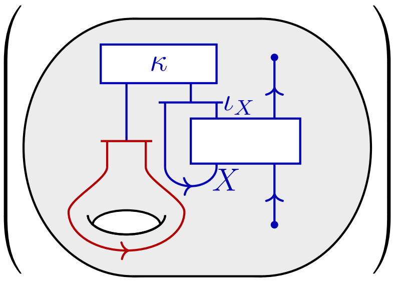
(b) handlebody after application of the "slide trick".

Figure 5.5: handlebody obtained after applying G_X to a handlebody bounding $\Sigma_0^{(2,2)}$.

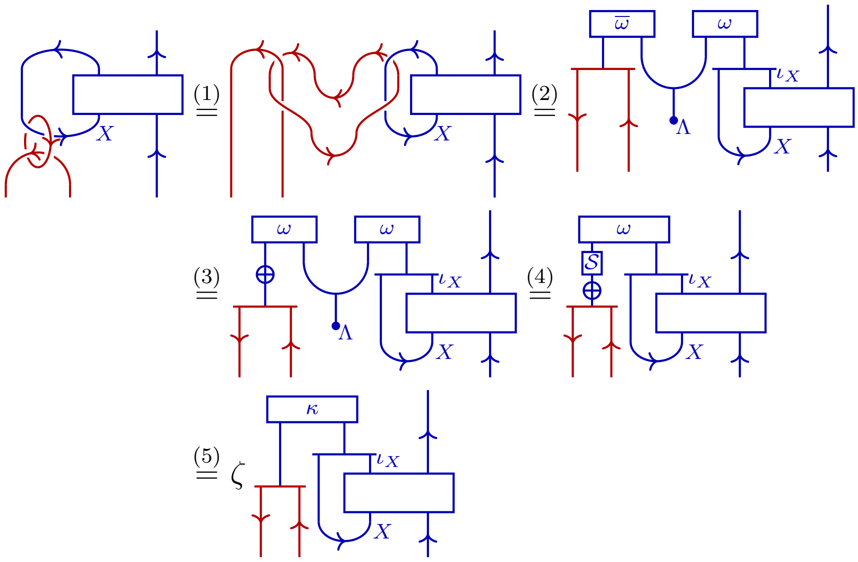
we obtain a handlebody with embedded ribbon graph as depicted in Figure 5.5a.

To translate the action of η_X to Hom-spaces we need to relate the graph in Figure 5.5a to a standard handlebody with bichrome graph as in Figure 3.2. To this end, we first apply the TFT $\hat{\mathrm{V}}_{\mathcal{C}}$ and then use the "slide trick" employed in [DGGPR1, Lem. 4.14] to modify the bichrome graph. The resulting handlebody with bichrome graph is depicted in Figure 5.5b. We have

$$\hat{\mathrm{V}}_{\mathcal{C}}\big(\text{Fig. 5.5a}\big) = \zeta^{-1}\,\hat{\mathrm{V}}_{\mathcal{C}}\big(\text{Fig. 5.5b}\big) = \hat{\mathrm{V}}_{\mathcal{C}}\left(\quad \right) . \qquad (5.1.19)$$



The extra factor ζ^{-1} in the first equality comes from a surgery computation, see [DGGPR1, Lem. 4.14]. The second equality follows from a computation in bichrome graphs:



(5.1.20)

Step 1 is isotopy invariance; step 2 amounts to the defining identities for the structure morphisms of the coend L from Section 2.1.1 together with expressing the integral Λ as a red cup; step 3 is (2.1.8); in step 4 we insert the definition of the S-transformation $\mathcal{S}$ in (2.1.10); step 5 is (2.1.13).

In this form it is evident that η_X is the universal dinatural transformation i_X from Proposition 2.2.5, after identifying the state spaces of the TFT with the Hom spaces of $\mathcal{C}$. This finishes the proof of Proposition 5.1.7. $\square$

Altogether we now have:

Theorem 5.1.9. For every modular tensor category $\mathcal{C}$, the 3d TFT

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{5.1.21}$$

of [DGGPR1] induces a chiral modular functor

$$\mathrm{Bl}^{\chi}_{\mathcal{C}} \colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}. \tag{5.1.22}$$

Moreover going through the construction carefully we immediately find:

Corollary 5.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be modular tensor categories which are equivalent as finite ribbon categories. Then the ribbon equivalence $\mathcal{C} \simeq \mathcal{D}$ induces an isomorphism $\mathrm{Bl}^{\chi}_{\mathcal{C}} \cong \mathrm{Bl}^{\chi}_{\mathcal{D}}$ of symmetric monoidal 2-functors.

Remark 5.1.11. (1) Our construction is related to the one of [De2] as both use the 3d TFTs of [DGGPR1] to construct symmetric monoidal 2-functors. However both the source and target 2-categories differ. Moreover, in [De2] functoriality is satisfied automatically and the non-trivial step in the construction is proving monoidality. In our case it is exactly the other way around with monoidality being built in and functoriality being non-trivial.

(2) In [Lyu1] Lyubashenko constructed a modular functor, albeit in a different formulation, from a modular tensor category $\mathcal{C}$ directly using generators and relations. Nevertheless we can still compare his projective MCG representations and gluing maps to the ones constructed above. The projective MCG representations of Lyubashenko and of the $3d$ TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ are isomorphic by the main result of [DGGPR2]. Moreover the gluing morphisms η_X from Proposition 5.1.7 are precisely the gluing morphisms from [Lyu2, Sec. 9.2] under the above isomorphism.

(3) An operadic formulation and classification of modular functors is given in [BW]. There it is also shown that Lyubashenko's modular functor, suitably interpreted in their framework, is the essentially unique chiral modular functor that can be constructed from a modular tensor category [BW, Cor. 8.3].

5.2 Modular functors and the Drinfeld centre

Let us now discuss how we can obtain an anomaly free and a full modular functor from a modular tensor category $\mathcal{C}$ and how this is related to the Drinfeld centre $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. To this end we will study how the chiral modular functor $\mathrm{Bl}^{\chi}_{\mathcal{C}}$ from Theorem 5.1.9 behaves when precomposed with the various functors discussed in Section 4.1.2 and Section 4.1.3.

5.2.1 Anomaly free modular functors

If we pull back a chiral modular functor along the forgetful functor $\mathcal{U}\colon \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$ discussed in Section 4.1.2 we get an anomaly free modular functor, i.e. the corresponding mapping class group representations will no longer be projective. Moreover, if we further pull back along the orientation double functor $\widehat{(-)}\colon \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \to \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1}$ we obtain a new chiral modular functor. Applying this procedure to the chiral modular functor $\mathrm{Bl}^{\chi}_{\mathcal{C}}$ of $\mathcal{C}$ we get the chiral modular functor $\mathrm{Bl}^{\chi}_{\mathcal{Z}(\mathcal{C})}$ of the Drinfeld centre $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$:

Proposition 5.2.1. Let $\mathcal{C}$ be a modular tensor category. There exists a braided monoidal 2-natural isomorphism filling the following diagram of symmetric monoidal

2-functors

$$
\begin{tikzcd}
\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \arrow[r, "\widehat{(-)}"] \arrow[drr, "\mathrm{Bl}^{\chi}_{\mathcal{Z}(\mathcal{C})}"'] & \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \arrow[r, "\mathcal{U}"] \arrow[d, phantom, "\overset{\cong}{\Leftarrow}"] & \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \arrow[d, "\mathrm{Bl}^{\chi}_{\mathcal{C}}"] \\
 & {} & \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}
\end{tikzcd} . \tag{5.2.1}
$$

Proof. Let us denote the composition

$$
\mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \overset{\widehat{(-)}}{\to} \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \overset{\mathcal{U}}{\to} \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \overset{\mathrm{Bl}^{\chi}_{\mathcal{C}}}{\to} \mathcal{P}\mathrm{rof}^{\mathrm{Lex}_{\Bbbk}} \tag{5.2.2}
$$

by $\widehat{\mathrm{Bl}}_{\mathcal{C}}$. First note that the equivalence $\mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathcal{Z}(\mathcal{C})$ induces an isomorphism $\mathrm{Bl}^{\chi}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}} \cong \mathrm{Bl}^{\chi}_{\mathcal{Z}(\mathcal{C})}$ by Corollary 5.1.10. It thus suffices to show that

$$
\widehat{\mathrm{Bl}}_{\mathcal{C}} \cong \mathrm{Bl}^{\chi}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}. \tag{5.2.3}
$$

On the level of linear categories we have $\mathcal{C} = \overline{\mathcal{C}}$, thus we immediately get $\widehat{\mathrm{Bl}}_{\mathcal{C}}(\mathit{\Gamma}) = (\mathcal{C} \boxtimes \mathcal{C})^{\pi_0(\mathit{\Gamma})} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\pi_0(\mathit{\Gamma})} = \mathrm{Bl}^{\chi}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(\mathit{\Gamma})$ for any object $\mathit{\Gamma} \in \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1}$. On the level of 1- and 2-morphisms the result is a direct consequence of the behaviour of the TFT discussed in Section 3.1.5. $\square$

Remark 5.2.2. Note that in the string-net construction the chiral modular functor of a Drinfeld centre is automatically anomaly free because the mapping class group acts geometrically on the string-nets [MSWY, Cor. 8.7].

5.2.2 Open-closed bordisms and full modular functors

Finally, let us consider the corresponding full modular functor:

$$
\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}} \colon \mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1} \xrightarrow{\widehat{(-)}} \mathrm{Bord}^{\circlearrowleft}_{2+\varepsilon,2,1} \xrightarrow{\mathcal{U}} \mathrm{Bord}^{\chi}_{2+\varepsilon,2,1} \xrightarrow{\mathrm{Bl}^{\chi}_{\mathcal{C}}} \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}. \tag{5.2.4}
$$

Since the orientation double functor factorises over the open-closed bordism category by (4.1.16) we have $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(I) = \mathcal{C}$ and $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(S^1) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathcal{Z}(\mathcal{C})$.

In the following we will be interested in this full modular functor. To this end, it will be beneficial to describe the action of $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}$ on a 1-morphism $\mathit{\Sigma} \colon \mathit{\Gamma} \to \mathit{\Gamma}'$ in $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$ a bit more explicitly. According to the construction of $\mathrm{Bl}^{\chi}_{\mathcal{C}}$ above we first glue discs with $\mathcal{C}$-labelled marked points into the boundary components of $\widehat{\mathit{\Sigma}}$. However, since a $\widehat{S^1} = S^1 \sqcup -S^1$-boundary component of $\widehat{\mathit{\Sigma}}$ comes from a single S^1-boundary component of $\mathit{\Sigma}$ this should be reflected in this gluing process.

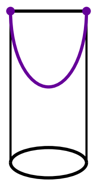
We already saw that $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(S^1) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$. This motivates labelling an S^1-boundary component with an object in $\mathcal{C} \times \overline{\mathcal{C}}$. The reason we use $\mathcal{C} \times \overline{\mathcal{C}}$ instead of $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ reflects that this gluing procedure is followed by an extension from the Cartesian to the Deligne product Definition 5.1.3.

As an illustrative application, we will now use $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}$ to give a topological proof of the monadicity of the Drinfeld centre as mentioned in Section 2.1.2:

Proposition 5.2.3. [BV2, Cor. 5.14] Let $\mathcal{C}$ be a modular tensor category, then the forgetful functor $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ has a two sided adjoint $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. Moreover the corresponding monad $U \circ F$ is naturally isomorphic to the central monad Z on $\mathcal{C}$

$$(U \circ F)(-) \cong Z(-) := \int^{X \in \mathcal{C}} X^* \otimes - \otimes X. \tag{5.2.5}$$

Proof. For the first part of the statement we will study the whistle bordism $W\colon S^1 \to I$ from Section 4.1.3



where the purple line indicates the free boundary. Its orientation double is given by a pair of pants bordism $\widehat{W}\colon S^1 \sqcup -S^1 \to S^1$. From this we get $\mathrm{bl}^{\chi}_{\mathcal{C}}(\widehat{W}) \cong \mathrm{Hom}_{\mathcal{C}}(- \otimes -, -)$ with the monoidal product viewed as a functor $- \otimes - \colon \mathcal{C} \times \overline{\mathcal{C}} \to \mathcal{C}$. Now recall from Section 2.1.2 that the extension of the monoidal product to the Deligne product $\otimes\colon \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{C}$ corresponds to the forgetful functor $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ under the equivalence $\mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathcal{Z}(\mathcal{C})$. Thus we get

$$\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(W) \cong \mathrm{Hom}_{\mathcal{C}}(U(-), -)\colon \mathcal{Z}(\mathcal{C}) \nrightarrow \mathcal{C}. \tag{5.2.6}$$

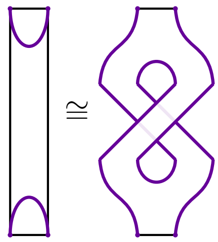
There is also a whistle bordism in the opposite direction $\widetilde{W}\colon I \to S^1$ in $\mathrm{Bord}^{\mathrm{oc}}_{2+\varepsilon,2,1}$. For this an analogous argument gives

$$\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(\widetilde{W}) \cong \mathrm{Hom}_{\mathcal{C}}(-, U(-))\colon \mathcal{C} \nrightarrow \mathcal{Z}(\mathcal{C}). \tag{5.2.7}$$

In particular the profunctors $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(W)$ and $\mathrm{Bl}^{\mathrm{full}}_{\mathcal{C}}(\widetilde{W})$ are adjoint 1-morphisms in $\mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk}$ by Lemma 2.2.9.

Moreover by the 2-equivalence $\mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}} \simeq \mathcal{L}\mathrm{ex}_{\Bbbk}$ we get a functor $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ such that $\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}(\widetilde{W}) \cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(F(-),-)$. In particular $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is a left-adjoint of $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ by definition. By an analogous argument F is also a right-adjoint. This proves the first part of Proposition 5.2.3.

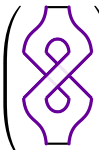
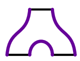
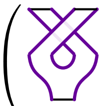
For the second part recall that there is a diffeomorphism between $W \sqcup_{S^1} \widetilde{W}$ and the composition between two flat pairs of pants and the braiding, see e.g. [Car, Sec. 3.2]:

 (5.2.8)

Applying $\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ to the left hand side we immediately get

$\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ () $\cong \mathrm{Hom}_{\mathcal{C}}((U \circ F)(-),-)$ (5.2.9)

by functoriality. For the right hand side first note that the double of a flat pair of pants is a closed pair of pants, and the double of the braiding in the open sector is the braiding in the closed sector. Composing the resulting profunctors finally leads to

$\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ () $\cong \mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ () $\diamond\, \mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ ()

$$
\begin{aligned}
&\cong \oint^{X \boxtimes Y \in \mathcal{C} \boxtimes \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(-, Y \otimes X) \\
&\cong \oint^{X \in \mathcal{C}} \oint^{Y \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(-, Y \otimes X) \\
&\cong \oint^{Y \in \mathcal{C}} \oint^{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(Y^* \otimes -, X) \\
&\cong \oint^{Y \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(Y^* \otimes - \otimes Y, -) \\
&\cong \mathrm{Hom}_{\mathcal{C}}\left(\int^{Y \in \mathcal{C}} Y^* \otimes - \otimes Y, - \right). \qquad (5.2.10)
\end{aligned}
$$

Here, the first and second isomorphism come from functoriality and definition of $\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$, respectively. The third step uses [FSS, Lem. 3.2] in the setting where all functors are left exact. The fourth isomorphism uses the Fubini theorem for left exact coends [Lyu2, Thm. B.2] as well as rigidity of $\mathcal{C}$. The last two isomorphisms are the Yoneda lemma and Proposition 2.2.5, respectively. By functoriality of $\mathrm{Bl}_{\mathcal{C}}^{\mathrm{full}}$ we thus obtain an isomorphism of profunctors

$$\mathrm{Hom}_{\mathcal{C}}((U \circ F)(-), -) \cong \mathrm{Hom}_{\mathcal{C}}\left(\int^{Y \in \mathcal{C}} Y^* \otimes - \otimes Y, -\right), \tag{5.2.11}$$

which by the Yoneda lemma implies the second part of Proposition 5.2.3. $\square$

Chapter 6

TFT construction of full 2d CFT

In this chapter we will discuss how to use certain 3-dimensional defect TFTs, obtained from the non-semisimple TFT of [DGGPR1], to construct a full conformal field theory, leading to the second main result of this thesis in the form of Theorem 6.6.2. This construction can be seen as a non-semisimple extension of [FFFS; FRSI; FRSII; FRSIII; FRSIV; FjFRS; FjFSt] with a particular emphasis on its topological nature.

We will start with a discussion of the defect TFTs to which our construction can be applied and how the labelling data for the world sheet category $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ is obtained. Afterwards, we will construct the data of a full CFT, as in Definition 4.3.1, by evaluating the 3d defect TFT on certain manifolds in Section 6.2. The rest of this chapter is devoted to checking that this data satisfies Definition 4.3.1, i.e. we will prove that it satisfies the axioms of a braided monoidal oplax natural transformation discussed in Section 4.3.

6.1 Allowed defect TFTs

As specified in the previous chapter $\mathcal{C}$ is a fixed modular tensor category and

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{6.1.1}$$

the version of the TFT constructed from $\mathcal{C}$ with admissibility condition on bordism components disjoint from the outgoing boundary viewed as defect TFT as in Section 3.2.

Instead of working with an explicit construction of a non-semisimple defect TFT we will assume there exists a defect TFT "based on" $\widehat{\mathrm{V}}_{\mathcal{C}}$ satisfying certain algebraic assumptions. The reason we do this is because we want to be able to apply our construction to possible defect TFTs for which we currently do not have an explicit construction. We will comment a bit more on such a hypothetical TFT in Section 8.3.4. Nonetheless, we do have two explicit non-semisimple defect TFTs in mind to which our construction applies, these will be discussed further below. Apart from this, taking the defect TFT as a "blackbox" also has the added benefit of highlighting the topological nature of our construction.

Let us now first make precise what we mean with 3d defect TFTs "based on" $\widehat{\mathrm{V}}_{\mathcal{C}}$.

Definition 6.1.1. Let

$$\mathrm{Z}_{\mathcal{C}} \colon \mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}}) \to \mathrm{vect}_{\Bbbk} \tag{6.1.2}$$

be a 3d defect TFT such that its defect data $\mathbb{D}_{\mathcal{C}}$ contains the defect data of $\widehat{\mathrm{V}}_{\mathcal{C}}$ as a subset in the sense of [CRS2, Sec. 2.3.1]. Let

$$\iota \colon \widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C}) \to \mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}}) \tag{6.1.3}$$

be the symmetric monoidal functor obtained from this inclusion of defect data. Then $\mathrm{Z}_{\mathcal{C}}$ is said to *extend* $\widehat{\mathrm{V}}_{\mathcal{C}}$ if there exists a monoidal natural isomorphism $\Phi \colon \mathrm{Z}_{\mathcal{C}} \circ \iota \Rightarrow \widehat{\mathrm{V}}_{\mathcal{C}}$. We call a pair $(\mathrm{Z}_{\mathcal{C}}, \Phi)$ an *extension* of $\widehat{\mathrm{V}}_{\mathcal{C}}$.

On the level of defect 3-categories from Remark 3.2.2 being $\mathrm{Z}_{\mathcal{C}}$ extending $\widehat{\mathrm{V}}_{\mathcal{C}}$ means that we can exhibit $B^2\mathcal{C}$ as a sub-3-category of $\mathcal{T}_{\mathrm{Z}_{\mathcal{C}}}$.

The simplest example of an extension $(\mathrm{Z}_{\mathcal{C}}, \Phi)$ is $(\widehat{\mathrm{V}}_{\mathcal{C}}, \mathrm{id})$ itself. A more interesting example can be obtained by introducing non-trivial surface defects into $\widehat{\mathrm{V}}_{\mathcal{C}}$ using the so-called orbifold construction [CRS1]. The first example will be discussed in detail in Chapter 7 while some aspects of the second example will be explained in Chapter 8.

From now on let $(\mathrm{Z}_{\mathcal{C}}, \Phi)$ be a fixed extension of $\widehat{\mathrm{V}}_{\mathcal{C}}$. Moreover, we will assume that D_3 is a singleton set corresponding to a unique phase of $\mathrm{Z}_{\mathcal{C}}$. This is because, as mentioned in the introduction, we want to construct full CFTs for which the chiral and anti-chiral sector are governed by the same VOA. The aforementioned algebraic assumptions on $\mathrm{Z}_{\mathcal{C}}$ will be discussed in detail when they come up. In the following we will often view $\widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C})$ as a (non-full) subcategory of $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ by identifying objects in $\widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C})$ with their image under ι in $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ and accordingly suppress the functor ι.

Remark 6.1.2. We want to note here that the 3d defect TFT $\mathrm{Z}_{\mathcal{C}}$ might also only be defined on a subcategory of $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ in analogy to how $\widehat{\mathrm{V}}_{\mathcal{C}}$ is only defined on such an admissible subcategory. However, all bordisms in $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ which will be needed for our considerations will automatically be in such a subcategory as they will always contain an outgoing boundary.

We could now use $\mathrm{Z}_{\mathcal{C}}$ to construct a chiral modular functor

$$\mathrm{Bl}_{\mathrm{Z}_{\mathcal{C}}}^{\chi} \colon \mathrm{Bord}_{2+\varepsilon,2,1}^{\chi} \to \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}} \tag{6.1.4}$$

in analogy to the previous chapter. However, this chiral modular functor will automatically be isomorphic to the chiral modular functor

$$\mathrm{Bl}_{\mathcal{C}}^{\chi}\colon \mathrm{Bord}_{2+\varepsilon,2,1}^{\chi} \to \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}} \tag{6.1.5}$$

constructed from $\widehat{\mathrm{V}}_{\mathcal{C}}$. To see this note that in the construction we only need surfaces and 3-bordisms which do not contain codimension 1-strata, i.e. we only work with $\widehat{\mathrm{Bord}}_{3,2}^{\chi}(\mathcal{C})$ and not all of $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$. This means we can use the natural isomorphism Φ to construct an isomorphism of chiral modular functors.

Next we need to discuss the allowed 2-dimensional defect data $\mathbb{D}^{2\mathrm{d}}$. As explained in the introduction we know that a full CFT should correspond to a surface defect in the 3d TFT $\mathrm{Z}_{\mathcal{C}}$. For this reason we will be interested in labelling data for topological world sheets such that we can view them as surface defects in the 3d defect TFT $\mathrm{Z}_{\mathcal{C}}$. Since we already assumed that D_3 is a singleton set we can use $\mathbb{D}^{2\mathrm{d}} = \mathbb{D}_{\mathcal{C}}$ directly. In particular we have a canonical choice for the transparent element $T \in D_2^{2\mathrm{d}}$ of $\mathbb{D}_{\mathcal{C}}$: the unique element in D_2 we get from the inclusion of $\mathcal{C}$ in $\mathbb{D}_{\mathcal{C}}$ as defect data.

Remark 6.1.3. In terms of the higher categories of defects this amounts to taking the 2-category of defects of the full CFT to be the endomorphism 2-category of the unique object in the 3-category $\mathcal{T}_{\mathrm{Z}_{\mathcal{C}}}$ of defects of $\mathrm{Z}_{\mathcal{C}}$. Moreover, the canonical choice for the transparent surface defect T is now simply the identity 1-morphism of the unique object of $\mathcal{T}_{\mathrm{Z}_{\mathcal{C}}}$.

Let $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ be the $(2,1)$-category of topological world sheets with labelling data obtained from $\mathbb{D}_{\mathcal{C}}$ and let us denote with $\mathrm{Bl}_{\mathcal{C}}$ the composition

$$\mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \xrightarrow{U} \mathrm{Bord}_{2+\varepsilon,2,1}^{\mathrm{oc}} \xrightarrow{\widehat{(-)}} \mathrm{Bord}_{2+\varepsilon,2,1}^{\chi} \xrightarrow{\mathrm{Bl}_{\mathcal{C}}^{\chi}} \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}}. \tag{6.1.6}$$

We will now construct a full CFT

$$\begin{tikzcd} \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \arrow[rr, bend left, "\Delta_{\Bbbk}"] \arrow[rr, bend right, "\mathrm{Bl}_{\mathcal{C}}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Cor}"] & & \mathcal{P}\mathrm{rof}_{\Bbbk}^{\mathcal{L}\mathrm{ex}} \end{tikzcd} \tag{6.1.7}$$

as in Definition 4.3.1 for the modular functor $\mathrm{Bl}_{\mathcal{C}}$.

6.2 Construction of full CFT

In this section we will explain how to construct the 1- and 2-morphism components of the full CFT Cor for $\mathrm{Bl}_{\mathcal{C}}$. As in [FFFS; FRSI], the basic idea is to obtain

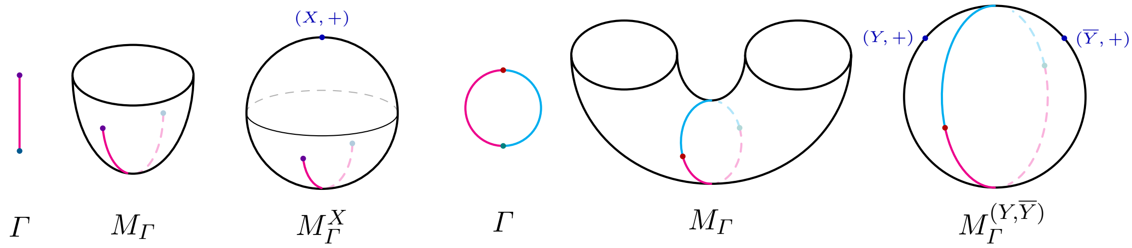


Figure 6.1: Connecting manifold M_Γ for Γ an interval (a) and a defect circle (b) with and without disc(s) glued in. Here the different colours are used to indicate different $\mathbb{D}$-labels of the strata except for the blue marked and labelled points which are always labelled by elements in $\mathcal{C}$.

the correlators by evaluating the $3d$-defect TFT $\mathrm{Z}_\mathcal{C}$ on the so-called *connecting manifolds*. In contrast to their work we will not only consider this on the level of world sheets and the resulting 3-manifolds but already one dimension lower in order to also obtain the field content.

6.2.1 Field content

Let $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_\mathcal{C})$ and denote with Γ the underlying compact 1-manifold. We want to construct a left exact profunctor $\mathrm{Cor}_\mathfrak{C} \colon \Delta_\Bbbk(\mathfrak{C}) \nrightarrow \mathrm{Bl}_\mathcal{C}(\mathfrak{C})$. By definition this is a left exact functor $\mathcal{C}^{\boxtimes \pi_0(\widehat{\Gamma})} \to \mathrm{vect}_\Bbbk$ since $\mathrm{Bl}_\mathcal{C}(\mathfrak{C}) = \mathrm{Bl}^\chi_\mathcal{C}(\widehat{\Gamma}) = \mathcal{C}^{\boxtimes \pi_0(\widehat{\Gamma})}$. To this end consider the *connecting manifold* $M_\mathfrak{C}$ of $\mathfrak{C}$ defined as the defect 2-manifold with underlying manifold

$$M_\Gamma = \Gamma \times [-1,1]/\sim \quad \text{with} \quad (p,t) \sim (p,-t) \quad \text{for} \quad p \in \partial\Gamma \tag{6.2.1}$$

and stratification induced from the inclusion $\Gamma \cong \Gamma \times \{0\} \hookrightarrow M_\Gamma$, see Figure 6.1 for an illustration.[1] Note that by construction $\partial M_\Gamma \cong \widehat{\Gamma}$. Now let $\underline{X} \in \mathcal{C}^{\times \pi_0(\widehat{\Gamma})}$. We construct an object $M_\Gamma^{\underline{X}} \in \mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_\mathcal{C})$ as in Section 5.1.1 by gluing discs $D_\Gamma^{\underline{X}}$ bounding $\widehat{\Gamma}$ with $\underline{X}$-labelled, positively oriented, marked points into the boundary of M_Γ, see Figure 6.1 for an illustration.[2] By a similar argument as for Lemma 5.1.2 this can be extended to a functor

$$\begin{aligned} \mathrm{cor}_\Gamma \colon \mathcal{C}^{\times \pi_0(\widehat{\Gamma})} &\to \mathrm{vect}_\Bbbk \\ \underline{X} &\mapsto \mathrm{Z}_\mathcal{C}(M_\Gamma^{\underline{X}}). \end{aligned} \tag{6.2.2}$$

[1] The name connecting manifold comes from the idea that it "connects" a manifold X with its orientation double $\widehat{X}$ without adding homotopical information.

[2] Choosing the marked points to be positively oriented is a convention, which we use to ensure that we get covariant functors.

For later use let us also consider the functor $\mathrm{cor}_\Gamma^\dagger$ obtained analogously as cor_Γ but for $-M_\Gamma$ instead, i.e. all marked points are negatively oriented leading to a contravariant functor.

Since we do not have an explicit construction of $\mathrm{Z}_\mathcal{C}$ we cannot guarentee that cor_Γ is left exact in general. This will give us to the first assumption on $\mathrm{Z}_\mathcal{C}$:

Assumption 1. *For any $\mathfrak{C} \in \mathfrak{W}\mathfrak{S}(\mathbb{D}_\mathcal{C})$, the functors $\mathrm{cor}_\Gamma \colon \mathcal{C}^{\times\pi_0(\widehat{\Gamma})} \to \mathrm{vect}_\Bbbk$ and $\mathrm{cor}_\Gamma^\dagger \colon (\mathcal{C}^{\mathrm{op}})^{\times\pi_0(\widehat{\Gamma})} \to \mathrm{vect}_\Bbbk$ are linear and left exact in each variable.*

As the notation suggests we further assume the extensions of cor_Γ and $\mathrm{cor}_\Gamma^\dagger$ to the Deligne product to be adjoint profunctors as in Lemma 2.2.9.

Under Assumption 1 we can now extend cor_Γ to the Deligne tensor product.

Definition 6.2.1 (Cor on objects)**.** For $\mathfrak{C} \in \mathfrak{W}\mathfrak{S}(\mathbb{D}_\mathcal{C})$ we define

$$\mathrm{Cor}_\mathfrak{C} \colon \Delta_\Bbbk(\mathfrak{C}) \nrightarrow \mathrm{Bl}_\mathcal{C}(\mathfrak{C}) \tag{6.2.3}$$

to be the functor induced by cor_Γ on the Deligne tensor product $\mathcal{C}^{\boxtimes\pi_0(\widehat{\Gamma})}$.

6.2.2 Correlators

Next let us construct the 2-morphism components of Cor. Let $\mathfrak{S} \colon \mathfrak{C} \to \mathfrak{C}$ be a 1-morphism in $\mathfrak{W}\mathfrak{S}(\mathbb{D}_\mathcal{C})$ and denote with $\Sigma \colon \Gamma \to \Gamma'$ its underlying open-closed bordism. The *connecting manifold* $M_\mathfrak{S}$ of $\mathfrak{S}$ is the defect 3-manifold (with corners) with underlying 3-manifold (with corners)

$$M_\Sigma = \Sigma \times [-1,1]/\sim \quad \text{with} \quad (p,t) \sim (p,-t) \quad \text{for} \quad p \in \partial^{\mathrm{fr}}\Sigma \tag{6.2.4}$$

and stratification induced from the embedding $\Sigma \cong \Sigma \times \{0\} \hookrightarrow M_\Sigma$, see Figure 6.2 for an illustration in the example of the 1-morphism from (4.2.11).

Note that the boundary of M_Σ naturally decomposes as

$$\partial M_\Sigma \cong -M_\Gamma \cup \widehat{\Sigma} \cup M_{\Gamma'} \tag{6.2.5}$$

and $\widehat{\Gamma} \sqcup \widehat{\Gamma}'$ as corner points. This can be seen by noting that

$$\partial(\Sigma \times [-1,1]) \cong \partial\Sigma \times [-1,1] \sqcup_{(\Gamma\sqcup\Gamma')\times\{\pm1\}} \Sigma \times \{\pm1\} \tag{6.2.6}$$

and tracking how the equivalence relation identifies points on the boundary, see Figure 6.2a and 6.2b for an illustration.

In particular, the boundary parametrisation $\partial\Sigma \cong -\Gamma \sqcup \Gamma'$ induces a parametrisation of the M_Γ and $M_{\Gamma'}$ boundary components. Using this we can turn M_Σ into a morphism in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_\mathcal{C})$ as follows: Let $\underline{X} \in \mathcal{C}^{\times\pi_0(\widehat{\Gamma})}$ and $\underline{Y} \in \mathcal{C}^{\times\pi_0(\widehat{\Gamma}')}$, and let

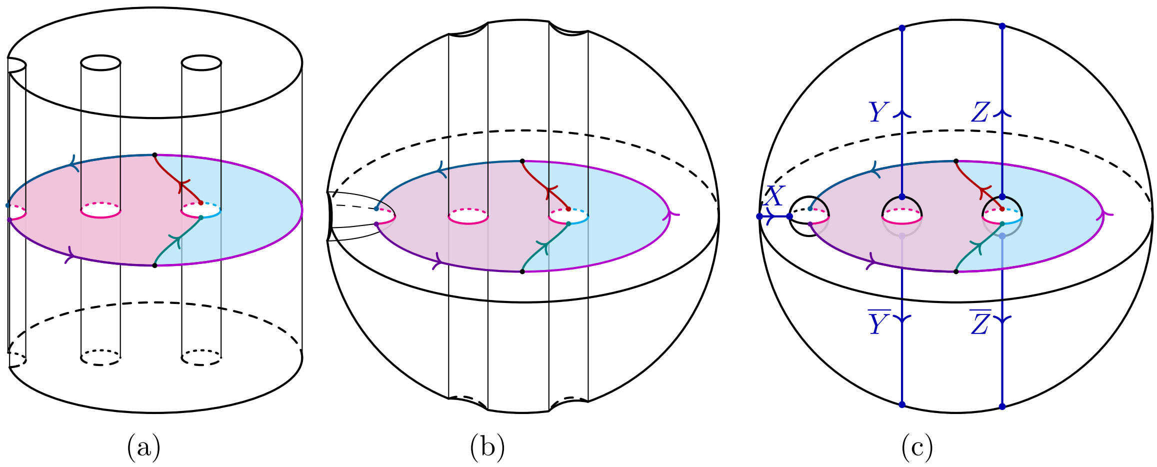


Figure 6.2: Schematic construction of the connecting bordism $M_{\Sigma}^{\underline{X},\underline{Y}}$ for Σ the 1-morphism in $\mathfrak{WG}(\mathbb{D}_{\mathcal{C}})$ from (4.2.11): (a) cylinder over Σ with Σ as surface defect, (b) the connecting manifold M_Σ with corners, (c) connecting bordism $M_{\Sigma}^{\underline{X},\underline{Y}}$;

$M_{\Gamma}^{\underline{X}}$ and $M_{\Gamma'}^{\underline{Y}}$ be the corresponding objects in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ obtained by gluing in discs as above. By construction these objects come with embeddings $M_\Gamma \hookrightarrow M_{\Gamma}^{\underline{X}}$ and $M_{\Gamma'} \hookrightarrow M_{\Gamma'}^{\underline{Y}}$. Now consider the cylinders $M_{\Gamma}^{\underline{X}} \times [-1,1]$ and $M_{\Gamma'}^{\underline{Y}} \times [-1,1]$ with standard boundary parametrisations as identity morphisms in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$. Using the embeddings $M_{\Gamma'} \hookrightarrow M_{\Gamma'}^{\underline{Y}}$ we want to view them as bordisms

$$\begin{aligned} M_{\Gamma}^{\underline{X}} \times [-1,1] &\colon M_{\Gamma}^{\underline{X}} \to M_\Gamma \sqcup_{\widehat{\Gamma}} D_{\Gamma}^{\underline{X}} \\ M_{\Gamma'}^{\underline{Y}} \times [-1,1] &\colon M_{\Gamma'}^{\underline{Y}} \to M_{\Gamma'} \sqcup_{\widehat{\Gamma}'} D_{\Gamma'}^{\underline{Y}} \end{aligned} \tag{6.2.7}$$

where $D_{\Gamma}^{\underline{X}}$ and $D_{\Gamma'}^{\underline{Y}}$ are the discs with marked points which were glued into M_Γ and $M_{\Gamma'}$, see Figure 6.3 for an illustration.

Using this parametrisation we can now glue these cylinders along their outgoing boundary to M_Σ and obtain

$$M_{\Sigma}^{\underline{X},\underline{Y}} := (M_{\Gamma}^{\underline{X}} \times [-1,1]) \sqcup_{M_\Gamma} M_\Sigma \sqcup_{M_{\Gamma'}} (M_{\Gamma'}^{\underline{Y}} \times [-1,1]). \tag{6.2.8}$$

By construction the boundary of $M_{\Sigma}^{\underline{X},\underline{Y}}$ is given by

$$\partial M_{\Sigma}^{\underline{X},\underline{Y}} \cong -M_{\Gamma}^{\underline{X}} \sqcup D_{\Gamma}^{\underline{X}} \sqcup_{\widehat{\Gamma}} \widehat{\Sigma} \sqcup_{\widehat{\Gamma}'} D_{\Gamma'}^{\underline{Y}} \sqcup M_{\Gamma'}^{\underline{Y}} \tag{6.2.9}$$

where $D_{\Gamma}^{\underline{X}}$ and $D_{\Gamma'}^{\underline{Y}}$ are the discs with marked points which were glued into M_Γ and $M_{\Gamma'}$. Technically we would need to choose a smoothing of $M_{\Sigma}^{\underline{X},\underline{Y}}$, however, as mentioned in the beginning of Chapter 3, we will instead work in the topological category here.

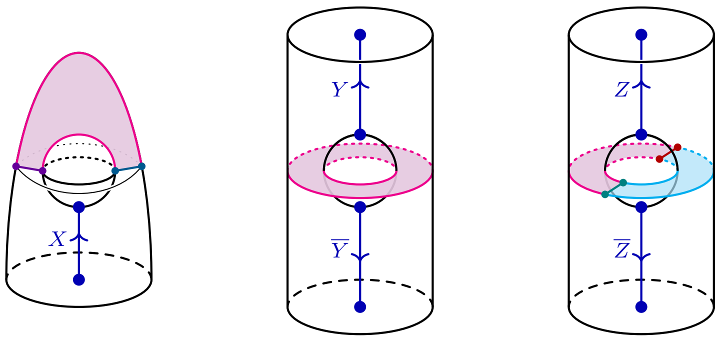


Figure 6.3: Cylinders over $M_{\Gamma}^{\underline{X}}$ and $M_{\Gamma'}^{\underline{Y}}$ for Γ and Γ' the source and target of the 1-morphism in $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ from (4.2.11).

Let $\widehat{\Sigma}^{\underline{X},\underline{Y}} := D_{\Gamma}^{\underline{X}} \sqcup_{\widehat{\Gamma}} \widehat{\Sigma} \sqcup_{\widehat{\Gamma}'} D_{\Gamma'}^{\underline{Y}}$ be the object in $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ obtained by gluing discs into the boundary components of $\widehat{\Sigma}$. Using this we can turn $M_{\Sigma}^{\underline{X},\underline{Y}}$ into a morphism

$$M_{\Gamma'}^{\underline{Y}} \sqcup -M_{\Gamma}^{\underline{X}} \to \widehat{\Sigma}^{\underline{X},\underline{Y}} \tag{6.2.10}$$

in $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$. Note here that the difference in orientation between $M_{\Gamma}^{\underline{X}}$ and $M_{\Gamma'}^{\underline{Y}}$ is a direct result from Γ being the incoming boundary and Γ' being the outgoing boundary of Σ. For an illustration of this whole procedure see Figure 6.2c, in particular the difference in the orientation of the line defects is induced by the difference in orientations of the incoming boundary. In the following we will call $M_{\Sigma}^{\underline{X},\underline{Y}}$ the *connecting bordism* of $\mathfrak{S}$.

Finally, observe that $\widehat{\Sigma}^{\underline{X},\underline{Y}} \in \mathrm{Bord}_{3,2}^{\chi}(\mathcal{C})$ is precisely the marked surface discussed in the description of the full modular functor from Section 5.2.2 and since $\mathrm{Z}_{\mathcal{C}}(\widehat{\Sigma}^{\underline{X},\underline{Y}}) \cong \widehat{\mathrm{V}}_{\mathcal{C}}(\widehat{\Sigma}^{\underline{X},\underline{Y}})$ the functor

$$\begin{aligned} \mathrm{bl}_Z(\widehat{\Sigma})\colon (\mathcal{C}^{\mathrm{op}})^{\times \pi_0(\widehat{\Gamma})} \times \mathcal{C}^{\times \pi_0(\widehat{\Gamma'})} &\to \mathrm{vect}_{\Bbbk} \\ (\underline{X}, \underline{Y}) &\mapsto \mathrm{Z}_{\mathcal{C}}(\widehat{\Sigma}^{\underline{X},\underline{Y}}) \end{aligned} \tag{6.2.11}$$

lifts to the full modular functor $\mathrm{Bl}_{\mathcal{C}}(\mathfrak{S})\colon (\mathcal{C}^{\mathrm{op}})^{\boxtimes \pi_0(\widehat{\Gamma})} \boxtimes (\mathcal{C})^{\boxtimes \pi_0(\widehat{\Gamma}')} \to \mathrm{vect}_{\Bbbk}$.

It is straightforward to check that the family of linear maps

$$\mathrm{cor}_{\Sigma}^{\underline{X},\underline{Y}}\colon \mathrm{Z}_{\mathcal{C}}(M_{\Gamma'}^{\underline{Y}} \sqcup -M_{\Gamma}^{\underline{X}}) \to \mathrm{Z}_{\mathcal{C}}(\widehat{\Sigma}^{\underline{X},\underline{Y}}) \tag{6.2.12}$$

defines a natural transformation

$$\mathrm{cor}_{\mathfrak{S}}\colon \mathrm{cor}_{\Gamma'} \otimes_{\Bbbk} \mathrm{cor}_{\Gamma}^{\dagger} \Rightarrow \mathrm{bl}_Z(\widehat{\Sigma}) \tag{6.2.13}$$

of functors from $(\mathcal{C})^{\times \pi_0(\widehat{\Gamma}')} \times (\mathcal{C}^{\mathrm{op}})^{\times \pi_0(\widehat{\Gamma})}$ to $\mathrm{vect}_{\Bbbk}$.

Definition 6.2.2 (Cor on morphisms)**.** For $\mathfrak{S}\colon \mathfrak{C} \to \mathfrak{C}$ a 1-morphism in $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ we define

$$\mathrm{Cor}_{\mathfrak{S}}\colon \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \Rightarrow \mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}) \tag{6.2.14}$$

to be the natural transformation induced by cor_{Σ} on the Deligne tensor product $(\mathcal{C})^{\boxtimes\pi_0(\widehat{\Gamma}')} \boxtimes (\mathcal{C}^{\mathrm{op}})^{\boxtimes\pi_0(\widehat{\Gamma})}$.

Remark 6.2.3. As explained in Section 3.2 the bordism category $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ contains extra data needed to cancel a gluing anomaly of the TFT $\mathrm{Z}_{\mathcal{C}}$. This data can be included in our construction in precisely the same way as in [FRSII, Sec. 3.1]. This is possible because for any object $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ the connecting manifold $M_{\mathfrak{C}}^{X}$ is topologically a finite disjoint union of 2-spheres and thus has trivial first homology. Thus the extra boundary components of our connecting bordism in comparison to the one of [FRSII, Sec. 3.1] do not lead to extra contributions.

The rest of this chapter will be devoted to proving that Cor satisfies the conditions of a braided monoidal oplax natural transformation under three more technical assumptions on $\mathrm{Z}_{\mathcal{C}}$. To this end it will be useful to recall from Section 4.3 that for every object $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ there exists an essentially unique object $\mathbb{F}_{\mathfrak{C}} \in \mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})$ such that

$$\mathrm{Cor}_{\mathfrak{C}}(-) \cong \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, -) \tag{6.2.15}$$

by left exactness of $\mathrm{Cor}_{\mathfrak{C}}(-)$.

6.3 Monoidality

Let us first discuss how to obtain the braided monoidal structure on Cor in the form of Π and Υ. For Υ we have $M_{\varnothing} = \varnothing$ which directly leads to $\mathrm{Cor}_{\varnothing} \cong \mathrm{id}_{\mathrm{Vect}}$ as desired. For $\Pi_{\mathfrak{C},\mathfrak{C}'}$ first note that $\mathrm{cor}_{\Gamma\sqcup\Gamma'}(-,\sim) \cong \mathrm{cor}_{\Gamma}(-) \otimes_{\Bbbk} \mathrm{cor}_{\Gamma'}(\sim)$ by monoidality of $\mathrm{Z}_{\mathcal{C}}$ and thus

$$\begin{aligned}\mathrm{Cor}_{\mathfrak{C}\sqcup\mathfrak{C}'}(-\boxtimes \sim) &\cong \mathrm{Cor}_{\mathfrak{C}}(-) \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}'}(\sim) \\ &\cong \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, -) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}(\mathbb{F}_{\mathfrak{C}'}, \sim).\end{aligned} \tag{6.3.1}$$

On the other hand a direct computation using the Yoneda lemma leads to

$$(\mathrm{Cor}_{\mathfrak{C}} \boxtimes \mathrm{id}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}) \diamond (\mathrm{id}_{\Delta_{\Bbbk}(\mathfrak{C})} \boxtimes \mathrm{Cor}_{\mathfrak{C}'})(-\boxtimes \sim) \cong \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})\boxtimes\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}(\mathbb{F}_{\mathfrak{C}} \boxtimes \mathbb{F}_{\mathfrak{C}'}, -\boxtimes \sim). \tag{6.3.2}$$

Combining this with the natural isomorphism

$$\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})\boxtimes\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}(- \boxtimes -, - \boxtimes -) \cong \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(-,-) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}(-,-) \tag{6.3.3}$$

coming from the universal property of the Deligne product [EGNO, Prop. 1.11.2] we finally get the desired isomorphism

$$\mathrm{Cor}_{\mathfrak{C}\sqcup\mathfrak{C}'}(-\boxtimes\sim) \cong (\mathrm{Cor}_{\mathfrak{C}} \boxtimes \mathrm{id}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C}')}) \diamond (\mathrm{id}_{\Delta_{\Bbbk}(\mathfrak{C})} \boxtimes \mathrm{Cor}_{\mathfrak{C}'})(-\boxtimes\sim). \tag{6.3.4}$$

For the modification axiom of $\Pi_{\mathfrak{C},\mathfrak{C}'}$, see e.g. [JY, Def. 4.4.2], we observe that for $\mathfrak{S}\colon \mathfrak{C}\to\mathfrak{C}'$ and $\widetilde{\mathfrak{S}}\colon \widetilde{\mathfrak{C}}\to\widetilde{\mathfrak{C}}'$ the connecting manifold of their disjoint union $M_{\Sigma\sqcup\widetilde{\Sigma}}$ is the disjoint union of the connecting manifolds $M_\Sigma \sqcup M_{\widetilde{\Sigma}}$. Since $\Pi_{\mathfrak{C},\mathfrak{C}'}$ is constructed using the monoidal structure of $\mathrm{Z}_{\mathcal{C}}$ this implies the required relation between $\mathrm{Cor}_{\mathfrak{S}\sqcup\widetilde{\mathfrak{S}}}$ and $\mathrm{Cor}_{\mathfrak{S}} \sqcup \mathrm{Cor}_{\widetilde{\mathfrak{S}}}$.

6.4 MCG covariance / 2-morphism naturality

Next let us discuss 2-morphism naturality in the form of commutativity of Diagram 4.3.4. Let $f\colon \mathfrak{S}\Rightarrow\mathfrak{S}'$ be a 2-morphism between 1-morphisms $\mathfrak{S},\mathfrak{S}'\colon\mathfrak{C}\to\mathfrak{C}'$ in $\mathfrak{W}\mathfrak{S}(\mathbb{D}_{\mathcal{C}})$. Since Diagram 4.3.4 consists of natural transformations we can do this component wise. Now all the functors and natural transformations in this diagram are specified via their action on pure tensors in the Deligne product. These actions are in turn induced from $\mathrm{Z}_{\mathcal{C}}$ being evaluated on various manifolds which leads us to study the diagram

$$\begin{tikzcd}
 & \mathrm{Z}_{\mathcal{C}}(\widehat{\Sigma}^{\underline{X},\underline{Y}}) \arrow[dd, "\mathrm{Z}_{\mathcal{C}}(\widehat{C_f}^{\underline{X},\underline{Y}})"] \\
\mathrm{Z}_{\mathcal{C}}(M_{\Gamma'}^{\underline{Y}}) \otimes_{\Bbbk} \mathrm{Z}_{\mathcal{C}}(-M_{\Gamma}^{\underline{X}}) \arrow[ur, "\mathrm{Z}_{\mathcal{C}}(M_{\Sigma}^{\underline{X},\underline{Y}})"] \arrow[dr, "\mathrm{Z}_{\mathcal{C}}(M_{\Sigma'}^{\underline{X},\underline{Y}})"'] & \\
 & \mathrm{Z}_{\mathcal{C}}(\widehat{\Sigma}'^{\underline{X},\underline{Y}})
\end{tikzcd} \tag{6.4.1}$$

in $\mathrm{vect}_{\Bbbk}$ with $\underline{X}\in\mathcal{C}^{\times\pi_0(\widehat{\Gamma})}$ and $\underline{Y}\in\mathcal{C}^{\times\pi_0(\widehat{\Gamma}')}$. Here $\widehat{C_f}^{\underline{X},\underline{Y}}$ is the mapping cylinder bordism induced from the mapping class f as in Section 5.1.2. Using functoriality of $\mathrm{Z}_{\mathcal{C}}$ we can instead check the diagram

$$\begin{tikzcd}
 & \widehat{\Sigma}^{\underline{X},\underline{Y}} \arrow[dd, "\widehat{C_f}^{\underline{X},\underline{Y}}"] \\
M_{\Gamma'}^{\underline{Y}} \sqcup -M_{\Gamma}^{\underline{X}} \arrow[ur, "M_{\Sigma}^{\underline{X},\underline{Y}}"] \arrow[dr, "M_{\Sigma'}^{\underline{X},\underline{Y}}"'] & \\
 & \widehat{\Sigma}'^{\underline{X},\underline{Y}}
\end{tikzcd} \tag{6.4.2}$$

in $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ for commutativity. In other words we need to find an isomorphism of defect 3-bordisms $M_{\Sigma'}^{\underline{X},\underline{Y}} \cong \widehat{C_f}^{\underline{X},\underline{Y}} \sqcup_{\widehat{\Sigma}^{\underline{X},\underline{Y}}} M_{\Sigma}^{\underline{X},\underline{Y}}$.

To this end first note that $f \times \mathrm{id}_{[-1,1]} \colon \Sigma \times [-1,1] \to \Sigma' \times [-1,1]$ descends to the quotients $\widetilde{f} \colon M_\Sigma \to M_{\Sigma'}$ because f is a morphism of world sheets and thus compatible with the free boundaries. Moreover, since f commutes with the boundary parametrisations we can extend $\widetilde{f}$ to $F \colon M_{\Sigma}^{\underline{X},\underline{Y}} \to M_{\Sigma'}^{\underline{X},\underline{Y}}$ which is an isomorphism of defect 3-manifolds because f takes the surface defect $\mathfrak{S}$ in $M_{\Sigma}^{\underline{X},\underline{Y}}$ to $\mathfrak{S}'$ in $M_{\Sigma'}^{\underline{X},\underline{Y}}$. Now finally note that $\widehat{C_f}^{\underline{X},\underline{Y}} \sqcup_{\widehat{\Sigma}^{\underline{X},\underline{Y}}} M_{\Sigma}^{\underline{X},\underline{Y}}$ differs from $M_{\Sigma}^{\underline{X},\underline{Y}}$ only in the outgoing boundary parametrisation and F is compatible with the one of $\widehat{C_f}^{\underline{X},\underline{Y}} \sqcup_{\widehat{\Sigma}^{\underline{X},\underline{Y}}} M_{\Sigma}^{\underline{X},\underline{Y}}$. Thus $F \colon \widehat{C_f}^{\underline{X},\underline{Y}} \sqcup_{\widehat{\Sigma}^{\underline{X},\underline{Y}}} M_{\Sigma}^{\underline{X},\underline{Y}} \to M_{\Sigma'}^{\underline{X},\underline{Y}}$ gives the desired isomorphism of defect 3-bordisms.

6.5 Factorisation / oplax naturality

We now want to study oplax naturality in the form of Diagram 4.3.5, or in other words the behaviour of Cor under the gluing of world sheets. Let $\mathfrak{S}_1 \colon \mathfrak{C}_1 \to \mathfrak{C}$ and $\mathfrak{S}_2 \colon \mathfrak{C} \to \mathfrak{C}_2$ be 1-morphisms in $\mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$. In analogy to Section 5.1.3 we can restrict our attention to the case where $|\pi_0(\Gamma)| = 1$ since gluing is a local procedure. This leaves us with two distinct cases to consider: $\mathfrak{C}$ is either a defect interval or a defect circle. We will refer to these cases as *boundary* and *bulk factorisation*, respectively.

As in the previous section we want to reduce Diagram 4.3.5 to a corresponding diagram in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ by considering it component wise and then use functoriality of $\mathrm{Z}_{\mathcal{C}}$. However, this is less straightforward here because Diagram 4.3.5 contains two morphisms whose components are not necessarily in the image of $\mathrm{Z}_{\mathcal{C}}$: The composition $\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}$ and the unit $\eta_{\mathrm{Cor}_{\mathfrak{C}}} \colon \mathrm{id}_{\mathrm{vect}_{\Bbbk}} \Rightarrow \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \diamond \mathrm{Cor}_{\mathfrak{C}}$ of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}_{\mathfrak{C}}^{\dagger}$.

Let us start with the composition $\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}$. For this recall that the horizontal composition of $\mathrm{Cor}_{\mathfrak{S}_1} \colon \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}_1}^{\dagger} \Rightarrow \mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}_1)$ and $\mathrm{Cor}_{\mathfrak{S}_2} \colon \mathrm{Cor}_{\mathfrak{C}_2} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \Rightarrow \mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}_2)$ is defined via the universal property of the coend as the unique natural transformation which makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Cor}_{\mathfrak{C}_2}(\underline{Z}) \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger}(\underline{Y}) \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}(\underline{Y}) \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}_1}^{\dagger}(\underline{X}) \arrow[r, "\mathrm{Cor}_{\mathfrak{S}_2}^{\underline{Y},\underline{Z}} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{S}_1}^{\underline{X},\underline{Y}}"] \arrow[d, "\mathrm{id} \otimes_{\Bbbk} \iota_{\underline{Y}} \otimes_{\Bbbk} \mathrm{id}"'] & \mathrm{Bl}(\mathfrak{S}_2)(\underline{Y},\underline{Z}) \otimes_{\Bbbk} \mathrm{Bl}(\mathfrak{S}_1)(\underline{X},\underline{Y}) \arrow[d, "I_{\underline{Y}}^{(\underline{X},\underline{Z})}"] \\
\mathrm{Cor}_{\mathfrak{C}_2}(\underline{Z}) \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}^{\dagger} \diamond \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}_1}^{\dagger}(\underline{X}) \arrow[r, dashed, "\exists!"] & (\mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}_2) \diamond \mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}_1))(\underline{X},\underline{Z})
\end{tikzcd} \tag{6.5.1}$$

Here $\underline{X} \in \mathcal{C}^{\times \pi_0(\widehat{\Gamma_1})}$, $\underline{Y} \in \mathcal{C}^{\times \pi_0(\hat{\Gamma})}$, $\underline{Z} \in \mathcal{C}^{\times \pi_0(\widehat{\Gamma_2})}$, and $\iota_{\underline{Y}}$ and $I_{\underline{Y}}^{(\underline{X},\underline{Z})}$ are the universal

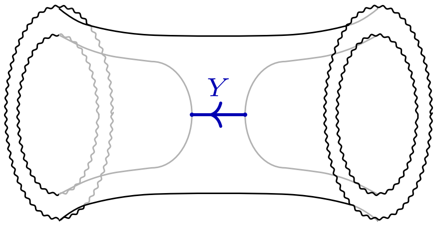


Figure 6.4: Local structure of the gluing bordism $G_{\underline{Y}}^{(\underline{X},\underline{Z})}$ for $\mathfrak{C}$ a defect interval. The incoming boundary surface of this bordism is drawn inside and the blue Y-labelled ribbon connects it to itself. The curly lines indicate interfaces at which the displayed piece is connected to other parts of the bordisms.

dinatural transformations. From Proposition 5.1.7 we know that the components of $I_{\underline{Y}}$ are obtained by evaluating the 3d TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ on (the double of) the gluing bordisms

$$G_{\underline{Y}}^{(\underline{X},\underline{Z})} \colon \widehat{\Sigma_2}^{(\underline{Y},\underline{Z})} \sqcup \widehat{\Sigma_1}^{(\underline{X},\underline{Y})} \to (\widehat{\Sigma_2} \sqcup_{\widehat{\Gamma}} \widehat{\Sigma_1})^{(\underline{X},\underline{Z})} \tag{6.5.2}$$

defined in Section 5.1.3. More concretely, the underlying manifold of $G_{\underline{Y}}^{(\underline{X},\underline{Z})}$ is given by

$$(\widehat{\Sigma_2}^{(\underline{Y},\underline{Z})} \sqcup \widehat{\Sigma_1}^{(\underline{X},\underline{Y})}) \times I/\sim \tag{6.5.3}$$

where $\sim$ identifies the discs glued into the components of $\widehat{\Gamma}$. For Γ an interval a local visualisation of $G_{\underline{Y}}^{(\underline{X},\underline{Z})}$ is given in Figure 6.4. For $\mathfrak{C}$ a defect circle one needs to use the double of the gluing bordism or equivalently the composition of the gluing bordisms for each component of $\widehat{\Gamma}$. In both cases $G_{\underline{Y}}^{(\underline{X},\underline{Z})}$ is a cylinder outside of the depicted local region. This allows us to extend its definition to the defect bordism category $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$. If the dinatural transformation $\iota_{\underline{Y}}$ is also induced by a family of gluing bordisms

$$\widetilde{G}_{\underline{Y}} \colon -M_{\Gamma}^{\underline{Y}} \sqcup M_{\Gamma}^{\underline{Y}} \to -M_{\Gamma} \sqcup_{\widehat{\Gamma}} M_{\Gamma}, \tag{6.5.4}$$

we can consider the corresponding diagram in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$:

$$\begin{tikzcd} M_{\Gamma_2}^{\underline{Z}} \sqcup -M_{\Gamma}^{\underline{Y}} \sqcup M_{\Gamma}^{\underline{Y}} \sqcup -M_{\Gamma_1}^{\underline{X}} \arrow[r, "M_{\Sigma_2}^{\underline{Y},\underline{Z}} \sqcup M_{\Sigma_1}^{\underline{X},\underline{Y}}"] \arrow[d, "\mathrm{id} \sqcup \widetilde{G}_{\underline{Y}} \sqcup \mathrm{id}"'] & \widehat{\Sigma_2}^{\underline{Y},\underline{Z}} \sqcup \widehat{\Sigma_1}^{\underline{X},\underline{Y}} \arrow[d, "G_{\underline{Y}}^{(\underline{X},\underline{Z})}"] \\ M_{\Gamma_2}^{\underline{Z}} \sqcup -M_{\Gamma} \sqcup_{\widehat{\Gamma}} M_{\Gamma} \sqcup -M_{\Gamma_1}^{\underline{X}} \arrow[r, dashed] & (\widehat{\Sigma_2} \sqcup_{\widehat{\Gamma}} \widehat{\Sigma_1})^{\underline{X},\underline{Z}} \end{tikzcd} \tag{6.5.5}$$

Moreover, let us further assume that we can find a bordism for the dashed arrow then functoriality of $\mathrm{Z}_{\mathcal{C}}$ and the uniqueness coming from the universal property

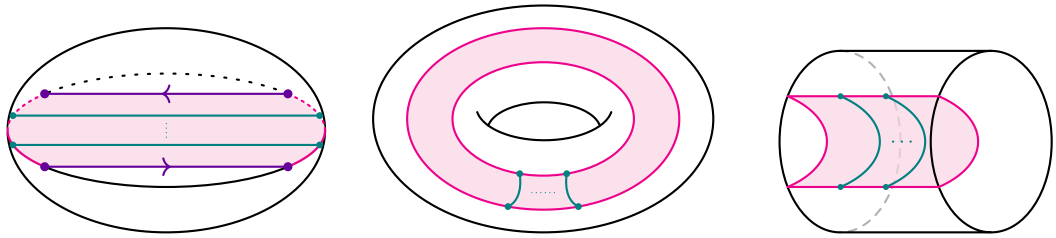


Figure 6.5: The unit morphism $\eta_{M_\Gamma}\colon \varnothing \to -M_{\Gamma'} \sqcup_{\widehat{\Gamma}'} M_{\Gamma'}$ for $\mathfrak{C}$ a defect interval (a) and a defect circle (b). As well as a cylinder (c) cut out of (b) to illustrate the stratification.

will guarantee that this bordism gets sent to the composition $\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}$. This motivates the next algebraic condition on $\mathrm{Z}_{\mathcal{C}}$:

Assumption 2. *For $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ a defect interval or circle, the universal dinatural transformation $\iota_{\underline{Y}}\colon \mathrm{Cor}_{\mathfrak{C}}(\underline{Y})^\dagger \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{C}}(\underline{Y}) \to \mathrm{Cor}_{\mathfrak{C}}^\dagger \diamond \mathrm{Cor}_{\mathfrak{C}}$ is induced by evaluating $\mathrm{Z}_{\mathcal{C}}$ on the family of gluing bordisms*

$$\widetilde{G}_{\underline{Y}}\colon -M_\Gamma^{\underline{Y}} \sqcup M_\Gamma^{\underline{Y}} \to -M_\Gamma \sqcup_{\widehat{\Gamma}} M_\Gamma \tag{6.5.6}$$

in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$.

Now for the second problem recall from Lemma 2.2.9 that $\eta_{\mathrm{Cor}_{\mathfrak{C}}}$ is given by the action of the constant functor $\mathbb{F}_{\mathfrak{C}}\colon \mathrm{vect}_{\Bbbk} \to \mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})$ on morphisms, i.e. by the linear map sending $1 \in \Bbbk$ to $\mathrm{id}_{\mathbb{F}_{\mathfrak{C}}}$ in $\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \mathbb{F}_{\mathfrak{C}}) \cong \mathrm{Cor}_{\mathfrak{C}}^\dagger \diamond \mathrm{Cor}_{\mathfrak{C}}$. Next note that the vector space $\mathrm{Cor}_{\mathfrak{C}}^\dagger \diamond \mathrm{Cor}_{\mathfrak{C}}$ is isomorphic to $\mathrm{Z}_{\mathcal{C}}(-M_\Gamma \sqcup_{\widehat{\Gamma}} M_\Gamma)$ by Assumption 2. We thus want to find a bordism $\eta_{M_\Gamma}\colon \varnothing \to -M_\Gamma \sqcup_{\widehat{\Gamma}} M_\Gamma$ such that $\mathrm{Z}_{\mathcal{C}}(\eta_{M_\Gamma})$ corresponds to $\mathrm{id}_{\mathbb{F}_{\mathfrak{C}}}$ under the isomorphism $\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \mathbb{F}_{\mathfrak{C}}) \cong \mathrm{Cor}_{\mathfrak{C}}^\dagger \diamond \mathrm{Cor}_{\mathfrak{C}}$. Moreover by monoidality it suffices to show this for $\mathfrak{C}$ a defect interval or a defect circle. This naturally leads us to the next condition on $\mathrm{Z}_{\mathcal{C}}$:

Assumption 3. *For $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$ connected the components of the unit $\eta_{\mathrm{Cor}_{\mathfrak{C}}}\colon \mathrm{id}_{\mathrm{vect}_{\Bbbk}} \Rightarrow \mathrm{Cor}_{\mathfrak{C}}^\dagger \diamond \mathrm{Cor}_{\mathfrak{C}}$ of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}_{\mathfrak{C}}^\dagger$ is given by the linear map $\mathrm{Z}_{\mathcal{C}}(\eta_{M_\Gamma})$ with $\eta_{M_\Gamma}\colon \varnothing \to -M_\Gamma \sqcup_{\widehat{\Gamma}} M_\Gamma$ as in Figure 6.5 under the isomorphism $\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \mathbb{F}_{\mathfrak{C}}) \cong \mathrm{Z}_{\mathcal{C}}(-M_\Gamma \sqcup_{\widehat{\Gamma}} M_\Gamma)$.*

Remark 6.5.1. The bordisms depicted in Figure 6.5 as candidates for the unit $\eta_{\mathrm{Cor}_{\mathfrak{C}}}$ of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}_{\mathfrak{C}}^\dagger$ can be informally motivated from once-extended 3d TFTs. Let us make this a bit more precise for Γ' a defect interval. The underlying manifold of M_Γ is a disc which we will view as the cup 1-morphism

$\varnothing \to S^1$ in the once-extended 3-dimensional bordism 2-category $\mathrm{Bord}_{3,2,1}$. Now recall from the conjectured generators and relations description of $\mathrm{Bord}_{3,2,1}$ discussed in [BDSV, Sec. 3] that this cup is one of the 1-morphism generators of $\mathrm{Bord}_{3,2,1}$ and its two-sided adjoint is given by a cap $S^1 \to \varnothing$. Moreover, the unit of this adjunction is given by a 3-ball, referred to as ν in [BDSV]. But this is precisely the 3-manifold underlying Figure 6.5 (a). For Γ a defect circle the idea is the same but one has to consider the composition of some of the generators because the underlying manifold of M_Γ is the composition of the cup generator with the up-side-down pair of pants generator. We will not discuss this point further here.

Under Assumptions 3 and 2 we are thus lead to study the following diagram in $\mathrm{Bord}_{3,2}^{\chi,\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$:

$$\begin{tikzcd}
M_{\Gamma_2}^{\underline{Z}} \sqcup -M_{\Gamma}^{\underline{Y}} \sqcup M_{\Gamma}^{\underline{Y}} \sqcup -M_{\Gamma_1}^{\underline{X}} \arrow[rr, "M_{\Sigma_2}^{\underline{Y},\underline{Z}} \sqcup M_{\Sigma_1}^{\underline{X},\underline{Y}}"] \arrow[d, "\mathrm{id}\sqcup \widetilde{G}_{\underline{Y}} \sqcup \mathrm{id}"'] & & \widehat{\Sigma_2}^{\underline{Y},\underline{Z}} \sqcup \widehat{\Sigma_1}^{\underline{X},\underline{Y}} \arrow[d, "G_{\underline{Y}}^{(\underline{X},\underline{Z})}"] \\
M_{\Gamma_2}^{\underline{Z}} \sqcup -M_{\Gamma} \sqcup_{\hat{\Gamma}} M_{\Gamma'} \sqcup -M_{\Gamma_1}^{\underline{X}} \arrow[rr, dashed] & & (\widehat{\Sigma_2} \sqcup_{\hat{\Gamma}} \widehat{\Sigma_1})^{\underline{X},\underline{Z}} \\
& M_{\Gamma_2}^{\underline{Z}} \sqcup -M_{\Gamma_1}^{\underline{X}} \arrow[ul, "\mathrm{id}\sqcup \eta_{M_\Gamma} \sqcup \mathrm{id}"] \arrow[ur, "M_{\Sigma_2 \sqcup_\Gamma \Sigma_1}^{\underline{X},\underline{Z}}"'] &
\end{tikzcd} \tag{6.5.7}$$

We will now show that this diagram commutes by studying the two cases mentioned above explicitly.

6.5.1 Boundary factorisation

Let us start with boundary factorisation, i.e. the case where $\mathfrak{C}$ is a defect interval. In contrast to [FjFSt, Sec. 4] we cannot assume that $\mathfrak{C}$ has no 0-strata in its interior as we do not have an explicit construction of the defect TFT $\mathrm{Z}_{\mathcal{C}}$. Instead we will restrict our attention to the case where $\mathfrak{C}$ has exactly one 0-stratum in the interior. The general case can be treated completely analogously but its pictorial presentation is more crowded.

The key observation to understand Diagram 6.5.7 is to note that all bordisms only differ in a neighbourhood of where the gluing procedure takes place. This is essentially a direct consequence of the locality of gluing, as in Section 5.1.3. In particular, it is sufficient to study Diagram 6.5.7 locally. To do this we will only

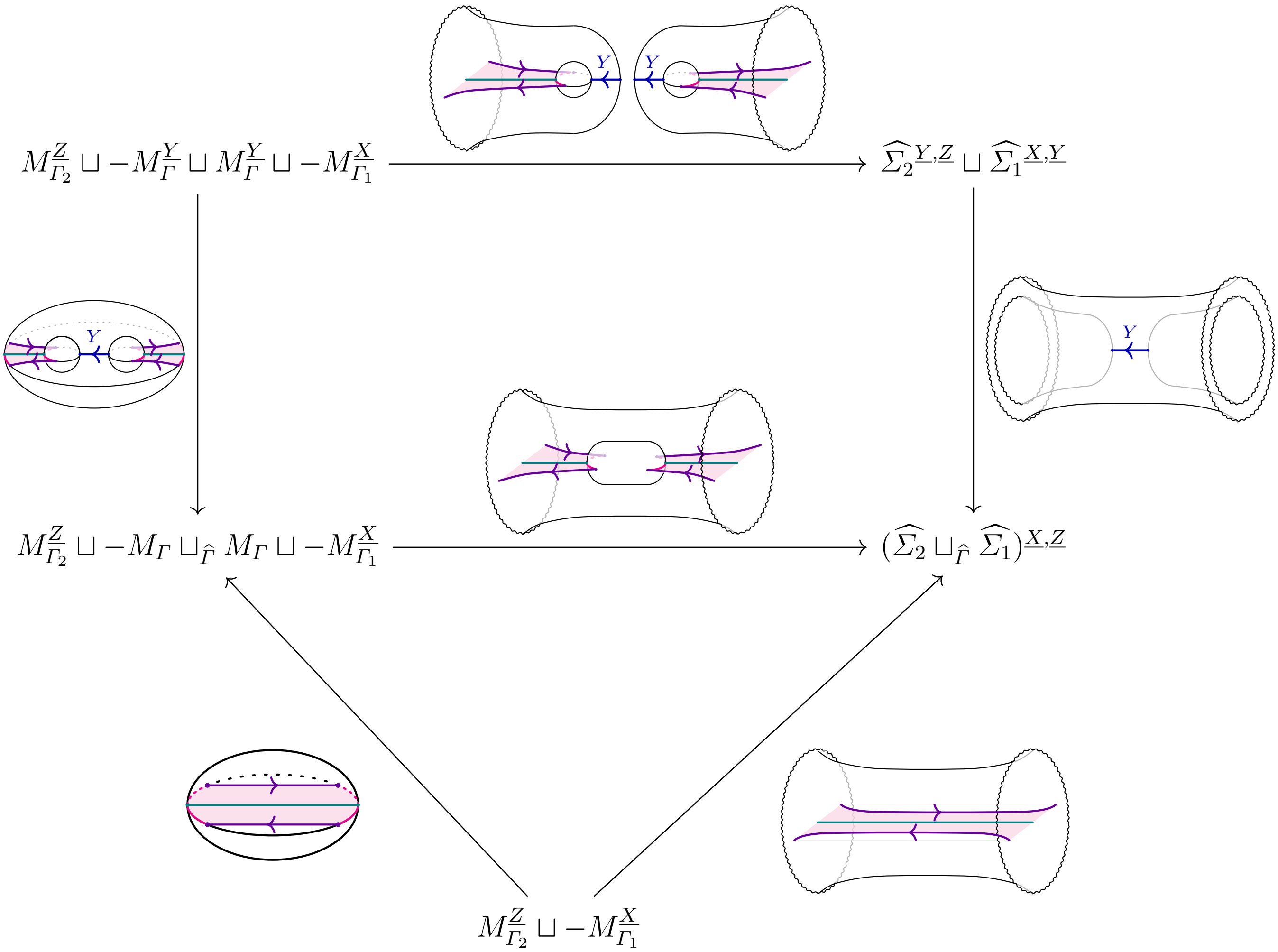
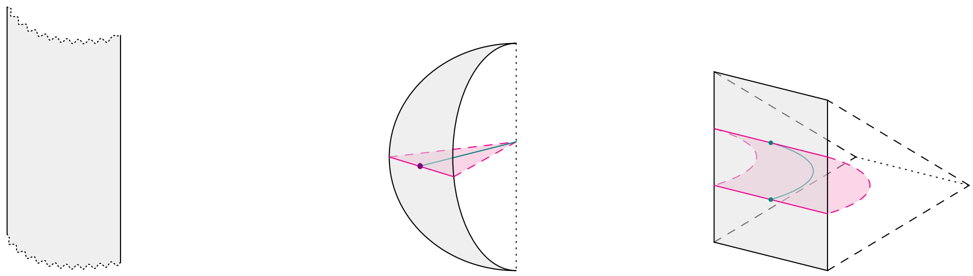


Figure 6.6: Diagram 6.5.7 for $\mathfrak{C}$ a defect interval in a neighbourhood of $\mathfrak{C}$. For both arrows on the left only the non-identity bordism components are drawn. The curly lines indicate interfaces at which the displayed piece is connected to other parts of the bordisms.

draw the corresponding local segments of the bordisms in analogy to the local illustration of the gluing bordism given in Figure 6.4. The resulting local version of Diagram 6.5.7 is illustrated in Figure 6.6.

Now first note that the illustrated bordism for the lower horizontal line indeed makes the square commute. As discussed above, this implies that this bordism represents the composition $\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}$. Informally we can obtain this bordism by cutting the left gluing bordism out of the composition of the top horizontal and the right gluing bordism. Finally, it is clear that with this bordism also the triangle commutes.



(a) (b) (c)

Figure 6.7: Examples of a cylindrical part of a defect surface (a), a defect ball (b), and (c) the defect solid torus from Figure 6.5b.

6.5.2 Bulk factorisation

Let us now come to bulk factorisation. As before we will focus on the case where $\mathfrak{C}$ is a defect circle with exactly one 0-stratum. To study Diagram 6.5.7 we will again restrict our attention to a local neighbourhood of the gluing process.

Before we continue with the local version of Diagram 6.5.7 it will be beneficial to introduce the "wedge presentation" of defect 2- and 3-manifolds building on the one introduced in [FjFRS, Sect. 5.1]. The idea behind this presentation is analogous to the presentation of a torus as a rectangle with opposite sides identified. In the wedge presentation we draw a horizontal disc as a disc sector with an identification of the legs of the sector is implied. As above we will use curly lines to indicate interfaces at which the illustrated piece is connected to other parts of the manifold and we will use dashed lines to indicate further identifications of the legs. Moreover, for clarity we will explicitly highlight the boundary components in this presentation using a light gray colouring. Examples of a cylindrical part of a defect surface, a defect ball, and the defect solid torus from Figure 6.5b are illustrated in Figure 6.7.

With this preparation we can now study Diagram 6.5.7. The local version in the wedge presentation is illustrated in Figure 6.8.

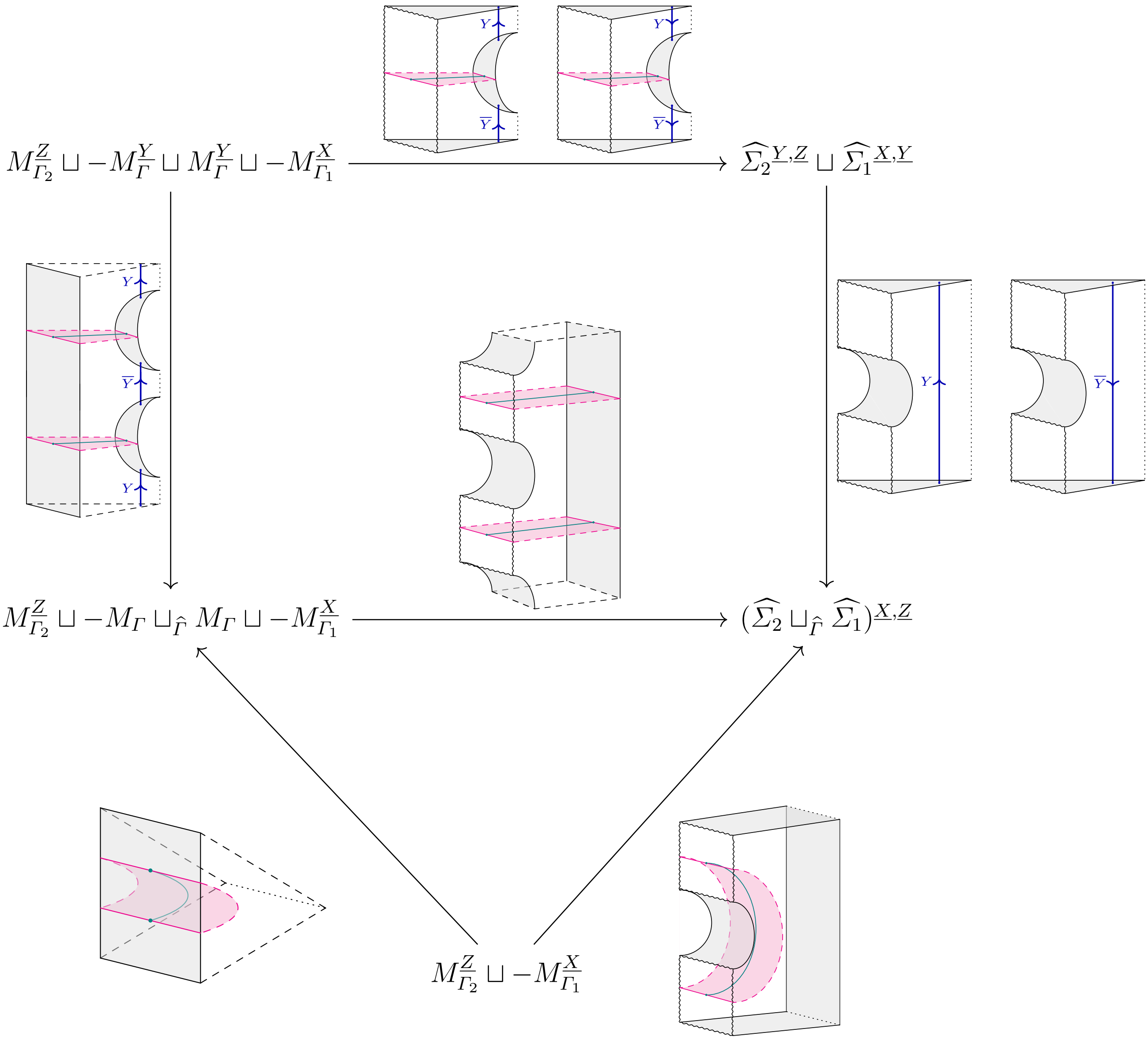
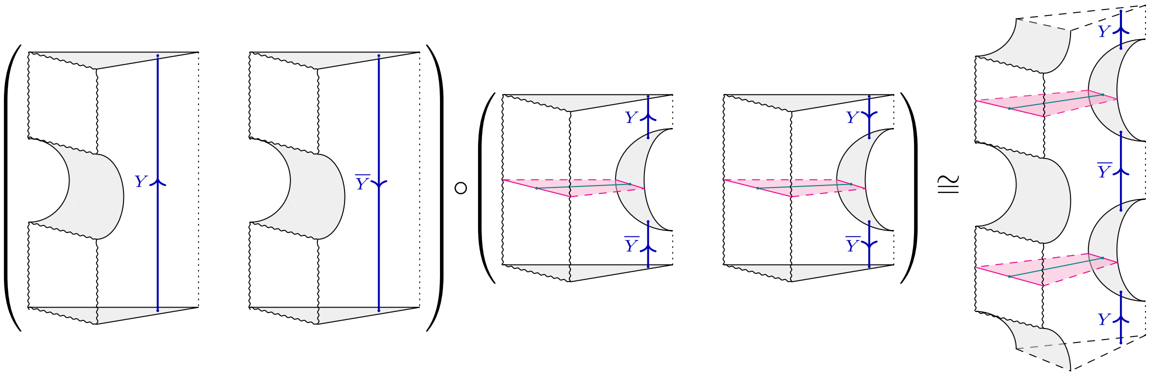


Figure 6.8: Wedge presentation of diagram 6.5.7 for $\mathfrak{C}$ a defect circle in a neighbourhood of $\mathfrak{C}$. For both arrows on the left only the non-identity bordism components are drawn.

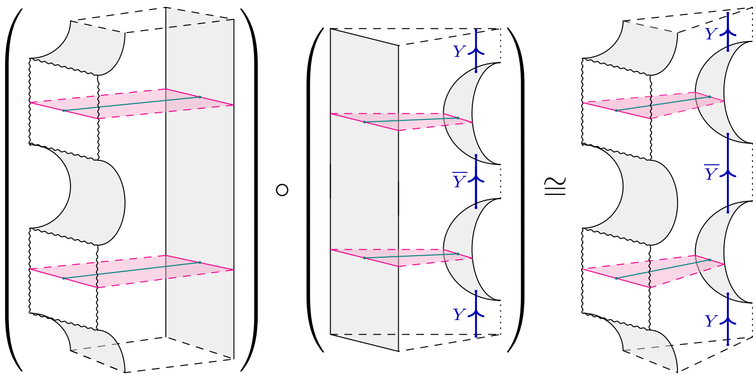
To compose bordisms in the wedge presentation we identify the corresponding gray shaded surface areas. In particular the composition of the top arrow with the right arrow amounts to



(6.5.8)

Note here that the top and bottom triangle are identified.

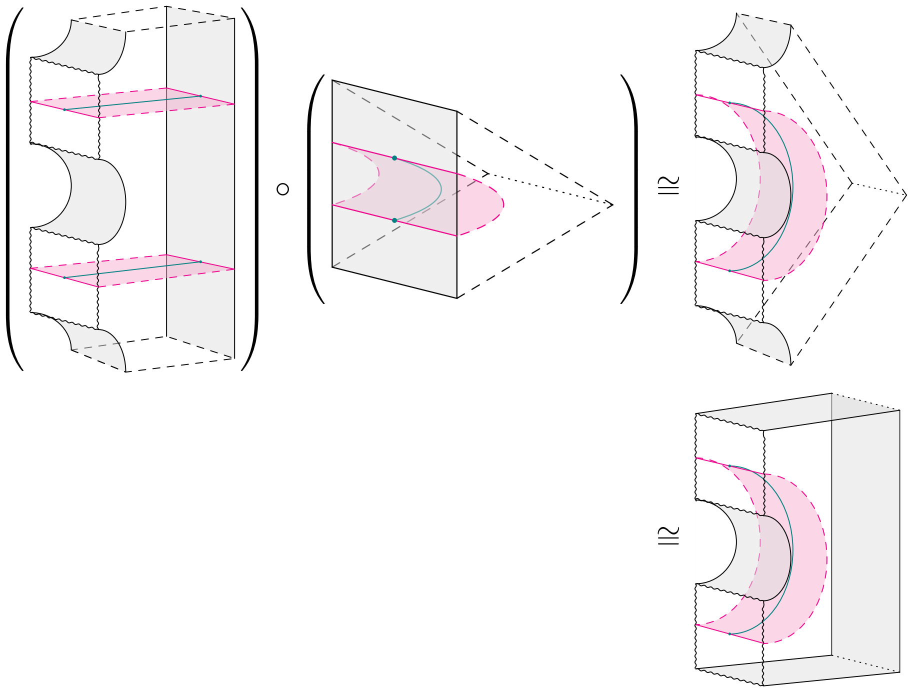
Analogously the composition of the left arrow with the dashed arrow in the middle is given by



(6.5.9)

Which shows that the dashed arrow indeed makes the square commute and thus represents the composition $\mathrm{Cor}_{\mathfrak{S}_2} \diamond \mathrm{Cor}_{\mathfrak{S}_1}$.

Finally, for the triangle we have



(6.5.10)

where the second isomorphism uses the identification of the rectangular regions, see also [FjFRS, Sec. 5.6]. With this we find that also the triangle commutes.

6.6 Non-degeneracy / oplax unitality

The final axiom we need to check is the oplax unitality axiom in the form of Equation (4.3.6). This is a purely algebraic condition on $\mathrm{Z}_{\mathcal{C}}$ and thus leads to our final assumption:

Assumption 4. *For any $\mathfrak{C} \in \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$, the components of the counit $\varepsilon_{\mathrm{Cor}_{\mathfrak{C}}} \colon \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}^{\dagger}_{\mathfrak{C}} \Rightarrow \mathrm{Bl}(\mathrm{id}_{\mathfrak{C}})$ of the adjunction $\mathrm{Cor}_{\mathfrak{C}} \dashv \mathrm{Cor}^{\dagger}_{\mathfrak{C}}$ are given by $\mathrm{Z}_{\mathcal{C}}(M^{X,Y}_{\Gamma \times I})$.*

Remark 6.6.1. For later use it is beneficial to note that $\varepsilon_{\mathrm{Cor}_{\mathfrak{C}}} \in \mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}^{\dagger}_{\mathfrak{C}}, \mathrm{Bl}(\mathrm{id}_{\mathfrak{C}}))$ gets sent to $\mathrm{id}_{\mathbb{F}_{\mathfrak{C}}} \in \mathrm{Hom}_{\mathrm{Bl}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \mathbb{F}_{\mathfrak{C}})$ under the isomorphism $\mathrm{Nat}(\mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}^{\dagger}_{\mathfrak{C}}, \mathrm{Bl}(\mathrm{id}_{\mathfrak{C}})) \cong \mathrm{Hom}_{\mathrm{Bl}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \mathbb{F}_{\mathfrak{C}})$ obtained by combining the isomorphisms 4.3.2 and 4.3.3 with unitality of the modular functor, i.e. $\mathrm{Bl}(\mathrm{id}_{\mathfrak{C}})(-,-) \cong \mathrm{Hom}_{\mathrm{Bl}(\mathfrak{C})}(-,-)$.

Putting everything together we arrive at the second main result of this thesis:

Theorem 6.6.2. Let

$$\mathrm{Z}_{\mathcal{C}} \colon \mathrm{Bord}^{\chi,\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}}) \to \mathrm{vect}_{\Bbbk} \tag{6.6.1}$$

be a 3d defect TFT extending the 3d TFT with embedded ribbon graphs

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{6.6.2}$$

of [DGGPR1] in the sense of Definition 6.1.1. If $\mathrm{Z}_{\mathcal{C}}$ satisfies Assumptions 1, 2, 3, and 4 then evaluating it on the connecting manifolds defines a full conformal field theory based on $\mathrm{Bl}_{\mathcal{C}}$, i.e. a braided monoidal oplax natural transformation

$$\begin{tikzcd} \mathfrak{WS}(\mathbb{D}_{\mathcal{C}}) \arrow[rr, bend left, "\Delta_{\Bbbk}"] \arrow[rr, bend right, "\mathrm{Bl}_{\mathcal{C}}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Cor}"] & & \mathcal{P}\mathrm{rof}^{\mathcal{L}\mathrm{ex}}_{\Bbbk} \end{tikzcd} . \tag{6.6.3}$$

Remark 6.6.3. For $\mathcal{C}$ semisimple our construction recovers the work of [FRSI] by considering the defect TFT constructed in [CRS1]. To be more precise the 2-morphism components of Cor recover their correlators while the 1-morphism components of Cor correspond to the vector spaces used in the analysis of the field content explained in [FRSIV, Sec. 3]. A slightly more detailed comparison will be given in Section 8.3.4. We also want to mention that the naturality proofs above are strongly informed by the topological considerations in the proofs given in [FjFRS; FjFSt]. In particular, our discussion of bulk factorisation is heavily inspired by the ideas and argument in [FjFRS, Sec. 5]. The reason the algebraic arguments in their proofs are absent from our discussion is because they are hidden in the defect TFT.

Chapter 7

Example: Diagonal CFT

In this chapter we study the simplest non-trivial situation to which our construction applies, the one with defect TFT given by $\mathrm{Z}_{\mathcal{C}} = \widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk}$. In CFT terminology this example corresponds to the so-called *diagonal/charge-conjugate/Cardy* case [FGSS]. As explained at the end of Section 3.2 the defect data $\mathbb{D}_{\mathcal{C}}$ is completely encoded in the modular tensor category $\mathcal{C}$ itself with the label set of 1-strata D_1 the objects of $\mathcal{C}$, the label set of 0-strata D_0 the morphisms of $\mathcal{C}$, and the other two label sets one element sets which we take to be $\{\mathbb{1}\}$. In particular the free boundaries are also labelled with objects in $\mathcal{C}$. Let us denote the corresponding world sheet 2-category by $\mathfrak{WS}_{\mathcal{C}} := \mathfrak{WS}(\mathbb{D}_{\mathcal{C}})$.

Before we can compute CFT quantities we first need to check that $\widehat{\mathrm{V}}_{\mathcal{C}}$ satisfies the conditions of Theorem 6.6.2. Assumptions 1 and 2 are Lemma 5.1.2 and Proposition 5.1.7, respectively. For Assumption 3 we need to compare the image of the bordisms depicted in Figure 6.5 for any defect interval and defect circle under $\widehat{\mathrm{V}}_{\mathcal{C}}$ with the identity of the corresponding field content. Due to this we will first compute the field content in Section 7.1 and then verify Assumption 3. Afterwards we will discuss Assumption 4 in detail in Section 7.2. Finally, in Section 7.3 we will compute some quantities of physical interest including boundary states as well as annulus partition functions and compare our results to the ones the proposed in [FGSS]. We will also compute the action of a line defect on bulk fields which to our knowledge has not been considered in the non-semisimple setting before.

7.1 Field content

Let us start with the field content. First note that monoidality of $\mathcal{C}$ allows us to reduce our discussion to understanding the case of a defect interval with no point defects in its interior and of a defect circle with one point defect. Informally this can be seen as the result of fusing the defects.

Boundary fields

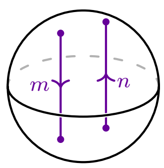
Let $m, n \in \mathcal{C}$ and let $I_{n,m} \in \mathfrak{WS}_{\mathcal{C}}$ be the defect interval with underlying manifold the standard interval $I = [0,1]$ oriented from 0 to 1, and free boundaries labelled with m at $\{0\} \subset I$ and n at $\{1\} \subset I$. For $X \in \mathcal{C}$ the corresponding connecting manifold $M^X_{I_{n,m}}$ is a three punctured sphere with the X and m puncture positively oriented and the n puncture negatively oriented. The reason for these orientations is because we want to think of as the embedded interval as ingoing while the X-labelled marked point should be outgoing. From this we get

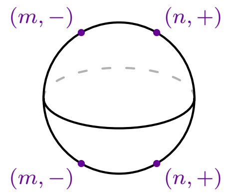
$$\begin{aligned} \widehat{\mathrm{V}}_{\mathcal{C}}\left(M^X_{I_{n,m}}\right) &\cong \mathrm{Hom}_{\mathcal{C}}(n, m \otimes X) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(m^* \otimes n, X), \end{aligned} \tag{7.1.1}$$

where we used that state spaces of the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ are isomorphic to morphism spaces in $\mathcal{C}$ [DGGPR1, Prop. 4.17]. We can now read off the boundary field content as

$$\mathbb{F}_{I_{n,m}} = m^* \otimes n. \tag{7.1.2}$$

For Assumption 3 we need to consider the bordism from Figure 6.5a. In our setting this reduces to

$\eta_{M_{I_{n,m}}} =$ 

$: \varnothing \longrightarrow$ 

(7.1.3)

It is now straightforward to see that $\mathrm{Z}_{\mathcal{C}}(\eta_{M_{I_{n,m}}})$ corresponds to $\mathrm{id}_{m^* \otimes n}$ under the isomorphism between the state spaces of $\widehat{\mathrm{V}}_{\mathcal{C}}$ with morphism spaces in $\mathcal{C}$, as desired.

Bulk and disorder fields

Let $k \in \mathcal{C}$ and let $S^1_k \in \mathfrak{WS}_{\mathcal{C}}$ be a defect circle with a single negatively oriented 0-stratum at $(0,-1) \in S^1 \subset \mathbb{R}^2$ labelled with k. For $(X, \overline{X}) \in \mathcal{C} \times \overline{\mathcal{C}} \cong \mathcal{C}^{\pi_0(\widehat{S^1})}$, see e.g. the paragraph at the end of Section 4.1 why $\overline{\mathcal{C}}$ is used. The connecting manifold $M^{(X,\overline{X})}_{S^1}$ is topologically a three-punctured 2-sphere. With this we compute

$$\begin{aligned} \widehat{\mathrm{V}}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{S^1_k}\right) &\cong \mathrm{Hom}_{\mathcal{C}}(k, X \otimes \overline{X}) \\ &\cong \mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(F(k), X \boxtimes \overline{X}) \end{aligned} \tag{7.1.4}$$

where $F\colon \mathcal{C} \to \mathcal{C} \boxtimes \overline{\mathcal{C}}$ is the two-sided adjoint of the monoidal product functor $\otimes\colon \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{C}$. From this we get

$$\mathbb{F}_{S^1_k} \cong F(k) \cong \int^{X \in \mathcal{C}} X^* \otimes k \boxtimes X \in \mathcal{C} \boxtimes \overline{\mathcal{C}}. \tag{7.1.5}$$

In particular for $k = \mathbb{1}$ we get the bulk field content

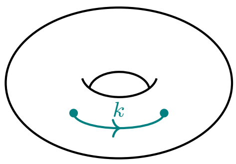
$$\mathbb{F}_{S^1_{\mathbb{1}}} \cong \int^{X \in \mathcal{C}} X^* \boxtimes X = L \in \mathcal{C} \boxtimes \overline{\mathcal{C}}. \tag{7.1.6}$$

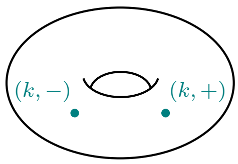
For $\mathcal{C}$ semisimple this is simply $L \cong \bigoplus_{i \in \mathrm{Irr}} i^* \boxtimes i$ where Irr is a set of representatives of simple objects in $\mathcal{C}$. With this we reproduced the known bulk fields for the rational full CFT with charge-conjugate partition function justifying the name also in the non-semisimple setting. More generally we want to highlight here that $\mathbb{F}_{I_{n,m}}$ and $\mathbb{F}_{S^1_{\mathbb{1}}}$ are precisely the field content proposed in [FGSS].

For Assumption 3 first recall the equivalence $\mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathcal{C}_{\mathrm{L}}$ to the category of right L-modules in $\mathcal{C}$ from Section 2.1.2. Under this equivalence $F(-)$ is simply the free functor $- \otimes \mathrm{L}$. Next note that $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M^{\dagger}_{S^1_k} \sqcup_{\widehat{S^1_k}} M_{S^1_k}\right) \cong \mathrm{Hom}_{\mathcal{C}}(k \otimes \mathrm{L}, k) \cong \mathrm{Hom}_{\mathrm{L}}(k \otimes \mathrm{L}, k \otimes \mathrm{L})$. Where the second isomorphism is explicitly given by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{L}}(X, k \otimes \mathrm{L}) &\to \mathrm{Hom}_{\mathcal{C}}(X, k) \\ f &\mapsto (\mathrm{id}_k \otimes \lambda) \circ f \end{aligned} \tag{7.1.7}$$

and exhibits $- \otimes \mathrm{L}$ as the right adjoint of the forgetful functor $\mathcal{C}_{\mathrm{L}} \to \mathcal{C}$. The bordism we have to consider is

$\eta_{S^1_k} =$ 

$: \varnothing \longrightarrow$ 

(7.1.8)

We can get $\widehat{\mathrm{V}}_{\mathcal{C}}\left(\eta_{S^1_k}\right)$ using the sliding trick employed in Section 5.1.3 for $X = \mathbb{1}$, or more precisely by setting $X = \mathbb{1}$ in (5.1.19). After a straightforward calculation in $\mathcal{C}$ this then leads to $\widehat{\mathrm{V}}_{\mathcal{C}}\left(\eta_{S^1_k}\right) = \lambda \otimes \mathrm{id}_k \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{L} \otimes k, k)$. Which is the image of $\mathrm{id}_{\mathrm{L} \otimes k} \in \mathrm{Hom}_{\mathrm{L}}(\mathrm{L} \otimes k, \mathrm{L} \otimes k)$ under (7.1.7).

7.2 Two point correlators

We now turn to study Assumption 4. To this end first recall Lemma 2.2.9 that the components of the counit $\varepsilon_{\mathrm{Cor}_{\mathfrak{C}}}\colon \mathrm{Cor}_{\mathfrak{C}} \otimes_{\Bbbk} \mathrm{Cor}^{\dagger}_{\mathfrak{C}} \Rightarrow \mathrm{Bl}_{\mathcal{C}}(\mathrm{id}_{\mathfrak{C}})$ are induced by the composition map

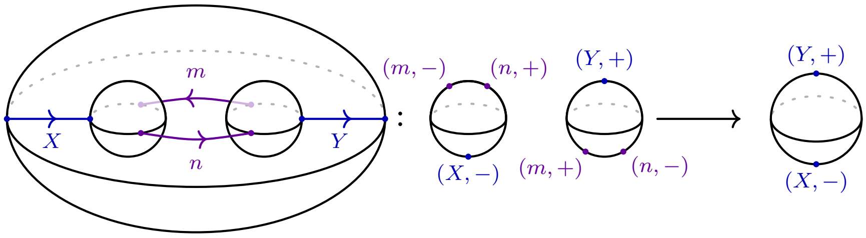
$$\mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\underline{X}, \mathbb{F}_{\mathfrak{C}}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\mathbb{F}_{\mathfrak{C}}, \underline{Y}) \to \mathrm{Hom}_{\mathrm{Bl}_{\mathcal{C}}(\mathfrak{C})}(\underline{X}, \underline{Y}). \tag{7.2.1}$$

7.2.1 Boundary

Let us start with $I_{n,m}$ the defect interval considered above. For $X, Y \in \mathcal{C}$ the connecting bordism

$$M_{\mathrm{id}_{I_{n,m}}}^{(X,Y)} \colon -M_{I_{n,m}}^X \sqcup M_{I_{n,m}}^Y \to \widehat{\mathrm{id}_{\mathrm{I_{n,m}}}}^{(X,Y)} \tag{7.2.2}$$

of $\mathrm{id}_{I_{n,m}}$ is given by



(7.2.3)

Here and below we will always use blue ribbons for "free labels" X, Y, i.e. the components of the natural transformation $\mathrm{cor}_{\mathrm{id}_{n,m}}$. Applying $\widehat{\mathrm{V}}_{\mathcal{C}}$ and using the isomorphism between TFT state spaces and Hom-spaces in $\mathcal{C}$ we immediately get the linear map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, m^* \otimes n) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(m^* \otimes n, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ f \otimes_{\Bbbk} g &\mapsto g \circ f. \end{aligned} \tag{7.2.4}$$

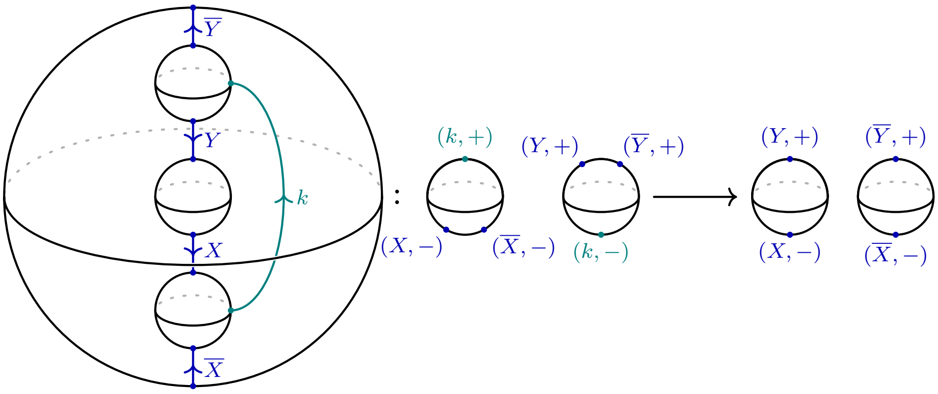
Thus $\mathrm{cor}_{\mathrm{id}_{n,m}}$ induces the same map as the composition map.

7.2.2 Bulk

Let $S^1_k \in \mathfrak{WS}_{\mathcal{C}}$ be the defect circle from above. For $(X, \overline{X}), (Y, \overline{Y}) \in \mathcal{C} \times \overline{\mathcal{C}}$ the connecting bordism

$$M_{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})} \colon -M_{S^1_k}^{(X,\overline{X})} \sqcup M_{S^1_k}^{(Y,\overline{Y})} \to \widehat{\mathrm{id}_{\mathrm{S^1_k}}}^{(X,\overline{X},Y,\overline{Y})} \tag{7.2.5}$$

of $\mathrm{id}_{S^1_k}$ is given by a bordism which has as underlying 3-manifold S^3 with two incoming and two outgoing 3-balls cut out. Since this does cannot be as nicely visualised, we instead draw it as a standard 3-ball with three smaller 3-balls cut out and read the middle interior boundary sphere as outgoing:



$$. \qquad (7.2.6)$$

Applying the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ we get a linear map

$$\widehat{\mathrm{V}}_{\mathcal{C}}\left(M^{(X,\overline{X},Y,\overline{Y})}_{\mathrm{id}_{S^1_k}}\right) : \widehat{\mathrm{V}}_{\mathcal{C}}\left(-M^{(X,\overline{X})}_{S^1_k} \sqcup M^{(Y,\overline{Y})}_{S^1_k}\right) \to \widehat{\mathrm{V}}_{\mathcal{C}}\left(\widehat{\mathrm{id}_{\mathrm{S}^1_{\mathrm{k}}}}^{(X,\overline{X},Y,\overline{Y})}\right) \qquad (7.2.7)$$

which we want to compare to the composition map

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(X\boxtimes\overline{X}, F(k))\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(F(k), Y\boxtimes\overline{Y}) \to \mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(X\boxtimes\overline{X}, Y\boxtimes\overline{Y}). \quad (7.2.8)$$

To this end it will again be useful to employ the equivalence $\mathcal{C}\boxtimes\overline{\mathcal{C}} \simeq \mathcal{C}_{\mathrm{L}}$. Under this equivalence the relevant adjunction isomorphisms are given by

$$\begin{aligned} \varphi\colon \mathrm{Hom}_{\mathcal{C}}(k, Y\otimes\overline{Y}) &\to \mathrm{Hom}_{\mathrm{L}}(k\otimes\mathrm{L}, Y\otimes\overline{Y}) \\ g &\mapsto \rho_{Y\otimes\overline{Y}}\circ(g\otimes\mathrm{id}_{\mathrm{L}}) \end{aligned} \qquad (7.2.9)$$

and

$$\begin{aligned} \psi\colon \mathrm{Hom}_{\mathcal{C}}(X\otimes\overline{X}, k) &\to \mathrm{Hom}_{\mathrm{L}}(X\otimes\overline{X}, k\otimes\mathrm{L}) \\ f &\mapsto \mathcal{D}^{-1}(f\otimes\mathrm{id}_{\mathrm{L}})\circ\delta^{\Lambda}_{X\otimes\overline{X}} \end{aligned} \qquad (7.2.10)$$

where $\rho_{Y\otimes\overline{Y}}$ is the canonical L action from Lemma 2.1.5 $\delta^{\Lambda}_{X\otimes\overline{X}}$ is the coaction on $X\otimes\overline{X}$ obtained by precomposing the canonical action $\rho_{X\otimes\overline{X}}$ with the Radford copairing $(S\otimes\mathrm{id})\circ\Delta\circ\Lambda$ from Proposition 2.1.2, and $\mathcal{D}$ is the choice of a square root of the modularity parameter ζ from Section 3.1.1. Note that introducing the factor $\mathcal{D}^{-1}$ is merely a convention and we choose this one because it will make the rest of our argument more transparent. Next recall the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_{\Bbbk}\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{X},\overline{Y}) \cong \mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(X\boxtimes\overline{X}, Y\boxtimes\overline{Y}) \cong \mathrm{Hom}_{\mathrm{L}}(X\otimes\overline{X}, Y\otimes\overline{Y}) \qquad (7.2.11)$$

coming from the universal property of the Deligne product [EGNO, Prop. 1.11.2] and the equivalence $\mathcal{C}\boxtimes\overline{\mathcal{C}}\simeq\mathcal{C}_{\mathrm{L}}$. Combining these isomorphisms with the isomorphisms between TFT state spaces and morphism spaces in $\mathcal{C}$ leads us to study commutativity of the following diagram:

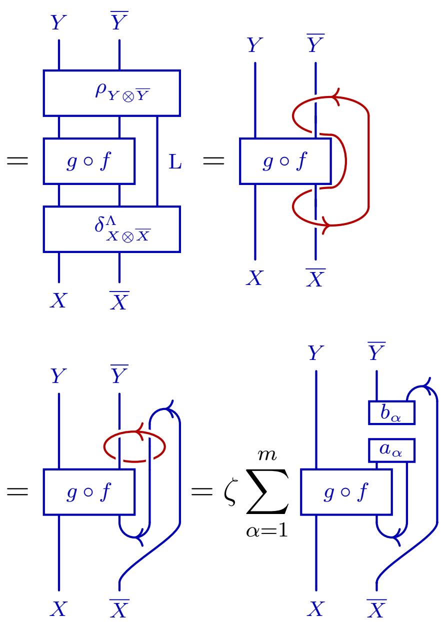
$$\begin{tikzcd}
\hat{\mathrm{V}}_{\mathcal{C}}\left(-M_{S^1_k}^{(X,\overline{X})}\sqcup M_{S^1_k}^{(Y,\overline{Y})}\right) \arrow[rr, "\hat{\mathrm{V}}_{\mathcal{C}}\left(M_{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})}\right)"] \arrow[d, "\cong"'] & & \hat{\mathrm{V}}_{\mathcal{C}}\left(\widehat{\mathrm{id}_{\mathrm{S}^1_{\mathrm{k}}}}^{(X,\overline{X},Y,\overline{Y})}\right) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{C}}(X\otimes\overline{X},k)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{C}}(k,Y\otimes\overline{Y}) \arrow[d, "\psi\otimes_{\Bbbk}\varphi"'] & & \mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_{\Bbbk}\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{X},\overline{Y}) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{L}}(X\otimes\overline{X},k\otimes\mathrm{L})\otimes_{\Bbbk}\mathrm{Hom}_{\mathrm{L}}(k\otimes\mathrm{L},Y\otimes\overline{Y}) \arrow[rr, "\circ"] & & \mathrm{Hom}_{\mathrm{L}}(X\otimes\overline{X},Y\otimes\overline{Y})
\end{tikzcd} \tag{7.2.12}$$

Let us first analyse the purely algebraic part. To this end consider the linear map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X\otimes\overline{X},k)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{C}}(k,Y\otimes\overline{Y}) &\to \mathrm{Hom}_{\mathrm{L}}(X\otimes\overline{X},Y\otimes\overline{Y})\\ f\otimes_{\Bbbk} g &\mapsto \mathcal{D}^{-1}\rho_{Y\otimes\overline{Y}}\circ(g\circ f\otimes\mathrm{id}_{\mathrm{L}})\circ\delta^{\Lambda}_{X\otimes\overline{X}}.\end{aligned} \tag{7.2.13}$$

In terms of bichrome graphs we can rewrite this as follows

$$\rho_{Y\otimes\overline{Y}}\circ(g\circ f\otimes\mathrm{id}_{\mathrm{L}})\circ\delta^{\Lambda}_{X\otimes\overline{X}} =$$



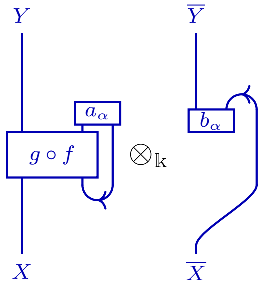
(7.2.14)

Where in the last step we used Lemma 2.1.8 to get $m\geqslant 0$, $a_\alpha\colon\overline{Y}\otimes\overline{X}^{*}\to\mathbb{1}$, and

$b_\alpha \colon \mathbb{1} \to \overline{Y} \otimes \overline{X}^*$. Thus the map

$$\Psi \colon \mathrm{Hom}_{\mathcal{C}}(X \otimes \overline{X}, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(k, Y \otimes \overline{Y}) \to \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\Bbbk} \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{X}, \overline{Y})$$

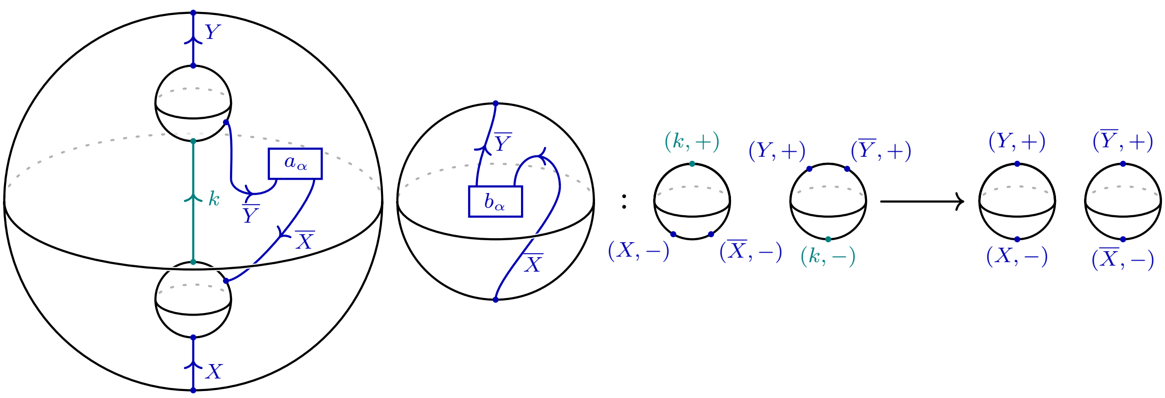
$$f \otimes_{\Bbbk} g \mapsto= \mathcal{D} \sum_{\alpha=1}^{m} \qquad \otimes_{\Bbbk} \qquad (7.2.15)$$



makes the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(X \otimes \overline{X}, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(k, Y \otimes \overline{Y}) & \xrightarrow{\quad \Psi \quad} & \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\Bbbk} \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{X}, \overline{Y}) \\
\Big\downarrow{\scriptstyle \psi \otimes_{\Bbbk} \varphi} & & \Big\downarrow{\scriptstyle \cong} \\
\mathrm{Hom}_{\mathrm{L}}(X \otimes \overline{X}, k \otimes \mathrm{L}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{L}}(k \otimes \mathrm{L}, Y \otimes \overline{Y}) & \xrightarrow{\quad \circ \quad} & \mathrm{Hom}_{\mathrm{L}}(X \otimes \overline{X}, Y \otimes \overline{Y})
\end{array}$$

commute. Next for a_α, b_α as above we define the family of bordisms $M_\alpha^{(X,\overline{X},Y,\overline{Y})}$:



$$. \qquad (7.2.16)$$

By construction this family satisfies $\Psi = \mathcal{D} \sum_{\alpha=1}^m \widehat{\mathrm{V}}_{\mathcal{C}} \left(M_\alpha^{(X,\overline{X},Y,\overline{Y})} \right)$. With this we can reformulate commutativity of Diagram 7.2.12 to the equality

$$\widehat{\mathrm{V}}_{\mathcal{C}} \left(M_{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})} \right) = \mathcal{D} \sum_{\alpha=1}^{m} \widehat{\mathrm{V}}_{\mathcal{C}} \left(M_\alpha^{(X,\overline{X},Y,\overline{Y})} \right) \qquad (7.2.17)$$

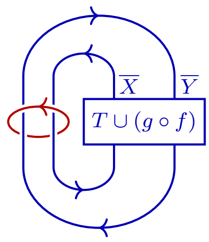
of linear maps. To show this equality we will compare the corresponding matrix elements. Recall from Section 3.1.4 that the state space $\widehat{\mathrm{V}}_{\mathcal{C}} \left(-M_{S^1_k}^{(X,\overline{X})} \sqcup M_{S^1_k}^{(Y,\overline{Y})} \right)$ is spanned by bordisms $B_{f,g}\colon \varnothing \to -M^{(X,\overline{X})}_{S^1_k} \sqcup M^{(Y,\overline{Y})}_{S^1_k}$ with underlying 3-manifold the disjoint union of two 3-balls and embedded ribbon graphs given by $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes \overline{X}, k)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(k, Y \otimes \overline{Y})$. For the dual state space $\hat{\mathrm{V}}_{\mathcal{C}}\left(\widehat{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})}\right)^*$ recall from [DGGPR1, Prop. 4.11] that it is generated by admissible bichrome graphs T, i.e. T containing at least one blue edge labelled with a projective object of $\mathcal{C}$, inside a fixed connected 3-manifold viewed as a bordism $N_T\colon \widehat{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})} \to \varnothing$.[1] In particular, we can choose the manifold underlying N_T to be given by $S^2 \times I$. With this we are lead to study the invariants of the closed 3-manifolds with admissible ribbon graphs given by:

$$A = N_T \bigsqcup_{\widehat{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})}} M^{(X,\overline{X},Y,\overline{Y})}_{\mathrm{id}_{S^1_k}} \bigsqcup_{-M^{(X,\overline{X})}_{S^1_k} \sqcup M^{(Y,\overline{Y})}_{S^1_k}} B_{f,g} \tag{7.2.18}$$

and

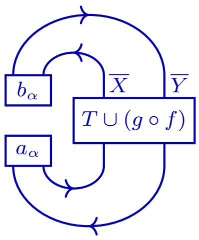
$$B_\alpha = N_T \bigsqcup_{\widehat{\mathrm{id}_{S^1_k}}^{(X,\overline{X},Y,\overline{Y})}} M^{(X,\overline{X},Y,\overline{Y})}_{\alpha} \bigsqcup_{-M^{(X,\overline{X})}_{S^1_k} \sqcup M^{(Y,\overline{Y})}_{S^1_k}} B_{f,g}. \tag{7.2.19}$$

First note that the manifolds underlying A and B_α are given by $S^2 \times S^1$ and S^3, respectively, and thus differ by index 1 surgery. In particular the two ribbons running along the core of the extra 1-handle of A can be chosen to be the ones labelled with $\overline{X}$ and $\overline{Y}$, so we find the following bichrome graph presentation of A in S^3:



(7.2.20)

where $T \cup (g \circ f)$ contains the admissible graph T and the graph corresponding to $(g \circ f)$. Analogously for B_α we have:



(7.2.21)

[1] Note that in our conventions the TFT $\hat{\mathrm{V}}_{\mathcal{C}}$ corresponds to the dual TFT $\check{\mathrm{V}}'_{\mathcal{C}}$ in [DGGPR1] and vice versa.

By Lemma 2.1.8 and since the surgery link for B has one less component then the one for A an analogous argument as in the proof of [DGGPR1, Prop. 4.10] leads to

$$\mathcal{D}\widehat{\mathrm{V}}_{\mathcal{C}}(A) = \zeta \sum_{\alpha=1}^{m} \widehat{\mathrm{V}}_{\mathcal{C}}(B_\alpha). \tag{7.2.22}$$

Since $\zeta = \mathcal{D}^2$ we get

$$\widehat{\mathrm{V}}_{\mathcal{C}}\left(M^{(X,\overline{X},Y,\overline{Y})}_{\mathrm{id}_{S^1_k}}\right) = \mathcal{D}\sum_{\alpha=1}^{m} \widehat{\mathrm{V}}_{\mathcal{C}}\left(M^{(X,\overline{X},Y,\overline{Y})}_{\alpha}\right). \tag{7.2.23}$$

as desired.

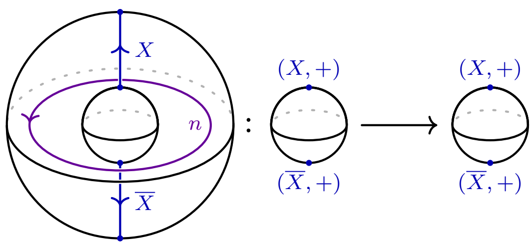
7.3 Some CFT quantities

In this section we compute some more quantities of physical interest including boundary states and annulus amplitudes for different boundary conditions and compare our results to the ones proposed in [FGSS]. Moreover, we will compute the action of a line defect on bulk fields.

7.3.1 Boundary states

Boundary states are bulk one point correlators on a disc with fixed boundary condition. In our formulation these correspond to the correlators for a world sheet $\mathfrak{D}$ with underlying surface a cylinder such that one boundary is a gluing boundary and the other one a free boundary labelled with a boundary condition $n \in \mathcal{C}$. If the gluing boundary is outgoing, i.e. $\mathfrak{D}^{\mathrm{out}}_n \colon \varnothing \to S^1$ we will call it an *outgoing* boundary state, and for an incoming gluing boundary $\mathfrak{D}^{\mathrm{in}}_n \colon S^1 \to \varnothing$ an *incoming* boundary state.

Let $(X, \overline{X}) \in \mathcal{C} \times \overline{\mathcal{C}}$, the connecting bordism $M^{(X,\overline{X})}_{\mathfrak{D}_n} \colon M^{(X,\overline{X})}_{S^1} \to \widehat{\mathfrak{D}_\mathfrak{n}}^{(X,\overline{X})}$ of $\mathfrak{D}^{\mathrm{out}}_n \colon \varnothing \to S^1$ is given by

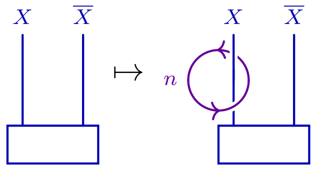


(7.3.1)

The connecting bordism for $\mathfrak{D}^{\mathrm{in}}_n \colon S^1 \to \varnothing$ differs from the one illustrated above only in the orientation of the $(X, \overline{X})$-labelled ribbons. We will only discuss the outgoing case in detail since the ingoing case can be treated completely analogously.

Applying the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ to $M_{\mathfrak{D}_n}^{(X,\overline{X})}$ and using the isomorphism between TFT state spaces and morphism spaces in $\mathcal{C}$ we get the linear map

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X})$$



. (7.3.2)

Next to get $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}}$ we use the adjunction $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X}) \cong \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, X \otimes \overline{X})$, where we view $X \otimes \overline{X}$ as an L-module with it's canonical L-module structure constructed from the half braiding on $X \otimes \overline{X}$ as in Lemma 2.1.5 to obtain a linear map

$$\mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, X \otimes \overline{X}) \to \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, X \otimes \overline{X}) \tag{7.3.3}$$

which we will also call $\widehat{\mathrm{V}}_{\mathcal{C}}(M_{D_n}^{(X,\overline{X})})$.[2] By definition $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}} \in \mathrm{Nat}(\mathrm{Cor}_{S^1}, \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_n^{\mathrm{out}}))$ is induced by the natural transformation with components $\widehat{\mathrm{V}}_{\mathcal{C}}(M_{\mathfrak{D}_n}^{(X,\overline{X})})$. In order to compare this with the proposed boundary states of [FGSS] we need to make this abstractly defined $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}}$ more concrete. To this end recall the linear isomorphisms

$$\begin{aligned} \mathrm{Nat}(\mathrm{Cor}_{S^1}, \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_n^{\mathrm{out}})) &\cong \mathrm{Nat}(\mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, -), \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, -)) \\ &\cong \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, \mathrm{L}) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L}) \end{aligned} \tag{7.3.4}$$

where in the first step we used $\mathrm{Cor}_{S^1} \cong \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, -) \cong \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_n^{\mathrm{out}})$, the Yoneda lemma in the second, and the free-forgetful adjunction $\mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, \mathrm{L}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L})$ in the last one. The space $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L})$ is precisely the one also considered in [FGSS, Sec. 3.2]. Unfortunately, we cannot simply compute the image of $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}}$ under this chain of isomorphisms because we do not have direct access to the L-component of the natural transformation which is needed for the Yoneda lemma. Instead we will work our way backwards by showing that the outgoing boundary state proposed in [FGSS, Sec. 3.2] induces the natural transformation $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}}$. According to [FGSS, Sec. 3.2] an outgoing boundary state should be described by the *cocharacter* $\check{\chi}_n = \iota_n \circ \mathrm{ev}_n \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L})$ of n from Section 2.1.2. Following the isomorphism above the cocharcter $\check{\chi}_n$ gets sent to the natural transformation with components

$$\begin{aligned} \Phi_n^Y \colon \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, Y) &\to \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, Y) \\ f &\mapsto f \circ \mu \circ (\check{\chi}_n \otimes \mathrm{id}_{\mathrm{L}}) \end{aligned} \tag{7.3.5}$$

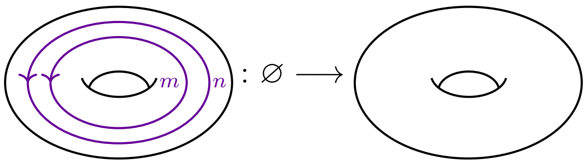
[2]It is important here that $\overline{X}$ is an object in $\overline{\mathcal{C}}$, i.e. we use the mirrored braiding on $\overline{X}$ to define the L-action.

for $Y \in \mathcal{C}_{\mathrm{L}}$. It now follows from a straightforward calculation that $\Phi_n^{X\otimes\overline{X}} = \widehat{\mathrm{V}}_{\mathcal{C}}(M^{(X,\overline{X})}_{D_n^{\mathrm{out}}})$, thus they both induce $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{out}}}$. As mentioned above, an analogous computation can be done for an incoming boundary state $\mathrm{Cor}_{\mathfrak{D}_n^{\mathrm{in}}}$. In this case we obtain a *character* $\chi_n \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}, \mathbb{1})$ precomposed with the modular S-transformation $\mathcal{S} \colon \mathrm{L} \to \mathrm{L}$ from Equation (2.1.10). With this we confirm the relation between boundary states and (co)characters postulated in [FGSS, Sec. 3.2].

7.3.2 Annulus amplitude

Next we compute the annulus amplitude in the form of the correlator for an annulus $\mathfrak{A}_{n,m}$ with boundary conditions $m, n \in \mathcal{C}$ and compare our results of [FGSS]. The connecting bordism of $\mathfrak{A}_{n,m}$ is given by

$$M_{\mathfrak{A}_{m,n}} =$$



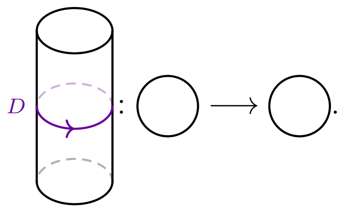
(7.3.6)

To compute $\widehat{\mathrm{V}}_{\mathcal{C}}(M_{\mathfrak{A}_{m,n}})$ we employ (5.1.19) again, this time for $X = m \otimes n$. We find that $\widehat{\mathrm{V}}_{\mathcal{C}}(M_{\mathfrak{A}_{m,n}}) \in \widehat{\mathrm{V}}_{\mathcal{C}}(T^2)$ corresponds to $\check{\chi}_{m\otimes n} \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L})$ under the isomorphisms $\widehat{\mathrm{V}}_{\mathcal{C}}(T^2) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{L}, \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathrm{L})$ where we used the Frobenius algebra structure on L for the second one. This again reproduces the results of [FGSS, Sec. 4]. To be more precise we recover their result for the *open-string channel*. The *closed-string channel* can be obtained by performing a modular S-transformation.

7.3.3 Line defect action on bulk fields

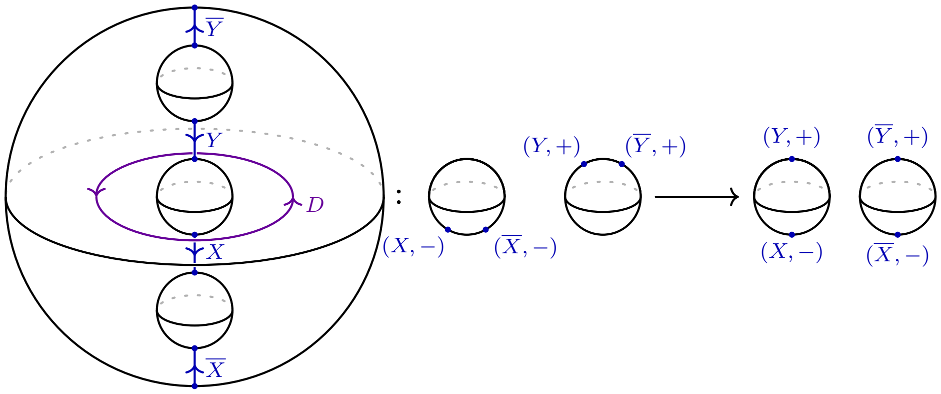
Finally, let us discuss how a line defect $D \in \mathcal{C}$ acts on the bulk fields $\mathbb{F}_{S^1}$. The action is induced by the world sheet with underlying surface the cylinder $S^1 \times I$ and D-labelled line defect at $S^1 \times \{1/2\}$

$$\mathfrak{O}_D =$$



(7.3.7)

The connecting bordism $M_{\mathfrak{D}_D}^{(X,\overline{X},Y,\overline{Y})}$ for this world sheet is given by



. (7.3.8)

By functoriality of $\widehat{\mathrm{V}}_{\mathcal{C}}$ the linear map $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M_D^{(X,\overline{X},Y,\overline{Y})}\right)$ can be obtained by composing the maps $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M_{\mathrm{id}_{S^1}}^{(X,\overline{X},Y,\overline{Y})}\right)$ and $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M_{\mathfrak{D}_D^{\mathrm{out}}}^{(Y,\overline{Y})}\right)$ from Sections 7.2.2 and 7.3.1. In particular, the natural transformation induced by $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M_{\mathfrak{D}_N}^{(X,\overline{X},Y,\overline{Y})}\right)$ is the composition

$$\mathrm{Cor}_{\mathfrak{D}_D} = \varepsilon_{\mathrm{Cor}_{S^1}} \circ (\mathrm{id}_{\mathrm{Cor}_{S^1}^\dagger} \otimes_{\Bbbk} \mathrm{Cor}_{\mathfrak{D}_D^{\mathrm{out}}}) \colon \mathrm{Cor}_{S^1}^\dagger \otimes_{\Bbbk} \mathrm{Cor}_{S^1} \Rightarrow \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_D). \tag{7.3.9}$$

Next note that by the Yoneda lemma we have $\mathrm{Nat}(\mathrm{Cor}_{S^1}^\dagger \otimes_{\Bbbk} \mathrm{Cor}_{S^1}, \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_D)) \cong \mathrm{Hom}_{\mathrm{L}}(\mathbb{F}_{S^1}, \mathbb{F}_{S^1})$, so instead of computing the corresponding natural transformation we will focus on the endomorphism of $\mathbb{F}_{S^1} \cong \mathrm{L}$ instead. We will call this endomorphism the *defect operator* associated to D and denote it with $\mathcal{O}_D$. To compute $\mathcal{O}_D$ first recall from Remark 6.6.1 that $\varepsilon_{\mathrm{Cor}_{S^1}}$ gets sent to $\mathrm{id}_{\mathrm{L}} \in \mathrm{Hom}_{\mathrm{L}}(\mathrm{L},\mathrm{L})$. In the discussion in Section 7.3.1 we saw that $\mathrm{Cor}_{\mathfrak{D}_D^{\mathrm{out}}}$ corresponds to $\mu \circ (\check{\chi}_D \otimes \mathrm{id}_{\mathrm{L}}) \colon \mathrm{L} \to \mathrm{L}$. Combining this we find

$$\mathcal{O}_D = \mu \circ (\check{\chi}_D \otimes \mathrm{id}_{\mathrm{L}}) \in \mathrm{Hom}_{\mathrm{L}}(\mathrm{L},\mathrm{L}). \tag{7.3.10}$$

As a self-consistency check note that we also could have decomposed $M_{\mathfrak{D}_N}^{(X,\overline{X},Y,\overline{Y})}$ using $M_{\mathfrak{D}_D^{\mathrm{in}}}^{(X,\overline{X})}$ instead of $M_{\mathfrak{D}_D^{\mathrm{out}}}^{(Y,\overline{Y})}$. By doing this we obtain the endomorphism $((\chi_D \circ \mathcal{S}) \otimes \mathrm{id}_{\mathrm{L}}) \circ \Delta_\Lambda$ where Δ_Λ is the Frobenius coalgebra structure on L from Proposition 2.1.2 and $\mathcal{S}$ is the modular S-transformation from Equation (2.1.10). Using the relation between characters, cocharacters, and the modular S-transformation from [FGSS, Sec. 2.4] a direct calculation gives $\mu \circ (\check{\chi}_D \otimes \mathrm{id}_{\mathrm{L}}) = ((\chi_D \circ \mathcal{S}) \otimes \mathrm{id}_{\mathrm{L}}) \circ \Delta_\Lambda$. Thus both decompositions of $M_{\mathfrak{D}_N}^{(X,\overline{X},Y,\overline{Y})}$ lead to the same endomorphism of L.

Remark 7.3.1. In the semisimple setting we have $\mathrm{L} \cong \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} i^* \boxtimes i$ as objects in $\mathcal{C}_{\mathrm{L}} \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$ where $\mathrm{Irr}(\mathcal{C})$ is a set of representatives of simple objects in $\mathcal{C}$. For

j a simple object the defect operator $\mathcal{O}_j$ acts on L by multiplication with the (normalised) S-matrix element $S_{j,i^*}/S_{0,i^*}$, see e.g. [FrFRS, Sec. 6.2].

We can compose two line defects by bringing them close to each other. To be more precise for a second line defect labelled by $E \in \mathcal{C}$ the composed line defect is labelled by $E \otimes D$. This is compatible with composition of endomorphisms since $\mathcal{O}_E \circ \mathcal{O}_D = \mathcal{O}_{E\otimes D}$ because $\mu \circ (\check{\chi}_E \otimes \check{\chi}_D) = \check{\chi}_{E\otimes D}$, see [Shi2, Thm. 3.10] or [FGSS, Sec. 2.3]. Finally, by reformulating [Shi2, Cor. 4.3] we find that the defect operators span the Grothendieck ring of $\mathcal{C}$:

Proposition 7.3.2. The subalgebra of $\mathrm{End}_{\mathrm{L}}(\mathrm{L})$ generated by the defect operators $(\mathcal{O}_D)_{D\in\mathcal{C}}$ is isomorphic (as an algebra) to the linearised Grothendieck ring $\mathrm{Gr}_{\Bbbk}(\mathcal{C})$ of $\mathcal{C}$.

Chapter 8

Outlook: Non-trivial surface defects

In this final chapter we want to briefly discuss some aspects of the second example mentioned in Section 6.1 and comment on further directions to pursue as well as possible relations to other approaches to algebraically construct full CFTs.

We will start with a very brief review of the algebraic structures relevant here including the connection between algebras in $\mathcal{C}$ and module categories over $\mathcal{C}$. After this we will provide a quick overview of the construction of non-trivial surface defects via the generalised orbifold procedure of [CRS1; CRS2]. In this setting we will then repeat some of the computations from Chapter 7. This is still work in progress, accordingly our exposition will be less detailed than in the previous chapter. Nonetheless we still want to include this example here as some of the general features are already worked out. Afterwards we will comment on what seems to be a key difference between the semisimple and the non-semisimple setting. In the final section 8.4 we will discuss some open questions as well as possible explicit examples one should consider next.

8.1 Algebras and module categories

Let us start with a quick recollection of (pivotal) module categories. We will be very brief here and refer the reader to [EGNO, Ch. 7] as well as [FS4]. A right $\mathcal{C}$-module category consists of a finite linear category $\mathcal{M}$, a functor $-\triangleleft-: \mathcal{M}\times\mathcal{C} \to \mathcal{M}$, exact in its first variable, as well as a mixed associator and a mixed unitor that obey mixed pentagon and triangle axioms. We will abbreviate all of this data and simply write $\mathcal{M}$ in the following. For two module categories $\mathcal{M}$ and $\mathcal{N}$ a module functor from $\mathcal{M}$ to $\mathcal{N}$ consists of a functor $F: \mathcal{M} \to \mathcal{N}$ together with a natural transformation $F(- \triangleleft -) \Rightarrow F(-) \triangleleft -$ satisfying a compatibility condition with the associators and unitors. Finally, one can also consider module natural transformations which need to be compatible with the above natural transformation. Altogether this forms a 2-category $\mathrm{Mod}(\mathcal{C})$.

There is a close relation between $\mathcal{C}$-module categories and algebra objects in $\mathcal{C}$. Let $A \in \mathcal{C}$ be an algebra object. For every left A-module M and every $X \in \mathcal{C}$ we can endow $M \otimes X$ with the structure of a left A-module. This endows the category ${}_A\mathcal{C}$ of left A-modules in $\mathcal{C}$ with the structure of a right $\mathcal{C}$-module category, see e.g. [EGNO, Sec. 7.8]. Under some technical conditions we also have the converse in the sense that for every (nice enough) module category $\mathcal{M}$ there exists an algebra $A \in \mathcal{C}$ such that $\mathcal{M} \simeq {}_A\mathcal{C}$ as $\mathcal{C}$-module categories, see [EGNO, Thm. 7.10.1] for the precise statement. The idea is to obtain the algebra using the *internal* or *inner* Hom functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$ which is defined via

$$\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}_{\mathcal{M}}(M,N)) \cong \mathrm{Hom}_{\mathcal{M}}(M \triangleleft X, N) \tag{8.1.1}$$

as the right adjoint of the action functor. Note that the inner Hom exists by the exactness of the action functor in the first argument and is automatically left exact. Using the adjunction we can endow $\underline{\mathrm{Hom}}_{\mathcal{M}}(M,M)$ with an algebra structure for any $M \in \mathcal{M}$ [EGNO, Sec. 7.9].

A module category is called *exact* if $\underline{\mathrm{Hom}}_{\mathcal{M}}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$ is exact in each variable.[1] We say an algebra $A \in \mathcal{C}$ is exact if its corresponding module category ${}_A\mathcal{C}$ is exact. For exact module categories the above theorem applies without further assumptions. Exactness of a module category, or equivalently the corresponding algebra, can be thought of as a semisimplicity condition relative to the base tensor category [CSZ].

For two exact $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ we can endow the category of right exact $\mathcal{C}$-module functors $\mathcal{R}\mathrm{ex}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ with the structure of an exact $\mathcal{Z}(\mathcal{C})$-module category using the half-braiding, see [FS4, Sec. 4.1] for details. The internal Hom for $\mathcal{R}\mathrm{ex}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ as a $\mathcal{Z}(\mathcal{C})$-module category is called the object of *internal natural transformations* or *internal* Nat and denoted by $\underline{\mathrm{Nat}}(F,G) \in \mathcal{Z}(\mathcal{C})$ for $F, G \in \mathcal{R}\mathrm{ex}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$.

Let us denote with $\mathrm{Mod}^{\mathrm{ex}}(\mathcal{C})$ the 2-category of exact $\mathcal{C}$-module categories, right exact module functors and module natural transformations. The above theorem can be upgraded to an equivalence of 2-categories via an Eilenberg-Watts type argument. More precisely one can show that the 2-functor

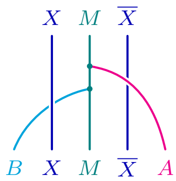
$$\begin{aligned} \mathrm{Alg}^{\mathrm{ex}}(\mathcal{C}) &\to \mathrm{Mod}^{\mathrm{ex}}(\mathcal{C}) \\ A &\mapsto {}_A\mathcal{C} \\ {}_BM_A &\mapsto M \otimes_A - \\ \varphi &\mapsto \varphi \otimes_A - \end{aligned} \tag{8.1.2}$$

[1] This is equivalent to the definition of exact module category from [EGNO, Sec. 7.5] by combining [Shi8, Def. 3.3] and [Shi8, Lem. 5.3].

where $\mathrm{Alg}^{\mathrm{ex}}(\mathcal{C})$ is the 2-category of exact algebras, bimodules and bimodule maps in $\mathcal{C}$, is an equivalence, see e.g. [EGNO, Prop. 7.11.1]. In particular we have equivalences ${}_B\mathcal{C}_A \simeq \mathcal{R}\mathrm{ex}_{\mathcal{C}}({}_A\mathcal{C}, {}_B\mathcal{C})$, where ${}_B\mathcal{C}_A$ denotes the category of B-A-bimodules in $\mathcal{C}$, for any pair A, B of exact algebras in $\mathcal{C}$. Using this equivalence ${}_B\mathcal{C}_A$ inherits a $\mathcal{Z}(\mathcal{C})$-action as well. Since $\mathcal{C}$ is modular this gives us a $\mathcal{C} \boxtimes \overline{\mathcal{C}}$-action on ${}_B\mathcal{C}_A$ which can be shown to be explicitely given by

$$\begin{aligned} {}_B\mathcal{C}_A \times \mathcal{C} \boxtimes \overline{\mathcal{C}} &\to {}_B\mathcal{C}_A \\ (M, X \boxtimes \overline{X}) &\mapsto X \otimes^+ M \otimes^- \overline{X} \end{aligned} \tag{8.1.3}$$

where $X \otimes^+ M \otimes^- \overline{X}$ is $X \otimes M \otimes \overline{X}$ with B-A-bimodule structure given by



. (8.1.4)

We will denote the inner Hom of ${}_B\mathcal{C}_A$ as a $\mathcal{C} \boxtimes \overline{\mathcal{C}}$-module category as $\underline{\mathrm{Hom}}_{B|A}$. The object $\underline{\mathrm{Hom}}_{A|A}(A, A)$ is naturally a commutative algebra and known as the *full centre* of the algebra A [DKR2].

From now on we will be interested in *special symmetric Frobenius algebras* and their module categories. A Frobenius algebra $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is called Δ-*separable* if $\mu \circ \Delta = \mathrm{id}_A$ and it is called symmetric if $\varepsilon \circ \mu = \varepsilon \circ \mu \circ \beta_{A,A} \circ (\mathrm{id}_A \otimes \vartheta_A)$ where β is the braiding and ϑ the ribbon twist of $\mathcal{C}$. A *special* symmetric Frobenius algebra is a Δ-separable symmetric Frobenius algebra with non-zero categorical dimension or equivalently $\varepsilon \circ \eta \neq 0$. On the level of module categories Δ-separability ensures exactness [SY, Lem. 5.3] while being symmetric endows it with the structure of a so-called *pivotal* module category. We will not need the precise definition of pivotal module category here and only want to note that this structure makes the inner Hom functor a two-sided adjoint of the action functor, see [Sch, Def. 5.2] and [Shi6, Def. 3.11] for the full definition. In particular, for A a symmetric Frobenius algebra we can now explicitly describe the inner Hom of ${}_A\mathcal{C}$ as

$$\underline{\mathrm{Hom}}_A(m, n) \cong m^* \otimes_A n \tag{8.1.5}$$

where $- \otimes_A -$ denotes the relative tensor product of A-modules and we used that m^* naturally carries the structure of a right A-module, see e.g. [EGNO, Lem. 7.8.24]. Moreover, it can be shown that for A, B special symmetric Frobenius algebras ${}_B\mathcal{C}_A$ inherits the structure of a pivotal module category over $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ [FS4, Cor. 19].

8.2 Surface defects from algebras

Let us now discuss how the algebraic objects discussed above can be used to construct non-trivial surface defects. More concretely, we will now to apply the construction of [CRS1] to the non-semisimple TFT

$$\widehat{\mathrm{V}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C}) \to \mathrm{vect}_{\Bbbk} \tag{8.2.1}$$

of [DGGPR1] to obtain a non-semisimple TFT $\mathrm{Z}_{\mathcal{C}}$ with non-trivial surface defects satisfying the requirements outlined in Section 6.1.

For our purposes we will only need the case where at most two 2-strata meet at an adjacent 1-stratum. Due to this we will not introduce the full defect data $\mathbb{D}_{\mathcal{C}}$ of [CRS1, Sec. 5] and instead restrict our attention to the following:

- $D_3 = \{*\}$
- $D_2 = \{$special symmetric Frobenius algebras in $\mathcal{C}\}$
- $D_1 = \{$Bimodules between special symmetric Frobenius algebras in $\mathcal{C}\}$
- $D_0 = \{$Bimodule maps in $\mathcal{C}\}$

The adjacency maps for bimodules are such that the algebra acting from the right is the source. In terms of higher categories of defects this data is encoded in the 2-category $\mathrm{FrobAlg}^{\mathrm{sym,\,sp}}$ of special symmetric Frobenius algebras, bimodules, and bimodule morphisms. The natural choice for the transparent element $T \in D_2$ is the monoidal unit $\mathbb{1}$. In terms of module categories the transparent element is $\mathcal{C}$ as a module category over itself.

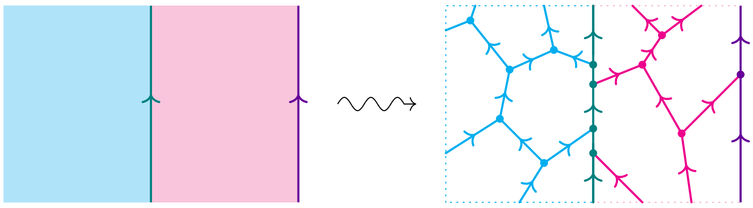
Next we turn to the construction of the defect TFT

$$\mathrm{Z}_{\mathcal{C}} \colon \mathrm{Bord}^{\chi;\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}}) \to \mathrm{vect}_{\Bbbk} \tag{8.2.2}$$

with this defect data. The idea is to obtain $\mathrm{Z}_{\mathcal{C}}$ by replacing every surface defect with a mesh of line defects labelled by the corresponding algebra object and then evaluating the resulting bordism with line defects with the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$. This way of constructing non-trivial surface defects is referred to as a *generalised orbifold* or *internal state sum* or *gauging* construction and goes back to [KS] where it was used to reinterpret the TFT construction of RCFT correlators [FRSI].

Let us make the evaluation process a bit more precise. The mesh of line defects is obtained as follows: First we pick a triangulation (or more generally a cell decomposition) of the surface defect. Next we consider the Poincaré dual graph of the triangulation. We now label edges of the graph with the algebra and the vertices with the multiplication or comultiplication, depending on the orientation.

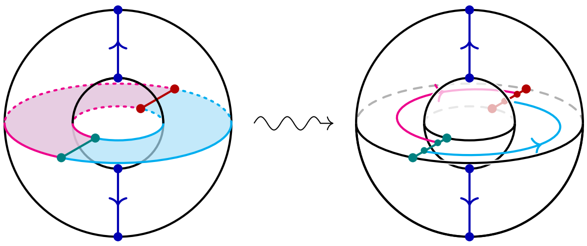
At the vertices where the graph intersects a line defect we use the bi(co)module structure.[2] Schematically this procedure can be visualised as follows:



After evaluating with $\widehat{\mathrm{V}}_{\mathcal{C}}$ the result does not depend on the choice of triangulation because of the Δ-separability and the Frobenius relations.

With this we get the action of $\mathrm{Z}_{\mathcal{C}}$ on closed 3-manifolds. For bordisms between non-trivial objects one needs to be more careful because the procedure outlined above is not independent of the induced triangulation on the boundary. To fix this one performs a colimit over all possible boundary triangulations. We will not go into more details on this point or prove that the whole procedure is well-defined and leads to a defect TFT and instead refer the reader to [CRS1, Sec. 5]. Note that for bordisms with only transparent 2-strata we simply get back $\widehat{\mathrm{V}}_{\mathcal{C}}$, thus the defect TFT $\mathrm{Z}_{\mathcal{C}}(\Sigma)$ is an extension of $\widehat{\mathrm{V}}_{\mathcal{C}}$ in the sense of Definition 6.1.1.

We also want to mention that for any object $\Sigma \in \mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ we can compute the state space $\mathrm{Z}_{\mathcal{C}}(\Sigma)$ as the image of the idempotent we get from performing the above procedure on the cylinder $\Sigma \times I$. For example for the defect sphere given in Figure 6.1b we have:



Note here that the boundary of the bordism on the right is an object in $\mathrm{Bord}_{3,2}^{\chi}(\mathcal{C})$.

Remark 8.2.1. The defect TFT $\mathrm{Z}_{\mathcal{C}}$ is technically not defined on all of $\mathrm{Bord}_{3,2}^{\chi;\mathrm{def}}(\mathbb{D}_{\mathcal{C}})$ as we still need to ensure admissibility in order to evaluate $\widehat{\mathrm{V}}_{\mathcal{C}}$. However, as already mentioned in Remark 6.1.2, all the bordism we care about here will have an outgoing boundary and will thus be automatically admissible. Due to this we will tacitly ignore this subtlety in the following.

Finally, one can slightly generalise this construction and work with *separable* Frobenius algebras instead of Δ-separable ones, see [Mul, Sec. 3.2].

[2] Recall here that every (bi)module over a Frobenius algebra is canonically also a (bi)comodule.

8.3 CFT computations

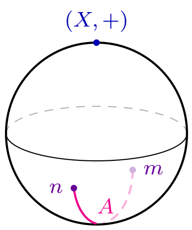
Let us now compute some CFT quantities. As in Chapter 7 we first need to check that $\mathrm{Z}_{\mathcal{C}}$ satisfies the conditions of Theorem 6.6.2. For Assumption 1 we essentially have to compute state spaces of defect spheres. This will be done in the next subsection. Assumption 2 can be proved in a similar way as Proposition 5.1.7, since this chapter is meant as an outlook we will not go into more details on this point here. Assumption 3 and 4 are more involved and we will only discuss them for boundary fields.

8.3.1 Field content

As in Section 7.1 we will be interested in the boundary and disorder fields corresponding to the field content for an interval and a defect circle with two point defects, respectively. Moreover, we will also restrict our attention to the setting of a defect interval with no point defects in its interior.

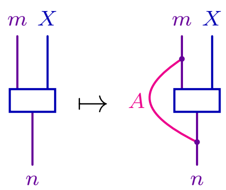
Boundary fields

Let $A \in \mathrm{FrobAlg}^{\mathrm{sym,sp}}$, and let $m, n \in {}_A\mathcal{C}$. Denote with $I^A_{n,m} \in \mathfrak{WG}_{\mathcal{C}}$ the defect interval with A labelling the interior $(0,1) \subset I$, m labelling $\{0\} \subset I$, and n labelling $\{1\} \subset I$. For $X \in \mathcal{C}$ the corresponding connecting manifold $M^X_{I^A_{n,m}}$ is given by:



As explained above in order to get the state space $\mathrm{Z}_{\mathcal{C}}\left(M^X_{I^A_{n,m}}\right)$ we have to compute the image of an idempotent on $\widehat{\mathrm{V}}_{\mathcal{C}}\left(M^X_{I_{n,m}}\right) \cong \mathrm{Hom}_{\mathcal{C}}(n, X \otimes m)$ where $M_{I_{n,m}}$ is the connecting manifold of the interval $I_{n,m}$ from Section 7.1. Going through the orbifold procedure described this idempotent is given by

$$P\colon \mathrm{Hom}_{\mathcal{C}}(n, m \otimes X) \to \mathrm{Hom}_{\mathcal{C}}(n, m \otimes X) \tag{8.3.1}$$



where we use the A-action on m and the A-coaction on n. Using the Frobenius relations and Δ-separability it is straightforward to verify that $\mathrm{im}(P) \cong \mathrm{Hom}_A(n, m \otimes X) \subset \mathrm{Hom}_{\mathcal{C}}(n, m \otimes X)$. Finally, recall that $\mathcal{C}_A$ is a right $\mathcal{C}$-module category with action given by tensoring with an object from the right. With this we get

$$\begin{aligned} \mathrm{Z}_{\mathcal{C}}\left(M^X_{I^A_{n,m}}\right) &\cong \mathrm{Hom}_A(n, m \otimes X) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}_A(m,n), X) \end{aligned} \tag{8.3.2}$$

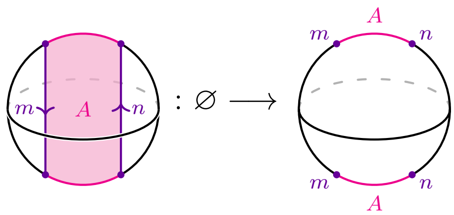
where we used that the inner hom functor $\underline{\mathrm{Hom}}_A$ is a two-sided adjoint of the action functor in our setting. We can now read of the boundary field content as

$$\mathbb{F}_{I^A_{n,m}} = \underline{\mathrm{Hom}}_A(m,n) \in \mathcal{C}. \tag{8.3.3}$$

Now to verify Assumption 1 for defect intervals we note that $\mathrm{Z}_{\mathcal{C}}\left(M^X_{I^A_{n,m}}\right) \cong \mathrm{Hom}_A(n, m \otimes X)$ is clearely linear and left exact in X.

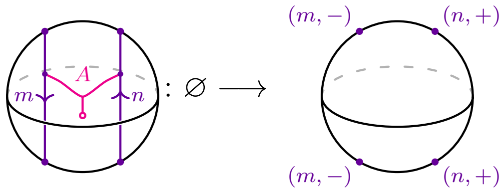
For Assumption 3 we have to compute the image of

$$\eta_{M_{I^A_{n,m}}} =$$



(8.3.4)

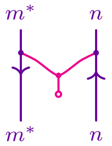
under the TFT $\mathrm{Z}_{\mathcal{C}}$. Going through the orbifold procedure we arrive at the bordism



(8.3.5)

in $\widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$. Applying $\widehat{\mathrm{V}}_{\mathcal{C}}$ to this leads us to the following morphism

$$P^A_{n,m^*} :=$$



$$: m^* \otimes n \to m^* \otimes n \tag{8.3.6}$$

in $\mathcal{C}$. Now by using Δ-separability of A it is straightforward to verify that P^A_{n,m^*} is an idempotent in $\mathcal{C}$. Moreover, from [CR1, Lem. 2.3] we know that the image of this idempotent is the relative tensor product

$$\mathrm{im}(P^A_{n,m^*}) \cong m^* \otimes_A n \tag{8.3.7}$$

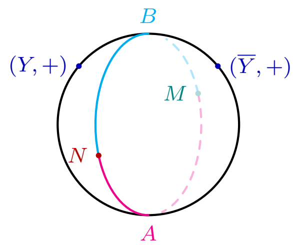
of A-modules. Finally, using the relation between the inner Hom and the relative tensor product

$$\underline{\mathrm{Hom}}_A(m,n) \cong m^* \otimes_A n \tag{8.3.8}$$

from above, we arrive at $\mathrm{Z}_{\mathcal{C}}(\eta_{M_{I^A_{n,m}}}) = \mathrm{id}_{\underline{\mathrm{Hom}}_A(m,n)}$.

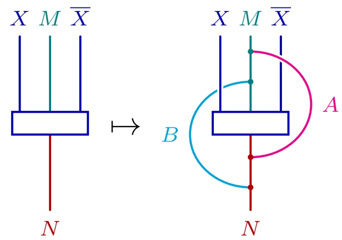
Bulk and disorder fields

Next we want to study the bulk and disorder fields. To this end let $A, B \in \mathrm{FrobAlg}^{\mathrm{sym,sp}}$ and let $M, N \in {}_B\mathcal{C}_A$. We denote with $S^1_{N,M}$ the defect circle with two 1-strata labelled by A and B, respectively, one positively oriented 0-stratum labelled with M, and one negatively oriented 0-stratum labelled with N. For $(X,\overline{X}) \in \mathcal{C} \times \overline{\mathcal{C}}$ the corresponding connecting manifold $M^{(X,\overline{X})}_{S^1_{N,M}}$ is given by the following defect sphere:



In analogy to before we now need to understand the linear idempotent:

$$Q\colon \mathrm{Hom}_{\mathcal{C}}(N, X \otimes M \otimes \overline{X}) \to \mathrm{Hom}_{\mathcal{C}}(N, X \otimes M \otimes \overline{X})$$



(8.3.9)

The image of Q is isomorphic to $\mathrm{Hom}_{B|A}(N, X \otimes^+ M \otimes^- \overline{X})$ where $X \otimes^+ M \otimes^- \overline{X}$ is the B-A-bimodule with underlying object $X \otimes M \otimes \overline{X}$ and bimodule structure as in (8.1.4). To compute the field content we use that ${}_B\mathcal{C}_A$ is a pivotal $\mathcal{C} \boxtimes \overline{\mathcal{C}}$-module category giving:

$$\begin{aligned} \mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{S^1_{N,M}}\right) &\cong \mathrm{Hom}_{B|A}(N, X \otimes^+ M \otimes^- \overline{X}) \\ &= \mathrm{Hom}_{B|A}(N, (X \boxtimes \overline{X}) \triangleright M) \\ &\cong \mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(\underline{\mathrm{Hom}}_{B|A}(M,N), X \boxtimes \overline{X}). \end{aligned} \tag{8.3.10}$$

Thus we have

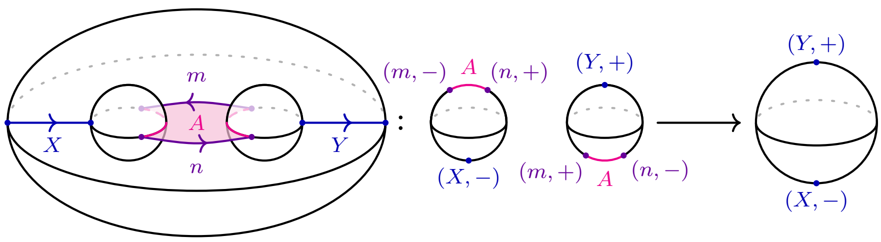
$$\mathbb{F}_{S^1_{N,M}} = \underline{\mathrm{Hom}}_{B|A}(M,N) \in \mathcal{C} \boxtimes \overline{\mathcal{C}}. \tag{8.3.11}$$

Moreover, using the Eilenberg-Watts equivalence ${}_B\mathcal{C}_A \simeq \mathrm{Rex}_{\mathcal{C}}({}_A\mathcal{C}, {}_B\mathcal{C})$ we get $\underline{\mathrm{Hom}}_{B|A}(M,N) \cong \underline{\mathrm{Nat}}(M \otimes_A -, N \otimes_A -)$ and thus obtain the proposed field content of [FS5].

Finally, note that $\mathrm{Hom}_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(\underline{\mathrm{Hom}}_{B|A}(M,N), X \boxtimes \overline{X})$ is clearly linear and left exact in X and $\overline{X}$ thus verifying Assumption 1 also for defect circles. Using monoidality of $\mathrm{Z}_{\mathcal{C}}$ this completely verifies Assumption 1.

8.3.2 Boundary two point correlator

We now turn to study Assumption 4 for the defect interval $I^A_{n,m}$ from above. Let $X, Y \in \mathcal{C}$. The connecting manifold $M^{X,Y}_{\mathrm{id}_{I^A_{n,m}}}$ is given by:



It is straightforward to see that the linear map we get from the orbifold procedure is given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, m^* \otimes n) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(m^* \otimes n, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f \otimes_{\Bbbk} g &\mapsto g \circ P^A_{n,m^*} \circ f \end{aligned} \tag{8.3.12}$$

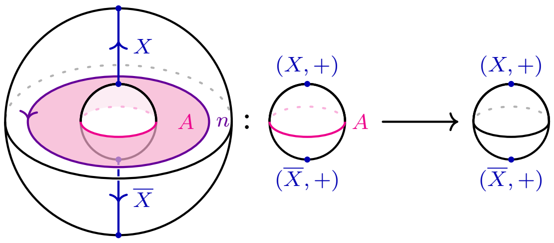
with P^A_{n,m^*} the idempotent used in the verification of Assumption 3. Moreover, since $m^* \otimes_A n \cong \mathrm{im}(P^A_{n,m^*})$ this map restricts to the composition map on the subspace $\mathrm{Hom}_{\mathcal{C}}(X, m^* \otimes_A n) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(m^* \otimes_A n, Y)$.

For the defect circle from above we did not work out the details yet but expect a combination of this type of argument combined with the argument in Section 7.2.2 to work.

8.3.3 Boundary states

Finally, let us discuss boundary states. As above these correspond to the correlators for a world sheet $\mathfrak{D}$ with underlying surface a cylinder such that one boundary is a gluing boundary and the other one a free boundary. Here the 2-stratum will now be labelled with a special symmetric Frobenius algebra, and the free boundary with a left A-module $n \in {}_A\mathcal{C}$. As before we need to distinguish between incoming and outgoing boundary states. For brevity we will only briefly discuss the case of outgoing boundary states to illustrate the differences to Section 7.3.1.

More precisely we consider the world sheet $\mathfrak{D}_{A,n} : \varnothing \to S^1_A$ where S^1_A denotes the A-labelled circle. Let $(X, \overline{X}) \in \mathcal{C} \times \overline{\mathcal{C}}$, the connecting bordism $M^{(X,\overline{X})}_{\mathfrak{D}_{A,n}} : M^{(X,\overline{X})}_{S^1_A} \to \widehat{\mathfrak{D}}^{(X,\overline{X})}_{A,n}$ of $\mathfrak{D}_{A,n} : \varnothing \to S^1_A$ is given by:



$$. \tag{8.3.13}$$

To get the linear map

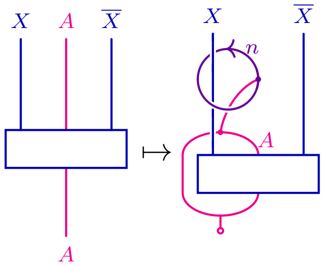
$$\mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{\mathfrak{D}_{A,n}}\right) : \mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{S^1_A}\right) \to \mathrm{Z}_{\mathcal{C}}(\widehat{\mathfrak{D}}^{(X,\overline{X})}_{A,n}) \tag{8.3.14}$$

we first note that

$$\begin{aligned} \mathrm{Z}_{\mathcal{C}}(\widehat{\mathfrak{D}}^{(X,\overline{X})}_{A,n}) &= \widehat{\mathrm{V}}_{\mathcal{C}}(\widehat{\mathfrak{D}}^{(X,\overline{X})}_{A,n}) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X}) \end{aligned} \tag{8.3.15}$$

as $\widehat{\mathfrak{D}}^{(X,\overline{X})}_{A,n} \in \widehat{\mathrm{Bord}}^{\chi}_{3,2}(\mathcal{C})$. Now according the orbifold construction we have that $\mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{\mathfrak{D}_{A,n}}\right)$ is given by the restriction of the linear map

$$\mathrm{Hom}_{\mathcal{C}}(A, X \otimes A \otimes \overline{X}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X})$$

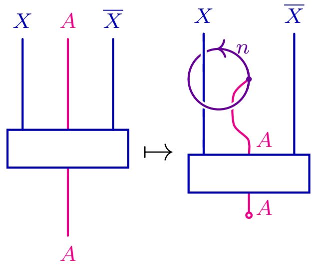


$$\tag{8.3.16}$$

to $\mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{S^1_A}\right) \cong \mathrm{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \overline{X})$. On $\mathrm{Hom}_{A|A}(A, X \otimes^+ A \otimes^- \overline{X})$ it is straightforward to further simplify this map using the Frobenius relations and

Δ-separability leading to:

$$\mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{\mathfrak{D}_{A,n}}\right) : \mathrm{Hom}_{A|A}(A, X \otimes^{+} A \otimes^{-} \overline{X}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \overline{X})$$



(8.3.17)

By definition $\mathrm{Cor}_{\mathfrak{D}_{A,n}} \in \mathrm{Nat}(\mathrm{Cor}_{S^1_A}, \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}_{A,n}))$ is now induced by the natural transformation with components $\mathrm{Z}_{\mathcal{C}}\left(M^{(X,\overline{X})}_{\mathfrak{D}_{A,n}}\right)$.

It would be useful to have a more concrete description of $\mathrm{Cor}_{\mathfrak{D}_{A,n}}$ similarly to the one derived in Section 7.3.1. More explicitely, we want to study the image of $\mathrm{Cor}_{\mathfrak{D}_{A,n}}$ under the chain of isomorphisms

$$\begin{aligned} \mathrm{Nat}(\mathrm{Cor}_{S^1_A}, \mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}^{\mathrm{out}}_n)) &\cong \mathrm{Nat}(\mathrm{Hom}_{\mathrm{L}}(\mathbb{F}_{S^1_A}, -), \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, -)) \\ &\cong \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, \mathbb{F}_{S^1_A}) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{F}_{S^1_A}) \end{aligned} \tag{8.3.18}$$

where we used $\mathrm{Cor}_{S^1_A} \cong \mathrm{Hom}_{\mathrm{L}}(\mathbb{F}_{S^1_A}, -)$, $\mathrm{Bl}_{\mathcal{C}}(\mathfrak{D}^{\mathrm{out}}_n) \cong \mathrm{Hom}_{\mathrm{L}}(\mathrm{L}, -)$, the Yoneda lemma, and the free-forgetful adjunction. In contrast to Section 7.3.1 we unfortunately do not have such a concrete description of $\mathrm{Cor}_{\mathfrak{D}_{A,n}}$ as an element in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{F}_{S^1_A})$ yet. However, there is a natural candidate in the form of the *internal cocharacter* generalising the cocharacter we encountered in Section 7.3.1 see [Shi4, Sec. 5.3] for the dual notion of relative cocharacters. The internal cocharacter is obtained as the composition $\mathbb{1} \to \underline{\mathrm{Hom}}_A(n,n) \to \int^{m \in {}_A\mathcal{C}} \underline{\mathrm{Hom}}_A(m,m) \cong \mathbb{F}_{S^1_A}$ where the first morphism is the unit of $\underline{\mathrm{Hom}}_A(n,n)$ as an algebra, and the second morphism the universal dinatural transformation of the coend. Making this idea more precise is unfortunately out of scope for this outlook as we would first need to make the description of $\mathbb{F}_{S^1_A}$ as a coend over the internal Hom stated above explicit [FS4, Thm. 18] and then use this and the natural Frobenius algebra structure on it to argue as in Section 7.3.1.

8.3.4 Relation to the semisimple setting and Morita invariance

Before we close with a discussion of open questions and possible future direction we want to make some comments on the relation and one a key difference to the original construction of [FRSI]. First we note that if $\mathcal{C}$ is semisimple then the above discussion with special, symmetric Frobenius algebras to label the surface defects recovers precisely the work of [FRSI]. This essentially follows by combining [DGGPR1, Rem. 4.9] with [KS] and [FSV]. See also the discussion in [FS5, Sec. 3.4] for how the description of the field content in the semisimple setting is encoded by the inner homs we encountered above. It turns out, that in this setting only the *Morita classes* of these algebras are seen by the construction [FSV]. This raises the question if this is still true beyond semisimplicity. The answer to this question is: no. In the non-semisimple setting Δ-separability is no longer a Morita invariant notion. To see this recall, e.g. from [EGNO, Exa. 7.10.2] that the monoidal unit $\mathbb{1} \in \mathcal{C}$ is Morita equivalent to $X \otimes X^*$, with algebra structure as in [EGNO, Exa. 7.8.4], for any $X \in \mathcal{C}$. Now note that the unit $\mathbb{1}$ is always Δ-separable as an algebra, $X \otimes X^*$ on the other hand is only Δ-separable if the categorical dimension $\dim_{\mathcal{C}}(X)$ is non-zero. Thus, for X projective $X \otimes X^*$ cannot be Δ-separable.

The work of [FS5] suggests a Morita invariant description of full CFTs in terms of exact pivotal module categories over $\mathcal{C}$ directly. These module categories correspond to exact symmetric Frobenius algebras in $\mathcal{C}$, by recent work of Shimizu [Shi8, Thm. 6.4] and this correspondence can be upgraded to an equivalence of 2-categories $\mathrm{FrobAlg}^{\mathrm{sym,\,ex}} \simeq \mathrm{Mod}^{\mathrm{piv}}(\mathcal{C})^{\mathrm{op}}$ as above. Moreover, as we already noted above, every special Frobenius algebra is automatically exact. Thus we can view $\mathrm{FrobAlg}^{\mathrm{sym,\,sp}}$ naturally as a 2-subcategory of $\mathrm{FrobAlg}^{\mathrm{sym,\,ex}}$. This observation raises the question of constructing a defect TFT with 2-strata labelled by exact symmetric Frobenius algebras which reproduces $\mathrm{Z}_{\mathcal{C}}$ for the special ones. The question of existence of such a hypothetical defect TFT was also one of the main motivations for making the construction in Chapter 6 largely independent of the explicit construction of the defect TFT used as an input.

8.4 Further directions

We conclude this outlook and thesis with some comments on further directions and open questions concerning the two main results of this thesis.

8.4.1 Modular functors

In Chapter 5 we start with a modular tensor category to construct a modular functor. It is natural to wonder if we can also go the other way around and start with a modular functor to obtain a modular tensor category. This question was already addressed in [BK], however there an issue with rigidity was not completely settled. In [BW] the question was addressed once more in a modern operadic setup and modular functors were finally classified by so-called ribbon Grothendieck

Verdier (GV) categories using factorisation homology. Ribbon GV-categories are similarly defined as ribbon tensor categories but come with a weaker notion of rigidity. Moreover, this weaker notion of rigidity is the natural duality structure on representation categories of VOAs [ALSW]. In [BW] it was also shown that Lyubashenko's modular functor, suitably interpreted in their framework, is the essentially unique modular functor that can be constructed from a modular tensor category [BW, Cor. 8.3].

The connection between this work and our constructions is a bit subtle as they use an operadic setup in contrast to our approach which is based on symmetric monoidal 2-categories. It is however expected that both approaches are equivalent via a 2-categorical version of the theory of monoidal envelopes, see [Ste] for recent lecture notes on the ∞-categorical version of this. Ignoring this subtlety for now, there is an even bigger problem: There is currently no TFT built from a ribbon GV-category! It would be interesting to investigate if it is possible to construct such a theory and, if this can be answered affirmatively, find an explicit construction.

In the operadic setting a string-net construction of Lyubashenko's modular functor for Drinfeld centres was given in [MSWY]. There it is also shown how to extend it to a full modular functor. In the semisimple setting it is known that the string-net and the surgery-TFT based full modular functor are isomorphic. This follows from the connection between string-net models and Turaev-Viro theories [Kir; Goo] combined with the connection between Turaev-Viro and Reshetikhin-Turaev theories [Bal; TV1; TV2]. It would be interesting to study these connections also in the non-semisimple setting and compare our modular functor to the string-net modular functor of [MSWY]. Turaev-Viro type theories have recently been generalised to the non-semisimple setting in [CGPV] using so-called chromatic maps, we are not aware of any comparison results between these and the 3d TFTs of [DGGPR1]. We expect the properties of the 3d TFT of [DGGPR1] we proved in Section 3.1 to be useful to study such a connection.

8.4.2 More CFT data: partition functions and OPE's

Building on our second main result, the construction of a full CFT, there are a few different directions which seem fruitful for further investigation.

First of all in Chapter 7 we computed some quantities of interest for the diagonal CFT. However, there are still quite a few interesting pieces missing notably the partition function, as well as boundary, bulk, and bulk-boundary OPEs. For all of these the resulting connecting bordisms are quite straightforward to obtain. However, computing the value of the TFT $\widehat{\mathrm{V}}_{\mathcal{C}}$ on these bordisms seems to be rather non-trivial in the non-semisimple setting. For example for the partition function, in the form of the torus correlator Cor_{T^2}, we have to compute $\widehat{\mathrm{V}}_{\mathcal{C}}(T^2 \times I)\colon \widehat{\mathrm{V}}_{\mathcal{C}}(\varnothing) \to \widehat{\mathrm{V}}_{\mathcal{C}}(T^2 \sqcup -T^2)$. In the semisimple setting one can compute the matrix coefficients of $\widehat{\mathrm{V}}_{\mathcal{C}}(T^2 \times I)$ by pairing it with the dual basis given by ribbons in two solid tori [FRSI, Sec. 5.3]. In the non-semisimple setting this approach becomes less straight-forward because $\widehat{\mathrm{V}}_{\mathcal{C}}$ is defined on a non-rigid category which results in not having a canonical basis for the dual state space.

For the OPE's we want to mention that the field content we obtained in Section 8.3 naturally carries the structure of symmetric Frobenius algebras in their respective categories. We expect to be able to recover this structure by computing the correlators for certain pair of pants world sheets. In particular, this should completetly determine the full CFT on genus zero. More generally, we expect to be able to prove the conjecture of [FS5] that the OPE's of the boundary and bulk fields are governed by their natural Frobenius algebra structures as inner Homs also beyond the diagonal case in the same way. In the semisimple setting the algebraic structures on boundary and bulk fields taking also higher genus contributions into account were classified using the notion of Cardy algebras [FFRS; KR; KLR]. In the non-semisimple setting a similar approach as in [KLR] was used in [FS2] to classify the algebraic structure of the bulk fields for a full CFT based on Lyubashenko's modular functor. It would be interesting to see if our construction can be used to combine these results to obtain a classification of full CFTs via a non-semisimple version of Cardy algebras.

Classification questions concerning full CFTs were also studied using the operadic approach in [Woi]. There the open sector of a full CFT for a modular functor coming from a GV-category was related to a GV version of symmetric Frobenius algebras. However, comparing these results to our construction outside the rigid setting would need the aforementioned TFT built from a ribbon GV-category making this question more out of reach for now.

8.4.3 Examples

Last but not least, it would be interesting to work out the details of our construction for specific examples of input modular tensor category $\mathcal{C}$. One starting point for this could be to compare our results to the ones of [FSSt1; FSSt2; FSSt3] in the setting where the modular tensor category is given as $\mathrm{Rep}(H)$, the representation category of some finite dimensional factorizable ribbon Hopf algebra H. In these papers the authors consider the bulk sector for the diagonal full CFT, as well as full CFTs obtained from twisting with a ribbon Hopf algebra automorphism. In particular, the bulk field content they obtain is isomorphic to the one we obtained in Section 7.1 by [FSSt1, App. A.1].

A concrete description of our construction in terms of Hopf algebraic data for the diagonal full CFT would also be useful to study examples from coming from the representation category of some VOA via some form of logarithmic Kazhdan-Lusztig correspondence [CLR; Len].

More concretely, it would be interesting to study the diagonal full CFT for the modular tensor categories coming from the so-called triplet models $\mathrm{Rep}(\mathcal{W}_p)$ (see e.g. [GN] and references therein) and the one of the even part of N-pairs of symplectic fermions $\mathcal{SF}(N, \beta)$ (see e.g. [DR; GR3] and references therein). The work of [FSSt1; FSSt2; FSSt3] is not quite sufficient to cover these examples as one needs to work with quasi-Hopf algebras instead. The reason for this is essentially because the naive Hopf algebras one would want to work with do not admit an R-matrix, see e.g. [Neg, Rem. 1.5]. For example the category $\mathrm{Rep}(\mathcal{W}_p)$ of the triplet is ribbon equivalent to the representation category of the small quantum group $u_q(\mathfrak{sl}_2)$ with $q = e^{i\pi/p}$ which only admits an R-matrix as a quasi-Hopf algebra [CGR; Neg; GN]. The bulk fields and partition function for the diagonal full CFT of the triplet model were constructed in [GR1; GR2] and it would be important to check if our construction reproduces their results.

The symplectic fermion category $\mathcal{SF}(N, \beta)$ also has a description description in terms of a quasi-Hopf algebra [GR3; FGR]. In particular for $p = 2$ and $N = 1$ the triplet and the symplectic fermion category are ribbon equivalent. The symplectic fermion category has the advantage that it always has four simple objects and thus also four indecomposable projectives, moreover many details of the 3d TFT have already been worked out in [BGR].

As far as we are aware, there are no known constructions for full CFTs based on either the triplet or the symplectic fermion VOA beyond the diagonal setting, see however [Flo] for possible modular invariant partition functions for the triplet. In our approach non-diagonal theories correspond to special symmetric Frobenius algebra or more generally exact pivotal module categories. In the Hopf algebra setting indecomposable exact module categories were classified in [AM] via certain left comodule algebras. This classification was further refined for pointed Hopf algebras and pointed quasi-Hopf algebras in [Mom; GM]. For $u_q(\mathfrak{sl}_2)$, with q an odd primitive root of unity, indecomposable exact module categories were also classified in [NSS]. However, none of these take pivotal structures on the module categories into account and we are not aware of any literature which does, see also [NSS, Sec. 5.7]. Moreover, to use the methods at our current disposal, we would first need to understand which of the module categories in the lists cited above come from special symmetric Frobenius algebras.

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